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Structure and Representations of Jordan Algebras

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Dedicated to
Adrian Albert

PREFACE

The purpose of this book is to give a comprehensive account of the structure and representation theory of Jordan algebras over a field of characteristic not two. It may be appropriate at this point to indicate the limits we have set ourselves and what lies immediately beyond these limits. In the first place, we note that a substantial part of the theory carries over to algebras over a commutative ring with 1 containing an element $\frac{1}{2}$ such that $\frac{1}{2}+\frac{1}{2}=1$. A reader who has need of a result of this generality will have no difficulty in ascertaining whether or not the corresponding result in the field case carries over.

A more serious and significant extension of the theory, which can now be made, is the passage from a linear theory to a quadratic one. In the case of a special Jordan algebra this amounts to the replacement of the Jordan product $a.b = \frac{1}{2}(ab+ba)$, which is bilinear, by the product aba, which is linear in b and quadratic in a. As the Jordan theory has developed it has become increasingly a quadratic theory based on the product $\{aba\} = bU_a = 2(b.a).a - b.a^{\cdot 2}$, where $a.b$ is the given bilinear product and $a^{\cdot 2} = a.a$. In a special Jordan algebra $\{aba\}$ coincides with aba. An important step in the transition to a quadratic theory was taken by the author in a paper which appeared in 1966 ([38] in the Bibliography), in which we gave an Artinian-like structure theory for Jordan algebras founded on axioms on quadratic ideals. These are defined by means of the composition $\{aba\}$. This theory is considered in detail in Chapter IV. The author's structure theory has been extended by McCrimmon in [6] and [14] to quadratic Jordan algebras over an arbitrary commutative ring with 1. A sketch of McCrimmon's theory can be found in our notes on "Further Results and Open Questions" at the end of this book.

A part of the Jordan theory can be founded also on the notion of inverses. This has been done in the book *Jordan Algebras* by Braun and Koecher and more recently by Springer in an unpublished paper which covers also the case of algebras of characteristic two. The connection between inverses and the quadratic composition aba in associative algebras is given by Hua's identity (p. 2). The drawback of a theory based on inverses is that one must be in a situation in which one has an ample supply of invertible elements. The theory seems to work best for finite-dimensional algebras with 1 over an infinite field. In this case the invertible elements form a Zariski open subset of the algebra. On the other hand, the theory does not apply very well to infinite-dimensional algebras or in arithmetic settings in which invertible elements may be scarce. For this reason we have preferred to base the theory on $a.b$ and $\{aba\}$.

A number of important applications of Jordan algebras have been found. The Jordan theory had its birth in an attempt by P. Jordan and subsequently by Jordan, von Neumann and Wigner to formulate the foundation of quantum mechanics in terms of the product $A.B = \frac{1}{2}(AB + BA)$ rather than the associative product AB. A very important area of applications of the Jordan theory, especially of exceptional Jordan algebras, is to exceptional Lie groups and algebras and related geometries. This is discussed in part in Chapter IX. Indications of additional results and the original papers containing these are given in the "Further Results etc." Another extremely interesting and promising area of applications is to real and complex analysis, particularly to homogeneous cones, Siegel half-spaces and automorphic functions. The most complete account of these extremely interesting results presently available is in Koecher's University of Minnesota mimeographed notes [4]. This aspect of the theory has been completely omitted in our discussion.

In the course of writing this book I have had invaluable help from Kevin McCrimmon—first, in carefully reading and criticising several versions of the manuscript, second, in communicating improved proofs of a number of important results and finally, in carefully reading the proofs. I am indebted also to Marshall Osborn for reading the proofs and to Michel Racine for help in compiling the Bibliography. Also I am greatly indebted to Jacques Tits who took time off from his own important researches on algebraic groups to derive, via the theory of algebraic groups, the elegant constructions of exceptional Jordan algebras which we have given in Chapter IX. I wish to record my sincere appreciation for all of these contributions.

The intention of writing this book had its inception with an invitation by the American Mathematical Society to give Colloquium Lectures at the 1955 summer meeting. The subject has changed enormously in the intervening years. Since there remain many natural open questions and possibilities for applications it is likely to change substantially also in the next decade.

Key West,
January 22, 1968

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CHAPTER I

FOUNDATIONS

In this book we shall be interested almost exclusively in (nonassociative) algebras over fields. We recall the definition: a vector space $\mathfrak{A}$ over a field Φ with a binary composition $(a, b) \rightarrow ab$ $(\in \mathfrak{A})$ satisfying

$$a(b+c) = ab+ac, \qquad (b+c)a = ba+ca, \tag{1}$$

$$\alpha(ab) = (\alpha a)b = a(\alpha b), \tag{2}$$

$a, b, c \in \mathfrak{A}$, $\alpha \in \Phi$. This notion is capable of generalization in two directions. In the first place one can consider algebras over commutative rings. Here Φ is a commutative ring with identity element 1 and $\mathfrak{A}$ is a (left) unital Φ-module with product ab satisfying (1) and (2). Algebras in this sense include nonassociative rings, which can be considered as algebras over the ring of integers. Another direction of generalization of the concept of an algebra is obtained by replacing the binary composition by one or more n-ary compositions where $n = 0, 1, 2, \cdots$. In particular, if we are given an algebra in the original sense we may occasionally be interested in other compositions (e.g., $a \rightarrow a^2$, $(a, b) \rightarrow ab - ba$, $(a, b) \rightarrow (ab)a$) which can be defined in terms of the given product ab. We may then study the "derived" algebras which are defined by means of these compositions.

As an illustration of this type of thing we consider a problem which arose in connection with a fundamental theorem in projective geometry (see *Artin's Geometric Algebra*, pp. 79–85 and 37–40). We are given two associative division rings Δ and Δ' and we wish to determine the mappings σ of Δ into Δ' which are additive, and preserve 1 and inverses. More precisely, we require

$$(a+b)^\sigma = a^\sigma + b^\sigma, \tag{3}$$

$$1^\sigma = 1, \tag{4}$$

$$\text{if} \quad a \neq 0 \quad \text{then} \quad a^\sigma \neq 0 \quad \text{and} \quad (a^{-1})^\sigma = (a^\sigma)^{-1}. \tag{5}$$

We claim that such a mapping necessarily satisfies

$$(aba)^\sigma = a^\sigma b^\sigma a^\sigma. \tag{6}$$

To see this we shall express aba in terms of sums, differences, and inverses. Let $a \neq 0$, $b \neq 0$, $a \neq b^{-1}$. Then

$$a^{-1}+(b^{-1}-a)^{-1}=a^{-1}[(b^{-1}-a)+a](b^{-1}-a)^{-1}=a^{-1}b^{-1}(b^{-1}-a)^{-1}.$$

Hence we have *Hua's identity*:

$$(a^{-1}+(b^{-1}-a)^{-1})^{-1}=(b^{-1}-a)ba=a-aba$$

or

$$a-(a^{-1}+(b^{-1}-a)^{-1})^{-1}=aba. \tag{7}$$

Clearly this relation and the conditions (3) and (5) imply (6) for $a \neq 0$, $b \neq 0$, $a \neq b^{-1}$. Evidently (6) holds if either $a=0$ or $b=0$, and if $a=b^{-1}$ then $aba = a$ and $a^\sigma=(b^\sigma)^{-1}=a^\sigma b^\sigma a^\sigma$ so $(aba)^\sigma=a^\sigma=a^\sigma b^\sigma a^\sigma$. We remark also that if we take $b=1$ in (6) and use (4) then we obtain $(a^2)^\sigma=(a^\sigma)^2$. Thus σ is a Jordan homomorphism of Δ into Δ' in the sense of the following

DEFINITION 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be associative rings. Then a mapping σ of $\mathfrak{A}$ into $\mathfrak{B}$ is called a* Jordan homomorphism *if*

$$(a+b)^\sigma = a^\sigma+b^\sigma, \tag{8}$$

$$(a^2)^\sigma = (a^\sigma)^2, \tag{9}$$

$$(aba)^\sigma = a^\sigma b^\sigma a^\sigma. \tag{10}$$

We shall now prove the following result which is due to Jacobson and Rickart: A Jordan homomorphism of a ring $\mathfrak{A}$ into any ring $\mathfrak{B}$ without zero divisors $\neq 0$ is either an associative homomorphism or an anti-homomorphism. We note first that $ab+ba=(a+b)^2-a^2-b^2$ and $abc+cba=(a+c)b(a+c) - aba - cbc$. Hence a Jordan homomorphism satisfies $(ab+ba)^\sigma= a^\sigma b^\sigma+b^\sigma a^\sigma$ and $(abc+cba)^\sigma=a^\sigma b^\sigma c^\sigma+c^\sigma b^\sigma a^\sigma$. Now consider the following relation:

$$\begin{aligned}&[(ab)^\sigma-a^\sigma b^\sigma][(ab)^\sigma-b^\sigma a^\sigma]\\ &= (ab)^\sigma(ab)^\sigma-a^\sigma b^\sigma(ab)^\sigma-(ab)^\sigma b^\sigma a^\sigma+a^\sigma(b^\sigma)^2a^\sigma\\ &= ((ab)^2)^\sigma-(ab(ab)+(ab)ba)^\sigma+(ab^2a)^\sigma\\ &= [(ab)^2-(ab)^2-ab^2a+ab^2a]^\sigma=0.\end{aligned}$$

Since $\mathfrak{B}$ has no zero divisors ($\neq 0$) we must have either $(ab)^\sigma=a^\sigma b^\sigma$ or $(ab)^\sigma=b^\sigma a^\sigma$ for the elements $a, b \in \mathfrak{A}$. We now proceed to show that σ is either a homomorphism or an anti-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, that is, either $(ab)^\sigma=a^\sigma b^\sigma$ for all a, b or $(ab)^\sigma=b^\sigma a^\sigma$ for all a, b. If this is not the case then we have elements a, b, c, d in $\mathfrak{A}$ such that $(ab)^\sigma=a^\sigma b^\sigma \neq b^\sigma a^\sigma$ and $(cd)^\sigma=d^\sigma c^\sigma \neq c^\sigma d^\sigma$. Let x be any element of $\mathfrak{A}$ and consider the element $(a(b+x))^\sigma$. This is either $a^\sigma(b+x)^\sigma$ or $(b+x)^\sigma a^\sigma$, so, by the additivity of σ and the distributive law, we have the following alternatives

$$(ab)^\sigma + (ax)^\sigma = \begin{cases} a^\sigma b^\sigma + a^\sigma x^\sigma \\ b^\sigma a^\sigma + x^\sigma a^\sigma. \end{cases}$$

The second of these and $(ax)^\sigma = x^\sigma a^\sigma$ gives $(ab)^\sigma = b^\sigma a^\sigma$ contrary to hypothesis. Hence either $(ax)^\sigma = a^\sigma x^\sigma$ or the first alternative holds, in which case, $(ab)^\sigma = a^\sigma b^\sigma$ gives $(ax)^\sigma = a^\sigma x^\sigma$. Hence in either case $(ax)^\sigma = a^\sigma x^\sigma$. In a similar fashion our hypotheses lead to three additional conclusions: $(xb)^\sigma = x^\sigma b^\sigma$, $(cx)^\sigma = x^\sigma c^\sigma$, $(xd)^\sigma = d^\sigma x^\sigma$ for all x in $\mathfrak{A}$. Now consider the element $((a + c)(b + d))^\sigma$ which is either $(a + c)^\sigma(b + d)^\sigma$ or $(b + d)^\sigma(a + c)^\sigma$. This gives the alternatives

$$(ab)^\sigma + (ad)^\sigma + (cb)^\sigma + (cd)^\sigma = \begin{cases} a^\sigma b^\sigma + a^\sigma d^\sigma + c^\sigma b^\sigma + c^\sigma d^\sigma \\ b^\sigma a^\sigma + d^\sigma a^\sigma + b^\sigma c^\sigma + d^\sigma c^\sigma. \end{cases}$$

Since $(ab)^\sigma = a^\sigma b^\sigma$, $(ad)^\sigma = a^\sigma d^\sigma$, $(cb)^\sigma = c^\sigma b^\sigma$ the first gives $(cd)^\sigma = c^\sigma d^\sigma$ contrary to hypothesis. Also $(cd)^\sigma = d^\sigma c^\sigma$, $(cb)^\sigma = b^\sigma c^\sigma$, $(ad)^\sigma = d^\sigma a^\sigma$ and the second alternative above give $(ab)^\sigma = b^\sigma a^\sigma$ again contrary to our hypotheses. Hence we see that σ is either a homomorphism or an antihomomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Since we have seen that an additive mapping of a division ring Δ into a division ring Δ' which preserves inverses is a Jordan homomorphism the result just proved implies Hua's theorem: If σ is an additive mapping of a division ring Δ into a division ring Δ' preserving inverses then σ is either a homomorphism or an antihomomorphism. We remark that *Hua's* result [1] and an earlier one due to *Ancochea* [1] led to an extensive investigation of Jordan homomorphisms of rings by *Kaplansky* [1], *Jacobson and Rickart* [1] and [2], *Herstein* [2], *Smiley* [4], *Martindale* [1]. We shall return to this subject in Chapters II and III.

1. **Special Jordan algebras and Jordan algebras.** If $\mathfrak{A}$ and $\mathfrak{B}$ are associative algebras over the commutative ring Φ (with 1) then the notion of a Jordan homomorphism σ of $\mathfrak{A}$ into $\mathfrak{B}$ is obtained by adding to the conditions (8), (9) and (10) of Definition 1 the condition $(\alpha a)^\sigma = \alpha a^\sigma$, $\alpha \in \Phi$. The notion of a Jordan homomorphism leads naturally to the introduction of new structures on the Φ-modules $\mathfrak{A}$ and $\mathfrak{B}$. These are given by the unary composition $a \to a^2$ and the binary composition $(a, b) \to aba$, and we define (tentatively) a *special Jordan algebra* to be a Φ submodule of an associative algebra $\mathfrak{A}$ over Φ which is closed under the compositions $a \to a^2$, $(a, b) \to aba$. A homomorphism σ of a special Jordan algebra $\mathfrak{J}$ over Φ into the special Jordan algebra $\mathfrak{K}$ over Φ is a Φ-module homomorphism of $\mathfrak{J}$ into $\mathfrak{K}$ such that $(a^2)^\sigma = (a^\sigma)^2$ and $(aba)^\sigma = a^\sigma b^\sigma a^\sigma$. Hence it is clear that a Jordan homomorphism of the associative algebra $\mathfrak{A}$ over Φ into $\mathfrak{B}$ over Φ is just a homomorphism of the special Jordan algebra $\mathfrak{A}$ (with the compositions a^2 and aba) into the special Jordan algebra $\mathfrak{B}$.

If $\mathfrak{A}$ is an associative algebra, $\mathfrak{A}$ gives rise to a special Jordan algebra by replacing the composition ab by the two compositions a^2 and aba. Another important class of special Jordan algebras is obtained as follows. Let $\mathfrak{A}$ be an associative algebra which possesses an involution J, that is, a Φ-endomorphism of $\mathfrak{A}$ such that $(ab)^J = b^J a^J$ and $a^{J^2} = a$. Let $\mathfrak{H}(\mathfrak{A}, J)$ denote the subset of $\mathfrak{A}$ of elements which are J-symmetric in the sense that $a^J = a$. It is clear that $\mathfrak{H}(\mathfrak{A}, J)$ is a Φ-submodule of $\mathfrak{A}$ and if $a, b \in \mathfrak{H}(\mathfrak{A}, J)$ then $(a^2)^J = (a^J)^2 = a^2$ and $(aba)^J = a^J b^J a^J = aba$ so $\mathfrak{H}(\mathfrak{A}, J)$ is a special Jordan algebra. It is interesting to study also the homomorphisms of these special Jordan algebras.

While it has just recently become possible to develop a structure theory of Jordan algebras for any characteristic based on the composition $(a, b) \to aba$ (see McCrimmon [6]), in this book we shall confine our attention to algebras over fields of characteristic not two. Unless otherwise stated *we make this restriction from now on.* Let $\mathfrak{A}$ be an associative algebra over a field Φ of characteristic $\neq 2$ and let $\mathfrak{J}$ be a special Jordan algebra defined by $\mathfrak{A}$ in the sense that $\mathfrak{J}$ is a subspace of $\mathfrak{A}$ closed under the compositions a^2 and aba. We have $\frac{1}{2}(ab + ba) = \frac{1}{2}(a + b)^2 - \frac{1}{2}a^2 - \frac{1}{2}b^2$ so it is clear that if $a, b \in \mathfrak{J}$ then

$$a \,.\, b = \tfrac{1}{2}(ab + ba) \in \mathfrak{J}. \tag{11}$$

Conversely, let $\mathfrak{J}$ be any subspace of $\mathfrak{A}$ which is closed under the composition $a \,.\, b$. Then if $a, b \in \mathfrak{J}$, $a^2 = a \,.\, a \in \mathfrak{J}$ and $aba = 2(b \,.\, a) \,.\, a - b \,.\, a^2 \in \mathfrak{J}$. It is therefore clear that our original definition of a special Jordan algebra (for Φ a field of characteristic $\neq 2$) is equivalent to the following

DEFINITION 2. *A* special Jordan algebra *is a subspace $\mathfrak{J}$ of an associative algebra over a field Φ of characteristic $\neq 2$ which is closed under the composition $a \,.\, b = \frac{1}{2}(ab + ba)$ where ab denotes the product in the associative algebra.*

There is a slightly different way of looking at this definition. Let $\mathfrak{A}$ be an arbitrary (nonassociative) algebra over a field Φ of characteristic $\neq 2$ with product composition denoted by ab. Set $a \,.\, b = \frac{1}{2}(ab + ba)$. Then it is clear from (1) and (2) that

$$a \,.\, (b + c) = a \,.\, b + a \,.\, c, \qquad (b + c) \,.\, a = b \,.\, a + c \,.\, a,$$

$$\alpha(a \,.\, b) = \alpha a \,.\, b = a \,.\, \alpha b.$$

Hence if we replace the given product in $\mathfrak{A}$ by the "derived" product $a \,.\, b$ then we obtain another algebra $\mathfrak{A}^+$. If the original algebra is associative then we call $a \,.\, b$ the *Jordan product* of a and b and $\mathfrak{A}^+$ and its subalgebras are special Jordan algebras. In fact, it is clear from Definition 2 that the special Jordan algebras are nothing more nor less than the subalgebras of the algebras $\mathfrak{A}^+$, $\mathfrak{A}$ associative.

There is another algebra which one can associate with a given algebra $\mathfrak{A}$. This is the algebra $\mathfrak{A}^-$ with product $[ab] = ab - ba$ in terms of the given product ab. If $\mathfrak{A}$ is associative then $\mathfrak{A}^-$ and its subalgebra are *Lie algebras* in the sense that the *Lie product* $[ab]$ satisfies the two identities

$$[aa] = 0, \tag{12}$$

$$[[ab]c] + [[bc]a] + [[ca]b] = 0. \tag{13}$$

It is a basic result in the theory of Lie algebras (a consequence of the Poincaré-Birkhoff-Witt theorem) that conversely every Lie algebra, that is, every algebra with product composition $[ab]$ satisfying (12) and (13), is isomorphic to a subalgebra of an algebra $\mathfrak{A}^-$, $\mathfrak{A}$ associative. Thus we have a characterization by identities of the class of algebras which are isomorphic to subalgebras of algebras $\mathfrak{A}^-$, $\mathfrak{A}$ associative.

It is natural to seek a similar characterization of the class of algebras isomorphic to special Jordan algebras. However, this turns out to be impossible, since, as we shall see in §3, there exist special Jordan algebras having homomorphic images which are not special. Now, it is clear that if $\mathscr{V}$ is a class of algebras defined by a set of identities then $\mathscr{V}$ is closed under homomorphic images, that is, if $\mathfrak{A} \in \mathscr{V}$ and $\bar{\mathfrak{A}}$ is a homomorphic image of $\mathfrak{A}$ then necessarily $\bar{\mathfrak{A}} \in \mathscr{V}$. Hence it is clear that one cannot characterize the class of special Jordan algebras by identities. On the other hand, it follows from a general theorem due to G. Birkhoff (P. Cohn, *Universal Algebra*, p. 169) that it is possible to characterize by identities the class of homomorphic images of special Jordan algebras. No such characterization has as yet been found, and results which we shall give later indicate that such a characterization is likely to be complicated. In lieu of this it is natural to determine some simple identities which hold for the product $a.b$ in any special Jordan algebra and to use these to define a larger class of algebras including the special Jordan algebras and their homomorphic images.

We therefore consider the problem of deriving identities for the composition $a.b$ in an associative algebra and we consider first one variable identities. We note that $a.a = \frac{1}{2}(a^2 + a^2) = a^2$ and by induction one sees that if $a^{\cdot k}$ is defined by $a^{\cdot 1} = a$, $a^{\cdot k} = a^{\cdot k-1}.a$ then $a^{\cdot k} = a^k$. Hence we have the rule: $a^{\cdot k}.a^{\cdot l} = a^{\cdot k+l}$, which is called *power associativity*. Next we seek two variable identities. One of these, the commutative law, $a.b = b.a$ is clear. We consider next the possibilities for two variable identities of degree one in a and degree two in b. In view of commutativity the monomials we have to consider are $(a.b).b$ and $a.b^{\cdot 2}$. We have $4(a.b).b = ab^2 + b^2a + 2bab$ and $2a.b^{\cdot 2} = ab^2 + b^2a$. Since there exist associative algebras in which $ab^2 + b^2a$ and bab are linearly independent no identities of the type desired exist. Next we consider the possibilities for identities of degree three in a and of degree 1 in b. Here the possible monomials are $a^{\cdot 3}.b$, $(a^{\cdot 2}.b).a$, $(a.b).a^{\cdot 2}$, $((a.b).a).a$. We have

$$\begin{aligned} 2a^{\cdot 3}.b &= a^3b + ba^3, \\ 4(a^{\cdot 2}.b).a &= a^3b + ba^3 + a^2ba + aba^2, \\ 4(a.b).a^{\cdot 2} &= aba^2 + a^2ba + a^3b + ba^3, \\ 8((a.b).a).a &= 3(a^2ba + aba^2) + a^3b + ba^3. \end{aligned}$$

This leads to two identities:

$$(a^{\cdot 2} . b) . a = a^{\cdot 2} . (b . a),$$

$$a^{\cdot 3} . b + 2((a . b) . a) . a = 3(a^{\cdot 2} . b) . a.$$

In a similar manner one can obtain the following identity in two a's and two b's

$$a^{\cdot 2} . b^{\cdot 2} - (a . b^{\cdot 2}) . a = 2((a . b) . a) . b - 2(a . b)^{\cdot 2}.$$

It can be seen (cf. §7) that all the identities we have indicated thus far are consequences of the two identities

(i) $$a . b = b . a,$$

(ii) $$(a^{\cdot 2} . b) . a = a^{\cdot 2} . (b . a).$$

Also there are no cubic identities which are of first degree in three variables which are not consequences of commutativity, (ex. 1 below). These considerations make it natural to consider the class of algebras defined by (i) and (ii) as a class which is simply defined and encompasses the class of special Jordan algebras. Accordingly, we give the following

DEFINITION 3. *A* Jordan algebra $\mathfrak{J}$ *is an algebra over a field* Φ *of characteristic* $\neq 2$ *with a product composition denoted as* $a . b$ *satisfying* (i) *and* (ii) *where* $a^{\cdot 2} = a . a$.

It is now natural to modify the earlier definition of a special Jordan algebra to the following

DEFINITION 2'. *A Jordan algebra* $\mathfrak{J}$ *is called* special *if there exists a monomorphism* σ *of* $\mathfrak{J}$ *into an algebra* $\mathfrak{A}^+$, $\mathfrak{A}$ *associative. Jordan algebras which are not special will be called* exceptional.

EXERCISES

1. Show that there are no identities involving only $(a . b) . c$, $(b . c) . a$ and $(c . a) . b$ which are valid in all special Jordan algebras.

2. Verify that the identity in two a's and two b's indicated above holds in any special Jordan algebra.

2. **Free special Jordan algebras. Cohn's theorem.** Before embarking on the abstract theory of Jordan algebras based on Definition 3 we consider again the special ones. For simplicity we assume that $\mathfrak{A}$ is an associative algebra with an identity element 1 and that $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}^+$ containing 1. Observe that since $1 . a = \frac{1}{2}(a + a) = a$, 1 is the identity also in $\mathfrak{J}$. We have seen that the closure of $\mathfrak{J}$ relative to $(a, b) \rightarrow a . b = \frac{1}{2}(ab + ba)$ implies that $\mathfrak{J}$ is closed also under $(a, b, c) \rightarrow \{abc\} \equiv \frac{1}{2}(abc + cba)$ (p. 4). Also we have seen that $a^{\cdot k} = a^k$ so $\mathfrak{J}$ is

closed under the unary composition $a \to a^k$, $k = 1, 2, \cdots$. We have $[[ab]c] = abc - bac - cab + cba = 2\{abc\} - 2\{bac\}$ so $[[a, b]c] \in \mathfrak{J}$ if $a, b, c \in \mathfrak{J}$. Also $[ab]^2 = (ab - ba)^2 = abab + baba - ab^2a - ba^2b = 2a.\{bab\} - \{ab^2a\} - \{ba^2b\} \in \mathfrak{J}$ if $a, b \in \mathfrak{J}$. The compositions given by $\{abc\}$, $[[ab]c]$, $[ab]^2$, a^k are examples of compositions under which $\mathfrak{J}$ is closed. We now consider the problem of determining all such compositions and expressing these in terms of the given compositions in the associative algebra $\mathfrak{A}$. This leads us to consider the free associative algebra with 1, $\Phi\{x_1, x_2, \cdots, x_r\}$, with r (free) generators $x_1, x_2, \cdots, x_r$, $r = 1, 2, \cdots$. We recall the definition of this object. We first define the free monoid (semigroup) with 1, $M(X)$, defined by $X = \{x_1, x_2, \cdots, x_r\}$. This is the set consisting of 1 and the (associative) "words" $x_{i_1}x_{i_2}\cdots x_{i_k}$, $i_j = 1, 2, \cdots, r$. We define multiplication in $M(X)$ by juxtaposition:

$$(x_{i_1}\cdots x_{i_k})(x_{j_1}\cdots x_{j_l}) = x_{i_1}\cdots x_{i_k}\, x_{j_1}\cdots x_{j_l} \quad \text{and} \quad 1w = w1 = w$$

for all $w \in M(X)$. We now define the free associative algebra with 1, $\Phi\{x_1, x_2, \cdots, x_r\}$, to be the vector space with basis $M(X)$ with multiplication: $(\sum \alpha_i w_i)(\sum \beta_j w_j) = \sum \alpha_i \beta_j w_i w_j$ for $\alpha_i, \beta_j \in \Phi$, $w_i, w_j \in M(X)$. Clearly this is an associative algebra. We now consider the corresponding Jordan algebra $\Phi\{x_1, x_2, \cdots, x_r\}^+$, and we define the *free special Jordan algebra with* 1 *on r (free) generators* x_i, $i = 1, \cdots, r$, to be the subalgebra $FSJ^{(r)}$ of $\Phi\{x_1, x_2, \cdots, x_r\}^+$ generated by 1 and the x_i. We shall also call the elements of $FSJ^{(r)}$ the *Jordan elements* (or polynomials) of $\Phi\{x_1, x_2, \cdots, x_r\}$.

The justification of the term "free" in the foregoing definition is the following fact. If $\mathfrak{A}$ is associative with 1 and $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}^+$ containing 1 and $y_1, y_2, \cdots, y_r$ are any elements of $\mathfrak{J}$ then there exists a unique homomorphism of $FSJ^{(r)}$ into $\mathfrak{J}$ sending $1 \to 1$, $x_i \to y_i$, $i = 1, \cdots, r$. To see this we recall that there exists a homomorphism η of $\Phi\{x_1, x_2, \cdots, x_r\}$ into $\mathfrak{A}$ such that $1 \to 1$, $x_i \to y_i$, $i = 1, \cdots, r$. Clearly η induces a homomorphism of $\Phi\{x_1, x_2, \cdots, x_r\}^+$ into $\mathfrak{A}^+$. Since the $y_i \in \mathfrak{J}$ the restriction of the homomorphism η of $\Phi\{x_1, \cdots, x_r\}^+$ to $FSJ^{(r)}$ is a homomorphism of $FSJ^{(r)}$ into $\mathfrak{J}$ such that $1 \to 1$, $x_i \to y_i$. Since 1 and the x_i generate $FSJ^{(r)}$ uniqueness of the homomorphism satisfying the stated conditions is clear.

If $p(x_1, \cdots, x_r)$ is any element of $\Phi\{x_1, \cdots, x_r\}$ then we denote the image $p(x_1, \cdots, x_r)^\eta$ in $\mathfrak{A}$ by $p(y_1, \cdots, y_r)$. If p is a Jordan element, that is, $p \in FSJ^{(r)}$ then $p(y_1, \cdots, y_r) \in \mathfrak{J}$. Hence the elements p of $FSJ^{(r)}$ define derived compositions under which $\mathfrak{J}$ is closed. It is clear that this gives all such derived compositions for the special Jordan algebras.

The free associative algebra $\Phi\{x_1, x_2, \cdots, x_r\}$ is a graded algebra in which the subspace of degree k, $k = 0, 1, 2, \cdots$, is the space of homogeneous elements of degree k, that is, the linear combinations of the monomials $x_{i_1}\cdots x_{i_k}$. If a and b are homogeneous Jordan elements then $a.b$ is a homogeneous Jordan element.

Hence the set of sums of homogeneous Jordan elements is a subalgebra of $FSJ^{(r)}$. Since this subalgebra contains 1 and the x_i it is clear that it coincides with $FSJ^{(r)}$. This shows that it suffices to determine the spaces $FSJ^{(r,k)}$ of homogeneous elements of degree k of $FSJ^{(r)}$, $k = 0, 1, 2, \cdots$. Another interesting problem is that of determining the dimensionality $d(r,k)$ of $FSJ^{(r,k)}$. This is finite since the space of homogeneous elements of degree k in $\Phi\{x_1, \cdots, x_r\}$ has dimensionality r^k.

The problems we have just indicated have been settled if $r \leqq 3$ by Cohn [1] who has also obtained an interesting partial result for $r > 3$. For this we introduce the *reversal operation* in $\Phi\{x_1, x_2, \cdots, x_r\}$, which is the linear mapping $a \to a^*$ in $\Phi\{x_1, x_2, \cdots, x_r\}$ such that $1^* = 1$ and $(x_{i_1} x_{i_2} \cdots x_{i_k})^* = x_{i_k} x_{i_{k-1}} \cdots x_{i_1}$. If a and b are monomials then one checks that $(ab)^* = b^* a^*$. Hence $a \to a^*$ is an involution in $\Phi\{x_1, x_2, \cdots, x_r\}$. Let $\mathfrak{H}$ be the set of *reversible* elements, that is, the set of *-symmetric elements. Then $\mathfrak{H}$ is a subalgebra of $\Phi\{x_1, \cdots, x_r\}^+$ containing 1 and the x_i. Hence $\mathfrak{H} \supseteq FSJ^{(r)}$. If $r \geqq 4$ then $\mathfrak{H} \supset FSJ^{(r)}$. In fact, the element $x_1x_2x_3x_4 + x_4x_3x_2x_1$ which is evidently in $\mathfrak{H}$ does not belong to $FSJ^{(r)}$. To see this it suffices to give a special Jordan algebra $\mathfrak{J}$ containing elements e_1, e_2, e_3, e_4 and not containing $e_1e_2e_3e_4 + e_4e_3e_2e_1$ where the product is the associative one. We now take $\mathfrak{A}$ to be the exterior algebra with canonical generators e_1, e_2, e_3, e_4. Then $\mathfrak{A}$ is generated by these elements and we have the relations $e_i^2 = 0$, $e_ie_j = -e_je_i$. It is well known that the 2^4 elements $e_1^{k_1}e_2^{k_2}e_3^{k_3}e_4^{k_4}$, $k_i = 0, 1$, form a basis for $\mathfrak{A}$. Let $\mathfrak{J}$ be the subspace of elements $\alpha 1 + \sum_1^4 \alpha_i e_i$ in $\mathfrak{A}$. Then $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}^+$ and $2e_1e_2e_3e_4 = e_1e_2e_3e_4 + e_4e_3e_2e_1 \notin \mathfrak{J}$. Hence $\mathfrak{J}$ has the desired properties.

If $a_1, a_2, \cdots, a_k$ are elements of an associative algebra then we shall now write $\{a_1a_2 \cdots a_k\} = \frac{1}{2}(a_1a_2 \cdots a_k + a_ka_{k-1} \cdots a_1)$. Then the elements $\{x_{i_1} \cdots x_{i_k}\} = \frac{1}{2}x_{i_1} \cdots x_{i_k} + \frac{1}{2}(x_{i_1} \cdots x_{i_k})^*$ span the vector space $\mathfrak{H}$ of reversible elements of $\Phi\{x_1, \cdots, x_r\}$. The elements $\{x_{i_1}x_{i_2}x_{i_3}x_{i_4}\}$ with $i_1 < i_2 < i_3 < i_4$ will be called *tetrads*. We can now state

COHN'S THEOREM. *The space $\mathfrak{H}$ of reversible elements of $\Phi\{x_1, x_2, \cdots, x_r\}$ coincides with the subalgebra $\mathfrak{H}'$ of $\Phi\{x_1, x_2, \cdots, x_r\}^+$ generated by* 1, *the x_i and all the tetrads.*

PROOF. Clearly $\mathfrak{H}' \subseteq \mathfrak{H}$. To show the reverse inequality it is enough to show that every element $\{x_{i_1}x_{i_2} \cdots x_{i_k}\} \in \mathfrak{H}'$. For $k = 1, 2$ this is clear and we may assume it for $k < n$. If $1 \leqq m < n$ we have

$$\begin{aligned} 4\ &\{x_{i_1}x_{i_2} \cdots x_{i_m}\} . \{x_{i_{m+1}} \cdots x_{i_n}\} \\ &= \{x_{i_1} \cdots x_{i_n}\} + \{x_{i_1} \cdots x_{i_m}x_{i_n} \cdots x_{i_{m+1}}\} \\ &\quad + \{x_{i_m} \cdots x_{i_1}x_{i_{m+1}} \cdots x_{i_n}\} + \{x_{i_m} \cdots x_{i_1}x_{i_n} \cdots x_{i_{m+1}}\}. \end{aligned}$$

Hence, by the induction assumption,

$$\begin{aligned}(14)\qquad &\{x_{i_1}\cdots x_{i_n}\} + \{x_{i_1}\cdots x_{i_m}x_{i}\ \cdots x_{i_{m+1}}\}\\ &+ \{x_{i_m}\cdots x_{i_1}x_{i_{m+1}}\cdots x_{i_n}\} + \{x_{i_m}\cdots x_{i_1}x_{i_n}\cdots x_{i_{m+1}}\}\equiv 0 \pmod{\mathfrak{H}'}.\end{aligned}$$

If we take $m = 1$ in this we obtain

$$(15)\qquad \{x_{i_1}\cdots x_{i_n}\}\equiv -\{x_{i_2}\cdots x_{i_n}x_{i_1}\} \pmod{\mathfrak{H}'}.$$

If n is odd we may iterate this to obtain $\{x_{i_1}\cdots x_{i_n}\}\equiv -\{x_{i_1}\cdots x_{i_n}\} \pmod{\mathfrak{H}'}$. Hence $\{x_{i_1}\cdots x_{i_n}\}\in\mathfrak{H}'$. Hence we may assume n is even and > 2. Then (14) with $m = 2$ and (15) imply that

$$\{x_{i_1}\cdots x_{i_n}\}\equiv -\{x_{i_2}x_{i_1}x_{i_3}\cdots x_{i_n}\} \pmod{\mathfrak{H}'}.$$

Now the cycle $(1\,2\cdots n)$ and the transposition (12) generate the symmetric group. Hence if n is even then the above equations imply that

$$(16)\qquad \{x_{i_1}\cdots x_{i_n}\}\equiv \pm\{x_{i_{1'}}\cdots x_{i_{n'}}\} \pmod{\mathfrak{H}'}$$

for any permutation $1', 2', \cdots, n'$ of $1, 2, \cdots, n$ where the sign is $+$ or $-$ according as the permutation is even or odd. It now follows that if n is even and $i_j = i_k$ for some $j \neq k$ then $\{x_{i_1}\cdots x_{i_n}\}\in\mathfrak{H}'$. We may therefore assume the i_j distinct. If $n = 4$ we have a permutation $1', 2', 3', 4'$ of $1, 2, 3, 4$ so that $i_{1'} < i_{2'} < i_{3'} < i_{4'}$. Then $\{x_{i_{1'}}x_{i_{2'}}x_{i_{3'}}x_{i_{4'}}\}$ is a tetrad and so is contained in $\mathfrak{H}'$. Then (16) implies that $\{x_{i_1}x_{i_2}x_{i_3}x_{i_4}\}\in\mathfrak{H}'$. Now let n be even $\geqq 6$ and consider (14) for $m = 4$. This and (16) imply that $\{x_{i_1}\cdots x_{i_n}\}\in\mathfrak{H}'$.

Evidently if $r < 4$ then there are no tetrads. Hence we have the following important

COROLLARY. *If $r \leqq 3$ then the free special Jordan algebra $FSJ^{(r)}$ coincides with the space $\mathfrak{H}$ of reversible elements of the free associative algebra $\Phi\{x_1, x_2, \cdots, x_r\}$.*

EXERCISES

1. Call an element of $\Phi\{x_1, \cdots, x_r\}$ a *Lie element* if it is contained in the subalgebra of the Lie algebra $\Phi\{x_1, \cdots, x_r\}^-$ generated by 1 and the x_i. Show that every homogeneous Lie element of odd degree is Jordan.

2. Let Φ be of characteristic two. Show that the space $\mathfrak{H}$ of reversible elements of $\Phi\{x_1, x_2, \cdots, x_r\}$ coincides with the smallest subspace $\mathfrak{H}'$ of $\Phi\{x_1, x_2, \cdots, x_r\}$ containing 1 and all the *n-tads* $\{x_{i_1}\cdots x_{i_n}\} = x_{i_1}\cdots x_{i_n} + x_{i_n}\cdots x_{i_1}$, $i_1 < i_2 < \cdots < i_n$ for $n \leqq r$ and closed under the composition aba.

3. Consider the free special Jordan algebra $FSJ^{(2)}$ generated by x, y and set $y_i = xy^ix$, $i = 1, 2, \cdots, r$. Note that $y_i \in FSJ^{(2)}$. Show that the canonical homomorphism of $FSJ^{(r)}$ into $FSJ^{(2)}$ such that $1 \to 1$, $x_i \to y_i$, $i = 1, \cdots, r$ is an

isomorphism. Hence $FSJ^{(2)}$ contains a subalgebra isomorphic to $FSJ^{(r)}$ for any $r = 1, 2, \cdots$.

4. Show that $\{x_1x_2x_3x_4\} \notin FSJ^{(4)}$ by noting that $\sum_{\pi \in S_4} (\operatorname{sign} \pi) \{x_{1\pi}x_{2\pi}x_{3\pi}x_{4\pi}\} \neq 0$ where S_4 is the symmetric group on $1, 2, 3, 4$. (Show that the same operator applied to a Jordan element gives 0.)

5. Let $\Phi\{x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}\}$ be the group algebra of the free group generated by $x_1, \cdots, x_r$. Define the reversal anti-automorphism as in $\Phi\{x_1, \cdots, x_r\}$. Show that the space $\mathfrak{H}$ of reversible elements coincides with the subalgebra of $\Phi\{x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}\}^+$ generated by $x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}$ and all the *tetrads* $\{x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}x_{i_3}^{\varepsilon_3}x_{i_4}^{\varepsilon_4}\}$ where $i_1 < i_2 < i_3 < i_4$ and $\varepsilon_j = \pm 1$.

6. (Faulkner). Let Φ be of characteristic $p \neq 0, 2$. Show that one has an identity of the form $[x_1x_2]^p = \sum [f_i(x_1, x_2), g_i(x_1, x_2)]$ in $\Phi\{x_1, x_2\}$ where f_i, g_i are Jordan elements in $\Phi\{x_1, x_2\}$.

3. **Homomorphic images of special Jordan algebras.** Suppose $\mathfrak{J}$ is a special Jordan algebra and has an identity element 1: $1 . a = a$, $a \in \mathfrak{J}$. Let σ be a monomorphism of $\mathfrak{J}$ into $\mathfrak{A}^+$, $\mathfrak{A}$ associative. Then $e = 1^\sigma$ satisfies $e^2 = e^{\cdot 2} = 1^\sigma . 1^\sigma = (1.1)^\sigma = 1^\sigma = e$; so e is an idempotent element in $\mathfrak{A}$. Clearly e acts as identity in the subalgebra $e\mathfrak{A}e$ of $\mathfrak{A}$. Now let $a \in \mathfrak{J}$ and consider the relation $a = 2(a . 1).1 - a .1$. Applying σ we obtain $a^\sigma = ea^\sigma e$. Hence the image $\mathfrak{J}^\sigma \subseteq e\mathfrak{A}e$. Since we can replace $\mathfrak{A}$ by $e\mathfrak{A}e$, clearly there is no loss in generality in assuming that 1^σ is an identity element for the associative algebra $\mathfrak{A}$. We shall do this from now on without further mention.

We shall now apply Cohn's Theorem to derive some important results on homomorphic images of special Jordan algebras (Cohn [1]). We prove first the following

LEMMA 1. *If $\mathfrak{K}$ is an ideal in $FSJ^{(r)}$ then $FSJ^{(r)}/\mathfrak{K}$ is special if and only if $\mathfrak{B} \cap FSJ^{(r)} = \mathfrak{K}$ where $\mathfrak{B}$ is the ideal in $\Phi\{x_1, \cdots, x_r\}$ generated by $\mathfrak{K}$.*

PROOF. Suppose first that $FSJ^{(r)}/\mathfrak{K}$ is special so that we have the monomorphism σ of $FSJ^{(r)}/\mathfrak{K}$ into $\mathfrak{A}^+$ where $\mathfrak{A}$ is associative with the identity element $(1 + \mathfrak{K})^\sigma$. If ν is the canonical homomorphism of $FSJ^{(r)}$ into $FSJ^{(r)}/\mathfrak{K}$ then $\tau = \nu\sigma$ is a homomorphism of $FSJ^{(r)}$ into $\mathfrak{A}^+$ with kernel $\mathfrak{K}$ sending 1 into 1. On the other hand, we have a homomorphism τ' of $\Phi\{x_1, x_2, \cdots, x_r\}$ into $\mathfrak{A}$ such that $1^{\tau'} = 1^\tau$ and $x_i^{\tau'} = x_i^\tau$. It is clear that the restriction of τ' to $FSJ^{(r)}$ is τ. Hence if $\mathfrak{D}$ is the kernel of τ' then $\mathfrak{D} \cap FSJ^{(r)} = \mathfrak{K}$ the kernel of τ. If $\mathfrak{B}$ is the ideal in $\Phi\{x_1, \cdots, x_r\}$ generated by $\mathfrak{K}$ then $\mathfrak{B} \subseteq \mathfrak{D}$. Hence $\mathfrak{B} \cap FSJ^{(r)} \subseteq \mathfrak{K}$. Since $\mathfrak{B} \cap FSJ^{(r)} \supseteq \mathfrak{K}$ is clear, we have $\mathfrak{B} \cap FSJ^{(r)} = \mathfrak{K}$.

Conversely, assume $\mathfrak{B} \cap FSJ^{(r)} = \mathfrak{K}$. Consider the Jordan algebra

$$(\Phi\{x_1, \cdots, x_r\} / \mathfrak{B})^+$$

and its subalgebra generated by $1 + \mathfrak{B}$ and $x_i + \mathfrak{B}$, $i = 1, \cdots, r$. This is special and isomorphic to $(FSJ^{(r)}+\mathfrak{B})/\mathfrak{B} = FSJ^{(r)}/(\mathfrak{B} \cap FSJ^{(r)}) = FSJ^{(r)}/\mathfrak{K}$. Hence $FSJ^{(r)}/\mathfrak{K}$ is special.

THEOREM 1. *Any homomorphic image of* $FSJ^{(2)}$ *is special.*

PROOF. We denote the generators by x, y rather than x_1, x_2. Let $\mathfrak{K}$ be an ideal in $FSJ^{(2)}$ and $\mathfrak{B}$ the ideal in $\Phi\{x, y\}$ generated by $\mathfrak{K}$. Let $\mathfrak{B}'$ be the set of linear combinations of elements which are products of x's and y's and of one element $k \in \mathfrak{K}$ (e.g., $x^2ky^3x^3y$). Then $\mathfrak{K} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$ and $\mathfrak{B}'$ is an ideal. Hence $\mathfrak{B}' = \mathfrak{B}$. Now suppose $u \in \mathfrak{B} \cap FSJ^{(2)}$. Then $u = \frac{1}{2}(u + u^*)$ since $FSJ^{(2)} \subseteq \mathfrak{H}$. Hence u is a linear combination of elements $v = \frac{1}{2}(m + m^*)$ where m is a product of x's and y's and a single $k \in \mathfrak{K}$. Now consider the free associative algebra $\Phi\{x, y, z\}$ with (free) generators x, y, z and the homomorphsm of this into $\Phi\{x, y\}$ such that $1 \to 1$, $x \to x$, $y \to y$, $z \to k$. Since $k^* = k$ it is clear that v is the image of an element $V = \frac{1}{2}(M + M^*)$ where M is a monomial in x, y, z which is of degree 1 in z. By the corollary to Cohn's Theorem, $V \in FSJ^{(3)}$. Moreover, since V is homogeneous of degree 1 in z it is clear that V is a linear combination of Jordan products of x's, y's and a single z. Hence v is a linear combination of Jordan products of x's, y's and k and so v is in the ideal in $FSJ^{(2)}$ generated by k. It follows now that $FSJ^{(2)} \cap \mathfrak{B} \subseteq \mathfrak{K}$. Since the reverse inequality is clear we have $FSJ^{(2)} \cap \mathfrak{B} = \mathfrak{K}$ and $FSJ^{(2)}/\mathfrak{K}$ is special.

We shall next use the criterion given in Lemma 1 to construct an example of a homomorphic image of a special Jordan algebra which is not special; namely, we have

THEOREM 2. *Let* $FSJ^{(3)}$ *be the free special Jordan algebra in three generators,* x, y, z, *and let* $\mathfrak{K}$ *be the ideal in* $FSJ^{(3)}$ *generated by* $k = x^{\cdot 2} - y^{\cdot 2}$. *Then* $FSJ^{(3)}/\mathfrak{K}$ *is exceptional.*

PROOF. The element $v = \{kxyz\} \in \mathfrak{B} \cap \mathfrak{H} = \mathfrak{B} \cap FSJ^{(3)}$. Hence our result will follow from Lemma 1 if we can show that $v \notin \mathfrak{K}$. Now assume $v \in \mathfrak{K}$. Then there exists a Jordan element $f(x, y, z, t)$ in the free special Jordan algebra $FSJ^{(4)}$ in four free generators x, y, z, t linear in t such that $v = f(x, y, z, x^2 - y^2)$. Since $FSJ^{(4)} \subseteq \mathfrak{H}$ it is clear from degree considerations and the form of v that we may assume that the Jordan element f is a linear combination of $\{txyz\}$, $\{xtyz\}$, $\{tyxz\}$, $\{ytxz\}$, $\{xytz\}$, $\{yxtz\}$, say, $f = \alpha_1\{txyz\} + \alpha_2\{xtyz\} + \alpha_3\{tyxz\} + \alpha_4\{ytxz\} + \alpha_5\{xytz\} + \alpha_6\{yxtz\}$, $\alpha_i \in \Phi$. Replacing t by $x^2 - y^2$ gives the relation

$$\begin{aligned}\{x^3yz\} - \{y^2xyz\} = {} & \alpha_1\{x^3yz\} - \alpha_1\{y^2xyz\} \\ & + \alpha_2\{x^3yz\} - \alpha_2\{xy^3z\} + \alpha_3\{x^2yxz\} - \alpha_3\{y^3xz\} \\ & + \alpha_4\{yx^3z\} - \alpha_4\{y^3xz\} + \alpha_5\{xyx^2z\} - \alpha_5\{xy^3z\} \\ & + \alpha_6\{yx^3z\} - \alpha_6\{yxy^2z\}.\end{aligned}$$

Comparing coefficients of $\{y^2xyz\}$ shows that $\alpha_1 = 1$. Then comparing coefficients of $\{x^3yz\}$ gives $\alpha_2 = 0$. Comparing coefficients of $\{x^2yxz\}$ gives $\alpha_3 = 0$. Then $\alpha_4 = 0$ since the coefficient of $\{y^3xz\}$ must be 0. Next using the coefficients of $\{xyx^2z\}$ and $\{yxy^2z\}$ we see that $\alpha_5 = \alpha_6 = 0$. Hence we must have $f = \{txyz\}$ Since this is not a Jordan element we have a contradiction.

Exercises

1. Show that if $\mathfrak{K}$ is the ideal in $FSJ^{(3)}$ generated by $x.y$ then $FSJ^{(3)}/\mathfrak{K}$ is exceptional.

2. (McCrimmon). Let $\mathfrak{K}$ be the ideal in $FSJ^{(3)}$ generated by $x.y-1$, $x^{\cdot 2}.y-x$. Show that $FSJ^{(3)}/\mathfrak{K}$ is special.

3. Let Φ be of characteristic two and consider the polynomial algebra $\Phi[x]$ $(=\Phi\{x\})$ in the indeterminate x. Let $\mathfrak{K}$ be the subspace with basis x^2, x^k, $k \geqq 4$. In $\Phi[x]/\mathfrak{K}$ define for $a, b \in \Phi[x]$, $(a + \mathfrak{K})^{\cdot 2} = a^2 + \mathfrak{K}$, $(a+\mathfrak{K}).(b+\mathfrak{K}).(a+\mathfrak{K}) = aba + \mathfrak{K}$. Show that these are well defined. Show that there exists no 1-1 linear mapping σ of $\Phi[x]/\mathfrak{K}$ into an associative algebra $\mathfrak{A}$ such that

$$((a+\mathfrak{K})^{\cdot 2})^\sigma = ((a+\mathfrak{K})^\sigma)^2$$

and

$$((a+\mathfrak{K}).(b+\mathfrak{K}).(a+\mathfrak{K}))^\sigma = (a+\mathfrak{K})^\sigma(b+\mathfrak{K})^\sigma(a+\mathfrak{K})^\sigma.$$

4. Let $\mathfrak{H}$ be the Jordan subalgebra of reversible elements in the group algebra $\Phi\{x, x^{-1}, y, y^{-1}\}$ of the free group generated by x, x^{-1}, y, y^{-1} (see ex. 5, p. 10). Show that any homomorphic image of $\mathfrak{H}$ is special.

4. **Some constructions of Jordan algebras.** In this section and the next we shall consider the sources of the most important examples of Jordan algebras. We have seen that if $\mathfrak{A}$ is an associative algebra (over a field Φ of characteristic $\neq 2$) then $\mathfrak{A}$ determines the Jordan algebra $\mathfrak{A}^+$. If $\mathfrak{A}$ is commutative as well as associative then $a.b = \frac{1}{2}(ab+ba) = ab$ so $\mathfrak{A}^+$ is identical with $\mathfrak{A}$. We remark also that the notions of commutative associative algebra and associative Jordan algebra are identical. If σ is a homomorphism of the associative algebra $\mathfrak{A}$ into the associative algebra $\mathfrak{B}$ then clearly σ is a homomorphism of $\mathfrak{A}^+$ into $\mathfrak{B}^+$. The same thing holds also if σ is an antihomomorphism since $(a.b)^\sigma = \frac{1}{2}(ab+ba)^\sigma = \frac{1}{2}(b^\sigma a^\sigma + a^\sigma b^\sigma) = a^\sigma . b^\sigma$. It is convenient to express these simple observations in the following way: Let $\mathscr{A}$ denote the category whose objects are associative algebras and whose morphisms are homomorphisms and antihomomorphisms. Let $\mathscr{J}$ denote the category of Jordan algebras with homomorphisms as morphisms. Then we obtain a functor from $\mathscr{A}$ into $\mathscr{J}$ by mapping an associative algebra $\mathfrak{A}$ into the Jordan algebra $\mathfrak{A}^+$ and a homomorphism or antihomomorphism σ of $\mathfrak{A}$ into $\mathfrak{B}$ into the same σ as mapping of $\mathfrak{A}^+$ into $\mathfrak{B}^+$.

A more important category and functor for Jordan algebras is obtained as follows. By an *associative algebra with involution* we mean a pair $(\mathfrak{A}, J)$ where $\mathfrak{A}$ is an associative algebra and J is an involution in $\mathfrak{A}$. A subalgebra (ideal) of $(\mathfrak{A}, J)$ is a subalgebra (ideal) $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B}^J = \mathfrak{B}$. If $(\mathfrak{B}, K)$ is a second associative algebra with involution then a *homomorphism* σ *of* $(\mathfrak{A}, J)$ *into* $(\mathfrak{B}, K)$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $J\sigma = \sigma K$.

If $(\mathfrak{A}, J)$ is an associative algebra with involution then the set $\mathfrak{H}(\mathfrak{A}, J)$ of J-symmetric elements of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^+$. If σ is a homomorphism of $(\mathfrak{A}, J)$ into $(\mathfrak{B}, K)$ then $a^J = a$ implies $a^{\sigma K} = a^{J\sigma} = a^\sigma$ so σ maps $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{H}(\mathfrak{B}, K)$. Clearly the restriction $\sigma' = \sigma \mid \mathfrak{H}(\mathfrak{A}, J)$ of σ to $\mathfrak{H}(\mathfrak{A}, J)$ is a homomorphism of $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{H}(\mathfrak{B}, K)$. Let $\mathscr{A}\mathscr{I}$ denote the category whose objects are associative algebras with involutions and whose morphisms are homomorphism of associative algebras with involutions. We now define a functor H from $\mathscr{A}\mathscr{I}$ into $\mathscr{J}$ by specifying that H maps an associative algebra with involution $(\mathfrak{A}, J)$ into the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ (subalgebra of $\mathfrak{A}^+$) and H maps a homomorphism σ of $(\mathfrak{A}, J)$ into $(\mathfrak{B}, K)$ into the restriction σ' of σ to $\mathfrak{H}(\mathfrak{A}, J)$.

If $\mathfrak{A}$ is an algebra we let $\mathfrak{A}^\circ$ denote the *opposite algebra*, that is, $\mathfrak{A}^\circ$ has the same vector space structure as $\mathfrak{A}$, but the product $a^\circ b$ in $\mathfrak{A}^\circ$ is $a^\circ b = ba$. Let $\mathfrak{S} = \mathfrak{A} \oplus \mathfrak{A}^\circ$, so $\mathfrak{S}$ is the set of pairs (a_1, a_2), $a_i \in \mathfrak{A}$ with the usual vector space structure and multiplication in $\mathfrak{S}$ is given by

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, b_2 a_2). \tag{17}$$

One checks directly that the mapping $J: (a_1, a_2) \to (a_2, a_1)$ is an involution in $\mathfrak{S}$. The set $\mathfrak{H}(\mathfrak{S}, J)$ of symmetric elements relative to J is the set $\{(a, a) \mid a \in \mathfrak{A}\}$. Let $\mathfrak{B}$ be a second algebra, $\mathfrak{B}^\circ$ its opposite, and $\mathfrak{T} = \mathfrak{B} \oplus \mathfrak{B}^\circ$ with the involution $K: (b_1, b_2) \to (b_2, b_1)$. If σ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ then $(a_1, a_2) \to (a_1{}^\sigma, a_2{}^\sigma)$ is a homomorphism of $(\mathfrak{S}, J)$ into $(\mathfrak{T}, K)$ and if σ is an antihomomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ then $(a_1, a_2) \to (a_2^\sigma, a_1^\sigma)$ is a homomorphism of $(\mathfrak{S}, J)$ into $(\mathfrak{T}, K)$. Now let $\mathfrak{A}$ be associative. Then $\mathfrak{A}^\circ$ and $\mathfrak{S} = \mathfrak{A} \oplus \mathfrak{A}^\circ$ are associative and $\mathfrak{H}(\mathfrak{S}, J) = \{(a, a) \mid a \in \mathfrak{A}\}$ is a Jordan algebra. It is clear that $a \to (a, a)$ is an isomorphism of $\mathfrak{A}^+$ onto $\mathfrak{H}(\mathfrak{S}, J)$. Hence we see that the Jordan algebras $\mathfrak{A}^+$, $\mathfrak{A}$ associative, obtained by our first construction can also be obtained by the second one (the algebras $\mathfrak{H}$ of symmetric elements).

Now let $\mathfrak{V}$ be a vector space over a field Φ which is equipped with a symmetric bilinear form f. Thus $f(x, y) \in \Phi$, $f(x, y) = f(y, x)$ and $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$, $\alpha f(x, y) = f(\alpha x, y)$, $x_1, x_2, x, y \in \mathfrak{V}$, $\alpha \in \Phi$. We now construct the vector space $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ which is a direct sum of $\mathfrak{V}$ and a one dimensional space $\Phi 1$ with basis 1. We define a product in $\mathfrak{J}$ by

$$(\alpha 1 + x) . (\beta 1 + y) = (\alpha\beta + f(x, y))1 + (\beta x + \alpha y) \tag{18}$$

for $\alpha, \beta \in \Phi$, $x, y \in \mathfrak{V}$. Since $f(x, y)$ is symmetric the product $a . b$ defined in $\mathfrak{J}$ is commutative. Also it is immediate that the distributive laws hold and $\gamma(a . b)$

$= \gamma a \,.\, b = a \,.\, \gamma b$, $\gamma \in \Phi$. Hence $\mathfrak{J}$ is an algebra. If $a = \alpha 1 + x$ then $a^{\cdot 2} = (\alpha^2 + f(x,x))^1 + 2\alpha x$ so $a^{\cdot 2} = (\alpha^2 + f(x,x))1 + 2\alpha a - 2\alpha^2 . 1$ and

$$a^{\cdot 2} = (f(x,x) - \alpha^2)1 + 2\alpha a$$

is a linear combination of 1 and a. Then the identity $a^{\cdot 2}.(a.b) = a.(a^{\cdot 2}.b)$ for $a, b \in \mathfrak{J}$ is an immediate consequence of $a.(a.b) = a.(a.b)$ and $1.(a.b) = a.(1.b)$. Hence $\mathfrak{J}$ is a Jordan algebra. We shall call $\mathfrak{J}$ *the Jordan algebra of the symmetric bilinear form f or of the quadratic form f where $f(x) = f(x, x)$.* We shall see later (§ 7.1) that $\mathfrak{J}$ is special.

The class of vector spaces $\mathfrak{V}$ equipped with symmetric bilinear forms f is a category whose objects are the pairs $(\mathfrak{V}, f)$ and whose morphisms: $(\mathfrak{V}, f) \underset{T}{\to} (\mathfrak{W}, g)$ are the linear mappings T of $\mathfrak{V}$ into $\mathfrak{W}$ such that $g(xT, yT) = f(x, y)$, $x, y \in \mathfrak{V}$. Such a T determines a homomorphism T' of the Jordan algebra $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ into the Jordan algebra $\mathfrak{K} = \Phi 1 \oplus \mathfrak{W}$ such that $(\alpha 1 + x)^{T'} = \alpha 1 + x^T$. In this way one obtains a functor from the category of $(\mathfrak{V}, f)$'s to the category of Jordan algebras, mapping $(\mathfrak{V}, f) \to \mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ and $T \to T'$.

Exercises

1. Call an associative algebra with involution $(\mathfrak{A}, J)$, $\mathfrak{A}$ with 1, *simple* if the only ideals in $(\mathfrak{A}, J)$ are $(\mathfrak{A}, J)$ and 0. Show that if $(\mathfrak{A}, J)$ is simple then either $\mathfrak{A}$ is simple or $\mathfrak{A}$ is a direct sum of two simple ideals which are exchanged by J.

2. Let $(\mathfrak{A}, J)$, $(\mathfrak{B}, K)$ be as in 1. with $\mathfrak{A}$ and $\mathfrak{B}$ not simple. Show that $(\mathfrak{A}, J)$ is isomorphic to $(\mathfrak{B}, K)$ if and only if the simple components of $\mathfrak{A}$ are isomorphic or anti-isomorphic to those of $\mathfrak{B}$.

In exercises 3–6, $\mathfrak{V}$ is a vector space equipped with a symmetric bilinear form f and $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ is the corresponding Jordan algebra.

3. Let $\mathfrak{N}$ be the radical of f, that is, $\mathfrak{N} = \mathfrak{V}^{\perp}$ the subspace of elements z such that $f(x, z) = 0$, $x \in \mathfrak{V}$. Show that $\mathfrak{N}$ is an ideal in $\mathfrak{J}$ and that $\mathfrak{J}/\mathfrak{N}$ is isomorphic to the Jordan algebra of the form induced by f in $\bar{\mathfrak{V}} = \mathfrak{V}/\mathfrak{N}$.

4. Show that $\mathfrak{J}$ is simple, that is, $\mathfrak{J}$ has no ideals $\neq \mathfrak{J}$, 0 if $\dim \mathfrak{V} > 1$ and $f(x, y)$ is nondegenerate.

5. Show that if $f(x, y)$ is nondegenerate and $\dim \mathfrak{V} > 1$ then $\mathfrak{V}$ is the subspace of $\mathfrak{J}$ spanned by all the associators $(a.b).c - a.(b.c)$, $a, b, c \in \mathfrak{A}$. Use this to show that a linear mapping σ of $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ into the Jordan algebra $\mathfrak{K} = \Phi 1 \oplus \mathfrak{W}$ of the nondegenerate symmetric bilinear form g on $\mathfrak{W}$ with $\dim \mathfrak{W} > 1$ is an isomorphism if and only if (1) $1^{\sigma} = 1$, (2) σ is bijective of $\mathfrak{V}$ onto $\mathfrak{W}$, (3) $g(x^{\sigma}, y^{\sigma}) = f(x, y)$, $x, y \in \mathfrak{V}$. Note that this shows that $\mathfrak{J}$ and $\mathfrak{K}$ are isomorphic if and only if f and g are equivalent symmetric bilinear forms and that the group of automorphisms of $\mathfrak{J}$ is isomorphic to the orthogonal group of $\mathfrak{V}$ relative to f.

6. Show that $\mathfrak{J}$ contains an idempotent element $e \neq 0, 1$ $(e^{\cdot 2} = e)$ if and only if there exists an x in $\mathfrak{V}$ such that $f(x, x) = 1$.

5. **Alternative algebras and Jordan algebras.** An algebra $\mathfrak{A}$ with multiplication ab is said to be *alternative* if the following identities hold in $\mathfrak{A}$:

$$a^2b = a(ab), \qquad ba^2 = (ba)a. \tag{19}$$

In any algebra we shall define the *associator*

$$[a, b, c] = (ab)c - a(bc). \tag{20}$$

Then the defining conditions for an alternative algebra are $[a, a, b] = 0$ and $[b, a, a] = 0$. If we replace a by $a + c$ in the first of these relations we obtain

$$0 = [a + c, a + c, b] = [a, a, b] + [c, c, b] + [a, c, b] + [c, a, b].$$

Hence $[a, c, b] + [c, a, b] = 0$. Similarly, the second relation gives $[b, a, c] + [b, c, a] = 0$. It follows from these two relations that the associator $[a, b, c]$ is invariant under even permutations of the arguments and changes sign under odd permutations. Also we have $[a, b, a] = -[a, a, b] = 0$ so that $(ab)a = a(ba)$ holds in any alternative algebra. Alternative algebras have a simple characterization in terms of associative algebras, namely, one has the theorem of Artin that an algebra is alternative if and only if all of its subalgebras generated by pairs of elements are associative. We shall not require this result so we do not prove it here. A simple proof can be found in Schafer's *An Introduction to Nonassociative Algebras*, p. 29.

In this section we shall establish two important connections between alternative and Jordan algebras. First we have

THEOREM 3. *If $\mathfrak{A}$ is alternative (of characteristic $\neq 2$) with an identity element 1 then $\mathfrak{A}^+$ is a special Jordan algebra.**

PROOF. We denote the linear mappings $x \to ax$, $x \to xa$ in $\mathfrak{A}$ by a_L and a_R respectively. Then the defining conditions (19) are

$$(a^2)_L = a_L{}^2, \qquad (a^2)_R = a_R{}^2. \tag{19'}$$

Now consider the mapping $a \to a_L$ of $\mathfrak{A}$ into $\text{Hom}(\mathfrak{A}, \mathfrak{A}) = \text{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$, the associative algebra of linear transformations in the vector space $\mathfrak{A}/\Phi$. It is clear from the definition of an algebra that $a \to a_L$ is linear. Also we obtain from (19′) that

$$\begin{aligned}(a + b)^2{}_L &= (a + b)_L{}^2 = (a_L + b_L)^2,\\ (a^2 + ab + ba + b^2)_L &= a_L{}^2 + a_Lb_L + b_La_L + b_L{}^2.\end{aligned}$$

* The hypothesis on the existence of the element 1 can be dropped since, as we shall see in the next section, any alternative algebra can be imbedded as a subalgebra of an alternative algebra with identity element.

Hence $(ab+ba)_L = a_L b_L + b_L a_L$ and consequently

$$(a\,.\,b)_L = a_L\,.\,b_L. \tag{21}$$

Since $\mathfrak{A}$ has an identity the mapping $a \to a_L$ is 1-1. Hence this is an isomorphism of $\mathfrak{A}^+$ into $\mathrm{Hom}(\mathfrak{A}, \mathfrak{A})^+$. Since $\mathrm{Hom}(\mathfrak{A}, \mathfrak{A})$ is associative this shows that $\mathfrak{A}^+$ is a special Jordan algebra.

The argument giving (21) can be applied also to the right multiplication a_R and yields

$$(a\,.\,b)_R = a_R\,.\,b_R. \tag{22}$$

We remark also that the existence of 1 was not used in the proofs of (21) and (22). We now proceed to derive some important identities for arbitrary alternative algebras due to R. Moufang. These are

$$(aba)x = a(b(ax)), \qquad aba = (ab)a = a(ba), \tag{23}$$

$$x(aba) = ((xa)b)a, \tag{24}$$

$$a(xy)a = (ax)(ya). \tag{25}$$

For (23) we have $(aba)x - a(b(ax)) = ((ab)a)x - a(b(ax)) = [ab, a, x] + (ab)(ax) - (ab)(ax) + [a, b, ax] = -[a, ab, x] - [a, ax, b] = -(a(ab))x + a((ab)x) - (a(ax))b + a((ax)b) = -(a^2b)x - (a^2x)b + a\{(ab)x + (ax)b\} = -[a^2, b, x] - a^2(bx) - [a^2, x, b] - a^2(xb) + a\{(ab)x + (ax)b\} = a\{-a(bx) - a(xb) + (ab)x + (ax)b\} = a\{[a,b,x] + [a,x,b]\} = 0$. The relation (24) is obtained in a similar manner. For (25) we have $a(xy)a - (ax)(ya) = (a(xy))a - (ax)(ya) = ((ax)y)a - [a,x,y]a - ((ax)y)a + [ax,y,a] = -[a,x,y]a - [y,ax,a] = -[a,x,y]a - (y(ax))a + y(axa) = -[a,x,y]a - (y(ax))a + ((ya)x)a$ (by (24)) $= \{-[a,x,y] + [y,a,x]\}a = 0$.

It is evident from the definition that any associative algebra is alternative. We shall now give a construction of the most important alternative algebras which are not associative, namely, the Graves-Cayley algebras of octonions. We begin with an arbitrary quaternion algebra $\mathfrak{Q}$ over the given field Φ (of characteristic $\neq 2$). We recall that such an algebra is associative, has an identity element 1 and two generators i, j with the defining relations $i^2 = \lambda 1$, $j^2 = \mu 1$, $ij = -ji$ where λ and μ are nonzero elements of Φ. The elements $1, i, j, k = ij$ form a basis for $\mathfrak{Q}$ and one has the multiplication table:

$$i^2 = \lambda 1, \qquad j^2 = \mu 1, \qquad k^2 = -\lambda\mu 1, \qquad ij = -ji = k,$$
$$jk = -kj = -\mu i, \qquad ki = -ik = -\lambda j$$

(together with $1a = a = a1$). We recall that $\mathfrak{Q}$ has an involution $a \to \bar{a}$ which we shall call the *standard* involution in $\mathfrak{Q}$, such that $\bar{a} = \alpha_0 1 - \alpha_1 i - \alpha_2 j - \alpha_3 k$ if $a = \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k$. Moreover, $a\bar{a} = n(a)1 = \bar{a}a$ where $n(a) = \alpha_0{}^2 - \lambda\alpha_1{}^2$

$-\mu\alpha_2^2 + \lambda\mu\alpha_3^2$ and $n(ab) = n(a)n(b)$. We call $n(a)$ the *norm* of the quaternion a. This is a quadratic form on $\mathfrak{Q}$ whose associated symmetric bilinear form $n(a,b) = \frac{1}{2}[n(a+b) - n(a) - n(b)] = \frac{1}{2}(a\bar{b} + b\bar{a})$ is nondegenerate. We have $a + \bar{a} = 2\alpha_0 1$ and we call $t(a) = 2\alpha_0$ the *trace* of a. The mapping $a \rightarrow t(a)$ is a linear function on $\mathfrak{Q}$.

Now let $\mathfrak{O} = \mathfrak{Q} \oplus \mathfrak{Q}$ the vector space direct sum of two copies of $\mathfrak{Q}$. We write the elements of $\mathfrak{O}$ as (a, b) where $a, b \in \mathfrak{Q}$. Let ν be a nonzero element of $\mathfrak{Q}$ and define a product in $\mathfrak{O}$ by

$$(26) \qquad (a,b)(c,d) = (ac + \nu\bar{d}b, da + b\bar{c})$$

where $a,b,c,d \in \mathfrak{Q}$. Since $\mathfrak{Q}$ is an algebra and $a \rightarrow \bar{a}$ is linear it is clear that the product defined in $\mathfrak{O}$ is bilinear, so $\mathfrak{O}$ is an algebra. We call $\mathfrak{O}$ the *algebra of octonions* (*Cayley algebra*, *Cayley-Dickson algebra*, *Cayley-Graves algebra*), defined by $\mathfrak{Q}$ and ν. (Note that $\mathfrak{O}$ is defined from Φ by three parameters λ, μ, ν.) It is clear from (26) that $(1,0)$ is an identity element for $\mathfrak{O}$ and that the subset of elements $(a,0)$ is a subalgebra of $\mathfrak{O}$ isomorphic under $(a,0) \rightarrow a$ to $\mathfrak{Q}$. We identify $\mathfrak{Q}$ with the corresponding subalgebra of $\mathfrak{O}$ and write a for the element $(a,0)$ of 0. Also we set $l = (0,1)$. Then (26) gives $bl = (b,0)(0,1) = (0,b) = l\bar{b}$. Hence every element of $\mathfrak{O}$ can be written in one and only one way as $a + bl$, a,b in $\mathfrak{Q}$.

For $x = a + bl$, $a, b \in \mathfrak{Q}$ we define $\bar{x} = \bar{a} - bl$. Clearly $x \rightarrow \bar{x}$ is a linear mapping of period two in $\mathfrak{O}$. For $y = c + dl$, $c, d \in \mathfrak{Q}$, then we have

$$\bar{y}\bar{x} = (\bar{c} - dl)(\bar{a} - bl) = (\bar{c}\bar{a} + \nu\bar{b}d) - (b\bar{c} + da)l,$$

$$\overline{xy} = (\bar{c}\bar{a} + \nu\bar{b}d) - (da + b\bar{c})l.$$

Hence $\overline{xy} = \bar{y}\bar{x}$ and $x \rightarrow \bar{x}$ is an involution in $\mathfrak{O}$. We have

$$(27) \qquad x + \bar{x} = a + bl + \bar{a} - bl = t(x)1$$

where $t(x) = t(a) \in \Phi$ and

$$(28) \qquad x\bar{x} = (a + bl)(\bar{a} - bl) = n(x)1 = \bar{x}x$$

where $n(x) = n(a) - \nu n(b) \in \Phi$. If $n(x,y) = \frac{1}{2}[n(x+y) - n(x) - n(y)]$ is the symmetric bilinear form associated with the quadratic form n on $\mathfrak{O}$ then the restriction of $n(x,y)$ to $\mathfrak{Q}$ is the symmetric bilinear form given by the norm form of $\mathfrak{Q}$. Since $n(x) = n(a) - \nu n(b)$ it is clear that $\mathfrak{Q}^{\perp} = \mathfrak{Q}l$ and that $n(x,y)$ is nondegenerate in $\mathfrak{O}$.

We claim that $\mathfrak{O}$ is an alternative algebra. We have to verify the associator conditions $[x,x,y] = 0$ and $[y,x,x] = 0$. Since $\overline{[x,y,z]} = -[\bar{z},\bar{y},\bar{x}]$ it is enough to show that $[x,x,y] = 0$. Clearly $[x,1,y] = 0$, and since $[x,y,z]$ is trilinear $[x,\alpha 1,y] = 0$ if $\alpha \in \Phi$. Hence $[x, x + \bar{x}, y] = [x, t(x)1, y] = 0$ and so $[x,x,y] = 0$ will follow if we can show that $[x,\bar{x},y] = 0$. Now

$$
\begin{aligned}
x(\bar{x}y) &= (a(\bar{a}c - v\bar{d}b) + v(a\bar{d} - c\bar{b})b) + ((d\bar{a} - b\bar{c})a + b(\bar{c}a - v\bar{b}d))l \\
&= (n(a) - vn(b))(c + dl)
\end{aligned}
$$

by associativity in $\mathfrak{Q}$. Hence $x(\bar{x}y) = n(x)y = (x\bar{x})y$ so $[x, \bar{x}, y] = 0$ and $\mathfrak{O}$ is alternative. Now we have $a(cl) = (ca)l$ and this may be different from $(ac)l$ since $\mathfrak{Q}$ is not commutative. Hence we see that $\mathfrak{O}$ is not associative.

It is clear from $x\bar{x} = n(x)1 = \bar{x}x$ that $n(\bar{x}) = n(x)$. We can now establish the main property of $n(x)$, namely, the composition law:

$$n(xy) = n(x)n(y). \tag{29}$$

We have

$$
\begin{aligned}
n(xy) = (xy)(\bar{y}\bar{x}) &= (xy)(\bar{y}(t(x)1 - x)) \\
&= t(x)(xy)\bar{y} - (xy)(\bar{y}x) \\
&= t(x)x(y\bar{y}) - x(y\bar{y})x \\
&= t(x)n(y)x - n(y)x^2 \\
&= n(y)x(t(x) - x) = n(y)x\bar{x} = n(y)n(x).
\end{aligned}
$$

Hence (29) is valid.

If $\mathfrak{A}$ is an arbitrary algebra then the *nucleus* $N(\mathfrak{A})$ of $\mathfrak{A}$ is the set of elements $n \in \mathfrak{A}$ such that $[n, x, y] = [x, n, y] = [x, y, n] = 0$ for all $x, y \in \mathfrak{A}$. The *center* $C(\mathfrak{A})$ of $\mathfrak{A}$ is the subset of $N(\mathfrak{A})$ of elements c such that $[c, x] \equiv cx - xc = 0$ for all $x \in \mathfrak{A}$.

In any algebra one has by direct verification the following associator identity:

$$
\begin{aligned}
&a[b, c, d] + [a, b, c]d \\
&\quad = [ab, c, d] - [a, bc, d] + [a, b, cd].
\end{aligned} \tag{30}
$$

This implies that if $n_1, n_2 \in N(\mathfrak{A})$ then $n_1 n_2 \in N(\mathfrak{A})$. It follows that $N(\mathfrak{A})$ is a subalgebra of $\mathfrak{A}$. If c_1 and $c_2 \in C(\mathfrak{A})$ then $(c_1 c_2)x = c_1(c_2 x) = c_1(x c_2) = (x c_2)c_1 = x(c_2 c_1) = x(c_1 c_2)$. Hence $C(\mathfrak{A})$ is a subalgebra of $N(\mathfrak{A})$ (containing 1 if $\mathfrak{A}$ has an identity element 1). Clearly $N(\mathfrak{A})$ is an associative subalgebra and $C(\mathfrak{A})$ is associative and commutative. We note also that (30) implies the following relation for $n \in N(\mathfrak{A})$, a, b, c arbitrary in $\mathfrak{A}$:

$$
\begin{aligned}
n[a, b, c] &= [na, b, c], \\
[an, b, c] &= [a, nb, c], \\
[a, bn, c] &= [a, b, nc], \\
[a, b, cn] &= [a, b, c]n.
\end{aligned} \tag{31}
$$

Now let $\mathfrak{A}$ be an arbitrary algebra with identity element 1 and let $\mathfrak{A}_n$ denote the algebra of $n \times n$ matrices with entries in $\mathfrak{A}$ with the usual definitions of addition, multiplication by elements of the base field Φ and row by column multiplication. As usual we let e_{ij} denote the matrix with a 1 in the (i,j) entry and 0's elsewhere. Then we have $e_{ij}e_{kl} = \delta_{jk}e_{il}$ ($\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$) and $\sum_1^n e_{ii}$ is the identity element 1 of $\mathfrak{A}_n$. We write diag $\{a_1, a_2, \cdots, a_n\}$ for the matrix with a_i in the (i,i) place and 0 in all (i,j)-places, $i \neq j$. The elements of the form diag $\{a, a, \cdots, a\}$, $a \in \mathfrak{A}$, form a subalgebra of $\mathfrak{A}_n$ isomorphic to $\mathfrak{A}$ under $a \rightarrow$ diag $\{a, a, \cdots, a\}$ and we identify this with $\mathfrak{A}$. Also we denote the element diag $\{a, \cdots, a\}$ by a. Then if $A = (a_{ij})$, $A = \sum a_{ij}e_{ij}$ and $(ae_{ij})(be_{kl}) = \delta_{jk}(ab)e_{il}$. It follows from this that $N(\mathfrak{A})_n$ the set of elements of the form $\sum n_{ij}e_{ij}$, $n_{ij} \in N(\mathfrak{A})$, is contained in the nucleus of $\mathfrak{A}_n$. In particular the e_{ij} are in $N(\mathfrak{A}_n)$.

If $\mathfrak{A}$ has an involution $x \rightarrow \bar{x}$ it is easy to check that $A = (a_{ij}) \rightarrow \bar{A}^t = (\bar{a}_{ij})^t$, where t denotes the transposed matrix, is an involution in $\mathfrak{A}_n$. We shall call this the *standard involution* in $\mathfrak{A}_n$ associated with the given involution in $\mathfrak{A}$. Let $\mathfrak{H}(\mathfrak{A}_n)$ denote the set of symmetric elements ($\bar{A}^t = A$) under the standard involution in $\mathfrak{A}_n$. Then it is clear that $\mathfrak{H}(\mathfrak{A}_n)$ is a subalgebra of $\mathfrak{A}_n{}^+$. We shall see later (§3.2) that if $\mathfrak{H}(\mathfrak{A}_n)$ is Jordan and $n \geqq 4$ then $\mathfrak{A}$ is necessarily associative (hence $\mathfrak{H}(\mathfrak{A}_n)$ is special) and if $\mathfrak{H}(\mathfrak{A}_3)$ is Jordan then $\mathfrak{A}$ is necessarily alternative with the symmetric elements of $\mathfrak{A}$ in the nucleus. Since $\mathfrak{A}_n$ is associative if $\mathfrak{A}$ is associative it is clear that if $\mathfrak{A}$ is associative then $\mathfrak{H}(\mathfrak{A}_n)$ is a Jordan algebra. We shall now prove that if $\mathfrak{A}$ is an alternative algebra with 1 and an involution $x \rightarrow \bar{x}$ such that the symmetric elements are in the nucleus then $\mathfrak{H}(\mathfrak{A}_3)$ is a Jordan algebra.[2]

We collect first some facts on associators which we shall need.

LEMMA 1. (i). *If* $\mathfrak{A}$ *is an arbitrary algebra* (char. $\neq 2$) *and* $[a,b,c]^+$ *denotes the associator* $(a.b).c - a.(b.c)$ *in* $\mathfrak{A}^+$ *then*

$$\begin{aligned} 4[a,b,c]^+ &= [a,b,c] - [c,b,a] + [b,a,c] \\ &\quad - [c,a,b] + [a,c,b] - [b,c,a] + [b[ac]], \quad ([ab] = ab - ba). \end{aligned} \tag{32}$$

(ii). *If* $\mathfrak{A}$ *is alternative then*

$$4[a,b,c]^+ = -2[a,b,c] + [b[ac]], \tag{33}$$

$$[d[a,b,c]] + [ab,c,d] + [bc,a,d] + [ca,b,d] = 0, \tag{34}$$

$$n[a,b,c] = [a,b,c]n, \quad \textit{if} \quad n \in N(\mathfrak{A}). \tag{35}$$

(iii). *If* $\mathfrak{A}$ *has an involution* $x \rightarrow \bar{x}$ *then*

$$\overline{[a,b,c]} = -[\bar{c},\bar{b},\bar{a}]. \tag{36}$$

(iv). *If* $\mathfrak{A}$ *is alternative and has an involution whose symmetric elements are in the nucleus then*

[2] The result in this generality was proved first by Sasser in [1]. We are indebted to K. McCrimmon for the proof which we shall give.

$$(37)\qquad [a,b,c] = -[\bar{a},b,c] = -[a,\bar{b},c] = -[a,b,\bar{c}],$$

$$(38)\qquad \overline{[a,b,c]} = -[a,b,c].$$

(v). *If $\mathfrak{A}$ is arbitrary and the notation a and e_{ij} are as above then*

$$(39)\qquad [ae_{ij}, be_{kl}, ce_{pq}] = [a,b,c]\, e_{ij}e_{kl}e_{pq}$$

for a,b,c in $\mathfrak{A}$ ($\subseteq \mathfrak{A}_n$).

PROOFS. (32) is obtained by direct verification and (33) follows from the alternating character of $[a,b,c]$ in an alternative algebra. To obtain (34) we linearize the Moufang identity $(ca)(bc) = (c(ab))c$ (see (25)) by replacing c by $c+d$ to obtain

$$(40)\qquad (ca)(bd) + (da)(bc) = (c(ab))d + (d(ab))c.$$

Then we have

$$\begin{aligned} d[a,b,c] &= d((ab)c) - d((a(bc)) \\ &= -[d,ab,c] + (d(ab))c - d(a(bc)) \\ &= -[ab,c,d] + (d(ab))c + [d,a,bc] - (da)(bc) \\ &= -[ab,c,d] - [bc,a,d] - (c(ab))d + (ca)(bd) \qquad \text{(by (40))} \\ &= -[ab,c,d] - [bc,a,d] + [c,a,b]d - ((ca)b)d + (ca)(bd) \\ &= -[ab,c,d] - [bc,a,d] + [a,b,c]d - [ca,b,d]. \end{aligned}$$

This is the same as (34). (35) is obtained by taking $d = n \in N(\mathfrak{A})$ in (34). (36) follows directly from the definition. For (37) we note that $a + \bar{a}$ is symmetric, hence in the nucleus. Hence $[a + \bar{a}, b, c] = 0$ which gives $[a,b,c] = -[\bar{a},b,c]$. The other parts are obtained in the same way. (38) is an immediate consequence of (36) and (37) and (39) is clear.

We prove next

LEMMA 2. *Let $\mathfrak{A}$ be an alternative algebra with 1 and involution with symmetric elements in the nucleus and let $A = (a_{ij}) \in \mathfrak{H}(\mathfrak{A}_3)$. Then $[A^2, A] = 2a$ where $a = [a_{12}, a_{23}, a_{31}]$.*

PROOF. We have

$$A = \sum_{i,j=1}^{3} a_{ij}e_{ij},\ \bar{a}_{ij} = a_{ji},\ [A^2, A] = [A,A,A] = \sum [a_{ij}, a_{jk}, a_{kl}]e_{il}$$

by (39) and the multiplication table of the matrix units e_{ij}. If $i = j$, $j = k$ or $k = l$ then $[a_{ij}, a_{jk}, a_{kl}] = 0$ since one of the arguments is symmetric and hence is in the nucleus. Also if $k = i$ then $a_{jk} = \bar{a}_{ij}$ and $[a_{ij}, \bar{a}_{ij}, a_{kl}] = -[a_{ij}, a_{ij}, a_{kl}] = 0$. Similarly, if $l = j$ then $[a_{ij}, a_{jk}, a_{kl}] = 0$. Hence the only possibly nonzero

$[a_{ij}, a_{jk}, a_{kl}]$ are those for which $i \neq j, \neq k,\ j \neq k, \neq l,\ k \neq l$. Since there are only three distinct indices the only possibly nonzero $[a_{ij}, a_{jk}, a_{kl}]$ are $[a_{ij}, a_{jk}, a_{ki}]$ with $i,j,k \neq$. Hence $[A^2, A] = \sum_{i,j,k\neq} [a_{ij}a_{jk}a_{ki}]e_{ii}$. We have $[a_{ij}, a_{jk}, a_{ki}] = [a_{jk}, a_{ki}, a_{ij}]$ and $[a_{ik}, a_{kj}, a_{ji}] = [\bar{a}_{ki}, \bar{a}_{jk}, \bar{a}_{ij}] = -[a_{ki}, a_{jk}, a_{ij}] = [a_{ij}, a_{jk}, a_{ki}]$. Hence $[a_{ij}, a_{jk}, a_{ki}] = [a_{12}, a_{23}, a_{31}] = a$ and $[A^2, A] = 2a(\sum e_{ii}) = 2a$.

We can now prove

THEOREM 4. *Let $\mathfrak{A}$ be an alternative algebra with* 1 *and involution $x \to \bar{x}$ such that the symmetric elements are in the nucleus. Then $\mathfrak{H}(\mathfrak{A}_3)$ is a Jordan algebra.*

PROOF. Since commutativity is clear all we have to do is verify that $(A^{\cdot 2}.B).A = A^{\cdot 2}.(B.A)$. This is the same as $[A,B,C]^+ = 0$ for $C = A^2$. By symmetry it is enough to show that the coefficients of e_{11} and e_{12} in $[A,B,C]^+$ are 0. Set $A = \sum a_{ij}e_{ij}$, $B = \sum b_{ij}e_{ij}$, $C = \sum c_{ij}e_{ij}$. Using (32), (39) and Lemma 2 we see that the coefficient of e_{11} in $[A,B,C]^+$ is a sum of associators and $-[b_{11}, \frac{1}{2}a]$ where a is an associator. Since $b_{11} = \bar{b}_{11} \in N(\mathfrak{A})$, (35) shows that $-[b_{11}, \frac{1}{2}a] = 0$. Hence the coefficient of e_{11} is a sum of associators; hence, by (38), this is skew. On the other hand, $[A,B,C]^+ \in \mathfrak{H}(\mathfrak{A}_3)$ which implies that the coefficient of e_{11} in this matrix is symmetric. Hence it is 0. Next we consider the coefficient of e_{12} in $[A,B,C]^+$. Since $[A,B,C]^+$ is linear in B it is enough to prove that the coefficient of e_{12} in $[A,B,C]^+$ is 0 for $B = b_{ii}e_{ii}$, $B = b_{13}e_{13} + b_{31}e_{31}$, $B = b_{23}e_{23} + b_{32}e_{32}$, $B = b_{12}e_{12} + b_{21}e_{21}$ where $b_{ij} = \bar{b}_{ji}$. In the first case, B is in the nucleus of $\mathfrak{A}_3$ so, by (32), the coefficient of e_{12} is a multiple of the coefficient of e_{12} in $[B, -2a]$ and this is 0 since $B = b_{ii}e_{ii}$. If $B = b_{13}e_{13} + b_{31}e_{31}$, the coefficient of e_{12} in $[A, B, C]$ is $\sum_{k,l} [a_{1k}, b_{kl}, c_{l2}] = [a_{11}, b_{13}, c_{32}] + [a_{13}, b_{31}, c_{12}] = [a_{13}, b_{31}, c_{12}]$. Similarly, the coefficients of e_{12} in $-[C,B,A]$, $[B,A,C]$, $-[C,A,B]$, $[A,C,B]$, $-[B,C,A]$ are respectively $-[c_{13}, b_{31}, a_{12}]$, $[b_{13}, a_{31}, c_{12}]$, 0, 0, $-[b_{13}, c_{31}, a_{12}]$. The coefficient of e_{12} in $[B[AC]] = -2[b_{13}e_{13} + b_{31}e_{31}, a]$ is 0. Hence the coefficient of e_{12} in $4[A,B,C]^+$ is

$$
\begin{aligned}
&[a_{13}, b_{31}, c_{12}] + [b_{13}, a_{31}, c_{12}] - [c_{13}, b_{31}, a_{12}] - [b_{13}, c_{31}, a_{12}] \\
&\quad = [a_{13}, b_{31}, c_{12}] + [\bar{b}_{31}, \bar{a}_{13}, c_{12}] - [c_{13}, b_{31}, a_{12}] - [\bar{b}_{31}, \bar{c}_{13}, a_{12}].
\end{aligned}
$$

This is 0 by (37) and the alternating character of $[a,b,c]$. A similar argument applies if $B = b_{23}e_{23} + b_{32}e_{32}$. Now let $B = b_{12}e_{12} + b_{21}e_{21}$. Then the reasoning we have used shows that the coefficient of e_{12} in $4[A,B,C]^+$ is

$$
\begin{aligned}
&[a_{12}, b_{21}, c_{12}] - [c_{12}, b_{21}, a_{12}] + [b_{12}, a_{21}, c_{12}] + [b_{12}, a_{23}, c_{32}] \\
&\quad - [c_{12}, a_{21}, b_{12}] - [c_{13}, a_{31}, b_{12}] + [a_{12}, c_{21}, b_{12}] + [a_{13}, c_{31}, b_{12}] \\
&\quad - [b_{12}, c_{21}, a_{12}] - [b_{12}, c_{23}, a_{32}] - 2[b_{12}, a].
\end{aligned}
$$

By (37) and the alternating character of $[a, b, c]$ this reduces to

$$(41)\quad \begin{aligned} &-2[c_{32}, a_{23}, b_{12}] - 2[c_{21}, a_{12}, b_{12}] - 2[c_{13}, a_{31}, b_{12}] \\ &- 2[b_{12}, [a_{12}, a_{23}, a_{31}]]. \end{aligned}$$

Now $c_{ij} = \sum_k a_{ik} a_{kj}$ so $[c_{ij}, a_{ji}, b_{12}] = [a_{ii} a_{ij}, a_{ji}, b_{12}] + [a_{ij} a_{jj}, a_{ji}, b_{12}] + [a_{ik} a_{kj}, a_{ji}, b_{12}]$ where $i, j, k \neq$. Since $a_{ii} \in N(\mathfrak{A})$, $[a_{ii} a_{ij}, a_{ji}, b_{12}] = a_{ii}[a_{ij}, a_{ji}, b_{12}]$ (by (31)) $= a_{ii}[a_{ij}, \bar{a}_{ij}, b_{12}] = 0$. Similarly, $[a_{ij} a_{jj}, a_{ji}, b_{12}] = 0$. Hence (41) becomes

$$\begin{aligned} &-2[a_{31} a_{12}, a_{23}, b_{12}] - 2[a_{23} a_{31}, a_{12}, b_{12}] \\ &-2[a_{12} a_{23}, a_{31}, b_{12}] - 2[b_{12}[a_{12}, a_{23}, a_{31}]]. \end{aligned}$$

This is 0 by (34). Hence the proof is complete.

Exercises

1. (McCrimmon). Let $U_a = a_L a_R$ in an alternative algebra $\mathfrak{A}$. Show that $U_a U_b U_a = U_{aba}$ for $a, b \in \mathfrak{A}$.

2. (McCrimmon). An element a of an alternative algebra $\mathfrak{A}$ with 1 is called *invertible* if there exists a b in $\mathfrak{A}$ such that $ab = 1 = ba$. Use the Moufang identities and exercise 1 to prove that the following conditions are equivalent: (1) a is invertible, (2) 1 is in the range of a_L and a_R, (3) a_L and a_R are invertible, (4) 1 is in the range of U_a, (5) U_a is invertible. Show that under these conditions the inverse is uniquely determined as $b = 1a_L{}^{-1} = 1a_R{}^{-1} = aU_a{}^{-1}$ and $b_L = a_L{}^{-1}$, $b_R = a_R{}^{-1}$, $U_b = U_a{}^{-1}$. Also show (using Artin's theorem) that if ab is invertible then a has a right inverse and b has a left inverse. Hence a is invertible if ab and ca are invertible.

3. Let $c_1, c_2, \cdots, c_k$ be elements of an alternative algebra with 1 and let $M = c_{1L} c_{2L} \cdots c_{kL}$, $N = c_{1R} c_{2R} \cdots c_{kR}$, $L = c_{1L} c_{1R} c_{2R} c_{2R} \cdots c_{kL} c_{kR}$. Show that for any x, y in the algebra we have

$$(xy)L = (xM)(yN).$$

Prove that if $(\cdots((c_1 c_2) c_3) \cdots c_k) = 1$ and $(c_k \cdots (c_3(c_2 c_1)) \cdots) = 1$ then $L = M = N$ is an automorphism of the algebra.

4. Show that an element a in the algebra of octonions is invertible if and only if $n(a) \neq 0$. Show that if $\bar{a} = -a$ then $n(1 + a) = n(1 - a)$ and this is $\neq 0$ unless $a^2 = 1$.

5. Show that if Φ has more than three elements then any octonion algebra over Φ has a basis $(u_1 = 1, u_2, \cdots, u_8)$ such that $n(u_i) = 1$.

6. Show that for any algebra with 1 the nucleus $N(\mathfrak{A}_n) = N(\mathfrak{A})_n$.

7. Let $A \in N(\mathfrak{A}_n)$ have an inverse A^{-1} in the associative algebra $N(\mathfrak{A}_n)$. Show that $X \to A^{-1}XA \equiv (A^{-1}X)A = A^{-1}(XA)$ is an automorphism in $\mathfrak{A}_n$. If $\mathfrak{A}$ has an involution $x \to \bar{x}$ and $\bar{A}^t = \pm A$ then $X \to A^{-1}\bar{X}^t A$ is an involution in $\mathfrak{A}_n$.

8. (Sasser). Let $\mathfrak{A}$ be an alternative algebra with 1 and an involution $x \to \bar{x}$ whose symmetric elements are in the nucleus. Prove that $\mathfrak{H}(\mathfrak{A}_2)$ is a special Jordan algebra.

6. **Varieties of algebras. Free algebras.** In this section we wish to make precise the notion of an algebra satisfying a set of identities. Associative, Lie, alternative and Jordan algebras are special cases and it is desirable, when possible, to treat these as special cases of a general theory of classes of algebras satisfying identities. We shall construct free algebras for such classes and formulate conditions on a given set of identities which insure that the algebras obtained by extending the base field remain in the given class and the same holds for the algebras obtained by adjoining an identity element.

We shall use the following terminology: a set S with a binary composition (associative binary composition) $(a, b) \to ab$ will be called a *monad* (*monoid*). It is customary in these definitions to assume the existence of an identity element. However, this is inconvenient for us so we shall use *monad* (*monoid*) *with* 1 when the existence of an identity element is assumed. We wish to give a construction first of a free monad generated by a given set. While this can be done directly it is perhaps clearer to base the construction on monoids and we shall follow this method. Let X be an arbitrary nonvacuous set and let $M(X)'$ denote the set of words or monomials $x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_k}$, $k \geqq 1$, formed of $x_{\alpha_i} \in X$. We define a product in $M(X)'$ by juxtaposition: $(x_{\alpha_1}\cdots x_{\alpha_k})(x_{\beta_1}\cdots x_{\beta_l}) = x_{\alpha_1}\cdots x_{\alpha_k}x_{\beta_1}\cdots x_{\beta_l}$.[3] This is clearly an associative composition so $M(X)'$ with its multiplication is a monoid. The monoid $M(X)'$ does not have an identity element. However, we can adjoin one by forming $M(X) = M(X)' \cup \{1\}$ where $1 \notin M(X)'$ and defining $1m = m = m1$, $m \in M(X)$. Then the associative law is valid also in $M(X)$, so this is a monoid with 1. The monoid $M(X)'$ $(M(X))$ has the following universal property. If S is any monoid (monoid with 1) then any mapping $x \to y$ of X into S has a unique extension to a homomorphism of $M(X)'$ $(M(X))$ into S (sending 1 into 1). Accordingly, we call $M(X)'$ $(M(X))$ the *free monoid* (*free monoid with* 1) generated by X. If Y is a subset of X then we define the *Y-degree* of $m = x_{\alpha_1}\cdots x_{\alpha_k}$ to be the number of $x_{\alpha_j} \in Y$ occuring in m. If $Y = \{x\}$ then we speak of the *x-degree* rather than the $\{x\}$-degree.

We remark next that if S is any monoid and p is a fixed element in S then we can define three p-compositions in S by the formulas

$$x_{p_L}y = pxy, \qquad x_{p_M}y = xpy, \qquad x_{p_R}y = xyp. \tag{42}$$

[3] More precisely, $M(X)'$ can be defined as the set of pairs (k, f) where k is a positive integer and f is a mapping of $\{1, 2, \cdots, k\}$ into X. Then $(k, f)\,(l, g) = (k + l, h)$ where $h(i) = f(i)$, $1 \leqq i \leqq k$ and $h(k + j) = g(j)$, $1 \leqq j \leqq l$.

The middle one of these is associative but not the other two and either of these can be used in the construction we now give of a free monad generated by a set. Let X be a nonvacuous set and form the free monoid $M(X,p)' \equiv M(X \cup \{p\})'$ generated by X and an element $p \notin X$. We introduce the right p-composition p_R in $M(X,p)'$ and let $N(X)'$ be the smallest submonad of $M(X,p)'$ relative to the composition p_R containing X. One sees easily that $N(X)'$ can be defined inductively by degree by the rules that $X \subseteq N(X)'$ and if $a, b \in N(X)'$ then $abp \in N(X)'$. It is clear from this that the X-degree of any $m \in N(X)'$ exceeds its p-degree by one. The elements of $N(X)'$ of degree > 1 have the form uvp where $u, v \in N(X)'$. It follows by induction on degree that if $w \in N(X)'$ then no proper right factor of w is of X-degree exceeding its p-degree. We use this remark to prove the following

LEMMA. *If $w \in N(X)'$ has degree > 1 then the representation of w as uvp, $u, v \in N(X)'$, is unique.*

PROOF. Let $w = uvp$, $u, v \in N(X),'$ and let v' be the right factor of least degree of uv with the property that its X-degree exceeds its p-degree. Since v has this property, v' is a right factor of v. Hence, by the result noted, we have $v' = v$. Hence v is uniquely determined. Since the cancellation law holds in $M(X)'$, u is also determined by w.

We now adjoin an identity element 1 to $N(X)'$ obtaining $N(X) = \{1\} \cup N(X)'$. We define the *degree* (or *Y-degree* for Y a subset of X) of $w \in N(X)'$ to be the X-degree (Y-degree) of w as element of $M(X,p)'$. Also we define the degree (Y-degree) of 1 to be 0. Let S be any monad (monad with 1) and let α be a mapping of X into S. We extend α to $N(X)'$ ($N(X)$) inductively on degree by requiring that if $w = uvp$, $u, v \in N(X)'$ then $w^\alpha = u^\alpha v^\alpha$ (and $1^\alpha = 1$). Since the writing of w as uvp is unique it is clear that α is single valued. Also α is a homomorphism of $N(X)'$ ($N(X)$) into S (sending 1 into 1). Moreover, α is uniquely determined by these properties. We shall call $N(X)'$ ($N(X)$) the *free monad* (*free monad with* 1) generated by X. From now on we revert to the usual notation uv for the product $u_{p_R}v$ in $N(X)'$.

If $\mathfrak{A}$ is an algebra (associative algebra) then $\mathfrak{A}$ is a monad (monoid) relative to its multiplication composition. We shall call this the *multiplicative monad* (*multiplicative monoid*) of $\mathfrak{A}$. Let S be any monad and Φ a field. Then we can form the vector space ΦS over Φ with S as basis and define a product in ΦS by

$$(43) \qquad (\sum \alpha_i s_i)(\sum \beta_j t_j) = \sum \alpha_i \beta_j (s_i t_j)$$

for $\alpha_i, \beta_j \in \Phi$, $s_i, t_j \in S$. Then ΦS is an algebra over Φ which is associative if S is associative and has an identity element if S has one. The mapping $s \rightarrow 1s$ is a monomorphism of S into the multiplicative monoid of ΦS. We shall identify S with its image $1S$, s with $1s$. If η is a homomorphism of S into the multiplicative monad of an algebra $\mathfrak{A}$ then $\sum \alpha_i s_i \rightarrow \sum \alpha_i s_i^\eta$ is an algebra homomorphism of

ΦS which extends η. This is unique. We now let S be one of the free systems $M(X)'$, $M(X)$, $N(X)'$, $N(X)$ and accordingly we denote the resulting algebras $\Phi M(X)'$ etc. by $\Phi\{X\}'$, $\Phi\{X\}$, $\Phi\{\{X\}\}'$, $\Phi\{\{X\}\}$ and call these respectively the *free associative algebra, free associative algebra with* 1, *free nonassociative algebra, free nonassociative algebra with* 1 *generated by the set* X.

In view of the universal property of $N(X)'$ and of ΦS for S a monad it is clear that if η is any mapping of X into an algebra $\mathfrak{A}$ then there exists a unique extension of η to a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$. Similar statements hold for the other free algebras we have defined.

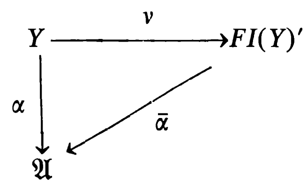
If $\mathfrak{A}$ is an algebra and f is an element of a free nonassociative algebra $\Phi\{\{X\}\}'$ then we say that $\mathfrak{A}$ *satisfies the identity* f (strictly speaking $f = 0$) or f *is an identity for* $\mathfrak{A}$ if f is mapped into 0 under every homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$. Clearly f is contained in a subalgebra of $\Phi\{\{X\}\}'$ generated by a finite subset $\{x_1, x_2, \cdots, x_n\}$ of X so f is a linear combination of products of the x_i. Accordingly, we write $f = f(x_1, x_2, \cdots, x_n)$. If $a_1, a_2, \cdots, a_n$ are arbitrary elements of $\mathfrak{A}$ then there exists a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ sending $x_i \to a_\iota$, $t = 1, \cdots, n$. The image of f under such a homomorphism is unique and we denote it as $f(a_1, a_2, \cdots, a_n)$. Hence $\mathfrak{A}$ satisfies the identity f if and only if $f(a_1, a_2, \cdots, a_n) = 0$ for all $a_i \in \mathfrak{A}$. If I is a subset of $\Phi\{\{X\}\}'$ then we shall call the class $\mathscr{V}(I)$ of algebras satisfying every identity $f \in I$ the *variety defined by* I. If $I = \{f\}$ where $f = (x_1x_2)x_3 - x_1(x_2x_3)$ then $\mathscr{V}(I)$ is the variety of associative algebras. If $I = \{f, g\}$ where $f = x_1{}^2$, $g = (x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2$ then $\mathscr{V}(I)$ is the variety of Lie algebras. If $I = \{f, g\}$, $f = x_1{}^2x_2 - x_1(x_1x_2)$, $g = x_2x_1{}^2 - (x_2x_1)x_1$ then $\mathscr{V}(I)$ is the variety of alternative algebras. If $I = \{f, g\}$, $f = x_1x_2 - x_2x_1$, $g = (x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1)$ then $\mathscr{V}(I)$ is the variety of Jordan algebras.

If $\mathfrak{A}$ is an algebra then the subset $\mathrm{Id}(\mathfrak{A})$ of identities $f \in \Phi\{\{X\}\}'$ satisfied by $\mathfrak{A}$ is an ideal in $\Phi\{\{X\}\}'$. It is clear that $\mathrm{Id}(\mathfrak{A})^\eta \subseteq \mathrm{Id}(\mathfrak{A})$ for every endomorphism η of the free nonassociative algebra $\Phi\{\{X\}\}'$. An ideal having this property is called a T-*ideal* in $\Phi\{\{X\}\}'$. Clearly the subsets I and I' of $\Phi\{\{X\}\}'$ define the same variety $\mathscr{V} = \mathscr{V}(I) = \mathscr{V}(I')$ if the T-ideals generated by I and I' are the same.

It is easily seen that in dealing with varieties of algebras $\mathscr{V}(I)$ there is no loss in generality in assuming that the set X is countably infinite, say $X = \{x_i\}$, $i = 1, 2, 3, \cdots$, and I is a subset of $\Phi\{\{X\}\}'$. At any rate, we shall do this from now on. Now let Y be any nonvacuous set and let $T(I, Y)$ be the set of images in $\Phi\{\{Y\}\}'$ of the set I under all homomorphisms of $\Phi\{\{X\}\}'$ into $\Phi\{\{Y\}\}'$. It is clear that $T(I, Y)$ is mapped into itself by every endomorphism of $\Phi\{\{Y\}\}'$. Hence the ideal $\mathfrak{T}(I, Y)$ in $\Phi\{\{Y\}\}'$ generated by $T(I, Y)$ is a T-ideal in $\Phi\{\{Y\}\}'$. Let $FI(Y)' = \Phi\{\{Y\}\}'/\mathfrak{T}(I, Y)$. Then $FI(Y)' \in \mathscr{V}(I)$. For, let η be a homomorphism of $\Phi\{\{X\}\}'$ into $FI(Y)'$. For each $x_i \in X$ let b_i be an element of $\Phi\{\{Y\}\}'$ such that $x_i{}^\eta = b_i + \mathfrak{T}(I, Y)$ and let τ be the homomorphism of $\Phi\{\{X\}\}'$ into

$\Phi\{\{Y\}\}'$ such that $x_i^{\tau} = b_i$. Then clearly $\eta = \tau\nu$ where ν is the canonical homomorphism $b \rightarrow b + \mathfrak{T}(I, Y)$ of $\Phi\{\{Y\}\}'$ onto $FI(Y)'$. If $f \in I$ then $f^{\tau} \in T(I, Y)$, so $f^{\eta} = f^{\tau\nu} = 0$. Thus $f^{\eta} = 0$ for every $f \in I$ and every homomorphism η of $\Phi\{\{X\}\}'$ into $FI(Y)'$, which means that $FI(Y)' \in \mathscr{V}(I)$. Next let $\mathfrak{A}$ be any algebra in $\mathscr{V}(I)$ and let α be a mapping of Y into $\mathfrak{A}$. We claim that there exists a homomorphism $\bar{\alpha}$ of $FI(Y)'$ into $\mathfrak{A}$ such that $\alpha = \nu\bar{\alpha}$, that is, the following diagram is commutative:

(44)



Let α' be the homomorphism of $\Phi\{\{Y\}\}'$ into $\mathfrak{A}$ which extends α. If $f \in I$ and τ is a homomorphism of $\Phi\{\{X\}\}'$ into $\Phi\{\{Y\}\}'$ then $f^{\tau\alpha'} = 0$ since $\tau\alpha'$ is a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ and $\mathfrak{A} \in \mathscr{V}(I)$. This shows that every element $f^{\tau} \in T(I, Y)$ is mapped into 0 by α'. Hence $\mathfrak{T}(I, Y) \subseteq \ker \alpha'$ and so we have a homomorphism $\bar{\alpha}$ of $FI(Y)' = \Phi\{\{Y\}\}'/\mathfrak{T}(I, Y)$ into $\mathfrak{A}$ such that $\alpha' = \nu\bar{\alpha}$. Then if $y \in Y$, $y^{\alpha} = y^{\alpha'} = y^{\nu\bar{\alpha}}$ so (44) is commutative. We remark that $\bar{\alpha}$ is unique since the y^{ν}, $y \in Y$, generate $FI(Y)'$. In view of the properties we have shown we shall call $FI(Y)'$ the *free I-algebra determined by the set Y*. If there exist nonzero algebras in $\mathscr{V}(I)$ then it is easy to construct, using direct sums, an algebra $\mathfrak{A}$ in $\mathscr{V}(I)$ having cardinal number $\geqq |Y|$ the cardinal number of Y. Then there exists a 1-1 mapping α of Y into $\mathfrak{A}$ and it follows from (44) that the restriction to Y of the canonical homomorphism ν of $\Phi\{\{Y\}\}'$ is 1-1. Then one can identify Y with its image in $FI(Y)'$. We shall do this and call $FI(Y)'$ the free I-algebra *generated* by Y. This is the case in the examples of the classes of alternative, Lie, etc. algebras which we singled out above. In the cases we noted we call $FI(Y)'$ the *free alternative, Lie*, etc. *algebra* generated by Y. We remark that if

$$I \equiv \{(x_1x_2)x_3 - x_1(x_2x_3)\}$$

so that $\mathscr{V}(I)$ is the class of associative algebras then the homomorphism of $FI(Y)'$ into $\Phi\{Y\}'$ which is the identity on Y is an isomorphism. Accordingly, we need not distinguish between these two algebras.

An element $f \in \Phi\{\{X\}\}'$ (or any one of the other free algebras $\Phi\{\{X\}\}$ etc.) is *homogeneous of degree j* in $x_i \in X$ if it is a linear combination of elements of $N(X)'$ all of which are of x_i-degree j. It is clear that for fixed x_i, $\Phi\{\{X\}\}'$ is a graded algebra relative to the subspaces $\mathfrak{H}(x_i,j)$ of elements which are homogeneous of degree j in x_i. This means that $\Phi\{\{X\}\}'$ is a direct sum of the spaces $\mathfrak{H}(x_i,j)$, $j = 0, 1, 2, \cdots$ and $\mathfrak{H}(x_i,j)\mathfrak{H}(x_i,k) \subseteq \mathfrak{H}(x_i,j+k)$.

We shall now formulate some simple conditions on the sets I defining varieties of algebras $\mathscr{V}(I)$. The first of these is the homogeneity condition:

H. Every $f \in I$ is homogeneous in every $x_i \in X$.

This condition is satisfied by the sets defining associative, Lie, alternative and Jordan algebras. Moreover, let $\mathfrak{T}$ be a T-ideal in $\Phi\{\{X\}\}'$ and let $f = f(x_1, x_2, \cdots, x_n) \in \mathfrak{T}$. Write $f = f_0^{(i)} + f_1^{(i)} + \cdots + f_r^{(i)}$ where $f_j^{(i)}$ is homogeneous of degree j in x_i. Let $\alpha \in \Phi$ and consider the endomorphism of $\Phi\{\{X\}\}'$ such that $x_i \to \alpha x_i$, $x_k \to x_k$ if $k \neq i$. This maps f into $f_0^{(i)} + \alpha f_1^{(i)} + \cdots + \alpha^r f_r^{(i)}$. It follows that if Φ contains $r+1$ distinct elements then every $f_j^{(i)} \in \mathfrak{T}$. Thus when Φ is infinite there is no loss in generality in assuming H.

We now introduce another element $y \notin X$ and form the free nonassociative algebra $\Phi\{\{Z\}\}'$ where $Z = X \cup \{y\}$. For each $i = 1, 2, \cdots$ let λ_i be the homomorphism of $\Phi\{\{X\}\}'$ into $\Phi\{\{Z\}\}'$ such that $x_i \to x_i + y$, $x_k \to x_k$, $k \neq i$. If $f \in \Phi\{\{X\}\}'$ has degree m_i in x_i then we can write

$$f(x_1, \cdots, x_{i-1}, x_i + y, x_{i+1}, \cdots, x_n) = \sum_{j=0}^{m_i} f_{ij}(x_1, \cdots, x_n, y) \tag{45}$$

where $f_{ij} = f_{ij}(x_1, \cdots, x_n, y)$ is homogeneous of degree j in y. The f_{ij} are uniquely determined and we have the mappings $D_{ij}: f \to f_{ij}$, $i = 1, 2, \cdots$, $j = 0, 1, 2, \cdots$ of $\Phi\{\{X\}\}'$ into $\Phi\{\{Z\}\}'$. The D_{ij} are linear and it is clear from (45) that

$$(fg)D_{ij} = \sum_{k=0}^{j} (fD_{ik})(gD_{i,j-k}). \tag{46}$$

Also, if we apply the homomorphism of $\Phi\{\{Z\}\}'$ into $\Phi\{\{X\}\}'$ sending $x_i \to x_i$, $y \to 0$ to (45) we obtain $f(x_1, \cdots x_n) = f_{i0} = fD_{i0}$. Hence $D_{i0} = 1$ and $\{D_{i0} = 1, D_{i1}, D_{i2}, \cdots\}$ is a higher derivation of infinite rank of $\Phi\{\{X\}\}'$ into $\Phi\{\{Z\}\}'$ (cf. Jacobson, *Lectures in Abstract Algebra*, vol. III, p. 191). In particular, $D_i \equiv D_{i1}$ is a derivation of $\Phi\{\{X\}\}'$ into $\Phi\{\{Z\}\}'$. This is characterized by the conditions

$$x_k D_i = \delta_{ik} y, \qquad i, k = 1, 2, \cdots. \tag{47}$$

If f is homogeneous of degree m_k in $x_k \neq x_i$ then the same holds for fD_{ij}, and if f is homogeneous of degree m_i in x_i and $j \leqq m_i$ then fD_{ij} is homogeneous of degree $m_i - j$ in x_i. Also in this case $fD_{im_i} = f(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n)$ and $fD_{ij} = 0$ if $j > m_i$. These assertions follow from (46) for the elements $f \in N(X)'$ if we use induction on the degree. Then they follow for all f by linearity of the operators D_{ij}.

We now apply a 1-1 mapping of the set Z onto X. The corresponding homomorphism of $\Phi\{\{Z\}\}'$ into $\Phi\{\{X\}\}'$ maps the elements $fD_{ij} = fD_{ij}(x_1, \cdots, x_n, y)$ into elements $fD_{ij}(x_{i_1}, \cdots, x_{i_n}, x_{i_{n+1}})$ of $\Phi\{\{X\}\}'$. The elements obtained in this way from f and by iterating the process will be called *linearizations of f*. For example, consider the element $g = (x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1)$ which is one of the two elements defining the class of Jordan algebras. We have

$$\begin{aligned} gD_{11} &= ((x_1y + yx_1)x_2)x_1 + (x_1{}^2x_2)y - (x_1y + yx_1)(x_2x_1) - x_1{}^2(x_2y), \\ gD_{12} &= (y^2x_2)x_1 + ((x_1y + yx_1)x_2)y - (x_1y + yx_1)(x_2y) - y^2(x_2x_1). \end{aligned} \tag{48}$$

Hence

$$\begin{aligned} g_1 &= ((x_1x_3 + x_3x_1)x_2)x_1 + (x_1{}^2x_2)x_3 - (x_1x_3 + x_3x_1)(x_2x_1) - x_1{}^2(x_2x_3), \\ g_2 &= (x_3{}^2x_2)x_1 + ((x_1x_3 + x_3x_1)x_2)x_3 - (x_1x_3 + x_3x_1)(x_2x_3) - x_3{}^2(x_2x_1) \end{aligned} \tag{49}$$

are linearizations of g. Also

$$\begin{aligned} g_1D_1 = g_1D_{11} &= ((yx_3 + x_3y)x_2)x_1 + ((x_1x_3 + x_3x_1)x_2)y \\ &+ ((x_1y + yx_1)x_2)x_3 - (yx_3 + x_3y)(x_2x_1) - (x_1x_3 + x_3x_1)(x_2y) \\ &- (x_1y + yx_1)(x_2x_3). \end{aligned} \tag{50}$$

Hence

$$\begin{aligned} h &= ((x_4x_3 + x_3x_4)x_2)x_1 - (x_4x_3 + x_3x_4)(x_2x_1) \\ &+ ((x_1x_3 + x_3x_1)x_2)x_4 - (x_1x_3 + x_3x_1)(x_2x_4) \\ &+ ((x_1x_4 + x_4x_1)x_2)x_3 - (x_1x_4 + x_4x_1)(x_2x_3) \end{aligned} \tag{51}$$

is a linearization of g. . .

Let $f = f(x_1, x_2, \cdots, x_n)$ be homogeneous of degree m_i in x_i, $i = 1, 2, \cdots, n$. Then we call $\sum_{m_i > 0}(m_i - 1)$ the *height* of f. Thus the height is 0 if and only if f is *multilinear*, that is, f is homogeneous of degree $\leqq 1$ in all the x's. It is clear from the results noted before that the linearizations of f are homogeneous and have heights $\leqq$ the height h of f. Moreover, the height of a linearization of f is h only if this element has the form $f(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ where the i_j are distinct. Now suppose Φ contains r distinct elements where $r \geqq m_i$, the degree of f in x_i, $i = 1, 2, \cdots$ and let $\mathfrak{T}$ be a T-ideal in $\Phi\{\{X\}\}'$ containing f. Let η be a homomorphism of $\Phi\{\{Z\}\}'$ into $\Phi\{\{X\}\}'$ mapping Z in a 1-1 way onto X. Then $\mathfrak{T}$ contains

$$f(x_1, \cdots, x_{i-1}, x_i + y, x_{i+1}, \cdots, x_n)^\eta - f(x_1, \cdots, x_n)^\eta - fD_{im_i}{}^\eta = \sum_{j=1}^{m_i - 1} fD_{ij}{}^\eta$$

and $fD_{ij}{}^\eta$ is homogeneous of degree j in y^η. Since there exist $m_i - 1$ distinct nonzero elements of Φ it follows that every linearization $fD_{ij}{}^\eta \in \mathfrak{T}$. Also it is clear that Φ contains as many elements as the degree of these linearizations in any of the x's. Hence it is clear that $\mathfrak{T}$ contains every linearization of f. We shall now formulate the following condition on the sets I of defining varieties $\mathscr{V}(I)$:

L. If $f \in I$ then every linearization of f is contained in the T-ideal $\mathfrak{T}$ generated by I.

The result we have just shown is that L is automatically satisfied if the elements of I are homogeneous in every x_i and Φ contains as many distinct elements as the degree of every $f \in I$ in every x_i. It is clear for the specification of the sets I defining associative, Lie and alternative algebras that these satisfy the conditions H and L. If we recall that for Jordan algebras the characteristic of the base field Φ is $\neq 2$, so that Φ contains at least three distinct elements we see also that the set I defining Jordan algebras also satisfies the conditions.

We shall now show that if the condition H and L hold for I then $\mathscr{V}(I)$ is closed under extension of base field in a sense which will be defined below. For this we require the following

LEMMA. *Let I' be a subset of $\Phi\{\{X\}\}'$ satisfying condition* H *and having the property that I' contains all linearizations of every $f \in I'$. Let $\mathfrak{A}$ be an algebra over Φ, (u_ι) a basis for $\mathfrak{A}/\Phi$. Then $\mathfrak{A} \in \mathscr{V}(I')$ if and only if $f(u_{\iota_1}, u_{\iota_2}, \cdots, u_{\iota_n}) = 0$ for all $f \in I'$ and all $u_{i_j} \in (u_\iota)$.*

PROOF. We have to show that the conditions given imply that $f(a_1, \cdots, a_n) = 0$ for all $a_i \in \mathfrak{A}$, $f \in I'$. We shall prove this by induction on the height of f. For this it is convenient to adjoin 0 to I' and call this an element of height -1. We now assume that $g(a_1, a_2, \cdots, a_m) = 0$ for all $a_j \in \mathfrak{A}$ and $g \in I'$ of height less than the height of f. We assume also that $f(a_1, \cdots, a_{r-1}, u_{\iota_r}, \cdots, u_{\iota_n}) = 0$ for all $a_j \in \mathfrak{A}$, $u_{\iota_k} \in (u_\iota)$. Let $\mathfrak{B}$ be the subset of elements $b \in \mathfrak{A}$ such that

$$f(a_1 \cdots, a_{r-1}, b, u_{\iota_{r+1}}, \cdots, u_{\iota_n}) = 0, \qquad a_j \in \mathfrak{A}, \qquad u_{\iota_k} \in (u_\iota).$$

Then $\mathfrak{B}$ contains the basis (u_ι). Also since f is homogeneous in x_r, $\mathfrak{B}$ is closed under multiplication by elements of Φ. Now let $a_j \in \mathfrak{A}$, $u_{\iota_k} \in (u_\iota)$, $b \in \mathfrak{B}$, and consider

$$\begin{aligned} f(a_1, \cdots, a_{r-1}, b + u_{\iota_s}, u_{\iota_{r+1}}, \cdots, u_{\iota_n}) &= f(a_1, \cdots, a_{r-1}, b, u_{\iota_{r+1}}, \cdots, u_{\iota_n}) \\ &+ f(a_1, \cdots, a_{r-1}, u_{\iota_r}, u_{\iota_{r+1}}, \cdots, u_{\iota_n}) \\ &+ \Sigma f_{rj}(a_1, \cdots, a_{r-1}, b, u_{i_{r+1}}, \cdots, u_{\iota_n}) \end{aligned}$$

where the f_{rj} are linearizations of f and have lower height than f. Our induction assumptions imply that $f(a_1, \cdots, a_{r-1}, b + u_{\iota_r}, u_{\iota_{r+1}}, \cdots, u_{\iota_n}) = 0$. Hence $b + u_{\iota_r} \in \mathfrak{B}$. This now implies that $\mathfrak{B} = \mathfrak{A}$. Thus $f(a_1, \cdots, a_r, u_{\iota_{r+1}}, \cdots, u_{\iota_n}) = 0$ and $f(a_1, \cdots, a_n) = 0$ by induction on r.

Let Γ be an extension field of Φ. Then the elements $f \in I$ can be regarded as elements of $\Gamma\{\{X\}\}'$. Hence these determine the class of algebras over Γ which satisfy the identities of I. We denote this as $\mathscr{V}(I, \Gamma)$ and the class determined by I and Φ as $\mathscr{V}(I, \Phi)$ ($= \mathscr{V}(I)$). Then we have the

THEOREM 5. *Let I be a subset of $\Phi\{\{X\}\}'$ satisfying the conditions H and L and let $\mathfrak{A} \in \mathscr{V}(I, \Phi)$. Then the extension algebra $\mathfrak{A}_\Gamma = \Gamma \otimes_\Phi \mathfrak{A} \in \mathscr{V}(I, \Gamma)$.*

PROOF. Let (u_ι) be a basis for $\mathfrak{A}/\Phi$, hence, for $\mathfrak{A}/\Gamma$ and let I' be the set of linearizations of the elements of I. Then $I' \supseteq I$, I' satisfies H and I' contains the linearizations of the elements of I', since, by definition, a linearization of a linearization of f is a linearization of f. Also since I satisfies L every $f \in I'$ is in the T-ideal generated by I. Hence $f(u_{\iota_1}, u_{\iota_2}, \cdots, u_{\iota_n}) = 0$ for all $u_{\iota_k} \in (u_\iota)$, since $\mathfrak{A} \in \mathscr{V}(I, \Phi)$. Then the lemma implies that $\mathfrak{A}_\Gamma \in \mathscr{V}(I, \Gamma)$.

In particular, this theorem shows that if $\mathfrak{A}$ is associative, Lie, alternative or Jordan then the same is true of $\mathfrak{A}_\Gamma$.

If $\mathfrak{A}$ is an algebra over Φ then we can adjoin an identity element 1 to $\mathfrak{A}$ by forming $\mathfrak{A}^* = \mathfrak{A} \oplus \Phi 1$ and extending the multiplication in $\mathfrak{A}$ to one in $\mathfrak{A}^*$ so that $1a^* = a^* = a^*1$ for $a^* \in \mathfrak{A}^*$. We shall now give a condition on a subset I of $\Phi\{\{X\}\}'$ that along with H and L will insure that if $\mathfrak{A} \in \mathscr{V}(I)$ then $\mathfrak{A}^* \in \mathscr{V}(I)$. This is

U. If $f = f(x_1, x_2, \cdots, x_n)$ is a linearization of an element of I then the elements obtained from f by replacing any number of the x's by 1, which are elements of $\Phi\{\{X\}\}$, are in $\Phi\{\{X\}\}'$ and in the T-ideal of $\Phi\{\{X\}\}'$ generated by I.

If f is multilinear its linearizations are equivalent to f in the sense that they have the form $f(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$, i_j distinct. Hence if I consists of multilinear elements, then I satisfies U if the condition given in U holds for every $f \in I$. For example, if $I = \{[x_1, x_2, x_3]\}$, the set defining associative algebras then $[1, x_2, x_3] = [x_1, 1, x_3] = [x_1, x_2, 1] = 0$ since 1 is in the nucleus of $\Phi\{\{X\}\}$. Hence I satisfies U. If $I = \{[x_1, x_1, x_2], [x_2, x_1, x_1]\}$, the set defining alternative algebras, then the linearizations are equivalent to the elements of I or to one of the two elements: $[x_1, x_2, x_3] + [x_2, x_1, x_3]$, $[x_3, x_1, x_2] + [x_3, x_2, x_1]$. Substituting $x_i = 1$ in any of these gives 0. Hence I satisfies condition U. Finally, let $I = \{[x_1, x_2], [x_1^2, x_2, x_1]\}$, the set defining Jordan algebras. One sees easily (cf. (48)–(51)) that the linearizations of the elements of I are equivalent to these or to one of the following two elements: $[x_1^2, x_3, x_2] + [x_1x_2 + x_2x_1, x_3, x_1]$, $[x_1x_2 + x_2x_1, x_3, x_4] + [x_1x_4 + x_4x_1, x_3, x_2] + [x_2x_4 + x_4x_2, x_3, x_1]$. Substitution of an $x_i = 1$ in these linearizations gives 0 or elements equivalent to $2[x_1, x_2, x_1]$, $2[x_1, x_2, x_3] + 2[x_3, x_2, x_1]$. Substitution of $x_i = 1$ in the latter gives 0. Now $[x_1, x_2, x_1] = [x_1, x_2]x_1 + [x_2x_1, x_1]$ and $[x_1, x_2, x_3] + [x_3, x_2, x_1] = [x_1x_2]x_3 + [x_3x_2]x_1 + [x_2x_1, x_3] + [x_2x_3, x_1]$ so $[x_1, x_2, x_1]$ and $[x_1, x_2, x_3] + [x_3, x_2, x_1]$ are in the T-ideal generated by $[x_1, x_2]$. Hence $I = \{[x_1, x_2], [x_1^2, x_2, x_1]\}$ satisfies U.

We now have

THEOREM 6. *Let I be a subset of $\Phi\{\{X\}\}'$ satisfying the conditions* H, L, U, *and let $\mathfrak{A}$ be an algebra in the class $\mathscr{V}(I)$. Then the algebra $\mathfrak{A}^* = \mathfrak{A} \oplus \Phi 1$ obtained by adjoining an identity element 1 to $\mathfrak{A}$ is in $\mathscr{V}(I)$.*

PROOF. Let (u_ι) be a basis for $\mathfrak{A}$, so $(u_\iota, 1)$ is a basis for $\mathfrak{A}^*$. Let I' be the set of linearizations of the elements of I so $I' \supseteq I$, I' satisfies H and every linearization of an element of I' is contained in I'. If $f = f(x_1, \cdots, x_n) \in I'$ then f evaluated at a subset of $(u_\iota, 1)$ amounts to the evaluation of a suitable polynomial g obtained by putting some of the $x_i = 1$ in f at a subset of (u_ι). By U, g is in the T-ideal generated by I. Since $\mathfrak{A} \in \mathscr{V}(I)$ the evaluation of g at the subset of (u_ι) gives 0. It follows from the Lemma that $\mathfrak{A}^* \in \mathscr{V}(I')$, so $\mathfrak{A}^* \in \mathscr{V}(I)$.

We have seen that the sets I defining associative, alternative, or Jordan algebras satisfy conditions H, L and U. Hence the adjunction of an identity element to an algebra of one of these classes gives an algebra in the same class. We remark also that if $\mathfrak{A}$ is a Lie algebra then $\mathfrak{A}^* = \Phi 1 \oplus \mathfrak{A}$ is not.

Let $f = f(x_1, x_2, \cdots, x_n) \in \Phi\{\{X\}\}'$ be homogeneous of degree m_i in x_i and consider the element $fD_i(x_1, \cdots, x_n, y)$ obtained by applying the derivation D_i to $f(x_1, \cdots, x_n)$. The usual argument via elements of $N(X)'$ and linearity shows that $fD_i(x_1, \cdots, x_n, x_i) = m_i f(x_1, \cdots, x_n)$. If the characteristic is 0 or is a prime exceeding m_i then this shows that f is contained in any T-ideal which contains a linearization $fD_i(x_{i_1}, \cdots, x_{i_n}, x_{i_{n+1}})$ of f. We can use this to prove the following sufficient condition that a class of algebras $\mathscr{V}(I)$ be definable by multilinear elements, that is, $\mathscr{V}(I) = \mathscr{V}(I')$ where I' is a set of multilinear elements of $\Phi\{\{X\}\}'$.

THEOREM 7. *Let I be a set of elements of $\Phi\{\{X\}\}'$ and assume the characteristic of Φ is either 0 or exceeds the degree of homogeneity of every $f \in I$ in every x_i. Then the T-ideal $\mathfrak{T}(I)$ generated by I is identical with the T-ideal $\mathfrak{T}(I')$ generated by a set of multilinear elements.*

PROOF. The result will follow if we can show that if $f \in I$ then there exists a multilinear element g such that the T-ideal $\mathfrak{T}(f)$ generated by f coincides with $\mathfrak{T}(g)$. If f is multilinear there is nothing to prove so we assume $f = f(x_1, x_2, \cdots, x_n)$ is homogeneous of degree $m_i > 1$ in x_i. Then $fD_i(x_{i_1}, \cdots, x_{i_n}, x_{i_{n+1}})$, $i_j \neq i_k$ for $j \neq k$, is a linearization of f of lower height than f. Since the hypothesis on Φ implies that there exist m_i distinct elements in Φ, all linearizations of f are in $\mathfrak{T}(f)$. In particular, $fD_i(x_1, \cdots, x_{i_n}, x_{i_{n+1}}) \in \mathfrak{T}(f)$. On the other hand, we have seen above that f is contained in the T-ideal generated by $fD_i(x_1, \cdots, x_{i_n}, x_{i_{n+1}})$. If we use induction on the height we may suppose that the T-ideal generated by $fD_i(x_1, \cdots, x_{i_n}, x_{i_{n+1}})$ coincides with $\mathfrak{T}(g)$ where g is multilinear. Hence $\mathfrak{T}(f) = \mathfrak{T}(g)$.

EXERCISES

1. Show that the number of elements of degree m in the free monad generated by n elements is given by the formula

$$N(n, m) = \frac{(2m-2)!}{m!(m-1)!} n^m.$$

2. Let $\mathscr{V}(I)$ be a variety of algebras defined by a set of identities $I \subseteq \Phi\{\{X\}\}'$.

Show that if $f \in \Phi\{\{X\}\}'$ is an identity for every algebra in $\mathscr{V}(I)$ then f is contained in the T-ideal generated by I.

3. Let Y' be a nonvacuous subset of Y, I a subset of $\Phi\{\{X\}\}'$ and $FI(Y)'$ the free I-algebra determined by Y. Show that the subalgebra of $FI(Y)'$ generated by the cosets of the elements of Y' is canonically isomorphic to the free I-algebra generated by Y'.

4. Prove that an element $f \in \Phi\{\{X\}\}'$ satisfies $fD_i = 0$ if and only if f is homogeneous of degree 0 in x_i. Use this to show that any nonzero T-ideal in $\Phi\{\{X\}\}'$ contains a nonzero multilinear element. (This implies that if $\mathfrak{A}$ satisfies an identity f where $f \neq 0$ in $\Phi\{\{X\}\}'$ then $\mathfrak{A}$ satisfies a nonzero multilinear identity.)

5. (Glennie). Let I be a set of multilinear elements in $\Phi\{\{X\}\}'$, $T(I)$ the set of elements $F = f(m_1, m_2. \cdots, m_n)$ where $f \in I$ and $m_j \in N(X)'$, $U(I)$ the set of elements of the form $m(x_1, \cdots, x_k, F)$ where $F \in T(I)$ and $m(x_1, \cdots, x_k, y)$ is an element of $N(Z)'$, $Z = X \cup \{y\}$ which is homogeneous of degree 1 in y. Show that the subspace $\Phi U(I)$ of $\Phi\{\{X\}\}'$ is the T-ideal $\mathfrak{T}(I)$ generated by I. Show that if I is finite and X is finite then the subset $U^{(r)}(I)$ of elements of $U(I)$ of degree $\leqq r$ is finite. Use this to prove that there is a finite procedure for deciding whether or not a given element $g \in \Phi\{\{X\}\}'$ belongs to $\mathfrak{T}(I)$. (This shows that there is an algorithm for deciding whether or not a given $g \in \Phi\{\{X\}\}'$ is an identity for the variety $\mathscr{V}(I)$, I a finite set of multilinear elements.)

6. Let $f(x_1, x_2, \cdots, x_{n+1}) \in \Phi\{\{X\}\}'$, $X = \{x_i\}$, $i = 1, 2, \cdots$, be homogeneous of degree m in x_{n+1}. Write x for $(x_1, \cdots, x_n)$ and define

$$\begin{aligned} f^{(m)}(x, x_{n+1}, \cdots, x_{n+m}) = {} & f(x, x_{n+1} + \cdots + x_{n+m}) \\ & - \sum_i f(x, x_{n+1} + \cdots + \hat{x}_{n+i} + \cdots + x_{n+m}) \\ & + \sum_{i<j} f(x, x_{n+1} + \cdots + \hat{x}_{n+i} + \cdots + \hat{x}_{n+j} + x_{n+m}) \\ & - \cdots + (-1)^{m-1} \sum_i f(x, x_{n+i}) \end{aligned}$$

where $\hat{}$ denotes the omission of the variable appearing below it. Prove that $f^{(m)}$ can be obtained by the following inductive process: $f^{(1)} = f(x, x_{n+1})$, $f^{(k+1)} = f^{(k)} D_{n+1}(x, x_{n+1}, \cdots, x_{n+k}, x_{n+k+1})$ (that is, the result of putting $y = x_{n+k+1}$ in the derivative $f^{(k)} D_{n+1}(x, x_{n+1}, \cdots, x_{n+k}, y)$). Prove that

$$f^{(m)} = f^{(m)}(x, x_{n+1}, \cdots, x_{n+m})$$

is homogeneous of degree 1 in every x_{n+i}, $1 \leqq i \leqq m$ and that

$$f^{(m)}(x, x_{n+1}, x_{n+1}, \cdots, x_{n+1}) = m!f.$$

7. An ideal $\mathfrak{T}$ in $\Phi\{\{X\}\}'$ is called *homogeneous in* x_i if the following condition holds: an element $f = \sum f_j$, f_j homogeneous of degree j in x_i, $\in \mathfrak{T}$ if and only if

every $f_j \in \mathfrak{T}$. Show that an ideal is homogeneous in x_i if and only if it has a set of generators which are homogeneous in x_i. Show that if I is a set satisfying H and L then the T-ideal generated by I is homogeneous in every x_i.

8. The class of *noncommutative Jordan algebra* is defined by the set of five identities: $x_1(x_1{}^2x_2) - x_1{}^2(x_1x_2)$, $(x_2x_1{}^2)x_1 - (x_2x_1)x_1{}^2$, $(x_1x_2)x_1 - x_1(x_2x_1)$, $(x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1)$, $(x_1x_2)x_1{}^2 - x_1(x_2x_1{}^2)$. Note that this set satisfies H, L, U if the characteristic is not two and that the class of noncommutative Jordan algebras can be defined by multilinear identities if the characteristic is not two or three. Show that if $\mathfrak{A}$ is a noncommutative Jordan algebra of characteristic not two then $\mathfrak{A}^+$ is a Jordan algebra.

9. Prove that every alternative algebra is a noncommutative Jordan algebra.

10. The class of *Malcev* algebras is defined by the following identities: $x_1{}^2$, $(x_1x_2)(x_1x_3) - ((x_1x_2)x_3)x_1 - ((x_2x_3)x_1)x_1 - ((x_3x_1)x_1)x_2$. Show that this class includes the class of Lie algebras. Show that if $\mathfrak{A}$ is alternative of characteristic not two then $\mathfrak{A}^-$ is a Malcev algebra.

7. **Basic Jordan identities. Jordan triple product.** Let $\mathfrak{J}$ be a Jordan algebra over a field Φ (of characteristic not two). We write the product in $\mathfrak{J}$ as $a . b$ and use this notation also for $\frac{1}{2}(ab + ba)$ in any algebra over Φ. Also we shall use the abbreviation $a_1 . a_2 . a_3 . \cdots . a_k$ for $(\cdots ((a_1 . a_2) . a_3) . \cdots a_k)$ and $a^{.k}$ for $a . a . \cdots . a$ with k factors. The linear mapping $x \to x . a$ $(= a . x)$ in $\mathfrak{J}$ will be denoted as R_a and if $\mathfrak{A}$ is any algebra with product ab then we write a_L for $x \to ax$, a_R for $x \to xa$ and $R_a = \frac{1}{2}(a_L + a_R)$. We have seen in §6 (eq. (51)) that

$$\begin{aligned} h &= ((x_4x_3 + x_3x_4)x_2)x_1 - (x_4x_3 + x_3x_4)(x_2x_1) \\ &\quad + ((x_1x_3 + x_3x_1)x_2)x_4 - (x_1x_3 + x_3x_1)(x_2x_4) \\ &\quad + ((x_1x_4 + x_4x_1)x_2)x_3 - (x_1x_4 + x_4x_1)(x_2x_3) \\ &= [x_4x_3 + x_3x_4, x_2, x_1] + [x_1x_3 + x_3x_1, x_2, x_4] + [x_1x_4 + x_4x_1, x_2, x_3] \end{aligned}$$

is a linearization of the defining identity $(x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1) = [x_1{}^2, x_2, x_1]$ for Jordan algebras. If we specialize $x_2 \to a$, $x_1 \to b$, $x_3 \to c$, $x_4 \to d$ in h and use commutativity and the fact that the characteristic is not two we obtain the following identity for any Jordan algebra:

$$\begin{aligned} &(a . b) . (c . d) + (a . d) . (b . c) + (a . c) . (b . d) \\ &\quad = a . (c . d) . b + a . (b . c) . d + a . (b . d) . c. \end{aligned} \tag{B$_1$}$$

This is equivalent to

$$[c . d, a, b] + [b . c, a, d] + [b . d, a, c] = 0. \tag{B$'_1$}$$

Also since $[a, b, c] = a . b . c - a . (b . c) = - c . b . a + c . (b . a) = - [c, b, a]$ we have also

$$[b, a, c . d] + [d, a, b . c] + [c, a, b . d] = 0. \tag{B$''_1$}$$

We now put

$$(52)\qquad \begin{aligned}(abcd) &= (a.b).(c.d)+(a.d).(b.c)+(a.c).(b.d),\\ (a;bcd) &= a.(b.c).d+a.(b.d).c+a.(c.d).b.\end{aligned}$$

Then (B_1) reads $(abcd) = (a;bcd)$. If we take into account the commutative law we see that $(abcd)$ is symmetric in all of its arguments while $(a;bcd)$ is symmetric in the last three. Hence (B_1) and commutativity imply

$$(53)\qquad (a;bcd) = (b;acd) = (c;abd) = (d;abc).$$

The equality of the first and third of these is equivalent to

$$(B_2)\qquad \begin{aligned}&a.(b.c).d+a.(b.d).c+a.(c.d).b\\ &= a.b.c.d+a.d.c.b+a.(b.d.c).\end{aligned}$$

By (B_1) and (B_2) we have also

$$(B_3)\qquad \begin{aligned}&a.b.c.d+a.d.c.b+a.(b.d.c)\\ &= (a.b).(c.d)+(a.c).(b.d)+(a.d).(b.c).\end{aligned}$$

In operator form the defining relation $(a^{\cdot 2}.b).a = a^{\cdot 2}(b.a)$ and commutativity give

$$(O)\qquad [R_aR_{a^{\cdot 2}}] = 0.$$

If we consider a as a variable in (B_1), (B_2) and (B_3) and b, c, d etc. as defining multiplications then we see that these identities are equivalent to the following operator identities:

$$(O_1)\qquad [R_{b.c}R_d]+[R_{b.d}R_c]+[R_{c.d}R_b] = 0,$$

$$(O_2)\qquad R_{b.d.c} = R_{b.c}R_d+R_{b.d}R_c+R_{c.d}R_b-R_bR_cR_d-R_dR_cR_b,$$

$$(O_3)\qquad R_{b.d.c} = R_dR_{b.c}+R_cR_{b.d}+R_bR_{c.d}-R_bR_cR_d-R_dR_cR_b.$$

If we interchange b and c in (O_2) and subtract from (O_2) we obtain

$$\begin{aligned}R_{b.d.c}-R_{b.(d.c)} &= R_cR_bR_d+R_dR_bR_c-R_bR_cR_d-R_dR_cR_b.\\ &= [[R_cR_b]R_d].\end{aligned}$$

Since R_a is linear in a, $R_{b.d.c}-R_{b.(d.c)} = R_{(b.d).c-b.(d.c)} = R_{[b,d,c]}$. Hence we have

$$(54)\qquad [[R_cR_b]R_d] = R_{[b,d,c]}.$$

This result shows that the set $R(\mathfrak{J})$ of multiplications R_a in a Jordan algebra $\mathfrak{J}/\Phi$ is a subspace of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ closed under the iterated Lie multiplication $[[AB]C]$. Such a subspace of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ is called a *Lie triple system of linear* *transformations in the vector space* $\mathfrak{J}$. The result (54) has another important interpretation which is more readily seen by writing (54) as

$$(54') \qquad [R_d[R_bR_c]] = R_{d[R_bR_c]}.$$

Now we recall that if $\mathfrak{A}$ is an arbitrary algebra then a derivation D in $\mathfrak{A}$ is a linear mapping $\mathfrak{A}$ satisfying $(xy)D = (xD)y + x(yD)$. This condition is equivalent to either one of the following operator conditions:

$$(55) \qquad [x_L, D] = (xD)_L, \qquad [y_R, D] = (yD)_R, \qquad x, y \in \mathfrak{A}.$$

We recall also that the set Der $\mathfrak{A}$ (more precisely $\mathrm{Der}_\Phi(\mathfrak{A}, \mathfrak{A})$) of derivations is a Lie algebra of linear transformations in $\mathfrak{A}/\Phi$, that is, a subalgebra of the Lie algebra $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})^-$. Moreover, this is restricted in the sense that $D^p \in \mathrm{Der}\,\mathfrak{A}$ for $D \in \mathrm{Der}\,\mathfrak{A}$ if the characteristic is $p \neq 0$ (Jacobson, *Lie Algebras*, p. 7). Now it is clear from (54′) and (55) that any mapping $[R_bR_c]$ in a Jordan algebra $\mathfrak{J}$ is a derivation. More generally, the mappings $\Sigma_i[R_{b_i}R_{c_i}]$ are derivations. We shall call the derivations of this form *inner* derivations of $\mathfrak{J}$. It is easy to check that the inner derivations form an ideal in Der $\mathfrak{J}$ (ex. 2 below) and it can be proved that this ideal, which we shall denote as Inder $\mathfrak{J}$, is closed under the p-power mapping if the characteristic is $p \neq 0$. (ex. 2 below and ex. 6 on p. 49).

We now put $b = a^{\cdot k}$, $k \geqq 1$, $c = d = a$ in O_2. This gives

$$R_{a^{\cdot k+2}} = 2R_{a^{\cdot k+1}}R_a + R_{a^{\cdot 2}}R_{a^{\cdot k}} - R_{a^{\cdot k}}R_a^{\,2} - R_a^{\,2}R_{a^{\cdot k}}.$$

We know that R_a and $R_{a^{\cdot 2}}$ commute, and the foregoing recursion formula implies that $R_{a^{\cdot k+2}}$, $k \geqq 1$ is contained in the subalgebra of $\mathrm{Hom}(\mathfrak{J}, \mathfrak{J})$ generated by R_a and $R_{a^{\cdot 2}}$. Hence $[R_{a^{\cdot k}}R_{a^{\cdot l}}] = 0$, $k, l \geqq 1$, which permits the simplification of the recursion formula to

$$(56) \qquad R_{a^{\cdot k+2}} = 2R_aR_{a^{\cdot k+1}} - (2R_a^{\,2} - R_{a^{\cdot 2}})R_{a^{\cdot k}}, \qquad k \geqq 1.$$

This recursion formula can be solved explicitly to give

$$(57) \qquad R_{a^{\cdot k}} = \tfrac{1}{2}\{(R_a + (R_{a^{\cdot 2}} - R_a^{\,2})^{\frac{1}{2}})^k + (R_a - (R_{a^{\cdot 2}} - R_a^{\,2})^{\frac{1}{2}})^k\}$$

which makes sense since the odd powers of $(R_{a^{\cdot 2}} - R_a^{\,2})^{\frac{1}{2}}$ cancel off. The formula can be established by induction on k.[4]

[4] To treat this rigorously we introduce the ring $\Phi[\xi, \eta, \zeta]$ where ξ, η are indeterminates and $\zeta^2 = \eta - \xi^2$. In this ring we define $f(1) = \xi$, $f(2) = \eta$ and $f(k+2) = 2\xi f(k+1) - (2\xi^2 - \eta)f(k)$. Then $f(k)$ is uniquely determined by these formulas and $f(k) \in \Phi[\xi, \eta]$. On the other hand, one sees by induction on k that if we set $g(k) = \frac{1}{2}\{(\xi + \zeta)^k + (\xi - \zeta)^k\}$ then $g(1) = \xi$, $g(2) = \eta$ and $g(k+2) = 2\xi g(k+1) - (2\xi^2 - \eta)g(k)$. Hence $g(k) = f(k) \in \Phi[\xi, \eta]$. Now we have the homomorphism of $\Phi[\xi, \eta]$ into $\mathrm{Hom}(\mathfrak{J}, \mathfrak{J})$ sending $\xi \to R_a$, $\eta \to R_{a^{\cdot 2}}$. Since (56) holds, our homomorphism maps $f(k)$ into $R_{a^{\cdot k}}$. We have the formula $f(k) = \frac{1}{2}\{(\xi + \zeta)^k + (\xi - \zeta)^k\}$ which reduces to a polynomial in ξ, η. Applying the homomorphism we obtain a formula for $R_{a^{\cdot k}}$ as a polynomial in R_a and $R_{a^{\cdot 2}}$.

A nonassociative algebra is called *power associative* if the subalgebras generated by single elements of the algebra are associative. This is equivalent to the identities $a^k a^l = a^{k+l}$, $k, l = 1, 2, \cdots$, if a^k is defined by $a^1 = a$, $a^k = a^{k-1}a$. Nearly all the interesting nonassociative algebras are power associative. In particular, we have

THEOREM 8. *Any Jordan algebra is power associative.*

PROOF. We shall prove $a^{\cdot k} . a^{\cdot l} = a^{\cdot k+l}$ by induction on l. For $l = 1$ this holds by definition of $a^{\cdot k}$. Now assume $a^{\cdot k} . a^{\cdot r} = a^{\cdot k+r}$. Then $a^{\cdot k} . a^{\cdot r+1} = a^{\cdot k} . (a^{\cdot r} . a) = a^{\cdot r} R_a R_{a^{\cdot k}} = a^{\cdot r} R_{a^{\cdot k}} R_a = (a^{\cdot r} . a^{\cdot k}) . a = a^{\cdot k+r} . a = a^{\cdot k+r+1}$. Hence $a^{\cdot k} . a^{\cdot l} = a^{\cdot k+l}$ for $k, l = 1, 2, 3, \cdots$.

An important composition in a Jordan algebra is given by the *Jordan triple product*

$$\{abc\} \equiv a . b . c + b . c . a - a . c . b. \tag{58}$$

If the algebra is special and $a . b = \frac{1}{2}(ab + ba)$ in terms of the associative product ab then

$$\begin{aligned} \{abc\} &= \tfrac{1}{4}(abc + bac + cab + cba + bca + cba + abc + acb \\ &\qquad - acb - cab - bac - bca) \\ &= \tfrac{1}{2}(abc + cba). \end{aligned}$$

In particular, $\{aba\} = aba$ in a special Jordan algebra. In any Jordan algebra we shall denote the linear mapping $x \to \{axb\}$ by $U_{a,b}$ and we abbreviate $U_{a,a} = U_a$. We have

$$U_{a,b} = R_a R_b + R_b R_a - R_{a.b}, \tag{59}$$

$$U_a = 2R_a^{\,2} - R_{a^{\cdot 2}}. \tag{60}$$

The recursion formula (56) can be written in terms of U_a as

$$R_{a^{\cdot k+2}} = 2R_a R_{a^{\cdot k+1}} - U_a R_{a^{\cdot k}}. \tag{57'}$$

Clearly the $U_{a^{\cdot k}}$ are contained in the subalgebra of $\mathrm{Hom}(\mathfrak{J}, \mathfrak{J})$ generated by R_a and $R_{a^{\cdot 2}}$. We note also that U_a is a quadratic form in a since for $\alpha \in \Phi$, $U_{\alpha a} = \alpha^2 U_a$ and

$$\tfrac{1}{2}(U_{a+b} - U_a - U_b) = U_{a,b} \tag{61}$$

is bilinear in a and b. We also have the following immediate consequence of the definition of $\{abc\}$:

$$\begin{aligned} \{abc\} &= \{cba\}, \\ \{1ab\} &= \{a1b\} = \{ab1\} = a . b \end{aligned}$$

if $\mathfrak{J}$ has an identity element 1, and

$$\{a^{\cdot k}a^{\cdot l}b\} = a^{\cdot k+l}.b.$$

Exercises

1. Verify that the derivations of any algebra form a Lie algebra of linear transformations and that this is restricted if the characteristic is $p \neq 0$.

2. Verify that the set Inder $\mathfrak{J}$ of inner derivations of a Jordan algebra $\mathfrak{J}$ is an ideal in Der $\mathfrak{J}$. Show that if $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}^+$, $\mathfrak{A}$ associative, then $x[R_aR_b] = \frac{1}{4}[x[ab]]$ (where $[ab] = ab - ba$). Use this, ex. 6, p. 10 and the p power formulas in associative algebras of characteristic p:

$$[ab^p] = [\cdots[ab]b]\cdots] \qquad (p \text{ times}),$$

$$(a+b)^p = a^p + b^p + \sum_{1}^{p-1} s_i(a,b)$$

where $s_i(a,b)$ are certain Lie elements in a and b (Jacobson's *Lie Algebras* pp. 186–187) to prove that if $\mathfrak{J}$ is a special Jordan algebra then Inder $\mathfrak{J}$ is a restricted subalgebra of Der $\mathfrak{J}$. (The extension of this to arbitrary Jordan algebras will be considered in ex. 6, p. 49.)

3. Show that $R_{a^{\cdot p}} = R_a^{\,p}$ in any Jordan algebra of characteristic $p \neq 0$.

4. Show that the following Jordan triple product identities hold in special Jordan algebras:

(62) $$\{aba\}^{\cdot 2} = \{a\{ba^{\cdot 2}b\}a\},$$

(63) $$\{\{aba\}c\{aba\}\} = \{a\{b\{aca\}b\}a\},$$

(64) $$\{\{\{aca\}cb\}ca\} = \{\{aca\}c\{acb\}\}.$$

5. Prove the following identities for arbitrary Jordan algebras:

$$\{abc\}.d = \{(a.d)bc\} - \{a(b.d)c\} + \{ab(c.d)\},$$

$$\{abc\}.d = \{a(b.c)d\} - \{(a.c)bd\} + \{c(a.b)d\},$$

$$\{a(b.c)d\} = \{abc\}.d - \{(a.d)bc\} + \{dcb\}.a.$$

8. **Strongly associative subalgebras. Jordan triple product identities.** In this section we shall derive some Jordan triple product identities which will be needed for the proof of an important general theorem on identities (Macdonald's theorem) which we shall give in §9. While these identities will be required only for the cases in which certain of the variables are powers of a single element, it appears to be of interest to generalize them to the situation in which these variables are in a strongly associative subalgebra which is defined as follows.

Definition 4. *A subalgebra* $\mathfrak{A}$ *of a Jordan algebra* $\mathfrak{J}$ *is called a* strongly associative subalgebra *if* $[R_a R_{a'}] = 0$ *for all* $a, a' \in \mathfrak{A}$ *where* R_a *denotes the multiplication* $(x \to x.a)$ *in* $\mathfrak{J}$ *determined by* a.

The condition is clearly equivalent to $[a, x, a'] = 0$ for $a, a' \in \mathfrak{A}$, $x \in \mathfrak{J}$. Specializing x to $\mathfrak{A}$ we see that $\mathfrak{A}$ is an associative subalgebra of $\mathfrak{J}$. If $\mathfrak{A}$ is any subalgebra of $\mathfrak{J}$ then we write $R_{\mathfrak{J}}(\mathfrak{A})$ for the set of multiplications R_a in $\mathfrak{J}$, $a \in \mathfrak{A}$. If S is a subset of $\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$, $\mathfrak{M}$ a vector space over Φ then we denote the subalgebra of $\mathrm{Hom}(\mathfrak{M}, \mathfrak{M})$ generated by S and the identity mapping 1 by S^*. In particular, $R_{\mathfrak{J}}(\mathfrak{A})^*$ is the subalgebra generated by 1 and the R_a, $a \in \mathfrak{A}$. Since an associative algebra is commutative if and only if it has a set of generators which commute, it is clear that $\mathfrak{A}$ is a strongly associative subalgebra of $\mathfrak{J}$ if and only if $R_{\mathfrak{J}}(\mathfrak{A})^*$ is a commutative algebra of linear transformations.

Examples. (1) If $a \in \mathfrak{J}$ then $[R_{a^{\cdot k}} R_{a^{\cdot l}}] = 0$. Hence any subalgebra $\mathfrak{A}$ of $\mathfrak{J}$ generated by a single element is a strongly associative subalgebra. Also if $\mathfrak{J}$ has an identity element then it is clear that the subalgebra $\Phi[a]$ generated by 1 and any $a \in \mathfrak{J}$ is a strongly associative subalgebra.

(2) If $\mathfrak{A}$ is a strongly associative subalgebra of $\mathfrak{J}$ then we have seen that $\mathfrak{A}$ is an associative subalgebra of $\mathfrak{J}$. We shall now give an example of a subalgebra which is associative but not strongly associative. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{M}$ the Jordan algebra of the symmetric bilinear for f on the vector space $\mathfrak{M}/\Phi$. If $x, y, z \in \mathfrak{M}$ then $y[R_x R_z] = x.y.z - x.(y.z) = f(x,y)z - f(y,z)x$. If $\mathfrak{N}$ is a subspace of $\mathfrak{M}$ then $\Phi 1 + \mathfrak{N}$ is a subalgebra of $\mathfrak{J}$ and if $\mathfrak{N}$ is totally isotropic then $[x, y, z] = 0$ for all $x, y, z \in \mathfrak{N}$ which implies that $\mathfrak{A} = \Phi 1 + \mathfrak{N}$ is an associative subalgebra. Now suppose there exist vectors $x, z \in \mathfrak{N}$, $y \in \mathfrak{M}$ such that $f(x,y) = 1, f(z,y) = 0$. This will be the case, for example, if $\mathfrak{M}$ has a basis $(u_1, \cdots, u_m, v_1, \cdots, v_m)$ such that

$$f(u_i, u_j) = 0 = f(v_i, v_j),$$

$$f(u_i, v_j) = \delta_{ij} = f(v_j, u_i).$$

Then we can take $\mathfrak{N} = \sum \Phi u_i$, $x = u_1$, $z = u_2$, $y = v_1$. Assuming the indicated condition, we have $y[R_x R_z] = f(x,y)z - f(y,z)x = z$. Hence $\mathfrak{A} = \Phi 1 + \mathfrak{N}$ is not a strongly associative subalgebra of $\mathfrak{J}$.

We shall now prove the following result on Jordan triple products.

Lemma 8. *Let* $\mathfrak{A}$ *be a strongly associative subalgebra of a Jordan algebra* $\mathfrak{J}$. *Then the following relations hold for* $a, a' \in \mathfrak{A}$, $b \in \mathfrak{J}$:

(65) $$U_{a.a',a} = U_a R_{a'},$$

(66) $$U_a U_{a'} = U_{a.a'},$$

(67) $$U_b R_a = 2R_b U_{a,b} - U_{a.b^{\cdot 2}},$$

(68) $$U_a U_{a',b} = 2R_a U_{a.a',b} - U_{a^{\cdot 2}.a'.b},$$

$$(69) \qquad 2U_{a.a',b}R_a = U_{a',b}U_a + U_{a^{\cdot 2}.a',b},$$

$$(70) \qquad U_{a,b}R_{a.a'} = U_{a^{\cdot 2}.a',b} + R_bR_{a'}U_a - U_{a.a',b}R_a.\text{[5]}$$

PROOF. Since we can adjoin an identity to $\mathfrak{J}$ and obtain a Jordan algebra, we may assume without loss of generality that $\mathfrak{J}$ has a 1 and $\mathfrak{A}$ contains 1. We now substitute $U_{x,y} = R_xR_y + R_yR_x - R_{x.y}$, $U_{x^{\cdot 2},y} = R_{x^{\cdot 2}}R_y + R_yR_{x^{\cdot 2}} - R_{x^{\cdot 2}.y}$ in $R_{x^{\cdot 2}}R_y + 2R_{x.y}R_x = R_yR_{x^{\cdot 2}} + 2R_xR_{x.y} = 2R_xR_yR_x + R_{x^{\cdot 2}.y}$ which are obtained from (B_1), (B_2) and (B_3). This gives

$$(71) \qquad U_{x^{\cdot 2},y} = 2U_{x,y}R_x - R_yU_x = 2R_xU_{x,y} - U_xR_y.$$

Linearizing the x gives

$$(72) \qquad \begin{aligned} U_{x.z,y} &= U_{x,y}R_z + U_{z,y}R_x - R_yU_{x,z} \\ &= R_zU_{x,y} + R_xU_{z,y} - U_{x,z}R_y. \end{aligned}$$

If we put $x = y = a$, $z = a'$ in the first of these and use the commutativity of $R_{\mathfrak{J}}(\mathfrak{A})^*$ we obtain (65). If we linearize (65) with respect to a in $\mathfrak{A}$ we obtain

$$(73) \qquad U_{a.a',a''} + U_{a''.a',a} = 2U_{a,a''}R_{a'}$$

for $a, a', a'' \in \mathfrak{A}$. We have

$$\begin{aligned} U_aU_{a'} &= 2U_aR_{a'}^2 - U_aR_{(a')^{\cdot 2}} \\ &= 2U_{a.a',a}R_{a'} - U_{a.(a')^{\cdot 2},a} && \text{(by (65))} \\ &= U_{(a.a').a',a} + U_{a.a'} - U_{a.(a')^{\cdot 2},a} && \text{(by (73))} \\ &= U_{a.a'} \end{aligned}$$

which proves (66). If we put $x = z = b$, $y = a$ in the second part of (72) we obtain $U_{b^{\cdot 2},a} = R_bU_{a,b} + R_bU_{a,b} - U_bR_a$ which is equivalent to (67). We now replace x by $x^{\cdot 2}$ in (72) to obtain

$$\begin{aligned} U_{x^{\cdot 2}.z,y} &= U_{x^{\cdot 2},y}R_z + U_{z,y}R_{x^{\cdot 2}} - R_yU_{x^{\cdot 2},z} \\ &= R_zU_{x^{\cdot 2},y} + R_{x^{\cdot 2}}U_{z,y} - U_{x^{\cdot 2},z}R_y. \end{aligned}$$

If we use (71) and the definition of U_x in these relations we obtain

$$\begin{aligned} U_{x^{\cdot 2}.z,y} &= 2U_{x,y}R_xR_z - R_yU_xR_z + U_{z,y}(2R_x^2 - U_x) \\ &\quad - 2R_yR_xU_{x,z} + R_yU_xR_z \\ &= 2U_{x,y}[R_xR_z] + 2U_{x,y}R_zR_x \\ &\quad + 2U_{z,y}R_x^2 - U_{z,y}U_x \\ &\quad + 2R_y[U_{x,z}R_x] - 2R_yU_{x,z}R_x. \end{aligned}$$

[5] These identities are due to Jacobson-Paige [1] and to Macdonald [1]. We are indebted to Professor Paul Cohn for the proofs we shall give.

Hence by (72),

$$U_{x^{\cdot 2}.z,y} = 2U_{x,y}[R_xR_z] + 2U_{x.z,y}R_x - U_{z,y}U_x + 2R_y[U_{x,z}R_x]. \tag{74}$$

Similarly,

$$U_{x^{\cdot 2}.z,y} = 2[R_zR_x]U_{x,y} + 2R_xU_{x.z,y} - U_xU_{z,y} + 2[R_xU_{x,z}]R_y. \tag{75}$$

If we put $x = a$, $z = a'$, $y = b$ in (75) and (74) we obtain (68) and (69) respectively. To obtain (70) we set $a' = 1$ in (69) and linearize with respect to a in $\mathfrak{A}$. This gives

$$U_{a,b}R_{a'} + U_{a',b}R_a = R_bU_{a,a'} + U_{a.a',b}.$$

Replace a' by $a.a'$ in this equation. Then we obtain

$$\begin{aligned} U_{a,b}R_{a.a'} &= R_bU_{a,a.a'} + U_{a^{\cdot 2}.a',b} - U_{a.a',b}R_a \\ &= R_bR_{a'}U_a + U_{a^{\cdot 2}.a',b} - U_{a.a',b}R_a \qquad \text{(by (65))} \end{aligned}$$

which is (70).

Exercise

1. Let $\mathfrak{J}$ be a subalgebra of a Jordan algebra $\mathfrak{A}^+$, $\mathfrak{A}$ associative. Show that $\mathfrak{J}$ is associative if and only if $[[ab]c] = 0$ for all $a, b, c \in \mathfrak{J}$. Let $\mathfrak{M}$ be a vector space over Φ, $E(\mathfrak{M})$ the exterior algebra over $\mathfrak{M}$. Show that $E(\mathfrak{M})^+$ is an associative Jordan algebra.

9. **Macdonald's theorem. Shirshov's theorem.** In this section we shall prove the most important general theorem on identities in Jordan algebras, namely, that any three variable identity which is of degree at most one in one of these and which holds for all special Jordan algebras is valid for all Jordan algebras. More precisely, let $f(x_1, x_2, x_3)$ be an element in the free nonassociative algebra generated by x_1, x_2, x_3 which is of degree $\leqq 1$ in x_3. Assume f is an identity for all special Jordan algebras. Then f is an identity for all Jordan algebras. For example, write $\{x_1x_2x_1\} = 2(x_2x_1)x_1 - x_2x_1{}^2$ and let $f = \{x_1x_2x_1\}^2 - \{x_1\{x_2x_1{}^2x_2\}x_1\}$, $g = \{\{x_1x_2x_1\}x_3\{x_1x_2x_1\}\} - \{x_1\{x_2\{x_1x_3x_1\}x_2\}x_1\}$. Then it is easily seen that f and g are identities for special Jordan algebras (ex. 4, §7). The first of these is of degree 0 in x_3, the second of degree 1. Hence, it will follow from the theorem that we shall prove that these are identities for all Jordan algebras.

It is convenient for the present considerations to deal with algebras having identity elements. Accordingly, we define the *free Jordan algebra with* 1 $FJ(Y)$ *generated by a* (nonvacuous) *set* Y as $FJ(Y) = \Phi\{\{Y\}\}/\mathfrak{T}(Y)$ where $\Phi\{\{Y\}\}$ is the free nonassociative algebra with 1 generated by Y and $\mathfrak{T}(Y)$ is the ideal in $\Phi\{\{Y\}\}$ generated by all the elements of the form $ab - ba$, $(a^2b)a - a^2(ba)$, $a, b \in \Phi\{\{Y\}\}$. One sees easily as in §7 that $FJ(Y)$ is a Jordan algebra with 1, that we can identify Y with its image in $FJ(Y)$ (under the canonical homomorphism ν) and that if α is any mapping of Y into a Jordan algebra $\mathfrak{J}$ with 1 then α has a unique extension to a homomorphism of $FJ(Y)$ into $\mathfrak{J}$ sending 1 into 1. If Y is finite and $|Y| = r$ we write $Y = \{y_1, y_2, \cdots, y_r\}$ and put $FJ(Y) = FJ^{(r)}$.

We recall the definition of the free special Jordan algebra with 1 $FSJ^{(r)}$. This is the subalgebra of $\Phi\{u_1, u_2, \cdots, u_r\}^+$ generated by 1 and the u_i where $\Phi\{u_1, u_2, \cdots, u_r\}$ is the free associative algebra with 1 generated by r distinct elements $u_1, u_2, \cdots, u_r$. We have the homomorphism σ of $FJ^{(r)}$ onto $FSJ^{(r)}$ such that $1^\sigma = 1$, $y_i^\sigma = u_i$, $1 \leqq i \leqq r$, whose kernel we denote as $\mathfrak{K}^{(r)}$. For $r = 3$ we use the notations x, y, z and u, v, w for y_1, y_2, y_3 and u_1, u_2, u_3 respectively. The main esult we shall prove in this section is

MACDONALD'S THEOREM. *Let $FJ^{(3)}$ be the free Jordan algebra with* 1 *generated by x, y, z, $FSJ^{(3)}$ the free special Jordan algebra with* 1 *generated by u, v, w, σ the homomorphism of $FJ^{(3)}$ onto $FSJ^{(3)}$ mapping $1 \to 1$, $x \to u$, $y \to v$, $z \to w$, $\mathfrak{K}^{(3)}$ the kernel of σ and let $\mathfrak{Z}$ be the subspace of elements of $FJ^{(3)}$ which are homogeneous of degree* 1 *in z. Then* $\mathfrak{K}^{(3)} \cap \mathfrak{Z} = 0$.

We shall see that this result will imply the results we stated above on identities. It will imply also an earlier result due to Shirshov that $\mathfrak{K}^{(2)} = 0$, or, equvalently, the free Jordan algebra $FJ^{(2)}$ is special. A precise definition of the set $\mathfrak{Z}$ in the theorem is that this is the image of the space of elements of $\Phi\{\{x, y, z\}\}$ homogeneous of degree one in z under the canonical homomorphism of $\Phi\{\{x, y, z\}\}$ onto $FJ^{(3)} = \Phi\{\{x, y, z\}\}/\mathfrak{T}$. It is immediate by induction on degree that any element of $\Phi\{\{x, y, z\}\}$ which is a product of x's, y's and one z has the form zP where P is product of left and right multiplications by elements which are products of x's and y's. It follows that the elements of $\Phi\{\{x, y, z\}\}$ which are homogeneous of degree one in z have the form zL where L is in the subalgebra of $\mathrm{Hom}(\Phi\{\{x, y, z\}\}, \Phi\{\{x, y, z\}\})$ generated by the multiplications b_L, b_R where b is in the subalgebra $\Phi\{\{x, y\}\}$ generated by x, y. This implies that $\mathfrak{Z} = zR_{FJ^{(3)}}(FJ^{(2)})^*$ where $R_{FJ^{(3)}}(FJ^{(2)})^*$ is the subalgebra of $\mathrm{Hom}(FJ^{(3)}, FJ^{(3)})$ generated by the R_c, $c \in FJ^{(2)}$ the subalgebra of $FJ^{(3)}$ generated by $1, x, y$.

We shall now formulate Macdonald's Theorem in an equivalent operator form. We observe first that if $\mathfrak{A}$ and $\mathfrak{B}$ are arbitrary algebras and σ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ then we have a homomorphism of the subalgebra $M(\mathfrak{A})$ of $\mathrm{Hom}(\mathfrak{A}, \mathfrak{A})$ generated by 1 and the a_L, a_R, $a \in \mathfrak{A}$ onto $M(\mathfrak{B})$ mapping $1 \to 1$, $a_L \to (a^\sigma)_L$, $a_R \to (a^\sigma)_R$. To see this let X and Y be two disjoint sets which are bijective with $\mathfrak{A}$ under mappings $a \to x_a \in X$, $a \to y_a \in Y$ and let $Z = X \cup Y$. We have homomorphism τ and τ' of the free associative algebra with 1 $\Phi\{Z\}$ onto $M(\mathfrak{A})$ and $M(\mathfrak{B})$ such that $1^\tau = 1$, $x_a^{\ \tau} = a_L$, $y_a^{\ \tau} = a_R$, $1^{\tau'} = 1$, $x_a^{\ \tau'} = (a^\sigma)_L$, $y_a^{\ \tau'} = (a^\sigma)_R$. If $F \in \Phi\{Z\}$

and $u \in \mathfrak{A}$ then one sees easily via monomials that $(uF^{\tau})^{\sigma} = u^{\sigma}F^{\tau'}$. Since σ is surjective this implies that if $F^{\tau} = 0$ then $F^{\tau'} = 0$. It follows that we have a homomorphism of $M(\mathfrak{A})$ onto $M(\mathfrak{B})$ such that $F^{\tau} \to F^{\tau'}$ and hence $1 \to 1$, $a_L \to (a^{\sigma})_L$, $a_R \to (a^{\sigma})_R$. In particular, the homomorphism σ of $FJ^{(3)}$ onto $FSJ^{(3)}$ gives rise to a homomorphism of $M(FJ^{(3)}) = R(FJ^{(3)})^*$ onto $R(FSJ^{(3)})^*$ sending $1 \to 1$, $R_a \to R_{a^{\sigma}}$, $a \in FSJ^{(3)}$. By restriction we obtain the homomorphism Σ of $R_{FJ^{(3)}}(FJ^{(2)})^*$, the subalgebra generated by the R_b, $b \in FJ^{(2)}$, (the subalgebra generated by $1, x, y$), onto $R_{FSJ^{(3)}}(FSJ^{(2)})^*$. Now suppose Σ is an isomorphism and let $a \in \mathfrak{Z} \cap \mathfrak{K}^{(3)}$. Then $a = zL$ where $L \in R_{FJ^{(3)}}(FJ^{(2)})^*$ and $wL^{\Sigma} = (zL)^{\sigma} = a^{\sigma} = 0$. Since there exists an endomorphism of $FSJ^{(3)}$ leaving 1, u and v fixed and mapping w into any element of $FSJ^{(3)}$, $wL^{\Sigma} = 0$ implies $bL^{\Sigma} = 0$ for all $b \in FSJ^{(3)}$. Thus $L^{\Sigma} = 0$ and $L = 0$ so $a = 0$. Hence we have the conclusion of Macdonald's Theorem: $\mathfrak{Z} \cap \mathfrak{K}^{(3)} = 0$. Conversely, assume this holds and let $L \in R_{FJ^{(3)}}(FJ^{(2)})^*$ satisfy $L^{\Sigma} = 0$. Then $wL^{\Sigma} = (zL)^{\sigma} = 0$ and $zL \in \mathfrak{Z} \cap \mathfrak{K}^{(3)}$. Hence $zL = 0$. Since we have an endomorphism of $FJ^{(3)}$ fixing $1, x, y$ and sending z into any $a \in FJ^{(3)}$ we see that $L = 0$. Hence we have shown that Macdonald's Theorem is equivalent to the operator result that Σ is an isomorphism of $R_{FJ^{(3)}}(FJ^{(2)})^*$ onto $R_{FSJ^{(3)}}(FSJ^{(2)})^*$. We shall prove this statement by choosing generators for the two algebras of operators and establishing the same defining relations for these generators. Our choice of generators will be based on the following

LEMMA 1. *Let $\mathfrak{J}$ be a Jordan algebra with* 1, *$\mathfrak{B}$ a subalgebra containing* 1, *X a set of generators for $\mathfrak{B}$ containing* 1. *Then the set of operators $U_{x,y}$, $x, y \in X$, is a set of generators for $R_{\mathfrak{J}}(\mathfrak{B})^*$.*

PROOF. We have $R_x = U_{x,1}$, $R_{x.y} = R_xR_y + R_yR_x - U_{x,y}$. Hence it suffices to show that the elements $R_{x.y}$, $x, y \in X$, generate $R_{\mathfrak{J}}(\mathfrak{B})^*$. Since X generates $\mathfrak{B}$ every element of $\mathfrak{B}$ is a linear combination of products of x's in X. Hence it is enough to show that if m is such a product then R_m is contained in the subalgebra $\mathfrak{X}$ of $\mathrm{Hom}(\mathfrak{J},\mathfrak{J})$ generated by the $R_{x.y}$, $x, y \in X$. Since $\mathfrak{B}$ is commutative it is enough to show that $R_m \in \mathfrak{X}$ for $m = m_1 . m_2 . m_3$ where m_i is a product of x's in $\mathfrak{X}$. Using induction on the number of factors we may assume that R_{m_i} and $R_{m_i . m_j} \in \mathfrak{X}$. By O_2 we have $R_{m_1 . m_2 . m_3} = R_{m_1 . m_2}R_{m_3} + R_{m_1 . m_3}R_{m_2} + R_{m_2 . m_3}R_{m_1} - R_{m_1}R_{m_3}R_{m_2} - R_{m_2}R_{m_3}R_{m_1}$ which is contained in $\mathfrak{X}$ by our induction assumption.

If we apply this lemma to $\mathfrak{J} = FJ^{(3)}$, $\mathfrak{B} = FJ^{(2)}$, which is generated by $X = \{1, x, y\}$, then we see that $R_{FJ^{(3)}}(FJ^{(2)})^*$ is generated by the five elements 1, $R_x, R_y, U_x, U_y, U_{x,y}$. However, it appears to be difficult to obtain a set of defining relations for these generators. On the other hand, we shall succeed in doing this for the infinite set of generators 1, $R_{x^{\cdot i}}$, $R_{y^{\cdot i}}$, $U_{x^{\cdot i}}$, $U_{y^{\cdot i}}$, $U_{x^{\cdot i}, y^{\cdot j}}$ where $i, j \geqq 1$. Accordingly, we introduce the free associative algebra with 1:

$$\mathfrak{F} = \Phi\{a_i, b_i, c_i, d_i, e_{ij}\}$$

(freely) generated by distinct elements $a_i, b_i, c_i, d_i, e_{ij}$, $i,j = 1,2,3\cdots$. Let τ be the homomorphism of $\mathfrak{F}$ into $R_{FJ^{(3)}}(FJ^{(2)})^*$ such that

$$(76)\qquad \begin{gathered} 1 \to 1,\ a_i \to R_{x^{\cdot i}},\ b_i \to R_{y^{\cdot i}}, \\ c_i \to U_{x^{\cdot i}},\ d_i \to U_{y^{\cdot i}},\ e_{ij} \to U_{x^{\cdot i}, y^{\cdot j}}, \end{gathered}$$

τ' the homomorphism of $\mathfrak{F}$ into $R_{FSJ^{(3)}}(FSJ^{(2)})^*$ such that

$$(77)\qquad \begin{gathered} 1 \to 1,\quad a_i \to R_{u^{\cdot i}},\quad b_i \to R_{v^{\cdot i}}, \\ c_i \to U_{u^{\cdot i}},\quad d_i \to U_{v^{\cdot i}},\quad e_{ij} \to U_{u^{\cdot i}, v^{\cdot j}}, \end{gathered}$$

Then τ and τ' are surjective and $\tau' = \tau\Sigma$ where Σ is the homomorphism we defined before of $R_{FJ^{(3)}}(FJ^{(2)})^*$ onto $F_{FSJ^{(3)}}(FSJ^{(2)})^*$. Let $\mathfrak{R} = \ker\tau$, $\mathfrak{R}' = \ker\tau'$. Then $\mathfrak{R}' \supseteq \mathfrak{R}$ and Σ is an isomorphism if and only if $\mathfrak{R}' = \mathfrak{R}$. We shall prove this by showing that both of these coincide with a third ideal $\mathfrak{G}$ of $\mathfrak{F}$ defined by generators.

LEMMA 2. *The following elements are contained in* $\mathfrak{R}$:

(i) $$[a_i, a_j],\quad [a_i, c_j],\quad [b_i, b_j],\quad [b_i, d_j],$$

(ii) $$2a_i a_j - a_{i+j} - a_{i-j}c_j,\quad 2b_i b_j - b_{i+j} - b_{i-j}d_j,\quad i \geqq j \geqq 1$$

where we take $a_0 = b_0 = 1$ *if* $i = j$,

(iii) $$c_i c_j - c_{i+j},\quad d_i d_j - d_{i+j},$$

(iv) $$c_i e_{jk} - 2a_i e_{i+j,k} + e_{2i+j,k},\quad d_i e_{jk} - 2b_i e_{j,i+k} - e_{j,k+2i},$$

(v) $$d_i a_j - 2b_i e_{ji} + e_{j,2i},\quad c_i b_j - 2a_i e_{ij} + e_{i,2j},$$

(vi) $$2e_{ij}a_k - e_{i-k,j}c_k - e_{i+k,j},\quad 2e_{ji}b_k - e_{j,i-k}d_k - e_{j,i+k},\qquad i > k,$$

(vii) $$2e_{ij}a_i - e_{2i,j} - b_j c_i,\quad 2e_{ji}b_i - e_{j,2i} - a_j d_i,$$

(viii) $$\begin{gathered} e_{ij}a_k - e_{i+k,j} - b_j a_{k-i}c_i + e_{kj}a_i, \\ e_{ji}b_k - e_{j,i+k} - a_j b_{k-i}c_i + e_{jk}b_i,\qquad i < k. \end{gathered}$$

PROOF. In view of the definition of $\mathfrak{R}$ and (76) the assertions are equivalent to the following identities for arbitrary Jordan algebras with 1:

(i′) $$[R_{p^{\cdot i}}R_{p^{\cdot j}}] = 0,\qquad [R_{p^{\cdot i}}, U_{p^{\cdot j}}] = 0,$$

(ii′) $$2R_{p^{\cdot i}}R_{p^{\cdot j}} = R_{p^{\cdot i+j}} + R_{p^{\cdot i-j}}U_{p^{\cdot j}},\quad i \geqq j,$$

(iii′) $$U_{p^{\cdot i}}U_{p^{\cdot j}} = U_{p^{\cdot i+j}},$$

(iv′) $$U_{p^{\cdot i}}U_{p^{\cdot j}, q^{\cdot k}} = 2R_{p^{\cdot i}}U_{p^{\cdot i+j}, q^{\cdot k}} - U_{p^{\cdot 2i+j}, q^{\cdot k+}},$$

(v′) $$U_{q^{\cdot i}}R_{p^{\cdot i}} = 2R_{q^{\cdot i}}U_{p^{\cdot j}, q^{\cdot i}} - U_{p^{\cdot j}, q^{\cdot 2i}},$$

$$\text{(vi}'\text{)} \qquad 2U_{p^{\cdot i},q^{\cdot j}}R_{p^{\cdot k}} = U_{p^{\cdot i-k},q^{\cdot j}}U_{p^{\cdot k}} + U_{p^{\cdot i+k},q^{\cdot j}}, \qquad i > k,$$

$$\text{(vii}'\text{)} \qquad 2U_{p^{\cdot i},q^{\cdot j}}R_{p^{\cdot i}} = U_{p^{\cdot 2i},q^{\cdot j}} + R_{q^{\cdot j}}U_{p^{\cdot i}},$$

$$\text{(viii}'\text{)} \qquad U_{p^{\cdot i},q^{\cdot j}}R_{p^{\cdot k}} = U_{p^{\cdot i+k},q^{\cdot j}} + R_{q^{\cdot j}}R_{p^{\cdot k-i}}U_{p^{\cdot i}} - U_{p^{\cdot k},q^{\cdot j}}R_{p^{\cdot i}}, \qquad i < k.$$

Now (i′) is clear and (ii′) for $i = j$ follows from the definition of U_a. Also (ii′) for $i > j$ is a special case of (vi′) if we allow $j = 0$ in (vi′). Now (vi′) with $j \geqq 0$ can be obtained by putting $a' = p^{\cdot i-k}$, $a = p^{\cdot k}$, $b = q^{\cdot j}$ in (69). To obtain (iii′) we put $a = p^{\cdot i}$, $a' = p^{\cdot j}$ in (66). For (iv′) we put $a = p^{\cdot i}$, $a' = p^{\cdot j}$, $b = q^{\cdot k}$ in (68) and for (v′) we put $b = q^{\cdot i}$, $a = p^{\cdot j}$ in (67). For (vii′) and (viii′) we put $a = p^{\cdot i}$, $b = q^{\cdot j}$, $a' = 1$ and $a = p^{\cdot i}$, $b = q^{\cdot j}$, $a' = p^{\cdot k-i}$ respectively in (70).

Now let $\mathfrak{G}$ be the ideal in $\mathfrak{F}$ generated by the elements (i)–(viii) given in Lemma 2. Then we have $\mathfrak{R}' \supseteq \mathfrak{R} \supseteq \mathfrak{G}$. We shall now show that there exists a set of elements $B = \{b\}$ in $\mathfrak{F}$ such that: (1) the cosets $\{b + \mathfrak{R}' \mid b \in B\}$ are linearly independent in $\mathfrak{F}/\mathfrak{R}'$ and (2) every coset $a + \mathfrak{G}$, $a \in \mathfrak{F}$, in $\mathfrak{F}/\mathfrak{G}$ is a linear combination of the cosets $b + \mathfrak{G}$, $b \in B$. The desired equality $\mathfrak{R}' = \mathfrak{R} = \mathfrak{G}$ will follow from these two facts since, if $\mathfrak{B}$ is the subspace of $\mathfrak{F}$ spanned by B, then $\mathfrak{F} = \mathfrak{B} + \mathfrak{G}$ and $\mathfrak{B} \cap \mathfrak{R}' = 0$. Then since $\mathfrak{R}' \supseteq \mathfrak{G}$, $\mathfrak{R}' = \mathfrak{F} \cap \mathfrak{R}' = (\mathfrak{B} \cap \mathfrak{R}') + (\mathfrak{G} \cap \mathfrak{R}') = \mathfrak{G}$. For the set B we take the set of monomials of the form PQR where P is a product of terms which are alternately a_i and b_j in the set of generators of $\mathfrak{F} = \Phi\{a_i, b_i, c_i, d_i, e_{ij}\}$, Q is any product of e's and R is a product of terms which are alternately c's and d's. We shall also include 1 in the set B. We introduce a bigrading in $\mathfrak{F}$ by defining an x-degree and y-degree of monomials in the generators by the following rules: 1 has x-degree and y-degree 0, a_i has x-degree i and y-degree 0, b_i has x-degree 0 and y-degree i, c_i has x-degree $2i$ and y-degree 0, d_i has x-degree 0 and y-degree $2i$, e_{ij} has x-degree i and y-degree j. Moreover, the x- and y-degrees of any monomial are the sums of the corresponding degrees of the factors. Let $\mathfrak{F}_{r,s}$ be the subspace of linear combinations of monomials of x-degree r and y-degree s. Then $\mathfrak{F} = \Sigma \oplus \mathfrak{F}_{r,s}$. Also let $B(r,s) = \mathfrak{F}_{r,s} \cap B$.

Lemma 3. *Every monomial in the generators* $a_i, b_i, \cdots$, *of*

$$\mathfrak{F} = \Phi\{a_i, b_i, c_i, d_i, e_{ij}\}$$

of x-degree r and y-degree s is congruent modulo $\mathfrak{G}$ to a linear combination of elements of $B(r,s)$.

Proof. We use induction on $r + s$. The result will follow if we can show that if $K' \in B$ and $K = K'a_i, K'b_i, K'c_i, K'd_i, K'e_{ij}$ has x-degree r and y-degree s then K is congruent to a linear combination of elements of $B(r,s)$ modulo $\mathfrak{G}$. We consider the various cases in the following order:

Case I. $K = K'c_i$, $K' \in B(r-2i, s)$. If K' ends in anything except a c_j then $K'c_i \in B(r,s)$. Hence we may assume $K' = K''c_j$, $K'' \in B(r-2i-2j, s)$. Then

$K = K''c_jc_i \equiv K''c_{i+j}$ (mod $\mathfrak{G}$) by (iii). It is clear that $K''c_{i+j} \in B(r,s)$ so the result holds in this case.

Case II. $K = K'd_i$ where $B' \in B(r, s-2i)$. This is exactly like Case I.

Case III. $K = K'e_{jk}$, $K' \in B(r-j, s-k)$. If $j+k = r+s$ then $K' = 1$ and the result is clear. We shall now treat this case by induction downward for $j+k$. If K' ends in an a, b or e then $K \in B(r,s)$. Now suppose $K' = K''c_i$ where $K'' \in B(r-j-2i, s-k)$. Then $K = K''c_ie_{jk} \equiv 2K''a_ie_{i+j,k} + K''e_{2i+j,k}$ by (iv). Since $i \geqq 1$, the downward induction on $j+k$ implies that $K''e_{2i+j,k}$ is a congruent modulo $\mathfrak{G}$ to a linear combination of elements of $B(r,s)$. The same argument can be applied to the term $K''a_ie_{i+j,k}$ after one applies the general induction assumption to conclude that $K''a_i$ is congruent to a linear combination of elements of $B(r-i-j, s-k)$. Hence the result holds if $K' = K''c_i$. The remaining case: $K' = K''d_i$ can be treated in exactly the same way.

Case IV. $K = K'a_i$, $K' \in B(r-i, s)$. If K' ends in a b then $K \in B(r,s)$. Next assume $K' = K''a_j$. Then $K = K''a_ja_i \equiv K''a_ia_j$ by (i) so we may assume $i \geqq j$. Then we can apply (ii) to obtain $K \equiv \frac{1}{2}K''a_{i+j} - \frac{1}{2}K''a_{i-j}c_j$. It is clear from the definition of the set B that $K''a_{i+j}$ and $K''a_{i-j}c_j \in B(r,s)$. Assume next that $K' = K''c_j$. Then $K = K''c_ja_i \equiv K''a_ic_j$ by (i). The induction hypothesis implies that $K''a_i$ is congruent modulo $\mathfrak{G}$ to a linear combination of elements of $B(r-2j, s)$. Hence $K''a_ic_j$ is congruent to a linear combination of elements of $B(r,s)$ by Case I. Next let $K' = K''d_j$ so $K = K''d_ja_i \equiv 2K''b_je_{ij} - K''e_{i,2j}$, by (v). The result then follows by induction and Case III. Finally, suppose $K' = K''e_{jk}$ so that $K = K''e_{jk}a_i$. We have to distinguish the three possibilities: $j > i$, $j = i$, $j < i$. If $j > i$ then we can apply (vi) to obtain $K \equiv \frac{1}{2}K''e_{j-i,k}c_i + \frac{1}{2}K''e_{i+j,k}$ (mod $\mathfrak{G}$). It is clear from the definition of B that $K''e_{j-i,k}c_i$ and $K''e_{i+j,k} \in B(r,s)$. If $j = i$ we use (vii) to obtain $K \equiv \frac{1}{2}K''e_{2i,k} + \frac{1}{2}K''b_kc_i$. Now $K''e_{2i,k} \in B(r,s)$ and $K''b_kc_i$ is congruent to a linear combination of elements of $B(r,s)$ by the induction hypothesis and Case I. If $j < i$ we apply (viii) to obtain $K \equiv K''e_{j+i,k} + K''b_ka_{i-j}c_j - K''e_{ik}a_j$. The first term is in $B(r,s)$. Induction and Case I shows that the second term is congruent to a linear combination of elements of $B(r,s)$. The last term $K''e_{ik}a_j$ falls under the first subcase of the three possibilities we are considering here. Hence the result is established for Case IV.

Case V. $K = K'b_i$. This is just like Case IV.

This concludes the proof.

It remains to prove

LEMMA 4. *The cosets* $\{b + \mathfrak{R}' \mid b \in B\}$ *are linearly independent.*

PROOF. This is equivalent to the linear independence of the set of operators $b^{\tau'}$ in $FSJ^{(3)}$. We shall prove this by showing that the set $\{wb^{\tau'} \mid b \in B\}$ is linearly independent. Let $\mathfrak{W}$ be the subspace of elements of $FSJ^{(3)}$ which are homogeneous of 1st degree in w and let $\mathfrak{W}_{r,s}$ be the subspace of these which are homogeneous of degree r in u and of degree s in v. Then $\mathfrak{W} = \Sigma \oplus \mathfrak{W}_{r,s}$ and $wb^{\tau'} \in \mathfrak{W}_{r,s}$ if and only if $b \in B(r,s)$, the subset of B of elements of x-degree r and y-degree s. Hence it is enough to show that all the subsets $\{wb^{\tau'} | b \in B(r,s)\}$ are linearly independent. We have seen that the subspace $\mathfrak{Z}$ of $FJ^{(3)}$ of elements homogeneous of degree 1 in z coincides with $zR_{FJ^{(3)}}(FJ^{(2)})^* = z\mathfrak{F}^{\tau}$. Hence $\mathfrak{W} = w\mathfrak{F}^{\tau'}$. Since every $f \in \mathfrak{F}$ is congruent modulo $\mathfrak{G}$, hence modulo $\mathfrak{R}' = \ker \tau'$, to a linear combination of elements of B it is clear that $\mathfrak{W}$ is spanned by the elements $wb^{\tau'}$, $b \in B$. Hence $\mathfrak{W}_{r,s}$ is spanned by the elements $wb^{\tau'}$, $b \in B(r,s)$. By Cohn's Theorem (p. 00), $FSJ^{(3)}$ is the subspace of reversible elements of $\Phi\{u,v,w\}$. Hence $\mathfrak{W}_{r,s}$ is the subspace of $\Phi\{u,v,w\}$ of reversible elements which are homogeneous of degrees $r, s, 1$ in u, v, w respectively. Since $\{wb^{\tau'} | b \in B(r,s)\}$ spans $\mathfrak{W}_{r,s}$ the required linear independence of $\{wb^{\tau'} | b \in B(r,s)\}$ will follow if we can show that the cardinal number $|B(r,s)|$ of $B(r,s)$ does not exceed $\dim \mathfrak{W}_{r,s}$. To do this we define a mapping λ of B into the set of monomials in u, v, w inductively on the degree of $b \in B$ in $a_k, b_k, c_k, d_k, e_{kl}$ by the following rules: (1) $1^\lambda = w$; (2) $b^\lambda = b'^\lambda u^k$ if $b = b'a_k$ and $b^\lambda = b'^\lambda v^k$ if $b = b'b_k$ where $b' \in B$; (3) $b^\lambda = u^k b'^\lambda v^l$ if $b = b'e_{kl}$, $b' \in B$ and ends with u, $b^\lambda = v^l b'^\lambda u^k$ if $b = b'e_{kl}$, $b' \in B$ and does not end with u; (4) $b^\lambda = u^k b'^\lambda u^k$ if $b = b'c_k$ and $b^\lambda = v^k b'^\lambda v^k$ if $b = b'd_k$, b' in B. It is clear from this definition that b^λ is of 1st degree in w and if $b \in B(r,s)$ then b^λ is of degree r in u and degree s in v. It is clear also from the definition and the form of the elements of B that b^λ is divisible right and left by u (v) only in case (4) and then if k is the largest integer such that b^λ is divisible right and left by $u^k(v^k)$ then $b = b'c_k$ ($= b'd_k$) where $b' \in B$. If b^λ is divisible on the left by u (v) and on the right by v (u) then we are in case (3): $b = b'e_{kl}$ where $b' \in B$ and ends with u (does not end with u). Then u^k is the highest power of u dividing b^λ on the left (right) and v^l is the highest power of v dividing b^λ on the right (left). If b^λ has the form $w \cdots$ then we are in case (1) or (2) and if $b = b'a_k$ ($b'b_k$) then u^k (v^k) is the highest power of u (v) dividing b^λ on the right. Now suppose $b, c \in B$ and $b^\lambda = c^\lambda$. Then the results just noted imply that either $b = c = 1$ or b and c are divisible on the right by the same generator a_k, b_k, c_k, d_k, e_k. If d is this generator then $b = b'd$, $c = c'd$ and $b'^\lambda = c'^\lambda$. Using induction on the degree we conclude that $b' = c'$ and hence $b = c$. Hence the mapping λ of B into the set of monomials in u, v, w is 1-1. We now define the *height* of a monomial in u and v to be the number of powers of u and v occurring in the monomial (e.g. the height of $u^i v^j u^k v^l u^m$ is five if all the exponents are positive). If $m = AwB$ where A and B are monomials in u and v then we call the height of $A(B)$ the *left height*, $lh(m)$ of m (*right height*, $rh(m)$, of m). It is clear from the definition of λ that $rh(b^\lambda) \geqq lh(b^\lambda)$ for all $b \in B$. Moreover if b contains a factor a_k or b_k and no factors c_k or d_k then $rh(b^\lambda) > lh(b^\lambda)$. Now suppose $(b^\lambda)^* = c^\lambda$ for b, $c \in B$ where m^* denotes the reversal of m. Then we claim that $(b^\lambda)^* = b^\lambda$. Using induction on the degree we can reduce the proof to the case in which b contains no factor of the form c_k or d_k. In this case the results noted on heights imply that b is not divisible by an a_k or a b_k. Hence either $b = 1$ or $b = e_{k_1 l_1} e_{k_2 l_2} \cdots e_{k_t l_t}$. Then $b^\lambda = \cdots v_{l_3} u_{k_2} v_{l_1} w u_{k_1} v_{l_2} u_{k_3} \cdots$ and it is clear that

$(b^\lambda)^* \notin B^\lambda$. Hence $b = 1$ and $(b^\lambda)^* = b^\lambda$. We now see that the mapping $b \to \frac{1}{2}(b^\lambda + (b^\lambda)^*)$ is 1-1 and consequently the restriction of this to $B(r,s)$ is 1-1. The images are distinct elements of the form $\frac{1}{2}(m + m^*)$ where m is a monomial of degree $r, s, 1$ in u, v, w. Hence these are linearly independent elements in the space $\mathfrak{W}_{r,s}$. This establishes $|B(r,s)| \leqq \dim \mathfrak{W}_{r,s}$ and completes the proof of the lemma.

We have now proved the following

THEOREM 9. *Let* $FJ^{(3)}$ $(FSJ^{(3)})$ *be the free (free special) Jordan algebra generated by* x, y, z (u, v, w) *and let* $\mathfrak{E} = R_{FJ^{(3)}}(FJ^{(2)})^*$ $(\mathfrak{E}_s = R_{FSJ^{(3)}}(FSJ^{(2)})^*$ *be the associative algebra of linear transformations in* $FJ^{(3)}$ $(FSJ^{(3)})$ *generated by the multiplications by elements of the subalgebra generated by* x, y (u, v). *Then the homomorphism* Σ *of* $\mathfrak{E}$ *onto* $\mathfrak{E}_S$ *induced by the homomorphism* σ *of* $FJ^{(3)}$ *onto* $FSJ^{(3)}$ *such that* $1 \to 1$, $x \to u$, $y \to v$, $z \to w$ *is an isomorphism. Moreover, if* $\mathfrak{F} = \Phi\{a_i, b_i, c_i, d_i, e_{ij}\}$ *is the free associative algebra with generators* $a_i, \cdots, e_{ij}$ *and* τ *is the homomorphism of* $\mathfrak{F}$ *onto* $\mathfrak{E}$ *defined by* (76) *then the kernels of* τ *and of* $\tau' = \tau\Sigma$ *coincide with the ideal* $\mathfrak{G}$ *in* $\mathfrak{F}$ *generated by the elements* (i) – (viii) *of Lemma* 2.

We remark also that we have obtained a basis for $\mathfrak{F}/\mathfrak{N}$, namely, the set of cosets $b + \mathfrak{N}$, $b \in B$. We showed at the beginning of our discussion that the first assertion of Theorem 9 is equivalent to Macdonald's Theorem (cf. p. 42). We shall show next that this result implies

SHIRSHOV'S THEOREM. *The free Jordan algebra* $FJ^{(2)}$ *with* 1 *and two generators is special.*

PROOF. We have to show that the kernel $\mathfrak{K}^{(2)}$ of $FJ^{(2)}$ onto $FSJ^{(2)}$ mapping $1 \to 1$, $x \to u$, $y \to v$ is 0. As before we consider $FJ^{(2)}$ and $FSJ^{(2)}$ as the subalgebras of $FJ^{(3)}$ and $FSJ^{(3)}$ which are generated by $1, x, y$ and $1, u, v$ respectively. Let $k \in \mathfrak{K}^{(2)}$. Then $k.z \in \mathfrak{K}^{(3)} \cap \mathfrak{Z}$. Hence $k.z = 0$ by Macdonald's Theorem. Now apply the endomorphism of $FJ^{(3)}$ such that $1 \to 1, x \to x$, $y \to y$, $z \to 1$. This maps $k.z$ into k. Hence $k = 0$ and so $\mathfrak{K}^{(2)} = 0$.

Shirshov's and Macdonald's Theorems taken together imply that if $f = f(x_1, x_2, x_3) \in \Phi\{\{x_1, x_2, x_3\}\}'$ is of degree one or 0 in x_3 and f is an identity for all special Jordan algebras with 1 then f is an identity for all Jordan algebras. To see this we write $f = f_0 + f_1$ where f_i is homogeneous of degree i in x_3. If we take the homomorphism of $\Phi\{\{x_1, x_2, x_3\}\}'$ into $FSJ^{(3)}$ such that $x_1 \to u$, $x_2 \to v$, $x_3 \to 0$ we see that f_0 and hence f_1 is an identity for $FSJ^{(3)}$. By Shirshov's and Macdonald's Theorems these are identities for $FJ^{(3)}$. It follows that they are identities for all Jordan algebras with 1 and hence for all Jordan algebras. We prove next the following extension of Shirshov's Theorem.

THEOREM 10. (SHIRSHOV-COHN). *Any Jordan algebra (with 1) generated by two elements (and 1) is special.*

PROOF. Since we can adjoin an identity element to a Jordan algebra to obtain a Jordan algebra, it is enough to prove that if $\mathfrak{J}$ is a Jordan algebra with 1 and $\mathfrak{J}$ is generated by 1 and two elements then $\mathfrak{J}$ is special. Now the hypothesis implies that $\mathfrak{J}$ is a homomorphic image of $FJ^{(2)}$. Since $FJ^{(2)}$ is special by Shirshov's Theorem, $\mathfrak{J}$ is a homomorphic image of $FSJ^{(2)}$. By Cohn's Theorem on homomorphic images of $FSJ^{(2)}$ (p. 11) $\mathfrak{J}$ is special.

EXERCISES

1. Let $\alpha(r,s)$ denote the cardinal number of the set $W(r,s)$ of elements of the form $\frac{1}{2}(m + m^*)$ where m is a monomial in u, v, w of degrees r, s, 1 in u, v, w respectively and let $\beta(r,s) = |B(r,s)|$. Show that

$$\alpha(r,s) = \tfrac{1}{2}\left\{(r+s+1)\binom{r+s}{r} + N\right\}$$

where $N = 0$ if either r or s is odd and $N = \binom{h+k}{k}$ if $r = 2h$, $s = 2k$. Show also that

$$\beta(r,s) =$$

$$\sum_{\substack{r_1+r_2+2r_3=r \\ s_1+s_2+2s_3=s \\ r_1,r_3,s_1,s_3\geqq 0,\, r_2,s_2>0}} \binom{r_1+s_1}{r_1}\binom{r_2+s_2-2}{r_2-1}\binom{r_3+s_3}{r_3} + \sum_{\substack{r_1+2r_3=r \\ s_1+2s_3=s \\ r_1,r_3,s_1,s_3\geqq 0}} \binom{r_1+s_1}{r_1}\binom{r_3+s_3}{r_3}.$$

Prove directly that $\alpha(r,s) = \beta(r,s)$. (Note that this gives an alternative proof of Lemma 4.)

2. Let $W(r,s)$ be as in exercise 1 and let $W'(r,s) \equiv \{wb^{\tau'} \mid b \in B(r,s)\}$. Show that the elements of $W'(r,s)$ are linear combinations of the elements of $W(r,s)$ with coefficients in the prime field which in the characteristic 0 case are rational numbers with denominators powers of two. Prove that the matrix C of the coefficients (for any orderings of $W(r,s)$ and $W'(r,s)$) is nonsingular. Prove that C^{-1} is integral in the characteristic 0 case and give a constructive method for determining C^{-1}.

3. Let $\Phi\{u,v\}$, $\Phi\{u',v'\}$ be free associative algebras with 1 and let $\mathfrak{P} = \Phi\{u,v\} \otimes \Phi\{u',v'\}$. Let π be the automorphism of $\mathfrak{P}$ which maps 1 into 1 and which exchanges u and u' and v and v'. Let $\mathfrak{E}'$ be the subalgebra of $\mathfrak{P}$ generated by the elements $a + a^{\pi}$, a a reversible element of $\Phi\{u,v\}$. Show that $\mathfrak{E}' \cong \mathfrak{E}_S$ where $\mathfrak{E}_S$ is the algebra defined in Theorem 9. Hence obtain a set of generators and defining relations for $\mathfrak{E}'$.

4. (Koecher). Let $\mathfrak{J}$ be a Jordan algebra with 1, $\mathfrak{A}$ a subalgebra generated by 1 and two elements a and b. Use Lemma 1 and the basic operator identities to show that $\mathfrak{A}$ is a strongly associative subalgebra of $\mathfrak{J}$ if and only if R_a, R_b and $R_{a.b}$ commute.

5. Show that the identities (62), (63) and (64) are valid for arbitrary Jordan algebras.

6. (Faulkner). Use Macdonald's Theorem and exercise 2 on p. 37 to prove that Inder $\mathfrak{J}$ for any Jordan algebra $\mathfrak{J}$ of characteristic $p \neq 0$ is closed under the p power mapping.

10. **s-identities. Exceptional character of $\mathfrak{H}(\mathfrak{O}_3)$.** It was first proved by Albert and Paige that the kernel $\mathfrak{R}^{(3)}$ of the canonical homomorphism of $FJ^{(3)}$ onto $FSJ^{(3)}$ is not 0. They did this by noting that $\mathfrak{H}(\mathfrak{O}_3)$ for $\mathfrak{O}$ the algebra of octonions is generated by three elements and by proving that $\mathfrak{H}(\mathfrak{O}_3)$ is not a homomorphic image of any special Jordan algebra. The first result implies that $\mathfrak{H}(\mathfrak{O}_3)$ is a homomorphic image of $FJ^{(3)}$ and the second rules out the isomorphism of $FJ^{(3)}$ and $FSJ^{(3)}$. A nonzero element of the kernel $\mathfrak{R}^{(r)}$ of the canonical homomorphism of $FJ^{(r)}$ onto $FSJ^{(r)}$ will be called an *s-identity*. Clearly such an element f gives an identity valid for all special Jordan algebras but not valid for all Jordan algebras. The theorem of Albert-Paige proved the existence of s-identities in three variables x, y, z. However, their proof did not provide any such identity in explicit form. Subsequently, Glennie ([1] and [2]) through an analysis of the Albert-Paige proof and also by using some simplifications and other methods constructed a number of three variable s-identities. We shall not give the reasoning which led him to these but shall confine ourselves to the verification of one of his s-identities and use this to prove the main result of Albert-Paige.

We show first that the identity

$$
\begin{aligned}
&2\{xzx\}.\{y\{zy^{\cdot 2}z\}x\} - 2\{yzy\}.\{x\{zx^2z\}y\} \\
&\quad = \{x\{z\{x\{yzy\}y\}z\}x\} - \{y\{z\{y\{xzx\}x\}z\}y\}
\end{aligned}
\tag{78}
$$

is valid in any special Jordan algebra. Using $a.b = \frac{1}{2}(ab+ba)$, $\{abc\} = \frac{1}{2}(abc+cba)$ in terms of the associative product we obtain

$$
\begin{aligned}
&4\{xzx\}.\{y\{zy^{\cdot 2}z\}x\} - 4\{yzy\}.\{x\{zx^{\cdot 2}z\}y\} \\
&= xzx(yzy^2zx + xzy^2zy) + (yzy^2zx + xzy^2zy)xzx \\
&\quad - yzy(yzx^2zy + yzx^2zx) - (xzx^2zy + yzx^2zx)yzy \\
&= xzxyzy^2zx + xzy^2zyxzx - yzyxzx^2zy - yzx^2zxyzy \\
&= xz(xyzy^2 + y^2zyx)zx - yz(yxzx^2 + x^2zxy)zy \\
&= 2\{x\{z\{x\{yzy\}y\}z\}x\} - 2\{y\{z\{y\{xzx\}x\}z\}y\}.
\end{aligned}
$$

Hence (78) holds for special Jordan algebras.

Next let $\mathfrak{A}$ be an arbitrary algebra with identity element 1 and an involution $x \to \bar{x}$. We consider the standard involution $X \to \bar{X}^t$ in $\mathfrak{A}_3$ (cf. §5). We use the notation e_{ij} for the (i,j) matrix unit and we put

$$a[ij] = ae_{ij} + \bar{a}e_{ji} = \bar{a}[ji] \tag{79}$$

for $a \in \mathfrak{A}$. Then $a[ij] \in \mathfrak{H}(\mathfrak{A}_3)$ the subalgebra of $\mathfrak{A}_3{}^+$ of symmetric elements under the standard involution in $\mathfrak{A}_3$. We have for $a, b \in \mathfrak{A}$ that

$$2a[ij] \cdot b[jk] = (ab)\,[ik] \quad \text{if} \quad i, j, k \neq . \tag{80}$$

We now consider the following three elements of $\mathfrak{H}(\mathfrak{A}_3)$:

$$\begin{aligned} X &= 1[12], \qquad Y = 1[23], \\ Z &= u[21] + v[13] + w[32] \end{aligned} \tag{81}$$

where u, v, w are arbitrary elements of $\mathfrak{A}$. Since X and Y are in the nucleus it is clear that any product in $\mathfrak{A}_3$ of X's and Y's and at most two Z's is independent of association (or parentheses). In particular, we have $\{XZX\} \equiv 2Z.X.X - Z.X^{\cdot 2} = XZX$. It follows that

$$\{XZX\} = u[12] \tag{82}$$

and similarly

$$\{YZY\} = w[23]. \tag{83}$$

Also, if $\{ABC\} = A.B.C + B.C.A - A.C.B$ then

$$2\{X\{ZY^{\cdot 2}Z\}Y\} = XZY^2ZY + YZY^2ZX.$$

The entry in the $(2,3)$-position of this matrix is the same as the $(2,3)$-entry of XZY^2ZY, since $X = 1[12]$. Taking into account the form of X and Y one sees that this is q where q is the element in the $(1,2)$ place of ZY^2Z. The latter is vw. Hence the $(2,3)$ entry of $2\{X\{ZY^{\cdot 2}Z\}Y\}$ is vw and so the $(1,3)$ entry of

$$\{XZX\}.2\{X\{ZY^{\cdot 2}Z\}Y\}$$

is $u(vw)$.

A similar calculation shows that the $(1,2)$-entry of $2\{Y\{ZX^{\cdot 2}Z\}X\}$ is uv so the $(1,3)$-entry of $\{YZY\}.2\{Y\{ZX^{\cdot 2}Z\}X\}$ is $(uv)w$. On the other hand, it is clear that for any U in $\mathfrak{A}_3$ the $(1,3)$-entries of XUX and YUY are 0. Hence if the identity (78) is to hold for $x = X, y = Y, z = Z$ then we must have $u(vw) = (uv)w$. Since u, v, w were arbitrary this must hold for all u, v, w in $\mathfrak{A}$. Hence we have proved

THEOREM 11 (ALBERT-PAIGE). *If $\mathfrak{H}(\mathfrak{A}_3)$ is a homomorphic image of a special Jordan algebra then $\mathfrak{A}$ is associative.*

We now specialize $\mathfrak{H}(\mathfrak{A}_3)$ to $\mathfrak{H}(\mathfrak{O}_3)$ where $\mathfrak{O}$ is a Cayley algebra. Then we know that $\mathfrak{H}(\mathfrak{O}_3)$ is a Jordan algebra and since $\mathfrak{O}$ is not associative, $\mathfrak{H}(\mathfrak{O}_3)$ is not a homomorphic image of a special Jordan algebra. Hence $\mathfrak{H}(\mathfrak{O}_3)$ is exceptional. Also the argument above shows that if we take $X = 1[12]$, $Y = 1[23]$, $Z = u[21] + v[13] + w[32]$ and choose u, v, w so that $[u, v, w] \neq 0$ then (78) does not hold for $x = X$, $y = Y$, $z = Z$. Hence we have

THEOREM 12. (GLENNIE). *The element*

$$2\{xzx\} . \{y\{zy^{\cdot 2}z\}x\} - 2\{yzy\} . \{x\{zx^{\cdot 2}z\}y\}$$
$$- \{x\{z\{x\{yzy\}y\}z\}x\} + \{y\{z\{y\{xzx\}x\}z\}y\}$$

is an s-identity in three variables x, y, z.

There are a number of other important results on s- identities which are known at the present time. It has been proved by Blattner [1] and by Glennie [1] that there are no s-identities of degree $\leqq 7$ in three variables. Also Glennie has shown that the following element of degree 8 is an s-identity:

$$4\{\{z\{xyx\}z\}y(x . z)\} - 2\{z\{x\{y(x.z)y\}x\}z\}$$
$$- 4\{(x . z)y\{x\{zyz\}x\}\} + 2\{x\{z\{y(x . z)y\}z\}x\}.$$

It can be checked that this holds for all special Jordan algebras but fails for

$$X = e_{11} - e_{33}, \qquad Y = 1[12] + 1[23],$$
$$Z = u[12] + v[13] + w[23]$$

in $\mathfrak{H}(\mathfrak{O}_3)$ if $[u, v, w] \neq 0$. Glennie has shown also that there are no s-identities in $\leqq 5$ variables which are of first degree in all of these.

EXERCISE

1. Show that if $u = i$, $v = j$, $w = l$ in $\mathfrak{O}$ as in §5 then the matrices X, Y, Z of (81) in $\mathfrak{H}(\mathfrak{O}_3, \gamma)$ generate this Jordan algebra.

11. **Inverses and zero divisors.** Let $\mathfrak{A}$ be an associative algebra with 1 and suppose that a and b are inverses in $\mathfrak{A}$: $ab = 1 = ba$. Then we have the Jordan relations $a . b = \frac{1}{2}(ab + ba) = 1$ and $a^{\cdot 2} . b = \frac{1}{2}(a^2b + ba^2) = a$. Conversely, assume a and b are elements which satisfy the Jordan relations

$$a . b = 1, \qquad a^{\cdot 2} . b = a. \tag{84}$$

Then we have $ab + ba = 2$ and $a^2b + ba^2 = 2a$. Hence $ab + aba = 2a = aba + ba^2$ and $a^2b = ba^2$. Then $2a = 2a^2b = 2ba^2$ so $a = a^2b = ba^2$ and $ab = ba^2b = ba$. This and $ab + ba = 2$ imply that $ab = 1 = ba$ so a and b are inverses. These considerations lead to the following

DEFINITION 5. *Let* $\mathfrak{J}$ *be a Jordan algebra with an identity element* 1. *Then an element* $a \in \mathfrak{J}$ *is called* invertible *with* b *as* an inverse *if the equations* (84) *hold in* $\mathfrak{J}$.

If we take into account the preceding result and the fact that the subalgebra generated by 1, a and b is special (Shirshov-Cohn Theorem) then we can conclude that b is invertible with a as inverse and if we define $a^{\cdot -n} = b^{\cdot n}$, $n > 0$, $a^{\cdot 0} = 1$ then we have $a^{\cdot k} . a^{\cdot l} = a^{\cdot k+l}$ for all integral k, l. These results can also be obtained easily from the basic Jordan identities and the recursion formula for $R_{a^{\cdot k}}$. We shall now derive these and some other basic properties of inverses which cannot be obtained just from the Shirshov-Cohn Theorem. The important tool for the derivation is the following fundamental identity:

$$U_a U_b U_a = U_{bU_a}. \tag{85}$$

This is equivalent to $\{a\{b\{axa\}b\}a\} = \{\{aba\}x\{aba\}\}$. Now, in a special Jordan algebra $\mathfrak{A}^+$, $\mathfrak{A}$ associative, the two sides of the relation we have just given become $a(b(axa)b)a$ and $(aba)x(aba)$ respectively. Hence (85) is valid for special Jordan algebras. Moreover, the corresponding identity in a, b and x is of the first degree in x. Hence it follows from Macdonald's Theorem that (85) holds for all Jordan algebras.

We are now ready to prove the following

THEOREM 13. *Let* $\mathfrak{J}$ *be a Jordan algebra with an identity* 1 *and let* $a, b \in \mathfrak{J}$. *Then*:

(1) *If* a *is invertible in* $\mathfrak{J}$ *with* b *as inverse then* b *is invertible with* a *as inverse.*

(2) *The following three conditions are equivalent*: (a) a *is invertible,* (b) 1 *is in the range of* U_a, (c) U_a^{-1} *exists.*

(3) *If* a *is invertible then it has a unique inverse and, in fact, this is the element* $b = aU_a^{-1}$.

(4) *If* a *and* b *are inverses then* $U_b = U_a^{-1}$ *and* $R_b = U_a^{-1} R_a$.

(5) $[R_{a^{\cdot k}}, R_{b^{\cdot l}}] = 0$ *for all* $k,l \geqq 0$. *Also if we define* $a^{\cdot -n} = b^{\cdot n}$ *for* $n > 0$, $a^{\cdot 0} = 1$ *then* $a^{\cdot k} . a^{\cdot l} = a^{\cdot k+l}$ *for all integral* k, l.

(6) a *and* b *are invertible if and only if* $\{aba\}$ *is invertible.*

PROOF. (1) If a is invertible with b as an inverse then the Jordan identity $[R_x R_{y.z}] + [R_y R_{x.z}] + [R_z R_{x.y}] = 0$ for $x = b$, $y = z = a$ gives $[R_b R_{a^{\cdot 2}}] + [R_a R_1] + [R_a R_1] = 0$. Since $R_1 = 1$ we have $[R_b R_{a^{\cdot 2}}] = 0$. Then $b^{\cdot 2} . a^{\cdot 2} = bR_b R_{a^{\cdot 2}} = bR_{a^{\cdot 2}} R_b = b . a^{\cdot 2} . b = a . b = 1$ and $b^{\cdot 2} . a = b^{\cdot 2} . (a^{\cdot 2} . b) = (b^{\cdot 2} . a^{\cdot 2}) . b = b$. This shows that a satisfies the conditions for an inverse of b; hence (1) is proved.

(2) If a is invertible with b as an inverse then $b^{\cdot 2} U_a = 2b^{\cdot 2} . a . a - b^{\cdot 2} . a^{\cdot 2} = 2b . a - 1 = 1$, using the result $b^{\cdot 2} . a^{\cdot 2} = 1$ established in (1). Hence 1 is in the range $\mathfrak{J} U_a$ of U_a. Hence (a) $\Rightarrow$ (b). Next assume $1 \in \mathfrak{J} U_a$ so we have a c in $\mathfrak{J}$

such that $1 = cU_a$. Then the operator $1 = U_1 = U_{cU_a} = U_aU_cU_a$ by (85). Hence U_a^{-1} exists and so (b) $\Rightarrow$ (c). Next assume (c) and let $b = aU_a^{-1}$. Then $a = bU_a$ and $1U_a = a^{\cdot 2} = aR_a = bU_aR_a = bR_aU_a$. Since U_a^{-1} exists this gives $1 = bR_a = a.b$. Similarly, $aU_a = a^{\cdot 3} = aR_{a^{\cdot 2}} = bU_aR_{a^{\cdot 2}} = bR_{a^{\cdot 2}}U_a$. Hence $a = b.a^{\cdot 2}$. Hence a is invertible with b as inverse and we have proved that (c) $\Rightarrow$ (a).

(3) If b is an inverse of a then $bU_a = 2b.a.a - b.a^{\cdot 2} = a$. Since U_a^{-1} exists by (2), this gives $b = aU_a^{-1}$ and b is unique.

In the remainder of the proof we assume a and b are inverses.

(4) We have $bU_a = a$ so by (85), $U_a = U_{bU_a} = U_aU_bU_a$. Since U_a^{-1} exists this gives $U_aU_b = 1 = U_bU_a$ and so $U_b = U_a^{-1}$. We had $[R_bR_{a^{\cdot 2}}]$ in the proof of (1). Also $[R_bU_a] = [R_bU_b^{-1}] = 0$. Since $U_a = 2R_a^2 - R_{a^{\cdot 2}}$ these two relations give $[R_bR_a^2] = 0$. We now put $x = y = a$, $z = b$ in the Jordan identity $R_xR_yR_z + R_zR_yR_x + R_{x.z.y} = R_{x.y}R_z + R_{x.z}R_y + R_{y.z}R_x$ to obtain $R_a^2R_b + R_bR_a^2 + R_a = R_{a^{\cdot 2}}R_b + R_a + R_a$. Since $R_a^2R_b = R_bR_a^2$ we get $(2R_a^2 - R_{a^{\cdot 2}})R_b = R_a$ so $U_aR_b = R_a$ and $R_b = U_a^{-1}R_a$.

(5) It is clear from (4) that the operators R_a, U_a, R_b and U_b commute. Since $R_{a^{\cdot k}}$ and $R_{b^{\cdot l}}$ for $k, l \geqq 0$ are polynomials in R_a, U_a, R_b and U_b we have $[R_{a^{\cdot k}}R_{b^{\cdot l}}] = 0$. Now if $k > 0$ then $a^{\cdot k}a^{\cdot -1} = aR_{a^{\cdot k-1}}R_{a^{\cdot -1}} = aR_{a^{\cdot -1}}R_{a^{\cdot k-1}} = 1R_{a^{\cdot k-1}} = a^{\cdot k-1}$. Assuming $a^{\cdot k}.a^{\cdot -(l-1)} = a^{\cdot k-l+1}$ for $l > 1$ we get from $[R_{a^{\cdot k}}R_b] = 0$ that $a^{\cdot k}.a^{\cdot -l} = a^{\cdot k}.(a^{\cdot -(l-1)}.a^{\cdot -1}) = (a^{\cdot k}a^{\cdot -(l-1)}).a^{\cdot -1} = a^{\cdot k-l+1}.a^{\cdot -1}$. If $k - l + 1 > 0$ this gives $a^{\cdot k-l}$ by what we proved first and the result is clear if $k - l + 1 = 0$ or < 0 by power associativity. The case $a^{\cdot k}.a^{\cdot l}$ where k and l are of the same sign is covered by power associativity. Hence (5) holds.

6) By the fundamental identity (85) we have $U_{\{aba\}} = U_aU_bU_a$. Now it is immediate that the linear transformation $U_aU_bU_a$ is invertible in $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ if and only if U_a and U_b are invertible. Hence (6) follows from (2).[6]

The second part of the preceding theorem shows that if the operator U_a is surjective then it is also injective. On the other hand, U_a may be injective without being surjective. Since U_a is linear the condition for this is that $bU_a = 0$ implies $b = 0$. In an associative algebra $bU_a = aba$. Hence in this case it is clear that U_a is injective if and only if a is neither a left nor a right zero divisor. Accordingly, we shall say that an element a of a Jordan algebra is a *zero divisor* if the mapping U_a is not injective in $\mathfrak{J}$. We shall also define a *Jordan division algebra* to be a Jordan algebra in which every $a \neq 0$ is invertible and a *Jordan integral domain* as a Jordan algebra in which there are no zero divisors except 0. If $\mathfrak{A}$ is an associative division algebra (integral domain) then $\mathfrak{A}^+$ is a Jordan division algebra (integral domain). We shall give other examples in the exercises.

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[6] The notion of inverses and ternary composition $\{abc\}$ were introduced by Jacobson in [19]. Theorem 12 was proved in this paper but the proof was considerably more complicated than the present one since it did not use (85) (which was conjectured in this paper). The present simple proof is due to McCrimmon [2].

If a is an algebraic element of a Jordan algebra $\mathfrak{J}$ (with 1) then it is clear that a is invertible in $\Phi[a]$, the subalgebra generated by 1 and a, if the minimum polynomial $f(\lambda)$ of a has a nonzero constant term. In fact, if $f(\lambda) = \lambda^n + \alpha_1\lambda^{\cdot n-1} + \cdots + \alpha_n$ and $\alpha_n \neq 0$ then $\alpha_n^{-1}(-a^{\cdot n-1} - \alpha_1 a^{\cdot n-2} - \cdots - \alpha_{n-1}1)$ is the inverse of a. On the other hand, if $\alpha_n = 0$ then $b = a^{\cdot n-1} + \alpha_1 a^{\cdot n-2} + \cdots + \alpha_{n-1}1 \neq 0$ and $bU_a = 0$. Hence a is a zero divisor and a cannot be invertible in $\mathfrak{J}$. It is well known (and easy) that if a is algebraic then the associative commutative algebra $\Phi[a]$ is a division algebra if and only if the minimum polynomial $f(\lambda)$ is irreducible in $\Phi[\lambda]$. Hence if $\mathfrak{J}$ is an *algebraic Jordan algebra* in the sense that every element of $\mathfrak{J}$ is algebraic then $\mathfrak{J}$ is a division algebra if and only if the minimum polynomials of all of its elements are irreducible.

Let $\mathfrak{U}$ be the set of invertible elements of the Jordan algebra $\mathfrak{J}$. Since we have $U_{a^{\cdot n}} = U_a^{\ n}$, $n = 1, 2, 3, \ldots$ and $U_{a^{\cdot 1}} = U_a^{\ -1}$, and a is invertible if and only if U_a is invertible in $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ it is clear that $a \in \mathfrak{U}$ if and only if $a^{\cdot n} \in \mathfrak{U}$ for $n = \pm 1, \pm 2, \pm 3, \ldots$. Also, by Theorem 13 (6), we see that $\{aba\} \in \mathfrak{U}$ if and only if a and $b \in \mathfrak{U}$. The set $\mathfrak{U}$ and especially the connected component of 1 of $\mathfrak{U}$ are important subsets of certain of the classical Jordan algebras. For example if $\mathfrak{J}$ is the Jordan algebra of real symmetric matrices with $a \,.\, b = \frac{1}{2}(ab + ba)$ then the component of 1 in $\mathfrak{U}$ is the set of positive definite symmetric matrices (exercise below). The set $U(\mathfrak{U})$ of mappings U_a, $a \in \mathfrak{U}$, is a set of 1-1 mappings of $\mathfrak{J}$ onto $\mathfrak{J}$. Clearly $U(\mathfrak{U})$ is closed under powers and under the composition $U_a U_b U_a$. The transformations U_a, $a \in \mathfrak{U}$, generate a group of linear transformations in the vector space $\mathfrak{J}/\Phi$.

Exercises

1. Let $\mathfrak{A}$ be an associative division algebra with an involution J. Show that $\mathfrak{H}(\mathfrak{A}, J)$, the subalgebra of $\mathfrak{A}^+$ of J-symmetric elements is a division algebra.

2. Let $\mathfrak{J}$ be the Jordan algebra of the symmetric bilinear form f in the vector space $\mathfrak{V}$ (§1.4). Show that $\mathfrak{J}$ is a division algebra if and only if $f(x, x)$ is not a square in Φ for any $x \in \mathfrak{V}$.

3. Let a, b be invertible elements of a Jordan algebra $\mathfrak{J}$ with 1 such that $a - b^{\cdot -1}$ is also invertible. Prove that $(a^{\cdot -1} + (b^{\cdot -1} - a)^{\cdot -1})$ is invertible and the Hua identity $(a^{\cdot -1} + (b^{\cdot -1} - a)^{\cdot -1})^{\cdot -1} = a - \{aba\}$ holds. Use this to show that if σ is a linear mapping of a Jordan division algebra Δ into a Jordan division algebra Δ' such that $1^\sigma = 1$ and if $a \neq 0$ then $a^\sigma \neq 0$ and $(a^{\cdot -1})^\sigma = (a^\sigma)^{\cdot -1}$ then σ is an algebra homomorphism.

4. Let a be an invertible element of a Jordan algebra $\mathfrak{J}$ with 1 and let D be a derivation of $\mathfrak{J}$. Show that $a^{\cdot -1}D = -(aD)U_a^{\ -1}$. Show also that if $\mathfrak{J}$ is a division algebra and D is a linear transformation in $\mathfrak{J}$ such that $a^{\cdot -1}D = -(aD)U_a^{-1}$ holds for all $a \neq 0$ then D is a derivation.

5. An element a of a Jordan algebra is called *quasi-invertible* if there exists a b in the algebra such that

$$a + b - a.b = 0$$

and

$$a + b - 2a.b - a^{\cdot 2} + a^{\cdot 2}.b = 0.$$

Then b is called a *quasi-inverse* of a. Show that for algebras with 1 this is equivalent to $1 - a$ is invertible with inverse $1 - b$. Show that the quasi-inverse is unique and that if a is quasi-invertible with quasi-inverse b then b is quasi-invertible with quasi-inverse a. Show also that if a and b are quasi-invertible then so is $\{aba\} - a^{\cdot 2} - 2a.b + 2a + b$.

6. Show that no idempotent $\neq 0$ is quasi-invertible and every nilpotent element is quasi-invertible. Show that for an algebraic Jordan algebra the following two conditions are equivalent: (1) every element is nilpotent; (2) every element is quasi-invertible.

7. Let $\mathfrak{J}$ be the Jordan algebra of $n \times n$ real symmetric matrices. Show that the connected component containing 1 of the set of invertible elements of $\mathfrak{J}$ is the set of positive definite matrices. Show the same thing for the Jordan algebras of complex hermitian and quaternionic hermitian matrices. Consider the same problem for the Jordan algebra of 3×3 hermitian octonion matrices where the octonion algebra is the one defined by the parameters $\lambda = \mu = \nu = -1$ in §5 (p. 16).

8. A Jordan algebra $\mathfrak{J}$ is called *regular* (à la von Neumann) if every $a \in \mathfrak{J}$ is contained in the range of U_a. Verify that $\Phi_n{}^+$, the Jordan algebra of all $n \times n$ matrices, is regular. Show also that $\mathfrak{H}(\Phi_n)$, the Jordan algebra of symmetric matrices, is regular. What about the exceptional Jordan algebra $\mathfrak{H}(\mathfrak{O}_3)$, $\mathfrak{O}$ an octonion algebra?

9. (Koecher). Let $\mathfrak{A}$ be a commutative (nonassociative) algebra with 1 over a field of characteristic $\neq 2, 5$ containing more than three elements. Define $U_a = 2R_a{}^2 - R_{a^2}$ in $\mathfrak{A}$. Show that if $U_{a^2} = U_a{}^2$ for all a in $\mathfrak{A}$ then $\mathfrak{A}$ is a Jordan algebra.

10. Let $\mathfrak{J}$ be a Jordan division algebra, $\mathfrak{D}$ a subalgebra of $\mathfrak{J}$ containing 1 such that (1) the division subalgebra of $\mathfrak{J}$ generated by $\mathfrak{D}$ is $\mathfrak{J}$, (2) $\mathfrak{D}$ has common multiple property: if a and b are nonzero elements of $\mathfrak{D}$ then $\mathfrak{D}U_a \cap \mathfrak{D}U_b \neq 0$. Show that every element of $\mathfrak{J}$ has the form $aU_b{}^{-1}$, $a, b \in \mathfrak{D}$, $b \neq 0$. Show that any isomorphism of $\mathfrak{D}$ into a Jordan division algebra has a unique extension to $\mathfrak{J}$.

11. (Koecher). Let $\mathfrak{A}$ be a vector space over a field of characteristic $\neq 2, 3$. Suppose there exists a symmetric bilinear mapping $(x, y) \to U_{x,y}$ of $\mathfrak{A}$ into the algebra of linear transformations of $\mathfrak{A}$ satisfying the following conditions (1) there exists an element 1 such that $U_{1,1} = 1$ and the mapping $x \to 1U_{1,x}$ is 1-1, (2) $U_{x,x}U_{y,y}U_{x,x} = U_{yU_{x,x}, yU_{x,x}}$. Show that a unique commutative multiplication can be defined on $\mathfrak{A}$ so that $U_{x,x} = 2R_x{}^2 - R_{x^{\cdot 2}}$ and this gives a Jordan structure on $\mathfrak{A}$.

12. **Homotopy and isotopy.** The ideas we shall consider in this section also have an associative background. We consider this first. Let $\mathfrak{A}$ be an associative algebra and let $a \in \mathfrak{A}$. Then we can define a middle a-multiplication in $\mathfrak{A}$ by $x_a y = xay$ (cf. §6). It is clear that this composition is associative, so we have a new associative algebra $(\mathfrak{A}, a)$ whose underlying vector space is the same as that for the given algebra $\mathfrak{A}$. We shall call $(\mathfrak{A}, a)$ the *a-homotope* of the associative algebra $\mathfrak{A}$. Next let b be another element of $\mathfrak{A}$ and consider the b-homotope of $(\mathfrak{A}, a)$. It is clear that the product $x_{a,b}y$ in this algebra is $xabay$. Hence the b-homotope of the a-homotope of $\mathfrak{A}$ is just the aba-homotope of $\mathfrak{A}$. In this sense homotopy is a transitive relation.

We assume next that $\mathfrak{A}$ has an identity element 1 and that a is invertible in $\mathfrak{A}$ with inverse a^{-1}. In this case we shall call $(\mathfrak{A}, a)$ the *a-isotope* of $\mathfrak{A}$. Since $a^{-1}ax = xaa^{-1} = x$ it is clear that a^{-1} is the identity element of $(\mathfrak{A}, a)$. Also a^{-2} has the inverse 1 in $(\mathfrak{A}, a)$ and if we put $b = a^{-2}$ in aba we obtain 1. This shows that the a^{-2}-isotope of $(\mathfrak{A}, a)$ is $\mathfrak{A}$ itself. Hence isotopy is a symmetric relation for algebras with identity elements. The relation is also reflexive since the isotope $(\mathfrak{A}, 1)$ is $\mathfrak{A}$ itself. We remark also that the a-isotope $(\mathfrak{A}, a)$ is isomorphic to $\mathfrak{A}$ since the mapping $x \to x^\alpha = xa^{-1}$ satisfies $x^\alpha{}_a y^\alpha = xa^{-1}aya^{-1} = xya^{-1} = (xy)^\alpha$.

We shall now carry this over first to the special Jordan algebra $\mathfrak{A}^+$ and then to arbitrary Jordan algebras. We consider the special Jordan algebra $(\mathfrak{A}, a)^+$ where $(\mathfrak{A}, a)$ is the a-homotope of $\mathfrak{A}$, a arbitrary. The product in this algebra is $x_{.a}y = \frac{1}{2}(x_a y + y_a x) = \frac{1}{2}(xay + yax) = \{xay\}$, the usual Jordan triple product in $\mathfrak{A}^+$. Now we have the Jordan identity $((x_{.a}x)_{.a}y)_{.a}x = ((x_{.a}x)_{.a}(y_{.a}x))$ in the special Jordan algebra $(\mathfrak{A}, a)^+$ and this gives the following triple product identity:

$$\{\{\{xax\}ay\}ax\} = \{\{xax\}a\{yax\}\} \tag{86}$$

in $\mathfrak{A}^+$. We now observe that this identity in the three variables x, y, a is of first degree in y. Hence, by Macdonald's Theorem, (86) is valid in every Jordan algebra. Now let $\mathfrak{J}$ be any Jordan algebra, a an element of $\mathfrak{J}$. Then we can define an a-multiplication in $\mathfrak{J}$ by

$$x_{.a}y = \{xay\}.$$

This is clearly bilinear in x and y and is commutative. Moreover the identity (86) is just the second Jordan identity for the product $_{.a}$. Hence we obtain the Jordan algebra $(\mathfrak{J}, a)$ which we shall call the *a-homotope* of $\mathfrak{J}$. It is clear from the definition and the foregoing remarks that the Jordan homotope $(\mathfrak{A}^+, a)$, $\mathfrak{A}$ associative, coincides with $(\mathfrak{A}, a)^+$ where $(\mathfrak{A}, a)$ is the a-homotope of $\mathfrak{A}$.

Next let b be a second element of $\mathfrak{A}$. Then the b-homotope of the a-homotope of $\mathfrak{A}^+$ has the multiplication

$$\begin{aligned} x_{.a,b}y &= (x_{.a}b)_{.a}y + (b_{.a}y)_{.a}x - (x_{.a}y)_{.a}b \\ &= \{\{xab\}ay\} + \{\{bay\}ax\} - \{\{xay\}ab\}. \end{aligned}$$

On the other hand, since the a-homotope of $\mathfrak{A}^+$ is the Jordan algebra $(\mathfrak{A}, a))^+$, the b-homotope of this algebra is the Jordan algebra $((\mathfrak{A}, a), b)^+$. We have seen that the b-homotope of the a-homotope of the associative algebra $\mathfrak{A}$ is $(\mathfrak{A}, aba)$. It follows that we also have $x_{.a,b}y = \{x\{aba\}y\}$. Hence we have the following identity

$$(87) \qquad \{x\{aba\}y\} = \{\{xab\}ay\} + \{\{bay\}ax\} - \{\{xay\}ab\},$$

valid for all special Jordan algebras. This is a four variable identity so Macdonald's Theorem is not immediately applicable. However, the symmetry of $\{xay\}$ in x and y shows that (87) is obtained by linearizing the three variable identity

$$(88) \qquad \{x\{aba\}x\} = 2\{\{xab\}ax\} - \{\{xax\}ab\}.$$

This three variable identity is of first degree in b. Hence it goes over to arbitrary Jordan algebras. Hence (87) holds for all Jordan algebras. This now gives the same result for Jordan homotopy which we had for associative algebras: homotopy is transitive and the b-homotope of the a-homotope of the Jordan algebra $\mathfrak{J}$ is the $\{aba\}$-homotope of $\mathfrak{J}$.

Next assume $\mathfrak{J}$ has a 1 and let a be invertible. Then we call $(\mathfrak{J}, a)$ the *a-isotope* of $\mathfrak{J}$.[7] We have $\{a^{\cdot -1}ax\} = a^{\cdot -1} . a . x + a . x . a^{\cdot -1} - a^{\cdot -1} . x . a = a^{\cdot -1} . a . x$ since the multiplications by a and $a^{\cdot -1}$commute. Hence $\{a^{\cdot -1}ax\} = x$ and $a^{\cdot -1}$ is the identity element of $(\mathfrak{J}, a)$. Now 1 and $a^{\cdot -2}$ are inverses in $(\mathfrak{J}, a)$ since $\{1aa^{\cdot -2}\} = a^{\cdot -1}$ and $\{\{1a1\}aa^{\cdot -2}\} = \{aaa^{\cdot -2}\} = 1$. Also $\{aa^{\cdot -2}a\} = 1$ so $\mathfrak{J}$ is the $a^{\cdot -2}$-isotope of $(\mathfrak{J}\ a)$. It is convenient to extend the notions of isotopy and homotopy to apply to different vector spaces. Accordingly, we give the following

DEFINITION 6. *Let $\mathfrak{J}$ and $\mathfrak{J}'$ be Jordan algebras. Then a mapping α of $\mathfrak{J}$ into $\mathfrak{J}'$ is called a* homotopy *if α is linear and there exists a $b \in \mathfrak{J}'$ such that $(x . y)^\alpha = \{x^\alpha b y^\alpha\}$. If $\mathfrak{J}$ and $\mathfrak{J}'$ have identity elements, and α is a linear isomorphism of the vector space $\mathfrak{J}$ onto $\mathfrak{J}'$ and b is invertible then we call α an* isotopy *of $\mathfrak{J}$ onto $\mathfrak{J}'$ and we say that $\mathfrak{J}$ and $\mathfrak{J}'$ are* isotopic.

This definition of homotopy amounts to that of a homomorphism of $\mathfrak{J}$ into the b-homotope of $\mathfrak{J}'$ and the definition of isotopy amounts to that of an isomorphism of $\mathfrak{J}$ onto the b-isotope of $\mathfrak{J}'$. Let α be a homotopy of $\mathfrak{J}$ into $\mathfrak{J}'$ and β a homotopy of $\mathfrak{J}'$ into $\mathfrak{J}''$ where we have $(x . y)^\alpha = \{x^\alpha b y^\alpha\}$, $x, y \in \mathfrak{J}, b \in \mathfrak{J}'$ and $(u . v)^\beta = \{u^\beta c v^\beta\}$, $u, v \in \mathfrak{J}'$, $c \in \mathfrak{J}''$. Then $(x . y)^{\alpha\beta} = \{x^\alpha b y^\alpha\}^\beta = \{\{x^{\alpha\beta} c b^\beta\} c y^{\alpha\beta}\} + \{\{y^{\alpha\beta} c b^\beta\} c x^{\alpha\beta}\} - \{\{x^{\alpha\beta} c y^{\alpha\beta}\} c b^\beta\} = \{x^{\alpha\beta}\{c b^\beta c\} y^{\alpha\beta}\}$, by (87). Hence the product of two homotopies is a homotopy. Now consider the class of Jordan algebras

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7 This concept was introduced in Jacobson [24]. The Jordan identity on which it is based was conjectured in Jacobson [19]. The notion of isotope has been called a *mutand* by Hel Braun and Koecher in their book, *Jordan Algebren* [1].

with 1. Then it is clear that the identity mapping in such an algebra is a homotopy (with $b = 1$). This and the result just noted shows that the class of Jordan algebras with 1, with morphisms defined as homotopies, is a category.

Let α be a homotopy of $\mathfrak{J}$ with 1 into $\mathfrak{J}'$ with 1, β a homotopy of $\mathfrak{J}'$ into $\mathfrak{J}$ and assume that α and β are inverses, that is, $\alpha\beta = 1_{\mathfrak{J}}$ the identity mapping on $\mathfrak{J}$ and $\beta\alpha = 1_{\mathfrak{J}'}$. If $(x\,.\,y)^{\alpha} = \{x^{\alpha}by^{\alpha}\}$ and $(u\,.\,v)^{\beta} = \{u^{\beta}cv^{\beta}\}$ where $x, y, c \in \mathfrak{J}$ and $u, v, b \in \mathfrak{J}'$ then α is a linear isomorphism of $\mathfrak{J}$ onto $\mathfrak{J}'$ and $x\,.\,y = (x\,.\,y)^{\alpha\beta} = \{x\{cb^{\beta}c\}y\}$. This implies that $\{cb^{\beta}c\} = 1$ in $\mathfrak{J}$ so 1 is in the range of U_c and c is invertible in $\mathfrak{J}$. By symmetry b is invertible in $\mathfrak{J}'$. Hence α and β are isotopies in the sense of Definition 6. Conversely, let α be an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$ and let b be as in the definition. Then there exists a c in $\mathfrak{J}$ such that $\{bc^{\alpha}b\} = 1'$ the identity element of $\mathfrak{J}'$. If $u, v \in \mathfrak{J}'$ and β is the inverse of the linear mapping α, then, by (87),

$$((u\,.\,v)^{\beta} - \{u^{\beta}cv^{\beta}\})^{\alpha} = u\,.\,v - \{u\{bc^{\alpha}b\}v\} = u\,.\,v - u\,.\,v = 0.$$

Hence $(u\,.\,v)^{\beta} = \{u^{\beta}cv^{\beta}\}$ so β is a homotopy of $\mathfrak{J}'$ onto $\mathfrak{J}$. Hence we have shown that α is an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$ if and only if α is a homotopy of $\mathfrak{J}$ into $\mathfrak{J}'$ and there exists a homotopy β of $\mathfrak{J}'$ into $\mathfrak{J}$ such that $\alpha\beta = 1_{\mathfrak{J}}$, $\beta\alpha = 1_{\mathfrak{J}'}$. This implies that isotopy is an equivalence relation in the class of Jordan algebras with 1. We note also that if α is an isomorphism of $(\mathfrak{J}, a)$ onto $\mathfrak{J}'$, α^{-1} is an isotopy of $\mathfrak{J}'$ onto $\mathfrak{J}$ and hence α is an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$.

Suppose again that a is arbitrary in $\mathfrak{J}$ and consider the homotope $(\mathfrak{J}, a)$. Let $R_c^{(a)}$ and $U_c^{(a)}$ denote the multiplication and U-operator determined by c in $(\mathfrak{J}, a)$. We have $xR_c^{(a)} = \{xac\} = x(R_aR_c - R_cR_a + R_{a.c})$, so

$$R_c^{(a)} = [R_aR_c] + R_{a.c}. \tag{89}$$

By definition, $U_c^{(a)} = 2(R_c^{(a)})^2 - R_{\{cac\}}{}^{(a)}$. To obtain a formula for this we put $b = x$, $x = c$ in (88) and obtain

$$\{c\{axa\}c\} = 2\{\{cax\}ac\} - \{\{cac\}ax\}.$$

Hence $2x(R_c^{(a)})^2 - xR_{\{cac\}}{}^{(a)} = xU_aU_c$ and so

$$U_c^{(a)} = U_aU_c. \tag{90}$$

Let α be a homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$. Since $\mathfrak{J}$ and $\mathfrak{J}'$ are commutative algebras of characteristic $\neq 2$ it is clear by linearization that a linear mapping of $\mathfrak{J}$ into $\mathfrak{J}'$ is a homomorphism if and only if $(x^{\cdot 2})^{\alpha} = (x^{\alpha})^{\cdot 2}$. The condition $(x\,.\,c)^{\alpha} = x^{\alpha}\,.\,c^{\alpha}$ can be written in operator form as

$$R_c\alpha = \alpha R_{c^{\alpha}}, \quad c \in \mathfrak{J}. \tag{91}$$

This implies $R_{c^{\cdot 2}}\alpha = \alpha R_{(c^{\alpha})^{\cdot 2}}$ and hence

$$U_c\alpha = \alpha U_{c^{\alpha}}, \qquad c \in \mathfrak{J}. \tag{92}$$

Next let α be a homotopy of $\mathfrak{J}$ into $\mathfrak{J}'$, so that α is a homomorphism of $\mathfrak{J}$ into a b-homotope $(\mathfrak{J}', b)$ of $\mathfrak{J}'$. Then $U_c\alpha = \alpha U_{c^\alpha}{}^{(b)}$ where $U_{c^\alpha}{}^{(b)}$ is the U-operator in $(\mathfrak{J}', b)$ defined by c^α. By (90), we have

$$U_c\alpha = \alpha U_b U_{c^\alpha}, \qquad c \in \mathfrak{J}, \tag{93}$$

or,

$$U_c\alpha = \beta U_{c^\alpha}, \qquad c \in \mathfrak{J}, \tag{93$'$}$$

where $\beta = \alpha U_b \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J}')$ (as vector spaces). Conversely, let α be any element in $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J}')$ such that there exists a $\beta \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J}')$ satisfying (93′). Assume also that $\mathfrak{J}$ has an identity element 1. Then if we apply the operators on the two sides of (93′) to 1, we obtain $(c^{\cdot 2})^\alpha = 1^\beta U_{c^\alpha} = \{c^\alpha b c^\alpha\}$ where $b = 1^\beta$. Then α is a homotopy of $\mathfrak{J}$ into $\mathfrak{J}'$.

Now assume $\mathfrak{J}$ and $\mathfrak{J}'$ have identity elements and let α be an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$. Then α is an isomorphism of $\mathfrak{J}$ onto $(\mathfrak{J}', b)$ where b is invertible in $\mathfrak{J}'$. Then (93′) holds and $\beta = \alpha U_b$ is bijective. On the other hand, let α be any bijective linear mapping of $\mathfrak{J}/\Phi$ onto $\mathfrak{J}'/\Phi$ such that there exists a $\beta \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J}')$ satisfying (93′). Putting $c = 1$ in (93′) we obtain $\alpha = \beta U_{1^\alpha}$. Since α is surjective this implies that U_{1^α} is surjective. Hence 1^α is invertible in $\mathfrak{J}'$ and U_{1^α} is bijective. Then $\beta = \alpha U_{1^\alpha}{}^{-1}$ is uniquely determined by α and, moreover, this mapping is also bijective. We have $1^\beta = 1^\alpha U_{1^\alpha}{}^{-1} = (1^\alpha)^{\cdot -1}$ so $b = 1^\beta$ is invertible. Then (93′) gives $(c^{\cdot 2})^\alpha = \{c^\alpha b c^\alpha\}$ which implies that α is an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$. We therefore see that a mapping α of $\mathfrak{J}$ onto $\mathfrak{J}'$ is an isotopy if and only if α is bijective and linear, and there exists a $\beta \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J}')$ such that (93′) holds. If we recall that c is invertible in $\mathfrak{J}$ if and only if U_c is invertible it is clear also by (93′) that c is invertible in $\mathfrak{J}$ if and only if its image c^α under the isotopy α is invertible in $\mathfrak{J}'$. Hence we see that the restriction of an isotopy to the set of invertible elements of $\mathfrak{J}$ is a bijection of this set onto the set of invertible elements of $\mathfrak{J}'$.

Now let $\mathfrak{J}' = \mathfrak{J}$ and consider the set $\Gamma(\mathfrak{J})$ of isotopies of $\mathfrak{J}$ onto itself. Clearly $\Gamma(\mathfrak{J})$ is a group of linear transformations in $\mathfrak{J}/\Phi$. Following Koecher ([4] or Braun and Koecher [1]) we shall call $\Gamma(\mathfrak{J})$ the *structure group* of $\mathfrak{J}$. We have seen that $\Gamma(\mathfrak{J})$ is the set of bijective linear mappings α in $\mathfrak{J}$ for which there exists a $\beta \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ satisfying (93′). Also, in this situation α is an isomorphism of $\mathfrak{J}$ onto its isotope $(\mathfrak{J}, b)$ where $b = 1^\beta = (1^\alpha)^{\cdot -1}$. Moreover $\beta = \alpha U_{1^\alpha}{}^{-1}$. If $1^\alpha = 1$ then α is an isomorphism of $\mathfrak{J}$ onto $(\mathfrak{J}, 1) = \mathfrak{J}$ so α is an automorphism of $\mathfrak{J}$. Conversely, $1^\alpha = 1$ for any automorphism. Hence it is clear that the group of automorphisms $\mathrm{Aut}\,\mathfrak{J}$ of $\mathfrak{J}$ is the subgroup of the structure group $\Gamma(\mathfrak{J})$ consisting of the elements having 1 as fixed point (isotropy subgroup of the element 1).

Let a be an invertible element of $\mathfrak{J}$ and put $\alpha = U_a$. Then the fundamental, identity $U_a U_c U_a = U_{cU_a}$ gives $U_c U_a = U_a{}^{-1} U_{cU_a}$ or $U_c\,\alpha = \alpha^{-1} U_{c^\alpha}$. Since $\alpha = U_a$ is bijective this implies that $U_a \in \Gamma(\mathfrak{J})$. We shall call the subgroup $\Gamma_1(\mathfrak{J})$ of $\Gamma(\mathfrak{J})$ generated by the U_a, a invertible, the *inner structure group* of $\mathfrak{J}$. Since

$U_a^{-1} = U_{a^{\cdot -1}}$ it is clear that the elements of $\Gamma_1(\mathfrak{J})$ are products of elements U_a, a invertible. By (93) we have $\alpha^{-1}U_a\alpha = U_b U_{a^\alpha}$ for any $\alpha \in \Gamma(\mathfrak{J})$. Here $b = (1^\alpha)^{\cdot -1}$ and a^α are invertible. Hence $\alpha^{-1}U_a\alpha \in \Gamma_1(\mathfrak{J})$ and $\Gamma_1(\mathfrak{J})$ is an invariant subgroup of $\Gamma(\mathfrak{J})$. We shall call the elements of $\text{Aut}\,\mathfrak{J} \cap \Gamma_1(\mathfrak{J})$ the *inner automorphisms* of $\mathfrak{J}$. Clearly this set is an invariant subgroup of Aut $\mathfrak{J}$.

If $\alpha = U_a$, a invertible, then α is an isomorphism of $\mathfrak{J}$ onto the isotope $(\mathfrak{J}, b)$ where $b = (1^\alpha)^{\cdot -1} = (a^{\cdot 2})^{\cdot -1} = a^{\cdot -2}$. Clearly this implies the following result which will be useful later on: If b is a square of an invertible element then $\mathfrak{J}$ and its isotope $(\mathfrak{J}, b)$ are isomorphic. Another important remark is that if $a^{\cdot 2} = 1$ then U_a is an automorphism since a is invertible and $1U_a = a^{\cdot 2} = 1$. Also $U_a^2 = U_{a^{\cdot 2}} = 1$

We shall show next that $\mathfrak{J}$ and any isotope $(\mathfrak{J}, a)$ have the same structure group. Thus let $\alpha \in \Gamma(\mathfrak{J})$ and suppose $U_c\alpha = \beta U_{c^\alpha}$, $c \in \mathfrak{J}$. Then if a is invertible, $U_c^{(a)}\alpha = U_a U_c \alpha = U_a \beta U_{c^\alpha} = (U_a \beta U_a^{-1}) U_a U_{c^\alpha} = \gamma U_{c^\alpha}^{(a)}$ where $\gamma = U_a \beta U_a^{-1}$. Hence $\alpha \in \Gamma((\mathfrak{J}, a))$. By symmetry, $\Gamma(\mathfrak{J}) = \Gamma((\mathfrak{J}, a))$. It is clear also from the relation $U_c^{(a)} = U_a U_c$ that $\Gamma_1(\mathfrak{J}) = \Gamma_1((\mathfrak{J}, a))$.

We recall that a (two-sided) *ideal* $\mathfrak{B}$ in a nonassociative algebra $\mathfrak{A}$ is a subspace of $\mathfrak{A}$ such that ab and $ba \in \mathfrak{B}$ for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$. Also $\mathfrak{A}$ is called *simple* if the only ideals in $\mathfrak{A}$ are $\mathfrak{A}$ and 0 and $\mathfrak{A}$ has a nontrivial multiplication, that is, there exist $a, b \in \mathfrak{A}$ such that $ab \neq 0$. If $\mathfrak{B}$ is an ideal in the Jordan algebra $\mathfrak{J}$ then it is clear from the definition of the product in the homotope $(\mathfrak{J}, a)$ that $\mathfrak{B}$ is an ideal in $(\mathfrak{J}, a)$. It follows that if $\mathfrak{J}$ has a 1 then $\mathfrak{J}$ and its isotopes have the same ideals. Also in this case $\mathfrak{J}$ simple implies $(\mathfrak{J}, a)$ simple for any isotope of $\mathfrak{J}$.

The notion of isotopy does not play an important role in the associative theory since isotopic algebras are isomorphic. On the other hand, for Jordan algebras this need not be the case, and isotopy gives a broader equivalence relation than isomorphism which is sometimes a more natural one.

We shall now give an important instance of isotopy which gives rise to isotopic nonisomorphic algebras. We assume that $\mathfrak{D}$ is an algebra with 1 and an involution $d \to \bar{d}$. We consider the standard involution $X \to \bar{X}^t$ in $\mathfrak{D}_n$ and let $\mathfrak{H}(\mathfrak{D}_n)$ be the subalgebra of $\mathfrak{D}_n^+$ of symmetric elements of $\mathfrak{D}_n$. Now let $A \in \mathfrak{H}(\mathfrak{D}_n) \cap N(\mathfrak{D}_n)$ where $N(\mathfrak{D}_n)$ is the nucleus of $\mathfrak{D}_n$ and assume A has an inverse A^{-1} in $N(\mathfrak{D}_n)$. Since $N(\mathfrak{D}_n)$ is an associative algebra A^{-1} is unique, which implies that $\overline{A^{-1}}^t = A^{-1}$, that is, $A^{-1} \in \mathfrak{H}(\mathfrak{D}_n)$. We have the involution $J_A \colon X \to A^{-1}\bar{X}^t A$ $(= A^{-1}(\bar{X}^t A) = (A^{-1}\bar{X}^t)A)$ in $\mathfrak{D}_n$ which gives the subalgebra $\mathfrak{H}(\mathfrak{D}_n, J_A)$ of $\mathfrak{D}_n^+$ of symmetric elements relative to this involution. Now $X \in \mathfrak{H}(\mathfrak{D}_n)$ if and only if $XA \in \mathfrak{H}(\mathfrak{D}_n, J_A)$ since $A^{-1}(\overline{XA})^t A = A^{-1}(A\bar{X}^t)A = \bar{X}^t A$. It follows from this that the mapping $X \to XA$ is a 1-1 linear mapping of $\mathfrak{H}(\mathfrak{D}_n)$ onto $\mathfrak{H}(\mathfrak{D}_n, J_A)$. Clearly the inverse of $X \to XA$ is the mapping $X \to XA^{-1}$. Now suppose that $\mathfrak{H}(\mathfrak{D}_n)$ is Jordan and consider the A-isotope $(\mathfrak{H}(\mathfrak{D}_n), A)$. The product $X ._A Y$ in this algebra is $X ._A Y = \{XAY\} = X . A . Y + A . Y . X - X . Y . A$. Since A is in the nucleus this reduces to $\frac{1}{2}(XAY + YAX)$ where parentheses are unnecessary. The image of $X ._A Y$ under the mapping $X \to XA$ is $\frac{1}{2}(XAYA + YAXA) = XA . YA$. Thus

$X \to XA$ is an isomorphism of $(\mathfrak{H}(\mathfrak{D}_n), A)$ onto $\mathfrak{H}(\mathfrak{D}_n, J_A)$ and since the former is a Jordan algebra the latter is a Jordan algebra. In a similar fashion one sees that if $\mathfrak{H}(\mathfrak{D}_n, J_A)$ is Jordan then the mapping $X \to XA^{-1}$ is an isomorphism of $(\mathfrak{H}(\mathfrak{D}_n, J_A), A^{-1})$ onto $\mathfrak{H}(\mathfrak{D}_n)$. Hence $\mathfrak{H}(\mathfrak{D}_n)$ is Jordan. We have therefore proved the following

THEOREM 14. *Let $\mathfrak{D}$ be an algebra with identity* 1 *and involution $d \to \bar{d}$ and let A be an element of $\mathfrak{H}(\mathfrak{D}_n)$ which is in the nucleus $N(\mathfrak{D}_n)$ and has an invoice A^{-1} in $N(\mathfrak{D}_n)$. Then $\mathfrak{H}(\mathfrak{D}_n)$ is Jordan if and only if $\mathfrak{H}(\mathfrak{D}_n, J_A)$ is Jordan. Also if the condition holds then $X \to XA$ is an isomorphism of the isotope $(\mathfrak{H}(\mathfrak{D}_n), A)$ onto $\mathfrak{H}(\mathfrak{D}_n, J_A)$ and $X \to XA^{-1}$ is an isomorphism of $(\mathfrak{H}(\mathfrak{D}_n, J_A), A^{-1})$ onto $\mathfrak{H}(\mathfrak{D}_n)$.*

Since we have seen in §1.5 that $\mathfrak{H}(\mathfrak{D}_3)$ is Jordan if $\mathfrak{D}$ is alternative with symmetric elements in the nucleus we have the

COROLLARY. *Let the notations be as in Theorem* 14. *Then $\mathfrak{H}(\mathfrak{D}_3, J_A)$ is Jordan if $\mathfrak{D}$ is alternative with symmetric elements in the nucleus.*

It is easy to give choices of A so that $\mathfrak{H}(\mathfrak{D}_n)$ is Jordan and $\mathfrak{H}(\mathfrak{D}_n, J_A)$ is an isotope which is not isomorphic to $\mathfrak{H}(\mathfrak{D}_n)$. For example, let R_n be the algebra of $n \times n$ matrices over the real field and let $\mathfrak{H}(\mathfrak{R}_n)$ be the Jordan algebra of symmetric matrices of R_n. If $A = (\alpha_{ij})$ is symmetric then the trace $\mathrm{tr} A^2 = \sum \alpha_{ij}{}^2$. Hence if $A \neq 0$ then $A^{\cdot 2} \neq 0$. Thus $\mathfrak{H}(R_n)$ has no nonzero nilpotent elements. On the other hand, we claim that if A is symmetric and invertible and is not definite (positive or negative) then $\mathfrak{H}(R_n, J_A)$ does contain nonzero nilpotent elements. This, of course, implies that $\mathfrak{H}(R_n) \not\cong \mathfrak{H}(R_n, J_A)$. To establish this we adopt the geometric point of view and consider in place of $\mathfrak{H}(R_n, J_A)$ the algebra of self-adjoint linear transformations in an n-dimensional vector space over the reals relative to the symmetric bilinear form f given by the matrix A. Then f is not definite. Hence there exists a vector $z \neq 0$ such that $f(z, z) = 0$. The mapping $Z: x \to f(x, z)z$ is linear and self-adjoint. Since A is nonsingular, f is nondegenerate, so there exists a vector u such that $f(u, z) \neq 0$. Then $xZ = f(x, z)z \neq 0$. On the other hand, $xZ^2 = f(x, z)f(z, z)z = 0$ so $Z^2 = 0$ and Z is a nonzero nilpotent element of our Jordan algebra.

EXERCISES

1. Show that any isotope of an exceptional Jordan algebra (with 1) is exceptional.
2. (Glennie). Prove that if $\mathfrak{H}(\mathfrak{D}_3, J_A)$ where A is a diagonal matrix is a homomorphic image of a special Jordan algebra then $\mathfrak{D}$ is associative.
3. Let $\mathfrak{J}$ be a Jordan algebra with 1. Show that if the homotope $(\mathfrak{J}, a)$ has an identity then a is invertible.

4. Show that $\mathfrak{H}(R_n)$ and $\mathfrak{H}(R_n, J_A)$ (notations as in the text) are isomorphic if and only if $A = B^2$, $B \in \mathfrak{H}(R_n)$. Also show that $X \to O^{-1}XO$, O orthogonal, is an inner automorphism of $\mathfrak{H}(R_n)$.

5. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a symmetric bilinear form f in $\mathfrak{V}$. Show that if a is an invertible element in $\mathfrak{J}$ then $(\mathfrak{J}, a)$ is also a Jordan algebra of a symmetric bilinear form. Show also that if $\mathfrak{J}$ is not a division algebra then a can be chosen so that $(\mathfrak{J}, a)$ has nonzero nilpotent elements ($z^n = 0$, $z \neq 0$).

6. Show that any element of the inner structure group $\Gamma_1(\mathfrak{J})$ of a Jordan algebra has the form $U_a{}^{(c)}$ where a, c are invertible elements.

CHAPTER II

ELEMENTS OF REPRESENTATION THEORY

The representation theory of Jordan algebras has two closely related aspects both of which are concerned with mappings of Jordan algebras into associative algebras. The first deals with homomorphisms of Jordan algebras into special ones. We shall now call a homomorphism of a Jordan algebra $\mathfrak{J}$ into an algebra $\mathfrak{G}^+$, $\mathfrak{G}$ an associative algebra with 1, an *associative specialization* of $\mathfrak{J}$ in $\mathfrak{G}$. This is a linear mapping σ of $\mathfrak{J}$ into $\mathfrak{G}$ such that

$$(1) \qquad (a\,.\,b)^\sigma = \tfrac{1}{2}(a^\sigma b^\sigma + b^\sigma a^\sigma), \qquad a, b \in \mathfrak{J}.$$

A second important type of mapping of a Jordan algebra $\mathfrak{J}$ into an associative algebra is the mapping $a \to R_a$ of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ where R_a is the multiplication $x \to x\,.\,a$ in $\mathfrak{J}$. Clearly $a \to R_a$ is linear and, moreover, the R_a satisfy $[R_a, R_{a^{\cdot 2}}] = 0$ and the identities which are consequences of this. More generally, if $\mathfrak{J}$ is a subalgebra of a Jordan algebra $\mathfrak{K}$ then we have the mapping $a \to R_a$ of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{K}, \mathfrak{K})$ where R_a is the multiplication in $\mathfrak{K}$ determined by the element $a \in \mathfrak{J}$. These examples lead to the notion of a *multiplication specialization* of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1, defined to be a linear mapping ρ of $\mathfrak{J}$ into $\mathfrak{G}$ such that

$$(2) \qquad \begin{aligned} [a^\rho, (a^{\cdot 2})^\rho] &= 0, \\ 2a^\rho b^\rho a^\rho + (a^{\cdot 2}\,.\,b)^\rho &= 2a^\rho (a\,.\,b)^\rho + b^\rho (a^{\cdot 2})^\rho. \end{aligned}$$

Since $[R_a R_{a^{\cdot 2}}] = 0$ and $2R_a R_b R_a + R_{a^{\cdot 2}.\,b} = 2R_a R_{a^{\cdot 2}.\,b} + R_b R_{a^{\cdot 2}}$ (by (O_3) on p. 34) it is clear that if $\mathfrak{J}$ is a subalgebra of the Jordan algebra $\mathfrak{K}$ then $a \to R_a$, $a \in \mathfrak{J}$, is multiplication specialization of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{K}, \mathfrak{K})$. Moreover, it can be shown that if $\mathfrak{G} = \mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ then a multiplication specialization ρ of $\mathfrak{J}$ in $\mathfrak{G}$ has the form $a \to R_a \,|\, \mathfrak{M}$ where $R_a \,|\, \mathfrak{M}$ is the restriction to a subspace $\mathfrak{M}$ of the multiplication by a in a suitable Jordan algebra $\mathfrak{K}$ containing $\mathfrak{J}$ and $\mathfrak{M}$. (The first equation in (2) alone would not be sufficient for this.)

It is rather remarkable that the notions of associative specialization and multiplication specialization which look so different formally are closely related and in a sense, the first of these concepts is a special case of the second one. The relation between the two notions is the following. Suppose σ_1 and σ_2 are two associative specializations in $\mathfrak{G}$ and assume that these commute in the sense that

$$(3) \qquad [a^{\sigma_1}, b^{\sigma_2}] = 0, \qquad a, b \in \mathfrak{J}.$$

Then the average $\rho = \frac{1}{2}(\sigma_1 + \sigma_2)$ is a multiplication specialization of $\mathfrak{J}$ in $\mathfrak{S}$. In particular, taking $\sigma_2 = 0$ we see that if σ is an associative specialization then $\rho = \frac{1}{2}\sigma$ is a multiplication specialization. Thus up to a factor the associative specializations are special cases of multiplication specializations.

The notion of a multiplication specialization of a Jordan algebra is a special case of that of a multiplication specialization of an algebra $\mathfrak{A}$ in a variety $\mathscr{V}(I)$ of algebras defined by a set of identities I. This is defined as a pair of linear mappings (λ, ρ) of $\mathfrak{A}$ into an associative algebra $\mathfrak{S}$ with 1 satisfying a set of conditions which can be determined explicitly from the identities and which are fulfilled by the left and right multiplications in $\mathfrak{A}$. We shall consider this more general notion also, both for its sake and since, as we shall see in Chapter VII, some important results in the Jordan case can be derived by reduction to multiplication specializations for alternative and associative algebras.

The concept of a multiplication specialization of $\mathfrak{A} \in \mathscr{V}(I)$ in $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ is equivalent to that of an I-bimodule $\mathfrak{M}$ for $\mathfrak{A}$. Such a bimodule gives rise to a split null extension $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M} \in \mathscr{V}(I)$. We shall consider also more general extensions of $\mathfrak{A}$ by a bimodule $\mathfrak{M}$ and obtain factor set conditions from the set of identities. In this connection and in other parts of the general consideration of varieties $\mathscr{V}(I)$ it will be convenient to assume the conditions H and L for the set of identities I which were formulated in §1.6.

A basic tool for the study of associative specializations of Jordan algebras and multiplication specializations of algebras in a variety $\mathscr{V}(I)$ is that of universal envelopes. This is a special case of a general situation for categories, which can be described as follows. We are given two categories $\mathscr{C}$ and $\mathscr{C}'$. For an object $a \in \mathscr{C}$ we have certain mappings ρ (or sets of these) called "representations" of a in objects $a' \in \mathscr{C}'$. If η' is a morphism $a' \to b'$ then $\rho\eta'$ is a representation of a in b'. Also if η is a morphism $b \to a$ in $\mathscr{C}$ then $\eta\rho$ is a representation of b in a'. The usual associativity and identity conditions are assumed. If $a \in \mathscr{C}$ a universal object for the representations of a is a pair (u', ι), $u' \in \mathscr{C}'$, ι a representation of a in u' such that if ρ is any representation of a in a' then there exists a unique morphism η' of u' making the following diagram commutative:

$$\begin{array}{ccc} a & \xrightarrow{\rho} & a' \\ {\scriptstyle\iota}\downarrow & \nearrow & \eta' \\ u' & & \end{array}$$

In important cases (such as the ones we shall consider) one can prove the existence of a universal representation (u', ι) for any $a \in \mathscr{C}$. Then u' is unique in a strong sense and we have an associated universal functor of $\mathscr{C}$ into $\mathscr{C}'$. There are a number of important properties possessed by such universal functors. However, we shall confine ourselves to deriving these only in the special cases we shall encounter. Occasionally, we shall call attention to the arguments which are purely functorial so that it will be unnecessary to repeat these in other situations in which analogous conditions hold.

1. **Associative specializations and universal envelopes.** If $\mathfrak{J}$ is a Jordan algebra over the field Φ and $\mathfrak{G}$ is an associative algebra with 1 over Φ then we have defined an associative specialization σ of $\mathfrak{J}$ in $\mathfrak{G}$ to be a homomorphism of $\mathfrak{J}$ into $\mathfrak{G}^+$. If ζ is a homomorphism or antihomomorphism of $\mathfrak{G}$ into another associative algebra $\mathfrak{G}'$ with 1 then $\sigma\zeta$ is an associative specialization of $\mathfrak{J}$ in $\mathfrak{G}'$. Also if η is a homomorphism of the Jordan algebra $\mathfrak{J}'$ into $\mathfrak{J}$ then $\eta\sigma$ is an associative specialization of $\mathfrak{J}'$ in $\mathfrak{G}$. Let Γ be an extension of the base field Φ of $\mathfrak{J}$ and $\mathfrak{G}$. Then σ has a unique extension to a linear mapping σ of the Jordan algebra $\mathfrak{J}_\Gamma = \Gamma \otimes_\Phi \mathfrak{J}$ into the associative algebra $\mathfrak{G}_\Gamma$ with 1. It is clear that the condition (1) for $a, b \in \mathfrak{J}$ carries over to $\mathfrak{J}_\Gamma$. Hence σ is an associative specialization of $\mathfrak{J}_\Gamma$ in $\mathfrak{G}_\Gamma$.

Definition 1. *If $\mathfrak{J}$ is a Jordan algebra, a pair $(S(\mathfrak{J}), \sigma_u)$ consisting of an associative algebra $S(\mathfrak{J})$ with 1 and an associative specialization of $\mathfrak{J}$ in $S(\mathfrak{J})$ is called a* special universal envelope *for $\mathfrak{J}$ if for any associative specialization σ of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1 there exists a unique homomorphism η of $S(\mathfrak{J})$ into $\mathfrak{G}$ sending 1 into 1 and such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{\ \sigma\ } & \mathfrak{G} \\ {\scriptstyle\sigma_u}\big\downarrow & \nearrow{\scriptstyle\eta} & \\ S(\mathfrak{J}) & & \end{array} \tag{4}$$

It is easy to prove the existence of a special universal envelope for any Jordan algebra. However, before doing this we shall prove the following theorem with many parts.

Theorem 1. (1) *If $(S(\mathfrak{J}), \sigma_u)$ and $(S'(\mathfrak{J}), \sigma_u')$ are special universal envelopes for $\mathfrak{J}$ then there exists a unique isomorpshism ι of $S(\mathfrak{J})$ onto $S'(\mathfrak{J})$ such that $\sigma_u' = \sigma_u \iota$.*

(2) *$S(\mathfrak{J})$ is generated by 1 and the image $\mathfrak{J}^{\sigma_u}$ of $\mathfrak{J}$ in $S(\mathfrak{J})$.*

(3) *There exists a unique involution π in $S(\mathfrak{J})$, called its main involution such that $a^{\sigma_u \pi} = a^{\sigma_u}$, $a \in \mathfrak{J}$.*

(4) *If ζ is a homomorphism of $\mathfrak{J}$ into a second Jordan algebra $\mathfrak{J}'$ and $(S(\mathfrak{J}), \sigma_u)$, $(S(\mathfrak{J}'), \sigma_u')$ are special universal envelopes for $\mathfrak{J}$ and $\mathfrak{J}'$ respectively, then there*

exists a unique homomorphism ζ_s *of* $S(\mathfrak{J})$ *into* $S(\mathfrak{J}')$ *such that* $1 \to 1$ *and the following diagram is commutative*

(5)
$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{\zeta} & \mathfrak{J}' \\ \sigma_u \big\downarrow & & \big\downarrow \sigma_u' \\ S(\mathfrak{J}) & \xrightarrow[\zeta_s]{} & S(\mathfrak{J}') \end{array}$$

Moreover, ζ_s *is a homomorphism of algebras with involution of* $(S(\mathfrak{J}), \pi)$ *into* $(S(\mathfrak{J}'), \pi')$ *where* π *and* π' *are the main involutions in* $S(\mathfrak{J})$ and $S(\mathfrak{J}')$.

(5) *If* $0 \to \mathfrak{J}'' \to_\eta \mathfrak{J} \to_\zeta \mathfrak{J}' \to 0$ *is exact and* $(S(\mathfrak{J}), \sigma_u)$, $(S(\mathfrak{J}'), \sigma_u')$ *are special universal envelopes for* $\mathfrak{J}$ *and* $\mathfrak{J}'$ *respectively, then* $S(\mathfrak{J}) \to_{\zeta s} S(\mathfrak{J}') \to 0$ *is exact and* $\ker \zeta_s$ *is the ideal in* $S(\mathfrak{J})$ *generated by* $\mathfrak{J}''^{\eta\sigma_u}$.

(6) *If* Γ *is an extension of the base field* Φ *of* $\mathfrak{J}$ *and* σ_u *is the linear extension to* $\mathfrak{J}_\Gamma$ *(into* $S(\mathfrak{J})_\Gamma$*) of* σ_u *then* $(S(\mathfrak{J})_\Gamma, \sigma_u)$ *is a special universal envelope for* $\mathfrak{J}_\Gamma$.

(7) *If* D *is a derivation in* $\mathfrak{J}$ *then there exists a unique derivation* D_u *in* $S(\mathfrak{J})$ *such that* $D\sigma_u = \sigma_u D_u$. *Moreover,* D_u *is a derivation for the algebra with involution* $(S(\mathfrak{J}), \pi)$ *in the sense that* $D_u \pi = \pi D_u$.

(8) *If* dim $\mathfrak{J} = n$ *then* $\dim S(\mathfrak{J}) \leqq 2^n$.

PROOFS. (1) If we apply the definition of $S(\mathfrak{J})$ with $\mathfrak{G} = S'(\mathfrak{J})$ we obtain a homomorphism ι of $S(\mathfrak{J})$ into $S'(\mathfrak{J})$ such that $\sigma_u' = \sigma_u \iota$. By symmetry, we have a homomorphism ι' of $S'(\mathfrak{J})$ into $S(\mathfrak{J})$ such that $\sigma_u = \sigma_u' \iota'$. Then $\sigma_u = \sigma_u \iota\iota'$. On the other hand, if $1_{S(\mathfrak{J})}$ is the identity mapping on $S(\mathfrak{J})$ then $\sigma_u = \sigma_u 1_{S(\mathfrak{J})}$. If we apply the definition of $(S(\mathfrak{J}), \sigma_u)$ taking $\mathfrak{G} = S(\mathfrak{J})$ we see that $1_{S(\mathfrak{J})}$ is the only homomorphism of $S(\mathfrak{J})$ into $S(\mathfrak{J})$ satisfying $\sigma_u = \sigma_u 1_{S(\mathfrak{J})}$. Hence $\iota\iota' = 1_{S(\mathfrak{J})}$. By symmetry $\iota'\iota = 1_{S'(\mathfrak{J})}$. Hence ι is an isomorphism of $S(\mathfrak{J})$ onto $S'(\mathfrak{J})$.

(2) Let $S'(\mathfrak{J})$ be the subalgebra of $S(\mathfrak{J})$ generated by 1 and $\mathfrak{J}^{\sigma_u}$. Then σ_u can be considered as an associative specialization of $\mathfrak{J}$ in $S'(\mathfrak{J})$ and $(S'(\mathfrak{J}), \sigma_u)$ is a special universal envelope for $\mathfrak{J}$. The injection mapping ι' of $S'(\mathfrak{J})$ into $S(\mathfrak{J})$ satisfies $\sigma_u = \sigma_u \iota'$. Hence by (1) ι' is an isomorphism of $S'(\mathfrak{J})$ onto $S(\mathfrak{J})$. Thus ι' is surjective and $S'(\mathfrak{J}) = S(\mathfrak{J})$.

(3) Let $\mathfrak{A}^\circ$ be the opposite algebra of $\mathfrak{A} = S(\mathfrak{J})$. Then $u \to u$ is an anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^\circ$. Hence σ_u is an associative specialization of $\mathfrak{J}$ in $\mathfrak{A}^\circ$ and there exists a homomorphism π' of $\mathfrak{A}$ into $\mathfrak{A}^\circ$ such that $1^{\pi'} = 1$ and $a^{\sigma_u \pi'} = a^{\sigma_u}$. If we compose π' with the anti-isomorphism $u \to u$ of $\mathfrak{A}^\circ$ onto $\mathfrak{A}$ we obtain an anti-homomorphism π of $\mathfrak{A}$ into $\mathfrak{A}$ such that $1^\pi = 1$ and $a^{\sigma_u \pi} = a^{\sigma_u}$, $a \in \mathfrak{J}$. Then π^2 is an endomorphism in $\mathfrak{A}$ such that $1^{\pi^2} = 1$, $a^{\sigma_u \pi^2} = a^{\sigma_u}$. Since 1 and the a^{σ_u} generate $\mathfrak{A}$ we have $\pi^2 = 1$ and π is an involution in $\mathfrak{A}$.

(4) If ζ is a homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$ then $\zeta\sigma_u'$ is an associative specialization of $\mathfrak{J}$ in $S(\mathfrak{J}')$. Hence, by definition of $(S(\mathfrak{J}), \sigma_u)$, there exists a unique homomor-

phism ζ_s of $S(\mathfrak{J})$ into $S(\mathfrak{J}')$ such that $1 \to 1$ and (5) is commutative. If $a \in \mathfrak{J}$ then $a^{\sigma_u \pi \zeta_s} = a^{\sigma_u \zeta_s} = a^{\zeta \sigma_u'} = a^{\zeta \sigma'_u \pi'} = a^{\sigma_u \zeta_s \pi'}$. Since 1 and the a^{σ_u} generate $S(\mathfrak{J})$ this implies that $\pi\zeta_s = \zeta_s\pi'$ so ζ_s is a homomorphism of $(S(\mathfrak{J}), \pi)$ into $(S(\mathfrak{J}'), \pi')$.

(5) The hypothesis implies that $\mathfrak{J}^{\zeta} = \mathfrak{J}'$. Hence $\mathfrak{J}^{\sigma_u \zeta_s} = \mathfrak{J}^{\zeta \sigma_u'} = \mathfrak{J}'^{\sigma_u'}$. Now 1 and $\mathfrak{J}'^{\sigma_u'}$ generate $S(\mathfrak{J}')$ and $S(\mathfrak{J})$ contains 1 and $\mathfrak{J}^{\sigma_u}$ so $S(\mathfrak{J})^{\zeta_s}$ contains the set of generators 1 and $\mathfrak{J}^{\sigma_u \zeta_s} = \mathfrak{J}'^{\sigma_u'}$. Since $S(\mathfrak{J})^{\zeta_s}$ is a subalgebra of $S(\mathfrak{J}')$ it follows that $S(\mathfrak{J}') = S(\mathfrak{J})^{\zeta_s}$. Hence $S(\mathfrak{J}) \to_{\zeta_s} S(\mathfrak{J}') \to 0$ is exact. We have $\mathfrak{J}''^{\eta \sigma_u \zeta_s} = \mathfrak{J}''^{\eta \zeta \sigma_u'} = 0$ so $\mathfrak{J}''^{\eta \sigma_u} \subseteq \ker \zeta_s$. Then the ideal $\mathfrak{K}$ in $S(\mathfrak{J})$ generated by $\mathfrak{J}''^{\eta \sigma_u}$ is contained in $\ker \zeta_s$. If ν denotes the canonical homomorphism $u \to u + \mathfrak{K}$ of $S(\mathfrak{J})$ onto $S(\mathfrak{J})/\mathfrak{K}$ then $\sigma_u \nu$ is an associative specialization of $\mathfrak{J}$. Since $\mathfrak{J}''^{\eta}$ is mapped into 0 by $\sigma_u \nu$ we have the associative specialization $a + \mathfrak{J}''^{\eta} \to a^{\sigma_u \nu}$ of $\mathfrak{J}/\mathfrak{J}''^{\eta}$ in $S(\mathfrak{J})/\mathfrak{K}$. Now the exactness of $0 \to \mathfrak{J}'' \to_{\eta} \mathfrak{J} \to_{\zeta} \mathfrak{J}' \to 0$ implies that $a^{\zeta} \to a + \mathfrak{J}''^{\eta}$ is an isomorphism of $\mathfrak{J}'$ onto $\mathfrak{J}/\mathfrak{J}''^{\eta}$. Hence $a^{\zeta} \to a^{\sigma_u \nu}$ is an associative specialization of $\mathfrak{J}'$ in $S(\mathfrak{J})/\mathfrak{K}$. Consequently, we have a homomorphism λ of $S(\mathfrak{J}')$ into $S(\mathfrak{J})/\mathfrak{K}$ such that $a^{\zeta \sigma_u' \lambda} = a^{\sigma_u \nu}$. Since ζ_s is surjective we have the isomorphism $u + \ker \zeta_s \to u^{\zeta_s}$ of $S(\mathfrak{J})/\ker \zeta_s$ onto $S(\mathfrak{J}')$ and hence we have the homomorphism $u + \ker \zeta_s \to u^{\zeta_s \lambda}$ of $S(\mathfrak{J})/\ker \zeta_s$ into $S(\mathfrak{J})/\mathfrak{K}$. In this homomorphism $a^{\sigma_u} + \ker \zeta_s \to a^{\sigma_u \zeta_s \lambda} = a^{\zeta \sigma_u' \lambda} = a^{\sigma_u \nu} = a^{\sigma_u} + \mathfrak{K}$. Since 1 and the a^{σ_u} generate $S(\mathfrak{J})$ this implies that the homomorphism sends $u + \ker \zeta_s \to u + \mathfrak{K}$. This implies that $\ker \zeta_s \subseteq \mathfrak{K}$. Since we had the inclusion $\mathfrak{K} \subseteq \ker \zeta_s$ we now have $\mathfrak{K} = \ker \zeta_s$.

(6) Let σ be an associative specialization of $\mathfrak{J}_{\Gamma}$ in an associative algebra $\mathfrak{G}/\Gamma$ with 1. Considering $\mathfrak{J}$ as a Φ-subalgebra of $\mathfrak{J}_{\Gamma}$ ($\mathfrak{J}$ identified with the set of elements $1 \otimes a$, $a \in \mathfrak{J}$) and $\mathfrak{G}$ an algebra over Φ then the restriction $\sigma | \mathfrak{J}$ of σ to $\mathfrak{J}$ is an associative specialization of $\mathfrak{J}$ in $\mathfrak{G}/\Phi$. Hence there exists a homomorphism η of $S(\mathfrak{J})$ into $\mathfrak{G}$ such that $1 \to 1$ and $\sigma_u \eta = \sigma | \mathfrak{J}$. Since $\mathfrak{G}$ is an algebra over Γ we can extend η to a homomorphism η of $S(\mathfrak{J})_{\Gamma}$ into $\mathfrak{G}$. Let σ_u denote the Γ-linear mapping of $\mathfrak{J}_{\Gamma}$ into $S(\mathfrak{J})_{\Gamma}$ which extends the associative specialization σ_u of $\mathfrak{J}$ in $S(\mathfrak{J})$. Then σ_u is an associative specialization of $\mathfrak{J}_{\Gamma}$ in $S(\mathfrak{J})_{\Gamma}$. The relation $\sigma_u \eta = \sigma | \mathfrak{J}$ in $\mathfrak{J}$ gives $\sigma_u \eta = \sigma$ in $\mathfrak{J}_{\Gamma}$. Since 1 and $\mathfrak{J}^{\sigma_u}$ generate $S(\mathfrak{J})$, 1 and $\mathfrak{J}_{\Gamma}{}^{\sigma_u}$ generate $S(\mathfrak{J})_{\Gamma}$. Hence η is the only homomorphism of $S(\mathfrak{J})_{\Gamma}$ into $\mathfrak{G}_{\Gamma}$ such that $1 \to 1$ and $\sigma_u \eta = \sigma$. This proves that $(S(\mathfrak{J})_{\Gamma}, \sigma_u)$ is a special universal envelope for $\mathfrak{J}_{\Gamma}$.

(7) Let D be a derivation in $\mathfrak{J}$. Let $\mathfrak{D}$ be the two-dimensional commutative associative algebra over Φ with identity 1 and basis $(1, t)$ such that $t^2 = 0$ and form $\mathfrak{D} \otimes_{\Phi} S(\mathfrak{J})$. With the obvious identifications we can consider $\mathfrak{D}$ and $S(\mathfrak{J})$ as subalgebras of $\mathfrak{D} \otimes S(\mathfrak{J})$ and the elements of $\mathfrak{D} \otimes S(\mathfrak{J})$ can be written uniquely in the form $x_1 + x_2 t$ where $x_i \in S(\mathfrak{J})$. A direct verification shows that the mapping $a \to a^{\delta} = a^{\sigma_u} + (aD)^{\sigma_u} t$ is an associative specialization of $\mathfrak{J}$ in $\mathfrak{D} \otimes S(\mathfrak{J})$. Hence we have a homomorphism of $S(\mathfrak{J})$ into $\mathfrak{D} \otimes S(\mathfrak{J})$ such that $1^{\eta} = 1$ and $a^{\sigma_u \eta} = a^{\sigma_u} + (aD)^{\sigma_u} t$. If $x \in S(\mathfrak{J})$ then we have $x^{\eta} = x + x' t$ since $1^{\eta} = 1$, $a^{\sigma_u \eta} = a^{\sigma_u} + (aD)^{\sigma_u} t$ and 1 and the a^{σ_u} generate $S(\mathfrak{J})$. The facts that $t^2 = 0$ and $tx = xt$, $x \in S(\mathfrak{J})$,

imply that $x \to x'$ is a derivation D_u in $S(\mathfrak{J})$. We have $a^{\sigma_u\eta} = a^{\sigma} + (aD)^{\sigma_u}t = a^{\sigma_u} + (a^{\sigma_u})D_u t$. Hence $D\sigma_u = \sigma_u D_u$. As in (4), it is clear that $D_u\pi = \pi D_u$.

(8) Let (u_ι), ι belonging to an ordered set, be a basis for $\mathfrak{J}$. Then we claim that every element of $S(\mathfrak{J})$ is a linear combination of 1 and the "standard" monomials $v_{\iota_1}v_{\iota_2}\cdots v_{\iota_r}$, $v_{\iota_j} = u_{\iota_j}{}^{\sigma_u}$, $\iota_1 < \iota_2 < \cdots < \iota_r$. Since $\mathfrak{A} = S(\mathfrak{J})$ is generated by 1 and $\mathfrak{J}^{\sigma_u}$ every element of $\mathfrak{A}$ is a linear combination of 1 and monomials in the $v_\iota = u_\iota{}^{\sigma_u}$. Hence it suffices to show that every monomial in the v_ι is a linear combination of standard monomials. We have the relations $v_\iota v_\kappa = -v_\kappa v_\iota + 2(u_\iota . u_\kappa)^{\sigma_u}$, by (1), and $(u_\iota . u_\kappa)^{\sigma_u}$ is a linear combination of the v's. This permits us to replace a product $v_\iota v_\kappa$ with $\iota > \kappa$ in a monomial in the v's by $v_\kappa v_\iota$ at the expense of a linear combination of monomials of lower formal degree than that of the given one. Also since $v_\iota{}^2 = (u_\iota{}^2)^{\sigma_u}$ we can apply the same procedure to a square $v_\iota{}^2$ occurring in a monomial. A succession of such reductions can be used to express any monomial in the v's as a linear combination of standard monomials in the v's (see Jacobson, *Lie Algebras*, p. 157 for a more formal argument). Now let $\dim \mathfrak{J} = n$ and assume $(u_1, u_2, \cdots, u_n)$ is a basis for $\mathfrak{J}/\Phi$. Then our result shows that every element of $\mathfrak{A}$ is a linear combination of 1 and the elements $v_{i_1}v_{i_2}\cdots v_{i_r}$, $i_1 < i_2 < \cdots < i_r$, $v_{i_j} = u_{i_j}{}^{\sigma_u}$. Hence $\dim \mathfrak{A} \leqq 2^n$.

In connection with (8) we remark that one can give examples in which the bound 2^n is attained. On the other hand, in the interesting cases $\dim \mathfrak{A} < 2^n$.

We shall now give a construction of a special universal envelope for any Jordan algebra $\mathfrak{J}$. For this purpose we form the tensor algebra $T(\mathfrak{J}) = \Phi 1 \oplus \mathfrak{J} \oplus \mathfrak{J}^{(2)} \oplus \mathfrak{J}^{(3)} \oplus \cdots$ where $\mathfrak{J}^{(i)} = \mathfrak{J} \otimes \mathfrak{J} \otimes \cdots \otimes \mathfrak{J}$, i factors. The associative algebra has the following universal mapping property: Let σ be a linear mapping $\mathfrak{J}$ into an associative algebra $\mathfrak{G}$ with 1. Then σ has a unique extension to a homomorphism of $T(\mathfrak{J})$ into $\mathfrak{G}$ mapping 1 into 1. Now let $\mathfrak{R}_s$ be the ideal in $T(\mathfrak{J})$ generated by all elements of the form $(a . b) - \frac{1}{2}(a \otimes b + b \otimes a)$, $a, b \in \mathfrak{J}$. Put $S(\mathfrak{J}) = T(\mathfrak{J})/\mathfrak{R}_s$ and $a^{\sigma_u} = a + \mathfrak{R}_s$, $a \in \mathfrak{J}$. Then $(a . b)^{\sigma_u} - \frac{1}{2}(a^{\sigma_u}b^{\sigma_u} + b^{\sigma_u}a^{\sigma_u}) = (a . b) - \frac{1}{2}(a \otimes b + b \otimes a) + \mathfrak{R}_s = \mathfrak{R}_s$ the 0-element of $S(\mathfrak{J})$. Hence σ_u is an associative specialization of $\mathfrak{J}$ in $S(\mathfrak{J})$. Next let σ be any associative specialization of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1 and let η' denote the extension of σ to a homomorphism of $T(\mathfrak{J})$ into $\mathfrak{G}$ mapping $1 \to 1$. Then $(a . b - \frac{1}{2}(a \otimes b + b \otimes a))^{\eta'} = (a . b)^{\sigma} - \frac{1}{2}(a^{\sigma}b^{\sigma} + b^{\sigma}a^{\sigma}) = 0$, if $a, b \in \mathfrak{J}$. Hence the generators of $\mathfrak{R}_s$ and consequently $\mathfrak{R}_s$ is mapped into 0 by η'. Hence we have the homomorphism η of $S(\mathfrak{J}) = T(\mathfrak{J})/\mathfrak{R}$ into $\mathfrak{G}$ such that $1 \to 1$ and $(a^{\sigma_u})^{\eta} = (a + \mathfrak{R}_s)^{\eta} = a^{\eta'} = a^{\sigma}$, $a \in \mathfrak{J}$. Thus $(S(\mathfrak{J}), \sigma_u)$ satisfies the definition of a special universal envelope. From now on we shall take the $(S(\mathfrak{J}), \sigma_u)$ we have just constructed as our standard one and refer to this as *the* special universal envelope of the Jordan algebra $\mathfrak{J}$.

The result (4) of Theorem 1 which we indicated in the diagram (5) gives a multiplicative property of the mapping $\zeta \to \zeta_s$. More precisely, suppose we have a homomorphism ζ of $\mathfrak{J}$ into $\mathfrak{J}'$ and a homomorphism λ of $\mathfrak{J}'$ into $\mathfrak{J}''$ and let $(S(\mathfrak{J}), \sigma_u)$, $(S(\mathfrak{J}'), \sigma_u')$, $(S(\mathfrak{J}''), \sigma_u'')$ be the special universal multiplication envelopes of $\mathfrak{J}$, $\mathfrak{J}'$ and $\mathfrak{J}''$ respectively. Then we have the homomorphism $\zeta\lambda$ of $\mathfrak{J}$ into $\mathfrak{J}''$ and $\zeta_s\lambda_s$ of $S(\mathfrak{J})$ into $S(\mathfrak{J}'')$ in the diagram

$$\begin{array}{ccccc} \mathfrak{J} & \xrightarrow{\zeta} & \mathfrak{J}' & \xrightarrow{\lambda} & \mathfrak{J}'' \\ \sigma_u\downarrow & & \sigma_u'\downarrow & & \sigma_u''\downarrow \\ S(\mathfrak{J}) & \xrightarrow[\zeta_s]{} & S(\mathfrak{J}') & \xrightarrow[\lambda_s]{} & S(\mathfrak{J}''). \end{array} \tag{6}$$

Since the two squares are commutative: $\zeta\sigma_u' = \sigma_u\zeta_s$, $\lambda\sigma_u'' = \sigma_u'\lambda_s$ it follows that the rectangle is commutative, that is, $\zeta\lambda\sigma_u'' = \sigma_u\zeta_s\lambda_s$. Since $(\zeta\lambda)_s$ is the unique homomorphism of $S(\mathfrak{J})$ into $S(\mathfrak{J}'')$ such that $(\zeta\lambda)\sigma_u'' = \sigma_u(\zeta\lambda)_s$ it follows that we have

$$(\zeta\lambda)_s = \zeta_s\lambda_s. \tag{7}$$

We note also that if $\mathfrak{J}' = \mathfrak{J}$ and $\zeta = 1_{\mathfrak{J}}$ in (5) then $S(\mathfrak{J}') = S(\mathfrak{J})$ and $\zeta_s = 1_{S(\mathfrak{J})}$. This and (7) imply that the mappings $\mathfrak{J} \to S(\mathfrak{J})$, $\zeta \to \zeta_s$ define a functor S from the category of Jordan algebras into the category of associative algebras with 1 where the morphisms in the latter are the homomorphisms mapping 1 into 1. We recall also that ζ_s is a homomorphism of algebras with involution of $(S(\mathfrak{J}), \pi)$ into $(S(\mathfrak{J}'), \pi')$, π and π' the main involutions in the algebras. Hence we can consider the second category to be that of associative algebras with 1 and involutions. We shall do this from now on and refer to the functor just defined as the *special universal functor* from the category of Jordan algebras to the category of associative algebras with 1 and involutions.

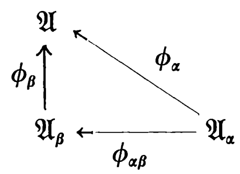
We recall the notion of a direct limit of algebras. We are given a partially ordered set $A = \{\alpha\}$ which is directed in the sense that for any given $\alpha, \beta \in A$ there exists a γ in A such that $\gamma \geqq \alpha$, $\gamma \geqq \beta$. Next we have a collection of algebras $\mathfrak{A}_\alpha$ indexed by the set A and if $\alpha \leqq \beta$ we have a homomorphism $\phi_{\alpha\beta}$ of $\mathfrak{A}_\alpha$ into $\mathfrak{A}_\beta$ satisfying the following conditions: (1) $\phi_{\alpha\alpha} = 1_{\mathfrak{A}_\alpha}$, (2) if $\alpha \leqq \beta \leqq \gamma$ then $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$. Let $\mathfrak{S}$ be the disjoint union of the sets $\mathfrak{A}_\alpha$ and define a relation $\sim$ in $\mathfrak{S}$ by $a_\alpha \sim a_\beta$ ($a_\alpha \in \mathfrak{A}_\alpha$, $a_\beta \in \mathfrak{A}_\beta$) if there exists a $\gamma \geqq \alpha, \beta$ such that $a_\alpha^{\phi_{\alpha\gamma}} = a_\beta^{\phi_{\beta\gamma}}$. This is an equivalence. Let $\mathfrak{A}$ denote the set of equivalence classes $[a_\alpha]$.

We define addition, multiplication by elements of Φ and a product in $\mathfrak{A}$ by:

$$\begin{aligned} [a_\alpha] + [b_\beta] &= [a_\alpha^{\phi_{\alpha\gamma}} + b_\beta^{\phi_{\beta\gamma}}] \quad \text{where } \gamma \geqq \alpha, \beta, \\ \rho[a_\alpha] &= [\rho a_\alpha], \qquad \rho \in \Phi, \\ [a_\alpha][b_\beta] &= [a_\alpha^{\phi_{\alpha\gamma}} b_\beta^{\phi_{\beta\gamma}}], \qquad \gamma \geqq \alpha, \beta. \end{aligned} \tag{8}$$

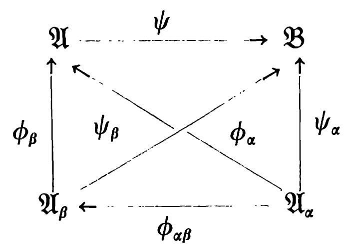
It is easy to check that these are well defined and give an algebra structure on $\mathfrak{A}$. We call $\mathfrak{A}$ the *direct limit* of the $\mathfrak{A}_\alpha$ and write $\mathfrak{A} = \varinjlim \mathfrak{A}_\alpha$. If all the $\mathfrak{A}_\alpha$ are in a variety $\mathscr{V}(I)$ of algebras defined by identities then the direct limit is in the variety $\mathscr{V}(I)$. We have the mapping ϕ_α of $\mathfrak{A}_\alpha$ into $\mathfrak{A}$ defined by $a_\alpha^{\phi_\alpha} = [a_\alpha]$. This is a homomorphism. Moreover, if $\alpha \leqq \beta$ then $a_\alpha^{\phi_{\alpha\beta}\phi_\beta} = [a_\alpha^{\phi_{\alpha\beta}}] = [a_\alpha] = a_\alpha^{\phi_\alpha}$. Hence the diagram

(9)



is commutative. Note also that every element of $\mathfrak{A}$ has the form $[a_\alpha]$ for some $a_\alpha \in \mathfrak{S}$. $\mathfrak{A} = \varinjlim \mathfrak{A}_\alpha$ has the following universal mapping property. Let $\mathfrak{B}$ be any algebra such that for each α there is a homomorphism ψ_α of $\mathfrak{A}_\alpha$ into $\mathfrak{B}$ such that if $\alpha \leqq \beta$ then $\phi_{\alpha\beta}\psi_\beta = \psi_\alpha$. Then there exists a unique homomorphism ψ of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\psi_\alpha = \phi_\alpha\psi$, that is, making the triangles $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_\alpha$ and $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_\beta$ in the following diagram commutative:

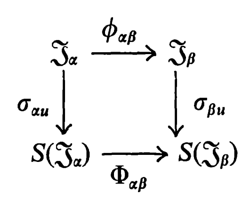
(10)



To define ψ let $a \in \mathfrak{A}$ and write $a = a_\alpha^{\phi_\alpha}$. Then we define $a^\psi = a_\alpha^{\psi_\alpha}$. If $a_\alpha^{\phi_\alpha} = a_\beta^{\phi_\beta}$ then we choose $\gamma \geqq \alpha, \beta$ so that $a_\alpha^{\phi_{\alpha\gamma}} = a_\beta^{\phi_{\beta\gamma}}$. Then $a_\alpha^{\phi_{\alpha\gamma}\psi_\gamma} = a_\beta^{\phi_{\beta\gamma}\psi_\gamma}$. Hence $a_\alpha^{\psi_\alpha} = a_\beta^{\psi_\beta}$ and so ψ is single-valued. Also ψ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and $a_\alpha^{\phi_\alpha\psi} = a^\psi = a_\alpha^{\psi_\alpha}$ so $\phi_\alpha\psi = \psi_\alpha$ as required. Since the $a_\alpha^{\phi_\alpha}$ generate $\mathfrak{A}$ it is clear that ψ is unique.

We now assume that every $\mathfrak{A}_\alpha = \mathfrak{J}_\alpha$ is Jordan. Then $\mathfrak{J} = \varinjlim \mathfrak{J}_\alpha$ is Jordan. Let $(S(\mathfrak{J}_\alpha), \sigma_{\alpha u})$ be the special universal envelope for $\mathfrak{J}_\alpha$. The homomorphism $\phi_{\alpha\beta}$, $\alpha \leqq \beta$, of $\mathfrak{J}_\alpha$ into $\mathfrak{J}_\beta$ gives a unique homomorphism $\Phi_{\alpha\beta}$ $(= \phi_{\alpha\beta s})$ of $S(\mathfrak{J}_\alpha)$ into $S(\mathfrak{J}_\beta)$ making the following diagram commutative:

(11)



We have $\Phi_{\alpha\alpha} = 1_{S(\mathfrak{J}_\alpha)}$, $\Phi_{\alpha\beta}\Phi_{\beta\gamma} = \Phi_{\alpha\gamma}$ if $\alpha \leqq \beta \leqq \gamma$. Hence we can define $\varinjlim S(\mathfrak{J}_\alpha)$.

Let Φ_α be the homomorphism of $S(\mathfrak{J}_\alpha)$ into $\varinjlim S(\mathfrak{J}_\alpha)$ such that $u_\alpha{}^\Phi = [u_\alpha]$. This is a homomorphism and $\Phi_{\alpha\beta}\Phi_\beta = \Phi_\alpha$ if $\alpha \leqq \beta$. Set $\Sigma_\alpha = \sigma_{\alpha u}\Phi_\alpha$. This is an associative specialization of $\mathfrak{J}_\alpha$ in $\varinjlim S(\mathfrak{J}_\alpha)$. We have $\phi_{\alpha\beta}\Sigma_\beta = \phi_{\alpha\beta}\sigma_{\beta u}\Phi_\beta = \sigma_{\alpha u}\Phi_{\alpha\beta}\Phi_\beta = \sigma_{\alpha u}\Phi_\alpha = \Sigma_\alpha$ if $\alpha \leqq \beta$. Hence we have a unique associative specialization Σ_u of $\mathfrak{J} = \varinjlim \mathfrak{J}_\alpha$ in $\varinjlim S(\mathfrak{J}_\alpha)$ such that $\Sigma_\alpha = \phi_\alpha\Sigma_u$.

We shall now prove that the special universal functor commutes with direct limits in the following sense:

THEOREM 2. *$(\varinjlim S(\mathfrak{J}_\alpha), \Sigma_u)$ is a special universal envelope for $\mathfrak{J} = \varinjlim \mathfrak{J}_\alpha$.*

PROOF. We have seen that Σ_u is an associative specialization of $\mathfrak{J}$ in $\varinjlim S(\mathfrak{J}_\alpha)$. Now let σ be an associative specialization of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1. Then $\sigma_\alpha = \phi_\alpha\sigma$ is an associative specialization of $\mathfrak{J}_\alpha$ in $\mathfrak{G}$. Hence we have a homomorphism η_α of $S(\mathfrak{J}_\alpha)$ into $\mathfrak{G}$ such that $1 \rightarrow 1$ and $\phi_\alpha\sigma = \sigma_\alpha = \sigma_{\alpha u}\eta_\alpha$. We claim that $\Phi_{\alpha\beta}\eta_\beta = \eta_\alpha$. Since 1 and $\mathfrak{J}_\alpha{}^{\sigma_{\alpha u}}$ generates $S(\mathfrak{J}_\alpha)$ it suffices to show that if $a_\alpha \in \mathfrak{J}_\alpha$ then $a_\alpha{}^{\sigma_{\alpha u}\Phi_{\alpha\beta}\eta_\beta} = a_\alpha{}^{\sigma_{\alpha u}\eta_\alpha}$. By (11), $\sigma_{\alpha u}\Phi_{\alpha\beta} = \phi_{\alpha\beta}\sigma_{\beta u}$ so $\sigma_{\alpha u}\Phi_{\alpha\beta}\eta_\beta = \phi_{\alpha\beta}\sigma_{\beta u}\eta_\beta = \phi_{\alpha\beta}\phi_\beta\sigma = \phi_\alpha\sigma = \sigma_{\alpha u}\eta_\alpha$. It follows that $\Phi_{\alpha\beta}\eta_\beta = \eta_\alpha$ and so by the universal property of $\varinjlim S(\mathfrak{J}_\alpha)$ we can define a homomorphism η of $\varinjlim S(\mathfrak{J}_\alpha)$ into $\mathfrak{G}$ such that $1 \rightarrow 1$ and $\eta_\alpha = \Phi_\alpha\eta$. Then if $a \in \mathfrak{J}$ we can write $a = a_\alpha{}^{\phi_\alpha}$, $a_\alpha \in \mathfrak{J}_\alpha$ for some α, and $a^{\Sigma_u\eta} = a_\alpha{}^{\phi_\alpha\Sigma_u\eta} = a_\alpha{}^{\Sigma_\alpha\eta}$ since $\Sigma_\alpha = \phi_\alpha\Sigma_u$. On the other hand, $a^\sigma = a_\alpha{}^{\phi_\alpha\sigma} = a_\alpha{}^{\sigma_{\alpha u}\eta_\alpha} = a_\alpha{}^{\sigma_{\alpha u}\Phi_\alpha\eta} = a_\alpha{}^{\Sigma_\alpha\eta}$. Hence $\sigma = \Sigma_u\eta$. Since the subalgebras $S(\mathfrak{J}_\alpha)^{\Phi_\alpha}$ generate $\varinjlim S(\mathfrak{J}_\alpha)$ and each $S(\mathfrak{J}_\alpha)$ is generated by 1 and $\mathfrak{J}_\alpha{}^{\sigma_{\alpha u}}$ it is clear that $\varinjlim S(\mathfrak{J}_\alpha)$ is generated by 1 and $\bigcup \mathfrak{J}_\alpha^{\sigma_{\alpha u}\Phi_\alpha} = \bigcup \mathfrak{J}_\alpha^{\Sigma_\alpha}$. Also $\mathfrak{J}_\alpha{}^{\Sigma_\alpha} = \mathfrak{J}_\alpha{}^{\phi_\alpha\Sigma_u} \subseteq \mathfrak{J}^{\Sigma_u}$. Hence 1 and $\mathfrak{J}^{\Sigma_u}$ generate $\varinjlim S(\mathfrak{J}_\alpha)$. It follows that the homomorphism η such that $1 \rightarrow 1$ and $\Sigma_u\eta = \sigma$ is unique. This proves that $(\varinjlim S(\mathfrak{J}_\alpha), \Sigma_u)$ is a special universal envelope for $\mathfrak{J} = \varinjlim \mathfrak{J}_\alpha$.

An important special case of a direct limit is the following. Let $\mathfrak{A}$ be an algebra and let F be the set of finite subsets of $\mathfrak{A}$ ordered by inclusion. Then F is directed. If $\alpha \in F$ we let $\mathfrak{A}_\alpha$ be the subalgebra generated by α, so $\{\mathfrak{A}_\alpha\}$ is the set of finitely generated subalgebras of $\mathfrak{A}$. If $\alpha \leqq \beta$ then $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$, and we let $\phi_{\alpha\beta}$ be the injection mapping of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_\beta$. It is clear that our conditions are satisfied and it is easy to see (using the universal property of the direct limit) that $\mathfrak{A}$ can be identified with $\varinjlim \mathfrak{A}_\alpha$. More generally, let $A = \{\alpha\}$, an ordered directed set, $\mathfrak{A}$ an algebra and suppose that we have a mapping $\alpha \rightarrow \mathfrak{A}_\alpha$ of A into the set of subalgebras of $\mathfrak{A}$ such that if $\alpha \leqq \beta$ then $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ and $\mathfrak{A} = \bigcup \mathfrak{A}_\alpha$. If $\alpha \leqq \beta$ then we let $\phi_{\alpha\beta}$ be the injection of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_\beta$. Then every $\phi_{\alpha\beta}$ is a monomorphism and $\mathfrak{A}$ can be identified with $\varinjlim \mathfrak{A}_\alpha$. We shall say that $\mathfrak{A}$ is a union of a directed set of subalgebras $\mathfrak{A}_\alpha$.

The notion of a special universal envelope permits us to replace the original definition of a special Jordan algebra by the following more definite condition.

THEOREM 3. *A Jordan algebra $\mathfrak{J}$ is special if and only if the associative specialization σ_u of $\mathfrak{J}$ in $S(\mathfrak{J})$ is 1-1.*

PROOF. The definition that $\mathfrak{J}$ is special is that there exists an associative algebra $\mathfrak{G}$ and a monomorphism σ of $\mathfrak{J}$ in $\mathfrak{G}^+$. Clearly we may assume that $\mathfrak{G}$ has a 1, so that σ is an associative specialization. Then $\sigma = \sigma_u \eta$ where η is a homomorphism of $S(\mathfrak{J})$ into $\mathfrak{G}$. Since σ has zero kernel the same is true of σ_u. Hence σ_u is 1-1. Conversely, if σ_u is 1-1 then this is a monomorphism of $\mathfrak{J}$ into $S(\mathfrak{J})^+$ and $\mathfrak{J}$ is special.

We can use this criterion and the commutativity of the special universal functor with direct limits to prove the following

THEOREM 4. *Let $\mathfrak{J}$ be a Jordan algebra which is a union of a directed set of special subalgebras $\mathfrak{J}_\alpha$. Then $\mathfrak{J}$ is special. In particular a Jordan algebra is special if every finitely generated subalgebra is special.*

PROOF. We identify $\mathfrak{J}$ with $\varinjlim \mathfrak{J}_\alpha$, so that the homomorphism ϕ_α of $\mathfrak{J}_\alpha$ into $\mathfrak{J}$ is the injection mapping. We know that $S(\mathfrak{J}) = \varinjlim S(\mathfrak{J}_\alpha)$ where the associative specialization of $\mathfrak{J}$ in $\varinjlim S(\mathfrak{J}_\alpha)$ is the mapping Σ_u defined above. Using the notations of the proof of Theorem 2, we have $\sigma_{\alpha u}\Phi_\alpha = \phi_\alpha \Sigma_u$. Let $a \in \mathfrak{J}$ satisfy $a^{\Sigma_u} = 0$. Then $a = a_\alpha^{\phi_\alpha}$ for some a_α in $\mathfrak{J}_\alpha$ and $0 = a^{\Sigma_u} = a_\alpha^{\phi_\alpha \Sigma_u} = a^{\sigma_{\alpha u}\Phi_\alpha}$. Then $a^{\sigma_{\alpha u}\phi_{\alpha\beta\mu}} = 0$ for some $\beta \geqq \alpha$. Hence $a_\alpha^{\phi_{\alpha\beta}\sigma_{\beta u}} = 0$. Since $\mathfrak{J}_\beta$ is special this gives $a_\alpha^{\phi_{\alpha\beta}} = 0$ and since $\phi_{\alpha\beta}$ is injective, $a_\alpha = 0$. Hence $a = 0$. Thus Σ_u is 1-1 and $\mathfrak{J}$ is special.

2. **Special universal envelopes for Jordan algebras with 1 and for direct sums.** Let σ be an associative specialization of the Jordan algebra $\mathfrak{J}$ in the associative algebra $\mathfrak{G}$ with 1. If $a \in \mathfrak{J}$ and n is a positive integer then $(a^n)^{\cdot\sigma} = (a^\sigma)^{\cdot n} = (a^\sigma)^n$. Also if $a, b, c \in \mathfrak{J}$ then $\{abc\}^\sigma = \{a^\sigma b^\sigma c^\sigma\} = \frac{1}{2}(a^\sigma b^\sigma c^\sigma + c^\sigma b^\sigma a^\sigma)$ in $\mathfrak{G}$ since $\{abc\}$ $a.b.c + b.c.a - a.c.b$. In particular, $\{aba\}^\sigma = a^\sigma b^\sigma a^\sigma$. If $e^2 = e$ in $\mathfrak{J}$ then $(e^\sigma)^2 = e^\sigma$ in $\mathfrak{G}$. Now suppose $e.a = 0$ for the idempotent element e of $\mathfrak{J}$. Then $\{eae\} = e.a.e + a.e.e - e.e.a = 0$. Hence we have $e^\sigma a^\sigma + a^\sigma e^\sigma = 0$, $(e^\sigma)^2 = e^\sigma$ and $e^\sigma a^\sigma e^\sigma = 0$. These imply that $e^\sigma a^\sigma = 0 = a^\sigma e^\sigma$. It follows that if $e_1, e_2, \cdots, e_n$ are orthogonal idempotent elements of $\mathfrak{J}$ ($e_i^{\cdot 2} = e_i$, $e_i . e_j = 0$ if $i \neq j$) then $e_1^\sigma, e_2^\sigma, \cdots, e_n^\sigma$ are orthogonal idempotents of $\mathfrak{G}$. Also one sees easily that $e.a = a$ implies $e^\sigma a^\sigma = a^\sigma = a^\sigma e^\sigma$ if e is idempotent.

Now assume $\mathfrak{J}$ has an identity element 1 and let $(S(\mathfrak{J}), \sigma_u)$ be the special universal envelope for $\mathfrak{J}$. Then $1^{\sigma_u} a^{\sigma_u} = a^{\sigma_u} = a^{\sigma_u} 1^{\sigma_u}$ in $S(\mathfrak{J})$ and since 1 and $\mathfrak{J}^{\sigma_u}$ generate $S(\mathfrak{J})$ it is clear that 1^{σ_u} is a central idempotent (= idempotent in the center) of $S(\mathfrak{J})$. It follows that if we put $z = 1 - 1^{\sigma_u}$ then $S(\mathfrak{J}) = \Phi z \oplus S_1(\mathfrak{J})$ where $S_1(\mathfrak{J})$ is the subalgebra of $S(\mathfrak{J})$ generated by $\mathfrak{J}^{\sigma_u}$ (without 1). Φz and $S_1(\mathfrak{J})$ are ideals and $z^2 = z$. Moreover, $\Phi z \neq 0$ since we have an associative specialization σ of $\mathfrak{J}$ in the one-dimensional algebra $\Phi = \Phi 1$ such that $a^\sigma = 0$, $a \in \mathfrak{J}$. If η is the homomorphism of $S(\mathfrak{J})$ into Φ such that $1 \to 1$ and $a^\sigma = a^{\sigma_u \eta}$, $a \in \mathfrak{J}$, then $\mathfrak{J}^{\sigma_u \eta} = 0$ and $S(\mathfrak{J})^\eta = \Phi$. Hence $(\Phi z)^\eta = \Phi$ and $\Phi z \neq 0$. The associative specialization σ_u can be regarded as an associative specialization of $\mathfrak{J}$ in $S_1(\mathfrak{J})$. When this is done σ_u is *unital* in the sense that 1^{σ_u} is the identity element of $S_1(\mathfrak{J})$. Now let σ be any unital associative specialization of $\mathfrak{J}$ in $\mathfrak{G}$ and let η be the homomorphism of $S(\mathfrak{J})$ into $\mathfrak{G}$ such that $1 \to 1$ and $a^\sigma = a^{\sigma_u \eta}$. Since $a^{\sigma_u} \in S_1(\mathfrak{J})$ and if η is the restriction of η to $S_1(\mathfrak{J})$ then $a^\sigma = a^{\sigma_u \eta}$ where σ_u is regarded as a mapping into $S_1(\mathfrak{J})$ and η as a mapping of $S_1(\mathfrak{J})$. Moreover, η is uniquely determined by this property. Thus we see that $(S_1(\mathfrak{J}), \sigma_u)$ is a *unital special universal envelope* in the sense of the definition obtained from Definition 1 by prefixing "unital" before "associative specialization" everywhere in the definition. We shall call $(S_1(\mathfrak{J}), \sigma_u)$ *the* unital special universal envelope for $\mathfrak{J}$. The properties (1)–(7) of $(S(\mathfrak{J}), \sigma_u)$ given in Theorem 1 carry over for unital special universal envelopes. In particular, we have the *main involution* π in $S_1(\mathfrak{J})$ such that $a^{\sigma_u \pi} = a^{\sigma_u}$, $a \in \mathfrak{J}$. This is the restriction to $S_1(\mathfrak{J})$ of the main involution in $S(\mathfrak{J})$. The analogue of property (4) of Theorem 1 is that if ζ is a homomorphism of a Jordan algebra $\mathfrak{J}$ with 1 into a Jordan algebra $\mathfrak{J}'$ with 1 such that $1 \to 1$ then there exists a unique homomorphism ζ_s of $S_1(\mathfrak{J})$ into $S_1(\mathfrak{J}')$ such that $a^{\sigma_u \zeta_s} = a^{\zeta \sigma_u'}$ where $(S_1(\mathfrak{J}'), \sigma'_u)$ is the unital special universal envelope of $\mathfrak{J}'$. Also this respects the main involutions. The argument used to establish (8) shows that if $\dim \mathfrak{J} = n$ then $\dim S_1(\mathfrak{J}) \leqq 2^{n-1}$. Since $S(\mathfrak{J}) = S_1(\mathfrak{J}) \oplus \Phi z$ we see that $\dim S(\mathfrak{J}) \leqq 2^{n-1} + 1$ if $\mathfrak{J}$ has an identity element.

Again assume $\mathfrak{J}$ has an identity element 1 and let $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ where the $\mathfrak{J}_i$ are ideals. Then $1 = 1_1 + 1_2$ where $1_i \in \mathfrak{J}_i$. Then $1_i^{\cdot 2} = 1_i$ and $1_i . 1_j = 0$ if $i \neq j$. Hence if $(S_1(\mathfrak{J}), \sigma_u)$ is the unital special universal envelope of $\mathfrak{J}$, then the elements $f_i = 1_i^{\sigma_u}$, $i = 1, 2$, are orthogonal idempotents in $\mathfrak{A} \equiv S_1(\mathfrak{J})$ and $f_1 + f_2 = 1$. Also f_i acts as identity for $\mathfrak{J}_i^{\sigma_u}$ and hence is the identity for the subalgebra $\mathfrak{A}_i$ generated by $\mathfrak{J}_i^{\sigma_u}$. Also $f_i a_j = 0 = a_j f_i$ if $a_j \in \mathfrak{J}_j^{\sigma_u}$ and $i \neq j$. Hence $f_i a_j = 0 = a_j f_i$ if $a_j \in \mathfrak{A}_j$. We have $\mathfrak{A}_1 \mathfrak{A}_2 = \mathfrak{A}_1 f_1 \mathfrak{A}_2 = 0$ and $\mathfrak{A}_2 \mathfrak{A}_1 = 0$. Hence $\mathfrak{A}_1 + \mathfrak{A}_2$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Since $\mathfrak{A}$ is generated by $\mathfrak{J}^{\sigma_u} = \mathfrak{J}_1^{\sigma_u} + \mathfrak{J}_2^{\sigma_u}$ we have $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$. Since f_1 is the identity for $\mathfrak{A}_1$ and annihilates $\mathfrak{A}_2$ we have $\mathfrak{A}_1 \cap \mathfrak{A}_2 = 0$. Hence the $\mathfrak{A}_i$ are ideals in $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$. Let σ_i denote the restriction of σ_u to $\mathfrak{J}_i$. Then σ_i maps $\mathfrak{J}_i$ into $\mathfrak{A}_i$ so we may consider this as a mapping of $\mathfrak{J}_i$ into $\mathfrak{A}_i$. Clearly, this is a unital associative specialization of $\mathfrak{J}_i$ in $\mathfrak{A}_i$. If $(S_1(\mathfrak{J}_i), \sigma_{iu})$ is the unital special universal envelope for $\mathfrak{J}_i$ then we have the homomorphism of $S_1(\mathfrak{J}_i)$ into $\mathfrak{A}_i$ such that $1 \to 1_i$ and $a_i^{\sigma_{iu}} \to a_i^{\sigma_i}$, $a_i \in \mathfrak{J}_i$, and consequently the homomorphism of $S_1(\mathfrak{J}_1) \oplus S_1(\mathfrak{J}_2)$ into $\mathfrak{A}$ such that $1 \to 1$, $a_1^{\sigma_{1u}} + a_2^{\sigma_{2u}} \to a_1^{\sigma_1} + a_2^{\sigma_2}$. On the other hand, $a_1 + a_2 \to a_1^{\sigma_{1u}} + a_2^{\sigma_{2u}}$ is a unital associative specialization of $\mathfrak{J}$. Hence we have the homomorphism of $S_1(\mathfrak{J})$ into $S_1(\mathfrak{J}_1) \oplus S_1(\mathfrak{J}_2)$ such that $1 \to 1, (a_1 + a_2)^{\sigma_u} = a_1^{\sigma_1} + a_2^{\sigma_2} \to a_1^{\sigma_{1u}} + + a_2^{\sigma_{2u}}$. Hence both homomorphisms are isomorphisms. It follows that we have an isomorphism of $S_1(\mathfrak{J}_i)$ onto $\mathfrak{A}_i$ mapping $1 \to 1$ and $a_i^{\sigma_{iu}} \to a_i^{\sigma_i}$. Hence $(\mathfrak{A}_i, \sigma_i)$ is a unital special envelope for $\mathfrak{J}_i$. It is clear that our result implies the following

THEOREM 5. *Let $\mathfrak{J}$ be a Jordan algebra with* 1, $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ *where $\mathfrak{J}_i$ is* *an ideal in* $\mathfrak{J}$, $(S_1(\mathfrak{J}_i), \sigma_{iu})$ *the unital special universal envelope for* $\mathfrak{J}_i$. *Then* $(S_1(\mathfrak{J}_1) \oplus S_1(\mathfrak{J}_2), \sigma_{1u} + \sigma_{2u})$ *where* $(a_1 + a_2)^{\sigma_{1u}+\sigma_{2u}} = a_1^{\sigma_{1u}} + a_2^{\sigma_{2u}}$, $a_i \in \mathfrak{J}_i$, *is a unital special universal envelope for* $\mathfrak{J}$.

3. **Examples.** In the first three examples we shall consider the Jordan algebras $\mathfrak{J}$ will be associative with 1. Hence these are associative commutative algebras with 1. Considering $\mathfrak{J}$ as Jordan algebra we write the product as $a.b$ and considering $\mathfrak{J}$ as associative algebra we write ab for $a.b$. We have $a.b = ab = \frac{1}{2}(ab + ba)$. Hence the identity mapping is a unital associative specialization of $\mathfrak{J}$ in $\mathfrak{J}$.

(1) $\mathfrak{J} = \Phi[a]$ the Jordan algebra with 1 generated by 1 and a single element a. Let σ be a unital associative specialization in $\mathfrak{G}$ of $\mathfrak{J}$. Then $(a^k)^\sigma = (a^{\cdot k})^\sigma = (a^\sigma)^{\cdot k} = (a^\sigma)^k$. Hence σ is a homomorphism of $\mathfrak{J}$ as associative algebra with 1 into $\mathfrak{G}$ with 1. It follows that $\mathfrak{J}$ and the identity mapping is a unital special universal envelope for $\mathfrak{J}$. This implies that $\dim S_1(\mathfrak{J}) = \dim \mathfrak{J} = n < 2^{\sigma-1} + 1$, the bound given above.

(2) $\mathfrak{J} = \Phi[a, b]$, the Jordan algebra with 1 generated by an invertible element a and its inverse b. If σ is a unital associative specialization of $\mathfrak{J} = \Phi[a, b]$ in $\mathfrak{G}$ then $a^\sigma . b^\sigma = 1$ and $(a^\sigma)^{\cdot 2} . b^\sigma = a^\sigma$. Hence, as we saw in §1.11, $a^\sigma b^\sigma = b^\sigma a^\sigma = 1$. It follows that σ is a homomorphism of $\mathfrak{J}$ as associative algebra. Consequently, $\mathfrak{J}$ and the identity mapping is a unital special universal envelope of the Jordan algebra $\mathfrak{J}$.

(3) Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ the Jordan algebra of the symmetric bilinear form $f = 0$ on $\mathfrak{B}$ and let $E(\mathfrak{B})$ be the exterior algebra based on $\mathfrak{B}$. $\mathfrak{J}$ is an associative Jordan algebra and the injection mapping $\alpha 1 + x \to \alpha 1 + x$, $x \in \mathfrak{B}$, of $\mathfrak{J} = \Phi 1 + \mathfrak{B}$ in $E(\mathfrak{B})$ is a unital associative specialization. This can be verified directly using the relations $x^2 = 0$, $xy = -yx$, $x, y \in \mathfrak{B}$, in $E(\mathfrak{B})$. It is known that if $\dim \mathfrak{B} = n - 1 < \infty$, so that $\dim \mathfrak{J} = n$, then $\dim E(\mathfrak{B}) = 2^{n-1}$. Since $\dim S_1(\mathfrak{J}) \leqq 2^{n-1}$ it is clear that the homomorphism of $S_1(\mathfrak{J})$ into $E(\mathfrak{B})$ such that $a^{\sigma_u} \to a$, $a \in \mathfrak{J}$, is an isomorphism. Hence $E(\mathfrak{B})$ and the injection mapping of $\mathfrak{J}$ is a unital special universal envelope of $\mathfrak{J}$ if $\mathfrak{B}$ is finite dimensional. The same result follows also for infinite dimensional $\mathfrak{B}$ by using Theorem 2 for unital special universal envelopes. These results are also special cases of the determination of unital special universal envelopes for Jordan algebras of symmetric bilinear forms which we shall consider in our next example. We observe that if $\dim \mathfrak{B} > 1$ then $E(\mathfrak{B}) \neq \mathfrak{J}$. Hence $\mathfrak{J}$ and the identity mapping is not a unital special universal envelope for the associative algebra $\mathfrak{J}$. We remark also that if $\mathfrak{B}$ is a trivial Jordan algebra, that is, $\mathfrak{B}$ is a vector space with product $x.y = 0$, then $E(\mathfrak{B})$ is the special universal envelope for $\mathfrak{J}$. Hence this gives an example in which the bound given in Theorem 1 (8) is reached.

(4) A generalization of (3) is obtained by taking $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$, the Jordan algebra of an arbitrary symmetric bilinear form f on $\mathfrak{B}$. We recall the definition

of the Clifford algebra $C(\mathfrak{V},f)$ of $\mathfrak{V}$ and f. This is $C(\mathfrak{V},f) = T(\mathfrak{V})/\mathfrak{K}$ where $T(\mathfrak{V})$ is the tensor algebra based on $\mathfrak{V}$ and $\mathfrak{K}$ is the ideal in $T(\mathfrak{V})$ generated by the elements $x \otimes x - f(x)1$, $f(x) = f(x,x)$. It is well known, and easily seen, that $C(\mathfrak{V},f)$ is characterized by the following universal mapping property: If σ is a linear mapping of $\mathfrak{V}$ into an associative algebra $\mathfrak{G}$ with 1 such that $(x^\sigma)^2 = f(x)1$ then there exists a unique homomorphism η of $C(\mathfrak{V},f)$ into $\mathfrak{G}$ such that $1^\eta = 1$ and $(x + \mathfrak{K})^\eta = x^\sigma$ (cf. Chevalley, *Algebraic Theory of Spinors*, p. 39). Now let σ_u denote the mapping $(\alpha 1 + x) \to (\alpha 1 + x)^{\sigma_u} \equiv \alpha 1 + x + \mathfrak{K}$, $x \in \mathfrak{V}$, of $\mathfrak{J}$ into $C(\mathfrak{V},f)$. Clearly this is linear and 1^{σ_u} is the identity element of $C(\mathfrak{V},f)$. Also, if $x, y \in \mathfrak{V}$ then $x^{\sigma_u} . y^{\sigma_u} = \frac{1}{2}(x + \mathfrak{K}).(y + \mathfrak{K}) = \frac{1}{2}(x \otimes y + y \otimes x) + \mathfrak{K} = \frac{1}{2}[(x + y) \otimes (x + y) - x \otimes x - y \otimes y] + \mathfrak{K} = \frac{1}{2}[f(x + y) - f(x) - f(y)]1 + \mathfrak{K} = f(x,y)1 + \mathfrak{K} = f(x,y)1^{\sigma_u} = (x \, . \, y)^{\sigma_u}$. It follows directly that σ_u is a unital associative specialization of $\mathfrak{J}$ in $C(\mathfrak{V},f)$. Moreover, if σ is any unital associative specialization of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ then we have $1^\sigma = 1$ and $(x^\sigma)^2 = (x^{\cdot 2})^\sigma = f(x)1$, $x \in \mathfrak{V}$. Hence, by the universal property of $C(\mathfrak{V},f)$, we have a unique homomorphism η of $C(\mathfrak{V},f)$ such that $1 \to 1$ and $(x + \mathfrak{K})^\eta = x^\sigma$. Then $(\alpha 1 + x)^{\sigma_u \eta} = (\alpha 1 + x + \mathfrak{K})^\eta = \alpha 1 + x^\sigma = (\alpha 1 + x)^\sigma$, which shows that $(C(\mathfrak{V},f), \sigma_u)$ is a unital special universal envelope for $\mathfrak{J}$. We remark also that the main involution in $C(\mathfrak{V},f)$ as usually defined (characterized by $1 \to 1$, $x^{\sigma_u} \to x^{\sigma_u}$) is the same as the main involution which we defined in $S_1(\mathfrak{J})$.

(5) Let $\mathfrak{J} = FJ^{(r)}$ the free Jordan algebra with 1 and r (free) generators $x_1, x_2, \cdots, x_r$, $\Phi\{u_1, u_2, \cdots, u_r\}$ the free associative algebra with 1 and r generators $u_1, u_2, \cdots, u_r$. It follows directly from the definition and the universal property of $\Phi\{u_1, u_2, \cdots, u_r\}$ that $\Phi\{u_1, u_2, \cdots, u_r\}$ and the homomorphism σ_u of $FJ^{(r)}$ into $\Phi\{u_1, \cdots, u_r\}^+$ such that $1 \to 1$ and $x_i \to u_i$, $1 \leqq i \leqq r$, is a unital special universal envelope of $FJ^{(r)}$. The main involution in $\Phi\{u_1, \cdots, u_r\}$ satisfies $u_i^{\,\pi} = u_i$. Hence this coincides with the reversal operator in $\Phi\{u_1, \cdots, u_r\}$.

Exercises

1. Prove the existence of a unital special universal envelope for any Jordan algebra $\mathfrak{J}$ with 1 by using Theorem 1 (5) (for the unital case) and the existence of such an envelope for free Jordan algebras with 1.

2. Let $\mathfrak{J}$ be an associative Jordan algebra $(S(\mathfrak{J}), \sigma_u)$ the special universal envelope for $\mathfrak{J}$. Show that for every $a, b \in \mathfrak{J}$, $[a^{\sigma_u}, b^{\sigma_u}]$ is in the center of $S(\mathfrak{J})$ and $[a^{\sigma_u}, b^{\sigma_u}]^2 = 0$.

3. Let $\mathfrak{J}$ be an associative Jordan algebra, $(S_1(\mathfrak{J}), \sigma_u)$ the unital special universal envelope for $\mathfrak{J}$. Show that $S_1(\mathfrak{J})$ contains a nil ideal $\mathfrak{N}$ such that the mapping $a \to a^{\sigma_u} + \mathfrak{N}$ is an isomorphism of $\mathfrak{J}$ as associative algebra onto $S_1(\mathfrak{J})/\mathfrak{N}$.

4. Let $\mathfrak{J} = \Phi[x,y]^+$ where $\Phi[x,y]$ is the usual commutative associative polynomial algebra in the indeterminates x, y, and let $\mathfrak{A} = \Phi\{u,v,w\}/\mathfrak{K}$ where $\Phi\{u,v,w\}$ is the free associative algebra with 1 generated by u, v, w and $\mathfrak{K}$ is the ideal generated by $[u,v] - w$, $[u,w], [v,w], w^2$. Show that $w + \mathfrak{K} \neq 0$ so $\mathfrak{A}$

is not commutative. Show that the linear mapping σ_u of $\Phi[x, y]$ into $\mathfrak{A}$ such that $1 \to \bar{1}$, $x^k y^l \to \bar{u}^{\cdot k} . \bar{v}^{\cdot l}$ where $\bar{a} = a + \mathfrak{K}$ is a unital associative specialization and $(\mathfrak{A}, \sigma_u)$ is universal.

5. Show that the unital special universal envelope of a purely inseparable field of characteristic $p \neq 0$ with basis $x^i y^j$, $0 \leqq i,j < p$, $x^p = \xi \in \Phi$, $y^p = \eta \in \Phi$ is not commutative.

4. **Associative algebras with involutions and Jordan algebras.** Let $\mathscr{A}\mathscr{I}_1$ denote the category of associative algebras with involution $(\mathfrak{A}, J)$ such that $\mathfrak{A}$ has identity element. The morphisms of $\mathscr{A}\mathscr{I}_1$ are the homomorphisms of algebras with involution mapping 1 into 1. Let $\mathscr{I}_1$ be the category of Jordan algebras with 1 in which the morphisms are homomorphisms mapping 1 into 1. If $(\mathfrak{A}, J) \in \mathscr{A}\mathscr{I}_1$ then $\mathfrak{H}(\mathfrak{A}, J) \in \mathscr{I}_1$ and if η is a morphism in $\mathscr{A}\mathscr{I}_1$ of $(\mathfrak{A}, J)$ into $(\mathfrak{B}, K)$ then the restriction η' of η to $\mathfrak{H}(\mathfrak{A}, J)$ is a morphism in $\mathscr{I}_1$ of $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{H}(\mathfrak{B}, K)$. In this way we obtain a functor H of $\mathscr{A}\mathscr{I}_1$ into $\mathscr{I}_1$ (cf. §I.4). On the other hand, we have the special universal functor S_1 of $\mathscr{I}_1$ into $\mathscr{A}\mathscr{I}_1$ which maps a $\mathfrak{J} \in \mathscr{I}_1$ into $(S_1(\mathfrak{J}), \pi)$ where $S_1(\mathfrak{J})$ is the unital special universal envelope of $\mathfrak{J}$ and π is its main involution. Moreover, if ζ is a homomorphism of $\mathfrak{J} \in \mathscr{I}_1$ into $\mathfrak{J}' \in \mathscr{I}_1$ such that $1 \to 1$ then S_1 maps ζ into the homomorphism ζ_s of $(S_1(\mathfrak{J}), \pi)$ into $(S_1(\mathfrak{J}'), \pi')$ where π' is the main involution in $S_1(\mathfrak{J}')$.

If $\mathfrak{J} \in \mathscr{I}_1$ then σ_u is a homomorphism of $\mathfrak{J}$ into $S_1(\mathfrak{J})^+$ and since $a^{\sigma_u \pi} = a^{\sigma_u}$ it is clear that σ_u can be considered also as a homomorphism of $\mathfrak{J}$ into $\mathfrak{H}(S_1(\mathfrak{J}), \pi) \in \mathscr{I}_1$. Also $1^{\sigma_u} = 1$ so σ_u is a morphism in the category $\mathscr{I}_1$ of Jordan algebras with 1. We have noted that $\mathfrak{J}$ is special if and only if the mapping σ_u of $\mathfrak{J}$ into $S(\mathfrak{J})$ is injective. Since $S_1(\mathfrak{J})$ is the subalgebra of $S(\mathfrak{J})$ generated by $\mathfrak{J}^{\sigma_u}$ it is clear that $\mathfrak{J}$ is special if and only if σ_u considered as a mapping into $S_1(\mathfrak{J})$ is injective. We shall now call a Jordan algebra with 1 *reflexive* if σ_u is surjective on $\mathfrak{H}(S_1(\mathfrak{J}), \pi)$. Thus if $\mathfrak{J}$ is special and reflexive then $\mathfrak{J}$ is isomorphic to a Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is an associative algebra with involution and $\mathfrak{A}$ has an identity element.

If $(\mathfrak{A}, J) \in \mathscr{A}\mathscr{I}_1$ then the injection mapping of $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{A}$ is a unital associative specialization of $\mathfrak{H}(\mathfrak{A}, J)$ in $\mathfrak{A}$. Hence we have a unique homomorphism ι_u of $S_1(\mathfrak{H}(\mathfrak{A}, J))$ into $\mathfrak{A}$ such that $a^{\sigma_u \iota_u} = a$, $a \in \mathfrak{H}(\mathfrak{A}, J)$. We have $a^{\sigma_u \pi \iota_u} = a^{\sigma_u \iota_u} = a$ and $a^{\sigma_u \iota_u J} = a$. Since the a^{σ_u} generate $S_1(\mathfrak{H}(\mathfrak{A}, J))$ this implies that ι_u is a homomorphism of algebras with involution mapping 1 into 1. Thus ι_u is a morphism in the category $\mathscr{A}\mathscr{I}_1$. Since $\mathfrak{H}(\mathfrak{A}, J)^{\sigma_u}$ generates $S_1(\mathfrak{H}(\mathfrak{A}, J))$, $\mathfrak{H}(\mathfrak{A}, J) = \mathfrak{H}(\mathfrak{A}, J)^{\sigma_u \iota_u}$ generates $S_1(\mathfrak{H}(\mathfrak{A}, J))^{\iota_u}$. Hence ι_u is surjective if and only if $\mathfrak{H}(\mathfrak{A}, J)$ generates $\mathfrak{A}$. We shall say that the associative algebra with involution $(\mathfrak{A}, J)$ is *perfect* if ι_u is an isomorphism of $(S_1(\mathfrak{H}(\mathfrak{A}, J)), \pi)$ onto $(\mathfrak{A}, J)$. This is the case if and only if $\mathfrak{A}$ and the injection mapping of $\mathfrak{H}(\mathfrak{A}, J)$ in $\mathfrak{A}$ is a unital special universal envelope for $\mathfrak{H}(\mathfrak{A}, J)$. More explicitly, $(\mathfrak{A}, J)$ is perfect if and only if any unital associative specialization of $\mathfrak{H}(\mathfrak{A}, J)$ has a unique extension to a homomorphism of $\mathfrak{A}$.

We now proceed to give some elementary properties of reflexivity and perfection. We prove first

THEOREM 6. *Any Jordan algebra $\mathfrak{J}$ with 1 which is generated by 1 and at most three elements is reflexive.*

PROOF. We have seen that for the free Jordan algebra $FJ^{(r)}$ with 1 and generators $x_1, x_2, \cdots, x_r$, the free associative algebra $\Phi\{u_1, u_2, \cdots, u_r\}$ and the unital associative specialization σ_u such that $x_i \to u_i$ is universal. Also the main involution in $\Phi\{u_1, u_2, \cdots, u_r\}$ is the reversal operator. The image of $FJ^{(r)}$ under σ_u is the free special Jordan algebra $FSJ^{(r)}$ and we know by Cohn's Theorem (p. 8) that this coincides with the space of reversible elements if and only if $r \leqq 3$. This means that $FJ^{(r)}$ is reflexive if and only if $r \leqq 3$. The proof of the theorem will now follow by showing that if $\mathfrak{J}$ is reflexive then any homomorphic image of $\mathfrak{J}$ is reflexive. Let ζ be a homomorphism of $\mathfrak{J}$ onto $\mathfrak{J}'$. Then we have the homomorphism ζ_s of $(S_1(\mathfrak{J}), \pi)$ onto $(S_1(\mathfrak{J}'), \pi')$ such that $a^{\sigma_u \zeta_s} = a^{\zeta \sigma_u'}$ where $(S_1(\mathfrak{J}), \sigma_u)$ and $(S_1(\mathfrak{J}'), \sigma_u')$ are the unital special universal envelopes, π and π' the main involutions. Let $a' \in \mathfrak{H}(S_1(\mathfrak{J}'), \pi')$. Then $a' = \frac{1}{2}(a' + a'^{\pi'})$ and $a' = a^{\zeta_s}$ for some $a \in S_1(\mathfrak{J})$. Then $a' = \frac{1}{2}(a' + a'^{\pi'}) = \frac{1}{2}(a^{\zeta_s} + a^{\zeta_s \pi'}) = \frac{1}{2}(a^{\zeta_s} + a^{\pi \zeta_s}) = b^{\zeta_s}$ where $b = \frac{1}{2}(a + a^{\pi}) \in \mathfrak{H}(S_1(\mathfrak{J}), \pi)$. If $\mathfrak{J}$ is reflexive then $b \in \mathfrak{J}^{\sigma_s}$. Hence $b = c^{\sigma_u}$ $c \in \mathfrak{J}$, and $a' = b^{\zeta_s} = c^{\sigma_u \zeta_s} = c^{\zeta \sigma_u'} \in \mathfrak{J}'^{\sigma_u'}$. Thus $\mathfrak{J}'$ is reflexive.

The theorem of Shirshov-Cohn (p. 48) states that any Jordan algebra with 1 and two generators (and 1) is special. Theorem 6 states that such an algebra is reflexive. Together the two results have the following consequence, which is a strengthening of Shirshov-Cohn:

COROLLARY 1. *Let $\mathfrak{J}$ be a Jordan algebra with 1 which is generated by 1 and two elements. Then $\mathfrak{J}$ is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{A}$ is an associative algebra with 1.*

For three generators we obtain in the same way the following

COROLLARY 2. *If $\mathfrak{J}$ is a special Jordan algebra with 1 generated by 1 and three elements then $\mathfrak{J}$ is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{A}$ is an associative algebra with 1.*

It is immediate from the definition that if $(\mathfrak{A}, J)$ is a perfect associative algebra with involution then the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ is reflexive. On the other hand, suppose $\mathfrak{J}$ is a Jordan algebra with 1 which is reflexive. Let σ be a unital associative specialization of $\mathfrak{H}(S_1(\mathfrak{J}), \pi)$ into the associative algebra $\mathfrak{G}$ with 1. Then $\sigma_u \sigma$ is a unital associative specialization of $\mathfrak{J}$. Hence there exists a homomorphism η of $S_1(\mathfrak{J})$ into $\mathfrak{G}$ such that $a^{\sigma_u \sigma} = a^{\sigma_u \eta}$, $a \in \mathfrak{J}$. If $h \in \mathfrak{H}(S_1(\mathfrak{J}), \pi)$ then the reflexivity of $\mathfrak{J}$ implies that $h = a^{\sigma_u}$ for some $a \in \mathfrak{J}$. Hence we have $h^{\sigma} = h^{\eta}$ so σ can be extended to the homomorphism η of $S_1(\mathfrak{J})$ into $\mathfrak{G}$. Also since $\mathfrak{H}(S_1(\mathfrak{J}), \pi) = \mathfrak{J}^{\sigma_u}$

generates $S_1(\mathfrak{J})$ it is clear that η is unique. Hence we see that if $\mathfrak{J}$ with 1 is reflexive then $(S_1(\mathfrak{J}),\pi)$ is perfect. We also have the following

THEOREM 7. *For any Jordan algebra $\mathfrak{J}$ with 1 the associative algebra with involution $(S_1(\mathfrak{J}),\pi)$ is isomorphic to an ideal direct summand of $(S_1(\mathfrak{H}(S_1(\mathfrak{J}),\pi),\pi))$.*

PROOF. Put $\mathfrak{A} = S_1(\mathfrak{J})$, $\mathfrak{K} = \mathfrak{H}(S_1(\mathfrak{J}),\pi)$, $\mathfrak{B} = S_1(\mathfrak{K})$ and let σ_u and τ_u be the given unital associative specializations of $\mathfrak{J}$ and $\mathfrak{K}$ in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Considering σ_u as a homomorphism of the Jordan algebra $\mathfrak{J}$ into the Jordan algebra $\mathfrak{K}$ we obtain a homomorphism Σ of $\mathfrak{A}$ into $\mathfrak{B}$ such that

$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{\sigma_u} & \mathfrak{K} \\ {\scriptstyle\sigma_u}\downarrow & & \downarrow{\scriptstyle\tau_u} \\ \mathfrak{A} & \xrightarrow[\Sigma]{} & \mathfrak{B} \end{array}$$

is commutative. Then if $a\in\mathfrak{J}$ we have $a^{\sigma_u\tau_u} = a^{\sigma_u\Sigma}$. Let ι_u be the homomorphism of $\mathfrak{B}=S_1(\mathfrak{K})$ into $\mathfrak{A}$ such that $k^{\tau_u\iota_u}=k$, $k\in\mathfrak{K}$. Now if $a\in\mathfrak{J}$ then $a^{\sigma_u}\in\mathfrak{K}=\mathfrak{H}(S_1(\mathfrak{J}_1)\,\pi)$. Hence $a^{\sigma_u} = a^{\sigma_u\tau_u\iota_u} = a^{\sigma_u\Sigma\iota_u}$. Since $\mathfrak{J}^{\sigma_u}$ generates $\mathfrak{A}$ this implies that $\Sigma\iota_u = 1_{\mathfrak{A}}$ for the homomorphisms Σ, ι_u of associative algebras with involution. Now put $\eta = \iota_u\Sigma$. Then η is a homomorphism of the algebra with involution $(\mathfrak{B},\pi)$ into itself and $\eta^2 = \iota_u\Sigma\iota_u\Sigma = \iota_u 1_{\mathfrak{A}}\Sigma = \iota_u\Sigma = \eta$. Hence we have the direct decomposition $\mathfrak{B} = \mathfrak{B}\eta \oplus \mathfrak{B}(1-\eta)$ where $\mathfrak{B}\eta$ and $\mathfrak{B}(1-\eta)$ are ideals of $(\mathfrak{B},\pi)$. It is immediate that Σ is an isomorphism of $(\mathfrak{A},\pi)$ onto $(\mathfrak{B}\eta,\pi)$. This proves the result.

An important feature of the properties: $\mathfrak{J}$ is special or reflexive and $(\mathfrak{A},J)$ is perfect is that these hold for an algebra obtained by a base field extension if and only if they hold for the given algebra. If $(\mathfrak{A},J)$ is an algebra with involution and Γ is an extension of the base field Φ of $\mathfrak{A}$ then the linear extension J of J to $\mathfrak{A}_\Gamma$ is an involution in $\mathfrak{A}_\Gamma$. Hence $(\mathfrak{A}_\Gamma,J)$ is an algebra with involution over Γ. If we identify $\mathfrak{A}$ with the Φ subalgebra $1\otimes\mathfrak{A}$ of $\mathfrak{A}_\Gamma = \Gamma\otimes\mathfrak{A}$ then $\mathfrak{H}(\mathfrak{A}_\Gamma,J)$ is the Γ-subspace spanned by $\mathfrak{H}(\mathfrak{A},J)$ $(= 1\otimes\mathfrak{H}(\mathfrak{A},J))$. Since $\mathfrak{H}(\mathfrak{A},J)$ and $\Gamma = \Gamma\otimes 1$ are linearly disjoint over Φ it is clear that $\mathfrak{H}(\mathfrak{A}_\Gamma,J) = \Gamma\mathfrak{H}(\mathfrak{A},J)$ can be identified with $\mathfrak{H}(\mathfrak{A},J)_\Gamma$. We shall call a property P of algebras (algebras with involution) *linear* if the validity of P for an algebra $\mathfrak{A}/\Phi$ (algebra with involution $(\mathfrak{A}/\Phi,J)$) is equivalent to the validity of P for $\mathfrak{A}_\Gamma$ $((\mathfrak{A}_\Gamma,J))$ for any extension field Γ/Φ. The result we indicated can now be stated in the following way.

THEOREM 8. *The property that a Jordan algebra is special or that a Jordan algebra with 1 is reflexive is linear. The property that an associative algebra with involution and 1 is perfect is linear.*

PROOF. Suppose $\mathfrak{J}$ (with 1) is special (reflexive). Then the associative specialization σ_u of $\mathfrak{J}$ in $S(\mathfrak{J})$ $(S_1(\mathfrak{J}))$ is injective (surjective on $\mathfrak{H}(S_1(\mathfrak{J}),\pi))$. Since $(S(\mathfrak{J})_\Gamma,\sigma_u)$ $((S_1(\mathfrak{J})_\Gamma,\sigma_u))$ is a special universal envelope (unital special universal envelope for $\mathfrak{J}_\Gamma$ and $\mathfrak{H}(S(\mathfrak{J})_\Gamma,\pi)=\mathfrak{H}(S(\mathfrak{J}),\pi)_\Gamma)$, σ_u is injective (surjective on $\mathfrak{H}(S(\mathfrak{J})_\Gamma,\pi))$. Hence $\mathfrak{J}_\Gamma$ is special (reflexive). Conversely, if $\mathfrak{J}_\Gamma$ is special (reflexive) then a reversal of the argument shows that $\mathfrak{J}$ is special (reflexive). Next let $(\mathfrak{A},J)$ be an associative algebra with involution and 1 which is perfect. Then $\mathfrak{A}$ and the injection mapping of $\mathfrak{H}(\mathfrak{A},J)$ in $\mathfrak{A}$ is a unital special universal envelope for $\mathfrak{H}(\mathfrak{A},J)$. Hence $\mathfrak{A}_\Gamma$ and the injection mapping of $\mathfrak{H}(\mathfrak{A}_\Gamma,J)$ in $\mathfrak{A}_\Gamma$ is a unital special universal envelope for $\mathfrak{H}(\mathfrak{A}_\Gamma,J)$ and so $(\mathfrak{A}_\Gamma,J)$ is perfect. Conversely, suppose $(\mathfrak{A}_\Gamma,J)$ is perfect and let σ be a unital associative specialization of $\mathfrak{H}(\mathfrak{A},J)$ in $\mathfrak{G}/\Phi$. The linear extension σ of σ to $\mathfrak{H}(\mathfrak{A},J)_\Gamma$ into $\mathfrak{G}_\Gamma$ is a unital associative specialization of $\mathfrak{H}(\mathfrak{A},J)_\Gamma$. Since we can identify $\mathfrak{H}(\mathfrak{A},J)_\Gamma$ with $\mathfrak{H}(\mathfrak{A}_\Gamma,J)$ we may consider σ as a unital associative specialization of $\mathfrak{H}(\mathfrak{A}_\Gamma,J)$. Then σ can be extended to a homomorphism of $\mathfrak{A}_\Gamma$ to $\mathfrak{G}_\Gamma$. This implies that the given mapping σ of $\mathfrak{H}(\mathfrak{A},J)$ can be extended to a homomorphism of $\mathfrak{A}$ into $\mathfrak{G}$. Since $\mathfrak{H}(\mathfrak{A}_\Gamma,J)$ generates $\mathfrak{A}_\Gamma$, $\mathfrak{H}(\mathfrak{A},J)$ generates $\mathfrak{A}$. Hence σ is unique and consequently $(\mathfrak{A},J)$ is perfect.

REMARKS. There are a number of places in the foregoing discussion where we could have referred to well-known facts in categories and functors. In particular, properties (1), (2) and (4) of Theorem 1 and Theorem 2 are well-known properties of universal functors. Also the two functors H and S which we have been considering in this section are adjoint functors: We have a natural equivalence of $\mathrm{Hom}_{\mathscr{J}_1}(\mathfrak{J},\mathfrak{H}(\mathfrak{A},J))$ and $\mathrm{Hom}_{\mathscr{A}\mathscr{J}_1}((S_1(\mathfrak{J}),\pi),(\mathfrak{A},J))$. Theorem 7 can be derived from this fact.

EXERCISES

1. Note that if $\mathfrak{A}$ is a commutative associative algebra then the identity mapping is an involution in $\mathfrak{A}$. Use this to give an example of a Jordan algebra $\mathfrak{H}(\mathfrak{A},J)$ which is not reflexive.

2. Show that $(S_1(FJ^{(r)}),\pi)$ is not perfect if $r \geqq 4$.

5. Bimodules for algebras in a variety $\mathscr{V}(I)$.

Let $\mathfrak{A}/\Phi$ be an algebra, $\mathfrak{M}/\Phi$ a vector space and suppose we have a pair of bilinear mappings $(a,u)\to au$, $(a,u)\to ua$, $a\in\mathfrak{A}$, $u\in\mathfrak{M}$, of $\mathfrak{A}\otimes\mathfrak{M}$ into $\mathfrak{M}$. Thus we have

$$(12)\qquad \begin{aligned} (a_1+a_2)u &= a_1u + a_2u, \qquad a(u_1+u_2) = au_1 + au_2, \\ \alpha(au) &= (\alpha a)u = a(\alpha u),\ \alpha(ua) = (\alpha u)a = u(\alpha a) \end{aligned}$$

for a, a_1, $a_2\in\mathfrak{A}$, u, u_1, $u_2\in\mathfrak{M}$ and $\alpha\in\Phi$. Now let $\mathfrak{E} = \mathfrak{A}\oplus\mathfrak{M}$ the vector space direct sum of $\mathfrak{A}$ and $\mathfrak{M}$ and define a product composition in $\mathfrak{E}$ by

$$(13)\qquad (a_1+u_1)(a_2+u_2) = a_1a_2 + a_1u_2 + u_1a_2.$$

It is clear that this product is bilinear, so $\mathfrak{E}$ is an algebra. Moreover, $\mathfrak{A}$ is a subalgebra and $\mathfrak{M}$ is an ideal in $\mathfrak{E}$ such that $\mathfrak{M}^2 = 0$. We shall call $\mathfrak{E}$ the *split null extension* of $\mathfrak{A}$ determined by the given bilinear mappings of $\mathfrak{A}$ and $\mathfrak{M}$. Now let I be a subset of the free nonassociative algebra $\Phi\{\{X\}\}'$ where $X = \{x_i\}$, $i = 1,2,3,\cdots$ (cf. p. 25) and let $\mathscr{V}(I)$ be the variety of algebras satisfying the identities in I. Let $\mathfrak{A} \in \mathscr{V}(I)$ and let $\mathfrak{M}$ be a vector space, $(a,u) \to au$, $(a,u) \to ua$ two bilinear mappings of $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$, as before. Then $\mathfrak{M}$ and the two bilinear compositions constitute an *I-bimodule* for $\mathfrak{A}$ if the split null extension $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ is in the variety $\mathscr{V}(I)$.[1] In particular, if I is the set defining associative, Lie, alternative or Jordan algebras then we say that $\mathfrak{M}$ is an *associative, Lie, alternative* or *Jordan bimodule* for $\mathfrak{A}$.

As in §1.6, we define $fD_i(x_1,\cdots,x_n,y)$ for $f = f(x_1,\cdots,x_n)$ as the part of $f(x_1,\cdots,x_{i-1},x_i + y,x_{i+1},\cdots,x_n)$ which is homogeneous of degree one in y. Thus we have

$$(14)\qquad \begin{aligned} &f(x_1,\cdots,x_{i-1},x_i + y,x_{i+1},\cdots,x_n) \\ &\qquad \equiv f(x_1,\cdots,x_n) + fD_i(x_1,\cdots,x_n,y) \pmod{(y)^2} \end{aligned}$$

where (y) is the ideal in $\Phi\{\{Z\}\}'$, $Z = X \cup \{y\}$, generated by y. We now introduce another infinite set $Y = \{y_i\}$ disjoint with X and put $W = X \cup Y$. Then it follows easily from (14) that

$$(15)\qquad \begin{aligned} f(x_1 + y_1,x_2 + y_2,\cdots,x_n + y_n) &\equiv f(x_1,\cdots,x_n) \\ &\quad + \sum fD_i(x_1,\cdots,x_n,y_i) \pmod{\mathfrak{Y}^2} \end{aligned}$$

where $\mathfrak{Y}$ is the ideal in $\Phi\{\{W\}\}'$ generated by the set Y.

We can now establish the following criterion for an I-bimodule.

THEOREM 9. *Let $\mathfrak{A} \in \mathscr{V}(I)$ and let $\mathfrak{M}$ be a vector space over Φ, $(a,u) \to au$ and $(a,u) \to ua$ bilinear compositions of $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$. Then $\mathfrak{M}$ with these compositions is an I-bimodule for $\mathfrak{A}$ if and only if we have the relations $fD_i(a_1,\cdots,a_n,u) = 0$ in $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ for all $a_i \in \mathfrak{A}$, $u \in \mathfrak{M}$ and $f \in I$.*

PROOF. $\mathfrak{M}$ with the given bilinear compositions is an I-bimodule if and only if the split null extension $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M} \in \mathscr{V}(I)$. If this is the case then $f(a_1,\cdots,a_{i-1},a_i + u,a_{i+1},\cdots,a_n) = 0$ for $a_k \in \mathfrak{A}$, $u \in \mathfrak{M}$, $f \in I$. By (14) and the fact that $\mathfrak{M}$ is an ideal such that $\mathfrak{M}^2 = 0$, we have $f(a_1,\cdots,a_{i-1},a_i + u,a_{i+1},\cdots,a_n) = fD_i(a_1,\cdots,a_n,u)$. Hence $fD_i(a_1,\cdots,a_n,u) = 0$. Conversely, suppose the indicated conditions hold. Then, by (15), we obtain

$$f(a_1 + u_1,\cdots,a_n + u_n) = \sum_i fD_i(a_1,\cdots,a_n,u_i) = 0.$$

1 This definition is due to Eilenberg [1].

This shows that $f(b_1, \cdots, b_n) = 0$ for all $b_i \in \mathfrak{E}$, $f \in I$. Hence $\mathfrak{E} \in \mathscr{V}(I)$ and $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$.

We now suppose that the conditions H and L of §1.6 are satisfied by the set I. We recall that this is the case for the sets defining associative, Lie, alternative and Jordan algebras. Now let $\mathfrak{A} \in \mathscr{V}(I)$, take $\mathfrak{M} = \mathfrak{A}$ as vector space and let au, ua, $a \in \mathfrak{A}$, $u \in \mathfrak{M}$ be the products as defined in the algebra $\mathfrak{A}$. Then if $f \in I$ condition L implies that $fD_i(a_1, \cdots, a_u, u) = 0$ for all $a_i \in \mathfrak{A}$, $u \in \mathfrak{M}$. Hence $\mathfrak{M} = \mathfrak{A}$ is an I-bimodule for $\mathfrak{A}$. We shall call this *the regular* bimodule for $\mathfrak{A}$.

EXAMPLES. (1) $\mathscr{V}(I)$, the variety of associative algebras. Here $I = \{f\}$ where $f = (x_1x_2)x_3 - x_1(x_2x_3)$. We have

$$
\begin{aligned}
fD_1 &= (yx_2)x_3 - y(x_2x_3),\\
fD_2 &= (x_1y)x_3 - x_1(yx_3),\\
fD_3 &= (x_1x_2)y - x_1(x_2y).
\end{aligned}
\tag{16}
$$

Hence $\mathfrak{M}$ is an associative bimodule for the associative algebra $\mathfrak{A}$ if and only if $(ua)b = u(ab)$, $(au)b = a(ub)$, $(ab)u = a(bu)$ for $a, b \in \mathfrak{A}$, $u \in \mathfrak{M}$. This is the usual definition of a bimodule (or two-sided module) for an associative algebra.

(2) $\mathscr{V}(I)$ the variety of Lie algebras. Here $I = \{f, g\}$ where $f = x_1{}^2$, $g = (x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2$. Then

$$
\begin{aligned}
fD_1 &= x_1y + yx_1,\\
gD_1 &= (yx_2)x_3 + (x_2x_3)y + (x_3y)x_2,\\
gD_2 &= (x_1y)x_3 + (yx_3)x_1 + (x_3x_1)y,\\
gD_3 &= (x_1x_2)y + (x_2y)x_1 + (yx_1)x_2.
\end{aligned}
\tag{17}
$$

The corresponding conditions for a Lie bimodule $\mathfrak{M}$ for the Lie algebra $\mathfrak{A}$ are $au = -ua$, $(ua)b + (ab)u + (bu)a = 0$, $(au)b + (ub)a + (ba)u = 0$, $(ab)u + (bu)a + (ua)b = 0$. These are equivalent to: $au = -ua$, $u(ab) = (ua)b - (ub)a$.

(3) $\mathscr{V}(I)$, the variety of alternative algebras. Here $I = \{f, g\}$ where $f = x_1{}^2x_2 - x_1(x_1x_2)$, $g = x_2x_1{}^2 - (x_2x_1)x_1$. Then

$$
\begin{aligned}
fD_1 &= (x_1y + yx_1)x_2 - x_1(yx_2) - y(x_1x_2),\\
fD_2 &= x_1{}^2y - x_1(x_1y),\\
gD_1 &= x_2(x_1y + yx_1) - (x_2x_1)y - (x_2y)x_1,\\
gD_2 &= yx_1{}^2 - (yx_1)x_1.
\end{aligned}
\tag{18}
$$

Hence the alternative bimodule conditions are: $(au + ua)b = a(ub) + u(ab)$, $a^2u = a(au)$, $b(au + ua) = (ba)u + (bu)a$, $ua^2 = (ua)a$, $a, b \in \mathfrak{A}$, $u \in \mathfrak{M}$.

(4) $\mathscr{V}(I)$, the variety of Jordan algebras. Here $I = \{f, g\}$ where $f = x_1x_2 - x_2x_1$ $g = (x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1)$. Then

$$\begin{aligned} fD_1 &= yx_2 - x_2y, \quad fD_2 = x_1y - yx_1, \\ gD_1 &= ((x_1y + yx_1)x_2)x_1 + (x_1{}^2x_2)y - (x_1y + yx_1)(x_2x_1) - x_1{}^2(x_2y) \\ gD_2 &= (x_1{}^2y)x_1 - x_1{}^2(yx_1). \end{aligned} \tag{19}$$

The corresponding bimodule conditions are $au = ua$, $((au + ua)b)a + (a^2b)u = (au + ua)(ba) + a^2(bu)$, $(a^2u)a = a^2(ua)$. These are equivalent to

$$\begin{aligned} au &= ua, \\ (ua^2)a &= (ua)a^2, \\ 2((ua)b)a + u(a^2b) &= 2(ua)(ab) + (ub)a^2. \end{aligned} \tag{20}$$

6. **Birepresentations for an algebra in a variety $\mathscr{V}(I)$.** If $(a,u) \to au$ is a bilinear mapping of $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$, then the mapping $u \to au$, $u \in \mathfrak{M}$ is an element a^λ of $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Moreover, the mapping $a \to a^\lambda$ is a linear mapping of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Conversely, if we are given a linear mapping $\lambda: a \to a^\lambda$ of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ then the mapping $(a,u) \to ua^\lambda$ is a bilinear mapping of $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$. Thus if we are given a pair of bilinear mappings $(a,u) \to au$ and $(a,u) \to ua$ of $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$ then we have a pair of linear mappings λ and ρ of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ and conversely. We shall now seek to formulate the I-bimodule conditions given in Theorem 9 as conditions on the mappings λ and ρ determined by the given bilinear compositions.

For this purpose we consider again the free nonassociative algebra $\Phi\{\{Z\}\}'$ where $Z = X \cup \{y\}$, $X = \{x_i\}$, $i = 1,2,3,\cdots$, $y \notin X$. The subalgebra of $\Phi\{\{Z\}\}'$ generated by X can be identified with $\Phi\{\{X\}\}'$. Let $\mathfrak{B}$ be the subalgebra of $\mathrm{Hom}_\Phi(\Phi\{\{Z\}\}',\Phi\{\{Z\}\}')$ generated by 1 and the multiplications a_L, a_R in $\Phi\{\{Z\}\}'$ by the elements $a \in \Phi\{\{X\}\}'$. As in §1.6, let $N(X)'$ be the (free) monad generated by the x's, so that $N(X)'$ is a basis for $\Phi\{\{X\}\}'$. Then clearly $\mathfrak{B}$ is generated by 1 and the multiplications m_L, m_R, $m \in N(X)'$. We now introduce a set P which is a disjoint union of two copies of $N(X)'$. Thus we assume we have two injective mappings $m \to l(m)$, $m \to r(m)$ of $N(X)'$ into P such that $P = l(N(X)') \cup r(N(X)')$ and $l(N(X)') \cap r(N(X)') = \varnothing$. Let $\Phi\{P\}$ be the free associative algebra with 1 generated by 1 and the set P. Then we have a homomorphism ν of $\Phi\{P\}$ onto $\mathfrak{B}$ such that $1 \to 1$, $l(m) \to m_L$, $r(m) \to m_R$, $m \in N(X)'$. More generally, let η be a homomorphism of $\Phi\{\{X\}\}'$ into an algebra $\mathfrak{A}$. Then we have the homomorphism η' of $\Phi\{P\}$ into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ such that $1 \to 1$, $l(m) \to (m^\eta)_L$, $r(m) \to (m^\eta)_R$. We have now the following

LEMMA 1. *There exists a homomorphism η'' of $\mathfrak{B} = \mathrm{Hom}_\Phi(\Phi\{\{Z\}\}', \Phi\{\{Z\}\}')$ into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ such that $1 \to 1$ and*

(21) $$m_L{}^{\eta''} = (m^\eta)_L, \qquad m_R{}^{\eta''} = (m^\eta)_R, \qquad m \in N(X)'.$$

PROOF. Let u be any element of $\mathfrak{A}$ and let (η, u) denote the homomorphism of $\Phi\{\{Z\}\}'$ into $\mathfrak{A}$ which coincides with η on $\Phi\{\{X\}\}'$ and sends y into u. We claim that

(22) $$(yQ^\nu)^{(\eta,u)} = uQ^{\eta'}$$

for every $Q \in \Phi\{P\}$. Since 1, $l(m)$, $r(m)$, $m \in N(X)'$ generate $\Phi\{P\}$ this will follow if we can show that if (22) holds for a $Q \in \Phi\{P\}$ then it holds for $Ql(m)$ and for $Qr(m)$, $m \in N(X)'$. Now if (22) holds then $y(Ql(m))^\nu = yQ^\nu l(m)^\nu = yQ^\nu m_L = m(yQ^\nu)$ and $(y(Ql(m))^\nu)^{(\eta,u)} = (m(yQ^\nu))^{(\eta,u)} = m^\eta(yQ^\nu)^{(\eta,u)} = m^\eta(uQ^{\eta'}) = (uQ^{\eta'})\,(m^\eta)_L = (uQ^{\eta'})l(m)^{\eta'} = u(Ql(m))^{\eta'}$. Similarly, $(y(Qr(m))^\nu)^{(\eta,u)} = u(Qr(m))^{\eta'}$. Hence (22) holds for all Q. Now suppose $Q^\nu = 0$. Then (22) implies that $uQ^{\eta'} = 0$ for all u. Hence $Q^{\eta'} = 0$. Thus the kernel of the homomorphism ν of $\Phi\{P\}$ into $\mathfrak{B}$ is contained in the kernel of the homomorphism η' of $\Phi\{P\}$ into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$. Hence we have a homomorphism η'' of $\mathfrak{B}$ into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ such that

$$\begin{array}{ccc} \Phi\{P\} & \xrightarrow{\nu} & \mathfrak{B} \\ & {}_{\eta'}\searrow & \downarrow{\scriptstyle \eta''} \\ & & \mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A}) \end{array}$$

is commutative. Then $1^{\eta''} = 1^{\nu\eta''} = 1^{\eta'} = 1$, $m_L{}^{\eta''} = l(m)^{\nu\eta''} = l(m)^{\eta'} = (m^\eta)_L$ and $m_R{}^{\eta''} = (m^\eta)_R$, as required in (21).

LEMMA 2. *ν is an isomorphism of $\Phi\{P\}$ onto $\mathfrak{B}$.*

PROOF. We take a 1-1 representation σ of $\Phi\{P\}$ by linear transformations in a vector space $\mathfrak{N}/\Phi$. For example, we can take $\mathfrak{N} = \Phi\{P\}$ and let Q^σ be the multiplication $A \to AQ$ in $\mathfrak{N}$. We now define bilinear compositions of $\Phi\{\{X\}\}' \times \mathfrak{N}$ into $\mathfrak{N}$ such that for any $m \in N(X)'$, $n \in \mathfrak{N}$ we have $(m, n) \to nl(m)^\sigma$ and $(m, n) \to nr(m)^\sigma$. This can be done since $N(X)'$ is a basis for $\Phi\{\{X\}\}'$. The bilinear compositions determine the split null extension $\mathfrak{E} = \Phi\{\{X\}\}' \oplus \mathfrak{N}$. Let η be the injection of $\Phi\{\{X\}\}'$ in $\mathfrak{E}$ and let η'' be the corresponding homomorphism of $\mathfrak{B}$ into $\mathrm{Hom}_\Phi(\mathfrak{E}, \mathfrak{E})$ given in Lemma 1. By definition of the multiplication in $\mathfrak{E}$, we have $n(m^\eta)_L = nl(m)^\sigma$, $n(m^\eta)_R = nr(m)^\sigma$ if $n \in \mathfrak{N}$, $m \in N(X)'$. Hence $m_L{}^{\eta''} = (m^\eta)_L$ and $m_R{}^{\eta''} = (m^\eta)_R$ map $\mathfrak{N}$ into itself and consequently the elements of $\mathfrak{B}^{\eta''}$ map $\mathfrak{N}$ into itself. Let $v^{\eta'''}$ denote the restriction to $\mathfrak{N}$ of $v^{\eta''}$. Then η''': $v \to v^{\eta'''}$ is a homomorphism of $\mathfrak{B}$ into $\mathrm{Hom}_\Phi(\mathfrak{N}, \mathfrak{N})$ such that $m_L{}^{\eta'''} = l(m)^\sigma$, $m_R{}^{\eta'''} = r(m)^\sigma$. Since σ is a monomorphism we have the homomorphism of $\mathfrak{B}$ into $\Phi\{P\}$ such that $1 \to 1$, $m_L \to l(m)$, $m_R \to r(m)$. Clearly this is the inverse of the homomorphism ν of $\Phi\{P\}$ onto $\mathfrak{B}$. Hence ν is an isomorphism.

It is clear from the definition of $\Phi\{\{Z\}\}'$ that any element q of this algebra which is homogeneous of degree one in y has the form yQ^{ν} where $Q \in \Phi\{P\}$. It is an easy consequence of Lemma 2 that Q is uniquely determined. We can now translate the conditions of Theorem 9 for a bimodule for a class of algebras defined by a set of identities into conditions on the associated mappings λ, ρ, as follows.

THEOREM 10. *Let $\mathfrak{A} \in \mathscr{V}(I)$ a variety of algebras defined by a set of identities $I \subseteq \Phi\{\{X\}\}'$. Let $\Phi\{P\}$ be the free associative algebra with 1 generated by 1 and the set P which is a disjoint union of two copies $l(N(X)')$ and $r(N(X)')$ of $N(X)'$, ν the homomorphism of $\Phi\{P\}$ into* $\mathrm{Hom}_{\Phi}(\Phi\{\{Z\}\}', \Phi\{\{Z\}\}')$ *such that $1 \to 1$, $l(m) \to m_L$, $r(m) \to m_R$ the multiplications in $\Phi\{\{Z\}\}'$ determined by the element $m \in N(X)'$. For each $f \in I$ and $i = 1,2,3,\cdots$ write $fD_i(x_1,\cdots,x_n,y) = yF_i^{\nu}$, $F_i \in \Phi\{P\}$. Let $\mathfrak{M}$ be a vector space over Φ, $(a,u) \to au$, $(a,u) \to ua$ bilinear compositions of $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$ and let a^{λ}, a^{ρ} for $a \in \mathfrak{A}$ be the linear mappings $u \to au$ and $u \to ua$ respectively in $\mathfrak{M}$. If η is a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ let $\tau = \tau(\eta)$ be the homomorphism of $\Phi\{P\}$ into* $\mathrm{Hom}_{\Phi}(\mathfrak{M},\mathfrak{M})$ *such that $1 \to 1$, $l(m)^{\tau} = m^{\eta\lambda}$, $r(m)^{\tau} = m^{\eta\rho}$. Then $\mathfrak{M}$ with the given bilinear compositions is an I-bimodule for $\mathfrak{A}$ if and only if $F_i^{\tau} = 0$ for every $f \in I$, $i = 1,2,\cdots$ and every homomorphism η.*

PROOF. Let $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ be the split null extension defined by the given bilinear mappings. If η is a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ then we have the homomorphism η' of $\Phi\{P\}$ into $\mathrm{Hom}_{\Phi}(\mathfrak{E},\mathfrak{E})$ such that $1 \to 1$, $l(m)^{\eta'} = (m^{\eta})_L$, $r(m)^{\eta'} = (m^{\eta})_R$ where $m \in N(X)'$ and L and R denote left and right multiplications in $\mathfrak{E}$. Since a^{λ} and a^{ρ} are the restrictions to $\mathfrak{M}$ of a_L and a_R, respectively, we have for $u \in \mathfrak{M}$, $ul(m)^{\tau} = um^{\eta\lambda} = u(m^{\eta})_L = ul(m)^{\eta'}$ and $ur(m)^{\tau} = ur(m)^{\eta'}$. Consequently, $uF^{\tau} = uF^{\eta'}$ for all $F \in \Phi\{P\}$. Now let $f \in I$ and let $fD_i = yF_i^{\nu}$ where $F_i \in \Phi\{P\}$. Consider the homomorphism (η,u) of $\Phi\{\{Z\}\}'$ into $\mathfrak{E}$ which extends η and maps y into u. Then, by (22), $(yF_i^{\nu})^{(\eta,u)} = uF_i^{\eta'} = uF_i^{\tau}$. Hence $(fD_i)^{(\eta,u)} = uF_i^{\tau}$. By Theorem 9, $\mathfrak{M}$ is an I-bimodule if and only if $(fD_i)^{(\eta,u)} = 0$ for all $f \in I$, all $i = 1,2,\cdots$, all homomorphisms η and all $u \in \mathfrak{M}$. Clearly this is equivalent to $F_i^{\tau} = 0$ for all $f \in I$, $i = 1,2,\cdots$ and all homomorphisms $\tau = \tau(\eta)$.

If $f \in \Phi\{\{X\}\}'$ then we shall call the elements $F_i \in \Phi\{P\}$ such that $fD_i(x_1,\cdots,x_n,y) = yF_i^{\nu}$ the *derived elements of f in $\Phi\{P\}$*. We can now give the following

DEFINITION 2. *Let $\mathfrak{A}$ be an algebra in a variety $\mathscr{V}(I)$ defined by a set of identities I (in $\Phi\{\{X\}\}'$) and let $\mathfrak{M}$ be a vector space. Then a pair of linear mappings (λ,ρ) of $\mathfrak{A}$ into* $\mathrm{Hom}_{\Phi}(\mathfrak{M},\mathfrak{M})$ *is called an I-*birepresentation *for $\mathfrak{A}$ if the following conditions hold: Given any homomorphism η of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$, let τ be the homomorphism of $\Phi\{P\}$ into* $\mathrm{Hom}_{\Phi}(\mathfrak{M},\mathfrak{M})$ *such that $1 \to 1$, $l(m)^{\tau} = m^{\eta\lambda}$, $r(m)^{\tau} = m^{\eta\rho}$, $m \in N(X)'$. Then $F_i^{\tau} = 0$ for every derived element F_i of every $f \in I$.*

It is clear from Theorem 10 that the notions of I-bimodule and I-birepresentation of an algebra $\mathfrak{A} \in \mathscr{V}(I)$ are equivalent in the following sense. If $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ then (λ, ρ) where $a^\lambda : u \to au$ and $a^\rho : u \to ua$, $a \in \mathfrak{A}$, $u \in \mathfrak{M}$, is an I-birepresentation for $\mathfrak{A}$. Conversely, if (λ, ρ) is an I-birepresentation of $\mathfrak{A}$ such that the domains of a^λ and a^ρ are $\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ then $\mathfrak{M}$ is an I-bimodule if we define $au = ua^\lambda$ and $ua = ua^\rho$.

We now consider the explicit form of the birepresentation conditions for the important classes we have singled out.

(1) *Associative algebras.* If we refer to (16) we obtain that the nonzero derived elements of $f = (x_1x_2)x_3 - x_1(x_2x_3)$ are

$$(23)\qquad \begin{aligned} F_1 &= r(x_2)r(x_3) - r(x_2x_3), \\ F_2 &= l(x_1)r(x_3) - r(x_3)l(x_1), \\ F_3 &= l(x_1x_2) - l(x_2)l(x_1). \end{aligned}$$

Let η be a homomorphism of $\Phi\{\{X\}\}'$ into an associative algebra $\mathfrak{A}$, (λ, ρ) a pair of linear mappings of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ and let τ be the homomorphism of $\Phi\{P\}$ into $\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ such that $1 \to 1$, $l(m)^\tau = m^{\eta\lambda}$, $r(m)^\tau = m^{\eta\rho}$. Then if $x_1^\eta = a$, $x_2^\eta = b$, $x_3^\eta = c$ we have $r(x_2)^\tau = x_2^{\eta\rho} = b^\rho$, $r(x_3)^\tau = c^\rho$, $r(x_2x_3)^\tau = (x_2x_3)^{\eta\rho} = (bc)^\rho$, $l(x_1)^\tau = a^\lambda$, $l(x_1x_2)^\tau = (ab)^\lambda$, $l(x_2)^\tau = b^\lambda$. With a slight change in notations this leads to the following conditions for an associative birepresentation of $\mathfrak{A}$:

$$(24)\qquad (ab)^\rho = a^\rho b^\rho, \qquad a^\rho b^\lambda = b^\lambda a^\rho, \qquad (ab)^\lambda = b^\lambda a^\lambda,$$

for all $a, b \in \mathfrak{A}$.

(2) *Lie algebras.* Using (17) we see that the nonzero derived elements of $f = x_1{}^2$ and $g = (x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2$ are

$$(25)\qquad \begin{aligned} F_1 &= l(x_1) + r(x_1), \\ G_1 &= r(x_2)r(x_3) + l(x_2x_3) + l(x_3)r(x_2), \\ G_2 &= l(x_1)r(x_3) + r(x_3)r(x_1) + l(x_3x_1), \\ G_3 &= l(x_1x_2) + l(x_2)r(x_1) + r(x_1)r(x_2). \end{aligned}$$

These lead to the following conditions for a birepresentation (λ, ρ) of a Lie algebra $\mathfrak{A}$:

$$(26)\qquad \begin{aligned} \lambda &= -\rho, \\ (ab)^\rho &= a^\rho b^\rho - b^\rho a^\rho, \end{aligned}$$

$a, b \in \mathfrak{A}$.

(3) *Alternative algebras.* By (18) the derived elements $\neq 0$ of $f = x_1{}^2x_2 - x_1(x_1x_2)$, $g = x_2x_1{}^2 - (x_2x_1)x_1$ are

$$
\begin{aligned}
F_1 &= (l(x_1) + r(x_1))r(x_2) - r(x_1x_2) - r(x_2)l(x_1),\\
F_2 &= l(x_1{}^2) - l(x_1)^2,\\
G_1 &= (l(x_1) + r(x_1))l(x_2) - l(x_2x_1) - l(x_2)r(x_1),\\
G_2 &= r(x_1{}^2) - r(x_1)^2.
\end{aligned} \tag{27}
$$

These give the alternative birepresentation conditions

$$
\begin{aligned}
a^{\lambda+\rho}b^\rho &= (ab)^\rho + b^\rho a^\lambda,\\
(a^2)^\lambda &= (a^\lambda)^2,\\
a^{\lambda+\rho}b^\lambda &= (ba)^\lambda + b^\lambda a^\rho,\\
(a^2)^\rho &= (a^\rho)^2.
\end{aligned} \tag{28}
$$

(4) *Jordan algebras.* By (19) the nonzero derived elements of the defining identities $f = x_1x_2 - x_2x_1$, $g = (x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1)$ are

$$
\begin{aligned}
F_1 &= r(x_2) - l(x_2),\\
F_2 &= l(x_1) - r(x_1),\\
G_1 &= (l(x_1) + r(x_1))r(x_2)r(x_1) + l(x_1{}^2x_2)\\
&\quad - (l(x_1) + r(x_1))r(x_2x_1) - l(x_2)l(x_1{}^2),\\
G_2 &= l(x_1{}^2)r(x_1) - r(x_1)l(x_1{}^2).
\end{aligned} \tag{29}
$$

These give the following Jordan birepresentation conditions:

$$
\begin{aligned}
\lambda &= \rho,\\
(a^2)^\rho a^\rho &= a^\rho(a^2)^\rho,\\
2a^\rho b^\rho a^\rho + (a^2b)^\rho &= 2a^\rho(ba)^\rho + b^\rho(a^2)^\rho.
\end{aligned} \tag{30}
$$

Exercise

1. Derive the bimodule and birepresentation conditions for noncommutative Jordan algebras (ex. 8, p. 33) and Malcev algebras (ex. 10, p. 33).

7. **Multiplication specializations and universal envelopes.** It is convenient to generalize the notion of an I-birepresentation for an algebra $\mathfrak{A}$ in a variety $\mathscr{V}(I)$ by replacing $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ by any associative algebra $\mathfrak{G}$ with 1. Using the notations of the last section, we define an *I-multiplication specialization* of $\mathfrak{A} \in \mathscr{V}(I)$ in the associative algebra $\mathfrak{G}$ with 1 to be a pair of linear mappings (λ, ρ) of $\mathfrak{A}$

into $\mathfrak{G}$ such that if η is any homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ and τ is the homomorphism of $\Phi\{P\}$ into $\mathfrak{G}$ such that $1 \to 1$, $l(m)^\tau = m^{\eta\lambda}$, $r(m)^\tau = m^{\eta\rho}$, $m \in N(X)'$, then $F_i^\tau = 0$ for all derived polynomials $F_i \in \Phi\{P\}$ of every $f \in I$. Thus an I-birepresentation is an I-multiplication specialization in $\mathfrak{G} = \mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. It is clear that if (λ, ρ) is an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{G}$ and μ is a homomorphism of $\mathfrak{G}$ into another associative algebra $\mathfrak{G}'$ with 1 such that $1 \to 1$ then $(\lambda\mu, \rho\mu)$ is an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{G}'$. In particular, this holds if μ is a unital representation of $\mathfrak{G}$, that is, a homomorphism of $\mathfrak{G}$ into an algebra $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that $1 \to 1$. It is clear also that if μ is a 1-1 unital representation of $\mathfrak{G}$ then a pair of linear mappings (λ, ρ) of $\mathfrak{A}$ into $\mathfrak{G}$ is an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{G}$ if and only if $(\lambda\mu, \rho\mu)$ is an I-birepresentation of $\mathfrak{A}$.

Now let ζ be a homomorphism of $\mathfrak{A} \in \mathscr{V}(I)$ into $\mathfrak{A}' \in \mathscr{V}(I)$ and let (λ, ρ) be an I-multiplication specialization of $\mathfrak{A}'$ in $\mathfrak{G}$. We claim that $(\zeta\lambda, \zeta\rho)$ is an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{G}$. By the remark just made and the fact that any associative algebra with 1 has a 1-1 unital representation, it is sufficient to consider the case in which $\mathfrak{G} = \mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$, that is, (λ, ρ) is an I-birepresentation of $\mathfrak{A}'$. In view of Theorem 10, this will follow if we can show that if $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}'$, then $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ relative to the compositions

$$(31) \qquad au = a^\zeta u, \qquad ua = ua^\zeta, \qquad a \in \mathfrak{A}, \qquad u \in \mathfrak{M}.$$

Clearly these compositions are bilinear so we have to show that the split null extension $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ determined by them is in $\mathscr{V}(I)$. Let η be the homomorphism of $\Phi\{\{Z\}\}'$ into $\mathfrak{E}$ such that $x_i \to a_i \in \mathfrak{A}$, $y \to u \in \mathfrak{M}$. Then it follows from (31) by induction on the degree that if $m(x_1, \cdots, x_n, y)$ is an element of $N(Z)'$ of first degree in y then $m(x_1, \cdots, x_n, y)^\eta = m(a_1, \cdots, a_n, u) = m(a_1{}^\zeta, \cdots, a_n{}^\zeta, u)$, which is an element of $\mathfrak{M}$. Hence if $g(x_1, \cdots, x_n, y)$ is an element of $\Phi\{\{Z\}\}'$ homogeneous of degree 1 in y then

$$(31') \qquad g(a_1, \cdots, a_n, u) = g(a_1{}^\zeta, \cdots, a_n{}^\zeta, u).$$

Now let $f(x_1, \cdots, x_n) \in I$. Then $fD_i(a_1, \cdots, a_n, u) = fD_i(a_1{}^\zeta, \cdots, a_n{}^\zeta, u) = 0$ since $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}'$. Hence, by Theorem 9, $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$.

We suppose next that (λ, ρ) is an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{G}$ and $\mathfrak{K}$ is an ideal of $\mathfrak{A}$ contained in $\ker\lambda \cap \ker\rho$. If $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{K}$ and $\bar{a} = a + \mathfrak{K}$, $a \in \mathfrak{A}$, then the linear mappings $\bar{\lambda}, \bar{\rho}$ of $\bar{\mathfrak{A}}$ in $\mathfrak{G}$ such that $\bar{a}^{\bar{\lambda}} = a^\lambda$, $\bar{a}^{\bar{\rho}} = a^\rho$ constitute an I-multiplication specialization of $\bar{\mathfrak{A}}$ in $\mathfrak{G}$. Again, it is enough to consider the case in which $\mathfrak{G} = \mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Then the assertion is equivalent to the following: If $\mathfrak{M}$ is an I-bimodule $\mathfrak{A}$ and $\mathfrak{K}$ is an ideal in $\mathfrak{A}$ such that $\mathfrak{K}\mathfrak{M} = 0 = \mathfrak{M}\mathfrak{K}$ then $\mathfrak{M}$ is an I-bimodule for $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{K}$ relative to the compositions $\bar{a}u = au$, $u\bar{a} = ua$, $a \in \mathfrak{A}$, $\bar{a} = a + \mathfrak{K}$. It is clear that $(\bar{a}, u) \to au$, $(\bar{a}, u) \to ua$ are bilinear. Also we have the homomorphism $a + u \to \bar{a} + u$ of the split null extension $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ onto the split null extension $\bar{\mathfrak{E}} = \bar{\mathfrak{A}} \oplus \mathfrak{M}$. Since $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$, $\mathfrak{E} \in \mathscr{V}(I)$. Hence $\bar{\mathfrak{E}} \in \mathscr{V}(I)$ and $\mathfrak{M}$ is an I-bimodule for $\bar{\mathfrak{A}}$.

We are now ready to consider the universal objects defined by the notion of an I-multiplication specialization. These are defined as follows.

DEFINITION 3. *Let $\mathfrak{A}$ be an algebra in a variety $\mathscr{V}(I)$ defined by a set of identities I. Then an associative algebra $\mathfrak{U}$ with identity element* 1 *together with an I-multiplication specialization (λ_u, ρ_u) of $\mathfrak{A}$ in $\mathfrak{U}$ is called a* universal I-multiplication envelope of $\mathfrak{A}$ *if for any I-multiplication specialization (λ, ρ) of $\mathfrak{A}$ in an associative algebra $\mathfrak{G}$ with* 1 *there exists a unique homomorphism μ of $\mathfrak{U}$ into $\mathfrak{G}$ such that $1 \to 1$ and $\lambda_u \mu = \lambda$, $\rho_u \mu = \rho$, that is, the following diagrams are commutative*:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\lambda_u} & \mathfrak{U} \\ {\scriptstyle \lambda}\downarrow & \swarrow {\scriptstyle \mu} & \\ \mathfrak{G} & & \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\rho_u} & \mathfrak{U} \\ {\scriptstyle \rho}\downarrow & \swarrow {\scriptstyle \mu} & \\ \mathfrak{G} & & \end{array}$$

We can now state the following basic result.

THEOREM 11. *There exists a universal I-multiplication envelope for any algebra $\mathfrak{A}$ in a variety $\mathscr{V}(I)$. Universal multiplication envelopes have the following properties*:

(1) *If $(\mathfrak{U}, \lambda_u, \rho_u)$ and $(\mathfrak{U}', \lambda_u', \rho_u')$ are universal I-multiplication envelopes for $\mathfrak{A}$ then there exists a unique isomorphism ι of $\mathfrak{U}$ onto $\mathfrak{U}'$ such that $\lambda_u' = \lambda_u \iota$, $\rho_u' = \rho_u \iota$.*

(2) *$\mathfrak{U}$ is generated by $\mathfrak{A}^{\lambda_u} \cup \mathfrak{A}^{\rho_u} \cup \{1\}$.*

(3) *If ζ is a homomorphism of $\mathfrak{A}$ into another algebra $\mathfrak{A}'$ in $\mathscr{V}(I)$ and $(\mathfrak{U}', \lambda_u', \rho_u')$ is an I-multiplication envelope for $\mathfrak{A}'$ then there exists a unique homomorphism ζ_u of $\mathfrak{U}$ into $\mathfrak{U}'$ such that $1 \to 1$ and the following diagrams are commutative*:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\zeta} & \mathfrak{A}' \\ {\scriptstyle \lambda_u}\downarrow & & \downarrow{\scriptstyle \lambda_u'} \\ \mathfrak{U} & \xrightarrow[\zeta_u]{} & \mathfrak{U}' \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\zeta} & \mathfrak{A}' \\ {\scriptstyle \rho_u}\downarrow & & \downarrow{\scriptstyle \rho_u'} \\ \mathfrak{U} & \xrightarrow[\zeta_u]{} & \mathfrak{U}' \end{array}$$

(4) *If $0 \to \mathfrak{A}'' \to_{\xi} \mathfrak{A} \to_{\zeta} \mathfrak{A}' \to 0$ is exact and $\mathfrak{U}$, $\mathfrak{U}'$ and ζ_u are as in* (3) *then $\mathfrak{U} \to_{\zeta_u} \mathfrak{U}' \to 0$ is exact and $\ker \zeta_u$ is the ideal in $\mathfrak{U}$ generated by $\mathfrak{A}''^{\xi\lambda_u} \cup \mathfrak{A}''^{\xi\rho_u}$.*

(5) *Suppose the set of identities I satisfies conditions H and L of §1.6. Let $\mathfrak{A} \in \mathscr{V}(I)$ and let Γ be an extension of the base field Φ. Then $(\mathfrak{U}_\Gamma, \lambda_u, \rho_u)$ where*

$\mathfrak{U}_\Gamma = \Gamma \otimes_\Phi \mathfrak{U}$ *and* λ_u, ρ_u *are the* Γ*-linear extensions of* λ_u, ρ_u *to* $\mathfrak{A}_\Gamma$ *is a universal I-multiplication envelope for* $\mathfrak{A}_\Gamma$.

(6) *Suppose the set of identities I satisfies conditions H, L and U of* §1.6. *Then the mappings* λ_u, ρ_u *of* $\mathfrak{A} \in \mathscr{V}(I)$ *into a universal I-multiplication envelope* $\mathfrak{U}$ *are* 1-1.

PROOF. To construct $(\mathfrak{U}, \lambda_u, \rho_u)$ for a given $\mathfrak{A}$ in $\mathscr{V}(I)$ we form $\mathfrak{B} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$ where $\mathfrak{A}_i$ is a vector space isomorphic to $\mathfrak{A}$ under an isomorphism $a \to a_i$, $i = 1,2$. Let $T(\mathfrak{B})$ be the tensor algebra $\Phi 1 \oplus \mathfrak{B} \oplus (\mathfrak{B} \otimes \mathfrak{B}) \oplus \cdots$ based on the vector space $\mathfrak{B}$. Let η be a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ and let η_i be the resultant of η and the mapping $a \to a_i$ regarded as a mapping of $\mathfrak{A}$ into $\mathfrak{B} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$. Then there exists a unique homomorphism $\psi = \psi(\eta)$ of $\Phi\{P\}$ into $\mathfrak{B}$ such that $1 \to 1$ and $l(m)^\psi = m^{\eta_1}$, $r(m)^\psi = m^{\eta_2}$, $m \in N(X)'$. Let $\mathfrak{R}$ be the ideal in $T(\mathfrak{B})$ generated by all F_i^{ψ} where the F_i are the derived elements $(\in \Phi\{P\})$ of the $f \in I$ and the ψ's are all the homomorphisms obtained by varying η. Set $\mathfrak{U} = T(\mathfrak{B})/\mathfrak{R}$ and let λ_u and ρ_u be the linear mappings of $\mathfrak{A}$ into $\mathfrak{U}$ such that $a^{\lambda_u} = a_1 + \mathfrak{R}$, $a^{\rho_u} = a_2 + \mathfrak{R}$. We proceed to show that $(\mathfrak{U}, \lambda_u, \rho_u)$ is a universal I-multiplication envelope for $\mathfrak{A}$. Let η be a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ and let $\tau = \tau(\eta)$ be the homomorphism of $\Phi\{P\}$ into $\mathfrak{U}$ such that $1 \to 1$, $l(m)^\tau = m^{\eta\lambda_u} = m^{\eta_1} + \mathfrak{R}$, $r(m)^\tau = m^{\eta\rho_u} = m^{\eta_2} + \mathfrak{R}$, $m \in N(X)'$. Since $l(m)^\psi = m^{\eta_1}$, $r(m)^\psi = m^{\eta_2}$ we have $l(m)^\tau = l(m)^\psi + \mathfrak{R}$, $r(m)^\tau = r(m)^\psi + \mathfrak{R}$. Since ψ and τ are homomorphisms of $\Phi\{P\}$ into $T(\mathfrak{B})$ and $\mathfrak{U} = T(\mathfrak{B})/\mathfrak{R}$ respectively and 1 and the $l(m)$ and $r(m)$ generate $\Phi\{P\}$, we have $F^\tau = F^\psi + \mathfrak{R}$ for any $F \in \Phi\{P\}$. In particular, if F_i is a derived element of $f \in I$ then $F_i^\tau = F_i^\psi + \mathfrak{R} = 0$, since $F_i^\psi \in \mathfrak{R}$. Thus (λ_u, ρ_u) is an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{U}$. Next let (λ, ρ) be any I-multiplication specialization of $\mathfrak{A}$ in an associative algebra $\mathfrak{G}$ with 1. We have a homomorphism μ' of $T(\mathfrak{B})$ into $\mathfrak{G}$ such that $1 \to 1$, $a_1^{\mu'} = a^\lambda$, $a_2^{\mu'} = a^\rho$, $a \in \mathfrak{A}$. Let η be a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$, τ the corresponding homomorphism of $\Phi\{P\}$ into $\mathfrak{G}$ such that $1 \to 1$, $l(m)^\tau = m^{\eta\lambda}$, $r(m)^\tau = m^{\eta\rho}$. Then $F_i^\tau = 0$ for all i and $f \in I$. We have $l(m)^{\psi\mu'} = m^{\eta_1\mu'} = (m^\eta)_1^{\mu'} = m^{\eta\lambda}$ and $r(m)^{\psi\mu'} = m^{\eta\rho}$. Hence $l(m)^{\psi\mu'} = l(m)^\tau$, $r(m)^{\psi\mu'} = r(m)^\tau$ so $\tau = \psi\mu'$. Then $F_i^\tau = 0$ for the derived elements F_i gives $F_i^{\psi\mu'} = 0$. Thus the ideal $\mathfrak{R}$ generated by the F_i^ψ is mapped into 0 by μ' and hence we have the homomorphism μ of $\mathfrak{U} = T(\mathfrak{B})/\mathfrak{R}$ into $\mathfrak{G}$ such that $1 \to 1$ and $a^{\lambda_u\mu} = (a_1 + \mathfrak{R})^\mu = a_1^{\mu'} = a^\lambda$, $a^{\rho_u\mu} = a^\rho$. Thus $\lambda_u\mu = \lambda$, $\rho_u\mu = \rho$. Since the elements 1, a_1, a_2 for $a \in \mathfrak{A}$ generate $T(\mathfrak{B})$ the elements 1, a^{λ_u}, a^{ρ_u} generate $\mathfrak{U}$. Consequently, the homomorphism μ is unique and we have proved that $(\mathfrak{U}, \lambda_u, \rho_u)$ is a universal I-multuplication envelope for $\mathfrak{A}$.

The properties (1)–(3) are strictly functorial and the proofs are the same as those of (1), (2) and (4) of Theorem 1. The proof of (4) is exactly like that of (5) of Theorem 1 since we have the necessary properties relating I-multiplication specializations and homomorphisms which we noted above. We now consider (5). Suppose the set I satisfies H and L of §1.6. Then we have seen that if $\mathfrak{A} \in \mathscr{V}(I)$

and Γ is an extension of the base field Φ of $\mathfrak{A}$ then $\mathfrak{A}_\Gamma \in \mathscr{V}(I,\Gamma)$ (Corollary 1 to Theorem 1.4). Let (λ,ρ) be an I-multiplication specialization of $\mathfrak{A}$ in $\mathfrak{G}$ and let λ,ρ be the Γ-linear extensions of these mappings to $\mathfrak{A}_\Gamma$ into $\mathfrak{G}_\Gamma$. We claim that (λ,ρ) is an I-multiplication specialization of $\mathfrak{A}_\Gamma$ into $\mathfrak{G}_\Gamma$. In view of the remarks made on I-multiplication specializations and birepresentations and the fact that $\mathrm{Hom}_\Gamma(\mathfrak{M}_\Gamma,\mathfrak{M}_\Gamma)$ can be identified with $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})_\Gamma$ if $\mathfrak{M}$ is a vector space over Φ, our claim will follow if we can show that if $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ then $\mathfrak{M}_\Gamma$ is an I-bimodule for $\mathfrak{A}_\Gamma$ relative to the extensions of the bilinear mappings of $\mathfrak{A}\times\mathfrak{M}$ into $\mathfrak{M}$. Hence let $\mathfrak{M}$ be an I-bimodule for $\mathfrak{A}$. Then the split null extension $\mathfrak{E} = \mathfrak{A}\oplus\mathfrak{M}\in\mathscr{V}(I)$. Then $\mathfrak{E}_\Gamma = \mathfrak{A}_\Gamma\oplus\mathfrak{M}_\Gamma\in\mathscr{V}(I,\Gamma)$ which implies that $\mathfrak{M}_\Gamma$ is an I-bimodule for $\mathfrak{A}_\Gamma$. The proof of Theorem 1 (6) can now be carried over to show that $(\mathfrak{U}_\Gamma,\lambda,\rho)$ is a universal I-multiplication envelope of $\mathfrak{A}_\Gamma$. We now consider (6). If I satisfies H and L then $\mathfrak{A}$ itself is an I-bimodule, the regular I-bimodule for $\mathfrak{A}$. Hence we have the *regular birepresentation* (L,R) where L and R are $a\to a_L$ $a\to a_R$ in $\mathrm{Hom}_\Phi(\mathfrak{A},\mathfrak{A})$. If $\mathfrak{A}$ has an identity element then L and R are 1-1. Hence, λ_u and ρ_u are 1-1. If $\mathfrak{A}$ does not have an identity element then by H,L, U, we can adjoin one to obtain $\mathfrak{A}^* = \Phi 1\oplus\mathfrak{A}$ which is in the class $\mathscr{V}(I)$ (Corollary 2 to Theorem 1.4). Then $a\to a_L$, $a\to a_R$ acting in $\mathfrak{A}^*$ are birepresentations of $\mathfrak{A}$. Since these linear mappings of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{A}^*,\mathfrak{A}^*)$ are 1-1 it follows again that λ_u and ρ_u are 1-1.

Property (3) of the foregoing theorem permits us to define a functor from the category $\mathscr{V}(I)$ of algebras satisfying a set of identities I to the category of associative algebras with 1. If $\mathfrak{A}\in\mathscr{V}(I)$ then we let $U(\mathfrak{A})$ be the associative algebra $\mathfrak{U}$ with 1 given in the construction above for a universal I-multiplication envelope. Then if ζ is a homomorphism of $\mathfrak{A}\in\mathscr{V}(I)$ into $\mathfrak{A}'\in\mathscr{V}(I)$ we have the homomorphism ζ_u of $U(\mathfrak{A})$ into $U(\mathfrak{A}')$ sending 1 into 1. The mappings we have indicated define a functor which we shall call the *universal multiplication functor* from $\mathscr{V}(I)$ to the category of associative algebras with 1.

Let $\mathfrak{A}\in\mathscr{V}(I)$, $(\mathfrak{U},\lambda_u,\rho_u)$ a universal I-multiplication envelope for $\mathfrak{A}$ and let $\mathfrak{M}$ be an I-bimodule for $\mathfrak{A}$. Then we have the homomorphism μ of $\mathfrak{U}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that $a^{\lambda_u\mu}$ is the mapping $u\to au$ and $a^{\rho_u\mu}$ the mapping $u\to ua$ in $\mathfrak{M}$. The homomorphism μ of $\mathfrak{U}$ permits us to endow $\mathfrak{M}$ with a right $\mathfrak{U}$-module structure in which $ub = ub^\mu$, $u\in\mathfrak{M}$, $b\in\mathfrak{U}$. This is a unital right module for $\mathfrak{U}$ since $1^\mu = 1$. Also we have $ua^{\lambda_u} = au$ and $ua^{\rho_u} = ua$. Since 1 and the a^{λ_u} and a^{ρ_u} generate $\mathfrak{U}$ these equations determine the module structure on $\mathfrak{M}$. Conversely, assume $\mathfrak{M}$ is a unital right module for $\mathfrak{U}$. Then we have the homomorphism μ of $\mathfrak{U}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that b^μ is $u\to ub$, $b\in\mathfrak{U}$. Then $(\lambda_u\mu,\rho_u\mu)$ is an I-birepresentation of $\mathfrak{A}$ in $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Consequently, $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ in which $au = ua^{\lambda_u\mu} = ua^{\lambda_u}$ and $ua = ua^{\rho_u\mu} = ua^{\rho_u}$. Thus we see that we have an equivalence between the notions of an I-bimodule for $\mathfrak{A}$ and of a unital right module for the associative algebra $\mathfrak{U}$.

Now assume that I satisfies conditions H and L. Let $\mathfrak{B}\in\mathscr{V}(I)$ and let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$. Then $a \to a_L$, $a \to a_R$, $a \in \mathfrak{A}$, a_L and a_R acting in $\mathfrak{B}$, constitute a birepresentation of $\mathfrak{A}$. Hence we have a homomorphism μ of $U(\mathfrak{A})$ into $\mathrm{Hom}_\Phi(\mathfrak{B},\mathfrak{B})$ such that $1 \to 1$, $a^{\lambda_u} \to a_L$, $a^{\rho_u} \to a_R$. We note also that we can construct an algebra $\mathfrak{B} \in \mathscr{V}(I)$ containing $\mathfrak{A}$ as a subalgebra so that the homomorphism μ is a monomorphism of $U(\mathfrak{A})$. For this purpose we take a 1-1 unital representation of $U(\mathfrak{A})$ in an algebra $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Accordingly, we may consider $\mathfrak{M}$ as a right $U(\mathfrak{A})$-module and as before as an I-bimodule for $\mathfrak{A}$. Then $\mathfrak{A}$ is a subalgebra of the split null extension $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{M}$ and one sees easily that the homomorphism μ of $U(\mathfrak{A})$ into $\mathrm{Hom}_\Phi(\mathfrak{B},\mathfrak{B})$ such that $1 \to 1$, $a^{\lambda_u} \to a_L$ (acting in $\mathfrak{B}$), $a^{\rho_u} \to a_R$ is 1-1.

Exercise

1. Show that the universal multiplication functor on a class $\mathscr{V}(I)$ commutes with direct limits in the sense that the analogue of Theorem 2 holds for multiplication specializations.

8. **Extension of algebras and factor sets.** If $\mathfrak{A}$ and $\mathfrak{M} \in \mathscr{V}(I)$ then we define an *I-extension* of $\mathfrak{A}$ by $\mathfrak{M}$ as a short exact sequence

$$0 \longrightarrow \mathfrak{M} \underset{\alpha}{\longrightarrow} \mathfrak{E} \underset{\beta}{\longrightarrow} \mathfrak{A} \longrightarrow 0 \tag{32}$$

such that $\mathfrak{E} \in \mathscr{V}(I)$. Two I-extensions are called *equivalent* if they can be imbedded in a commutative diagram

$$0 \longrightarrow \mathfrak{M} \begin{array}{c} \overset{\alpha}{\nearrow} \quad \mathfrak{E} \quad \overset{\beta}{\searrow} \\ \gamma \downarrow \\ \underset{\alpha'}{\searrow} \quad \mathfrak{E}' \quad \underset{\beta'}{\nearrow} \end{array} \mathfrak{A} \longrightarrow 0. \tag{33}$$

It follows from this that γ is an isomorphism of $\mathfrak{E}$ onto $\mathfrak{E}'$. The extension (32) is called *inessential* (or *split*) if there exists a homomorphism $\delta: \mathfrak{A} \to \mathfrak{E}$ such that $\delta\beta = 1_{\mathfrak{A}}$ the identity mapping on $\mathfrak{A}$. If this is the case we have the vector space decomposition $\mathfrak{E} = \mathfrak{A}^\delta \oplus \mathfrak{M}^\alpha$ and $\mathfrak{A}^\delta$ is a subalgebra of $\mathfrak{E}$ isomorphic to $\mathfrak{A}$. If $\mathfrak{M}$ is a trivial algebra in the sense that $\mathfrak{M}^2 = 0$ (necessarily contained in $\mathscr{V}(I)$) then the extension of $\mathfrak{A}$ by $\mathfrak{M}$ will be called a *null* extension. We shall now analyze these extensions assuming that the set of identities I satisfies conditions H and L of §1.6.

We note first that (32) implies that α is injective. Hence we may identify $\mathfrak{M}$ and $\mathfrak{M}^\alpha$ and so take α to be the injection of $\mathfrak{M}$ in $\mathfrak{E}$. Since $\mathfrak{M}$ has a complementary vector subspace in $\mathfrak{E}$ there exists a linear mapping δ of $\mathfrak{A}$ into $\mathfrak{E}$ such that $\delta\beta = 1_{\mathfrak{A}}$. Then we have the vector space decomposition $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{A}^\delta$. If $a, b \in \mathfrak{A}$ then

$(a^\delta b^\delta - (ab)^\delta)^\beta = ab - ab = 0$. Since (32) is exact this implies that $a^\delta b^\delta - (ab)^\delta \in \mathfrak{M}$. Hence we have

$$a^\delta b^\delta = (ab)^\delta + h(a,b) \tag{34}$$

where $h(a,b) \in \mathfrak{M}$. It is clear that $(a,b) \to h(a,b)$ is a bilinear mapping of $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{M}$.

We shall show next that $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ relative to the bilinear products

$$au = a^\delta u, \qquad ua = ua^\delta, \qquad a \in \mathfrak{A}, \qquad u \in \mathfrak{M} \tag{35}$$

where the right-hand sides are the products in $\mathfrak{E}$. Since I satisfies H and L, we can consider $\mathfrak{E}$ as I-bimodule for $\mathfrak{E}$ relative to the algebra products. Since $\mathfrak{M}$ is an ideal $\mathfrak{M}$ is a sub-bimodule of $\mathfrak{E}$ and since $\mathfrak{M}^2 = 0$, $\mathfrak{M}$ is an I-bimodule for $\bar{\mathfrak{E}} = \mathfrak{E}/\mathfrak{M}$ relative to $\bar{e}u = eu$, $u\bar{e} = ue$, $e \in \mathfrak{E}$, $\bar{e} = e + \mathfrak{M}$. Since (32) is exact $\bar{e} \to e^\beta$ is an isomorphism of $\bar{\mathfrak{E}} = \mathfrak{E}/\mathfrak{M}$ onto $\mathfrak{A}$. Hence $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ relative to $e^\beta u = eu$, $ue^\beta = ue$. If we take $e = a^\delta$ in this we obtain $au = a^{\delta\beta}u = e^\beta u = eu = a^\delta u$ and $ua = ua^\delta$. Hence $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ relative to (35).

We shall now derive the conditions on $h(a,b)$ which are imposed by the identities in the set I. Let η be a homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$. Then we define a mapping (η,h) of $\Phi\{\{X\}\}'$ into the $\mathfrak{A}$-bimodule $\mathfrak{M}$ as follows: (η,h) is Φ-linear, $x_i^{(\eta,h)} = 0$, $(x_ix_j)^{(\eta,h)} = h(x_i^\eta, x_j^\eta)$, and if M is a monomial of degree $\geqq 3$ and $M = M_1M_2$ is the (unique) factorization of M as a product of two monomials then

$$M^{(\eta,h)} = M_1{}^\eta M_2{}^{(\eta,h)} + M_1{}^{(\eta,h)}M_2{}^\eta + h(M_1{}^\eta, M_2{}^\eta). \tag{36}$$

Let η be the homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$ such that $x_i \to a_i$ and let η' be the homomorphism of $\Phi\{\{X\}\}'$ into $\mathfrak{E}$ such that $x_i \to a_i{}^\delta$. Then we claim that

$$f(a_1{}^\delta, \cdots, a_n{}^\delta) = f(a_1, \cdots, a_n)^\delta + f^{(\eta,h)} \tag{37}$$

for all $f = f(x_1, \cdots, x_n) \in \Phi\{\{X\}\}'$. This is clear if f is of degree 1 since in this case $f^{(\eta,h)} = 0$. Also, it holds if f is a monomial of degree two, since $f^{(\eta,h)} = h(x_i^\eta, x_j^\eta) = h(a_i, a_j)$ and (34) holds. Now assume (37) holds for the monomials M_1 and M_2 and let $M = M_1M_2$. Then

$$\begin{aligned} M(a_1^\delta, \cdots, a_n^\delta) &= M_1(a_1^\delta, \cdots, a_n^\delta)M_2(a_1{}^\delta, \cdots, a_n{}^\delta) \\ &= (M_1(a_1, \cdots, a_n)^\delta + M_1^{(\eta,h)})(M_2(a_1, \cdots, a_n)^\delta + M_2^{(\eta,h)}) \\ &= M(a_1, \cdots, a_n)^\delta + h(M_1(a_1, \cdots, a_n), M_2(a_1, \cdots, a_n)) \\ &\quad + M_1(a_1, \cdots, a_n)M_2{}^{(\eta,h)} + M_1{}^{(\eta,h)}M_2(a_1, \cdots, a_n) \end{aligned}$$

by (34), (35) and $\mathfrak{M}^2 = 0$. Hence, by (36),

$$M(a_1^{\delta}, \cdots, a_n^{\delta}) = M(a_1, \cdots, a_n)^{\delta} + M^{(\eta,h)}$$

which implies (37) for all $f \in \Phi\{\{X\}\}'$.

Now suppose $f \in I$. Then since $\mathfrak{E}$ and $\mathfrak{A} \in \mathscr{V}(I)$, $f(a_1^{\delta}, \cdots, a_n^{\delta}) = 0$ and $f(a_1, \cdots, a_n) = 0$. Hence (37) implies that

$$f^{(\eta,h)} = 0 \tag{38}$$

for every homomorphism η of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$. This leads us to formulate the following

DEFINITION 4. *Let $\mathfrak{A} \in \mathscr{V}(I)$ and let $\mathfrak{M}$ be an I-bimodule for $\mathfrak{A}$. Then a bilinear mapping h of $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{M}$ is called an* I-factor set *for $\mathfrak{A}$ in $\mathfrak{M}$ if* (38) *holds for all $f \in I$ and all homomorphisms η of $\Phi\{\{X\}\}'$ into $\mathfrak{A}$.*

Our result is that we if have an extension (32) of $\mathfrak{A} \in \mathscr{V}(I)$ by $\mathfrak{M}$ such that $\mathfrak{M}^2 = 0$ and α is the injection mapping then if δ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{E}$ such that $\delta\beta = 1_{\mathfrak{A}}$, then $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ via (35) and h defined by $h(a,b) = a^{\delta}b^{\delta} - (ab)^{\delta}$ is an I-factor set for $\mathfrak{A}$ in $\mathfrak{M}$. Conversely, suppose $\mathfrak{A} \in \mathscr{V}(I)$, $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$ and h is an I-factor set of $\mathfrak{A}$ in $\mathfrak{M}$. Let $\mathfrak{E}$ be a vector space containing $\mathfrak{M}$ such that we have a linear isomorphism δ of $\mathfrak{A}$ into $\mathfrak{E}$ such that $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{A}^{\delta}$. Then we define an algebra structure on $\mathfrak{E}$ by defining

$$(a^{\delta} + u)(b^{\delta} + v) = (ab)^{\delta} + h(a,b) + av + ub, \tag{39}$$

$a, b \in \mathfrak{A}$, $u, v \in \mathfrak{M}$. Clearly $\mathfrak{M}$ is an ideal in $\mathfrak{E}$ such that $\mathfrak{M}^2 = 0$. Hence, by (15),

$$f(a_1^{\delta} + u_1, \cdots, a_n^{\delta} + u_n) = f(a_1^{\delta}, \cdots, a_n^{\delta}) + \sum_{i=1}^{n} fD_i(a_1^{\delta}, \cdots, a_n^{\delta}, u_i)$$

for $f \in \Phi\{\{X\}\}'$, $a_i \in \mathfrak{A}$, $u_i \in \mathfrak{M}$. Also (37) holds as before for η the homomorphism such that $x_i^{\eta} = a_i$. Hence

$$f(a_1^{\delta} + u_1, \cdots, a_n^{\delta} + u_n) = f(a_1, \cdots, a_n)^{\delta} + f^{(\eta,h)} + \sum f D_i(a_1^{\delta}, \cdots, a_n^{\delta}, u_i).$$

If $f \in I$, $f(a_1, \cdots, a_n) = 0$ and $\sum fD_i(a_1^{\delta}, \cdots, a_n^{\delta}, u_i) = 0$ since (38) gives $a^{\delta}u = au$, $ua^{\delta} = ua$ and $\mathfrak{M}$ is an I-bimodule for $\mathfrak{A}$. Also $f^{(\eta,h)} = 0$ by hypothesis. Hence $f(a_1^{\delta} + u_1, \cdots, a_n^{\delta} + u_n) = 0$ and $\mathfrak{E} \in \mathscr{V}(I)$.

We now replace δ by another linear mapping δ' of $\mathfrak{A}$ into $\mathfrak{E}$ such that $\delta'\beta = 1_{\mathfrak{A}}$. Such a δ' has the form $\delta + \mu$ where μ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}$. Then $a^{\delta'}u = a^{\delta}u$, $ua^{\delta'} = ua^{\delta}$ so the bimodule structure on $\mathfrak{M}$ is unchanged. The bilinear mapping h is replaced by h' where

$$h'(a,b) = a^{\delta'}b^{\delta'} - (ab)^{\delta'} = (a^\delta + a^\mu)(b^\delta + b^\mu) - (ab)^\delta - (ab)^\mu$$
$$= h(a,b) - (ab)^\mu + ab^\mu + a^\mu b.$$

Accordingly, we call the I-factor sets h and h' of $\mathfrak{A}$ in $\mathfrak{M}$ *equivalent* if there exists a linear mapping μ of $\mathfrak{A}$ into $\mathfrak{M}$ such that

$$h'(a,b) = h(a,b) - (ab)^\mu + ab^\mu + a^\mu b, \tag{40}$$

$a, b \in \mathfrak{A}$. This is an equivalence relation in the set of factor sets of $\mathfrak{A}$ in $\mathfrak{M}$. The set of equivalence classes thus determined is denoted as $H^2(I, \mathfrak{A}, \mathfrak{M})$. It is now easy to complete the proof of the following

THEOREM 12. *If $\mathfrak{A} \in \mathscr{V}(I)$ and $\mathfrak{M}$ is an I-bimodule then we have a bijection of $H^2(I, \mathfrak{A}, \mathfrak{M})$ with the set of equivalence classes of null I-extensions of $\mathfrak{A}$ by $\mathfrak{M}$ such that the associated bimodule structure on $\mathfrak{M}$ (by* (35)) *is the given one. In this correspondence the equivalence class of* 0 *in $H^2(I, \mathfrak{A}, \mathfrak{M})$ corresponds to the isomorphism class of inessential extensions.*

We leave the rest of the details to the reader (cf. Cartan-Eilenberg's *Homological Algebra*, p. 295). We now list the factor set conditions in the cases which we have singled out throughout the discussion.

(1) *Associative algebras.* Let η be a homomorphism of $\Phi\{\{X\}\}'$ such that $x_1{}^\eta = a$, $x_2{}^\eta = b$, $x_3{}^\eta = c$, h a bilinear mapping of $(\mathfrak{A}, \mathfrak{A})$ into $\mathfrak{M}$. Then $(x_1x_2)^{(\eta,h)} = h(a,b)$, $x_3{}^{(\eta,h)} = 0$, $((x_1x_2)x_3)^{(\eta,h)} = h(a,b)c + h(ab,c)$, $x_1{}^{(\eta,h)} = 0$, $(x_2x_3)^{(\eta,h)} = h(b,c)$, $(x_1(x_2x_3))^{(\eta,h)} = ah(b,c) + h(a,bc)$. Hence the factor set condition $f^{(\eta,h)} = 0$ for $f = (x_1x_2)x_3 - x_1(x_2x_3)$ is

$$h(a,b)c + h(ab,c) = ah(b,c) + h(a,bc). \tag{41}$$

(2) *Lie algebras.* Let η be as in (1). Then $x_1{}^{2(\eta,h)} = h(a,a)$, $((x_1x_2)x_3)^{(\eta,h)} = h(a,b)c + h(ab,c)$. Hence the factor set conditions $f^{(\eta,h)} = 0$ for $f = x_1{}^2$ and $g^{(\eta,h)} = 0$ for $g = (x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2$ are

$$\begin{gathered} h(a,a) = 0, \\ h(a,b)c + h(ab,c) + h(b,c)a + h(bc,a) + h(c,a)b + h(ca,b) = 0. \end{gathered} \tag{42}$$

(3) *Alternative algebras.* Let η be as before. Then $(x_1{}^2x_2)^{(\eta,h)} = h(a,a)b + h(a^2,b)$, $(x_1(x_1x_2))^{(\eta,h)} = ah(a,b) + h(a,ab)$, $(x_2x_1{}^2)^{(\eta,h)} = bh(a,a) + h(b,a^2)$, $((x_2x_1)x_1)^{(\eta,h)} = h(b,a)a + h(ba,a)$. Hence the factor set conditions given by $f = x_1{}^2x_2 - x_1(x_1x_2)$ and $g = x_2x_1{}^2 - (x_2x_1)x_1$ are

$$\begin{aligned} h(a,a)b + h(a^2,b) &= ah(a,b) + h(a,ab), \\ bh(a,a) + h(b,a^2) &= h(b,a)a + h(ba,a). \end{aligned} \tag{43}$$

(4) *Jordan algebras.* Let η be as before. Then $(x_1x_2)^{(\eta,h)} = h(a,b)$,

$(x_2x_1)^{(\eta,h)} = h(b, a)$, $((x_1{}^2x_2)x_1)^{(\eta,h)} = (h(a, a)b)a + h(a^2, b)a + h(a^2b, a)$, $(x_1{}^2(x_2x_1))^{(\eta,h)} = a^2h(b,a) + h(a, a)(ba) + h(a^2, ba)$. Hence the conditions given by $f = x_1x_2 - x_2x_1$ and $g = (x_1{}^2x_2)x_1 - x_1{}^2(x_2x_1)$ are

$$\begin{aligned} h(a,b) &= h(b,a), \\ (h(a,a)b)a + h(a^2,b)a + h(a^2b,a) &= a^2h(b,a) + h(a,a)(ba) + h(a^2,ba). \end{aligned} \tag{44}$$

9. **Jordan bimodules and universal multiplication envelopes.** We shall now specialize the theory of bimodules and multiplication specializations to the Jordan case. As before, we denote the Jordan product as $a.b$, $a^{\cdot 2}$, etc. Correspondingly, we denote the products in a bimodule $\mathfrak{M}$ for $\mathfrak{J}$ as $a.u$ and $u.a$, $a \in \mathfrak{J}$, $u \in \mathfrak{M}$. The conditions on these given in (20) are $a.u = u.a$ and

$$\begin{aligned} (u.a).a^{\cdot 2} &= (u.a^{\cdot 2}).a, \\ 2((u.a).b).a + u.(a^{\cdot 2}.b) &= 2(u.a).(a.b) + (u.b).a^{\cdot 2}. \end{aligned} \tag{45}$$

Since $a.u = u.a$ we shall generally drop the composition $a.u$. Similarly, if (λ, ρ) is a multiplication specialization of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1 then $\lambda = \rho$. Hence we may drop λ. The conditions on ρ given by (30) are

$$\begin{aligned} [a^\rho, a^{\cdot 2\rho}] &= 0, \\ 2a^\rho b^\rho a^\rho + (a^{\cdot 2}.b)^\rho &= 2a^\rho(a.b)^\rho + b^\rho(a^{\cdot 2})^\rho. \end{aligned} \tag{46}$$

From now on we shall understand by a multiplication specialization of a Jordan algebra $\mathfrak{J}$ in $\mathfrak{G}$ a linear mapping ρ of $\mathfrak{J}$ into $\mathfrak{G}$ such that (46) holds. Clearly this is equivalent to the earlier definition. We make a corresponding alteration in the definition of a universal multiplication envelope $(U(\mathfrak{J}), \rho_u)$.

We can now give a simpler construction of a universal envelope than the one provided in the proof of Theorem 11. For this we begin with the tensor algebra $T(\mathfrak{J})$ and we let $\mathfrak{R}$ be the ideal in this algebra generated by the elements $a \otimes a^{\cdot 2} - a^{\cdot 2} \otimes a$, $2a \otimes b \otimes a + a^{\cdot 2}.b - 2a \otimes a.b - b \otimes a^{\cdot 2}$, $a, b \in \mathfrak{J}$. Let $U(\mathfrak{J}) = T(\mathfrak{J})/\mathfrak{R}$, $a^{\rho_u} = a + \mathfrak{R}$, $a \in \mathfrak{J}$. Then one sees, as in the proof of Theorem 11, that $(U(\mathfrak{J}), \rho_u)$ is a universal multiplication envelope for $\mathfrak{J}$. We shall take this to be our standard construction and we shall call $(U(\mathfrak{J}), \rho_u)$ *the* universal multiplication envelope for $\mathfrak{J}$. Also we write the associative product in $U(\mathfrak{J})$ as xy. Since the identities defining Jordan algebras satisfy conditions H, L and U, the mapping ρ_u is 1-1. Accordingly, we may identify $a \in \mathfrak{J}$ with its image a^{ρ_u} in $U(\mathfrak{J})$ and $\mathfrak{J}$ with the corresponding subset of $U(\mathfrak{J})$. We shall usually do this. Then the defining property of $U(\mathfrak{J})$ is that if ρ is a multiplication specialization of $\mathfrak{J}$ in $\mathfrak{G}$ then ρ can be extended in a unique way to a homomorphism of $U(\mathfrak{J})$ into $\mathfrak{G}$ sending 1 into 1. Similarly, any bimodule for $\mathfrak{J}$ can be regarded as a unital right module for $U(\mathfrak{J})$ and conversely.

For the subset $\mathfrak{J}$ of $U(\mathfrak{J})$ we have the Jordan product $a \,.\, b \in \mathfrak{J}$ and the associative product $ab \in U(\mathfrak{J})$. In this connection one should note that $a \,.\, b$ is not the same as $\frac{1}{2}(ab + ba)$ as defined in $\mathfrak{J}$. Instead one has the following basic relations connecting these compositions which are obtained from (46):

$$(46') \qquad \begin{aligned} [a, a^{\cdot 2}] &= 0 \\ 2aba + a^{\cdot 2} \,.\, b &= 2a(a \,.\, b) + b(a^{\cdot 2}). \end{aligned}$$

These relations have the following consequences:

$$(47) \qquad \begin{aligned} [a, b \,.\, c] + [b, c \,.\, a] + [c, a \,.\, b] &= 0, \\ abc + cba + (a \,.\, c) \,.\, b &= a(b \,.\, c) + b(a \,.\, c) + c(a \,.\, b). \\ [[ca]b] &= [a, b, c]. \end{aligned}$$

These correspond to the identities $[R_a R_{b.c}] + [R_b R_{c.a}] + [R_c R_{a.b}] = 0$, $R_a R_b R_c + R_c R_b R_a + R_{a.c.b} = R_a R_{b.c} + R_b R_{c.a} + R_c R_{a.b}$, $[[R_c R_a] R_b] = R_{[a,b,c]}$ and can be derived from (46′) by linearization and subtraction. However, it is easier to obtain these relations and others by noting that it is sufficient to prove

$$(47') \qquad \begin{aligned} [a^\rho, (b \,.\, c)^\rho] + [b^\rho, (c \,.\, a)^\rho] + [c^\rho, (a \,.\, b)^\rho] &= 0, \\ a^\rho b^\rho c^\rho + c^\rho b^\rho a^\rho + ((a \,.\, c) \,.\, b)^\rho &= a^\rho (b \,.\, c)^\rho + b^\rho (a \,.\, c)^\rho + c^\rho (a \,.\, b)^\rho, \\ [[c^\rho a^\rho] b^\rho] &= [a, b, c]^\rho \end{aligned}$$

for any birepresentation ρ. This follows from the remarks preceding the proof of Theorem 11. If we form the split null extension $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ where $\mathfrak{M}$ is the bimodule determined by ρ then a^ρ is the restriction to $\mathfrak{M}$ of R_a, $a \in \mathfrak{J}$. Since we have the basic identities: $[R_a, R_{b.c}] + [R_b R_{a.c}] + [R_c R_{a.b}] = 0$, $R_a R_b R_c + R_c R_b R_a + R_{(a.c).b} = R_a R_{b.c} + R_b R_{a.c} + R_c R_{a.b}$, $[[R_c R_a] R_b] = R_{[a,b,c]}$ in $\mathfrak{E}$, it is clear that (47′) and hence (47) hold. In a similar fashion we see that $[a^{\cdot k}, a^{\cdot l}] = 0$ and we have the recursion formula

$$(48) \qquad a^{\cdot k+2} = 2aa^{\cdot k+1} + (a^{\cdot 2} - 2a^2) a^{\cdot k}, \qquad k \geqq 1.$$

In general, any operator identity for Jordan algebras yields a corresponding relation in the universal multiplication envelopes. As another example we note that if we set $u(a) = 2a^2 - a^{\cdot 2}$ then we have

$$(49) \qquad u(a) u(b) u(a) = u(bU_a), \qquad a, b \in \mathfrak{J}$$

in $U(\mathfrak{J})$. This comes from the identity $U_a U_b U_a = U_{bU_a}$ in any Jordan algebra. Similarly, we have $u(a^{\cdot k}) = u(a)^k$, $k = 1, 2, \cdots$.

In addition to the properties (i)–(vi) of Theorem 11, all of which are valid in the Jordan case, we have the following

THEOREM 13. *Let $\mathfrak{J}$ be a Jordan algebra, $\mathfrak{U} = U(\mathfrak{J})$ the universal multiplication envelope for $\mathfrak{J}$ (identified with a subset of $\mathfrak{U}$). Then*: (i) *$\mathfrak{U}$ has a unique involution π such that $a^\pi = a$, $a \in \mathfrak{J}$.* (ii) *Any derivation D in $\mathfrak{J}$ has a unique extension to a derivation D in $\mathfrak{U}$. Moreover, $D\pi = \pi D$.* (iii) *If* $\dim \mathfrak{J}/\Phi = n < \infty$ *then* $\dim \mathfrak{U} \leqq \binom{2n+1}{n}$.

PROOF. (i) Let $\mathfrak{U}^0$ be the opposite algebra of $\mathfrak{U}$. Then the relations (46′) give the following relations in $\mathfrak{U}^0$:

$$[a, a^{\cdot 2}] = 0,$$

$$2aba + a^{\cdot 2}. b = 2(a.b)a + a^{\cdot 2}b.$$

Also, by (47), $2(a.b)a + a^{\cdot 2}b = 2a(a.b) + ba^{\cdot 2}$. Hence we have $2aba + a^{\cdot 2}.b = 2a(a.b) + ba^{\cdot 2}$ in $\mathfrak{U}^0$. Thus the mapping $a \to a$, $a \in \mathfrak{J}$, in $\mathfrak{U}^0$ is a multiplication specialization of $\mathfrak{J}$ in $\mathfrak{U}^0$. Hence we have a homomorphism π of $\mathfrak{U}$ into $\mathfrak{U}^0$ such that $1 \to 1$ and $a \to a$, $a \in \mathfrak{J}$. By definition of $\mathfrak{U}^0$, we have an anti-isomorphism of $\mathfrak{U}^0$ onto $\mathfrak{U}$ such that $1 \to 1$, $a \to a$. Hence we have an anti-homomorphism π of $\mathfrak{U}$ into $\mathfrak{U}$ such that $1 \to 1$, $a \to a$, $a \in \mathfrak{J}$. It follows that $\pi^2 = 1$ and it is clear that π is characterized by the property $a^\pi = a$, $a \in \mathfrak{J}$, since 1 and $\mathfrak{J}$ generate $\mathfrak{U}$.

(ii) As in the proof of Theorem 1 (7), let $\mathfrak{D}$ be the commutative associative algebra over Φ with basis $(1, t)$ such that $t^2 = 0$ and consider $U(\mathfrak{J}) \otimes \mathfrak{D}$. With the usual identifications, the elements of this algebra are uniquely representable in the form $x + yt$, $x, y \in U(\mathfrak{J})$, and $yt = ty$. Let D be a derivation in $\mathfrak{J}$ and define a^ρ for $a \in \mathfrak{J}$ as $a^\rho = a + (aD)t$. Then $(a^{\cdot 2})^\rho = a^{\cdot 2} + (a^{\cdot 2}D)t = a^{\cdot 2} + 2(a.aD)t$ and $[a^\rho, (a^{\cdot 2})^\rho] = ([aD, a^{\cdot 2}] + 2[a, a.aD])t = 0$, by (47). Also $2a^\rho b^\rho a^\rho + (a^{\cdot 2}.b)^\rho - 2a^\rho(a.b)^\rho - b^\rho(a^{\cdot 2})^\rho = (2(aD)ba + 2a(bD)a + 2ab(aD) + (a^{\cdot 2}.b)D - 2aD(a.b) - 2a(a.b)D - (bD)a^{\cdot 2} - b(a^{\cdot 2}D))t = (2(aD)ba + 2a(bD)a + 2ab(aD) - 2aD(a.b) + a^{\cdot 2}.bD + 2a.aD.b - 2a(aD.b) - 2a(a.bD) - (bD)a^{\cdot 2} - 2b(a.aD)t = (2a(bD)a + a^{\cdot 2}.bD - 2a(a.bD) - (bD)a^{\cdot 2})t$ (by (47)) $= 0$, by (46′)). Thus $a \to a^\rho$ is a multiplication specialization of $\mathfrak{J}$ in $U(\mathfrak{J}) \otimes \mathfrak{D}$ so this can be extended to a homomorphism of $U(\mathfrak{J})$ into $U(\mathfrak{J}) \otimes \mathfrak{D}$ sending $1 \to 1$. Since 1 and $\mathfrak{J}$ generate $U(\mathfrak{J})$ it is clear that this homomorphism has the form $x \to x + x't$. Then $x \to x'$ is a derivation D in $U(\mathfrak{J})$ which extends D. Since $1D\pi = 0 = 1\pi D$ and $aD\pi = aD = a\pi D$, $a \in \mathfrak{J}$, it is clear that $D\pi = \pi D$.

(iii) We shall show first that if (u_ι) is a basis for $\mathfrak{J}$ then every element of $\mathfrak{U}$ is a linear combination of the monomials

$$u_{\iota_1}{}^2 u_{\iota_2}{}^2 \cdots u_{\iota_r}{}^2 u_{\iota_{r+1}} u_{\iota_{r+2}} \cdots u_{\iota_s} \tag{50}$$

where the ι_j are distinct and satisfy

$$\iota_1 < \iota_2 < \cdots < \iota_r, \tag{51}$$
$$\iota_{r+1} < \iota_{r+3} < \iota_{r+5} < \cdots,$$
$$\iota_{r+2} < \iota_{r+4} < \iota_{r+6} < \cdots.$$

By (46′) and (47) we have the relations $2aba = 2a(a\,.\,b) + ba^{\cdot 2} - a^{\cdot 2}.\,b$, $abc = -cba + a(b\,.\,c) + b(a\,.\,c) + c(a\,.\,b) - (a\,.\,c).\,b$, $ab^2 = -b^2a + ab^{\cdot 2} + 2b(a\,.\,b) - (a\,.\,b).\,b$ for $a, b, c \in \mathfrak{J}$. Now consider a monomial in the u_ι. The middle relation just noted permits us to move a $c = u_\iota$ in such a monomial two places to the left at the expense of a linear combination of monomials in the u's of lower formal degree. If a monomial contains a u_ι in three places then by moving two of these to the left we can replace the monomial by one of the same formal degree containing a factor $u_\iota u_\kappa u_\iota$ plus a linear combination of monomials in the u_i of lower formal degree. Then the first relation noted above shows that our monomial is a linear combination of monomials in the u_ι of lower formal degree. Hence we may assume the monomial contains a u_ι in at most two positions and if it contains u_ι in two places then these are consecutive. Then the last relation shows that the factor u_ι^2 can be permuted with any u_κ preceding it at the expense of a linear combination of monomials of lower formal degree. We may therefore suppose that the factors u_ι^2 occur at the beginning and that the subscripts for these are in increasing order. Then the remaining u_ι in the monomial can be arranged by our process of moving two places to the left so that the conditions (51) are achieved. Since 1 and the u_ι generate $\mathfrak{U}$ it follows that every element of $\mathfrak{U}$ is a linear combination of 1 and the standard monomials in the u_ι. Now suppose the basis is finite: $(u_1, u_2, \cdots, u_n)$. Then it is clear from (51) that the number of distinct standard monomials together with 1 does not exceed

$$N = \sum_{s=0}^{n} \binom{n}{s} \sum_{r=0}^{s} \binom{s}{r} \binom{s-r}{[\frac{1}{2}(s-r)]} \tag{52}$$

where $[i]$ denotes the largest integer in i. Since

$$\binom{s}{r}\binom{s-r}{[\frac{1}{2}(s-r)]} = \begin{cases} \dfrac{s!}{r!\left(\dfrac{s-r}{2}\right)!\left(\dfrac{s-r}{2}\right)!} & \text{if } s-r \text{ is even,} \\[2ex] \dfrac{s!}{r!\left(\left[\dfrac{s-r}{2}\right]+1\right)!\left[\dfrac{s-r}{2}\right]!} & \text{if } s-r \text{ is odd.} \end{cases}$$

$f(s) \equiv \sum_{r=0}^{s} \binom{s}{r} \cdot \binom{s-r}{[\frac{1}{2}(s-r)]}$ is the sum of the coefficients of the terms in $(x+y+z)^s$ for which the exponents of y and z are equal and those for which the exponent of y exceeds that of z by 1. This is the same as the sum of the constant term and the coefficient of x in $(1 + x + 1/x)^s$. Hence N is the sum of the constant term and the coefficient of x in

$$\sum_{s=0}^{n} \binom{n}{s} \left(1 + x + \frac{1}{x}\right)^s = \left(2 + x + \frac{1}{x}\right)^n.$$

This is the same as the sum of the coefficients of x^n and x^{n+1} in $(x^2 + 2x + 1)^n = (x + 1)^{2n}$. Hence

$$N = \binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+1}{n}$$

and $\dim \mathfrak{U} \leqq N = \binom{2n+1}{n}$.

REMARK. It can be shown that if $\mathfrak{J}$ is trivial, that is, $a \,.\, b = 0$ for all a, b in $\mathfrak{J}$ then the bound given for $\dim \mathfrak{U}$ is attained.

EXERCISES

1. Show that if the characteristic is $\neq 3$ then a linear mapping ρ of a Jordan algebra $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1 is a multiplication specialization if and only if

(i) $[a^\rho, a^{\cdot 2\rho}] = 0$,

(ii) $[[c^\rho a^\rho] b^\rho] = [a, b, c]^\rho$,

(iii) $(a^\rho)^3 = \frac{3}{2} a^\rho (a^{\cdot 2})^\rho - \frac{1}{2}(a^{\cdot 3})^\rho$.

2. Prove that the universal multiplication envelope for a finite-dimensional alternative algebra is finite dimensional.

10. **Associative specializations and multiplicative specializations.** In this section we consider a basic relation between associative specializations and multiplication specializations and its consequences. Let σ_1 and σ_2 be associative specializations of a Jordan algebra $\mathfrak{J}$ in an associative algebra with 1. Then we say that σ_1 and σ_2 *commute* if

$$[a^{\sigma_1}, b^{\sigma_2}] = 0 \tag{53}$$

for all $a, b \in \mathfrak{J}$. We now have

THEOREM 14. *If σ_1 and σ_2 are associative specializations of $\mathfrak{J}$ in $\mathfrak{G}$ which commute then the average $\rho = \frac{1}{2}(\sigma_1 + \sigma_2)$ is a multiplication specialization of $\mathfrak{J}$ in $\mathfrak{G}$.*

PROOF. One can verify this by a direct calculation. However, it is more instructive to establish the result in the following way. We note first since $\mathfrak{G}$ has a 1-1 unital representation it is enough to prove the result for the case $\mathfrak{G} = \mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$, $\mathfrak{M}$ a vector space over Φ. Let $(S(\mathfrak{J}), \sigma_u)$ be the special universal envelope for $\mathfrak{J}$. Since σ_1 is an associative specialization of $\mathfrak{J}$ we have the homomorphism of $\mathfrak{A} = S(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{M}, \mathfrak{M})$ such that $1 \to 1$ and $a^{\sigma_u} \to a^{\sigma_1}$, $a \in \mathfrak{J}$.

Consequently, we can consider $\mathfrak{M}$ as a unital right $\mathfrak{A}$-module in which $ua^{\sigma_u} = ua^{\sigma_1}$. We have the associative specialization σ_2 of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$, hence the homomorphism of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that $1 \to 1$ and $a^{\sigma_u} \to a^{\sigma_2}$. If we precede this with the main involution in $\mathfrak{A}$ we obtain an anti-homomorphism of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that $1 \to 1$, $a^{\sigma_u} \to a^{\sigma_2}$. Consequently, $\mathfrak{M}$ is a unital left module for $\mathfrak{A}$ in which $a^{\sigma_u}u = ua^{\sigma_2}$. We have $(a^{\sigma_u}u)b^{\sigma_u} = a^{\sigma_u}(ub^{\sigma_u})$, $a, b \in \mathfrak{J}$, since σ_1 and σ_2 commute. Since $\mathfrak{J}^{\sigma_u}$ and 1 generate $\mathfrak{A}$ this implies $(cu)d = c(ud)$, $c, d \in \mathfrak{A}$. Thus $\mathfrak{M}$ is an associative bimodule for $\mathfrak{A}$. Hence this gives rise to the split null extension $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ which is an associative algebra. Consider the Jordan algebra $\mathfrak{E}^+$. This contains $\mathfrak{M}$ as ideal and $\mathfrak{A}^+$ as a subalgebra. Also $\mathfrak{J}^{\sigma_u}$ is a subalgebra of $\mathfrak{A}^+$, hence of $\mathfrak{E}^+$. We have the birepresentation $a^{\sigma_u} \to \bar{R}_{a^{\sigma_u}}$ on $\mathfrak{J}^{\sigma_u}$ where $\bar{R}$ denotes the restriction of R to $\mathfrak{M}$. Since $a \to a^{\sigma_u}$ is a homomorphism of $\mathfrak{J}$ as Jordan algebra, $a \to \bar{R}_{a^{\sigma_u}}$ is a birepresentation of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Now $u\bar{R}_{a^{\sigma_u}} = \frac{1}{2}(ua^{\sigma_u} + a^{\sigma_u}u) = \frac{1}{2}u(a^{\sigma_1} + a^{\sigma_2})$. Hence $a \to \frac{1}{2}(a^{\sigma_1} + a^{\sigma_2})$ is a birepresentation of $\mathfrak{J}$.

If we take $\sigma_2 = 0$ in Theorem 14 we obtain the

COROLLARY. *If σ is an associative specialization of a Jordan algebra then $\frac{1}{2}\sigma$ is a multiplication specialization.*

The type of bimodule which we constructed in the foregoing proof is called special. More generally, we shall say that a bimodule $\mathfrak{M}$ for $\mathfrak{J}$ is *special* if there exists a monomorphism of $\mathfrak{M}$ into a bimodule $\mathfrak{N}$ such that if $v \in \mathfrak{N}$, $a \in \mathfrak{J}$, then $v.a = \frac{1}{2}v(a^{\sigma_1} + a^{\sigma_2})$ where σ_1 and σ_2 are commuting associative specializations of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{N},\mathfrak{N})$.

We now consider the tensor product $S(\mathfrak{J}) \otimes S(\mathfrak{J})$ and put

$$a'' = \tfrac{1}{2}(a^{\sigma_u} \otimes 1 + 1 \otimes a^{\sigma_u}), \qquad a \in \mathfrak{J}. \tag{54}$$

Clearly $a \to a^{\sigma_u} \otimes 1$ and $a \to 1 \otimes a^{\sigma_u}$ are associative specializations of $\mathfrak{J}$ in $S(\mathfrak{J}) \otimes S(\mathfrak{J})$. Since $[a^{\sigma_u} \otimes 1, 1 \otimes b^{\sigma_u}] = 0$ these commute. Hence $\rho'' : a \to a''$, which is their average, is a multiplication specialization. Let $S''(\mathfrak{J})$ be the subalgebra of $S(\mathfrak{J}) \otimes S(\mathfrak{J})$ generated by 1 and $\mathfrak{J}'' = \{a'' \mid a \in \mathfrak{J}\}$. We shall call $(S''(\mathfrak{J}), \rho'')$ the *squared special universal envelope* of the Jordan algebra $\mathfrak{J}$. Let $\rho = \frac{1}{2}(\sigma_1 + \sigma_2)$ where σ_1 and σ_2 are commuting associative specializations of $\mathfrak{J}$ in $\mathfrak{G}$. It follows from the basic property of the tensor product of algebras and the fact that we have a homomorphism of $S(\mathfrak{J})$ into $\mathfrak{G}$ mapping $1 \to 1$ and $a^{\sigma_u} \to a^{\sigma_i}$ that we have a homomorphism of $S(\mathfrak{J}) \otimes S(\mathfrak{J})$ into $\mathfrak{G}$ such that $1 \to 1$ and $a^{\sigma_u} \otimes b^{\sigma_u} \to a^{\sigma_1}b^{\sigma_2}$. The restriction of this to $S''(\mathfrak{J})$ maps $a'' = \frac{1}{2}(a^{\sigma_u} \otimes 1 + 1 \otimes a^{\sigma_u})$ into $\frac{1}{2}(a^{\sigma_1} + a^{\sigma_2}) = a^\rho$. Since 1 and $\mathfrak{J}''$ generate $S''(\mathfrak{J})$ it is clear that this homomorphism is unique. This establishes the universal property of $(S''(\mathfrak{J}), \rho'')$ in the class of averages of commuting associative specializations. Also it is clear that if $\mathfrak{M}$ is a special bimodule for $\mathfrak{J}$ and ρ is the corresponding birepresentation then we have a homomorphism of $S''(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that $1 \to 1$ and

$a'' \to a^\rho$, $a \in \mathfrak{J}$. If σ is an associative specialization of $\mathfrak{J}$ in the associative algebra $\mathfrak{G}$ with 1 and b_L and b_R denote the left and right multiplications by $b \in \mathfrak{G}$ and $\mathfrak{G}$ then $a \to a^{\sigma_1} \equiv (a^\sigma)_R$ and $a \to a^{\sigma_2} \equiv (a^\sigma)_L$ are commuting associative specializations of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$. Hence their average is a multiplication specialization and this coincides with the mapping $a \to R_{a^\sigma} = \frac{1}{2}({a^\sigma}_R + {a^\sigma}_L)$ where R_b is the multiplication in $\mathfrak{G}^+$. It follows that we have a homomorphism of $S''(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{G},\mathfrak{G})$ such that $1 \to 1$ and $a'' \to R_{a^\sigma}$. More generally, let σ be a homomorphism of $\mathfrak{J}$ in any special Jordan algebra $\mathfrak{K}$. Then the result just proved implies that we have a homomorphism of $S''(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{K},\mathfrak{K})$ such that $1 \to 1$, $a'' \to R_{a^\sigma}$, $a \in \mathfrak{J}$. If $\mathfrak{J}$ itself is special we have the indicated homomorphism of $S''(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$.

The universal property of $(U(\mathfrak{J}), \rho_u)$ implies that we have an exact sequence

$$U(\mathfrak{J}) \underset{\xi}{\to} S''(\mathfrak{J}) \to 0 \tag{55}$$

where $1^\xi = 1$ and $a^\xi = a''$, $a \in \mathfrak{J}$ ($\subseteq U(\mathfrak{J})$). Of particular interest is the situation in which ξ is an isomorphism. In this case the study of $U(\mathfrak{J})$ is reduced to that of $S''(\mathfrak{J})$ which is determined by $S(\mathfrak{J})$ and the specialization of $\mathfrak{J}$ in $S(\mathfrak{J})$. Since $\mathfrak{J}$ is a subset of $U(\mathfrak{J})$ it is clear that if ξ is an isomorphism then the mapping $a \to a''$, $a \in \mathfrak{J}$, is injective. Since $a'' = \frac{1}{2}(a^{\sigma_u} \otimes 1 + 1 \otimes a^{\sigma_u})$ this implies that σ_u is injective and hence that $\mathfrak{J}$ is special. Accordingly, we shall say that $\mathfrak{J}$ is *strongly special* if ξ is an isomorphism. We shall see later (§7.4 and ex. 3, p. 110) that there exist Jordan algebras which are special but not strongly special. On the positive side we note that if every bimodule for $\mathfrak{J}$ is special then $\mathfrak{J}$ is strongly special. For, let $U(\mathfrak{J})$ be considered as right $U(\mathfrak{J})$-module in the usual way, so that $u \to u_R$ is the corresponding representation. Then $U(\mathfrak{J})$ is a bimodule for $\mathfrak{J}$ with corresponding birepresentation $a \to a_R$. Hence the hypothesis implies that this bimodule is special and consequently we have a homomorphism of $S''(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(U(\mathfrak{J}), U(\mathfrak{J}))$ such that $1 \to 1$, $a'' \to a_R$, $a \in \mathfrak{J}$. Since the homomorphism $u \to u_R$ is a monomorphism we have a homomorphism of $S''(\mathfrak{J})$ into $U(\mathfrak{J})$ such that $1 \to 1$, $a'' \to a$, $a \in \mathfrak{J}$. Clearly, this is the inverse of the mapping ξ so ξ is an isomorphism.

The tensor product $S(\mathfrak{J}) \otimes S(\mathfrak{J})$ has an automorphism ε of period two such that $u \otimes v \to v \otimes u$. We shall call ε the *exchange automorphism* in $S(\mathfrak{J}) \otimes S(\mathfrak{J})$. The subalgebra $\mathfrak{V}$ of ε-fixed points is the set of elements of the form $\Sigma(u_i \otimes v_i + v_i \otimes u_i)$. If $\dim S(\mathfrak{J}) = m$ then $\dim \mathfrak{V} = m(m+1)/2$. The generators 1 and $a'' = a^{\sigma_u} \otimes 1 + 1 \otimes a^{\sigma_u}$, $a \in \mathfrak{J}$, of $S''(\mathfrak{J})$ are evidently contained in $\mathfrak{V}$. Accordingly, we have the exact sequence

$$0 \to S''(\mathfrak{J}) \underset{i}{\to} \mathfrak{V} \tag{56}$$

where i is the injection. Again the desirable situation is that in which i is an isomorphism, that is, $S''(\mathfrak{J})$ coincides with the subalgebra of fixed elements under the exchange automorphism.

11. **Universal multiplication envelopes for Jordan algebras with identity elements.** Let e be an idempotent element in $\mathfrak{J}$ and put $a = b = e$ in the second equation in (46′). This gives $2e^3 + e = 3e^2$ in $U(\mathfrak{J})$. Hence we have

$$e(e-1)(2e-1) = 0. \tag{57}$$

Put $E_0 = (e-1)(2e-1)$, $E_1 = e(2e-1)$, $E_{\frac{1}{2}} = -4e(e-1)$. Then

$$E_0 + E_1 + E_{\frac{1}{2}} = 1 \tag{58}$$

and, by (57), $E_iE_j = 0$ if $i \neq j$. This and (58) imply that the E_i are idempotent elements of $\mathfrak{U} = U(\mathfrak{J})$. Hence these are orthogonal idempotents. Next let a be an element of $\mathfrak{U}$ such that $a.e = a$. Then if we put $b = c = e$ in the first equation in (47) then we obtain $[a,e] = 0$ and hence $[a,E_i] = 0$. We also obtain the same commutativities if $a.e = 0$.

We now suppose that $\mathfrak{J}$ has an identity. In order to distinguish this from the identity element of $\mathfrak{U}$ we denote this as c. Then $c^{\cdot 2} = c$ so the foregoing considerations are applicable. They show that if $C_0 = (c-1)(2c-1)$, $C_1 = c(2c-1)$, $C_{\frac{1}{2}} = -4c(c-1)$ then these are central idempotents of $\mathfrak{U}$ which are orthogonal and have sum 1. Consequently, we have the decomposition

$$\mathfrak{U} = \mathfrak{U}_0 \oplus \mathfrak{U}_{\frac{1}{2}} \oplus \mathfrak{U}_1, \qquad \mathfrak{U}_i = \mathfrak{U}C_i \tag{59}$$

and $\mathfrak{U}C_i$ is an ideal in $\mathfrak{U}$. If $x \in \mathfrak{U}$ we put $x_i = xC_i$. Since $cC_0 = c(c-1)(2c-1) = 0$, $(c-1)C_1 = 0$ and $(c-\frac{1}{2})C_{\frac{1}{2}} = 0$, we have $c_0 = 0$, $c_1 = C_1$ and $c_{\frac{1}{2}} = \frac{1}{2}C_{\frac{1}{2}}$. Since $x \to x_i$ is a homomorphism of $\mathfrak{U}$ it is clear that $a \to a_i$, $a \in \mathfrak{J}$, is a multiplication specialization of $\mathfrak{J}$. If $i = 0$ we obtain a multiplication specialization into $\mathfrak{U}_0$ mapping the identity element c of $\mathfrak{J}$ into 0. If $i = \frac{1}{2}$ the multiplication specialization maps c into $\frac{1}{2}C_{\frac{1}{2}}$ and $C_{\frac{1}{2}}$ is the identity of $\mathfrak{U}_{\frac{1}{2}}$. Finally for $i = 1$ we obtain a *unital* multiplication specialization in the sense that c is mapped into the identity element of $\mathfrak{U}_1$.

Let ρ be a multiplication specialization of $\mathfrak{J}$ into $\mathfrak{G}$ with 1 such that $c^\rho = 0$. Then if we take $b = c$ in the second equation of (47′) we obtain $a^\rho = 0$. The same equation with $c^\rho = \frac{1}{2}$ gives

$$(a.b)^\rho = a^\rho b^\rho + b^\rho a^\rho. \tag{60}$$

In particular, we have $a_0 = 0$, $a \in \mathfrak{J}$ and

$$(a.b)_{\frac{1}{2}} = a_{\frac{1}{2}}b_{\frac{1}{2}} + b_{\frac{1}{2}}a_{\frac{1}{2}}. \tag{61}$$

Now let ρ be any unital multiplication specialization of $\mathfrak{J}$ and consider the homomorphism μ of $\mathfrak{U}$ such that $1 \to 1 = c^\rho$ and $a \to a^\rho$, $a \in \mathfrak{J}$. Since $C_0 = (c-1)(2c-1)$ and $C_{\frac{1}{2}} = -4c(c-1)$ we have $C_0{}^\mu = 0 = C_{\frac{1}{2}}{}^\mu$. Hence $C_1{}^\mu = 1^\mu = 1$ and the restriction μ_1 of μ to $\mathfrak{U}_1$ is a homomorphism sending the identity C_1 of $\mathfrak{U}_1$ into 1. Also $\mathfrak{U}_0{}^\mu = 0$, $\mathfrak{U}_{\frac{1}{2}}{}^\mu = 0$, so $a_1{}^{\mu_1} = a^\mu = a^\rho$. Since

$\mathfrak{J}_1 = \{a_1 \mid a \in \mathfrak{J}\}$ generates $\mathfrak{U}_1$ it is clear that μ_1 is uniquely determined by $a_1^{\mu_1} = a^\rho$. We shall now call $\mathfrak{U}_1 = U_1(\mathfrak{J})$ and the unital multiplication specialization $a \to a_1$ the *universal unital multiplication envelope* of $\mathfrak{J}$ since, as we have just shown, if ρ is any unital multiplication specialization of $\mathfrak{J}$ then there exists a unique homomorphism μ_1 of $\mathfrak{U}_1$ such that $a_1^{\mu_1} = a^\rho$.

We shall show next that $\mathfrak{U}_0 + \mathfrak{U}_{\frac{1}{2}}$ is canonically isomorphic to $S(\mathfrak{J})$ where $(S(\mathfrak{J}), \sigma_u)$ is the special universal envelope of $\mathfrak{J}$. First we recall that $\frac{1}{2}\sigma_u$ is a multiplication specialization of $\mathfrak{J}$. Hence we have the homomorphism μ of $\mathfrak{U}$ into $S(\mathfrak{J})$ such that $1 \to 1$ and $a \to \frac{1}{2}a^{\sigma_u}$. Then $C_1{}^\mu = (c(2c-1))^\mu = \frac{1}{2}c^{\sigma_u}(c^{\sigma_u} - 1) = 0$ since c^{σ_u} is an idempotent element. Hence $\mathfrak{U}_1{}^\mu = 0$ and we have the homomorphism μ' of $\mathfrak{U}_{\frac{1}{2}} + \mathfrak{U}_0$ into $S(\mathfrak{J})$ such that $C_{\frac{1}{2}} + C_0 \to 1$, $a_{\frac{1}{2}} + a_0 \to \frac{1}{2}a^{\sigma_u}$. On the other hand, we have $a_0 = 0$ and $(a\,.\,b)_{\frac{1}{2}} = a_{\frac{1}{2}}b_{\frac{1}{2}} + b_{\frac{1}{2}}a_{\frac{1}{2}}$, $a, b \in \mathfrak{J}$. Hence $a \to 2(a_{\frac{1}{2}} + a_0)$ is an associative specialization of $\mathfrak{J}$ in $\mathfrak{U}_{\frac{1}{2}} + \mathfrak{U}_0$. Consequently, we have the homomorphism η of $S(\mathfrak{J})$ into $\mathfrak{U}_{\frac{1}{2}} + \mathfrak{U}_0$ such that $1 \to C_{\frac{1}{2}} + C_0$, $a^{\sigma_u} \to 2(a_{\frac{1}{2}} + a_0)$. It follows that η and μ' are inverses, so η is an isomorphism of $S(\mathfrak{J})$ onto $\mathfrak{U}_{\frac{1}{2}} + \mathfrak{U}_0$.

We summarize our results in the following

THEOREM 15. *Let $\mathfrak{J}$ be a Jordan algebra with identity element c and let $\mathfrak{U} = U(\mathfrak{J})$ be the universal multiplication envelope of $\mathfrak{J}$ where we consider $\mathfrak{J}$ as imbedded in $\mathfrak{U}$. Put $C_0 = (c-1)(2c-1)$, $C_1 = c(2c-1)$, $C_{\frac{1}{2}} = -4c(c-1)$. Then the C_i are central idempotents in $\mathfrak{U}$ which are orthogonal and have sum 1 and $\mathfrak{U} = \mathfrak{U}_0 \oplus \mathfrak{U}_1 \oplus \mathfrak{U}_{\frac{1}{2}}$ where $\mathfrak{U}_i = \mathfrak{U}C_i$ is an ideal. Moreover, if x_i denotes the component of $x \in \mathfrak{U}$ in $\mathfrak{U}_i$ then $\mathfrak{U}_1$ and $a \to a_1$ is a universal unital multiplication envelope, and $\mathfrak{U}_0 + \mathfrak{U}_{\frac{1}{2}}$ and $a \to 2(a_0 + a_{\frac{1}{2}})$ is a special universal envelope for $\mathfrak{J}$.*

It is clear also, by referring to §2, that $\mathfrak{U}_{\frac{1}{2}}$ and the mapping $a \to 2a_{\frac{1}{2}}$ is a unital special universal envelope for $\mathfrak{J}$. The argument used to establish the dimensionality inequality in Theorem 13 shows that $\dim \mathfrak{U}_1 \leqq \binom{2n-1}{n}$ if $\dim \mathfrak{J} = n < \infty$. We have also seen that $\dim S_1(\mathfrak{J}) = \dim \mathfrak{U}_{\frac{1}{2}} \leqq 2^{n-1}$ and $\dim \mathfrak{U}_0 = 1$. Hence we have the following improved inequality for the dimensionality of the universal multiplication envelope of a Jordan algebra with identity element:

$$\dim \mathfrak{U} \leqq \binom{2n-1}{n} + 2^{n-1} + 1. \tag{62}$$

Let $(S_1(\mathfrak{J}), \sigma_u)$ be the unital special universal envelope of $\mathfrak{J}$ and set $a_1'' = \frac{1}{2}(a^{\sigma_u} \otimes 1 + 1 \otimes a^{\sigma_u})$ an element of $S_1(\mathfrak{J}) \otimes S_1(\mathfrak{J})$. Let $S_1(\mathfrak{J})$ be the subalgebra genrated by the a_1'', $a \in \mathfrak{J}$. Then $a \to a_1''$ is a unital multiplication specialization in $S_1''(\mathfrak{J})$. Hence we have the exact sequence

$$\mathfrak{U}_1 \underset{\xi_1}{\to} S_1''(\mathfrak{J}) \to 0.$$

where ξ_1 is the homomorphism such that $a_1 \to a_1''$. If we use the decompositions $\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_{\frac{1}{2}} \oplus \mathfrak{U}_0$ and $S(\mathfrak{J}) = S_1(\mathfrak{J}) \oplus \Phi z$ we see easily that ξ_1 is an isomorphism if and only if the homomorphism ξ of (55) is an isomorphism. In other words, an algebra $\mathfrak{J}$ with identity element is strongly special if and only if ξ_1 is an isomorphism. If $\rho = \frac{1}{2}(\sigma_1 + \sigma_2)$ where the σ_i are commuting unital associative specializations then we have a homomorphism of $S_1''(\mathfrak{J})$ such that $a_1'' \rightarrow a^\rho$, $a \in \mathfrak{J}$. We shall call $(S_1''(\mathfrak{J}), \rho'')$ where ρ'' is the mapping $a \rightarrow a_1''$ the *unital squared special universal envelope* of $\mathfrak{J}$. We have called a bimodule $\mathfrak{M}$ for $\mathfrak{J}$ special if it is isomorphic to a sub-bimodule of a bimodule $\mathfrak{N}$ in which $v \,.\, a = \frac{1}{2}v(a^{\sigma_1} + a^{\sigma_2})$ where the σ_i are commuting associative specializations in $\mathrm{Hom}_\Phi(\mathfrak{N}, \mathfrak{N})$. We call $\mathfrak{M}$ *unital special* if the σ_i are unital. We have seen that if every bimodule of a Jordan algebra is special then $\mathfrak{J}$ is strongly special (p. 101). The same argument shows that a Jordan algebra $\mathfrak{J}$ with identity element is strongly special if every unital bimodule for $\mathfrak{J}$ is unital special.

Let e and f be orthogonal idempotent elements of $\mathfrak{J}$. As before, we set $u(a) = 2a^2 - a^{\cdot 2}$, $a \in \mathfrak{J}$, which corresponds to the operator U_a. We have noted that $u(aU_b) = u(b)u(a)u(b)$ and $u(a^{\cdot k}) = u(a)^k$. Also we have $u(a + b) - u(a) - u(b) = 2u(a, b)$ where $u(a, b) = ab + ba - a \,.\, b = u(b, a)$. Now $e^{\cdot 2} = e$, $f^{\cdot 2} = f$ imply that $u(e)$ and $u(f)$ are idempotent elements of $\mathfrak{U} = U(\mathfrak{J})$. Also $e \,.\, f = 0$ implies that e and f commute and since $u(e) = 2e^2 - e$ and $u(f) = 2f^2 - f$, it is clear that $u(e)$ and $u(f)$ commute. Since $\{efe\} = fU_e = 0$ we have $u(e)u(f)u(e) = 0$. Hence, by the commutativity of $u(e)$ and $u(f)$ we have $u(e)u(f) = 0 = u(f)u(e)$. Now $g = e + f$ is an idempotent and $g \,.\, e = e = \{geg\}$, $g \,.\, f = f = \{gfg\}$. Hence $u(g)$ is an idempotent which commutes with $u(e)$ and $u(f)$ and $u(g)u(e)u(g) = u(e)$, $u(g)u(f)u(g) = u(f)$ which give $u(g)u(e) = u(e) = u(e)u(g)$, $u(g)u(f) = u(f) = u(f)u(g)$. Now $u(g) = u(e) + u(f) + 2u(e,f)$ and $u(e,f) = ef + fe - e \,.\, f = 2ef$. Hence $u(e) = u(g)u(e) = u(e) + 4efu(e)$. Hence $4efu(e) = 0$. Similarly, $u(e)(4ef) = 0$, $4efu(f) = 0 = u(f)(4ef)$. Then $u(g)^2 = u(g)$ gives $(4ef)^2 = 4ef$ so the three elements $u(e)$, $u(f)$, $4ef$ are orthogonal idempotents in $\mathfrak{U}$.

Suppose again that $\mathfrak{J}$ has identity element c and consider $\mathfrak{U}_1 = \mathfrak{U}C_1$. To avoid confusion with other indices we avoid using our earlier notation x_1 for the component $xC_1 = C_1x$ in $\mathfrak{U}_1$ of $x \in \mathfrak{U}$. Here, as above, $C_1 = 2c^2 - c$ is the identity of $\mathfrak{U}_1$. Let $e_1, e_2, \cdots, e_n$ be orthogonal idempotents in $\mathfrak{J}$ such that $\sum e_i = c$. Since the elements $u(e_i)$, $4e_ie_j$, $i < j$, are orthogonal idempotents in $\mathfrak{U}$ it is clear that the $n(n + 1)/2$ elements $E_{ii} = u(e_i)C_1$, $E_{ij} = 4e_ie_jC_1 = 2u(e_i, e_j)C_1$, $i \neq j$, are orthogonal idempotents in $\mathfrak{U}_1$. We note also that since $u(a, b) = ab + ba - ab$ is linear in a and b and $u(a + b) = u(a) + u(b) + 2u(a, b)$ we can prove by induction on n that $u(\sum a_i) = \sum u(a_i) + 2 \sum_{i<j} u(a_i, a_j)$. In particular, if $\sum e_i = c$ then $u(c) = \sum u(e_i) + 2 \sum_{i<j} u(e_i, e_j)$. Also, since $cC_1 = C_1$, $u(c)C_1 = 2(cC_1)^2 - cC_1 = C_1$ the identity of $\mathfrak{U}_1$. Hence

$$C_1 = \sum_{i \leqq j} E_{ij} \tag{64}$$

and the E_{ij} are orthogonal idempotents in $\mathfrak{U}_1$.

We now suppose that $\mathfrak{J}$ with identity element c is a direct sum of two ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Write $c = 1_1 + 1_2$, $1_i \in \mathfrak{J}_i$. Then 1_i is the identity of $\mathfrak{J}_i$ and $1_i . a_j = 0$ if $a_j \in \mathfrak{J}_j$ and $i \neq j$. Hence $[1_i, a_i] = 0$ and $[1_i, a_j] = 0$ which implies that 1_i is in the center of $\mathfrak{U}$. It follows that the elements $u(1_i)$, $i = 1,2$, and $2u(1_1, 1_2)$ are central idempotents of $\mathfrak{U}$ and $C_{11} = u(1_1)C_1$, $C_{22} = u(1_2)C_1$, $C_{12} = 2u(1_1, 1_2)C_1 = C_{21}$ are central idempotents of $\mathfrak{U}_1$. Hence $\mathfrak{U}_1 = \mathfrak{U}_{11} \oplus \mathfrak{U}_{22} \oplus \mathfrak{U}_{12}$ where $\mathfrak{U}_{ij} = \mathfrak{U}_1 C_{ij}$ is an ideal in $\mathfrak{U}_1$. We shall now prove

THEOREM 16. *Let $\mathfrak{J}$ be a Jordan algebra with identity element c and suppose $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ where the $\mathfrak{J}_i$ are ideals. Let $\mathfrak{U}_1$ be the unital universal multiplication envelope*[2] *of $\mathfrak{J}$. Then $\mathfrak{U}_1$ is a direct sum of three ideals isomorphic respectively to the unital universal multiplication envelopes of the $\mathfrak{J}_i$, $i = 1,2$, and $S_1(\mathfrak{J}_1) \otimes S_1(\mathfrak{J}_2)$ where $S_1(\mathfrak{J}_i)$ is the unital special universal envelope for $\mathfrak{J}_i$.*

PROOF. We have $\mathfrak{U}_1 = \mathfrak{U}_{11} \oplus \mathfrak{U}_{22} \oplus \mathfrak{U}_{12}$ as above. Let $a_{ij} = aC_1C_{ij}$ be the component in $\mathfrak{U}_{ij}$ of aC_1, $a \in \mathfrak{J}$. Then $a \to a_{ij}$ is a unital multiplication specialization of $\mathfrak{J}$. We have $(1_1)_{11} = 1_1 C_{11} = 1_1(2(1_1)^2 - 1_1)C_1 = (2(1_1)^2 - 1_1)C_1 = C_{11}$ since 1_1 is an idempotent in $\mathfrak{J}$ so $2(1_1)^3 - 3(1_1)^2 + 1_1 = 0$ in $\mathfrak{U}$. It follows that the restriction of $a \to a_{11}$ to $\mathfrak{J}_1$ is a unital multiplication specialization of $\mathfrak{J}_1$. Hence we have the homomorphism of the unital universal envelope $U_1(\mathfrak{J}_1)$ of $\mathfrak{J}_1$ into $\mathfrak{U}_{11}$ such that $(a_1)_1 \to (a_1)_{11} = a_1 C_1 C_{11}$ where $a_1 \in \mathfrak{J}_1$ and $a_1 \to (a_1)_1$ is the multiplication specialization of $\mathfrak{J}_1$ in $U_1(\mathfrak{J}_1)$. We note next that we have the unital multiplication specialization $a = a_1 + a_2 \to (a_1)_1$, hence the homomorphism η of the unital universal envelope $U_1(\mathfrak{J})$ into $U_1(\mathfrak{J}_1)$ such that $aC_1 \to (a_1)_1$. Clearly, $(1_2C_1)^\eta = 0$ since the component of 1_2 in $\mathfrak{J}_1$ is 0. Hence $C_{22}{}^\eta = (2(1_2C_1)^2 - (1_2C_1))^\eta = 0$ and $C_{12}{}^\eta = 4((1_2C_1)(1_1C_1))^\eta = 0$. Then $\mathfrak{U}_{22}{}^\eta = 0 = \mathfrak{U}_{12}{}^\eta$ and we have the homomorphism of $\mathfrak{U}_{11}$ into $U_1(\mathfrak{J}_1)$ such that $(a_1)_{11} \to (a_1)_1$. It follows that $(a_1)_1 \to (a_1)_{11}$ is an isomorphism of $U_1(\mathfrak{J}_1)$ onto $\mathfrak{U}_{11}$. Similarly, $U_1(\mathfrak{J}_2)$ and $\mathfrak{U}_{22}$ are isomorphic. We now note that $(1_1)_{12} = 1_1C_1C_{12} = 1_1C_1(4(1_1 1_2)C_1) = 4(1_1C_1)^2(1_2C_1) = 4(1_1C_1)^2(C_1 - 1_1C_1) = 4(1_1C_1)^2 - 4(1_1C_1)^3 = 4(1_1C_1)^2 - 6(1_1C_1)^2 + 2(1_1C_1) = -2(1_1C_1)^2 + 2(1_1C_1) = 2(1_1C_1)(C_1 - (1_1C_1)) = 2(1_1C_1)(1_2C_1) = \frac{1}{2}C_{12}$. Similarly, $(1_2)_{12} = \frac{1}{2}C_{12}$. As we have seen before (p. 103), this implies that $a_i \to 2(a_i)_{12}$ is a unital associative specialization of $\mathfrak{J}_i$ in $\mathfrak{U}_{12}$ so we have the homomorphism of $S_1(\mathfrak{J}_i)$ into $\mathfrak{U}_{12}$ such that $a_i^{\sigma_u} \to 2(a_i)_{12}$, $a_i \in \mathfrak{J}_i$. Since $[a_1 . 1_1, a_2] + [1_1 . a_2, a_1] + [a_1 . a_2, 1_1] = 0$ in $\mathfrak{U}_1$, $[a_1, a_2] = 0$ and $[(a_1)_{12}, (a_2)_{12}] = 0$ in $\mathfrak{U}_1$. It follows that we have the homomorphism of $S_1(\mathfrak{J}_1) \otimes S_1(\mathfrak{J}_2)$ into $\mathfrak{U}_{12}$ such that $a_1{}^{\sigma_u} \otimes a_2{}^{\sigma_u} \to 4(a_1)_{12}(a_2)_{12}$. We note next that we have the associative specializations $a_1 + a_2 \to a_1{}^{\sigma_u} \otimes 1$, $a_1 + a_2 \to 1 \otimes a_2{}^{\sigma_u}$ which give rise to the unital multiplication specialization $a_1 + a_2 \to \frac{1}{2}(a_1{}^{\sigma_u} \otimes 1 + 1 \otimes a_2{}^{\sigma_u})$ of $\mathfrak{J}$ in $S_1(\mathfrak{J}_1) \otimes S_1(\mathfrak{J}_2)$. From this we obtain

[2] Of course, strictly speaking $\mathfrak{U}_1$ and $a \to aC_1$ is this envelope. However, it is convenient to refer to $\mathfrak{U}_1$ also as the envelope and to use this terminology for the algebra parts of the other envelopes.

the homomorphism λ of $\mathfrak{U}_1$ into $S_1(\mathfrak{J}_1)\otimes S_1(\mathfrak{J}_2)$ such that $(a_1+a_2)C_1 \to \frac{1}{2}(a_1{}^{\sigma_u}\otimes 1 + 1\otimes a_2{}^{\sigma_u})$. Now $C_{11}{}^{\lambda} = (21_1{}^2 - 1_1)^{\lambda}C_1{}^{\lambda} = 2(\frac{1}{4}(1_1\otimes 1)) - \frac{1}{2}(1_1\otimes 1) = 0$ and similarly $C_{22}{}^{\lambda} = 0$. Hence we have the homomorphism of $\mathfrak{U}_{12}$ into $S_1(\mathfrak{J}_1)\otimes S_1(\mathfrak{J}_2)$ such that $aC_1C_{12} \to \frac{1}{2}(a_1{}^{\sigma_u}\otimes 1 + 1\otimes a_2{}^{\sigma_u})$ for $a = a_1 + a_2$, $a_i \in \mathfrak{J}_i$. Then $(a_1)_{12} \to \frac{1}{2}(a_1{}^{\sigma_u}\otimes 1)$ and $(a_2)_{12} \to \frac{1}{2}(1\otimes a_2{}^{\sigma_u})$ so $(a_1)_{12}(a_2)_{12} \to \frac{1}{4}(a_1{}^{\sigma_u}\otimes a_2{}^{\sigma_u})$ in this homomorphism. It follows that this is the inverse of the homomorphism we had before of $S_1(\mathfrak{J}_1)\otimes S_1(\mathfrak{J}_2)$ into $\mathfrak{U}_{12}$. Hence these are isomorphic and the proof is complete.

EXERCISES

1. Let $\mathfrak{J}$ be a Jordan algebra with identity element 1, c an invertible element in $\mathfrak{J}$, $U_1(\mathfrak{J})$ and $a \to a_1$ a unital universal multiplication envelope for $\mathfrak{J}$. Show that $U_1(\mathfrak{J})$ and the mapping

$$a \to a_1{}^{(c)} = [c_1, a_1] + (c\,.\,a)_1$$

constitute a unital universal multiplication envelope for the isotope $(\mathfrak{J}, c)$.

2. Let $\mathfrak{M}$ be a bimodule for a Jordan algebra with identity element and define $\mathfrak{M}_i$ for $i = 0, 1, \frac{1}{2}$ by $\mathfrak{M}_i = \{x_i \mid x_i\,.\,c = ix_i\}$. Prove that the $\mathfrak{M}_i$ are sub-bimodules, that $x_0\,.\,a = 0$ for every $x_0 \in \mathfrak{M}_0$, $a \in \mathfrak{J}$, $x_{\frac{1}{2}}u(a) = 0$ for every $x_{\frac{1}{2}} \in \mathfrak{M}_{\frac{1}{2}}$, $a \in \mathfrak{J}$.

3. Define unital multiplication specializations and unital universal envelopes for associative algebras with 1 in the obvious way. Show that if $\mathfrak{A}$ is an associative algebra with 1, $\mathfrak{A}^0$ the opposite algebra then $\mathfrak{A}^e \equiv \mathfrak{A}^0 \otimes \mathfrak{A}$ and the mappings $a \to a^{\lambda} = a\otimes 1$, $a \to a^{\rho} = 1\otimes a$ constitute a unital universal multiplication envelope for $\mathfrak{A}$.

12. **Some basic criteria.** In this section we shall derive some useful conditions that a Jordan bimodule be special and that the homomorphisms ξ, i of (55) and (56) be isomorphisms. We prove first

THEOREM 17. *Let $\mathfrak{J}$ be a special Jordan algebra, $\mathfrak{M}$ a bimodule for $\mathfrak{J}$. Then $\mathfrak{M}$ is special if and only if the split null extension $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ is a special Jordan algebra.*

PROOF. Suppose first that $\mathfrak{M}$ is special. Then we have an isomorphism μ of $\mathfrak{M}$ into a bimodule $\mathfrak{N}$ for $\mathfrak{J}$ such that $v\,.\,a = \frac{1}{2}(va^{\sigma_1} + va^{\sigma_2})$ where σ_1 and σ_2 are commuting associative specializations of $\mathfrak{J}$ in $\mathrm{Hom}_{\Phi}(\mathfrak{N}, \mathfrak{N})$. We showed in the proof of Theorem 14 that $\mathfrak{N}$ is a unital associative bimodule for $S(\mathfrak{J})$ such that $va^{\sigma_u} = va^{\sigma_1}$, $a^{\sigma_u}v = va^{\sigma_2}$, $a \in \mathfrak{J}$. Consider the mapping $a + u \to a^{\sigma_u} + u^{\mu}$, $a \in \mathfrak{J}$, $u \in \mathfrak{M}$ of $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ into $\mathfrak{F} = S(\mathfrak{J}) \oplus \mathfrak{N}$. Since $\mathfrak{J}$ is special, σ_u is 1-1. Hence $a + u \to a^{\sigma_u} + u^{\mu}$ is 1-1. A direct verification shows that this mapping is a homomorphism of $\mathfrak{E}$ into $\mathfrak{F}^+$. Hence it is an isomorphism and $\mathfrak{E}$ is special. Conversely, assume $\mathfrak{E}$ is special. Then we have an isomorphism σ of $\mathfrak{E}$ into $\mathfrak{G}^+$ where $\mathfrak{G}$ is associative. If L and R denote the left and right multiplications in $\mathfrak{G}$ then we have the associative specializations $a \to a^{\sigma_1} = (a^\sigma)_L$, $a \to a^{\sigma_2} = (a^\sigma)_R$ of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{G},\mathfrak{G})$. These commute and their average is $a \to R_{a^\sigma}$ where $R_c = \frac{1}{2}(c_L + c_R)$ is the (right) multiplication in the Jordan algebra $\mathfrak{G}^+$. Hence $\mathfrak{G}$ is a bimodule for $\mathfrak{J}$ so that $v.a \equiv vR_{a^\sigma} = \frac{1}{2}(va^{\sigma_1} + va^{\sigma_2})$, $v \in \mathfrak{G}$, $a \in \mathfrak{J}$. Now consider the restriction of σ to $\mathfrak{M}$. If $u \in \mathfrak{M}$, $a \in \mathfrak{J}$, we have $(u.\,a)^\sigma = \frac{1}{2}(u^\sigma a^\sigma + a^\sigma u^\sigma)$ $= \frac{1}{2}(u^\sigma a^{\sigma_1} + u^\sigma a^{\sigma_2}) = u^\sigma.a$. Hence the restriction of σ to $\mathfrak{M}$ is an isomorphism of this $\mathfrak{J}$-bimodule into $\mathfrak{G}$ as $\mathfrak{J}$-bimodule and so $\mathfrak{M}$ is special.

Theorem 17 is valid also for Jordan algebras with 1 and unital bimodules. In this case the statement is that a unital bimodule $\mathfrak{M}$ for $\mathfrak{J}$ is special if and only if the split null extension $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ is a special Jordan algebra. The proof is the same as that of Theorem 17. We shall see in Chapter III (Corollary to the Coordinatization Theorem, p. 137) that there exist important Jordan algebras $\mathfrak{J}$ with 1 such that every Jordan algebra $\mathfrak{K}$ with 1 containing $\mathfrak{J}$ as a subalgebra with the same identity element is special. For these we have the following

COROLLARY. *Let $\mathfrak{J}$ be a Jordan algebra with* 1 *such that every Jordan algebra $\mathfrak{K}$ with* 1 *having $\mathfrak{J}$ as subalgebra with the same identity element is special. Then $\mathfrak{J}$ is strongly special.*

PROOF. If $\mathfrak{M}$ is a unital bimodule for $\mathfrak{J}$ then the hypothesis implies that the split null extension $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ is special. Hence $\mathfrak{M}$ is unital special by Theorem 17. Now we have seen that if every unital bimodule for an algebra is unital special then $\mathfrak{J}$ is strongly special. Thus the conclusion holds.

We shall derive next a condition that the fact that the canonical homomorphism ξ of $U(\mathfrak{J})$ onto $S''(\mathfrak{J})$ is an isomorphism carries over to a homomorphic image $\bar{\mathfrak{J}}$ of $\mathfrak{J}$. Thus suppose we have the exact sequence

$$\mathfrak{J} \underset{\zeta}{\to} \bar{\mathfrak{J}} \to 0. \tag{65}$$

Then we have the exact sequences

$$U(\mathfrak{J}) \underset{\zeta_u}{\to} U(\bar{\mathfrak{J}}) \to 0, \qquad S(\mathfrak{J}) \underset{\zeta_s}{\to} S(\bar{\mathfrak{J}}) \to 0. \tag{66}$$

The homomorphism ζ_s gives rise to the homomorphism $\zeta_s \otimes \zeta_s$ of $S(\mathfrak{J}) \otimes S(\mathfrak{J})$ into $S(\bar{\mathfrak{J}}) \otimes S(\bar{\mathfrak{J}})$ and, by restriction, we obtain the homomorphism ζ'' of $S''(\mathfrak{J})$ into $S''(\bar{\mathfrak{J}})$. Then

$$S''(\mathfrak{J}) \underset{\zeta''}{\to} S''(\bar{\mathfrak{J}}) \to 0 \tag{67}$$

is exact. Also we have the homomorphism ξ of $U(\mathfrak{J})$ onto $S''(\mathfrak{J})$ and a similar homomorphism $\bar{\xi}$ of $U(\bar{\mathfrak{J}})$ onto $S''(\bar{\mathfrak{J}})$. It follows directly from the definitions that

$$(68)\qquad \begin{array}{ccc} U(\mathfrak{J}) & \xrightarrow{\zeta_u} & U(\bar{\mathfrak{J}}) \\ {\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\xi}} \\ S''(\mathfrak{J}) & \xrightarrow[\zeta'']{} & S''(\bar{\mathfrak{J}}) \end{array}$$

is commutative. It is well known that the kernel $\ker(\zeta_s \otimes \zeta_s) = S(\mathfrak{J}) \otimes \ker\zeta_s + \ker\zeta_s \otimes S(\mathfrak{J})$. Hence

$$(69)\qquad \ker\zeta'' = S''(\mathfrak{J}) \cap (S(\mathfrak{J}) \otimes \ker\zeta_s + \ker\zeta_s \otimes S(\mathfrak{J}))\,.$$

Let $b \in \ker\zeta$. Since $\mathfrak{J} \subseteq U(\mathfrak{J})$ and the given multiplication specialization of $\mathfrak{J}$ in $U(\mathfrak{J})$ and of $\bar{\mathfrak{J}}$ in $U(\bar{\mathfrak{J}})$ are the injection mappings, we have $b^{\zeta_u} = 0$. Hence, by (68), $b''^{\zeta''} = b^{\xi\zeta''} = 0$. Thus $b'' \in \ker\zeta''$ and so the ideal $\mathfrak{K}''$ in $S''(\mathfrak{J})$ generated by $(\ker\zeta)'' = \{b'' \mid b \in \ker\zeta\}$ satisfies

$$(70)\qquad \mathfrak{K}'' \subseteq \ker\zeta''$$

We can now prove

THEOREM 18. *Let $\mathfrak{J}$ be a Jordan algebra, ζ a homomorphism of $\mathfrak{J}$ onto $\bar{\mathfrak{J}}$. Suppose the canonical homomorphism of $U(\mathfrak{J})$ onto $S''(\mathfrak{J})$ is an isomorphism. Then the same thing is true of the canonical homomorphism $\bar{\xi}$ of $U(\bar{\mathfrak{J}})$ onto $S''(\bar{\mathfrak{J}})$ if and only if the ideal $\mathfrak{K}''$ in $S''(\mathfrak{J})$ generated by* $(\ker\zeta)''$ *coincides with* $\ker\zeta''$ *where ζ'' is the homomorphism of $S''(\mathfrak{J})$ onto $S''(\bar{\mathfrak{J}})$ induced by ζ.*

PROOF. Assume first that $\mathfrak{K}'' = \ker\zeta''$ and let $\bar{v} \in \ker\bar{\xi}$. Let v be an element of $U(\mathfrak{J})$ such that $\bar{v} = v^{\zeta_u}$. Then $v^{\zeta_u\bar{\xi}} = 0$ and the commutativity of (68) gives $v^{\xi\zeta''} = 0$ so $v^{\xi} \in \ker\zeta''$. Hence $v^{\xi} \in \mathfrak{K}''$. Since ξ is an isomorphism of $U(\mathfrak{J})$ onto $S''(\mathfrak{J})$ it follows that v is in the ideal in $U(\mathfrak{J})$ generated by $\ker\zeta$. By Theorem 11(4), we know that $\ker\zeta_u$ is the ideal in $U(\mathfrak{J})$ generated by $\ker\zeta$. Hence $v \in \ker\zeta_u$ so $\bar{v} = v^{\zeta_u} = 0$. This shows that $\bar{\xi}$ is an isomorphism. In the same way we see that if $\mathfrak{K}'' \neq \ker\zeta''$ then $\mathfrak{K}'' \subset \ker\zeta''$, by (70). Hence we can choose a $v'' \in \ker\zeta''$, $\notin \mathfrak{K}''$. Then $v'' = v^{\xi}$ for a $v \in U(\mathfrak{J})$. Since $v^{\xi} \notin \mathfrak{K}''$ and ξ is an isomorphism, v is not contained in the ideal in $U(\mathfrak{J})$ generated by $\ker\zeta$. Hence $v \notin \ker\zeta_u$. Thus $v^{\zeta_u} \neq 0$ while $v^{\zeta_u\bar{\xi}} = v^{\xi\zeta''} = v''^{\zeta''} = 0$. Hence $\bar{\xi}$ is not an isomorphism.

We remark that in view of (69) and (70) the condition $\mathfrak{K}'' = \ker\zeta''$ of the theorem is equivalent to

$$(71)\qquad \mathfrak{K}'' \supseteq S''(\mathfrak{J}) \cap (S(\mathfrak{J}) \otimes \ker\zeta_s + \ker\zeta_s \otimes S(\mathfrak{J}))\,.$$

We remark also that Theorem 18 and its proof carry over word for word to the unital envelopes $U_1(\mathfrak{J})$ and $S_1''(\mathfrak{J})$ if $\mathfrak{J}$ is a Jordan algebra with identity element.

We obtain next two useful sufficient conditions for a Jordan algebra with 1 that $S_1''(\mathfrak{J})$ coincides with the subalgebra $\mathfrak{B}_1$ of $S_1(\mathfrak{J}) \otimes S_1(\mathfrak{J})$ of fixed points under the exchange automorphism.

THEOREM 19. (1) *Let* $\mathfrak{J}$ *be a Jordan algebra with* 1 *which is reflexive and assume that* $\mathfrak{S}(S_1(\mathfrak{J}),\pi) = [\mathfrak{H}(S_1(\mathfrak{J}),\pi),\ \mathfrak{H}(S_1(\mathfrak{J}),\pi)]$ *where* $\mathfrak{S}(S_1(\mathfrak{J}),\pi)$ *and* $\mathfrak{H}(S_1(\mathfrak{J}),\pi)$ *are the sets of skew and symmetric elements of* $S_1(\mathfrak{J})$ *relative to* π. *Then* $S''(\mathfrak{J})$ *coincides with the subalgebra* $\mathfrak{B}_1$ *of* $S_1(\mathfrak{J})\otimes S_1(\mathfrak{J})$ *of fixed elements under the exchange automorphism.* (2) *The same conclusion holds if* $\mathfrak{J}$ *has a* 1 *and is generated by* 1 *and two generators* x_1, x_2.[3]

PROOF. (1) Since $x\otimes y + y\otimes x = (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y) - (xy\otimes 1 + 1\otimes xy)$ it is clear that $\mathfrak{B}_1$ is generated by the elements $x\otimes 1 + 1\otimes x$, $x\in S_1(\mathfrak{J})$. Hence it is enough to show that the elements $x\otimes 1 + 1\otimes x$, $x\in S_1(\mathfrak{J})$, are contained in $S_1''(\mathfrak{J})$. Since $\mathfrak{J}$ is reflexive any $x\in\mathfrak{H}(S_1(\mathfrak{J}),\pi)$ has the form a^{σ_u}, $a\in\mathfrak{J}$. Then $x\otimes 1 + 1\otimes x = a^{\sigma_u}\otimes 1 + 1\otimes a^{\sigma_u} = a_1''\in S_1''(\mathfrak{J})$. Also since $S_1(\mathfrak{J}) = \mathfrak{H}(S_1(\mathfrak{J}),\pi) + \mathfrak{S}(S_1(\mathfrak{J}),\pi)$ it is enough to show that every $x\otimes 1 + 1\otimes x$ with $x\in\mathfrak{S}(S_1(\mathfrak{J}),\pi)$ is contained in $S_1''(\mathfrak{J})$. Now our second hypothesis is that if $x\in\mathfrak{S}(S_1(\mathfrak{J}),\pi)$ then x is a sum of elements $[yz]$, $y,z\in\mathfrak{H}(S_1(\mathfrak{J}),\pi)$. Since $[y\otimes 1 + 1\otimes y,\ z\otimes 1 + 1\otimes z] = [yz]\otimes 1 + 1\otimes [yz]\in S_1''(\mathfrak{J})$ the result is clear. (2) More generally, let $\mathfrak{J}$ be generated by $1, x_1, x_2, \cdots, x_r$. Then we shall show that $\mathfrak{B}_1$ is generated by $1 = 1\otimes 1$ and the elements

(i) $x_i^{\sigma_u}\otimes 1 + 1\otimes x_i^{\sigma_u}$,

(ii) $x_i^{\sigma_u}.x_j^{\sigma_u}\otimes 1 + 1\otimes x_i^{\sigma_u}.x_j^{\sigma_u}$, $\quad i\leqq j$,

(iii) $x_i^{\sigma_u}x_j^{\sigma_u}x_k^{\sigma_u}\otimes 1 + 1\otimes x_i^{\sigma_u}x_j^{\sigma_u}x_k^{\sigma_u}$, $\quad i<j<k$.

Since (i) and (ii) are contained in $S_1''(\mathfrak{J})$ and (iii) is nonexistent if $r=2$ this will imply that $S_1''(\mathfrak{J}) = \mathfrak{B}_1$. Now $S_1(\mathfrak{J})$ is generated by $\mathfrak{J}^{\sigma_u}$ and $\mathfrak{J}$ is generated by 1 and the x_i. Since σ_u is a homomorphism of $\mathfrak{J}$ into $S_1(\mathfrak{J})^+$ it follows that $S_1(\mathfrak{J})$ is generated by 1 and the elements $x_i^{\sigma_u}$. Also we have seen in the proof of (1) that $\mathfrak{B}_1$ is generated by the elements $x\otimes 1 + 1\otimes x$, $x\in S_1(\mathfrak{J})$. Hence it suffices to show that if p is a product (in the associative algebra $S_1(\mathfrak{J})$) of m elements $x_i^{\sigma_u}$ then $p\otimes 1 + 1\otimes p$ is in the subalgebra $\mathfrak{B}_1'$ of $\mathfrak{B}_1$ generated by the elements (i), (ii) and (iii). For $m=1$ this is clear. Also the result is clear for $m=2$ since $ab = a.b + \frac{1}{2}[ab]$ and $[ab]\otimes 1 + 1\otimes [ab] = [a\otimes 1 + 1\otimes a,\ b\otimes 1 + 1\otimes b]$. Also this commutator relation for $a = x_i^{\sigma_u}x_j^{\sigma_u}$, $b = x_k^{\sigma_u}$ and the relation

$$\begin{aligned}(abc + cba)\otimes 1 + 1\otimes(abc + cba) = {} & (ab\otimes 1 + 1\otimes ab)(c\otimes 1 + 1\otimes c)\\ & - (a\otimes c + c\otimes a)(b\otimes 1 + 1\otimes b)\\ & + (cb\otimes 1 + 1\otimes cb)(a\otimes 1 + 1\otimes a)\end{aligned}$$

for $a = x_i^{\sigma_u}$, $b = x_j^{\sigma_u}$, $c = x_k^{\sigma_u}$ imply that if i',j',k' is a permutation of i,j,k then

[3] The second result and the more general result which is given in the proof below are due to McCrimmon.

$$x_i^{\sigma_u} x_j^{\sigma_u} x_k^{\sigma_u} \otimes 1 + 1 \otimes x_i^{\sigma_u} x_j^{\sigma_u} x_k^{\sigma_u}$$
$$\equiv \pm\, x_{i'}^{\sigma_u} x_{j'}^{\sigma_u} x_{k'}^{\sigma_u} \otimes 1 \pm 1 \otimes x_{i'}^{\sigma_u} x_{j'}^{\sigma_u} x_{k'}^{\sigma_u} \pmod{\mathfrak{B}_1'}$$

where the sign is $+$ or $-$ according as the permutation is even or odd. If two of the indices are equal this implies that $p \otimes 1 + 1 \otimes p \in \mathfrak{B}_1'$ for $p = x_i^{\sigma_u} x_j^{\sigma_u} x_k^{\sigma_u}$. The same result holds for distinct indices since the elements (iii) are contained in $\mathfrak{B}_1'$. This proves the case $m = 3$. For $m > 3$ we obtain the result by induction using the relation

$$\begin{aligned} 2(abcd \otimes 1 + 1 \otimes abcd) = {} & (ab \otimes 1 + 1 \otimes ab)(cd \otimes 1 + 1 \otimes cd) \\ & - (a \otimes c + c \otimes a)(b \otimes d + d \otimes b) \\ & + (ad \otimes 1 + 1 \otimes ad)(cb \otimes 1 + 1 \otimes cb) \\ & + [b \otimes 1 + 1 \otimes b, adc \otimes 1 + 1 \otimes adc] \\ & + (ab \otimes 1 + 1 \otimes ab)(cd \otimes 1 + 1 \otimes cd) \\ & - (a \otimes cd + cd \otimes a)(b \otimes 1 + 1 \otimes b) \\ & + (cdb \otimes 1 + 1 \otimes cdb)(a \otimes 1 + 1 \otimes a) \\ & - (cd \otimes 1 + 1 \otimes cd)(ba \otimes 1 + 1 \otimes ba) \\ & + (ba \otimes c + c \otimes ba)(d \otimes 1 + 1 \otimes d) \\ & - (bad \otimes 1 + 1 \otimes bad)(c \otimes 1 + 1 \otimes c). \end{aligned}$$

Exercises

1. (McCrimmon). Let $\mathfrak{H}$ and $\mathfrak{S}$ be the sets of symmetric and skew elements under the reversal operator in the free associative algebra with 1, $\Phi\{x, y\}$. Show that $\mathfrak{S} \neq [\mathfrak{H}, \mathfrak{H}]$.

2. (Cohn). Let $FSJ^{(2)}$ be the free special Jordan algebra with 1 and two generators x, y and let $\mathfrak{J} = FSJ^{(2)}/\mathfrak{K}$ where $\mathfrak{K}$ is the ideal generated by $x^{\cdot 2} - y^{\cdot 2}$ Also let $FSJ^{(3)}$ be the free special Jordan algebra with 1 and generators x, y, z, $\mathfrak{L}$ the ideal in $FSJ^{(3)}$ generated by $x^{\cdot 2} - y^{\cdot 2}$ and all the symmetric elements of $\Phi\{x, y, z\}$ which are homogeneous of degree two in z. Show that $\mathfrak{E} = FSJ^{(3)}/\mathfrak{L}$ is not special. Show that $\mathfrak{E}$ is a split null extension of $\mathfrak{J}$ by a bimodule which is not special.

3. (Cohn). Let $\mathfrak{J}$ be a Jordan algebra with two generators which is strongly special (def. on p. 101). Show that any bimodule for $\mathfrak{J}$ is special. Use this to show that the $\mathfrak{J}$ given in exercise 2 is special but not strongly special.

13. **Examples and applications.** In this section we shall consider the universal multiplication envelopes for the Jordan algebras considered in §3. All of these have iden-

tity elements. Hence in view of the decomposition $\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_0 \oplus \mathfrak{U}_{\frac{1}{2}}$ and the fact that we have already determined the algebras $\mathfrak{U}_{\frac{1}{2}} \cong S_1(\mathfrak{J})$ it suffices to determine $\mathfrak{U}_1$.

(1) $\mathfrak{J} = \Phi[a]$ the Jordan algebra with 1 generated by 1 and the single element a. Suppose first that $a = x$ is transcendental. We have seen that $\mathfrak{J} = \Phi[x]$ and the identity mapping form a unital special universal envelope for $\mathfrak{J}$. The main involution π is the identity mapping. We recall that $\Phi[x] \otimes \Phi[x]$ is isomorphic to $\Phi[x, y]$, x, y algebraically independent, under the isomorphism such that $x \otimes 1 \to x$, $1 \otimes x \to y$. Hence we may consider $\mathfrak{U}_1'' = S_1''(\mathfrak{J})$ as the subalgebra of $\Phi[x, y]$ generated by the elements $\frac{1}{2}(f(x) + f(y))$, $f(x)$ a polynomial in x. The automorphism in $\Phi[x, y]$ corresponding to the exchange automorphism is the one which exchanges x and y. Hence the set of fixed points is the subalgebra of symmetric polynomials. Since the main involution in $\Phi[x]$ is the identity mapping the set of skew elements $\mathfrak{S} = 0$. Hence the condition $\mathfrak{S} = [\mathfrak{H}, \mathfrak{H}]$ of Theorem 19 (1) is fulfilled and consequently $\mathfrak{U}_1''$ coincides with the subalgebra of symmetric polynomials. Next consider $\mathfrak{U}_1$ the unital universal multiplication envelope. As in the considerations preceding (48), it is clear that 1, $x_1, (x^{\cdot 2})_1$ (a_1 the image of $a \in \mathfrak{J}$ in $\mathfrak{U}_1$) commute and generate $\mathfrak{U}_1 : \mathfrak{U}_1 = \Phi[x_1, (x^{\cdot 2})_1]$. The canonical homomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$ maps $x_1 \to \frac{1}{2}(x + y), (x^{\cdot 2})_1 \to \frac{1}{2}(x^2 + y^2)$. It is well known that the symmetric polynomials $x + y$, $x^2 + y^2$ are algebraically independent. Hence the same is true of x_1 and $(x^{\cdot 2})_1$, and the canonical homomorphism is an isomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$.

Next assume a is algebraic and let $\mu(x) \in \Phi[x]$ be its minimum polynomial. We have the homomorphism $f(x) \to \overline{f(x)} = f(a)$ of $\mathfrak{J} = \Phi[x]$ into $\bar{\mathfrak{J}} \equiv \Phi[a]$ whose kernel is the principal ideal $(\mu(x))$. Let $\bar{\mathfrak{U}}_1 = U_1(\bar{\mathfrak{J}})$, $\bar{\mathfrak{U}}_1'' \equiv S_1''(\bar{\mathfrak{J}})$. We know that $\Phi[a]$ and the identity mapping constitute a unital special universal envelope for $\bar{\mathfrak{J}} = \Phi[a]$. Also, as in the case of $\Phi[x]$, it is clear that $\bar{\mathfrak{U}}_1''$ coincides with the subalgebra of $\Phi[a] \otimes \Phi[a]$ of fixed points under the exchange automorphism. Since $\dim \Phi[a] = n$, the degree of $\mu(x)$, it is clear that $\dim \bar{\mathfrak{U}}_1'' = n(n+1)/2$. We now consider the kernel $\mathfrak{K}$ of the homomorphism of $\mathfrak{U}_1'' = \Phi[x + y, x^2 + y^2]$ onto $\bar{\mathfrak{U}}_1''$ corresponding to the homomorphism $f(x) \to \overline{f(x)}$. We note that $\mathfrak{K} = (\mu(x), \mu(y)) \cap \Phi[x + y, x^2 + y^2]$ where $(\mu(x), \mu(y))$ is the ideal in $\Phi[x, y]$ generated by $\mu(x)$ and $\mu(y)$. This is clear from (69) and the fact that $(\mu(x), \mu(y))$ is the ideal in $\Phi[x, y]$ corresponding to $(\mu(x)) \otimes \Phi[x] + \Phi[x] \otimes (\mu(x))$ in the isomorphism of $\Phi[x, y]$ with $\Phi[x] \otimes \Phi[x]$. It is clear that the symmetric polynomials

$$\mu_0 = \mu(x) + \mu(y), \qquad \mu_1 = x\mu(x) + y\mu(y) \in \mathfrak{K}. \tag{72}$$

We claim that these generate $\mathfrak{K}$. Thus let $g(x, y) \in \mathfrak{K}$. Then $g(x, y) = p(x, y)\mu(x) + q(x, y)\mu(y)$ where $p(x, y)$, $q(x, y) \in \Phi[x, y]$. Then $g(x, y) = g(y, x) = \frac{1}{2}[g(x, y) + g(y, x)]$ gives $g(x, y) = r(x, y)\mu(x) + r(y, x)\mu(y)$ where $r(x, y) = \frac{1}{2}[p(x, y) + q(y, x)]$. Thus $g(x, y)$ is a linear combination of polynomials $x^k y^l \mu(x) + y^k x^l \mu(y)$, $k, l = 0, 1, 2, \cdots$. We have

$$\begin{aligned}(x + y)[x^k\mu(x) + y^k\mu(y)] &= x^{k+1}\mu(x) + y^{k+1}\mu(y)\\ &\quad + xy[x^{k-1}\mu(x) + y^{k-1}\mu(y)],\end{aligned}$$

which implies, by induction on k, that $x^k\mu(x) + y^k\mu(y)$ is in the ideal $\mathfrak{K}'$ in $\Phi[x + y, x^2 + y^2]$ generated by μ_0 and μ_1. Also

$$y^k\mu(x) + x^k\mu(y) = (x^k + y^k)\mu_0 - [x^k\mu(x) + y^k\mu(y)] \in \mathfrak{K}'.$$

Hence

$$\begin{aligned}x^ky^l\mu(x) + y^kx^l\mu(y) &= x^ly^l[x^{k-l}\mu(x) + y^{k-l}\mu(y)] \quad \text{if} \quad k \geqq l\\ &= x^ky^k[y^{l-k}\mu(x) + x^{l-k}\mu(y)] \quad \text{if} \quad k < l\end{aligned}$$

is in $\mathfrak{K}'$. Hence $\mathfrak{K} = \mathfrak{K}'$ the ideal in $\Phi[x + y, x^2 + y^2]$ generated by μ_0 and μ_1.

Now $\mu(x)$ and $x\mu(x)$ are in the kernel of the homomorphism of $\Phi[x]$ onto $\mathfrak{J}$. The image of these elements in $\mathfrak{U}_1'' = S_1''(\Phi[x])$ are $\mu(x)'' = \frac{1}{2}[\mu(x) + \mu(y)] = \mu_0$, $(x\mu(x))'' = \frac{1}{2}[x\mu(x) + y\mu(y)] = \mu_1$. Since these elements generate $\mathfrak{K}$ we see that $\mathfrak{K} \subseteq \mathfrak{K}''$ the ideal generated by the elements $f(x)''$, $f(x) \in (\mu(x))$. Since the canonical homomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$ is an isomorphism it now follows from Theorem 18 that the same result holds for $\overline{\mathfrak{U}}_1$ and $\overline{\mathfrak{U}}_1''$. It follows that $\dim U_1(\Phi[a]) = \dim \overline{\mathfrak{U}}_1'' = n(n + 1)/2$.

We shall now apply the isomorphism of $U_1(\Phi[a])$ with $\overline{\mathfrak{U}}_1''$ to determine the minimum polynomials of the elements a_1 and $u(a)_1$, $u(a) = 2a^2 - a^{\cdot 2}$, in $S_1''(\Phi[a])$. In view of this isomorphism these are the same as the minimum polynomials of the elements $\frac{1}{2}(a \otimes 1 + 1 \otimes a)$ and $2(\frac{1}{4})(a \otimes 1 + 1 \otimes a)^2 - \frac{1}{2}(a^2 \otimes 1 + 1 \otimes a^2) = a \otimes a$ in $\Phi[a] \otimes \Phi[a]$. We require first the following

LEMMA. *Let $\Phi[z_i]$, $i = 1,2$, be an associative (and commutative) algebra generated by 1 and a nilpotent element z_i of index n_i ($z_i^{n_i} = 0$, $z_i^{n_i - 1} \neq 0$). Then the element $z_1 \otimes 1 + 1 \otimes z_2$ in $\Phi[z_1] \otimes \Phi[z_2]$ is nilpotent of index $N = N(n_1, n_2)$ where $N(n_1, n_2)$ is either $n_1 + n_2 - 1$ or the smallest positive integer $N < n_1 + n_2 - 1$ such that all the binomial coefficients $\binom{N}{i} = 0$ for $i < n_1$ and $N - i < n_2$, provided such integers N exist. Also if $\rho_1, \rho_2 \in \Phi$ then $z_1 \otimes z_2 + \rho_1(1 \otimes z_2) + \rho_2(z_1 \otimes 1)$ is nilpotent of index $M = M(n_1, n_2, \rho_1, \rho_2)$ where $M(n_1, n_2, 0, 0) = \min(n_1, n_2)$, $M(n_1, n_2, 0, \rho_2) = n_1$ if $\rho_2 \neq 0$, $M(n_1, n_2, \rho_1, 0) = n_2$ if $\rho_1 \neq 0$ and $M(n_1, n_2, \rho_1, \rho_2)$ for $\rho_1 \neq 0$, $\rho_2 \neq 0$ is either $n_1 + n_2 - 1$ or is the smallest positive integer $M < n_1 + n_2 - 1$ such that*

$$Q(k, l, \rho_1, \rho_2) = \sum_{\substack{k_1 + k_2 + k_3 = M\\ k_1 + k_3 = k\\ k_1 + k_2 = l}} \frac{M!}{k_1!\,k_2!\,k_3!}\rho_1^{k_2}\rho_2^{k_3} \tag{73}$$

is 0 for all $k < n_1$ and $l < n_2$, provided such integers exist.

PROOF. We recall that the elements $z_1{}^i \otimes z_2{}^j$, $0 \leqq i < n_1$, $0 \leqq j < n_2$ form a basis for $\Phi[z_1] \otimes \Phi[z_2]$. We have $(z_1 \otimes 1 + 1 \otimes z_2)^N = \sum \binom{N}{i} z_1{}^i \otimes z_2^{N-i} = 0$ if $N = n_1 + n_2 - 1$, since in this case either $z_1{}^i = 0$ or $z_2{}^{N-i} = 0$. If $N < n_1 + n_2 - 1$ then $\sum \binom{N}{i} z_1{}^i \otimes z_2{}^{N-i} = 0$ if and only if $\binom{N}{i} = 0$ for all $i < n_1$, and $N - i < n_2$, by the linear independence of the elements $z_1^i \otimes z_2^j$, $0 \leqq i < n_1$, $0 \leqq j < n_2$. This proves the first statement. To establish the second we note first that $(z_1 \otimes z_2)^M = z_1{}^M \otimes z_2{}^M = 0$ if and only if $M \geqq \min(n_1, n_2)$. Hence $M(n_1, n_2, 0, 0) = \min(n_1, n_2)$. Next let $\rho_1 = 0$, $\rho_2 \neq 0$. Then $z_1 \otimes z_2 + \rho_1(1 \otimes z_2) + \rho_2(z_1 \otimes 1) = z_1 \otimes (z_2 + \rho_2 1)$. Since $\rho_2 \neq 0$ and z_2 is nilpotent $z_2 + \rho_2 1$ is invertible in $\Phi[z_2]$. Hence $(z_1 \otimes (z_2 + \rho_2 1))^M = z_1{}^M \otimes (z_2 + \rho_2 1)^M = 0$ if and only if $z_1{}^M = 0$. Hence the minimum M for this is $M = n_1$. This proves the second statement for $\rho_1 = 0, \rho_2 \neq 0$. By symmetry, we have the case $\rho_1 \neq 0$, $\rho_2 = 0$. Now assumes $\rho_1 \neq 0$, $\rho_2 \neq 0$ and consider

$$
\begin{aligned}
&(z_1 \otimes z_2 + \rho_1(1 \otimes z_2) + \rho_2(z_1 \otimes 1))^M \\
&= \sum_{k_1 + k_2 + k_3 = M} \frac{M!}{k_1! k_2! k_3!} \rho_1{}^{k_2} \rho_2{}^{k_3} (z_1{}^{k_1 + k_3} \otimes z_2{}^{k_1 + k_2}).
\end{aligned}
$$

This is 0 if $M \geqq n_1 + n_2 - 1$ since in this case $2k_1 + k_2 + k_3 = M + k_1 \geqq n_1 + n_2 - 1$ and so either $k_1 + k_3 \geqq n_1$ or $k_1 + k_2 \geqq n_2$. On the other hand, if $M < n_1 + n_2 - 1$ then the foregoing formula and the linear independence of the $z_1^i \otimes z_2^j$, $0 < i < n_1$ $0 \leqq j < n_2$, imply that $(z_1 \otimes z_2 + \rho_1(1 \otimes z_2) + \rho_2(z_1 \otimes 1))^M = 0$ if and only if $Q(k, l, \rho_1, \rho_2) = 0$ for all $k < n_1$, $l < n_2$. This completes the proof.

THEOREM 20. *Let $\mathfrak{J} = \Phi[a]$, where a is algebraic with minimum polynomial $\mu(x)$ which factors as*

$$
\mu(x) = \prod_1^r (x - \rho_i)^{n_i}, \quad \rho_i \neq \rho_j \quad \text{if} \quad i \neq j \tag{74}
$$

in a splitting field P/Φ. Then the polynomials

$$
\nu(x) = L.C.M.(x - \tfrac{1}{2}(\rho_i + \rho_j))^{N(n_i, n_j)} \tag{75}
$$

$$
\pi(x) = L.C.M.(x - \rho_i \rho_j)^{M(n_i, n_j, \rho_i, \rho_j)}, \tag{76}
$$

where $N(n_i, n_j)$ and $M(n_i, n_j, \rho_i, \rho_j)$ are as defined in the lemma, have coefficients in Φ and these are respectively the minimum polynomials of a_1 and $u(a)_1$ in $U_1(\mathfrak{J})$.[4]

PROOF. We have seen that the minimum polynomials of a_1 and $u(a)_1$ coincide with the minimum polynomials respectively of $\frac{1}{2}(a \otimes 1 + 1 \otimes a)$ and $a \otimes a$ in $\Phi[a] \otimes \Phi[a]$. Since these are unchanged on extension of the base field we may

[4] The result on $\nu(x)$ in a slightly different form and with a different proof is due to Mills [1].

consider these elements in $P[a] \otimes P[a]$ where $P[a] = P \otimes \Phi[a]$ and P/Φ is a splitting field in which we have the factorization (74). Then, by elementary algebra, $P[a] = P[z_1] \oplus P[z_2] \oplus \cdots \oplus P[z_r]$ where $P[z_i]$ is an ideal with identity element 1_i which is generated by 1_i and a nilpotent z_i of index n_i. Moreover, we have $a = a_1 + a_2 + \cdots + a_r$ where $a_i = \rho_i 1_i + z_i$. Then $P[a] \otimes P[a] = \Sigma \oplus P[z_i] \otimes P[z_j]$ and the minimum polynomials of $\frac{1}{2}(a \otimes 1 + 1 \otimes a)$ and $a \otimes a$ are respectively the least common multiples of those of the components of these elements in the ideals $P[z_i] \otimes P[z_j]$. These components are $\frac{1}{2}(\rho_i + \rho_j)(1_i \otimes 1_j) + \frac{1}{2}(z_i \otimes 1_j + 1_i \otimes z_j)$ and $\rho_i\rho_j(1_i \otimes 1_j) + \rho_i(1_i \otimes z_j) + \rho_j(z_i \otimes 1_j) + (z_i \otimes z_j)$. Since the minimum polynomial of an element of the form $\rho 1 + z$ where z is nilpotent is $(x - \rho)^k$ where k is the index of nilpotency of z, it is clear from the lemma that the minimum polynomials of $\frac{1}{2}(\rho_i + \rho_j)(1_i \otimes 1_j) + \frac{1}{2}(z_i \otimes 1_j + 1_i \otimes z_j)$ and $\rho_i\rho_j(1_i \otimes 1_j) + \rho_i(1_i \otimes z_j) + \rho_j(z_i \otimes 1_j) + (z_i \otimes z_j)$ are respectively $(x - \frac{1}{2}(\rho_i + \rho_j))^{N(n_i,n_j)}$ and $(x - \rho_i\rho_j)^{M(n_i,n_j,\rho_i,\rho_j)}$. Hence the minimum polynomials of $\frac{1}{2}(a \otimes 1 + 1 \otimes a)$ and $a \otimes a$ are $\nu(x)$ and $\pi(x)$ and these have coefficients in Φ.

(2) $\mathfrak{J} = \Phi[a, a^{\cdot -1}]$ where $\mathfrak{J}$ is a Jordan algebra with 1 generated by 1 and an invertible element a and its inverse $a^{\cdot -1}$. If a is algebraic we know that $\mathfrak{J} = \Phi[a]$ so the preceding considerations apply. Hence we assume $a = x$ is transcendental. We have seen in §3 that $\mathfrak{J}$ and the identity mapping form a unital special universal envelope for $\mathfrak{J} = \Phi[x, x^{\cdot -1}]$. The main involution is the identity mapping so the only skew element is 0. We can identify $\Phi[x, x^{\cdot -1}] \otimes \Phi[x, x^{\cdot -1}]$ in the obvious way with $\Phi[x, x^{-1}, y, y^{-1}]$ the subalgebra generated by x, y, x^{-1}, y^{-1} in the field of rational expressions in the algebraically independent elements x, y. By Theorem 19 (1), we may identify $\mathfrak{U}_1'' = S_1''(\mathfrak{J})$ with the subalgebra of $\Phi[x, y, x^{-1}, y^{-1}]$ of elements which are symmetric in x and y. We have $(x^{\cdot k})'' = \frac{1}{2}(x^k + y^k)$, $k = 0, \pm 1, \pm 2, \cdots$ and these elements generate $\mathfrak{U}_1''$ since the $x^{\cdot k}$ form a basis for $\Phi[x, x^{\cdot -1}]$.

We now consider the unital universal multiplication envelope $\mathfrak{U}_1$ of $\Phi[x, x^{\cdot -1}]$. We recall that if a and b are inverses in a Jordan algebra with 1 then R_a, R_b, U_a and U_b commute and we have the relations $U_b = U_a^{-1}$, $R_b = R_a U_b = R_a U_a^{-1}$ (§1.11). We know also that R_a, U_a, U_a^{-1} generate the algebra of linear transformations generated by all $R_{a^{\cdot k}}$, $k = 0, \pm 1, \pm 2, \cdots$. As we have seen (§9), these results carry over to $\mathfrak{U}_1$. They imply that $\mathfrak{U}_1$ is a commutative algebra generated by x_1, $u(x_1)$ and $u(x_1)^{-1}$. The images of these elements under the canonical homomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$ are respectively $\frac{1}{2}(x + y)$, xy and $(xy)^{-1}$. Since $x + y$ and xy are algebraically independent it follows that the canonical homomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$ is an isomorphism and that $\mathfrak{U}_1'' = \Phi[x + y, xy, (xy)^{-1}]$.

The commutativity of $S_1(\mathfrak{J})$ and of $\mathfrak{U}_1$ for any $\Phi[a, a^{\cdot -1}]$ imply the commutativity of the universal multiplication envelope $\mathfrak{U}$ of any Jordan algebra generated by an invertible element and its inverse.

(3) $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a symmetric bilinear form f on $\mathfrak{V}$. We saw in §3 that the Clifford algebra $C(\mathfrak{V},f)$ is a unital special universal envelope for $\mathfrak{J}$. We shall now give a similar construction for the unital universal multiplication envelope $\mathfrak{U}_1(\mathfrak{J})$. For this we give the following

DEFINITION 5. *Let $\mathfrak{V}/\Phi$ be a vector space, f a symmetric bilinear form on $\mathfrak{V}$. Let $T(\mathfrak{V})$ be the tensor algebra based on $\mathfrak{V}$, $\mathfrak{L}$ the ideal in $T(\mathfrak{V})$ generated by the elements $x \otimes y \otimes x - f(x,y)x$, $x, y \in \mathfrak{V}$. Then $D(\mathfrak{V},f) = T(\mathfrak{V})/\mathfrak{L}$ is called the* meson algebra *of f.*[5]

Let ρ be a linear mapping of $\mathfrak{V}/\Phi$ into an associative algebra $\mathfrak{G}$ with 1 such that $x^\rho y^\rho x^\rho = f(x,y)x^\rho$, $x, y \in \mathfrak{V}$. Then there is a unique extension of ρ to a homomorphism ρ of $T(\mathfrak{V})$ into $\mathfrak{G}$ such that $1^\rho = 1$. In this homomorphism $(x \otimes y \otimes x - f(x,y)x)^\rho = x^\rho y^\rho x^\rho - f(x,y)x^\rho = 0$. Hence we have an induced homomorphism of $D(\mathfrak{V},f)$ into $\mathfrak{G}$ such that $\overline{\alpha 1 + x} \equiv \alpha 1 + x + \mathfrak{L} \to \alpha 1 + x^\rho$, $\alpha \in \Phi$, $x \in \mathfrak{V}$. We note next that if $a = \alpha 1 + x$ and $b = \beta 1 + y$, $\alpha, \beta \in \Phi$, $x, y \in \mathfrak{V}$, then $a^{\cdot 2} = (\alpha^2 + f(x, x))1 + 2\alpha x$, $a.b = (\alpha\beta + f(x, y))1 + \alpha y + \beta x$, $a^{\cdot 2}.b = (\alpha^2\beta + \beta f(x, x) + 2\alpha f(x, y))1 + \alpha^2 y + f(x, x)y + 2\alpha\beta x$. Also $\bar{x}\bar{y}\bar{x} = f(x, y)\bar{x}$ implies that

$$\bar{a}\bar{b}\bar{a} = \alpha^2\beta 1 + 2\alpha\beta\bar{x} + \beta\bar{x}^2 + \alpha(\bar{x}\bar{y} + \bar{y}\bar{x}) + f(x,y)\bar{x}.$$

A direct verification now shows that

$$2\bar{a}\bar{b}\bar{a} + \overline{a^{\cdot 2}.b} = 2\bar{a}(\overline{a.b}) + \bar{b}(\overline{a^{\cdot 2}}).$$

Since $[\bar{a}, \overline{a^{\cdot 2}}] = [\alpha 1 + \bar{x}, (\alpha + f(x,x))1 + 2\alpha\bar{x}] = 0$, it is clear that $a \to \bar{a}$ is a unital multiplication specialization of $\mathfrak{J}$ in $D(\mathfrak{V},f)$. Now if ρ is any unital multiplication specialization of $\mathfrak{J}$ then $2x^\rho y^\rho x^\rho + (x^{\cdot 2}.y)^\rho = 2x^\rho(x.y)^\rho + y^\rho(x^{\cdot 2})^\rho$ gives $x^\rho y^\rho x^\rho = f(x,y)x^\rho$ for $x, y \in \mathfrak{V}$. Hence the result noted before shows that we have a unique homomorphism of $D(\mathfrak{V},f)$ into $\mathfrak{G}$ such that $\overline{\alpha 1 + x} \to (\alpha 1 + x)^\rho$. Hence $D(\mathfrak{V},f)$ and the mapping $a \to \bar{a}$ constitute a unital universal multiplication envelope for $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$.

If we take $C(\mathfrak{V},f)$ and $D(\mathfrak{V},f)$ for our choices of unital special universal envelope and unital universal multiplication envelope, then $\mathfrak{U}_1'' = S_1''(\mathfrak{J})$ becomes the subalgebra of $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$ generated by 1 and the elements $(x + \mathfrak{K}) \otimes 1 + 1 \otimes (x + \mathfrak{K})$, $x \in \mathfrak{V}$. The canonical homomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$ becomes the homomorphism of the meson algebra $D(\mathfrak{V},f)$ into $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$ mapping $x + \mathfrak{L} \to \{\frac{1}{2}(x + \mathfrak{K}) \otimes 1 + 1 \otimes (x + \mathfrak{K})\}$. We shall see later (§7.2)—after we give a self-contained development of the theory of Clifford algebras — that this homomorphism is an isomorphism.

(4) $\mathfrak{J} = FJ^{(2)}$ the free Jordan algebra with 1 and two generators x, y. Let $\Phi\{u, v\}$ be free associative with 1 and generators u, v. Then we saw in §3 that

[5] These algebras have been used in physics in meson theory. They were introduced in this connection by Duffin [1] and Kemmer [1] and [2].

$\Phi\{u,v\}$ and the homomorphism σ_{1u} of $FJ^{(2)}$ into $\Phi\{u,v\}$ such that $1 \to 1$, $x \to u$, $y \to v$ is a unital special universal envelope for $FJ^{(2)}$. The main involution in $\Phi\{u,v\}$ is the reversal operator.

Let $FJ^{(3)}$ be the free Jordan algebra with 1 and three generators x,y,z, $\mathfrak{E}$ the subalgebra of $\mathrm{Hom}_\Phi(FJ^{(3)},FJ^{(3)})$ generated by the R_b, $b \in FJ^{(2)}$ ($\subseteq FJ^{(3)}$, cf. §1.9). We claim that $(\mathfrak{E},R)$ where R is the mapping $b \to R_b$, $b \in FJ^{(2)}$, R_b acting in $FJ^{(3)}$, is a unital universal multiplication envelope for $FJ^{(2)}$. It is enough to show that if ρ is a unital multiplication specialization of $FJ^{(2)}$ in an associative algebra $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ then there exists a unique homomorphism of $\mathfrak{E}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ such that $R_b \to b^\rho$. Also it is enough to prove existence of the homomorphism since uniqueness will then follow from the fact that the R_b generate $\mathfrak{E}$. Now if we form the split null extension $\mathfrak{E}'$ of $\mathfrak{J}$ by the bimodule $\mathfrak{M}$, then it follows from Lemma 1 of §6 that we have a homomorphism of $\mathfrak{E}$ into $\mathrm{Hom}_\Phi(\mathfrak{E}',\mathfrak{E}')$ such that $R_b \to R_b'$ where R_b' is the multiplication in $\mathfrak{E}'$ determined by $b \in FJ^{(2)}$. By restriction we obtain the required homomorphism such that $R_b \to b^\rho$.

Let $FSJ^{(3)}$ be the free special Jordan algebra with generators u,v,w, $\mathfrak{E}_s$ the subalgebra of $\mathrm{Hom}_\Phi(FSJ^{(3)},FSJ^{(3)})$ generated by the elements R_c, c in $FSJ^{(2)}$ (generated by u,v). By the operator form of Macdonald's Theorem we have the isomorphism of $\mathfrak{E}$ onto $\mathfrak{E}_s$ such that $R_b \to R_{b\sigma_{1u}}$. Since $FSJ^{(3)}$ is special we have the homomorphism of $\mathfrak{U}_1'' = S_1''(FJ^{(2)})$ onto $\mathfrak{E}_s$ such that $b_1'' \to R_{b\sigma_{1u}}$. Hence we have the homomorphism of $\mathfrak{U}_1''$ onto $\mathfrak{E}$ such that $b_1'' \to R_b$. It follows that the canonical homomorphism of $\mathfrak{U}_1$ onto $\mathfrak{U}_1''$ is an isomorphism. Also by Theorem 19 (2), $\mathfrak{U}_1''$ coincides with the subalgebra of $S(FJ^{(2)}) \otimes S(FJ^{(2)}) = \Phi\{u,v\} \otimes \Phi\{u,v\}$ of fixed elements under the exchange automorphism.

Exercises

1. Show that the polynomial $\nu(x)$ of Theorem 20 is L.C.M. $(x - \frac{1}{2}(\rho_i + \rho_j))^{n_i+n_j-1}$ if the field is of characteristic 0. Show that if $\mu(x)$ has distinct roots then the same is true of $\nu(x)$ and $\pi(x)$ (any characteristic). Show that if a is nilpotent of index n then $a_1^{2n-1} = 0$ and $u(a)_1^n = 0$. Show also that if a is unipotent ($= 1 + a$ nilpotent) then a_1 and $u(a)_1$ are unipotent.

2. Let $\mathfrak{J} = \Phi[a]$ with $\mu(x)$ as minimum polynomial as in Theorem 20. Determine the minimum polynomial of $(a^{\cdot 2})_1 - (a_1)^2$ in $U_1(\mathfrak{J})$. Show that if a is unipotent then $(a^{\cdot 2})_1 - (a_1)^2$ is nilpotent.

3. Let $\mathfrak{J}$ be a Jordan algebra with basis $(e_1, e_2, \cdots, e_n)$ where the e_i are nonzero orthogonal idempotents such that $\sum e_i = 1$ the identity element of $\mathfrak{J}$. Show that $\dim U_1(\mathfrak{J}) = n(n+1)/2$ and determine $U_1(\mathfrak{J})$.

Chapter III

PEIRCE DECOMPOSITIONS AND JORDAN MATRIX ALGEBRAS

In this chapter we shall develop some of the main tools for the structure theory of Jordan algebras. These concern the Peirce decomposition of a Jordan algebra relative to a finite set of orthogonal idempotent elements and the study of Jordan matrix algebras. If e is an idempotent element in a Jordan algebra $\mathfrak{J}$ then the multiplication R_e determined by e satisfies the equation $R_e(2R_e - 1)(R_e - 1) = 0$. Hence we can decompose $\mathfrak{J}$ as $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_0 \oplus \mathfrak{J}_{\frac{1}{2}}$ where $\mathfrak{J}_i$ is the space of the characteristic root i of R_e. This decomposition is analogous to the two-sided Peirce decomposition of an associative algebra relative to an idempotent in the algebra. More generally, if $\mathfrak{J}$ has an identity element $1 = \sum_1^n e_i$ where the e_i are orthogonal idempotents ($e_i^2 = e_i$, $e_i . e_j = 0$ if $i \neq j$) then we have the Peirce decomposition $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$ where

$$\mathfrak{J}_{ii} = \{x_{ii} \mid x_{ii} . e_i = x_{ii}\} \text{ and } \mathfrak{J}_{ij} = \{x_{ij} \mid x_{ij} . e_i = \tfrac{1}{2} x_{ij} = x_{ij} . e_j\}$$

if $i \neq j$. This is analogous to the two-sided Peirce decomposition of an associative algebra $\mathfrak{A}$ as $\mathfrak{A} = \sum \mathfrak{A}_{ij}$, $\mathfrak{A}_{ij} = e_i \mathfrak{A} e_j$, relative to the finite set of orthogonal idempotents e_i with $\sum e_i = 1$. There are a number of important multiplicative properties of the Peirce spaces $\mathfrak{J}_{ij}$. These are sorted out in Lemma 2 and 3 of §1.

If $\mathfrak{D}$ is an algebra with 1 and an involution j ($d \to \bar{d}$) then the algebra $\mathfrak{D}_n$ of $n \times n$ matrices with entries in $\mathfrak{D}$ has the involution $X \to \bar{X}^t$ which is called the standard involution (associated with j) (cf §1.5). More generally, if a is a diagonal matrix $\operatorname{diag}\{a_1, a_2, \cdots, a_n\}$ whose diagonal entries a_i are j-symmetric elements of the nucleus of $\mathfrak{D}$ which have inverses in this nucleus, then we can define the involution J_a: $X \to a^{-1}\bar{X}^t a$ in $\mathfrak{D}_n$. We call involutions of this type canonical involutions in the matrix algebra $\mathfrak{D}_n$. The set $\mathfrak{H}(\mathfrak{D}_n, J_a)$ of matrices $A \in \mathfrak{D}_n$ such that $A^{J_a} = A$ is a subalgebra of $\mathfrak{D}_n^+$ which is the vector space $\mathfrak{D}_n$ considered as an algebra relative to the multiplication $A . B = \frac{1}{2}(AB + BA)$. The algebras $\mathfrak{H}(\mathfrak{D}_n, J_a)$ which are Jordan will be called Jordan matrix algebras. These play a role analogous to that of the matrix algebras $\mathfrak{D}_n$ in the associative theory.

One of the main results of this chapter is a characterization of Jordan matrix algebras $\mathfrak{H}(\mathfrak{D}_n, J_a)$ of order $n \geqq 3$ in terms of existence of certain types of idempotents. The result giving this is called the Coordinatization Theorem for Jordan algebras, since it is somewhat reminiscent of the introduction of coordinates in projective geometry. This result will play an important role in the structure theory which will be developed in Chapters IV and V.

We shall establish in this chapter also some basic results on representation theory of Jordan matrix algebras. We shall show that if $\mathfrak{D}$ is associative and $n \geqq 3$ then the associative algebra $\mathfrak{D}_n$ with canonical involution J_a is perfect in the sense of §2.4. We shall show also that the study of unital bimodules for a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n. J_a)$ with $n \geqq 3$ can be reduced to that of certain types of bimodules with involution for the coefficient algebra $(\mathfrak{D},j)$. The Coordinatization Theorem and the determination of the ideals and certain subalgebras of the algebras $\mathfrak{H}(\mathfrak{D}_n, J_a)$ will be needed to carry out this reduction.

1. **Peirce decompositions.** Let e be an idempotent element of a Jordan algebra $\mathfrak{J}$. Then we have seen in §2.11 that e satisfies $e(e-1)(2e-1) = 0$ in the universal multiplication envelope $U(\mathfrak{J})$ and that the elements

$$(1) \qquad E_0 = (e-1)(2e-1), \qquad E_1 = e(2e-1), \qquad E_{\frac{1}{2}} = -4e(e-1)$$

are orthogonal idempotents in $U(\mathfrak{J})$ such that $E_0 + E_1 + E_{\frac{1}{2}} = 1$. It is clear also from the equation satisfied by e that

$$(2) \qquad E_0 e = 0, \qquad E_1(e-1) = 0, \qquad E_{\frac{1}{2}}(e - \tfrac{1}{2}1) = 0.$$

If we use the canonical homomorphism of $U(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ (sending $1 \to 1$ and $a \to R_a$, $a \in \mathfrak{J}$) then we see that the elements

$$(3) \quad P_0 = (R_e - 1)(2R_e - 1), \qquad P_1 = R_e(2R_e - 1), \qquad P_{\frac{1}{2}} = -4R_e(R_e - 1)$$

are orthogonal idempotents in $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ such that $P_0 + P_1 + P_{\frac{1}{2}} = 1$ and $P_0 R_e = 0$, $P_1(R_e - 1) = 0$, $P_{\frac{1}{2}}(R_e - \frac{1}{2}1) = 0$. Since the P_i are orthogonal projections in $\mathfrak{J}$ with sum 1 we have

$$(4) \qquad \mathfrak{J} = \mathfrak{J}_0(e) \oplus \mathfrak{J}_1(e) \oplus \mathfrak{J}_{\frac{1}{2}}(e), \qquad \mathfrak{J}_i(e) = \mathfrak{J}P_i.$$

We shall call the decomposition (4) of $\mathfrak{J}$ (as a direct sum of the subspaces $\mathfrak{J}_i(e)$) the *Peirce decomposition of* $\mathfrak{J}$ *relative to the idempotent e*. If $x_i \in \mathfrak{J}_i = \mathfrak{J}_i(e)$, so that $x_i = xP_i$, $x \in \mathfrak{J}$, then $x_i.e = x_iR_e = xP_iR_e = xP_i(R_e - i1 + i1) = xP_i(R_e - i1) + ixP_i = ixP_i = ix_i$. It follows from the decomposition (4) that

$$(5) \qquad \mathfrak{J}_i(e) = \{x_i \mid x_i \in \mathfrak{J}, \quad x_i . e = ix_i\}.$$

We note also that $P_1 = 2R_e^2 - R_e = U_e$ so

$$(6) \qquad \mathfrak{J}_1(e) = \mathfrak{J}U_e.$$

We shall now derive the main properties of the Peirce components $\mathfrak{J}_i$. The proofs will be based on the basic identities

$$(\mathrm{B}_1)\qquad \begin{aligned}&(a.b).(c.d)+(a.d).(b.c)+(a.c).(b.d)\\ &\quad = a.(c.d).b+a.(b.c).d+a.(b.d).c,\end{aligned}$$

$$(\mathrm{B}_2)\qquad \begin{aligned}&a.(b.c).d+a.(b.d).c+a.(c.d).b\\ &\quad = a.b.c.d+a.d.c.b+a.(b.d.c),\end{aligned}$$

$$(\mathrm{B}_3)\qquad \begin{aligned}&a.b.c.d+a.d.c.b+a.(b.d.c)\\ &\quad = (a.b).(c.d)+(a.c).(b.d)+(a.d).(b.c)\end{aligned}$$

(see §1.7).

LEMMA 1. *Let* $\mathfrak{J}=\mathfrak{J}_0\oplus\mathfrak{J}_1\oplus\mathfrak{J}_{\frac{1}{2}}$ *be the Peirce decomposition of the Jordan algebra* $\mathfrak{J}$ *relative to the idempotent e. Then:* (i) $\mathfrak{J}_0^{\cdot 2}\subseteq\mathfrak{J}_0$, $\mathfrak{J}_1^{\cdot 2}\subseteq\mathfrak{J}_1$, $\mathfrak{J}_0\cdot\mathfrak{J}_1=0$, (ii) $\mathfrak{J}_{\frac{1}{2}}^{\cdot 2}\subseteq\mathfrak{J}_0+\mathfrak{J}_1$, $\mathfrak{J}_{\frac{1}{2}}\cdot(\mathfrak{J}_0+\mathfrak{J}_1)\subseteq\mathfrak{J}_{\frac{1}{2}}$, (iii) $x_{\frac{1}{2}}.(a_i.b_i)=x_{\frac{1}{2}}.a_i.b_i+x_{\frac{1}{2}}.b_i.a_i$ *if* $x_{\frac{1}{2}}\in\mathfrak{J}_{\frac{1}{2}}$, $a_i,b_i\in\mathfrak{J}_i$, $i=0,1$. *Also* $[R_{a_0},R_{a_1}]=0$ *if* $a_i\in\mathfrak{J}_i$.

PROOF. Let $a\in\mathfrak{J}_i$, $c\in\mathfrak{J}_j$ so $a.e=ia$, $c.e=jc$, $i,j=0,1,\frac{1}{2}$. Put $b=d=e$ in (B_3) to obtain $2a.e.c.e+a.(c.e)=2a.e.(c.e)+a.c.e$. This gives

$$(1-2i)(a.c)R_e=j(1-2i)a.c. \tag{7}$$

If $i=j=0$ we obtain $(a.c)R_e=0$ and if $i=j=1$ we have $(a.c)R_e=a.c$. Hence the first two relations in (i) hold. Also $i=0,j=1$ and $i=1,j=0$ in (7) give $(a.c)R_e=a.c$ and $(a.c)R_e=0$. Hence $a.c=0$ and the last relation in (i) holds. Similarly, $i=0,1$ and $j=\frac{1}{2}$ in (7) give the second relation in (ii). Next, put $c=d=e$ in (B_3) to obtain $a.b.e.e+a.e.e.b+a.(b.e.e)=a.b.e+2a.e.(b.e)$. Then if $a,b\in\mathfrak{J}_{\frac{1}{2}}$ we obtain $a.b(R_e^2-R_e)=0$. Writing $a.b=c_0+c_1+c_{\frac{1}{2}}$ where $c_i\in\mathfrak{J}_i$ we obtain $0=(a.b)(R_e^2-R_e)=c_{\frac{1}{2}}(\frac{1}{4}-\frac{1}{2})$. Hence $c_{\frac{1}{2}}=0$ and $a.b\in\mathfrak{J}_0+\mathfrak{J}_1$. Hence the first assertion in (ii) holds. Next take $a=x_{\frac{1}{2}}\in\mathfrak{J}_{\frac{1}{2}}$, $b=e$, $c,d\in\mathfrak{J}_1$ in (B_3). This gives $x_{\frac{1}{2}}.(c.d)=x_{\frac{1}{2}}.c.d+x_{\frac{1}{2}}.d.c$. Similarly, the same relation with $c,d\in\mathfrak{J}_0$ gives $x_{\frac{1}{2}}.(c.d)$, $=x_{\frac{1}{2}}.c.d+x_{\frac{1}{2}}.d.c$. This proves the first assertion in (iii). Now let $a_i\in\mathfrak{J}_i$, $i=0,1$ and consider the basic operator identity $[R_aR_{b.c}]+[R_bR_{a.c}]+[R_cR_{a.b}]=0$ for $a=e$, $b=a_0$, $c=a_1$. This gives $[R_{a_0}R_{a_1}]=0$ and completes the proof.

We suppose next that $\mathfrak{J}$ has an identity element 1 and

$$1=e_1+e_2+\cdots+e_n \tag{8}$$

where

$$e_i^{\cdot 2}=e_i,\qquad e_i.e_j=0,\qquad i\neq j. \tag{9}$$

Then, as in §2.11, the $n(n+1)/2$ elements $E_{ii}=u(e_i)C_1$, $E_{ij}=E_{ji}=4e_ie_jC_1=2u(e_i,e_j)C_1$, $i\neq j$, of $U_1(\mathfrak{J})$, where C_1 is the identity element of $U_1(\mathfrak{J})$, are orthogonal idempotent elements and $\sum_{i\leq j}E_{ij}=C_1$. The corresponding elements in $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ are $P_{ii}=U_{e_i}$, $P_{ij}=4R_{e_i}R_{e_j}=2U_{e_i,e_j}$ which satisfy

(10) $$\sum_{i \leqq j} P_{ij} = 1, \qquad P_{ij}^2 = P_{ij}, \qquad P_{ij}P_{kl} = 0$$

if $\{i,j\} \neq \{k,l\}$. Hence we have

(11) $$\mathfrak{J} = \sum_{i \leqq j} \oplus \mathfrak{J}_{ij}, \qquad \mathfrak{J}_{ij} = \mathfrak{J}P_{ij}.$$

We shall call (11) the *Peirce decomposition of* $\mathfrak{J}$ *relative to the idempotents* e_i (orthogonal and with sum 1).

If e is any idempotent in $\mathfrak{J}$ then $e_1 = e$, $e_2 = 1 - e$ are orthogonal idempotents in $\mathfrak{J}$ with sum 1. Then $P_{11} = U_{e_1} = P_1$ determined by e as in (3). Hence $\mathfrak{J}_{11} = \mathfrak{J}_1(e)$. Similarly, we have $P_{22} = U_{e_2} = 2R_{1-e}{}^2 - R_{1-e} = 2(1 - R_e)^2 - 1 + R_e = (R_e - 1)(2R_e - 1) = P_0$ and $P_{12} = 2U_{e_1,e_2} = 4R_{e_1}R_{e_2} = 4R_e(1 - R_e) = P_{\frac{1}{2}}$. Hence $\mathfrak{J}_{22} = \mathfrak{J}_0(e)$ and $\mathfrak{J}_{12} = \mathfrak{J}_{21} = \mathfrak{J}_{\frac{1}{2}}(e)$. By symmetry we have also $\mathfrak{J}_{12} = \mathfrak{J}_{21} = \mathfrak{J}_{\frac{1}{2}}(e_2)$.

Again let $1 = e_1 + e_2 + \cdots + e_n$ where the e_i are orthogonal idempotents. Put $e_i' = 1 - e_i = \sum_{j \neq i} e_j$. Then

(12) $$\begin{aligned} \mathfrak{J}_1(e_i) &= \mathfrak{J}U_{e_i} = \mathfrak{J}_{ii}, \\ \mathfrak{J}_0(e_i) &= \mathfrak{J}U_{e_i'} = \sum_{j \neq i} \mathfrak{J}U_{e_j} + \sum_{\substack{k \neq l \\ k,l \neq i}} \mathfrak{J}(2U_{e_k,e_l}), \\ &= \sum_{j,k \neq i} \mathfrak{J}_{jk}, \\ \mathfrak{J}_{\frac{1}{2}}(e_i) &= \mathfrak{J}U_{e_i,e_i'} = \sum_{j \neq i} \mathfrak{J}_{ij}. \end{aligned}$$

It is clear from these formulas that

(13) $$\begin{aligned} \mathfrak{J}_{ii} &= \mathfrak{J}_1(e_i) = \{x_{ii} \mid x_{ii} . e_i = x_{ii}\}, \\ \mathfrak{J}_{ij} &= \mathfrak{J}_{\frac{1}{2}}(e_i) \cap \mathfrak{J}_{\frac{1}{2}}(e_j) = \{x_{ij} \mid x_{ij} . e_i = \tfrac{1}{2}x_{ij} = x_{ij} . e_j\} \text{ if } i \neq j. \end{aligned}$$

Let $m \leqq n$ and set $e = \sum_{j=1}^m e_j$. Then $\mathfrak{J}_1(e) = \mathfrak{J}U_e$ is a subalgebra of $\mathfrak{J}$ with e as identity element. Also $e . e_j = e_j$ if $1 \leqq j \leqq m$ so $e_j \in \mathfrak{J}_1(e)$ and $e = \sum_1^m e_j$ is a decomposition of the identity e of $\mathfrak{J}_1(e)$ as a sum of orthogonal idempotents in this subalgebra. We have $U_e = U_{\Sigma e_j} = \sum_{j=1}^m U_{e_j} + \sum_{j<k=1}^m 2U_{e_j,e_k}$ and the operators U_{e_j}, $2U_{e_j,e_k}$ are orthogonal idempotents with sum U_e. Hence $U_eU_{e_j} = U_{e_j}$ and $U_e(2U_{e_j,e_k}) = 2U_{e_j,e_k}$. Hence the Peirce component $\mathfrak{J}U_eU_{e_j} = \mathfrak{J}U_{e_j}$ and $\mathfrak{J}U_eU_{e_j,e_k} = \mathfrak{J}U_{e_j,e_k}$. Thus $\mathfrak{J}_1(e) = \sum_{j \leqq k=1}^m \mathfrak{J}_{jk}$ is the Peirce decomposition of $\mathfrak{J}_1(e)$ relative to the orthogonal idempotents $e_1, e_2, \cdots, e_m$.

We can now prove

LEMMA 2. *Let* $\mathfrak{J}$ *be a Jordan algebra with* $1 = \sum_1^n e_i$ *where the* e_i *are orthogonal idempotents and let* $\mathfrak{J}_{ij}$, $i = 1, \cdots, n$ $(\mathfrak{J}_{ij} = \mathfrak{J}_{ji})$ *be the Peirce components of* $\mathfrak{J}$ *relative to the* e_i. *Then*:
(i) $\mathfrak{J}_{ii}{}^{.2} \subseteq \mathfrak{J}_{ii}$, $\mathfrak{J}_{ij} \cdot \mathfrak{J}_{ii} \subseteq \mathfrak{J}_{ij}$, $\mathfrak{J}_{ij}{}^{.2} \subseteq \mathfrak{J}_{ii} + \mathfrak{J}_{jj}$, $\mathfrak{J}_{ii} \cdot \mathfrak{J}_{jj} = 0$ *if* $i \neq j$, (ii) $\mathfrak{J}_{ij} \cdot \mathfrak{J}_{jk} \subseteq \mathfrak{J}_{ik}$, $\mathfrak{J}_{ij} \cdot \mathfrak{J}_{kk} = 0$, $\mathfrak{J}_{ij} \cdot \mathfrak{J}_{kl} = 0$ *if* i,j,k,l *are* $\neq$.

PROOF. Let $i \neq j$ and put $e = e_i + e_j$, $\mathfrak{B} = \mathfrak{J}U_e$. Then $\mathfrak{B} = \mathfrak{J}_{ii} + \mathfrak{J}_{jj} + \mathfrak{J}_{ij}$ is the Peirce decomposition of $\mathfrak{B}$ relative to e_i, e_j. We have $\mathfrak{J}_{ii} = \mathfrak{B}_1(e_i)$, $\mathfrak{J}_{jj} = \mathfrak{B}_0(e_i)$ $= \mathfrak{J}_{ij} = \mathfrak{B}_{\frac{1}{2}}(e_i)$. Hence (i) follows from (i) and (ii) of Lemma 1. Next let i,j,k be $\neq$ and put $f = e_i + e_j + e_k$, $\mathfrak{C} = \mathfrak{J}U_f$. Then $\mathfrak{C} = \mathfrak{J}_{ii} + \mathfrak{J}_{jj} + \mathfrak{J}_{kk} + \mathfrak{J}_{ij} + \mathfrak{J}_{ik} + \mathfrak{J}_{jk}$ is the Peirce decomposition of $\mathfrak{C}$ relative to e_i, e_j, e_k ($\mathfrak{J}_{pq} = \mathfrak{C}_{pq}$). Also let $e = e_i + e_j$. Then e is an idempotent contained in $\mathfrak{C}$ and $\mathfrak{C}_1(e) = \mathfrak{J}_{ii} + \mathfrak{J}_{ij} + \mathfrak{J}_{jj}$, $\mathfrak{C}_0(e) = \mathfrak{J}_{kk}$ and $\mathfrak{C}_{\frac{1}{2}}(e) = \mathfrak{J}_{ik} + \mathfrak{J}_{jk}$. Since $\mathfrak{C}_0(e) . \mathfrak{C}_1(e) = 0$ we have $\mathfrak{J}_{ij} . \mathfrak{J}_{kk} = 0$. Since $\mathfrak{C}_{\frac{1}{2}}(e) . \mathfrak{C}_1(e) \subseteq \mathfrak{C}_{\frac{1}{2}}(e)$ we have $\mathfrak{J}_{ij} . \mathfrak{J}_{jk} \subseteq \mathfrak{J}_{ik} + \mathfrak{J}_{jk}$ and since $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$, $\mathfrak{J}_{jk} = \mathfrak{J}_{kj}$ we have also $\mathfrak{J}_{ij} . \mathfrak{J}_{jk} = \mathfrak{J}_{kj} . \mathfrak{J}_{ji} \subseteq \mathfrak{J}_{ki} + \mathfrak{J}_{ji}$. Hence $\mathfrak{J}_{ij} . \mathfrak{J}_{jk} \subseteq (\mathfrak{J}_{ik} + \mathfrak{J}_{jk}) \cap (\mathfrak{J}_{ki} + \mathfrak{J}_{ji}) = \mathfrak{J}_{ik}$. It remains to prove the last statement in (ii). For this we take $g = e_i + e_j + e_k + e_l$ and $\mathfrak{D} = \mathfrak{J}U_g$. Then $\mathfrak{D} = \mathfrak{J}_{ii} + \mathfrak{J}_{jj} + \mathfrak{J}_{kk} + \mathfrak{J}_{ll} + \mathfrak{J}_{ij} + \mathfrak{J}_{ik} + \mathfrak{J}_{il} + \mathfrak{J}_{jk} + \mathfrak{J}_{jl} + \mathfrak{J}_{kl}$. Also $e = e_i + e_j$ is an idempotent contained in $\mathfrak{D}$ and $\mathfrak{J}_{ij} \subseteq \mathfrak{D}_1(e)$ while $\mathfrak{J}_{kl} \subseteq \mathfrak{D}_0(e)$. Hence $\mathfrak{D}_0(e) . \mathfrak{D}_1(e) = 0$ gives $\mathfrak{J}_{ij} . \mathfrak{J}_{kl} = 0$.

We shall list next some important formulas for products of elements in the Peirce components.

LEMMA 3. *Let $\mathfrak{J}$, 1 and the e_i be as in Lemma 2, let $x_{ii} \in \mathfrak{J}_{ii}$, $x_{ij} \in \mathfrak{J}_{ij}$ etc. and assume i,j,k,l are $\neq$. Then*

(PD1) $x_{ij}^{.2} . e_i . x_{ij} = x_{ij}^{.2} . e_j . x_{ij}$,

(PD1′) $2x_{ij} . y_{ij} . e_i . x_{ij} + x_{ij}^{.2} . e_i . y_{ij} = 2x_{ij} . y_{ij} . e_j . x_{ij} + x_{ij}^{.2} . e_j . y_{ij}$,

(PD2) $[x_{ii}, y_{ij}, z_{jj}] = 0$,

(PD3) $x_{ii} . y_{ii} . z_{ij} = x_{ii} . z_{ij} . y_{ii} + y_{ii} . z_{ij} . x_{ii}$,

(PD4) $[x_{ij}, y_{ij}, z_{jj}] = (e_j - e_i) . (x_{ij} . z_{jj} . y_{ij})$,

(PD5) $x_{ij} . y_{ij} . z_{jj} = x_{ij} . z_{jj} . y_{ij} . e_j + y_{ij} . z_{jj} . x_{ij} . e_j$,

(PD6) $[x_{ij}, y_{jj}, z_{ij}] . e_i = 0$,

(PD7) $[x_{ii}, y_{ij}, z_{jk}] = 0$,

(PD8) $[x_{ij}, y_{jj}, z_{jk}] = 0$,

(PD9) $[x_{ij}, y_{jk}, z_{kk}] = 0$,

(PD10) $x_{ij} . y_{ij} . z_{jk} = x_{ij} . z_{jk} . y_{ij} + y_{ij} . z_{jk} . x_{ij}$,

(PD11) $[x_{ij}, y_{jk}, z_{ki}] = (e_k - e_j) . (x_{ij} . z_{ki} . y_{jk})$,

(PD12) $[x_{ij}, y_{jk}, z_{ki}] . e_i = 0$,

(PD13) $[x_{ij}, y_{jk}, z_{kl}] = 0$.

REMARK. The first six of these formulas involve just two (distinct) indices and so refer to two idempotents. Formulas (PD7)–(PD12) refer to three idempotents and (PD13) refers to four idempotents.

PROOF. To obtain (PD1) we begin with the basic Jordan identity $x_{ij}^{\cdot 2}.e_i.x_{ij} = x_{ij}^{\cdot 2}.(e_i.x_{ij}) = \frac{1}{2}x_{ij}^{\cdot 2}.x_{ij}$. Also we have $x_{ij}^{\cdot 2} = x_{ii} + x_{jj}$ where $x_{ii} \in \mathfrak{J}_{ii}$ and $x_{jj} \in \mathfrak{J}_{jj}$. Then $x_{ij}^{\cdot 2}.e_i = x_{ii}$, $x_{ij}^{\cdot 2}.e_j = x_{jj}$ and the foregoing relation becomes $x_{ii}.x_{ij} = \frac{1}{2}x_{ii}.x_{ii} + \frac{1}{2}x_{jj}.x_{ij}$. Hence $x_{ii}.x_{ij} = x_{jj}.x_{ij}$ and so $x_{ij}^{\cdot 2}.e_i.x_{ij} = x_{ij}^{\cdot 2}.e_j.x_{ij}$ which is (PD1). (PD1′) is obtained by linearization of (PD1). Precisely, let $f(x_1,x_2) = (x_1{}^2x_2)x_1$ in the free nonassociative algebra $\Phi\{\{X\}\}'$ (§1.6). Then $fD_1 = ((x_1y + yx_1)x_2)x_1 + (x_1{}^2x_2)y$ is a linearization of f and, by (PD1), $f(x_{ij}, e_i - e_j) = 0$ for all $x_{ij} \in \mathfrak{J}_{ij}$. Since the degree of f in x_1 is three and Φ contains three distinct elements it follows as on p. 23 that $fD_1(x_{ij}, e_i - e_j, y_{ij}) = 0$ for all $x_{ij}, y_{ij} \in \mathfrak{J}_{ij}$. This is equivalent to (PD1′). (PD2) and (PD3) are obtained by applying Lemma 1 (iii) to the Jordan algebra $\mathfrak{B} = \mathfrak{J}_{ii} + \mathfrak{J}_{jj} + \mathfrak{J}_{ij}$ with identity $e_i + e_j$, idempotent e_i and $\mathfrak{B}_1(e_i) = \mathfrak{J}_{ii}$, $\mathfrak{B}_0(e_i) = \mathfrak{J}_{jj}$, $\mathfrak{B}_{\frac{1}{2}}(e_i) = \mathfrak{J}_{ij}$. The same considerations can be applied to obtain some of the other formulas. However, we shall derive (PD4)–(PD13) directly using the basic identities (B_1)–(B_3) and Lemma 2. We indicate the steps required. For (PD4) we put $a = z_{jj}$, $b = x_{ij}$, $c = y_{ij}$, $d = e_j$ in (B_3). For (PD5) multiply (PD4) by e_j. For (PD6) multiply (PD4) by e_i and interchange y and z. For (PD7) put $a = x_{ii}$, $b = y_{ij}$, $c = z_{jk}$, $d = e_i$ in (B_3). For (PD8) take $a = y_{jj}$, $b = x_{ij}$, $c = z_{jk}$, $d = e_i$ in (B_3). For (PD9) take $a = x_{ij}$, $b = y_{jk}$, $c = z_{kk}$, $d = e_k$ in (B_3). For (PD10) put $a = x_{ij}$, $b = y_{ij}$, $c = z_{jk}$, $d = e_i$ in (B_2). For (PD11) put $a = x_{ij}$, $b = z_{ki}$, $c = y_{jk}$, $d = e_k$ in (B_3). To obtain (PD12) multiply (PD11) by e_i. For (PD13) take $a = x_{ij}$, $b = y_{jk}$, $c = z_{kl}$, $d = e_i$ in (B_3).

Let e_1 and e_2 be nonzero orthogonal idempotents in $\mathfrak{J}$ (with 1) and put $v = e_1 + e_2$. Then e_1 and $e_2 \in \mathfrak{J}U_v$ which is a subalgebra of $\mathfrak{J}$ with identity element v, and $\mathfrak{J}U_v = \mathfrak{J}U_{e_1} \oplus \mathfrak{J}U_{e_2} \oplus \mathfrak{J}U_{e_1,e_2}$ is the Peirce decomposition of $\mathfrak{J}U_v$ with respect to e_1 and e_2. We shall now call e_1 and e_2 *connected* orthogonal idempotents if there exists an element u_{12} in $\mathfrak{J}U_{e_1,e_2}$ which is invertible in $\mathfrak{J}U_v$. We shall call the orthogonal idempotents e_1 and e_2 *strongly connected* if there exists an element $u_{12} \in \mathfrak{J}U_{e_1,e_2}$ such that $u_{12}^{\cdot 2} = e_1 + e_2$. Evidently this implies connectedness of e_1 and e_2. Clearly the relations of connectedness and strong connectedness are symmetric. We shall now prove transitivity of these relations in the following

LEMMA 4. *Let e_1, e_2, e_3 be nonzero orthogonal idempotent elements such that e_1 and e_2 are connected (strongly connected) and e_2 and e_3 are connected (strongly connected). Then e_1 and e_3 are connected (strongly connected).*

PROOF. We prove first the statement on connectedness. By hypothesis there exists an element $u_{12} \in \mathfrak{J}U_{e_1,e_2}$ which is invertible in $\mathfrak{B} = \mathfrak{J}U_{e_1} + \mathfrak{J}U_{e_2} + \mathfrak{J}U_{e_1,e_2}$. We prove first that the inverse u_{21} of u_{12} in $\mathfrak{B}$ is contained in $\mathfrak{J}U_{e_1,e_2}$. We put $\mathfrak{B}_{11} = \mathfrak{J}U_{e_1}$, $\mathfrak{B}_{22} = \mathfrak{J}U_{e_2}$, $\mathfrak{B}_{12} = \mathfrak{J}U_{e_1,e_2}$, so $\mathfrak{B} = \mathfrak{B}_{11} \oplus \mathfrak{B}_{22} \oplus \mathfrak{B}_{12}$ is the Peirce decomposition of the algebra $\mathfrak{B}$ with identity element v relative to the orthogonal idempotents e_1, e_2. Now it follows directly from Lemma 2 and

(PD5) that if $x_{12} \in \mathfrak{B}_{12}$ then $\mathfrak{B}_{12}U_{x_{12}} \subseteq \mathfrak{B}_{12}$, $\mathfrak{B}_{11}U_{x_{12}} \subseteq \mathfrak{B}_{22}$ and $\mathfrak{B}_{22}U_{x_{12}} \subseteq \mathfrak{B}_{11}$. Hence, if x_{12} is invertible in $\mathfrak{B}$ then $\mathfrak{B}_{12}U_{x_{12}} = \mathfrak{B}_{12}$ and $\mathfrak{B}_{12}U_{x_{12}}{}^{-1} = \mathfrak{B}_{12}$. In particular, $u_{21} = u_{12}U_{u_{12}}{}^{-1} \in \mathfrak{B}_{12}$. Similarly, the connectedness of e_2 and e_3 implies that we have elements u_{23}, u_{32} in $\mathfrak{J}U_{e_2,e_3}$ which are inverses in $\mathfrak{C} = \mathfrak{J}U_{e_2} + \mathfrak{J}U_{e_3} + \mathfrak{J}U_{e_2,e_3}$. Now put $u_{13} = 2u_{12}.u_{23}$, $u_{31} = 2u_{32}.u_{21}$. Then u_{13} and $u_{31} \in \mathfrak{J}U_{e_1,e_3}$ by Lemma 2 applied to $\mathfrak{D} = \mathfrak{J}U_w$, $w = e_1 + e_2 + e_3$. We shall show that u_{13} and u_{31} are inverses in $\mathfrak{J}U_{e_1} + \mathfrak{J}U_{e_3} + \mathfrak{J}U_{e_1,e_3}$ which will prove the connectedness of e_1 and e_3. We recall that the universal multiplication envelope of a Jordan algebra with 1 generated by an invertible element and its inverse is commutative (p. 102). It follows that if a and b are inverses in a subalgebra of a Jordan algebra $\mathfrak{J}$ then R_a and R_b commute in $\mathfrak{J}$. Applying this to u_{23} and u_{32} we obtain $u_{23}.u_{31} = 2u_{23}.(u_{32}.u_{21}) = u_{23}.(u_{32}.u_{21}) + u_{32}.(u_{23}.u_{21}) = (u_{23}.u_{32}).u_{21}$ (PD10) $= (e_2 + e_3).u_{21} = \frac{1}{2}u_{21}$. Then

$$(u_{13}.u_{31}).e_1 = 2((u_{12}.u_{23}).u_{31}).e_1 = 2(u_{12}.(u_{23}.u_{31})).e_1 \quad \text{(PD12)}$$

$$= (u_{12}.u_{21}).e_1 = e_1.$$

By symmetry, $(u_{13}.u_{31}).e_3 = e_3$. Hence $u_{13}.u_{31} = e_1 + e_3$. Next we have $u_{21}.u_{13} = \frac{1}{2}u_{23}$ by the same argument used to prove $u_{23}.u_{31} = \frac{1}{2}u_{21}$. Then $(u_{13}{}^{.2}.e_1).u_{31} = 2(u_{13}{}^{.2}.e_1).(u_{32}.u_{21}) = 2((u_{13}{}^{.2}.e_1).u_{21}).u_{32}$ (PD9) $= 2(u_{13}{}^{.2}.u_{21}).u_{32} = 4(u_{13}.(u_{13}.u_{21})).u_{32}$ (PD10) $= 2(u_{13}.u_{23}).u_{32} = (u_{13}.u_{23}).u_{32} + (u_{13}.u_{32}).u_{23} = u_{13}.(u_{23}.u_{32}) = u_{13}.(e_2 + e_3) = \frac{1}{2}u_{13}$ where in the last part we have used the commutativity of $R_{u_{23}}$ and $R_{u_{32}}$ and (PD10). Similarly, we obtain $(u_{13}{}^{.2}.e_3).u_{31} = \frac{1}{2}u_{13}$. Hence $u_{13}{}^{.2}.u_{31} = u_{13}$. Since we had $u_{13}.u_{31} = e_1 + e_3$ it follows that u_{13} and u_{31} are inverses in $\mathfrak{J}U_{e_1} + \mathfrak{J}U_{e_3} + \mathfrak{J}U_{e_1,e_3}$ as required. Now assume e_1 and e_2 and e_2 and e_3 are strongly connected. Then we have elements $u_{12} \in \mathfrak{J}U_{e_1,e_2}$ and $u_{23} \in \mathfrak{J}U_{e_2,e_3}$ such that $u_{12}{}^{.2} = e_1 + e_2$, $u_{23}{}^{.2} = e_2 + e_3$. Hence in the foregoing discussion we can take $u_{21} = u_{12}$, $u_{32} = u_{23}$. Then the argument shows that the inverse of $u_{13} = 2u_{12}.u_{23}$ in $\mathfrak{J}U_{e_1} + \mathfrak{J}U_{e_3} + \mathfrak{J}U_{e_1,e_3}$ is $u_{31} = 2u_{32}.u_{21}$. Hence $u_{13}{}^{.2} = e_1 + e_3$ and e_1 and e_3 are strongly connected.

We shall need the following result on connected idempotents later.

Lemma 5. *Let $\mathfrak{J}$ be a Jordan algebra with $1 = \sum_{i=1}^{n} e_i$ where the e_i are nonzero connected orthogonal idempotents and let $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ be the Peirce decomposition of $\mathfrak{J}$ relative to the e_i. Let u_{1j}, u_{j1}, $j > 1$, be elements of $\mathfrak{J}_{1j}$ which are inverses in $\mathfrak{J}_{11} + \mathfrak{J}_{jj} + \mathfrak{J}_{1j}$ and put*

$$f_1 = e_1 = g_1, \quad f_j = u_{j1}{}^{.2}.e_j, \quad g_j = u_{1j}{}^{.2}.e_j, \quad j > 1, \tag{14}$$
$$f = \sum_1^n f_i, \quad g = \sum_1^n g_i.$$

Then f and g are inverses and the elements f_i are orthogonal idempotents in the *isotope* $(\mathfrak{J}, g)$ *and* $\Sigma f_i = f$ *the identity of* $(\mathfrak{J}, g)$. *Moreover,* u_{j1} *is in the Peirce space* $(\mathfrak{J}, g)_{1j}$ *relative to the* f_i *and* $u_{j1} \cdot_g u_{j1} = f_1 + f_j$.

REMARK. In a less precise form the hypothesis is that the e_i are connected idempotents with sum 1 in $\mathfrak{J}$ and the conclusion is that the f_i are strongly connected idempotents with sum the identity element in the isotope $(\mathfrak{J}, g)$ of $\mathfrak{J}$.

PROOF. Clearly $u_{1j}^{\cdot 2}$ and $u_{j1}^{\cdot 2}$ are inverses in $\mathfrak{J}_{11} + \mathfrak{J}_{jj} + \mathfrak{J}_{1j}$. Since these elements are contained in $\mathfrak{J}_{11} + \mathfrak{J}_{jj}$ and $\mathfrak{J}_{11} \cdot \mathfrak{J}_{jj} = 0$ it follows that $f_j = u_{j1}^{\cdot 2} . e_j$ and $g_j = u_{1j}^{\cdot 2} . e_j$ are inverses in $\mathfrak{J}_{jj}$, $j > 1$. This implies that $f = \Sigma f_i$ and $g = \Sigma g_i$ are inverses in $\mathfrak{J}$. Hence we can form the isotope $(\mathfrak{J}, g)$ whose identity element is f. We have

$$\begin{aligned} \{f_i g f_i\} &= \{f_i(\Sigma g_k) f_i\} = \{f_i g_i f_i\} = f_i, \\ \{f_i g f_j\} &= \{f_i g_i f_j\} + \{f_i g_j f_j\} = 0, \qquad i \neq j. \end{aligned} \tag{15}$$

Hence the f_i are orthogonal idempotents in $(\mathfrak{J}, g)$ and their sum is the identity element f. We have for $j > 1$,

$$\begin{aligned} \{u_{j1} g f_1\} &= \{u_{j1} e_1 e_1\} + \{u_{j1} g_j e_1\} \\ &= u_{j1} . e_1 . e_1 + e_1 . e_1 . u_{j1} - u_{j1} . e_1 . e_1 \\ &\quad + u_{j1} . g_j . e_1 + g_j . e_1 . u_{j1} - u_{j1} . e_1 . g_j \\ &= \tfrac{1}{2} u_{j1} \quad \text{(Lemma 2)} \\ \{u_{j1} g f_j\} &= \{u_{\ 1} e_1 f_j\} + \{u_{j1} g_j f_j\} \\ &= u_{j1} . e_1 . f_j + e_1 . f_j . u_{j1} - u_{j1} . f_j . e_1 \\ &\quad + u_{j1} . g_j . f_j + g_j . f_j . u_{j1} - u_{j1} . f_j . g_j \\ &= e_j . u_{j1} \quad \text{(Lemma 2 and commutativity of } R_{f_j} \text{ and } R_{g_j}) \\ &= \tfrac{1}{2} u_{j1}. \end{aligned}$$

Hence $u_{j1} \in (\mathfrak{J}, g)_{1j}$. Also

$$\begin{aligned} \{u_{j1} g u_{j1}\} &= \{u_{j1} e_1 u_{j1}\} + \{u_{j1} g_j u_{j1}\} \\ &= 2e_1 . u_{j1} . u_{j1} - e_1 . u_{j1}^{\cdot 2} + 2g_j . u_{j1} . u_{j1} - g_j . u_{j1}^{\cdot 2} \\ &= u_{j1}^{\cdot 2} - e_1 . u_{j1}^{\cdot 2} + 2u_{1j}^{\cdot 2} . e_j . u_{j1} . u_{j1} - u_{1j}^{\cdot 2} . e_j . u_{j1}^{\cdot 2} \\ &= e_j . u_{j1}^{\cdot 2} + 2e_j . u_{j1} . u_{1j}^{\cdot 2} . u_{j1} - u_{1j}^{\cdot 2} . e_j . (u_{j1}^{\cdot 2} . e_j) \\ &\qquad \text{(commutativity of } R_{u_{j1}} \text{ and } R_{u_{1j}^{\cdot 2}}, \text{ Lemma 2)} \\ &= f_j + e_1 + e_j - e_j \\ &= f_1 + f_j. \end{aligned}$$

Hence $u_{j1} \cdot_g u_{j1} = f_1 + f_j$ and the proof is complete.

Exercises

1. Show that the set of restrictions of the $R_{x_{ii}}$, $x_{ii} \in \mathfrak{J}_{ii}$ to $\mathfrak{J}_{ij}$, $i \neq j$, form a subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{J}_{ij}, \mathfrak{J}_{ij})^+$.
2. Verify that $\mathfrak{J}_{ij}{}^{\cdot 2}.e_i$ is an ideal in $\mathfrak{J}_{ii}$.
3. Let $\mathfrak{J}$ be a Jordan algebra with 1 containing nonzero connected idempotents e_1 and e_2. Let $u_{12} \in \mathfrak{J}U_{e_1,e_2}$ be an invertible element in $\mathfrak{J}U_{e_1} + \mathfrak{J}U_{e_2} + \mathfrak{J}U_{e_1,e_2}$. Verify that $\{e_1 u_{12} e_1\} = 0$ and use this to show that there exists an isotope of $\mathfrak{J}$ in which e_1 is nilpotent.

2. **Jordan matrix algebras.** Let $\mathfrak{D}$ be an arbitrary algebra of characteristic not two with identity element 1 and involution $j: d \to \bar{d}$ and let $\mathfrak{D}_n$ be the algebra of $n \times n$ matrices with entries in $\mathfrak{D}$. We have called the involution $X \to \bar{X}^t$ the *standard* involution in $\mathfrak{D}_n$ (§1.5). More generally, we shall consider involutions $J_a: X \to a^{-1}\bar{X}^t a$ where

(16) $$a = \mathrm{diag}\{a_1, a_2, \cdots, a_n\}$$

and the a_i satisfy: $\bar{a}_i = a_i$, $a_i \in N(\mathfrak{D})$, the nucleus of $\mathfrak{D}$, and $a_i{}^{-1}$ exists in the associative algebra $N(\mathfrak{D})$. We shall call such involutions *canonical* and we let $\mathfrak{H}(\mathfrak{D}_n, J_a)$ denote the subalgebra of $\mathfrak{D}_n^+$ of elements symmetric relative to J_a. We remark that in the present notation the standard involution is J_1, where 1 is the identity matrix. Moreover, we shall usually write $\mathfrak{H}(\mathfrak{D}_n)$ for $\mathfrak{H}(\mathfrak{D}_n, J_1)$.

We shall now introduce a notation for certain elements of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ which will be used throughout this book. Let e_{ij}, $i,j = 1, \cdots, n$, be the usual matrix units in $\mathfrak{D}_n$, and if $x \in \mathfrak{D}$ then we identify x with the diagonal matrix in $\mathfrak{D}_n$ all of whose diagonal entries are x. Then xe_{ij} is the matrix with x in the (i,j)-position and 0's elsewhere. Now let $x \in \mathfrak{D}$ and let $i,j = 1, 2, \cdots, n$. Then we put

(17) $$x[ij] = xe_{ij} + (xe_{ij})^J = xe_{ij} + (a_j{}^{-1}\bar{x}a_i)e_{ji}.$$

Since the characteristic is not two $\mathfrak{H}(\mathfrak{D}_n, J_a)$ coincides with the set of elements $X + X^{J_a}$. It follows that every element of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is a sum of elements $x[ij]$, $x \in \mathfrak{D}$, $i,j = 1, \cdots, n$. We now put $\mathfrak{H}_{ij} = \{x[ij] \mid x \in \mathfrak{D}\}$. Then $\mathfrak{H}_{ij} = \mathfrak{H}_{ji}$ and $\mathfrak{H}(\mathfrak{D}_n, J_a) = \sum_{i \leq j = 1}^n \oplus \mathfrak{H}_{ij}$. We leave it to the reader to verify the following multiplication rules in $\mathfrak{H}(\mathfrak{D}_n, J_a)$:

(18) $$2x[ij].y[jk] = xy[ik], \qquad i,j,k \neq ,$$

(19) $$2x[ii].y[ij] = (xy + (a_i^{-1}\bar{x}a_i)y)[ij], \qquad i \neq j,$$

(20) $$2x[ij].y[ji] = xy[ii] + yx[jj], \qquad i \neq j,$$

(21) $$2x[ii].y[ii] = (x + a_i{}^{-1}\bar{x}a_i).(y + a_i{}^{-1}\bar{y}a_i)[ii].$$

Also we have

(22) $$x[ij] = (a_j{}^{-1}\bar{x}a_i)[ji]$$

and

(23) $$x[ij]\,.\,y[kl] = 0, \quad \text{if} \quad \{i,j\} \cap \{k,l\} = \varnothing.$$

Clearly these formulas give every product $x[ij]\,.\,y[kl]$ for all possible pairs (i,j), (k,l). If the involution is standard, that is, $a = 1$, then the foregoing formulas simplify to the following

(18′) $$2x[ij]\,.\,y[jk] = xy[ik], \qquad i,j,k \neq ,$$

(19′) $$2x[ii]\,.\,y[ij] = (x + \bar{x})y[ij], \qquad i \neq j,$$

(20′) $$2x[ij]\,.\,y[ji] = xy[ii] + yx[jj], \qquad i \neq j,$$

(21′) $$2x[ii]\,.\,y[ii] = (x + \bar{x})\,.\,(y + \bar{y})[ii],$$

(22′) $$x[ij] = \bar{x}[ji],$$

(23′) $$x[ij]\,.\,y[kl] = 0 \quad \text{if} \quad \{i,j\} \cap \{k,l\} = \varnothing.$$

We now assume that $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is Jordan. Let $e_i = \frac{1}{2}[ii]$. Then, by (21) and (23), the e_i are orthogonal idempotents in $\mathfrak{H}(\mathfrak{D}_n, J_a)$. Moreover, $e_i = e_{ii}$ so $\sum e_i = 1$. By (21) and (19), $e_i\,.\,x[ii] = x[ii]$ and $e_i\,.\,x[ij] = \frac{1}{2}x[ij]$. By (23), $e_i\,.\,x[jk] = 0$ if $i \neq j$ and $i \neq k$. It follows that the $\mathfrak{H}_{ij}$ are the Peirce spaces of $\mathfrak{H} = \mathfrak{H}(\mathfrak{D}_n, J_a)$ relative to the idempotents e_i. We shall now show that the fact that $\mathfrak{H}$ is a Jordan algebra implies that $\mathfrak{D}$ is associative if $n \geqq 4$ and $\mathfrak{D}$ is alternative with its symmetric elements in the nucleus if $n = 3$. Assume first that $n \geqq 4$. Let x, y, z be arbitrary in $\mathfrak{D}$, i, j, k, l unequal indices. Then $4(x[ij]\,.\,y[jk])\,.\,z[kl] = (xy)z[il]$ and $4x[ij]\,.\,(y[jk]\,.\,z[kl]) = x(yz)[il]$, by (18). On the other hand, since $x[ij] \in \mathfrak{H}_{ij}$, $y[jk] \in \mathfrak{H}_{jk}$, $z[kl] \in \mathfrak{H}_{kl}$ and these are the Peirce spaces of $\mathfrak{H}$ relative to the e_i, (PD13) implies that $(x[ij]\,.\,y[jk])\,.\,z[kl] = x[ij]\,.\,(y[jk]\,.\,z[kl])$. Hence $(xy)z = x(yz)$ and $\mathfrak{D}$ is associative. Next assume $n = 3$. We have shown in §1.12 that $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is Jordan if and only if $\mathfrak{H}(\mathfrak{D}_n)$ is Jordan. Hence we may assume that the involution in $\mathfrak{D}_3$ is standard. Let x, y, z be arbitrary in $\mathfrak{D}$, i, j, k distinct indices. Then $2(x[ij]\,.\,y[jk])\,.\,z[kk] = 2x[ij]\,.\,(y[jk]\,.\,z[kk])$ by (PD9). Using (18′) and (19′) this gives $(xy)(z + \bar{z}) = x(y(z + \bar{z}))$, so $[x, y, z + \bar{z}] = 0$. By (PD10) we have $4x[ij]\,.\,y[ij]\,.\,z[jk] = 4x[ij]\,.\,z[jk]\,.\,y[ij] + 4y[ij]\,.\,z[jk]\,.\,x[ij]$. Using (22′), (20′), (19′) and (18′) this gives $((\bar{y}x + \bar{x}y)z)[jk] = (\bar{y}(xz) + \bar{x}(yz))[jk]$. Hence $[\bar{y}, x, z] + [\bar{x}, y, z] = 0$. If we take $x = y$ this gives $[\bar{x}, x, z] = 0$. Applying the involution to $[x, y, z + \bar{z}] = 0$ gives $[z + \bar{z}, \bar{y}, \bar{x}] = 0$ which is the same as $[z + \bar{z}, y, x] = 0$. Then $[x + \bar{x}, x, z] = 0$ and since $[\bar{x}, x, z] = 0$ we have $[x, x, z] = 0$. Applying the involution gives $[z, x, x] = 0$. Hence $\mathfrak{D}$ is alternative. Then $[x, y, z + \bar{z}] = 0$ implies also $[x, z + \bar{z}, y] = 0$. Since we had $[z + \bar{z}, x, y] = 0$ it is clear that $z + \bar{z}$ is in the nucleus. This implies that the symmetric elements are in the nucleus. Hence we have shown that necessary conditions that $\mathfrak{H}(\mathfrak{D}_n, J_a)$ with $n \geqq 3$ is Jordan are that $\mathfrak{D}$ is associative if $n \geqq 4$ and $\mathfrak{D}$ is alternative with self-adjoint elements in the nucleus if $n = 3$. In the Corollary to Theorem 1.14 we proved that these conditions are also sufficient. Hence we have

THEOREM 1. *Let $\mathfrak{D}$ be an algebra over a field of characteristic not two with an identity element and an involution j, and let J_a be a canonical involution in $\mathfrak{D}_n$. Then $\mathfrak{H}(\mathfrak{D}_n, J_a)$ for $n \geqq 3$ is Jordan if and only if either $\mathfrak{D}$ is associative or $n = 3$ and $\mathfrak{D}$ is alternative with symmetric elements in the nucleus.*

We shall call the algebras $\mathfrak{H}(\mathfrak{D}_n, J_a)$ which are Jordan, *Jordan matrix algebras.* Also we shall call the integer n the *order* of the algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ (whether Jordan or not). The conditions for $\mathfrak{H}(\mathfrak{D}_n, J_a)$ to be Jordan if $n \geqq 3$ are given in the foregoing theorem. Necessary and sufficient conditions for the Jordan property for the case $\mathfrak{H}(\mathfrak{D}_2, J_a)$ can also be given. Since these will not be required in the sequel we shall be content just to state these in an exercise (ex. 4 below). (For $n = 1$ there seems to be nothing more to state than the Jordan condition $a^{\cdot 2}.b.a = a^{\cdot 2}.(b.a)$ for $a = d + \bar{d}$, $b = e + \bar{e}$, d, e in $\mathfrak{D}$.)

We shall indicate next two important special cases of Jordan matrix algebras. First, let $\mathfrak{D}$ be an arbitrary associative algebra with 1 and let $\mathfrak{E} = \mathfrak{D} \oplus \mathfrak{D}^\circ$ where $\mathfrak{D}^\circ$ is the opposite algebra of $\mathfrak{D}$. Thus $\mathfrak{E}$ is the set of pairs (a_1, a_2), a_1, a_2 in $\mathfrak{D}$ with the multiplication $(a_1, a_2)(b_1, b_2) = (a_1 b_1, b_2 a_2)$ (cf. §1.4). We have the involution $j: (a_1, a_2) \to (a_2, a_1)$ in $\mathfrak{E}$ which gives rise to the standard involution J_1 in $\mathfrak{E}_n$ and to the Jordan matrix algebra $\mathfrak{H}(\mathfrak{E}_n) = \mathfrak{H}(\mathfrak{E}_n, J_1)$. Next we consider $\mathfrak{F} = \mathfrak{D}_n \oplus \mathfrak{D}_n{}^\circ$ where $\mathfrak{D}_n{}^\circ$ is the opposite algebra of $\mathfrak{D}_n$. As we saw in §1.4, if J is the involution $(A_1, A_2) \to (A_2, A_1)$ in $\mathfrak{F}$ then $\mathfrak{H}(\mathfrak{F}, J)$ is the set of elements (A, A) and the mapping $A \to (A, A)$ is an isomorphism of $\mathfrak{D}_n{}^+$ onto $\mathfrak{H}(\mathfrak{F}, J)$. Let (A_1, A_2) be an element of $\mathfrak{F}$ and write $A_1 = (a_{1ij})$, $A_2 = (a_{2ij})$ where a_{kij} is the (i,j)-entry of A_k. Let $(A_1, A_2)^\eta$ be the element of $\mathfrak{E}_n$ whose (i,j)-entry is the element (a_{1ij}, a_{2ji}) of $\mathfrak{E} = \mathfrak{D} \oplus \mathfrak{D}^\circ$. Then one verifies directly that η is an isomorphism of $(\mathfrak{F}, J)$ onto $(\mathfrak{E}_n, J_1)$. Hence $\mathfrak{H}(\mathfrak{E}_n) \cong \mathfrak{H}(\mathfrak{F}, J) \cong \mathfrak{D}_n{}^+$. Thus we see that the Jordan algebra $\mathfrak{D}_n{}^+$, where $\mathfrak{D}$ is an arbitrary associative algebra with 1, is isomorphic to the Jordan matrix algebra $\mathfrak{H}(\mathfrak{E}_n)$ where $\mathfrak{E} = \mathfrak{D} \oplus \mathfrak{D}^\circ$ and j is the usual involution exchanging the two factors.

We consider next the matrix algebra Φ_{2n} and the involution $J_S: X \to S^{-1} X^t S$ in Φ_{2n} where

$$S = \mathrm{diag}\{Q, Q, \cdots, Q\} \tag{24}$$

and

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{25}$$

A matrix which is symmetric relative to J_S ($S^{-1} A^t S = A$) will be called *symplectic symmetric.* We shall show that the Jordan algebra $\mathfrak{H}(\Phi_{2n}, J_S)$ consisting of these matrices is isomorphic to a Jordan matrix algebra.

Let $\mathfrak{D} = \Phi_2$ and let

$$i_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad i_2 = Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{26}$$

Then one checks that $(1, i_1, i_2, i_1 i_2)$ is a basis for $\mathfrak{D}$ and we have the basic multiplication formulas: $i_1{}^2 = 1$, $i_2{}^2 = -1$, $i_1 i_2 = -i_2 i_1$. Hence $\mathfrak{D}$ is a quaternion algebra and in $\mathfrak{D}$ we have the standard involution $j: \alpha_0 + \alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_1 i_2 \to \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 i_1 i_2$, $\alpha_j \in \Phi$. Since

$$\begin{aligned} Q^{-1} i_1{}^t Q &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -i_1, \\ Q^{-1} i_2{}^t Q &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i_2, \end{aligned} \tag{27}$$

it is clear that j is the involution $a \to Q^{-1} a^t Q$ in $\mathfrak{D}$.

We now consider $\mathfrak{D}_n = (\Phi_2)_n$ and let J_1 be the standard involution in $\mathfrak{D}_n$ corresponding to the involution j in $\mathfrak{D}$. If $A \in \mathfrak{D}_n$ then A is an $n \times n$ matrix whose entries are 2×2 matrices with entries in Φ. Hence A determines a $2n \times 2n$ matrix A^η in Φ_{2n}. It is well known and clear from block multiplication of matrices that $A \to A^\eta$ is an isomorphism of $\mathfrak{D}_n$ onto Φ_{2n}. It is immediate also that η is an isomorphism of $(\mathfrak{D}_n, J_1)$ onto (Φ_{2n}, J_S). Hence the Jordan algebra $\mathfrak{H}(\Phi_{2n}, J_S)$ of symplectic symmetric matrices is isomorphic to a Jordan matrix algebra. We remark also that it is well known and easily seen that (Φ_{2n}, J_S) is isomorphic to $(\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V}), J)$ where $\mathfrak{V}$ is a $2n$-dimensional vector space over Φ and J is the involution $A \to A^*$, A^* the adjoint (or transpose) of A relative to a nondegenerate skew bilinear form f on $\mathfrak{M}$.

Let J_a be the canonical involution in $\mathfrak{D}_n$ defined by the involution j in $\mathfrak{D}$ and the diagonal matrix $a = \mathrm{diag}\{a_1, a_2, \cdots, a_n\}$ where $a_i = \bar{a}_i$ is in the nucleus of $\mathfrak{D}$ and $a_i{}^{-1}$ exists in the nucleus. Then one checks directly that the mapping $k: d \to a_1{}^{-1} \bar{d} a_1$ is an involution in $\mathfrak{D}$ and J_a coincides with the canonical involution J_b defined by the involution k in $\mathfrak{D}$ and the diagonal matrix

$$b = \mathrm{diag}\{1, a_1{}^{-1} a_2, \cdots, a_1{}^{-1} a_n\}.$$

Hence there is no loss in generality in assuming that $a_1 = 1$ and we shall do this from now on.

If $(\mathfrak{D}, j)$ is an algebra with involution then, as in the associative case considered in §1.4, we define an *ideal* of $(\mathfrak{D}, j)$ to be an ideal of $\mathfrak{D}$ which is invariant under j. Similarly, a *subalgebra* of $(\mathfrak{D}, j)$ is a subalgebra of $\mathfrak{D}$ invariant under j. If $\mathfrak{E}$ is an ideal (subalgebra) of $\mathfrak{D}$ then the set $\mathfrak{E}_n$ of $n \times n$ matrices with entries in $\mathfrak{E}$ is an ideal (subalgebra) of $\mathfrak{D}_n$ and hence $\mathfrak{H} \cap \mathfrak{E}_n$ is an ideal (subalgebra) of $\mathfrak{H} = \mathfrak{H}(\mathfrak{D}_n, J_a)$. We shall now prove

THEOREM 2. *Let $\mathfrak{H} = \mathfrak{H}(\mathfrak{D}_n, J_a)$ be a Jordan matrix algebra of order $n \geqq 3$ defined by the canonical involution J_a in $\mathfrak{D}_n$ such that $a_1 = 1$. Then the mapping $\mathfrak{E} \to \mathfrak{H} \cap \mathfrak{E}_n$ is a lattice isomorphism of the lattice of subalgebras $\mathfrak{E}$ of $(\mathfrak{D},j)$ containing the elements a_i and a_i^{-1} onto the lattice of subalgebras of $\mathfrak{H}$ containing the elements $1[ij]$, $i,j = 1,\cdots,n$. Also the mapping $\mathfrak{E} \to \mathfrak{H} \cap \mathfrak{E}_n$ is a lattice isomorphism of the lattice of ideals $\mathfrak{E}$ of $(\mathfrak{D},j)$ onto the lattice of ideals of $\mathfrak{H}$. In this $\mathfrak{K} = \mathfrak{H} \cap \mathfrak{E}_n$ satisfies $\mathfrak{K}^{\cdot 2} = 0$ if and only if $\mathfrak{E}^2 = 0$.*

PROOF. In order to treat the case of ideals and subalgebras simultaneously we assume that $\mathfrak{K}$ is a subspace of $\mathfrak{H}$ which is mapped into itself by Jordan multiplication by the elements $1[ij]$, $i,j = 1,\cdots,n$. Let $\mathfrak{E}$ denote the subset of elements of $\mathfrak{D}$ which are entries of the matrices $\in \mathfrak{K}$. Let $k = \sum d_{ij}e_{ij}$, $d_{ij} \in \mathfrak{D}$, be in $\mathfrak{K}$. Then $kU_{e_{ii}} = d_{ii}e_{ii} \in \mathfrak{K}$ since $U_{e_{ii}} = 2R_{e_{ii}}{}^2 - R_{e_{ii}}$ and $e_{ii} = \frac{1}{2}[ii]$. Also $kU_{e_{ii}} \in \mathfrak{H}$ so $d_{ii}e_{ii} = a_i^{-1}\bar{d}_{ii}a_ie_{ii}$ and $d_{ii}e_{ii} = \frac{1}{2}d_{ii}[ii]$. By (19) and the hypothesis on $\mathfrak{K}$, $d_{ii}[ij] = d_{ii}[ii].1[ij] \in \mathfrak{K}$ for any $j \neq i$. Also we have that $2k\,U_{e_{ii},e_{jj}} = d_{ij}e_{ij} + d_{ji}e_{ji} \in \mathfrak{K}$ for any $i \neq j$ since $U_{e_{ii},e_{jj}} = R_{e_{ii}}R_{e_{jj}} + R_{e_{jj}}R_{e_{ii}}$. Since $d_{ij}e_{ij} + d_{ji}e_{ji} \in \mathfrak{H}$ it is clear that $d_{ij}e_{ij} + d_{ji}e_{ji} = d_{ij}[ij]$. We have therefore shown that $\mathfrak{E}$ is the set of elements $d \in \mathfrak{D}$ such that $d[ij] \in \mathfrak{K}$ for some pair (i,j) with $i \neq j$. If $d[ij] \in \mathfrak{K}$ and $k \neq i,j$ then $d[ik] = 2d[ij].1[jk] \in \mathfrak{K}$ and $d[kj] = 2(1[ki].d[ij]) \in \mathfrak{K}$. It follows that $d[kl] \in \mathfrak{K}$ for every (k,l) with $k \neq l$. Also

$$d[kk] = 2d[kl].1[lk].\tfrac{1}{2}[kk] \in \mathfrak{K}.$$

It is now clear that $\mathfrak{E}$ is a subspace of $\mathfrak{D}$ and if $d \in \mathfrak{E}$ then $d[ij] \in \mathfrak{K}$ for every i,j. Clearly $\mathfrak{K} \subseteq \mathfrak{E}_n \cap \mathfrak{H}$ holds by the definition of the set $\mathfrak{E}$. On the other hand, if $k = \sum d_{ij}e_{ij} \in \mathfrak{E}_n \cap \mathfrak{H}$ then every $d_{ij} \in \mathfrak{E}$ and hence $d_{ij}[ij] \in \mathfrak{K}$. Also since $k = \sum d_{ij}e_{ij} \in \mathfrak{H}$, $k = \sum_{i<j} d_{ij}[ij] + \frac{1}{2}\sum_i d_{ii}[ii]$ and this is contained in $\mathfrak{K}$. Hence $\mathfrak{K} = \mathfrak{E}_n \cap \mathfrak{H}$. Now let $\mathfrak{K}$ be a subalgebra of $\mathfrak{H}$ containing the elements $1[ij], i,j = 1,\cdots,n$. Then our results apply to $\mathfrak{K}$ so $\mathfrak{K} = \mathfrak{E}_n \cap \mathfrak{H}$ where $\mathfrak{E}$ is the set of entries in $\mathfrak{D}$ of the matrices belonging to $\mathfrak{K}$ and $\mathfrak{E}$ is a subspace. Moreover, if $d \in \mathfrak{E}$ then $d[ij] \in \mathfrak{K}$ for all i,j. It is now clear from (18) that $\mathfrak{E}$ is a subalgebra of $\mathfrak{D}$. Also, since $a_1 = 1$ and $u_{j1} = e_{j1} + a_1^{-1}a_je_{1j} = e_{j1} + a_je_{1j} \in \mathfrak{K}$, $a_j \in \mathfrak{E}$. Similarly, since $u_{1j} \in \mathfrak{K}$, $a_j^{-1} \in \mathfrak{E}$. Finally, if $d \in \mathfrak{E}$ then $d[12] = a_2^{-1}\,\bar{d}a_1[21] \in \mathfrak{K}$. Hence $a_2^{-1}\bar{d}a_1 \in \mathfrak{E}$ and $\bar{d} \in \mathfrak{E}$. Thus $\mathfrak{E}$ is a subalgebra of $(\mathfrak{D},j)$ containing the elements a_i and a_i^{-1}. The first statement of the theorem is now clear. A similar argument yields the second statement of the theorem. The last statement is clear from the multiplication formula (18).

Let $(\mathfrak{E},k)$ be a second algebra with involution k and identity element 1 and let K_b be the canonical involution in $\mathfrak{E}_n$ defined by k and $b = \operatorname{diag}\{b_1 = 1, b_2,\cdots,b_n\}$. If $e \in \mathfrak{E}$ we put $e\{ij\} = ee_{ij} + b_j^{-1}e^*b_ie_{ji}$, $e^* = e^k$. Suppose η is a homomorphism of $(\mathfrak{D},j)$ into $(\mathfrak{E},k)$ such that $a_i^{\eta} = b_i$, $i = 1,2,\cdots,n$. Then it is immediate that the mapping $(d_{ij}) \to (d_{ij}^{\eta})$ is a homomorphism of $(\mathfrak{D}_n, J_a)$ into $(\mathfrak{E}_n, K_b)$. Hence the restriction σ of this mapping to $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is a homomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_a)$

into $\mathfrak{H}(\mathfrak{E}_n, J_b)$. Also it is clear that $1[ij]^\sigma = 1\{ij\}$ for $i,j = 1,2,\cdots,n$. Now suppose $n \geqq 3$ and assume $\mathfrak{H}(\mathfrak{D}_n, J_a)$ and $\mathfrak{H}(\mathfrak{E}_n, K_b)$ are Jordan. Let σ be a homomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ into $\mathfrak{H}(\mathfrak{E}_n, K_b)$ such that $1[ij]^\sigma = 1\{ij\}$, $i,j = 1,\cdots,n$. We have $e_{ii}{}^\sigma = e_{ii}$ so the Peirce space $\mathfrak{H}(\mathfrak{D}_n, J_a)_{ij}$ relative to the e_{ii} in $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is mapped into the Peirce space $\mathfrak{H}(\mathfrak{E}_n, K_b)_{ij}$. It follows that for every pair (i,j) we have a mapping η_{ij} of $\mathfrak{D}$ into $\mathfrak{E}$ such that $d[ij]^\sigma = d^{\eta_{ij}}\{ij\}$. Clearly η_{ij} is linear and $1^{\eta_{ij}} = 1$. By (18) and the corresponding formula in $\mathfrak{E}_n$ we have for $a, b \in \mathfrak{D}$ and i,j,k distinct indices that $(ab)^{\eta_{ik}}\{ik\} = ab[ik]^\sigma = 2(a[ij] . b[jk])^\sigma = 2a[ij]^\sigma . b[jk]^\sigma = 2a^{\eta_{ij}}\{ij\} . b^{\eta_{jk}}\{jk\} = a^{\eta_{ij}}b^{\eta_{jk}}\{ik\}$. Hence $(ab)^{\eta_{ik}} = a^{\eta_{ij}}b^{\eta_{jk}}$. If we put $b = 1$ in this we obtain $a^{\eta_{ik}} = a^{\eta_{ij}}$ and if we put $a = 1$ we obtain $b^{\eta_{ik}} = b^{\eta_{jk}}$. It follows that $\eta_{ij} = \eta_{kl}$ for any (i,j), (k,l) such that $i \neq j$, $k \neq l$. If we call this linear mapping η, then the relation $(ab)^{\eta_{ik}} = a^{\eta_{ij}}b^{\eta_{jk}}$ gives $(ab)^\eta = a^\eta b^\eta$, so η is a homomorphism of $\mathfrak{D}$ into $\mathfrak{E}$ such that $1^\eta = 1$. If $i \neq 1$ we have for $d \in \mathfrak{D}$: $d[i1]^\sigma = d^\eta\{i1\} = b_1{}^{-1}(d^\eta)^* b_i\{1i\} = (d^\eta)^* b_i\{1i\}$ and $d[i1]^\sigma = a_1{}^{-1}\bar{d}a_i[1i]^\sigma = \bar{d}a_i[1i]^\sigma = (\bar{d}a_i)^\eta\{1i\}$. Hence $(d^\eta)^* b_i = (\bar{d}a_i)^\eta = \bar{d}^\eta a_i{}^\eta$. If we take $d = 1$ in this we obtain $b_i = a_i{}^\eta$. Then $(d^\eta)^* = \bar{d}^\eta$ so η is a homomorphism of $(\mathfrak{D}, j)$ into $(\mathfrak{E}, k)$ sending a_i into b_i for $i = 1,2,\cdots,n$. Then $(d_{ij}) \to (d_{ij}{}^\eta)$ is a homomorphism of $(\mathfrak{D}_n, J_a)$ into $(\mathfrak{E}_n, K_b)$ so its restriction τ to $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is a homomorphism into $\mathfrak{H}(\mathfrak{E}_n, K_b)$. If $i \neq j$ and $d \in \mathfrak{D}$ then $d[ij]^\sigma = d^\eta\{ij\} = d^\eta e_{ij} + b_j{}^{-1}(d^\eta)^* b_i e_{ji}$ and $d[ij]^\tau = (de_{ij} + a_j{}^{-1}\bar{d}a_i e_{ji})^\tau = d^\eta e_{ij} + (a_j{}^{-1}\bar{d}a_i)^\eta e_{ji} = d^\eta e_{ij} + b_j{}^{-1}\bar{d}^\eta b_i e_{ji} = d^\eta e_{ij} + b_j{}^{-1}(d^\eta)^* b_i e_{ji}$. Thus $d[ij]^\sigma = d[ij]^\tau$. Now it is easily seen from (20) and (21) that the elements $d[ij]$, $d \in \mathfrak{D}$, $i \neq j$ generate $\mathfrak{H}(\mathfrak{D}_n, J_a)$. Hence $\sigma = \tau$ and we have proved the following homomorphism theorem for Jordan matrix algebras.

THEOREM 3. *Let $\mathfrak{H}(\mathfrak{D}_n, J_a)$ and $\mathfrak{H}(\mathfrak{E}_n, K_b)$ be Jordan matrix algebras of order $n \geqq 3$ determined by canonical involutions J_a and K_b respectively where the first of these is defined by an involution j in $\mathfrak{D}$ and a diagonal matrix a such that $a_1 = 1$ and the second is defined by an involution k in $\mathfrak{E}$ and a diagonal matrix b with $b_1 = 1$. Suppose η is a homomorphism of $(\mathfrak{D}, j)$ into $(\mathfrak{E}, k)$ such that $a_i{}^\eta = b_i$, $i = 1,\cdots,n$. Then the restriction σ to $\mathfrak{H}(\mathfrak{D}_n, J_a)$ of the mapping $(d_{ij}) \to (d_{ij}{}^\eta)$ of $\mathfrak{D}_n$ is a homomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ into $\mathfrak{H}(\mathfrak{E}_n, K_b)$ such that $1[ij]^\sigma = 1\{ij\}$ where $d[ij]$ in $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is defined by* (17) *and $e\{ij\}$ is defined in a similar fashion in $\mathfrak{H}(\mathfrak{E}_n, K_b)$. Conversely, assume σ is a homomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ into $\mathfrak{H}(\mathfrak{E}_n, K_b)$ such that $1[ij]^\sigma = 1\{ij\}$, $i,j = 1,\cdots,n$. Then there exists a homomorphism η of $(\mathfrak{D}, j)$ into $(\mathfrak{E}, k)$ such that $a_i^\eta = b_i$, $i = 1,\cdots,n$, and such that σ is the restriction to $\mathfrak{H}(\mathfrak{D}_n, J_a)$ of the mapping $(d_{ij}) \to (d_{ij}{}^\eta)$ of $\mathfrak{D}_n$.*

If $(\mathfrak{D}, j)$ is an algebra with involution we let $\mathfrak{S}(\mathfrak{D}, j)$ denote the set of skew elements of $\mathfrak{D}$. This is a subspace closed under the composition $[ab] \equiv ab - ba$. Also, if $a \in \mathfrak{S}(\mathfrak{D}, j)$ and $b \in \mathfrak{H}(\mathfrak{D}, j)$ then $[ab] \in \mathfrak{H}(\mathfrak{D}, j)$, and if $a, b \in \mathfrak{H}(\mathfrak{D}, j)$ then $[ab] \in \mathfrak{S}(\mathfrak{D}, j)$. In other words, if $[\mathfrak{A}\mathfrak{B}]$ denotes the subspace spanned by all $[ab]$, a in the subspace $\mathfrak{A}$ and b in the subspace $\mathfrak{B}$ then $[\mathfrak{S}(\mathfrak{D}, j), \mathfrak{S}(\mathfrak{D}, j)] \subseteq \mathfrak{S}(\mathfrak{D}, j)$ and $[\mathfrak{S}(\mathfrak{D}, j), \mathfrak{H}(\mathfrak{D}, j)] \subseteq \mathfrak{H}(\mathfrak{D}, j)$ and $[\mathfrak{H}(\mathfrak{D}, j), \mathfrak{H}(\mathfrak{D}, j)] \subseteq \mathfrak{S}(\mathfrak{D}, j)$. In certain considerations it is important to know that $[\mathfrak{H}(\mathfrak{D},j),\ \mathfrak{H}(\mathfrak{D},j)] = \mathfrak{S}(\mathfrak{D},j)$. We shall now establish the following useful sufficient condition for this to hold for a matrix algebra with canonical involution.

THEOREM 4. *Let $(\mathfrak{D},j)$ be an algebra with involution and identity element, J_a a canonical involution in $\mathfrak{D}_n$ determined by a diagonal matrix a with $a_1 = 1$. Assume $n \geqq 3$. Then $\mathfrak{S}(\mathfrak{D}_n, J_a) = [\mathfrak{H}(\mathfrak{D}_n, J_a),\ \mathfrak{H}(\mathfrak{D}_n, J_a)]$ if*

$$\mathfrak{S}(\mathfrak{D},j) = [\mathfrak{S}(\mathfrak{D},j), \mathfrak{S}(\mathfrak{D},j)] + [\mathfrak{H}(\mathfrak{D},j), \mathfrak{H}(\mathfrak{D},j)].$$

PROOF. If $d \in \mathfrak{D}$ and e_{ij} is the usual matrix unit then $de_{ij}^{J_a} = a_j^{-1}\bar{d}a_i e_{ji}$. Hence every element of $\mathfrak{S}(\mathfrak{D}_n, J_a)$ is a sum of elements of the form $de_{ij} - a_j^{-1}\bar{d}a_i e_{ji}$. If $d[ij] \in \mathfrak{H}(\mathfrak{D}_n, J_a)$ is defined as usual then $[d[ij],\ 1[jk]] = de_{ik} - a_k^{-1}\bar{d}a_i e_{ki}$ if i,j,k are distinct. Hence every element $de_{ik} - a_k^{-1}\bar{d}a_i e_{ki}$ with $i \neq k$ is contained in $[\mathfrak{H}(\mathfrak{D}_n, J_a),\ \mathfrak{H}(\mathfrak{D}_n, J_a)]$. If $i \neq j$ then $[d[ij],\ 1[ji]] = (d - a_i^{-1}\bar{d}a_i)e_{ii} - (d - a_j^{-1}\bar{d}a_j)e_{jj}$. This and the result just noted imply that $\mathfrak{S}(\mathfrak{D}_n, J_a) = [\mathfrak{H}(\mathfrak{D}_n, J_a),\ \mathfrak{H}(\mathfrak{D}_n, J_a)] + \mathfrak{S}(\mathfrak{D},j)e_{11}$ where $\mathfrak{S}(\mathfrak{D},j)e_{11} = \{qe_{11} \mid q \in \mathfrak{S}(\mathfrak{D},j)\}$. If $p_1, p_2 \in \mathfrak{H}(\mathfrak{D},j)$ then $[p_1 p_2]e_{11} = [p_1 e_{11}, p_2 e_{11}] \in [\mathfrak{H}(\mathfrak{D}_n, J_a),\ \mathfrak{H}(\mathfrak{D}_n, J_a)]$ and if $q_1, q_2 \in \mathfrak{S}(\mathfrak{D},j)$ then we can write $q_1 = d_1 - \bar{d}_1$, $q_2 = d_2 - \bar{d}_2$, $d_i \in \mathfrak{D}$, and we have

$$[q_1 q_2]e_{11} = [[d_1[1j],\ 1[j1]],\ [d_2[1k],\ 1[k1]]] \tag{28}$$

for $1,j,k$ distinct. Put $[d_2[1k],\ 1[k1]] = L$ so that $L \in \mathfrak{S}(\mathfrak{D}_n, J_a)$. Since $1[j1]$ is in the nucleus of $\mathfrak{D}_n$ the Jacobi identity: $[[d_1[1j], 1[j1]], L] = [[d_1[1j], L], 1[j1]] + [d_1[1j], [1[j1], L]]$ is valid. Since $[d_1[1j], L]$ and $[1[j1], L] \in \mathfrak{H}(\mathfrak{D}_n, J_a)$ it is now clear that $[q_1 q_2]e_{11} \in [\mathfrak{H}(\mathfrak{D}_n, J_a),\ \mathfrak{H}(\mathfrak{D}_n, J_a)]$ and hence $[\mathfrak{H}(\mathfrak{D},j), \mathfrak{H}(\mathfrak{D},j)]e_{11} + [\mathfrak{S}(\mathfrak{D},j),\ \mathfrak{S}(\mathfrak{D},j)]e_{11} \subseteq [\mathfrak{H}(\mathfrak{D}_n, J_a), \mathfrak{H}(\mathfrak{D}_n, J_a)]$. Since $\mathfrak{S}(\mathfrak{D},j) = [\mathfrak{H}(\mathfrak{D},j), \mathfrak{H}(\mathfrak{D},j)] + [\mathfrak{S}(\mathfrak{D},j), \mathfrak{S}(\mathfrak{D},j)]$ we now have that $\mathfrak{S}(\mathfrak{D}_n, J_a) = [\mathfrak{H}(\mathfrak{D}_n, J_a), \mathfrak{H}(\mathfrak{D}_n, J_a)] + \mathfrak{S}(\mathfrak{D},j)e_{11} = [\mathfrak{H}(\mathfrak{D}_n, J_a),\ \mathfrak{H}(\mathfrak{D}_n, J_a)]$.

EXERCISES

1. Show that Theorems 2 and 3 are valid without the hypothesis that $\mathfrak{H}(\mathfrak{D}_n, J_a)$ and $\mathfrak{H}(\mathfrak{E}_n, J_b)$ are Jordan.

2. Let $\mathfrak{D}$ be an arbitrary algebra with 1, $\mathfrak{E} = \mathfrak{D} \oplus \mathfrak{D}^{\circ}$, j the involution $(a_1, a_2) \to (a_2, a_1)$. Show that any ideal in $(\mathfrak{E},j)$ has the form $\mathfrak{K} \oplus \mathfrak{K}^j$ where $\mathfrak{K}$ is an ideal in $\mathfrak{D}$, and that $\mathfrak{K} \to \mathfrak{K} \oplus \mathfrak{K}^j$ is a bijection of the set of ideals of $\mathfrak{D}$ and the set of ideals of $(\mathfrak{E},j)$. Show that if $n \geqq 3$ then $\mathfrak{K} \to \mathfrak{K}_n$ is a bijection of the set of ideals of $\mathfrak{D}$ and the set of ideals of $\mathfrak{D}_n{}^+$.

3. Let $\mathfrak{V}$ be an n-dimensional vector space, f a symmetric bilinear form such that

$$f(u_i, u_{n-j+1}) = \delta_{ij}$$

$i,j = 1, \cdots, n$ for a basis $(u_1, u_2, \cdots, u_n)$ of $\mathfrak{V}/\Phi$. Note that f is nondegenerate and has maximal Witt index. Let $\mathfrak{H}^{(n)}$ be the Jordan algebra of self-adjoint linear transformations relative to f. Let E and F be the linear transformations in $\mathfrak{B}$ defined by

$$u_1E = 0, \quad u_{i+1}E = i(i-n)u_i, \quad 1 \leqq i \leqq n-1,$$

$$u_iF = u_{i+1}, \quad 1 \leqq i \leqq n-1, \quad u_nF = 0.$$

Verify that $E, F \in \mathfrak{H}^{(n)}$ and that if $[A,B,C] = A.B.C - A.(B.C)$ then

$$2[E,E,F] = E, \quad 2[F,F,E] = F, \quad E^n = 0 = F^n$$

and that E and F generate $\mathfrak{H}^{(n)}$.

4. (Sasser). Let $\mathfrak{D}$ be an algebra of characteristic $\neq 2$ with identity element 1 and involution j and let $\mathfrak{H}(\mathfrak{D}_2)$ be the subalgebra of $\mathfrak{D}_2{}^+$ of symmetric elements under the standard involution. Prove that $\mathfrak{H}(\mathfrak{D}_2)$ is Jordan if and only if the following relations hold in $\mathfrak{D}$:

$$\begin{aligned} [h,a,k] &= 0, \\ [h,h,a] &= 0, \\ \overline{[a,\bar{a},h]} &= -[a,\bar{a},h], \\ \overline{[a,h,b]} &= -[a,h,b], \\ [a,\bar{a},a] &= 0 \end{aligned}$$

where $a, b \in \mathfrak{D}$, $h, k \in \mathfrak{H}(\mathfrak{D}, j)$ and $\bar{a} = a^j$.

5. (Sasser). Let $\mathfrak{D}$ be an algebra with 1 of characteristic $\neq 2$. Show that $\mathfrak{D}_n{}^+$, $n \geqq 2$, is Jordan if and only if $\mathfrak{D}$ is associative.

3. **Coordinatization theorems.** Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_a)$ be a Jordan matrix algebra determined by the canonical involution J_a in $\mathfrak{D}_n$ where $a = \mathrm{diag}\{a_1, a_2, \cdots, a_n\}$, as in §2. Also, as in §2, we write $x[ij] = xe_{ij} + (xe_{ij})^{J_a}$, $e_i = \frac{1}{2}[ii] = e_{ii}$. Then the e_i are orthogonal idempotents in $\mathfrak{J}$ and $\sum_1^n e_i = 1$. Moreover, if $\mathfrak{J}_{ij} = \{x[ij] \mid x \in \mathfrak{D}\}$ then $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$ is the Peirce decomposition of $\mathfrak{J}$ relative to the e_i. Now put $u_{1j} = 1[1j]$, $u_{j1} = 1[j1]$, $j = 2, 3, \cdots, n$. Then it follows from (20) and (19) that

$$u_{1j}.u_{j1} = e_1 + e_j, \quad u_{1j}^{.2}.u_{j1} = u_{1j}. \tag{29}$$

Hence u_{1j} and u_{j1} are inverses in the subalgebra $\mathfrak{J}_{11} + \mathfrak{J}_{jj} + \mathfrak{J}_{1j}$ whose identity is $e_1 + e_j$ and, consequently, e_1 and e_j are connected idempotents (cf. §1). If $a = 1$, so the involution in $\mathfrak{D}_n$ is standard, then $u_{j1} = u_{1j}$ and

$$u_{1j}^{.2} = e_1 + e_j. \tag{30}$$

Hence e_1 and e_j are strongly connected orthogonal idempotents. In this section we shall show that the situations we have just indicated characterize Jordan matrix algebras and Jordan matrix algebras defined by standard involutions respectively. We consider the latter case first and we shall prove the following

THEOREM 5 (STRONG COORDINATIZATION THEOREM). *Let $\mathfrak{J}$ be a Jordan algebra with* 1 *which is a sum of $n \geqq 3$ strongly connected nonzero idempotent elements. Then $\mathfrak{J}$ is isomorphic to a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ defined by a standard involution. More precisely, assume $1 = \sum_1^n e_i$ where the e_i are nonzero orthogonal idempotent elements and $n \geqq 3$. Assume also that there exist elements u_{1j}, $j = 2, \cdots, n$, such that*

$$e_1 . u_{1j} = \tfrac{1}{2}u_{1j} = e_j . u_{1j}, \tag{31}$$

$$u_{1j}^{\cdot 2} = e_1 + e_j. \tag{32}$$

Then there exists a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ given by a standard involution in $\mathfrak{D}_n$ and an isomorphism η of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n)$ such that $e_i^\eta = \tfrac{1}{2}[ii]$, $u_{1j}^\eta = 1[1j]$.

The proof will be based on a number of lemmas.

LEMMA 1. *Let $u_{ij} \in \mathfrak{J}_{ij}$, $i \neq j$, satisfy $u_{ij}^{\cdot 2} = e_i + e_j$. Set $p_{ij} = 1 - e_i - e_j + u_{ij} = \sum_{k \neq i,j} e_k + u_{ij}$, $U_{(ij)} = U_{p_{ij}}$, $R_{ij} = R_{u_{ij}}$. Then $U_{(ij)}$ is an automorphism of period two in $\mathfrak{J}$ with the following properties*:

(1) *$U_{(ij)}$ interchanges the idempotents e_i and e_j and the spaces $\mathfrak{J}_{ii}$ and $\mathfrak{J}_{jj}$, and maps $\mathfrak{J}_{ij}$ onto itself,*

(2) *$U_{(ij)}$ is the identity mapping on every $\mathfrak{J}_{kl}$, $k, l \neq i,j$,*

(3) *$U_{(ij)}$ interchanges $\mathfrak{J}_{ik}$ and $\mathfrak{J}_{jk}$ if $k \neq i,j$ and coincides with $2R_{ij}$ on these two spaces.*

PROOF. We have $p_{ij}^{\cdot 2} = (\sum_{k \neq i,j} e_k + u_{ij})^{\cdot 2} = \sum_{k \neq i,j} e_k + u_{ij}^{\cdot 2} = \sum_{k \neq i,j} e_k + e_i + e_j = 1$. Hence $U_{(ij)} = U_{p_{ij}}$ is an automorphism of period two in $\mathfrak{J}$ (§1.12, p. 60). We have $e_i U_{(ij)} = 2e_i . p_{ij} . p_{ij} - e_i = u_{ij} . p_{ij} - e_i = u_{ij} . u_{ij} - e_i = e_i + e_j - e_i = e_j$. Since $U_{(ij)}^2 = 1$ we have $e_j U_{(ij)} = e_i$. Since $\mathfrak{J}_{ii} = \mathfrak{J}_1(e_i)$, $\mathfrak{J}_{jj} = \mathfrak{J}_1(e_j)$ and $\mathfrak{J}_{ij} = \mathfrak{J}_{\frac{1}{2}}(e_i) \cap \mathfrak{J}_{\frac{1}{2}}(e_j)$ and $U_{(ij)}$ is an automorphism, it is clear that $U_{(ij)}$ interchanges $\mathfrak{J}_{ii}$ and $\mathfrak{J}_{jj}$ and maps $\mathfrak{J}_{ij}$ onto itself. Let $x \in \mathfrak{J}_{kl}$, $k, l \neq i,j$. Then

$$\begin{aligned} xU_{(ij)} &= 2x . p_{ij} . p_{ij} - x = 2x . \Big(\sum_{k \neq i,j} e_k + u_{ij}\Big) . \Big(\sum_{k \neq i,j} e_k + u_{ij}\Big) - x \\ &= 2x - x = x. \end{aligned}$$

Next let $x \in \mathfrak{J}_{ik}$, $k \neq i,j$. Then

$$\begin{aligned} xU_{(ij)} &= 2x . \Big(\sum_{l \neq i,j} e_l + u_{ij}\Big) . \Big(\sum_{l \neq i,j} e_l + u_{ij}\Big) - x \\ &= (x + 2x . u_{ij}) . \Big(\sum_{l \neq i,j} e_l + u_{ij}\Big) - x \\ &= \tfrac{1}{2}x + x . u_{ij} + x . u_{ij} + 2x . u_{ij} . u_{ij} - x \\ &= \tfrac{1}{2}x + 2x . u_{ij} + x . u_{ij}^{\cdot 2} - x \quad \text{(PD10)} \\ &= \tfrac{1}{2}x + 2x . u_{ij} + \tfrac{1}{2}x - x \\ &= x(2R_{ij}). \end{aligned}$$

By symmetry, it follows that $U_{(ij)} = 2R_{ij}$ on $\mathfrak{J}_{jk}$. Since $\mathfrak{J}_{ik}.\mathfrak{J}_{ij} \subseteq \mathfrak{J}_{jk}$ and $\mathfrak{J}_{jk}.\mathfrak{J}_{ij} \subseteq \mathfrak{J}_{ik}$ and $U_{(ij)}$ is an automorphism it is clear that $\mathfrak{J}_{jk}$ and $\mathfrak{J}_{ik}$ are interchanged by $U_{(ij)}$.

We apply this lemma to the u_{1j} which are given in the statement of the theorem. This gives $n-1$ automorphisms $U_{(1j)}$ of period two in $\mathfrak{J}$. Set $u_{j1} = u_{1j}$, $j > 1$ and $u_{ij} = u_{i1}U_{(1j)}$ if $1, i, j \neq$. Then, by Lemma 1 (3), $u_{ij} = 2u_{i1}R_{1j} = 2u_{i1}.u_{1j}$. Hence $u_{ij} = u_{ji}$ for all i,j $(i \neq j)$. Since $U_{(1j)}$ is an automorphism such that $(e_1 + e_i)U_{1j} = e_i + e_j$ and $u_{i1}^{.2} = e_1 + e_i$ we have $u_{ij}^{.2} = e_i + e_j$. Hence Lemma 1 is applicable to all the u_{ij} and the corresponding mappings $U_{(ij)}$ and R_{ij}. Also we have $p_{ij} = p_{i1}U_{(1j)}$, so $U_{(ij)} = U_{p_{ij}} = U_{p_{i1}U_{(1j)}} = U_{(1j)}U_{(1i)}U_{(1j)} = U_{(1i)}U_{(1j)}U_{(1i)}$ because of the symmetry in i and j.

LEMMA 2. *Let Σ denote the group of automorphisms in $\mathfrak{J}$ generated by the $U_{(1j)}$, $j = 2, \cdots, n$. Then we have a unique isomorphism $\pi \to U_\pi$ of the symmetric group S_n onto Σ such that $(1j) \to U_{(1j)}$. We have $e_iU_\pi = e_{i\pi}$ for any $i = 1, 2, \cdots, n$ and any $\pi \in S_n$. Moreover, if π and $\pi' \in S_n$ and $i\pi = i\pi'$, $j\pi = j\pi'$ ($i = j$ allowed) then U_π and $U_{\pi'}$ have the same effect on the Peirce space $\mathfrak{J}_{ij}$.*

PROOF. Let $\mathfrak{F}$ be the free group with $n-1$ generators $x_2, x_3, \cdots, x_n$, θ the homomorphism of $\mathfrak{F}$ onto S_n such that $x_j \to (1j)$. It is known that the kernel $\mathfrak{K}$ of θ is the invariant subgroup of $\mathfrak{F}$ generated by the elements

$$x_j^2,\ (x_jx_k)^3,\ (x_jx_kx_jx_l)^2, \qquad j,k,l \neq$$

(Burnside's *Theory of Groups of Finite Order*, second edition, Dover reprint, p. 464). Now we have $U_{(1j)}{}^2 = 1$, $(U_{(1j)}U_{(1k)})^3 = (U_{(1j)}U_{(1k)}U_{(1j)})(U_{(1k)}U_{(1j)}U_{(1k)}) = U_{(jk)}U_{(jk)} = 1$ and $(U_{(1j)}U_{(1k)}U_{(1j)}U_{(1l)})^2$

$$= (U_{(1j)}U_{(1k)}U_{(1j)})U_{(1l)}(U_{(1j)}U_{(1k)}U_{(1j)})U_{(1l)} = U_{(jk)}U_{(1l)}U_{(jk)}U_{(1l)}.$$

We have $U_{(jk)} = U_{p_{jk}}$, $U_{1l} = U_{p_{1l}}$ and $U_{(jk)}U_{(1l)}U_{(jk)} = U_{p_{jk}}U_{p_{1l}}U_{p_{jk}} = U_{p_{1l}U_{jk}} = U_{p_{1l}}$ by Lemma 1 (2). Hence $U_{(jk)}U_{(1l)}U_{(jk)}U_{(1l)} = U_{(1l)}{}^2 = 1$. Hence $\mathfrak{K}$ is mapped into 1 in the homomorphism of $\mathfrak{F}$ onto Σ such that $x_j \to U_{(1j)}$. Consequently, we have a homomorphism of S_n onto Σ such that $(1j) \to U_{(1j)}$. If U_π denotes the image of $\pi \in S_n$ in this homomorphism then U_π is a product of $U_{(1j)}$. Hence, by Lemma 1, U_π maps $\{e_i\}$ into itself. Moreover, since $U_{(1j)}$ interchanges e_1 and e_j and leaves the other e_i fixed we have $e_iU_{(1j)} = e_{i(1j)}$, $i = 1, 2, \cdots, n$. Consequently, $e_iU_\pi = e_{i\pi}$ for any i and $\pi \in S_n$. It is clear from this that $\pi \to U_\pi$ is 1-1. Also uniqueness is clear since the transpositions $(1j)$ generate S_n. If $1, i, j \neq$ and $\pi = (ij)$ then $U_\pi = U_{(1i)}U_{(1j)}U_{(1i)} = U_{(ij)}$ as defined before. Now let π and π' satisfy $i\pi = i\pi'$, $j\pi = j\pi'$. Then $\pi'' = \pi'\pi^{-1}$ fixes i and j. Hence this is a product of transpositions (kl), $k, l \neq i, j$. Hence, by Lemma 1, $U_{\pi''}$, which is a product of $U_{(kl)}$, is the identity on $\mathfrak{J}_{ij}$. Hence π and π' have the same effect on $\mathfrak{J}_{ij}$.

We now let $\mathfrak{D}$ be the Peirce space $\mathfrak{J}_{12}$ and we define an algebra structure on $\mathfrak{D}$ by the multiplication

$$xy = 2xU_{(23)} \cdot yU_{(13)}. \tag{33}$$

LEMMA 3. *$\mathfrak{D}$ is an algebra with identity element u_{12} and the mapping $d \to \bar{d} = dU_{(12)}$ is an involution in $\mathfrak{D}$.*

PROOF. It is clear that xy is bilinear and $xy \in \mathfrak{D}$ since $xU_{(23)} \in \mathfrak{J}_{13}$ and $yU_{(13)} \in \mathfrak{J}_{23}$. Hence $\mathfrak{D}$ is an algebra. Put $y = u_{12}$ in (33). This gives $xu_{12} = 2x\,U_{(23)} \cdot u_{12}U_{(13)} = 2xU_{(23)} \cdot u_{23} = xU_{(23)}{}^2 = x$ since $xU_{(23)} \in \mathfrak{J}_{13}$ and $U_{(23)}$ coincides with $2R_{u_{23}}$ on this space. Hence u_{12} is a right identity element for $\mathfrak{D}$. We have $U_{(23)}U_{(12)} = U_{(12)}U_{(13)}$ and $U_{(13)}U_{(12)} = U_{(12)}U_{(23)}$ since $(23)(12) = (12)(13)$ and $(13)(12) = (12)(23)$ in S_n. Hence $\overline{xy} = 2(xU_{(23)} \cdot yU_{(13)})U_{(12)} = 2xU_{(23)}U_{(12)} \cdot yU_{(13)}U_{(12)} = 2xU_{(12)}U_{(13)} \cdot yU_{(12)}U_{(23)} = 2\bar{y}U_{(23)} \cdot \bar{x}U_{(13)} = \bar{y}\bar{x}$. Hence $x \to \bar{x}$ is an involution. It follows that $\mathfrak{D}$ has the left identity $\bar{u}_{12}$. This must coincide with u_{12}.

For each ordered pair of indices i, j we shall now define mappings $x \to x_{ij}$ of $\mathfrak{D}$ into the Peirce space $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$. If $i \neq j$ we let π be a permutation such that $1\pi = i$, $2\pi = j$ and if $i = j$ then we let π be a permutation such that $1\pi = i$. Then we define

$$x_{ij} = xU_\pi, \qquad i \neq j, \qquad x_{ii} = 2xR_{12}R_{e_1}U_\pi. \tag{34}$$

It is clear that $x \to x_{ij}$ ($i = j$ allowed) is linear. If $i \neq j$ we have $x_{ji} = xU_{(12)}U_\pi = \bar{x}_{ij}$. Hence

$$x_{ji} = \bar{x}_{ij} \tag{35}$$

holds for $i \neq j$. To establish this for $i = j$ we note that since $u_{12}{}^{\cdot 2} = e_1 + e_2$ and this is the identity on $\mathfrak{J}_{11} + \mathfrak{J}_{12} + \mathfrak{J}_{22}$, we see by Theorem 20 of §2.13 (or directly) that $R_{12}{}^3 = R_{12}$ on $\mathfrak{J}_{11} + \mathfrak{J}_{12} + \mathfrak{J}_{22}$. Hence $U_{(12)}R_{12} = (2R_{12}{}^2 - 1)R_{12} = R_{12}$ on $\mathfrak{J}_{11} + \mathfrak{J}_{12} + \mathfrak{J}_{22}$. Hence if $x \in \mathfrak{J}_{12}$ then $\bar{x}_{ii} = 2xU_{(12)}R_{12}R_{e_1}U_\pi = 2xR_{12}R_{e_1}U_\pi = x_{ii}$.

We note next that if $i \neq j$ then $x_{ij}U_{\pi'} = xU_\pi U_{\pi'} = xU_{\pi\pi'} = x_{i\pi', j\pi'}$. Also $x_{ii}U_{\pi'} = 2xR_{12}R_{e_1}U_\pi U_{\pi'} = 2xR_{12}R_{e_1}U_{\pi\pi'} = x_{i\pi', i\pi'}$. Hence for all i, j we have

$$x_{ij}U_{\pi'} = x_{i\pi', j\pi'}. \tag{36}$$

If we now denote the identity element u_{12} of $\mathfrak{D}$ as 1, then we have $1_{ii} = 2u_{12}{}^{\cdot 2} . e_1U_\pi = 2e_1U_\pi = 2e_i$ and if $j > 2$, then $1_{1j} = u_{12}U_{(2j)} = 2u_{12} . u_{2j} = 4u_{12} . (u_{12} . u_{1j}) = u_{1j}U_{(12)}{}^2 = u_{1j}$. Since $1_{12} = u_{12}$ is clear, we have

$$1_{ii} = 2e_i, \qquad 1_{1j} = u_{1j}, \qquad j \geqq 2. \tag{37}$$

If $1\pi = i$, $2\pi = j$ then U_π is an automorphism of $\mathfrak{J}$ which permutes the Peirce spaces $\mathfrak{J}_{kl}$, $k \neq l$. Since it maps $\mathfrak{D} = \mathfrak{J}_{12}$ into $\mathfrak{J}_{ij}$ it is clear that the mapping $x \to x_{ij}$, $i \neq j$, is a bijective linear mapping of $\mathfrak{D} = \mathfrak{J}_{12}$ onto $\mathfrak{J}_{ij}$. We shall now prove the following formulas for x, y in $\mathfrak{D}$:

$$2x_{ij}\,.\,y_{jk} = (xy)_{ik}, \tag{38}$$

$$2x_{ii}\,.\,y_{ij} = ((x+\bar{x})y)_{ij}, \tag{39}$$

$$2x_{ij}\,.\,y_{ji}\,.\,e_i = (xy)_{ii}, \tag{40}$$

$$2x_{ii}^{\cdot 2} = (x+\bar{x})^2{}_{ii}, \tag{41}$$

if i, j, k are unequal. We note first that the definition (33) gives $(xy)_{12} = 2x_{13}\,.\,y_{32}$. Now let π be a permutation such that $1\pi = i$, $2\pi = k$, $3\pi = j$. Then we have $(xy)_{12}U_\pi = 2x_{13}U_\pi\,.\,y_{32}U_\pi$. Hence $(xy)_{ik} = 2x_{ij}\,.\,y_{jk}$, which proves (38). The argument just used shows also that it is enough to prove (39)–(41) for $i = 1, j = 2$. If $x \in \mathfrak{D} = \mathfrak{J}_{12}$ then $2x_{13}\,.\,1_{31}\,.\,e_1 = 2xU_{(23)}\,.\,u_{13}\,.\,e_1 = 4x\,.\,u_{23}\,.\,u_{13}\,.\,e_1 = 4x\,.\,(u_{23}\,.\,u_{13})\,.\,e_1$ (PD12) $= 2x\,.\,1_{13}U_{(23)}\,.\,e_1 = 2x\,.\,1_{12}\,.\,e_1 = x_{11}$. Hence

$$2x_{13}\,.\,1_{31}\,.\,e_1 = x_{11}. \tag{42}$$

We now have $2x_{11}\,.\,y_{12} = 4x_{13}\,.\,1_{31}\,.\,e_1\,.\,y_{12} = 4x_{13}\,.\,1_{31}\,.\,y_{12} = 4x_{13}\,.\,(1_{31}\,.\,y_{12}) + 4\,1_{31}\,.\,(x_{13}\,.\,y_{12})$ (PD10) $= 2x_{13}\,.\,y_{32} + 2(1_{31}\,.\,(\bar{x}y)_{32})$ ((38), (35)) $= (xy)_{12} + (\bar{x}y)_{12} = ((x+\bar{x})y)_{12}$. This implies (39). Next we have $x_{11}\,.\,y_{11} = 4x_{12}\,.\,1_{21}\,.\,e_1\,.\,(y_{13}\,.\,1_{31}\,.\,e_1) = 4x_{12}\,.\,1_{21}\,.\,(y_{13}\,.\,1_{31}) = -\,4x_{12}\,.\,1_{31}\,.\,(1_{21}\,.\,y_{13}) - 4x_{12}\,.\,y_{13}\,.\,(1_{21}\,.\,1_{31}) + 4x_{12}\,.\,1_{21}\,.\,y_{13}\,.\,1_{31} + 4x_{12}\,.\,1_{31}\,.\,y_{13}\,.\,1_{21} + 4x_{12}\,.\,(1_{31}\,.\,1_{21}\,.\,y_{13}) = -\,x_{32}\,.\,y_{23} - (\bar{x}y)_{23}\,.\,1_{32} + 2x_{11}\,.\,y_{13}\,.\,1_{31} + (yx)_{12}\,.\,1_{21} + x_{12}\,.\,y_{12}$. By (39), this gives

$$\begin{aligned} x_{11}\,.\,y_{11} = -\,x_{32}\,.\,y_{23} - (\bar{x}y)_{23}\,.\,1_{32} &+ ((x+\bar{x})y)_{13}\,.\,1_{31} \\ &+ (yx)_{12}\,.\,1_{21} + x_{12}\,.\,y_{12}. \end{aligned} \tag{43}$$

If we multiply this by e_3 we obtain

$$x_{32}\,.\,y_{23}\,.\,e_3 = -\,(\bar{x}y)_{23}\,.\,1_{32}\,.\,e_3 + ((x+\bar{x})y)_{13}\,.\,1_{31}\,.\,e_3.$$

Applying $2U_{(13)}$ to both sides we obtain, by (42) and (35), $2x_{12}\,.\,y_{21}\,.\,e_1 = -\,(\bar{y}x)_{11} + (\bar{y}(x+\bar{x}))_{11} = (\bar{y}\bar{x})_{11} = (xy)_{11}$. Hence (40) is valid. Next we multiply (43) by $2e_1$ and set $y = x$. This gives $2x_{11}^{\cdot 2} = ((x+\bar{x})x)_{11} + (x^2)_{11} + 2x_{12}^{\cdot 2}\,.\,e_1 = ((x+\bar{x})x + x^2 + x\bar{x})_{11} = ((x+\bar{x})x + \bar{x}^2 + x\bar{x})_{11} = (x+\bar{x})^2{}_{11}$. This implies (41).

We can now give the

Proof of Theorem 5. Let $\mathfrak{D}$ be the algebra $\mathfrak{J}_{12}$ with the product (33), and consider the algebra $\mathfrak{H}(\mathfrak{D}_n)$. We use the notation $x[ij] = xe_{ij} + \bar{x}e_{ji}$, $x \in \mathfrak{D}$, and the formulas (18′)–(23′) of §2. If $i \neq j$ then we have seen that $x \to x_{ij}$ is a 1–1 linear mapping of $\mathfrak{D}$ onto $\mathfrak{J}_{ij}$. Also $x_{ij} = \bar{x}_{ji}$. It follows that $x_{ij} \to x[ij]$ is a 1–1 linear mapping η_{ij} of $\mathfrak{J}_{ij}$ onto the subspace of $\mathfrak{H}(\mathfrak{D}_n)$ of elements $x[ij]$, $x \in \mathfrak{D}$, and $\eta_{ij} = \eta_{ji}$. Now let $y \in \mathfrak{J}_{11}$. Then $yU_{(12)} \in \mathfrak{J}_{22}$ so $yU_{(12)}\,.\,e_1 = 0$. This gives $2y\,.\,u_{12}\,.\,u_{12}\,.\,e_1 = y\,.\,u_{12}^{\cdot 2}\,.\,e_1 = y$. Since $x = y\,.\,u_{12} \in \mathfrak{J}_{12}$ this shows that any $y \in \mathfrak{J}_{11}$ has the form $x_{11} = 2x\,.\,u_{12}\,.\,e_1$. Also since $x_{11} = \bar{x}_{11}$ we may write $y = x_{11}$

where $\bar{x} = x$. If we put $a = u_{12}$, $b = e_1$ in $2R_aR_bR_a + R_{b.a^{\cdot 2}} = 2R_aR_{a.b} + R_bR_{a^{\cdot 2}}$ we obtain $2R_{12}R_{e_1}R_{12} + R_{e_1} = R_{12}{}^2 + R_{e_1}(R_{e_1} + R_{e_2})$. Since $e_1 + e_2$ is the identity on $\mathfrak{J}_{11} + \mathfrak{J}_{12} + \mathfrak{J}_{22}$ this gives $2xR_{12}R_{e_1}R_{12} = xR_{12}{}^2$ if $x \in \mathfrak{J}_{12}$. Hence if $x \in \mathfrak{J}_{12}$ and $x_{11} = 2xR_{12}R_{e_1} = 0$ then $xR_{12}{}^2 = 0$. Then $\bar{x} = xU_{(12)} = 2x.u_{12}.u_{12} - x = -x$. Hence $x \to x_{11}$ is 1-1 on the subspace of symmetric elements of $\mathfrak{D}$. Our results imply that the restriction η_{ii} of the mapping $x \to x_{ii}$ to the space of symmetric elements of $\mathfrak{D}$ is 1-1 and surjective on $\mathfrak{J}_{ii}$. Hence the mapping η_{ii}: $x_{ii} \to x[ii]$ is a 1-1 linear mapping of $\mathfrak{J}_{ii}$ onto the subspace of $\mathfrak{H}(\mathfrak{D}_n)$ of elements $x[ii]$. It follows that if we let η be the linear mapping of $\mathfrak{J}$ which coincides with η_{ij} on $\mathfrak{J}_{ij}$, $i,j = 1, 2, \cdots$, then η is a bijection of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n)$. A comparison of (35), (38)–(41) with (18′)–(23′) and properties of the Peirce decomposition show that η is an isomorphism of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n)$. Since $\mathfrak{J}$ is Jordan, clearly $\mathfrak{H}(\mathfrak{D}_n)$ is a Jordan matrix algebra. Also, by (37), $e_i = (\frac{1}{2})_{ii} \to \frac{1}{2}[ii]$ and $u_{ij} = 1_{1j} \to 1[1j]$. Hence the theorem is proved.

We shall now extend the foregoing result to the following

COORDINATIZATION THEOREM. *Let $\mathfrak{J}$ be a Jordan algebra with* 1 *which is a sum of $n \geqq 3$ connected nonzero orthogonal idempotent elements. Then $\mathfrak{J}$ is isomorphic to a Jordan matrix algebra. More precisely, assume $1 = \sum_1^n e_i$, $n \geqq 3$, where the e_i are orthogonal idempotents $\neq 0$ and let $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$ be the Peirce decomposition relative to the e_i. Assume that for $j = 2, 3, \cdots, n$ there exists an element $u_{1j} \in \mathfrak{J}_{1j}$ which is invertible in $\mathfrak{J}_{11} + \mathfrak{J}_{jj} + \mathfrak{J}_{1j}$. Then there exists a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ and an isomorphism ζ of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n, J_a)$ such that $e_i{}^\zeta = \frac{1}{2}[ii]$, $u_{1j}{}^\zeta = 1[1j]$* (cf. (17)).

PROOF. We shall obtain a reduction of the result to the preceding one via isotopy.[1] Let u_{j1} be the inverse of u_{1j} relative to $e_1 + e_j$. As in Lemma 5 of §1, we introduce the elements $f_1 = e_1 = g_1$, $f_j = u_{j1}{}^{\cdot 2}.e_j$, $g_j = u_{1j}{}^{\cdot 2}.e_j$, $j \geqq 2$, $f = \sum_1^n f_i$, $g = \sum_1^n g_i$ (cf. (14)). Then, by Lemma 1.5, f and g are inverses in $\mathfrak{J}$ and the f_i are orthogonal idempotent elements of the isotope $(\mathfrak{J}, g)$ such that $\sum f_i = f$ is the identity of $(\mathfrak{J}, g)$. Moreover, u_{j1} is in the Peirce space $(\mathfrak{J}, g)_{1j}$ relative to the f_i and $u_{j1} \cdot_g u_{j1} = f_1 + f_j$. Thus the f_i and u_{j1} satisfy the conditions for the e_i and u_{1j} given in Theorem 5. Consequently, we have an isomorphism η of $(\mathfrak{J}, g)$ onto a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ such that $f_i{}^\eta = e_{ii}$, $u_{j1}{}^\eta = e_{1j} + e_{j1}$. Then η is an isomorphism of $\mathfrak{J}$, which is the $f^{\cdot 2}$-isotope of $(\mathfrak{J}, g)$, onto $(\mathfrak{H}(\mathfrak{D}_n), (f^{\cdot 2})^2)$. We have $f^{\cdot 2} = \sum f_i{}^{\cdot 2}$ and $\{f_i{}^{\cdot 2} g f_i\} = \{f_i{}^{\cdot 2} g_i f_i\} = f_i{}^{\cdot 2}$ since f_i, $g_i \in \mathfrak{J}_{ii}$ and these are inverses in $\mathfrak{J}_{ii}$. Hence $f_i^{\cdot 2}$ is in the Peirce space $(\mathfrak{J}, g)_{ii}$ relative to the f_i. It follows that $(f_i{}^{\cdot 2})^\eta = a_i e_{ii}$ where a_i is a symmetric element of $\mathfrak{D}$. Also $f_1{}^{\cdot 2} = e_1$ so $(f_1{}^{\cdot 2})^\eta = e_{11}$. Hence $(f^{\cdot 2})^\eta = a = \operatorname{diag}\{a_1, a_2, \cdots, a_n\}$, $a_1 = 1$, and η is an isomorphism of $\mathfrak{J}$ onto the isotope $(\mathfrak{H}(\mathfrak{D}_n), a)$ of $\mathfrak{H}(\mathfrak{D}_n)$. We recall also that the

[1] Theorem 5 is due to the author ([17]). The Coordinatization Theorem was formulated by the author and proved by D. Sasser in his Yale dissertation. The original proof was direct and quite long. The present reduction to the first result was given by the author in [30].

mapping $X \to Xa$ is an isomorphism of the isotope $(\mathfrak{H}(\mathfrak{D}_n), a)$ onto the Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ where J_a is the canonical involution $X \to a^{-1}\bar{X}^t a$ (§1.12, p. 60). Hence we have the isomorphism $\zeta: x \to x^\eta a$ of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n, J_a)$. It remains to verify that $e_i^\zeta = \frac{1}{2}[ii]$ and $u_{1j}{}^\zeta = 1[1j]$. We have $u_{j1}{}^\eta = e_{1j} + e_{j1}$ so $u_{j1}{}^\zeta = (e_{1j} + e_{j1})a = e_{j1} + a_j e_{1j}$ (since $a_1 = 1$). Also we have $f_i{}^\eta = e_{ii}$, $(f_i^{\cdot 2})^\eta = a_i e_{ii}$. We now note that

$$(44)\qquad \begin{aligned} \{e_i g f_i\} &= \{e_i g_i f_i\} = e_i, \\ \{f_i^{\cdot 2} g f_i\} &= \{f_i^{\cdot 2} g_i f_i\} = f_i^{\cdot 2}, \\ \{e_i g f_i^{\cdot 2}\} &= \{e_i g_i f_i^{\cdot 2}\} = f_i, \\ \{\{e_i g e_i\} g f_i^{\cdot 2}\} &= \{g_i g_i f_i^{\cdot 2}\} = e_i. \end{aligned}$$

These relations follow from the fact that f_i and $g_i \in \mathfrak{J}_{ii}$ are inverses in this subalgebra with identity e_i. On the other hand, (44) show that $f_i^{\cdot 2}$ and $e_i \in (\mathfrak{J}, g)_{ii}$ where $(\mathfrak{J}, g)_{ij}$ are the Peirce spaces of $(\mathfrak{J}, g)$ relative to the f_i, and $f_i^{\cdot 2}$ and e_i are inverses in $(\mathfrak{J}, g)_{ii}$. Since $f_i{}^\eta = e_{ii}$ and $(f_i^{\cdot 2})^\eta = a_i e_{ii}$ it follows that $e_i^\eta = a_i^{-1} e_{ii}$ and $e_i^\zeta = e_i^\eta a = e_{ii}$. Also since u_{1j} is the inverse of u_{j1} relative to $e_1 + e_j$ and $u_{j1}{}^\zeta = e_{j1} + a_j e_{1j}$ it follows that $u_{1j}{}^\zeta = e_{1j} + a_j^{-1} e_{j1} = 1[1j]$. This completes the proof.

The Jordan matrix algebras $\mathfrak{H}(\mathfrak{D}_n, J_a)$ with $n \geqq 4$ are necessarily special since in this case $\mathfrak{D}$ must be associative. It follows from the Coordinatization Theorem that any Jordan algebra with $1 = \sum_1^n e_i$, $n \geqq 4$, e_i nonzero connected orthogonal idempotents is special. Moreover, these conditions carry over to any Jordan algebra having the given Jordan algebra as subalgebra with the same identity. Hence we have the following

COROLLARY. *Let $\mathfrak{J}$ be a Jordan algebra with $1 = \sum_1^n e_i$ $n \geqq 4$, e_i nonzero connected orthogonal idempotents. Then every Jordan algebra containing $\mathfrak{J}$ as subalgebra with the same identity element is special.*

4. **Perfection of certain classes of associative algebras with involutions.** We recall that an associative algebra with involution $(\mathfrak{A}, J)$ is called perfect if $\mathfrak{A}$ and the injection mapping of $\mathfrak{H} = \mathfrak{H}(\mathfrak{A}, J)$ in $\mathfrak{A}$ constitute a unital special universal envelope for the Jordan algebra $\mathfrak{H}$ (§2.4). This means that given any homomorphism σ of $\mathfrak{H}$ into $\mathfrak{G}^+$ where $\mathfrak{G}$ is an associative algebra with 1 such that $1^\sigma = 1$, then σ has a unique extension to a homomorphism of $\mathfrak{A}$ into $\mathfrak{G}$. In this section we shall prove perfection for associative algebras with involution of the form $(\mathfrak{D}_n, J_a)$ where J_a is a canonical involution and $n \geqq 3$. We shall establish also perfection for $(\mathfrak{D}_2, J_a)$ provided that the set $\mathfrak{H}(\mathfrak{D}, j)$ of symmetric elements of $\mathfrak{D}$ generates $\mathfrak{D}$. The result for $n \geqq 3$ is due to Jacobson and Rickart in [2]. We shall derive this and also the result for $n = 2$ as corollaries of a more general theorem which has been proved recently by Martindale in [1].

In this theorem one considers an arbitrary associative algebra with involution $(\mathfrak{A}, J)$ such that $\mathfrak{A}$ has an identity element $1 = \sum_1^n e_i$ where the e_i are orthogonal idempotent elements in $\mathfrak{A}$ ($e_i^2 = e_i, e_ie_j = 0$ if $i \neq j$) such that $e_i \in \mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{A}e_i\mathfrak{A} = \mathfrak{A}$, $i = 1, 2, \cdots, n$. The last condition is that the (two-sided) ideal in $\mathfrak{A}$ generated by every e_i is $\mathfrak{A}$ itself. In particular, this holds if $\mathfrak{A}$ is a simple algebra. Martindale's theorem asserts that under the given hypotheses, $(\mathfrak{A}, J)$ is perfect if either $n \geqq 3$ or $n = 2$ and the set $\mathfrak{H} \cap e_i\mathfrak{A}e_i$ of symmetric elements of $e_i\mathfrak{A}e_i$ generates the associative algebra $e_i\mathfrak{A}e_i$ for $i = 1, 2, \cdots, n$.

Now let $(\mathfrak{D}, j)$ be an associative algebra with involution and identity element 1 and let $\mathfrak{A} = \mathfrak{D}_n$ the $n \times n$ matrix algebra over $\mathfrak{D}$ and J_a a canonical involution defined by $a = \operatorname{diag}\{a_1, a_2, \cdots, a_n\}$ where $a_i = \bar{a}_i$ is invertible in $\mathfrak{D}$. Using the usual $x[ij]$ notation of (17) we put $e_i = \frac{1}{2}[ii]$ so that $e_i = e_{ii}$ where $e_{ij}, i, j = 1, 2, \cdots, n$ are the matrix units. Then the e_i are orthogonal idempotents in $\mathfrak{A}$ contained in $\mathfrak{H} = \mathfrak{H}(\mathfrak{D}_n, J_a)$ and $\sum_1^n e_i = 1$. If $d \in \mathfrak{D}$ we have $de_{jk} = de_{ji}e_ie_{ik} \in \mathfrak{D}_n e_i \mathfrak{D}_n$. Hence the e_i satisfy the first set of conditions of Martindale's theorem. Clearly $e_i\mathfrak{A}e_i$ is the set of matrices de_{ii}, $d \in \mathfrak{D}$ and such a matrix is in $\mathfrak{H}$ if and only if $a_i^{-1}\bar{d}a_i = d$ or $a_id = \bar{d}a_i \in \mathfrak{H}(\mathfrak{D}, j)$. In particular, a_ie_{ii} and $a_i^{-1}e_{ii} \in \mathfrak{H} \cap e_i\mathfrak{A}e_i$. It follows that $\mathfrak{H} \cap e_i\mathfrak{A}e_i$ generates $e_i\mathfrak{A}e_i$ if and only if $\mathfrak{H}(\mathfrak{D}, j)$ generates $\mathfrak{D}$. Hence Martindale's theorem will imply that $(\mathfrak{D}_n, J_a)$ is perfect if $n \geqq 3$, and $(\mathfrak{D}_2, J_a)$ is perfect if $\mathfrak{H}(\mathfrak{D}, j)$ generates $\mathfrak{D}$.

Now let $(\mathfrak{A}, J)$ be an associative algebra with involution satisfying the hypotheses of Martindale's theorem relative to the orthogonal idempotents e_i. We have the associative two-sided Peirce decomposition of $\mathfrak{A}$ as $\mathfrak{A} = \sum_{i,j=1}^n \oplus \mathfrak{A}_{ij}$ where $\mathfrak{A}_{ij} = e_i\mathfrak{A}e_j$. The condition $\mathfrak{A}e_j\mathfrak{A} = \mathfrak{A}$ for $j = 1, 2, \cdots, n$ evidently implies that $\mathfrak{A}_{ij}\mathfrak{A}_{jk} = \mathfrak{A}_{ik}$. Also we have $\mathfrak{A}_{ij}\mathfrak{A}_{kl} = 0$ if $j \neq k$ since $e_je_k = 0$. Since $e_i^J = e_i$ it is clear that if $a_{ij} \in \mathfrak{A}_{ij}$ then $a_{ij}^J \in \mathfrak{A}_{ji}$. It is evident also that the e_i are orthogonal idempotents in the Jordan algebra $\mathfrak{A}^+$. Since $2xU_{e_i,e_j} = e_ixe_j + e_jxe_i \in \mathfrak{A}_{ij} + \mathfrak{A}_{ji}$ it is clear that the Peirce space $\mathfrak{A}_{ij}^+ \equiv \mathfrak{A}U_{e_i,e_j} = \mathfrak{A}_{ij} + \mathfrak{A}_{ji}$. Since $e_i \in \mathfrak{H} = \mathfrak{H}(\mathfrak{A}, J)$ the Peirce decomposition of $\mathfrak{H}$ relative to the e_i is $\mathfrak{H} = \sum_{i \leqq j} \oplus \mathfrak{H}_{ij}$ where $\mathfrak{H}_{ij} = \mathfrak{H} \cap \mathfrak{A}_{ij}^+ = \mathfrak{H} \cap (\mathfrak{A}_{ij} + \mathfrak{A}_{ji})$.

Let σ be a homomorphism of $\mathfrak{H}$ into $\mathfrak{G}^+$ where $\mathfrak{G}$ is an associative algebra with 1. We assume also that $1^\sigma = 1$. Put $g_i = e_i^\sigma$, $i = 1, 2, \cdots, n$. Then the g_i are orthogonal idempotents in $\mathfrak{G}$ (cf. §2.2, p. 72) and $\sum_1^n g_i = 1$. Hence we have the two-sided Peirce decomposition $\mathfrak{G} = \sum \oplus \mathfrak{G}_{ij}$, $\mathfrak{G}_{ij} = g_i\mathfrak{G}g_j$, of the associative algebra $\mathfrak{G}$ and the Peirce decomposition of the Jordan algebra $\mathfrak{G}^+$ as $\mathfrak{G}^+ = \sum \oplus \mathfrak{G}_{ij}^+$ where $\mathfrak{G}_{ij}^+ = \mathfrak{G}_{ij} + \mathfrak{G}_{ji}$. Since $e_i^\sigma = g_i$ it is clear that $\mathfrak{H}_{ij}^\sigma \subseteq \mathfrak{G}_{ij}^+ = \mathfrak{G}_{ij} + \mathfrak{G}_{ji}$. Now let $i \neq j$ and let $x_{ij} \in \mathfrak{A}_{ij}$. Then $x_{ij} + x_{ij}^J \in \mathfrak{A}_{ij}^+ \cap \mathfrak{H} = \mathfrak{H}_{ij}$ so $(x_{ij} + x_{ij}^J)^\sigma = y_{ij} + y_{ji}$ where y_{ij} and y_{ji} are uniquely determined elements of $\mathfrak{G}_{ij}$ and $\mathfrak{G}_{ji}$ respectively. We use this to define a mapping η_{ij} of $\mathfrak{A}_{ij}$ into $\mathfrak{G}_{ij}$ for $i \neq j$ by

$$x_{\ j}^{\eta_{ij}} = y_{ij}. \tag{45}$$

It is clear that η_{ij} is linear. Since $x_{ij}^{J^2} = x_{ij}$ and $x_{ij}^J \in \mathfrak{A}_{ji}$ we have $(x_{ij}^J)^{\eta_{ji}} = y_{ji}$. Hence

$$(x_{ij} + x_{ij}^J)^\sigma = x_{ij}^{\eta_{ij}} + (x_{ij}^J)^{\eta_{ji}}, \qquad i \neq j. \tag{46}$$

We now prove the following

LEMMA. *If $x_{ij} \in \mathfrak{A}_{ij}$, $x_{ji} \in \mathfrak{A}_{ji}$ etc. and i,j,k are distinct then*

$$(x_{ij}x_{ji} + x_{ji}^J x_{ij}^J)^\sigma = x_{ij}^{\eta_{ij}} x_{ji}^{\eta_{ji}} + (x_{ji}^J)^{\eta_{ij}}(x_{ij}^J)^{\eta_{ji}}. \tag{47}$$

$$(x_{ij}x_{jk})^{\eta_{ik}} = x_{ij}^{\eta_{ij}} x_{jk}^{\eta_{jk}}. \tag{48}$$

If $h_{ii} \in \mathfrak{H}_{ii}$ and $h_{jj} \in \mathfrak{H}_{jj}$ then

$$\begin{aligned} (h_{ii}x_{ij})^{\eta_{ij}} &= h_{ii}^\sigma x_{ij}^{\eta_{ij}}, \\ (x_{ij}h_{jj})^{\eta_{ij}} &= x_{ij}^{\eta_{ij}} h_{jj}^\sigma. \end{aligned} \tag{49}$$

PROOF. For the proof of (47) we consider

$$\begin{aligned} &(x_{ij} + x_{ij}^J)(x_{ji} + x_{ji}^J) + (x_{ji} + x_{ji}^J)(x_{ij} + x_{ij}^J) \\ &\quad = (x_{ij}x_{ji} + x_{ji}^J x_{ij}^J) + (x_{ji}x_{ij} + x_{ij}^J x_{ji}^J). \end{aligned}$$

Applying σ we obtain

$$\begin{aligned} &(x_{ij}x_{ji} + x_{ji}^J x_{ij}^J)^\sigma + (x_{ji}x_{ij} + x_{ij}^J x_{ji}^J)^\sigma \\ &= (x_{ij} + x_{ij}^J)^\sigma (x_{ji} + x_{ji}^J)^\sigma + (x_{ji} + x_{ji}^J)^\sigma (x_{ij} + x_{ij}^J)^\sigma \\ &= (x_{ij}^{\eta_{ij}} + (x_{ij}^J)^{\eta_{ji}})(x_{ji}^{\eta_{ji}} + (x_{ji}^J)^{\eta_{ij}}) \\ &\quad + (x_{ji}^{\eta_{ji}} + (x_{ji}^J)^{\eta_{ij}})(x_{ij}^{\eta_{ij}} + (x_{ij}^J)^{\eta_{ji}}) \\ &= (x_{ij}^{\eta_{ij}} x_{ji}^{\eta_{ji}} + (x_{ji}^J)^{\eta_{ij}}(x_{ij}^J)^{\eta_{ji}}) \\ &\quad + (x_{ji}^{\eta_{ji}} x_{ij}^{\eta_{ij}} + (x_{ij}^J)^{\eta_{ji}}(x_{ji}^J)^{\eta_{ij}}). \end{aligned}$$

Comparison of the parts in $\mathfrak{G}_{ii}$ yields (47). Next assume i,j,k unequal, $x_{ij} \in \mathfrak{A}_{ij}$, $x_{jk} \in \mathfrak{A}_{jk}$. Then $x_{ij}x_{jk} \in \mathfrak{A}_{ik}$ and $x_{ij}x_{jk} + x_{jk}^J x_{ij}^J = (x_{ij} + x_{ij}^J)(x_{jk} + x_{jk}^J) + (x_{jk} + x_{jk}^J)(x_{ij} + x_{ij}^J)$. Hence

$$\begin{aligned} (x_{ij}x_{jk} + x_{jk}^J x_{ij}^J)^\sigma &= (x_{ij} + x_{ij}^J)^\sigma (x_{jk} + x_{jk}^J)^\sigma \\ &\quad + (x_{jk} + x_{jk}^J)^\sigma (x_{ij} + x_{ij}^J)^\sigma \\ &= (x_{ij}^{\eta_{ij}} + (x_{ij}^J)^{\eta_{ji}})(x_{jk}^{\eta_{jk}} + (x_{jk}^J)^{\eta_{kj}}) \\ &\quad + (x_{jk}^{\eta_{jk}} + (x_{jk}^J)^{\eta_{kj}})(x_{ij}^{\eta_{ij}} + (x_{ij}^J)^{\eta_{ji}}) \\ &= x_{ij}^{\eta_{ij}} x_{jk}^{\eta_{jk}} + (x_{jk}^J)^{\eta_{kj}}(x_{ij}^J)^{\eta_{ji}} \end{aligned}$$

By definition of the η's this gives (48). For (49) we consider $h_{ii}(x_{ij} + x_{ij}^J) + (x_{ij} + x_{ij}^J)h_{ii} = h_{ii}x_{ij} + x_{ij}^J h_{ii}$ and apply σ. This gives

$$(h_{ii}x_{ij} + x_{ij}{}^J h_{ii})^\sigma = h_{ii}{}^\sigma(x_{ij} + x_{ij}{}^J)^\sigma + (x_{ij} + x_{ij}{}^J)^\sigma h_{ii}{}^\sigma.$$

The definition of η_{ij} then gives the first equation in (49). The second is obtained in the same way.

We are now ready to prove

THEOREM 6. (MARTINDALE). *Let* $(\mathfrak{A}, J)$ *be an associative algebra with involution and identity element* $1 = \sum_1^n e_i$ *where the* e_i *are orthogonal idempotents such that* (1) $e_i \in \mathfrak{H} = \mathfrak{H}(\mathfrak{A}, J)$ *and* (2) $\mathfrak{A}e_i\mathfrak{A} = \mathfrak{A}$, $i = 1, 2, \cdots, n$. *Then* $(\mathfrak{A}, J)$ *is perfect if either* $n \geqq 3$ *or* $n = 2$ *and* $\mathfrak{H} \cap e_i\mathfrak{A}e_i$ *generates* $e_i\mathfrak{A}e_i$ *for* $i = 1, 2$.

PROOF. Suppose first that $n \geqq 3$. Assume we have elements $x_{ij}{}^{(p)} \in \mathfrak{A}_{ij}$, $x_{ji}{}^{(p)} \in \mathfrak{A}_{ji}$, $i \neq j$, $p = 1, 2, \cdots, m$ such that $\sum_p x_{ij}{}^{(p)} x_{ji}{}^{(p)} = 0$. We claim that $\sum_p x_{ij}{}^{(p)\eta_{ij}} x_{ji}{}^{(p)\eta_{ji}} = 0$. Let $k \neq i, j$ and let $a_{ik} \in \mathfrak{A}_{ik}$. Then, by (48),

$$\begin{aligned}\sum_p x_{ij}{}^{(p)\eta_{ij}} x_{ji}{}^{(p)\eta_{ji}} a_{ik}{}^{\eta_{ik}} &= \sum_p x_{ij}{}^{(p)\eta_{ij}} (x_{ji}{}^{(p)} a_{ik})^{\eta_{jk}} \\ &= \sum (x_{ij}{}^{(p)} x_{ji}{}^{(p)} a_{ik})^{\eta_{ik}} = 0.\end{aligned} \tag{50}$$

Now $\mathfrak{A}_{ik}\mathfrak{A}_{ki} = \mathfrak{A}_{ii}$ implies that $\frac{1}{2} e_i = \sum_q a_{ik}^{(q)} a_{ki}^{(q)}$ for suitable $a_{ik}^{(q)} \in \mathfrak{A}_{ik}$, $a_{ki}^{(q)} \in \mathfrak{A}_{ki}$. Then

$$e_i = \tfrac{1}{2} e_i + \tfrac{1}{2} e_i^J = \sum_q (a_{ik}{}^{(q)} a_{ki}{}^{(q)} + a_{ki}{}^{(q)J} a_{ik}{}^{(q)J}). \tag{51}$$

Hence, by (47),

$$g_i = e_i^\sigma = \sum_q (a_{ik}{}^{(q)\eta_{ik}} a_{ki}{}^{(q)\eta_{ki}} + a_{ki}{}^{(q)J\eta_{ik}} a_{ik}{}^{(q)J\eta_{ki}}). \tag{52}$$

Since $a_{ik}{}^{(q)}$, $a_{ki}{}^{(q)J} \in \mathfrak{A}_{ik}$ it follows from (50) and (52) that $(\sum_p x_{ij}{}^{(p)\eta_{ij}} x_{ji}{}^{(p)\eta_{ji}}) g_i = 0$. Since the term in the parenthesis is in $\mathfrak{G}_{ii}$ and g_i is the identity in $\mathfrak{G}_{ii}$ we have $\sum x_{ij}{}^{(p)\eta_{ij}} x_{ji}{}^{(p)\eta_{ji}} = 0$. The result just established implies that there exists a unique linear mapping η_{ii} of $\mathfrak{A}_{ii} = \mathfrak{A}_{ij}\mathfrak{A}_{ji}$ into $\mathfrak{G}_{ii}$ such that

$$(x_{ij}x_{ji})^{\eta_{ii}} = x_{ij}{}^{\eta_{ij}} x_{ji}{}^{\eta_{ji}}. \tag{53}$$

The mapping η_{ii} has been defined by means of the index $j \neq i$. Let $\eta_{ii}{}'$ be the mapping similarly determined by $k \neq i, j$. We can write $x_{ji} = \sum_q x_{jk}{}^{(q)} x_{ki}{}^{(q)}$, $x_{jk}{}^{(q)} \in \mathfrak{A}_{jk}$, $x_{ki}{}^{(q)} \in \mathfrak{A}_{ki}$. Then

$$\begin{aligned}(x_{ij}x_{ji})^{\eta_{ii}'} &= \left(\sum_q (x_{ij}x_{jk}{}^{(q)}) x_{ki}{}^{(q)}\right)^{\eta'_{ii}} \\ &= \sum_q (x_{ij}x_{jk}{}^{(q)})^{\eta_{ik}} (x_{ki}{}^{(q)})^{\eta_{ki}} \\ &= \sum_q x_{ij}{}^{\eta_{ij}} x_{jk}{}^{(q)\eta_{jk}} x_{ki}{}^{(q)\eta_{ki}} \\ &= \sum_q x_{ij}{}^{\eta_{ij}} (x_{jk}{}^{(q)} x_{ki}{}^{(q)})^{\eta_{ji}} \\ &= x_{ij}{}^{\eta_{ij}} x_{ji}{}^{\eta_{ji}} \\ &= (x_{ij}x_{ji})^{\eta_{ii}}.\end{aligned}$$

Hence η_{ii} is independent of $j \neq i$. Now let η be the linear mapping of $\mathfrak{A}$ into $\mathfrak{G}$ which coincides with η_{ij} on $\mathfrak{A}_{ij}$, $i,j = 1, 2, \cdots, n$. We claim that η is a homomorphism of $\mathfrak{A}$ into $\mathfrak{G}$. Taking into account the products which are automatically 0 since $\mathfrak{A}_{ij}$ and $\mathfrak{G}_{ij}$ are Peirce spaces of $\mathfrak{A}$ and $\mathfrak{G}$ respectively it is enough to verify the homomorphism property for the following products: $x_{ij}x_{jk}$, $x_{ij}x_{ji}$, $x_{ii}x_{ij}$, $x_{ij}x_{jj}$, $x_{ii}y_{ii}$ where i, j, k are distinct, $x_{ij} \in \mathfrak{A}_{ij}$ etc. The first two are clear by (48) and (53). To settle the third it is enough to consider the case in which $x_{ii} = x_{ik}x_{ki}$, $x_{ik} \in \mathfrak{A}_{ik}$, $x_{ki} \in \mathfrak{A}_{ki}$ (since $\mathfrak{A}_{ii} = \mathfrak{A}_{ik}\mathfrak{A}_{ki}$). Then $(x_{ii}x_{ij})^\eta = (x_{ik}x_{ki}x_{ij})^\eta = x_{ik}^\eta(x_{ki}x_{ij})^\eta = x_{ik}^\eta x_{ki}^\eta x_{ij}^\eta = (x_{ik}x_{ki})^\eta x_{ij}^\eta = x_{ii}^\eta x_{ij}^\eta$. The next case follows in a similar manner. For the last case we again assume $x_{ii} = x_{ik}x_{ki}$. Then $(x_{ii}y_{ii})^\eta = (x_{ik}x_{ki}y_{ii})^\eta = x_{ik}^\eta(x_{ki}y_{ii})^\eta = x_{ik}^\eta x_{ki}^\eta y_{ii}^\eta = (x_{ik}x_{ki})^\eta y_{ii}^\eta = x_{ii}^\eta y_{ii}^\eta$. We show next that η is an extension of σ. Since every element of $\mathfrak{H}$ has the form $a + a^J$, $a \in \mathfrak{A}$, and since $\mathfrak{A}_{ii} = \mathfrak{A}_{ij}\mathfrak{A}_{ji}$ it is clear that every element of $\mathfrak{H}$ is a sum of elements of the form $x_{ij} + x_{ij}^J$ and $x_{ij}x_{ji} + x_{ji}^J x_{ij}^J$ where $i \neq j$, $x_{ij} \in \mathfrak{A}_{ij}$ and $x_{ji} \in \mathfrak{A}_{ji}$. It is now clear from (46) and (47) that η coincides with σ on $\mathfrak{H}$. Now $\mathfrak{H}$ generates $\mathfrak{A}$ since $(x_{ij} + x_{ij}^J)e_j = x_{ij}$, $i \neq j$ and the $\mathfrak{A}_{ij}$, $i \neq j$ generate $\mathfrak{A}$. It follows that η is the only extension of σ to a homomorphism of $\mathfrak{A}$. This completes the proof for the case $n \geqq 3$.

Now assume $n = 2$. In this case the supplementary hypothesis implies that every element of $\mathfrak{A}_{ii}$, $i = 1, 2$, is a sum of products of elements $h_{ii} \in \mathfrak{H} \cap \mathfrak{A}_{ii}$. Suppose we have a relation of the form $\Sigma_p m_p = 0$ where each m_p is a product $h_1 h_2 \cdots h_q$ where $h_j \in \mathfrak{H} \cap \mathfrak{A}_{ii}$. Then we claim that we have the relation $\Sigma m_p' = 0$ where $m_p' = h_1^\sigma h_2^\sigma \cdots h_q^\sigma$. To see this we let $x_{ik} \in \mathfrak{A}_{ik}$, $k \neq i$. Then, by iterating (49), we obtain $m_p' x_{ik}^{\eta_{ik}} = (m_p x_{ik})^{\eta_{ik}}$ and this implies that $\Sigma_p m_p' x_{ik}^{\eta_{ik}} = 0$. It then follows from (52) that $(\Sigma_p m_p')g_i = 0$. Since $\Sigma_p m_p' \in \mathfrak{G}_{ii}$ and g_i is the identity for $\mathfrak{G}_{ii}$ this gives $\Sigma_p m_p' = 0$. Our result implies that we have a homomorphism η_{ii} of the associative algebra $\mathfrak{A}_{ii}$ into $\mathfrak{G}_{ii}$ which coincides with σ on $\mathfrak{A}_{ii} \cap \mathfrak{H}$. Now let η be the linear mapping of $\mathfrak{A}$ into $\mathfrak{G}$ such that $\eta = \eta_{ij}$ on $\mathfrak{A}_{ij}$, $i,j = 1, 2$. Then the definition of η implies that $(x_{ii}y_{ii})^\eta = x_{ii}^\eta y_{ii}^\eta$ if $x_{ii}, y_{ii} \in \mathfrak{A}_{ii}$. Also, by (49) and the definition of η, we have $(h_{ii}x_{ij})^\eta = h_{ii}^\eta x_{ij}^\eta$ and $(x_{ij}h_{jj})^\eta = x_{ij}^\eta h_{jj}^\eta$ if $x_{ij} \in \mathfrak{A}_{ij}$, $h_{ii} \in \mathfrak{A}_{ii} \cap \mathfrak{H}$, $h_{jj} \in \mathfrak{A}_{jj} \cap \mathfrak{H}$. Since $\mathfrak{A}_{ii} \cap \mathfrak{H}$ generates $\mathfrak{A}_{ii}$ it follows that $(x_{ii}x_{ij})^\eta = x_{ii}^\eta x_{ij}^\eta$ and $(x_{ij}x_{jj})^\eta = x_{ij}^\eta x_{jj}^\eta$ if $x_{ij} \in \mathfrak{A}_{ij}$, $x_{ii} \in \mathfrak{A}_{ii}$, $x_{jj} \in \mathfrak{A}_{jj}$. Hence to prove η is an algebra homomorphism it remains to show that $(x_{ik}x_{ki})^\eta = x_{ik}^\eta x_{ki}^\eta$ if $x_{ik} \in \mathfrak{A}_{ik}$, $x_{ki} \in \mathfrak{A}_{ki}$ and $i \neq k$. Let $a_{ik} \in \mathfrak{A}_{ik}$, $a_{ki} \in \mathfrak{A}_{ki}$. Then by the results already established we have

$$\begin{aligned}(x_{ik}x_{ki})^\eta a_{ik}^\eta a_{ki}^\eta &= (x_{ik}x_{ki}a_{ik})^\eta a_{ki}^\eta \\ &= x_{ik}^\eta (x_{ki}a_{ik})^\eta a_{ki}^\eta = x_{ik}^\eta (x_{ki}a_{ik}a_{ki})^\eta.\end{aligned}$$

By (52), this implies that $(x_{ik}x_{ki})^\eta g_i = x_{ik}^\eta (x_{ki}e_i)^\eta$. Since $x_{ki} \in \mathfrak{A}_{ki}$ and $(x_{ik}x_{ki})^\eta \in \mathfrak{G}_{ii}$ we have $x_{ki}e_i = x_{ki}$ and $(x_{ik}x_{ki})^\eta g_i = (x_{ik}x_{ki})^\eta$. This shows that $(x_{ik}x_{ki})^\eta = x_{ik}^\eta x_{ki}^\eta$ and completes the proof of the homomorphism property of η. It is clear that

η is an extension of σ. Also, as before, $\mathfrak{A}$ is generated by $\mathfrak{H}$. Hence η is unique and the proof is complete for $n = 2$.

At the beginning of our discussion we showed that Theorem 6 has the following consequence.

COROLLARY 1. *Let $(\mathfrak{D},j)$ be an associative algebra with involution and identity element, J_a a canonical involution in $\mathfrak{D}_n$. Then $(\mathfrak{D}_n, J_a)$ is perfect if either $n \geqq 3$ or $n = 2$ and $\mathfrak{H}(\mathfrak{D},j)$ generates $\mathfrak{D}$.*

We remark that the supplementary condition or some alternative of it really is necessary to insure the validity of the result for the case $n = 2$. An example which shows this is $(\mathfrak{D}_2, J_1)$ where $(\mathfrak{D},j)$ is a quaternion algebra with standard involution. It is easily seen that in this case $\mathfrak{H}(\mathfrak{D}_2)$ is isomorphic to the Jordan algebra of a symmetric bilinear form f in a five dimensional space $\mathfrak{M}$. It follows that the unital special universal envelope of $\mathfrak{H}(\mathfrak{D}_2)$ is isomorphic to the Clifford algebra $C(\mathfrak{M},f)$ (§2.3). Since $\dim C(\mathfrak{M},f) = 2^5 = 32$ (§7.1) and $\dim \mathfrak{D}_2 = 16$ it is clear that $\mathfrak{D}_2$ is not a unital special universal envelope for $\mathfrak{H}(\mathfrak{D}_2)$ (see also exercise 2 below). Hence $(\mathfrak{D}_2, J_1)$ is not perfect.

Theorem 6 can also be applied to Jordan algebras of the form $\mathfrak{A}^+$, $\mathfrak{A}$ associative, using the usual trick of replacing $\mathfrak{A}^+$ by $\mathfrak{H}(\mathfrak{B}, J)$ where $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A}^\circ$ and J is the involution $(a_1, a_2) \to (a_2, a_1)$. One obtains in this way the following result.

COROLLARY 2. *Let $\mathfrak{A}$ be an associative algebra with $1 = \sum_1^n e_i$ where the e_i are orthogonal idempotent elements in $\mathfrak{A}$ such that $\mathfrak{A}e_i\mathfrak{A} = \mathfrak{A}$, $i = 1, 2, \cdots, n$. Let $\mathfrak{A}^\circ$ be the opposite algebra and $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A}^\circ$. Assume $n \geqq 3$. Then $\mathfrak{B}$ and the mapping $a \to (a, a)$ of $\mathfrak{A}$ into $\mathfrak{B}$ is a unital special universal envelope for $\mathfrak{A}^+$.*

PROOF. We have the isomorphism of $a \to (a, a)$ of $\mathfrak{A}^+$ into $\mathfrak{B}^+$ whose image is $\mathfrak{H}(\mathfrak{B}, J)$ the set of symmetric elements of $\mathfrak{B}$ under the involution J. If we set $f_i = (e_i, e_i)$ then the $f_i \in \mathfrak{H}(\mathfrak{B}, J)$ and these are orthogonal idempotents with sum 1. Moreover, since $\mathfrak{A}e_i\mathfrak{A} = \mathfrak{A}$ we have $\mathfrak{B}f_i\mathfrak{B} = \mathfrak{B}$. Hence the conditions of Theorem 6 are fulfilled and so $\mathfrak{B}$ and the mapping $a \to (a, a)$ constitute a unital special universal envelope for $\mathfrak{A}^+$.

Martindale's theorem and its corollaries can also be used to give important results on derivations. The theorem itself has the following consequence.

COROLLARY 3. *Let $(\mathfrak{A}, J)$ be as in Theorem 6. Then any derivation D of $\mathfrak{H}(\mathfrak{A}, J)$ has a unique extension to a derivation of the associative algebra $\mathfrak{A}$. Moreover, $JD = DJ$.*

PROOF. This is an immediate consequence of property (7) of Theorem 2.1 and the fact proved in Theorem 6 that $\mathfrak{A}$ and the injection constitute a unital special universal envelope for $\mathfrak{H}(\mathfrak{A}, J)$. Also the main involution in $\mathfrak{A}$ coincides with J. Hence $DJ = JD$ holds for the extension D of D to $\mathfrak{A}$.

It is natural to define a *derivation D of an algebra with involution* $(\mathfrak{A}, J)$ to be a derivation of $\mathfrak{A}$ which commutes with J. Then Corollary 3 states that under the hypothesis of the theorem, every derivation of $\mathfrak{H}(\mathfrak{A}, J)$ has a unique extension to a derivation of $(\mathfrak{A}, J)$. The converse is general: If D is a derivation of $(\mathfrak{A}, J)$ then the restriction of D to $\mathfrak{H}(\mathfrak{A}, J)$ is a derivation. This is immediate from the definitions.

Another consequence for derivations of Theorem 6 is the following

COROLLARY 4. *Let $\mathfrak{A}$ be an associative algebra satisfying the conditions of Corollary 2. Then any derivation D of $\mathfrak{A}^+$ is a derivation of $\mathfrak{A}$.*

PROOF. Let $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A}^\circ$, J the involution $(a_1, a_2) \to (a_2, a_1)$ as before. The derivation D in $\mathfrak{A}^+$ gives the derivation $(a, a) \to (aD, aD)$ in $\mathfrak{H}(\mathfrak{B}, J)$. By Corollary 3, we have a derivation D' of $\mathfrak{B}$ such that $(a, a)D' = (aD, aD)$. The ideal $\mathfrak{A}$ in $\mathfrak{B}$ consisting of the elements $(a, 0)$ is mapped into itself by D' since $\mathfrak{A}^2 = \mathfrak{A}$, so $\mathfrak{A}D' = \mathfrak{A}^2 D' \subseteq \mathfrak{A}(\mathfrak{A}D') + (\mathfrak{A}D')\mathfrak{A} \subseteq \mathfrak{A}$. Similarly, the ideal $\mathfrak{A}^\circ$ of elements $(0, a)$ is mapped into itself by D'. Hence we can write $(a, 0)D' = (aD_1, 0)$, $(0, a)D' = (0, aD_2)$ where the D_i are linear mappings of $\mathfrak{A}$ into $\mathfrak{A}$. Since $(a, 0) + (0, a) = (a, a)$ we have $(aD, aD) = (a, a)D' = (a, 0)D' + (0, a)D' = (aD_1, 0) + (0, aD_2) = (aD_1, aD_2)$. Hence $D_1 = D_2 = D$ and

$$\begin{aligned} ((ab)D, (ba)D) &= (ab, ba)D' = ((a, a)(b, b))D' \\ &= ((a, a)D')(b, b) + (a, a)((b, b)D') \\ &= (aD, aD)(b, b) + (a, a)(bD, bD) \\ &= ((aD)b + a(bD),\ b(aD) + (bD)a). \end{aligned}$$

Thus $(ab)D = a(bD) + (aD)b$ and D is a derivation in $\mathfrak{A}$.

EXERCISES

1. (Jacobson and Rickart). Let $\mathfrak{D}$ be an arbitrary associative algebra with 1. Show that Corollary 2 holds for $\mathfrak{A} = \mathfrak{D}_2$.

2. Let $\mathfrak{Q}$ be a quaternion algebra with standard involution and basis $(1, i, j, k)$ such that $i^2 = \alpha 1$, $j^2 = \beta 1$, $ij = -ji = k$, α, β in Φ. Let σ be the linear mapping of $\mathfrak{H}(\mathfrak{Q}_2)$ such that $e_{ii}{}^\sigma = e_{ii}$, $1[12]^\sigma = 1[12]$, $i[12]^\sigma = j[12]$, $j[12]^\sigma = i[12]$, $k[12]^\sigma = k[12]$. Show that σ is a homomorphism of $\mathfrak{H}(\mathfrak{Q}_2)$ which can not be extended to a homomorphism of $\mathfrak{Q}_2$.

5. **Unital bimodules for Jordan matrix algebras.** In this section we shall establish a basic equivalence between the theory of unital bimodules for a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ with $n \geqq 3$ and certain types of bimodules with involution for the coefficient algebra $(\mathfrak{D}, j)$. These are defined as follows.

DEFINITION 1. *Let $(\mathfrak{D}, j)$ be an algebra with involution and identity element* 1.

A unital bimodule with involution *for* $(\mathfrak{D},j)$ *is a pair* $(\mathfrak{N},j)$ *such that* $\mathfrak{N}$ *is a unital bimodule for* $\mathfrak{D}$ *and* $j: v \to \bar{v}$ *is a linear transformation in* $\mathfrak{N}$ *such that* $j^2 = 1$ *and*

$$(54) \qquad \overline{dv} = \bar{v}\bar{d}, \qquad \overline{vd} = \bar{d}\bar{v}, \qquad d \in \mathfrak{D}, \qquad v \in \mathfrak{N}.$$

If $\mathfrak{D}$ *is associative (alternative) then* $(\mathfrak{N},j)$ *is called* associative (alternative) *if* $\mathfrak{N}$ *is an associative (alternative) bimodule for* $\mathfrak{D}$. *In the alternative case* $(\mathfrak{N},j)$ *is* Jordan admissible *if*

$$(55) \qquad (d_1d_2)v = d_1(d_2v), \qquad (d_1v)d_2 = d_1(vd_2), \qquad v(d_1d_2) = (vd_1)d_2$$

holds whenever $\bar{v} = v$, $\bar{d}_1 = d_1$ *or* $\bar{d}_2 = d_2$ $(d_i \in \mathfrak{D},\ v \in \mathfrak{N})$.

Let $\mathfrak{F} = \mathfrak{D} \oplus \mathfrak{N}$ the split null extension of $(\mathfrak{D},j)$ by the unital bimodule with involution $(\mathfrak{N},j)$. Then we let j be the linear mapping in $\mathfrak{F}$ which extends the given linear mappings j on $\mathfrak{D}$ and $\mathfrak{N}$. Then $(\mathfrak{F},j)$ is an algebra with involution and identity element 1, the identity of $\mathfrak{D}$. If $\mathfrak{D}$ is associative (alternative) then $\mathfrak{F}$ is associative (alternative) if and only if $\mathfrak{N}$ is associative (alternative). In the alternative case if the symmetric elements of $\mathfrak{D}$ are in the nucleus then the symmetric elements of $\mathfrak{F}$ are in the nucleus if and only if $(\mathfrak{N},j)$ is Jordan admissible. These conclusions are immediate from the definitions.

If $(\mathfrak{N},j)$ is a bimodule with involution for $(\mathfrak{D},j)$ a *sub-bimodule* $\mathfrak{N}'$ of $(\mathfrak{N},j)$ is a sub-bimodule of $\mathfrak{N}$ such that $\mathfrak{N}'j = \mathfrak{N}'$. If $(\mathfrak{N}',j)$ is a second bimodule with involution for $(\mathfrak{D},j)$ then a homomorphism η of $(\mathfrak{N},j)$ into $(\mathfrak{N}',j)$ is a homomorphism of the bimodule $\mathfrak{N}$ into the bimodule $\mathfrak{N}'$ such that $\bar{v}^\eta = \overline{v\eta}$ for all $v \in \mathfrak{N}$.

Now let $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_a)$ be a Jordan matrix algebra defined by the algebra with 1 and involution $(\mathfrak{D},j)$ and the canonical involution J_a such that $a_1 = 1$ for the diagonal matrix a. We assume first that $n \geqq 4$. Then $\mathfrak{D}$ is associative. Let $(\mathfrak{N},j)$ be a unital associative bimodule with involution for $(\mathfrak{D},j)$, $\mathfrak{F}$ the split null extension $\mathfrak{D} \oplus \mathfrak{N}$. Then $(\mathfrak{F},j)$ is an associative algebra with involution and identity element 1, the identity of $\mathfrak{D}$. We can form the Jordan matrix algebra $\mathfrak{E} \equiv \mathfrak{H}(\mathfrak{F}_n, J_a)$ which contains $\mathfrak{J}$ as a subalgebra. Also $\mathfrak{E}$ contains the ideal $\mathfrak{M} \equiv \mathfrak{N}_n \cap \mathfrak{E}$ which is just the set of matrices of $\mathfrak{E}$ whose entries are in the ideal $(\mathfrak{N},j)$ of $(\mathfrak{F},j)$. Then $\mathfrak{M}$ is a unital Jordan bimodule for $\mathfrak{J}$ relative to the multiplication defined in $\mathfrak{E}$. We shall call $\mathfrak{M}$ the $\mathfrak{J}$-bimodule *associated* with the given bimodule with involution $(\mathfrak{N},j)$ of $(\mathfrak{D},j)$. Since $\mathfrak{F} = \mathfrak{D} \oplus \mathfrak{N}$ we have $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$. Also $\mathfrak{N}^2 = 0$ in $\mathfrak{F}$ implies $\mathfrak{M}^{\cdot 2} = 0$ in $\mathfrak{E}$ so $\mathfrak{E}$ is the split null extension of $\mathfrak{J}$ by its bimodule $\mathfrak{M}$.

Let $(\mathfrak{N}',j)$ be a second unital associative bimodule with involution for $(\mathfrak{D},j)$, $\mathfrak{F}'$ the corresponding split null extension, $\mathfrak{E}' = \mathfrak{H}(\mathfrak{F}_n', J_a)$, $\mathfrak{M}' = \mathfrak{E}' \cap \mathfrak{N}_n'$. Then $\mathfrak{E}' = \mathfrak{J} \oplus \mathfrak{M}'$ is the split null extension of $\mathfrak{J}$ by the Jordan bimodule $\mathfrak{M}'$. Let η be a homomorphism of $(\mathfrak{N},j)$ into $(\mathfrak{N}',j)$ and let η also denote the linear extension of this mapping to a mapping of $\mathfrak{F} = \mathfrak{D} \oplus \mathfrak{N}$ into $\mathfrak{F}' = \mathfrak{D} \oplus \mathfrak{N}'$ which is the identity on $\mathfrak{D}$. Then η is a homomorphism of the algebra with involution $(\mathfrak{F},j)$

into $(\mathfrak{F}',j)$ which leaves the entries of the diagonal matrix a fixed. Hence, by Theorem 3, the mapping $(f_{ij}) \to (f_{ij}^{\eta})$ is a homomorphism of $\mathfrak{E} = \mathfrak{H}(\mathfrak{F}_n, J_a)$ into $\mathfrak{E}' = \mathfrak{H}(\mathfrak{F}_n', J_a)$. Evidently this leaves the elements of $\mathfrak{J}$ fixed. Hence the restriction σ to $\mathfrak{M}$ of the homomorphism of $\mathfrak{E}$ is a $\mathfrak{J}$-bimodule homomorphism of $\mathfrak{M}$ into $\mathfrak{M}'$. Thus every homomorphism η of $(\mathfrak{N},j)$ into $(\mathfrak{N}',j)$ gives rise to a homomorphism σ of the associated bimodule $\mathfrak{M}$ into the associated $\mathfrak{M}'$. It is clear that the mappings $(\mathfrak{N},j) \to \mathfrak{M}$, $\eta \to \sigma$ define a functor from the category of unital associative bimodules with involution for $(\mathfrak{D},j)$ into the category of unital Jordan bimodules for $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_a)$.

We shall now study the properties of this functor. We show first that if $(\mathfrak{N},j) \to \mathfrak{M}$ and $(\mathfrak{N}',j) \to \mathfrak{M}'$, that is, $(\mathfrak{N},j)$ and $(\mathfrak{N}',j)$ are unital associative bimodules with involutions for $(\mathfrak{D},j)$ and $\mathfrak{M}$ and $\mathfrak{M}'$ are the associated unital Jordan bimodule for $\mathfrak{J}$ then any homomorphism of $\mathfrak{M}$ into $\mathfrak{M}'$ is obtained by applying the functor to a homomorphism of $(\mathfrak{N},j)$ into $(\mathfrak{N}',j)$. Thus let σ be a homomorphism of the $\mathfrak{J}$-bimodule $\mathfrak{M}$ into the $\mathfrak{J}$-bimodule $\mathfrak{M}'$. Then we obtain the algebra homomorphism of $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ into $\mathfrak{E}' = \mathfrak{J} \oplus \mathfrak{M}'$ which is the identity on $\mathfrak{J}$ and is σ on $\mathfrak{M}$. This maps $1[ij]$, $i,j = 1,\cdots,n$, into itself. Hence, by Theorem 3, we have a homomorphism η of $(\mathfrak{F},j)$ into $(\mathfrak{F}',j)$ such that the given algebra homomorphism has the form $(f_{ij}) \to (f_{ij}^{\eta})$. Clearly η is the identity on $\mathfrak{D}$ so the restriction η of η to $\mathfrak{N}$ is a homomorphism of $(\mathfrak{N},j)$ into $(\mathfrak{N}',j)$. It is clear that the given homomorphism σ of $\mathfrak{M}$ into $\mathfrak{N}$ is obtained by applying the functor we defined to the homomorphism η of $(\mathfrak{N},j)$ into $(\mathfrak{N}',j)$. An immediate consequence of this is that $(\mathfrak{N},j)$ and $(\mathfrak{N}',j)$ are isomorphic if and only if $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic.

We consider next the sub-bimodules of $(\mathfrak{N},j)$ and its associated $\mathfrak{J}$-bimodule $\mathfrak{M}$. Since $(\mathfrak{F},j)$ and $\mathfrak{E}$ are the split null extensions of $(\mathfrak{D},j)$ and $\mathfrak{J}$ by $(\mathfrak{N},j)$ and $\mathfrak{M}$ respectively, it is clear that the sub-bimodules of $(\mathfrak{N},j)$ and $\mathfrak{M}$ are just the ideals of $(\mathfrak{F},j)$ contained in $\mathfrak{N}$ and of $\mathfrak{E}$ contained in $\mathfrak{M}$. Hence, by Theorem 2, the functorial mapping we defined gives a lattice isomorphism of the lattice of sub-bimodules of $(\mathfrak{N},j)$ relative to $(\mathfrak{D},j)$ onto the lattice of sub-bimodules of $\mathfrak{M}$ relative to $\mathfrak{J}$.

To complete our reduction of the theory of unital bimodules for $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_a)$, $n \geqq 4$, to that of $(\mathfrak{D},j)$ we shall now show that every unital bimodule for $\mathfrak{J}$ is isomorphic to a bimodule associated with a unital associative bimodule for $(\mathfrak{D},j)$. Let $\mathfrak{M}$ be a unital bimodule for $\mathfrak{J}$ and form $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ the split null extension of $\mathfrak{J}$ by $\mathfrak{M}$. Then $\mathfrak{E}$ contains the elements $e_i = \frac{1}{2}[ii]$ and $u_{1j} = 1[1j]$, $j = 2,\cdots,n$, which are contained in $\mathfrak{J}$. Hence, by the Coordinatization Theorem, there exists a Jordan matrix algebra $\mathfrak{H}(\mathfrak{F}_n', J_{a'})$ and an isomorphism ζ of $\mathfrak{E}$ onto $\mathfrak{H}(\mathfrak{F}_n', J_{a'})$ such that $e_i^{\zeta} = \frac{1}{2}\{ii\}$, $u_{1j}^{\zeta} = 1\{1j\}$. Here we let j' be the involution in $\mathfrak{F}'$, $J_{a'}$ a canonical involution corresponding to j' and a diagonal matrix a' with first entry $a_1' = 1$. Also $\mathfrak{F}'$ is associative, and if $f' \in \mathfrak{F}'$ then $f'\{ij\} = f'e_{ij} + (f'e_{ij})^{J_{a'}}$. Since $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$, we have $\mathfrak{E}^{\zeta} = \mathfrak{H}(\mathfrak{F}_n', J_{a'}) = \mathfrak{J}^{\zeta} \oplus \mathfrak{M}^{\zeta}$. Moreover, $\mathfrak{M}^{\zeta}$ is an ideal in $\mathfrak{E}^\zeta$ such that $(\mathfrak{M}^\zeta)^{\cdot 2} = 0$ and $\mathfrak{J}^\zeta$ is a subalgebra containing all the elements $1\{ij\} = 1[ij]^\zeta$. Hence, by Theorem 2, $\mathfrak{M}^\zeta = \mathfrak{N}_n \cap \mathfrak{H}(\mathfrak{F}_n', J_{a'})$ where $\mathfrak{N}$ is an ideal in $(\mathfrak{F}', j')$ such that $\mathfrak{N}^2 = 0$. Also, $\mathfrak{J}^\zeta = \mathfrak{D}_n' \cap \mathfrak{H}(\mathfrak{F}_n', J_{a'})$ where $\mathfrak{D}'$ is a subalgebra of $(\mathfrak{F}', j')$ containing the a_i', and since $\mathfrak{H}(\mathfrak{F}_n', J_{a'}) = \mathfrak{J}^\zeta \oplus \mathfrak{M}^\zeta$, $\mathfrak{F}' = \mathfrak{D}' \oplus \mathfrak{N}$. We note next that the isomorphism ζ restricted to $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_a)$ maps $1[ij]$ into $1\{ij\}$. Hence, by Theorem 3 applied to the restriction of ζ to $\mathfrak{J}$ and its inverse, there exists an isomorphism η of $(\mathfrak{D}, j)$ into $(\mathfrak{D}', j')$ such that $a_i^\eta = a_i'$ and $(d_{ij})^\zeta = (d_{ij}^\eta)$ for $(d_{ij}) \in \mathfrak{J}$. We now replace $\mathfrak{D}'$ by $\mathfrak{D}$ and consider $\mathfrak{N}$ as bimodule for $\mathfrak{D}$ by defining $dv = d^\eta v$, $vd = vd^\eta$, $v \in \mathfrak{N}$, $d \in \mathfrak{D}$. If we call the restriction of j' to $\mathfrak{N}$, j, we see that $(\mathfrak{N}, j)$ is a unital associative bimodule with involution for $(\mathfrak{D}, j)$ and if $\mathfrak{F} = \mathfrak{D} \oplus \mathfrak{N}$ is the corresponding split null extension then we have the isomorphism $\lambda\colon d + v \to d^\eta + v$ of $(\mathfrak{F}, j)$ onto $(\mathfrak{F}', j')$ sending $a_i \to a_i'$. We now have the isomorphism $(f_{ij}) \to (f_{ij}^\lambda)$ of $\mathfrak{H}(\mathfrak{F}_n, J_a)$ onto $\mathfrak{H}(\mathfrak{F}_n', J_{a'})$ which coincides with ζ on $\mathfrak{J}$ and is the identity on $\mathfrak{N}_n \cap \mathfrak{H}(\mathfrak{F}_n, J_a) = \mathfrak{N}_n \cap \mathfrak{H}(\mathfrak{F}_n', J_{a'}) = \mathfrak{M}^\zeta$. It follows that $\mathfrak{M}^\zeta$ is the Jordan bimodule for $\mathfrak{J}$ associated with the unital associative bimodule with involution $(\mathfrak{N}, j)$. Moreover, the bimodule composition $a.x$ for $a \in \mathfrak{J}$, $x \in \mathfrak{M}^\zeta$, coincides with the product $a^\zeta.x$ as defined in the Jordan algebra $\mathfrak{H}(\mathfrak{F}_n', J_{a'})$. Hence $u \to u^\zeta$ is an isomorphism of $\mathfrak{M}$ as $\mathfrak{J}$-bimodule onto the bimodule associated with $(\mathfrak{N}, j)$.

The case $n = 3$ can be treated in a similar fashion. Here $(\mathfrak{D}, j)$ is alternative with involution and identity element 1 and the symmetric elements of $\mathfrak{D}$ are in the nucleus. Let $(\mathfrak{N}, j)$ be a unital Jordan admissible (alternative) bimodule with involution for $(\mathfrak{D}, j)$, $(\mathfrak{F}, j)$ the corresponding split null extension. Then $(\mathfrak{F}, j)$ is alternative with involution and 1 and symmetric elements in the nucleus. Hence $\mathfrak{H}(\mathfrak{F}_n, J_a)$ is Jordan and contains $\mathfrak{J}$ as subalgebra. The ideal $\mathfrak{M} = \mathfrak{N}_n \cap \mathfrak{H}(\mathfrak{F}_n, J_a)$ is a Jordan bimodule for $\mathfrak{J}$ which we shall call the *associated* bimodule to $(\mathfrak{N}, j)$. All the results we proved for the case $n \geqq 4$ carry over. Hence in this case we have a reduction of the study of the unital bimodules for $\mathfrak{H}(\mathfrak{D}_3, J_a)$ to unital Jordan admissible bimodules $(\mathfrak{N}, j)$ for $(\mathfrak{D}, j)$.

6. **Some remarks on the universal envelopes of Jordan matrix algebras.** We consider first the universal envelopes for Jordan matrix algebras of the form $\mathfrak{H}(\mathfrak{D}_n, J_a)$ where $\mathfrak{D}$ is associative and $n \geqq 3$. Since these algebras have identities it is sufficient to consider the universal envelopes $S_1(\mathfrak{H}(\mathfrak{D}_n, J_a))$ and $U_1(\mathfrak{H}(\mathfrak{D}_n, J_a))$. By Corollary 1 to Theorem 6, $S_1(\mathfrak{H}(\mathfrak{D}_n, J_a)) = \mathfrak{D}_n$. We consider next $U_1(\mathfrak{H}(\mathfrak{D}_n, J_a))$ and here we assume that $n \geqq 4$. By the Corollary to the Coordinatization Theorem (p. 138) every Jordan algebra with 1 containing $\mathfrak{H}(\mathfrak{D}_n, J_a)$ as subalgebra with the same identity is special. This and Theorem 2.17 imply that every unital bimodule for $\mathfrak{H}(\mathfrak{D}_n, J_a)$, $n \geqq 4$, is special. Then we can conclude (Corollary to Theorem 2.17) that $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is strongly special. Hence the canonical homomorphism of $U_1(\mathfrak{H}(\mathfrak{D}_n, J_a))$ onto the unital squared special envelope $S_1''(\mathfrak{H}(\mathfrak{D}_n, J_a))$ is an isomorphism. Thus we may replace the former algebra by the latter. We consider next

the question of the coincidence of $S_1''(\mathfrak{H}(\mathfrak{D}_n, J_a))$ with the subalgebra $\mathfrak{B}_1$ of $S_1(\mathfrak{H}(\mathfrak{D}_n, J_a)) \otimes S_1(\mathfrak{H}(\mathfrak{D}_n, J_a))$ of fixed elements of the exchange automorphism. Here we assume $n \geqq 3$ and $\mathfrak{D}$ associative. The theorems which are available to use for deriving sufficient conditions for $S_1''(\mathfrak{H}(\mathfrak{D}_n, J_a)) = \mathfrak{B}_1$ are Theorems 2.19 and Theorem 4. By Theorem 2.19 (1) and Theorem 4 we obtain the sufficient condition that $\mathfrak{S}(\mathfrak{D},j) = [\mathfrak{S}(\mathfrak{D},j), \mathfrak{S}(\mathfrak{D},j)] + [\mathfrak{H}(\mathfrak{D},j), \mathfrak{H}(\mathfrak{D},j)]$. If we recall that $S_1(\mathfrak{H}(\mathfrak{D}_n, J_a)) = \mathfrak{D}_n$ and that $S_1''(\mathfrak{H}(\mathfrak{D}_n, J_a)) = U_1(\mathfrak{H}(\mathfrak{D}_n, J_a))$ if $n \geqq 4$ we obtain the following

THEOREM 7. *Let $(\mathfrak{D},j)$ be an associative algebra with involution and identity element, J_a a canonical involution in $\mathfrak{D}_n$ defined by the diagonal matrix a. Assume $\mathfrak{S}(\mathfrak{D},J) = [\mathfrak{S}(\mathfrak{D},j), \mathfrak{S}(\mathfrak{D},j)] + [\mathfrak{H}(\mathfrak{D},j), \mathfrak{H}(\mathfrak{D},j)]$. Then $S_1''(\mathfrak{H}(\mathfrak{D}_n, J_a))$ is the subalgebra $\mathfrak{B}_1$ of $\mathfrak{D}_n \otimes \mathfrak{D}_n$ of fixed elements of the exchange automorphism if $n \geqq 3$ and $U_1(\mathfrak{H}(\mathfrak{D}_n, J_a)) \cong \mathfrak{B}_1$ if $n \geqq 4$.*

We consider next the application of Theorem 2.19 (2). For this purpose one needs to know that $\mathfrak{H}(\mathfrak{D}_n, J_a)$ is generated by 1 and two elements. A simple sufficient condition for this, which we shall use in Chapter VII is the following

THEOREM 8. *Let $\mathfrak{H}(\mathfrak{D}_n, J_1)$ be a Jordan matrix algebra defined by a standard involution J_1. Assume $n \geqq 3$ and the base field infinite. Then $\mathfrak{H}(\mathfrak{D}_n, J_1)$ is generated by 1 and two elements if the algebra with involution $(\mathfrak{D},j)$ is generated by 1 and $(n-1)(n-2)/2$ elements.*

PROOF. The hypothesis is that there exist $(n-1)(n-2)/2$ elements which we label as u_{ij}, $i < j = 2, 3, \cdots, n$ such that the smallest subalgebra of $(\mathfrak{D},j)$ containing these and 1 is $\mathfrak{D}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be distinct elements of Φ and put $a = \sum \alpha_i e_{ii} = \sum(\alpha_i/2)[ii]$, $b = \sum_{j=2}^{n} 1[1j] + \sum_{i<j=2}^{n} u_{ij}[ij]$. Then $\Phi[a]$ contains the idempotents e_{ii} and the algebra $\mathfrak{H}'$ generated by 1, a and b contains the elements $1[1j] = 2bU_{e_{11}, e_{jj}}$ and $u_{ij}[ij] = 2b\ U_{e_{ii}, e_{jj}}$. Hence by Theorem 2, $\mathfrak{H}'$ has the form $\mathfrak{D}_n' \cap \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $\mathfrak{D}'$ is a subalgebra of $(\mathfrak{D},j)$ containing 1 (and the a_i). Evidently $\mathfrak{D}'$ contains the u_{ij} since these appear as entries of the matrix b contained in $\mathfrak{H}'$. Hence $\mathfrak{D}' = \mathfrak{D}$ and $\mathfrak{H}' = \mathfrak{H}$.

This result, Theorem 2.19 (2) and the considerations preceding Theorem 7 imply the following

THEOREM 9. *Let $(\mathfrak{D},j)$ be an associative algebra with involution and identity element over an infinite field which is generated by 1 and $(n-1)(n-2)/2$ elements, $n \geqq 3$. Then $S_1''(\mathfrak{H}(\mathfrak{D}_n, J_1))$ is the subalgebra $\mathfrak{B}_1$ of $\mathfrak{D}_n \otimes \mathfrak{D}_n$ of fixed elements of the exchange automorphism and $U_1(\mathfrak{H}(\mathfrak{D}_n, J_1)) \cong \mathfrak{B}_1$ if $n \geqq 4$.*

7. **Lifting of idempotents and Jordan matrix algebras.** Let $\mathfrak{J}$ be a Jordan algebra with 1, $\mathfrak{N}$ an ideal in $\mathfrak{J}$ which is *nil* in the sense that every element $z \in \mathfrak{N}$ is nilpotent ($z^{\cdot n} = 0$ for some n). Let $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$ and let $\bar{e} = e + \mathfrak{N}$ be a nonzero idempotent element of $\bar{\mathfrak{J}}$. Then for the representative e of the coset $\bar{e}$ we have $e^{\cdot 2} - e = z \in \mathfrak{N}$, which shows that e is an algebraic element of $\mathfrak{J}$. Moreover, since

$0 \neq \bar{e} = \bar{e}^{\cdot 2} = \bar{e}^{\cdot 3} = \cdots$ it is clear that e is not nilpotent. We shall now show that we can choose an element $f \in \bar{e}$, that is, $f \equiv e \pmod{\mathfrak{N}}$ such that $f^{\cdot 2} = f\ (\neq 0)$. For this we require the following well-known and useful

LEMMA 1. *Let $\Phi[a]$ be an associative algebra with 1 generated by 1 and a single algebraic element a which is not nilpotent. Then there exists a polynomial $g(x) \in \Phi[x]$, x an indeterminate, with 0 constant term such that $f = g(a)$ is a nonzero idempotent element. If $\mathfrak{A}$ is an associative algebra generated by a single algebraic element which is not nilpotent then $\mathfrak{A}$ contains a nonzero idempotent element.*

PROOF. It is clear that the second statement follows from the first by adjoining an identity element to $\mathfrak{A}$ to obtain $\Phi[a]$ with 1. To prove the first statement let $f(x)$ be a nonzero polynomial of least degree such that $f(0) = 0$ and $f(a) = 0$. Write $f(x) = x^k h(x)$ where $h(0) \neq 0$ and $k \geqq 1$. Since a is not nilpotent it is clear that the degree, $\deg h(x) > 0$. Then there exist polynomials $p(x) \neq 0$, $q(x) \neq 0$ with $\deg p(x) < \deg h(x)$ such that $p(x)x^k + q(x)h(x) = 1$. Then if $g(x) = p(x)x^k$ we have $g(0) = 0$, $g(x) \neq 0$ and $\deg g(x) < \deg f(x)$. Hence $e = g(a) \neq 0$. Also $g(x) + q(x)h(x) = 1$ and $f(x) = x^k h(x)$ give $g(x)^2 \equiv g(x) \pmod{f(x)}$. Hence $e^2 = g(a)^2 = g(a) = e$ as required.

We remark that the second statement of the lemma implies that if $\mathfrak{A}$ is a power associative algebra containing an algebraic element which is not nilpotent then $\mathfrak{A}$ contains an idempotent $e \neq 0$. We can now prove the following

LEMMA 2. *Let $\mathfrak{J}$ be a Jordan algebra with 1, $\mathfrak{N}$ a nil ideal in $\mathfrak{J}$. Let $\bar{e} = e + \mathfrak{N}$ be a nonzero idempotent in $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$. Then there exists an idempotent f in $\mathfrak{J}$ such that $\bar{f} = \bar{e}$.*

PROOF. We have seen that e is algebraic and not nilpotent. Hence, by Lemma 1 applied to the subalgebra generated by 1 and e, there exists a polynomial $g(x) = \beta_1 x + \beta_2 x^2 + \cdots + \beta_r x^r$, β_i in Φ, such that $f = g(e) = \beta_1 e + \beta_2 e^{\cdot 2} + \cdots + \beta_r e^{\cdot r}$ is a nonzero idempotent. Since $\bar{e} = \bar{e}^{\cdot 2} = \cdots = \bar{e}^{\cdot r}$ we have $\bar{f} = \beta \bar{e}$ where $\beta = \Sigma \beta_i \in \Phi$. Since $f^{\cdot 2} = f$, $\bar{f}^{\cdot 2} = \bar{f}$ so $\beta^2 \bar{e}^{\cdot 2} = \beta \bar{e}$. Hence $\beta^2 = \beta$ and either $\beta = 0$ or $\beta = 1$. If $\beta = 0$ then $\bar{f} = 0$ and f is nilpotent contrary to $f^{\cdot 2} = f \neq 0$. Thus $\beta = 1$ and $\bar{f} = \bar{e}$.

We now extend Lemma 2 to the following

LEMMA 3. *Let $\mathfrak{J}$ and $\mathfrak{N}$ be as in Lemma 1. Let $\bar{e}_i = e_i + \mathfrak{N}$, $i = 1, 2, \cdots, n$, be nonzero orthogonal idempotents in $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$. Then there exist orthogonal idempotents f_i in $\mathfrak{J}$ such that $\bar{f}_i = \bar{e}_i$, $i = 1, 2, \cdots, n$. Moreover, if $\Sigma\, \bar{e}_i = \bar{1}$ then $\Sigma\, f_i = 1$.*

PROOF. The case $n = 1$ of the first statement is covered by Lemma 1. Now assume that for $1 \leqq r < n$ we have orthogonal idempotents f_j in $\mathfrak{J}$ such that

$\bar{f}_j = \bar{e}_j$, $j = 1,\cdots,r$. Let $u_1 = \sum_1^r f_j$, $u_2 = 1 - u_1$ so u_1 and u_2 are orthogonal idempotents such that $u_1 + u_2 = 1$. Let $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{22}$ be the Peirce decomposition of $\mathfrak{J}$ relative to u_1 and u_2. Since $u_1 . f_j = f_j$ we have that $f_j \in \mathfrak{J}_{11}$, $j = 1,\cdots,r$. Now put $e = e_{r+1}U_{u_2}$, so $e \in \mathfrak{J}_{22}$. Also $e = e_{r+1}U_{u_2} = 2e_{r+1} . (1 - \sum_1^r f_j) . (1 - \sum_1^r f_j) - e_{r+1} . (1 - \sum_1^r f_j) \equiv e_{r+1} \pmod{\mathfrak{N}}$ since $\bar{f}_j = \bar{e}_j$ and $\bar{e}_j . \bar{e}_{r+1} = 0$. Hence $\bar{e}^{\cdot 2} = \bar{e}$ and $\bar{e} \neq 0$. Now $\mathfrak{N}_{22} = \mathfrak{N} \cap \mathfrak{J}_{22}$ is a nil ideal in the Jordan algebra $\mathfrak{J}_{22}$ with identity u_2 and $e^{\cdot 2} \equiv e \pmod{\mathfrak{N}_{22}}$ and $e \not\equiv 0 \pmod{\mathfrak{N}_{22}}$. Hence, by the case of a single idempotent, there exists an idempotent f_{r+1} in $\mathfrak{J}_{22}$ such that $f_{r+1} \equiv e \pmod{\mathfrak{N}_{22}}$. Then $f_{r+1} \equiv e_{r+1} \pmod{\mathfrak{N}}$ and we have completed the inductive step of the proof. This proves the first statement. Now suppose $\sum_1^n \bar{e}_i = \bar{1}$. Then $\sum_1^n \bar{f}_i = \bar{1}$ and if $u = \sum_1^n f_i$ then $u^{\cdot 2} = u$ and $\bar{u} = \bar{1}$. Hence $u = 1 + z$ where $z \in \mathfrak{N}$ and $u^{\cdot 2} = u$ gives $z + z^{\cdot 2} = 0$. Then $z = -z^{\cdot 2} = z^{\cdot 3} = -z^{\cdot 4} = \cdots = 0$ and $u = 1$.

We shall show next that the relations of connectedness and strong connectedness for orthogonal idempotents can be lifted from $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$ to $\mathfrak{J}$. For this we require the following

LEMMA 4. *Let $\mathfrak{J}$ be a Jordan algebra with* 1 *and nil ideal $\mathfrak{N}$. Then*: (1) *If $u \in \mathfrak{J}$ and $\bar{u} = u + \mathfrak{N}$ is invertible in $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$ then u is invertible in $\mathfrak{J}$.* (2) *If $u \equiv 1 \pmod{\mathfrak{N}}$, so $u = 1 - 4z$ where $z \in \mathfrak{N}$ and $z^{\cdot n+1} = 0$ then there exists an element w of the form $\sum_1^n \beta_i z^{\cdot i}$ such that $(1 - 2w)^{\cdot 2} = u = 1 - 4z$.* (3) *If $u^{\cdot 2} \equiv 1 \pmod{\mathfrak{N}}$ then there exists an element v which is a linear combination of odd powers of u such that $v \equiv u \pmod{\mathfrak{N}}$ and $v^{\cdot 2} = 1$.*

PROOF. (1) Since $\bar{u}$ is invertible, $\bar{\mathfrak{J}}U_{\bar{u}}$ contains the identity $\bar{1} = 1 + \mathfrak{N}$ of $\bar{\mathfrak{J}}$. Hence there exists a $v \in \mathfrak{J}$ such that $\{uvu\} = vU_u = 1 - z$ where $z \in \mathfrak{N}$. Clearly $1 - z$ is invertible with inverse $1 + z + z^{\cdot 2} + \cdots + z^{\cdot n}$ if $z^{\cdot n+1} = 0$. Thus $\{uvu\}$ is invertible in $\mathfrak{J}$ and hence u and v are invertible (Theorem 1.13).

(2) The condition $(1 - 2w)^{\cdot 2} = 1 - 4z$ is equivalent to $w - w^{\cdot 2} = z$ so if $w = \sum \beta_i z^{\cdot i}$ then the condition will hold if $\beta_1 = 1$ and $\beta_i = \sum_{k=1}^{i-1} \beta_k \beta_{i-k}$, $i = 2,\cdots,n$. Evidently these have a solution in Φ (even in integers in the prime field). Hence (2) holds.

(3) We have $u^{\cdot 2} = 1 - 4z$ where $z \in \mathfrak{N}$. Hence, by (2) there exists an element $1 - 2w$, $w = \sum \beta_i z^{\cdot i}$, such that $(1 - 2w)^{\cdot 2} = 1 - 4z$. Clearly, $1 - 2w$ is invertible with inverse $1 + 2w + (2w)^{\cdot 2} + \cdots$ and $v = u(1 - 2w)^{\cdot -1} = u \pmod{\mathfrak{N}}$. Also, since all the elements with which we are dealing are polynomials in u, we have $v^{\cdot 2} = u^{\cdot 2}(1 - 2w)^{\cdot -2} = 1$. Since $z, w \in \Phi[u^{\cdot 2}]$ it is clear that v has the required form.

We can now prove

LEMMA 5. *Let $\mathfrak{J}$ be a Jordan algebra with* 1, *$\mathfrak{N}$ a nil ideal in $\mathfrak{J}$, e_1 and e_2 nonzero orthogonal idempotents in $\mathfrak{J}$. Then e_1 and e_2 are connected* (*strongly connected*) *in $\mathfrak{J}$ if and only if $\bar{e}_1 = e_1 + \mathfrak{N}$ and $\bar{e}_2 = e_2 + \mathfrak{N}$ are connected* (*strongly connected*) *in $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$.*

PROOF. If e_1 and e_2 are connected (strongly connected) then it is clear that $\bar{e}_1$ and $\bar{e}_2$ are connected (strongly connected). Conversely, assume $\bar{e}_1$ and $\bar{e}_2$ are connected in $\bar{\mathfrak{J}}$. Then there exists an element $\bar{u}_{12} \in \bar{\mathfrak{J}}U_{\bar{e}_1,\bar{e}_2}$ which is invertible in the Jordan algebra $\bar{\mathfrak{J}}U_{\bar{v}}$ whose identity is $\bar{v} = \bar{e}_1 + \bar{e}_2$. We have $\bar{u}_{12} = u_{12} + \mathfrak{N}$ and if we put $v_{12} = 2u_{12}U_{e_1,e_2}$ then $\bar{v}_{12} = \bar{u}_{12}$ since $2U_{\bar{e}_1,\bar{e}_2}$ acts as identity on $\bar{\mathfrak{J}}U_{\bar{e}_1,\bar{e}_2}$. Now $\mathfrak{J}U_v$, $v = e_1 + e_2$, is a Jordan algebra with identity v and $\bar{\mathfrak{J}}U_{\bar{v}}$ is a homomorphic image of $\mathfrak{J}U_v$ with nil kernel. Since $\bar{v}_{12} = \bar{u}_{12}$ is invertible in $\bar{\mathfrak{J}}U_{\bar{v}}$ it follows that v_{12} is invertible in $\mathfrak{J}U_v$. Since $v_{12} \in \mathfrak{J}U_{e_1,e_2}$ this shows that e_1 and e_2 are connected in $\mathfrak{J}$. If $\bar{e}_1$ and $\bar{e}_2$ are strongly connected then we may assume that the element $\bar{u}_{12}$ in the foregoing argument satisfies $\bar{u}_{12}^{\cdot 2} = \bar{e}_1 + \bar{e}_2$. Then Lemma 4 (3) is applicable to show that we can choose v_{12} in $\mathfrak{J}_{12}$ so that $v_{12}^{\cdot 2} = e_1 + e_2$. Then e_1 and e_2 are strongly connected.

We can now prove

THEOREM 10. *Let $\mathfrak{J}$ be a Jordan algebra with* 1 *containing a nil ideal $\mathfrak{N}$ such that $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N} \cong \mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})$ ($\mathfrak{H}(\bar{\mathfrak{D}}_n)$) a Jordan matrix algebra (determined by a standard involution) of order $n \geqq 3$. Then $\mathfrak{J}$ is isomorphic to a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ ($\mathfrak{H}(\mathfrak{D}_n)$) where the ideal in $\mathfrak{H}(\mathfrak{D}_n, J_a)$ ($\mathfrak{H}(\mathfrak{D}_n)$) corresponding to $\mathfrak{N}$ has the form $\mathfrak{M}_n \cap \mathfrak{H}(\mathfrak{D}_n, J_a)$ ($\mathfrak{M}_n \cap \mathfrak{H}(\mathfrak{D}_n)$) where $\mathfrak{M}$ is an ideal in $(\mathfrak{D},j)$ and $\bar{\mathfrak{D}} \cong \mathfrak{D}/\mathfrak{M}$ as algebras with involution.*

PROOF. Since the proof for the case $\mathfrak{H}(\bar{\mathfrak{D}}_n)$ is very similar to that of the case of a general Jordan matrix algebra $\mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})$ we shall give the proof only for the case $\mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})$. Let τ be an isomorphism of $\bar{\mathfrak{J}}$ onto $\mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})$ and let $\bar{e}_i, i = 1, \cdots, n$, be elements of $\bar{\mathfrak{J}}$ such that $\bar{e}_i^{\tau} = \frac{1}{2}[ii]$ (in $\mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})$). By Lemma 3, we may assume that the elements e_i of $\mathfrak{J}$ such that $\bar{e}_i = e_i + \mathfrak{N}$ are orthogonal idempotents such that $\Sigma e_i = 1$. By the proof of Lemma 4 we can choose an element $u_{1j}, j = 2, \cdots, n$ in the Peirce space $\mathfrak{J}_{1j}$ such that $\bar{u}_{1j}^{\tau} = 1[1j]$ and u_{1j} is invertible in $\mathfrak{J}_{11} + \mathfrak{J}_{jj} + \mathfrak{J}_{1j}$ with inverse u_{j1}. Also $\bar{u}_{j1}^{\tau} = 1[j1]$ since $1[j1]$ is the inverse of $1[1j]$ in the Peirce space $\mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})_{11} + \mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})_{jj} + \mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})_{1j}$. By the Coordinatization Theorem, there exists a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ and an isomorphism ζ of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n, J_a)$ such that $e_i^{\zeta} = \frac{1}{2}[ii]$, $u_{1j} = 1[1j]$ (in $\mathfrak{H}(\mathfrak{D}_n, J_a)$). Let ν denote the canonical homomorphism of $\mathfrak{J}$ onto $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$. Then $\sigma = \zeta^{-1}\nu\tau$ is a homomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ onto $\mathfrak{H}(\bar{\mathfrak{D}}_n, J_{\bar{a}})$ such that $\frac{1}{2}[ii]^{\sigma} = \frac{1}{2}[ii]$, $1[1j]^{\sigma} = 1[1j]$. It follows that $1[j1]^{\zeta} = 1[j1]$ and, in general, $1[ij]^{\sigma} = 1[ij]$. Hence, by Theorem 3, there exists a homomorphism η of $(\mathfrak{D},j)$ onto $(\bar{\mathfrak{D}},j)$ such that σ is the restriction to $\mathfrak{H}(\mathfrak{D}_n, J_a)$ of the mapping $(d_{ij}) \to (d_{ij}^{\eta})$ of $\mathfrak{D}_n$ onto $\bar{\mathfrak{D}}_n$. If $\mathfrak{M}$ is the kernel of η then $\mathfrak{M}$ is an ideal in $(\mathfrak{D},j)$ and $(\bar{\mathfrak{D}},j) \cong (\mathfrak{D},j)/\mathfrak{M}$. Moreover, the kernel of σ is $\mathfrak{H}(\mathfrak{M}_n, J_a) = \mathfrak{H}(\mathfrak{D}_n, J_a) \cap \mathfrak{M}_n$. Hence $\mathfrak{N}^{\zeta} = \mathfrak{H}(\mathfrak{M}_n, J_a)$ and the proof is complete.

CHAPTER IV

JORDAN ALGEBRAS WITH MINIMUM CONDITIONS ON QUADRATIC IDEALS

It is well known that one of the most satisfactory segments of the theory of associative rings is the theory of right (left) Artinian rings. These are defined as the rings which satisfy the minimum, or equivalently, the descending chain condition for right (left) ideals. If such a ring is semisimple in the sense that it has no nonzero nilpotent ideals then one has the classical Wedderburn-Artin structure theorems: I. Any semisimple right (left) Artinian ring has an identity element and is a direct sum of a finite number of minimal right (left) Artinian rings and conversely; II. A right (left) Artinian ring is simple if and only if it is isomorphic to a matrix ring Δ_n where Δ is a division ring. If one recalls the relation between linear transformations and matrices, it is clear that an alternative formulation of the second structure theorem is that an associative ring is right (left) Artinian and simple if and only if it is isomorphic to a ring $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ where $\mathfrak{M}$ is a finite-dimensional vector space over a division ring.

We recall also that the class of semisimple (left or right) Artinian rings has the following important module characterization: These are just the associative rings with 1 having the property that all unital left or right modules are completely reducible. Moreover, the isomorphism classes of irreducible right (left) modules are in 1-1 correspondence with the simple components of the ring, that is, with the uniquely determined simple ideals $\mathfrak{A}_i$ in the decomposition of the ring $\mathfrak{A}$ as $\mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$. In particular, if $\mathfrak{A}$ is simple Artinian then all irreducible right (left) $\mathfrak{A}$-modules are isomorphic. This result leads easily to the uniqueness of n and of the isomorphism class of the division ring Δ in the second Wedderburn-Artin theorem: $\mathfrak{A} \cong \Delta_n$.

The Wedderburn-Artin structure theorems are due to Wedderburn for the case of finite-dimensional algebras. The extension to rings satisfying the minimum and maximum conditions for right (left) ideals was given by Artin and the removal of the superfluous maximum condition in this connection is due to C. Hopkins. A structure theory of finite-dimensional Jordan algebras analogous to Wedderburn's associative theory was developed by Albert in three papers ([7], [9] and [17]). Until quite recently there was no analogue in the Jordan theory of Artin's associative structure theory, though it has been clear for some time that an essential step for developing such a theory was the formulation of a suitable analogue or substitute for the associative concept of a one-sided ideal. Recently a notion of this sort, called quadratic ideals, was introduced by Topping ([1]) for Jordan algebras of operators in Hilbert space. This was appropriated to the general case by the author, and served as the basic concept for the development of an analogue for Jordan algebras of Artin's associative structure theory (in [38]). The hypotheses for this consist of existence of 1, the validity of certain minimum conditions on quadratic ideals, and an assumption of "nondegeneracy", which is a substitute for the hypothesis of nonexistence of nilpotent ideals $\neq 0$. The author's structure theory for simple algebras contained two gaps: (1) determination of the structure of simple algebras of "capacity" two, and (2) determination of Jordan division algebras. The first of these was filled in a completely satisfactory manner by Osborn in [6]. The second remains an outstanding problem.

The theory which we shall present in this chapter deals, as usual in this book, with algebras over a field of characteristic not two. The final step in the development of an Artinian-like structure theory for Jordan rings was taken quite recently by McCrimmon ([6] and [14]). Since no assumptions are made on the additive group, his theory encompasses also a "good" structure theory for Jordan algebras of characteristic two. This follows the lines indicated at the beginning of this book (p. 4). Since we have decided at the outset to simplify the discussion by sticking to the characteristic not two theory, we shall indicate McCrimmon's results only in a sketchy fashion in the section on "Further Results and Open Questions" at the end of this volume.

1. **Quadratic ideals.** We now introduce the main concept on which our structure theory will be based. This is defined as follows:

DEFINITION 1. *A subspace* $\mathfrak{B}$ *of a Jordan algebra* $\mathfrak{J}$ *is called a* quadratic ideal *in* $\mathfrak{J}$ *if* $\mathfrak{J}U_b \subseteq \mathfrak{B}$ *for every* $b \in \mathfrak{B}$.

Since $U_{b_1+b_2} = U_{b_1} + U_{b_2} + 2U_{b_1,b_2}$ and $U_b = U_{b,b}$ it is clear that a subspace $\mathfrak{B}$ is a quadratic ideal if and only if $\{b_1ab_2\} = aU_{b_1,b_2} \in \mathfrak{B}$ for $a \in \mathfrak{J}$, $b_1, b_2 \in \mathfrak{B}$. This condition is equivalent to: $\mathfrak{B}$ is a subalgebra of the a-homotope $(\mathfrak{J}, a)$ for every $a \in \mathfrak{J}$. If $\mathfrak{J}$ has an identity element 1 then this implies that any quadratic ideal is a subalgebra of $\mathfrak{J}$. In any case, if $\mathfrak{B}$ is a quadratic ideal then $\{b_1b_2b_3\} \in \mathfrak{B}$ for all $b_i \in \mathfrak{B}$. Since $b_1 . b_2 . b_3 = \frac{1}{2}\{b_1b_2b_3\} + \frac{1}{2}\{b_2b_1b_3\}$ we have also that $b_1 . b_2 . b_3 \in \mathfrak{B}$ if $b_i \in \mathfrak{B}$.

Clearly any ideal of $\mathfrak{J}$ is a quadratic ideal. If $\mathfrak{A}$ is an associative algebra then in the Jordan algebra $\mathfrak{A}^+$ we have $xU_b = bxb$ in terms of the associative product. It is clear from this that any left or right ideal of $\mathfrak{A}$ is a quadratic ideal of $\mathfrak{A}^+$. If $\mathfrak{B}$ is a quadratic ideal in the Jordan algebra $\mathfrak{J}$ and $a \in \mathfrak{J}$ then $\mathfrak{B}U_a$ is a quadratic ideal. For, if $c \in \mathfrak{B}U_a$ then $c = bU_a$, $b \in \mathfrak{B}$, and $\mathfrak{J}U_c = \mathfrak{J}U_{bU_a} = \mathfrak{J}U_aU_bU_a \subseteq \mathfrak{B}U_a$.

In particular, $\mathfrak{J}U_a$ is a quadratic ideal for any $a \in \mathfrak{J}$. We shall call this the *principal quadratic ideal determined by a*. An element b will be called an *absolute zero divisor* (*in* $\mathfrak{J}$) if $U_b = 0$, or equivalently, the principal quadratic ideal $\mathfrak{J}U_b = 0$. In this case Φb is a quadratic ideal in $\mathfrak{J}$. If b is an absolute zero divisor then bU_a is an absolute zero divisor for any a in $\mathfrak{J}$ since $U_{bU_a} = U_a U_b U_a = 0$.

If e is an idempotent element in $\mathfrak{J}$ then the principal quadratic ideal $\mathfrak{J}U_e$ determined by e coincides with the Peirce space $\mathfrak{J}_1(e) = \{x \mid x . e = x\}$. Then the quadratic ideal $\mathfrak{J}U_e$ is a subalgebra of $\mathfrak{J}$ having e as identity element. Moreover, if $\mathfrak{B}$ is a quadratic ideal of the subalgebra $\mathfrak{J}U_e$ then $\mathfrak{B}$ is a quadratic ideal of $\mathfrak{J}$ since, if $b \in \mathfrak{B}$, then $b = bU_e$ and $\mathfrak{J}U_b = \mathfrak{J}U_{bU_e} = \mathfrak{J}U_e U_b U_e \subseteq \mathfrak{B}U_e = \mathfrak{B}$. Similarly, one sees that if $b \in \mathfrak{J}U_e$ is an absolute zero divisor in $\mathfrak{J}U_e$ then b is an absolute zero divisor in $\mathfrak{J}$.

Again, let e be idempotent and let $\mathfrak{J} = \mathfrak{J}_0(e) \oplus \mathfrak{J}_{\frac{1}{2}}(e) \oplus \mathfrak{J}_1(e)$ be the corresponding Peirce decomposition. By Lemma 1 of §1, we have $\mathfrak{J}_0 . \mathfrak{J}_1 = 0$ and $x . b^{\cdot 2} = 2x . b . b$ if $x \in \mathfrak{J}_{\frac{1}{2}}$ and $b \in \mathfrak{J}_0$. It follows that $\mathfrak{J}_1 U_b = 0$ and $\mathfrak{J}_{\frac{1}{2}} U_b = 0$. Hence $\mathfrak{J}U_b = \mathfrak{J}_0 U_b$ if $b \in \mathfrak{J}_0$. This implies that any quadratic ideal of the subalgebra $\mathfrak{J}_0$ is a quadratic ideal of $\mathfrak{J}$. In particular, $\mathfrak{J}_0$ is a quadratic ideal of $\mathfrak{J}$. It is clear also that if b is an absolute zero divisor in $\mathfrak{J}_0$ then it is an absolute zero divisor in $\mathfrak{J}$.

It is clear that the intersection of any set of quadratic ideals is a quadratic ideal. On the other hand, if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are quadratic ideals then $\mathfrak{B}_1 + \mathfrak{B}_2$ need not be a quadratic ideal (see exercise 2 below). If $\{\mathfrak{B}\}$ is a collection of quadratic ideals which is totally ordered by inclusion then it is clear that $\bigcup \mathfrak{B}$ is a quadratic ideal. Of particular importance in our theory will be the minimal quadratic ideals. A quadratic ideal $\mathfrak{B}$ of $\mathfrak{J}$ is called *minimal* if $\mathfrak{B} \neq 0$ and there exists no quadratic ideal $\mathfrak{C}$ such that $\mathfrak{B} \supset \mathfrak{C} \supset 0$. The possibilities for these is given in the following theorem.

THEOREM 1. *If* $\mathfrak{B}$ *is a minimal quadratic ideal in a Jordan algebra* $\mathfrak{J}$ *then one has one of the following possibilities*: I. $\mathfrak{B} = \Phi b$ *where* b *is an absolute zero divisor.* II. $\mathfrak{B} = \mathfrak{J}U_b$ *for every nonzero* $b \in \mathfrak{B}$ *and* $\mathfrak{B}^{\cdot 2} = 0$. III. $\mathfrak{B} = \mathfrak{J}U_e$ *for an idempotent* e *and* $\mathfrak{B}$ *is a division subalgebra of* $\mathfrak{J}$.

PROOF. Suppose $\mathfrak{B}$ contains an element $b \neq 0$ such that $U_b = 0$. Then Φb is a nonzero quadratic ideal contained in $\mathfrak{B}$ so $\mathfrak{B} = \Phi b$ and we have case I. Now assume that for every $b \in \mathfrak{B}$ we have $\mathfrak{J}U_b \neq 0$. Then $\mathfrak{J}U_b$ is a nonzero quadratic ideal contained in $\mathfrak{B}$ so $\mathfrak{J}U_b = \mathfrak{B}$ for every $b \neq 0$ in $\mathfrak{B}$. Also $\mathfrak{B}U_b$ is a quadratic ideal contained in $\mathfrak{B}$ so either $\mathfrak{B}U_b = 0$ or $\mathfrak{B}U_b = \mathfrak{B}$. Suppose there exists a $b \neq 0$ in $\mathfrak{B}$ such that $\mathfrak{B}U_b = 0$ and let $b' \in \mathfrak{B}$. Since $\mathfrak{B} = \mathfrak{J}U_b$, $b' = aU_b$ for $a \in \mathfrak{J}$. Then $\mathfrak{B}U_{b'} = \mathfrak{B}U_{aUb} = \mathfrak{B}U_b U_a U_b = 0$. Thus $\mathfrak{B}U_b = 0$ for all $b \in \mathfrak{B}$ and hence $b^{\cdot 3} = 0$ for every $b \in \mathfrak{B}$. We prove next that $b^{\cdot 2} \in \mathfrak{B}$ for all $b \in \mathfrak{B}$. Now in any special Jordan algebra we have the identity

(1) $$\{xyx\}^{\cdot 2} = \{x\{yx^{\cdot 2}y\}x\}$$

since $\{xyx\} = xyx$ in $\mathfrak{A}^+$, $\mathfrak{A}$ associative (see ex. 4 p. 40). Since this is a two variable identity it follows from Shirshov's Theorem that it holds for all Jordan algebras. Now let $b \in \mathfrak{B}$. Then there exists an $a \in \mathfrak{J}$ such that $b = \{bab\}$, so $b^{\cdot 2} = \{b\{ab^{\cdot 2}a\}b\} \in \mathfrak{J}U_b \subseteq \mathfrak{B}$. Now suppose $b^{\cdot 2} \neq 0$. Since $b^{\cdot 2} \neq 0$ there exists a $c \in \mathfrak{J}$ such that $b = \{b^{\cdot 2}cb^{\cdot 2}\}$. Then $b^{\cdot 2} = \{b^{\cdot 2}\{cb^{\cdot 4}c\}b^{\cdot 2}\} = 0$ since $b^{\cdot 3} = 0$. This contradiction shows that $b^{\cdot 2} = 0$ for all $b \in \mathfrak{B}$. Since $\mathfrak{J}$ is a commutative algebra of characteristic $\neq 2$ this implies that $\mathfrak{B}^{\cdot 2} = 0$. It remains to consider the remaining alternative for $\mathfrak{B}$: $\mathfrak{B} = \mathfrak{B}U_b$ for every $b \neq 0$ in $\mathfrak{B}$. Take $b \neq 0$ in $\mathfrak{B}$. Then there exists a $c \in \mathfrak{B}$ such that $b = cU_b$. This implies that $U_b = U_bU_cU_b$ so $E = U_bU_c$ and $F = U_cU_b$ are idempotent linear transformations in $\mathfrak{J}$ which map $\mathfrak{B}$ into itself and have surjective restrictions to $\mathfrak{B}$. Hence these restrictions are the identity mapping $1_{\mathfrak{B}}$ on $\mathfrak{B}$, so $\bar{U}_b\bar{U}_c = 1_{\mathfrak{B}} = \bar{U}_c\bar{U}_b$ where $\bar{U}$ denotes the restriction of U to $\mathfrak{B}$. Consider $f = \{cb^{\cdot 2}c\} \in \mathfrak{B}$. We have $\bar{U}_f = \bar{U}_c\bar{U}_{b^{\cdot 2}}\bar{U}_c = 1_{\mathfrak{B}}$ and $e = f^{\cdot 2} = \{cb^{\cdot 2}c\}^{\cdot 2} = \{c\{b^{\cdot 2}c^{\cdot 2}b^{\cdot 2}\}c\} \in \mathfrak{B}$, by (1). Hence $e^{\cdot 2} = f^{\cdot 4} = f^{\cdot 2}\bar{U}_f = f^{\cdot 2} = e$ so e is an idempotent contained in $\mathfrak{B}$. Also $e \neq 0$ since $\bar{U}_e = \bar{U}_{f^{\cdot 2}} = \bar{U}_f{}^2 = 1_{\mathfrak{B}}$. Then $\mathfrak{B} = \mathfrak{J}U_e = \mathfrak{J}_1(e)$ the Peirce 1-space of e. Hence $\mathfrak{B}$ is a subalgebra of $\mathfrak{J}$. If $d \in \mathfrak{B}$ and $d \neq 0$ then $\mathfrak{B}U_d = \mathfrak{B}$ contains e. This implies that d is invertible in $\mathfrak{B}$ and so $\mathfrak{B}$ is a division algebra. Hence we have case III.

COROLLARY. *A Jordan algebra $\mathfrak{J}$ is a division algebra if and only if $\mathfrak{J}U_b = \mathfrak{J}$ for every $b \neq 0$ in $\mathfrak{J}$.*

PROOF. If $\mathfrak{J}$ is a division algebra and $b \neq 0$ in $\mathfrak{J}$ then b is invertible and U_b is bijective. Hence $\mathfrak{J}U_b = \mathfrak{J}$. Conversely, the condition $\mathfrak{J}U_b = \mathfrak{J}$, $b \neq 0$, implies that $\mathfrak{J}$ is a minimal quadratic ideal of type III. Then $\mathfrak{J}$ is a division algebra.

All three types of minimal quadratic ideals indicated in the theorem exist. To obtain one of type I let $\mathfrak{J}$ be a Jordan algebra in which $a_1 . a_2 . a_3 = 0$, $a_i \in \mathfrak{J}$. (For example, we can take $\mathfrak{J}$ to be a trivial algebra: $a . b = 0$ for all $a, b \in \mathfrak{J}$.) Then $U_b = 0$ for all b in $\mathfrak{J}$ and so Φb for $b \neq 0$ is a minimal quadratic ideal of type I. Next let $\mathfrak{J} = \Phi_n{}^+$ with the usual matrix units e_{ij} as basis. Then $\mathfrak{J}U_{e_{12}} = e_{12}\mathfrak{J}e_{12} = \Phi e_{12}$ so Φe_{12} is a minimal quadratic ideal. Since $e_{12}{}^{\cdot 2} = 0$ it is clear that this is of type II. For type III we take $\mathfrak{B} = \Phi e_{11} = \mathfrak{J}U_{e_{11}}$ in $\mathfrak{J} = \Phi_n{}^+$. More generally, in any Jordan algebra $\mathfrak{J}$, any quadratic ideal of the form $\mathfrak{J}U_e = \mathfrak{J}_1(e)$ where e is a nonzero idempotent such that $\mathfrak{J}_1(e)$ is a division algebra is minimal.

We shall call a Jordan algebra $\mathfrak{J}$ *nondegenerate* if $\mathfrak{J}$ contains no absolute zero divisors $\neq 0$. Thus $\mathfrak{J}$ is nondegenerate if and only if $U_a = 0$ in $\mathfrak{J}$ implies $a = 0$. We shall obtain next a sufficient condition that a nondegenerate Jordan algebra with 1 contain an idempotent $e \neq 0, 1$. For this purpose we need to recall Lemma 1 of §3.7: If $\Phi[a]$ is an associative algebra generated by 1 and a which is algebraic and not nilpotent then there exists a polynomial $g(x)$ with 0 constant term such that $e = g(a)$ is an idempotent $\neq 0$. We remark that if $f(x)$ is a polynomial $\neq 0$ of least degree with 0 constant term such that $f(a) = 0$ then we may assume $\deg g(x) < \deg f(x)$. This is clear from the division algorithm. We shall require also

LEMMA 1. *Let $\mathfrak{J}$ be a Jordan algebra with 1 and elements a, b such that $a^{\cdot 2} = 0 = b^{\cdot 2}$, $2a . b = 1$. Then $\mathfrak{J}$ contains an idempotent $e \neq 0, 1$.*

PROOF. One sees directly that $1, a, b$ are linearly independent. Choose α, β in Φ such that $\alpha\beta = 1$ and put $f = 1 + \alpha a + \beta b$. Then $f^{\cdot 2} = 2f$. Hence $e = \frac{1}{2}f$ is an idempotent $\neq 0, 1$.

We can now prove

THEOREM 2. *Let $\mathfrak{J}$ be a nondegenerate Jordan algebra with 1 and assume $\mathfrak{J}$ contains minimal quadratic ideals. Then either $\mathfrak{J}$ is a division algebra or $\mathfrak{J}$ contains an idempotent $\neq 0, 1$.*

PROOF. Let $\mathfrak{B}$ be a minimal quadratic ideal in $\mathfrak{J}$. Then $\mathfrak{B}$ is not of type I, since $\mathfrak{J}$ is nondegenerate. If $\mathfrak{B}$ is of type III either $\mathfrak{J} = \mathfrak{B}$ is a division algebra or we have an idempotent $e \neq 0, 1$. Hence we may assume $\mathfrak{B}$ is of type II. Also we have $\mathfrak{B}^{\cdot 2} = 0$. Now choose $b \neq 0$ in $\mathfrak{B}$ and $a \in \mathfrak{J}$ so that $b = \{bab\}$. Let $c = a . b$. We have the identity

$$(2) \qquad (a . b)^{\cdot 2} = \tfrac{1}{2}a . \{bab\} + \tfrac{1}{4}\{ba^{\cdot 2}b\} + \tfrac{1}{4}\{ab^{\cdot 2}a\}$$

in any Jordan algebra. This can be verified directly for special Jordan algebras and concluded for arbitrary Jordan algebras from the fact that the identity involves only two variables. In the present situation (2) gives $c^{\cdot 2} = \frac{1}{2}c + \frac{1}{4}\{ba^{\cdot 2}b\}$. Since $\{ba^{\cdot 2}b\} \in \mathfrak{B}$ we have $(2c^{\cdot 2} - c)^{\cdot 2} = 0$. It follows from Lemma 1 of §3.7 quoted above that either $c^{\cdot 2} = 0$ or there exists an idempotent $e \neq 0$ of the form $\alpha c + \beta c^{\cdot 2} + \gamma c^{\cdot 3}$, $\alpha, \beta, \gamma \in \Phi$. In the first case we have $-2c = \{ba^{\cdot 2}b\}$ so $c \in \mathfrak{B}$. Then $b = aU_b = 2(a . b) . b - a . b^{\cdot 2} = 2c . b = 0$ since $\mathfrak{B}^{\cdot 2} = 0$. Hence we have the second case: $e = \alpha c + \beta c^{\cdot 2} + \gamma c^{\cdot 3}$ is a nonzero idempotent. Suppose $e = 1$. Then c is a root of a cubic equation with nonzero constant term. Since $(2c^{\cdot 2} - c)^{\cdot 2} = 0$ This implies that $(2c - 1)^{\cdot 2} = 4c^{\cdot 2} - 4c + 1 = 0$. Then $\{ba^{\cdot 2}b\} = 4c^{\cdot 2} - 2c = 2a . b - 1$. Since $b^{\cdot 2} = 0$, $xU_b = 2(x . b) . b$ so the foregoing equation gives $2(a^{\cdot 2} . b) . b = 2a . b - 1$ or $2a' . b = 1$ for $a' = a - a^{\cdot 2} . b$. Then $2(a' . b) . b = b$ so $a'U_b = b$ and replacing a by a' permits us to assume that $2a . b = 1$. Then $c = \frac{1}{2}$ and $\{ba^{\cdot 2}b\} = 0$. Hence $\{aba\}^{\cdot 2} = 0$ by (1). Also the symmetry of the left-hand side of (2) in a and b implies that $\{aba\} . b = a . \{bab\}$. Hence $2\{aba\} . b = 2a . \{bab\} = 2a . b = 1$ and $\{aba\}U_b = 2(\{aba\} . b) . b = b$. Hence replacing a by $\{aba\}$ we have $a^{\cdot 2} = 0 = b^{\cdot 2}$ and $2a . b = 1$. Then we obtain an idempotent $\neq 0, 1$ by Lemma 1.

EXERCISES

1. Show that if $\mathfrak{B}$ is an ideal and $\mathfrak{C}$ is a quadratic ideal then $\mathfrak{B} + \mathfrak{C}$ is a quadratic ideal. Show that if v is a homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$ and $\mathfrak{B}$ is a quadratic ideal in $\mathfrak{J}$ then $\mathfrak{B}^v$ is a quadratic ideal in $\mathfrak{J}^v$. Show also that if $\mathfrak{B}'$ is a quadratic ideal in $\mathfrak{J}'$ then the inverse image of $\mathfrak{B}'$ is a quadratic ideal in $\mathfrak{J}$. Show that if $\mathfrak{B}$ is a quadratic ideal in a Jordan algebra with 1 such that $\mathfrak{B} \neq \mathfrak{J}$ then $\mathfrak{B}U_a$ is a quadratic ideal $\neq \mathfrak{J}$ for any $a \in \mathfrak{J}$.

2. Let $\mathfrak{J} = \Phi\{x, y\}^+$ where $\Phi\{x, y\}$ is the free associative algebra with 1 generated by x and y. Show that $\mathfrak{J}U_x + \mathfrak{J}U_y$ is not a quadratic ideal in $\mathfrak{J}$.

3. A quadratic ideal $\mathfrak{B}$ in $\mathfrak{J}$ is called *maximal* if $\mathfrak{J} \supset \mathfrak{B}$ and there is no quadratic ideal $\mathfrak{C}$ such that $\mathfrak{J} \supset \mathfrak{C} \supset \mathfrak{B}$. Show that every proper quadratic ideal in a Jordan algebra with 1 can be imbedded in a maximal quadratic ideal.

4. (Topping). Let $\mathfrak{J}$ be a Jordan algebra with 1 and let $\mathfrak{R}$ denote the intersection of all maximal quadratic ideals of $\mathfrak{J}$. Show that if $z \in \mathfrak{R}$ then $1 - z$ is invertible.

5. Show that if z is an algebraic element contained in the quadratic ideal $\mathfrak{R}$ of exercise 4 then z is nilpotent.

6. Determine the quadratic ideals of $\Phi[x]^+$, x an indeterminate and show that this Jordan algebra has no minimal quadratic ideals.

7. Let $\mathfrak{J}$ be a Jordan algebra $\neq 0$ such that $\mathfrak{J}$ and 0 are the only quadratic ideals in $\mathfrak{J}$. Show that either $\mathfrak{J} = \Phi b$ with $b^{\cdot 2} = 0$ or $\mathfrak{J}$ is a division algebra.

2. **Axioms and first structure theorem.** We shall say that a Jordan algebra $\mathfrak{J}$ *satisfies the minimum conditions* (*for quadratic ideals*) if (1) the minimum condition holds for quadratic ideals $\mathfrak{J}U_e$ determined by idempotent elements and (2) every quadratic ideal $\mathfrak{J}U_e$ where $e^{\cdot 2} = e \neq 0$ contains a minimal quadratic ideal. Condition (1) is that if C is any collection of quadratic ideals of the form $\mathfrak{J}U_e$, $e^{\cdot 2} = e$, ordered by inclusion, then C contains minimal elements. This is equivalent to: there exist no infinite properly descending sequences $\mathfrak{J}U_{e_1} \supset \mathfrak{J}U_{e_2} \supset \mathfrak{J}U_{e_3} \supset \cdots$ where $e_i{}^{\cdot 2} = e_i$. We shall now investigate the structure of Jordan algebras satisfying the following axioms:

(i) $\mathfrak{J}$ has an identity element 1.
(ii) $\mathfrak{J}$ is nondegenerate.
(iii) $\mathfrak{J}$ satisfies the minimum conditions.

We note first that if $\mathfrak{J}$ satisfies these axioms and e is a nonzero idempotent in $\mathfrak{J}$ then the Peirce quadratic ideal $\mathfrak{J}_1(e) = \mathfrak{J}U_e$ also satisfies the axioms. It is clear that (i) holds in $\mathfrak{J}U_e$ and we have seen that if $b \in \mathfrak{J}U_e$ is an absolute zero divisor in $\mathfrak{J}U_e$ then b is an absolute zero divisor in $\mathfrak{J}$. Hence (ii) holds in $\mathfrak{J}U_e$. Now let $f^{\cdot 2} = f \in \mathfrak{J}U_e$. Then $f = fU_e$, so $U_f = U_eU_fU_e$ and $\mathfrak{J}U_eU_f = \mathfrak{J}U_f$. It is clear from this equation and the fact that any quadratic ideal of $\mathfrak{J}U_e$ is a quadratic ideal of $\mathfrak{J}$ that if $\mathfrak{J}$ satisfies the minimum conditions then so does $\mathfrak{J}U_e$.

An idempotent e in a Jordan algebra will be called *primitive* if $e \neq 0$ and e can not be written as $e = e_1 + e_2$ where the e_i are nonzero orthogonal idempotent elements. If $e = e_1 + e_2$ where the e_i are orthogonal idempotents then $e \cdot e_i = e_i$ so $e_i \in \mathfrak{J}_1(e)$. On the other hand, if e_1 is an idempotent element contained in $\mathfrak{J}_1(e)$ then $e_2 = e - e_1$ is an idempotent orthogonal to e_1 and $e = e_1 + e_2$. It follows that a nonzero idempotent e is primitive if and only if the subalgebra $\mathfrak{J}_1(e)$ contains no idempotent $\neq 0, e$. Clearly the identity element is the only nonzero idempotent in a Jordan division algebra. Hence e is primitive if $\mathfrak{J}_1(e)$ is a division algebra. We shall call an idempotent e *completely primitive* if $\mathfrak{J}_1(e)$ is a division algebra. We now give the following

DEFINITION 2. *A Jordan algebra with* 1 *is said to have a (finite)* capacity *if* $1 = \Sigma_1^n e_i$ *where the* e_i *are completely primitive orthogonal idempotents in* $\mathfrak{J}$. *The minimum n for which this holds will be called* the capacity *of* $\mathfrak{J}$.

It is clear that $\mathfrak{J}$ has capacity 1 if and only if $\mathfrak{J}$ is a division algebra and $\mathfrak{J}$ has capacity two if and only if $1 = e_1 + e_2$ where the e_i are completely primitive orthogonal idempotents in $\mathfrak{J}$. We now prove

THEOREM 3. *Any Jordan algebra* $\mathfrak{J}$ *satisfying axioms* (i)–(iii) *has a finite capacity.*

PROOF. Let $\mathfrak{J}U_e$, $e^{\cdot 2} = e \neq 0$, be a minimal element in the collection of quadratic ideals of the form $\mathfrak{J}U_f$, $f^{\cdot 2} = f \neq 0$. We claim that e is completely primitive. Otherwise, since the axioms hold for $\mathfrak{J}U_e$, we can apply Theorem 2 to conclude that $\mathfrak{J}U_e$ contains an idempotent $f \neq 0, e$. Then $\mathfrak{J}U_f \subset \mathfrak{J}U_e$ contrary to the minimality of $\mathfrak{J}U_e$. Thus $\mathfrak{J}$ contains completely primitive idempotents. Now let $e_1, e_2, \cdots, e_r$ be orthogonal completely primitive idempotents. Put $f_i = 1 - \Sigma_1^i e_j$. Then f_i is an idempotent and

$$f_i \cdot f_{i+1} = \left(1 - \sum_1^i e_j\right)\left(1 - \sum_1^{i+1} e_k\right) = 1 - \sum_1^i e_j - \sum_1^{i+1} e_k + \sum_1^i e_j = f_{i+1} \neq f_i.$$

Hence $\mathfrak{J}U_{f_1} \supset \mathfrak{J}U_{f_2} \supset \cdots \supset \mathfrak{J}U_{f_r}$, If $f_r = 0$ then $1 = \Sigma_1^r e_j$ and the theorem is proved. If $f_r \neq 0$ then $\mathfrak{J}U_{f_r} \neq 0$ satisfies the axioms so this contains a completely primitive idempotent e_{r+1}. Then $f_i \cdot e_{r+1} = f_{i-1} \cdot e_{r+1} = e_{r+1}$ for $i \leqq r$ gives $e_i \cdot e_{r+1} = 0$. Hence $e_1, e_2, \cdots, e_{r+1}$ is a set of orthogonal completely primitive idempotents and $\mathfrak{J}U_{f_1} \supset \mathfrak{J}U_{f_2} \supset \cdots \supset \mathfrak{J}U_{f_{r+1}}$ for $f_{r+1} = 1 - \Sigma_1^{r+1} e_j$. By (iii), this process must terminate with an $f_n = 0$, which proves the theorem.

Our analysis of the algebras satisfying the axioms will rest on some results on algebras of capacity two which we shall now derive. Suppose first that $\mathfrak{J}$ is a Jordan algebra with $1 = e_1 + e_2$ where the e_i are orthogonal idempotents and let $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{22}$ be the corresponding Peirce decomposition. Let $a_{11}, b_{11} \in \mathfrak{J}_{11}, x_{12} \in \mathfrak{J}_{12}$, etc. Then, by (PD3) (p. 121), $a_{11} \cdot b_{11} \cdot x_{12} = a_{11} \cdot x_{12} \cdot b_{11}$

$+ b_{11}.x_{12}.a_{11}$. Since $\mathfrak{J}_{12}.\mathfrak{J}_{11} \subseteq \mathfrak{J}_{12}$ the multiplication $R_{a_{11}}$ maps $\mathfrak{J}_{12}$ into itself and, denoting the restriction of $R_{a_{11}}$ to $\mathfrak{J}_{12}$ by $\bar{R}_{a_{11}}$, the relation (PD3) can be written as

$$\bar{R}_{a_{11}.b_{11}} = \bar{R}_{a_{11}}\bar{R}_{b_{11}} + \bar{R}_{b_{11}}\bar{R}_{a_{11}}. \tag{3}$$

Now put $V_{a_{11}} = 2\bar{R}_{a_{11}}$. Then we have

$$V_{a_{11}.b_{11}} = V_{a_{11}}.V_{b_{11}} \equiv \tfrac{1}{2}(V_{a_{11}}V_{b_{11}} + V_{b_{11}}V_{a_{11}}). \tag{4}$$

Thus the mapping

$$a_{11} \to V_{a_{11}} \equiv 2\bar{R}_{a_{11}} \tag{5}$$

is a homomorphism of $\mathfrak{J}_{11}$ into the special Jordan algebra $\mathrm{Hom}_\Phi(\mathfrak{J}_{12},\mathfrak{J}_{12})^+$. Similarly, we have the homomorphism $a_{22} \to V_{a_{22}} = 2\bar{R}_{a_{22}}$, the restriction to $\mathfrak{J}_{12}$ of $2R_{a_{22}}$, $a_{22} \in \mathfrak{J}_{22}$, into $\mathrm{Hom}_\Phi(\mathfrak{J}_{12},\mathfrak{J}_{12})^+$. By (PD4), we have $[a_{11},x_{12},a_{22}] = 0$ and this implies that

$$[V_{a_{11}},V_{a_{22}}] = 0 \tag{6}$$

if $a_{11} \in \mathfrak{J}_{11}, a_{22} \in \mathfrak{J}_{22}$. Now suppose that $\mathfrak{J}$ has capacity two, the e_i are completely primitive and $\mathfrak{J}_{12} \neq 0$. Then $\mathfrak{J}_{ii}$ is a division algebra and hence is simple. Consequently, the homomorphism $a_{ii} \to V_{a_{ii}}$ of $\mathfrak{J}_{ii}$ is a monomorphism. It follows that $\mathfrak{J}_{ii}$ is special and the image $V(\mathfrak{J}_{ii})$ is a division subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{J}_{12},\mathfrak{J}_{12})^+$. We recall that if $\mathfrak{A}$ is an associative algebra with 1 then a and b are inverses in $\mathfrak{A}$ if and only if they are inverses in $\mathfrak{A}^+$ (§1.11). It follows from this that if $a_{ii} \neq 0$ then $V_{a_{ii}}$ is an invertible linear transformation of $\mathfrak{J}_{12}$. In fact, $V_{a_{11}}V_{a_{11}^{\cdot -1}} = 1 = V_{a_{11}^{\cdot -1}}V_{a_{11}}$. We now prove

Lemma 1. *Let $\mathfrak{J}$ be a Jordan algebra of capacity two, $1 = e_1 + e_2$ where the e_i are orthogonal absolutely primitive idempotents, $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{22}$ the corresponding Peirce decomposition. Let $a_{11} \in \mathfrak{J}_{11}$, $a_{12} \in \mathfrak{J}_{22}$. Then:*

(1) *Either a_{12} is invertible or $a_{12}^{\cdot 2} = 0$.*

(2) *$U_{a_{12}}$ maps $\mathfrak{J}_{ii}$ into $\mathfrak{J}_{jj}$, $i \neq j$, and if $a_{12}^{\cdot 2} = 0$ then $\mathfrak{J}_{ii}U_{a_{12}} = 0$.*

(3) *If $a_{ii} \neq 0$, $a_{12} \neq 0$ then $a_{ii}.a_{12} \neq 0$ and if $a_{ii} \neq 0$ and a_{12} is invertible then $a_{ii}.a_{12}$ is invertible.*

Proof. (1) We know that $a_{12}^{\cdot 2} = a_1 + a_2$ where $a_i = a_{12}^{\cdot 2}.e_i \in \mathfrak{J}_{ii}$. By (PD1), we have $a_{12}^{\cdot 2}.e_1.a_{12} = a_{12}^{\cdot 2}.e_2.a_{12}$. Hence $a_1.a_{12} = a_2.a_{12}$. Suppose $a_1.a_{12} = 0$. Then $a_2.a_{12} = 0$ and $a_{12}^{\cdot 3} = (a_1 + a_2).a_{12} = 0$. Then $a_1^{\cdot 2} + a_2^{\cdot 2} = a_{12}^{\cdot 4} = 0$. Since $\mathfrak{J}_{ii}$ is a division algebra this implies $a_i = 0$ and $a_{12}^{\cdot 2} = 0$. Next assume $a_1.a_{12} \neq 0$. Then $a_2.a_{12} \neq 0$ and so $a_i \neq 0$. Then $a_{12}^{\cdot 2} = a_1 + a_2$ is invertible. Hence a_{12} is invertible. This proves (1).

(2) By symmetry, it is enough to prove (2) for $i = 1$. Clearly

$$a_{11}U_{a_{12}} = 2a_{11}.a_{12}.a_{12} - a_{11}.a_{12}^{\cdot 2} \in \mathfrak{J}_{11} + \mathfrak{J}_{22}.$$

We have $a_{11}U_{a_{12}}.e_1 = 2a_{11}.a_{12}.a_{12}.e_1 - a_{11}.a_{12}{}^{.2}$ and

$$a_{11}.a_{12}{}^{.2} = 2a_{11}.a_{12}.a_{12}.e_1,$$

by (PD5). Hence $a_{11}U_{a_{12}}.e_1 = 0$ and $a_{11}U_{a_{12}} = a_{11}U_{a_{12}}.e_2 \in \mathfrak{J}_{22}$. Now assume $a_{12}{}^{.2} = 0$. Then $a_{11}U_{a_{12}} = 2a_{11}.a_{12}.a_{12}$ and we have $a_{11}.a_{12}.a_{12}.e_1 = 0$. By (PD1′) with $x_{12} = a_{12}$ and $y_{12} = a_{12}.a_{11}$ we obtain $2a_{12}.(a_{11}.a_{12}).e_1.a_{12} = 2a_{12}.(a_{11}.a_{12}).e_2.a_{12}$. Since $a_{11}.a_{12}.a_{12}.e_1 = 0$ this gives

$$a_{11}.a_{12}.a_{12}.e_2.a_{12} = 0.$$

Let $a_{22} = a_{11}.a_{12}.a_{12}.e_2$. Then $a_{22} \in \mathfrak{J}_{22}$. If $a_{22} \neq 0$ then $a_{12} \neq 0$ and $V_{a_{22}}$ is invertible. Then $a_{22}.a_{12} \neq 0$ contrary to what we have established. Thus $a_{22} = a_{11}.a_{12}.a_{12}.e_2 = 0$. Since $a_{11}.a_{12}.a_{12}.e_1 = 0$ this gives

$$a_{11}U_{a_{12}} = 2a_{11}.a_{12}.a_{12} = 0.$$

(3) Again, we may assume $i = 1$. The argument we have just used on a_{12} and a_{22} shows that if $a_{11} \neq 0$ and $a_{12} \neq 0$ then $a_{11}.a_{12} \neq 0$. To prove the next assertion that $a_{11}.a_{12}$ is invertible if $a_{11} \neq 0$ and a_{12} is invertible, it is equivalent by (1) and the fact that $V_{a_{11}}$ and $V_{a_{11}{}^{.-1}}$ are inverses to showing, that if $a_{12}{}^{.2} = 0$ then $(a_{11}.a_{12})^{.2} = 0$. For this we use (2) to obtain $4(a_{11}.a_{12})^{.2} = a_{12}{}^{.2}U_{a_{11}} + a_{11}{}^{.2}U_{a_{12}} + 2a_{11}.a_{11} = 0$ since $\mathfrak{J}_{11}U_{a_{12}} = 0$ by (2).

We recall that e_1 and e_2 are called connected if there exists an invertible element $a_{12} \in \mathfrak{J}_{12}$. If this is not the case then, by Lemma 1 (1), $a_{12}{}^{.2} = 0$ for all $a_{12} \in \mathfrak{J}_{12}$. Then $a_{12}.b_{12} = 0$ if $a_{12}, b_{12} \in \mathfrak{J}_{12}$. Hence we see that either e_1 and e_2 are connected or $\mathfrak{J}_{12}{}^{.2} = 0$. We now prove

THEOREM 4. *Let $\mathfrak{J}$, e_1, e_2, $\mathfrak{J}_{ij}$ be as in Lemma 1 and let*

$$\mathfrak{N} = \{z_{12} \in \mathfrak{J}_{12} \mid z_{12}.\mathfrak{J}_{12} = 0\}.$$

Then $\mathfrak{N}$ is an ideal in $\mathfrak{J}$ such that $\mathfrak{N}^{.2} = 0$ and $\mathfrak{N}$ coincides with the set of absolute zero divisors of $\mathfrak{J}$. If $\mathfrak{N} = 0$ (equivalently: $\mathfrak{J}$ is nondegenerate) then $\mathfrak{J}$ is simple if and only if $\mathfrak{J}_{12} \neq 0$. If $\mathfrak{J}_{12} = 0$ then $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{22}$ and $\mathfrak{J}_{ii}$ is an ideal in $\mathfrak{J}$. The algebra $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$ is nondegenerate and is of capacity two.

PROOF. Let $z_{12} \in \mathfrak{N}$, $a_{12} \in \mathfrak{J}_{12}$, $a_{11} \in \mathfrak{J}_{11}$. By (PD5),

$$z_{12}.a_{11}.a_{12}.e_1 = z_{12}.a_{12}.a_{11} - a_{12}.a_{11}.z_{12}.e_1 = 0,$$

since $a_{12}.a_{11} \in \mathfrak{J}_{12}$. By (PD6), $z_{12}.a_{11}.a_{12}.e_2 = z_{12}.(a_{11}.a_{12}).e_2 = 0$. Hence $z_{12}.a_{11}.a_{12} = 0$ for all $a_{12} \in \mathfrak{J}_{12}$. Thus $z_{12}.a_{11} \in \mathfrak{N}$ and, similarly, $z_{12}.a_{22} \in \mathfrak{N}$ for all a_{22} in $\mathfrak{J}_{22}$. It follows that $\mathfrak{N}$ is an ideal such that $\mathfrak{N}^{.2} = 0$. This implies that every element of $\mathfrak{N}$ is an absolute zero divisor. Conversely, let $z = z_1 + z_2$

$+z_{12}$, $z_i \in \mathfrak{J}_i$, $z_{12} \in \mathfrak{J}_{12}$, be an absolute zero divisor in $\mathfrak{J}$. Since $\mathfrak{J}$ has an identity element $1, 0 = 1U_z = z^{\cdot 2}$, so the condition for an absolute zero divisor z is $x.z.z = 0$, $x \in \mathfrak{J}$. It is clear also that if z is an absolute zero divisor, then the same is true of zU_a, $z \in \mathfrak{J}$, since $U_{zU_a} = U_a U_z U_a = 0$. Hence $z_i = zU_{e_i}$ is an absolute zero divisor. Since $\mathfrak{J}_{ii}$ is a division algebra this gives $z_i = 0$. Then $z = z_{12} \in \mathfrak{J}_{12}$ and $z_{12}^{\cdot 2} = 0$, $x.z_{12}.z_{12} = 0$, $x \in \mathfrak{J}$. By (PD1′), we obtain $a_{12}.z_{12}.e_1.z_{12} = a_{12}.z_{12}.e_2.z_{12}$ for $a_{12} \in \mathfrak{J}_{12}$. Since $a_{12}.z_{12}.z_{12} = 0$ this implies that $a_{12}.z_{12}.e_i.z_{12} = 0$. By Lemma 1 (3), we conclude from this that $a_{12}.z_{12}.e_i = 0$. Hence $a_{12}.z_{12} = 0$ and $z_{12} \in \mathfrak{N}$. Thus $\mathfrak{N}$ is the set of absolute zero divisors. Now assume $\mathfrak{N} = 0$. If $\mathfrak{J}_{12} = 0$ then $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{22}$ and since $\mathfrak{J}_{11}.\mathfrak{J}_{22} = 0$ it is clear that the $\mathfrak{J}_{ii}$ are ideal. Suppose next that $\mathfrak{N} = 0$ and $\mathfrak{J}_{12} \neq 0$. Let $\mathfrak{B}$ be an ideal $\mathfrak{J}$. Since $\mathfrak{J}_{ii} = \mathfrak{J}U_{e_i}$, $\mathfrak{J}_{12} = \mathfrak{J}U_{e_1,e_2}$, $1 = U_{e_1} + U_{e_2} + U_{e_1,e_2}$, and $\mathfrak{B}$ is an ideal, we have $\mathfrak{B} = \mathfrak{B}_{11} + \mathfrak{B}_{12} + \mathfrak{B}_{22}$ where $\mathfrak{B}_{ij} = \mathfrak{B} \cap \mathfrak{J}_{ij}$. Clearly $\mathfrak{B}_{ii}$, is an ideal in $\mathfrak{J}_{ii}$ and since $\mathfrak{J}_{ii}$ is a division algebra, either $\mathfrak{B}_{ii} = 0$ or $\mathfrak{B}_{ii} = \mathfrak{J}_{ii}$. In the latter case $\mathfrak{B} \supseteq \mathfrak{J}_{12} = e_i.\mathfrak{J}_{12}$. Since $\mathfrak{N} = 0$ there exists an $a_{12} \in \mathfrak{J}_{12}$ such that $a_{12}^{\cdot 2} \neq 0$. Then a_{12} is invertible and $a_{12} \in \mathfrak{B}$. Hence $\mathfrak{B} = \mathfrak{J}$. Thus if $\mathfrak{B} \neq \mathfrak{J}$ we must have $\mathfrak{B}_{ii} = 0$ and $\mathfrak{B} = \mathfrak{B}_{12}$. It is clear also that $\mathfrak{B}_{12}^{\cdot 2} = 0$ since, otherwise, $\mathfrak{B}$ contains an invertible element. Hence $\mathfrak{B}$ is an ideal such that $\mathfrak{B}^{\cdot 2} = 0$ and consequently every element of $\mathfrak{B}$ is an absolute zero divisor. Then $\mathfrak{B} = 0$. Then either $\mathfrak{B} = 0$ or $\mathfrak{B} = \mathfrak{J}$ and $\mathfrak{B}$ is simple. Now let $\mathfrak{J}$ be arbitrary again and let $\mathfrak{N}$ be the ideal of absolute zero divisors, $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{N}$. If we apply the canonical homomorphism $x \to \bar{x}$ of $\mathfrak{J}$ onto $\bar{\mathfrak{J}}$ we obtain the idempotents $\bar{e}_i$ such that $\bar{e}_1 + \bar{e}_2 = \bar{1}$, $\bar{e}_1.\bar{e}_2 = 0 = \bar{e}_2.\bar{e}_1$ and the Peirce decomposition $\bar{\mathfrak{J}} = \bar{\mathfrak{J}}_{11} \oplus \bar{\mathfrak{J}}_{12} \oplus \bar{\mathfrak{J}}_{22}$. It is clear also that $\bar{\mathfrak{J}}_{ii}$ is isomorphic to $\mathfrak{J}_{ii}$ and so is a division algebra. Hence $\bar{\mathfrak{J}}$ is of capacity two. Let $\bar{z}$ be an absolute zero divisor of $\bar{\mathfrak{J}}$. Then $\bar{z} = \bar{z}_{12}$, where $z_{12} \in \mathfrak{J}_{12}$ satisfies $a_{12}.z_{12} \in \mathfrak{N}$ for all $a_{12} \in \mathfrak{J}_{12}$. Then $a_{12}.z_{12}.b_{12} = 0$ for all $a_{12}, b_{12} \in \mathfrak{J}_{12}$. In particular, $z_{12}^{\cdot 3} = 0$ which, by Lemma 1 (1) implies that $z_{12}^{\cdot 2} = 0$. Also, $a_{12}U_{z_{12}} = a_{12}.z_{12}.z_{12} = 0$ and by Lemma 1 (2) $a_{ii}U_{z_{12}} = 0$ for all $a_{ii} \in \mathfrak{J}_{ii}$. Hence z_{12} is an absolute zero divisor and $\bar{z} = \bar{z}_{12} = 0$. Thus $\bar{\mathfrak{N}} = 0$ and $\bar{\mathfrak{J}}$ is nondegenerate.

We can now prove our first main result on nondegenerate Jordan algebras satisfying the minimum conditions.

First Structure Theorem. *Let $\mathfrak{J}$ be a Jordan algebra satisfying axioms (i)–(iii). Then $\mathfrak{J}$ is a direct sum of a finite number of ideals which are simple algebras satisfying the axioms.*

Proof. By Theorem 3, $1 = \sum_1^n e_i$ where the e_i are absolutely primitive idempotents. Let $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ be the corresponding Peirce decomposition. If $n = 1$, $\mathfrak{J}$ is simple and there is nothing to prove. Hence assume $n > 1$. Let $\mathfrak{B}$ be an ideal in $\mathfrak{J}$. As in the proof of Theorem 4, $\mathfrak{B} = \sum \mathfrak{B}_{ij}$ where $\mathfrak{B}_{ij} = \mathfrak{B} \cap \mathfrak{J}_{ij}$. Let $i \neq j$. Then $\mathfrak{B}_{ii} + \mathfrak{B}_{ij} + \mathfrak{B}_{jj}$ is an ideal in $\mathfrak{J}U_{e_i+e} = \mathfrak{J}_{ii} + \mathfrak{J}_{ij} + \mathfrak{J}_{jj}$ and since $e_i + e_j$ is an idempotent, $\mathfrak{J}U_{e_i+e_j}$ satisfies the axioms and is of capacity two. In particular, $\mathfrak{J}U_{e_i+e_j}$ has no absolute zero divisors. Hence, by Theorem 4, either $\mathfrak{J}U_{e_i+e_j}$ is simple or $\mathfrak{J}_{ij}=0$ and $\mathfrak{J}U_{e_i+e_j}=\mathfrak{J}_{ii}+\mathfrak{J}_{jj}$. In the first case, either $\mathfrak{B}_{ii}+\mathfrak{B}_{ij}+\mathfrak{B}_{jj}=\mathfrak{J}U_{e_i+e_j}$ or $\mathfrak{B}_{ii}+\mathfrak{B}_{ij}+\mathfrak{B}_{jj}=0$. In the second case, $\mathfrak{B}_{ij}=0$, and $\mathfrak{B}_{ii}+\mathfrak{B}_{jj}$ is either 0, $\mathfrak{J}_{ii}$, $\mathfrak{J}_{jj}$ or $\mathfrak{J}_{ii}+\mathfrak{J}_{jj}$. This implies that $\mathfrak{B} = \Sigma'\mathfrak{J}_{kl}$ a sum of Peirce spaces $\mathfrak{J}_{kl}$ such that if $\mathfrak{J}_{kl}\neq 0\subseteq\mathfrak{B}$ then $\mathfrak{J}_{kk}$ and $\mathfrak{J}_{ll}\subseteq\mathfrak{B}$. Let $\mathfrak{B}' = \Sigma\mathfrak{J}_{ij}$ the sum of the Peirce spaces not contained in $\mathfrak{B}$. Then $\mathfrak{J}=\mathfrak{B}\oplus\mathfrak{B}'$ and we proceed to show that $\mathfrak{B}'$ is an ideal. We recall that if we define $\mathfrak{J}_{ji}=\mathfrak{J}_{ij}$ for $j>i$ then we have the following multiplication table for the Peirce spaces: $\mathfrak{J}_{ij}\mathfrak{J}_{kl}=0$ if $\{i,j\}\cap\{k,l\}=\varnothing$, $\mathfrak{J}_{ij}\cdot\mathfrak{J}_{jk}\subseteq\mathfrak{J}_{ik}$ if i,j,k are distinct, $\mathfrak{J}_{ii}\cdot\mathfrak{J}_{ij}=\mathfrak{J}_{ij}$ and $\mathfrak{J}_{ij}^{\cdot 2}\subseteq\mathfrak{J}_{ii}+\mathfrak{J}_{jj}$. Hence the fact that $\mathfrak{B}'$ is an ideal will follow if we can show that there is no index k such that there exists a nonzero $\mathfrak{J}_{kl}\subseteq\mathfrak{B}$ and a nonzero $\mathfrak{J}_{ki}\subseteq\mathfrak{B}'$. Suppose the contrary. Then $\mathfrak{J}_{kk}\subseteq\mathfrak{B}$ and $\mathfrak{J}_{ki}=\mathfrak{J}_{kk}\cdot\mathfrak{J}_{ki}\subseteq\mathfrak{B}$ which contradicts $\mathfrak{J}_{ki}\subseteq\mathfrak{B}'$. Thus we have shown that every ideal $\mathfrak{B}$ in $\mathfrak{J}$ has a complementary ideal. The argument shows also that the number of ideals is finite. Now let $\mathfrak{B}=\mathfrak{J}_1$ be a minimal ideal and write $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{B}'$ where $\mathfrak{B}'$ is an ideal. Then $\mathfrak{B}'$ satisfies the axioms and any ideal of $\mathfrak{B}'$ is an ideal in $\mathfrak{B}$ since $\mathfrak{J}_1.\mathfrak{B}'=0$. Hence, if $\mathfrak{B}'\neq 0$, then $\mathfrak{B}'=\mathfrak{J}_2\oplus\mathfrak{B}''$ where $\mathfrak{J}_2$ is a minimal ideal in $\mathfrak{J}$. Then $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{J}_2\oplus\mathfrak{B}''$. Since there are only a finite number of ideals this process yields a decomposition $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{J}_2\oplus\cdots\oplus\mathfrak{J}_s$ where every $\mathfrak{J}_k$ is a minimal ideal. Then $\mathfrak{J}_k$ is simple since the direct decomposition implies that $\mathfrak{J}_i.\mathfrak{J}_k=0$ if $i\neq k$, and this implies that any ideal of $\mathfrak{J}_k$ is an ideal of $\mathfrak{J}$. This completes the proof.

It is easy to see that the decomposition $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{J}_2\oplus\cdots\oplus\mathfrak{J}_s$ given in the theorem is unique. In general, if $\mathfrak{J}$ is a Jordan algebra with 1 and $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{J}_2\oplus\cdots\oplus\mathfrak{J}_s$ where the $\mathfrak{J}_i$ are simple ideals then this decomposition is unique (except for order of the $\mathfrak{J}_i$). In this case we call the $\mathfrak{J}_i$, $i=1,2,\cdots,s$, the *simple components* of $\mathfrak{J}$.

3. **Determination of a class of alternative algebras with involution.** The First Structure Theorem reduces the consideration of the algebras satisfying axioms (i)–(iii) to that of the simple ones in this class. An essential step in the determination of these will be the determination of the alternative algebras with involution $(\mathfrak{D},j)$ satisfying the following conditions: (1) $(\mathfrak{D},j)$ is simple as algebra with involution in the sense that the only ideals in $\mathfrak{D}$ which are invariant under j are $\mathfrak{D}$ and 0, (2) $\mathfrak{D}$ has an identity element, (3) the nonzero symmetric elements of $\mathfrak{D}$ are invertible in the nucleus. The last condition is that if $h=\bar{h}\neq 0$ then $h\in N(\mathfrak{D})$, the nucleus of $\mathfrak{D}$, and h is invertible in $N(\mathfrak{D})$. Let Δ be an associative division algebra, Δ° its opposite and $\mathfrak{D}=\Delta\oplus\Delta^\circ$. Let j be *exchange involution* $(a,b)\to(b,a)$ in $\mathfrak{D}$. Then it is clear that $(\mathfrak{D},j)$ satisfies conditions (1), (2) and (3). Next let $(\mathfrak{D},j)$ be an associative division algebra with involution. Then $(\mathfrak{D},j)$ satisfies the conditions. We shall see in a moment that another class of algebras satisfying the conditions are the composition algebras which are defined as follows.

DEFINITION 3. *An algebra with involution* $(\mathfrak{D},j)$ *is called a* composition algebra *if* (1) $\mathfrak{D}$ *is alternative with* 1, (2) $x\bar{x}=Q(x)1=\bar{x}x$ *where* $Q(x)$ *is a quadratic* *form whose associated symmetric bilinear form* $Q(x,y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$ *is nondegenerate.*[1]

We consider first the problem of determining these algebras. Hence let $(\mathfrak{D},j)$ be a composition algebra. We have $Q(x,y)1 \equiv \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]1 = \frac{1}{2}[(x+y)(\bar{x}+\bar{y}) - x\bar{x} - y\bar{y}] = \frac{1}{2}(x\bar{y} + y\bar{x})$. Hence $2Q(x,y)1 = x\bar{y} + y\bar{x}$ and similarly $Q(x,y) = \bar{y}x + \bar{x}y$. Hence

$$x\bar{y} + y\bar{x} = \bar{y}x + \bar{x}y = 2Q(x,y)1. \tag{7}$$

Taking $y = 1$ we obtain

$$x + \bar{x} = 2Q(x,1)1 \tag{8}$$

so that $x + \bar{x} \in \Phi 1$. Also $2Q(xz,y)1 - 2Q(x,y\bar{z})1 = (xz)\bar{y} + y(\bar{z}\bar{x}) - x(z\bar{y}) - (y\bar{z})\bar{x} = [x,z,\bar{y}] - [y,\bar{z},\bar{x}] = -[x,z,y] - [y,z,x] = 0$ by $[x,y,z+\bar{z}] = 0$ and the alternative law. Hence $Q(xz,y) = Q(x,y\bar{z})$. By (7) we have

$$Q(x,y) = Q(\bar{x},\bar{y}). \tag{9}$$

Hence $Q(zx,y) = Q(\bar{x}\bar{z},\bar{y}) = Q(\bar{x},\bar{y}z) = Q(x,\bar{z}y)$. Thus we have

$$Q(xz,y) = Q(x,y\bar{z}) = Q(z,\bar{x}y). \tag{10}$$

The alternative law and $x + \bar{x} \in \Phi 1$ give $x(\bar{x}y) = Q(x)y$ and $(yx)\bar{x} = Q(x)y$. Linearization then gives

$$x(\bar{z}y) + z(\bar{x}y) = 2Q(x,z)\,y = (yx)\bar{z} + (yz)\bar{x}. \tag{11}$$

Now suppose $\mathfrak{B}$ is a finite dimensional nonisotropic subalgebra of $(\mathfrak{D},j)$. Thus $\mathfrak{B}$ is a subalgebra of $\mathfrak{D}$ invariant under j and the restriction of $Q(x,y)$ to $\mathfrak{B}$ is nondegenerate. We assume also that $\mathfrak{B} \subset \mathfrak{D}$ and $\mathfrak{B}$ contains 1. Since $\mathfrak{B}$ is a finite dimensional nonisotropic subspace it is well known that $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{B}^{\perp}$, and $\mathfrak{B}^{\perp} \neq 0$ since $\mathfrak{B} \neq \mathfrak{D}$. We can choose a nonisotropic vector q in $\mathfrak{B}^{\perp}$. Then $Q(q) = -\mu \neq 0$. Since $1 \in \mathfrak{B}$, $Q(q,1) = 0$, which together with (8), gives $q + \bar{q} = 0$ or $\bar{q} = -q$. Then

$$q^2 = -q\bar{q} = \mu 1 \neq 0. \tag{12}$$

If $a \in \mathfrak{B}$ then $Q(a,q) = 0$ and (7) give

$$aq = q\,\bar{a},\ a \in \mathfrak{B}. \tag{13}$$

If $a,b \in \mathfrak{B}$ then $Q(aq,b) = Q(q,\bar{a}b) = 0$. Hence $\mathfrak{B}q = \{xq \mid x \in \mathfrak{B}\} \subseteq \mathfrak{B}^{\perp}$ and $\mathfrak{C} = \mathfrak{B} + \mathfrak{B}q = \mathfrak{B} \oplus \mathfrak{B}q$. If $a,b \in \mathfrak{B}$ then $Q(aq,bq) = Q((aq)\bar{q},b) = Q(a(q\bar{q}),b)$

[1] This definition is different from the customary one which drops the involution and the hypotheses that $\mathfrak{D}$ is alternative but adds the composition law: $Q(xy) = Q(x)\,Q(y)$. However, it is easily seen that the two definitions are equivalent. See Jacobson [22].

$= -\mu Q(a,b)$. This implies that $x \to xq$ is a linear isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}q$ and that $\mathfrak{B}q$ is not isotropic. Hence $\dim \mathfrak{B}q = \dim \mathfrak{B}$, $\dim \mathfrak{C} = 2\dim \mathfrak{B}$ and $\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{B}q$ is not isotropic. Now put $x = a$, $y = \bar{b}$, $z = q$ in (11). This and (13) give

$$a(bq) = (ba)q, \qquad (aq)b = (a\bar{b})q. \tag{14}$$

Also, by Moufang's identities, we have $(aq)(bq) = (q\bar{a})(bq) = q(\bar{a}b)q = (\bar{b}a)q^2 = \mu\bar{b}a$. Hence

$$(aq)(bq) = \mu\bar{b}a. \tag{15}$$

Altogether, if $a,b,c,d \in \mathfrak{B}$, then

$$(a + bq)(c + dq) = (ac + \mu\,\bar{d}b) + (da + b\bar{c})q. \tag{16}$$

Moreover, $\overline{a + bq} = \bar{a} - q\bar{\bar{b}} = \bar{a} - bq \in \mathfrak{C}$. Thus we see that $\mathfrak{C}$ is a finite dimensional nonisotropic subalgebra of $\mathfrak{D}$ containing $\mathfrak{B}$. Let $x = a + bq$, $y = dq$ where $a,b,d \in \mathfrak{B}$. We have $[\bar{x},x,y] = 0$. Since $(\bar{x}x)y$ is a Φ multiple of $y = dq$ and $\bar{x}(xy) = (\bar{a} - bq)(\mu\bar{d}b + (da)q)$ this implies that $\bar{a}(\bar{d}b) = (\bar{a}\bar{d})b$. Thus we see that $\mathfrak{B}$ is necessarily an associative subalgebra of $\mathfrak{D}$

We now choose $\mathfrak{B} \equiv \mathfrak{B}_0 = \Phi 1$. If $\mathfrak{B}_0 \subset \mathfrak{D}$ then the argument gives a two-dimensional nonisotropic subalgebra $\mathfrak{C} \equiv \mathfrak{B}_1$ of $(\mathfrak{D},j)$ with basis $(1,q_1)$ such that $q_1{}^2 = \mu_1 1 \neq 0, \mu_1 \in \Phi$. If $\mathfrak{B}_1 \subset \mathfrak{D}$ we repeat the argument obtaining $\mathfrak{C} = \mathfrak{B}_2$ which is four dimensional. Moreover, the multiplication formula (16) shows that $\mathfrak{B}_2$ is a quaternion algebra with the standard involution. If $\mathfrak{B}_2 \subseteq \mathfrak{D}$ we repeat and obtain $\mathfrak{C} = \mathfrak{B}_3$ which is an algebra of octonions with standard involution (cf. §1.5). This is alternative but not associative. Hence the argument shows that we can not have $\mathfrak{B}_3 \subset \mathfrak{D}$. Thus $\mathfrak{D} = \mathfrak{B}_3$ and we have proved the following

THEOREM 5. *Any composition algebra $(\mathfrak{D},j)$ is finite dimensional and is isomorphic to one of the following*: I. Φ, $j = 1$; II. $\Phi[q]$ *a two-dimensional (commutative and associative) algebra with basis $(1,q)$ such that $\bar{q} = -q$*; III. *a quaternion algebra with standard involution*; IV. *an octonion algebra with standard involution.*

It is clear that, conversely, all of the algebras I–IV listed in the theorem are composition algebras. Now, any composition algebra $(\mathfrak{D},j)$ is an alternative algebra with involution and identity element. Moreover, $(\mathfrak{D},j)$ is simple, since if $\mathfrak{B}$ is an ideal $\neq 0$ in $(\mathfrak{D},j)$ then $\mathfrak{B}$ contains either a symmetric element $\neq 0$ or an element $v = -\bar{v} \neq 0$. In the first case, it is clear that $\mathfrak{B}$ contains 1, so $\mathfrak{B} = \mathfrak{D}$. In the second case, we choose w so that $Q(v,w) \neq 0$. Then, by (7), $\mathfrak{B}$ contains $v\bar{w} + w\bar{v} = 2Q(v,w)1$ so again $\mathfrak{B} = \mathfrak{D}$. Since the symmetric elements of $(\mathfrak{D},j)$ are the elements of Φ it is clear that $(\mathfrak{D},j)$ satisfies the conditions (1), (2), (3) given above. Moreover, it is clear that this is true also if $(\mathfrak{D},j)$ is a composition algebra over any extension field Γ of Φ. We shall now show that these algebras with involution, together with the two types given before ($\mathfrak{D} = \Delta \oplus \Delta^{\circ}$, Δ an associative division algebra with exchange involution j, and $(\mathfrak{D},j)$ where $\mathfrak{D}$ is an associative division algebra) exhaust the possibilities for algebras satisfying (1)–(3). The proof of this will require some properties of Peirce decompositions of alternative algebras, which we shall now derive.

Let $\mathfrak{D}$ be an alternative algebra with 1 and let e_1 be an idempotent element in $\mathfrak{D}$. Then $e_2 = 1 - e_1$ is an idempotent orthogonal to e_1 and $e_1 + e_2 = 1$. By the alternative laws we have ${e_{iL}}^2 = (e_i^2)_L = e_{iL}$, ${e_{iR}}^2 = e_{iR}$, $e_{iL}e_{iR} = e_{iR}e_{iL}$. Since $e_{1L} + e_{2L} = 1$, $e_{1R} + e_{2R} = 1$, e_{1L}, e_{2L} are orthogonal idempotents and e_{1R}, e_{2R} are orthogonal idempotents. Also these four linear transformations commute. Hence the operators $e_{1L}e_{1R}$, $e_{1L}e_{2R}$, $e_{2L}e_{1R}$, $e_{2L}e_{2R}$ are orthogonal idempotents and $(e_{1L} + e_{2L})(e_{1R} + e_{2R}) = 1$ shows that if we put $E_{ij} = e_{iL}e_{jR}$ then $E_{11} + E_{12} + E_{21} + E_{22} = 1$. Then we have the decomposition

$$\mathfrak{D} = \mathfrak{D}_{11} \oplus \mathfrak{D}_{12} \oplus \mathfrak{D}_{21} \oplus \mathfrak{D}_{22} \tag{17}$$

where $\mathfrak{D}_{ij} = \mathfrak{D}E_{ij} = e_i\mathfrak{D}e_j = \{e_ide_j \mid d \in \mathfrak{D}\}$ (parentheses unnecessary). We shall call (17) the *Peirce decomposition* of $\mathfrak{D}$ relative to the idempotent e_1 (or the pair of orthogonal idempotents e_1 and $e_2 = 1 - e_1$). It is immediate from the relations $E_{ij}e_{kL} = \delta_{ik}E_{ij}$, $E_{ij}e_{kR} = \delta_{jk}E_{ij}$ that $\mathfrak{D}_{ij} = \{x_{ij} \mid e_ix_{ij} = x_{ij} = x_{ij}e_j\}$.

We can now establish the following relations:

$$\mathfrak{D}_{ij}\mathfrak{D}_{jk} \subseteq \mathfrak{D}_{ik}, \tag{18}$$

$$\mathfrak{D}_{ij}\mathfrak{D}_{ij} \subseteq \mathfrak{D}_{ji}, \tag{19}$$

$$\mathfrak{D}_{ij}\mathfrak{D}_{kl} = 0 \quad \text{if} \quad j \neq k, \qquad (i,j) \neq (k,l), \tag{20}$$

$$[\mathfrak{D}_{ij}, \mathfrak{D}_{jk}, \mathfrak{D}_{ki}] = 0 \quad \text{if} \quad (i,j,k) \neq (i,i,i). \tag{21}$$

Here $[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$ denotes the space spanned by the associators $[a,b,c]$, $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, $c \in \mathfrak{C}$. To prove (18), let $x_{ij} \in \mathfrak{D}_{ij}$, $y_{jk} \in \mathfrak{D}_{jk}$ and consider $e_i(x_{ij}y_{jk})$ and $(x_{ij}y_{jk})e_k$. We have $e_i(x_{ij}y_{jk}) = (e_ix_{ij})y_{jk} - [e_i, x_{ij}, y_{jk}] = x_{ij}y_{jk} + [x_{ij}, e_i, y_{jk}]$ and $[x_{ij}, e_i, y_{jk}] = (x_{ij}e_i)y_{jk} - x_{ij}(e_iy_{jk}) = \delta_{ij}x_{ij}y_{jk} - \delta_{ij}x_{ij}y_{jk} = 0$. Hence $e_i(x_{ij}y_{jk}) = x_{ij}y_{jk}$. Similarly, $(x_{ij}y_{jk})e_k = x_{ij}y_{jk}$. Hence (18) holds. In view of (18) it is enough to prove (19) for $i \neq j$. Then, if $x_{ij}, y_{ij} \in \mathfrak{D}_{ij}$, $e_j(x_{ij}y_{ij}) = -[e_j, x_{ij}, y_{ij}] = [x_{ij}, e_j, y_{ij}] = x_{ij}y_{ij}$ and, similarly, $(x_{ij}y_{ij})e_i = x_{ij}y_{ij}$. Hence (19) holds. For (20), let $x_{ij} \in \mathfrak{D}_{ij}$, $y_{kl} \in \mathfrak{D}_{kl}$ and consider $x_{ij}y_{kl} = (x_{ij}e_j)y_{kl} = [x_{ij}, e_j, y_{kl}] = -[y_{kl}, e_j, x_{ij}] = 0$ unless either $j = i$ or $l = j$. In the first case, we have $x_{ii}y_{kl} = x_{ii}(e_ky_{kl}) = -[x_{ii}, e_k, y_{kl}] = [y_{kl}, e_k, x_{ii}] = 0$ unless $l = k$. Then $e_k(x_{ii}y_{kk}) = -[e_k, x_{ii}, y_{kk}] = [x_{ii}, e_k, y_{kk}] = -x_{ii}y_{kk}$. Since the characteristic is not two and the characteristic roots of e_{kL} are 0 and 1, this implies that $x_{ii}y_{kk} = 0$. If $l = j$ then $x_{ij}y_{kl} = x_{ij}y_{kj}$ and since we are excluding the case $k = i$ we must have $i = j$. Then $x_{ij}y_{kl} = x_{ii}y_{ki} = 0$, as before. Hence (20) holds. Next consider $[x_{ij}, y_{jk}, z_{ki}]$

where $x_{ij} \in \mathfrak{D}_{ij}$, $y_{jk} \in \mathfrak{D}_{jk}$, $z_{ki} \in \mathfrak{D}_{ki}$. Two of the three indices are equal. Since the associator is alternating and $(i,j,k) \neq (i,i,i)$ it is enough to consider $[x_{ii}, y_{ij}, z_{ji}]$ where $i \neq j$. Then $[x_{ii}, y_{ij}, z_{ji}] = -[y_{ij}, x_{ii}, z_{ji}] = 0$, by (20). Thus (21) holds.

The Peirce space relations (18)–(21) have the following consequence:

LEMMA 1. *$\mathfrak{D}' \equiv \mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12} + \mathfrak{D}_{21} + \mathfrak{D}_{21}\mathfrak{D}_{12}$ is an ideal in $\mathfrak{D}$.*

PROOF. By symmetry it is enough to show that $\mathfrak{D}'$ is a left ideal. Moreover, symmetry considerations show that it is sufficient to verify that $\mathfrak{D}_{11}\mathfrak{D}' \subseteq \mathfrak{D}'$ and $\mathfrak{D}_{12}\mathfrak{D}' \subseteq \mathfrak{D}_{12}$. We have

$$\begin{aligned}
\mathfrak{D}_{11}(\mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12} + \mathfrak{D}_{21} + \mathfrak{D}_{21}\mathfrak{D}_{12}) &\subseteq \mathfrak{D}_{11}(\mathfrak{D}_{12}\mathfrak{D}_{21}) + \mathfrak{D}_{11}\mathfrak{D}_{12} \\
&\subseteq (\mathfrak{D}_{11}\mathfrak{D}_{12})\mathfrak{D}_{21} + \mathfrak{D}_{12} \subseteq \mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12} \\
\mathfrak{D}_{12}(\mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12} + \mathfrak{D}_{21} + \mathfrak{D}_{21}\mathfrak{D}_{12}) &\subseteq \mathfrak{D}_{12}\mathfrak{D}_{12} + \mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12}(\mathfrak{D}_{21}\mathfrak{D}_{12}) \\
&\subseteq \mathfrak{D}_{21} + \mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12},
\end{aligned}$$

which prove the required inclusions.

We are now ready to prove the following

THEOREM 6. *Let $(\mathfrak{D},j)$ be a simple alternative algebra with involution and identity element such that every nonzero symmetric element is invertible in the nucleus and let Γ be the subset of $\mathfrak{D}$ of symmetric elements in the center. Then Γ is a subfield of $\mathfrak{D}$ and we have the following possibilities for $(\mathfrak{D},j)$*: I. *$\mathfrak{D} = \Delta \oplus \Delta^{\circ}$, Δ an associative division algebra, j the exchange involution*; II. *$(\mathfrak{D},j)$ an associative division algebra with involution*; III. *$(\mathfrak{D},j)$ a composition algebra over Γ.*[2]

PROOF. Let $\mathfrak{H}(\mathfrak{D},j)$, $N(\mathfrak{D})$ and $C(\mathfrak{D})$ be the set of symmetric elements, the nucleus of $\mathfrak{D}$, and the center of $\mathfrak{D}$ respectively. We recall that $N(\mathfrak{D})$ is an associative subalgebra of $\mathfrak{D}$ and $C(\mathfrak{D})$, which is the subset of elements $c \in N(\mathfrak{D})$ such that $ac = ca$, $a \in \mathfrak{D}$, is a commutative associative subalgebra. Since $\mathfrak{D}$ is alternative, $N(\mathfrak{D}) = \{n \mid [n,a,b] = 0,\ a, b \in \mathfrak{D}\}$. We recall also that alternativity implies that if $n \in N(\mathfrak{D})$ then

$$n[a,b,c] = [na,b,c] = [an,b,c] \tag{22}$$

(eqs. (31) and (35) of Chapter I). It follows that if $a \in N(\mathfrak{D})$ $(C(\mathfrak{D}))$ and a is invertible in $\mathfrak{D}$ in the sense that there exists a $b \in \mathfrak{D}$ such that $ab = 1 = ba$ then $b \in N(\mathfrak{D})$ $(C(\mathfrak{D}))$. Our hypothesis is that if $h \neq 0$ is in $\mathfrak{H}(\mathfrak{D},j)$ then $h \in N(\mathfrak{D})$ and h

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[2] This result was first proved by using a theorem of Kleinfeld's determining the simple alternative algebras and a theorem of Osborn's, in which the hypotheses are as in Theorem 6 plus associativity. Moreover, Osborn's proof (see Jacobson [38]) makes use of a theorem of Herstein's on the generation of $\mathfrak{D}$ by its symmetric elements. The present self contained proof is due to McCrimmon.

is invertible in $N(\mathfrak{D})$. It is clear also that $N(\mathfrak{D})^j = N(\mathfrak{D})$ and $C(\mathfrak{D})^j = C(\mathfrak{D})$. Let $\Gamma = C(\mathfrak{D}) \cap \mathfrak{H}(\mathfrak{D},j)$. Then Γ is a subalgebra of $\mathfrak{D}$ containing 1 and every nonzero element of Γ is invertible in Γ. Hence Γ is a subfield of $C(\mathfrak{D})$ and it is clear that $\mathfrak{D}$ can be regarded as an algebra over Γ. Since nothing is changed in replacing the base field Φ by Γ we may assume $\Gamma = \Phi$ ($= \Phi 1$). This normalization gives the additional property that if $\gamma \in C(\mathfrak{D})$ and $\bar{\gamma} = \gamma$ then $\gamma \in \Phi$.

Since $a + \bar{a} \in \mathfrak{H}(\mathfrak{D},j) \subseteq N(\mathfrak{D})$ the alternative law gives $[a, \bar{a}, b] = 0$, $a, b \in \mathfrak{D}$. We note next that the following four conditions are equivalent: (i) $a\bar{a}$ is invertible, (ii) a has a right inverse, (iii) $\bar{a}a$ is invertible, (iv) a has a left inverse. Assume (i). Then we have a $b \in \mathfrak{D}$ such that $(a\bar{a})b = 1$, which gives $a(\bar{a}b) = 1$. Hence (ii) holds. Next assume this: $ab = 1$ and suppose $\bar{a}a$ is not invertible. Since $\bar{a}a \in \mathfrak{H}(\mathfrak{D},j)$ this implies $\bar{a}a = 0$. Then $0 = (\bar{a}a)b = \bar{a}(ab) = \bar{a}$ contrary to $ab = 1$. Hence $\bar{a}a$ is invertible. Now assume this. Then the left-right dual of the implication (i) $\Rightarrow$ (ii) implies that a has a left inverse. Similarly, dualizing the implication (ii) $\Rightarrow$ (iii) shows that (iv) $\Rightarrow$ (i). Hence the four conditions are equivalent. Now let $\mathfrak{Z}$ be the complement of the set satisfying these conditions. Then $\mathfrak{Z}$ is the set of elements z such that $z\bar{z} = 0 = \bar{z}z$. If $d \in \mathfrak{D}$ and $z \in \mathfrak{Z}$ then $(dz)(\overline{dz}) = (dz)(\bar{z}(d + \bar{d} - d))$ $= (dz)(\bar{z}(d + \bar{d})) - (dz)(\bar{z}d) = ((dz)\bar{z})(d + \bar{d}) - d(z\bar{z})d = 0$, by Moufang's identities and fact that $d + \bar{d}$ and $z + \bar{z} \in N(\mathfrak{D})$. Hence $dz \in \mathfrak{Z}$ and, similarly, $zd \in \mathfrak{Z}$. It is clear also that $\mathfrak{Z}$ is closed under multiplication by elements of the base field. Hence if $\mathfrak{Z}$ is closed under addition then $\mathfrak{Z}$ is an ideal.

We now note that if $N(\mathfrak{D}) = \mathfrak{D}$ and $\mathfrak{Z} = 0$ then $\mathfrak{D}$ is associative and every nonzero element of $\mathfrak{D}$ is invertible. Then we have case II. Next suppose $\mathfrak{H}(\mathfrak{D},j)$, which is contained in $N(\mathfrak{D})$, is contained in $C(\mathfrak{D})$. Then $\mathfrak{H}(\mathfrak{D},j) \subseteq \Phi = C(\mathfrak{D}) \cap \mathfrak{H}(\mathfrak{D},j)$. Then if $a \in \mathfrak{D}$, $a\bar{a} = \alpha$ is a symmetric element. Hence $\alpha \in \Phi$. Similarly, $\bar{a}a = \beta \in \Phi$ and either $\alpha = 0 = \beta$ or $\alpha \neq 0$, $\beta \neq 0$. In the latter case, $\bar{a}a\bar{a} = \alpha\bar{a} = \beta\bar{a}$, which implies that $\alpha = \beta$. If we put $\alpha = Q(a)$ then it is immediate that $Q(a)$ is a quadratic form on $\mathfrak{D}$. Then (9) and (10) (which are valid without the assumption of nondegeneracy) show that the radical $\mathfrak{D}^{\perp}$ of the symmetric bilinear form $Q(a,b)$ $= \frac{1}{2}[Q(a + b) - Q(a) - Q(b)]$ is an ideal of $(\mathfrak{D},j)$. Since $Q(1,1) = Q(1) = 1$, simplicity of $(\mathfrak{D},j)$ implies that $\mathfrak{D}^{\perp} = 0$. Hence $(\mathfrak{D},j)$ is a composition algebra over Φ and we have case III.

Now assume $\mathfrak{D}$ is not simple. Then it is immediate that $\mathfrak{D} = \Delta \oplus \Delta^j$ where Δ is an ideal (ex. 1, p. 14). This structure and the fact that the nonzero symmetric elements are invertible in the nucleus imply that Δ is an associative division algebra. Thus we have case I.

Next assume $\mathfrak{D}$ is simple and $\mathfrak{Z}$ is closed under addition. Then $\mathfrak{Z}$ is an ideal not containing 1. Hence $\mathfrak{Z} = 0$. If $\mathfrak{D}$ is associative we have case II. Hence assume $\mathfrak{D}$ is not associative. By (22), we see that if $n \in N(\mathfrak{D})$ and $a \in \mathfrak{D}$ then $[n, a] = na$ $- an \in N(\mathfrak{D})$. Thus

$$[N(\mathfrak{D}), \mathfrak{D}] \subseteq N(\mathfrak{D}). \tag{23}$$

A direct verification shows also that $[ab,n] = [a,n]b + a[b,n]$ if $a, b \in \mathfrak{D}$ and $n \in N(\mathfrak{D})$. Hence, by (22) and (23), $[a[b,n] + [a,n]b, c, d] = [[ab,n],c,d] = 0$. Using (22) and (23) again we obtain

$$[a,n][b,c,d] + [b,n][a,c,d] = 0. \tag{24}$$

If $a \in N(\mathfrak{D})$ this gives $[a,n][b,c,d] = 0$. Since $\mathfrak{D}$ is not associative we may assume $u = [b,c,d] \neq 0$. Then $u\bar{u}$ has an inverse m in $N(\mathfrak{D})$ and $[a,n]u = 0$ gives $[a,n](u\bar{u}) = 0$ and $([a,n](u\bar{u}))m = [a,n]((u\bar{u})m) = [a,n] = 0$. If $a \notin N(\mathfrak{D})$ we choose c, d so that $[a,c,d] \neq 0$ and put $b = a$ in (24). This gives $[a,n][a,c,d] = 0$ which implies $[a,n] = 0$. Hence $[a,n] = 0$ for all a so $C(\mathfrak{D}) = N(\mathfrak{D})$. Then $\mathfrak{H}(\mathfrak{D},j) \subseteq C(\mathfrak{D})$ and we have case III.

Finally, assume $\mathfrak{D}$ simple and $\mathfrak{Z}$ is not closed under addition. Then we have $z, w \in \mathfrak{Z}$ such that $z + w$ is invertible. Then we may assume that $z + w = 1$, $z\bar{z} = 0 = \bar{z}z$, $w\bar{w} = 0 = \bar{w}w$. Then $\bar{z} + \bar{w} = 1$ and $w = w1 = w(\bar{z} + \bar{w}) = w\bar{z} = (w + z)\bar{z} = \bar{z}$. Hence $z + \bar{z} = 1$ and $z^2 = z^2 + z\bar{z} = z$, $\bar{z}^2 = \bar{z}$ and $z\bar{z} = 0 = \bar{z}z$. Thus z and $\bar{z}$ are orthogonal idempotents. Put $e_1 = z$, $e_2 = \bar{z}$ and let $\mathfrak{D} = \mathfrak{D}_{11} \oplus \mathfrak{D}_{12} \oplus \mathfrak{D}_{21} \oplus \mathfrak{D}_{22}$ be the Peirce decomposition relative to e_1, e_2. By Lemma 1, $\mathfrak{D}_{12}\mathfrak{D}_{21} + \mathfrak{D}_{12} + \mathfrak{D}_{21} + \mathfrak{D}_{21}\mathfrak{D}_{12}$ is an ideal in $\mathfrak{D}$. If this is 0, $\mathfrak{D}_{12} + \mathfrak{D}_{21} = 0$ and $\mathfrak{D} = \mathfrak{D}_{11} \oplus \mathfrak{D}_{22}$, $\mathfrak{D}_{11}\mathfrak{D}_{22} = 0 = \mathfrak{D}_{22}\mathfrak{D}_{11}$. This is impossible by the simplicity of $\mathfrak{D}$. Hence the ideal indicated coincides with $\mathfrak{D}$. Consequently, $\mathfrak{D}_{12}\mathfrak{D}_{21} = \mathfrak{D}_{11}$, $\mathfrak{D}_{21}\mathfrak{D}_{12} = \mathfrak{D}_{22}$ and there exists $u \in \mathfrak{D}_{12}$, $v \in \mathfrak{D}_{21}$ such that $uv \neq 0$. Then $w = uv \in \mathfrak{D}_{11}$ and $\bar{w} \in \mathfrak{D}_{22}$. Since $t = w + \bar{w} \in \mathfrak{H}(\mathfrak{D},j)$ and $t \neq 0$, t^{-1} exists in $N(\mathfrak{D})$. It is immediate that $t^{-1} = t_1 + t_2$ where $t_i \in \mathfrak{D}_{ii}$ and $u(vt_1) = (uv)t_1 = e_1$ (using (21)). Hence we may assume that $uv = e_1$. Now let $s \in \mathfrak{D}_{12} = e_1\mathfrak{D}\bar{e}_1$. Then it is clear from (18), (19) and (20) that $s\mathfrak{D} \subseteq \mathfrak{D}_{11} + \mathfrak{D}_{12} + \mathfrak{D}_{21}$ so s does not have a right inverse. Then $s + \bar{s} \in \mathfrak{D}_{11} + \mathfrak{D}_{22}$ is symmetric and is not invertible. Hence $\bar{s} = -s$. Similarly, every element of $\mathfrak{D}_{21}$ is skew and, consequently, if $h \in \mathfrak{H}(\mathfrak{D},j)$ then $h = a + \bar{a}$, $a \in \mathfrak{D}_{11}$. If $s \in \mathfrak{D}_{12} + \mathfrak{D}_{21}$ then $\bar{s} = -s$ and $[s,h] \in (\mathfrak{D}_{12} + \mathfrak{D}_{21}) \cap \mathfrak{H}(\mathfrak{D},j) = 0$. Hence $\overline{[s,h]} = -[s,h]$. This gives $hs = sh$. We shall now use the elements u, v determined above to show that h commutes with every element $d \in \mathfrak{D}_{11}$. Let $k = d + \bar{d}$. Then $\bar{d} \in \mathfrak{D}_{22}$ and $k \in N(\mathfrak{D})$. Then $d = ke_1$ and $hd = h(ke_1) = (hk)e_1 = (hk)(uv) = h(k(uv)) = h((ku)v)$. Since $k \in \mathfrak{H}(\mathfrak{D},j)$ and $u \in \mathfrak{D}_{12}$ the result just established shows that $ku = uk$. Hence $hd = h((uk)v) = (h(uk))v$ since $h \in \mathfrak{D}_{11} + \mathfrak{D}_{22}$, $uk \in \mathfrak{D}_{12}$, $v \in \mathfrak{D}_{21}$ and (20) and (21) hold. Also we have $h(uk) = (uk)h$. Hence $hd = ((uk)h)v = (uk)(hv) = (ku)(vh) = ((ku)v)h = (k(uv))h = (ke_1)h = dh$. Thus $hd = dh$ if $d \in \mathfrak{D}_{11}$ and, similarly, $hd = dh$ if $d \in \mathfrak{D}_{22}$. Hence $h \in C(\mathfrak{D})$ and so $\mathfrak{H}(\mathfrak{D},j) \subseteq C(\mathfrak{D})$. Then $(\mathfrak{D},j)$ is a composition algebra.

It is clear that the two classes of algebras I and II defined in Theorem 6 have no common members. On the other hand, it is immediate that the one and two-dimensional composition algebras (over an extension field of the base field) and the quaternion division algebras are in the classes I or II. Finally, it is clear that the octonion algebras are not in I or II. In order to obtain our final determination of the algebras with involution satisfying (1)–(3) we now determine the composition algebras which are not division algebras.

If the element a of the composition algebra $(\mathfrak{D},j)$ has the inverse a^{-1} ($aa^{-1} = 1 = a^{-1}a$, see ex. 2, p. 22) then we have $1 = Q(1) = Q(aa^{-1}) = Q(a)Q(a^{-1})$. Hence the existence of an inverse of a implies $Q(a) \neq 0$. On the other hand, if this condition holds then $a^{-1} = Q(a)^{-1}\bar{a}$ satisfies $aa^{-1} = 1 = a^{-1}a$. An alternative algebra $\mathfrak{A}$ is called a *division algebra* if $\mathfrak{A}$ has an identity element and every $a \neq 0$ in $\mathfrak{A}$ is invertible. The classification of composition division algebra can be made only for special base fields (see the exercises below). On the other hand, as we shall now show, any two composition $(\mathfrak{D},j)$, $(\mathfrak{D}',j')$ of the same dimensionality (over Φ) which are not division algebras are isomorphic.

THEOREM 7. *The following conditions on a composition algebra $(\mathfrak{D},j)$ are equivalent*: (1) $Q(x) = 0$ *for some* $x \neq 0$ *in* $\mathfrak{D}$; (2) $\mathfrak{D}$ *has zero divisors* $\neq 0$ ($ab = 0$, $a \neq 0$, $b \neq 0$); (3) $\mathfrak{D}$ *is not a division algebra*; (4) Q *has maximum Witt index which is positive. Moreover, any two composition algebras of the same dimensionality* (*over* Φ) *satisfying the conditions are isomorphic.*

PROOF. (1) $\Leftrightarrow$ (2). If $Q(x) = 0$ for some $x \neq 0$ then we have $x\bar{x} = 0$ and $\bar{x} \neq 0$. Hence we have zero divisors $\neq 0$. Conversely, if $ab = 0$, $a \neq 0$, $b \neq 0$ then $0 = Q(0) = Q(a)Q(b)$ implies either $Q(a) = 0$ or $Q(b) = 0$. (1) $\Leftrightarrow$ (3). This is trivial and has been noted above. (1) $\Leftrightarrow$ (4). The statement that Q has positive Witt index is equivalent to $Q(x) = 0$ for some $x \neq 0$. Hence (4) $\Rightarrow$ (1). Now assume (1) and let $\mathfrak{D}_0 = \Phi 1^{\perp}$, so that this is the subspace of elements satisfying $a + \bar{a} = 0$. Since $Q(1) = 1$ it is clear that $\mathfrak{D}_0 \neq 0$ and $\mathfrak{D}_0$ is not isotropic. We shall now show that there exists an $a \in \mathfrak{D}_0$ such that $a^2 = 1$. Let $x = \alpha 1 + x_0$, $x_0 \in \mathfrak{D}_0$, $\alpha \in \Phi$, satisfy $Q(x) = 0$. Then $\alpha^2 = x_0{}^2$ and if $\alpha \neq 0$, we can take $a = \alpha^{-1}x_0$ and obtain $\bar{a} = -a$, $a^2 = 1$. If $\alpha = 0$ then we have $x_0 \in \mathfrak{D}_0$, $x_0 \neq 0$ and $Q(x_0) = 0$. Since the restriction of $Q(x, y)$ to $\mathfrak{D}_0$ is nondegenerate, for any α in Φ there exists an $x_0 \in \mathfrak{D}_0$ such that $Q(x_0) = \alpha$. Taking $\alpha = -1$ we obtain $a \in \mathfrak{D}_0$ such that $Q(a) = -1$. Then $a^2 = 1$. Let $\mathfrak{D}_1 = \Phi[a]$. Then $a^2 = 1$ and $(\mathfrak{D}_1,j)$ is a nonisotropic subalgebra of $(\mathfrak{D},j)$ with orthogonal basis $(1, a)$ such that $Q(1) = 1$, $Q(a) = -1$. Then the restriction of Q to $\mathfrak{D}_1$ is nondegenerate and has Witt index 1. If $\mathfrak{D}_1 = \mathfrak{D}$ then (4) holds. Moreover, it is clear from the argument that any two two-dimensional composition algebras satisfying the conditions have bases with the same multiplication table and so are isomorphic. If $\mathfrak{D}_1 \subset \mathfrak{D}$ then we obtain the subalgebra $\mathfrak{D}_2 = \mathfrak{D}_1 \oplus \mathfrak{D}_1 q$ where $q \in \mathfrak{D}_1{}^{\perp}$ and $q^2 = \mu 1 \neq 0$, as in the proof of Theorem 5. We have seen that $Q(xq, yq) = \mu Q(x, y)$ if $x, y \in \mathfrak{D}_1$ and this implies that the restriction of Q to $\mathfrak{D}_1 q$ is nondegenerate and has Witt index 1. Since $\mathfrak{D}_1 \perp \mathfrak{D}_1 q$, the restriction of Q to $\mathfrak{D}_2$ has Witt index two. Also since $\mathfrak{D}_1 q$ contains isotropic vectors, we can choose an element b in this space such that $b^2 = 1$ and replacing q by this element we may assume $\mu = 1$. This determines the multiplication in $\mathfrak{D}_2 = \mathfrak{D}_1 + \mathfrak{D}_1 q$. Hence it is clear that any two four-dimensional composition algebras satisfying the conditions are isomorphic. If $\mathfrak{D}_2 \subset \mathfrak{D}$ we repeat the argument. This shows that (1) ⇒ (4) and also proves the last statement of the theorem.

The composition algebras satisfying the conditions of the theorem (of dimensions 2,4 and 8) will be called *split* composition algebras. It is clear that for dimension 2 such an algebra is a direct sum of two copies of Φ and j exchanges the two components. If the dimension is 4 one sees easily that $(\mathfrak{D},j) \cong (\Phi_2, J)$ where J is the involution

$$X \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{in} \quad \Phi_2 \quad \text{(see p. 128).}$$

If the base field is algebraically closed then any composition algebra $(\mathfrak{D},j)$ with $\dim \mathfrak{D} > 1$ is split since any quadratic form on a finite-dimensional vector space of dimensionality > 1 over an algebraically closed field has maximal Witt index.

It is clear that Theorems 5, 6 and 7 and the results noted about the converses give the following

THEOREM 8. *Let $(\mathfrak{D},j)$ be a simple alternative algebra with involution and identity element such that every nonzero symmetric element of $\mathfrak{D}$ is invertible in the nucleus and let Γ be the subset of $\mathfrak{D}$ of symmetric elements of the center. Then Γ is a subfield of $\mathfrak{D}$ and we have the following possibilities for $(\mathfrak{D},j)$:* I. $\mathfrak{D} = \Delta \oplus \Delta^{\circ}$, *$\Delta$ an associative division algebra, j the exchange involution*; II. $(\mathfrak{D},j)$ *an associative division algebra with involution*; III. *a split quaternion algebra over Γ ($\cong \Gamma_2$) with standard involution*; IV. *an algebra of octonions over Γ with standard involution. Conversely, any algebra in the classes* I–IV *satisfy the given conditions.*

EXERCISES

1. Let $(\mathfrak{D},j)$, $(\mathfrak{D}',j')$ be composition algebras. Show that any isomorphism of $\mathfrak{D}$ into $\mathfrak{D}'$ mapping $1 \to 1$ is an isomorphism of $(\mathfrak{D},j)$ into $(\mathfrak{D}',j')$.

2. Let $(\mathfrak{D},j)$ be a composition algebra, $(\mathfrak{B},j)$ a nonisotropic subalgebra. Use Witt's theorem to prove that any isomorphism of $\mathfrak{B}$ into $\mathfrak{D}$ can be extended to an automorphism of $\mathfrak{D}$.

3. Show that two composition algebras are isomorphic if and only if the associated quadratic forms Q are equivalent.

4. Let $(\mathfrak{D},j)$ be a composition algebra, Q the associated quadratic form. Define $\{a,b,c\} = Q(a,[bc])$ for $a,b,c \in \mathfrak{D}$, $[ab] = ab - ba$. Show that $\{a,b,c\}$ is alternating in the three arguments a,b,c. Show also that a 1-1 linear mapping η of $\mathfrak{D}$ onto $\mathfrak{D}$ is an automorphism if and only if $Q(a^\eta, b^\eta) = Q(a,b)$, $\{a^\eta, b^\eta, c^\eta\} = \{a,b,c\}$, $a,b,c \in \mathfrak{D}$.

5. Show that the only right (left) ideals in an octonion algebra $\mathfrak{O}$ are $\mathfrak{O}$ and 0.

6. Determine all the composition algebras over the field R of real numbers. (There are seven of these.)

7. Show that every octonion algebra over a p-adic field is split. Use this and Hasse's theorem on quadratic forms to show that if Φ is an algebraic number field with t real conjugate fields then there are at most 2^t nonisomorphic octonion algebras over Φ. (This number is exact. For example, if Φ is the field of rational numbers, then there are two nonisomorphic octonion algebras over Φ.)

4. **Simple Jordan algebras of capacity two.** Let $\mathfrak{J}$ be a simple Jordan algebra of capacity two. Then $\mathfrak{J}$ has the identity $1 = e_1 + e_2$ where the e_i are completely primitive orthogonal idempotents. If $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{22}$ is the corresponding Peirce decomposition, then the $\mathfrak{J}_{ii}$ are division algebras and $\mathfrak{J}_{12} \neq 0$. Moreover, by Theorem 4, $z_{12} = 0$ is the only element of $\mathfrak{J}_{12}$ such that $z_{12} \cdot \mathfrak{J}_{12} = 0$. If $a_{ii} \in \mathfrak{J}_{ii}$ and $V_{a_{ii}}$ denotes the restriction of $2R_{a_{ii}}$ to $\mathfrak{M} \equiv \mathfrak{J}_{12}$ then $a_{ii} \to V_{a_{ii}}$ is a monomorphism of $\mathfrak{J}_{11}$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})^+$ (see (5)). Also, by (6), $[V_{a_{11}}, V_{a_{22}}] = 0$ if $a_{ii} \in \mathfrak{J}_{ii}$ Let $S_1(\mathfrak{J}_{ii})$ be the unital special universal envelope for $\mathfrak{J}_{ii}$. Then we can identify $\mathfrak{J}_{ii}$ with its image in $S_1(\mathfrak{J}_{ii})$ and then $\mathfrak{J}_{ii}$ becomes a subalgebra of $S_1(\mathfrak{J}_{ii})^+$. We have the main involution π in $S_1(\mathfrak{J}_{ii})$ which is characterized by: $a_{ii}^{\pi} = a_{ii}$ if $a_{ii} \in \mathfrak{J}_{ii} \subseteq S_1(\mathfrak{J}_{ii})$. Since $a_{ii} \to V_{a_{ii}}$ is a homomorphism of Jordan algebras mapping $1 \to 1$ this can be extended uniquely to a homomorphism v of $S_1(\mathfrak{J}_{ii})$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$. Also we can regard $\mathfrak{M}$ as a right module for the associative algebra $S_1(\mathfrak{J}_{ii})$, which is characterized by:

(25) $$xa_{ii} = xV_{a_{ii}} = 2x \cdot a_{ii}, \qquad x \in \mathfrak{M}, \qquad a_{ii} \in \mathfrak{J}_{ii}.$$

Since $[V_{a_{11}}, V_{a_{22}}] = 0$ we have

(26) $$(xa_1)a_2 = (xa_2)a_1 \quad \text{if} \quad a_i \in S_1(\mathfrak{J}_{ii}).$$

We have seen that e_1 and e_2 are connected idempotents, that is, there exist invertible elements in $\mathfrak{J}_{12}$. Let u be any such element and put $u_{21} = u$, $u_{12} = u^{\cdot -1}$. As in Lemma 5 of §3.1 put $f_1 = e_1 = g_1$, $f_2 = u_{21}^{\cdot 2} \cdot e_2$, $g_2 = u_{12}^{\cdot 2} \cdot e_2$, $f = f_1 + f_2$, $g = g + g_2$. Then $g = f^{\cdot -1}$ and the isotope $\tilde{\mathfrak{J}} \equiv (\mathfrak{J}, g)$ has f as identity element and has the orthogonal idempotents f_1, f_2 such that $f_1 + f_2 = f$. We shall now consider how much of our data is unaltered in passing from $\mathfrak{J}, e_1, e_2$ to $\tilde{\mathfrak{J}}, f_1, f_2$. Let $\tilde{U}_a$ denote the U-operator determined by a in $\tilde{\mathfrak{J}}$. By (90) of Chapter I, we have $\tilde{U}_a = U_g U_a$. Hence $\tilde{\mathfrak{J}}\tilde{U}_{f_1} = \mathfrak{J}U_g U_{e_1} = \mathfrak{J}U_{e_1} = \mathfrak{J}_{11}$ and $\tilde{\mathfrak{J}}\tilde{U}_{f_2} = \mathfrak{J}U_g U_{f_2} = \mathfrak{J}U_{f_2}$. Since $f_2 \in \mathfrak{J}_{22}$ and is invertible in $\mathfrak{J}_{22}$, we have $f_2 = f_2 U_{e_2}$, $U_{f_2} = U_{e_2} U_{f_2} U_{e_2}$ and $\mathfrak{J}U_{f_2} = \mathfrak{J}U_{e_2} U_{f_2} U_{e_2} = \mathfrak{J}U_{e_2} = \mathfrak{J}_{22}$. If $a_{ii} \in \mathfrak{J}_{ii}$, then it is immediate from the definition of $U_{a_{11},a_{22}}$ that $\mathfrak{J}_{11} U_{a_{11},a_{22}} = 0 = \mathfrak{J}_{22} U_{a_{11},a_{22}}$ and $\mathfrak{J}_{12} U_{a_{11},a_{12}} \subseteq \mathfrak{J}_{12}$. Hence $\mathfrak{J}U_{a_{11},a_{22}} = \mathfrak{J}_{12} U_{a_{11},a_{22}} \subseteq \mathfrak{J}_{12}$. Then $\tilde{\mathfrak{J}}\tilde{U}_{f_1,f_2} = \mathfrak{J}U_g U_{f_1,f_2} \subseteq \mathfrak{J}_{12}$. It follows that the Peirce decomposition $\tilde{\mathfrak{J}} = \tilde{\mathfrak{J}}_{11} \oplus \tilde{\mathfrak{J}}_{12} \oplus \tilde{\mathfrak{J}}_{22}$ of $\mathfrak{J}$ relative to f_1 and f_2 coincides with that relative to e_1, e_2: $\tilde{\mathfrak{J}}_{ij} = \mathfrak{J}_{ij}$.

If $a_{11}, b_{11} \in \tilde{\mathfrak{J}}_{11} = \mathfrak{J}_{11}$ then $a_{11 \cdot g} b_{11} = \{a_{11} g b_{11}\} = \{a_{11} e_1 b_{11}\} = a_{11} \cdot b_{11}$.

Hence $\tilde{\mathfrak{J}}_{11}$ is identical with $\mathfrak{J}_{11}$ also as algebra (subalgebras of $\tilde{\mathfrak{J}}$ and $\mathfrak{J}$ respectively). If $a_{22}, b_{22} \in \tilde{\mathfrak{J}}_{22} = \mathfrak{J}_{22}$ then $a_{22.g}\, b_{22} = \{a_{22} g\, b_{22}\} = \{a_{22} g_2 b_{22}\} = a_{22.g_2} b_{22}$. Hence the algebra $\tilde{\mathfrak{J}}_{22}$ is the isotope $(\mathfrak{J}_{22}, g_2)$ of $\mathfrak{J}_{22}$. Thus $\tilde{\mathfrak{J}}_{11}$ and $\tilde{\mathfrak{J}}_{22}$ are division algebras and $\tilde{\mathfrak{J}}$ is a Jordan algebra of capacity two. It is clear also that $\tilde{\mathfrak{J}}$ is simple (since an algebra $\mathfrak{J}$ and its isotope $(\mathfrak{J}, g)$ have the same ideals). If $x \in \tilde{\mathfrak{M}} = \tilde{\mathfrak{J}}_{12}$ and $a_{11} \in \tilde{\mathfrak{J}}_{11}$ then $x_{.g} a_{11} = \{x e_1 a_{11}\} + \{x g_2 a_{11}\}$. Since $\{x g_2 a_{11}\} = 0$ by (PD2) and $\{x e_1 a_{11}\} = x \,.\, a_{11}$, $x_{.g} a_{11} = x \,.\, a_{11}$. Clearly, this implies that if $\mathfrak{M}$ is made into a right module for $S_1(\tilde{\mathfrak{J}}_{11}) = S_1(\mathfrak{J}_{11})$ as in (25) then this right module is identical with that defined by $\mathfrak{J}$. In other words, if $\tilde{\nu}$ is the corresponding homomorphism of $S_1(\mathfrak{J}_{11})$ into $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ then $\tilde{\nu} = \nu$.

We recall also that $u_{21.g} u_{21} = f$ the identity of $\tilde{\mathfrak{J}}$. Since $u = u_{21}$ was any invertible element of $\mathfrak{J}$ the results show that given any invertible element $u \in \mathfrak{J}_{12}$ we may pass to an isotope in which this element has square the identity. Moreover, this change leaves the Peirce decomposition unchanged, the algebra $\mathfrak{J}_{11}$ unchanged and the $S_1(\mathfrak{J}_{11})$ right module $\mathfrak{M} = \mathfrak{J}_{12}$ unchanged.

Now assume we have a $u \in \mathfrak{J}_{12}$ such that $u^{\cdot 2} = 1$. Then we know that $\eta = U_u$ is an automorphism of period two in $\mathfrak{J}$ which interchanges the e_i and the $\mathfrak{J}_{ii}$ and maps $\mathfrak{J}_{12}$ into itself (Lemma 1 of §3.3). The restriction of η to $\mathfrak{J}_{11}$ has a unique extension to an isomorphism of $S_1(\mathfrak{J}_{11})$ onto $S_1(\mathfrak{J}_{22})$. We denote this again as η. If $a_{11} \in \mathfrak{J}_{11}$ and $x \in \mathfrak{M}$, then $(x a_{11})^\eta = (2x \,.\, a_{11})^\eta = 2x^\eta \,.\, a_{11}{}^\eta = x^\eta a_{11}{}^\eta$. Since $\mathfrak{J}_{11}$ generates $S_1(\mathfrak{J}_{11})$ this implies that

$$(27) \qquad (xa)^\eta = x^\eta a^\eta, \qquad x \in \mathfrak{M}, \qquad a \in S_1(\mathfrak{J}_{11}).$$

Since $u^{\cdot 2} = 1$, $R_u{}^3 = R_u$. Hence $u^\eta = 2u^{\cdot 3} - u = u$ and $x^\eta \,.\, u = (2x R_u{}^2 - x) R_u = 2x R_u{}^3 - x R_u = x R_u$. Hence

$$(28) \qquad (u \,.\, x)^\eta = u \,.\, x^\eta = u \,.\, x, \qquad x \in \mathfrak{J}.$$

Now let $x_1, x_2, \cdots, x_m \in \mathfrak{J}_{11}$ and consider the element $a = x_1 x_2 \cdots x_m \in S_1(\mathfrak{J}_{11})$. We have

$$\begin{aligned} u a^\pi &= u x_m x_{m-1} \cdots x_1 = 2^m u \,.\, x_m \,.\, x_{m-1} \,.\, \cdots \,.\, x_1 \\ &= 2^m u \,.\, x_m{}^\eta \,.\, x_{m-1} \cdots x_1 \quad \text{(by (28))} \\ &= 2^m u \,.\, x_{m-1} \,.\, \cdots \,.\, x_1 \,.\, x_m{}^\eta \quad \text{(by (PD2))}. \end{aligned}$$

Repetition of this gives

$$\begin{aligned} u a^\pi &= 2^m u \,.\, x_1{}^\eta \,.\, x_2{}^\eta \,.\, \cdots \,.\, x_m{}^\eta \\ &= u x_1{}^\eta x_2{}^\eta \cdots x_m{}^\eta \\ &= u a^\eta \,. \end{aligned}$$

Combining this with (27) we obtain

$$(29) \qquad ua^{\pi} = ua^{\eta} = (ua)^{\eta},$$

for $a = x_1x_2\cdots x_m$, since $u^{\eta} = u$. Since the products $x_1x_2\cdots x_m$ span $S_1(\mathfrak{J}_{11})$, (29) holds for all $a \in S_1(\mathfrak{J}_{11})$. We remark also that the derivation of (29) was based on the formulas $u^{\eta} = u$, $ua_{11} = ua_{11}{}^{\eta}$, $a_{11} \in \mathfrak{J}_{11}$. Hence this is valid for any u satisfying these conditions. Also if $v \in \mathfrak{M}$ satisfies $v^{\eta} = -v$ and $va_{11} = va_{11}{}^{\eta}$, $a_{11} \in \mathfrak{J}_{11}$ then the same argument shows that

$$(30) \qquad va^{\pi} = va^{\eta} = -(va)^{\eta}.$$

A consequence of (29) is

$$(31) \qquad ua = 0,\ a \in S_1(\mathfrak{J}_{11}), \Rightarrow uS_1(\mathfrak{J}_{11})a = 0.$$

To see this let $b \in S_1(\mathfrak{J}_{11})$. Then $ub^{\pi}a = ub^{\eta}a = uab^{\eta}$ (by (26)) $= 0$. Since $S_1(\mathfrak{J}_{11})^{\pi} = S_1(\mathfrak{J}_{11})$ this gives (31).

We can now prove

LEMMA 1. *If* $x^{\eta} = x \in \mathfrak{M}$ *then*

$$(32) \qquad x = ua_{11} \ \textit{where}\ a_{11} = x.u.e_1 \in \mathfrak{J}_{11}.$$

If $h^{\pi} = h \in S_1(\mathfrak{J}_{11})$ *then*

$$(33) \qquad xh = xa_{11}, \quad x \in uS_1(\mathfrak{J}_{11}) \quad \textit{where} \quad a_{11} = uh.u.e_1 \in \mathfrak{J}_{11}.$$

PROOF. If $x^{\eta} = x$, we have $2x.u.u - x = x$ so $x.u.u = x$. On the other hand, $x.u = x.u.e_1 + x.u.e_2$ and $x.u.u = x.u.e_1.u + x.u.e_2.u = x.u.e_1.u + (x.u.e_1)^{\eta}.u = 2x.u.e_1.u$ (by (28)) $= ua_{11}$ where $a_{11} = 2x.u.e_1 \in \mathfrak{J}_{11}$. Hence $x = x.u.u = ua_{11}$ which proves (32). In view of (31) it is sufficient to prove (33) for $x = u$. Then $y = uh$ satisfies $y^{\eta} = (uh)^{\eta} = uh^{\pi} = uh = y$, by (29). Hence, by (32), $y = uh = ua_{11}$, $a_{11} = uh.u.e_1$, as required.

We show next that

$$(34) \qquad x.ya.e_2 = xa^{\pi}.y.e_2, \qquad x, y \in \mathfrak{M}, \quad a \in S_1(\mathfrak{J}_{11}).$$

For $a = a_{11} \in \mathfrak{J}_{11}$ this is (PD6). Iteration gives the result for $a = x_1x_2\cdots x_m$, $x_i \in \mathfrak{J}_{11}$, which implies (34) for all $a \in S_1(\mathfrak{J}_{11})$. We prove next

LEMMA 2. $\mathfrak{M} = uS_1(\mathfrak{J}_{11}) + \mathfrak{M}^*$ *where* $\mathfrak{M}^* = \{v \in \mathfrak{M} \mid v^{\eta} = -v,\ va_{11}{}^{\eta} = va_{11},\ a_{11} \in \mathfrak{J}_{11}\}$.

PROOF. If $v^{\eta} = v \in \mathfrak{M}$ then $v \in uS_1(\mathfrak{J}_{11})$, by (31). Now assume $v^{\eta} = -v \in \mathfrak{M}$. Let $a_{11} \in \mathfrak{J}_{11}$ and put $w = va_{11} - va_{11}{}^{\eta}$. Then $w^{\eta} = -va_{11}{}^{\eta} + va_{11} = w$ so $w = uc_{11}$, $c_{11} \in \mathfrak{J}_{11}$. If $w = 0$ for all a_{11} then $va_{11}{}^{\eta} = va_{11}$, $a_{11} \in \mathfrak{J}_{11}$ and $v \in \mathfrak{M}^*$. We shall now complete the proof by showing that if $w \neq 0$ for some a_{11} then $v \in uS_1(\mathfrak{J}_{11})$. Hence suppose $w = va_{11} - va_{11}{}^{\eta} \neq 0$ for some $a_{11} \in \mathfrak{J}_{11}$. Then $w = v.b = uc_{11}$ where $b = 2a_{11} - 2a_{11}{}^{\eta}$ and $c_{11} \in \mathfrak{J}_{11}$. Clearly $c_{11} \neq 0$ and since u is invertible, w is invertible by Lemma 1 of §2. We claim that also v is invertible.

Otherwise, by Lemma 1 of §2, $v^{\cdot 2}=0$ and $\mathfrak{J}_{11}U_v=0=\mathfrak{J}_{22}U_v$. Since $b=2a_{11}-2a_{11}{}^{\pi}\in\mathfrak{J}_{11}+\mathfrak{J}_{22}$ this implies that $bU_v=0=b^{\cdot 2}U_v$. Then, by (2), $w^{\cdot 2}=(v.b)^{\cdot 2}=\frac{1}{2}b.bU_v+\frac{1}{4}b^{\cdot 2}U_v+\frac{1}{4}v^{\cdot 2}U_b=0$, which contradicts the fact that w is invertible. Accordingly, v is invertible. Put $d_{11}=a_{11}{}^{\pi}U_v{}^{-1}\in\mathfrak{J}_{11}$ (Lemma 1 of §2), $v_{11}=v^{\cdot 2}.e_1\in\mathfrak{J}_{11}$. Then $va_{11}{}^{\pi}=v(d_{11}U_v)=(vd_{11})U_v=2d_{11}R_vU_v=2d_{11}U_{v,v^{\cdot 2}}$ (by (65) of Chapter I) $=2\{vd_{11}v^{\cdot 2}\}=2\{vd_{11}v_{11}\}+2\{vd_{11}(v^{\cdot 2}.e_2)\}=2\{vd_{11}v_{11}\}$ since $\{a_{12}a_{11}a_{22}\}=0$, by (PD2). Since $2\{vd_{11}v_{11}\}=vd_{11}v_{11}$, by (PD3), we have $va_{11}{}^{\pi}=vd_{11}a_{11}$. Hence

$$w=va_{11}-va_{11}{}^{\pi}=vs,\qquad s=a_{11}-d_{11}a_{11}\in S_1(\mathfrak{J}_{11}). \tag{35}$$

Since w is invertible, $0\neq w^{\cdot 2}.e_2=vs.vs.e_2=vss^{\pi}.v.e_2$, by (34). Hence $ss^{\pi}\neq 0$. Then $(a_{11}-d_{11}v_{11})(a_{11}-v_{11}d_{11})\neq 0$ which gives

$$0\neq a_{11}{}^2-2\{d_{11}v_{11}a_{11}\}+\{d_{11}v_{11}{}^2d_{11}\}\in\mathfrak{J}_{11}$$

since $\mathfrak{J}_{11}$ is a subalgebra of $S_1(\mathfrak{J}_{11})^+$. Similarly, $s^{\pi}s\in\mathfrak{J}_{11}$. Since $ss^{\pi}\neq 0$ and $\mathfrak{J}_{11}$ is a division algebra ss^{π} is invertible in $\mathfrak{J}_{11}$ and hence in $S_1(\mathfrak{J}_{11})$. Since $(ss^{\pi})s=s(s^{\pi}s)$, $s^{\pi}s\neq 0$ and so this element is invertible in $S_1(\mathfrak{J}_{11})$. Consequently, s is invertible. Then, by (35), $v=ws^{-1}=uc_{11}s^{-1}\in uS_1(\mathfrak{J}_{11})$. This completes the proof.

We shall now establish the main properties of $\mathfrak{D}\equiv S_1(\mathfrak{J}_{11})^{\nu}$, which is the subalgebra of $\mathrm{Hom}_{\Phi}(\mathfrak{M},\mathfrak{M})$ generated by the $V_{a_{11}}$, $a_{11}\in\mathfrak{J}_{11}$.

LEMMA 3. (1) *The main involution π of $S_1(\mathfrak{J}_{11})$ induces an involution j in $\mathfrak{D}$.* (2) $\mathfrak{J}_{11}{}^{\nu}=\mathfrak{H}(\mathfrak{D},j)$. (3) $(\mathfrak{D},j)$ *is simple.*

PROOF. (1). This will follow by showing that $\ker\nu$ is invariant under π. Thus let $k\in\ker\nu$. By (29), $uk^{\pi}=0$. Now, this implies that $vk^{\pi}=0$ for any invertible $v\in\mathfrak{M}$. For, we can determine an isotope $\tilde{\mathfrak{J}}$ of $\mathfrak{J}$ using $v=u_{21}$, as above. Since $\mathfrak{M}$ as right module for $S_1(\mathfrak{J}_{11})$ is unchanged in passing to the isotope we see that $vk^{\pi}=0$. Now let $z\in\mathfrak{M}$ and assume z is not invertible. Then $z^{\cdot 2}=0$. We claim that either $z+u$ or $z-u$ is invertible. Otherwise, $(z+u)^{\cdot 2}=0=(z-u)^{\cdot 2}$. These give $z.u=0$ and $u^{\cdot 2}=0$, contrary to $u^{\cdot 2}=1$. Since $z=(z+u)-u=(z-u)+u$ we see that any element $z\in\mathfrak{M}$ which is not invertible is a sum of two invertible elements. Then $zk^{\pi}=0$ and $k^{\pi\nu}=0$, which proves (1).

(2) It suffices to show that if $h=h^{\pi}\in S_1(\mathfrak{J}_{11})$ then there exists an $a_{11}\in\mathfrak{J}_{11}$ such that $h^{\nu}=a_{11}{}^{\nu}$. By Lemma 1 and (33) there exists an $a_{11}\in\mathfrak{J}_{11}$ such that $xh=xa_{11}$ for all $x\in uS_1(\mathfrak{J}_{11})$. The argument just used with the isotope determined by the invertible element $v\in\mathfrak{M}$ shows that if v is any such element, then there exists a $b_{11}\in\mathfrak{J}_{11}$ such that $yh=yb_{11}$ for all $y\in vS_1(\mathfrak{J}_{11})$. Assume first that for every $x_{11}\neq 0$ in $\mathfrak{J}_{11}$, $u+vx_{11}$ is not invertible. Then $(u+vx_{11})^{\cdot 2}=0=(u-vx_{11})^{\cdot 2}$. Hence $1+(vx_{11})^{\cdot 2}=0=u.vx_{11}$ for all $x_{11}\neq 0$ in $\mathfrak{J}_{11}$. Then $-1=(vx_{11})^{\cdot 2}=4(v.x_{11})^2=2v.vU_{x_{11}}+v^{\cdot 2}U_{x_{11}}+x_{11}{}^{\cdot 2}U_v$. Taking $x_{11}=e_1$ in

$0 = 1 + (vx_{11})^{\cdot 2}$ we obtain $v^{\cdot 2} = -1$ and since $v \in \mathfrak{J}_{12}$, $vU_{x_{11}} = 0$ by (PD3). Hence $x_{11}^{\cdot 2}U_v = x_{11}^{\cdot 2} - 1$. Since $x_{11}^{\cdot 2}U_v \in \mathfrak{J}_{22}$ by Lemma 1 of §2 this implies that $x_{11}^{\cdot 2} = e_1$ for every $x_{11} \neq 0$ in $\mathfrak{J}_{11}$. Then if $x_{11} \neq 0, -e_1, (x_{11} + e_1)^{\cdot 2} = e_1$ gives $2x_{11} = -e_1$. Hence $\mathfrak{J}_{11}$ is the field Z_3 of three elements, $S_1(\mathfrak{J}_{11}) = Z_3$, and the result is clear. We now assume that there exists an $x_{11} \neq 0$ in $\mathfrak{J}_{11}$ such that $u + vx_{11}$ is invertible. Then we have a $c_{11} \in \mathfrak{J}_{11}$ such that $(u + vx_{11})h = (u + vx_{11})c_{11} = ua_{11} + vx_{11}b_{11}$. Hence $u(a_{11} - c_{11}) = vx_{11}(c_{11} - b_{11})$. If $c_{11} - b_{11} \neq 0$ then this element, as well as x_{11}, is invertible in $S_1(\mathfrak{J}_{11})$, which implies that $v \in uS_1(\mathfrak{J}_{11})$. Then $vh = va_{11}$. If $c_{11} = b_{11}$ then our relation gives $a_{11} = c_{11}$. Hence $b_{11} = a_{11}$ and again $vh = va_{11}$. Thus we see that this relation holds for every invertible v in $\mathfrak{M}$. Since we saw in the proof of (1) that $\mathfrak{M}$ is spanned by invertible elements, this implies that $h^{\nu} = a_{11}^{\nu}$.

(3) Let $\mathfrak{B}$ be an ideal in $(\mathfrak{D},j)$ such that $\mathfrak{B} \subset \mathfrak{D}$ and let $b \in \mathfrak{B}$. Since $b + \bar{b}$ and $b\bar{b}$ ($\bar{b} = b^j$) are in $\mathfrak{H}(\mathfrak{D},j) = \mathfrak{J}_{11}^{\nu}$ these elements are either invertible or 0 in $\mathfrak{D}$. Since $\mathfrak{B} \subset \mathfrak{D}$ it follows that $\bar{b} = -b$, $b^2 = 0$. It follows from (29) and the homomorphism ν of $(S_1(\mathfrak{J}_{11}),\pi)$ onto $(\mathfrak{D},j)$ that $u\bar{b} = (ub)^{\eta}$. Hence $(ub)^{\eta} = -ub$, which gives $ub.u.u = 0$. Then $ub.u = ubR_{u^{\cdot 3}} = 0$. Since $\mathfrak{B}$ is an ideal we can replace b by db, where d is any element of $\mathfrak{D}$, to obtain $udb.u = 0$. Then $udb.u.e_2 = 0$ which implies, by (34), that $ud.ub.e_2 = 0$. Applying η then gives $u\bar{d}.ub.e_1 = 0$. Since d is arbitrary we have $ud.ub.e_1 = 0$. Hence

$$ub.ud = 0, \qquad b \in \mathfrak{B}, \qquad d \in \mathfrak{D}. \tag{36}$$

Next let $v \in \mathfrak{M}^*$ where the subspace $\mathfrak{M}^*$ is defined in Lemma 2. Then $v^{\eta} = -v$ and $va_{11}{}^{\eta} = va_{11}$, $a_{11} \in \mathfrak{J}_{11}$. Hence, by (30), $va^{\pi} = va^{\eta} = -v^{\eta}a^{\eta} = -(va)^{\eta}$ if $a \in S_1(\mathfrak{J}_{11})$. Hence we have $(vb)^{\eta} = -v\bar{b} = vb$. Then by (32), $vb = ub_{11}$ where $b_{11} \in \mathfrak{J}_{11}$. Since $vb.vb.e_2 = v.vb\bar{b}.e_2 = 0$, vb is not invertible, which implies that $b_{11} = 0$. Hence $vb = 0$ and $ub.v.e_2 = u.v\bar{b}.e_2 = -u.vb.e_2 = 0$. Applying η gives $ub.v.e_1 = 0$ since $(ub)^{\eta} = -ub$ and $v^{\eta} = -v$. Hence

$$ub.v = 0. \tag{37}$$

By Lemma 2 and (36) and (37), $ub.\mathfrak{J}_{12} = 0$. Since $\mathfrak{J}$ is simple we have $ub = 0$. Replacing b by db gives $udb = 0$ and hence $uS_1(\mathfrak{J}_{11})b = 0$. Since we had $vb = 0$ for all $v \in \mathfrak{M}^*$ we obtain $\mathfrak{M}b = 0$ and $b = 0$. Hence $\mathfrak{B} = 0$ and $(\mathfrak{D},j)$ is simple.

Lemma 3 shows that $(\mathfrak{D},j)$ is a simple associative algebra with involution such that every nonzero symmetric element is invertible (since it is contained in $\mathfrak{J}_{11}{}^{\nu}$). Then Theorem 8 gives the structure of $(\mathfrak{D},j)$, where the algebras of octonions are eliminated since they are not associative. Also since $(\mathfrak{D},j)$ is a homomorphic image of $(S_1(\mathfrak{J}_{11}),\pi)$ and the latter algebra is generated by its symmetric elements it is clear that $(\mathfrak{D},j)$ is generated by $\mathfrak{H}(\mathfrak{D},j)$. It follows that $(\mathfrak{D},j)$ can not be a quaternion algebra (over its center) with standard involution. We remark also that if $\mathfrak{D}$ is commutative then $\mathfrak{H}(\mathfrak{D},j)$ is a subalgebra. Hence in this case the condition that $\mathfrak{H}(\mathfrak{D},j)$ generates $\mathfrak{D}$ implies that j is the identity mapping and $\mathfrak{D}$ is a field. Thus we see that the possibilities for $(\mathfrak{D},j)$ are: $(\mathfrak{D},j)$ is a field with $j=1$; $\mathfrak{D}=\Delta\oplus\Delta^{\circ}$, where Δ is a division algebra which is not commutative and j is the exchange involution; $\mathfrak{D}=\Delta$ a division algebra which is not commutative and $(\mathfrak{D},j)$ is not a quaternion algebra over its center with standard involution.[3] In all but the first of these cases it is clear that $\mathfrak{D}$ contains an invertible skew element s. We now observe that this implies that $\mathfrak{M}=u\mathfrak{D}=uS_1(\mathfrak{J}_{11})$. For, let $v\in\mathfrak{M}^*$ where $\mathfrak{M}^*$ is defined in Lemma 2. Then, as in the proof of Lemma 3 (3) (for vb), $(vs)^{\eta}=vs$. Hence, by Lemma 1, $vs\in u\mathfrak{D}$. Then $v=(vs)s^{-1}\in u\mathfrak{D}$ and $\mathfrak{M}^*\subseteq u\mathfrak{D}$ so $\mathfrak{M}=u\mathfrak{D}$.

We can now determine the structure of the simple Jordan algebras of capacity two in the following theorem which is due to Osborn ([6]).

THEOREM 9. *Let $\mathfrak{J}$ be a simple Jordan algebra of capacity two. Then either $\mathfrak{J}$ is isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form f on a vector space over an extension field Γ of the base field Φ, such that $f(x,x)=1$ for some x, or $\mathfrak{J}$ is isomorphic to a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_2,J_a)$ where $(\mathfrak{D},j)$ is either of the form $\Delta\oplus\Delta^{\circ}$, Δ an associative division algebra which is not commutative, j, the exchange involution or $\mathfrak{D}$ is a division algebra which is not commutative and is not a quaternion algebra over its center with j the standard involution.*

PROOF. We assume first that $\mathfrak{M}=\mathfrak{J}_{12}$ contains an element u such that $u^{\cdot 2}=1$. Then $\mathfrak{D}=S(\mathfrak{J}_{11})^{\nu}$ has the involution j such that the symmetric elements are those of $\mathfrak{J}_{11}{}^{\nu}$. We have seen that $(\mathfrak{D},j)$ is either as specified in the statement of the theorem or is a field with j the identity mapping. Assume first that $(\mathfrak{D},j)$ is as stated in the theorem. Then we have seen that $\mathfrak{M}=u\mathfrak{D}$, so $\mathfrak{J}=\mathfrak{J}_{11}\oplus u\mathfrak{D}\oplus\mathfrak{J}_{11}{}^{\eta}$. Hence any element of $\mathfrak{J}$ can be written in one and only one way in the form $a_{11}+ud+b_{11}{}^{\eta}$ where a_{11}, $b_{11}\in\mathfrak{J}_{11}$ and $d\in\mathfrak{D}$. Moreover, by (31), d is uniquely determined. We have the following multiplication formulas:

$$a_{11}.ud=\tfrac{1}{2}uda_{11}{}^{\nu} \tag{38}$$

$$a_{11}{}^{\eta}.ud=\tfrac{1}{2}ua_{11}{}^{\nu}d, \tag{39}$$

$$ub.ud.e_1=\tfrac{1}{2}c_{11},\qquad c_{11}{}^{\nu}=\bar{b}d+\bar{d}b, \tag{40}$$

$$ub.ud.e_2=\tfrac{1}{2}d_{11}{}^{\eta},\qquad d_{11}{}^{\nu}=b\bar{d}+d\bar{b}. \tag{41}$$

Here a_{11}, c_{11}, $d_{11}\in\mathfrak{J}_{11}$, $b,d\in\mathfrak{D}$. The first is clear from the definition of $a_{11}{}^{\nu}=V_{a_{11}}$. For the second, we have $a_{11}{}^{\eta}.ud=(a_{11}.u\bar{d})^{\eta}=\tfrac{1}{2}(u\bar{d}a_{11}{}^{\nu})^{\eta}=\tfrac{1}{2}ua_{11}{}^{\nu}d$.

[3] It is easily seen from the Cartan-Brauer-Hua theorems on automorphisms and derivations (Jacobson, *Structure of Rings*, pp. 186–187) that, conversely, in all the cases indicated $\mathfrak{D}$ is generated by $\mathfrak{H}(\mathfrak{D},j)$.

For (40), $ub.ud.e_1 = (u\bar{b}.u\bar{d}.e_2)^\eta = (u\bar{b}d.u.e_2)^\eta = (u.u\bar{d}b.e_2)^\eta = \frac{1}{2}(u.u(\bar{b}d+\bar{d}b).e_2)^\eta = \frac{1}{2}(u.u(\bar{b}d+\bar{d}b).e_1) = \frac{1}{2}u.uc_{11}{}^\nu.e_1 = u.c_{11}.u.e_1 = \frac{1}{2}c_{11}$(PD5). The last relation is obtained in a similar fashion. These formulas and the fact that $a_{11}\to a_{11}{}^\nu$ is an isomorphism of $\mathfrak{J}_{11}$ onto $\mathfrak{H}(\mathfrak{D},j)$ imply that the mapping

$$a_{11}+ud+b_{11}{}^\eta \to \begin{pmatrix} a_{11}{}^\nu & \bar{d} \\ d & b_{11}{}^\nu \end{pmatrix}$$

is an isomorphism of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_2,J_1)$. If $\mathfrak{J}$ does not contain an element $u\in\mathfrak{J}_{12}$ such that $u^{\cdot 2}=1$ then the isotope $\tilde{\mathfrak{J}}$ determined by any invertible $u\in\mathfrak{J}_{12}$ has this property, so $\tilde{\mathfrak{J}}\cong\mathfrak{H}(\mathfrak{D}_2,J_1)$. As in the proof of the Coordinatization Theorem (p. 137), this implies that $\mathfrak{J}\cong\mathfrak{H}(\mathfrak{D}_2,J_a)$ for some canonical involution J_a.

Now assume $\mathfrak{D}$ is a field, $j=1$. Then $\mathfrak{J}_{11}{}^\nu=\mathfrak{H}(\mathfrak{D},j)=\mathfrak{D}$ and $\mathfrak{J}_{11}$ is an associative Jordan algebra. Again assume $\mathfrak{J}_{12}$ contains u such that $u^{\cdot 2}=1$. If $a_{11}\in\mathfrak{J}_{11}$, $a\in S_1(\mathfrak{J}_{11})$ then $uaa_{11}=ua_{11}a$ (by commutativity of $\mathfrak{D}$) $=ua_{11}{}^\eta a = uaa_{11}{}^\eta$. Since $va_{11}=va_{11}{}^\eta$, $v\in\mathfrak{M}^*$, as defined in Lemma 2 we have $wa_{11}=wa_{11}{}^\eta$ $w\in\mathfrak{M}$. Since $\mathfrak{J}_{11}{}^\eta=\mathfrak{J}_{22}$, $\mathfrak{J}_{22}{}^\eta=\mathfrak{J}_{11}$ we can write $\mathfrak{J}_{11}+\mathfrak{J}_{22}=\Gamma\oplus\Sigma$ where $\Gamma=\{a_{11}+a_{11}{}^\eta\,|\,a_{11}\in\mathfrak{J}_{11}\}$ and $\Sigma=\{a_{11}-a_{11}{}^\eta\,|\,a_{11}\in\mathfrak{J}_{11}\}$ are the sets of symmetric and skew elements of $\mathfrak{J}_{11}+\mathfrak{J}_{22}$ under η. Then $a_{11}\to a_{11}+a_{11}{}^\eta$ is an isomorphism of $\mathfrak{J}_{11}$ onto Γ. We claim that Γ is contained in the center of $\mathfrak{J}$. The nontrivial verifications required to establish this are: (1) $[b_{11},x_{12},a_{11}+a_{11}{}^\eta] = [b_{11},x_{12},a_{11}]=0$, since $\mathfrak{D}$ is commutative, $[b_{22},x_{12},a_{11}+a_{11}{}^\eta]=0$ by symmetry; (2) $[b_{12},x_{11},a_{11}+a_{11}{}^\eta] = b_{12}.x_{11}.a_{11} - b_{12}.(x_{11}.a_{11}) + b_{12}.x_{11}.a_{11}{}^\eta = b_{12}.x_{11}.a_{11} - b_{12}.(x_{11}.a_{11}) + b_{12}.x_{11}.a_{11} = 0$, by (PD3) and the commutativity of $\mathfrak{D}$, $[b_{12},x_{22},a_{11}+a_{11}{}^\eta]=0$, by symmetry; (3) $[b_{12},x_{12},a_{11}+a_{11}{}^\eta] = (e_1-e_2).(b_{12}.a_{11}.x_{12})+(e_2-e_1).(b_{12}.a_{11}{}^\eta.x_{12})$ (by (PD4)) $=0$. Here the notations are as usual. Since Γ is an associative field containing 1 and contained in the center, we can consider $\mathfrak{J}$ as algebra over Γ. We have $\mathfrak{J}=\Gamma\oplus\Sigma\oplus\mathfrak{M}$. If $x_{12}\in\mathfrak{M}$ then $(x_{12}{}^{\cdot 2})^\eta.e_2.x_{12}=(x_{12}{}^{\cdot 2}.e_1)^\eta.x_{12} = x_{12}{}^{\cdot 2}.e_1.x_{12}=x_{12}{}^{\cdot 2}.e_2.x_{12}$, by (PD1). Hence $(x_{12}{}^{\cdot 2})^\eta.e_2=x_{12}{}^{\cdot 2}.e_2$ and, similarly, $(x_{12}{}^{\cdot 2}).e_1=x_{12}{}^{\cdot 2}.e_1$. Thus $(x_{12}{}^{\cdot 2})^\eta=x_{12}{}^{\cdot 2}$ so $x_{12}{}^{\cdot 2}\in\Gamma$. This and $\Sigma.\mathfrak{M}=0$ imply that $\mathfrak{J}/\Gamma$ is the Jordan algebra of a quadratic form f on the vector space $\mathfrak{N}=\Sigma+\mathfrak{M}$. The nondegeneracy of f is clear from the simplicity of $\mathfrak{J}$. Since $(e_1-e_2)^{\cdot 2}=1$ we have $f(e_1-e_2)=1$. This completes the proof assuming the existence of a $u\in\mathfrak{J}_{12}$ such that $u^{\cdot 2}=1$. The general case can be reduced to this via isotopy (see ex. 5, p. 62).

It is easy to verify that, conversely, any algebra of the type specified in the theorem is a simple Jordan algebra of capacity two. We leave the verification of this to the reader.

EXERCISE

1. Show that $\mathfrak{H}(\mathfrak{D}_2,J_a)$ is isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form if $(\mathfrak{D},j)$ is any one of the following: $\mathfrak{D}=\Delta+\Delta^\circ$, Δ com-

mutative, j, the exchange involution; $\mathfrak{D}$ a field; $(\mathfrak{D},j)$ a quaternion algebra over its center, j, standard.

5. **Second structure theorem.** We recall that an associative algebra is called (right) *Artinian* if it satisfies the minimum condition for right ideals. One has the classical Wedderburn-Artin structure theorem: An associative algebra $\mathfrak{A}$ is simple Artinian if and only if $\mathfrak{A} \cong \Delta_n$ where Δ is a division algebra or, equivalently, $\mathfrak{A} \cong \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ where $\mathfrak{M}$ is a finite-dimensional vector space over a division algebra Δ. We now define a *simple Artinian algebra with involution* $(\mathfrak{A},J)$ to be a simple associative algebra with involution such that $\mathfrak{A}$ is Artinian. Since a simple algebra with involution either has the form $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is simple, or $(\mathfrak{A},J)$ where $\mathfrak{A}$ is simple it follows that the simple Artinian algebras with involution are the algebras $\Delta_n \oplus \Delta_n{}^\circ$, J the exchange involution, or the algebras Δ_n with J any involution. In both cases, Δ is a division algebra and Δ_n can be replaced by $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$, $\mathfrak{M}$ an n-dimensional vector space over Δ. The form of the involutions in the algebra $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ is well known (see, for example, the author's *Lectures in Abstract Algebra.* II, p. 270). The result is the following: Either $\Delta = \Gamma$ a field (extension field of Φ), $\mathfrak{M}$ is even dimensional over Γ, and J is the adjoint (or transpose) mapping in $\mathrm{Hom}_\Gamma(\mathfrak{M},\mathfrak{M})$ relative to a nondegenerate skew symmetric (alternate) bilinear form f in $\mathfrak{M}/\Gamma$; or Δ has an involution j, and J is the adjoint mapping on $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ relative to a nondegenerate hermitian form in $\mathfrak{M}/\Delta$ associated with j $(d \to \bar{d})$. In the first case, f is bilinear and $f(x,x) = 0$, $x \in \mathfrak{M}/\Gamma$. In the second case, $f(x_1 + x_2, y) = f(x_1,y) + f(x_2,y)$, $f(x, y_1 + y_2) = f(x,y_1) + f(x,y_2)$, $f(dx,y) = df(x,y)$, $f(x,dy) = f(x,y)\bar{d}$, $d \in \Delta$. In both cases, $f(x,y) \in \Delta$ ($= \Gamma$ in the first case) and nondegeneracy is defined as usual. If f is skew there exists a basis $(u_1,u_2,\cdots,u_{2n})$ (where $2n = \dim\mathfrak{M}$) for $\mathfrak{M}/\Gamma$ such that the matrix $(f(u_i,u_j))$ of f relative to this basis has the form

$$S = \mathrm{diag}\{Q,Q,\cdots,Q\}, \quad Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{42}$$

If $l \in \mathrm{Hom}_\Gamma(\mathfrak{M},\mathfrak{M})$ has the matrix L relative to the basis (u_i) then the adjoint l^* relative to f $(f(xl,y) = f(x,yl^*))$ has the matrix $S^{-1}L^tS$ relative to this basis. It follows that if J is the adjoint mapping $l \to l^*$ then $(\mathrm{Hom}_\Gamma(\mathfrak{M},\mathfrak{M}), J)$ is isomorphic to (Γ_{2n}, J_s) where S is the involution $L \to S^{-1}L^tS$ in Γ_{2n}. We have seen also that (Γ_{2n}, J_s) is isomorphic to $(\mathfrak{D}_n, J_1)$ where $\mathfrak{D} = \Gamma_2$ the split quaternion algebra over Γ, involution standard and J_1 is the standard involution in $\mathfrak{D}_n$ (p. 128).

If f is a nondegenerate hermitian form on $\mathfrak{M}/\Delta$ where (Δ,j) is a division algebra with involution, then there exists a basis $(u_1,u_2,\cdots,u_n)$ for $\mathfrak{M}/\Delta$ such that the matrix of f relative to this basis is

$$a = \mathrm{diag}\{a_1,a_2,\cdots,a_n\}, \quad \bar{a}_i = a_i. \tag{43}$$

Then the adjoint l^* of $l \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ relative to f has the matrix $a^{-1}\bar{L}^t a$ relative to (u_i), where L is the matrix of l. Hence if J is the involution $l \to l^*$ then $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ is isomorphic to (Δ_n, J_a) where J_a is the canonical involution defined by a.

It is well known that any left ideal in $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})(\dim \mathfrak{M} = n < \infty)$ is a principal left ideal $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})e$ generated by an idempotent element e (Jacobson, *Lectures*. II, ex. 4, p. 232). This implies another well-known fact, namely $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ is regular in the sense of von Neumann, which means that for any $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ there exists an x in $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ such that $axa = a$. To see this we consider the left ideal $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})a = \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})e$. Then $a = be$, $b \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$, so $a = ae$. Also $e = xa$, $x \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. Hence $a = ae = axa$. If $\mathfrak{A}$ is any associative algebra then $axa = xU_a$ in $\mathfrak{A}^+$. This leads us to call an arbitrary Jordan algebra $\mathfrak{J}$ *regular* if $a \in \mathfrak{J}U_a$ for any $a \in \mathfrak{J}$. Thus if $\mathfrak{A}$ is associative then $\mathfrak{A}$ is regular if and only if the Jordan algebra $\mathfrak{A}^+$ is regular, and this is the case for $\mathfrak{A} = \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. It is clear that a regular Jordan algebra is nondegenerate, that is it contains no absolute zero divisors $\neq 0$.

We are now ready to prove our second main structure theorem on Jordan algebras with minimum conditions on quadratic ideals.

Second Structure Theorem. *The following conditions on a Jordan algebra $\mathfrak{J}$ are equivalent:* (α) *$\mathfrak{J}$ is a simple algebra satisfying the axioms* (i)–(iii) (*of* §2); (β) *$\mathfrak{J}$ is either a division algebra, a Jordan algebra of a nondegenerate symmetric bilinear form in a vector space $\mathfrak{M}$ over an extension field Γ with* $\dim \mathfrak{M}/\Gamma > 1$, *a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ where $n \geqq 2$ and $(\mathfrak{D}, j)$ is either $\Delta \oplus \Delta^\circ$ where Δ is an associative division algebra, j the exchange involution, an associative division algebra with involution, a split quaternion algebra over an extension field, standard involution, an algebra of octonions over an extension field, standard involution* (*only if* $n = 3$); (γ) *$\mathfrak{J}$ is either a division algebra, a Jordan algebra of a nondegenerate symmetric bilinear form in a vector space $\mathfrak{M}/\Gamma$, Γ an extension field,* $\dim \mathfrak{M}/\Gamma > 1$, *a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ where $\mathfrak{D}$ is an octonion algebra over an extension field with standard involution and J_γ is a canonical involution, or an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is simple Artinian with involution.*

Proof. $(\alpha) \Rightarrow (\beta)$. Let $1 = \sum_1^n e_i$ where the e_i are completely primitive idempotents and let $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$ be the the corresponding Peirce decomposition (Theorem 3). If $n = 1$, $\mathfrak{J}$ is a division algebra and if $n = 2$, (β) follows from Theorem 9. Hence assume $n \geqq 3$. Let $i \neq j$ and assume $\mathfrak{J}_{ij} \neq 0$. By Theorem 4 and the fact that $\mathfrak{J}U_{e_i,e_j} = \mathfrak{J}_{ii} + \mathfrak{J}_{ij} + \mathfrak{J}_{jj}$ satisfies the axioms (i)–(iii), we see that $\mathfrak{J}_{ij}{}^{\cdot 2} \neq 0$ so e_i and e_j are connected (Lemma 1 of §2). Thus either e_i and e_j are connected or $\mathfrak{J}_{ij} = 0$. Let $e_2, e_3, \cdots, e_m$ be the complete set of e_i connected with e_1 and put $\mathfrak{J}' = \sum_{k,l=1}^m \mathfrak{J}_{kl}$. Since connectedness is a transitive relation (Lemma 4 of §3.1) every $\mathfrak{J}_{ki}$ with $k \leqq m$ and $i > m$ is 0. This implies that $\mathfrak{J}'$ is an ideal. Hence $\mathfrak{J}' = \mathfrak{J}$ so all the e_i are connected. Then, by the Coordinatization Theorem (§3.3), $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_n, J_a)$. Also the proof of the Coordinatization Theorem (p. 137) shows that we may assume $a_1 = 1$ for the diagonal matrix $a = \mathrm{diag}\{a_1, a_2, \cdots, a_n\}$. For the sake of simplicity we identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{D}_n, J_a)$. Then $\mathfrak{J}_{ij} = \{d[ij] \mid d \in \mathfrak{D}\}$ where $d[ij] = de_{ij} + a_j^{-1} \bar{d} a_i e_{ji}$ ((17) on p. 125) where e_{ij} are the usual matrix units. Since $a_1 = 1$, $\mathfrak{J}_{11} \cong \mathfrak{H}(\mathfrak{D}, j)$ where $(\mathfrak{D}, j)$ is the coordinate algebra with involution. Since $\mathfrak{J}$ is simple $(\mathfrak{D}, j)$ is simple (Theorem 3.2). We know also that $\mathfrak{D}$ is associative if $n \geqq 4$ and is alternative with $\mathfrak{H}(\mathfrak{D}, j) \subseteq N(\mathfrak{D})$ if $n = 3$. Since $\mathfrak{H}(\mathfrak{D}, j) \cong \mathfrak{J}_{11}$ which is a Jordan division algebra and $\mathfrak{H}(\mathfrak{D}, j) \subseteq N(\mathfrak{D})$, it follows that every nonzero element of $\mathfrak{H}(\mathfrak{D}, j)$ has an inverse in $N(\mathfrak{D})$. Thus $(\mathfrak{D}, j)$ satisfies the hypotheses of Theorem 8 and, moreover, $\mathfrak{D}$ is associative if $n \geqq 4$. Hence, the statements made about $(\mathfrak{D}, j)$ in (β) are clear from Theorem 8.

$(\beta) \Rightarrow (\gamma)$. The assertion here is that associative algebras with involution $(\mathfrak{D}_n, J_a)$ specified in (β) are simple Artinian. If $\mathfrak{D} = \Delta \oplus \Delta^\circ$, j the exchange involution, then $\mathfrak{D}_n = \Delta_n \oplus (\Delta^\circ)_n$ and the two components are simple by the Wedderburn-Artin theorem. Also it is immediate that J_a exchanges the two simple components. Hence $(\mathfrak{D}_n, J_a)$ is simple. Since $\Delta_n \oplus (\Delta^\circ)_n$ is Artinian the result holds in this case. In the remaining cases, $\mathfrak{D}_n$ is simple Artinian.

$(\gamma) \Rightarrow (\alpha)$. It is clear that any Jordan division algebra is simple and satisfies the axioms. Next let $\mathfrak{J}$ be the Jordan algebra $\Gamma 1 \oplus \mathfrak{M}$ of a nondegenerate symmetric bilinear form f on $\mathfrak{M}/\Gamma$ where Γ/Φ is a field and $\dim \mathfrak{M}/\Gamma > 1$. The simplicity of $\mathfrak{J}$ (as algebra over Φ) is immediate (cf. ex. 4, p. 14). It is immediate also that $\mathfrak{J}$ is a division algebra if and only if $f(x, x)$ is not a square for every $x \neq 0$ in $\mathfrak{M}$ (ex. 6, p. 14). Now assume this condition does not hold. Then we have $u \in \mathfrak{M}$ such that $f(u, u) = 1$. Put $e_1 = \frac{1}{2}(1 + u)$, $e_2 = \frac{1}{2}(1 - u)$. Then the e_i are orthogonal idempotents and $e_1 + e_2 = 1$. The conditions that $a = \alpha 1 + x$, $\alpha \in \Gamma$, $x \in \mathfrak{M}$, is in $\mathfrak{J}_{11} = \mathfrak{J}_1(e_1)$ are $\frac{1}{2}\alpha + \frac{1}{2} f(x, u) = \alpha$, $\frac{1}{2} x + \frac{1}{2} \alpha u = x$. These give $x = \alpha u$, $a = 2\alpha e_1$. Hence $\mathfrak{J}_{11} = \Gamma e_1$ and similarly $\mathfrak{J}_{22} = \Gamma e_2$ so the e_i are completely primitive and so $\mathfrak{J}$ is simple of capacity two. Hence $\mathfrak{J}$ is nondegenerate (Theorem 4). Since $\mathfrak{J}$ contains 1 it remains to show that $\mathfrak{J}$ satisfies the minimum conditions. The computation just used for e_1 shows that if e is any idempotent $\neq 0, 1$ in $\mathfrak{J}$ then $\mathfrak{J} U_e \cong \Gamma e$ is a division algebra. Hence $\mathfrak{J} U_e$ is a minimal quadratic ideal and the conditions in axiom (iii) are clear.

We consider next the exceptional Jordan algebra $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $\mathfrak{O}$ is an algebra of octonions over an extension field Γ/Φ, j is the standard involution in $\mathfrak{O}$ and J_γ is the canonical involution in $\mathfrak{O}_3$ defined by $\gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_i \neq 0$ in Γ. Clearly, $\mathfrak{O}$ and $\mathfrak{J}$ can be considered as algebras over Γ. We now note that if $\mathfrak{J}$ is any Jordan algebra over a field Γ and $\mathfrak{J}/\Gamma$ satisfies axioms (i)–(iii) then $\mathfrak{J}/\Phi$ satisfies these axioms for any subfield Φ of Γ. This is obvious for (i) and (ii). It is clear also that the sets $\mathfrak{J} U_a$, $a \in \mathfrak{J}$, are the principal quadratic ideals of $\mathfrak{J}/\Phi$ as well as of $\mathfrak{J}/\Gamma$. Hence if the minimum condition holds for the principal quadratic ideals of $\mathfrak{J}/\Gamma$ of the form $\mathfrak{J}U_e$, $e^{\cdot 2} = e$, then this condition holds also in $\mathfrak{J}/\Phi$. Next assume $\mathfrak{J}$ is nondegenerate. Then Theorem 1 implies that a subspace $\mathfrak{B}/\Phi(\mathfrak{B}/\Gamma)$ of $\mathfrak{J}/\Phi(\mathfrak{J}/\Gamma)$ is a minimal quadratic ideal if and only if $\mathfrak{J}U_b = \mathfrak{B}$ for every $b \neq 0$ in $\mathfrak{B}$. This implies that $\mathfrak{J}/\Phi$ and $\mathfrak{J}/\Gamma$ have the same minimal quadratic ideals, so the validity of (iii) for $\mathfrak{J}/\Gamma$ implies (iii) for $\mathfrak{J}/\Phi$. We observe next that since quadratic ideals are subspaces, it is clear that any finite-dimensional algebra satisfies axiom (iii). In particular, $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, which is 27 dimensional over Γ, satisfies (iii) as algebra over Γ and hence as algebra over Φ. It is clear also that this $\mathfrak{J}$ has an identity element so (i) holds. It remains to prove nondegeneracy. We use the usual notation: $x[ij] = xe_{ij} + \gamma_j^{-1}\gamma_i\bar{x}e_{ji}$. Let $z = \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3 + u[12] + v[23] + w[31]$, $\zeta_i \in \Gamma$, $u, v, w \in \mathfrak{O}$ be an absolute zero divisor in $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. Then $zU_{e_i} = \zeta_i e_i$ is an absolute zero divisor in $\mathfrak{J}U_{e_i} = \Gamma e_i$, so $\zeta_i = 0$. Now $zU_{1[12]} = u[12]U_{1[12]}$ since $v[23]U_{1[12]} = 0 = w[31]U_{1[12]}$, by (PD10). Also $u[12]U_{1[12]} = 1[12]u[12]1[12] = \gamma_2^{-1}\gamma_1\bar{u}[12]$. Hence $\bar{u}[12]$ and consequently $u[12]$ and $u[21]$ are absolute zero divisors. By (2), $(x.u[12])^{\cdot 2} = 0$ for every $x \in \mathfrak{J}$ (since $u[12]^{\cdot 2} = 1U_{u[12]} = 0$ and $U_{u[12]} = 0$). In particular, $u[12].\bar{d}[21] = \frac{1}{2}(u\bar{d} + d\bar{u})e_{11} + \frac{1}{2}(\bar{u}d + \bar{d}u)e_{22}$ is nilpotent. Hence $u\bar{d} + d\bar{u} = 0$, $d \in \mathfrak{D}$. This implies $u = 0$ by the nondegeneracy of $Q(a, b)$. Similarly, $v = 0$ and $w = 0$. Hence $z = 0$ is the only absolute zero divisor in $\mathfrak{H}(\mathfrak{O}_3, I_\gamma)$ so this is a nondegenerate algebra. Since $(\mathfrak{O}, j)/\Phi$ is simple by Theorem 8, $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ is simple by Theorem 3.2. Hence $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ is a simple algebra satisfying the axioms.

Now assume $(\mathfrak{A}, J)$ is a simple Artinian algebra with involution. If $\mathfrak{A}$ is not simple then $\mathfrak{A} \cong \Delta_n \oplus \Delta_n{}^\circ$ where Δ is an associative division algebra. Then $\mathfrak{H}(\mathfrak{A}, J) \cong \Delta_n{}^+ \cong \mathrm{Hom}_\Delta(\mathfrak{M}, \mathfrak{M})^+$ where $\mathfrak{M}$ is an n-dimensional vector space over Δ. Clearly, axiom (i) holds and we have seen that $\mathfrak{J} = \mathrm{Hom}_\Delta(\mathfrak{M}, \mathfrak{M})^+$ is regular. Hence the nondegeneracy condition (ii) holds for $\mathfrak{J}$. Now let e be an idempotent in $\mathfrak{J} = \mathrm{Hom}_\Delta(\mathfrak{M}, \mathfrak{M})^+$. Then e is a projection on $\mathfrak{M}e$, that is, e maps $\mathfrak{M}$ into $\mathfrak{M}e$ and is the identity on $\mathfrak{M}e$. If f is a second idempotent in $\mathfrak{J}$ and $\mathfrak{J}U_f \subseteq \mathfrak{J}U_e$ then $f = efe$ and $ef = f = fe$. Hence $\mathfrak{M}f \subseteq \mathfrak{M}e$. Moreover, if $\mathfrak{M}f = \mathfrak{M}e$, then for any $x \in \mathfrak{M}$, $xe = yf$, $y \in \mathfrak{M}$. Then $xef = yf^2 = yf = xe$. Hence $ef = e$ and since $ef = f$ we have $e = f$. Thus $\mathfrak{J}U_e \supset \mathfrak{J}U_f$ implies $\mathfrak{M}e \supset \mathfrak{M}f$. This and the finiteness of dimensionality of $\mathfrak{M}/\Delta$ imply the minimum condition for quadratic ideal of the form $\mathfrak{J}U_e, e^{\cdot 2} = e$. We note next that if e is an idempotent then $e\mathrm{Hom}_\Delta(\mathfrak{M}, \mathfrak{M})e \cong \mathrm{Hom}_\Delta(\mathfrak{M}e, \mathfrak{M}e)$. This is well known and is immediate from the decomposition $\mathfrak{M} = \mathfrak{M}e \oplus \mathfrak{M}(1 - e)$. If $\mathfrak{M}e$ is one dimensional then $\mathrm{Hom}_\Delta(\mathfrak{M}e, \mathfrak{M}e) \cong \Delta$. Then $\mathfrak{J}U_e \cong \Delta^+$ which is a Jordan division algebra and $\mathfrak{J}U_e$ is a minimal quadratic ideal in $\mathfrak{J}$. Now let e be any nonzero idempotent in $\mathfrak{J}$ and let $\mathfrak{N}$ be a one-dimensional subspace of $\mathfrak{M}e$. Let $\mathfrak{N}'$ be a complementary subspace of $\mathfrak{N}$ in $\mathfrak{M}$ containing $\mathfrak{M}(1 - e)$ and let f be the projection on $\mathfrak{N}$ determined by the decomposition $\mathfrak{M} = \mathfrak{N} \oplus \mathfrak{N}'$. Then $\mathfrak{N} = \mathfrak{M}f \subseteq \mathfrak{M}e$ so $fe = f$. Since $\mathfrak{N}' = \mathfrak{M}(1 - f) \supseteq \mathfrak{M}(1 - e)$ we have $(1 - e)(1 - f) = 1 - e$ which implies $ef = f$. Hence $efe = f$ and $\mathfrak{J}U_f \subseteq \mathfrak{J}U_e$. Since f is a projection on the one-dimensional $\mathfrak{N}$, $\mathfrak{J}U_f$ is a division algebra so

$\mathfrak{J}U_f$ is minimal. Hence we have shown that any quadratic ideal of the form $\mathfrak{J}U_e$, $e^{\cdot 2} = e$, contains a minimal quadratic ideal. This completes the verification of the axioms (i)–(iii) for $\mathfrak{J}$. If $\dim\mathfrak{M} = 1$, $\mathfrak{J} = \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ is a division algebra. If $\dim\mathfrak{M} = 2$ one sees easily that the capacity is two and the simplicity follows from Theorem 4. If $\dim\mathfrak{M} \geqq 3$ we use the isomorphism $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_n, J_a)$ and the simplicity of $(\mathfrak{D},j)$ to conclude simplicity of $\mathfrak{J}$.

The remaining simple Artinian algebras with involution have the form $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ where $\mathfrak{M}$ and Δ are as before and J is the adjoint mapping relative to a nondegenerate hermitian or skew symmetric bilinear form f. The simplicity follows as in the case just considered and, of course, (i) holds. To prove (ii) we prove $\mathfrak{J} = \mathfrak{H}(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ is regular. Thus let $a \in \mathfrak{J}$. Then, by the regularity of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$, there exists an $x \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ such that $axa = a$. Since $a^J = a$, we can replace x by $y = \frac{1}{2}(x + x^J)$ in this equation. This shows that $\mathfrak{J}$ is regular. We note next that the mapping $e \to \mathfrak{M}e$ is bijective of the set $\mathfrak{P}$ of idempotents of $\mathfrak{J} = \mathfrak{H}(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ with the set of nonisotropic subspaces of $\mathfrak{M}$. For, if $e \in \mathfrak{P}$ then $\mathfrak{M} = \mathfrak{M}e \oplus \mathfrak{M}(1 - e)$ and $\mathfrak{M}(1 - e) = (\mathfrak{M}e)^\perp$ which implies that $\mathfrak{M}e$ is not isotropic. Conversely, if $\mathfrak{N}$ is not isotropic then $\mathfrak{M} = \mathfrak{N} \oplus \mathfrak{N}^\perp$ and the projection e determined by this decomposition is selfadjoint since $f(xe, y) = f(x, ye)$ holds if x and y are either in $\mathfrak{N}$ or in $\mathfrak{N}^\perp$. It is clear also that $e \to \mathfrak{M}e$, $\mathfrak{N} \to e$ are inverses. If e and $f \in \mathfrak{P}$ and $\mathfrak{M}e \supseteq \mathfrak{M}f$ then $fe = f$ and $ef = (fe)^J = f^J = f$. Hence $f = efe$ and $\mathfrak{J}U_e \supseteq \mathfrak{J}U_f$. The converse follows by retracing the steps. It is now clear that we have the minimum condition for quadratic ideals $\mathfrak{J}U_e$, $e \in \mathfrak{P}$. The nonzero minimal elements of the set of these quadratic ideals correspond to the minimal nonisotropic subspaces $\neq 0$ of $\mathfrak{M}$. If the form f is hermitian these are one dimensional and if f is skew then any nonzero minimal nonisotropic subspace is two dimensional and has a basis (u,v) such that $f(u,u) = 0 = f(v,v)$, $f(u,v) = 1 = -f(v,u)$. We claim that in either case, if e is the corresponding projection then $\mathfrak{J}U_e$ is a division algebra so $\mathfrak{J}U_e$ is a minimal quadratic ideal. We note first that $\mathfrak{J}U_e = \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})U_e \cap \mathfrak{J}$. If f is hermitian $\mathfrak{M}e$ is one dimensional and, as we saw above, $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})U_e$ is a division algebra. Then if $a \neq 0$ is in $\mathfrak{J}U_e$ there exists a $b \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})U_e$ such that $a \,.\, b = e$, $a^{\cdot 2} \,.\, b = a$. Applying J gives $a \,.\, b^J = e$, $a^{\cdot 2} \,.\, b^J = a$. Hence $b = b^J \in \mathfrak{J}$ by the uniqueness of the inverse. Thus $b \in \mathfrak{J}U_e = \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})U_e \cap \mathfrak{J}$ and $\mathfrak{J}U_e$ is a division algebra. Next assume f skew, so $\Delta = \Gamma$ is a field and $\mathfrak{M}e$ has a basis (u,v) such that $f(u,u) = 0 = f(v,v)$, $f(u,v) = 1 = -f(v,u)$. If $a \in \mathfrak{J}U_e$ then $a = eae$ and $\mathfrak{N}a \subseteq \mathfrak{N}$ for $\mathfrak{N} = \mathfrak{M}e$. Also since $\mathfrak{N}^\perp = \mathfrak{M}(1 - e)$ it is clear that $\mathfrak{N}^\perp a = 0$. We have $ua = \alpha u + \beta v$, $va = \gamma a + \delta v$ where $\alpha, \beta, \gamma, \delta \in \Gamma$. Since a is selfadjoint relative to f, $f(ua,u) = f(u,ua) = -f(ua,u) = 0$, so $f(ua,u) = 0$. Similarly, $f(va,v) = 0$ and $f(ua,v) = f(u,va)$. These relations imply that $\beta = 0 = \gamma, \alpha = \delta$. Then $a - \alpha e$ annihilates $\mathfrak{N}$ as well as $\mathfrak{N}^\perp$. Hence $a = \alpha e$ and $\mathfrak{J}U_e = \Gamma e$, which is a division algebra. Since any nonisotropic subspace $\neq 0$ contains a minimal one, it is now clear that in both cases any quadratic ideal of the form

$\mathfrak{J}U_e$, $e^{\cdot 2} = e$, contains a minimal quadratic ideal. This completes the verification of the axioms in this case and of the implication $(\gamma) \Rightarrow (\alpha)$.

The two structure theorems reduce the determination of the algebras satisfying the axioms to the determination of the Jordan division algebras. If $\mathfrak{J}$ is a Jordan algebra of a symmetric bilinear form f such that $f(x,x) \neq 1$ for all $x \in \mathfrak{M}$ then $\mathfrak{J}$ is a division algebra. Also the Jordan algebras Δ^+ where Δ is an associative division algebra and the algebras $\mathfrak{H}(\Delta, j)$ where (Δ, j) is an associative division algebra with involution are Jordan division algebras. Finally, Albert has constructed some examples of exceptional Jordan division algebras. These are all "forms" of the exceptional algebra $\mathfrak{H}(\mathfrak{O}_3, J_1)$ in the sense that they become this algebra on extension of the center (which is a field) to its algebraic closure. We shall see in the next chapter that every finite-dimensional Jordan division algebra is of one of the types just indicated. However, it seems likely that other types of examples of Jordan division algebras of infinite dimensionalities over their centers exist. (See the notes on Chapter I in "Further results and Open Question".)

Exercises

1. Assuming that the Jordan algebra of a quadratic form is special (to be proved in §7.1) show that any exceptional simple Jordan algebra satisfying the axioms is either a division algebra or is isomorphic to an algebra $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{O}$ an algebra of octonions.

2. Use the Second Structure Theorem to prove that if $\mathfrak{J}$ is a simple Jordan algebra satisfying the axioms and $\mathfrak{J}$ is of capacity n, then for any decomposition of $1 = \sum_1^m f_j$, where the f_j are orthogonal completely primitive idempotents one has $m = n$.

6. **Special universal envelopes. Isomorphisms and derivations of special simple Jordan algebras satisfying the axioms.** We shall now consider the problem of classifying the simple algebras which we determined in the last section. We pass over the cases of division algebras and the exceptional algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. The problem for finite-dimensional special division algebras will be considered in the next chapter and the classification problem for the algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ will be considered in Chapter IX. There remain the Jordan algebras of symmetric bilinear forms and the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is simple Artinian with involution and the capacity $n \geqq 2$. The problem of classification of Jordan algebras of symmetric bilinear forms can be reduced immediately to that of equivalence of forms (ex. 5, p. 14). We have nothing to add to the well-known results on these so we shall pass to the final case of interest: the Jordan algebras $\mathfrak{H}(\mathfrak{A}, J)$ of capacity $\geqq 2$ where $(\mathfrak{A}, J)$ is simple Artinian with involution.

We consider the algebras $(\mathfrak{A}, J)$ first in their matrix form: $(\mathfrak{D}_n, J_a)$, where $n \geqq 2$, $(\mathfrak{D}, j)$ is either an associative division algebra with involution or a direct sum of an associative division algebra and its opposite with exchange involution, and J_a is a canonical involution in $\mathfrak{D}_n$. If $n = 2$ the definition of $(\mathfrak{D},j)$ in the proof of Osborn's theorem (p. 176) shows that we may assume $\mathfrak{D}$ is generated by $\mathfrak{H}(\mathfrak{D},j)$ ($= S_1(\mathfrak{J}_{11})^\nu$). We shall therefore suppose this is the case if $n = 2$. Then the hypotheses of Corollary 1 to Martindale's Theorem (p. 141) are fulfilled and so the associative algebra with involution $(\mathfrak{D}_n, J_a)$ is perfect. This means that $\mathfrak{D}_n$ and the injection mapping constitute a unital special universal envelope $(S_1(\mathfrak{H}(\mathfrak{D}_n, J_a), \sigma_u)$ for the Jordan algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$. Thus any homomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_a)$ into $\mathfrak{G}^+$ where $\mathfrak{G}$ is an associative algebra with 1 has a unique extension to a homomorphism of $\mathfrak{D}_n$ into $\mathfrak{G}$ such that $1 \to 1$. It follows from Theorem 2.1 (4) (the functorial character of S_1) that any isomorphism between two of the Jordan algebras $\mathfrak{H}(\mathfrak{D}_n, J_a)$ has a unique extension to an isomorphism of the corresponding algebras with involution $(\mathfrak{D}_n, J_a)$. In this way the classification problem is reduced to that of the simple algebras with involution $(\mathfrak{D}_n, J_a)$. Since it is clear that one of these algebras with involution in which $\mathfrak{D}_n$ is not simple cannot be isomorphic with any in which $\mathfrak{D}_n$ is simple, we can separate the two cases: $\mathfrak{D}_n$ not simple, $\mathfrak{D}_n$ simple. In the first case $\mathfrak{D}_n = \Delta_n \oplus \Delta_n{}^{J_a}$ where Δ_n is a simple ideal. The isomorphisms between two algebras with involution of this type correspond to the isomorphisms and antiisomorphisms of the simple component Δ_n.

We now switch to the geometric point of view in which Δ_n is replaced by $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$, $\mathfrak{M}$ an n-dimensional vector space over the division algebra Δ. Let $\mathfrak{N}$ be an m-dimensional vector space over the division algebra E. Then it is well known that $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ and $\mathrm{Hom}_E(\mathfrak{N},\mathfrak{N})$ are isomorphic if and only if $n = m$ and $\Delta \cong E$. Moreover, if σ is an isomorphism of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ onto $\mathrm{Hom}_E(\mathfrak{N},\mathfrak{N})$ then there exists a semilinear isomorphism of $\mathfrak{M}/\Delta$ onto $\mathfrak{N}/E$ with associated isomorphism s of Δ/Φ onto E/Φ such that $a^\sigma = S^{-1}aS$, $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ (Jacobson, *Lectures*. II, pp. 235–237). Next let $\mathfrak{N}' = \mathrm{Hom}_E(\mathfrak{N}, E)$ the right vector space over E of linear functions on $\mathfrak{N}$(to E). Then $\mathfrak{N}'$ can be regarded as a left vector space over E° by putting $\varepsilon y' = y'\varepsilon$ where the ε on the right is in E and that on the left is in E°. If $b \in \mathrm{Hom}_E(\mathfrak{N},\mathfrak{N})$ then b has a transpose linear transformation b' in the right vector space $\mathfrak{N}'/E$. This can be regarded as a linear transformation in $\mathfrak{N}'/E^\circ$. Then $b \to b'$ is an antiisomorphism of $\mathrm{Hom}_E(\mathfrak{N},\mathfrak{N})$ onto $\mathrm{Hom}_{E^\circ}(\mathfrak{N}',\mathfrak{N}')$. Hence if $a \to a^\sigma$ is an antiisomorphism of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ onto $\mathrm{Hom}_E(\mathfrak{N},\mathfrak{N})$ then $a \to (a^\sigma)'$ is an isomorphism of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ onto $\mathrm{Hom}_{E^\circ}(\mathfrak{N}',\mathfrak{N}')$. The form of this has been indicated and this gives the form of σ.

We consider next the isomorphisms of the algebras with involution $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ where J is the adjoint relative to a hermitian or skew symmetric form f. Thus we have $f(xa, y) = f(x, ya^J)$, $x, y \in \mathfrak{M}$, $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. We note first that f is determined by J up to a right multiplier in Δ. Let g be a second nondegenerate hermitian or skew form such that $g(xa, y) = g(x, ya^J)$, $x, y \in \mathfrak{M}$, $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. Consider the linear mappings $x \to g(x,u)v$ and $x \to \varepsilon g(x,v)u$ where $\varepsilon = 1$ or -1 according as g is hermitian or skew. Since

$$g(g(x,u)v,y) = g(x,u)g(v,y),$$

$$g(x,\varepsilon g(y,v)u) = g(x,u)g(v,y),$$

these are adjoints relative to g, hence, relative to f, so we have $f(g(x,u)v,y) = f(x,\varepsilon g(y,v)u)$. This gives $g(x,u)f(v,y) = f(x,u)\varepsilon\overline{g(y,v)}$ for all $x,y,u,v \in \mathfrak{M}$. Now we can choose v,y so that $f(v,y) = 1$ and obtain $g(x,u) = f(x,u)\gamma$, $x,u \in \mathfrak{M}$.

Let $\mathfrak{N}/\mathrm{E}$ be a second finite-dimensional vector space over a division algebra with involution, g a nondegenerate hermitian or skew hermitian form on $\mathfrak{N}$, $K\colon b \to b'$ the adjoint mapping determined by g. Let σ be an isomorphism of the algebra with involution $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}),J)$ onto $(\mathrm{Hom}_\mathrm{E}(\mathfrak{N},\mathfrak{N}),K)$. Since σ is an isomorphism of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ onto $\mathrm{Hom}_\mathrm{E}(\mathfrak{N},\mathfrak{N})$ there exists a semilinear isomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$ such that $a^\sigma = S^{-1}aS$, $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. Let $x,y \in \mathfrak{M}$ and let s be the isomorphism of Δ onto E associated with S. Then one checks directly that the mapping f' defined by $f'(x,y) = g(xS,yS)^{S^{-1}}$ is a nondegenerate hermitian or skew form on $\mathfrak{M}/\Delta$. Since $Sa^\sigma = aS$ and $J\sigma = \sigma K$ we have

$$\begin{aligned} g(xaS,yS) &= g(xSa^\sigma,yS) = g(xS,ySa^{\sigma K}) \\ &= g(xS,ySa^{J\sigma}) = g(xS,ya^JS). \end{aligned}$$

Hence $g(xaS,yS)^{S^{-1}} = g(xS,ya^JS)^{S^{-1}}$ and so $f'(xa,y) = f'(x,ya^J)$. This implies that there exists a $\lambda \neq 0$ in Δ such that

$$f'(x,y)^S = g(xS,yS) = f(x,y)^S\lambda^S. \tag{44}$$

A form f on $\mathfrak{M}/\Delta$ and a form g on $\mathfrak{N}/\mathrm{E}$ will be called *equivalent* if there exists a semilinear isomorphism S of $\mathfrak{M}/\Delta$ onto $\mathfrak{N}/\mathrm{E}$ and a nonzero $\lambda \in \Delta$ such that (44) holds for all $x,y \in \mathfrak{M}$. Here s is the isomorphism of Δ onto E associated with S. One sees easily that this is an equivalence relation. Our result is that any isomorphism of $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}),J)$ onto $(\mathrm{Hom}_\mathrm{E}(\mathfrak{N},\mathfrak{N}),K)$ has the form $A \to S^{-1}AS$ where S is an equivalence of f with g.

We now consider the derivations of our Jordan algebras and again consider first the case of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})^+$. For simplicity we now assume $\dim\mathfrak{M}/\Delta \geqq 3$. Then we know by Corollary 4 to Martindale's Theorem that any derivation of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})^+$ is a derivation of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$, so we consider the latter. If d is a derivation in the division algebra Δ/Φ then we define a *d-semiderivation* (or d-differential transformation) D of $\mathfrak{M}$ to be an additive mapping in $\mathfrak{M}$ such that $(\rho x)D = \rho(xD) + \rho^d x, \rho \in \Delta$, $x \in \mathfrak{M}$. If $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ and D is a semiderivation then a direct verification shows that $[a,D] = aD - Da \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. It is clear that the mapping $a \to [a,D]$ of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ into itself is Φ-linear. Also $[ab,D] = [a,D]b + a[b,D]$ for $a,b \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. Hence $a \to [a,D]$ is a derivation in $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. It is known that, conversely, every derivation in $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ has the form $a \to [a,D]$, D a d-semiderivation of $\mathfrak{M}/\Delta$ (Jacobson's *Structure of*

Rings, p. 87). Our results now show that the derivations of $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})^+$ with $\dim\mathfrak{M}/\Delta \geqq 3$ have the form $a \to [a, D]$, D a semiderivation in $\mathfrak{M}/\Delta$.

Next let Δ have an involution and let f be a nondegenerate hermitian or skew form associated with this involution. The nondegeneracy of f implies that if $g: x \to g(x)$ is a linear function on $\mathfrak{M}$ then there exists a unique vector y_g in $\mathfrak{M}$ such that $f(x, y_g) = g(x)$ for all $x \in \mathfrak{M}$. Let D be a d-semiderivation in $\mathfrak{M}/\Delta$ and consider $g_y(x) \equiv f(xD, y) - f(x, y)^d$. If $\rho \in \Delta$ then

$$\begin{aligned} g_y(\rho x) &= f((\rho x)D, y) - f(\rho x, y)^d \\ &= f(\rho(xD) + \rho^d x, y) - (\rho f(x, y))^d \\ &= \rho f(xD, y) + \rho^d f(x, y) - \rho^d f(x, y) - \rho f(x, y)^d \\ &= \rho g_y(x). \end{aligned}$$

Also it is clear that $g_y(x_1 + x_2) = g_y(x_1) + g_y(x_2)$, so g_y is a linear function on $\mathfrak{M}/\Delta$. Hence there exists a unique vector z in $\mathfrak{M}$ such that $g_y(x) = f(x, z)$, $x \in \mathfrak{M}$. We now have the mapping $y \to z$ in $\mathfrak{M}$, which we denote as D^J and call the *adjoint* of D relative to f. By definition, we have

$$f(xD, y) - f(x, yD^J) = f(y, x)^d. \tag{45}$$

If we apply j to (45) we obtain

$$f(yD^J, x) - f(y, xD) = f(y, x)^{\bar{d}} \tag{46}$$

where $\bar{d}$ is the derivation $-jdj$ in Δ. This gives

$$f((\rho y)D^J - \rho^{\bar{d}} y - \rho(yD^J), x) = 0$$

for $x, y \in \mathfrak{M}$, $\rho \in \Delta$. Since f is nondegenerate we obtain $(\rho y)D^J = \rho^{\bar{d}} y + \rho(yD^J)$. Since D^J is clearly additive this shows that D^J is a $\bar{d}$-semiderivation of $\mathfrak{M}$. Then (46) shows that D is the adjoint of D^J: $D^{J^2} = D$. If $d = 0$ then $D \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ and D^J is the adjoint of D relative to f in the usual sense. If $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ and D is a d-semiderivation in $\mathfrak{M}$, then $[a, D] \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ and

$$\begin{aligned} &f(x[a, D], y) - f(x, y[D^J, a^J]) \\ &= f(xaD - xDa, y) - f(x, yD^J a^J - ya^J D^J) \\ &= f(xa, yD^J) + f(xa, y)^d - f(xD, ya^J) \\ &\quad - f(x, yD^J a^J) + f(x, ya^J D^J) \\ &= f(x, yD^J a^J) + f(x, ya^J)^d - f(x, ya^J D^J) \\ &\quad - f(x, ya^J)^d - f(x, yD^J a^J) + f(x, ya^J D^J) \\ &= 0. \end{aligned}$$

Hence $[a, D]^J = [D^J, a^J]$.

Now let δ be a derivation in the algebra with involution $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ in the sense that δ is a derivation in $\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ which commutes with J. Then there exists a d-semiderivation D in $\mathfrak{M}$ such that $a^\delta = [a, D]$, $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$. Hence $a^{\delta J} = a^{J\delta} = [a^J, D]$ and $a^{\delta J} = [a, D]^J = [D^J, a^J] = -[a^J, D^J]$. Thus $D + D^J$ commutes with every linear transformation a^J and consequently $D + D^J = \lambda_L$ where λ_L is the multiplication $x \to \lambda x$, $\lambda \in \Delta$, in $\mathfrak{M}$. If $\mu \in \Delta$ then $\rho \to [\mu, \rho]$ is a derivation in Δ and $(\rho x)\mu_L = \mu(\rho x) = \rho\mu x + [\mu, \rho]x = \rho(x\mu_L) + [\mu, \rho]x$. Hence μ_L is a semiderivation in $\mathfrak{M}$ corresponding to the derivation ad μ: $\rho \to [\mu, \rho]$ in Δ. Since $f(\mu x, y) - f(x, \bar{\mu} y) = [\mu, f(x, y)]$ we see that $\mu_L{}^J = \bar{\mu}_L$. It is clear also that if $a \in \mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M})$ then $[a, \mu_L] = 0$. Now we have $D + D^J = \lambda_L$ which implies that $\lambda_L{}^J = \lambda_L$ and $\bar{\lambda} = \lambda$. Put $E = D + \mu_L$ where $\mu = -\frac{1}{2}\lambda$. Then E is an e-semiderivation for $e = d - \mathrm{ad}\,\mu$ and $E^J + E = D + \mu_L + D^J + \mu_L = D + D^J - \lambda_L = 0$. Also $a\delta = [a, D] = [a, E]$. Hence we have shown that every derivation of $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$ has the form $a \to [a, E]$ where E is a semiderivation of $\mathfrak{M}$ which is skew relative to J: $E^J = -E$. Conversely, if E is any skew semiderivation then $a \to [a, E]$ is a derivation of $(\mathrm{Hom}_\Delta(\mathfrak{M},\mathfrak{M}), J)$.

In conclusion, our results reduce the problem of classifying the simple Jordan algebras we determined and of studying their derivations to well-defined problems on associative algebras. In particular, the isomorphism problem is reduced to one of classifying division algebras and classifying hermitian forms. For finite dimensional algebras over special fields, a fair amount of information on these problems is known.

7. **Alternative set of axioms.** In the next chapter we shall develop a structure theory for finite-dimensional Jordan algebras without the assumption that the algebras have identity elements. In order to be able to apply the results of this chapter it is necessary to modify our axioms (i)–(iii) by omitting the axiom (i) on existence of an identity element. We shall now show that we obtain the same class of algebras if we replace (i) by the following axiom on the quadratic ideals of the form $\mathfrak{J}_0(e)$:

(iv) The minimum condition holds for quadratic ideals of form $\mathfrak{J}_0(e)$, $e^{\cdot 2} = e$ and if $\mathfrak{J}_0(e) \neq 0$ then $\mathfrak{J}_0(e)$ contains a minimal quadratic ideal.

Thus (iv) is a strengthening of the minimum conditions given in (iii) which, consists of adding the hypotheses made in (iii) on the quadratic ideals $\mathfrak{J}_1(e) = \mathfrak{J}U_e$ to the quadratic ideals $\mathfrak{J}_0(e)$. If $\mathfrak{J}$ has an identity element then $\mathfrak{J}_0(e) = \mathfrak{J}_1(1-e)$ so (iii) and (iv) are equivalent for Jordan algebras with 1. We shall now show that (ii), (iii) and (iv) imply (i) so the axioms (ii), (iii) and (iv) also characterize the class of Jordan algebras which we have studied in this chapter. The proof of this fact will be based on the standard trick of adjoining an identity element to an algebra.

Let $\mathfrak{J}$ be a Jordan algebra and let $\mathfrak{J}' = \Phi 1 \oplus \mathfrak{J}$ be the algebra obtained by adjoining the identity element 1 to $\mathfrak{J}$. Then we know that $\mathfrak{J}'$ is Jordan (p. 31).

If $b' = \beta 1 + b$, $\beta \in \Phi$, $b \in \mathfrak{J}$, then $b^{\cdot 2} = \beta 1 + (2\beta b + b^{\cdot 2})$ so $(b')^{\cdot 2} = 0$ implies $\beta = 0$ and $b' = b \in \mathfrak{J}$. It follows that $\mathfrak{J}'$ has no absolute zero divisors $\neq 0$ if $\mathfrak{J}$ has none. Next assume $(b')^{\cdot 2} = b'$. Then $\beta = 0$ or $\beta = 1$. In the first case, $b' = b \in \mathfrak{J}$ and in the second, $b^{\cdot 2} = -b$ so $f = -b$ is an idempotent in $\mathfrak{J}$ and $b' = 1 - f$. If $e^{\cdot 2} = e \in \mathfrak{J}$, then $\mathfrak{J}_1'(e) = \mathfrak{J}_1' U_e = \mathfrak{J}_1 U_e$ since $1 U_e = e$. Hence $\mathfrak{J}_1'(e) = \mathfrak{J}_1(e)$. If $e' = 1 - f$, $f^{\cdot 2} = f \in \mathfrak{J}$, then $\mathfrak{J}_1'(e') = \mathfrak{J}_0(f) + \Phi e'$. Clearly the minimum condition holds for quadratic ideals of this form if and only if it holds for quadratic ideals of $\mathfrak{J}$ of the form $\mathfrak{J}_0(f)$, $f^{\cdot 2} = f$.

Assume $\mathfrak{J}$ has no absolute zero divisors, so $\mathfrak{J}'$ has none. Let $\mathfrak{B}$ be a minimal quadratic ideal of $\mathfrak{J}$. By Theorem 1, $\mathfrak{B} = \mathfrak{J} U_b$ where $b \in \mathfrak{B}$ and $\mathfrak{B}$ is a subalgebra of $\mathfrak{J}$. Then $\mathfrak{J}' U_b = \mathfrak{J} U_b + \Phi b^{\cdot 2} = \mathfrak{B}$ so $\mathfrak{B}$ is a quadratic ideal of $\mathfrak{J}'$, which is clearly minimal. If $e' = 1 - f$, $f^{\cdot 2} = f \in \mathfrak{J}$ then $\mathfrak{J}_1'(e') = \mathfrak{J}_0(f) + \Phi e'$. Hence if $\mathfrak{J}_0(f)$ contains a minimal quadratic ideal of $\mathfrak{J}$ then $\mathfrak{J}_1'(e')$ contains a minimal quadratic ideal of $\mathfrak{J}'$. If $\mathfrak{J}_0(f) = 0$ then $\mathfrak{J}_1'(e') = \Phi e'$ is a minimal quadratic ideal of $\mathfrak{J}'$.

We can now prove

THEOREM 10. *If $\mathfrak{J}$ satisfies* (ii), (iii) *and* (iv), *then $\mathfrak{J}$ has an identity element, so the structure theorems hold for $\mathfrak{J}$.*

PROOF. The foregoing results show that $\mathfrak{J}' = \Phi 1 \oplus \mathfrak{J}$ satisfies (i), (ii), (iii). Hence, by the proof of the First Structure Theorem. every ideal in $\mathfrak{J}'$ has a complementary ideal. In particular, $\mathfrak{J}' = \mathfrak{J} \oplus \mathfrak{B}$ where $\mathfrak{B}$ is an ideal. Then $\mathfrak{J}$ has an identity element.

Axioms (iii) and (iv) are evidently satisfied if $\mathfrak{J}$ is finite dimensional over Φ.

CHAPTER V

STRUCTURE THEORY FOR FINITE-DIMENSIONAL JORDAN ALGEBRAS

The structure theory of Jordan algebras was initiated by Jordan, von Neumann and Wigner in [1] in which they gave a complete determination of the finite-dimensional Jordan algebras over the reals which are formally real in the sense that a relation $\sum a_i^{\cdot 2} = 0$ holds in such an algebra only if every $a_i = 0$. They showed that these algebras are direct sums of simple ones and they determined all the simple ones. The structure theory of finite-dimensional Jordan algebras over arbitrary fields has been developed b Albert in three papers [7], [9], [17]. In these he gave a definition of solvability and the radical and showed that a semisimple algebra, defined as one with 0 radical, is a direct sum of simple ideals. Moreover, he determined the finite-dimensional simple Jordan algebras over an algebraically closed field (with a small exception which was settled by the author in [19]). The extension of the determination and classification of special simple finite-dimensional Jordan algebras to the case of a nonalgebraically closed field is due to Kalisch [1] and F. D. Jacobson and N. Jacobson [1]. The problem of classifying exceptional simple Jordan algebras over arbitrary base fields has been studied by Albert and Jacobson [1], by Albert [25], [32] and by Springer [4] (see Chapter IX).

In this chapter we shall give Albert's theory of the radical of a finite-dimensional Jordan algebra and we shall show that his definition of semisimplicity is equivalent to nondegeneracy (= nonexistence of absolute zero divisors $\neq 0$) which we introduced in the last chapter. As a consequence, we shall obtain Albert's theorem that a finite-dimensional semisimple Jordan algebra is a direct sum of simple ones as a corollary to the First Structure Theorem of Chapter IV. We can also deduce from the Second Structure Theorem Albert's determination of the finite-dimensional simple Jordan algebras over an algebraically closed field and, more generally, the determination of the finite-dimensional simple Jordan algebras over an arbitrary field which are not division algebras. However, to get at the division algebras we shall require a base field extension argument which is applicable also to arbitrary finite-dimensional simple Jordan algebras. Moreover, this method can be made independent of the Second Structure Theorem by substituting for it a determination of another important class of Jordan algebras — the reduced simple Jordan algebras.

We shall follow the procedure just indicated in this chapter. We shall first reduce the study of the finite-dimensional simple Jordan algebras to those which are central in the sense that their centers coincide with the base field. We shall show that a finite-dimensional central simple Jordan algebra over Φ is one of the following types: I. a Jordan algebra of a nondegenerate symmetric bilinear form, II. a Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite-dimensional central simple associative algebra with involution, III. a Jordan algebra $\mathfrak{J}$ such that $\mathfrak{J}_\Omega$, for Ω the algebraic closure of Φ, is isomorphic to $\mathfrak{H}(\mathfrak{O}_3, J_1)$ where $\mathfrak{O}$ is the (uniquely determined) octonion algebra over Ω. The classification of the algebras in I is equivalent to that of symmetric bilinear forms. The classification of the algebras of type II is equivalent to the classification of the associative algebras with involution. The algebras in III are the only finite-dimensional central simple exceptional Jordan algebras. We shall postpone the study of these to Chapter IX.

1. **Ideals and associator ideals.** A subspace $\mathfrak{B}$ of a Jordan algebra $\mathfrak{J}$ will be called an *associator ideal* in $\mathfrak{J}$ if any associator $[a_1, a_2, a_3] \in \mathfrak{B}$ whenever any one of the $a_i \in \mathfrak{B}$. Clearly any ideal is an associator ideal. If σ is a homomorphism of $\mathfrak{J}$ then, of course, the kernel of σ is an ideal. On the other hand, if ρ is a multiplication specialization of $\mathfrak{J}$ in an associative algebra $\mathfrak{G}$ with 1 then we know that $[a,b,c]^\rho = [[c^\rho, a^\rho], b^\rho]$ (ex. 1, p. 99). This evidently implies that the kernel of any multiplication specialization is an associator ideal. The kernel of a multiplication specialization need not be an ideal. For example, let $\mathfrak{T}$ be the subalgebra of $\Phi_2{}^+$ of triangular matrices $\alpha e_{11} + \beta e_{22} + \gamma e_{12}, \alpha, \beta, \gamma \in \Phi$, and let $\mathfrak{J} = \Phi e_{11} + \Phi e_{22}$, $\mathfrak{M} = \Phi e_{12}$. Then $(\alpha e_{11} + \beta e_{22}) . e_{12} = \frac{1}{2}(\alpha + \beta) e_{12}$ so $\mathfrak{M}$ relative to the multiplication defined in $\mathfrak{T}$ is a bimodule for $\mathfrak{J}$. The formula just indicated shows that the kernel of the associated birepresentation is $\Phi(e_{11} - e_{22})$. This is not an ideal in $\mathfrak{J}$.

If $a, b, c \in \mathfrak{J}$ then $[a,b,c] = a.b.c - c.b.a = -[c,b,a]$. Also $[a,b,c] + [b,c,a] + [c,a,b] = a.b.c - c.b.a + b.c.a - a.c.b + c.a.b - b.a.c = 0$. This implies that a subspace $\mathfrak{B}$ is an associator ideal if and only if $[b, a_1, a_2] \in \mathfrak{B}$ for all $b \in \mathfrak{B}$ and $a_i \in \mathfrak{J}$. If $\mathfrak{B}$ is an associator ideal then the subspace $\mathfrak{B} + \mathfrak{B}.\mathfrak{J}$ is an ideal since if $b \in \mathfrak{B}$ and $a_1, a_2 \in \mathfrak{J}$ then $b.a_1.a_2 = a_1.a_2.b + [b, a_1, a_2] \in \mathfrak{B}.\mathfrak{J} + \mathfrak{B}$. In particular, if $\mathfrak{B}$ is the kernel of a multiplication specialization then $\mathfrak{B} + \mathfrak{J}.\mathfrak{B}$ is an ideal.

If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are ideals (associator ideals) then $\mathfrak{B}_1 \cap \mathfrak{B}_2$ and $\mathfrak{B}_1 + \mathfrak{B}_2$ are ideals (associator ideals). The product $\mathfrak{B}_1 . \mathfrak{B}_2$ of two ideals need not be an ideal. We consider now some multiplicative properties of ideals in the following

LEMMA 1. (1) *If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are ideals then $\mathfrak{B}_1 . \mathfrak{B}_2$ is an associator ideal; hence $\mathfrak{B}_1 . \mathfrak{B}_2 + \mathfrak{B}_1 . \mathfrak{B}_2 . \mathfrak{J}$ is an ideal.* (2) *If $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ are ideals then $\mathfrak{B}_1 . \mathfrak{B}_2 . \mathfrak{B}_3 + \mathfrak{B}_2 . \mathfrak{B}_3 . \mathfrak{B}_1 + \mathfrak{B}_3 . \mathfrak{B}_1 . \mathfrak{B}_2$ is an ideal.* (3) *If $\mathfrak{B}$ is an ideal then $\mathfrak{B}^{\cdot 3} = \mathfrak{B} . \mathfrak{B}^{\cdot 2}$ is an ideal.*

PROOF. (1) Let $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$, $a_1, a_2 \in \mathfrak{J}$. Then

$$\begin{aligned}[b_1 . b_2, a_1, a_2] &= b_1 . b_2 . a_1 . a_2 - b_1 . b_2 . (a_1 . a_2)\\ &= b_1 . a_2 . (b_2 . a_1) + b_1 . a_1 . (b_2 . a_2)\\ &- b_1 . a_2 . a_1 . b_2 - b_1 . (b_2 . a_2 . a_1) \quad (\text{by } B_3, \text{p. } 34) \in \mathfrak{B}_1 . \mathfrak{B}_2 .\end{aligned}$$

Hence $\mathfrak{B}_1 . \mathfrak{B}_2$ is an associator ideal and $\mathfrak{B}_1 . \mathfrak{B}_2 + \mathfrak{B}_1 . \mathfrak{B}_2 . \mathfrak{J}$ is an ideal. (2) Let $b_i \in \mathfrak{B}_i$, $i = 1, 2, 3$, and let $a \in \mathfrak{J}$. Then

$$\begin{aligned} b_1 . b_2 . b_3 . a = &- b_1 . a . b_3 . b_2 - b_1 (b_2 . a . b_3)\\ &+ b_1 . b_2 . (a . b_3) + b_1 . a . (b_2 . b_3)\\ &+ b_1 . b_3 . (a . b_2)\\ \in \mathfrak{B}_1 . \mathfrak{B}_2 . \mathfrak{B}_3 &+ \mathfrak{B}_2 . \mathfrak{B}_3 . \mathfrak{B}_1 + \mathfrak{B}_3 . \mathfrak{B}_1 . \mathfrak{B}_2 .\end{aligned}$$

This implies that $\mathfrak{B}_1 . \mathfrak{B}_2 . \mathfrak{B}_3 + \mathfrak{B}_2 . \mathfrak{B}_3 . \mathfrak{B}_1 + \mathfrak{B}_3 . \mathfrak{B}_1 . \mathfrak{B}_2$ is an ideal and proves (2). Evidently (3) is a special case of this.

LEMMA 2. *Let $\mathfrak{B}$ be an ideal, $\mathfrak{S}$ an m ($< \infty$) dimensional subspace of $\mathfrak{J}$. Define $\mathfrak{B}_0 = \mathfrak{B}$, $\mathfrak{B}_k = \mathfrak{B}_{k-1} . \mathfrak{B} . \mathfrak{S}$. Then $\mathfrak{B}_{m+1} \subseteq \mathfrak{B}^{\cdot 2}$.*

PROOF. Let $b, b_1, b_2, \cdots, b_k \in \mathfrak{B}$, $a_1, a_2, \cdots, a_k \in \mathfrak{J}$, $k > 1$, and consider the product $b . b_1 . a_1 . b_2 . a_2 \cdots b_k . a_k = b R_{b_1} R_{a_1} \cdots R_{b_k} R_{a_k}$. We claim that if two of the a_i are equal then this product is contained in $\mathfrak{B}^{\cdot 2}$. We assume first that $a_k = a_i = a$ for $i < k$. If $i = k-1$ then the product has the form $dR_a R_{b_k} R_a$ where $d \in \mathfrak{B}^{\cdot 2}$. We have $2dR_a R_{b_k} R_a = d(-R_{a^{\cdot 2} . b_k} + 2R_{a . b_k} R_a + R_{a^{\cdot 2}} R_{b_k})$. Since $\mathfrak{B}$ is an ideal it is clear that $dR_{a^{\cdot 2} . b_k}$ and $dR_{a^{\cdot 2}} R_{b_k} \in \mathfrak{B}^{\cdot 2}$. Also $dR_{a . b_k} \in \mathfrak{B}^{\cdot 3}$ and this is an ideal. Hence $dR_{a . b_k} R_a \in \mathfrak{B}^{\cdot 3} \subseteq \mathfrak{B}^{\cdot 2}$. Thus $dR_a R_{b_k} R_a \in \mathfrak{B}^{\cdot 2}$. Now let $i = k - r$, $r > 1$. Then our product has the form $dR_a R_{b'} R_{a'} R_{b''} \cdots R_a$ where $d \in \mathfrak{B}^{\cdot 2}, b', b'' \in \mathfrak{B}$. Then $dR_a R_{b'} R_{a'} = d(-R_{a'} R_{b'} R_a - R_{a . a' . b'} + R_{a . b'} R_{a'} + R_{a' . b'} R_a + R_{a . a'} R_b)$ and since $dR_{a . a' . b'}$, $dR_{a . b'} R_{a'}$, $dR_{a' . b'} R_a$ and $dR_{a . a'} R_b \in \mathfrak{B}^{\cdot 2}$ our product is congruent modulo the ideal $\mathfrak{B}^{\cdot 3}$ to $-dR_{a'} R_{b'} R_a R_{b''} \cdots R_a$. Hence we see by induction on r that any product $b . b_1 . a_1 \cdots b_k . a_k$ in which $a_k = a_{k-r}, r \geqq 1$, is contained in $\mathfrak{B}^{\cdot 2}$. This result and the fact that $\mathfrak{B}^{\cdot 3}$ is an ideal implies that if $a_i = a_j$ for $i < j < k$ then the product is in $\mathfrak{B}^{\cdot 3}$. Hence we have established the claim made at the outset. Now let $(s_1, s_2, \cdots, s_m)$ be a basis for $\mathfrak{S}/\Phi$. Then it is clear from the definition of the $\mathfrak{B}_k$ that any element of $\mathfrak{B}_{m+1}$ is a linear combination of products of the form $b . b_1 . s_{i_1} . b_2 . s_{i_2} . \cdots . b_{m+1} . s_{i_{m+1}}$ where $b, b_j \in \mathfrak{B}$ and $i_j \in \{1, 2, \cdots, m\}$. Since two of the s's are equal the result established shows that the product indicated is in $\mathfrak{B}^{\cdot 2}$. Hence $\mathfrak{B}_{m+1} \subseteq \mathfrak{B}^{\cdot 2}$.

We can now prove the following useful result.

THEOREM 1. *Let $\mathfrak{B}$ be an ideal in the Jordan algebra $\mathfrak{J}$ of finite codimension m in $\mathfrak{J}$. Define*

(1) $\mathfrak{B}^{[0]} = \mathfrak{B}, \quad \mathfrak{B}^{[k]} = \mathfrak{B}^{[k-1]}.\mathfrak{B} + \mathfrak{B}^{[k-1]}.\mathfrak{B}.\mathfrak{J}, \quad k \geqq 1,$

(2) $\mathfrak{B}^{(0)} = \mathfrak{B}, \quad \mathfrak{B}^{(k)} = \mathfrak{B}^{(k-1)}.\mathfrak{B}^{(k-1)} + \mathfrak{B}^{(k-1)}.\mathfrak{B}^{(k-1)}.\mathfrak{J}, \quad k \geqq 1.$

Then $\mathfrak{B}^{[k]}$ *and* $\mathfrak{B}^{(k)}$ *are ideals in* $\mathfrak{J}$ *and* $\mathfrak{B}^{[m+1]}$ *and* $\mathfrak{B}^{(m+1)}$ *are contained in* $\mathfrak{B}^{\cdot 2}$.

PROOF. The fact that $\mathfrak{B}^{[k]}$ and $\mathfrak{B}^{(k)}$ are ideals follows from Lemma 1 (1). Since $\dim \mathfrak{J}/\mathfrak{B} = m$ we can write $\mathfrak{J} = \mathfrak{B} + \mathfrak{S}$ where $\mathfrak{S}$ is a subspace such that $\dim \mathfrak{S} = m$. Then one proves, by induction on k, that if $\mathfrak{B}_k$ is defined as in Lemma 2 then $\mathfrak{B}^{[k]} \subseteq \mathfrak{B}_k + \mathfrak{B}^{\cdot 2}$. It follows from Lemma 2 that $\mathfrak{B}^{[m+1]} \subseteq \mathfrak{B}^{\cdot 2}$. Also, by induction, we have $\mathfrak{B}^{(k)} \subseteq \mathfrak{B}^{[k]}$. Hence $\mathfrak{B}^{(m+1)} \subseteq \mathfrak{B}^{\cdot 2}$.

2. **Solvable ideals.** We define the powers $\mathfrak{J}^{\cdot 2^k}$ of a Jordan algebra $\mathfrak{J}$ by $\mathfrak{J}^{\cdot 2^0} = \mathfrak{J}$, $\mathfrak{J}^{\cdot 2^k} = (\mathfrak{J}^{\cdot 2^{k-1}})^{\cdot 2}$. Clearly $\mathfrak{J}^{\cdot 2}$ is an ideal and all the $\mathfrak{J}^{\cdot 2^k}$ are subalgebras. However, the $\mathfrak{J}^{\cdot 2^k}$ with $k > 1$ need not be ideals. We shall call $\mathfrak{J}$ *solvable* if there exists an integer N such that $\mathfrak{J}^{\cdot 2^N} = 0$. It is clear that any subalgebra and any homomorphic image of a solvable algebra is solvable. Moreover, we have the following

LEMMA 1. (1) *If* $\mathfrak{J}$ *contains a solvable ideal* $\mathfrak{B}$ *such that* $\mathfrak{J}/\mathfrak{B}$ *is solvable then* $\mathfrak{J}$ *is solvable.* (2) *If* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are solvable ideals then* $\mathfrak{B}_1 + \mathfrak{B}_2$ *is a solvable ideal.*

PROOF. (1) Assume $(\mathfrak{J}/\mathfrak{B})^{\cdot 2^N} = 0$. It is clear from the definition of the powers that this implies that $\mathfrak{J}^{\cdot 2^N} \subseteq \mathfrak{B}$. Also $\mathfrak{J}^{\cdot 2^{N+M}} \subseteq \mathfrak{B}^{\cdot 2^M}$. Since $\mathfrak{B}$ is solvable this proves that $\mathfrak{J}$ is solvable. (2) We know that $\mathfrak{B}_1 + \mathfrak{B}_2$ is an ideal and $(\mathfrak{B}_1 + \mathfrak{B}_2)/\mathfrak{B}_2 \cong \mathfrak{B}_1/(\mathfrak{B}_1 \cap \mathfrak{B}_2)$. Since the right-hand side is a homomorphic image of $\mathfrak{B}_1$ it is solvable. Hence $(\mathfrak{B}_1 + \mathfrak{B}_2)/\mathfrak{B}_2$ and $\mathfrak{B}_2$ are solvable. Then $\mathfrak{B}_1 + \mathfrak{B}_2$ is solvable by (1).

It is clear from Lemma 1 (2) that if $\mathfrak{J}$ satisfies the maximum condition for ideals then $\mathfrak{J}$ contains a solvable ideal $\mathfrak{R}$ such that $\mathfrak{R}$ contains every solvable ideal of $\mathfrak{J}$. In particular, this holds for finite-dimensional $\mathfrak{J}$. In this case we call $\mathfrak{R}$ the *radical* of $\mathfrak{J}$ (or rad $\mathfrak{J}$). This is analogous to the definition of the radical of a finite-dimensional associative algebra as the maximal nilpotent ideal. If $\mathfrak{R} = 0$, or, equivalently, $\mathfrak{J}$ has no nonzero solvable ideals then we shall call $\mathfrak{J}$ *semisimple.* If $\mathfrak{R}$ is the radical of $\mathfrak{J}$ and $\bar{\mathfrak{S}}$ is the radical of $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{R}$ then $\bar{\mathfrak{S}} = \mathfrak{S}/\mathfrak{R}$ where $\mathfrak{S}$ is an ideal in $\mathfrak{J}$. Since $\mathfrak{S}/\mathfrak{R}$ and $\mathfrak{R}$ are solvable $\mathfrak{S}$ is solvable. Hence $\mathfrak{S} \subseteq \mathfrak{R}$ and $\bar{\mathfrak{S}} = 0$. Thus $\mathfrak{J}/\mathfrak{R}$ is semisimple if $\mathfrak{R}$ is the radical of $\mathfrak{J}$.

The main result we shall obtain in this section is that if $\mathfrak{J}$ is finite-dimensional and $\mathfrak{J}$ is considered as imbedded in its universal multiplication envelope $U(\mathfrak{J})$ in the usual way (§2.9) then the radical $\mathfrak{R}$ of $\mathfrak{J}$ is contained in the radical $\mathfrak{N}$ of $U(\mathfrak{J})$. The proof will be based on several lemmas.

LEMMA 2 (PENICO). *Let* $\mathfrak{J}$ *be a finite-dimensional Jordan algebra,* $\mathfrak{B}$ *a solvable ideal in* $\mathfrak{J}$ *and define* $\mathfrak{B}^{(k)}$ *by* $\mathfrak{B}^{(0)} = \mathfrak{B}$, $\mathfrak{B}^{(k)} = \mathfrak{B}^{(k-1)}.\mathfrak{B}^{(k-1)}$

$+\mathfrak{B}^{(k-1)}.\mathfrak{B}^{(k-1)}.\mathfrak{J}$. *Then the* $\mathfrak{B}^{(k)}$ *are ideals,* $\mathfrak{B}^{(k)} \supseteq \mathfrak{B}^{(k+1)}$, $(\mathfrak{B}^{(k)})^{\cdot 2} \subseteq \mathfrak{B}^{(k+1)}$ *and there exists an integer* M *such that* $\mathfrak{B}^{(M)} = 0$.

PROOF. Everything except the last statement is clear by Lemma 1 of §1. Also, by Theorem 1, if $\dim \mathfrak{J} = n$ then $\mathfrak{B}^{(n+1)} \subseteq \mathfrak{B}^{\cdot 2}$. We claim that for $r = 1, 2, \cdots$, we have $\mathfrak{B}^{(r(n+1))} \subseteq \mathfrak{B}^{\cdot 2^r}$. Thus assume this holds for r. Then $\mathfrak{B}^{((r+1)(n+1))} = \mathfrak{B}^{(r(n+1)+n+1)} = (\mathfrak{B}^{(r(n+1))})^{(n+1)}$ since the definition of the $\mathfrak{B}^{(k)}$ gives $(\mathfrak{B}^{(k)})^{(j)} = \mathfrak{B}^{(k+j)}$. Hence $\mathfrak{B}^{((r+1)(n+1))} = (\mathfrak{B}^{(r(n+1))})^{(n+1)} \subseteq (\mathfrak{B}^{(r(n+1))})^{\cdot 2} \subseteq (\mathfrak{B}^{\cdot 2^r})^{\cdot 2} = \mathfrak{B}^{\cdot 2^{r+1}}$. Hence $\mathfrak{B}^{(r(n+1))} \subseteq \mathfrak{B}^{\cdot 2^r}$ for all $r = 1, 2, \cdots$. Since $\mathfrak{B}$ is solvable this implies that there exists an integer M such that $\mathfrak{B}^{(M)} = 0$.

If $\mathfrak{B} \neq 0$ is solvable then $\mathfrak{B}^{(0)} \neq 0$ in the sequence of $\mathfrak{B}^{(k)}$. Let M be minimal such that $\mathfrak{B}^{(M)} = 0$. Then $\mathfrak{B} = \mathfrak{B}^{(0)} \supset \mathfrak{B}^{(1)} \supset \cdots \supset \mathfrak{B}^{(M)} = 0$ since $\mathfrak{B}^{(k)} = \mathfrak{B}^{(k+1)}$ implies that $\mathfrak{B}^{(k+1)} = (\mathfrak{B}^{(k)})^{(1)} = (\mathfrak{B}^{(k+1)})^{(1)} = \mathfrak{B}^{(k+2)}$. Then $\mathfrak{C} = \mathfrak{B}^{(1)}$ is an ideal properly contained in $\mathfrak{B}$ such that $\mathfrak{B}^{\cdot 2} \subseteq \mathfrak{C}$. Also $\mathfrak{D} = \mathfrak{B}^{(M-1)}$ is a nonzero ideal such that $\mathfrak{D}^{\cdot 2} = 0$.

Let $\mathfrak{B}$ be a solvable ideal in the finite-dimensional Jordan algebra $\mathfrak{J}$ and let $U(\mathfrak{J})$ be the universal multiplication envelope of $\mathfrak{J}$. We assume that $\mathfrak{J} \subseteq U(\mathfrak{J})$. Then we shall show that $\mathfrak{B}$ generates a nilpotent ideal in $U(\mathfrak{J})$. We shall prove this by induction on $\dim \mathfrak{B}$ and we may assume $\mathfrak{B} \neq 0$. Then we have an ideal $\mathfrak{C}$ of $\mathfrak{J}$ such that $\mathfrak{B} \supset \mathfrak{C}$ and $\mathfrak{B}^{\cdot 2} \subseteq \mathfrak{C}$. Let $\mathfrak{V}$ be the ideal in $U(\mathfrak{J})$ generated by $\mathfrak{C}$. Then the induction hypothesis implies that $\mathfrak{V}$ is nilpotent. Also we know that $U(\mathfrak{J})/\mathfrak{V}$ and the mapping $a + \mathfrak{C} \to a + \mathfrak{V}$, $a \in \mathfrak{J}$, constitute a universal multiplication envelope for $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{C}$ (Theorem 2.11). The ideal $\bar{\mathfrak{B}} = \mathfrak{B}/\mathfrak{C}$ in $\bar{\mathfrak{J}}$ satisfies $\bar{\mathfrak{B}}^{\cdot 2} = 0$. Suppose we know that $\bar{\mathfrak{B}}$ generates a nilpotent ideal in the universal multiplication envelope $U(\bar{\mathfrak{J}})$ of $\bar{\mathfrak{J}}$. Then the set of cosets $b + \mathfrak{V}$, $b \in \mathfrak{B}$, generates a nilpotent ideal in $U(\mathfrak{J})/\mathfrak{V}$. This ideal has the form $\mathfrak{W}/\mathfrak{V}$ where $\mathfrak{W}$ is the ideal in $U(\mathfrak{J})$ generated by $\mathfrak{B}$. Since $\mathfrak{W}/\mathfrak{V}$ and $\mathfrak{V}$ are nilpotent it follows that $\mathfrak{W}$ is nilpotent. Thus the ideal in $U(\mathfrak{J})$ generated by $\mathfrak{B}$ is nilpotent. The argument therefore shows that it is enough to prove the result for the case in which $\mathfrak{B}^{\cdot 2} = 0$. This will be achieved in the following two lemmas.

LEMMA 3. *Let* $\mathfrak{J}$ *be a finite-dimensional Jordan algebra such that* $\mathfrak{J}^{\cdot 2} = 0$. *Then* $\mathfrak{J}$ *generates a nilpotent ideal in* $U(\mathfrak{J})$.

PROOF. We note that $\mathfrak{J}$ is associative and we prove first that if $\mathfrak{J}$ is a finite-dimensional associative Jordan algebra then $U(\mathfrak{J})$ contains a nilpotent ideal $\mathfrak{N}$ such that $U(\mathfrak{J})/\mathfrak{N}$ is commutative. We have the relation $[[ca]b] = [a, b, c]$ in $U(\mathfrak{J})$ for $a, b, c \in \mathfrak{J}$ (eq. (47), p. 96). Since $\mathfrak{J}$ is associative this implies that every commutator $[ca], c, a \in \mathfrak{J}$, is in the center of $U(\mathfrak{J})$. Next we consider the relation $2aba + a^{\cdot 2} . b = ba^{\cdot 2} + 2a(a . b)$ (eq. (46′), p. 96). Taking commutators with b and using the fact that this operation is a derivation, we obtain $2[a, b]ba + 2ab[a, b] + [a^{\cdot 2} . b, b] = b[a^{\cdot 2}, b] + 2[a, b](a . b) + 2a[a . b, b]$. Since commutators of elements of $\mathfrak{J}$ are in the center, commutation again with b gives

$4[a,b]^2 b = 4[a,b][a.b,b]$. Then commutation with a gives $4[a,b]^3 = 0$ and $[a,b]^3 = 0$. Since $U(\mathfrak{J})$ is finite-dimensional associative it follows that the commutators generate a nilpotent ideal $\mathfrak{N}$ in $U(\mathfrak{J})$. Clearly $U(\mathfrak{J})/\mathfrak{N}$ is commutative. Next we note that $\mathfrak{J}^{\cdot 2} = 0$ and $2aba + a^{\cdot 2}.b = ba^{\cdot 2} + 2a(a.b)$ imply that $a^3 = 0$. Hence the cosets $a + \mathfrak{N}$ generate a nilpotent ideal in $U(\mathfrak{J})/\mathfrak{N}$ and the elements a generate a nilpotent ideal in $U(\mathfrak{J})$.

LEMMA 4. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra, $\mathfrak{B}$ a nonzero ideal in $\mathfrak{J}$ such that $\mathfrak{B}^{\cdot 2} = 0$. Let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{J}$ such that $(u_1, u_2, \cdots, u_m)$ is a basis for $\mathfrak{B}$. Then any monomial $u_{j_1} u_{j_2} \cdots u_{j_l}$ in $U(\mathfrak{J})$ in which $k + m$ of the $u_j \in \mathfrak{B}$ is a linear combination of monomials in the u_i in which the first k of the u's $\in \mathfrak{B}$.*

PROOF. This is clear if $l = k + m$; hence we may use induction on the formal degree l of the monomial $M = u_{j_1} u_{j_2} \cdots u_{j_l}$. Now suppose the first h u's in M are in $\mathfrak{B}$ but $u_{j_{h+1}} \notin \mathfrak{B}$. Then there is nothing to prove if $h \geqq k$ so for a given l we may use a downward induction on h and we may assume $h < k$. Then there are at least $m + 1$ u_{j_q}'s $\in \mathfrak{B}$ with $q > h + 1$ and hence M has the form

$$\cdots u \cdots u \cdots \tag{3}$$

where $u = u_i \in \mathfrak{B}$ and the first displayed u in (3) occurs after the $(h + 1)$-th place. Let $\mathfrak{C}$ denote the subspace of monomials $u_{i_1} u_{i_2} \cdots u_{i_k} \cdots$ where the u_{i_p}, $1 \leqq p \leqq k$, are in $\mathfrak{B}$. Suppose first that the two displayed u's in (3) are consecutive and let v denote the u_{j_q} in (3) preceding the first u. If we use the relation $abc = -cba - a.c.b + a(b.c) + b(a.c) + c(a.b)$ (eq. (47), p. 96) in $U(\mathfrak{J})$ for $a = v$, $b = c = u$ we obtain $vu^2 = -u^2 v + 2u(u.v)$. Since $u.v \in \mathfrak{B}$ and $u.v = 0$ if $v \in \mathfrak{B}$, a succession of replacements of the type indicated shows that $M \equiv \pm M' \pmod{\mathfrak{C}}$ where M' is a monomial in the u's having $h + 2$ u_i's $\in \mathfrak{B}$ at the beginning and containing $m + k$ u_i's $\in \mathfrak{B}$. Then the downward induction on h implies that $M' \in \mathfrak{C}$; hence $M \in \mathfrak{C}$. Next assume the two displayed u's in (3) are separated by a single $v = u_{j_q}$. Then we can use the relation $2uvu + u^{\cdot 2}.v = 2u(u.v) + v(u^{\cdot 2})$, which gives $uvu = u(u.v)$, to express M as a linear combination of monomials of formal degree $< l$ all of which have $k + m$ factors $u_j \in \mathfrak{B}$. Then induction on l implies that $M \in \mathfrak{C}$. Finally, supposed the displayed u's in (3) are separated by at least two $u_i \neq u$. Then we can use the relation $vwu = -uwv - v.u.w + u(v.w) + v(u.w) + w(u.v)$ to show that $M \equiv -M' \pmod{\mathfrak{C}}$ where M' is obtained from M by moving the second u in (3) two places to the left. A succession of moves of this type gives a reduction to one of the first two cases. Hence $M \in \mathfrak{B}$ in all cases.

We can now complete the proof of

THEOREM 2. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra, $\mathfrak{B}$ a solvable ideal in $\mathfrak{J}$. Then $\mathfrak{B}$ generates a nilpotent ideal in $U(\mathfrak{J})$.*

PROOF. We have reduced the proof to the case $\mathfrak{B}^{\cdot 2} = 0$. Now it follows from Lemma 3 that there exists an integer k such that the product of any k elements of $\mathfrak{B}$ is 0. Let $(u_1, \cdots, u_n)$ be a basis for $\mathfrak{J}$ such that $(u_1, \cdots, u_m)$ is a basis for $\mathfrak{B}$. Then Lemma 4 implies that any monomial $u_{j_1} \cdots u_{j_l}$ in which $k + m$ of the u's are in $\mathfrak{B}$ is 0. This implies that if $\mathfrak{B}$ is ideal generated by $\mathfrak{B}$ then $\mathfrak{B}^{k+m} = 0$.

Since the radical of a finite-dimensional Jordan algebra is the maximal solvable ideal and since the radical of a finite-dimensional associative algebra is the maximal nilpotent ideal it is clear that Theorem 2 can be stated in an equivalent form that the radical $\mathfrak{R}$ of $\mathfrak{J}$ is contained in the radical $\mathfrak{N}$ of $U(\mathfrak{J})$.

A nonassociative algebra is called *nilpotent* if there exists an integer N such that every product (in any association) of N elements of the algebra is 0. It is evident that if $\mathfrak{J}$ is a nilpotent Jordan algebra then $\mathfrak{J}$ is solvable. On the other hand, we have

COROLLARY 1 (ALBERT). *Any finite-dimensional solvable Jordan algebra $\mathfrak{J}$ is nilpotent.*

PROOF. Since we have a homomorphism of $U(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ mapping $1 \to 1$ and $a \to R_a$, $a \in \mathfrak{J}$, it follows from Theorem 2 that there exists an integer N such that $R_{a_1} R_{a_2} \cdots R_{a_N} = 0$ for any $a_i \in \mathfrak{J}$. Now it is immediate by induction on r that a product of 2^r elements of $\mathfrak{J}$ associated in any way can be written in the form $aR_{a_1}R_{a_2} \cdots R_{a_r}$ for suitable a, $a_i \in \mathfrak{J}$. Hence any product of 2^N elements of $\mathfrak{J}$ is 0 and $\mathfrak{J}$ is nilpotent.

COROLLARY 2. *Let $\mathfrak{J}$ and $\mathfrak{B}$ be as in the theorem, $(S(\mathfrak{J}), \sigma_u)$ the special universal envelope of $\mathfrak{J}$. Then $\mathfrak{B}^{\sigma_u}$ generates a nilpotent ideal in $S(\mathfrak{J})$.*

PROOF. We have the homomorphism of $U(\mathfrak{J})$ onto $S(\mathfrak{J})$ such that $1 \to 1$, $a \to \frac{1}{2} a^{\sigma_u}$. Since a nilpotent ideal is mapped into a nilpotent ideal under a surjective homomorphism, the result follows from Theorem 2.

In the same way we can obtain the analogous statements to Theorem 2 and Corollary 2 for the unital universal multiplication envelope and the unital special universal envelope for a Jordan algebra $\mathfrak{J}$ with identity element.

3. **Finite-dimensional nil algebras.** If $\mathfrak{J}$ is a solvable Jordan algebra, say $\mathfrak{J}^{\cdot 2^N} = 0$, then $a^{\cdot 2^N} = 0$ for every $a \in \mathfrak{J}$. Hence $\mathfrak{J}$ is a *nil algebra* in the sense that every element of $\mathfrak{J}$ is nilpotent. We shall now prove that for finite-dimensional Jordan algebras one has the converse: nillity implies solvability. For this we need the following

LEMMA. *Let $\mathfrak{B}$ be a subalgebra of a Jordan algebra $\mathfrak{J}$, x an element of $\mathfrak{J}$, $x \notin \mathfrak{B}$, such that $\mathfrak{B} . x \subseteq \mathfrak{B}$. Then there exists a $y \in \mathfrak{J}$, $y \notin \mathfrak{B}$, such that $\mathfrak{B} . y \subseteq \mathfrak{B}$ and $\mathfrak{B} . y^{\cdot 2} \subseteq \mathfrak{B}$.*

PROOF. Let $b_1, b_2, b_3 \in \mathfrak{B}$. Then we shall show that

(4) $$x^{\cdot 2} . (b_1 . b_2) \in \mathfrak{B},$$

(5) $$(x^{\cdot 2} . b_1) . b_2 \in \mathfrak{B},$$

(6) $$(x^{\cdot 2} . b_1) . (x^{\cdot 2} . b_2) . b_3 \in \mathfrak{B}.$$

For the proofs of these we shall use the associator identity: $[a,b,c] = -[c,b,a]$ which is immediate by commutativity, and (B_1') on p. 33: $[c.d,a,b] + [b.c,a,d] + [b.d,a,c] = 0$. For (4) we have $x^{\cdot 2}.(b_1.b_2) = [x,x,b_1.b_2] + x.(x.(b_1.b_2)) = [b_1.x,x,b_2] + [b_2.x,x,b_1] + x.(x.(b_1.b_2)) \in \mathfrak{B}$ since $\mathfrak{B}^{\cdot 2} \subseteq \mathfrak{B}$ and $x.\mathfrak{B} \subseteq \mathfrak{B}$. For (5) we have $x^{\cdot 2}.b_1.b_2 = [x^{\cdot 2},b_1,b_2] + x^{\cdot 2}.(b_1.b_2) = x^{\cdot 2}.(b_1.b_2) - 2[x.b_2,b_1,x] \in \mathfrak{B}$ by the hypotheses and (4). For (6) we have $(x^{\cdot 2}.b_1).(x^{\cdot 2}.b_2).b_3 = [x^{\cdot 2}, b_1, x^{\cdot 2}.b_2].b_3 + (x^{\cdot 2}.(b_1.(x^{\cdot 2}.b_2))).b_3$. Using (5) twice we see that the second term is in $\mathfrak{B}$. The first term equals $-2[x,b_1,x^{\cdot 2}.b_2.x].b_3 = -2[x,b_1,b_2.x.x^{\cdot 2}].b_3$ (by $[R_xR_{x^{\cdot 2}}]=0) \in \mathfrak{B}$, by (5). Now if $\mathfrak{B}.x^{\cdot 2} \subseteq \mathfrak{B}$ then we take $y = x$. Otherwise, there exists a $b_1 \in \mathfrak{B}$ such that $y = b_1.x^{\cdot 2} \notin \mathfrak{B}$. Then (5) and (6) show that $y.b$ and $y^{\cdot 2}.b \in \mathfrak{B}$ for all $b \in \mathfrak{B}$, as required.

The condition on y in Lemma 1 can be written in operator form as $\mathfrak{B}R_y \subseteq \mathfrak{B}$ and $\mathfrak{B}R_{y^{\cdot 2}} \subseteq \mathfrak{B}$. Since $R_{y^{\cdot k}}$ is a polynomial in R_y and $R_{y^{\cdot 2}}$ this implies that $\mathfrak{B}R_{y^{\cdot k}} \subseteq \mathfrak{B}$, $k = 1,2,3,\cdots$. Now let $\mathfrak{Y}$ be the set of polynomials $\beta_1 y + \beta_2 y^{\cdot 2} + \beta_3 y^{\cdot 3} + \cdots$, $\beta_i \in \Phi$. Then it is clear that $\mathfrak{C} = \mathfrak{B} + \mathfrak{Y}$ is a subalgebra containing $\mathfrak{B}$ as an ideal. We can now prove

THEOREM 3 (ALBERT). *Any finite-dimensional nil algebra is solvable.*

PROOF. Let $\mathfrak{J}$ be finite-dimensional nil and let $\mathfrak{B}$ be a maximal solvable subalgebra of $\mathfrak{J}$. Consider the subalgebra $\mathfrak{R}$ of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ of linear combinations of elements $R_{b_1}R_{b_2}\cdots R_{b_k}$, $b_i \in \mathfrak{B}$. It follows from Theorem 2 that $\mathfrak{R}$ is a nilpotent algebra of linear transformations in $\mathfrak{J}$ mapping $\mathfrak{B}$ into itself. Hence if $\mathfrak{B} \subset \mathfrak{J}$ then there exists an $x \in \mathfrak{J}$, $\notin \mathfrak{B}$ such that $xR_b \in \mathfrak{B}$ for all $b \in \mathfrak{B}$. This is the same as $x.\mathfrak{B} \subseteq \mathfrak{B}$. Hence by the Lemma and the remark above we have a subalgebra $\mathfrak{C} = \mathfrak{B} + \mathfrak{Y}$ such that $\mathfrak{C} \supset \mathfrak{B}$, $\mathfrak{B}$ is an ideal in $\mathfrak{C}$ and $\mathfrak{Y}$ is the set of polynomials $\beta_1 y + \beta_2 y^{\cdot 2} + \cdots$. Since y is nilpotent it is clear that $\mathfrak{Y}$ is solvable. Now $\mathfrak{C}/\mathfrak{B} \cong \mathfrak{Y}/(\mathfrak{Y} \cap \mathfrak{B})$ is solvable and $\mathfrak{B}$ is solvable. Hence $\mathfrak{C}$ is solvable, contrary to the maximality of $\mathfrak{B}$. Thus $\mathfrak{J} = \mathfrak{B}$ is solvable.

An important consequence of Theorem 3 is the

COROLLARY *Any finite-dimensional algebra $\mathfrak{J}$ which is not solvable contains a nonzero idempotent element.*

PROOF. Since $\mathfrak{J}$ is not solvable it contains an element y which is not nilpotent. The subalgebra $\mathfrak{Y} = \{\beta_1 y + \beta_2 y^{\cdot 2} + \cdots\}$ is associative and can be imbedded in an algebra $\Phi[y]$ with 1. Then, by Lemma 1 on p. 149, $\mathfrak{Y}$ contains a nonzero idempotent element.

EXERCISES

1. Show that Lemma 1 of §2 is valid for nil ideals. Use this to prove that any Jordan algebra $\mathfrak{J}$ contains a nil ideal $\mathfrak{N}$ which contains every nil ideal of $\mathfrak{J}$. Show that $\mathfrak{J}/\mathfrak{N}$ has no nonzero nil ideals.
2. Give an example of a nil algebra which is not solvable.

4. **Reduced Jordan algebras.** An idempotent e in a Jordan algebra $\mathfrak{J}$ will be called *absolutely primitive* if $e \neq 0$ and every element of $\mathfrak{J}_1(e) = \mathfrak{J}U_e$ has the form $\alpha e + z$ where $\alpha \in \Phi$ and z is nilpotent. This condition implies primitivity since $(\alpha e + z)^{\cdot 2} = \alpha e + z$ gives $(\alpha - \alpha^2)e = z^{\cdot 2} + (2\alpha - 1)z$. Since the right-hand side is nilpotent we get $\alpha^2 = \alpha$ or $\alpha = 0, 1$. Then $z = \pm z^{\cdot 2} = \pm z^{\cdot 3} = \cdots = 0$ and $\alpha e + z = e$ or 0. Thus the only idempotents in $\mathfrak{J}_1(e)$ are 0 and e. A Jordan algebra $\mathfrak{J}$ will be called *reduced* if $\mathfrak{J}$ has an identity 1 and $1 = \sum_1^n e_i$ where the e_i are absolutely primitive idempotents. The set $\{e_i\}$ appearing in this definition will be called a *reducing set of idempotents* for $\mathfrak{J}$.

THEOREM 4. *Any finite-dimensional Jordan algebra with* 1 *over an algebraically closed field is reduced.*

PROOF. We observe that any set of nonzero orthogonal idempotents is linearly independent. Hence any such set has cardinality $\leq \dim \mathfrak{J}$. Let $\{e_i \mid i = 1, \cdots, n\}$ be a maximal set of nonzero orthogonal idempotents. If $e = \sum e_i \neq 1$ then $e_{n+1} = 1 - \sum e_i$ is an idempotent $\neq 0$ orthogonal to all the e_i since $e_{n+1} \in \mathfrak{J}_0(e)$ and the $e_i \in \mathfrak{J}_1(e)$ and $\mathfrak{J}_0(e) . \mathfrak{J}_1(e) = 0$. Hence $\sum_1^n e_i = e = 1$. Similarly, if $e_i = e_i' + e_i''$ where e_i', e_i'' are orthogonal idempotents then $e_1, \cdots, e_{i-1}, e_i', e_i'', \cdots, e_n$ are orthogonal idempotents. Hence $e_i' = 0$ or $e_i'' = 0$ which shows that the e_i are primitive. We now note that any primitive idempotent in a finite-dimensional Jordan algebra over an algebraically closed field is absolutely primitive. There is no loss in generality in assuming $e = 1$ is primitive and we do this. Let $a \in \mathfrak{J}$ and let $f(x)$ be the minimum polynomial of a. Suppose $f(x) = f_1(x)f_2(x)$ where $\deg f_i(x) > 0$ and $(f_1(x), f_2(x)) = 1$. Then there exist polynomials $g_i(x)$ such that $\deg g_i(x) < \deg f_i(x)$ and $f_1(x)g_2(x) + f_2(x)g_1(x) = 1$. Then $e_1 = g_1(a)f_2(a) \neq 0$, $e_2 = g_2(a)f_1(a) \neq 0$, $e_1 + e_2 = 1$ and $e_1 . e_2 = 0$ since $g_1(x)f_2(x)g_2(x)f_1(x) \equiv 0 \pmod{f(x)}$. Hence $e_i^{\cdot 2} = e_i$ and 1 is not primitive. Thus we see that $f(x)$ can not be factored as a product of proper relatively prime factors. Since the base field is algebraically closed this implies that $f(x) = (x - \alpha)^k$. Then $(a - \alpha 1)^k = 0$ and $a = \alpha 1 + (a - \alpha 1) = \alpha 1 + z$ where z is nilpotent. Hence 1 is absolutely primitive, which completes the proof.

If e is an idempotent $\neq 0$ in a finite-dimensional Jordan algebra with 1 then the first part of the foregoing proof shows that we can write $e = e_1 + e_2 + \cdots + e_m$ where the e_j are primitive orthogonal idempotents. Also if $f = 1 - e \neq 0$ then $f = e_{m+1} + \cdots + e_n$ where the e_j are primitive orthogonal idempotents. Hence, in any case, we can find a set of primitive orthogonal idempotents $e_1, \cdots, e_n$ such that $\sum_1^m e_j = e$ and $\sum_1^n e_i = 1$. If the base field is algebraically closed then the e_i are absolutely primitive so that these form a reducing set for $\mathfrak{J}$. We shall obtain next a very useful result on the structure of $\mathfrak{J}_1(e)$ for e absolutely primitive. Again we may as well assume $e = 1$ and we have the following:

THEOREM 5. (ALBERT-JACOBSON-MCCRIMMON). *Let $\mathfrak{J}$ be a Jordan algebra with 1 such that every element of $\mathfrak{J}$ has the form $\alpha 1 + z$, $\alpha \in \Phi$, z nilpotent. Then $\mathfrak{J} = \Phi 1 \oplus \mathfrak{N}$ where $\mathfrak{N}$ is the set of nilpotent elements and is an ideal in $\mathfrak{J}$.*

PROOF. If $\alpha \neq 0$ and z is nilpotent then $\alpha 1 - \alpha z$ has the inverse $\alpha^{-1}(1 + z + z^{\cdot 2} + \cdots)$. Hence the set of elements which are not invertible in $\mathfrak{J}$ coincides with the set of $\mathfrak{N}$ nilpotent elements. It is enough to show that $\mathfrak{N}$ is a subalgebra since this will imply $\mathfrak{J} = \Phi 1 + \mathfrak{N}$. Then clearly $\mathfrak{N}$ is an ideal. Now it is clear that $\mathfrak{N}$ is closed under multiplication by elements of Φ and under squaring. Hence to prove $\mathfrak{N}$ a subalgebra it is enough, because of the commutativity of multiplication, to prove $\mathfrak{N}$ is closed under addition. If this is not the case, then we have $z_1, z_2 \in \mathfrak{N}$ such that $z_1 + z_2 = 1 - 4z$ where z is nilpotent. Then there exists an element $u = 1 - 2w$ such that $u^{\cdot 2} = 1 - 4z$ and $u^{\cdot -1}$ exists (Lemma 4(2), p. 150). We now apply the operator $U_{u^{\cdot -1}}$ to both sides of $z_1 + z_2 = 1 - 4z$ and obtain $z_1 U_{u^{\cdot -1}} + z_2 U_{u^{\cdot -1}} = (1 - 4z)U_{u^{\cdot -1}} = u^{\cdot 2} U_{u^{\cdot -1}} = 1 U_u U_{u^{\cdot -1}} = 1$. Moreover, since z_i is not invertible, $z_i U_{u^{\cdot -1}} = \{u^{\cdot -1} z_i u^{\cdot -1}\}$ is not invertible. Hence $w_i = z_i U_{u^{\cdot -1}}$ is nilpotent and we have the relation $w_1 + w_2 = 1$. This is impossible since w_1 is nilpotent and $w_1 = 1 - w_2$ which is invertible. This shows that $\mathfrak{N}$ is a subalgebra and the proof is complete.

Now let $\mathfrak{J}$ be a reduced Jordan algebra, $E = \{e_i \mid i = 1, 2, \cdots, n\}$ a reducing set of idempotents for $\mathfrak{J}$. Let $\mathfrak{J} = \sum_{i \leq j} \oplus \mathfrak{J}_{ij}$ be the Peirce decomposition relative to the e_i. Then, by Theorem 5, applied to the algebra $\mathfrak{J}_{ii}, \mathfrak{J}_{ii} = \Phi e_i + \mathfrak{N}_i$ where $\mathfrak{N}_i$ is a nil ideal in $\mathfrak{J}_{ii}$ and, in fact, $\mathfrak{N}_i$ is the set of nilpotent elements of $\mathfrak{J}_{ii}$. If $i \neq j$ and $a_{ij} \in \mathfrak{J}_{ij}$ then $a_{ij}^{\cdot 2} = \alpha_i e_i + \alpha_j e_j + z_i + z_j$ where $\alpha_i, \alpha_j \in \Phi$, $z_i \in \mathfrak{N}_i$, $z_j \in \mathfrak{N}_j$, and z_i and z_j are nilpotent. By (PD1), $a_{ij}^{\cdot 2} . e_i . a_{ij} = a_{ij}^{\cdot 2} . e_j . a_{ij}$, which gives $(\alpha_i - \alpha_j) a_{ij} = 2(z_j - z_i) . a_{ij}$. Now $z_j - z_i$ is nilpotent; hence $R_{z_j - z_i}$ is nilpotent and the last equation implies that $\alpha_i = \alpha_j$ (even if $a_{ij} = 0$). Hence we have for $i \neq j$ and $a_{ij} \in \mathfrak{J}_{ij}$ that

$$a_{ij}^{\cdot 2} = \alpha(e_i + e_j) + z_i + z_j \tag{7}$$

where $\alpha \in \Phi$, $z_i \in \mathfrak{N}_i$, $z_i \in \mathfrak{N}_j$.

Now let $a \in \mathfrak{J}$ and write

$$a = \sum_1^n (\alpha_i e_i + z_i) + \sum_{i<j} a_{ij} \tag{8}$$

where $\alpha_i \in \Phi$, $z_i \in \mathfrak{N}_i$ and $a_{ij} \in \mathfrak{J}_{ij}$. The decomposition (8) is unique. We now define the *E-trace* of $a \in \mathfrak{J}$ for the reducing set of idempotents E as

(9) $$t_E(a) = \sum_1^n \alpha_i .$$

The mapping $t_E: a \to t_E(a)$ of $\mathfrak{J}$ into Φ is a linear mapping. Also $t_E(e_i) = 1$ and $t_E(1) = n$. The E-trace defines a bilinear mapping, which we denote again by t_E, such that

(10) $$t_E(a, b) = t_E(a, b).$$

Since $\mathfrak{J}$ is commutative it is clear that t_E is a symmetric bilinear form. A bilinear form f on $\mathfrak{J}$ is called *associative* if $f(a\,.\,b, c) = f(a, b\,.\,c)$ for all $a, b, c \in \mathfrak{J}$. We now prove

THEOREM 6. *If $\mathfrak{J}$ is a reduced Jordan algebra, E a reducing set of idempotents for $\mathfrak{J}$ then the E-trace bilinear form t_E is associative.*

PROOF Since $t_E(a, b) = t_E(a.\,b)$, $t_E(a\,.\,b, c) - t_E(a, b\,.\,c) = t_E(a\,.\,b\,.\,c) - t_E(a\,.(b\,.\,c))$ $= t_E([a, b, c])$. Hence we have to show that $t_E([a, b, c]) = 0$ for every associator $[a, b, c]$. It is enough to show this for a, b, c in Peirce spaces. Now a product of three elements in Peirce spaces is either 0 or is in a Peirce space $\mathfrak{J}_{ij}$, $i \neq j$, except in the following three cases: I. The three elements are in a $\mathfrak{J}_{ii}$. II. Two of the elements are in $\mathfrak{J}_{ij}$, $i \neq j$ and the third is in $\mathfrak{J}_{jj}$. III. The elements are in $\mathfrak{J}_{ij}$, $\mathfrak{J}_{jk}$, $\mathfrak{J}_{ik}$ respectively where i, j, k are unequal. Since $t_E(a_{ij}) = 0$ if $a_{ij} \in \mathfrak{J}_{ij}$, $i \neq j$, it is enough to prove the associators have 0 E-trace in the three cases I, II, III. Since $[a, b, c] = -[c, b, a]$ and $[a, b, c] + [b, c, a] + [c, a, b] = 0$ it is enough to prove

I. $t_E([a_{ii}, b_{ii}, c_{ii}]) = 0, \quad a_{ii}, b_{ii}, c_{ii} \in \mathfrak{J}_{ii}$

II. $t_E([a_{ij}, b_{ij}, c_{jj}]) = 0, \quad a_{ij}, b_{ij} \in \mathfrak{J}_{ij}, c_{jj} \in \mathfrak{J}_{ji}, \quad i \neq j.$

III. $t_E([a_{ij}, b_{jk}, c_{ki}]) = 0, \quad a_{ij} \in \mathfrak{J}_{ij}, b_{jk} \in \mathfrak{J}_{jk}, c_{ik} \in \mathfrak{J}_{ik}, \quad i, j, k \neq .$

The first of these is clear since $a_{ii} = \alpha e_i + z_i$, $\alpha_i \in \Phi$, $z_i \in \mathfrak{R}_i$, etc., so $[a_{ii}, b_{ii}, c_{ii}] \in \mathfrak{R}_i$. For II we use (PD4) which gives $[a_{ij}, b_{ij}, c_{jj}] = (e_j - e_i).(a_{ij}.c_{jj}\,.\,b_{ij})$. By (7) and linearization we have $a_{ij}\,.\,c_{jj}\,.\,b_{ij} = \alpha(e_i + e_j) + z_i + z_j$ where $\alpha \in \Phi$, $z_k \in \mathfrak{J}_{kk}$. Hence $[a_{ij}, b_{ij}, c_{jj}] = \alpha(e_j - e_i) + z_j - z_i$ and $t_E([a_{ij}, b_{ij}, c_{jj}]) = 0$ by the definition of t_E. For III we use (PD11) which gives $[a_{ij}, b_{jk}, c_{ki}] = (e_k - e_j)\,.\,a_{ij}.c_{ki}.b_{jk}$. Then the same argument used for II shows that $t_E([a_{ij}, b_{jk}, c_{ki}]) = 0$.

If f is an associative symmetric bilinear form on a Jordan algebra $\mathfrak{J}$ and $\mathfrak{B}$ is an ideal in $\mathfrak{J}$ then the orthogonal complement $\mathfrak{B}^\perp$ of $\mathfrak{B}$ relative to f is an ideal; for, if $b \in \mathfrak{B}$, $c \in \mathfrak{B}^\perp$ and $x \in \mathfrak{J}$ then $f(b, x\,.\,c) = f(b\,.\,x, c) = 0$ so $x\,.\,c \in \mathfrak{B}^\perp$. In particular, $\mathfrak{J}^\perp$, the radical of f is an ideal in $\mathfrak{J}$. Define $f(a) = f(a, 1)$. Then f is linear and $f([a, b, c]) = f(a\,.\,b\,.\,c - a\,.(b\,.\,c), 1) = f(a\,.\,b, c) - f(a, b\,.\,c) = 0$. Now let $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ be a Peirce decomposition of $\mathfrak{J}$ with 1 relative to orthogonal idempotents e_i and let $a_{ij} \in \mathfrak{J}_{ij}$, $i \neq j$. Then $4[a_{ij}, e_i, e_j] = 4a_{ij}\,.\,e_i\,.\,e_j - 4a_{ij}.(e_i\,.\,e_j)$ $= a_{ij}$. Hence $f(a_{ij}) = 0$. If a, b are in different Peirce spaces then $a\,.\,b$ is either 0 or is in a Peirce space $\mathfrak{J}_{ij}$ with $i \neq j$. Then $f(a, b) = f(a\,.\,b, 1) = f(a\,.\,b) = 0$.

Thus we see that the spaces $\mathfrak{J}_{ij}$ are mutually orthogonal relative to the bilinear form f. This implies that $\mathfrak{J}^{\perp} = \sum_{i \leq j}(\mathfrak{J}_{ij}{}^{\perp} \cap \mathfrak{J}_{ij})$. If $f = t_E$ the E-trace defined before relative to a reducing set of idempotents e_i of a reduced Jordan algebra $\mathfrak{J}$, then it is immediate from the definition of t_E that $\mathfrak{J}_{ii}{}^{\perp} \cap \mathfrak{J}_{ii} = \mathfrak{N}_i$ the ideal of nilpotent elements of $\mathfrak{J}_{ii}$. Also since $a_{ij} . b_{ij} = \alpha(e_i + e_j) + z_i + z_j$, $z_k \in \mathfrak{N}_k$, it is clear that for $i \neq j$, $\mathfrak{J}_{ij}{}^{\perp} \cap \mathfrak{J}_{ij} = \{z_{ij} \in \mathfrak{J}_{ij} \mid z_{ij} . a_{ij} \in \mathfrak{N}_i + \mathfrak{N}_j,\ a_{ij} \in \mathfrak{J}_{ij}\}$.

5. **Structure of finite-dimensional semisimple Jordan algebras.** We are now ready to link the theory of finite-dimensional Jordan algebras based on the notion of the radical defined as the maximal solvable ideal with the structure theory based on the axioms (i)–(iv) of the last chapter. The connecting link for the two theories is the following

THEOREM 7. *If $\mathfrak{J}$ is a finite-dimensional Jordan algebra then any absolute zero divisor z in $\mathfrak{J}$ generates a nil ideal.*

PROOF. We reduce the proof first to the case in which $z^{\cdot 2} = 0$. Let $\mathfrak{B}$ be the ideal generated by z, $\mathfrak{C}$ the ideal generated by $z^{\cdot 2}$ so $\mathfrak{B} \supseteq \mathfrak{C}$. Consider $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{C}$. The elements $\bar{z} = z + \mathfrak{C}$ is an absolute zero divisor in $\bar{\mathfrak{J}}$ and $(\bar{z})^{\cdot 2} = 0$. The ideal generated by this element is $\bar{\mathfrak{B}} = \mathfrak{B}/\mathfrak{C}$. Hence, assuming the theorem holds with the additional hypothesis that $z^{\cdot 2} = 0$, we can conclude that $\bar{\mathfrak{B}}$ is a nil ideal. Next we note that $z^{\cdot 2}$ is an absolute zero divisor in $\mathfrak{J}$ and $(z^{\cdot 2})^{\cdot 2} = 0$ since $z^{\cdot 3} = 0$. Hence again we may assume that the ideal $\mathfrak{C}$ generated by $z^{\cdot 2}$ is nil. Since $\mathfrak{B}/\mathfrak{C}$ and $\mathfrak{C}$ are nil it follows that $\mathfrak{B}$ is nil (ex. 1, p. 197). We now assume z is an absolute zero divisor such that $z^{\cdot 2} = 0$. In this case z is an absolute zero divisor in the Jordan algebra $\mathfrak{J}' = \Phi 1 \oplus \mathfrak{J}$ obtained by adjoining the identity element 1 to $\mathfrak{J}$. Hence we may assume that $\mathfrak{J}$ has an identity element. Now it is clear that if z is an absolute zero divisor in an algebra $\mathfrak{J}/\Phi$ then z is an absolute zero divisor in $\mathfrak{J}_\Gamma = \Gamma \otimes_\Phi \mathfrak{J}$ for any extension field Γ/Φ. Also the ideal generated by z in $\mathfrak{J}_\Gamma$ is $\mathfrak{B}_\Gamma$ where $\mathfrak{B}$ is the ideal generated by z in $\mathfrak{J}$. Evidently $\mathfrak{B}_\Gamma$ nil implies $\mathfrak{B}$ nil. Hence we see that it is sufficient to prove the theorem assuming $\mathfrak{J}$ has a 1 and the base field Φ is algebraically closed and we do this from now on. If $\mathfrak{B}$ is not a nil algebra then $\mathfrak{B}$ contains an idempotent element $e \neq 0$ (Corollary to Theorem 3) and $\mathfrak{J}$ contains a reducing set of idempotents $E = \{e_1, e_2, \cdots, e_n\}$ such that $e_1 + e_2 + \cdots + e_m = e$. Then $\mathfrak{B}$ contains the absolutely primitive idempotent $e_1 = e . e_1$. Let t_E be the E-trace bilinear form associated with E, $\mathfrak{J}^{\perp}$ the radical of this form. We proceed to show that $z \in \mathfrak{J}^{\perp}$. We recall that if z is an absolute zero divisor then so is zU_a, $a \in \mathfrak{J}$, and also $(z . a)^{\cdot 2} = 0$ for all $a \in \mathfrak{J}$ (by (2) on p. 156 and the fact that $z^{\cdot 2} = 0$). Write $z = \sum_{i \leq j} z_{ij}$ where $z_{ij} \in \mathfrak{J}_{ij}$. Then $z_{ii} = zU_{e_i}$ is an absolute zero divisor and so z_{ii} is nilpotent. Then $z_{ii} \in \mathfrak{N}_i$ the ideal of nilpotent elements of $\mathfrak{J}_{ii}$. Hence $z_{ii} \in \mathfrak{J}^{\perp}$. Also $zU_{e_i + e_j} = z_{ii} + z_{jj} + z_{ij}$ for $i \neq j$ is an absolute zero divisor. If $a_{ij} \in \mathfrak{J}_{ij}$ then $a_{ij} . z_{ij} = \gamma(e_i + e_j) + w_i + w_j$ where $\gamma \in \Phi$, $w_k \in \mathfrak{N}_k$. Then $a_{ij} . (z_{ii} + z_{jj} + z_{ij}) = d_{ij} + \gamma(e_i + e_j) + w_i + w_j$ where $d_{ij} \in \mathfrak{J}_{ij}$.

Also the square of this element is 0 and this implies that $\gamma d_{ij} = -(w_i + w_j).d_{ij}$. Since $R_{w_i+w_j}$ is nilpotent this gives either $\gamma = 0$ or $d_{ij} = 0$. If $d_{ij} = 0$, $a_{ij}.(z_{ii} + z_{jj} + z_{ij}) = \gamma(e_i + e_j) + w_i + w_j$ and the nilpotency of this element implies $\gamma = 0$. Hence, in any case, $\gamma = 0$, so $a_{ij}.z_{ij} = w_i + w_j$ where $w_k \in \mathfrak{N}_k$, and $t_E(a_{ij}, z_{ij}) = 0$ for all $a_{ij} \in \mathfrak{J}_{ij}$. Then $z_{ij} \in \mathfrak{J}^{\perp}$ and $z \in \mathfrak{J}^{\perp}$. Now $\mathfrak{J}^{\perp}$ is an ideal containing z. Hence $\mathfrak{J}^{\perp} \supseteq \mathfrak{B}$ and consequently $e_1 \in \mathfrak{J}^{\perp}$. Since $t_E(e_1, e_1) = t_E(e_1) = 1$ this is impossible. Hence $\mathfrak{B}$ is nil.

Corollary 1. *A finite-dimensional Jordan algebra $\mathfrak{J}$ is semisimple if and only if it is nondegenerate* (*p.* 155).

Proof. Suppose $\mathfrak{J}$ contains a solvable ideal $\mathfrak{B} \neq 0$. Then by the remarks following Lemma 2 on p. 193, $\mathfrak{J}$ contains an ideal $\mathfrak{C} \neq 0$ such that $\mathfrak{C}^{\cdot 2} = 0$. Then any element of $\mathfrak{C}$ is an absolute zero divisor. Conversely, assume $\mathfrak{J}$ contains an absolute zero divisor $z \neq 0$. Then, by Theorem 7, $\mathfrak{J}$ contains a nil ideal $\mathfrak{B} \neq 0$. Since the dimensionality is finite this is solvable. Hence $\mathfrak{J}$ is not semisimple.

It is now clear that if $\mathfrak{J}$ is finite-dimensional semisimple then $\mathfrak{J}$ satisfies axioms (ii), (iii), (iv) of §4.7 and consequently axiom (i) of §4.2. The structure theory of the last chapter is now applicable. In particular, we can obtain from the First Structure Theorem the following

Corollary 2 (Albert). *Any finite-dimensional semisimple Jordan algebra has an identity element and is a direct sum of ideals which are simple algebras.*

If $\mathfrak{J}$ is an algebraic Jordan division algebra then the minimum polynomial $\mu(\lambda)$ of any $a \in \mathfrak{J}$ is irreducible. Hence, if the base field is algebraically closed, then $\mu(\lambda) = \lambda - \alpha$ and $a = \alpha 1 \in \Phi 1$. Thus $\mathfrak{J} = \Phi$ $(= \Phi 1)$ is the only algebraic Jordan division algebra over the algebraically closed field Φ. Since any finite-dimensional Jordan algebra is algebraic it follows that the only finite-dimensional Jordan division algebra over the algebraically closed field Φ is Φ itself. This trivial remark and the determination of the finite-dimensional simple associative algebras with involution permit us to deduce Albert's determination of the finite-dimensional simple Jordan algebras over an algebraically closed field from the Second Structure Theorem of Chapter IV. The latter also yields all the finite-dimensional simple Jordan algebras over an arbitrary field which are not division algebras. In the remainder of this chapter we shall give another method for determining the finite-dimensional simple Jordan algebras over an arbitrary field which will yield also the special division algebras and will reduce the problem of the determination of the finite-dimensional exceptional division algebras to a problem on the "forms" of $\mathfrak{H}(\mathfrak{O}_3, J_1)$ where $\mathfrak{O}$ is an octonion algebra over an algebraically closed field.

We shall determine first (in §6) the reduced simple Jordan algebras over an arbitrary field Φ. Since any finite-dimensional Jordan algebra with 1 over an algebraically closed field is reduced this will give a determination of the finite-dimensional simple Jordan algebras over algebraically closed fields. The corresponding problem for arbitrary fields can be reduced to the determination of the simple algebras which are central, that is, the centers coincide with the base field. We shall consider this in §7.

EXERCISES

1. Let $\mathfrak{J}$ be a Jordan algebra with 1, $\mathfrak{B}$ a maximal quadratic ideal in $\mathfrak{J}$ (definition in Exercise 3 p. 157) and a an invertible element in $\mathfrak{J}$. Show that $\mathfrak{B}U_a$ is a maximal quadratic ideal in $\mathfrak{J}$. Use this to prove, via the Zariski topology, that if $\mathfrak{J}$ is finite-dimensional over an infinite field and $\mathfrak{R}'$ is the intersection of all the maximal quadratic ideals of $\mathfrak{J}$, then $\mathfrak{R}'U_a \subseteq \mathfrak{R}'$ for every $a \in \mathfrak{J}$. Hence prove that $\mathfrak{R}'$ is an ideal.

2. Let $\mathfrak{B}$ be a maximal quadratic ideal and $\mathfrak{N}$ a nil ideal in a Jordan algebra $\mathfrak{J}$ with 1. Show that $\mathfrak{B} \supseteq \mathfrak{N}$.

3. Prove that the ideal $\mathfrak{R}'$ of Exercise 1 coincides with $\operatorname{rad}\mathfrak{J}$ of the finite-dimensional Jordan algebra $\mathfrak{J}$ with 1 (cf. exs. 4, 5 on p. 157).

4. Extend the result of Exercise 3 to finite-dimensional algebras without 1 using the notion of quasi-invertible elements (Exercise 5, p. 55)·

6. **Reduced simple Jordan algebras.** Let $\mathfrak{J}$ be a simple reduced Jordan algebra, $E = \{e_i \mid i = 1, 2, \cdots, n\}$ a reducing set of idempotents, $\mathfrak{J} = \sum_{i \leq j} \oplus \mathfrak{J}_{ij}$ the corresponding Peirce decomposition and t_E the E-trace bilinear form. We have seen in §4 that the radical $\mathfrak{J}^{\perp}$ of t_E coincides with $\sum_{i \leq j}(\mathfrak{J}_{ij} \cap \mathfrak{J}_{ij}^{\perp})$ and that $\mathfrak{J}_{ii} \cap \mathfrak{J}_{ii}^{\perp}$ is the ideal $\mathfrak{N}_i$ of nilpotent elements of $\mathfrak{J}_{ii}$. Also for $i \neq j$, $\mathfrak{J}_{ij} \cap \mathfrak{J}_{ij}^{\perp}$ $= \{z_{ij} \in \mathfrak{J}_{ij} \mid z_{ij} . \mathfrak{J}_{ij} \subseteq \mathfrak{N}_i + \mathfrak{N}_j\}$. We know also that $\mathfrak{J}^{\perp}$ is an ideal, so by simplicity, either $\mathfrak{J}^{\perp} = 0$ or $\mathfrak{J}^{\perp} = \mathfrak{J}$. Since $t_E(e_1, e_1) = t_E(e_1) = 1$ it is clear that we cannot have $\mathfrak{J}^{\perp} = \mathfrak{J}$. Hence $\mathfrak{J}^{\perp} = 0$ and so t_E is nondegenerate. We now see that every $\mathfrak{N}_i = 0$; hence $\mathfrak{J}_{ii} = \Phi e_i$ and $z_{ij} = 0$ is the only element of $\mathfrak{J}_{ij}$, $i \neq j$, satisfying $z_{ij} . \mathfrak{J}_{ij} = 0$. Suppose for some $i \neq j$ we have $\mathfrak{J}_{ij} \neq 0$. Then there exists an $a_{ij} \in \mathfrak{J}_{ij}$ such that $a_{ij}^{\cdot 2} \neq 0$. Otherwise, linearization gives $a_{ij} . b_{ij} = 0$ for all a_{ij}, b_{ij} and this has been ruled out. We have seen that $a_{ij}^{\cdot 2} = \alpha_{ij}(e_i + e_j)$, $\alpha_{ij} \in \Phi$. It is clear from this that if $a_{ij}^{\cdot 2} \neq 0$ then a_{ij} is invertible in $\mathfrak{J}_{ii} + \mathfrak{J}_{ij} + \mathfrak{J}_{jj}$ and consequently e_i and e_j are connected. Thus we see that if $\mathfrak{J}_{ij} \neq 0$ then e_i and e_j are connected. The simple argument used at the beginning of the proof of the Second Structure Theorem (p. 179) now shows that all the e_i are connected (cf. p. 180). We now distinguish the cases $n = 1$, $n = 2$ and $n \geqq 3$.

$n = 1$. In this case evidently $\mathfrak{J} = \Phi 1$.

$n = 2$. Let $\mathfrak{B} = \Phi(e_1 - e_2) + \mathfrak{J}_{12}$. If $a_{12} \in \mathfrak{J}_{12}$ we have $a_{12}^{\cdot 2} = \alpha(e_1 + e_2) = \alpha 1$ and $(e_1 - e_2) . a_{12} = 0$. Hence if $x = \beta(e_1 - e_2) + a_{12}$ then $x^{\cdot 2} = f(x)1$ where $f(x) = \beta^2 + \alpha \in \Phi$. It is immediate that $f(x)$ is a quadratic form on $\mathfrak{B}$ and that $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ is the Jordan algebra of this form. Since $\mathfrak{J}$ is simple, f is nondegenerate and $\dim \mathfrak{B} > 1$. Also $f(e_1 - e_2) = 1$ so f represents 1.

$n \geqq 3$. Since we have $n \geqq 3$ connected orthogonal idempotents e_i with $\sum e_i = 1$ it is clear from the Coordinatization Theorem that $\mathfrak{J}$ is a Jordan matrix algebra. More precisely, let $u_{1j}, j \geqq 2$, be an element in $\mathfrak{J}_{1j}$ such that $u_{1j}^{\cdot 2} \neq 0$. Then we can write $u_{1j}^{\cdot 2} = \gamma_j^{-1}(e_1 + e_j)$ where $\gamma_j \in \Phi$. Then u_{1j} is invertible in $\mathfrak{J}_{11} + \mathfrak{J}_{1j} + \mathfrak{J}_{jj}$ with inverse $u_{j1} = \gamma_j u_{1j}$. By the Coordinatization Theorem we can identify $\mathfrak{J}$ with a Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n, J_a)$ where $a = \text{diag}\{a_1, a_2, \cdots, a_n\}$, $a_i = \bar{a}_i$, $a_1 = 1$. Moreover, we can identify e_i with e_{ii}, u_{1j} with $e_{1j} + a_j^{-1}e_{j1}$. Since $\gamma_j^{-1}(e_1 + e_j) = u_{1j}^{\cdot 2} = (e_{1j} + a_j^{-1}e_{j1})^{\cdot 2} = a_j^{-1}(e_1 + e_j)$ we have $a_j = \gamma_j$. Hence if we put $\gamma = \{1, \gamma_2, \cdots, \gamma_n\}$ then $\gamma = a$ and our Jordan matrix algebra is $\mathfrak{H}(\mathfrak{D}_n, J_\gamma)$. Now let $x \in \mathfrak{D}$ and consider $x[12] = xe_{12} + \gamma_2^{-1}\bar{x}e_{21}$, $y[12] = ye_{12} + \gamma_2^{-1}e_{21}$. Since $x[12]$, $y[12] \in \mathfrak{J}_{12}$ and $a_{12}^{\cdot 2} = \alpha(e_1 + e_2)$, $\alpha \in \Phi$, for any $a_{12} \in \mathfrak{J}_{12}$ we have $x[12] . y[12] = f(x,y)(e_1 + e_2)$ where $f(x,y) \in \Phi$. On the other hand, $x[12] . y[12] = (xe_{12} + \gamma_2^{-1}\bar{x}e_{21}) \cdot (ye_{12} + \gamma_2^{-1}\bar{y}e_{21}) = \frac{1}{2}\gamma_2^{-1}(x\bar{y} + y\bar{x})e_1 + \frac{1}{2}\gamma_2^{-1}(\bar{x}y + \bar{y}x)e_2$. Hence

$$(11) \qquad 2\gamma_2 f(x,y) = x\bar{y} + y\bar{x} = \bar{x}y + \bar{y}x.$$

Then $x\bar{x} = Q(x)1 = \bar{x}x$ in $\mathfrak{D}$ where $Q(x) = \gamma_2 f(x,x)$ is a quadratic form on $\mathfrak{D}$. We have seen that if $x[12] \neq 0$ then there exists a $y[12]$ such that $x[12] \cdot y[12] \neq 0$. This condition is equivalent to nondegeneracy of Q. Hence we see that $(\mathfrak{D}, j)$, $j : d \to \bar{d}$, is a composition algebra and we have proved

THEOREM 8. *Any reduced simple Jordan algebra is of one of the following types*: (i) $\mathfrak{J} = \Phi 1$, (ii) $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$, *the Jordan algebra of a nondegenerate symmetric bilinear form f on* $\mathfrak{V}$ *such that* $\dim \mathfrak{V} > 1$ *and there exists an* $x \in \mathfrak{V}$ *such that* $f(x,x) = 1$, (iii) $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_n, J_\gamma)$, $n \geqq 3$, *where* $(\mathfrak{D}, j)$ *is a composition algebra which is associative if* $n \geqq 4$, *and* $\gamma = \text{diag}\{1, \gamma_2, \cdots, \gamma_n\}$, $\gamma_i \neq 0$ *in* Φ.

We recall that the composition algebras were determined in Theorem 4.5 (p. 164). In particular, we saw that their dimensionalities over Φ are 1, 2, 4 or 8. It follows from Theorem 8 that the dimensionalities of the reduced simple Jordan algebras with reducing set of idempotents of cardinality $n \geqq 3$ are finite. Hence it is clear that we have the following

COROLLARY 1. *Any reduced simple Jordan algebra is either finite-dimensional or is a Jordan algebra of a nondegenerate symmetric bilinear form on an infinite-dimensional vector space.*

It is easily verified that the converse of Theorem 8 is valid also. This is trivial for the algebras in (i) and is immediate for those in (ii) (cf. p. 180). The Jordan matrix algebras $\mathfrak{H}(\mathfrak{D}_n, J_\gamma)$, $n \geqq 3$, in (iii) are simple since the composition algebras are simple (cf. p. 170). The fact that these are reduced is clear from the Peirce decomposition of $\mathfrak{H}(\mathfrak{D}_n, J_\gamma)$ relative to the idempotents $e_i = \frac{1}{2}[ii]$.

We recall that we have called a composition algebras $(\mathfrak{D}, j)$ split if it is not division algebra. Such an algebra is uniquely determined (up to isomorphism) by its dimensionality (Theorem 4.7). If $\dim\mathfrak{D} = 2$ then $\mathfrak{D} = \Phi \oplus \Phi$ and j exchanges the two components. If $\dim\mathfrak{D} = 4$ then $(\mathfrak{D}, j) \cong (\Phi_2, J)$ where J is the involution

$$X \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{(see p. 128).}$$

Now consider a reduced Jordan algebra $\mathfrak{H}(\mathfrak{D}_n, J_\gamma)$ where $n \geqq 3$ and $(\mathfrak{D}, j)$ is a split composition algebra and $\gamma = \operatorname{diag}\{1, \gamma_2, \cdots, \gamma_n\}$, $\gamma_i \neq 0$ in Φ. If $a \in \mathfrak{D}$ and $j > 1$ then $a[1j] = ae_{1j} + \gamma_j^{-1}\bar{a}e_{j1} \in \mathfrak{J}_{1j}$ and $a[1j]^{\cdot 2} = \gamma_j^{-1}Q(a)(e_1 + e_j)$. Since Q has positive Witt index we can choose $a \in \mathfrak{D}$ so that $Q(a) = \gamma_i$. If we do this for $j > 1$ and replace $u_{1j} \in \mathfrak{J}_{1j}$ in the Coordinatization Theorem by this element then we may assume that all the γ's are 1, which means that the involution is the standard one J_1. The new choice of u_{1j} alters the composition algebra. However, it is clear that the dimensionality is unchanged since it coincides with $\dim\mathfrak{J}_{ij}$, $i \neq j$. Also the norm form Q for the new composition algebra is a multiple of the quadratic form $f(x)$ obtained by writing $x[12]^{\cdot 2} = f(x)(e_1 + e_2)$ and hence is a multiple of the norm form of $(\mathfrak{D}, j)$. Hence the new composition algebra is split and so is isomorphic to $(\mathfrak{D}, j)$. Thus $\mathfrak{H}(\mathfrak{D}_n, J_\gamma) \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$ so we may always assume the involution is standard if $(\mathfrak{D}, j)$ is split. A Jordan algebra $\mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is a split composition algebra will be called a *split* Jordan algebra.

If the base field is algebraically closed then any composition algebra $(\mathfrak{D}, j)$ with $\dim\mathfrak{D} > 1$ is split since any quadratic form on a finite-dimensional vector space of dimensionality > 1 over an algebraically closed field has maximal Witt index. Also, in the algebraically closed case, in $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_\gamma)$, if $u_{1j}^{\cdot 2} = \gamma_j^{-1}(e_1 + e_j)$ then $v_{ij} = \gamma_j^{\frac{1}{2}}u_{1j}$ satisfies $v_{1j}^{\cdot 2} = e_1 + e_j$. Hence it is clear that for any composition algebra $(\mathfrak{D}, j)$ over an algebraically closed field, $\mathfrak{H}(\mathfrak{D}_n, J_\gamma) \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$. Thus we may assume that the involution is standard. We recall also that if $\dim\mathfrak{D} = 2$ then $\mathfrak{H}(\mathfrak{D}_n, J_1) \cong \mathfrak{D}_n{}^+$ and if $\dim\mathfrak{D} = 4$ then $\mathfrak{H}(\mathfrak{D}_n, J_1) \cong \mathfrak{H}(\Phi_{2n}, J_S)$ where J_S is $X \to S^{-1}X^tS$, $S = \operatorname{diag}\{Q, Q, \cdots, Q\}$ where $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Now suppose $\mathfrak{J}$ is a finite-dimensional simple Jordan algebra over the algebraically closed field Φ. Let $\mathfrak{S}$ be a solvable ideal in $\mathfrak{J}$. Then either $\mathfrak{S} = 0$ or $\mathfrak{S} = \mathfrak{J}$. If $\mathfrak{S} = \mathfrak{J}$ then $\mathfrak{S}^{\cdot 2} \subset \mathfrak{S}$ is an ideal so $\mathfrak{J}^{\cdot 2} = \mathfrak{S}^{\cdot 2} = 0$. This contradicts the definition of simplicity. Hence $\mathfrak{S} = 0$ and $\mathfrak{J}$ is semisimple. Consequently $\mathfrak{J}$ has an identity and $\mathfrak{J}$ is reduced (Theorem 4). Our determination of reduced simple Jordan algebras over an algebraically closed field now yields the following

COROLLARY 2 (ALBERT). *Let $\mathfrak{J}$ be a finite-dimensional simple Jordan algebra over an algebraically closed field Φ. Then we have the following possibilities for $\mathfrak{J}$: (1) $\mathfrak{J} = \Phi$, (2) $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ the Jordan algebra of a nondegenerate sym-*

metric bilinear form f in a finite-dimensional vector space $\mathfrak{V}$ *such that* $\dim \mathfrak{V} > 1$, (3) $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_1)$, $n \geqq 3$, *where* $(\mathfrak{D}, j)$ *is a composition algebra of dimension* 1, 2 *or* 4 *if* $n \geqq 4$ *and of dimensions* 1, 2, 4, *and* 8 *if* $n = 3$.

Conversely the algebras listed are simple Jordan algebras.

Exercises

1. Show that if $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ is the Jordan algebra of a symmetric bilinear form f of positive Witt index on the finite-dimensional vector space $\mathfrak{V}$ then $\mathfrak{J}$ contains nonzero nilpotent elements. Hence prove that if the base field is the field of real numbers then $\mathfrak{J}$ contains nonzero nilpotent elements if and only if f is not definite (positive or negative).

2. Show that if $\mathfrak{M}$ is a finite-dimensional vector space over a division algebra Δ with involution j and f is a nondegenerate hermitian form of positive Witt index or a nondegenerate skew form (Δ commutative, $j = 1$) on $\mathfrak{M}$, then the Jordan algebra $\mathfrak{H}$ of self-adjoint linear transformations in $\mathfrak{M}$ over Δ contains nonzero nilpotent elements. (Hint: See p. 61 where a special case of this is proved.)

3. Prove that if $\mathfrak{D}$ is a composition algebra and $\gamma = \{1, -1, 1\}$ then $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ contains nonzero nilpotent elements. (Hint: Consider the subalgebra $\mathfrak{H}(\Phi_3, J_\gamma)$ and apply Exercise 2.) Use this to show that if $\mathfrak{O}$ is a split octonion algebra then $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ contains nonzero nilpotent elements.

4. A Jordan algebra $\mathfrak{J}$ over the field R of real numbers is called *formally real* if $a^{\cdot 2} + b^{\cdot 2} = 0$ in $\mathfrak{J}$ implies $a = 0$ and $b = 0$. Show that if $\mathfrak{J}$ has an identity 1 and is formally real and algebraic (every element is algebraic) then the minimum polynomial of every $a \in \mathfrak{J}$ has only real roots. Hence show that if such a $\mathfrak{J}$ is a division algebra then $\mathfrak{J} = R1$ is one dimensional over R.

5. Show that any finite-dimensional formally real Jordan algebra over R is semisimple and hence has an identity element and is a direct sum of simple ideals.

6. Show that any finite-dimensional formally real simple Jordan algebra $\mathfrak{J}$ over R is reduced. Use this and Exercises 1–3 to prove the following *theorem of Jordan, von Neumann and Wigner*: Any finite-dimensional simple formally real Jordan algebra over R is isomorphic to one of the following: I. $\mathfrak{J} = R1$, II. $\mathfrak{J} = R1 \oplus \mathfrak{V}$ the Jordan algebra of a positive definite symmetric bilinear form f on $\mathfrak{V}$, III. $\mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is either R, C the field of complex numbers with the usual involution, Q Hamilton's quaternion division algebra with the standard involution, or if $n = 3$, O the (unique) octonion division algebra over R with its standard involution and J_1 is the standard involution in $\mathfrak{D}_n$.

Remark. Formally real Jordan algebras were introduced by Jordan, von Neumann and Wigner in [1]. The defining property which they used, which is the same as that in the Artin-Schreier theory of formally real fields, is that $\sum_{i=1}^{n} a_i^{\cdot 2} = 0$ implies every $a_i = 0$. The above definition with $n = 2$ is due to

Braun and Koecher [1]. Further results on these algebras can be found in Chapter 11 of their book.

7. **Finite-dimensional central simple Jordan algebras.** With any algebra $\mathfrak{A}/\Phi$ we can associate two associative algebras of linear transformations. The first of these is the *multiplication algebra* $M(\mathfrak{A})$ which is defined to be the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ generated by the identity mapping 1 and the left and right multiplications $a_L, a_R, a \in \mathfrak{A}$. The second is the *centroid* $C(\mathfrak{A})$ of $\mathfrak{A}$ which is the centralizer of $M(\mathfrak{A})$ in $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$. Thus $C(\mathfrak{A})$ is the set of linear mappings γ in $\mathfrak{A}/\Phi$ such that $\gamma m = m\gamma$ for every $m \in M(\mathfrak{A})$. Since 1 and the a_L and a_R generate $M(\mathfrak{A})$ it is clear that $C(\mathfrak{A})$ is the set of linear mappings γ such that $a_L\gamma = \gamma a_L$, $a_R\gamma = \gamma a_R$ or, equivalently, $(ax)\gamma = a(x\gamma)$, $(xa)\gamma = (x\gamma)a, a, x \in \mathfrak{A}$. The algebra $\mathfrak{A}$ can be considered in the natural way as (right) module for $M(\mathfrak{A})$ and for $C(\mathfrak{A})$. It is clear that the submodules of $\mathfrak{A}$ as $M(\mathfrak{A})$-module are just the ideals of $\mathfrak{A}$. It follows that $\mathfrak{A}$ is simple if and only if $\mathfrak{A}^2 \neq 0$ and $\mathfrak{A}$ is irreducible as $M(\mathfrak{A})$-module. In this case we have $\mathfrak{A}^2 = \mathfrak{A}$ since $\mathfrak{A}^2$ is an ideal in $\mathfrak{A}$. If $a, b \in \mathfrak{A}$ and $\gamma, \delta \in C(\mathfrak{A})$ then $(ab)\gamma\delta = ((a\gamma)b)\delta = (a\gamma)(b\delta)$ and $(ab)\delta\gamma = (a(b\delta))\gamma = (a\gamma)(b\delta)$. Hence $(ab)(\gamma\delta - \delta\gamma) = 0$ and consequently if $\mathfrak{A}^2 = \mathfrak{A}$ then $C(\mathfrak{A})$ is commutative. In particular, we see that the centroid of a simple algebra is commutative. Also it is clear from Schur's lemma applied to $M(\mathfrak{A})$ that the centroid $C(\mathfrak{A})$ of a simple algebra is a division algebra. Hence $C(\mathfrak{A})$ is a field if $\mathfrak{A}$ is simple. Also it is clear that $C(\mathfrak{A})$ contains the set of mappings $x \to \alpha x$, $\alpha \in \Phi$, which can be identified with Φ. We shall now call a simple algebra $\mathfrak{A}$ *central* if $C(\mathfrak{A})$ coincides with the set of mappings $x \to \alpha x$, $\alpha \in \Phi$, in $\mathfrak{A}$.

If $\mathfrak{A}$ is simple with centroid $\Gamma = C(\mathfrak{A}) \supseteq \Phi$ then we can consider $\mathfrak{A}$ as left Γ-module by defining $\gamma a = a\gamma$, $\gamma \in \Gamma$, $a \in \mathfrak{A}$. This is clear since Γ is commutative. Hence $\mathfrak{A}$ is a vector space over Γ. Moreover, we have

$$\gamma(ab) = (\gamma a)b = a(\gamma b), \tag{12}$$

which shows that $\mathfrak{A}$ is an algebra over Γ. Since $\Gamma \supseteq \Phi$ any ideal in $\mathfrak{A}/\Gamma$ is an ideal in $\mathfrak{A}/\Phi$. Also $\mathfrak{A}^2 = \mathfrak{A}$. Hence it is clear that $\mathfrak{A}/\Gamma$ is simple. Also it is immediate that Γ is also the centroid of $\mathfrak{A}/\Gamma$. Thus any simple algebra can be considered as a central simple algebra over its centroid. In this sense we can reduce the consideration of simple algebras to central simple ones, and for those we have the following fundamental property:

THEOREM 9. *Let $\mathfrak{A}/\Phi$ be central simple. Then $\mathfrak{A}_P$ is simple for any extension field* P/Φ.

We refer the reader to the author's *Lie Algebras*, p. 292 for the proof of this theorem.

We now suppose that the algebra $\mathfrak{A}/\Phi$ has an identity element 1. Then, of course, $\mathfrak{A}^2 = \mathfrak{A}$ so the centroid $C(\mathfrak{A})$ is commutative. Let $\gamma \in C(\mathfrak{A})$ and put $c = 1\gamma$.

Then $ac = a(1\gamma) = a\gamma$, $ca = (1\gamma)a = (1a)\gamma = a\gamma$. Hence $\gamma = c_L = c_R$. Also $[a, b, c] = [a, b, 1\gamma] = ab(1\gamma) - a(b(1\gamma)) = ab\gamma - ab\gamma = 0$ and, similarly, $[a, c, b] = 0$, $[c, a, b] = 0$. This and $c_L = c_R$ show that $1\Gamma = \{1\gamma \mid \gamma \in C(\mathfrak{A})\} \subseteq \mathfrak{C}$ the center of $\mathfrak{A}$. Conversely, if $c \in \mathfrak{C}$ then $\gamma = c_L = c_R$ is in the centroid since $[a, b, c] = 0$ gives $(ab)\gamma = a(b\gamma)$ and $[c, a, b] = 0$ gives $(ab)\gamma = (a\gamma)b$. Thus $\mathfrak{C} = 1\Gamma$. Since $\mathfrak{A}$ has an identity element it is clear that the mapping $a \to a_R$ of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ is injective. Also we have $(cd)_R = c_R d_R$ if $c, d \in \mathfrak{C}$ since $a(cd) = (ac)d$. It follows that $c \to c_R$ is an isomorphism of $\mathfrak{C}$ onto the centroid, and we can identify the center and centroid for algebras with 1. In particular, our results show that if $\mathfrak{A}$ is a simple algebra with 1 then the center $\mathfrak{C}$ is a field and $\mathfrak{A}$ can be regarded as a central simple algebra over $\mathfrak{C}$.

We now consider the problem of classifying the finite-dimensional central simple Jordan algebras. Let $\mathfrak{J}/\Phi$ be such an algebra and let Ω be the algebraic closure of the base field Φ. Then $\mathfrak{J}_\Omega$ is a finite-dimensional simple Jordan algebra over the algebraically closed field Ω and hence $\mathfrak{J}_\Omega$ is one of the algebras listed in Corollary 2 to Theorem 8. We shall now use this information to determine $\mathfrak{J}$.

First, it is clear that if $\mathfrak{J}_\Omega = \Omega 1$ then $\dim \mathfrak{J}/\Phi = 1$ and $\mathfrak{J} = \Phi 1$. Next assume that $\mathfrak{J}_\Omega = \Omega 1 \oplus \tilde{\mathfrak{B}}$ the Jordan algebra of a nondegenerate symmetric bilinear form $\tilde{f}$ on the finite-dimensional vector space $\tilde{\mathfrak{B}}/\Omega$ with $\dim \tilde{\mathfrak{B}} \geqq 2$. It is easy to verify that $\tilde{\mathfrak{B}}$ is the subspace of $\tilde{\mathfrak{J}} = \mathfrak{J}_\Omega$ spanned by the associators $[\tilde{a}, \tilde{b}, \tilde{c}]$, $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{\mathfrak{J}}$ (ex. 5, p. 14). Since the associator is trilinear it is clear that if $\mathfrak{B}$ denotes the subspace of $\mathfrak{J}$ spanned by the $[a, b, c]$, $a, b, c \in \mathfrak{J}$ then $\tilde{\mathfrak{B}} = \Omega\mathfrak{B}$. Then $\dim \mathfrak{B} = \dim \mathfrak{J} - 1$ and $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ since $\mathfrak{J}_\Omega = \Omega 1 \oplus \tilde{\mathfrak{B}}$. Now let $x, y \in \mathfrak{B}$. Then $x . y = \tilde{f}(x, y)1$ where $\tilde{f}(x, y) \in \Omega$. Since $x . y \in \mathfrak{J}$ and $\mathfrak{J} \cap \Omega 1 = \Phi 1$ it follows that $\tilde{f}(x, y) \in \Phi$. Thus the restriction f of $\tilde{f}$ to $\mathfrak{B}$ is a symmetric bilinear form on $\mathfrak{B}$. It is clear that this is nondegenerate. Hence $\mathfrak{J}$ is the Jordan algebra of a nondegenerate symmetric bilinear form on the finite-dimensional space $\mathfrak{B}$ with $\dim \mathfrak{B} \geqq 2$. The classification of these algebras is equivalent to the classification of the corresponding symmetric bilinear forms (cf. ex. 5, p. 14).

It remains to consider the algebras $\mathfrak{J}$ such that $\mathfrak{J}_\Omega = \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $n \geqq 3$ and $(\mathfrak{D}, j)$ is a (split) composition algebra which is associative if $n \geqq 4$. We postpone the case $n = 3$, $(\mathfrak{D}, j)$, an algebra of octonions, to Chapter IX. These are exceptional algebras. The remaining algebras are special (Theorem 2.8, p. 78). We shall reduce the study of these algebras to simple associative algebras with involution. For this we require some concepts for algebras with involution which are analogous to those we considered for ordinary algebras at the beginning of this section.

Let $(\mathfrak{A}, J)$ be an algebra with involution. Then we define the *multiplication algebra* $M(\mathfrak{A}, J)$ of $(\mathfrak{A}, J)$ to be the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ generated by $1, J, a_L, a_R, a \in \mathfrak{A}$. We define the *centroid* $C(\mathfrak{A}, J)$ of $(\mathfrak{A}, J)$ to be the centralizer of $M(\mathfrak{A}, J)$ in $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$. Since J is an involution we have $(ab)^J = b^J a^J$ which implies that $b_R J = J(b^J)_L$ and $a_L J = J(a^J)_R$. It follows that if $m \in M(\mathfrak{A})$

then $J^{-1}mJ \in M(\mathfrak{A})$. Also $J^2 = 1$. This implies that $M(\mathfrak{A}, J)$ is the set of linear transformations of the form $m_1 + m_2 J$ where $m_i \in M(\mathfrak{A})$. Clearly $C(\mathfrak{A}, J)$ is the subalgebra of $C(\mathfrak{A})$ of elements commuting with J. If $\mathfrak{A}^2 = \mathfrak{A}$ then $C(\mathfrak{A})$ and hence $C(\mathfrak{A}, J)$ is commutative. The algebra with involution $(\mathfrak{A}, J)$ is simple if and only if $\mathfrak{A}^2 \neq 0$ and $\mathfrak{A}$ is an irreducible module for $M(\mathfrak{A}, J)$. In this case the centroid $C(\mathfrak{A}, J)$ is a field. We shall call $(\mathfrak{A}, J)$ *central simple* if $(\mathfrak{A}, J)$ is simple and $C(\mathfrak{A}, J)$ coincides with the set of mappings $x \to \alpha x$, α in the base field Φ of $\mathfrak{A}$. The proof that if $\mathfrak{A}$ is central simple over Φ then $\mathfrak{A}_{\mathrm{P}}$ is simple for every extension field P/Φ (*Lie Algebras*, p. 292) can be carried over almost word for word to show that if $(\mathfrak{A}, J)$ is central simple then $(\mathfrak{A}_{\mathrm{P}}, J)$ is simple. We therefore omit this proof.

We define the *center* of an algebra with involution $(\mathfrak{A}, J)$ to be the subalgebra of the center $\mathfrak{C}$ of $\mathfrak{A}$ of fixed elements under J. If $\mathfrak{A}$ has an identity element then we have seen that the centroid is the set of elements $c_L = c_R$, c in the center of $\mathfrak{A}$. It follows that if $\mathfrak{A}$ has an identity element then the centroid of the algebra with involution $(\mathfrak{A}, J)$ is the set of elements $c_L = c_R$, c in the center of $(\mathfrak{A}, J)$. In this way we can identify the centroid with the center of $(\mathfrak{A}, J)$ if $\mathfrak{A}$ has a 1.

We consider next the finite-dimensional central simple associative algebras with involution. If $(\mathfrak{A}, J)$ satisfies these conditions then either $\mathfrak{A}$ is simple or $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is simple. In both cases $\mathfrak{A}$ has an identity element. If $\mathfrak{A}$ is simple the center P of $\mathfrak{A}$ is a field and the restriction of J to P is an automorphism such that $J^2 = 1$. Hence either J is the identity on P or J is an automorphism of period two in P. In the first case, J is called an involution of *first kind* and in the second case, an involution of *second kind*. Since $(\mathfrak{A}, J)$ is central simple the subfield of P of fixed elements under J is the base field Φ. Hence $\mathrm{P} = \Phi$ if J is of first kind, and P is a quadratic field extension of Φ if J is of second kind. If $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is simple and c is an element of the center of $\mathfrak{B}$ then $c + c^J$ is an element of the center of $(\mathfrak{A}, J)$. Since the center of $(\mathfrak{A}, J)$ is Φ it is clear that the center of $\mathfrak{B}$ is Φ, that is, $\mathfrak{B}$ is central simple. Then the center of $\mathfrak{A}$ can be identified with the two-dimensional algebra $\Phi \oplus \Phi$ and J interchanges the two components of the center. If Ω is the algebraic closure of Φ then $(\mathfrak{A}_\Omega, J)$ is simple and the dimensionality of the center of $\mathfrak{A}_\Omega$ over Ω is the same as that of the center P/Φ. Also since Ω is algebraically closed the only possibilities for the center of $\mathfrak{A}_\Omega$ are Ω and $\Omega \oplus \Omega$. The first case occurs if $\mathfrak{A}$ is central simple and the second case if either $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ or $\mathfrak{A}$ is simple and J is of second kind.

We now consider the possibilities for $(\mathfrak{A}_\Omega, J)$. By the pre-Wedderburn theorem, the finite-dimensional simple associative algebras over Ω are the algebras Ω_n, $n = 1, 2, \cdots$, and conversely. It is known also that if J is an involution in Ω_n then $(\Omega_n, J) \cong (\Omega_n, J_1)$ or $(\Omega_n, J) \cong (\Omega_n, J_S)$ where J_S is the involution $X \to S^{-1}X^tS$ and $S = \operatorname{diag}\{Q, Q, \cdots, Q\}$, $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus in the second case n is even, and we shall write $2n$ for n in this case. Hence any finite-dimensional simple associative algebra with involution over Ω is isomorphic to one of the following:

A. $(\Omega_n \oplus \Omega_n, J)$ where J is the exchange involution $(X, Y^t) \to (Y, X^t)$, B. (Ω_n, J_1), C. (Ω_{2n}, J_S). Algebras in different classes A, B or C are not isomorphic and algebras in the same class are not isomorphic if the integers n are different for the two. We have seen before that $(\Omega_n \oplus \Omega_n, J) \cong (\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is the (split) two-dimensional composition algebra and $(\Omega_{2n}, J_S) \cong (\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is the (split) four-dimensional composition algebra. Hence we see that any finite-dimensional simple associative algebra with involution over Ω is isomorphic to one of the algebras $(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is an associative composition algebra. Moreover, isomorphism of two of these holds if and only if the composition algebras are isomorphic and the integers n are equal. We shall call the invariant n the *degree* of $(\mathfrak{D}_n, J_1)$.

If $(\mathfrak{A}_\Omega, J) \cong (\mathfrak{D}_n, J_1)$ then we shall call n the *degree* of the algebra with involution $(\mathfrak{A}, J)$ over Φ. Also we shall say that $(\mathfrak{A}, J)$ is of *type* A, B or C according as the composition algebra is two dimensional, one dimensional or four dimensional.

Now let $\mathfrak{J}$ be a Jordan algebra over Φ such that $\mathfrak{J}_\Omega \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is an associative composition algebra over Ω and $n \geqq 3$. We claim that n is an invariant of $\mathfrak{J}$. This will follow if we can show that if $(\mathfrak{D}, j)$ and $(\mathfrak{D}', j')$ are composition algebras over Ω then the isomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_1)$ and $\mathfrak{H}(\mathfrak{D}'_{n'}, J_1)$ for $n, n' \geqq 3$ imply that $n = n'$. By Corollary 1 to Martindale's theorem (p. 141), $(\mathfrak{D}_n, J_1)$ and $(\mathfrak{D}'_{n'}, J_1)$ are perfect. Hence the isomorphism of $\mathfrak{H}(\mathfrak{D}_n, J_1)$ and $\mathfrak{H}(\mathfrak{D}'_{n'}, J_1)$ implies that $(\mathfrak{D}_n, J_1)$ and $(\mathfrak{D}_{n'}, J_1)$ are isomorphic, and we have seen that this implies $n = n'$. We shall now call the integer $n \geqq 3$ the *degree* of $\mathfrak{J}$. In the next chapter we shall see that this can be defined in a more direct manner as the degree of the generic minimum polynomial of $\mathfrak{J}$ (§6.4).

We are now ready to establish the connection between finite-dimensional special central simple Jordan algebras and central simple associative algebras with involutions. The key result is the following

THEOREM 10. (1) *Let $(\mathfrak{A}, J)$ be a finite-dimensional central simple associative algebra with involution of degree $n \geqq 3$. Then $(\mathfrak{A}, J)$ is perfect and $\mathfrak{H}(\mathfrak{A}, J)$ is a finite-dimensional central simple Jordan algebra of degree n.* (2) *Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $n \geqq 3$. Then $\mathfrak{J}$ is reflexive and $(S_1(\mathfrak{J}), \pi)$, where $S_1(\mathfrak{J})$ is the unital special universal envelope and π is its main involution, is a finite-dimensional central simple associative algebra with involution of degree n.*

PROOF. (1) Let $(\mathfrak{A}, J)$ be a finite-dimensional central simple associative algebra with involution of degree $n \geqq 3$, Ω the algebraic closure of the base field Φ. Then $(\mathfrak{A}_\Omega, J)$ has the form $(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is an associative composition algebra over Ω and J_1 is the standard involution. Since $n \geqq 3$, $(\mathfrak{D}_n, J_1)$ is perfect and $\mathfrak{H}(\mathfrak{D}_n, J_1) = \mathfrak{H}(\mathfrak{A}_\Omega, J)$ is simple. Since perfection is a linear property (p. 78) the first statement implies that $(\mathfrak{A}, J)$ is perfect. Since any ideal in $\mathfrak{H}(\mathfrak{A}, J)$ extends to an ideal in $\mathfrak{H}(\mathfrak{A}, J)_\Omega \cong \mathfrak{H}(\mathfrak{A}_\Omega, J)$ it is clear that $\mathfrak{H}(\mathfrak{A}, J)$ is simple. The center of a finite-dimensional simple algebra with 1 over an algebraically closed field Ω coincides with Ω. Now it is clear that if $\mathfrak{C}$ is the center of $\mathfrak{H}(\mathfrak{A}, J)$ then $\mathfrak{C}_\Omega$ is the center of $\mathfrak{H}(\mathfrak{A}_\Omega, J)$. Since $\mathfrak{C}_\Omega = \Omega$ is one dimensional over Ω, $\mathfrak{C}/\Phi$ is one dimensional. Hence $\mathfrak{H}(\mathfrak{A}, J)$ is central simple. Since $\mathfrak{H}(\mathfrak{A}_\Omega, J) \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$ the degree of $\mathfrak{H}(\mathfrak{A}, J)$ is n. (2) Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $n \geqq 3$. Then $\mathfrak{J}_\Omega = \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $\mathfrak{D}$ is an associative composition algebra. Since $(\mathfrak{D}_n, J_1)$ is perfect $\mathfrak{J}_\Omega = \mathfrak{H}(\mathfrak{D}_n, J_1)$ is reflexive and hence $\mathfrak{J}$ is reflexive (p. 77 and p. 78). Also $(S_1(\mathfrak{J}_\Omega), \pi) \cong (\mathfrak{D}_n, J_1)$ and $(\mathfrak{D}_n, J_1)$ is a simple algebra with involution since $\mathfrak{H}(\mathfrak{D}_n, J_1)$ is simple. It follows that $(S_1(\mathfrak{J}), \pi)$ is central simple and also it is clear that its degree is n.

We can now prove the main result on the classification of finite-dimensional special central simple Jordan algebras of degree $\geqq 3$.

THEOREM 11. *A Jordan algebra is finite-dimensional special central simple of degree $n \geqq 3$ if and only if its is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite-dimensional central simple associative algebra with involution of degree n. If $(\mathfrak{A}, J)$, $(\mathfrak{B}, K)$ are finite-dimensional central simple associative algebras with involutions of degree $\geqq 3$ then these are isomorphic if and only if the Jordan algebras $\mathfrak{H}(\mathfrak{A}, J)$, $\mathfrak{H}(\mathfrak{B}, K)$ are isomorphic.*

PROOF. If $\mathfrak{J}$ is a special finite-dimensional central simple Jordan algebra of degree $n \geqq 3$ then $\mathfrak{J}$ is isomorphic to a Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is finite-dimensional central simple associative algebra with involution of degree n, by the second part of the preceding theorem. The converse follows from the last statement of the first part of the same theorem. If $(\mathfrak{B}, K)$ is a second associative algebra with involution of the type indicated then an isomorphism of $\mathfrak{H}(\mathfrak{A}, J)$ onto $\mathfrak{H}(\mathfrak{B}, K)$ can be extended uniquely to an isomorphism of $(\mathfrak{A}, J)$ onto $(\mathfrak{B}, K)$ since these algebras with involution are perfect. Hence $\mathfrak{H}(\mathfrak{A}, J) \cong \mathfrak{H}(\mathfrak{B}, K)$ implies $(\mathfrak{A}, J) \cong (\mathfrak{B}, K)$. The converse is clear.

This result reduces the problem of classifying the special finite-dimensional central simple Jordan algebras of degree $\geqq 3$ to the same problem for finite-dimensional central simple associative algebras with involution of degree $\geqq 3$. As we have seen in §4.6 the latter amounts to the problem of classifying the algebras (without involutions) and of classifying hermitian forms. For special base fields, for example the reals or p-adic fields, these problems are easily solved. In these cases one gets a complete classification of the corresponding Jordan algebras. The case of the field of real numbers will be indicated in the exercises below.

EXERCISES

1. Show that if $\mathfrak{J}$ is a special finite-dimensional division algebra over Φ, then $\mathfrak{J}$ is isomorphic to one of the following: (1) the Jordan algebra of a non-

degenerate symmetric bilinear form f on a finite-dimensional vector space $\mathfrak{B}/\Gamma$ where $\dim \mathfrak{B}/\Gamma > 1$, Γ is a finite-dimensional field extension of Φ, and $f(x,x) = 1$ has no solution in $\mathfrak{B}$, (2) a Jordan algebra Δ^+ where Δ is a finite-dimensional associative division algebra over Φ, (3) a Jordan algebra $\mathfrak{H}(\Delta, J)$ where (Δ, J) is a finite-dimensional associative division algebra with involution.

In the exercises 2–7 we shall sketch the classification of special central simple Jordan algebras of finite dimension over the field R of real numbers. For this we shall need two classical results: (1) the theorem of Frobenius that the finite-dimensional division algebras over R are R, $C = R(-1)^{\frac{1}{2}}$ and Q, Hamilton's quaternion algebra over R; (2) the theorem that the only automorphisms of Q/R are the inner ones: $x \to a^{-1}xa$.

2. Use Exercise 1 to show that every finite-dimensional special central Jordan division algebra over R is of degree 1 or 2. Hence show that any such algebra is either R or is the Jordan algebra of a negative definite quadratic form on a vector space $\mathfrak{B}/R$ such that $1 < \dim \mathfrak{B}/R < \infty$.

3. Show that if $\mathfrak{B}$ is a finite-dimensional vector space over Q then any involution in $\mathrm{Hom}_Q(\mathfrak{B}, \mathfrak{B})$ is the adjoint mapping relative to a nondegenerate hermitian or skew hermitian form f associated with the standard involution $x \to \bar{x}$ in Q.

4. Let f be a nondegenerate symmetric bilinear form in a finite-dimensional vector space $\mathfrak{B}$ over R or a nondegenerate hermitian form in a finite-dimensional vector space $\mathfrak{B}$ over C or over Q (standard involution). Show that $\mathfrak{B}$ has an orthogonal basis $(u_1, u_2, \cdots, u_n)$ such that

$$f(u_1, u_1) = \cdots = f(u_p, u_p) = 1,$$
$$f(u_{p+1}, u_{p+1}) = \cdots = f(u_n, u_n) = -1.$$

5. Prove that if f is a nondegenerate skew hermitian form in a finite-dimensional vector space $\mathfrak{B}$ over Q relative to the standard involution in Q then $\mathfrak{B}$ has an orthogonal basis $(u_1, u_2, \cdots, u_n)$ such that $f(u_i, u_i) = q$ where q is any element of Q such that $\bar{q} = -q$.

6. Prove that any finite-dimensional central simple associative algebra with involution over R is isomorphic to one of the following:

I. $R_n \oplus R_n{}^0$, involution exchanging the two components.

II. $Q_n \oplus Q_n{}^0$, involution exchanging the two components.

III. R_n, involution $X \to \gamma_p{}^{-1} X^t \gamma_p$ where

$$\gamma_p = \mathrm{diag}\{\overbrace{1, 1, \cdots, 1}^{p}, -1, -1, \cdots -1\}$$

and $p = 0, 1, \cdots, [n/2]$.

IV. C_n, involution $X \to \gamma_p{}^{-1} \bar{X}^t \gamma_p$ where γ_p and p are as in III.

V. Q_n, involution $X \to \gamma_p{}^{-1} \bar{X}^t \gamma_p$ where γ_p and p are as in III.

VI. R_{2n}, involution $X \to S^{-1} X^t S$ where

$$S = \text{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

VII. Q_n, involution $X \to D^{-1}\bar{X}^t D$ where

$$D = \text{diag}\{q, q, \cdots, q\}$$

and $\bar{q} = -q \in Q$.

Prove that no two of the algebras with involution listed are isomorphic. Determine the capacity and degree in all cases.

7. Classify the finite-dimensional central simple Jordan algebras of capacities $\geqq 2$ over R.

8. Let $\mathfrak{A}_i$, $i = 1, 2$, be a simple algebra over Φ with centroid Γ_i. Show that if η is an isomorphism of $\mathfrak{A}_1/\Phi$ onto $\mathfrak{A}_2/\Phi$ then η is a semilinear transformation of $\mathfrak{A}_1/\Gamma_1$ onto $\mathfrak{A}_2/\Gamma_2$. Show that if D is a derivation in $\mathfrak{A}_1/\Phi$ then D is a semiderivation of $\mathfrak{A}_1/\Gamma_1$.

CHAPTER VI

GENERIC MINIMUM POLYNOMIALS, TRACES AND NORMS

The algebras we shall consider in this chapter will be assumed to be finite dimensional with identity elements and to be strictly power associative in the sense that $\mathfrak{A}_P$ is power associative for every extension field P of the base field Φ. We shall define the generic minimum polynomial, trace and norm of an element of such an algebra. These notions generalize the notions of the characteristic polynomial, trace and determinant of a matrix. We shall establish some important multiplicative properties of the generic norm and of the irreducible factors of the generic norm polynomial. For example, for Jordan algebras we shall show that $n(\{aba\}) = n(a)^2 n(b)$ for the generic norm n. Moreover, this multiplicative property characterizes the products of irreducible factors $p(x)$ of $n(x)$ which are normalized in the sense that $p(1) = 1$. The generic trace t determines a generic trace bilinear form $t(a, b) = t(ab)$. If $\mathfrak{A}$ is associative or Jordan this is symmetric and associative in the sense that $t(ab, c) = t(a, bc)$.

A finite-dimensional Jordan (associative) algebra $\mathfrak{A}$ is called separable if $\mathfrak{A}_\Omega$ is semisimple for Ω the algebraic closure of the base field Φ of $\mathfrak{A}$. We shall see that $\mathfrak{A}$ is separable if and only if the generic trace bilinear form of $\mathfrak{A}$ is nondegenerate.

If $\mathfrak{J}$ and $\mathfrak{J}'$ are finite-dimensional Jordan algebras with 1 then a bijective linear mapping η of $\mathfrak{J}$ into $\mathfrak{J}'$ is called a norm similarity if $n'(a^\eta) = \rho n(a)$ for the generic norms n and n' on $\mathfrak{J}$ and $\mathfrak{J}'$, ρ a fixed nonzero element of the base field. Isotopic Jordan algebras are norm similar and the converse holds for separable Jordan algebras. Of particular interest is the group $M(\mathfrak{J})$ of norm similarities of $\mathfrak{J}$ onto $\mathfrak{J}$ and an analogous Lie algebra $\mathfrak{M}(\mathfrak{J})$. We shall determine these for $\mathfrak{J}$ special central simple.

The notions of generic minimum polynomials, traces and norms are classical notions for associative algebras (see, for example, Jacobson's *Theory of Rings* pp. 111). The extension of these notions to strictly power associative algebras has been given by the author in [25] and [32]. Some of the proofs we shall give are simplifications due to McCrimmon in [2]. An important technique in this connection is that of the differential calculus of rational mappings of finite-dimensional vector spaces. This important tool for the study of finite-dimensional algebras has been developed in the book by Braun and Koecher [1]. We mention also that the theory of this chapter has recently been generalized by McCrimmon in [10] to (infinite-dimensional) generically algebraic algebras.

1. **Differentiation of rational expressions.** Let $\Phi[\xi,\eta] \equiv \Phi[\xi_1,\cdots,\xi_n,\eta_1,\cdots,\eta_n]$ be the algebra of polynomials in independent indeterminates $\xi_1,\cdots,\xi_n,\eta_1,\cdots,\eta_n$ and let $\Phi(\xi,\eta)\equiv\Phi(\xi_1,\cdots,\xi_n,\eta_1,\cdots,\eta_n)$ be its field of fractions. Let $\Phi[\xi]\equiv\Phi[\xi_1,\cdots,\xi_n]$ $\Phi(\xi)\equiv\Phi(\xi_1,\cdots,\xi_n)$ be the subalgebra and subfield respectively of $\Phi(\xi,\eta)$ generated by 1 and the ξ_i. The notions of degree and homogeneity of an element $f(\xi,\eta)$ $\in\Phi[\xi,\eta]$ in a subset of the set of indeterminates $\{\xi_1,\cdots,\xi_n,\eta_1,\cdots,\eta_n\}$ are clear. Now consider an element $f(\xi)\in\Phi[\xi]$. We can write

$$
\begin{aligned}
f(\xi+\eta) &\equiv f(\xi_1+\eta_1,\xi_2+\eta_2,\cdots,\xi_n+\eta_n) \\
&= f(\xi)+\Delta^{\eta}{}_{\xi} f + R(f)
\end{aligned}
\tag{1}
$$

where $\Delta^{\eta}_{\xi} f$ is homogeneous of degree 1 in the η_i and $R(f)$ is a sum of polynomials in the ξ's and η's which are homogeneous of degree $\geqq 2$ in the η's. It is immediate from the definition that the mapping

$$
f \to \Delta^{\eta}{}_{\xi} f \tag{2}
$$

is a derivation of $\Phi[\xi]$ into $\Phi[\xi,\eta]$. If $\partial/\partial\xi_i$ denotes the derivation in $\Phi[\xi]$ which is characterized by $\partial\xi_j/\partial\xi_i=\delta_{ij}, i,j=1,2,\cdots,n$, then $\sum\eta_i\,\partial/\partial\xi_i$ is a derivation of $\Phi[\xi]$ into $\Phi[\xi,\eta]$ which, by (1), coincides with Δ^{η}_{ξ} on the ξ_j. Since 1 and the ξ_j generate $\Phi[\xi]$ it follows that we have

$$
\Delta^{\eta}{}_{\xi} f = \sum_{i=1}^{n} \eta_i \frac{\partial f}{\partial \xi_i} \tag{3}
$$

for all $f\in\Phi[\xi]$. The derivation $f\to\Delta^{\eta}{}_{\xi} f$ has a unique extension to a derivation of $\Phi(\xi)$ into $\Phi(\xi,\eta)$. We denote the image of $f\in\Phi(\xi)$ again by $\Delta^{\eta}{}_{\xi} f$ and we call this the *directional derivative of f in the direction* $(\eta_1,\cdots,\eta_n)$. Again, one has the formula (3) expressing $\Delta^{\eta}{}_{\xi}$ in terms of the $\partial/\partial\xi_i$, which shows that $\Delta^{\eta}{}_{\xi} f\in\Phi(\xi)[\eta]$ the subalgebra of polynomials in the η's with coefficients in $\Phi(\xi)$.

Let $\Phi(\sigma,\tau)=\Phi(\sigma_1,\cdots,\sigma_r,\tau_1,\cdots,\tau_r)$ be the field of fractions of $\Phi[\sigma,\tau]$ $\equiv\Phi[\sigma_1,\cdots,\sigma_r,\tau_1,\cdots,\tau_r]$ where $\sigma_1,\cdots,\sigma_r,\tau_1,\cdots,\tau_r$ are independent indeterminates. Let $f_1,\cdots,f_r\in\Phi(\xi_1,\cdots,\xi_n)$ and let F be the homomorphism of $\Phi(\sigma)\equiv\Phi(\sigma_1,\cdots,\sigma_r)$ into $\Phi(\xi)$ such that $1^F=1$ and $\sigma_j{}^F=f_j$, $j=1,2,\cdots,r$. Let D be the resultant of F and the derivation of $\Phi(\xi)$ into $\Phi(\xi,\eta)$ such that $f\to\Delta^{\eta}{}_{\xi} f$. Then D is a linear mapping of $\Phi(\sigma)$ into $\Phi(\xi,\eta)$ such that $(gh)D=g^F(hD)+(gD)h^F$. Next let D' be the linear mapping of $\Phi(\sigma)$ into $\Phi(\xi,\eta)$ which is the resultant of the derivation $g\to\Delta^{\tau}{}_{\sigma} g$ and the homomorphism F' of $\Phi(\sigma,\tau)$ into $\Phi(\xi,\eta)$ such that $1^{F'}=1$, $\sigma_j{}^{F'}=f_j$, $\tau_j{}^{F'}=\Delta^{\eta}{}_{\xi} f_j$, $j=1,2,\cdots,r$. Then D' is linear and $(gh)D'=g^F(hD')+(gD')h^F$. Hence $D''=D-D'$ is linear and $(gh)D''=g^F(hD'')$

$+ (gD'')h^F$. Also $\sigma_j D = \Delta^\eta{}_\xi f_j$ and $\sigma_j D' = \Delta^\eta{}_\xi f_j$ so $\sigma_j D'' = 0$. Also $1D'' = 0$. Since $ff^{-1} = 1$ gives $f^{-1}D'' = -(fD'')(f^F)^{-2}$ it is clear that the kernel of D'' is a subfield of $\Phi(\sigma)/\Phi$. Since the σ_j are in this kernel and these generate $\Phi(\sigma)$ we have $D'' = 0$ and $D = D'$. This gives the *chain rule*: If $q = q(\sigma_1, \cdots, \sigma_r) \in \Phi(\sigma)$ and $f_1, \cdots, f_r \in \Phi(\xi)$ then

$$\begin{aligned} \Delta^\eta{}_\xi q(f_1, \cdots, f_1) &= \Delta^{\tau'}{}_{\sigma'} q, \\ \sigma'_j &= f_j(\xi_1, \cdots, \xi_n), \qquad \tau'_j = \Delta^\eta{}_\xi f_j \end{aligned} \tag{4}$$

where $\Delta^{\tau'}{}_{\sigma'} q$ denotes the element of $\Phi(\xi, \eta)$ obtained from $\Delta^\tau{}_\sigma q$ by the specializations $\sigma_j \to \sigma'_j = f_j(\xi_1, \cdots, \xi_n)$, $\tau_j \to \tau'_j = \Delta^\eta{}_\xi f_j$, $1 \leqq j \leqq n$.

Exercise

1. Show that any derivation in $\Phi(\xi_1, \cdots, \xi_n)$ has the form $f \to \Delta^g{}_\xi f$ where $g = (g_1(\xi), \cdots, g_n(\xi))$. Denoting this as D_g show that $D_g + D_h = D_{g+h}$ and for $\alpha \in \Phi$, $\alpha D_g = D_{\alpha g}$ where $g + h = (g_1(\xi) + h_1(\xi), \cdots, g_n(\xi) + h_n(\xi))$, $\alpha g = (\alpha g_1(\xi), \cdots, \alpha g_n(\xi))$. Show that $[D_g, D_h] = D_k$ where

$$k_i(\xi) = \sum_j \left(h_j(\xi) \frac{\partial g_i}{\partial \xi_j} - g_j(\xi) \frac{\partial h_i}{\partial \xi_j} \right).$$

2. **Differential calculus of rational mappings.** Let $\mathfrak{V}$ be a finite dimensional vector space over an infinite field Φ and let $\mathfrak{V}^*$ be the space of linear functions on $\mathfrak{V}$, that is, the space of linear mappings of $\mathfrak{V}$ into the base field Φ. We recall that one defines the algebra $\mathfrak{P} = \mathfrak{P}(\mathfrak{V})$ of *polynomial functions* on $\mathfrak{V}$ to be the subalgebra of mappings of $\mathfrak{V}$ into Φ which is generated by 1 and the linear functions on $\mathfrak{V}$ (cf. Chevalley [2], p. 26). Here addition, multiplication by $\alpha \in \Phi$ and multiplication of functions are defined by

$$(f + g)(v) = f(v) + g(v), \ \alpha f(v) = \alpha(f(v)), \ (fg)(v) = f(v)g(v),$$

where we are using the usual functional notation: $f(v)$, the image of v under f. Let $(v_1, v_2, \cdots, v_n)$ be a basis for $\mathfrak{V}/\Phi$, $(v_1^*, v_2^*, \cdots, v_n^*)$ the dual basis for $\mathfrak{V}^*$, that is, v_i^* is the linear function on $\mathfrak{V}$ such that $v_i^*(v_j) = \delta_{ij}$, $j = 1, \cdots, n$. Since $(v_1^*, v_2^*, \cdots, v_n^*)$ span $\mathfrak{V}^*$ it is clear that these elements are generators of $\mathfrak{P}(\mathfrak{V})$. Since the algebra of functions is commutative, $\mathfrak{P}(\mathfrak{V})$ is commutative. Hence we can define a homomorphism $\eta = \eta(v_i)$ of the polynomial algebra $\Phi[\xi_1, \cdots, \xi_n]$, ξ_i indeterminates, onto $\mathfrak{P}(\mathfrak{V})$ such that $1^\eta = 1$, $\xi_i{}^\eta = v_i^*$. Then $v_i^*(\sum_1^n \alpha_j v_j) = \alpha_i$ for $\alpha_i \in \Phi$ and, consequently, if $f = f(\xi_1, \cdots, \xi_n) \in \Phi[\xi_1, \cdots, \xi_n]$ then $f^\eta(\sum \alpha_j v_j) = f(\alpha_1, \cdots, \alpha_n)$. Hence we see that $\mathfrak{P}$ coincides with the set of mappings

$$f^\eta\colon \sum \alpha_i v_i \to f(\alpha_1, \cdots, \alpha_n), \qquad f \in \Phi[\xi_1, \cdots, \xi_n]. \tag{5}$$

Since Φ is infinite, $f(\alpha_1, \cdots, \alpha_n) = 0$ for all $\alpha_i \in \Phi$ implies $f = 0$. Hence the homomorphism η is an isomorphism and the elements $\xi_i{}^\eta = v_i^*$, $i = 1, \cdots, n$, are algebraically independent generators of $\mathfrak{P}$.

If $f \in \mathfrak{P}$ then a vector $z \in \mathfrak{V}$ is called a zero of f if $f(z) = 0$. The set of common zeros of a set of polynomial functions on $\mathfrak{V}$ is called an algebraic subset of $\mathfrak{V}$. It is well known that these sets can be taken to be the closed sets of a topology called the Zariski topology of $\mathfrak{V}$ (Chevalley [3], p. 169).

If $f \neq 0$, that is, f is not the mapping $x \rightarrow 0$ then the set $O_f = \{x \mid f(x) \neq 0\}$ is a nonvacuous Zariski open subset of $\mathfrak{V}$. Any finite number of such subsets have a nonvacuous intersection. More precisely, if $f_1, \cdots, f_k \in \mathfrak{P}$ and $f_j \neq 0$ then $O_{f_1} \cap \cdots \cap O_{f_k} = O_{f_1 \cdots f_k}$ and this is a nonvacuous open set in $\mathfrak{V}$ since $f_1 \cdots f_k \neq 0$. If $f \in \mathfrak{P}$ and $f(z) = 0$ for all z in a nonvacuous Zariski open subset of $\mathfrak{V}$ then $f = 0$.

If g and h are polynomial functions on $\mathfrak{V}$ and $h \neq 0$ then the mapping $x \rightarrow g(x)h(x)^{-1}$ is defined on the nonvacuous Zariski open set O_h. We denote this mapping as gh^{-1} and call it a *rational function* on $\mathfrak{V}$ (even though the domain of definition may not be the whole of $\mathfrak{V}$). It is convenient to identify two such functions if they are identical on a nonvacuous Zariski open subset of $\mathfrak{V}$ and we shall do this from now on. With this identification it is immediate that the set $\mathfrak{R} = \mathfrak{R}(\mathfrak{V})$ of rational functions is a subfield of the algebra of functions. Moreover, $\mathfrak{R}(\mathfrak{V})$ is the field of fractions of the algebra $\mathfrak{P}$ of polynomial functions. The isomorphism η of $\Phi[\xi_1, \cdots, \xi_n]$ onto $\mathfrak{P}$ defined before by a basis $(v_1, v_2, \cdots, v_n)$ for $\mathfrak{V}/\Phi$ has a unique extension to an isomorphism η of $\Phi(\xi_1, \cdots, \xi_n)$ onto $\mathfrak{R}$. In this the element $f = gh^{-1} \in \Phi(\xi_1, \cdots, \xi_n)$ is mapped into the rational mapping f^η such that $a = \sum_1^n \alpha_i v_i \rightarrow f(\alpha_1, \cdots, \alpha_n)$ $\equiv g(\alpha_1, \cdots, \alpha_n) h(\alpha_1, \cdots, \alpha_n)^{-1}$. It is clear also that the elements $(v_1^*, v_2^*, \cdots, v_n^*)$ of the dual basis of (v_i) are an algebraically independent set of generators of the field $\mathfrak{R}(\mathfrak{V})$.

It is clear from the isomorphism η of $\Phi(\xi_1, \cdots, \xi_n)$ onto $\mathfrak{R}$ that any mapping of the set of generators $\{v_1^*, \cdots, v_n^*\}$ into $\mathfrak{R}$ has a unique extension to a derivation in $\mathfrak{R}/\Phi$. Now let $a \in \mathfrak{V}$. Then the mapping $l \rightarrow l(a)$ of $\mathfrak{V}^*$ into Φ is the linear mapping of $\mathfrak{V}^*$ into Φ which extends the mapping $v_i{}^* \rightarrow v_i^*(a) = \alpha_i$, $a = \sum \alpha_i u_i$, of $\{v_i^*, \cdots, v_n^*\}$ into Φ. These mappings can be regarded as mappings into $\mathfrak{R} \supseteq \Phi$. Hence the mapping $v_i{}^* \rightarrow \alpha_i$ can be extended to a derivation $f \rightarrow \Delta^a f$ of $\mathfrak{R}$. Clearly this is an extension of the linear mapping $l \rightarrow l(a)$ of $\mathfrak{V}^*$. Hence we see that the mapping $l \rightarrow l(a) \in \Phi \subseteq \mathfrak{R}$ has a unique extension to the derivation $f \rightarrow \Delta^a f$ in $\mathfrak{R}$. $\Delta^a f$ is a rational function on $\mathfrak{V}$ whose value $(\Delta^a f)(c)$ at c will also be denoted as $\Delta^a{}_c f$. We shall call this the *directional derivative of f at c in the direction a.* The derivation $f \rightarrow \Delta^a f$ gives rise, via the isomorphism η defined by $\xi_i \rightarrow v_i^*$ to a derivation in $\Phi(\xi_1, \cdots, \xi_n)$. Since $\Delta^a v_i^* = \alpha_i$ the corresponding derivation in $\Phi(\xi) = \Phi(\xi_1, \cdots, \xi_n)$ maps ξ_i into α_i. Hence it maps $f \in \Phi(\xi)$ into $\sum_1^n (\partial f / \partial \xi_i) \alpha_i$. This is the resultant of the derivation $f \rightarrow \Delta^\eta{}_\xi f$ of $\Phi(\xi)$ into $\Phi(\xi, \eta)$ defined in §1 with the homomorphism of $\Phi(\xi,\eta)$ into $\Phi(\xi)$ such that $\xi_i \to \xi_i$, $\eta_i \to \alpha_i$. We denote $\sum_1^n (\partial f/\partial \xi)_i \alpha_i$ also as $\Delta^a f$ where f here is an element of $\Phi(\xi)$. It is clear from this formula and the usual formulas for differentiation of rational expressions that $\Delta^a f$ is defined at every c at which f is defined.

If $l \in \mathfrak{B}^*$ and $a \in \mathfrak{B}$ then $\Delta^a l = l(a)$, by definition of Δ^a. If b is a second element of $\mathfrak{B}$ then $\Delta^{a+b} l = l(a+b) = l(a) + l(b) = \Delta^a l + \Delta^b l$. Since $\mathfrak{B}^*$ generates $\mathfrak{R}(\mathfrak{B})$ this implies that $\Delta^{a+b} = \Delta^a + \Delta^b$. Similarly, if $\beta \in \Phi$ then $\Delta^{\beta a} = \beta \Delta^a$. Hence the mapping $a \to \Delta^a$ is a linear mapping of $\mathfrak{B}$ into the algebra of derivations of $\mathfrak{R}$ and for a fixed $f \in \mathfrak{R}$, $a \to \Delta^a{}_f$ is a linear mapping of $\mathfrak{B}$ into $\mathfrak{R}$. Finally, for a fixed $f \in \mathfrak{R}$ and $c \in \mathfrak{B}$, $a \to \Delta^a{}_c f$ is a linear mapping $\Delta_c f$ of $\mathfrak{B}$ into Φ, that is, an element of $\mathfrak{B}^*$. We call this element of $\mathfrak{B}^*$ the *differential* at c of f.

Since Δ^a is a derivation in $\mathfrak{R}$ we have

$$\Delta^a{}_c(f+g) = \Delta^a{}_c f + \Delta^a{}_c g, \tag{6}$$

$$\Delta^a{}_c(\gamma f) = \gamma \Delta^a{}_c f, \tag{7}$$

$$\Delta^a{}_c(fg) = f(c)\Delta^a{}_c g + (\Delta^a{}_c f) g(c), \tag{8}$$

$$\Delta^a{}_c f^{-1} = -(\Delta^a{}_c f) f^{-2}(c), \qquad \text{if } f(c) \neq 0 \tag{9}$$

for $f, g \in \mathfrak{R}$ and $\gamma \in \Phi$. If b is a second element of $\mathfrak{B}$ then we have

$$\Delta^b{}_c \Delta^a f = \Delta^a{}_c \Delta^b f. \tag{10}$$

This follows since the derivations $f \to \partial f/\partial \xi_i$ in $\Phi(\xi)$ commute. Hence the derivations $f \to \sum \alpha_i \partial f/\partial \xi_i$ and $f \to \sum \beta_i \partial f/\partial \xi_i$ commute if $\alpha_i, \beta_i \in \Phi$. Since the corresponding derivations in $\mathfrak{R}$ are Δ^a and Δ^b for $a = \sum \alpha_i u_i$, $b = \sum \beta_i u_i$ it is clear that (10) holds.

If $f \in \mathfrak{R}$ we define the *logarithmic derivative of* $f \neq 0$ *in the direction* a as

$$\Delta^a \log f = f^{-1} \Delta^a f. \tag{11}$$

We have the formulas

$$\Delta^a \log fg = \Delta^a \log f + \Delta^a \log g, \tag{12}$$

$$\Delta^b \Delta^a \log f = [f(\Delta^b \Delta^a f) - \Delta^a f \Delta^b f] f^{-2}. \tag{13}$$

For a fixed c and f it is clear that the mapping $(a,b) \to \Delta^b{}_c \Delta^a f$ is a bilinear form on $\mathfrak{B}$. By (10) this is symmetric. Similarly, the formula (13) shows that $\Delta^b{}_c \Delta^a \log f = \Delta^b{}_c (f^{-1} \Delta^a f)$ for fixed c and f is a symmetric bilinear form on $\mathfrak{B}/\Phi$.

Now let $\mathfrak{W}$ be a second finite-dimensional vector space. Let $f = gh^{-1}$ where g and h are polynomial functions on $\mathfrak{B}$ and let $w \in \mathfrak{W}$. Then $x \to f(x)w$ is a mapping of the (Zariski) open subset O_h defined by $h(x) \neq 0$ into $\mathfrak{W}$. The space generated by such mappings consists of the mappings of the form $x \to \sum f_j(x) w_j$ where the f_j are rational functions on $\mathfrak{B}$. We shall call these *rational mappings* *on* $\mathfrak{V}$ (more precisely, of an open subset of $\mathfrak{V}$ on which all the f_j are defined) in $\mathfrak{W}$ and we denote their totality as $\mathfrak{R}(\mathfrak{V},\mathfrak{W})$. If all the f_j in $x \to \sum f_j(x)w_j$ are polynomial functions then this mapping is called a *polynomial mapping* of $\mathfrak{V}$ into $\mathfrak{W}$ (defined on all of $\mathfrak{V}$). The set of these will be denoted as $\mathfrak{P}(\mathfrak{V},\mathfrak{W})$. If $w_1,\cdots,w_r$ are linearly independent elements of $\mathfrak{V}$ and $\sum_1^r f_j(x)w_j = 0$ for all x at which the f_j are defined then the f_j are 0. This implies that $\mathfrak{R}(\mathfrak{V},\mathfrak{W})(\mathfrak{P}(\mathfrak{V},\mathfrak{W}))$ can be identified with $\mathfrak{R}(\mathfrak{V}) \otimes_\Phi \mathfrak{W}$ $(\mathfrak{P}(\mathfrak{V}) \otimes \mathfrak{W})$. If $(w_1,w_2,\cdots,w_r)$ is a basis for $\mathfrak{W}/\Phi$ then it is clear that $\mathfrak{R}(\mathfrak{V},\mathfrak{W})$ is the set of mappings $x \to \sum_1^r f_j(x)w_j$ where the f_j are rational functions. If F is a polynomial or rational mapping on $\mathfrak{V}$ and $F(z) = 0$ for all z in a nonvacuous open subset of $\mathfrak{V}$ then $F = 0$.

Let $a \in \mathfrak{V}$ and consider the derivation $f \to \Delta^a f$ in $\mathfrak{R}$ corresponding to a. Since the space $\mathfrak{R}(\mathfrak{V},\mathfrak{W})$ of rational mappings of $\mathfrak{V}$ into $\mathfrak{W}$ is the same as $\mathfrak{R}(\mathfrak{V}) \otimes_\Phi \mathfrak{W}$ we can define a linear mapping $F \to \Delta^a F$ in $\mathfrak{R}(\mathfrak{V},\mathfrak{W})$ by specifying that if F is $x \to \sum f_j(x)w_j$, $f_j \in \mathfrak{R}$, $w_j \in \mathfrak{W}$, then $\Delta^a F$ is $x \to \sum(\Delta^a{}_x f_j)w_j$. As in the special case of rational functions ($\mathfrak{W} = \Phi$), we denote the value $\Delta^a F(c)$ of $\Delta^a F$ at c also as $\Delta^a{}_c F$. Since $\Delta^a{}_c f_j = \sum_{i=1}^n \alpha_i(\partial f_j/\partial \xi_i)_{\xi_i=\gamma_i}$ if $a = \sum_1^n \alpha_i v_i$ and $c = \sum \gamma_i v_i$ we have $\Delta^a{}_c F = \sum_{i,j} \alpha_i(\partial f_j/\partial \xi_i)_{\xi_i=\gamma_i} w_j$. Since $a \to \Delta^a{}_c f_j$ is a linear mapping of $\mathfrak{V}$ into Φ, $a \to \Delta^a{}_c F = \sum_j \Delta^a{}_c f_j v_j$ is a linear mapping of $\mathfrak{V}$ into $\mathfrak{W}$. We shall call this mapping the *differential* $\Delta_c F$ *of* F *at* c. The explicit formula for $\Delta^a_c F$ shows that if $(v_1,v_2,\cdots,v_n)$ is a basis for $\mathfrak{V}$, as before, and $(w_1,w_2,\cdots,w_r)$ is a basis for $\mathfrak{W}$ then the matrix of $\Delta_c F$ relative to this pair of bases is the $n \times r$ Jacobian matrix $((\partial f_j/\partial \xi_i)_{\xi_i=\gamma_i})$, $i = 1,\cdots,n, j = 1,\cdots,r$. It is immediate from the definition of $\Delta^a{}_c F$ and (6)–(8) that if $F, G \in \mathfrak{R}(\mathfrak{V},\mathfrak{W})$, $f \in \mathfrak{R}$, $\gamma \in \Phi$, then

$$\Delta^a{}_c(F + G) = \Delta^a{}_c F + \Delta^a{}_c G, \tag{14}$$

$$\Delta^a{}_c(\gamma F) = \gamma \Delta^a{}_c F, \tag{15}$$

$$\Delta^a{}_c(fF) = f(c)\Delta^a{}_c F + (\Delta^a{}_c f)F(c). \tag{16}$$

Let F be a rational mapping on $\mathfrak{V}$ into $\mathfrak{W}$ and g a rational function on $\mathfrak{W}$. If $(v_1,\cdots,v_n)$ and $(w_1,\cdots,w_r)$ are bases for $\mathfrak{V}$ and $\mathfrak{W}$ then F has the form $\sum_1^n \alpha_i v_i \to \sum_1^r f_j(\alpha_1,\cdots,\alpha_n)w_j$ and g has the form $\sum_1^r \beta_j w_j \to q(\beta_1,\cdots,\beta_r)$ where $f_1,\cdots,f_r$ are uniquely determined elements of $\Phi(\xi_1,\cdots,\xi_n)$ and q is a uniquely determined element of $\Phi(\sigma_1,\cdots,\sigma_r)$, ξ_i,σ_j indeterminates. We now define the composite function $g \circ F$ on $\mathfrak{V}$ as

$$\sum_1^n \alpha_i v_i \to q(f_1(\alpha_1,\cdots,\alpha_n),\cdots,f_r(\alpha_1,\cdots,\alpha_n)). \tag{17}$$

One can check directly that this is independent of the choice of the bases and it is clear that $g \circ F$ is a rational function. Moreover, it is clear that if c is an element of $\mathfrak{V}$ such that F is defined at c and g is defined at $F(c)$, then $g \circ F$ is defined at c and $(g \circ F)(c) = g(F(c))$. (The set of such c's may be vacuous.) In this situation the directional derivative $\Delta^a{}_c\, g \circ F$ is defined for any a. By (4) and the formula for the directional derivative we obtain

$$\Delta^a{}_c\, g \circ F = \Delta^b{}_d\, g, \quad b = \Delta^a{}_c F, \quad d = F(c). \tag{18}$$

This states that the differential of $g \circ F$ at c is the resultant of the differential of F at c and the differential of g at $F(c)$.

Again let F: $\sum_1^n \alpha_i v_i \to \sum_1^r f_j(\alpha_1, \cdots, \alpha_n) w_j$ be a rational mapping of $\mathfrak{V}$ into $\mathfrak{W}$ and let G be a rational mapping of $\mathfrak{W}$ into the vector space $\mathfrak{S}$ with basis $(s_1, \cdots, s_t)$. Assume G has the form $\sum_1^r \beta_j w_j \to \sum_{k=1}^t g_k(\beta_1, \cdots, \beta_r) s_k$ where the $g_k \in \Phi(\sigma_1, \cdots, \sigma_r)$, σ_i indeterminates. Then we define the composite mapping $G \circ F$ by

$$\sum_1^n \alpha_i v_i \to \sum_{k=1}^t g_k(f_1(\alpha_1, \cdots, \alpha_n), \cdots, f_r(\alpha_1, \cdots, \alpha_n))\, s_k. \tag{19}$$

It is clear from the case of composite functions ($\mathfrak{S} = \Phi$) that this is independent of bases and is a rational mapping. Also as in the case $\mathfrak{S} = \Phi$, if c is an element of $\mathfrak{V}$ such that F is defined at c and G is defined at $F(c)$, then we have the chain rule

$$\Delta^a{}_c G \circ F = \Delta^b{}_d G, \quad b = \Delta^a{}_c F, \quad d = F(c). \tag{20}$$

A rational mapping F on $\mathfrak{V}$ is called *homogeneous of degree* $m = 0, \pm 1, \pm 2, \cdots$ if for any c at which F is defined and any $\lambda \neq 0$ in Φ, F is defined at λc and $F(\lambda c) = \lambda^m F(c)$. It is easily seen (ex. 1 below) that if f is a homogeneous rational function on $\mathfrak{V}$ then f has the form $\sum_1^n \alpha_i v_i \to g(\alpha_1, \cdots, \alpha_n)\, h(\alpha_1, \cdots, \alpha_n)^{-1}$ where g and h are homogeneous polynomials (in all the ξ_i) in $\Phi[\xi_1, \cdots, \xi_n]$. Let f be a homogeneous rational function on $\mathfrak{V}$ of degree m and let c be an element of $\mathfrak{V}$ at which f is defined. The function $\lambda \to f(\lambda c)$ on Φ is a rational function which is the composite $f \circ \lambda c$ where λc is the mapping $\lambda \to \lambda c$, and the defining equation gives $f \circ \lambda = f(c)\lambda^m$. Hence by (18) we have

$$mf(c) = \Delta^1{}_1 f \circ \lambda = \Delta^c{}_c f \tag{21}$$

which is *Euler's differential equation.*

If $f \in \Phi(\xi_1, \cdots, \xi_n)$ satisfies $\partial f / \partial \xi_1 = 0$ then $f \in \Phi(\xi_2, \cdots, \xi_n)$ if the characteristic is 0 and $f \in \Phi(\xi_1{}^p, \xi_2, \cdots, \xi_n)$ if the characteristic is p. It follows that if f is a rational function such that $\Delta^a f = 0$ for all $a \in \mathfrak{V}$ then f is a constant (function) if the characteristic is 0 and for characteristic p, f has the form $g \circ \tilde{p}$ where $\tilde{p}$ has the form $\sum \alpha_i u_i \to \sum \alpha_i{}^p u_i$ relative to some basis $(v_1, \cdots, v_n)$ for $\mathfrak{V}$. The converse is clear also.

If f is a rational function on $\mathfrak{V}$ and T is a linear mapping of $\mathfrak{V}$ into itself then

f is called an *invariant* (*semi-invariant*) under T if for any c at which f is defined, f is defined at cT and $f(cT) = f(c)$ ($f(cT) = \gamma f(c)$ for a fixed $\gamma \neq 0$ in Φ). f is a *Lie invariant* (*Lie semi-invariant*) under T if $\Delta^{cT}{}_c f = 0$ ($\Delta^{cT}{}_c f = \rho f$, $\rho \in \Phi$) for all c at which f is defined. It is clear that the set of invertible T which have a given f as invariant (semi-invariant) form a subgroup of the linear group on $\mathfrak{V}$. On the other hand, the set of $T \in \mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})$ having f as Lie invariant (Lie semi-invariant) is a subalgebra of the Lie algebra $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})^-$ which is restricted (that is is closed under pth powers) if Φ is of characteristic $p \neq 0$. To see this we note that if $T \in \mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})$ then the mapping $g \to g^{D_T}$ where $g^{D_T}(x) = \Delta^{xT}{}_x g$ is a derivation in $\mathfrak{R} = \mathfrak{R}(\mathfrak{V})$ over Φ. This is clear from equations (6), (7) and (8). If $l \in \mathfrak{V}^*$ then $l^{D_T}(x) = \Delta^{xT}{}_x l = l(xT)$ since $\Delta^a l = l(a)$ for $l \in \mathfrak{V}^*$. Hence $l^{D_T} = lT^*$ where T^* is the transpose of T ($lT^*(x) = l(xT)$). If $T_1, T_2 \in \mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})$ and $\alpha \in \Phi$ then $l^{D_{T_1+T_2}} = l(T_1 + T_2)^* = l(T_1^* + T_2^*) = lT_1^* + lT_2^* = l^{D_{T_1}} + l^{D_{T_2}} = l^{D_{T_1}+D_{T_2}}$. Hence $l^{D_{T_1+T_2}} = l^{D_{T_1}+D_{T_2}}$. Similarly, $(\alpha T_1)^* = \alpha T_1^*$ gives $l^{D_{\alpha T}} = l^{\alpha D_T}$, $[T_1 T_2]^* = [T_2^* T_1^*]$ gives $l^{D[T_1 T_2]} = l^{[D_{T_2} D_{T_1}]}$ and $(T_1^*)^p = (T_1{}^p)^*$ gives $l^{(D_T)^p} = l^{(D_T)^p}$. Since 1 and $\mathfrak{V}^*$ generate $\mathfrak{R}/\Phi$ these formulas show that $T \to D_T$ is an antihomomorphism of the Lie algebra (restricted Lie algebra) $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})^-$ (if the characteristic is p) into the Lie algebra (restricted Lie algebra) of derivations of $\mathfrak{R}/\Phi$. It is clear from this that if $f \in \mathfrak{R}$, the elements $T \in \mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})$ satisfying $f^{D_T} = 0$ or $f^{D_T} = \rho(T) f$, $\rho(T) \in \Phi$, form a subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})^-$ as Lie algebra (restricted Lie algebra). Since $f^{D_T}(c) = \Delta^{cT}{}_c f$ this is just the set of T having f as Lie invariant or Lie semi-invariant.

If F is a rational mapping on $\mathfrak{V}$ to $\mathfrak{W}$ and F is given by $\sum_1^n \alpha_i v_i \to \sum_1^r f_j(\alpha_1, \cdots, \alpha_n) w_j$ in terms of the basis where the $f_j \in \Phi(\xi_1, \cdots, \xi_n)$ then the same formula can be used to define a rational mapping on $\mathfrak{V}_\Gamma$ into $\mathfrak{W}_\Gamma$ where Γ is an extension field of Φ. It is clear that this is independent of the choices of the bases. We shall call this the extension of F to $\mathfrak{V}_\Gamma$ and shall usually denote it as F also. Clearly if F is a polynomial mapping then so is its extension.

Let $\mathfrak{A}$ be a finite-dimensional algebra over Φ and let F and G be rational mappings on $\mathfrak{V}$ to $\mathfrak{A}$. For bases $(v_1, \cdots, v_n)$, $(w_1, \cdots, w_r)$ of $\mathfrak{V}$ and $\mathfrak{A}$ respectively these have the form $F\colon \sum_1^n \alpha_i v_i \to \sum_1^r f_g(\alpha_1, \cdots, \alpha_n) w_j$ and $G\colon \sum_1^n \alpha_i v_i \to \sum_1^r g_j(\alpha_1, \cdots, \alpha_n) w_j$ where $f_j, g_j \in \Phi(\xi_1, \cdots, \xi_n)$. Also we have the multiplication table $w_j w_k = \sum_l \gamma_{jkl} w_l$, $\gamma_{jkl} \in \Phi$, for the basis (w_j). Then $(\sum f_j(\alpha) w_j)(\sum g_k(\alpha) w_k) = \sum \gamma_{jkl} f_j(\alpha) g_k(\alpha) w_l$ and so the mapping $x \to F(x)G(x)$ is a rational mapping on $\mathfrak{V}$ into $\mathfrak{A}$. We denote this as FG. Evidently, we have $FG\colon \sum \alpha_i v_i \to \sum h_j(\alpha_1, \cdots, \alpha_n) w_j$ where

$$h_j = \sum_{k,l} \gamma_{klj} f_k g_l, \qquad j, k, l = 1, \cdots, r. \tag{22}$$

It is immediate from this and (6)–(8) that

$$\Delta^a{}_c FG = F(c) \Delta^a{}_c G + (\Delta^a{}_c F) G(c). \tag{23}$$

EXERCISES

1. Let f_i, $i = 1,2$, be a homogeneous rational mapping on $\mathfrak{V}$ of degree m_i. Show that $f_1 f_2$, and $f_1 f_2^{-1}$ ($f_2 \neq 0$) are homogeneous of degree $m_1 + m_2$ and $m_1 - m_2$ respectively. Use a Vandermonde determinant argument to show that any homogeneous rational mapping on $\mathfrak{V}$ has the form $\sum \alpha_i u_i \to g(\alpha_1, \cdots, \alpha_n) h(\alpha_1, \cdots, \alpha_n)^{-1}$ where g and h are homogeneous polynomials in $\Phi[\xi_1, \cdots, \xi_n]$.

2. Let $\Delta^{\xi}{}_{\xi} f$ for $f \in \Phi(\xi_1, \cdots, \xi_n)$ be the image of $\Delta^{\eta}{}_{\xi} f$ under $\eta_i \to \xi_i$. Thus $\Delta^{\xi}{}_{\xi} f = \sum \xi_i \partial f / \partial \xi_i$. Verify that if $f = \xi_1{}^{m_1} \xi_2{}^{m_2} \cdots \xi_n{}^{m_n}$, $m_i \geqq 0$, $\sum m_i = m$, then $\Delta^{\xi}{}_{\xi} f = mf$. Use this and ex. 1 to give another proof of Euler's theorem.

3. Let f be a homogeneous polynomial function of degree m on $\mathfrak{V}$. Show that for fixed $a \in \mathfrak{V}$, $\Delta^a f$ is a homogeneous polynomial function of degree $m-1$. Show that if the characteristic is 0 or $p > m$ then $(1/m!)\Delta^{a_1}\Delta^{a_2} \cdots \Delta^{a_m} f$ is a constant and $f(a_1, a_2, \cdots, a_m) = (1/m!)\Delta^{a_1}\Delta^{a_2} \cdots \Delta^{a_m} f$ is a symmetric multilinear form in $a_1, a_2, \cdots, a_m$ such that $f(x, x, \cdots, x) = f(x)$.

4. Let f and the characteristic be as in 3. Show that f is a Lie invariant under T if and only if

$$f(a_1 T, a_2, \cdots, a_m) + f(a_1, a_2 T, a_3, \cdots, a_m) + \cdots + f(a_1, \cdots, a_{m-1}, a_m T) = 0.$$

3. **Generic minimum polynomial of a strictly power associative algebra.** Let $\mathfrak{A}$ be a finite-dimensional power associative algebra with 1 over the field Φ (which may be finite). We recall that power associativity means $a^r a^s = a^{r+s}$ if a^r is defined by $a^1 = a$, $a^r = a^{r-1} a$. Equivalently, the condition is that every subalgebra $\Phi[a]$ generated by 1 and a single element a is associative (hence also commutative). Let λ be an indeterminate and let $\mu_a(\lambda) \in \Phi[\lambda]$ be the minimum polynomial of a. As usual, we denote the right multiplication $x \to xa$ in $\mathfrak{A}$ by a_R. If $b \in \Phi[a]$ then b_R maps $\Phi[a]$ into itself and we let $\overline{b_R}$ denote the restriction of b_R to $\Phi[a]$. Since $b \to \overline{b_R}$ is a monomorphism of $\Phi[a]$ into $\mathrm{Hom}_\Phi(\Phi[a], \Phi[a])$ the minimum polynomial $\mu_b(\lambda)$ of b is the same as the minimum polynomial $\mu_{\overline{b_R}}(\lambda)$ of $\overline{b_R}$. Also this is a factor of the characteristic polynomial of $\overline{b_R}$ which in turn is a factor of the characteristic polynomial $f_b(\lambda)$ of b_R. Hence

$$\mu_b(\lambda) \,|\, f_b(\lambda). \tag{24}$$

We recall also that

$$f_b(\lambda) = \det(\lambda 1 - (\beta)) \tag{25}$$

where (β) is a matrix of b_R relative to some basis of $\mathfrak{A}/\Phi$. If b is a generator of $\Phi[a]$, that is, $\Phi[b] = \Phi[a]$, then $\deg \mu_{\overline{b_R}}(\lambda) = \dim \Phi[a]$ and $(1, b, \cdots, b^{m-1})$, where $m = \deg \mu_b(\lambda)$, is a basis for $\Phi[a]$. In this case $\deg \mu_b(\lambda)$ is the same as the degree of the characteristic polynomial $f_{\overline{b_R}}(\lambda)$ of $\overline{b_R}$ and since $\mu_b(\lambda) \,|\, f_{\overline{b_R}}(\lambda)$ we have

(26) $$\mu_b(\lambda) = f_{\overline{b_R}}(\lambda)$$

if b is a generator of $\Phi[a]$. In particular, this holds if $b = a$.

If Φ is infinite then we have the polynomial mappings $p_r\colon a \to a^r$, $r = 1,2,\cdots$ in $\mathfrak{A}$ and the power associativity of $\mathfrak{A}$ is equivalent to the requirement that for every $r,s = 1,2,\cdots$ the polynomial mapping $p_{r+s} - p_r p_s$ is the 0 mapping. Now it is clear that if Γ is an extension field of Φ then the polynomial mapping $p_{r+s} - p_r p_s$ in $\mathfrak{A}_\Gamma$ is the extension to $\mathfrak{A}_\Gamma$ of the corresponding mapping in $\mathfrak{A}$. Hence the mappings indicated are also 0 in $\mathfrak{A}_\Gamma$ and so $\mathfrak{A}_\Gamma$ is power associative. For Φ finite it may happen that $\mathfrak{A}$ is power associative but $\mathfrak{A}_\Gamma$ is not power associative for some Γ/Φ. We shall now call an algebra *strictly power associative* if $\mathfrak{A}_\Gamma$ is power associative for every extension field Γ/Φ.

We now assume that $\mathfrak{A}/\Phi$ is finite-dimensional strictly power associative with 1 and $(u_1, u_2, \cdots, u_n)$ is a basis for $\mathfrak{A}/\Phi$. Let $P = \Phi(\xi_1, \xi_2, \cdots, \xi_n)$ where the ξ_i are algebraically independent over Φ and consider the algebra $\mathfrak{A}_P$. We shall call the element $x = \sum_1^n \xi_i u_i$ of $\mathfrak{A}_P$ a *generic element* of $\mathfrak{A}$. Let $m_x(\lambda)$ be the minimum polynomial of x (in $\mathfrak{A}_P$) so $m_x(\lambda) \in P[\lambda]$ and has leading coefficient 1. By (24), $m_x(\lambda) \mid f_x(\lambda)$ the characteristic polynomial of the linear transformation x_R in $\mathfrak{A}_P$. Now (u_i) is also a basis for $\mathfrak{A}_P$ and if $u_i u_j = \sum_k \gamma_{ijk} u_k$ is the multiplication table for this basis then $u_i x = \sum_{j,k} \gamma_{ijk} \xi_j u_k$ so that the matrix of x_R relative to (u_i) is (ξ_{ik}) where $\xi_{ik} = \sum_j \gamma_{ijk} \xi_j$. It is clear from this that the characteristic polynomial $f_x(\lambda)$ is an element of $\Phi[\xi_1, \cdots, \xi_n, \lambda]$ which is homogeneous of degree n in $\xi_1, \cdots, \xi_n$ and λ. Also the coefficient of λ^n in $f_x(\lambda)$ is 1. It follows from Gauss' lemma that $m_x(\lambda)$ which is a factor of $f_x(\lambda)$ in $\Phi(\xi_1, \cdots, \xi_n)[\lambda]$ has the form

(27) $$m_x(\lambda) = \lambda^m - \sigma_1(\xi)\lambda^{m-1} + \sigma_2(\xi)\lambda^{m-2} - \cdots + (-1)^m \sigma_m(\xi)$$

where $\sigma_j(\xi) = \sigma_j(\xi_1, \cdots, \xi_n) \in \Phi[\xi_1, \cdots, \xi_n]$. Moreover, since any factor of a homogeneous polynomial is homogeneous, $m_x(\lambda)$ is homogeneous of degree m in $\xi_1, \cdots, \xi_n$ and λ, which implies that $\sigma_j(\xi)$ is homogeneous of degree j in the ξ's.

It is clear by induction on k that if $x^0 = 1$ then

(28) $$x^k = \sum_i \rho_{ki}(\xi) u_i, \qquad k = 0,1,2,\cdots$$

$(\rho_{1i} = \xi_i)$ where $\rho_{ki}(\xi) \in \Phi[\xi] = \Phi[\xi_1, \cdots, \xi_n]$ is homogeneous of degree k in the ξ's. The relation $m_x(x) = 0$ is equivalent to the following set of relations in $\Phi[\xi]$:

(29) $$\rho_{mi}(\xi) - \sigma_1(\xi)\rho_{m-1,i}(\xi) + \cdots + (-1)^m \sigma_m(\xi)\rho_{0i} = 0$$

$i = 1,2,\cdots,n$. Now let $a = \sum_1^n \alpha_i u_i \in \mathfrak{A}$. Then the specialization $\xi_i = \alpha_i$ in (28) gives $a^k = \sum \rho_{ki}(\alpha) u_i$ and the same specialization in (29) gives $\rho_{mi}(\alpha) - \sigma_1(\alpha)\rho_{m-1,i}(\alpha) + \cdots + (-1)^m \sigma_m(\alpha)\rho_{0i} = 0$. These relations imply that if we put

(30) $$m_a(\lambda) = \lambda^m - \sigma_1(a)\lambda^{m-1} + \cdots + (-1)^m \sigma_m(a)$$

where $\sigma_i(a) = \sigma_i(\alpha)$ then we have $m_a(a) = 0$ in $\mathfrak{A}$. We shall call the polynomial $m_a(\lambda) \in \Phi[\lambda]$ the *generic minimum polynomial of a* and the elements $\sigma_1(a)$ and $\sigma_m(a)$ respectively the *generic trace* and *generic norm* of a. Also we shall call the degree m of all the $m_a(\lambda)$ (or of $m_x(\lambda)$ in λ) the *degree* of the algebra $\mathfrak{A}$. We shall see later that this coincides with the degree of a finite dimensional central simple Jordan algebra as defined in the last chapter (§5.7). Since $m_a(a) = 0$ we have $\mu_a(\lambda) \mid m_a(\lambda)$ if $\mu_a(\lambda)$ is the minimum polynomial of a.

The generic minimum polynomial has been defined by means of the basis $(u_1, u_2, \cdots, u_n)$ for $\mathfrak{A}/\Phi$. We proceed to establish independence of this choice of basis. Accordingly, let $(v_1, v_2, \cdots, v_n)$ be a second basis for $\mathfrak{A}/\Phi$ and consider the generic element $y = \sum_1^n \xi_i v_i$ associated with the basis. We have $v_i = \sum \mu_{ij} u_j$ where $\mu_{ij} \in \Phi$ and (μ_{ij}) is nonsingular, and $y = \sum \xi_i v_i = \sum \xi_i \mu_{ij} u_j$. The specialization argument used before shows that if $m_y(\lambda)$ is the polynomial obtained from $m_x(\lambda)$ by replacing ξ_j by $\eta_j = \sum_i \xi_i \mu_{ij}$ then $m_y(y) = 0$. Hence the degree of the minimum polynomial of y in $\mathfrak{A}_{\mathrm{P}}$ is $\leqq m$. By symmetry, it is m and so it is the polynomial $m_y(\lambda)$. This implies that for any $a = \sum_i \beta_i v_i = \sum_{i,j} \beta_i \mu_{ij} u_j$ the generic minimum polynomials of a defined by x and by y are the same.

We note next that $m_a(\lambda)$ is unchanged on extension of the base field Φ. By this we mean that if Γ is an extension field of Φ and we identify $\mathfrak{A}$ as usual with a Φ-subalgebra of $\mathfrak{A}_\Gamma$ then $m_a(\lambda)$ is the same whether calculated in $\mathfrak{A}$ or in $\mathfrak{A}_\Gamma$. To see this we note that since (u_i) is also a basis for $\mathfrak{A}_\Gamma$ then, assuming, as we may, that the ξ_i are algebraically independent over Γ, we have that $x = \sum \xi_i u_i$ is a generic element of $\mathfrak{A}_\Gamma$. Moreover, it is clear from properties of the tensor product that the minimum polynomial of an element is unaltered on extension of the base field. Hence $m_x(\lambda)$ is also the minimum polynomial of x as element of $\mathfrak{A}_{\Gamma(\xi)} = \Gamma(\xi) \otimes_\Phi \mathfrak{A}$, $\mathrm{P} = \Phi(\xi)$. Clearly this implies that the generic minimum polynomial of a is unchanged on extension of Φ to Γ.

The relation $m_a(a) = 0$ can be written in the form

$$n(a)1 = a[\sigma_{m-1}(a)1 - \sigma_{m-2}(a)a + \cdots + (-1)^{m-1}a^{m-1}] \tag{31}$$

where $n(a) \equiv \sigma_m(a)$ is the generic norm. We now put

$$\operatorname{adj} a = \sigma_{m-1}(a)1 - \sigma_{m-2}(a)a + \cdots + (-1)^{m-1}a^{m-1} \tag{32}$$

in analogy with the adjoint of a matrix. Then we have the relation

$$a(\operatorname{adj} a) = n(a)1 = (\operatorname{adj} a)a. \tag{33}$$

We have defined the generic minimum polynomial for a in any finite-dimensional strictly power associative algebra with 1. In deriving properties of these polynomials and their coefficients we may generally assume the base field is infinite since the generic minimum polynomial is unchanged under extension of the base field. For infinite Φ, $a \to \sigma_i(a)$ is a homogeneous polynomial function of degree

i on $\mathfrak{A}$. Since the minimum polynomial $\mu_a(\lambda)$ of $a \in \mathfrak{A}$ is a factor of $m_a(\lambda)$ it is clear that the degree of $\mu_a(\lambda)$, which is the same as the dimensionality of $\Phi[a]$, does not exceed the degree m of the algebra. We shall call a of *maximal degree* if $\deg \mu_a(\lambda) = m$. This is the case if and only if $(1, a, \cdots, a^{m-1})$ is a basis for $\Phi[a]/\Phi$. If $x = \sum \xi_i u_i$ is generic then the fact that $(1, x, \cdots, x^{m-1})$ is a basis for $P[x]$, $P = \Phi(\xi_1, \cdots, \xi_n)$ implies that the $m \times n$ matrix (ρ_{ik}) given by (28) is of rank m. If $D_1(\xi), \cdots, D_r(\xi)$ are the nonzero determinants of order m of this matrix then $a = \sum \alpha_i u_i \in \mathfrak{A}$ is of maximal degree if and only if $D_i(a) = \check{D}_i(\alpha) \neq 0$ for some i. Hence if Φ is infinite then the set of elements of maximal degree is a nonvacuous Zariski open subset of $\mathfrak{A}$.

We now write $t(a)$ for the generic trace $\sigma_1(a)$ and we prove first the following

THEOREM 1. *Let $\mathfrak{A}$ be a finite-dimensional strictly power associative algebra with* 1, $m_a(\lambda)$, $t(a)$, $n(a)$ *the generic minimum polynomial, trace and norm of* $a \in \mathfrak{A}$, m *the degree of* $\mathfrak{A}$. *Then*:

(i) $t(\alpha a) = \alpha t(a)$, $\alpha \in \Phi$, $t(a + b) = t(a) + t(b)$,

(ii) $n(\alpha a) = \alpha^m n(a)$, $n(ab) = n(a)n(b)$

if a and b are in the subalgebra $\Phi[c]$ generated by 1 *and a third element c.*

(iii) $n(\lambda 1 - a) = m_a(\lambda)$. *Here $\lambda 1 - a$ is an element of $\mathfrak{A}_{\Phi(\lambda)}$, λ an indeterminate.*

(iv) $t(1) = m$, $n(a) = 1$.

(v) *Every irreducible factor of $m_a(\lambda)$ is a factor of the minimum polynomial $\mu_a(\lambda)$ of a.*

(vi) *If η is an isomorphism or anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$ then* $m_{a^\eta}(\lambda) = m_a(\lambda)$.

(vii) *If Φ is infinite then the coefficients of the generic minimum polynomial are Lie invariant under the derivations of $\mathfrak{A}$: $\Delta^{aD}{}_a \, \sigma_i = 0$ if D is a derivation of $\mathfrak{A}$.*

PROOF. (i) and the first part of (ii) are clear since $\sigma_1(\xi)$ is homogeneous of degree one in the ξ's and $\sigma_m(\xi)$ is homogeneous of degree m in the ξ's. For the rest of the proof we assume, as we may without loss of generality, that Φ is infinite. To prove the second part of (ii) we assume first that c is a fixed element of $\mathfrak{A}$ of maximal degree and consider the subalgebra $\Phi[c]$. Let a be an element of maximal degree in $\Phi[c]$ (and hence in $\mathfrak{A}$). Since $b = 1$ has the property that ab is of maximal degree in $\Phi[c]$ it is clear that the set of b such that ab is of maximal degree in $\Phi[c]$ is a nonvacuous Zariski open subset of $\Phi[c]$. Hence the set of elements $b \in \Phi[c]$ such that b and ab are of maximal degree in $\Phi[c]$ is open in $\Phi[c]$. Let b be such an element. Then $n(b) = \det \overline{b_R}$, $n(ab) = \det \overline{(ab)_R}$ as well as $n(a) = \det \overline{a_R}$ where $\overline{u_R}$ for $u \in \Phi[c]$ is the restriction to $\Phi[c]$ of u_R. This is clear since for these elements the generic minimum polynomials coincide with the minimum polynomials and with the characteristic polynomials of the restriction to $\Phi[c]$ of the corresponding right multiplications. We have $\overline{(ab)_R} = \overline{a_R}\,\overline{b_R}$. Hence $n(ab) = n(a)n(b)$ follows from the multiplicative property of determinants. Since this holds for all b in an open subset of $\Phi[c]$ it holds for all b in $\Phi[c]$. Now $n(ab) = n(a)n(b)$ holds for a in $\Phi[c]$ of maximal degree and b arbitrary in $\Phi[c]$. Hence it holds for all $a, b \in \Phi[c]$. Now let $\alpha_0, \alpha_1, \cdots, \alpha_{m-1}$, $\beta_0, \cdots, \beta_{m-1}$ be fixed elements of Φ and let c be variable in $\mathfrak{A}$. Consider the polynomial function $c \to n(ab) - n(a)n(b)$ where $a = \alpha_0 1 + \alpha_1 c + \cdots + \alpha_{m-1}c^{m-1}$, $b = \beta_0 1 + \beta_1 c + \cdots + \beta_{m-1}c^{m-1}$. This function has the value 0 for all c in an open subset of $\mathfrak{A}$. Hence it is 0 for all $c \in \mathfrak{A}$. Then $n(ab) = n(a)n(b)$ holds for all c and fixed α_i, β_i. Since any pair of elements $a, b \in \Phi[c]$ can be obtained as $a = \sum \alpha_i c^i$, $b = \sum \beta_i c^i$ by a suitable choice of the α_i, β_i in Φ it is now clear that $n(ab) = n(a)n(b)$ holds for all $a, b \in \Phi[c]$ for any c, which proves the second part of (ii). Now let a be of maximal degree and λ an indeterminate, so $m_a(\lambda) = \det(\lambda 1 - \overline{a_R})$ and $n(a) = \det \overline{a_R}$ when $\overline{a_R}$ denotes the restriction of a_R to $\Phi[a]$. It is easily seen that $\lambda 1 - a$ is of maximal degree in $\mathfrak{A}_{\Phi[\lambda]}$, so $n(\lambda 1 - a) = \det \overline{(\lambda 1 - a)_R}$ the determinant of the restriction of $(\lambda 1 - a)_R$ to $\Phi(\lambda)[\lambda 1 - a] = \Phi(\lambda)[a]$. Thus

$$m_a(\lambda) = \det(\lambda 1 - \overline{a_R}) = \det(\overline{\lambda 1 - a_R}) = n(\lambda 1 - a).$$

Since the set of elements of maximal degree is open we conclude that $m_a(\lambda) = n(\lambda 1 - a)$ for all a, so (iii) holds. We now consider the relation $m_a(\lambda) = n(\lambda 1 - a)$ for $a = 0$. Since $n(\lambda 1) = n(1)\lambda^m$ by the first part of (ii) and $m_a(\lambda)$ is a polynomial in λ with leading coefficient 1, we see that $n(1) = 1$. Also $m_1(\lambda) = n(\lambda 1 - 1) = n((\lambda - 1)1) = (\lambda - 1)^m n(1) = (\lambda - 1)^m = \lambda^m - m\lambda^{n-1} + \cdots$. Hence $t(1) = m$. Thus (iv) is valid. Next we prove (v). Let $a \in \mathfrak{A}$ and let $\mu_a(\lambda) \in \Phi[\lambda]$ be the minimum polynomial. We introduce the commutative associative polynomial ring $\Phi[a][\lambda]$ of polynomials in the indeterminate λ with coefficients in $\Phi[a]$. This contains $\Phi[\lambda]$ as subring and it contains the element $\lambda - a$. By the division algorithm in $\Phi[a][\lambda]$ we can write $\mu_a(\lambda) = (\lambda - a)Q + R$ where $Q, R \in \Phi[a][\lambda]$ and $R \in \Phi[a]$. We apply to this relation the homomorphism of $\Phi[a][\lambda]$ which is the identity on $\Phi[a]$ and sends $\lambda \to a$. This gives $0 = 0 + R$ so we have $R = 0$ and

$$\mu_a(\lambda) = (\lambda - a)Q, \qquad Q \in \Phi[a][\lambda]. \tag{34}$$

Next we consider the algebra $\mathfrak{A}_{\Phi(\lambda)}$ which has the basis (u_i) where this is a basis for $\mathfrak{A}/\Phi$, so any element of $\mathfrak{A}_{\Phi(\lambda)}$ has the form $\sum \lambda_i u_i$ where $\lambda_i \in \Phi(\lambda)$. $\mathfrak{A}_{\Phi(\lambda)}$ contains $\mathfrak{A}$ as Φ-subalgebra and it also contains the Φ-subalgebra $\Phi(\lambda)1$ of $\Phi(\lambda)$ multiples of $1 = \sum \gamma_i u_i$. Clearly $a \in \mathfrak{A}$ commutes with every element of $\Phi(\lambda)1$. Hence we have a homomorphism η of $\Phi[a][\lambda]$ into $\mathfrak{A}_{\Phi(\lambda)}$ such that $\beta(\lambda)^\eta = \beta(\lambda)1$ if $\beta(\lambda) \in \Phi[\lambda]$ and $a^\eta = a$. Applying this to (34) we obtain

$$\mu_a(\lambda)1 = (\lambda 1 - a)Q^\eta. \tag{35}$$

Since $\lambda^\eta = \sum \gamma_i \lambda u_i$ and $a^\eta = \sum \alpha_i u_i$, $\alpha_i \in \Phi$ it is clear that $Q^\eta = \sum \lambda_i u_i$ where $\lambda_i \in \Phi[\lambda]$. Hence $n(Q^\eta) \in \Phi[\lambda]$ since it is obtained by putting $\xi_i = \lambda_i$ in $n(\xi_1, \xi_2, \cdots, \xi_n)$. On the other hand, it is clear that $\mu_a(\lambda)1$, $\lambda 1 - a$ and $Q^\eta \in \Phi(\lambda)[a]$ the $\Phi(\lambda)$-subalgebra of $\mathfrak{A}_{\Phi(\lambda)}$ generated by 1 and a. Hence the second part of (ii) is applicable. Since n is homogeneous of degree m and $n(1) = 1$ we obtain from (35) that $\mu_a(\lambda)^m = n(\lambda 1 - a)n(Q^\eta)$. By (iii) this gives

$$\mu_a(\lambda)^m = m_a(\lambda)\, q(\lambda) \tag{36}$$

where $q(\lambda) = n(Q^\eta) \in \Phi[\lambda]$. Clearly (36) implies that every irreducible factor of $m_a(\lambda)$ is a factor of $\mu_a(\lambda)$. Hence (v) holds. Next let η be an isomorphism or anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. It is clear that for the minimum polynomials we have $\mu_a(\lambda) = \mu_{a^\eta}(\lambda)$, $a \in \mathfrak{A}$. Hence if a has maximal degree then $m_a(\lambda) = \mu_a(\lambda) = \mu_{a^\eta}(\lambda)$. This implies that $\deg \mathfrak{A}' \geqq \deg \mathfrak{A}$. Then, by symmetry, $\deg \mathfrak{A}' = \deg \mathfrak{A}$ and a^η has maximal degree. Then $m_a(\lambda) = m_{a^\eta}(\lambda)$. Now $a \to \sigma_j(a^\eta)$ is a polynomial function on $\mathfrak{A}/\Phi$ and the result just proved shows that this coincides with $\sigma_j(a)$ if a is in the open subset of elements of maximal degree. Hence $\sigma_j(a^\eta) = \sigma_j(a)$ for all $a \in \mathfrak{A}$ and all j. This proves (vi). Let D be a derivation in $\mathfrak{A}/\Phi$. Then applying D to $m_a(a) = 0$ we obtain

$$a^m D - \sigma_1(a)(a^{m-1}D) + \cdots + (-1)^{m-1}\sigma_{m-1}(a)(aD) = 0. \tag{37}$$

If f is any polynomial mapping of $\mathfrak{A}$ into itself then we have the polynomial mapping $a \to \Delta^{aD}{}_a f$ of $\mathfrak{A}$. If we apply this to $p_k : a \to a^k$ we obtain $\Delta^{aD}{}_a p_k = a^k D$. This is clear for $k = 1$ since p_1 is the linear mapping $a \to a$ which implies that $\Delta_a^{aD} p_1 = aD$. Then if $\Delta_a^{aD} p_{k-1} = a^{k-1}D$, we have by (23) that $\Delta_a^{aD} p_k = \Delta_a^{aD} p_{k-1} p_1 = (a^{k-1}D)a + a^{k-1}(aD) = a^k D$. The relation $m_a(a) = 0$ is equivalent to the functional relation $p_m - \sigma_1 p_{m-1} + \cdots + (-1)^m \sigma_m = 0$. Applying Δ_a^{aD} to this gives

$$\begin{aligned} a^m D - {} & \sigma_1(a)a^{m-1}D - (\Delta_a^{aD}\sigma_1)a^{m-1} + \sigma_2(a)a^{m-2}D \\ & + (\Delta_a^{aD}\sigma_2)a^{m-2} - \cdots = 0. \end{aligned} \tag{38}$$

Subtracting this from (37) gives

$$(\Delta_a^{aD}\sigma_1)a^{m-1} - (\Delta_a^{aD}\sigma_2)a^{m-2} + \cdots = 0. \tag{39}$$

If a is of maximal degree this gives $\Delta_a^{aD}\sigma_j = 0$. This implies that $\Delta_a^{aD}\sigma_j = 0$ for all a and j and proves (vii).(1)

COROLLARY 1. (1) *If $m_a(\lambda)$ has distinct roots (in its splitting field) then $m_a(\lambda) = \mu_a(\lambda)$.* (2) *An element $a \in \mathfrak{A}$ is nilpotent if and only if $m_a(\lambda) = \lambda^m$. The generic trace of a nilpotent element is 0.* (3) *If e is an idempotent $\neq 0, 1$ then $m_a(\lambda) = (\lambda - 1)^k \lambda^{m-k}$ where $0 < k < m$ and accordingly $t(e) = k$.*

(1) This part of the theorem is due to Tits ([5]).

PROOF. The results (1) and (2) are immediate consequences of (v). Now let e be an idempotent $\neq 0, 1$. Then the minimum polynomial of e is $\lambda^2 - \lambda = \lambda(\lambda-1)$. It follows from (v) that $m_e(\lambda) = (\lambda - 1)^k\lambda^{m-k}$, $0 < k < m$. Then $t(e) = k$.

COROLLARY 2. *An element a has an inverse in $\Phi[a]$ if and only if $n(a) \neq 0$.*

PROOF. a has an inverse in $\Phi[a]$ if and only if its minimum polynomial $\mu_a(\lambda)$ has a nonzero constant term. This is equivalent to $\lambda \nmid \mu_a(\lambda)$. Hence, by (v), a has an inverse in $\Phi[a]$ if and only if $\lambda \nmid m_a(\lambda)$ and this is equivalent to $n(a) \neq 0$.

We shall call a finite-dimensional strictly power associative algebra with 1 a *division algebra* if every $a \neq 0$ has an inverse in $\Phi[a]$. This is equivalent to our earlier definition for the special case of Jordan algebras. Evidently Corollary 2 implies

COROLLARY 3. *$\mathfrak{A}$ is a division algebra if and only if $n(a) \neq 0$ for every $a \neq 0$ in $\mathfrak{A}$.*

By Theorem 1 (i), $a \to t(a)$ is a linear function on $\mathfrak{A}$. This gives rise to an important bilinear form $(a, b) \to t(a, b) \equiv t(ab)$ on $\mathfrak{A}$. We shall call this the *generic trace (bilinear) form* on $\mathfrak{A}$ and we have the following

COROLLARY 4. *The generic trace form of a finite-dimensional associative or Jordan algebra $\mathfrak{A}$ with 1 is symmetric and associative, that is,*

$$t(a,b) = t(b,a), \qquad t(ab,c) = t(a,bc), \qquad a,b,c \in \mathfrak{A}. \tag{40}$$

PROOF. We may assume Φ is infinite and we suppose first that $\mathfrak{A}$ is associative. Then $t(ab, c) - t(a, bc) = t([a, b, c]) = 0$. Now Theorem 1(vii) and the linearity of the trace imply that $t(aD) = 0$ for every derivation D. Since $\mathfrak{A}$ is associative $a \to [a, b]$ is a derivation. Hence we have $t([a, b]) = 0$ and $t(a, b) = t(b, a)$. Next assume $\mathfrak{A}$ Jordan. Then $t(a, b) = t(b, a)$ is clear from the commutativity of $\mathfrak{A}$. Also in this case we know that $b \to [a, b, c]$ is a derivation. Hence $t([a, b, c]) = 0$ which gives the associativity of the generic trace form.

Let $\mathfrak{B}$ be a subalgebra containing 1 of the finite-dimensional strictly power associative algebra $\mathfrak{A}$ with 1. Let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{A}$, $(v_1, v_2, \cdots, v_r)$ a basis for $\mathfrak{B}$, $x = \sum_1^n \xi_i u_i$, $y = \sum_1^r \xi_j v_j$ the corresponding generic elements and let $m_x(\lambda)$ be the minimum polynomial of x in $\mathfrak{A}_P$, $P = \Phi(\xi_1, \cdots, \xi_n)$, $m_{y|\mathfrak{B}}(\lambda)$ the minimum polynomial of y in $\mathfrak{B}_P$. Then $m_x(\lambda)$ is the generic minimum polynomial in $\mathfrak{A}_P$ of x and $m_{y|\mathfrak{B}}(\lambda)$ is that of y in $\mathfrak{B}_P$. If $v_j = \sum \mu_{ji} u_i$, $\mu_{ji} \in \Phi$ then $y = \sum \eta_i u_i$ where $\eta_i = \sum_{j=1}^r \xi_j \mu_{ji}$ and the generic minimum polynomial $m_y(\lambda)$ of y as element of $\mathfrak{A}_P$ is obtained by the specialization $\xi_i \to \eta_i$ in $m_x(\lambda)$. By Theorem 1(v), $m_y(\lambda)$ and $m_{y|\mathfrak{B}}(\lambda)$ have the same irreducible factors in $P[\lambda]$. Since these are polynomials in λ and $\xi_1, \xi_2, \cdots, \xi_r$ and have leading coefficients of $\lambda = 1$ it follows that $m_y(\lambda)$ and $m_{y|\mathfrak{B}}(\lambda)$ have the same irreducible factors in $\Phi[\lambda, \xi_1, \cdots, \xi_r]$. We state this result as

COROLLARY 5. *Let $\mathfrak{A}$ be a strictly power associative algebra with* 1, *$\mathfrak{B}$ a subalgebra containing* 1, *$(u_1, \cdots, u_n)$ a basis for $\mathfrak{A}$, $(v_1, \cdots, v_r)$ a basis for $\mathfrak{B}$, $x = \sum_1^n \xi_i u_i$, $y = \sum_1^r \xi_j v_j$ the corresponding generic elements. Let $m_y(\lambda)$ be the generic minimum polynomial of y as element of $\mathfrak{A}_P$, $P = \Phi(\xi_1, \cdots, \xi_n)$, $m_{y|\mathfrak{B}}(\lambda)$ the generic minumum polynomial of y as element of $\mathfrak{B}_P$. Then $m_{y|\mathfrak{B}}(\lambda)$ is a factor of $m_y(\lambda)$ in $\Phi[\xi_1, \cdots, \xi_n, \lambda]$ and these have the same irreducible factors in $\Phi[\xi_1, \cdots, \xi_n, \lambda]$.*

Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$ where the $\mathfrak{A}_i$ are ideals and let $a = a_1 + a_2$, $a_i \in \mathfrak{A}_i$. Then if $f(\lambda) = \lambda^q + \alpha_1 \lambda^{q-1} + \cdots + \alpha_q \in \Phi[\lambda]$, $f(a) = f(a_1) + f(a_2)$ where $f(a_i) = a_i^q + \alpha_1 a_i^{q-1} + \cdots + \alpha_q 1_i$, 1_i the identity element of $\mathfrak{A}_i$ $(1 = 1_1 + 1_2)$. Since $f(a) = 0$ if and only if $f(a_i) = 0$, $i = 1, 2$, it is clear that the minimum polynomial of a in $\mathfrak{A}$ is the least common multiple of the minimum polynomials of the a_i in $\mathfrak{A}_i$. Let $(u_1, \cdots, u_n)$ be a basis for $\mathfrak{A}/\Phi$ such that $(u_1, \cdots, u_r)$ is a basis for $\mathfrak{A}_1$ and $(u_{r+1}, \cdots, u_n)$ is a basis for $\mathfrak{A}_2$. Then the generic element $x = \sum_1^n \xi_i u_i = y + z$ where $y = \sum_1^r \xi_j u_j$ and $z = \sum_{r+1}^n \xi_k u_k$ are generic elements of $\mathfrak{A}_1$, $\mathfrak{A}_2$ respectively. Let $m^{(1)}{}_y(\lambda)$ and $m^{(2)}{}_z(\lambda)$ be the minimum polynomials of y and z in $\mathfrak{A}_{1P}$ and $\mathfrak{A}_{2P}$ respectively where $P = \Phi(\xi_1, \cdots, \xi_n)$. Then $m^{(1)}{}_y(\lambda) = \lambda^{m_1} - \sigma^{(1)}{}_1(\xi_1, \cdots, \xi_r)\lambda^{m_1 - 1} + \cdots + (-1)^{m_1}\sigma^{(1)}{}_{m_1}(\xi_1, \cdots, \xi_r)$ where $\sigma^{(1)}{}_l(\xi_1, \cdots, \xi_r)$ is homogeneous of degree l in $\xi_1, \cdots, \xi_r$ and $\sigma^{(1)}{}_{m_1}(\xi_1, \cdots, \xi_r) \neq 0$ (since $\sigma^{(1)}_{m_1}(1_1) = 1$). Similarly, $m^{(2)}{}_z(\lambda) = \lambda^{m_2} - \sigma^{(2)}{}_1(\xi_{r+1}, \cdots, \xi_n)\lambda^{m_2 - 1} + \cdots + (-1)^{m_2}\sigma^{(2)}{}_{m_2}(\xi_{r+1}, \cdots, \xi_n)$ where $\sigma^{(2)}_q(\xi_{r+1}, \cdots, \xi_n)$ is homogeneous of degree q in $\xi_{r+1}, \cdots, \xi_n$ and $\sigma^{(2)}{}_{m_2}(\xi_{r+1}, \cdots, \xi_n) \neq 0$. We claim that $m^{(1)}{}_y(\lambda)$ and $m^{(2)}{}_z(\lambda)$ are relatively prime in $P[\lambda]$. Otherwise, these have a common factor of the form $\lambda^h + \tau_1(\xi_1, \cdots, \xi_n)\lambda^{h-1} + \cdots + \tau_h(\xi_1, \cdots, \xi_n)$ in $\Phi[\lambda, \xi_1, \cdots, \xi_n]$ where $h \geqq 1$ and $\tau_i(\xi_1, \cdots, \xi_n)$ is homogeneous of degree i in $\xi_1, \cdots, \xi_n$. Putting $\lambda = 0$ implies that $\tau_h(\xi_1, \cdots, \xi_n)$ is a common factor of $\sigma^{(1)}{}_{m_1}(\xi_1, \cdots, \xi_r)$ and $\sigma^{(2)}{}_{m_2}(\xi_{r+1}, \cdots, \xi_n)$ which is impossible. Hence we see that $m^{(1)}{}_y(\lambda)$ and $m^{(2)}{}_z(\lambda)$ are relatively prime in $P[\lambda]$ so $m^{(1)}{}_y(\lambda) m^{(2)}{}_z(\lambda)$ is a least common multiple of these polynomials. As we saw before, this is the minimum polynomial of x. This result and the definitions clearly imply the first statement in

THEOREM 2. *Let $\mathfrak{A}$ be as in Theorem 1 and suppose $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$ where $\mathfrak{A}_i$ is an ideal with identity element 1_i. If $a = a_1 + a_2$ where $a_i \in \mathfrak{A}_i$ then the generic minimum polynomial $m_a(\lambda)$ of a in $\mathfrak{A}$ is $m^{(1)}{}_{a_1}(\lambda) m^{(2)}{}_{a_2}(\lambda)$ where $m^{(1)}{}_{a_i}(\lambda)$ is the generic minimum polynomial of a_i in $\mathfrak{A}_i$. Also $n(a) = n^{(1)}(a_1)n^{(2)}(a_2)$ where $n^{(i)}(a_i)$ is the generic norm of a_i in $\mathfrak{A}_i$ and $t(a_i) = t^{(i)}(a_i)$ where $t^{(i)}(a_i)$ is the generic trace of a_i in $\mathfrak{A}_i$.*

PROOF. $n(a) = n^{(1)}(a_1)n^{(2)}(a_2)$ and $t(a) = t^{(1)}(a_1) + t^{(2)}(a_2)$ are immediate consequences of the first result. If we take $a_2 = 0$ in the second of these we obtain $t(a_1) = t^{(1)}(a_1)$ and similarly $t(a_2) = t^{(2)}(a_2)$.

The last statement implies also that the trace bilinear form on $\mathfrak{A}_i$ is the restriction to $\mathfrak{A}_i$ of the trace bilinear form on $\mathfrak{A}$. It is clear also that Theorem 2 has an immediate extension to the situation in which $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where the $\mathfrak{A}_i$ are ideals.

EXERCISES

1. Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1, t its generic trace. Define $t(a,b,c) = t(a.b.c)$. Show that $t(a,b,c)$ is symmetric and trilinear. Show that if the characterisic is not three and η is a linear mapping in $\mathfrak{J}$ having the first three coefficients σ_i of the generic minimum polynomial as invariants $(\sigma_i(a^\eta) = \sigma_i(a),\ i = 1,2,3)$ then $t(a^\eta, b^\eta, c^\eta) = t(a,b,c)$. Hence show that if η is a linear mapping in $\mathfrak{J}$ having n as invariant and sending 1 into 1 and $|\Phi| > m$, the degree of $\mathfrak{J}$ then $t(a^\eta, b^\eta, c^\eta) = t(a,b,c)$.

2. Let $\mathfrak{J}$ and $\mathfrak{J}'$ be finite-dimensional Jordan algebras with 1 over an infinite field and let η be a 1-1 linear mapping of $\mathfrak{J}$ onto $\mathfrak{J}'$ such that $1^\eta = 1$ and if c is invertible in $\mathfrak{J}$ then c^η is invertible in $\mathfrak{J}'$ and $(c^{\cdot -1})^\eta = (c^\eta)^{\cdot -1}$. Show that η is an isomorphism.

3. State an analogue of exercise 2 for derivations.

4. **Some important examples.** Let $\mathfrak{A}$ be a finite-dimensional strictly power associative algebra with 1, $x = \sum_1^n \xi_i u_i$ a generic element of $\mathfrak{A}$, $m_x(\lambda)$ its minimum polynomial. Since $m_x(\lambda) = \lambda^m - \sigma_1(\xi)\lambda^{m-1} + \cdots + (-1)^m \sigma_m(\xi)$ where $\sigma_j(\xi) \in \Phi[\xi_1, \cdots, \xi_n]$, ξ_i indeterminates, it is clear that the discriminant $\delta(x)$ of $m_x(\lambda)$ is in $\Phi[\xi_1, \cdots, \xi_n]$ (Jacobson, *Lectures in Abstract Algebra*, vol. III, p. 92). We shall call $\mathfrak{A}$ *unramified* if $\delta(x) \neq 0$. The following lemma gives a criterion for this and is useful for determining the generic minimum polynomials for the algebras we shall consider.

LEMMA 1. *Let $\mathfrak{A}$ be a finite-dimensional strictly power associative algebra with 1, m the degree of $\mathfrak{A}$. Let I be a set of nonzero orthogonal idempotents in $\mathfrak{A}$ $(e_i^2 = e_i \neq 0,\ e_i e_j = 0,\ i \neq j)$. Then the cardinal number $|I| \leqq m$ and $\mathfrak{A}$ is unramified if and only if $\mathfrak{A}_\Omega$, where Ω is the algebraic closure of the base field Φ of $\mathfrak{A}$, contains m nonzero orthogonal idempotents.*

PROOF We note first that if $e_1, e_2, \cdots, e_r$ are nonzero orthogonal idempotents and $\alpha_i \neq \alpha_j$ are in Φ then it is easily seen that $\mu_a(\lambda) = \prod_1^r (\lambda - \alpha_i)$ is the minimum polynomial of $a = \sum \alpha_i e_i$. Since $\mu_a(\lambda) \mid m_a(\lambda)$ we have $m \geqq r$. Now if $e_1, \cdots, e_r$ are given then we can choose distinct α_i in an extension field of Φ. Since the degree is unchanged on extension of the base field it is clear that the first statement holds. Now suppose we have m distinct orthogonal idempotents $\neq 0$ in $\mathfrak{A}_\Omega$. Then we have an $a \in \mathfrak{A}_\Omega$ such that $\mu_a(\lambda)$ has m distinct roots. Then $m_a(\lambda) = \mu_a(\lambda)$ by Theorem 1 (v). Hence $m_a(\lambda)$ has distinct roots, which implies that the discriminant of $m_a(\lambda)$ is not zero. Since $m_a(\lambda)$ is obtained by specializing $\xi_i = \alpha_i \in \Phi$, the discriminant of $m_a(\lambda)$ is $\delta(a)$ which is obtained by making the same specialization in $\delta(x)$. Since $\delta(a) \neq 0$ we have $\delta(x) \neq 0$ and $\mathfrak{A}$ is unramified. Conversely, assume

$\mathfrak{A}$ unramified. Then we can choose a in $\mathfrak{A}_\Omega$ so that $\delta(a) \neq 0$. This implies that $m_a(\lambda)$ has m distinct roots. Then $\mu_a(\lambda) = m_a(\lambda)$ by Corollary 1 to Theorem 1 This implies that $\Omega[a]$ contains m nonzero orthogonal idempotents.

We now consider our examples.

A. Φ_n, the algebra of $n \times n$ matrices over Φ. If (e_{ij}), $i,j = 1,\cdots,n$, is the usual matrix basis, ξ_{ij} indeterminates, then by the Hamilton-Cayley theorem the generic element $X = \sum \xi_{ij} e_{ij}$ is a root of the characteristic polynomial $f_X(\lambda) = \det(\lambda 1 - X)$. Hence $m_X(\lambda)$, which is a factor of $f_X(\lambda)$, has degree $\leqq n$. On the other hand, Φ_n contains the n orthogonal idempotents e_{ii}, $i = 1,2,\cdots,n$. Hence $\deg m_X(\lambda) \geqq n$. It follows that Φ_n is unramified and $m_X(\lambda) = f_X(\lambda)$.

We note next that if the characteristic is $\neq 2$ then for the Jordan algebra $\Phi_n{}^+$ we also have $m_X(\lambda) = f_X(\lambda)$ since the Jordan power $X^{\cdot n} = X^n$. In both cases Φ_n and $\Phi_n{}^+$, $m_A(\lambda)$ is the characteristic polynomial of the matrix A, $t(A)$ is the ordinary trace of a and $n(A) = \det A$.

B. $\mathfrak{H}(\Phi_n, J_1)$ the Jordan algebra of ordinary $n \times n$ symmetric matrices over Φ (characteristic not two). Since $\mathfrak{H}(\Phi_n, J_1)$ contains the e_{ii} the argument used in A shows that $\mathfrak{H}(\Phi_n, J_1)$ is unramified and $m_A(\lambda)$ for A in this Jordan algebra is the characteristic polynomial of A. Also $t(A)$ is the usual trace and $n(A) = \det A$.

C. $\mathfrak{H}(\Phi_{2n}, J_S)$ the Jordan algebra of $2n \times 2n$ matrices which are symmetric under the involution $X \to S^{-1} X^t S$ where $S = \operatorname{diag}\{Q, Q, \cdots, Q\}$, $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The condition $S^{-1} A^t S = A$ that $A \in \mathfrak{H}(\Phi_{2n}, J_S)$ is equivalent to $(SA)^t = A^t S^t = -A^t S = -SA$, that is, SA is skew. In order to obtain the generic minimum polynomial for $\mathfrak{H}(\Phi_{2n}, J_S)$ we need to consider a factorization property of the Pfaffian of a skew symmetric matrix.

Let ξ_{ij}, $i < j = 1,\cdots,2n$, be indeterminates and consider the ring $Z[\xi_{ij}]$, Z the ring of integers. Consider the generic skew symmetric matrix $X = (\xi_{ij})$ where $\xi_{ij} = 0$ and $\xi_{ji} = -\xi_{ij}$ if $i < j$. Let X_{ij} be the cofactor of ξ_{ji} in X. Then $X_{ii} = 0$ since the determinant of a skew symmetric matrix of odd order is 0. Also if $i \neq j$ then $X_{ij} = -X_{ji}$ since X is skew. We shall now prove the following

LEMMA 2. *$\det X$ is a square in $\mathfrak{X} = Z[\xi_{ij}]$ and if we let $\operatorname{Pf} X$ denote a square root in X of $\det X$ then $\operatorname{Pf} X \mid X_{kl}$ in $\mathfrak{X}$ for all k, l.*

PROOF. We have the matrix relation $X \operatorname{adj} X = (\det X)1$ where $\operatorname{adj} X = (X_{ij})$. This gives the systems of equations

$$\sum_{k=1}^{2n} \xi_{lk} X_{kj} = \delta_{lj}(\det X), \qquad l = 1,2,\cdots,2n \tag{41}$$

for any $j = 1,2,\cdots,2n$. Let i be an index $\neq j$ and delete the ith equation in this set. If we take into account the fact that $X_{jj} = 0$ we obtain $2n - 1$ linear equations for the $2n-1$ elements X_{kj}, $k \neq j$. The determinants of the coefficients of these equations is the minor of ξ_{ij} in X. This is $(-1)^{i+j}$ cofactor of $\xi_{ij} = (-1)^{i+j} X_{ji} = (-1)^{i+j+1} X_{ij}$. Hence if we solve the equations for X_{ij} by Cramer's rule we obtain

$$(42) \qquad (-1)^{i+j+1} X_{ij}{}^2 = (\det X)\Delta_{ij}$$

where $\Delta_{ij} \in \mathfrak{X}$. For $i = 1$, $j = 2$ this becomes $X_{12}{}^2 = (\det X)\Delta_{12}$ and one sees that Δ_{12} is the determinant of the skew symmetric matrix obtained by deleting the first and second rows and columns from X. Assuming this is a square and using unique factorization in $\mathfrak{X}$ we see that $\det X$ is a square in $\mathfrak{X}$. This and the fact that if $n = 1$ then $X = \begin{pmatrix} 0 & \xi_{12} \\ -\xi_{12} & 0 \end{pmatrix}$, so $\det X = \xi_{12}^2$, proves the first statement by induction on n. Now put $\det X = (\mathrm{Pf}X)^2$. Then (42) becomes $(-1)^{i+j+1} X_{ij}{}^2 = (\mathrm{Pf}X)^2 \Delta_{ij}$ which implies that $\mathrm{Pf}X \mid X_{ij}$ if $i \neq j$. Since $X_{ii} = 0$ this proves the second statement.

It is clear that if $S = \mathrm{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots\right\}$ as before, then $\det S = 1$. Hence the specialization of the ξ_{ij} in Z which gives S gives $\mathrm{Pf}S = \pm 1$. We now choose the determination of the sign of $\mathrm{Pf}X$ so that $\mathrm{Pf}S = 1$ and from now on call this $\mathrm{Pf}X$. If A is a $2n \times 2n$ matrix $A = (\alpha_{ij})$ with the α_{ij} in a commutative ring and $\alpha_{ii} = 0$, $\alpha_{ji} = -\alpha_{ij}$ then we can obtain A by the specialization $\xi_{ij} \to \alpha_{ij}$, $i < j$ of X. Under this the element $\mathrm{Pf}X$ is mapped into an element $\mathrm{Pf}A$ which we shall call the *Pfaffian of* (the skew matrix with diagonal elements 0) A. Evidently we have $(\mathrm{Pf}A)^2 = \det A$.

The fact that $\mathrm{Pf}X \mid X_{ij}$ for X generic and X_{ij} the cofactor of ξ_{ji} in X implies that $X_{ij} = Y_{ij}(\mathrm{Pf}X)$ where $Y_{ij} \in \mathfrak{X}$. We can now cancel $\mathrm{Pf}X$ in the relation $X \operatorname{adj} X = (\det X)1$ and obtain for $Y = (Y_{ij})$ that

$$(43) \qquad XY = \mathrm{Pf}(X)1.$$

Now let $A \in \mathfrak{H}(\Phi_{2n}, J_S)$ and let λ be an indeterminate. Then $S\lambda - SA$ is skew with entries in $\Phi[\lambda]$. Specialization of (43) now gives

$$(44) \qquad (S\lambda - SA)B(\lambda) = \mathrm{Pf}(S\lambda - SA)1$$

where $B(\lambda) \in \Phi[\lambda]_{2n}$. We can use this relation and the usual proof of the Hamilton-Cayley theorem (Jacobson, *Lectures* vol. II, p. 101) to prove that A is a root of the equation $\mathrm{Pf}(S\lambda - SA) = 0$. Also this has the leading coefficient $\mathrm{Pf}S = 1$. Clearly we have also $\mathrm{Pf}(S\lambda - SX) = 0$ for X generic in $\mathfrak{H}(\Phi_{2n}, J_S)$. Hence $m_X(\lambda) \mid \mathrm{Pf}(S\lambda - SX)$ and $\deg m_X(\lambda) \leqq n$. On the other hand, it is clear from the fact that $A \in \mathfrak{H}(\Phi_{2n}, J_S)$ if SA is skew that the elements $e_1 = e_{11} + e_{22}$, $e_2 = e_{33} + e_{44}, \cdots$ are contained in $\mathfrak{H}(\Phi_{2n}, J_S)$. Clearly we obtain in this way n nonzero orthogonal idempotents. Hence, by Lemma 1, $\mathfrak{H}(\Phi_{2n}, J_S)$ is unramified and the generic minimum polynomial of $A \in \mathfrak{H}(\Phi_{2n}, J_S)$ is

$$(45) \qquad \mathrm{Pf}(S\lambda - SA).$$

This implies that the generic norm $n(A) = \mathrm{Pf}(SA)$. Also since $\mathrm{Pf}(S\lambda - SA)^2 = \det(S\lambda - SA) = \det(\lambda 1 - A)$ it is clear that the generic trace $t(A) = \frac{1}{2}\mathrm{tr}\, A$, $\mathrm{tr}\, A$ the usual trace.

D. $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ the Jordan algebra of the symmetric bilinear form f on $\mathfrak{B}$. It is easy to see that if $\dim \mathfrak{B} > 0$ and $a = \alpha 1 + u$, $\alpha \in \Phi$, $u \in \mathfrak{B}$, then $m_a(\lambda) = \lambda^2 - 2\alpha\lambda + (\alpha^2 - f(u,u))$. Hence $t(a) = 2\alpha$ and $n(a) = \alpha^2 - f(u,u)$. It is easy to see also that if $f \neq 0$ then $\mathfrak{J}$ is unramified.

E. $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $\mathfrak{O}$ is an algebra of octonions and $\gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_i \neq 0$ in Φ. We consider a generic element of this Jordan algebra. This has the form

$$X = \begin{bmatrix} \xi_1 & z & \gamma_1^{-1}\gamma_3\bar{y} \\ \gamma_2^{-1}\gamma_1\bar{z} & \xi_2 & x \\ y & \gamma_3^{-1}\gamma_2\bar{x} & \xi_3 \end{bmatrix} \tag{46}$$

where the ξ_i are indeterminates and x, y, z are generic elements of $\mathfrak{O}$ defined respectively by the indeterminates $\xi_4, \cdots, \xi_{11}$; $\xi_{12}, \cdots, \xi_{19}$; $\xi_{20}, \cdots, \xi_{27}$. We introduce the matrix

$$X \times X = X^{\cdot 2} - \tau(X)X + \tfrac{1}{2}(\tau(X^{\cdot 2}) - \tau(X)^2)1 \tag{47}$$

where $\tau(X) = \xi_1 + \xi_2 + \xi_3$. A direct calculation shows that

$$X \times X =$$

$$\begin{bmatrix} \xi_2\xi_3 - \gamma_3^{-1}\gamma_2 n(x) & \gamma_1^{-1}\gamma_2\bar{y}\bar{x} - \xi_3 z & xz - \gamma_1^{-1}\gamma_3\xi_3\bar{y} \\ xy - \gamma_2^{-1}\gamma_1\xi_3\bar{z} & \xi_1\xi_3 - \gamma_1^{-1}\gamma_3 n(y) & \gamma_2^{-1}\gamma_3\bar{z}\bar{y} - \xi_1 x \\ \gamma_3^{-1}\gamma_1\bar{z}\bar{x} - \xi_3 y & yz - \gamma_3^{-1}\gamma_2\xi_1\bar{x} & \xi_1\xi_2 - \gamma_2^{-1}\gamma_1 n(z) \end{bmatrix} \tag{48}$$

where $n(x)$ denotes the given quadratic form on $\mathfrak{O}$ $(x\bar{x} = n(x)1 = \bar{x}x)$. Direct multiplication of (48) by X gives

$$(X \times X).X = (\det X)1 \tag{49}$$

where

$$\det X = \xi_1\xi_2\xi_3 - \gamma_3^{-1}\gamma_2 n(x) - \gamma_1^{-1}\gamma_3 n(y) - \gamma_2^{-1}\gamma_1 n(z) + t((zx)y) \tag{50}$$

where $t(x)1 = x + \bar{x}$. By (47) and (49) we have the following analogue of the Hamilton-Cayley theorem for $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ which is due to Freudenthal [1]

$$X^{\cdot 3} - \tau(X)X^{\cdot 2} + \tfrac{1}{2}(\tau(X)^2 - \tau(X^{\cdot 2}))X - (\det X)1 = 0. \tag{51}$$

It is clear that $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ contains three nonzero orthogonal idempotent elements. Hence $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ is unramified and the generic minimum polynomial of

$$A = \begin{bmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\bar{b} \\ \gamma_2^{-1}\gamma_1\bar{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\bar{a} & \alpha_3 \end{bmatrix} \tag{52}$$

is

$$m_A(\lambda) = \lambda^3 - t(A)\lambda^2 + \tfrac{1}{2}(t(A)^2 - t(A^{\cdot 2}))\lambda - n(A) \tag{53}$$

where the generic trace $t(A) = \alpha_1 + \alpha_2 + \alpha_3$ and the generic norm is

$$\begin{aligned} n(A) = \det A = {} & \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2\alpha_1 n(a) - \gamma_1^{-1}\gamma_3\alpha_2 n(b) - \gamma_2^{-1}\gamma_1\alpha_3 n(c) \\ & + t((ca)b). \end{aligned} \tag{54}$$

We note also that

$$t(A^{\cdot 2}) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2(\gamma_3^{-1}\gamma_2 n(x) + \gamma_1^{-1}\gamma_3 n(y) + \gamma_2^{-1}\gamma_1 n(z)). \tag{55}$$

The results we have just derived give us the generic minimum polynomials of all finite-dimensional simple Jordan algebras over algebraically closed fields. We have seen in §5.6 that a complete list of these algebras is the following: (1) Φ, (2) the Jordan algebras of nondegenerate symmetric bilinear forms in vector spaces of finite dimensionalities > 1, (3) the Jordan matrix algebras $\mathfrak{H}(\mathfrak{D}_n, J_1)$ with $n \geqq 3$ and $(\mathfrak{D}, j)$ a composition algebra which is associative if $n > 3$. We have seen also that $\mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is the split two-dimensional composition algebra is isomorphic to $\Phi_n^{\,+}$ and $\mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is a split quaternion algebra is isomorphic to the algebra $\mathfrak{H}(\Phi_{2n}, J_S)$ which we considered in example C above. Thus it is clear that we have determined the generic minimum polynomials of all the finite dimensional simple Jordan algebras with $\dim \mathfrak{J} > 1$ over algebraically closed fields. Since the generic minimum polynomial is unchanged under extension of the base field it is clear that, in principle, we have determined the generic minimum polynomials of all finite-dimensional central simple Jordan algebras. If $\mathfrak{J}$ is a Jordan algebra over Φ such that $\mathfrak{J}_\Omega \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$ where Ω is the algebraic closure of Φ, $(\mathfrak{D}, j)$ is a composition algebra, and $n \geqq 3$ then we defined the degree of $\mathfrak{J}$ in §5.7 to be n (p. 209). It is clear from our results that this is the same as the degree of $\mathfrak{J}$ as defined in this chapter.

Exercises

1. Let $\mathfrak{O}$ be an algebra of octonions over Φ. Show that $m_a(\lambda) = \lambda^2 - t(a)\lambda + n(a)$ where $a + \bar{a} = t(a)1$, $a\bar{a} = n(a)1$.

2. Prove that in all of the examples A, B, C, D, E above the generic norm $n(x)$, $x = \sum \xi_i u_i$ a generic element, is irreducible in $\Phi[\xi_1, \xi_2, \cdots, \xi_n]$.

3. Let $\mathfrak{J}$ be a finite-dimensional semisimple Jordan algebra over an algebraically closed field, $n(x) = n(\xi_1, \cdots, \xi_n)$ the generic norm. Show that the structure of $\mathfrak{J}$ can be deduced from the factorization of $n(x)$ into irreducible factors in $\Phi[\xi_1, \cdots, \xi_n]$. By this we mean that if $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ where the $\mathfrak{J}_i$ are simple ideals then the $\mathfrak{J}_i$ are in 1-1 correspondence with the irreducible factors of $n(x)$ and the isomorphism class of $\mathfrak{J}_i$ is determined by the corresponding irreducible factor.

4. Note that in example E, $A \times A = \operatorname{adj} A$. Call an element A in $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ of *rank* 1 if $A \neq 0$ and $A \times A = 0$. Show that this is equivalent to: either A is nilpotent of index two or $A = \alpha E$ where $\alpha \neq 0$ in Φ and $E^{\cdot 2} = E$, $t(E) = 1$.

5. Show that the associative algebra of all triangular matrices $\sum_{i \leq j} \alpha_{ij} e_{ij}$ is unramified and has degree n (the size of the matrices).

5. **Multiplicative properties of the generic norm.** In this section we shall derive the important multiplicative properties of the generic norm. For these we require the following two lemmas.

LEMMA 1. *Let $\mathfrak{V}$ be an n-dimensional vector space over an algebraically closed field Ω with basis $(u_1, u_2, \cdots, u_n)$. Let $Q(\xi_1, \cdots, \xi_n) \in \Omega[\xi_1, \cdots, \xi_n]$ be of positive degree and let Q be the polynomial function $a = \sum \alpha_i u_i \to Q(a) = Q(\alpha_1, \cdots, \alpha_n)$. Suppose T is a linear isomorphism of $\mathfrak{V}$ onto $\mathfrak{V}$ which maps $O_Q = \{a \mid Q(a) \neq 0\}$ onto itself. Let $Q_1(\xi_1, \cdots, \xi_n), \cdots, Q_r(\xi_1, \cdots, \xi_n)$ be the distinct irreducible factors of $Q(\xi_1, \cdots, \xi_n)$ in $\Omega[\xi_1, \cdots, \xi_n]$ (determined up to nonzero multipliers in Ω). Then there exists a permutation π of $1, 2, \cdots, r$ and nonzero elements $\rho_j \in \Omega$ such that*

$$Q_j(xT) = \rho_j Q_{j\pi}(x), \qquad x \in \mathfrak{V}, \qquad j = 1, \cdots, r. \tag{56}$$

Moreover, if $Q_0(\xi_1, \cdots, \xi_n) = \prod_1^r Q_j(\xi_1, \cdots, \xi_n)$ then

$$Q_0(xT) = \rho Q(x), \qquad x \in \mathfrak{V}, \qquad \rho = \Pi \rho_j. \tag{57}$$

PROOF. Let $u_i T = \sum \tau_{ik} u_k$. Then $aT = \sum \alpha_i' u_i$ where $\alpha_i' = \sum_k \alpha_k \tau_{ki}$ if $a = \sum \alpha_i u_i$. If $f(\xi_1, \cdots, \xi_n) \in \Phi[\xi_1, \cdots, \xi_n]$ then we define $f^T(\xi_1, \cdots, \xi_n) = f(\xi_1', \cdots, \xi_n')$ where $\xi_i' = \sum \xi_k \tau_{ki}$. Then $f(\xi_1, \cdots, \xi_n) \to f^T(\xi_1, \cdots, \xi_n)$ is an automorphism of $\Phi[\xi_1, \cdots, \xi_n]/\Phi$ so $f(\xi_1, \cdots, \xi_n)$ is irreducible if and only if $f^T(\xi_1, \cdots, \xi_n)$ is irreducible. It follows from this that $Q_j{}^T(\xi_1, \cdots, \xi_n)$, $j = 1, \cdots, r$, are the distinct irreducible factors of $Q^T(\xi_1, \cdots, \xi_n)$. On the other hand, $Q^T(a) = Q(aT)$ is clear from our definitions. Hence by the hypothesis of the lemma, $Q^T(\xi_1, \cdots, \xi_n)$ and $Q(\xi_1, \cdots, \xi_n)$ have the same set of zeroes in Ω. Since the base field is algebraically closed we can invoke the Hilbert Nullstellensatz (Jacobson, *Lectures*, vol. III, p. 254) to conclude that $Q^T(\xi_1, \cdots, \xi_n)$ and $Q(\xi_1, \cdots, \xi_n)$ have the same irreducible factors in $\Phi[\xi_1, \cdots, \xi_n]$. Hence there exists a permutation π of $\{1, \cdots, r\}$ and $\rho_j \neq 0$ in Φ such that $Q_j{}^T(\xi_1, \cdots, \xi_n) = \rho_j Q_{j\pi}(\xi_1, \cdots, \xi_n)$. This is equivalent to (56). Evidently (57) follows from this.

LEMMA 2 (MCCRIMMON). *Let* $Q(\xi_1,\cdots,\xi_n)$ *be of positive degree in* $\Phi[\xi_1,\cdots,\xi_n]$ *and let* $M_i(\xi_1,\cdots,\xi_{2n})$, $1 \leqq i \leqq n$, *be elements of* $\Phi[\xi_1,\cdots,\xi_{2n}]$, ξ*'s indeterminates, such that* $M_i(\xi_1,\cdots,\xi_{2n})$ *is of total degree one in* $\xi_{n+1},\cdots,\xi_{2n}$. *Assume also that there exists* γ_i, $1 \leqq i \leqq n$, *in* Φ *such that* $M_i(\gamma_1,\cdots,\gamma_n,\xi_{n+1},\cdots,\xi_{2n}) = \xi_{n+i}$ *and a polynomial* $m(\xi_1,\cdots,\xi_n) \in \Phi[\xi_1,\cdots,\xi_n]$ *such that*

$$(58)\qquad Q(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n})) = m(\xi_1,\cdots,\xi_n)Q(\xi_{n+1},\cdots,\xi_{2n}).$$

Then for each irreducible factor $Q_j(\xi_1,\cdots,\xi_n)$ *of* $Q(\xi_1,\cdots,\xi_n)$ *in* $\Phi[\xi_1,\cdots,\xi_n]$ *there exists a polynomial* $m_j(\xi_1,\cdots,\xi_n) \in \Phi[\xi_1,\cdots,\xi_n]$ *such that*

$$(59)\qquad Q_j(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n})) = m_j(\xi_1,\cdots,\xi_n)\,Q_j(\xi_{n+1},\cdots,\xi_{2n}).$$

PROOF. Let $Q_1(\xi_1,\cdots,\xi_n),\cdots,Q_r(\xi_1,\cdots,\xi_n)$ be the distinct irreducible factors of $Q(\xi_1,\cdots,\xi_n)$. Clearly $Q_j(\xi_{n+1},\cdots,\xi_{2n})$ is irreducible in $\Phi[\xi_1,\cdots,\xi_{2n}]$ and, by (58), this is a factor of $Q(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n}))$ which is a product of a nonzero element of Φ and the polynomials $Q_k(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n}))$. Hence there exists a j' and a polynomial $m_j(\xi_1,\cdots,\xi_{2n}) \in \Phi[\xi_1,\cdots,\xi_{2n}]$ such that

$$(60)\quad Q_{j'}(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n})) = m_j(\xi_1,\cdots,\xi_{2n})\,Q_j(\xi_{n+1},\cdots,\xi_{2n}).$$

Putting $\xi_i = \gamma_i$, $1 \leqq i \leqq n$, in this gives

$$Q_{j'}(\xi_{n+1},\cdots,\xi_{2n}) = m_j(\gamma_1,\cdots,\gamma_n,\xi_{n+1},\cdots,\xi_{2n})\,Q_j(\xi_{n+1},\cdots,\xi_{2n})$$

which implies that $j' = j$ in (60). Since the total degree of every $M_i(\xi_1,\cdots,\xi_{2n})$ in $\xi_{n+1},\cdots,\xi_{2n}$ is one, the total degree in $\xi_{n+1},\cdots,\xi_{2n}$ of $Q_j(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,$ $M_n(\xi_1,\cdots,\xi_{2n}))$ does not exceed the total degree d_j of $Q_j(\xi_1,\cdots,\xi_n)$. It follows from (60) that the total degree of $m_j(\xi_1,\cdots,\xi_{2n})\,Q_j(\xi_{n+1},\cdots,\xi_{2n})$ in $\xi_{n+1},\cdots,\xi_{2n}$ does not exceed d_j. This implies that $m_j(\xi_1,\cdots,\xi_{2n})$ is of degree 0 in $\xi_{n+1},\cdots,\xi_{2n}$. Hence $m_j(\xi_1,\cdots,\xi_{2n}) = m_j(\xi_1,\cdots,\xi_n) \in \Phi[\xi_1,\cdots,\xi_n]$. Then (59) is valid.

If $\mathfrak{M} = \mathfrak{A}$ is an algebra and $1 = \sum_1^n \gamma_i u_i$, (u_i) a basis then the polynomial $Q(\xi_1,\cdots,\xi_n) \in \Phi[\xi_1,\cdots,\xi_n]$ will be called *normalized* if $Q(1) = Q(\gamma_1,\cdots,\gamma_n) = 1$. This is the case for the generic norm polynomial $n(\xi_1,\cdots,\xi_n) = n(x)$. If $Q(\xi_1,\cdots,\xi_n)$ is normalized then $Q_j(1) \neq 0$ for every irreducible factor $Q_j(\xi_1,\cdots,\xi_n)$ of $Q(\xi_1,\cdots,\xi_n)$ in $\Phi[\xi_1,\cdots,\xi_n]$. It follows that any normalized polynomial of positive degree has a unique factorization as a product of normalized irreducible polynomials.

We can now prove

THEOREM 3. *Let* $\mathfrak{A}$ *be a strictly power associative algebra with* 1 *over the field* Φ, $(u_1,\cdots,u_n)$ *a basis for* $\mathfrak{A}/\Phi$, $x = \sum \xi_i u_i$ *a generic element*, $n(\xi_1,\cdots,\xi_n) = n(x)$ *the generic norm, m the degree of* $\mathfrak{A}$. *Then*:

(i) $Q(ab) = Q(a)Q(b)$ *if a and b are in an associative subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$ *and* $Q(\xi_1,\cdots,\xi_n)$ *is a product of normalized irreducible factors of* $n(\xi_1,\cdots,\xi_n)$.

(ii) $Q(\{aba\}) = Q(a)^2Q(b)$ *if* $\mathfrak{A}$ *is Jordan and* $Q(\xi_1,\cdots,\xi_n)$ *is as in* (i).

(iii) *If* $Q(\xi_1, \cdots, \xi_n)$ *is a homogeneous polynomial of degree* q *such that* $Q(1) = 1$ *and*

$$Q(a(\operatorname{adj} a)) = Q(a)Q(\operatorname{adj} a) \tag{61}$$

for all $a \in \mathfrak{A}$ *and the cardinal number* $|\Phi| > mq$ *then* $Q(\xi_1, \cdots, \xi_n)$ *is a product of normalized irreducible factors of the generic norm* $n(\xi_1, \cdots, \xi_n)$.

(iv) *If* $\mathfrak{A}$ *is Jordan and* $Q(\xi_1, \cdots, \xi_n)$ *is a homogeneous polynomial of degree* q *such that* $Q(1) = 1$ *and*

$$Q(\{a(\operatorname{adj} a)a\}) = Q(a)^2 Q(\operatorname{adj} a) \tag{62}$$

for all $a \in \mathfrak{A}$ *and* $|\Phi| > (m+1)q$, *then* $Q(\xi_1, \cdots, \xi_n)$ *is the product of normalized irreducible factors of the generic norm.*

Proof. We may assume Φ algebraically closed in (i) and (ii). Also we may assume that the associative subalgebra $\mathfrak{B}$ in (i) contains 1.

(i) Let $(v_1, \cdots, v_r)$ be a basis for $\mathfrak{B}/\Phi$, $v_k = \sum_1^n \beta_{ki} u_i$. Then $y = \sum_1^r \xi_k v_k = \sum \xi_k \beta_{ki} u_i$ is a generic element of $\mathfrak{B}$. Then $n(y) = n(\sum \xi_k \beta_{k1}, \cdots, \sum \xi_k \beta_{kr}) \in \Phi[\xi_1, \cdots, \xi_r]$ and $Q(\sum \xi_k \beta_{k1}, \cdots, \sum \xi_k \beta_{kr})$ is a product of normalized irreducible factors of $n(y)$ in $\Phi[\xi_1, \cdots, \xi_r]$. It is enough to prove $n_j(ab) = n_j(a) n_j(b)$, $a, b \in \mathfrak{B}$, for every normalized irreducible factor $n_j(\xi_1, \cdots, \xi_r)$ of $n(y)$ in $\Phi[\xi_1, \cdots, \xi_r]$. Let $\mathfrak{U} = \{a \in \mathfrak{B} \mid n(a) \neq 0\}$. Then $\mathfrak{U}$ is the set of invertible elements of the associative algebra $\mathfrak{B}$ and the linear mapping $x \to xb$ in $\mathfrak{B}$ for $b \in \mathfrak{U}$ is a linear isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}$ mapping $\mathfrak{U}$ onto $\mathfrak{U}$. Hence, by Lemma 1, if $n_0(\xi_1, \cdots, \xi_n)$ is the product of the distinct normalized irreducible factors $n_j(\xi_1, \cdots, \xi_r)$ of $n(\xi_1, \cdots, \xi_r)$ then there exists a nonzero ρ in Ω such that $n_0(ab) = \rho n_0(a) n_0(b)$, $a \in \mathfrak{B}$, $b \in \mathfrak{U}$. Putting $a = 1$ gives $n_0(b) = \rho$. Hence we have

$$n_0(ab) = n_0(a) n_0(b) \tag{63}$$

for $a \in \mathfrak{B}$, $b \in \mathfrak{U}$. Since $\mathfrak{U}$ is a Zariski open subset we have (63) for all $a, b \in \mathfrak{A}$. Now put $z = \sum_{k=1}^r \xi_{r+k} v_k$, $v_k v_l = \sum \delta_{klq} v_q$. Then $yz = \sum M_k(\xi_1, \cdots, \xi_{2r}) v_k$ where $M_k(\xi_1, \cdots, \xi_{2r}) \in \Phi[\xi_1, \cdots, \xi_{2r}]$ and is of total degree one in $\xi_{r+1}, \cdots, \xi_{2r}$. Also if $1 = \sum \delta_k v_k$ then $M_k(\delta_1, \cdots, \delta_r, \xi_{r+1}, \cdots, \xi_{2r}) = \xi_{r+k}$ since $1z = z$. We have $n_0(M_1(\alpha_1, \cdots, \alpha_{2r}), \cdots, M_r(\alpha_1, \cdots, \alpha_{2r})) = n_0(\alpha_1, \cdots, \alpha_r) n_0(\alpha_{r+1}, \cdots, \alpha_{2r})$ for all $\alpha_i \in \Phi$. Hence $n_0(M_1(\xi_1, \cdots, \xi_{2r}), \cdots, M_r(\xi_1, \cdots, \xi_{2r})) = n_0(\xi_1, \cdots, \xi_r) n_0(\xi_{r+1}, \cdots, \xi_{2r})$. It follows from Lemma 2 that for each $n_j(\xi_1, \cdots, \xi_r)$ there exists a polynomial $m_j(\xi_1, \cdots, \xi_r)$ such that

$$n_j(M_1(\xi_1, \cdots, \xi_{2r}), \cdots, M_r(\xi_1, \cdots, \xi_{2r})) = m_j(\xi_1, \cdots, \xi_r) n_j(\xi_{r+1}, \cdots, \xi_{2r}).$$

This implies that $n_j(ab) = m_j(a) n_j(b)$. Putting $b = 1$ gives $n_j(a) = m_j(a)$. Hence $n_j(ab) = n_j(a) n_j(b)$.

(ii) The proof of this is similar to that of (i). Here we let $n_1(\xi_1, \cdots, \xi_n), \cdots, n_s(\xi_1, \cdots, \xi_n)$ be the distinct normalized irreducible factors of $n(\xi_1, \cdots, \xi_n) = n(x)$

and we put $n_0(\xi_1,\cdots,\xi_n)=\prod_1^s n_j(\xi_1,\cdots,\xi_n)$. It is enough to show $n_j(\{aba\}) = n_j(a)^2 n_j(b)$ for all $a,b\in\mathfrak{A}$ and $j = 1,2,\cdots,s$. Let $\mathfrak{U}$ be the set of invertible elements so $\mathfrak{U} = \{b \mid n(b)\neq 0\}$. If $a\in\mathfrak{U}$ then U_a is a linear isomorphism of $\mathfrak{A}$ mapping $\mathfrak{U}$ onto $\mathfrak{U}$. Hence as in the proof of (i) we have, by Lemma 1, that there is a non zero ρ in Φ such that $n_0(\{aba\}) = n_0(bU_a) = \rho n_0(b)$ for all $b\in\mathfrak{A}$. Putting $b = 1$ gives $\rho = n_0(a^2) = n_0(a)^2$, by (i), so $n_0(\{aba\}) = n_0(b)n_0(a)^2$ holds for all $b\in\mathfrak{A}$ and all $a\in\mathfrak{U}$. It follows that this holds for all $a,b\in\mathfrak{A}$ and this implies that if we take $x = \sum_1^n\xi_i u_i$, $y = \sum_1^n\xi_{n+i}u_i$ then $\{xyx\} = \sum_i M_i(\xi_1,\cdots,\xi_{2n})u_i$ and $n_0(\xi_1,\cdots,\xi_n)^2 n_0(\xi_{n+1},\cdots,\xi_{2n}) = n_0(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n}))$. Clearly $M_i(\xi_1,\cdots,\xi_{2n})$ is of total degree one in $\xi_{n+1},\cdots,\xi_{2n}$ and if $1 = \sum\gamma_i u_i$ then $\{1y1\} = y$ gives $M_i(\gamma_1,\cdots,\gamma_n,\xi_{n+1},\cdots,\xi_{2n}) = \xi_{n+i}$. Hence, by Lemma 2, there exists a polynomial $m_j(\xi_1,\cdots,\xi_n)$ such that $n_j(M_1(\xi_1,\cdots,\xi_{2n}),\cdots,M_n(\xi_1,\cdots,\xi_{2n})) = m_j(\xi_1,\cdots,\xi_n)n_j(\xi_{n+1},\cdots,\xi_{2n})$. This gives $n_j(\{aba\}) = m_j(a)n_j(b)$. Taking $b=1$ gives $m_j(a) = n_j(a^2)$. Since $n_j(a^2) = n_j(a)^2$ by (i) this gives $n_j(\{aba\}) = n_j(a)^2 n_j(b)$ which completes the proof of (ii).

(iii) Since $a(\operatorname{adj} a) = n(a)1$ and $Q(1) = 1$ we have $Q(a)Q(\operatorname{adj} a) = n(a)^q$. If $x = \sum\xi_i u_i$ then it is clear from the definition that $\operatorname{adj} x = \sum M_i(\xi_1,\cdots,\xi_n)u_i$ where $M_i(\xi_1,\cdots,\xi_n)$ is homogeneous of degree $m-1$ in the ξ's. Hence $Q(\operatorname{adj} x)$ is homogeneous of degree $q(m-1)$ in the ξ's and $F(\xi_1,\cdots,\xi_n) = Q(x)Q(\operatorname{adj} x)-n(x)^q$ is homogeneous of degree qm in the ξ's. Also we see that $F(\alpha_1,\cdots,\alpha_n) = 0$ for $\alpha_i\in\Phi$. Since $|\Phi| > qm$, the usual proof of the theorem on zeros of a polynomial (Jacobson, *Lectures*, vol. I p. 112) shows that we have $F(\xi_1,\cdots,\xi_n) = 0$. Hence $Q(x)Q(\operatorname{adj} x) = n(x)^q$ and this implies that $Q(\xi_1,\cdots,\xi_n)$ is a product of irreducible factors of $n(\xi_1,\cdots,\xi_n)$. Since $Q(1) = 1$ we can take these to be normalized.

(iv) Since $\operatorname{adj} a\in\Phi[a]$ and $a(\operatorname{adj} a) = n(a)1$ we have $\{a(\operatorname{adj} a)a\} = n(a)a$ and our hypothesis gives $n(a)^q Q(a) = Q(a)^2 Q(\operatorname{adj} a)$. As in (iii) this gives $n(x)^q Q(x) = Q(x)^2 Q(\operatorname{adj} x)$. Since $Q(1) = 1$, $Q(x)\neq 0$ and $Q(\xi_1,\cdots,\xi_n)$ can be cancelled in this relation. This gives $n(x)^q = Q(x)Q(\operatorname{adj} x)$ which implies that $Q(\xi_1,\cdots,\xi_n)$ is a product of irreducible factors of $n(\xi_1,\cdots,\xi_n)$ and these can be taken to be normalized.

Exercises

1. Let $\mathfrak{A}$ be a finite-dimensional strictly power associative algebra with 1, $\mathfrak{B}$ a subalgebra containing 1, $(v_1,\cdots,v_r)$ a basis for $\mathfrak{B}$, $y = \sum\xi_j v_j$ a generic element of $\mathfrak{B}$. Let $n(y)$ and $n_{\mathfrak{B}}(y)$ denote the generic norms of y in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Show that $n(y)$ and $n_{\mathfrak{B}}(y)$ have the same irreducible factors in $\Phi[\xi_1,\cdots,\xi_i]$.

2. Show that if $\mathfrak{A}$ is alternative then $Q(ab) = Q(a)Q(b)$ for $a,b\in\mathfrak{A}$ if Q is a normalized irreducible factor of $n(x) = n(\xi_1,\cdots,\xi_n)$, $x = \sum\xi_i u_i$ generic.

3. Let $\mathfrak{A}$ be alternative with basis $(u_1,\cdots,u_n)$, $Q(\xi_1,\cdots,\xi_n)$ a nonzero homogeneous polynomial of degree q in $\Phi[\xi_1,\cdots,\xi_n]$ such that $Q(ab) = Q(a)Q(b)$, $a,b\in\mathfrak{A}$. Assume $|\Phi| > q$. Show that $Q(\xi_1,\cdots,\xi_n)$ is a product of normalized

irreducible factors of $n(\xi_1, \cdots, \xi_n) = n(x)$, $x = \sum \xi_i u_i$. Show that the same conclusion holds if $\mathfrak{A}$ is Jordan, $Q(\{aba\}) = Q(a)^2 Q(b)$ and $|\Phi| > 2q$.

4. Let $\mathfrak{A} = \Phi_n$, e_{ij}, $i,j = 1, \cdots, n$, the usual matrix basis, ξ_{ij} indeterminates and let Q be a nonzero homogeneous polynomial in the ξ's such that $Q(AB) = Q(A)Q(B)$ $A, B \in \Phi_n$. Assume also that $|\Phi| > \deg Q$. Show that Q is a power of $\det X$, $X = \sum \xi_{ij} e_{ij}$.

6. Separable algebras.

DEFINITION 1. *A finite-dimensional algebra* $\mathfrak{A}/\Phi$ *is called* separable *if* $\mathfrak{A}_{\mathrm{P}}$ *is a direct sum of simple ideals for every extension field* P/Φ.

We note first that if $\mathfrak{A}/\Phi$ is a field then $\mathfrak{A}/\Phi$ is separable in the sense of Definition 1 if and only if it is a separable extension field in the usual sense. It is well known that if $\mathfrak{A}/\Phi$ is a separable extension field then $\mathfrak{A}_{\mathrm{P}}$ is a direct sum of fields (Jacobson, *Lectures*, vol. III, p. 87). On the other hand, if $\mathfrak{A}/\Phi$ is an inseparable extension field and Ω is the algebraic closure of Φ then $\mathfrak{A}_\Omega$ contains a nonzero nilpotent element (loc. cit. p. 197). Hence $\mathfrak{A}_\Omega$ is not a direct sum of simple ideals since this structure for a commutative associative algebra implies that the algebra is a direct sum of fields, and this is impossible if the algebra contains a nonzero nilpotent element. Thus if $\mathfrak{A}/\Phi$ is an inseparable finite-dimensional field extension then $\mathfrak{A}/\Phi$ is not a separable algebra. We recall also that if an algebra is central simple then $\mathfrak{A}_{\mathrm{P}}$ is simple for any P (p. 00). Hence any finite-dimensional central simple algebra is separable. It is clear also that a direct sum of a finite number of separable algebras is separable.

Now let $\mathfrak{A}/\Phi$ be an arbitrary finite-dimensional algebra and let $M(\mathfrak{A})$ and $C(\mathfrak{A})$ be the multiplication algebra and centroid of $\mathfrak{A}$ as defined in §5.7. Thus $M(\mathfrak{A})$ is the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A},\mathfrak{A})$ generated by 1 and the $a_L, a_R, a \in \mathfrak{A}$, and $C(\mathfrak{A})$ is the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A},\mathfrak{A})$ of elements γ which commute with every $m \in M(\mathfrak{A})$. Suppose $\mathfrak{A}$ has an identity element 1 and $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where $\mathfrak{A}_i$ is an ideal in $\mathfrak{A}$. Then $\mathfrak{A}_i\mathfrak{A}_j = 0$ if $i \neq j$ and if $1 = \sum_{i=1}^{s} 1_i$ where $1_i \in \mathfrak{A}_i$ then 1_i is the identity element of $\mathfrak{A}_i$. Also $M(\mathfrak{A})$ is the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A},\mathfrak{A})$ generated by the $a_L, a_R, a \in \mathfrak{A}$. It follows easily that $\mathfrak{M} = M(\mathfrak{A}) = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots \oplus \mathfrak{M}_s$ where $\mathfrak{M}_i$ is the subalgebra generated by the elements a_{iL}, a_{iR}, $a_i \in \mathfrak{A}_i$ and $\mathfrak{M}_i$ is an ideal in $\mathfrak{M}$. Moreover, $\mathfrak{M}_i$ annihilates every $\mathfrak{A}_j, j \neq i$, and the mapping $m_i \to m_i | \mathfrak{A}_i$, the restriction of m_i to $\mathfrak{A}_i$, is an isomorphism of $\mathfrak{M}_i$ onto the multiplication algebra $M(\mathfrak{A}_i)$. Identifying $\mathfrak{M}_i$ with $M(\mathfrak{A}_i)$ (m_i with its restriction) we can say that $M(\mathfrak{A})$ is a direct sum of the $M(\mathfrak{A}_i)$. We have $\mathfrak{A}_i = \mathfrak{A}\mathfrak{M}_i$ and if $\gamma \in C(\mathfrak{A})$ then $\mathfrak{A}_i\gamma = \mathfrak{A}\mathfrak{M}_i\gamma = (\mathfrak{A}\gamma)\mathfrak{M}_i \subseteq \mathfrak{A}\mathfrak{M}_i = \mathfrak{A}_i$. Thus γ maps $\mathfrak{A}_i$ into itself and the restriction of γ to $\mathfrak{A}_i$ is contained in $C(\mathfrak{A}_i)$. On the other hand, if $\gamma_i \in C(\mathfrak{A}_i)$ for $i = 1, 2, \cdots, s$ and γ is the linear mapping in $\mathfrak{A}$ which coincides with γ_i on $\mathfrak{A}_i$ then $\gamma \in C(\mathfrak{A}_i)$. It follows that $C(\mathfrak{A})$ is isomorphic to $C(\mathfrak{A}_1) \oplus C(\mathfrak{A}_2) \oplus \cdots \oplus C(\mathfrak{A}_s)$.

If $\mathfrak{A}$ is simple then it is known that $C(\mathfrak{A})$ is an extension field of Φ and $M(\mathfrak{A})$

is the algebra of linear transformations of $\mathfrak{A}$ considered as a vector space over $C(\mathfrak{A})$ (Jacobson, *Lie Algebras*, p. 293). Thus $M = M(\mathfrak{A})$ is isomorphic to the algebra $\mathfrak{C}_r$ of $r \times r$ matrices over $\mathfrak{C} = C(\mathfrak{A})$. Here $r = \dim \mathfrak{A}/\mathfrak{C}$. We have also that $\mathfrak{C}_r \cong \mathfrak{C} \otimes_\Phi \Phi_r$. Hence $M(\mathfrak{A}) \cong \mathfrak{C} \otimes \Phi_r$ where $\mathfrak{C}$ is the centroid and $r = \dim \mathfrak{A}/\mathfrak{C}$. If $\mathfrak{A}$ has an identity element then we know also that $\mathfrak{C}$ can be identified with the center of $\mathfrak{A}$.

Now suppose $\mathfrak{A}$ has an identity 1 and $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where $\mathfrak{A}_i$ is a simple ideal. Then our results show that $M(\mathfrak{A}) = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots \oplus \mathfrak{M}_s$ where $\mathfrak{M}_i$ is an ideal isomorphic to $\mathfrak{C}_i \otimes \Phi_{r_i}$ where $\mathfrak{C}_i$ is the center of $\mathfrak{A}_i$ and $r_i = \dim \mathfrak{A}_i/\mathfrak{C}_i$. We can now prove

THEOREM 4. *Let $\mathfrak{A}/\Phi$ be a finite-dimensional algebra with* 1. *Then the following conditions on $\mathfrak{A}$ are equivalent*: (1) *$\mathfrak{A}$ is separable.* (2) *$M(\mathfrak{A})$ is a direct sum of ideals $\mathfrak{M}_i$ where $\mathfrak{M}_i \cong \Gamma_i \otimes \Phi_{r_i}$ where Γ_i/Φ is a separable field.* (3) *$\mathfrak{A}$ is a direct sum of simple ideals whose centers are separable fields.* (4) *$M(\mathfrak{A})$ is separable.*

PROOF. Assume $\mathfrak{A}$ is separable. Then $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where $\mathfrak{A}_i$ is a simple ideal. If $\mathfrak{C}_i$ is the center of $\mathfrak{A}_i$ then we have seen that $M(\mathfrak{A})$ is a direct sum of ideals $\mathfrak{M}_i$, $i = 1, \cdots, s$, where $\mathfrak{M}_i \cong \mathfrak{C}_i \otimes \Phi_{r_i}$. Now it is easily seen that if P is an extension of the base field Φ then on identifying the elements $m \in M(\mathfrak{A})$ with their P-linear extensions to $\mathfrak{A}_{\mathrm{P}}$, $M(\mathfrak{A}_{\mathrm{P}}) = M(\mathfrak{A})_{\mathrm{P}} = \mathrm{P} \otimes_\Phi M(\mathfrak{A})$. Since $\mathfrak{A}_{\mathrm{P}}$ is a direct sum of simple ideals it follows that $M(\mathfrak{A})_{\mathrm{P}}$ is a direct sum of ideals of the form $\Gamma \otimes \mathrm{P}_r$ where Γ/P is a field. This implies that every $\mathfrak{C}_{i\mathrm{P}}$ is a direct sum of fields. If P is taken to be the algebraic closure Ω of Φ then this implies that $\mathfrak{C}_i/\Phi$ is separable. Hence we have proved that (1) implies (2) and (3). We note next that it is clear from the results preceding the statement of the theorem that (3) implies (2). Now assume (2) and let P be an extension field of Φ. Then $\Gamma_{i\mathrm{P}}$ is a direct sum of fields. Hence $M(\mathfrak{A})_{\mathrm{P}}$ is a direct sum of simple algebras. Since $M(\mathfrak{A}_{\mathrm{P}}) = M(\mathfrak{A})_{\mathrm{P}}$ we have this structure for $M(\mathfrak{A}_{\mathrm{P}})$. It follows that $\mathfrak{A}_{\mathrm{P}}$ as right module for $M(\mathfrak{A}_{\mathrm{P}})$ is completely reducible. Since the submodules of $\mathfrak{A}_{\mathrm{P}}$ as $M(\mathfrak{A}_{\mathrm{P}})$ module are the ideals of $\mathfrak{A}_{\mathrm{P}}$ this implies that $\mathfrak{A}_{\mathrm{P}}$ is a direct sum of simple ideals. Since P is arbitrary it follows that $\mathfrak{A}$ is separable. Hence (2) implies (1) and consequently also (3) and so (1), (2) and (3) are equivalent. If we apply the equivalence of (1) and (3) for $M(\mathfrak{A})$ in place of $\mathfrak{A}$ we see also that (4) is equivalent to the other conditions.

We shall now consider finite-dimensional associative or Jordan algebras with 1 and we shall show that such an algebra is separable if and only if its generic trace bilinear form is nondegenerate. We require first the following

LEMMA (DIEUDONNÉ). *Let $\mathfrak{A}$ be a finite-dimensional nonassociative algebra satisfying the following conditions*: (1) *there exists a nondegenerate symmetric* *associative bilinear form f on* $\mathfrak{A}$, (2) $\mathfrak{A}$ *contains no ideal* $\mathfrak{N} \neq 0$ *such that* $\mathfrak{N}^2 = 0$. *Then* $\mathfrak{A}$ *is a direct sum of simple ideals.*

PROOF. It is immediate from the symmetry and associativity of f that if $\mathfrak{B}$ is an ideal in $\mathfrak{A}$ then the orthogonal complement $\mathfrak{B}^\perp$ of $\mathfrak{B}$ relative to f is an ideal. Then $\mathfrak{N} = \mathfrak{B} \cap \mathfrak{B}^\perp$ is an ideal. If $z, w \in \mathfrak{N}$ and $a \in \mathfrak{A}$ we have $f(zw, a) = f(z, wa) = 0$ since $z \in \mathfrak{B}$ and $wa \in \mathfrak{B}^\perp$. Since f is nondegnerate this implies that $zw = 0$. Hence $\mathfrak{N}^2 = 0$ and so $\mathfrak{N} = 0$. Thus $\mathfrak{B} \cap \mathfrak{B}^\perp = 0$ and $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^\perp$. We have therefore shown that every ideal in $\mathfrak{A}$ has a complementary ideal. This implies, as is well known, that $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where the $\mathfrak{A}_i$ are simple ideals.

We shall now prove the following important criterion.

THEOREM 5. *Let* $\mathfrak{A}$ *be a finite-dimensional algebra with* 1 *which is either associative or Jordan. Then* $\mathfrak{A}$ *is separable if and only if its generic trace form t is nondegenerate.*

PROOF. Assume first that t is nondegenerate. By Corollary 4 to Theorem 1, t is symmetric and associative. Hence the first hypothesis of Dieudonné's lemma is satisfied for the bilinear form $f = t$. Now let $\mathfrak{N}$ be an ideal such that $\mathfrak{N}^2 = 0$. Since nilpotent elements have generic trace 0 (Corollary 1 to Theorem 1) we have $t(a, z) = t(az) = 0$ if $a \in \mathfrak{A}$ and $z \in \mathfrak{N}$. Hence $z = 0$ by the nondegeneracy of t and so $\mathfrak{N} = 0$. It now follows from Dieudonné's lemma that $\mathfrak{A}$ is a direct sum of simple ideals. Now let P be an extension field of Φ. Then the generic trace form t on $\mathfrak{A}_P$ is the P-bilinear extension of the generic trace form on $\mathfrak{A}$. Hence this is nondegenerate and so $\mathfrak{A}_P$ is a direct sum of simple ideals. Thus $\mathfrak{A}$ is separable. Conversely, assume $\mathfrak{A}$ is separable and let Ω be the algebraic closure of Φ. Then $\mathfrak{A}_\Omega$ is a direct sum of simple ideals. We have seen in §3 that these ideals are orthogonal and the restriction of the generic trace form of $\mathfrak{A}$ to the ideal is the generic trace form of this ideal. Hence to prove nondegeneracy of t on $\mathfrak{A}_\Omega$ it is sufficient to assume that $\mathfrak{A}_\Omega$ is simple. Since t is symmetric and associative $\mathfrak{A}_\Omega^\perp$ is an ideal. Hence either t is nondegenerate or $t = 0$. Hence the nondegeneracy of t on $\mathfrak{A}_\Omega$ will follow if we can show that $t \neq 0$ on $\mathfrak{A}_\Omega$. To verify this we use the structure theory. First, assume $\mathfrak{A}$ is associative. Then $\mathfrak{A}_\Omega \cong \Omega_n$ by the Wedderburn theorem. We have seen in §4 that t is the usual trace of the matrix (in Ω_n). If e_{ij} is the usual matrix unit then we have $t(e_{11}) = 1$. Hence t is nondegenerate in this case. Next assume $\mathfrak{A}$ is Jordan. Then the possibilities for a simple algebra $\mathfrak{A}_\Omega$ are: $\mathfrak{A}_\Omega = \Omega 1$; $\mathfrak{A}_\Omega = \Omega 1 \oplus \mathfrak{B}$ the Jordan algebra of a nondegenerate symmetric bilinear form on a finite-dimensional vector space $\mathfrak{B}$ with $\dim \mathfrak{B} > 1$; $\Omega_n{}^+$ with $n \geqq 3$; $\mathfrak{H}(\Omega_n, J_1)$ with $n \geqq 3$; $\mathfrak{H}(\Omega_{2n}, J_S)$, $n \geqq 3$; $\mathfrak{H}(\mathfrak{O}_3, J_1)$. The notations here are those in §4. Now it is clear that t is nondegenerate in the first case: $\mathfrak{A}_\Omega = \Omega 1$. In the second case the degree is two so $t(1, 1) = t(1) = 2 \neq 0$. In the remaining cases one sees directly from the determination of the generic trace in §4 that $t \neq 0$. Hence we see that if $\mathfrak{A}$ is separable then the generic trace form on $\mathfrak{A}_\Omega$ is nondegenerate. Then the generic trace form on $\mathfrak{A}$ is nondegenerate also.

7. Norm similarity of Jordan algebras. The group of norm preserving linear transformations.

DEFINITION 2. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be finite-dimensional strictly power associative algebras with* 1. *Then a bijective linear mapping η of $\mathfrak{A}$ into $\mathfrak{A}'$ is called a* norm similarity *of $\mathfrak{A}$ onto $\mathfrak{A}'$ if $n'(a^\eta) = \rho n(a)$ for all $a \in \mathfrak{A}$ where n and n' are the generic norm in $\mathfrak{A}$ and $\mathfrak{A}'$ respectively and ρ is a nonzero element of the base field. The element ρ is called the* multiplier *of η.*

It is clear that the identity mapping is a norm similarity on $\mathfrak{A}$, that if η is a norm similarity of $\mathfrak{A}$ onto $\mathfrak{A}'$ with multiplier ρ and ζ is norm similarity of $\mathfrak{A}'$ onto $\mathfrak{A}''$ with multiplier σ then $\eta\zeta$ is a norm similarity of $\mathfrak{A}$ onto $\mathfrak{A}''$ with multiplier $\rho\sigma$ and η^{-1} is a norm similarity of $\mathfrak{A}'$ onto $\mathfrak{A}$ with multiplier ρ^{-1}. Hence norm similarity is an equivalence relation and the set of norm similarities of $\mathfrak{A}$ onto $\mathfrak{A}$ is a subgroup $\mathrm{M}(\mathfrak{A})$ of the group of bijective linear transformations of $\mathfrak{A}$. The norm similarities of $\mathfrak{A}$ onto $\mathfrak{A}$ with multiplier 1 form an invariant subgroup $\mathrm{M}_1(\mathfrak{A})$ of $\mathrm{M}(\mathfrak{A})$ called the *norm preserving group* of $\mathfrak{A}$. Since $n(a^\eta) = n(a)$ for any automorphism or antiautomorphism, $\mathrm{M}_1(\mathfrak{A})$ contains the group of automorphisms and antiautomorphisms of $\mathfrak{A}$.

Let m' be the degree of $\mathfrak{A}'$, m the degree of $\mathfrak{A}$ and let η be a norm similarity of $\mathfrak{A}$ onto $\mathfrak{A}'$. Then $n'(1^\eta) = \rho$ and if $\beta \neq 0$ is in Φ then $n'((\beta 1)^\eta) = \rho n(\beta 1)$ gives $\beta^{m'} = \beta^m$ so $\beta^{|m-m'|} = 1$ for all $\beta \neq 0$ in Φ. If $m' \neq m$ this implies that $|\Phi| \leqq |m - m'| + 1$. We shall usually assume that $|\Phi|$ exceeds the degrees of the algebras under consideration. This will imply that if $\mathfrak{A}$ and $\mathfrak{A}'$ are norm similar then they have the same degree. If this degree is m then $n'(x^\eta) - \rho n(x)$ for $x = \sum \xi_i u_i$ generic for $\mathfrak{A}$ is a homogeneous polynomial of degree $\leqq m$ in the ξ's. Morever, the norm similarity $n'(a^\eta) = \rho n(a)$, $a \in \mathfrak{A}$, and $|\Phi| > m$ implies that $n'(x^\eta) = \rho n(x)$. This implies that the linear extension η of η to $\mathfrak{A}_\Gamma$ into $\mathfrak{A}'_\Gamma$, Γ an extension field of Φ, is a norm similarity.

We shall now study norm similarity for Jordan algebras and we shall see that this notion is closely related to that of isotopy defined in §1.12. We recall that if $\mathfrak{J}$ and $\mathfrak{J}'$ are Jordan algebras with 1 then an isotopy α is a bijective linear mapping of $\mathfrak{J}$ into $\mathfrak{J}'$ such that $(x.y)^\alpha = \{x^\alpha b y^\alpha\}$ where b is a fixed invertible element of $\mathfrak{J}'$ and $x, y \in \mathfrak{J}$. This is just an isomorphism of $\mathfrak{J}$ onto the b-isotope $(\mathfrak{J}', b)$ of $\mathfrak{J}'$. We recall also that a mapping α of $\mathfrak{J}$ into $\mathfrak{J}'$ is an isotopy if and only if α is bijective and linear and there exists a $\beta \in \mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J}')$ such that

$$U_c\alpha = \beta U_{c^\alpha} \tag{64}$$

holds for all $c \in \mathfrak{J}$. In this case 1^α is an invertible element of $\mathfrak{J}'$ and the element

b in the definition of isotopy is the inverse $(1^\alpha)^{-1}$. Also it is clear from the definition that an isotopy α is an isomorphism if and only if $1^\alpha = 1$.

We shall show first that an isotopy of a finite-dimensional Jordan algebra $\mathfrak{J}$ with 1 onto a second such algebra is necessarily a norm similarity. For this we require the following well-known result.

LEMMA. *If Ω is an algebraically closed field of characteristic $\neq 2$ and $\Omega[a]$ is the commutative associative algebra with* 1 *generated by* 1 *and an invertible element a, then there exists an invertible element b in $\Omega[a]$ such that $b^2 = a$.*

PROOF. Let $\mu_a(\lambda)$ be the minimum polynomial of a. Then $\mu_a(\lambda) = \prod_1^s (\lambda - \rho_i)^{e_i}$ where the ρ_i are distinct and nonzero. Then $\mathfrak{A} = \Omega[a] = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where $\mathfrak{A}_i$ has the identity 1_i and is generated by this element and the element a_i such that $a = a_1 + a_2 + \cdots + a_s$, $a_i \in \mathfrak{A}_i$. Moreover, the minimum polynomial of a_i in $\mathfrak{A}_i$ is $(\lambda - \rho_i)^{e_i}$. This implies that $a_i = \rho_i 1_i + z_i$ where $z_i^{n_i} = 0$. By Lemma 4 on p.150, there exists an element $1_i + w_i$, w_i nilpotent in $\mathfrak{A}_i$, such that $(1_i + w_i)^2 = 1_i + \rho_i^{-1} z_i$. If $\sigma_i = \rho_i^{\frac{1}{2}}$ a square root of ρ_i in Ω then $(\sigma_i(1 + w_i))^2 = \rho_i 1_i + z_i$ and $b = \sum_1^r \sigma_i(1 + w_i)$ is a square root of a in $\Omega[a]$. Clearly b is invertible.

We can now prove

THEOREM 6. *Let $\mathfrak{J}$ and $\mathfrak{J}'$ be finite-dimensional Jordan algebras with* 1 *and let α be an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$. Then α is a norm similarity.*

PROOF. We consider first the relation between the generic norm of a Jordan algebra $\mathfrak{J}$ and its isotope $(\mathfrak{J}, a)$ defined by the invertible element a. Denoting the generic norm of b in $(\mathfrak{J}, a)$ by $n^{(a)}(b)$ we claim that

$$n^{(a)}(b) = n(a)n(b). \tag{65}$$

To prove this we may assume the base field is algebraically closed. Then the lemma implies that there exists a square root c of a in $\mathfrak{J}$ so we have $(\mathfrak{J}, a) = (\mathfrak{J}, c^{\cdot 2})$. Then U_c is an isomorphism of $(\mathfrak{J}, a)$ onto $\mathfrak{J}$ (see p.155). The image of b under this isomorphism is $\{cbc\}$. Hence, by Theorem 1 (vi), $n^{(a)}(b) = n(\{cbc\})$. By Theorem 3, $n(\{cbc\}) = n(c)^2 n(b) = n(c^{\cdot 2})n(b) = n(a)n(b)$. Hence $n^{(a)}(b) = n(a)n(b)$ which is (65). Now let α be an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$. Then α is an isomorphism of $\mathfrak{J}$ onto an isotope $(\mathfrak{J}', b)$ of $\mathfrak{J}'$. If n and n' are the generic norms in $\mathfrak{J}$ and $\mathfrak{J}'$ respectively then $n(a) = n'^{(b)}(a^\alpha)$ where $n'^{(b)}$ is the generic norm in $(\mathfrak{J}', b)$. By (65) applied to $\mathfrak{J}'$ we have $n'^{(b)}(a^\alpha) = n'(b)n'(a^\alpha)$. Hence $n(a) = n'(b)n'(a^\alpha)$ and $n'(a^\alpha) = \rho n(a)$ where $\rho = n'(b)^{-1}$.

We shall prove next that the converse of Theorem 6 holds provided that the algebras are separable and Φ has enough elements. For this purpose we shall derive some important differentiation formulas for the generic norm in any finite dimensional Jordan algebra over an infinite field.

First, let $\mathfrak{V}$ be a finite-dimensional vector space over an infinite field and let f be a polynomial function on $\mathfrak{V}$ defined by an element $f(\xi_1, \cdots, \xi_n) \in \Phi[\xi_1, \cdots, \xi_n]$ of degree m. Let a be a fixed vector in $\mathfrak{V}$. Then it is clear that there exist uniquely determined polynomial functions $f_1, f_2, \cdots, f_m$ such that

$$f(c + \lambda a) = f(c) + f_1(c)\lambda + \cdots + f_m(c)\lambda^m \tag{66}$$

for all $c \in \mathfrak{V}$ and $\lambda \in \Phi$. We indicate (66) more briefly as

$$f(c + \lambda a) \equiv f(c) + f_1(c)\lambda \pmod{\lambda^2}. \tag{66'}$$

It is immediate that $f \to f_1$ is a derivation in the algebra $\mathfrak{P}$ of polynomial functions. Moreover, if $f = l$ is linear then f_1 is the constant function $c \to a$. It follows that

$$f_1 = \Delta^a f \quad \text{and} \quad f_1(c) = \Delta^a{}_c f. \tag{67}$$

Now let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1 over the infinite field Φ, n and t the generic norm and trace on $\mathfrak{J}$. These are homogeneous polynomial functions on $\mathfrak{J}$ of degrees m and one, respectively, where the degree of $\mathfrak{J}$ is m. As usual, we have the generic trace form $t(a, b) = t(a.b)$. We shall show first that for arbitrary $a \in \mathfrak{J}$ and invertible $c \in \mathfrak{J}$ we have

$$\Delta^a{}_c n = n(c)t(c^{\cdot -1}, a). \tag{68}$$

To see this we begin with the formula $n(\lambda 1 - a) = m_a(\lambda) = \lambda^m - t(a)\lambda^{m-1} + \cdots$. We may assume the base field is algebraically closed and then we have a $d \in \mathfrak{J}$ such that $d^{\cdot 2} = c$. Then

$$\begin{aligned} n(c + \lambda a) &= n((1 + \lambda a U_d^{-1})U_d) \\ &= n(c)n(1 + \lambda a U_d^{-1}) \\ &\equiv n(c) + n(c)t(aU_d^{-1})\lambda \pmod{\lambda^2}, \end{aligned} \tag{69}$$

since $n(1 + \lambda b) \equiv 1 + t(b)\lambda \pmod{\lambda^2}$ is an immediate consequence of $n(\lambda 1 - a) = \lambda^m - t(a)\lambda^{m-1} + \cdots$. Hence by what we showed above, $\Delta^a{}_c n = n(c)t(aU_d^{-1}) = n(c)t\{d^{\cdot -1}ad^{\cdot -1}\} = n(c)t(c^{\cdot -1}, a)$ by the associativity of t. Hence (68) holds. Now by the definition of the logarithmic derivative (68) can be written also as

$$\Delta^a{}_c \log n = t(c^{\cdot -1}, a). \tag{68'}$$

For later application it is useful to note also that since $c^{\cdot -1} = n(c)^{-1} \operatorname{adj} c$, (68) is also equivalent to

$$\Delta^a{}_c n = t(\operatorname{adj} c, a) \tag{68''}$$

which is valid for all a and c.

We now use the customary notation x for the identity mapping $c \to c$ and $x^{\cdot -1}$ for $c \to c^{\cdot -1}$ which is a rational mapping on $\mathfrak{J}$ into $\mathfrak{J}$ defined on the open set of invertible elements. Then we claim that

$$\Delta^a{}_c x^{\cdot -1} = -aU_c{}^{-1}. \tag{70}$$

To see this we differentiate the relation

$$\{xx^{\cdot -1}x\} = x \tag{71}$$

to obtain

$$a = \{c(\Delta^a{}_c x^{\cdot -1})c\} + \{ac^{\cdot -1}c\} + \{cc^{\cdot -1}a\}.$$

Since $\{ac^{\cdot -1}c\} = \{cc^{\cdot -1}a\} = a$ this gives $(\Delta^a{}_c x^{\cdot -1})U_c = -a$ which is equivalent to (70).

Next we compute $\Delta^b{}_c\Delta^a \log n = \Delta^b{}_c t(x^{\cdot -1}, a)$. Since t is bilinear the chain rule and (70) gives

$$-\Delta^b{}_c\Delta^a \log n = t(bU_c{}^{-1}, a). \tag{72}$$

We can now prove

THEOREM 7. *Let $\mathfrak{J}$ and $\mathfrak{J}'$ be finite-dimensional Jordan algebras with* 1 *such that $\mathfrak{J}'$ is separable. Assume also that $|\Phi| > \deg m, \deg m'$. Then*: (1) *If η is a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ mapping* 1 *into the identity element* $1'$ *of $\mathfrak{J}'$ then η is an isomorphism.* (2) *If η is a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ then η is an isotopy and $\mathfrak{J}$ is separable.*

PROOF. The condition on $|\Phi|$ implies that the linear extension of η to $\mathfrak{J}_\Gamma$ is a norm similarity. Hence we may assume Φ is infinite and employ the formulas developed above. (1) Let n', t' be the generic norm and trace on $\mathfrak{J}'$. It is clear that η maps the set of invertible elements of $\mathfrak{J}$ onto the set of invertible elements of $\mathfrak{J}'$. Let a be arbitrary in $\mathfrak{J}$, c invertible in $\mathfrak{J}$. Then, by (68), and the chain rule (20) we obtain

$$\Delta^a{}_c n' \circ \eta = n'(c^\eta)t'((c^\eta)^{\cdot -1}, a^\eta).$$

Hence $\Delta^a{}_c \log n' \circ \eta = t'((c^\eta)^{\cdot -1}, a^\eta)$. Similarly, we obtain $\Delta^a{}_c(x^\eta)^{\cdot -1} = -a^\eta U_{c^\eta}^{-1}$. Hence

$$-\Delta^b{}_c\Delta^a \log n' \circ \eta = t'(b^\eta U_{c^\eta}{}^{-1}, a^\eta). \tag{73}$$

On the other hand, we are given that $n' \circ \eta = \rho n$ where $\rho \neq 0$ is in Φ. Hence $\Delta^a \log n' \circ \eta = \Delta^a \log \rho n = \Delta^a \log n$ (by (12)). It follows from (72) and (73) that we have

$$t(bU_c{}^{-1}, a) = t'(b^\eta U_{c^\eta}{}^{-1}, a^\eta) \tag{74}$$

for all a, b and invertible c. Also we are given that 1^η is the identity of $\mathfrak{J}'$. Hence, taking $c = 1$ in (74), we obtain $t(b, a) = t'(b\eta, a\eta)$ $(\equiv t'(b^\eta, a^\eta))$ which permits us to rewrite (74) as

$$(75) \qquad t'(bU_c^{-1}\eta, a\eta) = t'(b\eta U_{c^\eta}^{-1}, a\eta).$$

Since $\mathfrak{J}'$ is separable, t' is nondegenerate. Hence (75) implies that $bU_c^{-1}\eta = b\eta U_{c^\eta}^{-1}$ for all b and invertible c. If we put $b = c$ in this we obtain $(c^{\cdot -1})^\eta = (c^\eta)^{\cdot -1}$. Hence $bU_{c^{\cdot -1}}\eta = bU_c^{-1}\eta = b\eta U_{c^\eta}^{-1} = b\eta U_{(c^\eta)^{\cdot -1}} = b\eta U_{(c^{\cdot -1})^\eta}$. Replacing c by $c^{\cdot -1}$ we obtain $bU_c\eta = b\eta U_{c^\eta}$. Putting $b = 1$ then $(c^{\cdot 2})^\eta = (c^\eta)^{\cdot 2}$ for invertible c. It follows that this holds for all c and hence η is an isomorphism. This completes the proof of (1).

(2) Again we have that c is invertible if and only if c^η is invertible. Hence 1^η is invertible so we can define the isotope $(\mathfrak{J}', (1^\eta)^{\cdot -1})$ whose identity element is 1^η. If n'' denotes the generic norm in this algebra then $n''(a^\eta) = n'((1^\eta)^{\cdot -1})n'(a^\eta) = \mu n(a)$, $\mu \neq 0$. Hence η is a norm similarity of $\mathfrak{J}$ onto $(\mathfrak{J}', (1^\eta)^{\cdot -1})$ and η maps 1 into the identity element of the second algebra. Hence, by (1), η is an isomorphism of $\mathfrak{J}$ onto an isotope of $\mathfrak{J}'$. This means that η is an isotopy of $\mathfrak{J}$ onto $\mathfrak{J}'$. It remains to show that $\mathfrak{J}$ is separable. This will follow by showing that if $\mathfrak{J}$ is separable then any isotope $(\mathfrak{J}, a)$ is separable. We have seen in §1.12 that $\mathfrak{J}$ and $(\mathfrak{J}, a)$ have the same ideals. Hence if $\mathfrak{J}$ is a direct sum of simple ideals then $(\mathfrak{J}, a)$ is a direct sum of simple ideals. Thus $\mathfrak{J}$ semisimple implies $(\mathfrak{J}, a)$ semisimple and if P is any extension of the base field then $\mathfrak{J}_P$ semisimple implies $(\mathfrak{J}_P, a) = (\mathfrak{J}, a)_P$ is semisimple. Hence $\mathfrak{J}$ separable implies $(\mathfrak{J}, a)$ is separable.

While Theorem 7 requires a restriction on the cardinality of the base field, the following corollary requires no such restriction.

COROLLARY. *Let $\mathfrak{J}$ be a finite-dimensional separable Jordan algebra over an arbitrary field; let $\eta = U_{a_1}U_{a_2}\cdots U_{a_r}$, $a_i \in \mathfrak{J}$, satisfy $1^\eta = 1$. Then η is an automorphism of $\mathfrak{J}$.*

PROOF. We have $1 = n(1) = n(1^\eta) = n(1U_{a_1}\cdots U_{a_r}) = \Pi n(a_i)^2$. Hence the a_i are invertible and η is a bijective linear mapping contained in $\mathrm{M}(\mathfrak{J})$ since $n(a^\eta) = \rho n(a)$ where $\rho = \Pi n(a_i)^2$. If Φ is infinite Theorem 7 implies that η is an automorphism. If Φ is finite we can extend Φ to an infinite field Γ and consider the linear mappings U_{a_i} in $\mathfrak{J}_\Gamma$. Clearly $\eta = U_{a_1}\cdots U_{a_r}$ is linear extension of the given η to $\mathfrak{J}_\Gamma$ and this is an automorphism. Hence the given η is an automorphism of $\mathfrak{J}$.

We recall that if $\mathfrak{V}$ and $\mathfrak{V}'$ are finite-dimensional vector spaces equipped with nondegenerate symmetric bilinear forms f and f' respectively then for any linear mapping T of $\mathfrak{V}$ into $\mathfrak{V}'$ we can define a unique linear mapping T^* of $\mathfrak{V}'$ into $\mathfrak{V}$ such that

$$(76) \qquad f'(xT, y') = f(x, y'T^*)$$

for all $x \in \mathfrak{V}$, $y' \in \mathfrak{V}'$. If T is bijective so is T^*. If $\mathfrak{V}''$ is another finite-dimensional vector space with nondegenerate symmetric bilinear form f'' and $U \in \mathrm{Hom}_\Phi(\mathfrak{V}', \mathfrak{V}'')$ then we have the element $U^* \in \mathrm{Hom}_\Phi(\mathfrak{V}'', \mathfrak{V}')$ defined by f' and f'' and $(TU)^* \in \mathrm{Hom}_\Phi(\mathfrak{V}'', \mathfrak{V})$ defined by f and f''. The definitions give $(TU)^* = U^*T^*$

Now let $\mathfrak{J}$ be a separable Jordan algebra with generic trace t. Since t is associative, $t(x.c, y) = t(x, y.c)$ which shows that $R_c^* = R_c$ (relative to t). Since $U_c = 2R_c^2 - R_{c^2}$ we have also $U_c^* = U_c$. Now suppose η is a norm similarity of $\mathfrak{J}$ onto a second separable Jordan algebra $\mathfrak{J}'$. Then the relation (74) is valid. Since η is bijective this gives

$$t'(b\eta U_{c^\eta}^{-1}, a') = t(bU_c^{-1}, a'\eta^{-1})$$

for all $b \in \mathfrak{J}$, $a' \in \mathfrak{J}'$. If we put $c = 1$ in this we obtain $\eta^{-1} = (\eta U_{1^\eta}^{-1})^* = U_{1^\eta}^{-1}\eta^*$. Hence

$$(77) \qquad \eta^* = U_{1^\eta}\eta^{-1}.$$

Since U_{1^η} and η^{-1} are norm similarities this equation shows that η^* is a norm similarity of $\mathfrak{J}'$ onto $\mathfrak{J}$. If $n'(a^\eta) = \rho n(a)$ then $n'(1^\eta) = \rho$ and (77) imply that $n(a'^{\eta^*}) = \rho n'(a')$, $a' \in \mathfrak{J}'$. In particular, if $n'(a^\eta) = n(a)$ then $n(a'^{\eta^*}) = n'(a')$.

We now let $\mathfrak{J}' = \mathfrak{J}$ and we consider the group $\mathrm{M}(\mathfrak{J})$ of norm similarities of the separable Jordan algebra $\mathfrak{J}$ onto itself. We assume also that $|\Phi| > \deg \mathfrak{J}$. Then Theorem 7 implies that $\mathrm{M}(\mathfrak{J})$ coincides with the structure group $\Gamma(\mathfrak{J})$, that is, the group of isotopies of $\mathfrak{J}$ onto $\mathfrak{J}$ (§1.12). We recall that $\Gamma(\mathfrak{J})$ contains as invariant subgroup the inner structure group $\Gamma_1(\mathfrak{J})$ which is defined to be the subgroup generated by the U_a, a invertible in $\mathfrak{J}$. We have defined the group $\mathrm{M}_1(\mathfrak{J})$ of norm preserving linear transformations of $\mathfrak{J}$ as the subgroup of $\mathrm{M}(\mathfrak{J})$ specified by $n(a^\eta) = n(a)$. We shall call the invariant subgroup $\mathrm{M}_1^{(1)}(\mathfrak{J}) = \mathrm{M}_1(\mathfrak{J}) \cap \Gamma_1(\mathfrak{J})$ the *inner norm preserving group* of $\mathfrak{J}$. Since $n(aU_c) = n(a)n(c)^2$ it is clear that $\mathrm{M}_1^{(1)}(\mathfrak{J})$ is the set of linear transformations in $\mathfrak{J}$ which have the form $U_{c_1}U_{c_2}\cdots U_{c_r}$ where $\prod_1^r n(c_i)^2 = 1$. The subset $\mathrm{M}_1^{(2)}(\mathfrak{J})$ of the transformations of the form $U_{c_1}U_{c_2}\cdots U_{c_r}$ such that $\prod_1^r n(c_i) = 1$ is a subgroup of $\mathrm{M}_1^{(1)}(\mathfrak{J})$.

If η is an automorphism of $\mathfrak{J}$ then $1^\eta = 1$ and $n(a^\eta) = n(a)$, $a \in \mathfrak{J}$. Hence $\mathrm{M}(\mathfrak{J}) \supseteq \mathrm{M}_1(\mathfrak{J}) \supseteq \mathrm{Aut}\,\mathfrak{J}$ where $\mathrm{Aut}\,\mathfrak{J}$ is the group of automorphisms of $\mathfrak{J}$. If $\mathfrak{J}$ is separable then it follows from Theorem 7 (1) that $\mathrm{Aut}\,\mathfrak{J}$ is the subgroup of $\mathrm{M}(\mathfrak{J})$ of η such that $1^\eta = 1$.

We have seen also that if $\mathfrak{J}$ is separable then the adjoint η^* of η relative to the generic trace form (which is nondegenerate) is in $\mathrm{M}(\mathfrak{J})$ for any $\eta \in \mathrm{M}(\mathfrak{J})$. The mapping $\eta \to \hat{\eta} = (\eta^*)^{-1}$ is an automorphism of period two in $\mathrm{M}(\mathfrak{J})$. By (77), we have $\hat{\eta} = \eta U_{1^\eta}^{-1}$. Hence $\hat{\eta} = \eta$ if η is an automorphism. On the other hand, if $\hat{\eta} = \eta$ then $U_{1^\eta} = 1$ which need not imply that $1^\eta = 1$. For example, if we take $\eta = -1$ then $\eta \in \mathrm{M}(\mathfrak{J})$ and $U_{1^\eta} = 1$ so $\hat{\eta} = \eta$. Hence the set of fixed points of $\mathrm{M}(\mathfrak{J})$ under the automorphism $\eta \to \hat{\eta}$ is a subgroup which contains properly the group of automorphisms.

If $\mathfrak{J}$ is finite-dimensional semisimple over an algebraically closed field then $\mathfrak{J}$

is separable. If $\eta \in \mathrm{M}(\mathfrak{J})$ then $1^\eta = a$ is invertible so $1^\eta = c^2$, c invertible. Then $U_c \in \mathrm{M}(\mathfrak{J})$ and $1U_c = 1^\eta$. Hence $\alpha = \eta U_c^{-1} \in \mathrm{M}(\mathfrak{J})$ and $1^\alpha = 1$. Then α is an automorphism. Thus in the case of an algebraically closed field we have that any element $\eta \in \mathrm{M}(\mathfrak{J})$ has the form $\eta = \alpha U_c$ where α is an automorphism.

Exercises

1. Show that $\Delta^a{}_1 \log n = t(a)$ holds for any finite-dimensional strictly power associative algebra with 1.

2. Let α be an isotopy of a Jordan algebra $\mathfrak{J}$ with 1 onto $\mathfrak{J}'$ with 1 and let $\mathfrak{B}$ be a quadratic ideal in $\mathfrak{J}$. Show that $\mathfrak{B}^\alpha$ is a quadratic ideal in $\mathfrak{J}'$. Assume $\mathfrak{J}$ and $\mathfrak{J}'$ are finite-dimensional separable and let η be a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$. Show that $\mathfrak{B}^\eta$ is a quadratic ideal.

3. Let $\mathfrak{J}$ be a separable Jordan algebra of characteristic $\neq 3$ and define $t(a,b,c) = t(a\,.\,b\,.\,c)$ on $\mathfrak{J}$ (cf. ex. 1, p. 229). Show that a bijective linear mapping η in $\mathfrak{J}$ is an automorphism if and only if $t(a^\eta, b^\eta) = t(a,b)$ and $t(a^\eta, b^\eta, c^\eta) = t(a,b,c)$, $a,b,c \in \mathfrak{J}$. Show also that $D \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ is a derivation if and only if $t(aD,b) + t(a,bD) = 0$ and $t(aD,b,c) + t(a,bD,c) + t(a,b,cD) = 0$.

4. Let $\mathfrak{J}$ be as in exercise 3. Show that if η is a bijective linear mapping of $\mathfrak{J}$ having the first three coefficients of the generic minimum polynomial as invariants then η is an automorphism.

5. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ the Jordan algebra of a nondegenerate symmetric bilinear form f on $\mathfrak{B}$, t and n the generic trace and norm on $\mathfrak{J}$. Let $n(a,b) = \frac{1}{2}[n(a+b) - n(a) - n(b)]$ the symmetric bilinear form corresponding to the quadratic form n. If $a = \alpha 1 + u$, $\alpha \in \Phi$, $u \in \mathfrak{B}$ then $t(a) = 2\alpha$ and $n(a) = \alpha^2 - f(u,u)$ (cf. §4). If a is invertible let S_a denote the symmetry in the vector space $\mathfrak{J}$ relative to the hyperplane orthogonal to a relative to n ($xS_a = x - 2n(x,a)n(a)^{-1}a$). Verify the formula $U_a = n(a)S_1S_a$. Use this result and the theorem of Cartan-Dieudonné on the decomposition of orthogonal transformations as products of symmetries (Artin's *Geometric Algebra*, p. 129) to show that any rotation in $\mathfrak{J}$ relative to n is a product of mappings $V_a \equiv n(a)^{-1}U_a$ where a is invertible. Show also that the group $\mathrm{M}_1^{(2)}(\mathfrak{J})$ defined to be the set of elements $U_{a_1}U_{a_2}\cdots U_{a_r}$ such that $\Pi n(a_i) = 1$ coincides with the reduced orthogonal group (definition in Artin's *Geometric Algebra*, p. 194).

6. Show that if $\mathfrak{J}$ is as in exercise 5 and $\dim \mathfrak{B} > 1$ then the only elements a in $\mathfrak{J}$ such that $U_a = 1$ are $a = \pm 1$. Hence show that if $\eta \in \mathrm{M}(\mathfrak{J})$ satisfies $\hat{\eta} = \eta$ then η is either an automorphism or the negative of an automorphism of $\mathfrak{J}$.

8. **Determination of M($\mathfrak{J}$) for $\mathfrak{J}$ special central simple of degree $\geqq 3$.** The Jordan algebras described in the title have the form $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite-dimensional central simple associative algebra with involution. Moreover, $\mathfrak{A}$ and the injection mapping constitute a unital special universal envelope of $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$. Hence $\mathfrak{J}$ generates the associative algebra $\mathfrak{A}$ and every automor-

phism of $\mathfrak{J}$ has a unique extension to an automorphism of $(\mathfrak{A}, J)$. We can distinguish three types of $(\mathfrak{A}, J)$:

I. $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^{\circ}$ where $\mathfrak{B}$ is central simple over the base field Φ and J is the involution $(a,b) \to (b,a)$, $a, b \in \mathfrak{B}$.

II. $\mathfrak{A}$ is simple with center P/Φ a quadratic extension field of Φ and the restriction of J to P is not the identity mapping.

III. $\mathfrak{A}$ is central simple over Φ.

We recall the theorem of Skolem-Noether that if $\mathfrak{A}$ is a finite-dimensional central simple Jordan algebra over Φ then every automorphism of $\mathfrak{A}/\Phi$ is inner, that is, it has the form $x \to c^{-1}xc$. It is clear that if $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where $\mathfrak{A}_i$ is finite dimensional simple then any automorphism of $\mathfrak{A}$ which is the identity on the center maps every $\mathfrak{A}_i$ onto itself and is the identity on the centers of the $\mathfrak{A}_i$. Hence, by the Skolem-Noether theorem applied to the $\mathfrak{A}_i$, it is an inner automorphism. If $(\mathfrak{A}, J)$ is central simple and $x \to c^{-1}xc$ is an automorphism of $(\mathfrak{A}, J)$ then $c^{-1}x^J c = c^J x^J (c^J)^{-1}$. This implies that cc^J is in the center of $\mathfrak{A}$. Since $cc^J \in \mathfrak{H}(\mathfrak{A}, J)$ we have $cc^J = \gamma 1$ where $\gamma \neq 0$ is in Φ. The converse is clear also: If $cc^J = \gamma 1 \neq 0$, γ in Φ, then $x \to c^{-1}xc$ is an automorphism of $(\mathfrak{A}, J)$. We shall call these *inner automorphisms of* $(\mathfrak{A}, J)$. Note that $x \to c^{-1}xc = \gamma^{-1}c^J xc$.

If $(\mathfrak{A}, J)$ is of type I or II then the center of $\mathfrak{A}$ is two dimensional. In the first case it is $\Phi \oplus \Phi$ and in the second case it is the field P/Φ. In both cases the group of auotmorphisms of P/Φ has order two. Also in either case there may exist automorphisms of $\mathfrak{A}/\Phi$ which induce an automorphism $\neq 1$ on the center. However, it is clear that any two such automorphisms have the same effect on the center and therefore they differ by an inner automorphism. Thus the group of inner automorphisms has index one or two in the group $\mathrm{Aut}\,\mathfrak{A}/\Phi$ of automorphisms of $\mathfrak{A}/\Phi$. If σ is an outer (noninner) automorphism of $\mathfrak{A}$ then σ and J have the same effect on the center of $\mathfrak{A}$. Hence $\sigma^{-1}J\sigma J$ is an automorphism which is the identity on the center and so is inner. Hence there exists a $c \in \mathfrak{A}$ such that $x^{\sigma J} = c^{-1}x^{J\sigma}c$ for all $x \in \mathfrak{A}$. Applying J we obtain $x^{\sigma} = c^J x^{J\sigma J}(c^J)^{-1}$ and replacing x by x^J in the first relation we obtain $x^{J\sigma J} = c^{-1}x^{\sigma}c$. Hence $x^{\sigma} = c^J c^{-1} x^{\sigma} c (c^J)^{-1}$ so $c^J c^{-1} = \gamma$ is in the center of $\mathfrak{A}$. Then $c^J = \gamma c$ and $c = \gamma^J c^J = \gamma^J \gamma c$ so $\gamma^J \gamma = 1$. If the center is a field P it follows from Hilbert's Satz 90 (Jacobson, *Lectures*, vol. III, p. 76) that $\gamma = \delta^J \delta^{-1}$, $\delta \in \mathrm{P}$. If the center is $\Phi \oplus \Phi$ then J has the form $(\alpha, \beta) \to (\beta, \alpha)$ in $\Phi \oplus \Phi$, so if $\gamma = (\alpha, \beta)$ satisfies $\gamma\gamma^J = 1$, then $\gamma = (\alpha, \alpha^{-1})$ and $\gamma = \delta^J \delta^{-1}$ where $\delta = (1, \alpha)$. Clearly we may replace c by $\delta^{-1}c$ and obtain $(\delta^{-1}c)^J = \delta^{-1}c$. Hence we may assume that the element c such that $x^{\sigma J} = c^{-1}x^{J\sigma}c$ satisfies $c^J = c$, that is, $c \in \mathfrak{H}(\mathfrak{A}, J)$.

We now consider the group $\mathrm{M}(\mathfrak{J})$ of norm similarities of $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is finite dimensional central simple associative with involution and the degree of $\mathfrak{J}$ is $\geqq 3$. Then $\mathfrak{J}$ is separable and Theorem 5 is applicable. Also if Ω is the algebraic closure of Φ, then $\mathfrak{J}_\Omega = \mathfrak{H}(\mathfrak{A}_\Omega, J)$ and these are the algebras whose generic minimum polynomials were determined in §4. The results obtained in §4 show that the generic norm of $a \in \mathfrak{H}(\mathfrak{A}_\Omega, J)$ as element of $\mathfrak{A}_\Omega$ either equals or is the square of the generic norm of a as element of $\mathfrak{H}(\mathfrak{A}_\Omega, J)$. It follows that this relation holds also for $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{A}$. We can use this remark to define certain elements of $\mathrm{M}(\mathfrak{J})$. First let a be an invertible element of $\mathfrak{A}$ and let $\gamma \neq 0$ be in Φ. Then the mapping

$$x \rightarrow x^\eta = \gamma a^J x a \tag{78}$$

in $\mathfrak{A}$ maps $\mathfrak{J}$ onto itself and is clearly a bijective linear mapping of $\mathfrak{J}$. We consider its linear extensions η to $\mathfrak{J}_\Omega$. Since the norm on $\mathfrak{A}$ is multiplicative it is clear that in $\mathfrak{J}_\Omega$ we have $n(x^\eta) = \pm \rho n(x)$ where $\rho = \gamma^m n(a^J a)$. Also we have $n(1^\eta) = \rho$ so the set of u such that $n(u^\eta) \neq -\rho n(u)$ is nonvacuous Zariski open subset of $\mathfrak{J}_\Omega$. On this set the polynomial function $x \rightarrow n(x^\eta) - \rho n(x)$ is 0. Hence we have $n(x^\eta) = \rho n(x)$ for all $x \in \mathfrak{J}$ and so $\eta \in \mathrm{M}(\mathfrak{J})$. Next let σ be an outer automorphism of $\mathfrak{A}/\Phi$ and let $c = c^J$ satisfy $x^{\sigma J} = c^{-1} x^{J\sigma} c$ for all $x \in \mathfrak{A}$. Then c is determined up to a nonzero multiplier in Φ. Let $x \in \mathfrak{J}$. Then $(c^{-1}x^\sigma)^J = x^{\sigma J} c^{-1} = c^{-1} x^{J\sigma} = c^{-1} x^\sigma$. Hence $c^{-1}x^\sigma \in \mathfrak{J}$ and

$$\zeta_{\sigma,c} : x \rightarrow c^{-1} x^\sigma \tag{79}$$

is a bijective linear mapping in $\mathfrak{J}$. The argument just used to prove that $\eta \in \mathrm{M}(\mathfrak{J})$ shows also that $n(c^{-1}x^\sigma) = n(c)^{-1} n(x)$ for all $x \in \mathfrak{J}$. Hence (79) defines an element of $\mathrm{M}(\mathfrak{J})$. It is clear that the set $\mathrm{M}'(\mathfrak{J})$ of elements η of the form (78) is a subgroup of $\mathrm{M}(\mathfrak{J})$. We claim also that no $\zeta_{\sigma,c}$ for σ outer is contained in $\mathrm{M}'(\mathfrak{J})$. Otherwise we have an invertible $a \in \mathfrak{A}$ and a $\gamma \neq 0$ in Φ such that $c^{-1}x^\sigma = \gamma a^J x a$, $x \in \mathfrak{J}$. Then $x^\sigma = bxa$ where $b = \gamma c a^J$ and $1^\sigma = 1$ gives $a = b^{-1}$ Hence $x^\sigma = bxb^{-1}$ for all x in $\mathfrak{J}$. Since $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ generates $\mathfrak{A}$ we have $x^\sigma = bxb^{-1}$ for all x in $\mathfrak{A}$ contrary to the assumption that σ is outer. Thus the existence of an outer automorphism of $\mathfrak{A}/\Phi$ implies that $\mathrm{M}(\mathfrak{J}) \supset \mathrm{M}'(\mathfrak{J})$. We can now state the following

THEOREM 8. *Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $m \geqq 3$ and assume $|\Phi| > m$. Let $\mathfrak{A}$ and the injection mapping of $\mathfrak{J}$ be a unital special universal envelope for $\mathfrak{J}$, J the main involution in $\mathfrak{A}$. Then $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ and the set $\mathrm{M}'(\mathfrak{J})$ of mappings η of the form* (78), *where a is an invertible element of $\mathfrak{A}$ and $\gamma \neq 0$ is in Φ, is a subgroup of index one or two in the group $\mathrm{M}(\mathfrak{J})$ of norm similarities of $\mathfrak{J}$ onto itself. Moreover, the index is two and if and only if there exist outer automorphisms of $\mathfrak{A}/\Phi$. In this case if σ is such an automorphism then $\zeta_{\sigma,c}$ defined by* (79) *is a representative of the coset $\neq \mathrm{M}'(\mathfrak{J})$ of $\mathrm{M}'(\mathfrak{J})$ in $\mathrm{M}(\mathfrak{J})$.*

PROOF. The fact that $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ is the reflexivitity of $\mathfrak{J}$ which we proved in §5.7. We shall now distinguish the three types of $(\mathfrak{A}, J)$ listed above.

TYPE I. Here $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^\circ$ and J is the mapping $(b_1, b_2) \to (b_2, b_1)$. Hence $\mathfrak{H}(\mathfrak{A}, J)$ is the set of elements of the form $(b, b), b \in \mathfrak{B}$. We have the isomorphism $(b, b) \to b$ of $\mathfrak{H}(\mathfrak{A}, J)$ onto $\mathfrak{B}^+$. Then the mapping in $\mathfrak{B}^+$ corresponding to η: $(x, x) \to \gamma(a, b)^J(x, x)(a, b)$ is $x \to \gamma bxa$ and every mapping of the form $x \to axb$ in $\mathfrak{B}^+$ is obtained in this way. We note also that $\mathfrak{A}$ has an outer automorphism if and only if $\mathfrak{B}$ has an antiautomorphism. For, if σ is an outer automorphism of $\mathfrak{A}$ then σ maps $\mathfrak{B}$ onto $\mathfrak{B}^\circ$ so we have an isomorphism of $\mathfrak{B}$ onto its opposite algebra $\mathfrak{B}^\circ$ and consequently we have an antiautomorphism of $\mathfrak{B}$. Conversely, if τ is an antiautomorphism of $\mathfrak{B}$ then $(x, y) \to (y^\tau, x^\tau)$ is an outer automorphism of $\mathfrak{A}$. Finally, it is clear that if τ is an antiautomorphism of $\mathfrak{B}$ then τ is not of the form $x \to axb, a, b \in \mathfrak{B}$. The theorem for Type I algebras will now follow by showing that if $\mathfrak{B}$ has no antiautomorphisms then $\mathrm{M}(\mathfrak{B}^+)$ is the set of mappings $x \to axb$, a and b invertible in $\mathfrak{B}$ and if $\mathfrak{B}$ has an antiautomorphism then $\mathrm{M}(\mathfrak{B}^+)$ consists of these mappings and the mappings $x \to ax^\tau b$, where a and b are invertible and τ is a fixed antiautomorphism of $\mathfrak{B}$. If $\zeta \in \mathrm{M}(\mathfrak{B}^+)$ then 1^ζ is invertible in $\mathfrak{B}^+$ and hence in $\mathfrak{B}$. Then $\eta: x \to 1^\zeta x$ is in $\mathrm{M}(\mathfrak{B}^+)$ and $\alpha = \zeta\eta^{-1} \in \mathrm{M}(\mathfrak{B}^+)$ and $1^\alpha = 1$. Hence, by Theorem 7, α is an automorphism of $\mathfrak{B}^+$. By §5.7, α is either an automorphism or an antiautomorphism of $\mathfrak{B}$. Since $\mathfrak{B}$ is central simple, in the first case α is inner and in the second case $\alpha = \tau\beta$ where τ is any fixed antiautomorphism and β is an inner automorphism. Accordingly, $\zeta = \alpha\eta$ or $\zeta = \tau\beta\eta$ has the form $x \to axb$ or the form $x \to ax^\tau b$.

TYPE II. Here we have $(\mathfrak{A}, J)$ where $\mathfrak{A}$ is simple with center a quadratic extension field P/Φ and the restriction of J to P is not the identity. If q is a skew element $\neq 0$ in P then every element of $\mathfrak{A}$ can be written in one and only one way in the form $x + yq$ where $x, y \in \mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$. This implies that as vector spaces we have $\mathfrak{A} \cong \mathrm{P} \otimes_\Phi \mathfrak{J}$. Hence $\mathfrak{A}^+/\mathrm{P}$ can be identified with $\mathfrak{J}_\mathrm{P}$. Now let $\zeta \in \mathrm{M}(\mathfrak{J})$ and let ζ be its linear extension to $\mathfrak{A}^+/\mathrm{P}$. Since $|\Phi| > m$, ζ is an element of $\mathrm{M}(\mathfrak{A}^+)$. Hence by the result established for the Type I algebras the mapping ζ of $\mathfrak{A}^+$ is either of the form $x \to axb$, a, b invertible elements of $\mathfrak{A}^+$ or of the form $x \to ax^\tau b$ where a and b are invertible and τ is any antiautomorphism of $\mathfrak{A}/\mathrm{P}$. In the first case, since ζ maps $\mathfrak{H}(\mathfrak{A}, J)$ onto itself we have $b^J x a^J = axb$, $x \in \mathfrak{J}$. Since $1 \in \mathfrak{J}$ this implies that $c = a^{-1}b^J = b(a^J)^{-1} = c^J$. Then c commutes with every $x \in \mathfrak{J}$ and hence also with every $x \in \mathfrak{A} = \mathrm{P}\mathfrak{J}$. Hence $c \in \mathrm{P}$ and since $c^J = c$, $c = \gamma^{-1} \in \Phi$ and $a = \gamma b^J$. Thus the given mapping ζ of $\mathfrak{J}$ has the form $x \to \gamma b^J x b$ so $\zeta \in \mathrm{M}'(\mathfrak{J})$ as defined above. Suppose next that the linear extension ζ of ζ to $\mathfrak{A}^+$ has the form $x \to ax^\tau b$ where τ is an antiautomorphism of $\mathfrak{A}/\mathrm{P}$. We now note that τ is an antiautomorphism of $\mathfrak{A}/\mathrm{P}$, if and only if $\sigma = J\tau$ is an outer automorphism of $\mathfrak{A}/\Phi$. Hence we may assume that $\tau = J\sigma$ where σ is any chosen outer automorphism of $\mathfrak{A}/\Phi$. If $x \in \mathfrak{J}$ we have $ax^\tau b = ax^{J\tau}b = ax^\sigma b \in \mathfrak{J}$. Then $(b^J)^{-1}ax^\sigma = (b^J)^{-1}(ax^\sigma b)b^{-1} \in \mathfrak{J}$ so $c^{-1}x^\sigma \in \mathfrak{J}$ for $c = a^{-1}b^J$. Then $c^{-1} = c^{-1}1^\sigma \in \mathfrak{J}$ and $c \in \mathfrak{J}$. We have $x^{\sigma J}c^{-1} = (c^{-1}x^\sigma)^J = c^{-1}x^\sigma = c^{-1}x^{J\sigma}$,

$x \in \mathfrak{J}$. Hence $x^{\sigma J} = c^{-1}x^{J\sigma}c$ holds for $x \in \mathfrak{J}$ and hence also for all $x \in \mathfrak{A}$. It follows that the mapping $x \rightarrow c^{-1}x^{\sigma}$ is a mapping $\zeta_{\sigma,c}$ defined by the given outer automorphism σ. Moreover, the given mapping ζ of $\mathfrak{J}$ has the form $x \rightarrow ax^{\sigma}b = b^J(c^{-1}x^{\sigma})b$ and so it is a product $\zeta_{\sigma,c}\eta$ where $\eta \in \mathrm{M}'(\mathfrak{J})$. This proves the result for Type II algebras.

TYPE III. Here $\mathfrak{A}$ is central simple over Φ. It is well known that there exists a finite-dimensional separable splitting field Γ/Φ (Jacobson, *Structure of Rings*, p. 181). Then $(\mathfrak{A}_\Gamma, J) = (\Gamma_n, J)$ and the involution J in Γ_n has the form $X \rightarrow S^{-1}X^tS$ where $S^t = \pm S$. If $S^t = -S$ then we may assume the entries of S are in the prime field. We can make the same assumption if $S^t = S$ provided we adjoin suitable square roots to Γ. Since the characteristic is $\neq 2$ these adjunctions do not destroy the separability over Φ. Hence we may suppose that Γ already has the required property and that, moreover, Γ/Φ is Galois. Let $G = \{1, s, \cdots, u\}$ be the Galois group of Γ/Φ. Let $(u_1, u_2, \cdots, u_{n^2})$ be a basis for $\mathfrak{A}/\Phi$, hence, also for $\mathfrak{A}_\Gamma = \Gamma_n$ over Γ, and let (e_{ij}) be a matrix basis for Γ_n/Γ. The multiplication constants determined by these two bases are contained in Φ. Hence if $s \in G$ then A_s: $\sum \gamma_k u_k \rightarrow \sum \gamma_k{}^s u_k$ and B_s: $X = \sum \gamma_{ij} e_{ij} \rightarrow X^s \equiv \sum \gamma_{ij}{}^s e_{ij}$ are automorphisms of $\mathfrak{A}_\Gamma$ as algebra over Φ and both are s-semilinear transformations of $\mathfrak{A}_\Gamma/\Gamma$. Now A_s is the identity on $\mathfrak{A}$ so it commutes with J on $\mathfrak{A}$. It follows also that it commutes with J on $\mathfrak{A}_\Gamma$ since J is the Γ-linear extension of J to $\mathfrak{A}_\Gamma$ and A_s is the s-semilinear extension of the identity on $\mathfrak{A}$. Also since $X^J = S^{-1}X^tS$ where the entries of S are in Φ it is clear that $B_sJ = JB_s$. Hence A_s and B_s are automorphisms of $(\mathfrak{A}_\Gamma/\Phi, J)$. Also it is clear from the definition of A_s for $s \in G$ that $A_sA_t = A_{st}$, $s, t \in G$, and that $\mathfrak{A}$, which is the set of Φ-linear combinations of the elements u_i, is the set of common fixed points $(x^{A_s} = x)$ of the A_s for every $s \in G$. Since A_s and B_s are both s-semilinear automorphisms of $\mathfrak{A}_\Gamma$ it is clear that $C_s = B_s{}^{-1}A_s$ is an automorphism of $(\mathfrak{A}_\Gamma/\Gamma, J)$. Since $\mathfrak{A}_\Gamma/\Gamma$ is central simple this implies that there exist $M_s \in \mathfrak{A}_\Gamma = \Gamma_n$ which are determined up to nonzero multipliers in Γ such that

$$X^{C_s} = M_s{}^{-1}XM_s, \qquad M_s{}^JM_s = \mu_s 1, \qquad \mu_s \in \Gamma. \tag{80}$$

Then $X^{A_s} = X^{B_sC_s} = M_s{}^{-1}X^sM_s$ and since $A_sA_t = A_{st}$ we have

$$M_s{}^tM_t = \mu_{s,t}M_{st} \tag{81}$$

for some $\mu_{s,t} \neq 0$ in Γ. Also since $\mathfrak{A} = \{x \in \Gamma_n \mid x^{A_s} = x, \ s \in G\}$ it is clear that $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J) = \{x \in \mathfrak{H}(\Gamma_n, J) \mid x^{A_s} = x, s \in G\}$.

Now let $\zeta \in \mathrm{M}(\mathfrak{J})$. Then $1^\zeta = D \in \mathfrak{J}$ and so $D \in \mathfrak{H}(\Gamma_n, J)$ and hence $S^{-1}D^tS = D$. Now let $X \in \mathfrak{H}(\Gamma_n, J)$ and consider the equation $X^JX = D$. This is equivalent to $X^tSX = SD$ and S and SD satisfy $S^t = \pm S$, $(SD)^t = \pm SD$. Hence in a suitable extension of Γ by square roots we can solve our equation for X. Since ζ is fixed in our discussion we may assume that the Galois extension field Γ chosen at the beginning has the property that $\mathfrak{H}(\Gamma_n, J)$ contains a matrix E such that $E^J E = D$. We now extend ζ linearly to Γ_n and obtain ζ which is contained in the group of norm similarities $\mathrm{M}(\mathfrak{H}(\Gamma_n, J))$. Also we have $1^\zeta = D = E^J E$ so if we let η be the mapping $X \to E^J X E$ in $\mathfrak{H}(\Gamma_n, J)$ then $\eta \in \mathrm{M}(\mathfrak{H}(\Gamma_n, J))$ and $1^\eta = 1^\zeta$. Hence $\alpha = \zeta\eta^{-1} \in \mathrm{M}(\mathfrak{H}(\Gamma_n, J))$ and satisfies $1^\alpha = 1$. Hence α is an automorphism of $\mathfrak{H}(\Gamma_n, J)$ and consequently there exists a matrix $F \in \Gamma_n$ such that $X^\alpha = F^{-1} X F$, $X \in \mathfrak{H}(\Gamma_n, J)$, and $F^J F = \rho 1$, $\rho \in \Gamma$. Also, since by adjunction of a square root we may assume $\rho = 1$, we may suppose this, too, holds in Γ and then we see that the extension ζ of ζ to $\mathfrak{H}(\Gamma_n, J)$ has the form $X \to C^J X C$, $C \in \Gamma_n$. We now apply the automorphism A_s of $(\Gamma_n/\Phi, J)$ to both sides of the equation $X^\zeta = C^J X C$, $X \in \mathfrak{J}$ to obtain $X^\zeta = C^{JA_s} X C^{A_s}$, $X \in \mathfrak{J}$. Taking into account the form of A_s we have

$$(M_s^{-1} C^{sJ} M_s) X (M_s^{-1} C^s M_s) = X^\zeta, \qquad X \in \mathfrak{J}.$$

Since $M_s^J M_s = \mu_s 1$ this gives

$$(M_s^{-1} C^s M_s)^J X (M_s^{-1} C^s M_s) = X^\zeta = C^J X C$$

for all $X \in \mathfrak{J}$. This implies that $M_s^{-1} C^s M_s C^{-1}$ commutes with every $X \in \mathfrak{J}$ and hence with every $X \in \mathfrak{H}(\Gamma_n, J)$. Since the subalgebra of Γ_n generated by $\mathfrak{H}(\Gamma_n, J)$ is Γ_n it now follows that we have an element $\rho_s \neq 0$ in Γ such that

$$C^{A_s} = M_s^{-1} C^s M_s = \rho_s C, \qquad s \in G. \tag{82}$$

Since A_s is s-semilinear and $A_s A_t = A_{st}$ this gives E. Noether's equations: $\rho_s^t \rho_t = \rho_{st}$ so by Noether's lemma (Jacobson, *Lectures*, vol. III, p. 75) there exists an element $\delta \neq 0$ in Γ such that $\rho_s = \delta^s \delta^{-1}$. Then $(\delta^{-1} C)^{A_s} = (\delta^s)^{-1} \rho_s C = \delta^{-1} C$. Since $\mathfrak{A}$ is the set of elements $a \in \Gamma_n$ such that $a^{A_s} = a$, $s \in G$, this shows that $a = \delta^{-1} C \in \mathfrak{A}$ and we have $x^\zeta = C^J x C = \delta^2 a^J x a, x \in \mathfrak{J}$. Applying this to 1 we see that $\gamma = \delta^2 \in \Phi$ and so $x^\zeta = \gamma a^J x a$ is contained in $\mathrm{M}^1(\mathfrak{J})$. Hence in this case $\mathrm{M}'(\mathfrak{J}) = \mathrm{M}(\mathfrak{J})$ which is what is asserted in the theorem. This completes the proof.

We note that the proof of the theorem shows also that if $\mathfrak{J} = \mathfrak{A}^+$ where $\mathfrak{A}$ is finite dimensional central simple associative of degree $\geqq 3$ then $\mathrm{M}(\mathfrak{J})$ consists of the mappings $x \to axb$ and $x \to ax^\sigma b$ where $n(a) \neq 0$, $n(b) \neq 0$ and σ is a fixed antiautomorphism of $\mathfrak{A}$. We have shown also that if $(\mathfrak{A}, J)$ is finite dimensional central simple with involution and $x \in \mathfrak{H}(\mathfrak{A}, J)$ then $n(\gamma a^J x a) = \gamma^m n(a^J a) n(x)$. Moreover, if σ is an outer automorphism of $\mathfrak{A}/\Phi$ and c is an invertible element of $\mathfrak{H}(\mathfrak{A}, J)$ such that $x^{\sigma J} = c^{-1} x^{J\sigma} c$, $x \in \mathfrak{A}$ then $n(c^{-1} x^\sigma) = n(c)^{-1} n(x)$. It follows from Theorem 6 that if the degree of $\mathfrak{H}(\mathfrak{A}, J) = m \geqq 3$ then $\mathrm{M}_1(J)$ consists of the mappings

$$x \to \gamma a^J x a \tag{83}$$

where $\gamma^m n(a^J a) = 1$ and the mappings

(84) $$x \to \gamma a^J c^{-1} x^\sigma a$$

where σ is an outer automorphism, c is as indicated and $\gamma^m n(c)^{-1} n(a^J a) = 1$.

Exercises

1. Show that if $\mathfrak{J}$ is as in Theorem 6 then the only elements η of $M(\mathfrak{J})$ such that $\hat{\eta} = (\eta^*)^{-1} = \eta$ are the automorphisms and negatives of automorphisms of $\mathfrak{J}$. Hence show that if the degree m is odd then the automorphism group is the subgroup of $M_1(\mathfrak{J})$ of elements η such that $\hat{\eta} = \eta$.

2. Let $M_1'(\mathfrak{J})$ denote the commutator subgroup of $M_1(\mathfrak{J})$. Prove that $M_1'(\Phi_m^+)$, $M_1^{(1)}(\Phi_m^+)$, $M_1^{(2)}(\Phi_m^+)$ for $m \geqq 3$ consist of the mappings $x \to axb$ where $\det ab = 1$ and (1) $\det a$ is a square for M_1', (2) $\det a = \pm 1$ for $M_1^{(1)}$, (3) $\det a = 1$ for $M_1^{(2)}$.

3. Let $\gamma = \operatorname{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$, $\gamma_i \neq 0$ in Φ. Show that any element of Φ_m such that $\det A = 1$ is a product of elements of $\mathfrak{H}(\Phi_m, J_\gamma)$. Use this to prove that if $m \geqq 3$ then $M_1'(\mathfrak{H}(\Phi_m, J_\gamma)) = M_1^{(2)}(\mathfrak{H}(\Phi_m, J_\gamma))$ and this is the set of mappings $x \to a^* x a$ where $\det a = 1$ and $a^* = a^{J_\gamma}$. Show also that $M_1^{(1)}(\mathfrak{H}(\Phi_m, J_\gamma))$ is the set of mappings $x \to a^* x a$ where $\det a = \pm 1$.

9. **The Lie algebras $\mathfrak{M}(\mathfrak{J})$, $\mathfrak{M}_0(\mathfrak{J})$ and Der $\mathfrak{J}$ for $\mathfrak{J}$ separable.** In this section we shall obtain the Lie algebra analogues of the group theoretic results of the last two sections.

If $\mathfrak{A}/\Phi$ is an algebra we denote the Lie algebra of derivations in $\mathfrak{A}$ by $\operatorname{Der}\mathfrak{A}$. Hence $\operatorname{Der}\mathfrak{A}$ is the subalgebra of the Lie algebra $\operatorname{Hom}_\Phi(\mathfrak{A},\mathfrak{A})^-$ of elements D such that $(ab)D = (aD)b + a(bD), a, b \in \mathfrak{A}$. Moreover, if (u_ι) is a basis for $\mathfrak{A}/\Phi$ then a $D \in \operatorname{Hom}_\Phi(\mathfrak{A},\mathfrak{A})$ is a derivation if and only if $(u_\iota u_\kappa)D = (u_\iota D)u_\kappa + u_\iota(u_\kappa D)$. Thus if $u_\iota D = \sum \alpha_{\iota\kappa} u_\kappa$, so that $(\alpha_{\iota\kappa})$ is the (row-finite) matrix of the linear transformation D relative to the basis (u_ι), and $u_\iota u_\kappa = \sum \gamma_{\iota\kappa\lambda} u_\lambda$, $\gamma_{\iota\kappa\lambda} \in \Phi$, then D is a derivation if and only if

$$\sum_\lambda \gamma_{\iota\kappa\lambda}\alpha_{\lambda\mu} = \sum_\lambda \alpha_{\iota\lambda}\gamma_{\lambda\kappa\mu} + \sum_\lambda \alpha_{\kappa\lambda}\gamma_{\iota\lambda\mu}$$

for all ι, κ, μ. It is clear from this that if P is an extension field of the base field Φ and we identify a linear mapping of $\mathfrak{A}/\Phi$ into itself with its linear extension to $\mathfrak{A}_P/P$ then every derivation of $\mathfrak{A}/\Phi$ is a derivation of $\mathfrak{A}_P/P$ and every derivation of $\mathfrak{A}_P/P$ is a P-linear combination of derivations in $\mathfrak{A}/\Phi$. In this way one can identify $\operatorname{Der}\mathfrak{A}_P$ with $(\operatorname{Der}\mathfrak{A})_P = P \otimes_\Phi \operatorname{Der}\mathfrak{A}$.

We recall that if $\mathfrak{J}$ is a Jordan algebra then the mapping $[R_a R_b]$, $a, b \in \mathfrak{J}$ is a derivation of $\mathfrak{J}$ (§1.7). The derivations of the form $\sum [R_{a_i} R_{b_i}]$, $a_i, b_i \in \mathfrak{J}$, are called inner and these constitute an ideal Inder $\mathfrak{J}$ in the Lie algebra Der $\mathfrak{J}$. It is clear that the inner derivations of $\mathfrak{J}_P$ are P-linear combinations of inner derivations of $\mathfrak{J}$. Hence we have $\operatorname{Inder}\mathfrak{J}_P = (\operatorname{Inder}\mathfrak{J})_P$.

We shall now study the derivation algebra of a separable Jordan algebra $\mathfrak{J}$. We have $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ where $\mathfrak{J}_i$ is simple with center $\mathfrak{C}_i$ a separable extension field of the base field Φ. If D is a derivation of $\mathfrak{J}$ then $\mathfrak{J}_i^{\cdot 2} = \mathfrak{J}_i$ gives $\mathfrak{J}_i D \subseteq (\mathfrak{J}_i D) \cdot \mathfrak{J}_i \subseteq \mathfrak{J}_i$ so D maps $\mathfrak{J}_i$ into itself and hence the restriction D_i of D to $\mathfrak{J}_i$ is a derivation in $\mathfrak{J}_i$. Conversely, if D_i, $i = 1, 2, \cdots, s$, is a derivation in $\mathfrak{J}_i$ then the linear mapping D of $\mathfrak{J}$ which coincides with D_i on $\mathfrak{J}_i$ is a derivation. It follows that $\mathrm{Der}\,\mathfrak{J} \cong \mathrm{Der}\,\mathfrak{J}_1 \oplus \mathrm{Der}\,\mathfrak{J}_2 \oplus \cdots \oplus \mathrm{Der}\,\mathfrak{J}_s$. Hence the study of $\mathrm{Der}\,\mathfrak{J}$ can be reduced to the case in which $\mathfrak{J}$ is simple with center $\mathfrak{C}$ a separable field. It is immediate from the definition of the center of a nonassociative algebra that any derivation of the algebra maps the center into itself and hence induces a derivation in the center. Let D be a derivation in the simple Jordan algebra $\mathfrak{J}$. Then $\mathfrak{C}D \subseteq \mathfrak{C}$. Since $\mathfrak{C}/\Phi$ is a separable field we have $\mathfrak{C} = \Phi(\theta)$ where the minimum polynomial $f(\lambda)$ of θ has distinct roots. If we apply D to the relation $f(\theta) = 0$ we obtain $f'(\theta)(\theta D) = 0$ where $f'(\lambda)$ is the usual formal derivative of the polynomial $f(\lambda)$. Since $f(\lambda)$ has distinct roots $f'(\lambda) \neq 0$ and since its degree is less than that of $f(\lambda)$, $f'(\theta) \neq 0$. Hence $f'(\theta)^{-1}$ exists in $\mathfrak{C}$ and $f'(\theta)(\theta D) = 0$ gives $\theta D = 0$. Hence $\mathfrak{C}D = 0$ and this implies that if we consider $\mathfrak{J}$ as an algebra over its center $\mathfrak{C}$ then D is a derivation of $\mathfrak{J}/\mathfrak{C}$. Since $\mathfrak{J}/\mathfrak{C}$ is central simple we have now reduced the consideration to central simple Jordan algebras.

Accordingly, we now suppose $\mathfrak{J}$ is a finite-dimensional central simple Jordan algebra over Φ. If the degree of $\mathfrak{J}$ is one, then $\mathfrak{J} = \Phi 1$ and $\mathrm{Der}\,\mathfrak{J} = 0$. We shall leave the consideration of the degree two case to the exercises and we shall postpone the study of $\mathrm{Der}\,\mathfrak{J}$ for $\mathfrak{J}$ exceptional central simple to Chapter IX. We now consider the remaining possibility: $\mathfrak{J}$ is finite dimensional special central simple of degree $m \geqq 3$. Then $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite-dimensional central simple associative algebra with involution. Moreover, we know that $\mathfrak{A}$ and the injection mapping of $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ in $\mathfrak{A}$ is a unital special universal envelope for $\mathfrak{J}$ (§5.7). Hence if D is a derivation of $\mathfrak{J}$ then D has a unique extension to a derivation of $(\mathfrak{A}, J)$, that is, a derivation of $\mathfrak{A}$ which commutes with J. We now have

THEOREM 9. *Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $m \geqq 3$, $(\mathfrak{A}, J)$ the corresponding finite-dimensional central simple associative algebra with involution, so that $\mathfrak{J}$ can be identified with $\mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{A}$ with the unital special universal envelope of $\mathfrak{J}$. Then any derivation of $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ has the form*

$$x \to [xd] \tag{85}$$

where $d^J = -d \in \mathfrak{A}$, and conversely. Moreover, every derivation of $\mathfrak{J}$ is inner (that is, of the form $\sum [R_{a_i} R_{b_i}]$) if and only if either the center of $\mathfrak{A}$ is Φ or the center is not Φ and the characteristic is either 0 or a prime p not dividing m.

PROOF. Let D be a derivation of $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$. Then D is the restriction to $\mathfrak{J}$ of a derivation of D of $(\mathfrak{A}, J)$. It is well known that every derivation of a finite-dimensional separable associative algebra is inner in the sense that it has the form $x \to [xd]$ for a suitable d in the algebra.(2) Hence there exists a $d \in \mathfrak{A}$ such that $xD = [xd]$, $x \in \mathfrak{J}$. Applying J we obtain also $xD = -[xd^J]$ so $d^J + d$ commutes with every $x \in \mathfrak{J}$. Since $\mathfrak{J}$ generates $\mathfrak{A}$ it follows that $d^J + d$ is in the center of $\mathfrak{A}$. Since $d^J + d \in \mathfrak{H}(\mathfrak{A}, J)$ we see that $d^J + d = \delta 1$ where $\delta \in \Phi$. Then $(d - \frac{1}{2}\delta 1)^J + (d - \frac{1}{2}\delta 1) = 0$ so $d - \frac{1}{2}\delta 1$ is skew. Since we can replace d by this element we may suppose that $d^J = -d$. Thus $xD = [xd]$ where $d^J = -d$. Let $\mathfrak{S}(\mathfrak{A}, J)$ denote the set of skew elements of $\mathfrak{A}$. This is a subalgebra of the Lie algebra $\mathfrak{A}^-$. Moreover, if $x \in \mathfrak{H}(\mathfrak{A}, J)$ and $d \in \mathfrak{S}(\mathfrak{A}, J)$ then $[xd] \in \mathfrak{H}(\mathfrak{A}, J)$. Also $[x.y, d] = x.[yd] + [xd].y$ so $x \to [xd]$ is a derivation of $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$. This completes the proof of the first statement of the theorem. Now let $x, a, b \in \mathfrak{J}$ and consider $x[R_a R_b] = x.a.b - x.b.a$. A direct calculation using $x.a = \frac{1}{2}(xa + ax)$ shows that $4x[R_a R_b] = [x[ab]]$. This and the first part of the theorem implies that every derivation of $\mathfrak{J}$ is inner if every $d \in \mathfrak{S}(\mathfrak{A}, J)$ has the form $\sum [a_i b_i]$ where $a_i, b_i \in \mathfrak{H}(\mathfrak{A}, J)$. We have $[\mathfrak{H}(\mathfrak{A}, J), \mathfrak{H}(\mathfrak{A}, J)] \subseteq \mathfrak{S}(\mathfrak{A}, J)$, so a sufficient condition that every derivation of $\mathfrak{J}$ is inner that $[\mathfrak{H}(\mathfrak{A}, J), \mathfrak{H}(\mathfrak{A}, J)] = \mathfrak{S}(\mathfrak{A}, J)$. We now suppose the base field Φ is algebraically closed and later we shall reduce the general case to this one. If Φ is algebraically closed then $(\mathfrak{A}, J) = (\mathfrak{D}_m, J_1)$ where $(\mathfrak{D}, j)$ is an associative composition algebra and J_1 is the standard involution in $\mathfrak{D}_m$. The possibilities for $(\mathfrak{D}, j)$ are $\mathfrak{D} = \Phi$, j the identity; $\mathfrak{D} = \Phi \oplus \Phi$, j the involution exchanging the two components; $\mathfrak{D}$, a (split) quaternion algebra, j standard. In the first and third cases the center of $\mathfrak{A} = \mathfrak{D}_m$ is Φ and in the second it is not. Now we have shown that a sufficient condition that $\mathfrak{S}(\mathfrak{A}, J) = [\mathfrak{H}(\mathfrak{A}, J), \mathfrak{H}(\mathfrak{A}, J)]$ for $(\mathfrak{A}, J) = (\mathfrak{D}_m, J_1)$, $m \geqq 3$, is that $\mathfrak{S}(\mathfrak{D}, j) = [\mathfrak{H}(\mathfrak{D}, j), \mathfrak{H}(\mathfrak{D}, j)] + [\mathfrak{S}(\mathfrak{D}, j), \mathfrak{S}(\mathfrak{D}, j)]$ (p. 148). If $\mathfrak{D} = \Phi$ then $\mathfrak{S}(\mathfrak{D}, j) = 0$ so this is clear. If $\mathfrak{D}$ is quaternion and j is standard one checks directly that $\mathfrak{S}(\mathfrak{D}, j) = [\mathfrak{S}(\mathfrak{D}, j), \mathfrak{S}(\mathfrak{D}, j)]$. Hence we see that the derivations are inner if the center of $\mathfrak{A}$ is Φ. Now assume that the center is not Φ, which is equivalent to assuming $\mathfrak{D} = \Phi \oplus \Phi$, j exchanging the two factors. In this case $\mathfrak{J} \cong \Phi_m{}^+$ $(m \geqq 3)$. By Corollary 4 to Theorem 3.6 (p. 144), any derivation of $\Phi_m{}^+$ is a derivation of Φ_m. Hence this has the form $x \to [xd]$. On the other hand, the inner derivations of $\Phi_m{}^+$ have the form $x \to [xf]$ where $f = \sum [a_i b_i]$. Hence all derivations will be inner if and only if every $d \in \Phi_m$ can be written in the form $\delta 1 + \sum [a_i b_i]$, $\delta \in \Phi$, $a_i, b_i \in \Phi_m$. If the characteristic is 0 or a prime p not dividing m then $d = (1/m)t(d)1 + d_0$ where $t(d)$ is the trace of the matrix d and $d_0 = d - (1/m)t(d)1$ has trace 0. Also a direct calculation with matrix units shows that every matrix of trace 0 is a sum of commutators (for any characteristic and any m). Hence if the characteristic is 0 or a prime not dividing m then any $d \in \Phi_m$ has the form $\delta 1 + \sum [a_i b_i]$ and so every derivation of $\Phi_m{}^+$ is inner in this case. On the other hand, suppose

(2) The proof can be reduced to the central simple case by the argument used at the beginning of this section. A proof in the central simple case is given in the author's *Structure of Rings*, p. 151.

the characteristic is a prime $p \mid m$. Then any element of the form $\delta 1 + \sum [a_i b_i]$ has trace 0. Hence if we choose d so that $t(d) \neq 0$ then $x \to [xd]$ is a derivation of $\Phi_m{}^+$ which is not inner. This completes the proof of the second statement of the theorem in the algebraically closed case. Now assume Φ is arbitrary and let $\mathfrak{J}$, $(\mathfrak{A}, J)$ be as before and let Ω be the algebraic closure of Φ. Let Inder $\mathfrak{J}$ denote the space of inner derivations of $\mathfrak{J}$. We have $\text{Der}\,\mathfrak{J}_\Omega = (\text{Der}\,\mathfrak{J})_\Omega$, $\text{Inder}\,\mathfrak{J}_\Omega = (\text{Inder}\,\mathfrak{J})_\Omega$. Hence every derivation of $\mathfrak{J}$ is inner if and only if this is the case for $\mathfrak{J}_\Omega$. Next we note that $\mathfrak{J}_\Omega = \mathfrak{H}(\mathfrak{A}_\Omega, J)$ and the center of $\mathfrak{A}_\Omega$ is Ω if and only if the center of $\mathfrak{A}$ is Φ. It is now clear that the result for $\mathfrak{J}$ follows from that for $\mathfrak{J}_\Omega$ and the proof is complete.

We now suppose that $\mathfrak{J}$ is an arbitrary finite-dimensional Jordan algebra with 1 over an infinite field Φ. Let $\mathfrak{M}_0(\mathfrak{J})$ and $\mathfrak{M}(\mathfrak{J})$ be the Lie algebras of linear transformations in $\mathfrak{J}$ which have the generic norm n as Lie invariant and Lie semi-invariant respectively (§2). The defining condition that $T \in \mathfrak{M}_0(\mathfrak{J})$ and $T \in \mathfrak{M}(\mathfrak{J})$ are respectively: $\Delta^{aT}{}_a\, n = 0$ and $\Delta^{aT}{}_a\, n = \rho(T) n(a), \rho(T) \in \Phi$. It is enough to have these conditions for $a = c$ invertible. In this case we have equation (68): $\Delta^a{}_c\, n = n(c) t(c^{\cdot -1}, a)$ which gives the conditions: $t(cT, c^{\cdot -1}) = 0$ for $T \in \mathfrak{M}_0(\mathfrak{J})$ and $t(cT, c^{\cdot -1}) = \rho(T)$ for $T \in \mathfrak{M}(\mathfrak{J})$. We have $\mathfrak{M}(\mathfrak{J}) \supseteq \mathfrak{M}_0(\mathfrak{J}) \supseteq \text{Der}\,\mathfrak{J}$ where the last inclusion is a special case of (vii) of Theorem 1. Let $\mathfrak{J}'$ denote the subspace of $\mathfrak{J}$ of elements a such that $t(a) = 0$ and let $R(\mathfrak{J})$ and $R(\mathfrak{J}')$ be the subspaces of $\text{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ of elements R_a, $a \in \mathfrak{J}$ and R_a, $a \in \mathfrak{J}'$, respectively. We have $t(cR_a, c^{\cdot -1}) = t(c\,.\,a, c^{\cdot -1}) = t(a, c\,.\,c^{\cdot -1}) = t(a, 1) = t(a)$. Hence $R_a \in \mathfrak{M}(\mathfrak{J})$ and $R_a \in \mathfrak{M}_0(\mathfrak{J})$ if and only if $t(a) = 0$. Hence $\mathfrak{M}(\mathfrak{J}) \supseteq R(\mathfrak{J})$ and $\mathfrak{M}_0(\mathfrak{J}) \supseteq R(\mathfrak{J}')$. If D is a derivation then $1D = 0$. Since $1R_a = a$ we see that $\text{Der}\,\mathfrak{J} \cap R(\mathfrak{J}) = 0$.

We now assume $\mathfrak{J}$ separable and we prove first

THEOREM 10. *Let $\mathfrak{J}$ be a finite-dimensional separable Jordan algebra over an infinite field Φ. Then a $D \in \text{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ is a derivation if and only if $D \in \mathfrak{M}(\mathfrak{J})$ and $1D = 0$.*

PROOF. We have seen that the conditions are necessary. Now assume $D \in \mathfrak{M}(\mathfrak{J})$ and $1D = 0$. Since $t(cD, c^{\cdot -1}) = \rho(D)$ for invertible c, if we put $c = 1$ we obtain $\rho(D) = 0$. Hence $D \in \mathfrak{M}_0(\mathfrak{J})$. We have $t(cD, c^{\cdot -1}) = t(cD\,.\,c^{\cdot -1}) = 0$. As usual, let x denote the identity mapping $a \to a$, xD its resultant with $D (= D \circ x)$ and $x^{\cdot -1}$ the mapping $c \to c^{\cdot -1}$, c invertible. Then we have $\Delta^a{}_c(xD\,.\,x^{-1}) = cD\,.\,\Delta^a{}_c x^{\cdot -1} + (\Delta^a{}_c xD)\,.\,c^{\cdot -1} = -\,cD\,.\,aU_c{}^{-1} + aD\,.\,c^{\cdot -1}$, by (70). Hence $-t(cD, aU_c{}^{-1}) + t(aD, c^{\cdot -1}) = t(\Delta^a{}_c(xD\,.\,x^{\cdot -1}))$. Now apply the chain rule to the composite function $t \circ (xD\,.\,x^{\cdot -1})$. This gives $\Delta^a{}_c(t \circ (xD\,.\,x^{-1})) = \Delta^b{}_d t$, $b = \Delta^a{}_c(xD\,.\,x^{-1})$, $d = cD\,.\,c^{\cdot -1}$, so $\Delta^a{}_c(t \circ (xD\,.\,x^{\cdot -1})) = t(b) = t(\Delta^a{}_c xD\,.\,x^{\cdot -1})$. On the other hand, $t(cD\,.\,c^{\cdot -1}) = 0$ for all invertible c. Hence $t(\Delta^a{}_c xD\,.\,x^{\cdot -1}) = 0$ and so the above relation gives

$$t(aD, c^{\cdot -1}) = t(cD, aU_c{}^{-1}) \tag{86}$$

for all a and invertible c. Since the generic trace form t is nondegenerate and U_c^{-1} is selfadjoint relative to t, this implies

$$cD = c^{\cdot -1}D^*U_c \tag{87}$$

where D^* is the adjoint of D relative to t.

Next we calculate

$$\begin{aligned}\Delta^1{}_c(U_x \circ D^* \circ x^{\cdot -1}) &= \Delta^1{}_c\{x(x^{\cdot -1}D^*)x\}\\ &= \{c(\Delta^1{}_c x^{\cdot -1}D^*)c\} + 2(c^{\cdot -1}D^*).c\\ &= -1U_c^{-1}D^*U_c + 2(c^{\cdot -1}D^*).c \qquad \text{(by (70))}.\end{aligned}$$

Also $U_x \circ D^* \circ x^{\cdot -1} = xD$ by (87) and $\Delta^1{}_c(xD) = 1D = 0$. Hence we have

$$2c^{\cdot -1}D^*.c = c^{-\cdot 2}D^*U_c. \tag{88}$$

Now consider

$$\begin{aligned}c^{\cdot 2}D - 2c.cD &= c^{\cdot -2}D^*U_c{}^2 - 2c.(c^{\cdot -1}D^*U_c)\\ &= (c^{\cdot -2}D^*U_c - 2c.c^{-1}D^*)U_c\\ &= 0, \quad \text{by (88)}.\end{aligned}$$

Hence $c^{2}D = 2c.cD$ and D is a derivation.

We can now prove

THEOREM 11. *If $\mathfrak{J}$ is finite dimensional separable over an infinite field then* $\mathfrak{M}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \mathrm{Der}\,\mathfrak{J}$ *and* $\mathfrak{M}_0(\mathfrak{J}) = R(\mathfrak{J}') \oplus \mathrm{Der}\,\mathfrak{J}$.

PROOF. Let $T \in \mathfrak{M}(\mathfrak{J})$. Then $D = T - R_{1T} \in \mathfrak{M}(\mathfrak{J})$ and $1D = 0$. Hence D is a derivation, by Theorem 10. Hence $\mathfrak{M}(\mathfrak{J}) = R(\mathfrak{J}) + \mathrm{Der}\,\mathfrak{J}$. Also we have seen that $R(\mathfrak{J}) \cap \mathrm{Der}\,\mathfrak{J} = 0$. Hence $\mathfrak{M}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \mathrm{Der}\,\mathfrak{J}$. Next let $T \in \mathfrak{M}_0(\mathfrak{J})$. Then $t(cT, c^{\cdot -1}) = 0$ gives for $c = 1$ that $t(1T) = 0$. Hence $1T \in \mathfrak{J}'$ and $R_{1T} \in R(\mathfrak{J}')$. As before, $T - R_{1T} = D$ is a derivation. This implies that $\mathfrak{M}_0(\mathfrak{J}) = R(\mathfrak{J}') \oplus \mathrm{Der}\,\mathfrak{J}$.

If D is a derivation then (87) is valid for any invertible c. This gives $c^{\cdot -1}D = cD^*U_c{}^{-1}$. On the other hand, since D is a derivation $c^{\cdot -1}D = -cDU_c{}^{-1}$. Hence $D^* = -D$ so every derivation is skew relative to the involution $A \to A^*$ in $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ where A^* is the adjoint of A relative to t. Also since t is an associative form, $R_a{}^* = R_a$. It follows from Theorem 11 that if $T \in \mathfrak{M}(\mathfrak{J})$ $(\mathfrak{M}_0(\mathfrak{J}))$ then $T^* \in \mathfrak{M}(\mathfrak{J})$ $(\mathfrak{M}_0(\mathfrak{J}))$ and $T \in \mathfrak{M}(\mathfrak{J})$ is a derivation if and only if $T^* = -T$. Also if $T \in \mathfrak{M}(\mathfrak{J})$ $(\mathfrak{M}_0(\mathfrak{J}))$ then $T = R_a$, $a \in \mathfrak{J}$ $(a \in \mathfrak{J}')$ if and only if $T^* = T$.

We now suppose that $\mathfrak{J}$ is special central simple of degree $m \geqq 3$ and we prove the following analogue (or infinitesimal form) of Theorem 8.

THEOREM 12. *Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $m \geqq 3$ over an infinite field, $(\mathfrak{A}, J)$ the corresponding finite-*

dimensional central simple associative algebra as in Theorem 9. Then any element of $\mathfrak{M}(\mathfrak{J})$ *has the form*

$$x \to a^J x + xa \tag{89}$$

where $a \in \mathfrak{A}$ *and conversely. Moreover, the condition that such an element is in* $\mathfrak{M}_0(\mathfrak{J})$ *is that* $t(a + a^J) = 0$.

PROOF. We have $\mathfrak{M}(\mathfrak{J}) = R(\mathfrak{J}) + \mathrm{Der}\,\mathfrak{J}$. Hence if $T \in \mathfrak{M}(\mathfrak{J})$, $T = R_c + D$ where $c \in \mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ and $D \in \mathrm{Der}\,\mathfrak{J}$. By Theorem 9, there exists a $d \in \mathfrak{S}(\mathfrak{A}, J)$ such that $xD = [xd]$. Hence $xT = \frac{1}{2}(cx + xc) + (xd - dx) = a^J x + xa$ where $a = \frac{1}{2}c + d$. The converse is clear since any $a \in \mathfrak{A}$ can be written as $a = \frac{1}{2}c + d$ where $c^J = c$ and $d^J = -d$. The condition that T of the form indicated is in $\mathfrak{M}_0(\mathfrak{J})$ is that $t(c) = 0$. Since $a = \frac{1}{2}c + d$, $a + a^J = c$, so this is equivalent to $t(a + a^J) = 0$.

If $\mathfrak{J}$ is any Jordan algebra then we have the basic identity $[[R_a R_b]R_c] = R_d$ where $d = [b, c, a]$. Now consider the set $\mathfrak{L}(\mathfrak{J})$ of linear transformation of the form $R_a + \sum [R_{b_j} R_{c_j}]$ where $a, b_j, c_j \in \mathfrak{J}$. Clearly, this is a subspace of $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$. Also, since $[R_a[R_b R_c]] = R_d$ and $[[R_b R_c], [R_e R_f]] = [[R_b[R_e R_f]]R_c] + [R_b[R_c[R_e R_f]]]$, by the Jacobi identity, it is clear that the set $\mathfrak{L}(\mathfrak{J})$ is a subalgebra of the Lie algebra $\mathrm{Hom}_\Phi(J, J)^-$. If $\mathfrak{J}$ is finite dimensional with 1 then $R_a \in \mathfrak{M}(\mathfrak{J})$ and so $\mathfrak{L}(\mathfrak{J}) \subseteq \mathfrak{M}(\mathfrak{J})$. Also $[[R_a R_b]R_c] = R_d$ where $d = [b, c, a]$ and $t(d) = 0$. This and the argument used for $\mathfrak{L}(\mathfrak{J})$ shows that the set $\mathfrak{L}_0(\mathfrak{J})$ of linear transformations of the form $R_a + \sum [R_{b_j} R_{c_j}]$, where $t(a) = 0$, is a subalgebra of the Lie algebra $\mathfrak{M}_0(\mathfrak{J})$. An immediate consequence of Theorem 9 and the decompositions $\mathfrak{M}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \mathrm{Der}\,\mathfrak{J}$, $\mathfrak{M}_0(\mathfrak{J}) = R(\mathfrak{J}') \oplus \mathrm{Der}\,\mathfrak{J}'$ for separable $\mathfrak{J}$ is the following

THEOREM 13. *Let* $\mathfrak{J}$ *be a finite-dimensional special central simple Jordan algebra of degree* $m \geqq 3$ *over an infinite field. Then* $\mathfrak{M}(\mathfrak{J}) = \mathfrak{L}(\mathfrak{J})$ *and* $\mathfrak{M}_0(\mathfrak{J}) = \mathfrak{L}_0(\mathfrak{J})$ *if and only if either the center of* $\mathfrak{A}$ *(the unital special universal envelope of* $\mathfrak{J}$*) is* Φ*, or the center properly contains* Φ *and the characteristic is either* 0 *or a prime* p *not dividing* m.

EXERCISES

1. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ be the Jordan algebra of a nondegenerate symmetric bilinear form on the vector space $\mathfrak{B}$ with $\dim \mathfrak{B} > 1$. Show that $\mathfrak{M}(\mathfrak{J}) = \mathfrak{L}(\mathfrak{J})$, $\mathfrak{M}_0(\mathfrak{J}) = \mathfrak{L}_0(\mathfrak{J})$ and that every derivation of $\mathfrak{J}$ is inner.
2. Show that $\mathfrak{M}_0(\mathfrak{J})$ acts irreduciblty on $\mathfrak{J}$ (that is, $\mathfrak{J}$ has no $\mathfrak{M}_0(\mathfrak{J})$-invariant subspace $\neq 0, \mathfrak{J}$) and that $\mathrm{Der}\,\mathfrak{J}$ acts irreducibly on $\mathfrak{B}$.

CHAPTER VII

REPRESENTATION THEORY FOR SEPARABLE JORDAN ALGEBRAS

The representation theory of a Jordan algebra is concerned with the study of the structure of the bimodules of $\mathfrak{J}$. Since any such bimodule is a right module for the universal multiplication envelope $U(\mathfrak{J})$ one can reduce the problem to the representation theory of the associative algebra $U(\mathfrak{J})$ provided one has enough information on $U(\mathfrak{J})$. Thus a central problem is the determination of the structure of $U(\mathfrak{J})$. In this chapter we consider this for finite-dimensional separable Jordan algebras $\mathfrak{J}$. One of the main results we shall obtain is that $\mathfrak{J}$ is separable if and only if $U(\mathfrak{J})$ is separable. Since a finite-dimensional associative algebra is semi-simple if and only if all its right modules are completely reducible, the separability of $U(\mathfrak{J})$ implies that every bimodule for $\mathfrak{J}$ is completely reducible. We shall obtain also the simple components of $U(\mathfrak{J})$, which is equivalent to obtaining the irreducible bimodules for $\mathfrak{J}$.

Our proof of the separability of $U(\mathfrak{J})$ for $\mathfrak{J}$ separable will rest heavily on the determination of the simple Jordan algebras over an algebraically closed field given in Chapter V. We note first that since $\mathfrak{J}$ has an identity element we have the decomposition $U(\mathfrak{J}) = \Phi z \oplus S_1(\mathfrak{J}) \oplus U_1(\mathfrak{J})$ where $z^2 = z \neq 0$, $S_1(\mathfrak{J})$ is the special unital universal envelope of $\mathfrak{J}$, and $U_1(\mathfrak{J})$ is the unital universal multiplication envelope. We can also reduce the discussion to the case $\mathfrak{J}$ simple and ultimately to the case $\mathfrak{J}$ (central) simple over an algebraically closed field. In the latter case we distinguish the cases: the degree, $\deg \mathfrak{J} = 1$, $\deg \mathfrak{J} = 2$ and $\deg \mathfrak{J} \geqq 3$. The first is trivial, since in this case $\dim \mathfrak{J} = 1$. The case $\deg \mathfrak{J} = 2$ amounts to the study of Clifford algebras and meson algebras (§2.3 and §2.13). If $\mathfrak{J}$ is finite-dimensional central simple of degree $n \geqq 3$ then $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D}, j)$ is a composition algebra. If $\mathfrak{D}$ is associative $S_1(\mathfrak{J}) \cong \mathfrak{D}_n$ since $(\mathfrak{D}_n, J_1)$ is a perfect algebra with involution. Also one can reduce the study of the bimodules of $\mathfrak{J}$ to that of the bimodules with involution for $(\mathfrak{D}, j)$. We determine the latter and prove complete reducibility for these. This gives the same information on $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$, $n \geqq 3$, and determines the structure of $U_1(\mathfrak{J})$.

The pattern of proof just indicated is used also to establish the other two main results of the present chapter: the radical splitting theorem (Albert-Penico-Taft theorem) and the theorem on derivations of separable algebras into bimodules (Jacobson-Harris theorem). The first of these asserts that if $\mathfrak{J}$ is a finite-dimensional

Jordan algebra with radical (= maximal solvable ideal) $\mathfrak{R}$ and $\mathfrak{J}/\mathfrak{R}$ is separable, then there exists a subalgebra $\mathfrak{S}$ of $\mathfrak{J}$ isomorphic to $\mathfrak{J}/\mathfrak{R}$. A special case of this is a "cohomology" result that any factor set of a separable Jordan algebra in a bimodule splits. The second result is that any derivation of a separable Jordan algebra into a bimodule is inner in a certain sense provided that the simple components have degrees over their centers prime to the characteristic. This result is applicable to proving a uniqueness theorem for the decomposition $\mathfrak{J} = \mathfrak{R} \oplus \mathfrak{S}$ where $\mathfrak{J}/\mathfrak{R}$ is separable and $\mathfrak{S} \cong \mathfrak{J}/\mathfrak{R}$. This result is proved only in the characteristic 0 case.

In the next chapter we shall prove all of the main theorems in the present one in the characteristic 0 case by Lie methods which do not require the detailed structure theory of semisimple Jordan algebras.

1. **Structure of Clifford algebras.** We recall that if $\mathfrak{J}$ is a Jordan algebra with identity element then the universal multiplication envelope $U(\mathfrak{J})$ is isomorphic to the direct sum of a one-dimensional algebra Φz $(z^2 = z)$, the universal unital multiplication envelope $U_1(\mathfrak{J})$, and the unital special universal envelope $S_1(\mathfrak{J})$ (§ 2.11). The problem of determining $U(\mathfrak{J})$ is therefore reduced to that of determining $U_1(\mathfrak{J})$ and $S_1(\mathfrak{J})$. Now let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ the Jordan algebra of the symmetric bilinear form f on the vector space $\mathfrak{B}$. Then we have seen that if $C(\mathfrak{B},f)$ is the Clifford algebra of $\mathfrak{B}$ relative to f then we can identify $S_1(\mathfrak{J})$ with $C(\mathfrak{B},f)$ and take the associative specialization of $\mathfrak{J}$ in $C(\mathfrak{B},f)$ to be given by $(\alpha 1 + x)^{\sigma_u} = \alpha 1 + x + \mathfrak{K}$ where $x \in \mathfrak{B}$ and $\mathfrak{K}$ is the ideal in the tensor algebra $T(\mathfrak{B})$ defining $C(\mathfrak{B},f)$ (§2.3). Thus $\mathfrak{K}$ is the ideal generated by all the elements $x \otimes x - f(x, x)1$, $x \in \mathfrak{B}$. We recall also that if $\dim \mathfrak{J} = n + 1 < \infty$ then $\dim S_1(\mathfrak{J}) \leqq 2^n$ (§ 2.2). Hence if $\dim \mathfrak{B} = n < \infty$ then $\dim C(\mathfrak{B},f) \leqq 2^n$.

If $\dim \mathfrak{B} = n < \infty$ and $(u_1, u_2, \cdots, u_n)$ is a basis for $\mathfrak{B}/\Phi$ then $\delta = \det(f(u_i, u_j))$, the determinant of the matrix $(f(u_i, u_j))$, is called a *discriminant* of the form f. f is nondegenerate if and only if $\delta \neq 0$. A change of basis replaces δ by $\delta\rho^2$ where ρ is a nonzero element of Φ.

We now proceed to determine the structure of the Clifford algebra $C(\mathfrak{B}, f)$. The key result for the method we shall use is the following

LEMMA 1. *Let $\mathfrak{N}$ be a $2r$-dimensional nonisotropic subspace of $\mathfrak{B}$. Then the subalgebra of $C(\mathfrak{B},f)$ generated by the image of $\mathfrak{N}$ and 1 is isomorphic to $C(\mathfrak{N},f)$ and $C(\mathfrak{B},f) \cong C(\mathfrak{N},f) \otimes C(\mathfrak{N}^\perp, (-1)^r\delta f)$ where δ is a discriminant of the restriction of f to $\mathfrak{N}$.*

PROOF. Let $(u_1, u_2, \cdots, u_{2r})$ be an orthogonal basis for $\mathfrak{N}/\Phi$ and put $v_i = u_i^{\sigma_u}$. We have $v_i v_j = -v_j v_i$, $i \neq j$, $v_i^2 = \beta_i 1$, $\beta_i = f(u_i, u_i) \neq 0$. Hence, if $c = v_1 v_2 \cdots v_{2r}$ then $v_i c = -c v_i$ and $c^2 = (-1)^r \delta 1$ where $\delta = \prod_1^{2r} \beta_i$ is a discriminant of the restriction of f to $\mathfrak{N}$. Since $\mathfrak{N}$ is finite-dimensional nonisotropic we have $\mathfrak{B} = \mathfrak{N} \oplus \mathfrak{N}^\perp$. Let $y \in \mathfrak{N}^\perp$ and put $z = y^{\sigma_u}$. Then $z v_i = -v_i z$, so $cz = zc$ and

$(zc)^2 = (-1)^r\delta f(y,y)$. Also $(zc)v_i = v_i(zc)$. Now consider the Clifford algebras $C(\mathfrak{N},f)$ and $C(\mathfrak{N}^\perp, (-1)^r\delta f)$ and denote the mappings of $\Phi 1 + \mathfrak{N}$ and $\Phi 1 + \mathfrak{N}^\perp$ into these algebras by σ_u' and σ_u'' respectively. We have the homomorphism of $C(\mathfrak{N}, f)$ into $C(\mathfrak{B},f)$ sending $(\alpha 1 + x)^{\sigma'_u} \to (\alpha 1 + x)^{\sigma_u}$; $x \in \mathfrak{N}$. Also if $y \in \mathfrak{N}^\perp$, $z = y^{\sigma_u}$, then the relation $(zc)^2 = (-1)^r\delta f(y,y)$ in $C(\mathfrak{B},f)$ implies that we have a homomorphism of $C(\mathfrak{N}^\perp, (-1)^r\delta f)$ into $C(\mathfrak{B},f)$ sending $(\alpha 1 + y)^{\sigma''_u} \to \alpha 1 + cz$. Since x^{σ_u}, $x \in \mathfrak{N}$, is a linear combination of the elements v_i it is clear that the elements of the image of $C(\mathfrak{N},f)$ under our homomorphism commute with the elements of the image of $C(\mathfrak{N}^\perp, (-1)^r\delta f)$ under the homomorphism of this algebra. Consequently, we have a homomorphism λ of $C(\mathfrak{N},f) \otimes C(\mathfrak{N}^\perp,(-1)^r\delta f)$ into $C(\mathfrak{B},f)$ such that

$$\lambda\colon x^{\sigma'_u} \otimes 1 + 1 \otimes y^{\sigma''_u} \to x^{\sigma_u} + cy^{\sigma_u}, \qquad x \in \mathfrak{N}, \qquad y \in \mathfrak{N}^\perp \tag{1}$$

and $1 \to 1$. We note next that we have the relations $u_i^{\sigma'_u}u_j^{\sigma'_u} = -u_j^{\sigma'_u}u_i^{\sigma'_u}$, $i \neq j, (u_i^{\sigma'_u})^2 = \beta_i 1$ in $C(\mathfrak{N},f)$ which imply that $c' = u_1^{\sigma'_u}u_2^{\sigma'_u}\cdots u_{2r}^{\sigma'_u}$ satisfies $(c')^2 = ((-1)^r\delta)1 \neq 0$. Hence c' is invertible in $C(\mathfrak{N},f)$. Also $c'x^{\sigma'_u} = -x^{\sigma'_u}c'$ if $x \in \mathfrak{N}$. Hence for $x \in \mathfrak{N}$, $y \in \mathfrak{N}^\perp$ we have

$$\begin{aligned}(x^{\sigma'_u}\otimes 1 + (c')^{-1}\otimes y^{\sigma''_u})^2 &= ((f(x,x) + (-1)^r\delta^{-1}(-1)^r\delta f(y,y)))1\\ &= (f(x,x) + f(y,y))1.\end{aligned} \tag{2}$$

Since $\mathfrak{B} = \mathfrak{N} \oplus \mathfrak{N}^\perp$ this and the universal property of $C(\mathfrak{B},f)$ implies that we have a homomorphism μ of $C(\mathfrak{B},f)$ into $C(\mathfrak{N},f) \otimes C(\mathfrak{N}^\perp, (-1)^r\delta f)$ sending $1 \to 1$ and

$$\mu\colon (x + y)^{\sigma_u} \to x^{\sigma'_u}\otimes 1 + (c')^{-1}\otimes y^{\sigma''_u}. \tag{3}$$

If follows from (1) and (3) and the fact that 1 and the elements $x^{\sigma_u} + y^{\sigma_u}$ generate $C(\mathfrak{B},f)$ while 1 and the elements $x^{\sigma'_u}\otimes 1 + 1 \otimes y^{\sigma''_u}$ generate $C(\mathfrak{N},f) \otimes C(\mathfrak{N}^\perp, (-1)^r\delta f)$ that λ and μ are inverses, so both are isomorphisms. This implies that the restriction of λ to $C(\mathfrak{N},f)$ is an isomorphism. Hence the the subalgebra of $C(\mathfrak{B},f)$ generated by 1 and $\mathfrak{N}^{\sigma_u}$ is isomorphic to $C(\mathfrak{N}, f)$ and $C(\mathfrak{B}, f) \cong C(\mathfrak{N},f) \otimes C(\mathfrak{N}^\perp, (-1)^r\delta f)$.

We can now prove our first main theorem on Clifford algebras.

THEOREM 1. *The associative specialization σ_u of $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ in $C(\mathfrak{B},f)$ is 1-1. (Equivalently, by Theorem 2.3, $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ is a special Jordan algebra.) If $\dim \mathfrak{B} = n < \infty$ then $\dim C(\mathfrak{B},f) = 2^n$.*

PROOF. Evidently if $\mathfrak{N}$ is a subspace of $\mathfrak{B}$ then $\mathfrak{K} = \Phi 1 + \mathfrak{N}$ is a subalgebra of $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ and $\mathfrak{K}$ is the Jordan algebra of the restriction of f to $\mathfrak{N}$. Hence the first statement for infinite-dimensional $\mathfrak{B}$ will follow from the result for finite-dimensional $\mathfrak{B}$ using Theorem 2.4. Hence we may assume $\mathfrak{B}$ is finite dimensional. We note next that the second statement of the theorem implies the first for finite-dimensional $\mathfrak{B}$. Let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{B}$. Then $(1, u_1, \cdots, u_n)$ is a basis for $\mathfrak{J}$ and we have seen in §2.2 that every element of $C(\mathfrak{B}, f) = S_1(\mathfrak{J})$ is a linear combination of the elements $v_1^{\varepsilon_1} v_2^{\varepsilon_2} \cdots v_n^{\varepsilon_n}$ where $v_i = u_i^{\sigma_u}$, $\varepsilon_i = 0, 1$ and $v_i^0 = 1$. Hence, if $\dim C(\mathfrak{B}, f) = 2^n$, then the 2^n elements $v_1^{\varepsilon_1} v_2^{\varepsilon_2} \cdots v_n^{\varepsilon_n}$ are linearly independent. À fortiori the elements $1, v_1, \cdots, v_n$ are linearly independent, which implies that the linear mapping σ_u is 1-1. As a final preliminary to the proof we obtain a reduction to the case of a nondegenerate f. For this, suppose $\mathfrak{B}^{\perp} \neq 0$ and write $\mathfrak{B} = \mathfrak{B}^{\perp} \oplus \mathfrak{N}$. Now form the vector space $\mathfrak{P} = \mathfrak{B} \oplus \mathfrak{B}'$ where $\dim \mathfrak{B}' = \dim \mathfrak{B}^{\perp}$ and extend f to a symmetric bilinear form on $\mathfrak{P}$ by choosing bases $(u_1, \cdots, u_r)$ for $\mathfrak{B}^{\perp}$, $(u_1', \cdots, u_r')$ for $\mathfrak{B}'$ and requiring that $f(u_i, u_j') = \delta_{ij}, f(u_i', u_j') = 0$, $f(u_i', y) = 0 = f(y, u_i')$ if $y \in \mathfrak{N}$. Then f is nondegenerate on $\mathfrak{P}$. Assuming the second statement holds for nondegenerate f we have that $\dim C(\mathfrak{P}, f) = 2^{n+r}$. It follows that if $(u_1, \cdots, u_n)$ is a basis for $\mathfrak{B}$ then $(u_1, \cdots, u_{n+r})$, where $u_{n+j} = u_j'$, is a basis for $\mathfrak{P}$ and the 2^{n+r} elements $v_1^{\varepsilon_1} v_2^{\varepsilon_2} \cdots v_{n+r}^{\varepsilon_{n+r}}$, $v_i = u_i^{\sigma_u}$, are linearly independent. Hence the elements $v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n}$ are linearly independent. This implies that $\dim C(\mathfrak{B}, f) = 2^n$. We now suppose that f is nondegenerate and that $\dim \mathfrak{B} = n < \infty$. We treat separately the cases $n = 1$ and $n = 2$. In the first case, $\mathfrak{B} = \Phi u$ where $f(u, u) = \beta \neq 0$. Then we form the algebra C' with basis $(1, v)$, 1 the identity element, and $v^2 = \beta 1$. This is associative. The linear mapping σ of $\mathfrak{B}$ into C' such that $u \rightarrow v$ satisfies $(x^{\sigma})^2 = f(x, x)1$. Hence we have a homomorphism of $C(\mathfrak{B}, f)$ onto C'. Since $\dim C' = 2$ we have $\dim C(\mathfrak{B}, f) \geqq 2$. Since we know that $\dim C(\mathfrak{B}, f) \leqq 2^{\dim \mathfrak{B}} = 2$ we have $\dim C(\mathfrak{B}, f) = 2$. Next suppose $\dim \mathfrak{B} = 2$ and let (u_1, u_2) be an orthogonal basis for $\mathfrak{B}$. Let $\beta_i = f(u_i, u_i) \neq 0$ and let C' be the quaternion algebra generated by v_1 and v_2 which satisfy $v_i^2 = \beta_i 1$, $v_1 v_2 = -v_2 v_1$. Let σ be the linear mapping of $\mathfrak{B}$ into C' such that $u_i \rightarrow v_i$, $i = 1, 2$. Then $(x^{\sigma})^2 = f(x, x)1$, $x \in \mathfrak{B}$, so we have a homomorphism of $C(\mathfrak{B}, f)$ onto C'. Then $\dim C(\mathfrak{B}, f) \geqq 2^2 = \dim C'$, and this implies that $\dim C(\mathfrak{B}, f) = 2^2$. We now assume $\dim \mathfrak{B} = n \geqq 3$ and we shall prove that $\dim C(\mathfrak{B}, f) = 2^n$ by induction on n. For this we decompose $\mathfrak{B}$ as $\mathfrak{B} = \mathfrak{N} \oplus \mathfrak{N}^{\perp}$ where $\mathfrak{N}$ is a two-dimensional nonisotropic subspace. By the induction assumption we have $\dim C(\mathfrak{N}, f) = 2^2$ and $\dim C(\mathfrak{N}^{\perp}, -\delta f) = 2^{n-2}$ if δ is a discriminant of f in $\mathfrak{N}$. By Lemma 1, $C(\mathfrak{B}, f) \cong C(\mathfrak{N}, f) \otimes C(\mathfrak{N}^{\perp}, -\delta f)$. Hence $\dim C(\mathfrak{B}, f) = 2^2 . 2^{n-2} = 2^n$, which completes the proof.

The first part of Theorem 1, that is, the fact that σ_u is 1-1 permits us to identify $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ with the corresponding subalgebra of $C(\mathfrak{B}, f)^+$. We shall do this from now on. In this way the associative specialization of $\mathfrak{J}$ in $C(\mathfrak{B}, f)$ is the identity mapping and the main involution π in $C(\mathfrak{B}, f)$ is characterized by the condition $x^{\pi} = x$, $x \in \mathfrak{B}$.

It is easy to see that $\mathfrak{R} = \mathfrak{B}^{\perp}$ generates a nil ideal $\mathfrak{N}$ in $C(\mathfrak{B}, f)$ and that $C(\mathfrak{B}, f)/\mathfrak{N} \cong C(\bar{\mathfrak{B}}, \bar{f})$ where $\bar{\mathfrak{B}} = \mathfrak{B}/\mathfrak{R}$ and $\bar{f}$ is the symmetric bilinear form on $\bar{\mathfrak{B}}$ induced by f on $\mathfrak{B}$. Since $\bar{f}$ is nondegenerate this reduces the study of the structure of $C(\mathfrak{B}, f)$ to the case of a nondegenerate form. We proceed now to determine the structure of $C(\mathfrak{V},f)$ for a nondegenerate f. For this purpose we shall require a special case of the following well-known

LEMMA 2. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are finite-dimensional central simple associative algebras then* $\mathfrak{A}\otimes\mathfrak{B}$ *is central simple.*

PROOF. Let Ω be the algebraic closure of the base field Φ. Since $\mathfrak{A}/\Phi$ is central simple, $\mathfrak{A}_\Omega$ is simple by Theorem 5.9. Hence, $\mathfrak{A}_\Omega\cong\Omega_n$. Similarly $\mathfrak{B}_\Omega\cong\Omega_m$. Hence $(\mathfrak{A}\otimes\mathfrak{B})_\Omega=\mathfrak{A}_\Omega\otimes\mathfrak{B}_\Omega\cong\Omega_n\otimes\Omega_m\cong\Omega_{mn}$. Since Ω_{nm} is simple this implies that $\mathfrak{A}\otimes\mathfrak{B}$ is simple. Since the center of Ω_{mn} is $\Omega 1$ and so is one dimensional over the base field Ω, the center of $\mathfrak{A}\otimes\mathfrak{B}$ is one dimensional over Φ and so it coincides with $\Phi 1$. Hence $\mathfrak{A}\otimes\mathfrak{B}$ is central.

We can now prove the second main theorem on Clifford algebras.

THEOREM 2. *Let* $C(\mathfrak{V},f)$ *be the Clifford algebra defined by a nondegenerate symmetric bilinear form* f *on* $\mathfrak{V}$. *If* $\dim\mathfrak{V}=n=2\nu$ *is even then* $C(\mathfrak{V},f)$ *is central simple and is isomorphic to a tensor product of quaternion algebras. If* $\dim\mathfrak{V}=n=2\nu+1$ *is odd then* $C(\mathfrak{V},f)$ *is isomorphic to a tensor product of a central simple algebra which is a tensor product of quaternion algebras and the center* $\mathfrak{C}$ *of* $C(\mathfrak{V},f)$. *Moreover,* $\mathfrak{C}$ *is a two-dimensional algebra* $\mathfrak{C}=\Phi[c]$ *where* $c^2=(-1)^\nu\delta$, δ *a discriminant of* f. *If* $\dim\mathfrak{V}$ *is infinite then* $C(\mathfrak{V},f)$ *is central simple.*

PROOF. Assume first that $n=2\nu$. If $\nu=1$ then the proof of Theorem 1 shows that $C(\mathfrak{V},f)$ is a quaternion algebra. Since any quaternion algebra is central simple we have the result in this case. We now use induction on ν and assume $\nu>1$. Let $\mathfrak{N}$ be a two-dimensional nonisotropic subspace of $\mathfrak{V}$. Then $C(\mathfrak{N},f)$ is a quaternion algebra and, by Lemma 1, $C(\mathfrak{V},f)\cong C(\mathfrak{N},f)\otimes C(\mathfrak{N}^\perp,-\delta f)$ where δ is a discriminant of the restriction of f to $\mathfrak{N}$. By the induction assumption $C(\mathfrak{N}^\perp,-\delta f)$ is isomorphic to a tensor product of quaternion algebras. Hence this is true also for $C(\mathfrak{V},f)$. Since quaternion algebras are central simple, the tensor product of a finite number of such algebras is central simple by Lemma 2. Next let $n=2\nu+1$ and let $\mathfrak{N}$ be a 2ν-dimensional nonisotropic subspace of $\mathfrak{V}$. Clearly $\mathfrak{N}^\perp$ is one dimensional, so $\mathfrak{N}^\perp=\Phi u$. If $f(u,u)=\beta$ and γ is a discriminant of the restriction of f to $\mathfrak{N}$ then $\delta=\beta\gamma$ is a discriminant of f. By Lemma 1, $C(\mathfrak{V},f)\cong C(\mathfrak{N},f)\otimes C(\mathfrak{N}^\perp,(-1)^\nu\gamma f)$ and, by the case $n=2\nu$, $C(\mathfrak{N},f)$ is isomorphic to a tensor product of quaternion algebras and is central simple. Now the proof of Theorem 1 shows that the Clifford algebra of $\mathfrak{N}^\perp=\Phi u$ relative to the form $(-1)^\nu\gamma f$ is isomorphic to an algebra $\Phi[c]$ with basis $(1,c)$ and $c^2=(-1)^\nu\gamma\beta=(-1)^\nu\delta$. On the other hand, since $C(\mathfrak{N},f)$ is central simple, it is easily seen that the center of $C(\mathfrak{N},f)\otimes C(\mathfrak{N}^\perp,(-1)^\nu\gamma f)$ is the commutative subalgebra $C(\mathfrak{N}^\perp,(-1)^\nu\gamma f)$ (identified with $1\otimes C(\mathfrak{N}^\perp,(-1)^\nu\gamma f)$. This proves the result for $n=2\nu+1$. Now assume $\dim\mathfrak{V}$ is infinite. We note that any finite-dimensional subspace

$\mathfrak{J}$ of $\mathfrak{B}$ can be imbedded in an even-dimensional nonisotropic subspace. We may assume that $\mathfrak{J} \cap \mathfrak{J}^\perp \neq 0$ and choose a nonzero z in this intersection. Then there exists a $w \in \mathfrak{B}$ such that $f(z,w) \neq 0$. It is immediate that $(\mathfrak{J} + \Phi w) \cap (\mathfrak{J} + \Phi w)^\perp \subset \mathfrak{J} \cap \mathfrak{J}^\perp$. A finite number of steps of this type yields a finite-dimensional nonisotropic subspace $\mathfrak{N} = \mathfrak{J} + \Phi w_1 + \Phi w_2 + \cdots + \Phi w_r$ containing $\mathfrak{J}$. If this is odd dimensional we can adjoin a nonisotropic vector in $\mathfrak{N}^\perp$ to obtain an even-dimensional nonisotropic space. If $\mathfrak{N}$ is even dimensional nonisotropic then $C(\mathfrak{N},f)$ is central simple. Hence the subalgebra of $C(\mathfrak{B},f)$ generated by 1 and $\mathfrak{N}$, which is a homomorphic image of $C(\mathfrak{N},f)$, is isomorphic to $C(\mathfrak{N},f)$ and so is central simple. It is now clear that any finite subset of $C(\mathfrak{B},f)$ can be imbedded in a central simple subalgebra containing 1. It is immediate that this implies that $C(\mathfrak{B},f)$ is central simple.

In the odd-dimensional case the center of $C(\mathfrak{B},f)$ has the form $\Phi[c]$ where the minimum equation of c is $c^2 = (-1)^\nu\delta$. If $(-1)^\nu\delta$ is a square of an element in Φ then $\Phi[c] \cong \Phi \oplus \Phi$ and consequently $C(\mathfrak{B},f)$ which is isomorphic to a tensor product of a central simple algebra and $\Phi \oplus \Phi$ is a direct sum of two isomorphic central simple algebras. If $(-1)^\nu\delta$ is not a square then $\Phi[c]$ is a quadratic field over Φ. In this case it is easy to see that $C(\mathfrak{B},f)$ is simple.

EXERCISES

1. Let $\dim \mathfrak{B} = 2\nu$ and let f be nondegenerate of maximal Witt index. Then $\mathfrak{B}$ has a basis $(u_1, u_2, \cdots, u_{2\nu})$ such that the matrix of f relative to this basis is $\operatorname{diag}\{1, -1, 1, -1, \cdots, 1, -1\}$. Show that $C(\mathfrak{B},f)$ is isomorphic to the tensor product of ν split quaternion algebras and hence that $C(\mathfrak{B},f) \cong \Phi_{2^\nu}$.

2. Let $\dim \mathfrak{B} = 2\nu + 1$ and let f be nondegenerate of maximal Witt index. Show that $C(\mathfrak{B},f) \cong \mathrm{P}_{2^\nu}$ where $\mathrm{P} = \Phi[c]$ has a basis $(1,c)$ with $c^2 = (-1)^\nu\delta$, δ a discriminant of f.

3. Let f be a symmetric bilinear form on $\mathfrak{B}$ which has the basis $(u_1, u_2, \cdots, u_{2\nu})$ such that the matrix of f relative to this basis is $\operatorname{diag}\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_\nu, \beta_\nu\}$. Let (α, β) denote the quaternion algebra with basis $(1, i, j, k)$ such that $i^2 = \alpha 1$, $j^2 = \beta 1$, $ij = -ji$. Show that $C(\mathfrak{B},f) \cong (\alpha_1, \beta_1) \otimes (\gamma_1\alpha_2, \gamma_1\beta_2) \otimes \cdots \otimes (\gamma_{\nu-1}\alpha_\nu, \gamma_{\nu-1}\beta_\nu)$ where $\gamma_i = -\alpha_i\beta_i$.

4. Let $\dim \mathfrak{B} = 2\nu$, f nondegenerate, π the main involution in $C(\mathfrak{B},f)$. Show that $\dim \mathfrak{H}(C(\mathfrak{B},f), \pi) = 2^{\nu-1}(2^\nu + 1)$ if $\nu \equiv 0, 1 \pmod 4$ and $\dim \mathfrak{H}(C(\mathfrak{B},f), \pi) = 2^{\nu-1}(2^\nu - 1)$ if $\nu \equiv 2, 3 \pmod 4$.

5. Let $\mathfrak{B}, f$ be as in 4. Assume, moreover, that f has maximal Witt index and $\nu \equiv 0, 1 \pmod 4$. Show that if we identify $C(\mathfrak{B},f)$ with Φ_{2^ν} using ex. 2, then π can be identified with the involution $X \to S^{-1}X^tS$ where $S = \operatorname{diag}\{1, -1, 1, -1, \cdots, 1, -1\}$.

2. **Structure of meson algebras.** The meson algebra $D(\mathfrak{B},f)$ defined by the vector space $\mathfrak{B}$ and the symmetric bilinear form f on $\mathfrak{B}$ is $D(\mathfrak{B},f) = T(\mathfrak{B})/\mathfrak{L}$

where $T(\mathfrak{B})$ is the tensor algebra based on $\mathfrak{B}$ and $\mathfrak{L}$ is the ideal in $T(\mathfrak{B})$ generated by the elements $x \otimes y \otimes x - f(x,y)x$, $x, y \in \mathfrak{B}$. We have seen in §2.13 that this can be taken to be the universal unital multiplication envelope $U_1(\mathfrak{J})$ for the Jordan algebra $\mathfrak{J} = \Phi \oplus \mathfrak{B}$ with the multiplication specialization ρ_u defined by $(\alpha 1 + x)^{\rho_u} = \alpha 1 + x + \mathfrak{L}$. We have seen also that the canonical homomorphism ξ of $U_1(\mathfrak{J})$ onto the squared special unital universal envelope $S_1''(\mathfrak{J})$ coincides with the homomorphism of $D(\mathfrak{B},f)$ into $C(\mathfrak{B},f) \otimes C(\mathfrak{B},f)$ sending $x^{\rho_u} \rightarrow \frac{1}{2}(x \otimes 1 + 1 \otimes x)$, $x \in \mathfrak{J}$, ($\mathfrak{J}$ identified with the corresponding subset of the Clifford algebra). For the sake of simplicity we assume throughout this section that $\mathfrak{B}$ is finite dimensional.

THEOREM 3. *The canonical homomorphism of the meson algebra* $D(\mathfrak{B},f)$ *into* $C(\mathfrak{B},f) \otimes C(\mathfrak{B},f)$ *is a monomorphism. Moreover,* $\dim D(\mathfrak{B},f) = \binom{2n+1}{n}$ *if* $\dim \mathfrak{B} = n$.

PROOF. Let $(u_1, u_2, \cdots, u_n)$ be an orthogonal basis for $\mathfrak{B}/\Phi$. Then we know that every element of $D(\mathfrak{B},f) = U_1(\mathfrak{J})$ is a linear combination of the elements

$$v_{k_1}^2 v_{k_2}^2 \cdots v_{k_r}^2 v_{k_{r+1}} v_{k_{r+2}} \cdots v_{k_s} \tag{4}$$

where $v_i \equiv u_i^{\rho_u}$, the k_i are distinct and satisfy $k_1 < k_2 < \cdots < k_r$; $k_{r+1} < k_{r+3} < \cdots$; $k_{r+2} < k_{r+4} < \cdots$ (§2.9, p. 97 and p. 103). The image of the element (4) under the hommorphism of $D(\mathfrak{B},f)$ into $C(\mathfrak{B},f) \otimes C(\mathfrak{B},f)$ is

$$w_{k_1}^2 w_{k_2}^2 \cdots w_{k_r}^2 w_{k_{r+1}} w_{k_{r+2}} \cdots w_{k_s} \tag{5}$$

where $w_i = u_i'' = \frac{1}{2}(u_i \otimes 1 + 1 \otimes u_i)$. We have seen also that the number of sets of indices $(k_1, \cdots, k_s)$ satisfying the indicated conditions is $\binom{2n+1}{n}$ (p. 99). It is clear from this that both assertions of the theorem will follow if we can show that the elements given by (5) associated with the set of indices $(k_1, \cdots, k_s)$ satisfying our conditions are linearly independent (and consequently distinct). In $C(\mathfrak{B},f)$ we have the relations $u_i u_j = -u_j u_i$, $i \neq j$, $u_i^2 = \beta_i 1$, $\beta_i = f(u_i, u_i)$. Also we have seen that the 2^n elements $u_{i_1} u_{i_2} \cdots u_{i_h}$, $i_1 < i_2 < \cdots < i_h$ form a basis for $C(\mathfrak{B},f)$. Here we allow $h = 0$ and agree that the corresponding element is 1. It follows that the 2^{2n} elements $u_{i_1} u_{i_2} \cdots u_{i_h} \otimes u_{j_1} u_{j_2} \cdots u_{j_k}$, $i_1 < i_2 < \cdots < i_h$, $j_1 < j_2 < \cdots < j_k$ form a basis for $C(\mathfrak{B},f) \otimes C(\mathfrak{B},f)$. We now examine the expression of the element (5) in terms of this basis. We note first that if no k_j in (5) has the value i then the expression of this element in terms of the basis $\{u_{i_1} \cdots u_{i_h} \otimes u_{j_1} \cdots u_{j_k}\}$ involves only those base elements for which every $i_p \neq i$ and every $j_q \neq i$. Next suppose $k_l = i$ for $r + 1 \leqq l \leqq s$. Then since $w_i = \frac{1}{2}(u_i \otimes 1 + 1 \otimes u_i)$ it is clear that the element (5) is a linear combination of the base elements $u_{i_1} \cdots u_{i_h} \otimes u_{j_1} \cdots u_{j_k}$ for which either one $i_j = i$ or one $j_q = i$ but not both. Finally, suppose $k_l = i$, $1 \leqq l \leqq r$. Then since $2w_i^2 = f(u_i, u_i)1 + u_i \otimes u_i$

it is clear that the expression of (5) in terms of our basis contains only base elements in which an $i_p = i$ and a $j_q = i$ and base elements in which no $i_p = i$ and no $j_q = i$. It follows from this that if we have a relation $A_i + B_i + C_i = 0$ where A_i is a linear combination of elements (5) in which one of the $k_l = i$ for $1 \leqq l \leqq r$, B_i a linear combination of elements (5) in which one of the $k_l = i$ for $r+1 \leqq l \leqq s$ and C_i is a linear combination of elements (5) in which no $k_l = i$ then $B_i = 0$ and $A_i + C_i = 0$. If $i = 1$ then $A_1 = w_1{}^2 A_1{}'$ where $A_1{}'$ is a linear combination of elements (5) in which no $k_l = 1$. Then it is clear from the form of $w_1{}^2$ that $A_1 + C_1 = w_1{}^2 A_1{}' + C_1 = 0$ implies that $A_1{}' = 0$ and $C_1 = 0$. This implies that if we have a nontrivial linear relation connecting the elements (5) then we have one in which either no term contains w_1 or every term contains w_1 but not $w_1{}^2$. By induction we see that we have a nontrivial relation in which for every i, $1 \leqq i \leqq n$, every term occurring in the relation is either of degree 0 or 1 in w_i. Thus if we have a nontrivial linear relation connecting the elements (5) then we have one of the form $\sum \alpha_{j_1 \ldots j_t} w_{j_1} w_{j_2} \cdots w_{j_t} = 0$ where the j's are a permutation of a subset of $\{1, \cdots, n\}$, which by reordering the basis, we may suppose is $\{1, \cdots, t\}$. Also, of course, we have $j_1 < j_3 < \cdots$, $j_2 < j_4 < \cdots$. We may assume also that t is minimal. Now the inequalities on the j's imply that w_1 occurs either in the first or second position in the terms $w_{j_1} w_{j_2} \cdots w_{j_t}$ occurring in our relation. The relation $xyx = f(x,y)x$ in $D(\mathfrak{V},f)$ and the orthogonality of the u_i imply that $w_i w_j w_i = 0$, if $i \neq j$, holds in $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$. Hence if we multiply our relation on the left by w_1, then after dropping the terms $w_1 w_j w_1 \cdots = 0$ we obtain a relation of the form $w_1{}^2 A = 0$ where A is a linear combination of $w_{j_2} w_{j_3} \cdots$ where $j_2, j_3, \cdots$ is a permutation of $2, 3, \cdots, t$ and the j's satisfy the usual inequalities. This implies that $A = 0$ which contradicts the minimality of t unless w_1 occurs in the second position in every term in our relation. Now suppose we already know that w_i, $1 \leqq i \leqq r$, $2r \leqq t$, occurs in the $2i$-position in all $w_{j_1} \cdots w_{j_t}$ having nonzero coefficients in our relation. Consider w_{r+1}. Either this occurs in the first or in the $2r+2$ position in the $w_{j_1} \cdots w_{j_t}$ in our relation. The latter possibility occurs only if $2r + 2 \leqq t$. Now linearization of the basic relation in $D(\mathfrak{V},f)$ gives $xyz + zyx = f(x,y)z + {}$ ${}+ f(y,z)x$. This and the orthogonality of the u_i imply that $w_i w_j w_k = -w_k w_j w_i$ if i, j, k are distinct. It follows from this and the relation $w_i w_j w_i = 0$ that if we multiply $w_{j_1} w_{j_2} \cdots w_{j_t}$ on the left by $w_{j_{2h}}$, $2h \leqq t$, then we obtain 0. This permits us to apply the argument used for w_1 to show that $2r + 2 \leqq t$ and w_{r+1} occurs in the $(2r+2)$-position in every $w_{j_1} w_{j_2} \cdots w_{j_u}$ in our relation. Since t is finite this argument leads to a contradiction. Hence we have proved the linear independence of the elements (5), which proves the theorem.

We shall determine next the structure of $D(\mathfrak{V},f)$ for $\mathfrak{V}$ finite dimensional and f nondegenerate, and we begin with the case $n = 2\nu$ is even. Then $C(\mathfrak{V},f)$ is central simple and hence also $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$ is central simple. Now we have the homomorphism of $C(\mathfrak{V},f)$ into $\mathfrak{E} = \text{Hom}(C(\mathfrak{V},f), C(\mathfrak{V},f))$ sending $1 \to 1$ and $x \to x_R$ for $x \in \mathfrak{V}$. Also, since we have an involution in $C(\mathfrak{V},f)$ sending $x \to x$ we have a homomorphism of $C(\mathfrak{V},f)$ into $\mathfrak{E}$ sending $1 \to 1$ and $x \to x_L$, $x \in \mathfrak{V}$. Since left and right multiplications commute, this gives a homomorphism of $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$ into $\mathfrak{E}$ mapping $1 \to 1$, $1 \otimes x \to x_R$, $x \otimes 1 \to x_L$, $x \in \mathfrak{V}$. Since $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$ is simple this is a monomorphism. Consequently, its restriction to $S_1''(\mathfrak{J})$, the subalgebra of $C(\mathfrak{V},f) \otimes C(\mathfrak{V},f)$ generated by 1 and the elements $\frac{1}{2}(x \otimes 1 + 1 \otimes x)$, $x \in \mathfrak{V}$, is a monomorphism. Combining this with the monomorphism given in Theorem 3 of $D(\mathfrak{V},f)$ onto $S_1''(\mathfrak{J})$, we obtain a monomorphism of $D(\mathfrak{V},f)$ into $\mathfrak{E}$ such that $1 \to 1$, $x^{\rho_u} \to \frac{1}{2}(x_L + x_R)$, $x \in \mathfrak{V}$. Thus to obtain the structure of $D(\mathfrak{V},f)$ it is equivalent to obtaining the structure of the subalgebra of $\mathfrak{E} = \mathrm{Hom}(C(\mathfrak{V},f), C(\mathfrak{V},f))$ generated by 1 and the elements $\frac{1}{2}(x_L + x_R)$, $x \in \mathfrak{V}$. Now it is clear that this algebra of linear transformations is the same as the enveloping algebra of linear transformations of $R_{C(\mathfrak{V},f)^+}(\mathfrak{J})$, the set of multiplications $R_a = \frac{1}{2}(a_L + a_R)$, $a \in \mathfrak{J}$, acting in the Jordan algebra $C(\mathfrak{V},f)^+$. We shall now obtain the structure of this algebra of linear transformations, hence of $D(\mathfrak{V},f)$, by determining certain subspaces of $C(\mathfrak{V},f)^+$ which are invariant under the R_x, $x \in \mathfrak{J}$.

For the moment we drop the assumptions that n is even and that f is nondegenerate. Let $x_i \in \mathfrak{V}$. Then we define $[x_1, \cdots, x_r]$ inductively by

$$
\begin{aligned}
[x_1] &= x_1, \\
[x_1, \cdots, x_{2k}] &= [[x_1, \cdots, x_{2k-1}], x_{2k}], \\
[x_1, \cdots, x_{2k+1}] &= 2[x_1, \cdots, x_{2k}] . x_{2k+1}.
\end{aligned} \tag{6}
$$

We now put $\mathfrak{V}^{[0]} = \Phi 1$, $\mathfrak{V}^{[r]}$, for $r > 0$, the space spanned by all the elements $[x_1, x_2, \cdots, x_r]$, $x_i \in \mathfrak{V}$.

LEMMA 1. $[x_1, \cdots, x_r]$ *is an alternating multilinear function of its arguments.*

PROOF. The multilinear character is clear, and this and the fact that the characteristic is not two imply that the alternating property is equivalent to: $[x_1, x_2, \cdots, x_r] = 0$ if $x_i = x_j$, $i \neq j$. For $r = 2$ this is clear since $[xx] = xx - xx = 0$. Now assume the result for r. To prove it for $r + 1$ it is enough, in view of the alternating character of $[x_1, \cdots, x_{r-1}]$, to prove that $[x_1, \cdots, x_{r-1}, x, x] = 0$. We put $[x_1, \cdots, x_{r-1}] = a$. Then we have the following two cases:

$$
\begin{aligned}
2[a . x, x] &= (ax + xa)x - x(ax + xa) \\
&= ax^2 - x^2 a = f(x,x)a - f(x,x)a = 0, \\
2[a, x] . x &= (ax - xa)x + x(ax - xa) \\
&= ax^2 - x^2 a = 0.
\end{aligned}
$$

This proves the alternating character of $[x_1, x_2, \cdots, x_{r+1}]$.

LEMMA 2. *If* $\dim \mathfrak{B} = n$ *then* $C(\mathfrak{B}, f) = \Phi 1 \oplus \mathfrak{B} \oplus \mathfrak{B}^{[2]} \oplus \cdots \oplus \mathfrak{B}^{[n]}$, $\dim \mathfrak{B}^{[i]} = \binom{n}{i}$ *and if* $(u_1, u_2, \cdots, u_n)$ *is an orthogonal basis for* $\mathfrak{B}$ *then the set* $\{[u_{i_1}, u_{i_2}, \cdots, u_{i_r}] \mid i_1 < i_2 < \cdots < i_r\}$ *is a basis for* $\mathfrak{B}^{[r]}$.

PROOF. Since $[x_1, x_2, \cdots, x_r]$ is multilinear and alternating it is clear that every element of $\mathfrak{B}^{[r]}$ is a linear combination of the elements $[u_{i_1}, u_{i_2}, \cdots, u_{i_r}]$, $i_1 < i_2 < \cdots < i_r$. It follows directly from the definition and induction on r that $[u_{i_1}, u_{i_2}, \cdots, u_{i_r}] = 2^{r-1} u_{i_1} u_{i_2} \cdots u_{i_r}$, $i_1 < i_2 < \cdots < i_r$. Since the set $\{u_{i_1} \cdots u_{i_r} \mid i_1 < i_2 < \cdots < i_r,\ r = 0, 1, \cdots, n\}$ is a basis for $C(\mathfrak{B}, f)$ all the statements of the lemma are clear.

LEMMA 3. *If* $y \in \mathfrak{B}^{[2k-1]}$ *and* $x \in \mathfrak{B}$ *then* $y.x \in \mathfrak{B}^{[2k-2]}$, $k = 1, 2, \cdots$.

PROOF. We have the following identity in any associative algebra:

$$x_1 \cdots x_{2k-1} . x = \sum_{h=0}^{2k-2} (-1)^h x_1 \cdots x_{2k-h-2} (x_{2k-h-1} . x) x_{2k-h} \cdots x_{2k-1} .$$

Since $u_i . x = f(u_i, x)1$ and the set of elements $\{u_{i_1} \cdots u_{i_r} \mid i_1 < i_2 < \cdots < i_r\}$ is a basis for $\mathfrak{B}^{[r]}$, this relation shows that $\mathfrak{B}^{[2k-1]} . \mathfrak{B} \subseteq \mathfrak{B}^{[2k-2]}$.

LEMMA 4. *If* $n = 2v$ *is even then*

$$\binom{2n+1}{n} = \sum_{k=0}^{v} \binom{n+1}{k}^2 = \sum_{k=0}^{v} \binom{n+1}{2k+1}^2 . \tag{7}$$

PROOF. For any n the relation $(1+x)^{2n+1} = (1+x)^n (1+x)^{n+1}$ gives the identity

$$\binom{2n+1}{n} = \sum_0^n \binom{n}{j} \binom{n+1}{n-j+1} \tag{8}$$

on comparing coefficients of x^{n+1}. Then if $n = 2v$

$$\begin{aligned} \sum_0^n \binom{n}{j}\binom{n+1}{n-j+1} &= 1 + \sum_1^v \binom{n+1}{n-j+1}\left(\binom{n}{j} + \binom{n}{n+1-j}\right) \\ &= 1 + \sum_1^v \binom{n+1}{j}\left(\binom{n}{j} + \binom{n}{j-1}\right) \\ &= 1 + \sum_1^v \binom{n+1}{j}^2 = \sum_0^v \binom{n+1}{j}^2 . \end{aligned}$$

Since $\binom{n+1}{2h} = \binom{n+1}{2(v-h)+1}$ this and (8) give

$$\binom{2n+1}{n} = \binom{n+1}{1}^2 + \binom{n+1}{3}^2 + \cdots + \binom{n+1}{n+1}^2$$

as required.

We are now ready to prove the following

THEOREM 4. *If* $\dim \mathfrak{B} = n = 2\nu$ *and* f *is nondegenerate then the meson algebra* $D(\mathfrak{B},f) \cong \sum_{k=0}^{\nu} \oplus \Phi_{\binom{n+1}{2k+1}}$.

PROOF. We have seen that $D(\mathfrak{B},f)$ is isomorphic to the algebra $\mathfrak{D}$ of linear transformations in $C(\mathfrak{B},f)^+$ generated by the elements R_a, $a \in \mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$. By Lemma 2, $C(\mathfrak{B},f) = \Phi 1 \oplus \mathfrak{B} \oplus \mathfrak{B}^{[2]} \oplus \cdots \oplus \mathfrak{B}^{[n]}$. By the definition (6) of $[x_1, x_2, \cdots, x_{2k+1}]$, the subspace $\mathfrak{B}^{[2k]}$ is mapped into $\mathfrak{B}^{[2k+1]}$ by R_x, $x \in \mathfrak{B}$, if $k < \nu$. By Lemma 1, $\mathfrak{B}^{[2\nu]} R_x = 0$. By Lemma 3, $\mathfrak{B}^{[2k-1]}$ is mapped into $\mathfrak{B}^{[2k-2]}$ if $x \in \mathfrak{B}$ and $k = 1, 2, \cdots$. It is clear from this that the subspaces $\Phi 1 + \mathfrak{B}$, $\mathfrak{B}^{[2]} + \mathfrak{B}^{[3]}, \cdots, \mathfrak{B}^{[2\nu-2]} + \mathfrak{B}^{[2\nu-1]}, \mathfrak{B}^{[2\nu]}$ are invariant under the R_a, $a \in \mathfrak{J}$. The dimensionalities of these spaces are respectively $\binom{n+1}{1}, \binom{n}{2} + \binom{n}{3} = \binom{n+1}{3}$, $\binom{n+1}{5}, \cdots, \binom{n+1}{2\nu+1}$. The dimensionality of the algebras of linear transformations in these spaces are therefore $\binom{n+1}{1}^2, \binom{n+1}{3}^2, \cdots, \binom{n+1}{2\nu+1}^2$. Hence we obtain the bound $\binom{n+1}{1}^2 + \binom{n+1}{3}^2 + \cdots + \binom{n+1}{2\nu+1}^2$ for $\dim \mathfrak{D}$. By Lemma 4 this is $\binom{2n+1}{n}$. On the other hand, by Theorem 3, $\dim \mathfrak{D} = \dim D(\mathfrak{B},f)$ $= \binom{2n+1}{n}$. Thus the bound is achieved. Clearly this implies that given any sequence $(A_0, A_1, \cdots, A_\nu)$ where A_i is a linear transformation in the ith space in the sequence $\Phi 1 + \mathfrak{B}, \mathfrak{B}^{[2]} + \mathfrak{B}^{[3]}, \cdots, \mathfrak{B}^{[2\nu]}$, then there exists a $D \in \mathfrak{D}$ such that the restrictions of D to the ith subspace is A_i. This implies that $\mathfrak{D} \cong \Phi_{\binom{n+1}{1}} \oplus \Phi_{\binom{n+1}{3}} \oplus \cdots \oplus \Phi_{\binom{n+1}{n+1}}$. Hence $D(\mathfrak{B},f)$ has the indicated structure.

We now assume $n = 2\nu - 1$, $\nu = 1, 2, \cdots$, f nondegenerate. In this case $C(\mathfrak{B},f)$ has the center $\Phi[c]$ where $c = u_1 u_2 \cdots u_n$, $(u_1, u_2, \cdots, u_n)$ an orthogonal basis for $\mathfrak{B}$. Hence $C(\mathfrak{B},f)$ is not central simple and so we can not identify $D(\mathfrak{B},f)$ with the algebra of linear transformations which we used in the foregoing case. To circumvent this difficulty we imbed $\mathfrak{B}$ in an $n+1 = 2\nu$ dimensional space $\mathfrak{W} = \mathfrak{B} \oplus \Phi u_{n+1}$ and extend f to f on $\mathfrak{W}$ so that $f(u_i, u_{n+1}) = 0 = f(u_{n+1}, u_i)$ for $i \leqq n$ and $f(u_{n+1}, u_{n+1}) = 1$. Then the 2^{n+1} elements $u_1^{\varepsilon_1} u_2^{\varepsilon_2} \cdots u_{n+1}^{\varepsilon_{n+1}}$, $\varepsilon_i = 0, 1$, form a basis for $C(\mathfrak{W},f)$ so it is clear that $C(\mathfrak{B},f)$ can be identified with the subalgebra of $C(\mathfrak{W},f)$ generated by 1 and the u_i, $i \leqq n$. Also we have $C(\mathfrak{W},f) = C(\mathfrak{B},f)$ $\oplus\, C(\mathfrak{B},f) u_{n+1}$. It is clear also that f is nondegenerate on $\mathfrak{W}$ and hence $C(\mathfrak{W},f)$ is central simple. The argument given in the even-dimensional case now implies that $D(\mathfrak{B},f)$ is isomorphic to the subalgebra of $\mathrm{Hom}_\Phi(C(\mathfrak{W},f), C(\mathfrak{W},f))$ generated by the R_a, $a \in \mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$, acting in $C(\mathfrak{W},f)$. Hence we consider the decomposition of $C(\mathfrak{W},f)$ into invariant subspaces relative to the R_a, $a \in \mathfrak{J}$.

If we define the spaces $\mathfrak{W}^{[i]}$ as before (for the $\mathfrak{V}^{[i]}$) then the result in the even-dimensional case implies that the spaces $\Phi 1 + \mathfrak{W}$, $\mathfrak{W}^{[2]} + \mathfrak{W}^{[3]}, \cdots, \mathfrak{W}^{[2\nu]}$ are invariant under R_a, $a \in \mathfrak{J}$. It is clear also that $C(\mathfrak{V},f)$ is invariant under R_a. Also $C(\mathfrak{V},f)u_{n+1}$ is invariant, since if $y \in C(\mathfrak{V},f)$ and $x \in \mathfrak{J}$, then $yu_{n+1} . x = \frac{1}{2}(xyu_{n+1} + yu_{n+1}x) = \frac{1}{2}(xyu_{n+1} - yxu_{n+1})$, by the orthogonality of x and u_{n+1}. Thus $yu_{n+1} . x \in C(\mathfrak{V},f)u_{n+1}$. Taking intersections we obtain the invariant subspaces $C(\mathfrak{V},f) \cap (\mathfrak{W}^{[2i]} + \mathfrak{W}^{[2i+1]})$ and $C(\mathfrak{V},f)u_{n+1} \cap (\mathfrak{W}^{[2i]} + \mathfrak{W}^{[2i+1]})$ where $\mathfrak{W}^{[0]} = \Phi 1$, $\mathfrak{W}^{[1]} = \mathfrak{W}$. Since $\{u_{i_1}u_{i_2}\cdots u_{i_r} \mid 1 \leqq i_1 < i_2 < \cdots < i_r \leqq n+1\}$ is a basis for $\mathfrak{W}^{[r]}$ and $\{u_{j_1}u_{j_2}\cdots u_{j_r} \mid 1 \leqq j_1 < j_2 < \cdots < j_k \leqq n,\ k = 1,2,\cdots,n\}$ is, a basis for $C(\mathfrak{V},f)$ it is clear that

$$
(9)\qquad
\begin{aligned}
C(\mathfrak{V},f) \cap (\mathfrak{W}^{[2i]} + \mathfrak{W}^{[2i+1]}) &= \mathfrak{V}^{[2i]} + \mathfrak{V}^{[2i+1]}, \\
C(\mathfrak{V},f)u_{n+1} \cap (\mathfrak{W}^{[2i]} + \mathfrak{W}^{[2i+1]}) &= \mathfrak{V}^{[2i-1]}u_{n+1} + \mathfrak{V}^{[2i]}u_{n+1}
\end{aligned}
$$

are invariant under R_a, $a \in \mathfrak{J}$. Also it is clear that $C(\mathfrak{W},f)$ is a direct sum of the spaces given in (9).

If $c = u_1u_2\cdots u_n$, as before, then c is an element of the center of $C(\mathfrak{V},f)$ and so the left multiplication $C = c_L$ in $C(\mathfrak{W},f)$ commutes with the a_L acting in $C(\mathfrak{W},f)$, $a \in \mathfrak{J}$. Clearly C commutes also with a_R, so $CR_a = R_aC$ holds in $\mathrm{Hom}_\Phi(C(\mathfrak{W},f), C(\mathfrak{W},f))$. Since $n = 2\nu - 1$, $c^2 = (-1)^{\nu-1}\delta 1$ where δ is a discriminant of f. Hence c is invertible and consequently C is invertible. It is clear from the basis for $\mathfrak{V}^{[r]}$ and the multiplication table for the u_i that C maps $\mathfrak{V}^{[r]}$, $0 \leqq r \leqq n$, onto $\mathfrak{V}^{[n-r]}$. Hence C maps $\mathfrak{V}^{[2i]} + \mathfrak{V}^{[2i+1]}$ onto $\mathfrak{V}^{[2(\nu-i-1)]} + \mathfrak{V}^{[2(\nu-i-1)+1]}$ and $\mathfrak{V}^{[2i-1]}u_{n+1} + \mathfrak{V}^{[2i]}u_{n+1}$ onto $\mathfrak{V}^{[2(\nu-i)-1]}u_{n+1} + \mathfrak{V}^{[2(\nu-i)]}u_{n+1}$. Since C commutes with the R_a, $a \in \mathfrak{J}$, the restriction of C to $\mathfrak{V}^{[2i]} + \mathfrak{V}^{[2i+1]}$ and to $\mathfrak{V}^{[2i-1]}u_{n+1} + \mathfrak{V}^{[2i]}u_{n+1}$ is a $\mathfrak{J}$-isomorphism (more precisely a $D(\mathfrak{V},f)$ module isomorphism). It follows that if ν is odd then $D(\mathfrak{V},f)$ is isomorphic to the associative algebra of linear transformations generated by the restrictions of the R_a, $a \in \mathfrak{J}$, to the space $\mathfrak{S} = \sum_{i=0}^{\nu} \mathfrak{V}^{[i]} + \sum_{j=0}^{\nu-1} \mathfrak{V}^{[j]}u_{n+1}$. If ν is even we may consider the restrictions of the R_a to the space $\mathfrak{S} = \sum_{j=0}^{\nu-1} \mathfrak{V}^{[j]} + \sum_{i=0}^{\nu} \mathfrak{V}^{[i]}u_{n+1}$. If ν is odd, C maps $\mathfrak{V}^{[\nu-1]}$ onto $\mathfrak{V}^{[\nu]}$ and if ν is even C maps $\mathfrak{V}^{[\nu-1]}u_{n+1}$ onto $\mathfrak{V}^{[\nu]}u_{n+1}$. These spaces have the same dimensionality $\binom{n}{\nu-1} = \binom{n}{\nu} = \frac{1}{2}\binom{n+1}{\nu}$.

Now suppose that $(-1)^{\nu-1}\delta = \rho^2$, $\rho \in \Phi$. We have $C^2 = (-1)^{\nu-1}\delta 1 = \rho^2 1$ and if ν is odd (even) then $\mathfrak{V}^{[\nu-1]} + \mathfrak{V}^{[\nu]}$ ($\mathfrak{V}^{[\nu-1]}u_{n+1} + \mathfrak{V}^{[\nu]}u_{n+1}$) is a direct sum of two spaces corresponding to the characteristic roots ρ and $-\rho$ of C and these have the same dimensionality $\frac{1}{2}\binom{n+1}{\nu}$. Moreover, these spaces are invariant under the R_a, $a \in \mathfrak{J}$. Besides these we have the invariant subspaces $\mathfrak{V}^{[0]} + \mathfrak{V}^{[1]}$, $\mathfrak{V}^{[2]} + \mathfrak{V}^{[3]}, \cdots, \mathfrak{V}^{[\nu-3]} + \mathfrak{V}^{[\nu-2]}$, $\mathfrak{V}^{[0]}u_{n+1}$, $\mathfrak{V}^{[1]}u_{n+1} + \mathfrak{V}^{[2]}u_{n+1}, \cdots, \mathfrak{V}^{[\nu-2]}u_{n+1} + \mathfrak{V}^{[\nu-1]}u_{n+1}$ ($\mathfrak{V}^{[0]} + \mathfrak{V}^{[1]}$, $\mathfrak{V}^{[2]} + \mathfrak{V}^{[3]}, \cdots, \mathfrak{V}^{[\nu-2]} + \mathfrak{V}^{[\nu-1]}$, $\mathfrak{V}^{[0]}u_{n+1}$,

$\mathfrak{V}^{[1]}u_{n+1} + \mathfrak{V}^{[2]}u_{n+1}, \cdots, \mathfrak{V}^{[\nu-3]}u_{n+1} + \mathfrak{V}^{[\nu-2]}u_{n+1})$ if ν is odd (even). The dimensionalities for the invariant subspaces we have listed are $\binom{n+1}{k}$, $k = 0, 1, \cdots, \nu - 1$, and $\frac{1}{2}\binom{n+1}{\nu}$ and $\frac{1}{2}\binom{n+1}{\nu}$. In this way we obtain the bound $\sum_{k=0}^{\nu-1}\binom{n+1}{k}^2 + \frac{1}{2}\binom{n+1}{\nu}^2$ for $\dim D(\mathfrak{V},f)$. On the other hand, we know that $\dim D(\mathfrak{V},f) = \binom{2n+1}{n}$, and we have the relation $\sum_{k=0}^{\nu-1}\binom{n+1}{k}^2 + \frac{1}{2}\binom{n+1}{\nu}^2 = \binom{2n+1}{n}$. For, by (8),

$$\begin{aligned}\binom{2n+1}{n} &= \sum_0^n \binom{n}{j}\binom{n+1}{n-j+1} \\ &= 1 + \sum_{k=1}^{\nu-1}\binom{n+1}{k}\left(\binom{n}{k} + \binom{n}{k-1}\right) + \frac{1}{2}\binom{n+1}{\nu}\left(\binom{n}{\nu} + \binom{n}{\nu}\right) \\ &= 1 + \sum_{k=1}^{\nu-1}\binom{n+1}{k}^2 + \frac{1}{2}\binom{n+1}{\nu}^2.\end{aligned}$$

Since the bound we obtained for $\dim D(\mathfrak{V},f)$ is attained it follows as in the even dimensional case that $D(\mathfrak{V},f) \cong \sum_{k=0}^{\nu-1} \oplus \Phi_{\binom{n+1}{k}} \oplus \Phi_{\frac{1}{2}\binom{n+1}{\nu}} \oplus \Phi_{\frac{1}{2}\binom{n+1}{\nu}}$.

Next assume $(-1)^{\nu-1}\delta$ is not a square in Φ. Then $P = \Phi[C]$ is a field. If ν is odd (even) then $\mathfrak{V}^{[\nu-1]} + \mathfrak{V}^{[\nu]}$ $(\mathfrak{V}^{[\nu-1]}u_{n+1} + \mathfrak{V}^{[\nu]}u_{n+1})$, which is mapped into itself by C, can be considered as a vector space over P. Since the dimensionality of this space over Φ is $\binom{n+1}{\nu}$ its dimensionality over P is $\frac{1}{2}\binom{n+1}{\nu}$. Then the dimensionality of the space of linear transformations of this space over P considered as a vector space over P is $\frac{1}{4}\binom{n+1}{\nu}^2$. Hence as a space over Φ the dimensionality is $\frac{1}{2}\binom{n+1}{\nu}^2$. Since the R_a, $a \in \mathfrak{J}$, commute with C it is clear that the restriction of R_a to $\mathfrak{V}^{[\nu-1]} + \mathfrak{V}^{[\nu]}$ or $\mathfrak{V}^{[\nu-1]}u_{n+1} + \mathfrak{V}^{[\nu]}u_{n+1}$ according as ν is odd or even is contained in the algebra of linear transformations of this space over P. If we take into account also the other invariant subspaces we obtain the bound $\sum_0^{\nu-1}\binom{n+1}{k}^2 + \frac{1}{2}\binom{n+1}{\nu}^2$ for the $\dim D(\mathfrak{V},f)$. Since this is attained we conclude as before that $D(\mathfrak{V},f) \cong \sum_0^{\nu-1} \oplus \Phi_{\binom{n+1}{k}} \oplus P_{\frac{1}{2}\binom{n+1}{\nu}}$.

The field P is isomorphic to the center of the Clifford algebra $C(\mathfrak{V},f)$ and it can be identified with this center. If $(-1)^{\nu-1}\delta$ is a square then the center P of the Clifford algebra is isomorphic to $\Phi \oplus \Phi$. Then $\Phi_{\frac{1}{2}\binom{n+1}{\nu}} \oplus \Phi_{\frac{1}{2}\binom{n+1}{\nu}} \cong P_{\frac{1}{2}\binom{n+1}{\nu}}$.

Hence we can state the results we have obtained in the following uniform way.

THEOREM 5. *If* $\dim\mathfrak{V} = n = 2\nu - 1$ *and* f *is nondegenerate then the meson algebra* $D(\mathfrak{V},f) \cong \sum_{k=0}^{\nu-1} \oplus \Phi_{\binom{n+1}{k}} \oplus \mathrm{P}_{\frac{1}{2}\binom{n+1}{\nu}}$ *where* P *is the center of the Clifford algebra.*

We remark finally that in the case $n = 2\nu$ we have $\binom{n+1}{2(\nu-h)+1} = \binom{n+1}{2h}$. Hence the structure of $D(\mathfrak{V},f)$ in this case as given in Theorem 4 can be described by $D(\mathfrak{V},f) \cong \sum_{j=0}^{\nu} \oplus \Phi_{\binom{n+1}{j}}$, which is similar to that given in Theorem 5.

EXERCISE

1. Let $\mathfrak{V}$ be an arbitrary vector space, f a symmetric bilinear form on $\mathfrak{V}$ and let $\bar{\mathfrak{V}} = \mathfrak{V}/\mathfrak{V}^{\perp}$. Let $\bar{f}$ denote the form on $\bar{\mathfrak{V}}$ determined by f ($\bar{f}(x+\mathfrak{V}^{\perp}, y+\mathfrak{V}^{\perp}) = f(x,y)$). Show that the elements z^{ρ_u}, $z \in \mathfrak{V}^{\perp}$ generate a nil ideal $\mathfrak{N}$ in $D(\mathfrak{V},f)$ and $D(\mathfrak{V},f)/\mathfrak{N} \cong D(\bar{\mathfrak{V}},\bar{f})$.

3. **Universal envelopes for finite-dimensional special central simple Jordan algebras of degree** $\geqq 3$. We recall that a Jordan algebra $\mathfrak{J}$ is finite-dimensional special central simple of degree $m \geqq 3$ if and only if $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite-dimensional central simple associative algebra with involution such that if Ω is the algebraic closure of the base field then $(\mathfrak{A}_{\Omega}, J) \cong (\mathfrak{D}_m, J_1)$ where $(\mathfrak{D},j)$ is a (split) associative composition algebra and J_1 is the standard involution (§5.7). We know also that if we identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{A}, J)$ then $\mathfrak{A}$ and the injection mapping constitute a unital special universal envelope for $\mathfrak{J}$. Then J is the main involution. We now consider the other basic universal envelopes $S_1''(\mathfrak{J})$ and $U_1(\mathfrak{J})$. For these we have the following

THEOREM 6. *If* $\mathfrak{J}$ *is a finite-dimensional central simple Jordan algebra of degree* $m \geqq 4$ *then* $\mathfrak{J}$ *is strongly special, that is, the canonical homomorphism of* $U_1(\mathfrak{J})$ *onto* $S_1''(\mathfrak{J})$ *is an isomorphism. Moreover, for any* $m \geqq 3$, $S_1''(\mathfrak{J})$ *coincides with the subalgebra* $\mathfrak{V}_1$ *of* $S_1(\mathfrak{J}) \otimes S_1(\mathfrak{J})$ *of fixed elements under the exchange automorphism.*

PROOF. It is clear that the hypotheses and conclusions are linear properties in the sense of §2.4. Hence it is enough to prove the assertions for an algebraically closed base field. Then we can take $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_m, J_1)$, $(\mathfrak{D},j)$ an associative composition algebra. If $m \geqq 4$, $\mathfrak{J}$ is strongly special by §3.6. If $\dim\mathfrak{D} = 1$ or 2 then $\mathfrak{D}$ is generated by 1 and a single element. Then $S_1''(\mathfrak{J}) = \mathfrak{V}_1$, by Theorem 3.9 (p. 148), if $m \geqq 3$. If $\dim\mathfrak{D} = 4$ so that $\mathfrak{D}$ is a quaternion algebra (the only other possibility for $(\mathfrak{D},j)$) then $\mathfrak{D}$ is generated by two elements. Then again Theorem 3.9 implies $S_1''(\mathfrak{J}) = \mathfrak{V}_1$ if $m \geqq 4$. The only case remaining is $m = 3$, $\mathfrak{D}$ quaternion. In this case Theorem 3.7 (p. 148) is applicable since it is easy to verify that $\mathfrak{S}(\mathfrak{D},j) = [\mathfrak{S}(\mathfrak{D},j), \mathfrak{S}(\mathfrak{D},j)]$.

We shall now determine the structure of $S_1''(\mathfrak{J})$. We distinguish the three cases: I. $S_1(\mathfrak{J})=\mathfrak{A}$ is central simple; II. $S_1(\mathfrak{J})=\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{B}^J$, $\mathfrak{B}$ an ideal. III. $S_1(\mathfrak{J})=\mathfrak{A}$ is simple with center P a quadratic extension of Φ, J an involution of second kind. In I, the dimensionality of $\mathfrak{A}$ is a square n^2 and in II and III it has the form $2n^2$. In all cases $n \geqq 3$ and we can identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{A}, J)$. Moreover, in II, $\mathfrak{H}(\mathfrak{A}, J) \cong \mathfrak{B}^+$ so we can also identify $\mathfrak{J}$ with $\mathfrak{B}^+$. By Theorem 6, $S_1''(\mathfrak{J}) = \mathfrak{B}_1$, the subalgebra of $\mathfrak{A} \otimes \mathfrak{A}$ of fixed elements under the exchange automorphism. Hence in I, $\dim S_1''(\mathfrak{J}) = n^2(n^2+1)/2$ and in II and III, $\dim S_1''(\mathfrak{J}) = n^2(2n^2+1)$. We remark also that if the base field is algebraically closed then I and II are the only possibilities. We prove first the following

THEOREM 7. *Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $m \geqq 3$, $\mathfrak{A} = S_1(\mathfrak{J})$, J the main involution in $\mathfrak{A}$.* (1) *If $\mathfrak{A}$ is central simple, so* $\dim \mathfrak{A} = n^2$, *then $S_1''(\mathfrak{J}) \cong \Phi_{n(n-1)/2} \oplus \Phi_{n(n+1)/2}$ and $\mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{S}(\mathfrak{A}, J)$ (the space of J-skew elements) are nonisomorphic irreducible unital bimodules for $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ under the action $x\,.\,h = \frac{1}{2}(xh + hx)$.* (2) *Let $\mathfrak{A} = \mathfrak{B}\oplus\mathfrak{B}^J$ and assume $\mathfrak{B}$ has an involution K. Then $S_1''(\mathfrak{J}) \cong \Phi_{n^2} \oplus \Phi_{n(n-1)/2} \oplus \Phi_{n(n-1)/2} \oplus \Phi_{n(n+1)/2} \oplus \Phi_{n(n+1)/2}$ and we have the following nonisomorphic irreducible unital bimodules for $\mathfrak{J} = \mathfrak{B}^+$:* (i) *$\mathfrak{B}$, with the action of $b \in \mathfrak{B}^+$ given by $x\,.\,b = \frac{1}{2}(xb + bx)$,* (ii) *$\mathfrak{H}(\mathfrak{B}, K)$, with the action of $b \in \mathfrak{B}^+$ given by $x\,.\,b = \frac{1}{2}(xb^K + bx)$,* (iii) *$\mathfrak{H}(\mathfrak{B}, K)$ with $x\,.\,b = \frac{1}{2}(xb + b^K x)$,* (iv) *$\mathfrak{S}(\mathfrak{B}, K)$ with $x\,.\,b = \frac{1}{2}(xb^K + bx)$,* (v) *$\mathfrak{S}(\mathfrak{B}, K)$ with $x\,.\,b = \frac{1}{2}(xb + b^K x)$.* (3) *Let $\mathfrak{A}$ be simple with center a quadratic field P, J of second kind and assume $\mathfrak{A}$/P has an involution K which commutes with J. Then $S_1''(\mathfrak{J}) \cong \Phi_{n^2} \oplus \mathrm{P}_{n(n-1)/2} \oplus \mathrm{P}_{n(n+1)/2}$ and we have the following nonisomorphic irreducible unital bimodules for $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$:* (i) *$\mathfrak{H}(\mathfrak{A}, J)$ with $x\,.\,h = \frac{1}{2}(xh + hx)$,* (ii) *$\mathfrak{H}(\mathfrak{A}, K)$ with $x\,.\,h = \frac{1}{2}(hx + xh^K)$* (iii) *$\mathfrak{S}(\mathfrak{A}, K)$ with $x\,.\,h = \frac{1}{2}(hx + xh^K)$.*

PROOF. (1) In this case $\mathfrak{A}$ is central simple and hence, as in the proofs of Theorem 4 and 5, we can identify $S_1''(\mathfrak{J})$ with the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ generated by the R_h, h in the subalgebra $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ of $\mathfrak{A}^+$. We have $\mathfrak{A} = \mathfrak{H}(\mathfrak{A}, J) \oplus \mathfrak{S}(\mathfrak{A}, J)$ and these two subspaces are invariant under the R_h, $h \in \mathfrak{J}$. If Ω is the algebraic closure of Φ then $(\mathfrak{A}_\Omega, J) \cong (\Omega_n, J_1)$ or $(\mathfrak{A}_\Omega, J) \cong (\mathfrak{Q}_{n/2}, J_1)$ where n is even and $\mathfrak{Q}$ is a quaternion algebra with standard involution. Accordingly, $\dim \mathfrak{H}(\mathfrak{A}, J) = n(n+1)/2$ and $\dim \mathfrak{S}(\mathfrak{A}, J) = \dim \mathfrak{A} - n(n+1)/2 = n^2 - n(n+1)/2 = n(n-1)/2$, or $\dim \mathfrak{H}(\mathfrak{A}, J) = n(n-1)/2$ and $\dim \mathfrak{S}(\mathfrak{A}, J) = n(n+1)/2$. In either case we obtain the bound $n^2(n+1)^2/4 + n^2(n-1)^2/4 = n^2(n^2+1)/2$ for $\dim S_1''(\mathfrak{J})$. On the other hand, Theorem 6 shows that $S_1''(\mathfrak{J}) = \mathfrak{B}_1$ and the latter has the dimensionality $n^2(n^2+1)/2$. The fact that the bound is attained implies, as in the proof of Theorem 4, that the algebras of linear transformations in $\mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{S}(\mathfrak{A}, J)$ generated by the restrictions of the R_h to these subspaces are the complete algebras $\mathrm{Hom}_\Phi(\mathfrak{H}, \mathfrak{H})$ and $\mathrm{Hom}_\Phi(\mathfrak{S}, \mathfrak{S})$.

Moreover, we have $S_1''(\mathfrak{J}) \cong \Phi_{n(n-1)/2} \oplus \Phi_{n(n+1)/2}$. It is clear that $\mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{S}(\mathfrak{A}, J)$ are irreducible relative to the R_h, $h \in \mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$, and these are not isomorphic.

(2) Now assume that $\mathfrak{J} = \mathfrak{B}^+$ where $\mathfrak{B}$ is finite-dimensional central simple of dimensionality n^2 and $\mathfrak{B}$ has the involution K. We now consider the 2×2 matrix algebra $\mathfrak{B}_2$ with matrix units e_{ij}, $i,j = 1,2$. This is central simple and we have the isomorphism $b \to h = be_{11} + b^K e_{22}$ of $\mathfrak{B}^+$ into $\mathfrak{B}_2{}^+$. Hence we can identify $S_1''(\mathfrak{J})$ with the algebra of linear transformations in $\mathfrak{B}_2$ generated by the R_h, $h = be_{11} + b^K e_{22}$. We now decompose $\mathfrak{B}_2$ as $\mathfrak{B}_2 = \mathfrak{B}e_{11} \oplus \mathfrak{B}e_{22} \oplus \mathfrak{H}(\mathfrak{B}, K)e_{12} \oplus \mathfrak{H}(\mathfrak{B}, K)e_{21} \oplus \mathfrak{S}(\mathfrak{B}, K)e_{12} \oplus \mathfrak{S}(\mathfrak{B}, K)e_{21}$. One checks that these are invariant under the R_h and hence are unital bimodules for $\mathfrak{J}$. Also it is clear that the bimodules $\mathfrak{B}e_{11}$ and $\mathfrak{B}e_{22}$ are isomorphic, so we may drop $\mathfrak{B}e_{22}$ and identify $S_1''(\mathfrak{J})$ with the subalgebra of the algebra of linear transformations in $\mathfrak{M} = \mathfrak{B}e_{11} + \mathfrak{B}e_{12} + \mathfrak{B}e_{21}$ generated by the R_h, $h = be_{11} + b^K e_{22}$. Using the five invariant subspaces $\mathfrak{B}e_{11}$, $\mathfrak{H}(\mathfrak{B}, K)e_{12}$, $\mathfrak{H}(\mathfrak{B}, \mathfrak{S})e_{21}$, $\mathfrak{S}((\mathfrak{B}, K)e_{12}$, $\mathfrak{S}(\mathfrak{B}, K)e_{21}$ we obtain the following bound for $\dim S_1''(\mathfrak{J})$: $n^4 + n^2(n-1)^2/4 + n^2(n+1)^2/4 + n^2(n-1)^2/4 + n^2(n+1)^2/4 = n^2(2n^2+1)$. Since $\dim S_1''(\mathfrak{J}) = \dim \mathfrak{B}_1' = n^2(2n^2+1)$ we can conclude that $S_1''(\mathfrak{J}) \cong \Phi_{n^2} \oplus \Phi_{n(n-1)/2} \oplus \Phi_{n(n-1)/2} \oplus \Phi_{n(n+1)/2} \oplus \Phi_{n(n+1)/2}$. Also it is clear that the five bimodules $\mathfrak{B}e_{11}, \cdots$ are irreducible. Moreover, no two of these are isomorphic since there exists an element in the algebra of linear transformations generated by the R_h which acts as the identity on one of the subspaces and annihilates the other. It is immediate also that the five subspaces are isomorphic respectively to the bimodules $\mathfrak{B}, \cdots$ specified in the statement of the theorem.

(3) Here $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{A}$ is simple with center a quadratic field P and J is an involution of the second kind. Also we are assuming that $\mathfrak{A}$/P has an involution K which commutes with J. Put $\theta = JK$. Then θ is an automorphism of period two in $\mathfrak{A}/\Phi$ whose restriction to P is also of period two. Let $G = \{1, \theta\}$ and form the crossed product $(\mathfrak{A}, G, 1)$. This is the set of elements of the form $a + bu$, a, $b \in \mathfrak{A}$ with the vector space composition as usual and multiplication given by $(a + bu)(c + du) = (ac + bd^\theta) + (ad + bc^\theta)u$. It is straightforward to check that $(\mathfrak{A}, G, 1)$ is a central simple associative algebra containing $\mathfrak{A}$ as a subalgebra. Hence $(\mathfrak{A}, G, 1)^+$ has the subalgebra $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ and $S_1''(\mathfrak{J})$ can be identified with the subalgebra of the algebra of linear transformations in $(\mathfrak{A}, G, 1)$ generated by the R_h, $h \in \mathfrak{H}(\mathfrak{A}, J)$. We now have the decomposition $(\mathfrak{A}, G, 1) = \mathfrak{H}(\mathfrak{A}, J) \oplus \mathfrak{S}(\mathfrak{A}, J) \oplus \mathfrak{H}(\mathfrak{A}, K)u \oplus \mathfrak{S}(\mathfrak{A}, K)u$. Clearly the first two are invariant under the R_h, $h \in \mathfrak{H}(\mathfrak{A}, J)$. Also if ξ is a J-skew element of P then $\mathfrak{S}(\mathfrak{A}, J) = \mathfrak{H}(\mathfrak{A}, J)\xi$ and $\mathfrak{S}(\mathfrak{A}, J)$ is isomorphic as $\mathfrak{J}$-bimodule to $\mathfrak{H}(\mathfrak{A}, J)$. If $x \in \mathfrak{H}(\mathfrak{A}, K)$ and $h \in \mathfrak{H}(\mathfrak{A}, J)$ then $xu \,.\, h = \frac{1}{2}(hx + xh^\theta)u = \frac{1}{2}(hx + xh^K)u \in \mathfrak{H}(\mathfrak{A}, K)u$. Hence $\mathfrak{H}(\mathfrak{A}, K)u$ is invariant under the R_h and it is clear that this is $\mathfrak{J}$-bimodule isomorphic to $\mathfrak{H}(\mathfrak{A}, K)$ with $x \,.\, h = \frac{1}{2}(hx + xh^K)$. Similarly $\mathfrak{S}(\mathfrak{A}, K)u$ is invariant under the R_h and is isomorphic to $\mathfrak{S}(\mathfrak{A}, K)$ under the action indicated. It is clear also that if $\mathfrak{M} = \mathfrak{H}(\mathfrak{A}, J) + \mathfrak{H}(\mathfrak{A}, K)u + \mathfrak{S}(\mathfrak{A}, K)u$

then $S_1''(\mathfrak{J})$ can be identified with the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ generated by the restrictions of the R_h, $h \in \mathfrak{J}$, to $\mathfrak{M}$. We note also that the spaces $\mathfrak{H}(\mathfrak{A}, K)u$, $\mathfrak{S}(\mathfrak{A}, K)u$ are P-spaces and the action of the R_h in these commutes with that of the $\rho \in \mathrm{P}$. It follows that we obtain the bound $n^4 + 2(\frac{1}{2}n(n-1))^2 + 2(\frac{1}{2}n(n+1))^2 = n^2(2n^2+1)$ for $\dim S_1''(\mathfrak{J})$. Since this is attained, we see as before that $S_1''(\mathfrak{J}) \cong \Phi_{n^2} \oplus \mathrm{P}_{n(n-1)/2} \oplus \mathrm{P}_{n(n+1)/2}$. The other assertion is clear also. This completes the proof.

We shall obtain next the structure of the algebra $S_1''(\mathfrak{J})$ without imposing the supplementary conditions of Theorem 7. For this purpose we recall some well-known results on simple associative algebras. First, we note that if $\mathfrak{A}$ is finite-dimensional simple over Φ then the center P of $\mathfrak{A}$ is a field and if $\dim \mathrm{P}/\Phi = r$ then $\dim \mathfrak{A}/\Phi = n^2 r$. If P is separable and Ω is the algebraic closure of Φ then $\mathfrak{A}_\Omega$ is isomorphic to a direct sum of r matrix algebras Ω_n (see, for example, Jacobson *Theory of Rings*, p. 115). It is an easy consequence of this and Theorem 6.4 that if $\mathfrak{A}$ and $\mathfrak{B}$ are finite-dimensional separable associative algebras then $\mathfrak{A} \otimes \mathfrak{B}$ is separable. Now assume that $\mathfrak{A}$ is finite-dimensional simple with center a Galois extension field P/Φ such that $\dim \mathrm{P}/\Phi = r$. It is known that $\mathrm{P} \otimes \mathrm{P} = \mathrm{P}^{(1)} \oplus \mathrm{P}^{(2)} \oplus \cdots \oplus \mathrm{P}^{(r)}$ where $\mathrm{P}^{(i)} \cong \mathrm{P}$ (Jacobson, *Theory of Rings*, p. 97). Since $\mathrm{P} \otimes \mathrm{P}$ is the center of $\mathfrak{A} \otimes \mathfrak{A}$ and $\mathfrak{A} \otimes \mathfrak{A}$ is separable it is clear that $\mathfrak{A} \otimes \mathfrak{A} = \mathfrak{A}^{(1)} \oplus \mathfrak{A}^{(2)} \oplus \cdots \oplus \mathfrak{A}^{(r)}$ where the center of $\mathfrak{A}^{(i)}$ is isomorphic to P. Then $\dim \mathfrak{A}^{(i)} = n_i^2 r$ and $\mathfrak{A}_\Omega^{(i)}$ is isomorphic to a direct sum of r matrix algebras Ω_{n_i}. On the other hand, $(\mathfrak{A} \otimes \mathfrak{A})_\Omega \cong \mathfrak{A}_\Omega \otimes \mathfrak{A}_\Omega$ is isomorphic to a direct sum of r^2 complete matrix algebras $\Omega_{n^2} \cong \Omega_n \otimes \Omega_n$. Since the decomposition of an algebra as a direct sum of simple ideals is unique we see that $n_i = n^2$. Thus $\mathfrak{A} \otimes \mathfrak{A}$ is isomorphic to a direct sum of r simple algebras having centers isomorphic to P and dimensionalities $n^4 r$. We shall require also the following

LEMMA 1. *Let $\mathfrak{A}$ be a central simple associative algebra of dimensionality $n^2 \geqq 9$ and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A} \otimes \mathfrak{A}$ of fixed points under the exchange automorphism. Then $\mathfrak{B}$ is a direct sum of two central simple algebras of dimensionalities $n^2(n-1)^2/4$ and $n^2(n+1)^2/4$ respectively.*

PROOF. Assume first that Φ is algebraically closed. Then $\mathfrak{A} = \Phi_n$, $n \geqq 3$, and we have the involution $X \to X^t$ in $\mathfrak{A}$. If $\mathfrak{J}$ is the Jordan algebra of symmetric elements under this involution then $\mathfrak{A}$ is a unital special universal envelope for $\mathfrak{J}$. Hence the first part of Theorem 7 shows that $\mathfrak{B} \cong \Phi_{n(n-1)/2} \oplus \Phi_{n(n+1)/2}$. Now let Φ be arbitrary and let Ω be its algebraic closure. Then $\mathfrak{A}_\Omega \cong \Omega_n$ and the subalgebra of $\mathfrak{A}_\Omega \otimes \mathfrak{A}_\Omega$ of fixed points under the exchange automorphism is isomorphic to $\Omega_{n(n-1)/2} \oplus \Omega_{n(n+1)/2}$. Hence $\mathfrak{B}_\Omega$ has this structure and consequently $\mathfrak{B}$ is separable. Now $\mathfrak{B}$ is not simple since this would imply that $\mathfrak{B}_\Omega$ is a direct sum of ideals of the same dimensionality. Since $\mathfrak{B}_\Omega \cong \Omega_{n(n-1)/2} \oplus \Omega_{n(n+1)/2}$ it now follows that $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ where $\mathfrak{B}_i$ is an ideal, and $\mathfrak{B}_{1\Omega} \cong \Omega_{n(n-1)/2}$ and $\mathfrak{B}_{2\Omega} \cong \Omega_{n(n+1)/2}$.

Then $\mathfrak{B}_1$ is central simple of dimensionality $n^2(n-1)^2/4$ and $\mathfrak{B}_2$ is central simple of dimensionality $n^2(n+1)^2/4$.

LEMMA 2. *Let η be an automorphism of period two in a finite-dimensional simple associative algebra $\mathfrak{A}$ with center* P *and let $\mathfrak{B}$ be the subalgebra of η-fixed points. If $\mathrm{P}_0 \equiv \mathfrak{B} \cap \mathrm{P} \subset \mathrm{P}$ then $\mathfrak{B}$ is simple with center P_0. If $\mathrm{P}_0 = \mathrm{P}$ then either $\mathfrak{B}$ is simple with center a quadratic extension field of* P, *or (assuming the characteristic is not two) $\mathfrak{B}$ is a direct sum of two simple ideals with centers isomorphic to* P.

PROOF. Assume first that $\mathrm{P}_0 \subset \mathrm{P}$. Since η maps P into itself and P_0 is the subfield of P of η-fixed points it follows by Galois theory that P is a separable quadratic extension of P_0. Since we may consider $\mathfrak{A}$ as an algebra over P_0 we may assume for the sake of simplicity of notation that $\mathrm{P}_0 = \Phi$ the base field. Let Ω be the algebraic closure of Φ. Then $\mathfrak{A}_\Omega \cong \Omega_n \oplus \Omega_n$ and the linear extension of η is an automorphism of period two in $\mathfrak{A}_\Omega$. It follows that η permutes the two simple components and consequently the algebra of η-fixed points of $\mathfrak{A}_\Omega$ is isomorphic to Ω_n. Then $\mathfrak{B}_\Omega \cong \Omega_n$ and $\mathfrak{B}$ is central simple over $\Phi = \mathrm{P}_0$. Next suppose $\mathrm{P}_0 = \mathrm{P}$. Then we may as well assume that $\mathrm{P} = \Phi 1$. By the Skolem-Noether theorem η is an inner automorphism so we have an element $c \in \mathfrak{A}$ such that $x^\eta = c^{-1}xc$, $x \in \mathfrak{A}$. Since $\eta^2 = 1$, $c^2 = \gamma \in \Phi 1$. Clearly $\mathfrak{B}$ is the centralizer of c in $\mathfrak{A}$. If γ is not a square in Φ then $\Phi[c]$ is a field and $\mathfrak{B}$ is simple with center $\Phi[c]$ (see, for example, Jacobson's *Theory of Rings*, p. 104). If c is a square then $\Phi[c] = \Phi[e]$ where e is an idempotent. We have the two sided Peirce decomposition $\mathfrak{A} = e\mathfrak{A}e \oplus e\mathfrak{A}(1-e) \oplus (1-e)\mathfrak{A}e \oplus (1-e)\mathfrak{A}(1-e)$. It is easy to see that $\mathfrak{B} = e\mathfrak{A}e \oplus (1-e)\mathfrak{A}(1-e)$, $e\mathfrak{A}e$ and $(1-e)\mathfrak{A}(1-e)$ are ideals in $\mathfrak{B}$. It is easy to see fom the Wedderburn theorem that these are central simple over $\Phi = \mathrm{P}$.

We are now ready to prove

THEOREM 8. *Let $\mathfrak{J}$ be a finite-dimensional special central simple Jordan algebra of degree $m \geq 3$, $\mathfrak{A} = S_1(\mathfrak{J})$, J the main involution in $\mathfrak{A}$. Then we have the following possibilities for the structure of $S_1''(\mathfrak{J})$:*

(1) *If $\mathfrak{A}$ is central simple, so $\dim \mathfrak{A} = n^2$, then $S_1''(\mathfrak{J}) \cong \Phi_{n(n-1)/2} \oplus \Phi_{n(n+1)/2}$.*

(2) *If $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is central simple and $\dim \mathfrak{B} = n^2$, then $S_1''(\mathfrak{J})$ is a direct sum of five central simple ideals, such that one of these is isomorphic to Φ_{n^2}, two are anti-isomorphic and have dimensionality $n^2(n-1)^2/4$ and two are anti-isomorphic and have dimensionality $n^2(n+1)^2/4$.*

(3) *If $\mathfrak{A}$ is simple with center a quadratic field P/Φ, $\dim \mathfrak{A} = 2n^2$, and J is of second kind, then $S_1''(\mathfrak{J})$ is a direct sum of three simple ideals such that one of these is isomorphic to Φ_{n^2} and the other two have centers isomorphic to* P *and dimensionalities $n^2(n-1)^2/2$ and $n^2/(n+1)^2/2$ respectively.*

PROOF. (1) This case has been settled in Theorem 7.

(2) We know that $S_1''(\mathfrak{J})$ coincides with the subalgebra $\mathfrak{B}_1$ of $\mathfrak{A} \otimes \mathfrak{A}$ of fixed points under the exhange automorphism ε. We have $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is central simple and $\mathfrak{A} \otimes \mathfrak{A} = (\mathfrak{B} \otimes \mathfrak{B}) \oplus (\mathfrak{B} \otimes \mathfrak{B}^J) \oplus (\mathfrak{B}^J \otimes \mathfrak{B}) \oplus (\mathfrak{B}^J \otimes \mathfrak{B}^J)$. The exchange automorphism ε maps the ideals $\mathfrak{B} \otimes \mathfrak{B}$ and $\mathfrak{B}^J \otimes \mathfrak{B}^J$ into themselves and interchanges the ideals $\mathfrak{B} \otimes \mathfrak{B}^J$ and $\mathfrak{B}^J \otimes \mathfrak{B}$. It follows that $\mathfrak{B}_1$ is a direct sum of the two ideals consisting of the fixed points under the exchange automorphism in $\mathfrak{B} \otimes \mathfrak{B}$ and $\mathfrak{B}^J \otimes \mathfrak{B}^J$ respectively and an ideal isomorphic to $\mathfrak{B} \otimes \mathfrak{B}^J$. By Lemma 1, the first of these is a direct sum of two ideals which are central simple algebras of dimensionalities $n^2(n-1)^2/4$ and $n^2(n+1)^2/4$ respectively. The same is true of the ideals obtained from $\mathfrak{B}^J \otimes \mathfrak{B}^J$. Moreover, since $\mathfrak{B}^J \otimes \mathfrak{B}^J$ is anti-isomorphic to $\mathfrak{B} \otimes \mathfrak{B}$ it is clear that we have the anti-isomorphism of the components asserted in the theorem. The last ideal, isomorphic to $\mathfrak{B} \otimes \mathfrak{B}^J$, is isomorphic to Φ_{n^2} since $\mathfrak{B} \otimes \mathfrak{B}^0 \cong \Phi_{n^2}$, by the theory of the Brauer group.

(3) Again $S_1''(\mathfrak{J}) = \mathfrak{B}_1$. If Ω is the algebraic closure of Φ then $S_1''(\mathfrak{J})_\Omega \cong S_1''(\mathfrak{J}_\Omega)$, and since $\mathfrak{J}_\Omega$ is as in (2), $S_1''(\mathfrak{J})_\Omega \cong \Omega_{n^2} \oplus \Omega_{n(n-1)/2} \oplus \Omega_{n(n-1)/2} \oplus \Omega_{n(n+1)/2} \oplus \Omega_{n(n+1)/2}$. Since $\mathfrak{A}$ is simple with center the (separable) quadratic field P we have $\mathfrak{A} \otimes \mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$ where $\mathfrak{A}_i$ is simple with center P_i isomorphic to P. If ε exchanges the two components $\mathfrak{A}_i$ then $\mathfrak{B}_1 \cong \mathfrak{A}_1 \cong \mathfrak{A}_2$ and $S_1''(\mathfrak{J})_\Omega \cong \mathfrak{B}_{1\Omega}$ is a direct sum of two simple ideals. This contradicts the fact that $S_1''(\mathfrak{J})_\Omega$ is a direct sum of five simple ideals. Hence ε maps each $\mathfrak{A}_i$ into itself and consequently it maps the center P_i of $\mathfrak{A}_i$ into itself. Clearly $\mathfrak{B}_1 = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ where $\mathfrak{B}_i$ is the set of ε-fixed points of $\mathfrak{A}_i$. If the restriction ε_i of ε to $\mathfrak{A}_i$ is the identity on P_i then either $\varepsilon_i = 1$ so $\mathfrak{B}_i = \mathfrak{A}_i$ or ε_i is of period two. In the latter case, by Lemma 2, either $\mathfrak{B}_i$ is simple with center a quadratic extension of P_i or $\mathfrak{B}_i$ is a direct sum of two simple ideals with centers isomorphic to P_i. These three possibilities for $\mathfrak{B}_i$ give rise to the following respective possibilities for $\mathfrak{B}_{i\Omega}$: (i) $\mathfrak{B}_{i\Omega}$ is a direct sum of two simple ideals of the same dimensionality, (ii) $\mathfrak{B}_{i\Omega}$ is a direct sum of four simple ideals of the same dimensionality, (iii) $\mathfrak{B}_{i\Omega}$ is a direct sum of four simple ideals having the same dimensionality by pairs. Next suppose ε_i is not the identity on P_i. Then, by Lemma 2, $\mathfrak{B}_i$ is central simple and we have (iv): $\mathfrak{B}_{i\Omega}$ is simple. If we take into account the possibilities (i)–(iv) and the fact that $S_1''(\mathfrak{J})_\Omega \cong \mathfrak{B}_{1\Omega} \oplus \mathfrak{B}_{2\Omega} \cong \Omega_{n^2} \oplus \Omega_{n(n-1)/2} \oplus \Omega_{n(n-1)/2} \oplus \Omega_{n(n+1)/2} \oplus \Omega_{n(n+1)/2}$ we see that one of the ideals, say $\mathfrak{B}_1$, is a direct sum of two simple ideals with centers isomorphic to P while $\mathfrak{B}_2$ is central simple. It is clear also that these ideals have dimensionalities $n^2(n-1)^2/2$ and $n^2(n+1)^2/2$. It remains to see that $\mathfrak{B}_2 \cong \Phi_{n^2}$. Now $\mathfrak{J}$ is special. Hence we have the canonical homomorphism of $S_1''(\mathfrak{J})$ onto the multiplication algebra $M(\mathfrak{J})$ sending $\frac{1}{2}(a \otimes 1 + 1 \otimes a) \to R_a$, $a \in \mathfrak{J}$. Also since $\mathfrak{J}$ is central simple of dimensionality n^2, $M(\mathfrak{J}) \cong \Phi_{n^2}$. Hence Φ_{n^2} is a homomorphic image of $S_1''(\mathfrak{J})$. This implies that one of the simple components of $S_1''(\mathfrak{J})$ is isomorphic to Φ_{n^2}. Clearly this must be $\mathfrak{B}_2$.

4. **Bimodules for composition algebras and reduced Jordan Algebras.** It is well

known that all modules for a finite-dimensional associative algebra are completely reducible if and only if the algebra is semisimple, that is, is a direct sum of simple ideals. Moreover, if $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_s$ where the $\mathfrak{A}_i$ are simple, then there are exactly s isomorphism classes of irreducible $\mathfrak{A}$-modules. If $\mathfrak{J}_j$ is a minimal right ideal of $\mathfrak{A}_j$ then $\{\mathfrak{J}_1, \mathfrak{J}_2, \cdots, \mathfrak{J}_s\}$ is a set of representatives of these isomorphism classes. If $\mathfrak{J}$ is a finite-dimensional Jordan algebra then we know that the bimodules for $\mathfrak{J}$ are just the right modules for the universal envelope $U(\mathfrak{J})$. The results of §§1–3 on the structure of $U(\mathfrak{J}) = S_1(\mathfrak{J}) \oplus U_1(\mathfrak{J}) \oplus \Phi z$ for $\mathfrak{J}$ finite-dimensional central simple show that if $\mathfrak{J}$ is not of degree three then $U(\mathfrak{J})$ is semisimple. Hence the associative theory shows that the bimodules for $\mathfrak{J}$ are completely reducible and gives a determination of the (representatives of the isomorphism classes of) irreducible bimodules for $\mathfrak{J}$. The gap in the degree three case is due to the fact that for some $\mathfrak{J}$ of this type (e.g. $\mathfrak{J}$ exceptional), $U_1(\mathfrak{J})$ and $S_1''(\mathfrak{J})$ are not isomorphic. We shall now give another method for dealing with unital bimodules for reduced simple Jordan algebras of degree $\geqq 3$ which will permit us to fill the gap we have indicated. This is based on the category isomorphism established in §3.5 of the category of unital bimodules for Jordan algebras of order $\geqq 3$ and that of certain bimodules for the coefficient algebra with involution. For reduced simple Jordan algebras of degree $\geqq 3$ this leads to the consideration of alternative bimodules with involution for composition algebras. We consider these first and we shall determine their structure by an inductive method which is due to McCrimmon in [4].

We recall that if $(\mathfrak{D}, j)$ is a composition algebra with involution j and $\dim \mathfrak{D} > 1$ then $\mathfrak{D}$ contains a composition subalgebra $\mathfrak{B}$ with $\dim \mathfrak{B} = \frac{1}{2} \dim \mathfrak{D}$ (p. 163). Moreover, $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{B}^{\perp}$ where $\mathfrak{B}^{\perp}$ is the orthogonal complement of $\mathfrak{B}$ relative to the symmetric bilinear form associated with the quadratic form Q such that $Q(x)1 = x\bar{x} = \bar{x}x$ (p. 163). We recall that $\mathfrak{D}$ is alternative and $\mathfrak{B}$ is associative and $\bar{v} = -v$, if $v \in \mathfrak{B}^{\perp}$. If q is any element of $\mathfrak{B}^{\perp}$ such that $Q(q) = \mu 1 \neq 0$ then $\mathfrak{B}^{\perp} = \mathfrak{B}q$ and we have the multiplication formula

$$(a + bq)(c + dq) = (ac + \mu \bar{d}b) + (da + b\bar{c})q \tag{10}$$

for $a, b, c, d \in \mathfrak{B}$. We note also that the construction of an algebra of octonions from a quaternion algebra (p. 16) can be duplicated for any associative composition algebra $(\mathfrak{B}, j)$. As in the special case, let μ be any nonzero element of Φ and let $\mathfrak{D}$ be the vector space $\mathfrak{B} \oplus \mathfrak{B}$ of pairs (a, b), $a, b \in \mathfrak{B}$ with multiplication defined, as in (10), by $(a,b)(c,d) = (ac + \mu \bar{d}b, da + b\bar{c})$. Also, define $(a,b)^j = (\bar{a}, -b)$. Then $(\mathfrak{D}, j)$ is a composition algebra containing $\mathfrak{B}$ as the subalgebra of elements $(a, 0)$ and $\mathfrak{B}^{\perp} = \{(0, a)\}$.

If $(\mathfrak{D}, j)$ is any composition algebra then the vector space $\mathfrak{D}$ considered in the usual way as a bimodule for $\mathfrak{D}$ together with the involution j constitute a unital alternative bimodule with involution for $(\mathfrak{D}, j)$. We shall call this the *regular bimodule with involution* for $(\mathfrak{D}, j)$ and denote it as reg $\mathfrak{D}$.

Next assume $\dim\mathfrak{D} > 1$ and let $(\mathfrak{B},j)$ be a composition subalgebra of $(\mathfrak{D},j)$ such that $\dim\mathfrak{B} = \frac{1}{2}\dim\mathfrak{D}$. Then it is clear that if we restrict the operators to $\mathfrak{B}$ then $\mathrm{reg}\,\mathfrak{D}$ can be considered as a bimodule with involution for $(\mathfrak{B},j)$. We have the decomposition $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{B}^{\perp}$ and the space $\mathfrak{B}$ is is a sub-bimodule for $(\mathfrak{B},j)$. Moreover, $\mathfrak{B}^{\perp} = \mathfrak{B}q$ where $q \in \mathfrak{B}^{\perp}$ and $Q(q) \neq 0$. By (10), we have $a(bq) = (ba)q$, $(bq)a = (b\bar{a})q$ for $a, b \in \mathfrak{B}$. Also $\overline{bq} = -bq$. Hence $\mathfrak{B}^{\perp}$ is also a sub-bimodule of reg $\mathfrak{D}$ considered as a bimodule for $(\mathfrak{B},j)$. It is clear that the $(\mathfrak{B},j)$-bimodule structure of $\mathfrak{B}^{\perp}$ is independent of the containing algebra. In fact, we can give a direct description of this bimodule as follows.

It is convenient to give this construction for an arbitrary composition algebra $(\mathfrak{D},j)$. Let $\mathfrak{D}$ be the underlying vector space and make $\mathfrak{D}$ into a bimodule with involution for $(\mathfrak{D},j)$ by defining the bimodule compositions $x \circ d$, $d \circ x$, $x \in \mathfrak{D}$, $d \in (\mathfrak{D},j)$ and $\bar{x}$ by

$$x \circ d = x\bar{d}, \quad d \circ x = xd, \quad \bar{x} = -x. \tag{11}$$

We shall call this bimodule with involution the *Cayley bimodule with involution*, $\mathrm{cay}\,\mathfrak{D}$, for $(\mathfrak{D},j)$. If $(\mathfrak{B},j)$ is an associative composition algebra then we can imbed $(\mathfrak{B},j)$ in a composition algebra $(\mathfrak{D},j)$ with $\dim\mathfrak{D} = 2\dim\mathfrak{B}$. If $q \in \mathfrak{B}^{\perp}$ satisfies $Q(q) \neq 0$ then we have seen that $\mathfrak{B}^{\perp} = \mathfrak{B}q$. It is immediate that $x \to xq$ is an isomorphism of $\mathrm{cay}\,\mathfrak{B}$ onto the sub-$(\mathfrak{B},j)$-bimodule $\mathfrak{B}^{\perp}$ of $\mathrm{reg}\,\mathfrak{D}$ considered as a bimodule with involution for $(\mathfrak{B},j)$. This implies that $\mathrm{cay}\,\mathfrak{B}$ is unital and alternative if $(\mathfrak{B},j)$ is an associative composition algebra. On the other hand, suppose $(\mathfrak{D},j)$ is not associative and let $a, b, c \in \mathfrak{D}$ satisfy $[a, b, c] \neq 0$. Consider the split null extension $\mathfrak{E} = \mathfrak{D} \oplus \mathrm{cay}\,\mathfrak{D}$ as algebra with involution. We write the elements of $\mathfrak{E}$ as (a,x), $a \in \mathfrak{D}$, $x \in \mathrm{cay}\,\mathfrak{D}$. Then $\overline{(a,x)} + (a,x) = (a + \bar{a}, 0) = \alpha(1,0)$ where $\alpha \in \Phi$. Hence if $u = (a,x)$ then $u + \bar{u}$ is in the nucleus of $\mathfrak{E}$ so the associator $[u,u,v] = 0$ if and only if $[\bar{u},u,v] = 0$. Now put $u = (\bar{c}, -a)$, $v = (\bar{b}, 0)$. Then a direct calculation shows that $[\bar{u},u,v] = (0, -[a,b,c]) \neq 0$. Hence $\mathrm{cay}\,\mathfrak{D}$ is not alternative. Thus $\mathrm{cay}\,\mathfrak{D}$ is alternative if and only if $\mathfrak{D}$ is associative.

Let $(\mathfrak{M},j)$ be a bimodule (with involution) for the algebra with involution $(\mathfrak{A},j)$ and put $j^* = -j$ where j is the operator in $\mathfrak{M}$. Then $x^{j^*} = -\bar{x}$. If $a \in \mathfrak{A}$ we have $(ax)^{j^*} = -\bar{x}\bar{a} = x^{j^*}\bar{a}$, $(xa)^{j^*} = -\bar{a}\bar{x} = \bar{a}x^{j^*}$. Hence $(\mathfrak{M},j^*)$ is a bimodule with involution for $(\mathfrak{A},j)$. We shall call this the *negative*, $-\mathfrak{M}$, of $(\mathfrak{M},j)$. In particular, we can apply this to $\mathrm{reg}\,\mathfrak{D}$ and $\mathrm{cay}\,\mathfrak{D}$ for the composition algebra $(\mathfrak{D},j)$ to obtain the bimodules $-\mathrm{reg}\,\mathfrak{D}$ and $-\mathrm{cay}\,\mathfrak{D}$.

If $(\mathfrak{M},j)$ is a bimodule for $(\mathfrak{A},j)$ then we shall say that $(\mathfrak{M},j)$ is *left free cyclic* if $(\mathfrak{M},j)$ contains an element u which is either symmetric or skew such that the mapping $a \to au$ of $\mathfrak{A}$ into $\mathfrak{M}$ is a linear isomorphism onto $\mathfrak{M}$. It is clear that if $(\mathfrak{D},j)$ is a composition algebra then the four bimodules $\mathrm{reg}\,\mathfrak{D}$, $-\mathrm{reg}\,\mathfrak{D}$, $\mathrm{cay}\,\mathfrak{D}$ and $-\mathrm{cay}\,\mathfrak{D}$ are left free cyclic. In all of these we can take $u = 1$. For $\mathrm{reg}\,\mathfrak{D}$ and $-\mathrm{cay}\,\mathfrak{D}$ this is symmetric and for $-\mathrm{reg}\,\mathfrak{D}$, $\mathrm{cay}\,\mathfrak{D}$ it is skew. We consider next the homomorphisms of the bimodules reg$\mathfrak{D}$, $-$ reg$\mathfrak{D}$, cay$\mathfrak{D}$, $-$ reg$\mathfrak{D}$. For this we require the following

LEMMA 1. *Let* $\mathfrak{M}$ *be a unital alternative bimodule for* $\mathfrak{D}$, $(\mathfrak{D},j)$ *a composition algebra.* (1) *Let* $a \in \mathfrak{D}$ *satisfy* $a^2 = \alpha 1 \neq 0$ *in* $\Phi 1$ *and let* $u \in \mathfrak{M}$ *satisfy* $au = ua$. *Then* $[a,u,b] = 0$ *for all* $b \in \mathfrak{D}$. (2) *Let* $a \in \mathfrak{D}$, $v \in \mathfrak{M}$ *satisfy* $av = v\bar{a}$. *Then* $a(vb) = v(\bar{a}b)$, $(bv)a = (b\bar{a})v$ for all $b \in \mathfrak{D}$.

PROOF. (1) We have $\alpha[a, u, b] = \alpha(au)b - \alpha a(ub) = \alpha(ua)b - a(u(a^2b)) = \alpha(ua)b - a(u(a(ab))) = \alpha(ua)b - (aua)(ab)$ (by one of Moufang's identities applied to the split null extension by $\mathfrak{M}$) $= \alpha(ua)b - (a^2u)(ab) = \alpha(ua)b - \alpha u(ab) = \alpha[u,a,b] = -\alpha[a,u,b]$. Hence $\alpha[a,u,b] = 0$ and $[a,u,b] = 0$. (2) We have $a(vb) - v(\bar{a}b) = a(vb) + v(ab) - v(ab) - v(\bar{a}b) = a(vb) + v(ab) - v(\alpha b)$ where $a + \bar{a} = \alpha 1$, α in Φ. Hence $a(vb) - v(\bar{a}b) = a(vb) + v(ab) - (\alpha v)b$. Since $[a,v,b] + [v,a,b] = 0$ we have $a(vb) + v(ab) = (av)b + (va)b$. Hence $a(vb) - v(\bar{a}b) = (av)b + (va)b - ((a + \bar{a})v)b = (va - \bar{a}v)b = 0$. In a similar fashion one proves the second relation: $(bv)a = (b\bar{a})v$.

LEMMA 2. *Let* $(\mathfrak{M},j)$ *be a unital alternative bimodule for the composition algebra* $(\mathfrak{D},j)$. (1) *Assume* u *is an element of* $\mathfrak{M}$ *such that* $au = ua$, $a \in \mathfrak{D}$ *and* $\bar{u} = u$ $(\bar{u} = -u)$. *Then the mapping* $a \to au$ *is a homomorphism of* reg$\mathfrak{D}$ $(-$reg$\mathfrak{D})$ *into* $\mathfrak{M}$. (2) *Assume* v *is an element of* $\mathfrak{M}$ *such that* $av = v\bar{a}$, $a \in \mathfrak{D}$ *and* $\bar{v} = v$ $(\bar{v} = -v)$. *Then* $a \to av$ *is a homomorphism of* $-$ cay$\mathfrak{D}$ (cay$\mathfrak{D}$) *into* $\mathfrak{M}$.

PROOF. (1) Since $\mathfrak{D}$ has a basis of elements a satisfying $a^2 = \alpha 1 \neq 0$, $\alpha \in \Phi$, Lemma 1 (1) implies that $[a,u,b]$ for all $a, b \in \mathfrak{D}$. Hence u is in the nucleus and consequently in the center of the split null extension $\mathfrak{E} = \mathfrak{D} \oplus \mathfrak{M}$. Now the mapping $a \to au$ of reg$\mathfrak{D}$ into $\mathfrak{M}$ sends $ab \to (ab)u = a(bu)$ and $ba \to (ba)u = b(au) = b(ua) = (bu)a$. Hence this is a bimodule homomorphism. If $\bar{u} = \pm u$ then $(\pm \bar{a})u = \bar{a}\bar{u} = \overline{ua} = \overline{au}$. This shows that the homomorphism is one of the bimodule with involution reg$\mathfrak{D}$ or $-$ reg$\mathfrak{D}$ according as $\bar{u} = u$ or $\bar{u} = -u$. (2) The mapping $a \to av$ of cay$\mathfrak{D}$ into $\mathfrak{M}$ sends $b \circ a = ab$, $b \in \mathfrak{D}$, $a \in$ cay$\mathfrak{D}$ into $(ab)u = v(\bar{b}\bar{a}) = b(v\bar{a}) = b(av)$, by Lemma 1 (2). Also $a \circ b = a\bar{b} \to (a\bar{b})v = (av)b$. Hence the mapping is a bimodule homomorphism. The result about the involutions follows as in (1).

It is clear that the sub-bimodules with involution of reg$\mathfrak{D}$ are the ideals of $(\mathfrak{D},j)$. Since $(\mathfrak{D},j)$ is simple, reg$\mathfrak{D}$ is an irreducible bimodule with involution. Evidently the same is true of $-$ reg$\mathfrak{D}$. The definition (11) and the fact that $j = -1$ for cay$\mathfrak{D}$ shows that the sub-bimodules of cay$\mathfrak{D}$ and $-$cay$\mathfrak{D}$ are the right ideals of $\mathfrak{D}$. Since we have to consider these bimodules only in the case $\mathfrak{D}$ associative and in these cases $\mathfrak{D}$ is a direct sum of irreducible right ideals, it is clear that all four of the bimodules with involution we are considering are direct sums of irreducible ones. Clearly this holds also for their homomorphic images.

We now suppose $\dim \mathfrak{D} > 1$ and $(\mathfrak{B}, j)$ is a composition subalgebra with $\dim \mathfrak{B} = \frac{1}{2} \dim \mathfrak{D}$. As before, we let q be an element of $\mathfrak{B}^{\perp}$ such that $Q(q) \neq 0$. Then $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{B}q$. We now prove

LEMMA 3. *Let (a_i) be an orthogonal basis for $\mathfrak{B}$ such that $a_1 = 1$. Then $\bar{a}_i = -a_i$, $i > 1$, $Q(a_i) = \lambda_i \neq 0$ and $Q(a_i q) \neq 0$. Let $\mathfrak{M}$ be a unital alternative bimodule for $\mathfrak{D}$. (1) If $u \in \mathfrak{M}$ satisfies $bu = ub$, $b \in \mathfrak{B}$, then $v = qu$ satisfies $bv = v\bar{b}$, $b \in \mathfrak{B}$. (2) Let $v \in \mathfrak{M}$ satisfy $bv = v\bar{b}$, $b \in \mathfrak{B}$. Then the elements $v_i = (a_i q)v + v(a_i q)$ satisfy $dv_i = v_i d$, $d \in \mathfrak{D}$, and the element $w = v + \frac{1}{2}\sum_i Q(a_i q)^{-1} (a_i q)v_i$ satisfies $dw = w\bar{d}$, $d \in \mathfrak{D}$.*

PROOF. The first statement is clear since a_i, $i > 1$, is orthogonal to 1, and $\Phi 1^{\perp}$ is the space of skew elements. Since Q is nondegenerate, we have $Q(a_i) \neq 0$ and $Q(a_i q) = Q(a_i)Q(q) \neq 0$. (1) If $u \in \mathfrak{M}$ satisfies $ub = bu$, $b \in \mathfrak{B}$, then by Lemma 1 (1), $[b, u, q] = 0$. Then $bv = b(qu) = (bq)u = (q\bar{b})u = q(\bar{b}u) = q(u\bar{b}) = (qu)\bar{b} = v\bar{b}$. (2) Let $v \in \mathfrak{M}$ satisfy $bv = v\bar{b}, b \in \mathfrak{B}$. Put $c_i = a_i q$ and $v_i = c_i v + v c_i$. The set $\{a_i, c_j\}$ is a basis for $\mathfrak{D}$. Hence to prove the assertion in (2) on $d \in \mathfrak{D}$, it is enough to prove these for $d = a_j$ and $d = c_j$. We consider first $d = a_j$. We have $a_j v_i = a_j(c_i v + v c_i) = a_j(c_i v) + a_j(v c_i) = a_j(c_i v) + v(\bar{a}_j c_i)$ (by Lemma 1 (2)) $= a_j(c_i v) + v(c_i a_j)$. Similarly, we can show that $v_i a_j = (a_j c_i)v + (v c_i)a_j$. It follows by alternativity that $v_i a_j = a_j v_i$ and hence $v_i b = b v_i$, $b \in \mathfrak{B}$. We note next that the elements $c_i \in \mathfrak{B}^{\perp}$ and $Q(c_i) \neq 0$. Hence $\mathfrak{B}^{\perp} = \mathfrak{B} c_i$ and consequently there exists elements $b_{ji} \in \mathfrak{B}$ such that $c_j = b_{ji} c_i$. Then $[c_j, v_i] = [b_{ji} c_i, v_i] = (b_{ji} c_i) v_i - v_i(b_{ji} c_i) = b_{ji}[c_i, v_i]$ by Lemma 1 (1) and the result just proved. Now $[c_i, v_i] = c_i(c_i v + v c_i) = -(c_i v + v c_i)c_i = (c_i^2 v - v c_i^2) = 0$ since $c_i^2 \in \Phi 1$. Hence $[c_j, v_i] = 0$, which completes the proof of the first statement in (2). Let $w = v + \frac{1}{2}\sum_i Q(c_i)^{-1} c_i v_i$. Since $a_j v_i = v_i a_j$, by Lemma 1 (1), we have $a_j(c_i v_i) = (a_j c_i)v_i = (c_i \bar{a}_j)v_i = c_i(\bar{a}_j v_i) = c_i(v_i \bar{a}_j) = (c_i v_i)\bar{a}_j$. Also by hypothesis, $a_j v = v a_j$. Hence $a_j w = w \bar{a}_j$. Again, by Lemma 1 (1), we have $c_j(c_i v_i) - (c_i v_i)\bar{c}_j = (c_j c_i - c_i \bar{c}_j)v_i$. Since $c_j = a_j q$ we have $c_j c_i = c_i \bar{c}_j$ if $i \neq j$ and $c_i^2 - c_i \bar{c}_i = 2c_i^2 = -2Q(c_i)1$. Hence $c_j(c_i v_i) - (c_i v_i)\bar{c}_j = -2Q(c_i)\delta_{ij} v_i$ and $c_j(\sum_i c_i v_i) - (\sum_i c_i v_i)\bar{c}_j = -2Q(c_j)v_j$. Also $c_j v - v\bar{c}_j = c_j v + v c_j = v_j$. Hence $c_j w = w \bar{c}_j$.

We can now prove the following

THEOREM 9. (1) *Any unital alternative bimodule with involution for a composition algebra $(\mathfrak{D}, j)$ is a sum of homomorphic images of one of the four bimodules* $\operatorname{reg}\mathfrak{D}$, $-\operatorname{reg}\mathfrak{D}$, $\operatorname{cay}\mathfrak{D}$, $-\operatorname{cay}\mathfrak{D}$, *where the last two can be dropped if $\mathfrak{D}$ is not associative.* (2) *Any unital alternative bimodule with involution for $(\mathfrak{D}, j)$ is completely reducible and any irreducible one is isomorphic to an irreducible component of* $\operatorname{reg}\mathfrak{D}$, $-\operatorname{reg}\mathfrak{D}$, $\operatorname{cay}\mathfrak{D}$ *or* $-\operatorname{cay}\mathfrak{D}$ *if $\mathfrak{D}$ is associative and to an irreducible component of* $\operatorname{reg}\mathfrak{D}$ *or* $-\operatorname{reg}\mathfrak{D}$ *if $\mathfrak{D}$ is not associative (hence an algebra of octonions).*

PROOF. Since the four bimodules with involution, $\operatorname{reg}\mathfrak{D}$, $-\operatorname{reg}\mathfrak{D}$, $\operatorname{cay}\mathfrak{D}$, $-\operatorname{cay}\mathfrak{D}$ are completely reducible, the complete reducibility of unital alternative bimodules with involution will follow from (1). We note also that if $\mathfrak{D}$ is an algebra of octonions then, as is easily verified, the only right ideals in $\mathfrak{D}$ are $\mathfrak{D}$ and 0 (ex. 5, p. 170). Hence cay $\mathfrak{D}$ and $-$cay $\mathfrak{D}$ are irreducible. Since these are not alternative it is clear that we can drop these if $\mathfrak{D}$ is not associative. Hence everything will follow if we can show that if $(\mathfrak{M},j)$ is a unital alternative bimodule for $(\mathfrak{D},j)$ then $\mathfrak{M}$ is a sum of homomorphic images of reg $\mathfrak{D}$, $-$reg $\mathfrak{D}$, cay $\mathfrak{D}$ or $-$cay $\mathfrak{D}$. We shall prove this by induction on dim $\mathfrak{D}$. If dim $\mathfrak{D} = 1$, $\mathfrak{D} = \Phi 1$ and $(\mathfrak{M},j)$ is just a vector space with a linear transformation j of period two. Then $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_{-1}$ where $\mathfrak{M}_i = \{x \mid x^j = ix\}$. Hence the result is clear in this case. Now assume dim $\mathfrak{D} > 1$ and let $(\mathfrak{B},j)$ be a composition subalgebra such that $\dim \mathfrak{B} = \frac{1}{2}\dim \mathfrak{D}$. Then we may assume that $(\mathfrak{M},j)$ as bimodule with involution for $(\mathfrak{B},j)$ is a sum of homomorphic images of reg $\mathfrak{B}$, $-$reg $\mathfrak{B}$, cay $\mathfrak{B}$, $-$cay $\mathfrak{B}$. If u is the image of 1 under a homomorphism of one of these bimodules with involution into $\mathfrak{M}$ then u is a generator of the image and $u^j = \pm u$, and either $bu = ub$, $b \in \mathfrak{B}$, or $bu = u\bar{b}$, $b \in \mathfrak{B}$. Our result will follow if we can show that any such u is contained in a homomorphic image of one of the basic bimodules reg $\mathfrak{D}$, $-$reg $\mathfrak{D}$, cay $\mathfrak{D}$, $-$cay $\mathfrak{D}$. Suppose first that $u = v$ satisfies $bv = v\bar{b}$, $b \in \mathfrak{B}$, $\bar{v} = \pm v$. Let v_i and w be the elements given in (2) of Lemma 3. Then $dv_i = v_i d$ and $dw = w\bar{d}$, $d \in \mathfrak{D}$. Also $v_i = (a_i q)v + v(a_i q)$ and $w = v + \frac{1}{2}\sum Q(a_i q)^{-1}(a_i q)v_i$. Hence $\bar{v}_i = \pm v(\overline{a_i q}) \pm (\overline{a_i q})v = \mp v(a_i q) \mp (a_i q)v = \mp v_i$ and $\bar{w} = \pm v \pm \frac{1}{2}\sum Q(a_i q)^{-1} v_i(a_i q) = \pm v \pm \frac{1}{2}\sum Q(a_i q)^{-1}(a_i q)v_i$. Hence, by Lemma 2, there is a homomorphism of one of our basic $\mathfrak{D}$-bimodules into $\mathfrak{M}$ mapping 1 into any v_i or into w. Moreover, it is clear from the definition of w that v is contained in the sum of these homomorphic images. Next assume $ub = bu$, $b \in \mathfrak{B}$, $\bar{u} = \pm u$. By Lemma 3 (1), $v = qu$ satisfies $vb = \bar{b}v$, $b \in \mathfrak{B}$. The same condition holds for the elements $v + \bar{v}$ and $v - \bar{v}$ which are respectively symmetric and skew. The result we have established shows that these elements are contained in a sum of homomorphic images of our four basic bimodules. Hence the same is true of $u = q^{-1}v = \frac{1}{2}q^{-1}(v + \bar{v}) + \frac{1}{2}q^{-1}(v - \bar{v})$. This completes the proof.

We shall now determine representatives of the isomorphism classes of irreducible unital bimodules for $(\mathfrak{D},j)$. Since the category isomorphism for bimodules of a Jordan matrix algebra is one with associative bimodules of the associative coefficient algebra if the order $n \geqq 4$ and with Jordan admissible bimodules if $n = 3$, we shall need to sort out these bimodules from our list of representatives. We shall distinguish the following cases for the structure of $(\mathfrak{D},j)$:

I. $\mathfrak{D} = \Phi 1$, $j = 1$. Here cay $\mathfrak{D}$ = reg $\mathfrak{D}$ is one dimensional and hence irreducible. Hence we have two nonisomorphic irreducible $(\mathfrak{D},j)$-bimodules: reg $\mathfrak{D}$ and $-$reg $\mathfrak{D}$.

II$_a$. $\mathfrak{D} = \Phi e_1 \oplus \Phi e_2$ where the e_i are orthogonal idempotents such that $e_1 + e_2 = 1$, $\bar{e}_1 = e_2$, $\bar{e}_2 = e_1$. We have seen that reg $\mathfrak{D}$ is irreducible as bimodule with involution. A direct verification shows that the linear mapping in $\mathfrak{D}$ which interchanges $e_1 + e_2$ and $e_1 - e_2$ is an isomorphism of reg $\mathfrak{D}$ onto $-$reg $\mathfrak{D}$. Hence we drop $-\text{reg}\,\mathfrak{D}$. For $\text{cay}\,\mathfrak{D}$ we have $j = -1$ so Φe_1 and Φe_2 are sub-bimodules. These are also sub-bimodules of $-\text{cay}\,\mathfrak{D}$. Clearly they are irreducible. Also it is immediate that no two of these are isomorphic. Hence we have five isomorphism classes of irreducible $(\mathfrak{D},j)$-bimodules. Since $\mathfrak{D}$ has a single generator $a = e_1 - e_2$ it is clear that all bimodules for $\mathfrak{D}$ are associative.

II$_b$. $\mathfrak{D}$ is a quadratic field $\Phi[l]$, $l^2 = \lambda 1$, $\bar{l} = -l$. $\text{reg}\,\mathfrak{D}$ is irreducible and isomorphic to $-\text{reg}\,\mathfrak{D}$ under the mapping $x \to lx$. $\text{cay}\,\mathfrak{D}$ is irreducible. We have three nonisomorphic $(\mathfrak{D},j)$-bimodules: $\text{reg}\,\mathfrak{D}$, $\text{cay}\,\mathfrak{D}$, $-\text{cay}\,\mathfrak{D}$. All are associative.

III$_a$. $\mathfrak{D}$ is a split quaternion algebra, j the standard involution. $\text{reg}\,\mathfrak{D}$ and $-\text{reg}\,\mathfrak{D}$ are irreducible, associative and not isomorphic. The last assertion can be seen by comparing dimensionalities of the symmetric elements of the two bimodules with involution. $\text{cay}\,\mathfrak{D}$ is a direct sum of two isomorphic irreducible bimodules. To see this let e_{ij}, $i,j = 1,2$ be a matrix basis for $\mathfrak{D} \cong \Phi_2$. Then $\mathfrak{D} = e_{11}\mathfrak{D} \oplus e_{22}\mathfrak{D}$ and the $e_{ii}\mathfrak{D}$ are minimal right ideals; hence irreducible sub-bimodules of $\text{cay}\,\mathfrak{D}$ ($j = -1$ for this). The mapping $x \to e_{21}x$ is an isomorphism of $e_{11}\mathfrak{D}$ onto $e_{22}\mathfrak{D}$. We now obtain the nonisomorphic irreducible bimodules $e_{11}\mathfrak{D}$ and $-e_{11}\mathfrak{D}$. It is easily seen that $\text{cay}\,\mathfrak{D}$ $(-\text{cay}\,\mathfrak{D})$ is not associative. This implies that $e_{11}\mathfrak{D}$ $(-e_{11}\mathfrak{D})$ is not associative. The first of these is Jordan admissible, the second not. Hence we have two irreducible associative bimodules and altogether we have three alternative Jordan admissible ones.

III$_b$. $\mathfrak{D}$, a quaternion division algebra, standard involution. As in III$_a$ we obtain two irreducible associative bimodules $\text{reg}\,\mathfrak{D}$, $-\text{reg}\,\mathfrak{D}$ and two irreducible alternative not-associative bimodules $\text{cay}\,\mathfrak{D}$, $-\text{cay}\,\mathfrak{D}$. The first is Jordan admissible, the second is not.

IV. $\mathfrak{D}$, an algebra of octonions, standard involution. We have two irreducible unital alternative bimodules: $\text{reg}\,\mathfrak{D}$ and $-\text{reg}\,\mathfrak{D}$. The first is Jordan admissible but not the second.

Now let $\mathfrak{J}$ be a reduced simple Jordan algebra of degree $n \geqq 3$. Then $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_r)$ where $(\mathfrak{D},j)$ is a composition algebra. Then $\mathfrak{J}$ is central simple and if $n \geqq 4$, the results of the last section (Theorems 6 and 8) show that $U_1(\mathfrak{J}) \cong S_1''(\mathfrak{J})$ and give the structure of this algebra. On the other hand, the category isomorphism between the category of unital Jordan bimodules for $\mathfrak{J}$ and the category of unital associative bimodules for the (associative) composition algebra and Theorem 9 imply that every unital bimodule for $\mathfrak{J}$ is completely reducible. This implies that $U_1(\mathfrak{J})$ is semisimple. Also the determination of the unital irreducible associative bimodules for $(\mathfrak{D},j)$ can be used to give a determination of the irreducible unital Jordan bimodules for $\mathfrak{J}$. This will give the structure of $U_1(\mathfrak{J})$. This method can be used to recover the results of §3. We leave the details to the reader and proceed to use the present method to investigate $U_1(\mathfrak{J})$, $\mathfrak{J}$ central simple of degree three, which we were unable to determine by the methods of §3.

THEOREM 10. *Let* $\mathfrak{J}$ *be a reduced simple Jordan algebra of degree three, so* $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ *where* $(\mathfrak{D},j)$ *is a composition algebra. Then we have the following possibilities for the structure of* $U_1(\mathfrak{J})$:

I $U_1(\mathfrak{J}) \cong \Phi_3 \oplus \Phi_6$ *if* $\mathfrak{D} = \Phi$.

$\mathrm{II_a}$ $U_1(\mathfrak{J}) \cong \Phi_9 \oplus \Phi_3 \oplus \Phi_3 \oplus \Phi_6 \oplus \Phi_6$ *if* $\mathfrak{D} \cong \Phi \oplus \Phi$.

$\mathrm{II_b}$ $U_1(\mathfrak{J}) \cong \Phi_9 \oplus \mathfrak{D}_3 \oplus \mathfrak{D}_6$ *if* $\mathfrak{D}/\Phi$ *is a quadratic field.*

$\mathrm{III_a}$ $U_1(\mathfrak{J}) \cong \Phi_{15} \oplus \Phi_{21} \oplus \Phi_6$ *if* $\mathfrak{D}$ *is a split quaternion algebra.*

$\mathrm{III_b}$ $U_1(\mathfrak{J}) \cong \Phi_{15} \oplus \Phi_{21} \oplus \mathfrak{D}_3$ *if* $\mathfrak{D}$ *is a quaternion division algebra.*

IV $U_1(\mathfrak{J}) \cong \Phi_{27}$ *if* $\mathfrak{D}$ *is an algebra of octonions.*

PROOF. Since the bimodules with involution for the coefficient algebra $(\mathfrak{D},j)$ are completely reducible the same is true, by §3.5, of the Jordan bimodules for $\mathfrak{J}$. Hence $U_1(\mathfrak{J})$ is semisimple and the simple components of $U_1(\mathfrak{J})$ are in 1–1 correspondence with the isomorphism classes of irreducible unital bimodules for $\mathfrak{J}$. The latter are the bimodules associated with the unital irreducible bimodules for $(\mathfrak{D},j)$ which are alternative Jordan admissible. We recall also that by the associative theory, if $\mathfrak{M}$ is a unital irreducible right module for $U_1(\mathfrak{J})$ (= unital irreducible bimodule for $\mathfrak{J}$) and Δ is the centralizer of $\mathfrak{M}$, that is, the algebra of linear transformations which commute with the mappings in $\mathfrak{M}$ corresponding to $U_1(\mathfrak{J})$ then Δ is a division algebra and the simple component of $U_1(\mathfrak{J})$ corresponding to $\mathfrak{M}$ is isomorphic to the matrix algebra Δ°_r where r is the dimensionality of $\mathfrak{M}$ considered as vector space over Δ°. If we apply these results we can easily obtain the list stated in the theorem. We consider IV first. Here we have just one irreducible Jordan admissible unital alternative bimodule for $(\mathfrak{D},j)$ and hence one irreducible unital bimodule for $\mathfrak{J}$. To find this it is unnecessary to go through the procedure indicated since $\mathfrak{J}$ itself as regular bimodule is unital irreducible by the simplicity of $\mathfrak{J}$. Also the centralizer is Φ since $\mathfrak{J}$ is central simple. Since $\dim \mathfrak{J}/\Phi = 27$ we have $U_1(\mathfrak{J}) \cong \Phi_{27}$. Next we consider $\mathrm{III_b}$: $\mathfrak{D}$ a quaternion division algebra. The results show that there are three unital irreducible bimodules for $\mathfrak{J}$ associated with $\operatorname{reg}\mathfrak{D}$, $-\operatorname{reg}\mathfrak{D}$ and $\operatorname{cay}\mathfrak{D}$ respectively. Since the first two are associative the corresponding split extensions are associative and the split extension by the associated bimodules for $\mathfrak{J}$ are special. Hence these are irreducible right modules for $S_1''(\mathfrak{J})$ and are therefore the ones we obtained in Theorem 7. These give the simple components Φ_{15} and Φ_{21} of $S_1''(\mathfrak{J})$ and of $U_1(\mathfrak{J})$. The remaining irreducible bimodule $\operatorname{cay}\mathfrak{D}$ is not associative and the associated bimodule $\mathfrak{M}$ for $\mathfrak{J}$ is not special. Since $\mathfrak{M} = \mathfrak{H}((\operatorname{cay}\mathfrak{D})_3, J_\gamma)$ and $j = -1$ it is immediate that $\dim \mathfrak{M}/\Phi = 12$. Since the actions of the algebra $\mathfrak{D}$ on $\operatorname{cay}\mathfrak{D}$ consist of right multiplications and since $\mathfrak{D}$ is associative it is clear that the centralizer of $\operatorname{cay}\mathfrak{D}$ is the algebra of left multiplications by elements of $\mathfrak{D}$. By the category isomorphism the same is true of $\mathfrak{M}$. Hence the simple component of $U_1(\mathfrak{J})$ corresponding to $\mathfrak{M}$ is isomorphic to $\mathfrak{D}_r$ where r is the dimensionality of $\mathfrak{M}/\Delta$. Since $\mathfrak{M}/\Phi = 12$ and $\Delta/\Phi = 4$ we have $r = 3$ which completes the proof in this case.

The consideration of III_a is almost identical with this and the remaining cases are easier, particularly, if we take into account our earlier results. We shall therefore omit the remaining details.

A comparison of Theorem 8 with Theorem 10 shows that the only reduced simple Jordan algebras of degree $\geqq 3$ which are not strongly special are the algebras $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ where $\mathfrak{D}$ is either a quaternion algebra or an algebra of octonions. It follows that all central simple Jordan algebras of degree $\geqq 3$ are strongly special with the exception of the exceptional Jordan algebras (which become $\mathfrak{H}(\mathfrak{D}_3, J_1)$, $\mathfrak{D}$ a split octonion algebra if the base field is extended to its algebraic closure Ω) and the algebras of degree three which are of *symplectic* type in the sense that $\mathfrak{J}_\Omega = \mathfrak{H}(\mathfrak{D}_3, J_1)$, $\mathfrak{D}$ a split quaternion algebra. The results of the last section give the structure of $U_1(\mathfrak{J}) \cong S_1''(\mathfrak{J})$ for the strongly special central simple algebras of degree $\geqq 3$. We remark also that our study of meson algebras shows that the central simple Jordan algebras of degree two are also strongly special (Theorem 3) and gives the structure of $U_1(\mathfrak{J})$ for these $\mathfrak{J}$. If $\mathfrak{J}$ is of degree 1 then $\mathfrak{J} = \Phi$ and everything is trivial. We now treat the missing case in the following

THEOREM 11. *If $\mathfrak{J}$ is a central simple Jordan algebra of degree three and symplectic type then $\mathfrak{J}$ is reduced; hence the structure of $U_1(\mathfrak{J})$ is given in Theorem* 10 (*cases* III_a *and* III_b). *If $\mathfrak{J}$ is an exceptional central simple Jordan algebra (finite-dimensional) then $U_1(\mathfrak{J}) \cong \Phi_{27}$.*

PROOF. For the first part we note that $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra with involution such that $(\mathfrak{A}_\Omega, J) \cong (\mathfrak{D}_3, J_1)$ where $(\mathfrak{D}, j)$ is a quaternion composition algebra. We have $\mathfrak{A} \cong \Delta_r$ where Δ is a central division algebra which has an involution. Also $\dim \mathfrak{A}/\Phi = 36 = r^2 s$ where $s = \dim \Delta/\Phi$. Since $\mathfrak{A} \otimes \mathfrak{A} \cong \mathfrak{A} \otimes \mathfrak{A}^\circ \cong \Phi_{36}$ it follows from the theory of the Brauer group (see, for example, Jacobson, *Theory of Rings*, p. 109) that s is a power of two. Hence the only possibilities for Δ are $\Delta = \Phi$ or Δ is a quaternion algebra. In the first case, $\mathfrak{A} \cong \Phi_6$ and since $\dim \mathfrak{H}(\mathfrak{A}, J) = 15$, $(\mathfrak{A}, J) \cong (\Phi_6, K)$ where K has the form $X \to S^{-1} X^t S$, $S = \mathrm{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$. Thus $(\mathfrak{A}, J) \cong (\mathfrak{D}_3, J_1)$ where $(\mathfrak{D}, j)$ is a split quaternion algebra with canonical involution. If Δ is a quaternion division algebra, then $(\mathfrak{A}, J) \cong (\Delta_3, K)$ where K has the form $X \to a^{-1} \bar{X}^t a$ where $x \to \bar{x}$ is the canonical involution in Δ and $a = \mathrm{diag}\{a_1, a_2, a_3\}$ where either $\bar{a}_i = a_i$ or $\bar{a}_i = -a_i$ for all i. In the first case $a_i = \gamma_i 1$, $\gamma_i \in \Phi$ and $\dim \mathfrak{H}(\Delta_3, K) = 15$. In the second case, it is immediate that $\dim \mathfrak{H}(\Delta_3, K) = 21$. Since $\dim \mathfrak{H}(\mathfrak{A}, J) = 15$ the second case is ruled out, so K is a canonical involution and $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ is reduced. Now suppose $\mathfrak{J}$ is finite-dimensional central simple and exceptional. Then $\mathfrak{J}_\Omega \cong \mathfrak{H}(\mathfrak{D}_3, J_1)$ where $(\mathfrak{D}, j)$ is an algebra of octonions. It follows from Theorem 10 that $U_1(\mathfrak{J})_\Omega \cong U_1(\mathfrak{J}_\Omega) \cong \Omega_{27}$.

Hence $U_1(\mathfrak{J})$ is simple. Since $\mathfrak{J}$ is central simple, $M(\mathfrak{J}) = \Phi_{27}$ and since this is a homomorphic image of $U_1(\mathfrak{J})$ we have $U_1(\mathfrak{J}) = \Phi_{27}$.

5. **Separability of $\mathfrak{J}$ and $U(\mathfrak{J})$.** We devote this section to the proof of the following fundamental theorem.

THEOREM 12. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with* 1. *Then the following three conditions on $\mathfrak{J}$ are equivalent*: (i) *$\mathfrak{J}$ is separable*, (ii) *$U(\mathfrak{J})$ is separable*, (iii) *$U_1(\mathfrak{J})$ is separable. Moreover, these conditions imply that $S_1(\mathfrak{J})$ is separable and the converse holds if $\mathfrak{J}$ is special.*

PROOF. Since a finite-dimensional Jordan (associative) algebra with 1 is separable if and only if the algebra obtained by extending the base field to its algebraic closure is semisimple we may assume the base field is algebraically closed and replace "separable" by "semisimple" in the statement of the theorem. Suppose first that $\mathfrak{J}$ is not semisimple so $\mathfrak{J}$ contains a nonzero solvable ideal $\mathfrak{N}$. Then, by Theorem 5.2, the images of $\mathfrak{N}$ under the canonical homomorphism of $\mathfrak{J}$ into $U(\mathfrak{J})$, $U_1(\mathfrak{J})$ and $S_1(\mathfrak{J})$ generates a nil ideal in these algebras. Moreover, the images in $U(\mathfrak{J})$ and $U_1(\mathfrak{J})$ are not zero and the same is true of the image in $S_1(\mathfrak{J})$ if $\mathfrak{J}$ is special. Hence $U(\mathfrak{J})$ and $U_1(\mathfrak{J})$ are not semisimple and $S_1(\mathfrak{J})$ is not semisimple if $\mathfrak{J}$ is special. Now assume $\mathfrak{J}$ is semisimple. We shall prove that $S_1(\mathfrak{J})$ and $U_1(\mathfrak{J})$ are semisimple, using induction on $\dim \mathfrak{J}$. This will imply that $U(\mathfrak{J}) \cong \Phi z \oplus S_1(\mathfrak{J}) \oplus U_1(\mathfrak{J})$ is semisimple. If $\dim \mathfrak{J} = 1$ then $S_1(\mathfrak{J}) = \Phi 1$ and $U_1(\mathfrak{J}) = \Phi 1$ so the result is clear. Now suppose $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ where the $\mathfrak{J}_i$ are nonzero ideals. By Theorem 2.5, $S_1(\mathfrak{J}) \cong S_1(\mathfrak{J}_1) \oplus S_1(\mathfrak{J}_2)$ and, by Theorem 2.16, $U_1(\mathfrak{J}) \cong U_1(\mathfrak{J}_1) \oplus U_1(\mathfrak{J}_2) \oplus S_1(\mathfrak{J}_1) \otimes S_1(\mathfrak{J}_2)$ and since $\dim \mathfrak{J}_i < \dim \mathfrak{J}$ we may assume that $S_1(\mathfrak{J}_1)$, $S_1(\mathfrak{J}_2)$, $U_1(\mathfrak{J}_1)$ and $U_1(\mathfrak{J}_2)$ are semisimple. Then $S_1(\mathfrak{J}_1) \otimes S_1(\mathfrak{J}_2)$ is semisimple and hence $S_1(\mathfrak{J})$ and $U_1(\mathfrak{J})$ are semisimple. Hence we may assume $\mathfrak{J}$ is simple and $\dim \mathfrak{J} > 1$. Then the degree $\deg \mathfrak{J} \geqq 2$ and if $\deg \mathfrak{J} = 2$, $S_1(\mathfrak{J})$ and $U_1(\mathfrak{J})$ are semisimple by the results on Clifford and meson algebras in §§1,2. If $\deg \mathfrak{J} \geqq 3$, $S_1(\mathfrak{J}) = 0$ if $\mathfrak{J}$ is exceptional, and if $\mathfrak{J}$ is special then $S_1(\mathfrak{J})$ is semisimple by Martindale's theorem (cf. §5.7). Also $U_1(\mathfrak{J})$ is semisimple by the results of the §§3, 4. This completes the proof.

REMARK. The foregoing proof of the second half of the theorem rests heavily on the structure theory and the results on the representation theory of §§1–4. In the next chapter we shall give a general proof of the result in the characteristic zero case which does not require these detailed results. This will be based on the theory of Lie algebras.

Theorem 12 has the following immediate consequence.

COROLLARY. *If $\mathfrak{J}$ is a finite-dimensional separable Jordan algebra then every bimodule for $\mathfrak{J}$ is completely reducible.*

6. **The Theorem of Albert-Penico-Taft.** In this section we consider the finite-

dimensional Jordan algebra extensions of separable Jordan algebras $\mathfrak{J}$. In §2.8 we defined a (Jordan) extension of a Jordan algebra $\mathfrak{J}$ by a Jordan algebra $\mathfrak{M}$ as a short exact sequence $0 \to \mathfrak{M} \underset{\alpha}{\to} \mathfrak{E} \underset{\beta}{\to} \mathfrak{J} \to 0$ where $\mathfrak{E}$ is Jordan. We now assume $\mathfrak{J}$, $\mathfrak{E}$ and $\mathfrak{M}$ are finite dimensional and $\mathfrak{J}$ is separable. Without loss of generality we may assume that $\mathfrak{M}$ is an ideal in $\mathfrak{E}$ and α is the injection mapping. Then $\mathfrak{J} \cong \mathfrak{E}/\mathfrak{M}$. We shall show that any extension of the type indicated splits. This is equivalent to showing that if $\mathfrak{E}$ is a finite-dimensional Jordan algebra, $\mathfrak{M}$ an ideal in $\mathfrak{E}$ such that $\mathfrak{E}/\mathfrak{M}$ is separable then there exists a subalgebra $\mathfrak{K}$ of $\mathfrak{E}$ such that $\mathfrak{E} = \mathfrak{K} \oplus \mathfrak{M}$ (cf. §2.8). This theorem for solvable $\mathfrak{M}$ was first proved by Albert for special Jordan algebras of characteristic 0, by Penico for arbitrary Jordan algebras of characteristic 0, and by Taft in the general case. We shall therefore call the splitting theorem for solvable $\mathfrak{M}$ the *Albert-Penico-Taft theorem.*

We shall now make a number of reductions of the general theorem to some special cases. For these it will be convenient to collect the following simple consequences of the structure theory of finite-dimensional Jordan algebras.

LEMMA 1. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with radical (= maximal solvable ideal)* $\operatorname{rad}\mathfrak{J}$. (1) *If $\mathfrak{B}$ is an ideal in $\mathfrak{J}$ then* $\operatorname{rad}\mathfrak{B} = \operatorname{rad}\mathfrak{J} \cap \mathfrak{B}$. *Hence* $\operatorname{rad}\mathfrak{B}$ *is an ideal in $\mathfrak{J}$.* (2) *If $\mathfrak{B}$ is an ideal such that $\mathfrak{J}/\mathfrak{B}$ is semisimple then the radicals of $\mathfrak{B}$ and $\mathfrak{J}$ coincide.* (3) *If $\mathfrak{B}$ is a semisimple ideal and $\mathfrak{J}/\mathfrak{B}$ is semisimple then $\mathfrak{J}$ is semisimple and $\mathfrak{B}$ is a direct summand of $\mathfrak{J}$.* (4) *If $\mathfrak{J}' = \Phi 1 \oplus \mathfrak{J}$ is the Jordan algebra obtained by adjoining an identity element 1 to $\mathfrak{J}$ then $\mathfrak{J}'$ and $\mathfrak{J}$ have the same radical.* (5) *If e is an idempotent in $\mathfrak{J}$ then* $\operatorname{rad}(\mathfrak{J}U_e) = (\operatorname{rad}\mathfrak{J})U_e$.

PROOF. (1) Clearly $\operatorname{rad}\mathfrak{J} \cap \mathfrak{B} \subseteq \operatorname{rad}\mathfrak{B}$ since $\operatorname{rad}\mathfrak{J} \cap \mathfrak{B}$ is a solvable ideal in $\mathfrak{B}$. Also $\mathfrak{B}/(\operatorname{rad}\mathfrak{J} \cap \mathfrak{B}) \cong (\mathfrak{B} + \operatorname{rad}\mathfrak{J})/\operatorname{rad}\mathfrak{J}$ which is an ideal in the semisimple algebra $\mathfrak{J}/\operatorname{rad}\mathfrak{J}$. Since any ideal in a semisimple Jordan algebra is a direct summand and hence is semisimple, it follows that $\mathfrak{B}/(\operatorname{rad}\mathfrak{J} \cap \mathfrak{B})$ is semisimple. This and $\operatorname{rad}\mathfrak{J} \cap \mathfrak{B} \subseteq \operatorname{rad}\mathfrak{B}$ imply that $\operatorname{rad}\mathfrak{B} = \operatorname{rad}\mathfrak{J} \cap \mathfrak{B}$. (2) Let $\mathfrak{B}$ be an ideal such that $\mathfrak{J}/\mathfrak{B}$ is semisimple. Since $(\mathfrak{B} + \operatorname{rad}\mathfrak{J})/\mathfrak{B} \cong \operatorname{rad}\mathfrak{J}/(\mathfrak{B} \cap \operatorname{rad}\mathfrak{J})$ this is a solvable ideal in $\mathfrak{J}/\mathfrak{B}$. Hence it is 0. Thus $\mathfrak{B} + \operatorname{rad}\mathfrak{J} = \mathfrak{B}$ and $\mathfrak{B} \supseteq \operatorname{rad}\mathfrak{J}$. (3) Let $\mathfrak{B}$ be a semisimple ideal and assume $\mathfrak{J}/\mathfrak{B}$ is semisimple. Then, by (2), $\mathfrak{B} \supseteq \operatorname{rad}\mathfrak{J}$. Since $\mathfrak{B}$ is semisimple this implies that $\operatorname{rad}\mathfrak{J} = 0$. Hence $\mathfrak{J}$ is semisimple. (4) Let $\mathfrak{J}' = \Phi 1 \oplus \mathfrak{J}$ the algebra obtained by adjoining the identity 1 to $\mathfrak{J}$. Then $\mathfrak{J}$ is an ideal in $\mathfrak{J}'$ so $\operatorname{rad}\mathfrak{J}$ is a solvable ideal in $\mathfrak{J}'$. Since $\mathfrak{J}'/\operatorname{rad}\mathfrak{J} \cong \Phi 1 \oplus (\mathfrak{J}/\operatorname{rad}\mathfrak{J})$ and $\mathfrak{J}/\operatorname{rad}\mathfrak{J}$ and $(\Phi 1 \oplus (\mathfrak{J}/\operatorname{rad}\mathfrak{J}))/(\mathfrak{J}/\operatorname{rad}\mathfrak{J}) \cong \Phi 1$ are semisimple, $\Phi 1 \oplus (\mathfrak{J}/\operatorname{rad}\mathfrak{J})$ is semisimple, by (3). Hence $\mathfrak{J}'/\operatorname{rad}\mathfrak{J}$ is semisimple and $\operatorname{rad}\mathfrak{J} = \operatorname{rad}\mathfrak{J}'$. (5) Let e be an idempotent in $\mathfrak{J}$. Then clearly $(\operatorname{rad}\mathfrak{J})U_e \subseteq \operatorname{rad}\mathfrak{J} \cap \mathfrak{J}U_e$ and if $z \in \operatorname{rad}\mathfrak{J} \cap \mathfrak{J}U_e$ then $z = zU_e \in (\operatorname{rad}\mathfrak{J})U_e$. Hence $(\operatorname{rad}\mathfrak{J})U_e = \operatorname{rad}\mathfrak{J} \cap \mathfrak{J}U_e$ is a solvable ideal in $\mathfrak{J}U_e$. To prove that this is the radical we have to show that $\mathfrak{J}U_e/(\operatorname{rad}\mathfrak{J})U_e$ is semisimple. Since $\mathfrak{J}U_e/(\operatorname{rad}\mathfrak{J})U_e \cong (\mathfrak{J}/\operatorname{rad}\mathfrak{J})U_{e'}$ where

$e' = e + \mathrm{rad}\,\mathfrak{J}$ it suffices to show that if $\mathfrak{J}$ is semisimple then $\mathfrak{J}U_e$ is semisimple. By Corollary 1 to Theorem 5.7, $\mathfrak{J}$ is semisimple if and only if it has no absolute zero divisor $\neq 0$. Also we know that any absolute zero divisor of $\mathfrak{J}U_e$ is an absolute zero divisor of $\mathfrak{J}$. Hence the result is clear.

Reduction I. To the Albert-Penico-Taft theorem. Let $\mathfrak{E}$ be a finite-dimensional Jordan algebra, $\mathfrak{M}$ an ideal in $\mathfrak{E}$ such that $\mathfrak{E}/\mathfrak{M}$ is separable. If $\mathfrak{R}$ is the radical of $\mathfrak{M}$ then, by Lemma 1 (2), $\mathfrak{R}$ is the radical of $\mathfrak{E}$ and $\mathfrak{E}/\mathfrak{R}$ is semisimple. Now $\mathfrak{M}/\mathfrak{R}$ is an ideal in $\mathfrak{E}/\mathfrak{R}$. Hence $\mathfrak{E}/\mathfrak{R} = (\mathfrak{M}/\mathfrak{R}) \oplus (\mathfrak{F}/\mathfrak{R})$ where $\mathfrak{F}/\mathfrak{R}$ is an ideal isomorphic to $(\mathfrak{E}/\mathfrak{R})/(\mathfrak{M}/\mathfrak{R}) \cong \mathfrak{E}/\mathfrak{M}$. Hence $\mathfrak{F}/\mathfrak{R}$ is separable and $\mathfrak{R}$ is solvable. Then if we assume the Albert-Penico-Taft theorem we obtain the existence of a subalgebra $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{F} = \mathfrak{K} \oplus \mathfrak{R}$. Then $\mathfrak{K} \cong \mathfrak{F}/\mathfrak{R} \cong \mathfrak{E}/\mathfrak{M}$. Moreover, $\mathfrak{K} \cap \mathfrak{M} \subseteq \mathfrak{F} \cap \mathfrak{M} \subseteq \mathfrak{R}$ since $\mathfrak{M}/\mathfrak{R} \cap \mathfrak{F}/\mathfrak{R} = 0$. Hence $\mathfrak{K} \cap \mathfrak{M} \subseteq \mathfrak{K} \cap \mathfrak{R} = 0$. This and $\dim \mathfrak{K} = \dim \mathfrak{E}/\mathfrak{M}$ imply that $\mathfrak{E} = \mathfrak{K} \oplus \mathfrak{M}$.

Reduction II. To null (= singular) extensions. We now assume $\mathfrak{E}$ contains a solvable ideal $\mathfrak{M}$ such that $\mathfrak{E}/\mathfrak{M}$ is separable. Suppose $\mathfrak{M}^{\cdot 2} \neq 0$. By Penico's lemma (Lemma 2 on p. 192) there exists a (solvable) ideal $\mathfrak{N}$ in $\mathfrak{E}$ such that $0 \subset \mathfrak{N} \subset \mathfrak{M}$. Then $\mathfrak{E}' = \mathfrak{E}/\mathfrak{N}$ has the solvable ideal $\mathfrak{M}' = \mathfrak{M}/\mathfrak{N}$ and $\mathfrak{E}'/\mathfrak{M}' \cong \mathfrak{E}/\mathfrak{M}$ is separable Also $\dim \mathfrak{E}' < \dim \mathfrak{E}$. Since the theorem we wish to prove is clear for one-dimensional algebras we may use induction on dimensionality to conclude that there exists a subalgebra $\mathfrak{F}'$ of $\mathfrak{E}'$ such that $\mathfrak{E}' = \mathfrak{F}' \oplus \mathfrak{M}'$. Now $\mathfrak{F}' = \mathfrak{F}/\mathfrak{N}$ where $\mathfrak{F}$ is a subalgebra of $\mathfrak{E}$ containing $\mathfrak{N}$ and $\mathfrak{F}/\mathfrak{N} = \mathfrak{F}' \cong \mathfrak{E}'/\mathfrak{M}' \cong \mathfrak{E}/\mathfrak{M}$ is separable. Also $\dim \mathfrak{F} = \dim \mathfrak{E}/\mathfrak{M} + \dim \mathfrak{N} < \dim \mathfrak{E}/\mathfrak{M} + \dim \mathfrak{M} = \dim \mathfrak{E}$. Hence, again, by induction on dimensionality, there exists a subalgebra $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{K} \cong \mathfrak{F}/\mathfrak{N} \cong \mathfrak{E}/\mathfrak{M}$. Since $\mathfrak{K}$ is semisimple and $\mathfrak{M}$ is solvable this implies that $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{K}$. This proves the result in the case $\mathfrak{M}^{\cdot 2} \neq 0$ so it remains to consider the case $\mathfrak{M}^{\cdot 2} = 0$.

Reduction III. To algebras $\mathfrak{E}$ with 1. Let $\mathfrak{E}$ contain an ideal $\mathfrak{M}$ such that $\mathfrak{M}^{\cdot 2} = 0$ and $\mathfrak{E}/\mathfrak{M}$ is separable. Let $\mathfrak{E}' = \Phi 1 \oplus \mathfrak{E}$ the Jordan algebra obtained by adjoining the identity element 1 to $\mathfrak{E}$. Then $\mathfrak{M}$ is a solvable ideal in $\mathfrak{E}'$ and $\mathfrak{E}'/\mathfrak{M} \cong \Phi 1 \oplus \mathfrak{E}/\mathfrak{M}$ is separable. Hence, assuming the theorem for algebras with 1 (and $\mathfrak{M}^{\cdot 2} = 0$) we obtain the existence of a subalgebra $\mathfrak{K}'$ of $\mathfrak{E}'$ such that $\mathfrak{E}' = \mathfrak{K}' \oplus \mathfrak{M}$. Since $\mathfrak{E} \supseteq \mathfrak{M}$ this gives $\mathfrak{E} = \mathfrak{E}' \cap \mathfrak{E} = (\mathfrak{E} \cap \mathfrak{K}') \oplus \mathfrak{M}$ and $\mathfrak{K} = \mathfrak{E} \cap \mathfrak{K}'$ is a subalgebra.

Reduction IV. To algebraically closed base fields. In §2.8 we associated with an extension $0 \to \mathfrak{M} \to_\alpha \mathfrak{E} \to_\beta \mathfrak{J} \to 0$, where $\mathfrak{M}$ is an ideal such that $\mathfrak{M}^{\cdot 2} = 0$ and α is the injection mapping, an equivalence class of (Jordan) factor sets for $\mathfrak{J}$ in $\mathfrak{M}$. Here $\mathfrak{M}$ is a Jordan bimodule for $\mathfrak{J}$ and a factor set for $\mathfrak{J}$ in $\mathfrak{M}$ is a bilinear mapping h $((a,b) \to h(a,b))$ of $\mathfrak{J}$ into $\mathfrak{M}$ satisfying the two conditions:

$$\begin{aligned} & h(a,b) = h(b,a), \\ (12) \qquad & h(a,a).b.a + h(a^{\cdot 2},b).a + h(a^{\cdot 2}.b,a) \\ & \qquad = h(a,b).a^{\cdot 2} + h(a,a)(a.b) + h(a^{\cdot 2},a.b). \end{aligned}$$

Given the extension as above we determine a linear mapping δ of $\mathfrak{J}$ into $\mathfrak{E}$ such that $\delta\beta = 1_{\mathfrak{J}}$. Then we have $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{J}^\delta$ and $h(a,b) = a^\delta b^\delta - (ab)^\delta \in \mathfrak{M}$. A change of δ to another linear mapping δ' of $\mathfrak{J}$ into $\mathfrak{E}$ such that $\delta'\beta = 1_{\mathfrak{J}}$ replaces the factor set h by the equivalent one h'. Here equivalence is defined by the existence of a linear mapping μ' of $\mathfrak{J}$ into $\mathfrak{M}$ such that $h'(a,b) = h(a,b) - (a\,.\,b)^\mu + a^\mu.\,b + b^\mu.\,a$. Conversely, if h' is any factor set for $\mathfrak{J}$ in $\mathfrak{M}$ equivalent to h this h' can be obtained by choosing a suitable δ. It is clear from the definition $h(a,b) = a^\delta b^\delta - (ab)^\delta$ that $\mathfrak{J}^\delta$ is a subalgebra if and only if $h = 0$. It is now clear that $0 \to \mathfrak{M} \to_\alpha \mathfrak{E} \to_\beta \mathfrak{J} \to 0$ splits if and only if given any factor set h defined by this extension then there exists a linear mapping μ of $\mathfrak{J}$ into $\mathfrak{M}$ such that

$$h(a,b) = (a\,.\,b)^\mu - a^\mu.\,b - b^\mu.\,a, \qquad a,b \in \mathfrak{J}. \tag{13}$$

Now let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{J}/\Phi$, $(v_1, \cdots, v_r)$ a basis for $\mathfrak{M}/\Phi$. Clearly (13) will hold for a linear mapping μ if and only if it holds for all $a = u_i$, $b = u_j$ in the basis for $\mathfrak{J}/\Phi$. Put $u_i^\mu = \sum_p \mu_{ip} v_p$, $h(u_i, u_j) = \sum_q \eta_{ijq} v_q$, $u_i.\,u_j = \sum_k \gamma_{ijk} u_k$, $v_p.\,u_i = \sum_q \delta_{piq} v_q$. Then (13) for $a = u_i$, $b = u_j$ is equivalent to the set of linear equations

$$\eta_{ijq} = \sum_k \gamma_{ijk}\mu_{kq} - \sum_p \mu_{ip}\delta_{pjq} - \sum_p \mu_{jp}\delta_{piq} \tag{14}$$

for the μ's in Φ. Here the coefficients η_{ijq}, δ_{pjq}, γ_{ijk} are elements of Φ determined by h, the multiplication table in $\mathfrak{J}$ and the multiplication table $v_p.\,u_i = \sum \delta_{piq} v_q$ for $\mathfrak{M}$ as $\mathfrak{J}$-bimodule. The solvability of (14) for the μ's in Φ is a necessary and sufficient condition that the extension splits.

Let Ω be the algebraic closure of Φ. Then the exact sequence $0 \to \mathfrak{M} \to_\alpha \mathfrak{E} \to_\beta \mathfrak{J} \to 0$ gives rise to the Jordan algebra exact sequence $0 \to \mathfrak{M}_\Omega \to_\alpha \mathfrak{E}_\Omega \to_\beta \mathfrak{J}_\Omega \to 0$ where α and β are the linear extensions of the given α and β. We have $\mathfrak{M}_\Omega^{\cdot 2} = 0$ so the new extension is a null extension. Also $\mathfrak{J}_\Omega$ is separable (equivalent to semisimple since the base field is algbraically closed). If h is a factor set determined by the extension $0 \to \mathfrak{M} \to \mathfrak{E} \to \mathfrak{J} \to 0$ then the linear extension of h to $\mathfrak{J}_\Omega$ into $\mathfrak{M}_\Omega$ is a factor set for $0 \to \mathfrak{M}_\Omega \to \mathfrak{E}_\Omega \to \mathfrak{J}_\Omega \to 0$. The bases (u_i), (v_p) are bases for $\mathfrak{J}_\Omega/\Omega$ and $\mathfrak{M}_\Omega/\Omega$ and we have the same coefficients η_{ijq}, δ_{pjq}, γ_{ijk} as before. Now assume the theorem holds in the algebraically closed case. Then the equations (14) have a solution (μ_{ip}), $\mu_{ip} \in \Omega$. Since these are linear equations with coefficients in Φ it follows that they have a solution also in Φ. Then the given extension splits. Hence it suffices to prove the theorem for algebraically closed base fields. We may assume also that $\mathfrak{E}$ has an identity element and $\mathfrak{M}^{\cdot 2} = 0$.

Reduction V. To $\mathfrak{J}$ simple. Suppose $\mathfrak{J}$ is not simple. Then $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ where $\mathfrak{J}_i$ is a simple ideal and $s > 1$. Then $\mathfrak{E}' = \mathfrak{E}/\mathfrak{M} = \mathfrak{E}_1' \oplus \mathfrak{E}_2' \oplus \cdots \oplus \mathfrak{E}_s'$ where $\mathfrak{E}_i' \cong \mathfrak{J}_i$. Let $1_i'$ be the identity of $\mathfrak{E}_i'$. Then the elements $1_i'$, $1 \leqq i \leqq s$, are orthogonal idempotents in $\mathfrak{E}'$ such that $\sum 1_i' = 1' = 1 + \mathfrak{M}$. By Lemma 3 of

§3.7 (p. 149) there exist orthogonal idempotents e_i, $1 \leqq i \leqq s$, in $\mathfrak{E}$ such that $\sum e_i = 1$ and $1_i' = e_i + \mathfrak{M}$. Let $\mathfrak{E} = \sum_{i \leqq j} \mathfrak{E}_{ij}$ be the Peirce decomposition of $\mathfrak{E}$ relative to the e_i. Then $\mathfrak{E}_{ii} = \mathfrak{E}U_{e_i}$ contains the ideal $\mathfrak{M}U_{e_i} = \mathfrak{M} \cap \mathfrak{E}U_{e_i}$ and $\mathfrak{E}_{ii}/\mathfrak{M}U_{e_i} \cong \mathfrak{E}'U_{1_i'} = \mathfrak{E}_i' \cong \mathfrak{J}_i'$. Since $\dim \mathfrak{J}_i < \dim \mathfrak{J}$, $\dim \mathfrak{E}_{ii} < \dim \mathfrak{E}$. Also $\mathfrak{J}_i$ is separable and $(\mathfrak{M}U_{e_i})^{\cdot 2} = 0$. Hence, using induction on dimensionality, we conclude the existence of a subalgebra $\mathfrak{K}_i$ of $\mathfrak{E}_{ii}$ such that $\mathfrak{K}_i \cong \mathfrak{J}_i$. Since $\mathfrak{E}_{ii} . \mathfrak{E}_{jj} = 0$ if $i \neq j$ we have $\mathfrak{K}_i . \mathfrak{K}_j = 0$. Hence $\mathfrak{K} \equiv \mathfrak{K}_1 + \mathfrak{K}_2 + \cdots + \mathfrak{K}_s$ is a subalgebra of $\mathfrak{E}$ isomorphic to $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$. Consequently, $\mathfrak{E} = \mathfrak{K} \oplus \mathfrak{M}$.

We have now reduced the proof to the case: Φ algebraically closed, $\mathfrak{E}$ has 1, $\mathfrak{M}^{\cdot 2} = 0$, $\mathfrak{J}$ is simple. We now distinguish the possible structures for $\mathfrak{J}$: I. $\mathfrak{J} \cong \Phi$. II. $\mathfrak{J}$ is the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space $\mathfrak{V}$ with $\dim \mathfrak{V} > 1$. III. $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $n \geqq 3$ and $(\mathfrak{D}, j)$ is a (split) composition algebra which is associative if $n \geqq 4$. The case I is trivial since $\mathfrak{E}$ has an identity element. To settle case II we prove

Lemma 2. *Let $\mathfrak{E}$ be a Jordan algebra with* 1 *containing an ideal $\mathfrak{M}$ such that $\mathfrak{M}^{\cdot 2} = 0$. Assume $\mathfrak{E}$ contains elements $v_1, v_2, \cdots, v_r$ such that $v_i^{\cdot 2} = 1$, $v_i . v_j = 0$ if $i \neq j$ and an element u such that $u^{\cdot 2} \equiv 1 \pmod{\mathfrak{M}}$, $v_i . u \equiv 0 \pmod{\mathfrak{M}}$, $i = 1, \cdots, r$. Then there exists an element v in $\mathfrak{E}$ such that $v \equiv u \pmod{\mathfrak{M}}$, $v . v_i = 0$ and $v^{\cdot 2} = 1$.*

Proof. Since $v_i^{\cdot 2} = 1$ $R_{v_i}^{\;3} = R_{v_i}$. Since $v_i . v_j = 0$ if $i \neq j$, $R_{v_i}^{\;2} R_{v_j} + R_{v_j} R_{v_i}^{\;2} + R_{v_i \cdot v_j \cdot v_i} = 2 R_{v_i} R_{v_i \cdot v_j} + R_{v_j} R_{v_i \cdot 2}$ gives $(1 - R_{v_i}^{\;2}) R_{v_j} = R_{v_j} R_{v_i}^{\;2}$. This and $(1 - R_{v_i}^{\;2}) R_{v_i} = 0$ imply that if $W = (1 - R_{v_1}^{\;2})(1 - R_{v_2}^{\;2}) \cdots (1 - R_{v_r}^{\;2})$ then $W R_{v_i} = 0$, $i = 1, 2, \cdots; r$. Put $w = uW$. Then $w \equiv u \pmod{\mathfrak{M}}$ since $u . v_i \equiv 0 \pmod{\mathfrak{M}}$. Also $w . v_i = 0$ since $W R_{v_i} = 0$. Since $2 R_w^{\;3} + R_{w^{\cdot 3}} = 3 R_w R_{w^{\cdot 2}}$ we have $w^{\cdot 3} . v_i = 0$. Now put $v = \frac{1}{2}(3w - w^{\cdot 3})$. Then $v . v_i = 0$. Also $w^{\cdot 2} \equiv u^{\cdot 2} \equiv 1 \pmod{\mathfrak{M}}$ so $w^{\cdot 3} \equiv w \pmod{\mathfrak{M}}$ and $v = \frac{1}{2}(3w - w^{\cdot 3}) \equiv w \equiv u \pmod{\mathfrak{M}}$. Since $u^{\cdot 2} \equiv 1 \pmod{\mathfrak{M}}$, $w^{\cdot 2} \equiv 1 \pmod{\mathfrak{M}}$. Then $w^{\cdot 2} = 1 + z$ where $z^{\cdot 2} = 0$ and $(w^{\cdot 2} - 1)^{\cdot 2} = w^{\cdot 4} - 2w^{\cdot 2} + 1 = 0$. Hence $v^{\cdot 2} = \frac{1}{4}(9w^{\cdot 2} - 6w^{\cdot 4} + w^{\cdot 6}) = 1$. Hence v satisfies the required conditions.

If $\mathfrak{J}$ is the Jordan algebra of the nondegenerate symmetric bilinear form f on a vector space over an algebraically closed field, then $\mathfrak{J}$ has a basis $(1, v_1, \cdots, v_n)$ such that $v_i^{\cdot 2} = 1$, $v_i . v_j = 0$ if $i \neq j$. Clearly this and Lemma 2 imply that if $\mathfrak{E}$ and $\mathfrak{M}$ are as in Lemma 2 and $\mathfrak{E}/\mathfrak{M} \cong \mathfrak{J}$ then $\mathfrak{E}$ contains a subalgebra isomorphic to $\mathfrak{J}$. Thus the splitting theorem holds in case II. To settle the remaining case we require the following result on alternative algebras with involution.

Lemma 3. *Let $(\mathfrak{F}, j)$ be an alternative algebra with* 1 *and involution over an algebraically closed field, $\mathfrak{N}$ an ideal in $(\mathfrak{F}, j)$ such that $\mathfrak{N}^2 = 0$ and $(\mathfrak{F}/\mathfrak{N}, j)$ is isomorphic to a composition algebra $(\mathfrak{D}, j)$. Then $(\mathfrak{F}, j)$ contains a subalgebra $(\mathfrak{L}, j)$ isomorphic to $(\mathfrak{D}, j)$.*

Proof. Let $(\mathfrak{H}, j)$ be a composition subalgebra of $(\mathfrak{F}, j)$. Since $(\mathfrak{H}, j)$ is simple the canonical homomorphism of $(\mathfrak{F},j)$ onto $(\mathfrak{F}/\mathfrak{N},j)$ maps $(\mathfrak{H},j)$ isomorphically onto the subalgebra $((\mathfrak{H}+\mathfrak{N})/\mathfrak{N},j)$ of $(\mathfrak{F}/\mathfrak{N},j)$. Assume $\mathfrak{H}+\mathfrak{N}\subset\mathfrak{F}$. Then if we recall the inductive construction of composition algebras given in the proof of Theorem 4.5, we see that there exists an element $u\in\mathfrak{F}$ such that $\bar{u}\equiv -u \pmod{\mathfrak{N}}$ (where $\bar{a}=a^j$), $uh\equiv\bar{h}u \pmod{\mathfrak{N}}$ for every $h\in\mathfrak{H}$ and $u^2\equiv 1 \pmod{\mathfrak{N}}$. Since we may replace u by $\frac{1}{2}(u-\bar{u})$ we may assume $\bar{u}=-u$. Moreover, we can choose a basis $(1,v_1,\cdots,v_r)$ for $\mathfrak{H}$ such that $\bar{v}_i=-v_i, v_i^2=1$ and $v_iv_j=-v_jv_i$, if $i\neq j$. We now consider the Jordan algebra $\mathfrak{F}^+$ which contains the ideal $\mathfrak{N}$ such that $\mathfrak{N}^{\cdot 2}=0$. Also we have $v_i^{\cdot 2}=1$, $v_i.v_j=\frac{1}{2}(v_iv_j+v_jv_i)=0$, $i\neq j$, and $u.v_i=\frac{1}{2}(uv_i+v_iu)=\frac{1}{2}(uv_i-\bar{v}_iu)\equiv 0 \pmod{\mathfrak{N}}$, $u^{\cdot 2}\equiv 1 \pmod{\mathfrak{N}}$. We can now apply Lemma 2 and its proof to $\mathfrak{E}=\mathfrak{F}^+$, $\mathfrak{M}=\mathfrak{N}$. Accordingly, we obtain the element $w=u(1-R_{v_1}{}^2)\cdots(1-R_{v_r}{}^2)$ and $v=\frac{1}{2}(3w-w^{\cdot 2})$ and we have $v\equiv u \pmod{\mathfrak{N}}$, $v.v_i=0$ and $v^{\cdot 2}=1$. Then $vv_i=-v_iv$ and consequently $vh=\bar{h}v$ for all $h\in\mathfrak{H}$. Also $v^2=1$. Since $\bar{u}=-u$ and $\bar{v}_i=-v_i$, $uR_{v_i}=\frac{1}{2}(uv_i+v_iu)$ is symmetric and $uR_{v_i}{}^2$ is skew. It follows that $\bar{w}=-w$ and $\bar{v}=-v$. Now consider the subspace $\mathfrak{H}+\mathfrak{H}v$. We claim that this is a composition subalgebra of $(\mathfrak{F},j)$. If $h,k\in\mathfrak{H}$ then $(hv)(kv)=(v\bar{h})(kv)=v(\bar{h}k)v$ (by one of Moufang's identities) $=((\bar{k}h)v)v=(\bar{k}h)v^2=\bar{k}h$. Also $k(hv)=((kv)v)(hv)=((v\bar{k})v)(hv)=(v\bar{k}v)(hv)=v(\bar{k}(v(hv))$ (by one of Moufang's identities) $=v(\bar{k}\bar{h})=(hk)v$. Finally, $\overline{(hv)k}=\bar{k}(\overline{hv})=-\bar{k}(v\bar{h})=-\bar{k}(hv)=-(h\bar{k})v$ so $(hv)k=v(k\bar{h})=(h\bar{k})v$. Since $(\mathfrak{H},j)$ is isomorphic to a proper subalgebra of a composition algebra it is associative. It follows from the proof of Theorem 4.5 and the multiplication formulas just derived that $\mathfrak{H}+\mathfrak{H}v$ is a homomorphic image of a composition algebra whose dimensionality is $2\dim\mathfrak{H}$. Hence $\mathfrak{H}+\mathfrak{H}v$ is a subalgebra of $(\mathfrak{F},j)$ isomorphic to a composition algebra of dimensionality $2\dim\mathfrak{H}$. Since we can begin our process with $\mathfrak{H}=\Phi 1$ this leads to a composition subalgebra of $(\mathfrak{F},j)$ isomorphic to $(\mathfrak{D},j)$.

We can now prove

LEMMA 4. *Let $\mathfrak{E}$ be a finite-dimensional Jordan algebra with 1 over an algebraically closed field containing an ideal $\mathfrak{M}$ such that $\mathfrak{M}^{\cdot 2}=0$ and $\mathfrak{E}/\mathfrak{M}\cong\mathfrak{H}(\mathfrak{D}_n,J_1)$ where $n\geqq 3$ and $(\mathfrak{D},j)$ is a composition algebra. Then $\mathfrak{E}$ contains a subalgebra $\mathfrak{K}$ isomorphic to $\mathfrak{H}(\mathfrak{D}_n,J_1)$.*

PROOF. We have seen in Theorem 3.10 that the hypotheses of the present lemma imply that $\mathfrak{E}\cong\mathfrak{H}(\mathfrak{F}_n,J_1)$ where the ideal corresponding to $\mathfrak{M}$ has the form $\mathfrak{H}(\mathfrak{N}_n,J_1)$, $\mathfrak{N}$ an ideal in $(\mathfrak{F},j)$ such that $(\mathfrak{F},j)/\mathfrak{N}\cong(\mathfrak{D},j)$. Since $\mathfrak{M}^{\cdot 2}=0$, $\mathfrak{N}^{\cdot 2}=0$ (Theorem 3.10). It is sufficient to show that there exists a subalgebra of $\mathfrak{H}(\mathfrak{F}_n,J_1)$ isomorphic to $\mathfrak{H}(\mathfrak{D}_n,J_1)$. By Lemma 3, there exists a subalgebra $(\mathfrak{L},j)$ of $(\mathfrak{F},j)$ isomorphic to $(\mathfrak{D},j)$. This gives a subalgebra of $\mathfrak{H}(\mathfrak{F}_n,J_1)$ isomorphic to $\mathfrak{H}(\mathfrak{D}_n,J_1)$.

Since Lemma 4 settles the case III of the structure of the simple algebra $\mathfrak{E}/\mathfrak{M}$ we have completed the proof of the following

THEOREM 13. *Let $\mathfrak{E}$ be a finite-dimensional Jordan algebra, $\mathfrak{M}$ an ideal in $\mathfrak{E}$ such that $\mathfrak{E}/\mathfrak{M}$ is separable. Then there exists a subalgebra $\mathfrak{K}$ of $\mathfrak{E}$ such that $\mathfrak{E} = \mathfrak{K} \oplus \mathfrak{M}$ (as vector spaces).*

As we have seen above and in §2.8, if $\mathfrak{M}^{\cdot 2} = 0$, this result is equivalent to the following one.

COROLLARY. *Let $\mathfrak{J}$ be a finite-dimensional separable Jordan algebra, $\mathfrak{M}$ a finite-dimensional bimodule for $\mathfrak{J}$, h a factor set of $\mathfrak{J}$ in $\mathfrak{M}$, that is, a bilinear mapping of $\mathfrak{J}$ in $\mathfrak{M}$ satisfying* (12). *Then there exists a linear mapping μ of $\mathfrak{J}$ into $\mathfrak{M}$ such that $h(a,b) = (a\,.\,b)^{\mu} - a^{\mu}.\,b - b^{\mu}.\,a$.*

7. **Derivations into bimodules.** If $\mathfrak{A}$ is an algebra and $\mathfrak{M}$ is a bimodule for $\mathfrak{A}$ then a *derivation D of $\mathfrak{A}$ into $\mathfrak{M}$* is a linear mapping $\mathfrak{A}$ into $\mathfrak{M}$ such that $(ab)D = a(bD) + (aD)b$ for all $a, b \in \mathfrak{A}$. If $\mathfrak{E} = \mathfrak{A} \oplus \mathfrak{M}$ is the split null extension of $\mathfrak{A}$ by $\mathfrak{M}$ then we extend D linearly to $\mathfrak{E}$ so that $\mathfrak{M}D = 0$. Then it is immediate that the extension D is derivation in the algebra $\mathfrak{E}$ mapping $\mathfrak{A}$ into $\mathfrak{M}$ and $\mathfrak{M}$ into 0. It is clear from this that the set Der $(\mathfrak{A},\mathfrak{M})$ of derivations of $\mathfrak{A}$ into $\mathfrak{M}$ coincides with the set of restrictions to $\mathfrak{A}$ of Der $\mathfrak{E} \cap \mathfrak{K}(\mathfrak{A},\mathfrak{M})$ where Der $\mathfrak{E}$ is the Lie algebra of derivations of $\mathfrak{E}$ and $\mathfrak{K}(\mathfrak{A},\mathfrak{M})$ is the subspace of $\mathrm{Hom}_{\Phi}(\mathfrak{E},\mathfrak{E})$ of mappings sending $\mathfrak{A}$ into $\mathfrak{M}$, $\mathfrak{M}$ into 0. Hence $\mathrm{Der}(\mathfrak{A},\mathfrak{M})$ is a subspace of $\mathrm{Hom}_{\Phi}(\mathfrak{A},\mathfrak{M})$. Also it is clear that if $D \in \mathrm{Der}\,\mathfrak{E} \cap \mathfrak{K}(\mathfrak{A},\mathfrak{M})$ and the restriction $D|\mathfrak{A}$ of D to $\mathfrak{A}$ is 0 then $D = 0$. Hence the linear mapping $D \to D|\mathfrak{A}$ of $\mathrm{Der}\,\mathfrak{E} \cap \mathfrak{K}(\mathfrak{A},\mathfrak{M})$ onto $\mathrm{Der}(\mathfrak{A},\mathfrak{M})$ is an isomorphism of vector spaces. In this way we can identify $\mathrm{Der}(\mathfrak{A},\mathfrak{M})$ with $\mathrm{Der}\,\mathfrak{E} \cap \mathfrak{K}(\mathfrak{A},\mathfrak{M})$.

Now let $\mathfrak{J}$ be Jordan and $\mathfrak{M}$ a (Jordan) bimodule for $\mathfrak{J}$ so that $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ is a Jordan algebra. Then we shall call the derivation D of $\mathfrak{J}$ into $\mathfrak{M}$ *inner* if the corresponding element of $\mathrm{Der}\,\mathfrak{E}$ (the linear extension such that $\mathfrak{M}D = 0$) is an inner derivation in $\mathfrak{E}$. Thus the condition is that there exist $v_i, w_i \in \mathfrak{E}$ such that $D = \sum_i [R_{v_i} R_{w_i}]$ or, equivalently, D is the mapping $x \to \sum_i [v_i, x, w_i]$. If we write $v_i = y_i + b_i$, $w_i = z_i + c_i$ where $y_i, z_i \in \mathfrak{M}$, b_i, $c_i \in \mathfrak{J}$, then $[R_{v_i} R_{w_i}] = [R_{b_i} R_{c_i}] + [R_{y_i} R_{c_i}] - [R_{z_i} R_{b_i}]$ since $[R_{y_i} R_{z_i}] = 0$ follows from $\mathfrak{M}\,.\,\mathfrak{E} \subseteq \mathfrak{M}$ and $\mathfrak{M}^{\cdot 2} = 0$. Hence $D = \sum [R_{b_i} R_{c_i}] + \sum [R_{y_i} R_{c_i}] - \sum [R_{z_i} R_{b_i}]$ and if $x \in \mathfrak{J}$ then the condition $xD \in \mathfrak{M}$ implies that $x \sum [R_{b_i} R_{c_i}] = 0$. Also $yD = 0$ for $y \in \mathfrak{M}$ implies $y \sum [R_{b_i} R_{c_i}] = 0$. Hence $\sum [R_{b_i} R_{c_i}] = 0$ and D has the form $\sum [R_{y_i} R_{b_i}]$ where the $y_i \in \mathfrak{M}$ and the $b_i \in \mathfrak{J}$. Conversely, it is clear that any derivation $D = \sum [R_{y_i} R_{b_i}]$, $y_i \in \mathfrak{M}$, $b_i \in \mathfrak{J}$ in $\mathfrak{E}$ maps $\mathfrak{J}$ into $\mathfrak{M}$ and $\mathfrak{M}$ into 0 so its restriction to $\mathfrak{J}$ is an inner derivation of $\mathfrak{J}$ into $\mathfrak{M}$. It is clear from the definition that the set $\mathrm{Inder}(\mathfrak{J},\mathfrak{M})$ of inner derivations is a subspace of $\mathrm{Der}(\mathfrak{J},\mathfrak{M})$. In fact, the identification of $\mathrm{Der}(\mathfrak{J},\mathfrak{M})$ with $\mathrm{Der}\,\mathfrak{E} \cap \mathfrak{K}(\mathfrak{J},\mathfrak{M})$ identifies $\mathrm{Inder}(\mathfrak{J},\mathfrak{M})$ with $\mathrm{Inder}\,\mathfrak{E} \cap \mathfrak{K}(\mathfrak{J},\mathfrak{M})$.

In this section we shall prove the following theorem on derivations of finite-dimensional separable Jordan algebras into bimodules.

THEOREM 14 (HARRIS). *Let $\mathfrak{J}$ be a finite-dimensional separable Jordan algebra such that the degrees of the simple components of $\mathfrak{J}$ over their centers are not divisible by the characteristic of the base field Φ and let $\mathfrak{M}$ be a finite-dimensional bimodule for $\mathfrak{J}$. Then every derivation of $\mathfrak{J}$ into $\mathfrak{M}$ is inner.*

We shall show also that if $\mathfrak{J}$ has a special simple component whose degree over its center is divisible by the characteristic then there exists a finite-dimensional bimodule $\mathfrak{M}$ for $\mathfrak{J}$ with a noninner derivation of $\mathfrak{J}$ into $\mathfrak{M}$. In Chapter IX we shall show that if $\mathfrak{J}$ is separable and is simple and exceptional then the derivations of $\mathfrak{J}$ into finite-dimensional bimodules are inner for all characteristics. In sum, we shall have that if $\mathfrak{J}$ is finite-dimensional separable then all derivations of $\mathfrak{J}$ into finite-dimensional bimodules are inner if and only if the degrees of the special simple components over their centers are not divisible by the characteristic. This sharper form of Theorem 14 is due also to Harris [2]. We remark also that the special case of this result for characteristic 0 was proved earlier by Jacobson in [9]. For this case we shall give a proof based on Lie algebras in the next chapter. On the other hand, the proof we shall give here of Harris' theorem will make use of the determination of the finite-dimensional simple algebras over an algebraically closed field, and some of the results on bimodules for these algebras which have been obtained in this chapter.

We shall show first that it suffices to prove Theorem 14 for algebraically closed base fields. The reduction of the general case to this one will be based on some well-known results on tensor products of algebras which we shall now derive. First, let $\mathfrak{A}$ be an arbitrary algebra over a field Γ which is an extension field of Φ and let P/Φ be another field extension of Φ. Let σ be an isomorphism of Γ/Φ into P/Φ. Then P is a right module for Γ relative to the composition $\rho\gamma = \rho\gamma^\sigma = \gamma^\sigma\rho$ for $\gamma \in \Gamma$, $\rho \in \mathrm{P}$. Hence we may form the tensor product $\mathrm{P} \otimes_\Gamma \mathfrak{A}$ considering $\mathfrak{A}$ as left Γ-module and P as right Γ-module in the manner just indicated. It is immediate that this tensor product is an algebra over P with the compositions $\rho(\sum \rho_i \otimes a_i) = \sum \rho\rho_i \otimes a_i$, $(\sum \rho_i \otimes a_i)(\sum \tau_j \otimes b_j) = \sum \rho_i\tau_j \otimes a_ib_j$, $\rho, \rho_i, \tau_j \in \mathrm{P}$, $a_i, b_j \in \mathfrak{A}$. We denote this algebra $\mathfrak{A}^\sigma{}_\mathrm{P}$. The special case in which $\mathrm{P} \supseteq \Gamma$ and σ is the injection is the usual algebra $\mathfrak{A}_P$ obtained by extending the base field Γ of $\mathfrak{A}$ to P. If (u_ι) is a basis for $\mathfrak{A}/\Gamma$ with multiplication table $u_\iota u_\kappa = \sum \gamma_{\iota\kappa\lambda}u_\lambda$, $\gamma_{\iota\kappa\lambda} \in \Gamma$, then the elements u_ι $(= 1 \otimes u_\iota)$ constitute a basis for $\mathfrak{A}^\sigma{}_\mathrm{P}$ over P and these have the multiplication table $u_\iota u_\kappa = \sum \gamma^\sigma_{\iota\kappa\lambda}u_\lambda$.

We now assume that Γ is finite-dimensional separable over Φ and Ω is the algebraic closure of Φ. Then there are n distinct isomorphisms, $\sigma_1, \cdots, \sigma_n$, of Γ/Φ into Ω/Φ where $n = \dim \Gamma/\Phi$. We can now state the following result.

LEMMA 1. *Let $\mathfrak{A}$ be an algebra over a field Γ, Φ a subfield of Γ such that Γ/Φ is finite-dimensional separable and let Ω be the algebraic closure of Φ, $\sigma_1, \cdots, \sigma_n$ the different isomorphisms of Γ/Φ into Ω/Φ. Then $(\mathfrak{A}/\Phi)_\Omega = \mathfrak{A}_1 \oplus \mathfrak{A}_2$*

$\oplus \cdots \oplus \mathfrak{A}_n$ *where* $\mathfrak{A}_i$ *is an ideal and is isomorphic as algebra over* Ω *to* $\mathfrak{A}_\Omega^{\sigma_i}$.

PROOF. We recall that $\Gamma_\Omega = \Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n$ where Ω_i is an ideal which is one dimensional over Ω: $\Omega_i = \Omega\varepsilon_i$, $\varepsilon_i^2 = \varepsilon_i \neq 0$, $\sum \varepsilon_i = 1$ (Jacobson, *Theory of Rings*, p. 97). We identify Γ with the Φ-subalgebra of Γ_Ω of elements $\gamma \equiv 1 \otimes \gamma$, $\gamma \in \Phi$, as usual. If $\gamma \in \Gamma$, $\gamma\varepsilon_i = \varepsilon_i\gamma = \gamma^{\sigma_i}\varepsilon_i$ where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are the distinct isomorphisms of Γ/Φ into Ω/Φ. Now consider $(\mathfrak{A}/\Phi)_\Omega$. This is an algebra over Ω and a left Γ-module such that $\gamma(\omega \otimes a) = \omega \otimes \gamma a$, $\gamma \in \Gamma$, $\omega \in \Omega$, $a \in \mathfrak{A}$. Since this action commutes with the multiplications by elements of Ω, $(\mathfrak{A}/\Phi)_\Omega$ is a left Γ_Ω-module such that $(\gamma \otimes \omega')(\omega \otimes a) = \omega'\omega \otimes \gamma a$, $\omega' \in \Omega$. It is immediate that relative to this composition $(\mathfrak{A}/\Phi)_\Omega$ is an algebra over the commutative ring Γ_Ω. Since $\Gamma_\Omega = \Omega\varepsilon_1 \oplus \Omega\varepsilon_2 \oplus \cdots \oplus \Omega\varepsilon_n$, $(\mathfrak{A}/\Phi)_\Omega = \varepsilon_1(\mathfrak{A}/\Phi)_\Omega \oplus \cdots \oplus \varepsilon_n(\mathfrak{A}/\Phi)_\Omega$ and $\varepsilon_i(\mathfrak{A}/\Phi)_\Omega$ is an ideal in $(\mathfrak{A}/\Phi)_\Omega$. We have $\varepsilon_i(\omega\gamma^{\sigma_i} \otimes a) = \gamma^{\sigma_i}\varepsilon_i(\omega \otimes a) = \gamma\varepsilon_i(\omega \otimes a) = \varepsilon_i(\omega \otimes \gamma a)$. Hence we have the (algebra) homomorphism of $\mathfrak{A}_\Omega^{\sigma_i}$ onto $\varepsilon_i(\mathfrak{A}/\Phi)_\Omega$ sending $\omega \otimes a \to \varepsilon_i(\omega \otimes a)$. We claim that this is an isomorphism. For this let (u_ι) be a basis for $\mathfrak{A}/\Gamma$, $(\lambda_1, \cdots, \lambda_n)$ a basis for Γ/Φ. Then $(\lambda_j u_\iota)$ is a basis for $\mathfrak{A}/\Phi$ and for $(\mathfrak{A}/\Phi)_\Omega$ over Ω and (λ_j) is a basis for Γ_Ω/Ω. Since $\lambda_j u_\iota = \lambda_j(1 \otimes u_\iota)$ the elements $\lambda_j(1 \otimes u_\iota)$ constitute a basis for $(\mathfrak{A}/\Phi)_\Omega$ over Ω. Since $(\lambda_1, \cdots, \lambda_n)$ and $(\varepsilon_1, \cdots, \varepsilon_n)$ are bases for Γ_Ω/Ω it follows that the elements $\varepsilon_j(1 \otimes u_\iota)$ constitute a basis for $(\mathfrak{A}/\Phi)_\Omega$ over Ω and for fixed j a basis for $\varepsilon_j(\mathfrak{A}/\Phi)_\Omega$ over Ω. It follows that the canonical homomorphism of $\mathfrak{A}_\Omega^{\sigma_i}$ onto $\varepsilon_i(\mathfrak{A}/\Phi)_\Omega$ is an isomorphism. Thus $(\mathfrak{A}/\Phi)_\Omega \cong \mathfrak{A}_\Omega^{\sigma_1} \oplus \cdots \oplus \mathfrak{A}_\Omega^{\sigma_n}$.

If $\mathfrak{A}_\Omega^{\sigma_i}$ and $\mathfrak{A}_\Omega^{\sigma_j}$ are as in the lemma then $\sigma_i^{-1}\sigma_j$ is an isomorphism of the subfield Γ^{σ_i} of Ω/Φ onto Γ^{σ_j}. Since Ω is the algebraic closure of Φ this can be extended to an automorphism τ in Ω/Φ. If (u_ι) is a basis for $\mathfrak{A}/\Gamma$ then the corresponding elements, which we write again as u_ι, constitute bases for $\mathfrak{A}_\Omega^{\sigma_i}$ and $\mathfrak{A}_\Omega^{\sigma_j}$ and we have the multiplication tables $u_\iota u_\kappa = \sum \gamma_{\iota\kappa\lambda}^{\sigma_i} u_\lambda$ and $u_\iota u_\kappa = \sum \gamma_{\iota\kappa\lambda}^{\sigma_j} u_\lambda$ in these algebras. Let η be the mapping $\sum \omega_\iota u_\iota \to \sum \omega_\iota^\tau u_\iota$, $\omega_\iota \in \Omega$, of $\mathfrak{A}_\Omega^{\sigma_i}$ into $\mathfrak{A}_\Omega^{\sigma_j}$. It is clear that this is τ-semilinear and bijective. Moreover, $(ab)^\eta = a^\eta b^\eta$. We shall call such a mapping a *τ-semilinear isomorphism* of the algebras. This, of course, can be defined in a similar manner for any two algebras over possibly different fields. If $\Omega \supseteq \Gamma$ is the algebraic closure of Γ then it is immediate also that the usual extension $(\mathfrak{A}/\Gamma)_\Omega$ is semilinearly isomorphic to $\mathfrak{A}^{\sigma_i}\Omega$. It is clear also that the semilinear isomorph of a central simple algebra is central simple. Hence if $\mathfrak{A}/\Gamma$ is central simple then the algebras $\mathfrak{A}_\Omega^{\sigma_i}$ are simpler over Ω. We remark that the same reasoning shows also that if $\mathfrak{A}$ is finite-dimensional strictly power associative then the same is true of the algebras $\mathfrak{A}_\Omega^{\sigma_i}$ and their degrees are the same as the degree of $\mathfrak{A}/\Gamma$.

We can now give the reduction of Theorem 14 to the algebraically closed case. We note first that since $\mathrm{Der}\,\mathfrak{E}_\Omega = (\mathrm{Der}\,\mathfrak{E})_\Omega$ for Ω an extension field of the base field of the algebra $\mathfrak{E}$ (p. 253), if $\mathfrak{J}$ is a Jordan algebra and $\mathfrak{M}$ is a bimodule then

$\mathrm{Der}(\mathfrak{J}_\Omega,\mathfrak{M}_\Omega) = \mathrm{Der}(\mathfrak{J},\mathfrak{M})_\Omega$. Also $\mathrm{Inder}(\mathfrak{J}_\Omega,\mathfrak{M}_\Omega) = (\mathrm{Inder}(\mathfrak{J},\mathfrak{M}))_\Omega$. Hence it is clear that all the derivations of $\mathfrak{J}$ into $\mathfrak{M}$ are inner, that is, $\mathrm{Der}(\mathfrak{J},\mathfrak{M}) = \mathrm{Inder}(\mathfrak{J},\mathfrak{M})$, if and only if all the derivations of $\mathfrak{J}_\Omega$ into $\mathfrak{M}_\Omega$ are inner. Now let $\mathfrak{J}$ satisfy the hypotheses of Theorem 14 and let $\mathfrak{M}$ be a finite-dimensional bimodule for $\mathfrak{J}$. Let $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ be the decomposition of $\mathfrak{J}$ into simple components. Let Γ_i be the center of $\mathfrak{J}_i$. Then Γ_i is separable and Lemma 1 and the remarks following it show that $\mathfrak{J}_{i\Omega}$ is a direct sum of simple ideals whose degrees over Ω are the same as the degree of $\mathfrak{J}_i$ over Γ_i. Hence $\mathfrak{J}_\Omega = \mathfrak{J}_{1\Omega} \oplus \mathfrak{J}_{2\Omega} \oplus \cdots \oplus \mathfrak{J}_{s\Omega}$ satisfies the hypotheses of Theorem 14 so it is clearly sufficient to prove the theorem for an algebraically closed base field. Accordingly, we assume from now on that Φ is algebraically closed.

Now suppose that $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots \oplus \mathfrak{M}_k$ where the $\mathfrak{M}_j$ are sub-bimodules. If E_j is the projection on $\mathfrak{M}_j$ determined by this decomposition and D is a derivation of $\mathfrak{J}$ into $\mathfrak{M}$ then $D_j = DE_j$ is a derivation of $\mathfrak{J}$ into $\mathfrak{M}_j$. It is immediate from the definitions that D is inner if and only if every D_j is inner. Since every $\mathfrak{J}$-bimodule is a direct sum of irreducible bimodules it is clear that it suffices to prove the theorem for irreducible bimudules. Since $\mathfrak{J}$ has an identity element 1 we have the decomposition $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_{\frac{1}{2}} \oplus \mathfrak{M}_1$ where $\mathfrak{M}_i = \{x_i \mid x_i . 1 = ix_i\}$ and the $\mathfrak{M}_i$ are sub-bimodules. This is an immediate consequences of the Peirce decomposition $\mathfrak{E} = \mathfrak{E}_0 \oplus \mathfrak{E}_{\frac{1}{2}} \oplus \mathfrak{E}_1$ of the split null extension $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ relative to the idempotent 1 in $\mathfrak{E}$ and the obvious relation $\mathfrak{M}_i = \mathfrak{M} \cap \mathfrak{E}_i$ for the ideal $\mathfrak{M}$ in $\mathfrak{E}$. Since $\mathfrak{J} \subseteq \mathfrak{E}_1$ we have $\mathfrak{M}_0 . \mathfrak{J} = 0$ so $\mathfrak{M}_0$ is a trivial bimodule in the sense that $x_0 . a = 0$, $x_0 \in \mathfrak{M}_0$, $a \in \mathfrak{J}$. If D is a derivation of $\mathfrak{J}$ into $\mathfrak{M}_0$ then $aD = (a.1)D = aD.1 + 1D.a = 0$. Hence D is inner. If D is a derivation of $\mathfrak{J}$ into $\mathfrak{M}_{\frac{1}{2}}$ then $aD = (1.a)D = 1D.a + aD.1 = 1D.a + \frac{1}{2}aD$ so $aD = 2(1D).a = 4[1, a, 1D]$. Hence D is inner. It now follows that it is enough to prove the theorem for $\mathfrak{M}$ unital and irreducible. We show next that we may assume $\mathfrak{J}$ simple. Thus, suppose $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ where the $\mathfrak{J}_i$ are ideals. Then the Peirce decomposition of $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ shows that $\mathfrak{M} = \mathfrak{M}_{11} \oplus \mathfrak{M}_{12} \oplus \mathfrak{M}_{22}$ where $\mathfrak{M}_{ii} = \{x_i \mid x_i . 1_i = x_i\}$, 1_i the identity of $\mathfrak{J}_i$ and $\mathfrak{M}_{12} = \{x_{12} \mid x_{12} . 1_1 = \frac{1}{2}x_{12} = x_{12} . 1_2\}$. Also $\mathfrak{M}_{ij} . \mathfrak{J}_i \subseteq \mathfrak{M}_{ij}$ and $\mathfrak{M}_{ii} . \mathfrak{J}_j = 0$ if $i \neq j$. Hence the $\mathfrak{M}_{ij}$ are sub-bimodules so if $\mathfrak{M}$ is irreducible then either $\mathfrak{M} = \mathfrak{M}_{11}$, $\mathfrak{M} = \mathfrak{M}_{22}$ or $\mathfrak{M} = \mathfrak{M}_{12}$. In the first case, $\mathfrak{M}$ is annihilated by $\mathfrak{J}_2$ and is a unital irreducible bimodule for $\mathfrak{J}_1$. It is clear that if a derivation D of $\mathfrak{J}$ into $\mathfrak{M}$ is inner as a derivation of $\mathfrak{J}_1$ then it is inner as derivation for $\mathfrak{J}$. Hence in this case, it is enough to prove D is inner as derivation of $\mathfrak{J}_1$. Similarly, if $\mathfrak{M} = \mathfrak{M}_{22}$ then it is enough to show that D is an inner derivation of $\mathfrak{J}_2$. If $\mathfrak{M} = \mathfrak{M}_{12}$ then, as above, $a_iD = 4[1_i, a_i, 1_iD]$ for $a_i \in \mathfrak{J}_i$. Also $1_1 . 1_2 = 0$ gives $0 = 1_1D . 1_2 + 1_2D . 1_1 = \frac{1}{2}(1_1D) + \frac{1}{2}(1_2D)$. Hence $1_1D = -1_2D$ and $a_iD = 2[1_i, a_i, (1_i - 1_j)D]$, $i \neq j$. Since $\mathfrak{M}$ is unital $[1, a, x] = 0$ for all $a \in \mathfrak{J}$, $x \in \mathfrak{M}$. Hence $[1_i, a, x] = -[1_j, a, x]$ if $i \neq j$ and

$$2[1_i, a_i, (1_i - 1_j)D] = [1_i - 1_j,\ a_i, (1_i - 1_j)D].$$

Thus $aD = [1_i - 1_j, a, (1_i - 1_j)D]$ for all $a \in \mathfrak{J}$ and so D is inner. It is now clear that we may assume $\mathfrak{J}$ simple.

We have now reduced the proof to the following situation: Φ is algebraically closed, $\mathfrak{M}$ is unital irreducible, $\mathfrak{J}$ is simple. The case $\mathfrak{J} = \Phi 1$ is trivial so we may assume also that the degree of $\mathfrak{J}$ is > 1. The results on the determination of the unital irreducible bimodules for $\mathfrak{J}$ in §§2, 3, 4 shows that unless $\mathfrak{J}$ is either exceptional or $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$ where $(\mathfrak{D},j)$ is a (split) quaternion algebra with involution then every unital irreducible bimodule $\mathfrak{M}$ for $\mathfrak{J}$ is special, that is, there exists a monomorphism η of $\mathfrak{M}$ into a bimodule $\mathfrak{N}$ for $\mathfrak{J}$ such that for $y \in \mathfrak{N}$, $a \in \mathfrak{J}$ we have $y\,.\,a = \frac{1}{2}y(a^{\sigma_1} + a^{\sigma_2})$ where σ_1 and σ_2 are commuting unital associative specializations of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(\mathfrak{N},\mathfrak{N})$. More precisely, we have the following results. If $\mathfrak{J}$ is of degree two, so $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a nondegenerate symmetric bilinear form f on $\mathfrak{V}$ and $\dim \mathfrak{V}$ is even, then every irreducible unital bimodule for $\mathfrak{J}$ is isomorphic to a sub-bimodule of $\mathfrak{N} = C(\mathfrak{V}, f)$ considered as a bimodule for $\mathfrak{J}$ so that $y\,.\,a = \frac{1}{2}(ya + ay)$ for $y \in C(\mathfrak{V},f)$, $a \in \mathfrak{J}$, where ya and ay are the associative products. Since $a \to a_R$ and $a \to a_L$ acting in $C(\mathfrak{V},f)$ are unital associative specializations of $\mathfrak{J}$ in $\mathrm{Hom}_\Phi(C(\mathfrak{V},f), C(\mathfrak{V},f))$ the result indicated is valid in this case. If $\dim \mathfrak{V}$ is odd the same result holds with $C(\mathfrak{V},f)$ replaced by $C(\mathfrak{W},f)$ where $\mathfrak{W} = \Phi u \oplus \mathfrak{V}$ and f is extended to $\mathfrak{W}$ so that $f(u,u) = 1$ and $f(u,v) = 0 = f(v,u)$, $v \in \mathfrak{V}$. Next assume that $\mathfrak{J}$ has degree $n \geqq 3$, so $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_1)$ where $(\mathfrak{D},j)$ is a composition algebra over the algebraically closed field Φ. The results of §§3, 4 show that if $\dim \mathfrak{D} = 1$ or 4, so $\mathfrak{D}_n$ is central simple, and $n > 3$ where $\dim \mathfrak{D} = 4$, then every unital irreducible bimodule for $\mathfrak{J}$ is isomorphic to a sub-bimodule of $\mathfrak{N} \equiv \mathfrak{D}_n$ considered $\mathfrak{J}$-bimodule with the action $y\,.\,a = \frac{1}{2}(ya + ay)$ for $y \in \mathfrak{D}_n$, $a \in \mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_1)$. If $n = 3$ and $\dim \mathfrak{D} = 4$ then the irreducible components of $\mathfrak{D}_n$ as $\mathfrak{J}$-bimodule give all the unital irreducible bimodules for $\mathfrak{J}$ except one, namely, the one obtained from an irreducible component of the Cayley bimodule for $(\mathfrak{D},j)$. If $\dim \mathfrak{D} = 2$ then $\mathfrak{H}(\mathfrak{D}_n, J_1) \cong \Phi_n{}^+$ and every unital irreducible $\mathfrak{J}$-bimodule is isomorphic to a sub-bimodule of $\mathfrak{N} \equiv \Phi_{2n}$ regarded as a bimodule for $\mathfrak{J} = \Phi_n{}^+$ so that for $y \in \Phi_{2n}$, $a \in \mathfrak{J}$ we have

$$(15) \qquad 2y\,.\,a = y \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix} y,$$

a^t the transpose of a. If $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$ where $(\mathfrak{D},j)$ is an octonion algebra then the regular bimodule $\mathrm{reg}\,\mathfrak{J}$ is the only irreducible unital bimodule for $\mathfrak{J}$ (p. 283). It is now clear that Theorem 14 will follow if we can show that if the degree of $\mathfrak{J}$ is not divisible by the characteristic then the derivations of $\mathfrak{J}$ into the corresponding bimodules $\mathfrak{N}$ just defined are inner and, moreover, the same holds for $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$, $\mathfrak{D}$ quaternion with $\mathfrak{M}$ the bimodule obtained from an irreducible component of $\mathrm{cay}(\mathfrak{D},j)$, and $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$, $\mathfrak{D}$ octonion with $\mathfrak{M} = \mathrm{reg}\,\mathfrak{J}$.

We shall consider the cases of the special bimodules first and for this we shall now develop some results which are of general interest. We suppose now that $\mathfrak{J}$ is any special Jordan algebra with 1 and $\mathfrak{N}$ is a unital special bimodule for $\mathfrak{J}$ such that $y.a = \frac{1}{2}(ya^{\sigma_1} + a^{\sigma_2})$ where σ_1 and σ_2 are commuting unital associative specializations of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{N},\mathfrak{N})$. Let $S_1(\mathfrak{J})$ be the unital special universal envelope for $\mathfrak{J}$, π the main involution in $S_1(\mathfrak{J})$. Since $\mathfrak{J}$ is special we can identify $\mathfrak{J}$ with the corresponding subspace of $S_1(\mathfrak{J})$. Then $\mathfrak{J}$ is a subalgebra of $S_1(\mathfrak{J})^+$ and π is characterized by $a^\pi = a$, $a \in \mathfrak{J}$. The $\mathfrak{J}$-bimodule $\mathfrak{N}$ can be regarded as an associative bimodule for $S_1(\mathfrak{J})$ so that $ay = ya^{\sigma_1}$, $ya = ya^{\sigma_2}$, $y \in \mathfrak{N}$, $a \in \mathfrak{J}$. Since $\mathfrak{J}$ generates $S_1(\mathfrak{J})$ this determines uniquely the actions of $S_1(\mathfrak{J})$ on $\mathfrak{N}$. Let $\mathfrak{F} = S_1(\mathfrak{J}) \oplus \mathfrak{N}$ the split null extension of $S_1(\mathfrak{J})$ by $\mathfrak{N}$, so $\mathfrak{F}$ is an associative algebra with 1. Let D be a derivation of $\mathfrak{J}$ into $\mathfrak{N}$. Then $\sigma : a \to a + aD$ is a linear mapping of $\mathfrak{J}$ into $\mathfrak{F}$. If $a, b \in \mathfrak{J}$ then $(a + aD).(b + bD) = a.b + a.bD + aD.b = a.b + (a.b)D$. Also $1D = 0$ so $1^\sigma = 1$. Hence σ is a unital associative specialization of $\mathfrak{J}$ in $\mathfrak{F}$ and so there exists a homomorphism of $S_1(\mathfrak{J})$ into $\mathfrak{F}$ which extends σ. It is clear that this has the form $x \to x + z$, $x \in S_1(\mathfrak{J})$, $z \in \mathfrak{N}$. Then we have the mapping $D_u : x \to z$ which is clearly linear. Since $x \to x + xD_u$ is a homomorphism of associative algebras, $(x + xD_u)(y + yD_u) = xy + (xy)D_u$. Since $\mathfrak{N}^2 = 0$ this gives $(xy)D_u = x(yD_u) + (xD_u)y$, so D_u is a derivation of $S_1(\mathfrak{J})$ into its bimodule $\mathfrak{N}$. It is clear also that D_u is an extension of the given derivation D of $\mathfrak{J}$ into $\mathfrak{N}$.

We recall that a derivation of an associative algebra $\mathfrak{A}$ into a bimodule $\mathfrak{N}$ is called inner if it has the form $x \to [x, d] = xd - dx$ for some $d \in \mathfrak{N}$. We have the following

LEMMA 2. *Let $\mathfrak{J}, \mathfrak{N}$, $S_1(\mathfrak{J})$ be as above. Then every derivation of $\mathfrak{J}$ into $\mathfrak{N}$ is inner if and only if*: (1) *every derivation of $S_1(\mathfrak{J})$ into $\mathfrak{N}$ is inner and* (2) $\mathfrak{N} = \mathfrak{Z} + [\mathfrak{N},\mathfrak{J}]$ *where* $\mathfrak{Z} = \{z \in \mathfrak{N} \mid [a,z] = 0,\ a \in \mathfrak{J}\}$ *and* $[\mathfrak{N},\mathfrak{J}]$ *is the subspace of* $\mathfrak{N}$ *spanned by the elements* $[y,a]$, $y \in \mathfrak{N}$, $a \in \mathfrak{J}$.

PROOF. Suppose first that (1) and (2) hold. Let D be a derivation of $\mathfrak{J}$ into $\mathfrak{N}$, D_u the extension D_u of D to a derivation of $S_1(\mathfrak{J})$ into $\mathfrak{N}$ as above. Then D_u is an inner derivation of $S_1(\mathfrak{J})$, so there exists a $d \in \mathfrak{N}$ such that $xD_u = [x, d]$, $x \in S_1(\mathfrak{J})$. Hence for all $a \in \mathfrak{J}$, $aD = [a, d]$. By (2), $d = z + \sum [y_i, b_i]$ where $z \in \mathfrak{Z}$, $y_i \in \mathfrak{N}$, $b_i \in \mathfrak{J}$. Then $aD = \sum_i [a, [y_i, b_i]]$. Now in $\mathfrak{F}^+$ we have $[y, a, b] = y.a.b - y.(a.b) = \frac{1}{4}[a, [y, b]]$. Hence $aD = 4 \sum [y_i, a, b_i]$ so D is an inner derivation of $\mathfrak{J}$ into $\mathfrak{N}$. Conversely, suppose every derivation of $\mathfrak{J}$ into $\mathfrak{N}$ is inner. Let D be a derivation of $S_1(\mathfrak{J})$ into $\mathfrak{N}$. Then the restriction of D to $\mathfrak{J}$ is a derivation of $\mathfrak{J}$ into $\mathfrak{N}$. Hence there exists elements $y_i \in \mathfrak{N}$, $b_i \in \mathfrak{J}$ such that $aD = 4 \sum [y_i, a, b_i]$, $a \in \mathfrak{J}$. Then $aD = [a, d]$ where $d = \sum [y_i, b_i] \in \mathfrak{N}$. Since $\mathfrak{J}$ generates $S_1(\mathfrak{J})$ it follows that D coincides with the inner derivation $x \to [x, d]$ of $S_1(\mathfrak{J})$ into $\mathfrak{N}$. Next let d be any element of $\mathfrak{N}$. Then the mapping $a \to [a, d]$ is a derivation of $\mathfrak{J}$ into $\mathfrak{N}$ so there exist $y_i \in N$, $b_i \in \mathfrak{J}$ such that $[a, d] = 4 \sum [y_i, a, b_i]$. This implies that $z = d - \sum [y_i, b_i]$ satisfies $[a, z] = 0$, $a \in \mathfrak{J}$. Thus, $z \in \mathfrak{Z}$ and $d = z + \sum [y_i b_i]$. Hence $\mathfrak{N} = \mathfrak{Z} + [\mathfrak{N},\mathfrak{J}]$.

We shall now apply this to the simple $\mathfrak{J}$ over an algebraically closed field of degree $\geqq 2$ and the $\mathfrak{R}$'s which we indicated before. We note first that in all cases $S_1(\mathfrak{J})$ is separable. It is well known that every derivation of a separable associative algebra into a finite-dimensional bimodule is inner (Cartan and Eilenberg, *Homological Algebra*, p. 179). Since $S_1(\mathfrak{J})$ is separable for finite-dimensional separable $\mathfrak{J}$ by Theorem 12, it follows that the first condition of Lemma 2 is satisfied in all cases. It remains to consider the validity of the second condition. Here we have to treat the various cases separately.

Case I. $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a nondegenerate symmetric bilinear form f on the even-dimensional space $\mathfrak{V}$. Here $\mathfrak{R} = C(\mathfrak{V},f)$ and $\mathfrak{Z} = \Phi 1$ since $C(\mathfrak{V},f)$ is central simple and $\mathfrak{J}$ generates $C(\mathfrak{V},f)$. Let $(u_1, u_2, \cdots, u_{2m})$ be an orthogonal basis for $\mathfrak{V}$. Then $u_iu_j = -u_ju_i$ if $i \neq j$ and 1 and the elements $u_{i_1}u_{i_2}\cdots u_{i_k}$, $i_1 < i_2 < \cdots < i_k$, constitute a basis for $C(\mathfrak{V},f)$. Since for k even and $\geqq 2$ and distinct j_i,

$$
\begin{aligned}
[u_{j_1}, u_{j_1}u_{j_2}\cdots u_{j_k}] &= 2f(u_{j_1}, u_{j_1})u_{j_2}\cdots u_{j_k}, \\
[u_{j_1}, u_{j_2}\cdots u_{j_k}] &= 2u_{j_1}\cdots u_{j_k}
\end{aligned}
\tag{16}
$$

it is clear that $\mathfrak{Z} + [\mathfrak{J}, C(\mathfrak{V},f)] = C(\mathfrak{V},f)$.

Case II. $\mathfrak{J}$ as in Case I but with $\dim \mathfrak{V}$ odd. Then $\mathfrak{R} = C(\mathfrak{W},f)$ where $\mathfrak{W} = \mathfrak{V} \oplus \Phi u_{2m}$. This has an orthogonal basis $(u_1, u_2, \cdots, u_{2m})$ where $(u_1, u_2, \cdots, u_{2m-1})$ is an orthogonal basis for $\mathfrak{V}$. If we take $j_1 \neq 2m$ in (16) we see that $C(\mathfrak{W},f) = \Phi 1 + \Phi u_1u_2\cdots u_{2m-1} + [\mathfrak{J}, C(\mathfrak{W},f)]$. Since 1 and $u_1u_2\cdots u_{2m-1} \in \mathfrak{Z}$ we again have $C(\mathfrak{W},f) = \mathfrak{Z} + [\mathfrak{J}, C(\mathfrak{W},f)]$.

Case III. $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_n, J_1)$, $n \geqq 3$, $(\mathfrak{D}, j)$ a one or four-dimensional composition algebra. Then $\mathfrak{R} = \mathfrak{D}_n$ and $\mathfrak{Z} = \Phi 1$ since $\mathfrak{D}_n$ is central simple and $\mathfrak{J}$ generates $\mathfrak{D}_n$. We shall show that $\mathfrak{D}_n = \mathfrak{Z} + [\mathfrak{J}, \mathfrak{D}_n]$ if and only if the characteristic p of Φ is not a divisor of n. We have $\mathfrak{J} = \mathfrak{H} = \mathfrak{H}(\mathfrak{D}_n, J_1)$ and $\mathfrak{D}_n = \mathfrak{H} \oplus \mathfrak{S}$ where $\mathfrak{S} \equiv \mathfrak{S}(\mathfrak{D}_n, J_1)$ is the space of J_1-skew elements. We have seen that $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{S}$ (p. 131). Hence $[\mathfrak{J}, \mathfrak{D}_n] = [\mathfrak{H}, \mathfrak{H}] + [\mathfrak{H}, \mathfrak{S}] = \mathfrak{S} + [\mathfrak{H}, [\mathfrak{H}, \mathfrak{H}]]$. Since $[\mathfrak{H}, [\mathfrak{H}, \mathfrak{H}]]$ is the subspace $\mathfrak{H}'$ of $\mathfrak{H}$ spanned by the associators $[a_1, a_2, a_3]$, $a_i \in \mathfrak{H}$, we have $[\mathfrak{J}, \mathfrak{D}_n] = \mathfrak{S} + \mathfrak{H}'$ and $\mathfrak{Z} + [\mathfrak{J}, \mathfrak{D}_n] = \mathfrak{S} + \Phi 1 + \mathfrak{H}'$. Hence $\mathfrak{D}_n = \mathfrak{Z} + [\mathfrak{J}, \mathfrak{D}_n]$ if and only if $\mathfrak{H} = \Phi 1 + \mathfrak{H}'$. We now put $e_i = \frac{1}{2}[ii]$ where $d[ij] = de_{ij} + \bar{d}e_{ji}$, $d \in \mathfrak{D}$, e_{ij} the usual matrix units. Then $E = \{e_i\}$ is a reducing set of primitive idempotents for $\mathfrak{J} = \mathfrak{H}$ and these define the E-trace $t_E(a) = \sum \alpha_i$ if $a = \sum \alpha_i e_i + \sum_{i<j} d_{ij}[ij]$. Then we have seen that t_E is associative (p. 199) so $\mathfrak{H}'$ is contained in the subspace of elements a such that $t_E(a) = 0$. Clearly the dimensionality of the latter space is $\dim \mathfrak{H} - 1$. If $i \neq j$, $d[ij] = 4[d[ij], e_i, e_j]$ and if $k \neq 1$ then $2[1[1k], 1[1k], e_1 = e_1 - e_k$. It follows that $\mathfrak{H}'$ coincides with the space of elements of E-trace 0 so $\dim \mathfrak{H}' = \dim \mathfrak{H} - 1$. Since $t_E(1) = n$ it is clear that $1 \in \mathfrak{H}'$ if and only if $p \mid n$. Thus $\mathfrak{H} = \Phi 1 + \mathfrak{H}'$ if and only if $p \nmid n$ and so $\mathfrak{D}_n = \mathfrak{Z} + [\mathfrak{J}, \mathfrak{D}_n]$ if and only if $p \nmid n$.

Case IV. $\mathfrak{J} = \Phi_n{}^+$, $n \geqq 3$. Here $\mathfrak{N} = \Phi_{2n}$ and the action of $\Phi_n{}^+$ on Φ_{2n} is given by (15). We identify $\mathfrak{J}$ with the subset of matrices $a^\sigma \equiv \operatorname{diag}\{a, a^t\}$. Then $y.a = y.a^\sigma$. Since $[a^\sigma, b^\sigma] = \operatorname{diag}\{[a,b], -[b,a]^t\}$ it is clear that $[\mathfrak{J}\mathfrak{J}]$ is the set of matrices $\operatorname{diag}\{a, -a^t\}$ with trace $\operatorname{tr} a = 0$. It is immediate from this that the subalgebra of Φ_{2n} generated by $\mathfrak{J}$ is the set of matrices $\operatorname{diag}\{a,b\}, a, b \in \Phi_n$. Hence $\mathfrak{Z}$ which is the subset of Φ_{2n} of matrices commuting with all a^σ is the set of matrices $\operatorname{diag}\{\alpha 1, \beta 1\}, \alpha, \beta \in \Phi, 1$ the identity of Φ_n. Now $[\mathfrak{J}, \Phi_{2n}]$ contains all the matrices

$$(17) \qquad \begin{pmatrix} [p,a] & qa^t - aq \\ ra - a^t r & [s, a^t] \end{pmatrix} = \left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix}\right].$$

It is easily seen from this that $[\mathfrak{J}, \Phi_{2n}]$ is the set of matrices $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with $\operatorname{tr} p = 0 = \operatorname{tr} s$. Since $\operatorname{tr} 1 = n$ for the identity 1 of Φ_{2n} it is clear that $\mathfrak{Z} + [\mathfrak{J}, \Phi_{2n}] = \Phi_{2n}$ if and only if $p \nmid n$.

The results just obtained and Lemma 2 imply that if $\mathfrak{J}$ and $\mathfrak{N}$ are as listed in the foregoing cases then all derivations of $\mathfrak{J}$ into $\mathfrak{N}$ are inner if and only if the degree of $\mathfrak{J}$ is not divisible by the characteristic. We have seen that this reduces the proof of Theorem 14 to the following two cases for Φ algebraically closed of characteristic $\neq 2,3$.

Case V. If $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$ where $(\mathfrak{D}, j)$ is quaternion and $\mathfrak{M}$ is an irreducible $\mathfrak{J}$-bimodule associated with an irreducible factor of $\operatorname{cay}\mathfrak{D}$ then every derivation of $\mathfrak{J}$ into $\mathfrak{M}$ is inner.

Case VI. If $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$ where $(\mathfrak{D}, j)$ is octonion and $\mathfrak{M} = \operatorname{reg}\mathfrak{D}$ then every derivation of $\mathfrak{J}$ into $\mathfrak{M}$ is inner. It is clear also that we may replace the irreducible bimodule $\mathfrak{M}$ in Case V by $\operatorname{cay}\mathfrak{D}$ itself. We shall give a proof which is applicable simultaneously to both cases. This will be based on a reduction to the consideration of bimodules for the coefficient algebra $(\mathfrak{D}, j)$. For this we shall need some basic results on derivations of alternative algebras (cf. Schafer's *An Introduction to Nonassociative Algebras*, pp. 77–80).

Let $\mathfrak{A}$ be an alternative algebra and denote the left and right multiplications on $\mathfrak{A}$ by the element $a \in \mathfrak{A}$ as a_L and a_R respectively. It is an immediate consequence of the alternating character of the associator $[a,b,c]$ that we have the operator identities

$$(18) \qquad \begin{aligned} [a_L, b_L] &= -[a,b]_L - 2[a_L, b_R], \\ [a_R, b_R] &= [a,b]_R - 2[a_L, b_R] \end{aligned}$$

in $\mathfrak{A}$. If the characteristic is $\neq 2$ then $\mathfrak{A}$ defines the Jordan algebra $\mathfrak{A}^+$ with product $a.b = \frac{1}{2}(ab + ba)$ and multiplication $R_a = \frac{1}{2}(a_L + a_R)$ (§1.5). We have seen also that the mappings $a \to a_R$ and $a \to a_L$ are associative specializations of $\mathfrak{A}^+$into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ (eg. (19′) on p. 15). It follows that $a \to \frac{1}{2}a_L$, $a \to \frac{1}{2}a_R$ are multiplication specializations and, consequently, we have

$$4[a,b,c]^+{}_L = [[c_L a_L]b_L], \qquad 4[a,b,c]^+{}_R = [[c_R a_R]b_R], \tag{19}$$

where $[a,b,c]^+ = (a\,.\,b)\,.\,c - a\,.\,(b\,.\,c)$, the associator in $\mathfrak{A}^+$ (cf. (47) on p. 96). We recall also that

$$4[a,b,c]^+ = -2[a,b,c] + [b[ac]] \tag{20}$$

((33) on p. 19). Hence

$$\begin{aligned} [[c_L a_L]b_L] &= -2[a,b,c]_L + [b[ac]]_L, \\ [[c_R a_R]b_R] &= -2[a,b,c]_R + [b[ac]]_R. \end{aligned} \tag{21}$$

The first equation in (18) implies that $[a_L, b_R - b_L] = [ab]_L + 3[a_L, b_R]$. If we put $D = b_R - b_L$ then $aD = [ab]$ and so our relation reads $[a_L, D] = (aD)_L + 3[a_L, b_R]$. Since a linear mapping D in an algebra is a derivation if and only if $[a_L, D] = (aD])_L$ or if and only if $[a_R, D] = (aD)_R$ for all $a \in \mathfrak{A}$ we see that $D = b_R - b_L$ is a derivation in $\mathfrak{A}$ if and only if $3[a_L, b_R] = 0$. Since $c[a_L, b_R] = [a,c,b]$ this is the case if and only if $3b$ is in the nucleus. Hence we see that if $b = g \in N(\mathfrak{A})$ the nucleus of $\mathfrak{A}$ then $D = g_R - g_L$ is a derivation and if the characteristic is not three then, conversely, if $D = g_R - g_L$ is a derivation in $\mathfrak{A}$ then g is in the nucleus. We shall show next that for any $a, b \in \mathfrak{A}$ of characteristic $\neq 2$, the operator

$$D_{a,b} \equiv [a,b]_R - [a,b]_L - 3[a_L, b_R] \tag{22}$$

is a derivation in $\mathfrak{A}$. We have

$$\begin{aligned} 2[c_R, D_{a,b}] &= 2[c_R, [a,b]_R - [a,b]_L - 3[a_L, b_R]] \\ &= 3[c_R, [ab]_R - 2[a_L b_R]] - [c_R, [ab]_R] - 2[c_R, [ab]_L] \\ &= 3[c_R, [a_R b_R]] - [c[ab]]_R \qquad \text{(by (18))} \\ &= -6[a,c,b]_R + 2[c[ab]]_R \qquad \text{(by (21))} \\ &= 2(cD_{a,b})_R. \end{aligned}$$

Hence $[c_R, D_{a,b}] = (cD_{a,b})_R$ and $D_{a,b}$ is a derivation. It follows that for arbitrary $a_i, b_i \in \mathfrak{A}$ (characteristic $\neq 2$) the mapping $\sum D_{a_i,b_i}$ is a derivation in $\mathfrak{A}$. Following Schafer we shall call any derivation of the form

$$\sum D_{a_i,b_i} + g_R - g_L, \qquad g \in N(\mathfrak{A}) \tag{23}$$

an *inner* derivation of $\mathfrak{A}$. It is useful to obtain two other formulas for the inner derivation $D_{a,b}$. First, direct substitution of (18) gives

$$\begin{aligned} [a_L, b_L] + [a_R, b_R] + [a_L b_R] &= [a,b]_R - [a,b]_L \\ &\quad - 3[a_L b_R] = D_{a,b}. \end{aligned} \tag{24}$$

Also we have $4[R_aR_b] = [a_Lb_L] + [a_Rb_R] + [a_Lb_R] + [a_Rb_L] = D_{a,b} + [a_Rb_L]$. Since $[a_Rb_L] = [a_Lb_R]$ by the alternative law we have $4[R_aR_b] = D_{a,b} + [a_Lb_R]$. If the characteristic is not three then (22) gives $-[a_Lb_R] = \frac{1}{3}(D_{a,b} - [a,b]_R + [a,b]_L)$. Hence

$$(25) \qquad 4[R_aR_b] = (2/3)D_{a,b} + (1/3)[ab]_R - (1/3)[ab]_L$$

if the characteristic of $\mathfrak{A}$ is $\neq 2, 3$.

If D is a derivation of $\mathfrak{A}$ then clearly D is a derivation of $\mathfrak{A}^+$. We have the following

LEMMA 3. *Let $\mathfrak{A}$ be an alternative algebra of characteristic $\neq 2,3$, D a derivation in $\mathfrak{A}$. Assume that D is an inner derivation of $\mathfrak{A}^+$. Then D is an inner derivation of $\mathfrak{A}$.*

PROOF. We are given that there exist elements $a_i, b_i \in \mathfrak{A}$ such that $D = 4\sum[R_{a_i}R_{b_i}]$. Then, by (25), $D = (2/3)\sum D_{a_i,b_i} + g_R - g_L$ where $g = (1/3)\sum[a_ib_i]$. Since $\sum D_{a_i,b_i}$ is a derivation, $g_R - g_L$ is a derivation and so $g \in N(\mathfrak{A})$. Then $D = (2/3)\sum D_{a_i,b_i} + g_R - g_L$ is an inner derivation of $\mathfrak{A}$.

Let $\mathfrak{N}$ be an alternative bimodule for $\mathfrak{A}$ and $\mathfrak{F} = \mathfrak{A} \oplus \mathfrak{N}$ the corresponding split null extension. We shall call a derivation D of $\mathfrak{A}$ into $\mathfrak{N}$ inner if its linear extension to $\mathfrak{F}$ such that $\mathfrak{N}D = 0$ is an inner derivation of $\mathfrak{F}$. It is easily seen, as in the Jordan case, that D is inner if and only if there exist elements $b_i \in \mathfrak{A}$, $y_i \in \mathfrak{N}$, $z \in \mathfrak{N} \cap N(\mathfrak{F})$, such that $D = \sum D_{b_i,y_i} + z_R - z_L$. We shall call $\mathfrak{N} \cap N(\mathfrak{F})$ the *nucleus* of the bimodule $\mathfrak{N}$. Clearly this is the set of elements $z \in \mathfrak{N}$ such that $[z, a, b] = 0, a, b \in \mathfrak{A}$.

If $\mathfrak{N}$ is an alternative bimodule for $\mathfrak{A}$ then $\mathfrak{N}$ is a Jordan bimodule for $\mathfrak{A}^+$ relative to the composition $x.a = \frac{1}{2}(xa + ax)$, $x \in \mathfrak{N}$, $a \in \mathfrak{A}$. It is clear that any derivation of $\mathfrak{A}$ into $\mathfrak{N}$ is a derivation of $\mathfrak{A}^+$ into $\mathfrak{N}$. Evidently, Lemma 3 implies that if D is an inner derivation of $\mathfrak{A}^+$ into $\mathfrak{N}$ then D is an inner derivation of $\mathfrak{A}$ into $\mathfrak{N}$.

We shall now apply these results to the alternative algebras and bimodules which are needed to treat the remaining Cases V and VI described above. Here we consider a composition algebra $(\mathfrak{D}, j)$ over a field of characteristic $\neq 2, 3$ which is either quaternion or octonion. Let $(\mathfrak{N}, j)$ be a bimodule with involution for $(\mathfrak{D}, j)$ which in the quaternion case is cay $\mathfrak{D}$ and in the octonion case is reg $\mathfrak{D}$. By a derivation D of $(\mathfrak{D}, j)$ into $(\mathfrak{N}, j)$ we shall mean a derivation of $\mathfrak{D}$ into $\mathfrak{N}$ such that $\bar{a}D = \overline{aD}$, $a \in \mathfrak{D}$. Then we have

LEMMA 4. *Let D be a derivation of $(\mathfrak{D}, j)$ into $(\mathfrak{N}, j)$. Then there exist elements $b_i \in \mathfrak{D}, y_i \in \mathfrak{N}$, $i = 1, 2, \cdots, r$, such that $\bar{b}_i = -b_i, \bar{y}_i = -y_i$, $\sum_1^r [y_i, b_i] = 0$ and $aD = \sum_1^r [y_i, a, b_i]^+$, $a \in \mathfrak{D}$.*

PROOF. Since $\mathfrak{D}^+$ is the Jordan algebra of a nondegenerate symmetric bilinear form and D is a derivation of $\mathfrak{D}^+$ into $\mathfrak{N}$ the result proved in Cases I and II shows that there exist elements $b_i \in \mathfrak{D}$, $y_i \in \mathfrak{N}$ such that $aD = \sum_1^r [y_i, a, b_i]^+$, $a \in \mathfrak{D}$. We can write $b_i = \beta_i 1 + c_i$ where $\bar{c}_i = -c_i$. Since $\mathfrak{N}$ is a unital bimodule for $\mathfrak{D}^+$ we have $[y_i, a, \beta_i 1]^+ = 0$, $a \in \mathfrak{D}$. Hence we may assume $\bar{b}_i = -b_i$. In the quaternion case $\bar{y} = -y$ for all $y \in \mathfrak{N}$ so $\bar{y}_i = -y_i$. In the octonion case we have $\mathfrak{N} = N(\mathfrak{N}) \oplus \mathfrak{N}_0$ where $N(\mathfrak{N})$ and $\mathfrak{N}_0$ are the images of $\Phi 1$ and the set of skew elements of $\mathfrak{D}$ under an isomorphism of $(\mathfrak{D},j)$ onto $(\mathfrak{N},j)$. It follows that $y_i = u_i + z_i$ where $u_i \in N(\mathfrak{N})$ and $\bar{z}_i = -z_i$. Then u_i is in the center of the split null extension $\mathfrak{F} = \mathfrak{D} \oplus \mathfrak{N}$ so $[u_i, a, b_i]^+ = 0$. Hence we may assume $\bar{y}_i = -y_i$. Since $aD = \sum [y_i, a, b_i]^+ = a \sum [R_{y_i} R_{b_i}]$, (25) gives $aD = \frac{1}{6} \sum D_{y_i, b_i} + a(g_R - g_L)$ where $g = \frac{1}{12} \sum [y_i b_i]$. Since $\sum D_{y_i, b_i}$ and D are derivations of $\mathfrak{F}$, $g_R - g_L$ is a derivation in $\mathfrak{F}$. Hence g is in the nucleus of $\mathfrak{N}$. On the other hand, $\bar{y}_i = -y_i$, $\bar{b}_i = -b_i$ imply that $\bar{g} = -g$. Since the elements of the nucleus of $\mathfrak{N}$ are symmetric we see that $g = 0$. Hence $\sum [y_i b_i] = 0$ as required.

Now let $\mathfrak{J}$ and $\mathfrak{M}$ be as in V and VI and let D be a derivation of $\mathfrak{J}$ into $\mathfrak{M}$. We have $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$ where $(\mathfrak{D},j)$ is either quaternion or octonion. The restriction of D to the subalgebra $\mathfrak{H}(\Phi_3, J_1)$ is a derivation of $\mathfrak{H}(\Phi_3, J_1)$ into $\mathfrak{M}$ as bimodule for this algebra. Since the characteristic is not three the result proved in Case III above implies that the restriction of D to $\mathfrak{H}(\Phi_3, J_1)$ is inner. Hence there exist $x_i \in \mathfrak{M}$, $a_i \in \mathfrak{H}(\Phi_3, J_1)$ so that $aD = \sum [x_i, a, a_i]$ for all $a \in \mathfrak{H}(\Phi_3, J_1)$. Thus if we subtract the inner derivation $x \to \sum [x_i, x, a_i]$ of $\mathfrak{J}$ from D we obtain a derivation annihilating $\mathfrak{H}(\Phi_3, J_1)$. Hence to prove that every derivation of $\mathfrak{J}$ into $\mathfrak{M}$ is inner it is enough to do this for the derivations D such that $\mathfrak{H}(\Phi_3, J_1)D = 0$. Let $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ the split null extension of $\mathfrak{J}$ by $\mathfrak{M}$. Then, as in §3.5, we may identify $\mathfrak{E}$ with the Jordan matrix algebra $\mathfrak{H}(\mathfrak{F}_3, J_1)$ where $\mathfrak{F} = \mathfrak{D} \oplus \mathfrak{N}$ is the split null extension of $(\mathfrak{D},j)$ by the unital bimodule $(\mathfrak{N},j)$. Moreover, by hypotheses, $(\mathfrak{N},j)$ is cay$\mathfrak{D}$ if $\mathfrak{D}$ is quaternion and $(\mathfrak{N},j)$ is reg$\mathfrak{D}$ if $\mathfrak{D}$ is octonion. Also $\mathfrak{M} = \mathfrak{H}(\mathfrak{F}_3, J_1) \cap \mathfrak{N}_3$ and we extend D to $\mathfrak{F} = \mathfrak{H}(\mathfrak{F}_3, J_1)$ so that $\mathfrak{M}D = 0$. Since $1[ii]D = 0$ it is clear that the Peirce component $\mathfrak{J}_{ij}$ of $\mathfrak{J}$ relative to the orthogonal idempotents $e_i = \frac{1}{2}[ii]$ is mapped by D into the Peirce space $\mathfrak{E}_{ij}$ of $\mathfrak{E}$ relative to the e_i. Hence for $x \in \mathfrak{D}$, $i \neq j$, $x[ij]D = x^{d_{ij}}[ij]$ where $x^{d_{ij}} \in \mathfrak{N}$. This defines a linear mapping $d_{ij}: x \to x^{d_{ij}}$ of $\mathfrak{D}$ into $\mathfrak{N}$. Since $2x[ij].1[jk] = x[ik]$ for distinct i,j,k and $1[jk]D = 0$ we see that $2x^{d_{ij}}[ij].1[jk] = x^{d_{ik}}[ik]$. Hence $d_{ij} = d_{ik}$ and, similarly, $d_{ij} = d_{kj}$. Hence $d_{ij} = d_{kl}$ if $i \neq j$ and $k \neq l$. Then the relation $2x[ij].y[jk] = xy[ik]$ implies that $d \equiv d_{ij}$ is a derivation of $\mathfrak{D}$ into $\mathfrak{N}$. Since $x[ji] = \bar{x}[ij]$, $x^d[ji] = x[ji]D = \bar{x}[ij]D = \bar{x}^d[ij] = \overline{\bar{x}^d}[ji]$. Hence $\bar{x}^d = \overline{x^d}$ so d is a derivation of $(\mathfrak{D},j)$ into $(\mathfrak{N},j)$. Hence, by Lemma 4, there exist elements $y_i \in \mathfrak{N}$, $b_i \in \mathfrak{D}$ such that $\bar{y}_i = -y_i$, $\bar{b}_i = -b_i$, $\sum [y_i b_i] = 0$ and $x^d = \sum [y_i, x, b_i]^+$ for all $x \in \mathfrak{D}$. Now let E be the inner derivation $a \to \sum_i [y_i[21], a, b_i[12]]$ of $\mathfrak{J}$ into $\mathfrak{M}$. We proceed to show that $E = D$ which will prove that D is inner. It is immediate from the multiplication formulas in $\mathfrak{H}(\mathfrak{D}_3, J_1)$ that this algebra is generated by the elements $x[12]$, $x \in \mathfrak{D}$ and $1[23]$.

Hence it is sufficient to show that $x[12]D = x[12]E$ and $1[23]E = 0$. Now, it is clear that $4[y[21], x[12], b[12]] = z[12]$ and direct calculation gives

$$z = (xy + \bar{y}\bar{x})b + b(yx + \bar{x}\bar{y}) - (b\bar{x} + x\bar{b})\bar{y} - \bar{y}(\bar{x}b + \bar{b}x).$$

If $\bar{y} = -y$, $\bar{b} = -b$ and $x = \xi 1 + w$ where $\xi \in \Phi$ and $\bar{w} = -w$ then this becomes

$$\begin{aligned} z &= (wy + yw)b + b(yw + wy) - (bw + wb)y - y(wb + bw) \\ &= 4[y, w, b]^{+} = 4[y, x, b]^{+}. \end{aligned}$$

Hence $[y[21], x[12]] = [y, x, b]^{+}[12]$ and $x[12]E = \sum[y_i[21], x[12], b_i[12]] = \sum[y_i, x, b]^{+}[12] = x^d[12] = x[12]D$. Also $4[y[21], 1[23], b[12]] = [b, y][23]$. Hence $1[23]E = \sum[y_i[21], 1[23], b_i[12]] = 0$. This completes the proof of Theorem 14.

We note also that the consideration of Cases III and IV above show that if a finite-dimensional separable Jordan algebra has a special simple component whose degree over its center is divisible by the characteristic then not all derivations of $\mathfrak{J}$ into finite-dimensional bimodules are inner.

If $\mathfrak{J}$ is a subalgebra of the Jordan algebra and D is a derivation of $\mathfrak{J}$ into $\mathfrak{K}$ in the sense that D is linear from $\mathfrak{J}$ to $\mathfrak{K}$ and $(a.b)D = aD.b + a.bD$, $a, b \in \mathfrak{J}$ then D is a derivation of $\mathfrak{J}$ into $\mathfrak{K}$ considered as $\mathfrak{J}$-bimodule with the composition $x.a$, $x \in \mathfrak{K}$, $a \in \mathfrak{J}$, as the product in $\mathfrak{K}$. If D is an inner derivation of $\mathfrak{J}$ into the $\mathfrak{J}$-bimodule $\mathfrak{K}$ then there exist $y_i \in \mathfrak{K}$, $b_i \in \mathfrak{J}$ such that $aD = \sum[y_i, a, b_i]$, $a \in \mathfrak{J}$. Evidently this implies the following

COROLLARY TO THEOREM 14. *Let $\mathfrak{J}$ be a separable Jordan algebra satisfying the condition of Theorem* 14. *Then every derivation of $\mathfrak{J}$ into a finite-dimensional Jordan algebra $\mathfrak{K}$ containing $\mathfrak{J}$ as subalgebra can be extended to an inner derivation of $\mathfrak{K}$.*

8. **Uniqueness of the Albert-Penico-Taft decomposition.** Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra, $\mathfrak{N} = \operatorname{rad}\mathfrak{J}$ and assume $\mathfrak{J}/\mathfrak{N}$ is separable so we have an Albert-Penico-Taft decomposition $\mathfrak{J} = \mathfrak{K} \oplus \mathfrak{N}$ where $\mathfrak{K}$ is a subalgebra of $\mathfrak{J}$. We shall now call such a $\mathfrak{K}$ an A.P.T. *factor* of $\mathfrak{J}$. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{J}$ and let $b \in \mathfrak{B}$. Then $b = b^{\eta} + bD$ where $b^{\eta} \in \mathfrak{K}$ and $bD \in \mathfrak{N}$ and η and D are linear mappings of $\mathfrak{B}$ into $\mathfrak{J}$ (or into $\mathfrak{K}$ and $\mathfrak{N}$ respectively). If $b_1, b_2 \in \mathfrak{B}$ then

$$\begin{aligned} b_1.b_2 &= (b_1.b_2)^{\eta} + (b_1.b_2)D \\ &= (b_1{}^{\eta} + b_1D).(b_2{}^{\eta} + b_2D) \\ &= b_1{}^{\eta}.b_2{}^{\eta} + (b_1D).b_2{}^{\eta} + b_1{}^{\eta}.b_2D + b_1D.b_2D. \end{aligned}$$

Since $\mathfrak{K}$ is a subalgebra and $\mathfrak{N}$ is an ideal we have

$$(26)\qquad \begin{aligned}(b_1 . b_2)^\eta &= b_1{}^\eta . b_2{}^\eta,\\ (b_1 . b_2)D &= b_1 D . b_2{}^\eta + b_1{}^\eta . b_2 D + b_1 D . b_2 D.\end{aligned}$$

Hence η is a homomorphism and consequently $\mathfrak{K}$ and $\mathfrak{N}$ are bimodules for $\mathfrak{B}$ relative to the composition $x * b = x . b^\eta$, $b \in \mathfrak{B}$. Assume first that $\mathfrak{N}^{\cdot 2} = 0$ (so $\mathfrak{J}$ is the split null extension of $\mathfrak{K}$ by $\mathfrak{N}$ as a $\mathfrak{K}$-bimodule). Then we have $(b_1 . b_2)D = b_1 D . b_2{}^\eta + b_1{}^\eta . b_2 D = b_1 D * b_2 + b_1 * b_2 D$. Hence D is a derivation of $\mathfrak{B}$ into $\mathfrak{N}$ as $\mathfrak{B}$-bimodule with the $*$ composition. Now assume that $\mathfrak{B}$ is separable and satisfies the condition of Harris' theorem (Theorem 14). Then D is an inner derivation of $\mathfrak{B}$ into $\mathfrak{N}$, which means that there exist elements $y_i \in \mathfrak{N}$, $d_i \in \mathfrak{B}$ such that $bD = \sum_i (y_i * b * d_i - y_i * (b . d_i))$. We have $y_i * b_i * d_i = y_i . b^\eta . d_i^\eta$ and $y_i * (b . d_i) = y_i . (b^\eta . d_i^\eta)$. Hence $y_i * b * d_i - y_i * (b . d_i) = b^\eta [R_{y_i}, R_{d_i{}^\eta}]$ and $bD = b^\eta \sum [R_{y_i} R_{d_i{}^\eta}]$. Now the mapping $E = \sum [R_{y_i} R_{d_i{}^\eta}]$ is an inner derivation in $\mathfrak{J}$ and since the $y_i \in \mathfrak{N}$, $\mathfrak{J}E \subseteq \mathfrak{N}$ and $\mathfrak{N}E = 0$ so $E^2 = 0$. These conditions imply that $\zeta = 1 + E$ is an automorphism with inverse $\zeta^{-1} = 1 - E$. Moreover, if $b \in \mathfrak{B}$ then

$$\begin{aligned} b^{\zeta^{-1}} &= b - bE = b^\eta + bD - (b^\eta + bD)E \\ &= b^\eta + bD - b^\eta E \\ &= b^\eta \in \mathfrak{K}. \end{aligned}$$

Hence ζ^{-1} is an automorphism mapping $\mathfrak{B}$ into the subalgebra $\mathfrak{B}^\eta$ of $\mathfrak{K}$. We therefore have the following theorem.

THEOREM 15. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra such that $\mathfrak{N}^{\cdot 2} = 0$ for $\mathfrak{N} = \mathrm{rad}\,\mathfrak{J}$ and $\mathfrak{J}/\mathfrak{N}$ is separable, and let $\mathfrak{J} = \mathfrak{K} \oplus \mathfrak{N}$ where $\mathfrak{K}$ is a subalgebra. Let $\mathfrak{B}$ be a separable subalgebra such that the degrees over their centers of the simple components of $\mathfrak{B}$ are not divisible by the characteristic. Then there exists an automorphism ζ of $\mathfrak{J}$ of the form $\zeta = 1 + E$ where E is an inner derivation such that $\mathfrak{J}E \subseteq \mathfrak{N}$ and $\mathfrak{N}E = 0$, such that $\mathfrak{B}^{\zeta^{-1}} \subseteq \mathfrak{K}$.*

We shall now extend this theorem to the case in which $\mathfrak{N}$ is arbitrary. For this we shall need to assume that the characteristic is 0. In this case it is well known (and easy to verify using Leibniz' rule) that if D is a nilpotent derivation in an algebra then $\exp D \equiv 1 + D + D^2/2! + \cdots + D^m/m!$, where $D^{m+1} = 0$, is an automorphism of the algebra. We recall that if $\mathfrak{J}$ is a finite-dimensional Jordan algebra then $\mathfrak{N} = \mathrm{rad}\,\mathfrak{J}$ generates a nilpotent ideal in $U(\mathfrak{J})$ (Theorem 5.2). It follows that any inner derivation of the form $\sum [R_{z_i} R_{a_i}]$, $z_i \in \mathfrak{N}$, $a_i \in \mathfrak{J}$, is nilpotent, and in the characteristic 0 case $\exp(\sum [R_{z_i} R_{a_i}])$ is an automorphism. Let H denote the subgroup of the group of automorphisms generated by these automorphisms. Then we have the following

THEOREM 16. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra over a field of characteristic* 0, $\mathfrak{K}$ *a subalgebra such that* $\mathfrak{J} = \mathfrak{K} \oplus \mathfrak{N}$, $\mathfrak{N} = \mathrm{rad}\,\mathfrak{J}$ *and let* $\mathfrak{B}$ *be a semisimple subalgebra of* $\mathfrak{J}$. *Then there exists an automorphism* $\eta \in \mathrm{H}$ *such that* $\mathfrak{B}^\eta \subseteq \mathfrak{K}$.

PROOF. The result is trivial if $\mathfrak{N} = 0$ so we may use induction on $\dim \mathfrak{N}$ and we may assume $\mathfrak{N} \neq 0$. Then $\mathfrak{N}$ contains an ideal $\mathfrak{S}$ of $\mathfrak{J}$ such that $\mathfrak{S} \neq 0$ and $\mathfrak{S}^{\cdot 2} = 0$. We consider the algebra $\bar{\mathfrak{J}} = \mathfrak{J}/\mathfrak{S}$ whose radical is $\bar{\mathfrak{N}} = \mathfrak{N}/\mathfrak{S}$. We have the decomposition $\bar{\mathfrak{J}} = \bar{\mathfrak{K}} \oplus \bar{\mathfrak{N}}$ where $\bar{\mathfrak{K}}$ is the subalgebra $(\mathfrak{K} + \mathfrak{S})/\mathfrak{S}$. Also $\bar{\mathfrak{B}} = (\mathfrak{B} + \mathfrak{S})/\mathfrak{S}$ is a semisimple subalgebra of $\mathfrak{J}$. Since $\dim \bar{\mathfrak{N}} < \dim \mathfrak{N}$ we can use the induction hypothesis to conclude that there is an automorphism $\bar{\eta}$ in the group $\bar{\mathrm{H}}$ defined for $\bar{\mathfrak{J}}$ such that $\bar{\mathfrak{B}}^{\bar{\eta}} \subseteq \bar{\mathfrak{K}}$. Now $\bar{\eta}$ is a product of automorphisms of the form $\exp(\sum [R_{\bar{z}_i}, R_{\bar{a}_i}])$, $\bar{z}_i = z_i + \mathfrak{S}$, $\bar{a}_i = a_i + \mathfrak{S}$ where $z_i \in \mathfrak{N}$, $a_i \in \mathfrak{J}$. Let η be the corresponding product of the automorphisms $\exp(\sum [R_{z_i} R_{a_i}])$ in $\mathfrak{J}$. Then $\eta \in \mathrm{H}$ and it is clear that $\mathfrak{B}^\eta \subseteq \mathfrak{K} \oplus \mathfrak{S}$. Now $\mathfrak{K} \oplus \mathfrak{S}$ is a subalgebra of $\mathfrak{J}$ whose radical is $\mathfrak{S}$ and $\mathfrak{B}^\eta$ is a semisimple subalgebra of $\mathfrak{K} + \mathfrak{S}$. Since $\mathfrak{S}^{\cdot 2} = 0$ the proof of Theorem 15 shows that there exists an automorphism $\zeta^{-1} = 1 - E$ where $E = \sum [R_{y_i} R_{b_i}]$, $y_i \in \mathfrak{S}$, $b_i \in \mathfrak{K} + \mathfrak{S}$ such that $\mathfrak{B}^{\eta\zeta^{-1}} \subseteq \mathfrak{K}$. Also $E^2 = 0$ so $\zeta^{-1} = \exp(-E)$. Hence $\zeta^{-1} \in \mathrm{H}$ and $\eta\zeta^{-1} \in \mathrm{H}$ and maps $\mathfrak{B}$ into $\mathfrak{K}$.

It is clear that any two A.P.T. factors of $\mathfrak{J}$ have the same dimensionality. Moreover, since $\mathfrak{N}$ is mapped into itself by every automorphism it is clear that the image of a A.P.T. factor under an automorphism is an A.P.T. factor. It is now clear that the following corollary is an immediate consequence of Theorem 16.

COROLLARY. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra over a field of characteristic* 0. *Then if* $\mathfrak{K}_1$ *and* $\mathfrak{K}_2$ *are A.P.T. factors of* $\mathfrak{J}$ *there exists an automorphism* $\eta \in \mathrm{H}$ *such that* $\mathfrak{K}_1{}^\eta = \mathfrak{K}_2$. *Also any semisimple subalgebra of* $\mathfrak{J}$ *is contained in an A.P.T. factor.*

EXERCISES

1. Let $\Phi[a]$ be the Jordan algebra with 1 generated by a single algebraic element a with minimum polynomial $\mu(\lambda)$ and let $\mathfrak{M}$ be a unital bimodule for $\Phi[a]$, $\mathfrak{E}$ the split null extension. If $f(\lambda) \in \Phi[\lambda]$ we let f be the polynomial mapping $x \to f(x)$ in $\mathfrak{E}$. Show that there exists a derivation of $\Phi[a]$ in $\mathfrak{M}$ sending a into $z \in \mathfrak{M}$ if and only if the directional derivative $\Delta^z{}_a \mu$ of $\mu(\lambda)$ in the direction z at a is 0.

2. Let $\mathfrak{H}$ be the Jordan algebra of $n \times n$ symmetric matrices over Φ, $\mathfrak{S}$ the space of $n \times n$ skew matrices, and consider $\mathfrak{S}$ as unital bimodule for $\mathfrak{S}$ relative to $s.h = \frac{1}{2}(sh + hs)$, $h \in \mathfrak{H}$, $s \in \mathfrak{S}$. Show that for fixed $k \in \mathfrak{H}$ the mapping $x \to [x, k] = xk - kx$, $x \in \mathfrak{H}$ is a derivation of $\mathfrak{H}$ into $\mathfrak{S}$. Show that if the minimum polynomial $\mu(\lambda)$ of the element $h \in \mathfrak{H}$ has degree n and s is any element of $\mathfrak{S}$ then there exists a derivation of the form indicated mapping h into s. Hence

show that $\Delta^{s}{}_{h^{\mu}} = 0$ and that any derivation of $\Phi[h]$ into $\mathfrak{S}$ can be extended to a derivation of $\mathfrak{H}$ into $\mathfrak{S}$. Does the last result hold without the hypothesis that $\deg \mu(\lambda) = n$?

3. Let $\mathfrak{E} = \mathfrak{H} \oplus \mathfrak{S}$ the split null extension of $\mathfrak{H}$ by $\mathfrak{S}$ defined in 2. Let $a \in \mathfrak{E}$ have minimum polynomial of degree n and satisfy $\Phi[a] \cap \mathfrak{S} = 0$. If $b \in \Phi[a]$ write $b = k + t$ where $k \in \mathfrak{H}$ and $t \in \mathfrak{S}$. Show that $b \to k$ is a monomorphism of $\Phi[a]$ into $\mathfrak{H}$ and that $k \to t$ is a derivation of $\Phi[h]$ into the bimodule $\mathfrak{S}$. Use this and exercise 2 to show that there exists an automorphism of $\mathfrak{E}$ sending a into h where $a = h + s$, $h \in \mathfrak{H}$, $s \in \mathfrak{S}$.

4. Let $\mathfrak{H}$, $\mathfrak{S}$ be as in 2. Show that there exist elements $h \in \mathfrak{H}$ and derivations of $\Phi[h]$ into $\mathfrak{S}$ which are not inner.

CHAPTER VIII

CONNECTIONS WITH LIE ALGEBRAS

With a given Jordan algebra $\mathfrak{J}$ one can associate a number of important Lie algebras. We have already encountered some of these before in special cases, for example, the Lie algebra Der $\mathfrak{J}$ and the Lie algebra of linear transformations having the generic norm as Lie invariant (§6.9). In this chapter we shall study several Lie algebras associated to a Jordan algebra. One of these is the Lie algebra of linear transformations generated by the multiplications R_a, $a \in \mathfrak{J}$, and related Lie algebras defined by the universal envelopes $U(\mathfrak{J})$, $U_1(\mathfrak{J})$ etc. Another important Lie algebra construction we shall consider is one which has been introduced by Tits in [4] and studied further by Koecher in [9]. The constructions we have indicated are interesting from the point of view of Lie theory since they provide realizations of some important classes of Lie algebras. However, our primary interest in these constructions in this chapter stems from the fact that they provide tools for establishing some of the main results of the last chapter in the characteristic 0 case without invoking a detailed knowledge of the classification of simple Jordan algebras and irreducible bimodules. Since this approach is based on results in Lie algebras which are known to be false in the characteristic $p \neq 0$ case it seems unlikely that the Lie algebra techniques developed here can be adapted in a simple manner to the characteristic p case.

It is well known that a fundamental concept in the theory of Lie algebras is that of a Cartan subalgebra of a Lie algebra. In this chapter we shall define an analogous concept for finite-dimensional Jordan algebras. We shall show that the main facts on Cartan subalgebras in the Lie case carry over and we shall apply the theory to obtain a general proof of the generic trace criterion for separability of Jordan algebras (Theorem 6.5). The notion of a Cartan subalgebra is based on the associator structure of a Jordan algebra. As we shall see in §1, this is a special case of a Lie triple system structure, which has a simple axiomatic characterization. Moreover, Lie triple systems give rise to universal enveloping Lie algebras. These Lie algebras defined by the associator structure of a Jordan algebra are interesting too, and are closely related to some of the Lie algebras we have indicated before.

1. **Associator structure of a Jordan algebra. Abstract Lie triple systems.** We recall that in any Jordan algebra $\mathfrak{J}$ we have the basic identity

$$[[R_c R_a] R_b] = R_{[a,b,c]} \tag{1}$$

where $[a,b,c]$ is the associator $[a,b,c] = a.b.c - a.(b.c)$ (e.g. (54) p. 34). As we noted before, this shows that the subspace $R(\mathfrak{J})$ of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ consisting of the linear transformations R_a, $a \in \mathfrak{J}$, is a Lie triple system of linear transformations of $\mathfrak{J}$, that is, a subspace of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ closed under the composition $[[AB]C]$. We shall now denote the iterated Lie product $[[CA]B]$ by $[ABC]$ and call this the *Lie triple product* of the linear transformations A, B, C. With this notation we can rewrite (1) as

$$R_{[a,b,c]} = [R_a R_b R_c]. \tag{1'}$$

It is clear that the Lie triple product is trilinear and we have

$$[ABA] = 0, \tag{2}$$

which for characteristic not two is equivalent to

$$[ABC] + [CBA] = 0. \tag{3}$$

Also, we have the Jacobi identity

$$[ABC] + [BCA] + [CAB] = 0. \tag{3}$$

Moreover, since $[ABC] = [[CA]B]$, $[D[ABC]E] = [[ED], [[CA]B]]$. We now write for the moment $[[ED], X] = X'$. Applying the Jacobi identity twice we obtain

$$[[CA]B]' = [[C'A]B] + [[CA']B] + [[CA]B']. \tag{4}$$

Hence we have following Lie triple product identity

$$[D[ABC]E] = [[DAE]BC] + [A[DBE]C] + [AB[DCE]]. \tag{4'}$$

In a similar fashion it is clear that if $\mathfrak{L}$ is any Lie algebra with product composition $[ab]$ then

$$[abc] = [[ca]b] \tag{5}$$

defines a trilinear composition on $\mathfrak{L}$ satisfying:

(i) $$[aba] = 0,$$

(ii) $$[abc] + [bca] + [cab] = 0,$$

(iii) $$[d[abc]e] = [[dae]bc] + [a[dbe]c] + [ab[dce]].$$

These observations lead us to introduce the following definitions.

DEFINITION 1. *A vector space* $\mathfrak{T}/\Phi$ *with a trilinear composition* $(a,b,c) \to [a,b,c]$ *satisfying* (i), (ii) *and* (iii) *is called an* (abstract) Lie triple system. *A linear mapping* D *in* $\mathfrak{T}/\Phi$ *is called a* derivation *if*

$$[abc]D = [(aD)bc] + [a(bD)c] + [ab(cD)]. \tag{6}$$

It is clear that (iii) states that the linear mapping $I_{d,e}\colon x \to [dxe]$ is a derivation in $\mathfrak{T}$. We shall call any derivation of the form $\sum I_{d_i,e_i}$ an *inner derivation* in $\mathfrak{T}$. Clearly the set Inder $\mathfrak{T}$ of inner derivations is closed under addition. Moreover, since

$$\alpha I_{d,e} = I_{\alpha d,e} = I_{d,\alpha e}, \tag{7}$$

Inder $\mathfrak{T}$ is a subspace of $\mathrm{Hom}_\Phi(\mathfrak{T},\mathfrak{T})$. It is trivial to verify that the set Der $\mathfrak{T}$ of derivations of $\mathfrak{T}$ is a subalgebra of the Lie algebra $\mathrm{Hom}_\Phi(\mathfrak{T},\mathfrak{T})^-$. If D is any derivation then (6) gives $[abc]D - [a(bD)c] = [(aD)bc] + [ab(cD)]$ which in operator form is

$$[I_{a,c}D] = I_{aD,c} + I_{a,cD}. \tag{8}$$

Hence it is clear that Inder $\mathfrak{T}$ is an ideal in Der $\mathfrak{T}$.

We now form the vector space

$$\mathfrak{H}(\mathfrak{T}) = \mathfrak{T} \oplus \mathrm{Der}\,\mathfrak{T}. \tag{9}$$

and we introduce the composition $[xy]$ in $\mathfrak{H}(\mathfrak{T})$ by

$$[a + D, b + E] = (aE - bD) + (I_{a,b} + [D, E]). \tag{10}$$

It is clear that this is a bilinear composition, so we have an algebra. Also since $I_{a,a} = 0$ by (i) and $[DD] = 0$ we have $[a + D, a + D] = 0$. Moreover, we have the Jacobi identity:

$$\begin{aligned}
[[a &+ D, b + E], c + F] + [[b + E, c + F], a + D] \\
&+ [[c + F, a + D], b + E] = \\
aEF &- bDF - [abc] - c[DE] \\
&+ I_{aE-bD,c} + [I_{a,b}F] + [[DE]F] + \\
bFD &- cED - [bac] - a[EF] \\
&+ I_{bF-cE,a} + [I_{b,c}, D] + [[EF]D] + \\
cDE &- aFE - [cba] - b[FD] \\
&+ I_{cD-aF,b} + [I_{c,a}, E] + [[FD]E] = 0,
\end{aligned}$$

since the Jacobi identity holds in Der $\mathfrak{T}$ and (iii) and (8) are valid. Thus $\mathfrak{H}(\mathfrak{T})$ is a Lie algebra. We shall call $\mathfrak{H}(\mathfrak{T})$ the *holomorph* of the Lie triple system $\mathfrak{T}$. If $a, b \in \mathfrak{T}$ then in $\mathfrak{H}(\mathfrak{T})$ we have, by (8), $[ab] = I_{a,b}$, $[aI_{b,c}] = aI_{b,c} = [bac] \in \mathfrak{T}$ and $[I_{a,b}I_{c,d}] = I_{[cad],b} + I_{a,[cbd]}$. Hence $[\mathfrak{T}\mathfrak{T}] = \mathrm{Inder}\,\mathfrak{T}$ and $SL(\mathfrak{T}) \equiv \mathfrak{T} + [\mathfrak{T}\mathfrak{T}]$ is the subalgebra of $\mathfrak{H}(\mathfrak{T})$ generated by $\mathfrak{T}$. Evidently $SL(\mathfrak{T}) = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$.

We have noted that if $\mathfrak{L}$ is a Lie algebra then the vector space $\mathfrak{L}$ is a Lie triple system relative to the composition $[abc] = [[ca]b]$. We denote this Lie triple system as $\mathfrak{L}^{(2)}$. Let $\mathfrak{T}$ be a subsystem of $\mathfrak{L}^{(2)}$ (= subspace closed under $[abc]$). Then we claim that $[\mathfrak{T}\mathfrak{T}]$ and $\mathfrak{T} + [\mathfrak{T}\mathfrak{T}]$ are subalgebras of $\mathfrak{L}$. Thus let $a, b\ c, d \in \mathfrak{T}$. Then $[ab] \in [\mathfrak{T}\mathfrak{T}]$, $[[ca]b] = [abc] \in \mathfrak{T}$ and $[[ab],[cd]] = [[[ab]c]d] + [c[[ab]d]]$ (by the Jacobi identity) $= [[bca]d] + [c[bda]] \in [\mathfrak{T}\mathfrak{T}]$. Clearly $\mathfrak{T} + [\mathfrak{T}\mathfrak{T}]$ is the subalgebra of $\mathfrak{L}$ generated by $\mathfrak{T}$.

If $\mathfrak{T}$ is a Lie triple system and $\mathfrak{L}$ is a Lie algebra then we shall call a homomorphism of $\mathfrak{T}$ into $\mathfrak{L}^{(2)}$ a *Lie specialization* of $\mathfrak{T}$ in $\mathfrak{L}$. An important instance of this is the injection mapping of $\mathfrak{T}$ into $SL(\mathfrak{T}) = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$. We shall call this the *standard* Lie specialization and call $SL(\mathfrak{T})$ the *standard Lie envelope* of $\mathfrak{T}$. We can easily construct a *universal Lie envelope* $(UL(\mathfrak{T}), \sigma_u)$ for any $\mathfrak{T}$, defined to be a Lie algebra $UL(\mathfrak{T})$ and a Lie specialization σ_u of $\mathfrak{T}$ in $UL(\mathfrak{T})$ such that if σ is any Lie specialization of $\mathfrak{T}$ in a Lie algebra $\mathfrak{L}$ then there exists a unique homomorphism η of $UL(\mathfrak{T})$ into $\mathfrak{L}$ such that $\sigma = \sigma_u \eta$. To construct this we begin with the Lie algebra $FL(\mathfrak{T})$ which is defined to be the subalgebra of the Lie algebra $T(\mathfrak{T})^-$ generated by $\mathfrak{T}$. Here, as usual, $T(\mathfrak{T})$ denotes the tensor algebra based on $\mathfrak{T}$. It is a well known and easy consequence of the Birkhoff-Witt theorem that the linear mapping σ of $\mathfrak{T}$ into $\mathfrak{L}$ has a unique extension to a homomorphism η' of $FL(\mathfrak{T})$ into $\mathfrak{L}$ (Jacobson, *Lie Algebras*, p. 168). It is clear that the elements $[abc] - (c \otimes a - a \otimes c) \otimes b + b \otimes (c \otimes a - a \otimes c)$ are in ker η'. Now let $\mathfrak{K}$ be the ideal in $FL(\mathfrak{T})$ generated by the elements just indicated $(a, b, c \in \mathfrak{T})$ and put $UL(\mathfrak{T}) = FL(\mathfrak{T})/\mathfrak{K}$, $a^{\sigma_u} = a + \mathfrak{K}$, $a \in \mathfrak{T}$. Then it is immediate that $(UL(\mathfrak{T}), \sigma_u)$ is a universal Lie envelope for $\mathfrak{T}$ (cf. p. 68). We have the usual properties of this universal object such as given in Theorem 2.1 (p. 65). Moreover, we have the following result.

THEOREM 1. *The homomorphism* σ_u *is a monomorphism and* $UL(\mathfrak{T}) = \mathfrak{T}^{\sigma_u} \oplus [\mathfrak{T}^{\sigma_u}, \mathfrak{T}^{\sigma_u}]$. *If* $\dim \mathfrak{T} = n < \infty$ *then* $\dim UL(\mathfrak{T}) \leqq n(n+1)/2$.

PROOF. As in Theorem 2.1, $UL(\mathfrak{T})$ is generated by $\mathfrak{T}^{\sigma_u}$. Since $\mathfrak{T}^{\sigma_u}$ is a subsystem of the Lie triple system $UL(\mathfrak{T})^{(2)}$ we have $UL(\mathfrak{T}) = \mathfrak{T}^{\sigma_u} + [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]$. Now consider the standard envelope $SL(\mathfrak{T}) = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$. We have the homomorphism η of $UL(\mathfrak{T})$ onto $SL(\mathfrak{T})$ sending a^{σ_u} into a, $a \in \mathfrak{T}$. This implies that σ_u is a monomorphism and $\mathfrak{T}^{\sigma_u} \cap [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}] = 0$. Hence $UL(\mathfrak{T}) = \mathfrak{T}^{\sigma_u} \oplus [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]$. If $\dim \mathfrak{T} = n < \infty$ then $\dim \mathfrak{T}^{\sigma_u} = n$. Moreover, since $[a^{\sigma_u} b^{\sigma_u}] = -[b^{\sigma_u} a^{\sigma_u}]$ we have a linear mapping of the space of skew symmetric tensors of degree two onto $[\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]$. Hence $\dim [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}] \leqq n(n-1)/2$. Then $\dim UL(\mathfrak{T}) \leqq n + n(n-1)/2 = n(n+1)/2$

The result noted above shows that if $\mathfrak{T}$ is a subsystem of $\mathfrak{L}^{(2)}$, $\mathfrak{L}$ a Lie algebra then $[\mathfrak{T}\mathfrak{T}]$ is a subalgebra of $\mathfrak{L}$. We shall call the subalgebra $[\mathfrak{T}\mathfrak{T}]$ ($[\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]$) of $SL(\mathfrak{T})$ ($UL(\mathfrak{T})$) the *complementary subalgebra* of $\mathfrak{T}$ in $SL(\mathfrak{T})$ ($UL(\mathfrak{T})$).

EXAMPLE. Let $\mathfrak{T}$ be the *trivial* Lie triple system consisting of a vector space with the ternary composition $[abc]=0$, $a,b,c\in\mathfrak{T}$. Then Inder $\mathfrak{T}=0$ and $SL(\mathfrak{T})=\mathfrak{T}$ considered as Lie algebra with the trivial composition $[ab]=0$. Let $\mathfrak{B}$ be the vector space of skew symmetric tensors of degree two based on $\mathfrak{T}$. Thus $\mathfrak{B}$ is the subspace of $\mathfrak{T}\otimes_\Phi\mathfrak{T}$ spanned by the vectors $[xy]=x\otimes y-y\otimes x$, $x,y\in\mathfrak{T}$ and $\dim\mathfrak{B}=n(n-1)/2$ if $\dim\mathfrak{T}=n$. Let $\mathfrak{L}=\mathfrak{T}\oplus\mathfrak{B}$ and define

$$[x_1+v_1, x_2+v_2]=[x_1x_2]=x_1\otimes x_2-x_2\otimes x_1$$

for $x_i\in\mathfrak{T}$, $v_i\in\mathfrak{B}$. Then it is immediate that this is a Lie algebra which together with the injection mapping of $\mathfrak{T}$ constitute a universal Lie envelope for $\mathfrak{T}$. If $\dim\mathfrak{T}=n$ then $\dim\mathfrak{L}=n(n+1)/2$.

If $\mathfrak{A},\mathfrak{B},\mathfrak{C}$ are subspaces of a Lie triple system $\mathfrak{T}$ then we denote the subspace spanned by all the products $[abc]$, $a\in\mathfrak{A}$, $b\in\mathfrak{B}$, $c\in\mathfrak{C}$, by $[\mathfrak{A}\mathfrak{B}\mathfrak{C}]$. It is clear from $[abc]= = -[cba]$ that $[\mathfrak{A}\mathfrak{B}\mathfrak{C}]=[\mathfrak{C}\mathfrak{B}\mathfrak{A}]$. Also, by the Jacobi identity (ii) $[\mathfrak{A}\mathfrak{B}\mathfrak{C}]\subseteq[\mathfrak{B}\mathfrak{C}\mathfrak{A}]+[\mathfrak{C}\mathfrak{A}\mathfrak{B}]$. A subspace $\mathfrak{B}$ of $\mathfrak{T}$ is called an *ideal* if $[\mathfrak{T}\mathfrak{T}\mathfrak{B}]\subseteq\mathfrak{B}$ or, equivalently, $[a_1a_2b]\in\mathfrak{B}$ if $b\in\mathfrak{B}$ and $a_i\in\mathfrak{T}$. If $\mathfrak{B}$ is an ideal then also $[\mathfrak{B}\mathfrak{T}\mathfrak{T}]=[\mathfrak{T}\mathfrak{T}\mathfrak{B}]\subseteq\mathfrak{B}$ and $[\mathfrak{T}\mathfrak{B}\mathfrak{T}]\subseteq[\mathfrak{B}\mathfrak{T}\mathfrak{T}]+[\mathfrak{T}\mathfrak{T}\mathfrak{B}]\subseteq\mathfrak{B}$. An ideal $\mathfrak{B}$ determines the *difference Lie triple system* $\mathfrak{T}/\mathfrak{B}$ which is the vector space $\mathfrak{T}/\mathfrak{B}$ with the composition $[a_1+\mathfrak{B}\ a_2+\mathfrak{B}\ a_3+\mathfrak{B}]=[a_1a_2a_3]+\mathfrak{B}$. We have the canonical homomorphism $a\to a+\mathfrak{B}$ onto $\mathfrak{T}/\mathfrak{B}$. The verification is immediate and implies that $\mathfrak{T}/\mathfrak{B}$ is indeed a Lie triple system. The subspace $\mathfrak{T}'=[\mathfrak{T}\mathfrak{T}\mathfrak{T}]$ is evidently an ideal. We shall call this the *derived Lie triple system* of $\mathfrak{T}$.

We consider again a Jordan algebra $\mathfrak{J}$ and we now note that $\mathfrak{J}$ together with the associator composition $[abc]\equiv a.b.c-a.(b.c)$ is a Lie triple system, that is, the composition is trilinear and the axioms (i), (ii) and (iii) are fulfilled. Since any Jordan algebra can be imbedded in a Jordan algebra with 1, it is enough to prove our assertion for $\mathfrak{J}$ with 1. Now, by (1), the linear mapping $a\to R_a$ of $\mathfrak{J}$ onto the Lie triple system of linear transformations $R(\mathfrak{J})$ is a homomorphism of the associator structure on $\mathfrak{J}$ (vector space $\mathfrak{J}$ with the associator composition). Since $\mathfrak{J}$ has an identity element $a\to R_a$ is an isomorphism. Hence it is clear that $[abc]$ is trilinear and satisfies the axioms for a Lie triple system. We shall call $\mathfrak{J}$, as Lie triple system relative to the associator composition, the *associator Lie triple system* $\mathfrak{J}$. As in the general case, we denote the associator ideal $[\mathfrak{J}\mathfrak{J}\mathfrak{J}]$ by $\mathfrak{J}'$ (cf. p. 190).

EXERCISES

1. Prove the following identities in Lie triple systems:

(11) $$[de[bca]]+[ce[adb]]+[ae[cdb]]+[be[dac]]=0,$$

(12) $$[fg[de[bca]]]+[eg[df[aeb]]]+[fg[ce[adb]]]+[cg[cf[bda]]] + Q + R = 0$$

where Q and R are obtained from the displayed terms by cyclic permutation of the pairs, (a,b), (c,d), (e,f).

2. Let ε be an automorphism of period two in a Lie algebra $\mathfrak{L}$ and let $\mathfrak{T} = \{a \mid a^{\varepsilon} = -a\}$. Show that $\mathfrak{T}$ is a Lie triple system (subsystem of $\mathfrak{L}^{(2)}$). Conversely, show that if $\mathfrak{T}$ is any Lie triple system then there exists an automorphism ε in $SL(\mathfrak{T})$ $(UL(\mathfrak{T}))$ such that $\mathfrak{T}$ $(\mathfrak{T}^{\sigma_u})$ is the characteristic space of the root -1 of ε in the Lie algebra $SL(\mathfrak{T})$ $(UL(\mathfrak{T}))$.

3. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a symmetric bilinear form f on $\mathfrak{V}$. Show that $\mathfrak{V}$ is a subsystem of the associator Lie triple system $\mathfrak{J}$ and that if $1 \leqq \dim \mathfrak{V} < \infty$ and f is nondegenerate then the Lie algebra $R(\mathfrak{V}) + [R(\mathfrak{V}), R(\mathfrak{V})]$ and the mapping $a \to R_a$ is a universal Lie envelope for $\mathfrak{V}$.

4. Let $\mathfrak{T}$ be a Lie triple system over a field Φ and let α be a nonzero element of Φ. Define the product $[abc]^{\alpha} \equiv \alpha[abc]$. Show that the vector space $\mathfrak{T}$ with this composition is a Lie triple system $(\mathfrak{T},\alpha)$. Show that $(\mathfrak{T},\alpha) \cong (\mathfrak{T},\beta)$ if $\beta = \rho^2\alpha$, ρ in Φ. Give an example such that $(\mathfrak{T},\alpha) \not\cong \mathfrak{T}$.

5. (Yamaguti). Let $\mathfrak{T}$ be a two-dimensional L.t.s. (= Lie triple system) with basis (u,v). Note that the multiplication in $\mathfrak{T}$ is determined by the formulas:

$$\begin{aligned} [uuv] &= \alpha u + \beta v, \\ [vvu] &= \gamma u + \delta v, \end{aligned} \tag{13}$$

$\alpha,\beta,\gamma,\delta \in \Phi$. Show that $\alpha = \delta$. Use this to prove that any two-dimensional L.t.s. over an algebraically closed field of characteristic $\neq 2$ is isomorphic to one of the three L.t.s. with basis (u,v) and multiplication table (13) where the possibilities for $\alpha,\beta,\gamma,\delta$ are:

I. $\alpha = \beta = \gamma = \delta = 0$, II. $\alpha = \gamma = \delta = 0$, $\beta = 1$, III. $\alpha = \delta = 2$, $\beta = \gamma = 0$.

6. Determine $SL(\mathfrak{T})$ and $UL(\mathfrak{T})$ for the L.t.s. I, II and III of exercise 5.

7. Let $\mathfrak{B}$ be an ideal in the L.t.s. $\mathfrak{T}$ which is a subsystem of $\mathfrak{L}^{(2)}$. Show that $\mathfrak{B} + [\mathfrak{B}\mathfrak{T}]$ is an ideal in $\mathfrak{T} + [\mathfrak{T}\mathfrak{T}]$ and $\mathfrak{B} + [\mathfrak{B}\mathfrak{B}]$ is an ideal in $\mathfrak{B} + [\mathfrak{B}\mathfrak{T}]$.

2. **Some results on completely reducible Lie algebras of linear transformations with applications to Jordan algebras.** In this section we shall recall some well-known results on completely reducible Lie algebras of linear transformations in a finite-dimensional vector space over a field of characteristic 0 and we shall apply these to obtain: (1) a proof of the semisimplicity of $U(\mathfrak{J})$ for $\mathfrak{J}$ a finite-dimensional semisimple Jordan algebra of characteristic 0, which is independent of the structure theory (cf. §7.5), and (2) a new criterion for semisimplicity in the characteristic 0 case. In this section, except in the exercises, we assume throughout that *all vector spaces and algebras are finite dimensional over fields of characteristic* 0. We remark first that under these circumstances semisimplicity and separability are equivalent notions for Jordan and associative algebras (cf. Theorem 6.4, p. 239).

We recall that a Lie algebra $\mathfrak{L}$ is called semisimple if it contains no nonzero abelian ideals. Any such algebra is a direct sum of simple ideals, and more generally, any representation of such an algebra is completely reducible (Jacobson, *Lie Algebras*, p. 71 and 79). A Lie algebra $\mathfrak{L}$ of linear transformations is completely reducible if and only if $\mathfrak{L} = \mathfrak{C} \oplus [\mathfrak{L}\mathfrak{L}]$ where $\mathfrak{C}$ is the center, $[\mathfrak{L}\mathfrak{L}]$ is semisimple and every element of $\mathfrak{C}$ is semisimple, that is, has minimum polynomial without multiple factors (loc. cit. p. 81). The last condition can be improved slightly to: the center has a basis $(C_1, C_2, \cdots, C_m)$ of semisimple linear transformations. To see this we have to show that if $C_1, C_2, \cdots, C_m$ commute and are semisimple then any $C = \sum \alpha_i C_i$, $\alpha_i \in \Phi$, is semisimple. It is sufficient to show this for algebraically closed base fields. Then semisimplicity is equivalent to diagonalizability (of the corresponding matrices) so the result follows from the well-known fact that diagonalizability of the C_i and commutativity implies simultaneous diagonalizability. A set of linear transformations is completely reducible if and only if its enveloping associative algebra (= associative algebra generated by 1 and the given set) is completely reducible. Also an associative algebra of linear transformations is completely reducible if and only if it is semisimple. It is immediate from these results and the fact that any finite-dimensional associative algebra is isomorphic to an algebra of linear transformations that we have the following

Criterion 1. Let $\mathfrak{A}$ be a finite-dimensional associative algebra with 1 over a field of characteristic 0, $\mathfrak{L}$ a subalgebra of the Lie algebra $\mathfrak{A}^-$ which together with 1 generates $\mathfrak{A}$. Then $\mathfrak{A}$ is semisimple if and only if $\mathfrak{L} = \mathfrak{C} \oplus [\mathfrak{L}\mathfrak{L}]$ where $\mathfrak{C}$ is the center of $\mathfrak{L}$, $[\mathfrak{L}\mathfrak{L}]$ is semisimple and $\mathfrak{C}$ has a basis of semisimple elements.

We shall now apply this result to the associative algebra $U(\mathfrak{J})$, $\mathfrak{J}$ a semisimple Jordan algebra. For this we require the

THEOREM 2. *Let $\mathfrak{L}$ be a finite-dimensional semisimple Lie algebra over a field of characteristic* 0, *$\mathfrak{T}$ a subsystem of the L.t.s. $\mathfrak{L}^{(2)}$ such that $\mathfrak{L} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$. Then $\mathfrak{L}$ and the injection mapping is a universal Lie envelope for $\mathfrak{T}$.*

PROOF. Let $(UL(\mathfrak{T}), \sigma_u)$ be a universal Lie envelope of $\mathfrak{T}$. Then we have the homomorphism η of $UL(\mathfrak{T})$ onto $\mathfrak{L}$ such that $\sigma_u\eta$ is the injection mapping of $\mathfrak{T}$ into $\mathfrak{L}$. We claim that ker η is the center $\mathfrak{C}$ of $UL(\mathfrak{T})$. First, let $c \in \mathfrak{C}$. Since $UL(\mathfrak{T}) = \mathfrak{T}^{\sigma_u} \oplus [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]$, $c = a^{\sigma_u} + \sum [b_i^{\sigma_u} c_i^{\sigma_u}]$ where $a, b_i, c_i \in \mathfrak{T}$. Applying η we see that $a + \sum [b_i c_i]$ is in the center of $\mathfrak{L}$. Since $\mathfrak{L}$ is semisimple, $a + \sum [b_i c_i] = 0$. Hence $c = a^{\sigma_u} + \sum [b_i^{\sigma_u} c_i^{\sigma_u}] \in \ker \eta$. Conversely, let $c = a^{\sigma_u} + \sum [b_i^{\sigma_u} c_i^{\sigma_u}] \in \ker \eta$. Then $a + \sum [b_i c_i] = 0$. Since $\mathfrak{L} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$ this gives $a = 0$ and $\sum [b_i c_i] = 0$. Then $\sum [c_i x\, b_i] = \sum [[b_i c_i] x] = 0$ for all $x \in \mathfrak{T}$. Hence $a^{\sigma_u} = 0$ and $\sum [c_i^{\sigma_u} x^{\sigma_u} b_i^{\sigma_u}] = \sum [[b_i^{\sigma_u} c_i^{\sigma_u}] x^{\sigma_u}] = 0$ for all $x \in \mathfrak{T}$. Since $\mathfrak{T}^{\sigma_u}$ generates $UL(\mathfrak{T})$ this implies that $c = a^{\sigma_u} + \sum [b_i^{\sigma_u} c_i^{\sigma_u}] = \sum [b_i^{\sigma_u} c_i^{\sigma_u}] \in \mathfrak{C}$. Since $\mathfrak{L} \cong UL(\mathfrak{T})/\ker \eta = UL(\mathfrak{T})/\mathfrak{C}$ it is clear that the center $\mathfrak{C}$ is the radical (= maximal solvable ideal) of $UL(\mathfrak{T})$. By the Levi decomposition theorem, $UL(\mathfrak{T}) = \mathfrak{C} \oplus \mathfrak{S}$, where $\mathfrak{S}$ is a subalgebra isomorphic to $UL(\mathfrak{T})/\mathfrak{C}$ and hence to $\mathfrak{L}$ (loc. cit. p. 91). Since $\mathfrak{L} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$ is semisimple, $[\mathfrak{L}\mathfrak{L}] = \mathfrak{L}$ (loc. cit. p. 72). Hence $[\mathfrak{T}\mathfrak{T}] + [[\mathfrak{T}\mathfrak{T}]\mathfrak{T}] + [[\mathfrak{T}\mathfrak{T}][\mathfrak{T}\mathfrak{T}]] = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$. Since $[[\mathfrak{T}\mathfrak{T}]\mathfrak{T}] = [\mathfrak{T}\mathfrak{T}\mathfrak{T}]$ this implies that $\mathfrak{T} = [\mathfrak{T}\mathfrak{T}\mathfrak{T}]$. Then $\mathfrak{T}^{\sigma_u} = [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}] = [[\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]\mathfrak{T}^{\sigma_u}]$ and $UL(\mathfrak{T}) = \mathfrak{T}^{\sigma_u} + [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}] = [[\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}]\mathfrak{T}^{\sigma_u}] + [\mathfrak{T}^{\sigma_u}\mathfrak{T}^{\sigma_u}] = [UL(\mathfrak{T}), UL(\mathfrak{T})]$. Since $UL(\mathfrak{T}) = \mathfrak{C} \oplus \mathfrak{S}$, $[UL(\mathfrak{T}), UL(\mathfrak{T})] = [\mathfrak{S}\mathfrak{S}] \subseteq \mathfrak{S}$. This implies that $\mathfrak{C} = 0$ so $\ker \eta = 0$ and η is an isomorphism. Accordingly, $\mathfrak{L}$ and the injection mapping of $\mathfrak{T}$ is a universal Lie envelope for $\mathfrak{T}$.

We can now prove the following result on the associator structure of a semisimple Jordan algebra.

THEOREM 3. *Let $\mathfrak{J}$ be a finite-dimensional semisimple Jordan algebra of characteristic* 0. *Then $\mathfrak{J} = \mathfrak{C} \oplus \mathfrak{J}'$ where $\mathfrak{C}$ is the center and $\mathfrak{J}' = [\mathfrak{J}\mathfrak{J}\mathfrak{J}]$ is the derived associator ideal. Moreover, the universal Lie envelope of $\mathfrak{J}'$ is semisimple.*

PROOF. We consider first the Lie triple system of linear transformations $R(\mathfrak{J})$. The Lie algebra of linear transformations generated by $R(\mathfrak{J})$ is $\mathfrak{L} = R(\mathfrak{J}) + [R(\mathfrak{J}), R(\mathfrak{J})]$. Since $\mathfrak{J}$ has a 1, $R(\mathfrak{J}) \cap [R(\mathfrak{J}), R(\mathfrak{J})] = 0$; for, if $a, b_i, c_i \in \mathfrak{J}$ then $1R_a = a$ and $1 \sum [R_{b_i} R_{c_i}] = 0$. Hence if $R_a = \sum [R_{b_i} R_{c_i}]$ then $a = 0$, $R_a = 0$ and $\sum [R_{b_i} R_{c_i}] = 0$. Thus $\mathfrak{L} = R(\mathfrak{J}) \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$. We note next that since $\mathfrak{J}$ is a direct sum of simple ideals and since the invariant subspace of $\mathfrak{J}$ relative to $R(\mathfrak{J})$ and to $\mathfrak{L}$ are the ideals, $\mathfrak{L}$ is a completely reducible Lie algebra of linear transformations. Hence $\mathfrak{L} = \mathfrak{S} \oplus [\mathfrak{L}\mathfrak{L}]$ where $\mathfrak{S}$ is the center of $\mathfrak{L}$. It is clear that $\mathfrak{S}$ is contained in the centroid of $\mathfrak{J}$. Since $\mathfrak{J}$ has 1, the centroid is $R(\mathfrak{C})$, the set of multiplications R_c, c in the center $\mathfrak{C}$ of $\mathfrak{J}$. Hence $\mathfrak{S} = R(\mathfrak{C})$ and $\mathfrak{L} = R(\mathfrak{C}) \oplus [\mathfrak{L}\mathfrak{L}]$. Since $\mathfrak{L} = R(\mathfrak{J}) \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$, $[\mathfrak{L}\mathfrak{L}] = [R(\mathfrak{J}) R(\mathfrak{J}) R(\mathfrak{J})] \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$. Since we have the Lie triple system homomorphism $a \to R_a$ of $\mathfrak{J}$ onto $R(\mathfrak{J})$ (cf. (1)), $[R(\mathfrak{J}) R(\mathfrak{J}) R(\mathfrak{J})] = R(\mathfrak{J})' = R(\mathfrak{J}')$. Hence $[\mathfrak{L}\mathfrak{L}] = R(\mathfrak{J}') \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$. Then $\mathfrak{L} = R(\mathfrak{C}) \oplus [\mathfrak{L}\mathfrak{L}] = R(\mathfrak{C}) \oplus R(\mathfrak{J}') \oplus [R(\mathfrak{J}), R(\mathfrak{J})] = R(\mathfrak{J}) \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$. Hence $R(\mathfrak{J}) = R(\mathfrak{C}) \oplus R(\mathfrak{J}')$. Since $\mathfrak{J}$ has a 1, $a \to R_a$ is a monomorphism. Hence $\mathfrak{J} = \mathfrak{C} \oplus \mathfrak{J}'$. The Lie algebra generated by $R(\mathfrak{J}')$ is $R(\mathfrak{J}') \oplus [R(\mathfrak{J}'), R(\mathfrak{J}')] = R(\mathfrak{J}') \oplus [R(\mathfrak{J}), R(\mathfrak{J})] = [\mathfrak{L}\mathfrak{L}]$ and this is semisimple. Hence, by Theorem 2, $[\mathfrak{L}\mathfrak{L}]$ and the mapping $a \to R_a$ is a universal Lie envelope for $\mathfrak{J}'$. This shows that the universal Lie envelope of the Lie triple system $\mathfrak{J}'$, is semisimple.

We can now give a proof of Theorem 7.12 in the characteristic 0 case which is independent of the structure theory. The result in this case is the following

THEOREM 4. *Let $\mathfrak{J}$ be a finite-dimensional semisimple Jordan algebra of characteristic* 0. *Then $U(\mathfrak{J})$ is semisimple.*

PROOF. We consider $\mathfrak{J}$ as imbedded in $U(\mathfrak{J})$ so $U(\mathfrak{J})$ is generated by $\mathfrak{J}$ and 1. We have the relation $[a, b, c] \equiv a \,.\, b \,.\, c - a \,.\, (b \,.\, c) = [[ca]b]$ in $U(\mathfrak{J})$ (eq. (47) on p. 96). This shows that $\mathfrak{J}$ is a subsystem of the Lie triple system $(U(\mathfrak{J})^-)^{(2)}$ and the composition given in $\mathfrak{J}$ as subsystem coincides with the associator composition in $\mathfrak{J}$. We show next that the elements of $\mathfrak{C}$ are semisimple. To see this we may suppose the base field is algebraically closed. Then, if $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$, $\mathfrak{J}_i$ a simple ideal, $\mathfrak{C}$ has the basis $(e_1, \cdots, e_s)$ where $e_i = 1_i$, the identity of $\mathfrak{J}_i$. Then the e_i are orthogonal idempotents. Hence $e_i(2e_i - 1)(e_i - 1) = 0$ and $e_i e_j = e_j e_i$ in $U(\mathfrak{J})$ (p. 104). Hence the e_i are commuting semisimple elements. This implies that every $c \in \mathfrak{C}$ is semisimple. It follows that the linear mapping $x \to [xc]$ in $U(\mathfrak{J})^-$ is semisimple and its restriction to $\mathfrak{L} = \mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$, the subalgebra of $U(\mathfrak{J})^-$ generated by $\mathfrak{J}$, is semisimple. On the other hand, $[[xc]c] = [ccx] = 0$, $x \in \mathfrak{J}$. Hence $C: x \to [xc]$ is nilpotent in $\mathfrak{L}$ ($C^3 = 0$). Hence $[xc] = 0$, $x \in \mathfrak{J}$, $c \in \mathfrak{C}$ and $\mathfrak{C}$ is contained in the center of $U(\mathfrak{J})$. Now, by Theorem 3, $\mathfrak{J} = \mathfrak{C} \oplus \mathfrak{J}'$. Hence, since $[\mathfrak{C}\mathfrak{J}'] = 0$, $\mathfrak{L} = \mathfrak{J} + [\mathfrak{J}\mathfrak{J}] = \mathfrak{C} + \mathfrak{J}' + [\mathfrak{J}'\mathfrak{J}']$. By Theorem 3, $\mathfrak{J}' + [\mathfrak{J}'\mathfrak{J}']$ is a homomorphic image of a semisimple Lie algebra and hence is semisimple. Then $\mathfrak{C} \cap (\mathfrak{J}' + [\mathfrak{J}'\mathfrak{J}']) = 0$ and $\mathfrak{L} = \mathfrak{C} \oplus (\mathfrak{J}' + [\mathfrak{J}'\mathfrak{J}'])$. Moreover, we have seen that the elements of $\mathfrak{C}$ are semisimple. Since 1 and $\mathfrak{L}$ generate $U(\mathfrak{J})$ it follows from the above Criterion 1 that $U(\mathfrak{J})$ is semisimple.

We consider next another important criterion for complete reducibility of a Lie algebra of linear transformations or, equivalently, as in Criterion 1, for semisimplicity of an associative algebra. For this we need to recall some definitions in the theory of Lie algebras. We recall first that a Lie algebra $\mathfrak{L}$ is split three-dimensional simple or an *A_1-Lie algebra*, if it has a basis (e, f, h) with the multiplication table

$$[eh] = 2e, \qquad [fh] = -2f, \qquad [ef] = h. \tag{14}$$

We shall call (e, f, h) a *canonical basis* for the A_1-Lie algebra. If $\mathfrak{A}$ is a finite-dimensional associative algebra with 1 and $\mathfrak{L}$ is an A_1-Lie subalgebra of $\mathfrak{A}^-$ then it is easily seen that the elements e and f in the canonical basis are nilpotent in $\mathfrak{A}$. If $\mathfrak{L}$ is any subalgebra of $\mathfrak{A}^-$ then we shall say that $\mathfrak{L}$ has the *imbedding property for nilpotent elements* if every nonzero element of $\mathfrak{L}$ which is nilpotent (in $\mathfrak{A}$) can be imbedded in a canonical basis (e, f, h) of an A_1-Lie subalgebra of $\mathfrak{L}$. If $\mathfrak{T}$ is a subsystem of the Lie triple system $(\mathfrak{A}^-)^{(2)}$ then we say that $\mathfrak{T}$ has the *imbedding property for nilpotent elements* if every nonzero nilpotent $f \in \mathfrak{T}$ can be imbedded in a canonical basis (e, f, h) of an A_1-Lie subalgebra of $\mathfrak{A}^-$ such the $e \in \mathfrak{T}$. This is equivalent to the existence of an element $e \in \mathfrak{T}$ such that $[eef] = 2e$, $[ffe] = 2f$.

It is well known that if $\mathfrak{A}$ is a finite-dimensional associative algebra with 1 over a field of characteristic 0 then any $a \in \mathfrak{A}$ can be written in one and only one way as $a = s + n$ where s and n are polynomials in $\mathfrak{A}$ and s is semisimple and n is nilpotent. The elements s and n are called the semisimple and nilpotent components of a. A subalgebra $\mathfrak{L}$ of the Lie algebra $\mathfrak{A}^-$ is called almost algebraic if $\mathfrak{L}$ contains the semisimple and nilpotent part of every $a \in \mathfrak{L}$ (loc. cit. p. 98).

We can now state the following criterion which is stated and proved in the author's *Lie Algebras*, p. 100, in a slightly different but equivalent form (referring to completely reducible algebras of linear transformations):

Criterion 2. Let $\mathfrak{A}$ be a finite-dimensional associative algebra with 1 over a field of characteristic 0, $\mathfrak{L}$ a subalgebra of $\mathfrak{A}^-$ such that $\mathfrak{A}$ is generated by 1 and $\mathfrak{L}$. Then if $\mathfrak{A}$ is semisimple, $\mathfrak{L}$ is almost algebraic and $\mathfrak{L}$ has the imbedding property for nilpotent elements. Conversely, assume $\mathfrak{L}$ has this imbedding property and the center of $\mathfrak{L}$ is almost algebraic. Then $\mathfrak{A}$ is semisimple.

We need to recall also the following result which is a key step in the proof of the foregoing criterion.

MOROZOV'S LEMMA. *Let $\mathfrak{L}$ be a finite-dimensional Lie algebra of characteristic* 0, *and suppose $\mathfrak{L}$ contains elements f,h such that $[fh] = -2f$ and $h \in [\mathfrak{L}f]$. Then f and h can be imbedded in a canonical basis (e,f,h) of an A_1-Lie subalgebra of $\mathfrak{L}$* (loc. cit. p. 98).

This lemma has the following consequences.

THEOREM 5. *Let $\mathfrak{A}$ be an associative algebra with* 1 *over a field of characteristic* 0, *$\mathfrak{L}$ a finite-dimensional subalgebra of $\mathfrak{A}^-$ having the imbedding property for nilpotent elements. Then*: (1) *Any subalgebra $\mathfrak{K}$ of $\mathfrak{L}$ which has a $\mathfrak{K}$-invariant complement $\mathfrak{M}$ in $\mathfrak{L}$, that is, a subspace $\mathfrak{M}$ such that $\mathfrak{L} = \mathfrak{K} \oplus \mathfrak{M}$ and $[\mathfrak{M}\mathfrak{K}] \subseteq \mathfrak{M}$, has the imbedding property for nilpotent elements.* (2) *Let $\mathfrak{T}$ be a subsystem of the Lie triple system $\mathfrak{L}^{(2)}$ such that there exists a subalgebra $\mathfrak{K}$ of $\mathfrak{L}$ satisfying $\mathfrak{L} = \mathfrak{T} \oplus \mathfrak{K}$ and*

$$[\mathfrak{T}\mathfrak{K}] \subseteq \mathfrak{T}, \qquad [\mathfrak{T}\mathfrak{T}] \subseteq \mathfrak{K}. \tag{15}$$

Then $\mathfrak{T}$ has the imbedding property for nilpotent elements.

PROOF. The proofs of the two statements are very similar and since (1) is proved in the author's *Lie Algebras*, p. 99, we shall just give the proof of (2). Accordingly, let f be a nilpotent element of $\mathfrak{T}$. By hypothesis there exist elements e,h in $\mathfrak{L}$ such that (e,f,h) satisfies the multiplication table (14). Write $e = e_1 + e_2$, $h = h_1 + h_2$ where $e_1, h_1 \in \mathfrak{T}$, $e_2, h_2 \in \mathfrak{K}$. Then $-2f = [f,h] = [fh_1] + [fh_2]$ and the conditions on $\mathfrak{T}$ and $\mathfrak{K}$ imply that $[fh_2] = -2f$. Also the conditions on $\mathfrak{T}$ and $\mathfrak{K}$ and $h_1 + h_2 = h = [ef] = [e_1f] + [e_2f]$ imply that $h_2 = [e_1f] \in [\mathfrak{L}f]$. Hence, by Morozov's Lemma, we may assume that $h = h_2 \in \mathfrak{K}$. Then $[e_1f] = h$ and $2e_1 + 2e_2 = 2e = [eh] = [e_1h] + [e_2h]$ give $[e_1h] = 2e_1$. Hence (e_1,f,h) gives the required imbedding of f in a canonical basis such that $e_1 \in \mathfrak{T}$.

It is well known that for any positive integer n then there exists a unique (up to similarity) irreducible representation of degree n (= dimensionality of the vector space) of the split Lie algebra A_1 with canonical basis (e,f,h). If we denote the linear transformations representing e,f,h in this representation by E_n, F_n, H_n

respectively then there exists a basis $(u_1, u_2, \cdots, u_n)$ for the underlying vector space $\mathfrak{V}_n$ such that

$$(16)\qquad \begin{aligned} u_1 E_n &= 0, \qquad u_{i+1}E_n = i(i-n)u_i, \qquad 1 \leqq i \leqq n-1, \\ u_i F_n &= u_{i+1}, \qquad 1 \leqq i \leqq n-1, \qquad u_n F_n = 0, \\ u_i H_n &= (n+1-2i)u_i \qquad \text{(loc. cit. p. 85).} \end{aligned}$$

We now introduce the bilinear form ϕ_n on $\mathfrak{V}_n/\Phi$ such that

$$(17)\qquad \phi_n(u_i, u_{n-j+1}) = \delta_{ij}.$$

In other words, the matrix of ϕ_n relative to the given basis is

$$(18)\qquad M_n = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}.$$

Since M_n is symmetric and nonsingular, ϕ_n is a nondegenerate symmetric bilinear form. Since $\mathfrak{W} \equiv \sum_1^\nu \Phi u_i$, where $\nu = [n/2]$, is a totally isotropic subspace, ϕ_n has maximal Witt index. It is easily seen that any nondegenerate symmetric bilinear form of maximal Witt index on an n-dimensional vector space is equivalent in the usual sense to a multiple of ϕ_n. Hence the Jordan algebras of selfadjoint linear transformations relative to any two such forms are isomorphic. Now let $\mathfrak{H}^{(n)}$ denote this Jordan algebra corresponding to the form ϕ_n. Thus $\mathfrak{H}^{(n)} \cong \mathfrak{H}(\Phi_n, J_{M_n})$ the Jordan algebra of matrices which are symmetric under the involution $X \to M_n X^t M_n^{-1}$.

By (16) and (17), $\phi_n(u_i E_n, u_{n-j+1}) = \phi_n(u_i, u_{n-j+1}E_n)$ and $\phi_n(u_i F_n, u_{n-j+1}) = \phi_n(u_i, u_{n-j+1}F_n)$. Hence $E_n, F_n \in \mathfrak{H}^{(n)}$. We now prove

LEMMA 1. *If $n > 1$ the linear transformations* 1 *and E_n and F_n defined in* (16) *generate the Jordan algebra $\mathfrak{H}^{(n)}$.*

PROOF. Let H_{ij} be the linear transformation such that

$$(19)\qquad u_i H_{ij} = u_j, \qquad u_{n+1-j}H_{ij} = u_{n+1-i}, \qquad u_l H_{ij} = 0, \qquad l \neq i,\ n+1-j.$$

Then the $H_{ij} \in \mathfrak{H}^{(n)}$ and the set $\{H_{ij} \mid i + j \leqq n+1\}$ is linearly independent. Since the number of elements in this set is $n(n+1)/2$ the set is a basis for $\mathfrak{H}^{(n)}$. By Cohn's Theorem (p. 8) $K = H_n^2 = [E_n F_n]^2$ is in the subalgebra $\mathfrak{H}'$ of $\mathfrak{H}^{(n)}$ generated by E_n and F_n. Now $u_i K = (n+1-2i)^2 u_i$ so $K = \sum_1{}^\nu \beta_i H_{ii}$ where $\nu = [(n+1)/2]$ and $\beta_i = (n+1-2i)^2$. Since the β_i are distinct and the H_{ii} are orthogonal idempotent elements it follows that every H_{ii} is a polynomial in K and so belongs to $\mathfrak{H}'$. Then $\mathfrak{H}'$ contains $H_{ii}F_n{}^{n+1-2i}H_{ii} = H_{i,n+1-i}$ if $2i < n+1$ and $H_{ii}F_n{}^{k-i}H_{kk} + H_{kk}F_n{}^{k-i}H_{ii} = H_{ik}$ if $k > i$ and $k \neq n+1-i$. Similarly, $H_{ii}E_n{}^{n+1-2i}H_{ii}$ is a nonzero multiple of $H_{n+1-i,i}$ for $2i < n+1$ and $H_{kk}E_n{}^{i-k}H_{ii}$

$+ H_{ii}E_n^{i-k}H_{kk}$ is a nonzero multiple of H_{ik} for $k < i$ and $i \neq n + 1 - k$. Hence $\mathfrak{H}'$ contains the basis $\{H_{ij} \mid i + j \leqq n + 1\}$ for $\mathfrak{H}^{(n)}$, so $\mathfrak{H}' = \mathfrak{H}^{(n)}$.

It is immediate from the Lie algebra Criterion 2 that if $\mathfrak{L}$ is a finite-dimensional Lie algebra of characteristic 0 then $\mathfrak{L}$ is semisimple if and only if every element $f \neq 0$ such that $\operatorname{ad} f$ is nilpotent in $\mathfrak{L}$ can be imbedded in an A_1-Lie subalgebra of $\mathfrak{L}$. We shall now derive an analogous result for Jordan algebras. For this we require the following

Definition 2. *Two elements u, v of a Jordan algebra will be called* associates *if they satisfy the associator relations*

$$[u, u, v] = 2u, \qquad [v, v, u] = 2v. \tag{20}$$

A Jordan algebra $\mathfrak{H}$ will be called an elementary semisimple *Jordan algebra if $\mathfrak{H} \cong \mathfrak{H}^{(n_1)} \oplus \mathfrak{H}^{(n_2)} \oplus \cdots \oplus \mathfrak{H}^{(n_s)}$ where the $\mathfrak{H}^{(k)}$ are defined as above, $1 \leqq n_1 < n_2 < \cdots < n_s$. The integer $m = n_s$ will be called the* index *of $\mathfrak{H}$.*

The defining conditions for associates are equivalent to $[uuv] = 2u$, $[vvu] = 2v$ in $U(\mathfrak{J})$ and these are the same as $[uv] = w$, $[uw] = 2u$, $[vw] = -2v$. Hence the condition is that the elements $(u, v, [uv])$ form a canonical basis for an A_1-Lie subalgebra of $U(\mathfrak{J})^-$. It follows that the elements u and v are nilpotent in $U(\mathfrak{J})$ and hence in $\mathfrak{J}$.

We now prove the following

Theorem 6. *Let $\mathfrak{H}$ be a finite-dimensional Jordan algebra over a field of characteristic 0 generated by 1 and two associates u and v with $v \neq 0$. Then u and v are nilpotent of the same index m and $\mathfrak{H}$ is an elementary semisimple Jordan algebra of index m.*

Proof. By Shirshov-Cohn's Theorem, $\mathfrak{H}$ is special, so we can identify $\mathfrak{H}$ with a subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})^+$ for a suitable finite-dimensional vector space $\mathfrak{V}/\Phi$, Since $\frac{1}{2}\sigma$ is a multiplication specialization if σ is an associative specialization we have the homomorphism of $U(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})$ mapping $1 \to 1$ and $a \to \frac{1}{2}a$. Since we have the relations $[uv] = w$, $[uw] = 2u$, $[vw] = -2v$ in $U(\mathfrak{J})^-$ the elements $e = \frac{1}{2}u$ and $f = \frac{1}{2}v$, $h = [ef]$ form a canonical basis for an A_1-subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})^-$ and, of course, e and f generate $\mathfrak{H}$ as subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})^+$. Since the representations of the Lie algebra A_1 are completely reducible we can write $\mathfrak{V} = \mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \cdots \oplus \mathfrak{V}_s$ where the $\mathfrak{V}_i$ are irreducible invariant subspaces of $\mathfrak{V}$ relative to $A_1 = \Phi e + \Phi f + \Phi h$ and $\dim \mathfrak{V}_i = n_i$. Since the induced representations in any two $\mathfrak{V}_i$ of the same dimensionality are similar we may, by dropping duplicates, assume that $n_1 < n_2 < \cdots < n_s$. Also since the representation of A_1 in a one-dimensional space is 0 and $f \neq 0$, we may assume $n_1 > 1$. Since the linear transformations E_n and F_n are nilpotent of index n it is clear that e and f and, hence, u and v are nilpotent of index $m \equiv n_s$. Let η_k be the restriction homomorphism of $\mathfrak{H}$ into the Jordan algebra $\mathrm{Hom}_\Phi(\mathfrak{V}_k, \mathfrak{V}_k)^+$. By Lemma 1, the image of $\mathfrak{H}$ under η_k is the Jordan algebra $\mathfrak{H}^{(n_k)}$. If $\mathfrak{B}_k = \ker \eta_k$ then clearly $\bigcap_1^s \mathfrak{B}_k = 0$, and $\mathfrak{B}_k$ is a maximal ideal in $\mathfrak{H}$, since $\mathfrak{H}^{(n_k)}$ is simple. If $\bigcap_{j \neq k} \mathfrak{B}_j = 0$ then we have an isomorphism of $\mathfrak{H}$ into the Jordan algebra of restriction of the elements of $\mathfrak{H}$ to $\sum_{j \neq k} \mathfrak{V}_j$. Hence we may assume that $\bigcap_{j \neq k} \mathfrak{B}_j \neq 0$ for $k = 1, 2, \cdots, s$. Then $\mathfrak{B}_k + \bigcap_{j \neq k} \mathfrak{B}_j = \mathfrak{H}$ for all k. This implies that the homomorphism η of $\mathfrak{H}$ into $\mathfrak{H}^{(n_1)} \oplus \mathfrak{H}^{(n_2)} \oplus \cdots \oplus \mathfrak{H}^{(n_s)}$ defined as $a \to (a^{(\eta_1)}, a^{(\eta_2)}, \cdots, a^{(\eta_s)})$ is an isomorphism.

We can now establish the following criterion for semisimplicity.

THEOREM 7. *A finite-dimensional Jordan algebra $\mathfrak{J}$ over a field of characteristic 0 is semisimple if and only if every nonzero nilpotent element v of $\mathfrak{J}$ can be imbedded in an elementary semisimple subalgebra.*

PROOF. *Sufficiency*. Assume $\mathfrak{J}$ has the property stated and let $\mathfrak{R} = \mathrm{rad}\, \mathfrak{J}$. If $\mathfrak{R} \neq 0$ let $v \neq 0$ be in $\mathfrak{R}$. Then v is nilpotent since $\mathfrak{R}$ is solvable. Hence there exists an elementary semisimple subalgebra $\mathfrak{H}$ of $\mathfrak{J}$ containing v. Since $\mathfrak{R}$ is a solvable ideal, $\mathfrak{H} \cap \mathfrak{R} = 0$ contrary to $v \in \mathfrak{H}$ and $v \in \mathfrak{R}$. Hence $\mathfrak{R} = 0$ and $\mathfrak{J}$ is semisimple.

Necessity. Assume $\mathfrak{J}$ is semisimple. Then $U(\mathfrak{J})$ is semisimple, by Theorem 4. Hence, by Criterion 2, the subalgebra $\mathfrak{L} = \mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ of $U(\mathfrak{J})^-$ has the imbedding property for nilpotent elements. Since $R(\mathfrak{J}) \cap [R(\mathfrak{J}), R(\mathfrak{J})] = 0$ and we have the homomorphism of $\mathfrak{L}$ on $R(\mathfrak{J}) + [R(\mathfrak{J}), R(\mathfrak{J})]$ which is 1-1 on $\mathfrak{J}$ we have $\mathfrak{L} = \mathfrak{J} \oplus [\mathfrak{J}\mathfrak{J}]$. Thus $\mathfrak{T} = \mathfrak{J}$, $\mathfrak{K} = [\mathfrak{J}\mathfrak{J}]$ and $\mathfrak{L}$ satisfy the hypotheses of (2) of Theorem 5. Hence the Lie triple system $\mathfrak{J}$ has the imbedding property for nilpotent elements, by the conclusion of Theorem 5 (2). Hence if v is a nilpotent element of $\mathfrak{J}$ then v is a nilpotent element of $U(\mathfrak{J})$ and hence there exists a $u \in \mathfrak{J}$ such that $[uuv] = 2u$ and $[vvu] = 2v$. Then we have the associator relations $[u, u, v] = 2u$, $[v, v, u] = 2v$ in $\mathfrak{J}$, that is, u and v are associates. Hence by Theorem 6, the subalgebra $\mathfrak{H}$ of $\mathfrak{J}$ generated by u and v is an elementary semisimple Jordan algebra and we have an imbedding of v in such a subalgebra.

EXERCISES

1. (Schafer). Let $\mathfrak{J}$ be finite-dimensional central simple of degree r over an arbitrary field. Show that $\mathfrak{J} = \mathfrak{C} \oplus \mathfrak{J}'$ if the characteristic of $\mathfrak{J}$ does not divide r.

2. Let $\mathfrak{J}$ be a special simple Jordan algebra satisfying the axioms of §4.2. Use the structure theorem for such algebras given in §4.5 to show that if u is a nilpotent element of such an algebra then there exists a nilpotent element v in $\mathfrak{J}$ such that the subalgebra generated by 1, u and v is elementary.

3. Let $\mathfrak{T}$ be a Lie triple system of linear transformations of a finite dimensional vector space $\mathfrak{V}$ into itself. Let $\mathfrak{W} = \mathfrak{T} \cap [\mathfrak{T}\mathfrak{T}]$. Show that $\mathfrak{W}$ is an ideal in $\mathfrak{T}$ and in the Lie algebra $[\mathfrak{T}\mathfrak{T}]$ and that $\mathfrak{V}\mathfrak{W}$ is a subspace of $\mathfrak{V}$ invariant under $\mathfrak{T}$. Assume every element of $\mathfrak{W}$ is nilpotent and $\mathfrak{V}$ is $\mathfrak{T}$-irreducible. Use Engel's theorem (Jacobson, *Lie Algebras*, p. 36) to show that $\mathfrak{W} = \mathfrak{T} \cap [\mathfrak{T}\mathfrak{T}] = 0$.

4. Let E and F be nonzero linear transformations of a finite-dimensional vector space $\mathfrak{V}$ over a field of characteristic 0 such that $[EEF] = 2E, [FFE] = 2F$. Show that $E \pm F$ are not nilpotent.

5. Use ex. 3 and 4 and Theorem 5 (2) to prove the following analogue of Engel's theorem for Lie triple systems: If $\mathfrak{T}$ is a Lie triple system of nilpotent linear transformations in a finite-dimensional vector space $\mathfrak{V}$ over a field of characteristic 0 then the associative algebra $\mathfrak{T}^*$ generated by $\mathfrak{T}$ (without 1) is nilpotent.

3. **Operator commutativity.** We shall say that the elements a, b of a Jordan algebra $\mathfrak{J}$ *operator commute* (or *o-commute*) *in* $\mathfrak{J}$ if $[R_a R_b] = 0$. This notion was first introduced by Jordan, von Neumann and Wigner in [1]. It is easy to give examples in which a and b, contained in a subalgebra $\mathfrak{K}$ of $\mathfrak{J}$, o-commute in $\mathfrak{K}$ but not in $\mathfrak{J}$ (ex. 2 below). If $\mathfrak{A}$ is a subset of $\mathfrak{J}$ then we denote the set of elements of $\mathfrak{J}$ which o-commute in $\mathfrak{J}$ with every $a \in \mathfrak{A}$ by $C_{\mathfrak{J}}(\mathfrak{A})$. It is clear that $C_{\mathfrak{J}}(\mathfrak{A})$ is a subspace of $\mathfrak{J}$ containing the center. Moreover, if $c_1, c_2, c_3 \in C_{\mathfrak{J}}(\mathfrak{A})$ and $a \in \mathfrak{A}$ then $[R_a, R_{[c_1, c_2, c_3]}] = [R_a [R_{c_1} R_{c_2} R_{c_3}]] = [R_a, [[R_{c_3} R_{c_1}], R_{c_2}]] = 0$. Hence $[c_1, c_2, c_3] \in C_{\mathfrak{J}}(\mathfrak{A})$, so $C_{\mathfrak{J}}(\mathfrak{A})$ is a subsystem of the associator Lie triple system $\mathfrak{J}$. However, this may not be a subalgebra of $\mathfrak{J}$ (ex. 2). It is immediate that if P is an extension of the base field Φ of $\mathfrak{J}$ and $\mathfrak{A}$ is a subset of $\mathfrak{J}$ then $C_{\mathfrak{J}_P}(\mathfrak{A}) = PC_{\mathfrak{J}}(\mathfrak{A})$, the P-subspace of $\mathfrak{J}_P$ spanned by $C_{\mathfrak{J}}(\mathfrak{A})$.

We shall now show that if $\mathfrak{B}$ is a finite-dimensional separable subalgebra of an arbitrary Jordan algebra $\mathfrak{J}$ then $C_{\mathfrak{J}}(\mathfrak{B})$ is a subalgebra of $\mathfrak{J}$ and if $\mathfrak{K}$ is a subalgebra of $\mathfrak{J}$ containing $\mathfrak{B}$ then $C_{\mathfrak{K}}(\mathfrak{B}) = C_{\mathfrak{J}}(\mathfrak{B}) \cap \mathfrak{K}$. The last equation is equivalent to: If $a \in \mathfrak{K}$ o-commutes in $\mathfrak{K}$ with every $b \in \mathfrak{B}$, then a o-commutes in $\mathfrak{J}$ with every $b \in \mathfrak{B}$.

LEMMA 1. *If e is an idempotent in* $\mathfrak{J}$ *then* $C_{\mathfrak{J}}(\{e\}) = \mathfrak{J}_0(e) + \mathfrak{J}_1(e)$ *where* $\mathfrak{J} = \mathfrak{J}_0(e) \oplus \mathfrak{J}_1(e) \oplus \mathfrak{J}_{\frac{1}{2}}(e)$ *is the Peirce decomposition of* $\mathfrak{J}$ *relative to e. Hence* $C_{\mathfrak{J}}(\{e\})$ *is a subalgebra.*

PROOF. We have the relation $[R_a R_{e^{\cdot 2}}] + 2[R_e R_{a.e}] = 0$. If we take a in $\mathfrak{J}_0(e)$ or in $\mathfrak{J}_1(e)$ this gives $[R_a R_e] = 0$. Hence $\mathfrak{J}_0(e) + \mathfrak{J}_1(e) \subseteq C_{\mathfrak{J}}(\{e\})$. Now let $a \in \mathfrak{J}_{\frac{1}{2}}(e)$. Then $eR_e R_a = \frac{1}{2}a$ and $eR_a R_e = \frac{1}{4}a$. Hence $[R_e R_a] = 0$ implies $a = 0$. It follows that $C_{\mathfrak{J}}(\{e\}) = \mathfrak{J}_0(e) + \mathfrak{J}_1(e)$. Since $\mathfrak{J}_0(e)$ and $\mathfrak{J}_1(e)$ are subalgebras and $\mathfrak{J}_0(e) \cdot \mathfrak{J}_1(e) = 0$ it is clear that $C_{\mathfrak{J}}(\{e\})$ is a subalgebra.

LEMMA 2. *If* $\mathfrak{B}$ *is a finite-dimensional semisimple Jordan algebra over an algebraically closed field then* $\mathfrak{B}$ *has a basis consisting of idempotent elements.*

PROOF. Suppose first that $\mathfrak{B} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a symmetric bilinear form f on $\mathfrak{V}$. Since $\mathfrak{B}$ is semisimple, f is nondegenerate and $\mathfrak{J}$ has a basis $(1, u_1, \cdots, u_m)$ such that $u_i . u_j = \delta_{ij} 1$. Then $u_i^{\cdot 2} = 1$ implies that $e_i = (1 + u_i)/2$ is idempotent. Hence $(1, e_1, \cdots, e_n)$ is a basis of idempotents for $\mathfrak{B}$. Suppose next that the base field is of characteristic 0. Then if $a \in \mathfrak{B}$, $a = \sum \alpha_i e_i + v$ where the e_i

are orthogonal idempotents, $\alpha_i \in \Phi$ and v is nilpotent. Hence it is enough to show that every nilpotent element of $\mathfrak{B}$ is a linear combination of idempotents. By Theorem 7, such an element is contained in an elementary semisimple subalgebra $\mathfrak{H}$. Hence it is enough to show that a Jordan algebra $\mathfrak{H}^{(m)}$, $m \geqq 2$, has a basis of idempotents. Since $\mathfrak{H}^{(m)} \cong \mathfrak{H}(\Phi_m, J_1)$ it is clear (using a Peirce decomposition of $\mathfrak{H}(\Phi_m, J_1)$) that $\mathfrak{H}^{(m)}$ is a vector space sum of Jordan algebras of nondegenerate symmetric bilinear forms. Since we have shown that these have a basis consisting of idempotents it is clear that $\mathfrak{H}^{(m)}$ has such a basis. Finally, suppose the characteristic is $p \neq 0$. In this case we have to use the classification of simple algebras over an algebraically closed field. Using this it is clear that any simple algebra over an algebraically closed field is either one dimensional or is a vector space sum of Jordan algebras of nondegenerate symmetric bilinear forms. It follows that any simple and hence any semisimple Jordan algebra over Φ has a basis of idempotents.

We can now prove

THEOREM 8. *Let $\mathfrak{J}$ be an arbitrary Jordan algebra, $\mathfrak{B}$ a finite-dimensional separable subalgebra. Then $C_{\mathfrak{J}}(\mathfrak{B})$ is a subalgebra and if $\mathfrak{K}$ is a subalgebra of $\mathfrak{J}$ containing $\mathfrak{B}$ then $C_{\mathfrak{K}}(\mathfrak{B}) = C_{\mathfrak{J}}(\mathfrak{B}) \cap \mathfrak{K}$.*

PROOF. It is easy to see that by an extension field argument we can reduce the proof to the case of an algebraically closed base field. Then $\mathfrak{B}$ has a basis $(e_1, e_2, \cdots, e_n)$ where the e_i are idempotents. Then

$$C_{\mathfrak{J}}(\mathfrak{B}) = \bigcap_1^n C_{\mathfrak{J}}(\{e_i\}) = \bigcap_1^n (\mathfrak{J}_0(e_i) + \mathfrak{J}_1(e_i))$$

which is a subalgebra. Also $C_{\mathfrak{K}}(\mathfrak{B}) = \bigcap_1^n(\mathfrak{K}_0(e_i) + \mathfrak{K}_1(e_i))$. It is clear that $(\mathfrak{J}_0(e_i) + \mathfrak{J}_1(e_i)) \cap \mathfrak{K} = \mathfrak{K}_0(e_i) + \mathfrak{K}_1(e_i)$. Hence $C_{\mathfrak{K}}(\mathfrak{B}) = C_{\mathfrak{J}}(\mathfrak{B}) \cap \mathfrak{K}$.

LEMMA 3. *Let $\mathfrak{J}$ be a Jordan algebra with 1, $\mathfrak{L}$ the Lie algebra of linear transformations in $\mathfrak{J}$ generated by the Lie triple system $R(\mathfrak{J})$. Then $\mathfrak{L} = R(\mathfrak{J}) \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$. If $\mathfrak{A}$ is a subset of $\mathfrak{J}$ we write $R_{\mathfrak{J}}(\mathfrak{A}) = \{R_a \mid a \in \mathfrak{A}\}$ where R_a acts in $\mathfrak{J}$. Then if $\mathfrak{B}$ is a subalgebra of $\mathfrak{J}$, $\mathfrak{K} = R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$ is the subalgebra of $\mathfrak{L}$ generated by $R_{\mathfrak{J}}(\mathfrak{B})$ and the centralizer $\mathfrak{M}$ of $\mathfrak{K}$ in $\mathfrak{L}$ has the form*

$$\mathfrak{M} = R_{\mathfrak{J}}(\mathfrak{C}) \oplus \mathfrak{N} \tag{21}$$

where $\mathfrak{C} = C_{\mathfrak{J}}(\mathfrak{B})$ and $\mathfrak{N}$ is a subalgebra of $\mathfrak{M}$ such that $[R_{\mathfrak{J}}(\mathfrak{C}), \mathfrak{N}] \subseteq R_{\mathfrak{J}}(\mathfrak{C})$ and $[R_{\mathfrak{J}}(\mathfrak{C}), R_{\mathfrak{J}}(\mathfrak{C})] \subseteq \mathfrak{N}$.

PROOF. The first statement is clear. Since $\mathfrak{B}$ is a subalgebra $R_{\mathfrak{J}}(\mathfrak{B})$ is a subsystem of the Lie triple system $R(\mathfrak{J})$. Hence the subalgebra of $\mathfrak{L}$ generated by $R_{\mathfrak{J}}(\mathfrak{B})$ is $\mathfrak{K} = R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$. It is clear also that the centralizer $\mathfrak{M}$ of $\mathfrak{K}$ in $\mathfrak{L}$ is the set of linear transformations $M \in \mathfrak{L}$ such that $[MR_b] = 0$, $b \in \mathfrak{B}$. Now $M = R_c + D$ where $c \in \mathfrak{J}$ and $D \in [R(\mathfrak{J}), R(\mathfrak{J})]$. Then $0 = [MR_b] = [R_cR_b]$

$+[DR_b]$ and $[DR_b] \in R(\mathfrak{J})$. Hence $[R_c R_b] = 0$ and $[DR_b] = 0$ for all $b \in \mathfrak{B}$. Thus $c \in \mathfrak{C} = C_{\mathfrak{J}}(\mathfrak{B})$ and $D \in \mathfrak{N} \equiv \mathfrak{M} \cap [R(\mathfrak{J}), R(\mathfrak{J})]$, which is a subalgebra of $\mathfrak{L}$ since $\mathfrak{M}$ and $[R(\mathfrak{J}), R(\mathfrak{J})]$ are subalgebras. Now it is clear that (21) holds where $\mathfrak{C} = C_{\mathfrak{J}}(\mathfrak{B})$ and $\mathfrak{N} = \mathfrak{M} \cap [R(\mathfrak{J}), R(\mathfrak{J})]$ is a subalgebra of $\mathfrak{L}$. If $c \in C_{\mathfrak{J}}(\mathfrak{B})$ and $D \in \mathfrak{N}$ then $[R_c D] \in [R(\mathfrak{J}), [R(\mathfrak{J}), R(\mathfrak{J})]] \subseteq R(\mathfrak{J})$ and $[R_c D] \in \mathfrak{M}$. Hence $[R_c D] = R_{c'}$ where $c' \in \mathfrak{C}$. Thus $[R_{\mathfrak{J}}(\mathfrak{C}), \mathfrak{N}] \subseteq R_{\mathfrak{J}}(\mathfrak{C})$. The inclusion $[R_{\mathfrak{J}}(\mathfrak{C}), R_{\mathfrak{J}}(\mathfrak{C})] \subseteq \mathfrak{N}$ is clear.

We prove next

THEOREM 9. *Let $\mathfrak{J}$ be a finite-dimensional semisimple Jordan algebra over a field of characteristic* 0, *$\mathfrak{B}$ a semisimple subalgebra of $\mathfrak{J}$. Then $C_{\mathfrak{J}}(\mathfrak{B})$ is a semisimple subalgebra of $\mathfrak{J}$.*

PROOF. Since $\mathfrak{J}$ is semisimple, the Lie algebra of linear transformations $\mathfrak{L} = R(\mathfrak{J}) + [R(\mathfrak{J}), R(\mathfrak{J})]$ is completely reducible. We consider next the Lie algebra of linear transformations $\mathfrak{K} = R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$ in $\mathfrak{J}$. Now $U(\mathfrak{B})$ is semisimple by Theorem 4 and we have a homomorphism of $U(\mathfrak{B})$ into $\mathrm{Hom}_{\Phi}(\mathfrak{J}, \mathfrak{J})$ sending $1 \to 1$ and $b \to R_b$, $b \in \mathfrak{B}$. The image under this homomorphism is a semisimple associative algebra and is generated by 1 and $\mathfrak{K}$. Hence $\mathfrak{K}$ is a completely reducible Lie algebra of linear transformations and, consequently, the centralizer $\mathfrak{M}$ of $\mathfrak{K}$ in $\mathfrak{L}$ is completely reducible (Jacobson, *Lie Algebras*, p. 102). By Criterion 2, $\mathfrak{M}$ has the imbedding property for nilpotent elements. By Lemma 3, $\mathfrak{M} = R_{\mathfrak{J}}(\mathfrak{C}) \oplus \mathfrak{N}$ where $\mathfrak{C} = C_{\mathfrak{J}}(\mathfrak{B})$ and $\mathfrak{N}$ is a subalgebra of $\mathfrak{M}$ such that $[R_{\mathfrak{J}}(\mathfrak{C}), \mathfrak{N}] \subseteq R_{\mathfrak{J}}(\mathfrak{C})$ and $[R_{\mathfrak{J}}(\mathfrak{C}), R_{\mathfrak{J}}(\mathfrak{C})] \subseteq \mathfrak{N}$. By Theorem 8, $\mathfrak{C}$ is a subalgebra, so $R_{\mathfrak{J}}(\mathfrak{C})$ is a Lie triple system of linear transformations. Thus $\mathfrak{T} = R_{\mathfrak{J}}(\mathfrak{C})$ satisfies the conditions in (2) of Theorem 5. Hence $R_{\mathfrak{J}}(\mathfrak{C})$ has the imbedding property for nilpotent elements. Now let v be a nilpotent element $\neq 0$ of the Jordan algebra $\mathfrak{C} = C_{\mathfrak{J}}(\mathfrak{B})$. Then R_v is a nonzero nilpotent of $R_{\mathfrak{J}}(\mathfrak{C})$. Hence there exists a $u \in \mathfrak{C}$ such that $[R_u R_u R_v] = 2R_u$, $[R_v R_v R_u] = 2R_v$. Then $[u, u, v] = 2u$ and $[v, v, u] = 2v$ so u and v are associates in the Jordan algebra $\mathfrak{C}$. Hence the subalgebra $\mathfrak{H}$ generated by 1, u and v is an elementary semisimple Jordan algebra and $\mathfrak{C}$ is semisimple by Theorem 7.

EXERCISES

1. Let $\mathfrak{J}$ be a special Jordan algebra and identify $\mathfrak{J}$ with the corresponding subspace of the special universal envelope $S(\mathfrak{J})$. Show that $a, b \in \mathfrak{J}$ o-commute in $\mathfrak{J}$ if and only if $[ab]$ is contained in the center of $S(\mathfrak{J})$.

2. Let Q be the field of rational numbers and let $\mathfrak{J}$ be the subalgebra of $Q_6{}^+$ generated by 1 and the matrices $a = e_{14} + e_{25} + 2e_{46}$, $b = e_{42} + e_{53} + e_{65}$. Verify that $[[ab]a] = 0$ and $[[ab]b] = 0$. Hence show that a and b o-commute in $\mathfrak{J}$. Show that a and $b^{\cdot 2}$ do not o-commute in $\mathfrak{C}$ and that a and b do not o-commute in $Q_6{}^+$.

4. **Derivations of semisimple Jordan algebras of characteristic 0 into bimodules.** In this section we shall prove by a Lie algebra method that every derivation of a finite-dimensional semisimple Jordan algebra of characteristic 0 into a finite-dimensional bimodule is inner. We recall that a more general result was given in Theorem 7.14 (Harris' theorem). The proof of the more general result required the determination of the finite-dimensional simple Jordan algebras over an algebraically closed field and their irreducible bimodules.

We note first that if $\mathfrak{J}$ is an arbitrary Jordan algebra, $U(\mathfrak{J})$ its universal multiplication envelope (containing $\mathfrak{J}$), then the subalgebra of the Lie algebra $U(\mathfrak{J})^-$ generated by $\mathfrak{J}$ is $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ and $\mathfrak{J} \cap [\mathfrak{J}\mathfrak{J}] = 0$. The first statement is clear since $\mathfrak{J}$ is a subsystem of the Lie triple system $(U(\mathfrak{J})^-)^{(2)}$. The second follows if $\mathfrak{J}$ has an identity element by observing that in this case the mapping $a \to R_a$ is 1-1. Moreover, we have the Lie algebra homomorphism of $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ onto $R(\mathfrak{J}) + [R(\mathfrak{J}), R(\mathfrak{J})]$ induced by the homomorphism of $U(\mathfrak{J})$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ such that $1 \to 1$, $a \to R_a$. Since $R(\mathfrak{J}) \cap [R(\mathfrak{J}), R(\mathfrak{J})] = 0$ and $a \to R_a$ is 1-1, we have $\mathfrak{J} \cap [\mathfrak{J}\mathfrak{J}] = 0$. If $\mathfrak{J}$ does not have an identity element then we can adjoin one and argue in the same way with the mapping $a \to R_a$ where R_a is the multiplication by $a \in \mathfrak{J}$ in the larger algebra. We can now give a Lie algebra proof of the following

THEOREM 10. *Let $\mathfrak{J}$ be a finite-dimensional semisimple Jordan algebra over a field of characteristic* 0, *$\mathfrak{M}$ a finite-dimensional bimodule for $\mathfrak{J}$. Then every derivation D of $\mathfrak{J}$ into $\mathfrak{M}$ is inner, that is, has the form $x \to \sum[b_i, x, z_i]$ where $b_i \in \mathfrak{J}$, $z_i \in \mathfrak{M}$.*

PROOF. As in the proof of Theorem 7.14, we may assume the base field is algebraically closed. We form the split null extension $\mathfrak{E} = \mathfrak{J} \oplus \mathfrak{M}$ and extend D linearly to $\mathfrak{E}$ so that $\mathfrak{M}D = 0$. Then the extended D is a derivation in $\mathfrak{E}$ mapping $\mathfrak{J}$ into $\mathfrak{M}$, $\mathfrak{M}$ into 0. Let $U(\mathfrak{E})$ be the universal multiplication envelope of $\mathfrak{E}$ Then D has a unique extension to a derivation D in $U(\mathfrak{E})$ (Theorem 2.13 (ii), p. 97). Clearly D is a derivation of the Lie algebra $U(\mathfrak{E})^-$ and this maps the subalgebra $\mathfrak{E} + [\mathfrak{E}\mathfrak{E}]$ of $U(\mathfrak{E})^-$ into itself. Let $\mathfrak{J}'$ be the associator ideal of $\mathfrak{J}$. Then $\mathfrak{L} = \mathfrak{J}' + [\mathfrak{J}'\mathfrak{J}']$ is a semisimple Lie algebra, by Theorem 3, and the restriction D of D to $\mathfrak{L}$ is a derivation of $\mathfrak{L}$ into the finite-dimensional Lie algebra $\mathfrak{N} = \mathfrak{E} + [\mathfrak{E}\mathfrak{E}]$ containing $\mathfrak{L}$ as a subalgebra. It follows that this can be extended to an inner derivation of $\mathfrak{N}$ (Jacobson, *Lie Algebras*, p. 80). Thus there exists an element $f \in \mathfrak{N}$ such that $xD = [xf]$ for all $x \in \mathfrak{J}'$. We have seen that $\mathfrak{N} = \mathfrak{E} \oplus [\mathfrak{E}\mathfrak{E}]$, and since $[\mathfrak{E}[\mathfrak{E}\mathfrak{E}]] \subseteq \mathfrak{E}$ and $xD \in \mathfrak{M} \subseteq \mathfrak{E}$ for all $x \in \mathfrak{J}'$ we may assume that $f \in [\mathfrak{E}\mathfrak{E}] = [\mathfrak{J}\mathfrak{J}] + [\mathfrak{J}\mathfrak{M}] + [\mathfrak{M}\mathfrak{M}]$. If $z, w \in \mathfrak{M}$, $x \in \mathfrak{J}'$ then $[x[zw]] = [[wz]x] = [zxw] = 0$ since $\mathfrak{M}^{\cdot 2} = 0$ in $\mathfrak{E}$. Hence we may assume $f \in [\mathfrak{J}\mathfrak{J}] + [\mathfrak{J}\mathfrak{M}]$. Also since $[\mathfrak{J}'[\mathfrak{J}\mathfrak{J}]] \subseteq \mathfrak{J}'$, $[\mathfrak{J}'[\mathfrak{J}\mathfrak{M}]] \subseteq \mathfrak{M}$ and $xD \in \mathfrak{M}$ if $x \in \mathfrak{J}$, we may assume $f \in [\mathfrak{J}\mathfrak{M}]$. Thus we have $f = \sum[b_i z_i]$, $b_i \in \mathfrak{J}$, $z_i \in \mathfrak{M}$ and $xD = [xf] = \sum[x[b_i z_i]] = -\sum[b_i x z_i]$, $x \in \mathfrak{J}'$. Hence if we subtract the inner derivation $x \to -\sum[b_i x z_i]$ from D we obtain a derivation E of $\mathfrak{J}$ into $\mathfrak{M}$ such that $\mathfrak{J}'E = 0$. It remains to prove E is inner. We now write $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ where the $\mathfrak{J}_i$ are the simple components which we assume are ordered so that $\dim \mathfrak{J}_i > 1$ if $1 \leqq i \leqq t$ and $\dim \mathfrak{J}_j = 1$ if $j > t$. Then $\mathfrak{J} = \mathfrak{K} \oplus \mathfrak{A}$ where $\mathfrak{K} = \mathfrak{J}_1 + \cdots + \mathfrak{J}_t$ and $\mathfrak{A} = \mathfrak{J}_{t+1} + \cdots + \mathfrak{J}_s$. Hence $\mathfrak{J}' = \mathfrak{K}' = \mathfrak{J}_1' + \cdots + \mathfrak{J}_t'$ and $\mathfrak{J}_i = \Phi 1_i \oplus \mathfrak{J}_i'$, where 1_i is the identity of $\mathfrak{J}_i$ (Theorem 3). Evidently, $\mathfrak{J}_i' \neq 0$ if $i \leqq t$ and the subalgebra of $\mathfrak{J}_i$ generated by $\mathfrak{J}_i'$ is an ideal. Hence the subalgebra of $\mathfrak{J}_i$ generated by $\mathfrak{J}_i'$ is $\mathfrak{J}_i$ and so the subalgebra of $\mathfrak{J}$ generated by $\mathfrak{J}'$ is $\mathfrak{K}$. Since $\mathfrak{J}'E = 0$ we have $\mathfrak{K}E = 0$. Let $(e_1, \cdots, e_{s-t})$ be a basis of $\mathfrak{A}$ consisting of orthogonal idempotent elements (the identities of the $\mathfrak{J}_j$) and suppose we have $e_1 E = \cdots = e_r E = 0$ for some $r < s - t$. The relation $e_{r+1}^{\cdot 2} = e_{r+1}$ gives $2e_{r+1}E \,.\, e_{r+1} = e_{r+1}E$ and $4e_{r+1}E \,.\, e_{r+1} \,.\, e_{r+1} = 2e_{r+1}E \,.\, e_{r+1} = e_{r+1}E$. Hence $e_{r+1}E = e_{r+1}F$ where F is the inner derivation $[R_{4e_{r+1}} R_{e_{r+1}E}]$ and $e_{r+1}E \in \mathfrak{M}$. If $x \in \mathfrak{K} + \sum_1^r \Phi e_i$ then $x \,.\, e_{r+1} = 0$ and $xE = 0$ give $x \,.\, e_{r+1}E = 0$. Hence $xF = 0$ so $G = E - F$ is a derivation of $\mathfrak{J}$ into $\mathfrak{M}$ such that $\mathfrak{K}G = 0$ and $e_j G = 0$ for $j \leqq r + 1$. It follows by induction on r that G and hence D is inner.

As we have seen in §7.7, Theorem 10 implies immediately that if $\mathfrak{K}$ is a finite-dimensional Jordan algebra over a field of characteristic 0 and $\mathfrak{J}$ is a semisimple subalgebra of $\mathfrak{K}$ then every derivation of $\mathfrak{J}$ into $\mathfrak{K}$ can be extended to an inner derivation of $\mathfrak{K}$ (Corollary on p. 303). In particular, every derivation of a finite-dimensional semisimple Jordan algebra of characteristic 0 is inner. We shall use this fact to prove the following

THEOREM 11. *If $\mathfrak{J}$ is a finite-dimensional semisimple Jordan algebra over a field of characteristic* 0 *then the Lie algebra* Der $\mathfrak{J}$ *of derivations in* $\mathfrak{J}$ *is completely reducible.*

PROOF. It is well known that the Lie algebra of derivations of any finite-dimensional algebra is almost algebraic (Jacobson's, *Lie Algebras*, ex. 8 on p. 54). Hen ceDer $\mathfrak{J}$ is almost algebraic. Now Der $\mathfrak{J} = [R(\mathfrak{J}), R(\mathfrak{J})]$, by the result noted, and $\mathfrak{L} = R(\mathfrak{J}) + \text{Der}\,\mathfrak{J}$ is the subalgebra of $\text{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})^-$ generated by $R(\mathfrak{J})$. Since $\mathfrak{L}$ is completely reducible it has the imbedding property for nilpotent elements. Since $\mathfrak{L} = R(\mathfrak{J}) \oplus \text{Der}\,\mathfrak{J}$ and $[R(\mathfrak{J}), \text{Der}\,\mathfrak{J}] \subseteq R(\mathfrak{J})$ the subalgebra Der $\mathfrak{J}$ has the imbedding property for nilpotent elements by Theorem 5 (1). Hence Der $\mathfrak{J}$ is completely reducible by Criterion 2.

5. **The Tits-Koecher construction of a Lie algebra from a Jordan algebra.** In this section we shall give a construction of a Lie algebra from an arbitrary Jordan algebra with 1, which has been given by Koecher in [9], and somewhat earlier and in a more general form by Tits ([4], cf. ex. 2, below). We begin the construction with the Lie algebra of linear transformations in $\mathfrak{J}$ generated by the multiplications R_a, $a \in \mathfrak{J}$. Since $R(\mathfrak{J})$ is a Lie triple system, this is

$$\mathfrak{L} = \{R_a + \sum [R_{b_i} R_{c_i}] \,|\, a, b_i\ c_i \in \mathfrak{J}\}.$$

The set $\{\sum[R_{b_i}R_{c_i}]\}$ is the Lie algebra Inder $\mathfrak{J}$ of inner derivations. Hence $\mathfrak{L} = R(\mathfrak{J}) + \text{Inder}\,\mathfrak{J}$. Since $1R_a = a$ and $1D = 0$ for any derivation D it is clear that $R(\mathfrak{J}) \cap \text{Inder}\,\mathfrak{J} = 0$. Hence $\mathfrak{L} = R(\mathfrak{J}) \oplus \text{Inder}\,\mathfrak{J}$ and so we have the linear mapping $\varepsilon\colon R_a + D \to \overline{R_a + D} = -R_a + D$, $a \in \mathfrak{J}$, $D \in \text{Inder}\,\mathfrak{J}$, in $\mathfrak{L}$, which is of period two. Since

$$\begin{aligned}[-R_a + D, -R_b + E] &= [R_aR_b] - R_{aE} + R_{bD} + [DE],\\ [R_a + D, R_b + E] &= [R_aR_b] + R_{aE} - R_{bD} + [DE]\end{aligned}$$

for $a, b \in \mathfrak{J}$, $D, E \in \text{Inder}\,\mathfrak{J}$ it is clear that ε is an automorphism of period two in $\mathfrak{L}$.

If $a, b \in \mathfrak{J}$ we define

$$a \triangle b = R_a{}^{(b)} = R_{a.b} - [R_aR_b]. \tag{22}$$

Hence $R_a{}^{(b)}$ is the multiplication by a in the b-homotope $(\mathfrak{J}, b)$ of $\mathfrak{J}$ (cf. p. 58). For $c \in \mathfrak{J}$, we have $c(a \triangle b) = c.(a.b) - c.a.b + c.b.a = \{abc\}$. Since $\{abc\} = \{cba\}$ this gives

$$c(a \triangle b) = a(c \triangle b). \tag{23}$$

Since $\overline{a \triangle b} = -R_{b.a} - [R_aR_b] = -R_b{}^{(a)}$ we have

$$\overline{a \triangle b} = -(b \triangle a) \quad\text{and}\quad c\overline{(a \triangle b)} = b\overline{(a \triangle c)}. \tag{24}$$

If $D \in \text{Inder}\,\mathfrak{J}$ then $[R_aD] = R_{aD}$. Hence

$$\begin{aligned}[a \triangle b, D] &= [R_{a.b} - [R_aR_b], D]\\ &= R_{aD.b} + R_{a.bD} - [R_{aD}R_b] - [R_aR_{bD}]\\ &= a \triangle bD + aD \triangle b.\end{aligned}$$

Also

$$\begin{aligned}&[a \triangle b, R_c] - aR_c \triangle b + a \triangle bR_c\\ &\quad= [R_{a.b}, R_c] - [[R_aR_b]R_c] - R_{a.c.b} + [R_{a.c}R_b]\\ &\qquad+ R_{a.(b.c)} - [R_aR_{b.c}]\\ &\quad= R_{[b,c,a]} - [R_bR_cR_a] = 0.\end{aligned}$$

Thus we have

$$[a \triangle b, L] = aL \triangle b + a \triangle b\bar{L} \tag{25}$$

for $a, b \in \mathfrak{J}$, $L \in \mathfrak{L}$.

We now form the vector space $\mathfrak{K} = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}$, where $\bar{\mathfrak{J}}$ is isomorphic to $\mathfrak{J}$ under a linear mapping $a \to \bar{a}$, and we define

$$\begin{aligned}[a_1 + \bar{b}_1 + L_1, a_2 + \bar{b}_2 + L_2] &= a_1L_2 - a_2L_1 + \overline{b_1\bar{L}_2} - \overline{b_2\bar{L}_1}\\ &\quad+ a_1 \triangle b_2 - a_2 \triangle b_1 + [L_1L_2].\end{aligned} \tag{26}$$

It is clear that this is skew symmetric and bilinear. Also we have

$$[[a_1 + \bar{b}_1 + L_1, a_2 + \bar{b}_2 + L_2], a_3 + \bar{b}_3 + L_3] = a_{123} + \bar{b}_{123} + L_{123}$$

where

$$\begin{aligned} a_{123} &= a_1 L_2 L_3 - a_2 L_1 L_3 - a_3(a_1 \triangle b_2) + a_3(a_2 \triangle b_1) - a_3[L_1 L_2], \\ b_{123} &= b_1 \bar{L}_2 \bar{L}_3 - b_2 \bar{L}_1 \bar{L}_3 - b_3(\overline{a_1 \triangle b_2}) + b_3(\overline{a_2 \triangle b_1}) - b_3[\bar{L}_1 \bar{L}_2], \\ L_{123} &= (a_1 L_2 - a_2 L_1) \triangle b_3 - a_3 \triangle (b_1 \bar{L}_2 - b_2 \bar{L}_1) \\ &\quad + [a_1 \triangle b_2 - a_2 \triangle b_1, L_3] + [[L_1 L_2] L_3]. \end{aligned}$$

Then

$$\begin{aligned} a_{123} + a_{231} + a_{312} = &- a_3(a_1 \triangle b_2) + a_3(a_2 \triangle b_1) - a_1(a_2 \triangle b_3) \\ &+ a_1(a_3 \triangle b_2) - a_2(a_3 \triangle b_1) + a_2(a_1 \triangle b_3) = 0, \end{aligned}$$

by (23). Similarly, (24) implies that $b_{123} + b_{231} + b_{312} = 0$. Also, by (25), we have

$$\begin{aligned} L_{123} = &(a_1 L_2 \triangle b_3 + a_1 L_3 \triangle b_2 - a_2 L_1 \triangle b_3 - a_2 L_3 \triangle b_1) \\ &+ (a_3 \triangle b_2 \bar{L}_1 + a_1 \triangle b_2 \bar{L}_3 - a_3 \triangle b_1 \bar{L}_2 - a_2 \triangle b_1 \bar{L}_3) \\ &+ [[L_1 L_2] L_3]. \end{aligned}$$

Hence by the Jacobi identity in $\mathfrak{L}$, we have $L_{123} + L_{231} + L_{312} = \sum_\sigma a_{1\sigma} L_{2\sigma} \triangle b_{3\sigma} - \sum_\sigma a_{1\sigma} L_{2\sigma} \triangle b_{3\sigma} + \sum_\sigma a_{1\sigma} \triangle b_{2\sigma} \bar{L}_{3\sigma} - \sum_\sigma a_{1\sigma} \triangle b_{2\sigma} \bar{L}_{3\sigma}$, where σ ranges over all the permutations of $1, 2, 3$. Hence $L_{123} + L_{231} + L_{312} = 0$. Thus the Jacobi identity holds in $\mathfrak{K}$ and $\mathfrak{K}$ is a Lie algebra.

It is clear from the multiplication formula (26) that

$$(27) \qquad [\mathfrak{J}\mathfrak{J}] = 0, \qquad [\bar{\mathfrak{J}}\bar{\mathfrak{J}}] = 0.$$

Also, we have

$$(28) \qquad [a, L] = aL, \qquad [\bar{a}, L] = \overline{a\bar{L}}.$$

Since $1 = R_1 \in \mathfrak{L}$ and $1^\varepsilon = -1$, it is clear that

$$(29) \qquad [\mathfrak{J}\mathfrak{L}] = \mathfrak{J}, \qquad [\bar{\mathfrak{J}}\mathfrak{L}] = \bar{\mathfrak{J}}.$$

Next we note that

$$(30) \qquad [a, \bar{b}] = a \triangle b.$$

Since the space spanned by the $a \triangle b$, $a, b \in \mathfrak{J}$ includes every $R_b = 1 \triangle b$ and every $[R_b R_a] = a \triangle b - R_{a.b}$ it is clear that

$$(31) \qquad [\mathfrak{J}\bar{\mathfrak{J}}] = \mathfrak{L}.$$

Let ε be the extension to $\mathfrak{K}$ of the automorphism ε on $\mathfrak{L}$ such that

$$a + \bar{b} + L \to \overline{a + \bar{b} + L} = b + \bar{a} + \bar{L}. \tag{32}$$

Clearly this is linear and of period two. By (30) and (24), $\overline{[a, \bar{b}]} = -b \triangle a = -[b, \bar{a}] = [\bar{a}, b]$. Also $\overline{[a, L]} = \overline{aL} = [\bar{a}, \bar{L}]$ and $\overline{[\bar{a}, L]} = a\bar{L} = [a, \bar{L}]$. It is clear from this and (27) that ε is an automorphism of $\mathfrak{K}$. The subalgebra of $\mathfrak{K}$ of ε-fixed points is $\mathfrak{F} = \{a + \bar{a} + D \mid a \in \mathfrak{J}, D \in \text{Inder}\,\mathfrak{J}\}$ and the Lie triple system (subsystem of $\mathfrak{K}^{(2)}$) of elements $x \in \mathfrak{K}$ such that $\bar{x} = -x$ is

$$\mathfrak{T} = \{a - \bar{a} + R_b \mid a, b \in \mathfrak{J}\}.$$

We have the multiplication formulas

$$[a + \bar{a}, D] = aD + \overline{aD}, \qquad [a + \bar{a}, b + \bar{b}] = -2[R_a R_b] \tag{33}$$

in $\mathfrak{F}$ and the Lie triple system formula:

$$\begin{aligned} &[a_1 - \bar{a}_1 + R_{b_1}, a_2 - \bar{a}_2 + R_{b_2}, a_3 - \bar{a}_3 + R_{b_3}] \\ &\quad = [[a_3 - \bar{a}_3 + R_{b_3}, a_1 - \bar{a}_1 + R_{b_1}], a_2 - \bar{a}_2 + R_{b_2}] \\ &\quad = a - \bar{a} + R_b \end{aligned} \tag{34}$$

where

$$\begin{aligned} a &= a_3 . b_1 . b_2 - a_1 . b_3 . b_2 + 2[a_1, a_2, a_3] + [b_1, a_2, b_3], \\ b &= [b_1, b_2, b_3] + 2[a_1, b_2, a_3] - 2a_3 . b_1 . a_2 + 2a_1 . b_3 . a_2. \end{aligned} \tag{35}$$

Now put

$$e = 1, \qquad f = 2(\bar{1}), \qquad h = 2R_1 \tag{36}$$

where 1 is the identity element of $\mathfrak{J}$ and the elements indicated are in $\mathfrak{K} = \mathfrak{J} + \bar{\mathfrak{J}} + \mathfrak{L}$. Then (28) and (30) show that $[ef] = h$, $[eh] = 2e$, $[fh] = -2f$. Hence if we define an A_1-Lie algebra and canonical basis for arbitrary characteristic $\neq 2$ as on p. 315, then (e, f, h) is a canonical basis for the A_1-subalgebra $\mathfrak{A} = \Phi e + \Phi f + \Phi h$ of $\mathfrak{K}$. If $D \in \text{Inder}\ \mathfrak{J}$, then $1D = 0$ and $[R_1, D] = 0$. Hence $[e, D] = [f, D] = [h, D] = 0$. Consequently, the subalgebra Inder$\mathfrak{J}$ centralizes $\mathfrak{A}$. If $a \in \mathfrak{J}$ we put $\mathfrak{M}_a = \Phi a + \Phi \bar{a} + \Phi R_a$. Then (28) and (30) show that $[\mathfrak{M}_a \mathfrak{A}] \subseteq \mathfrak{M}_a$ and $\mathfrak{M}_a$ as $\mathfrak{A}$-module is isomorphic to $\mathfrak{A}$. Clearly, $\mathfrak{K} = \text{Inder}\,\mathfrak{J} + \sum_{a \in \mathfrak{J}} \mathfrak{M}_a$. Hence $\mathfrak{K}$ as $\mathfrak{A}$-module (with respect to the module product for $x \in \mathfrak{K}$, $b \in \mathfrak{A}$ as $[x, b]$) is a sum of the submodule Inder $\mathfrak{J}$ and three-dimensional submodules isomorphic to $\mathfrak{A}$. It follows that the centralizer of $\mathfrak{A}$ in $\mathfrak{K}$ is Inder $\mathfrak{J}$.

Let c be an invertible element of $\mathfrak{J}$ and consider the isotope $\mathfrak{J}^{(c)} \equiv (\mathfrak{J}, c)$ of $\mathfrak{J}$ and the Lie algebras $\mathfrak{L}^{(c)} = R(\mathfrak{J}^{(c)}) + \text{Inder}\,\mathfrak{J}^{(c)}$ and $\mathfrak{K}^{(c)} = \mathfrak{J}^{(c)} \oplus \bar{\mathfrak{J}}^{(c)} \oplus \mathfrak{L}^{(c)}$ defined by $\mathfrak{J}^{(c)}$. We write $R_a^{(c)}$ for the multiplication by a in $\mathfrak{J}^{(c)}$, $\triangle^{(c)}$ for the multiplication $\triangle$ on $\mathfrak{J}^{(c)}$ and $\varepsilon^{(c)}$ for the automorphism $R_a^{(c)} + \sum [R_{b_i}{}^{(c)} R_{c_i}{}^{(c)}]$

$\to -R_a^{(c)} + \sum[R_{b_i}^{(c)}R_{c_i}^{(c)}]$. We have $R_a^{(c)} = R_{a.c} - [R_aR_c] \in \mathfrak{L}$. Hence $\mathfrak{L}^{(c)} \subseteq \mathfrak{L}$. Since isotopy is a symmetric relation we have $\mathfrak{L} = \mathfrak{L}^{(c)}$. Also, by definition $a \triangle^{(c)} b = R_a^{(c,b)}$, the multiplication by a in the b-homotope of $\mathfrak{J}^{(c)}$. We have seen that the b-homotope of the c-homotope of $\mathfrak{J}$ is the $\{cbc\}$-homotope of $\mathfrak{J}$ (p. 57). Hence we have

$$a \triangle^{(c)} b = a \triangle \{cbc\} = a \triangle bU_c. \tag{37}$$

We obtain next the relation between the automorphism ε and $\varepsilon^{(c)}$ in $\mathfrak{L} = \mathfrak{L}^{(c)}$. We have $(a \triangle c)^{\varepsilon(c)} = (R_a^{(c)})^{\varepsilon(c)} = -R_a^{(c)} = -a \triangle c$. By (24), $(a \triangle c)^{\varepsilon} = -(c \triangle a)$. Now it is immediate that the identity $\{c\{bca\}c\} = \{\{cbc\}ac\}$ is valid for special Jordan algebras. Hence, by Macdonald's Theorem it holds for arbitrary Jordan algebras. In operator form the identity reads $(a \triangle c)U_c = U_c(c \triangle a)$. Hence the foregoing equations show that

$$L^{(\varepsilon^c)} = U_cL^{\varepsilon}U_c^{-1} \tag{38}$$

holds for every $L = a \triangle c$. Since ε and $\varepsilon^{(c)}$ are automorphisms and the elements $a \triangle c = R_a^{(c)}$ generate $\mathfrak{L}^{(c)} = \mathfrak{L}$ it is clear that (38) holds for all $L \in \mathfrak{L}$.

We can now prove

THEOREM 12 (KOECHER). *The mapping* $a + \bar{b} + L \to a + \overline{bU_c} + L$ *is an isomorphism of* $\mathfrak{K}^{(c)}$ *onto* $\mathfrak{K}$.

PROOF. Since U_c is linear and bijective in $\mathfrak{J}$ it is clear that the indicated mapping is linear and bijective. In view of the multiplication formulas (27), (28) and (30) and the fact that $\mathfrak{L}^{(c)} = \mathfrak{L}$, it is enough to verify that $a \triangle^{(c)} b = a \triangle bU_c$ and $bL^{\varepsilon^{(c)}}U_c = bU_cL^{\varepsilon}$ for $b \in \mathfrak{J}$, $L \in \mathfrak{L}$. These hold by (37) and (38) so the proof is complete.

EXERCISES

1. Let $\mathfrak{D}$ be a subalgebra of Der $\mathfrak{J}$ containing Inder $\mathfrak{J}$. Show that $R(\mathfrak{J}) + \mathfrak{D}$ is a subalgebra of the Lie algebra $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})^-$ and $R_a + D \to \overline{R_a + D} = -R_a + D$ is an automorphism in this algebra. Consider the vector space $\mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus (R(\mathfrak{J}) + \mathfrak{D})$ and define $[a_1 + \bar{b}_1 + L_1, a_2 + \bar{b}_2 + L_2]$, $a_i \in \mathfrak{J}$, $\bar{b}_i \in \bar{\mathfrak{J}}$, $L_i \in R(\mathfrak{J}) + \mathfrak{D}$ as in (26). Show that this is a Lie algebra and show that the foregoing results on $\mathfrak{K}$ with the exception of Theorem 12 carry over to this Lie algebra.

2. (Tits) Let $\mathfrak{A}$ be a three-dimensional simple Lie algebra, $\mathfrak{J}$ a Jordan algebra and $\mathfrak{D}$ a subalgebra of Der $\mathfrak{J}$ containing Inder $\mathfrak{J}$. If $a, b \in \mathfrak{A}$ put $(a,b) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}\, a)(\mathrm{ad}\, b)$ (half the Killing form). Let $\mathfrak{N} = \mathfrak{D} \oplus (\mathfrak{J} \otimes \mathfrak{A})$ and define for $D_i \in \mathfrak{D}$, $a_i \in \mathfrak{J}$, $u_i \in \mathfrak{A}$,

$$\begin{aligned}[D_1 + a_1 \otimes u_1, D_2 + a_2 \otimes u_2] &= [D_1, D_2] + (u_1, u_2)[R_{a_1}R_{a_2}] \\ &\quad + a_1D_2 \otimes u_1 - a_2D_1 \otimes u_2 + a_1 . a_2 \otimes [u_1, u_2].\end{aligned} \tag{39}$$

Show that this defines a Lie algebra structure on $\mathfrak{N}$. Prove that if $\mathfrak{A}$ is split and $\mathfrak{J}$ has an identity 1 then $\mathfrak{N}$ is isomorphic to the Lie algebra defined in ex.1.

3. Show that the center of $\mathfrak{K}$ is 0.

4. Let $\mathfrak{J}$ be a Jordan algebra with 1, $\mathfrak{K} = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}$ the Lie algebra defined by $\mathfrak{J}$ as in the text. Verify that $(\operatorname{ad} a)^3 = 0$ in $\mathfrak{K}$ for every $a \in \mathfrak{J}$.

5. Let $\mathfrak{J}$, $\mathfrak{K}$, $\mathfrak{L}$ be as in 4. and assume the characteristic is $p \neq 0$. Show that $\mathfrak{L} = R(\mathfrak{J}) + \operatorname{Inder}\mathfrak{J}$ is a restricted Lie algebra of linear transformations in $\mathfrak{J}$. Show that a p-mapping $x \to x^{[p]}$ can be introduced in $\mathfrak{K}$ in one and only one way so that: (1) $\mathfrak{K}$ with this mapping is a restricted Lie algebra, (2) $\mathfrak{L}$ with $L^{[p]} = L^p$ is a subalgebra of $\mathfrak{K}$ as restricted Lie algebra. Show that for the p-mapping satisfying these conditions we have $a^{[p]} = 0$ for all $a \in \mathfrak{J}$.

6. Let D be a derivation in $\mathfrak{J}$ ($\mathfrak{J}$ arbitrary with 1). Show that the mapping $a + \bar{b} + L \to aD + \overline{bD} + [L, D]$ is a derivation in $\mathfrak{K}$.

7. Let $\mathfrak{J}$ be an arbitrary Jordan algebra and let $\mathfrak{T} = \mathfrak{J} \oplus \bar{\mathfrak{J}}$ where $\varepsilon\colon a \to \bar{a}$ is a linear isomorphism of $\mathfrak{J}$ onto $\bar{\mathfrak{J}}$. Define

$$[a_1 + \bar{b}_1, a_2 + \bar{b}_2, a_3 + \bar{a}_3] = a + \bar{b}$$

where a and b are given by the formulas (35). Prove that $\mathfrak{T}$ with this composition is a Lie triple system.

6. **Subalgebras and ideals of $\mathfrak{K} = \mathfrak{K}(\mathfrak{J})$.** Let $\mathfrak{J}$ be an arbitrary Jordan algebra with 1. We now denote the Lie algebra of linear transformations $R(\mathfrak{J}) + \operatorname{Inder}\mathfrak{J}$ in $\mathfrak{J}$ by $\mathfrak{L}(\mathfrak{J})$ and the extension $\mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$ constructed in the last section by $\mathfrak{K}(\mathfrak{J})$. We have the subalgebra $\mathfrak{A} = \Phi e + \Phi f + \Phi h$ where $e = 1, f = 2(\bar{1}), h = 2R_1$ and the automorphism of period two $\varepsilon\colon a + \bar{b} + L \to \overline{a + \bar{b} + L} = b + \bar{a} + \bar{L}$, where $\bar{R}_a = -R_a$. We shall consider first the subalgebras of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ containing $\mathfrak{A}$, where by a subalgebra of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ we mean a subalgebra of $\mathfrak{K}(\mathfrak{J})$ which is invariant under ε.

First, let $\mathfrak{B}$ be a subalgebra of $\mathfrak{J}$ containing 1 and let $l_1(\mathfrak{B})$ be the subalgebra of $\mathfrak{L}(\mathfrak{J})$ generated by the subspace $R_{\mathfrak{J}}(\mathfrak{B}) \equiv \{R_b \mid b \in \mathfrak{J}\}$, ($R_b$ acting in $\mathfrak{J}$). If $b_1, b_2, b_3 \in \mathfrak{B}$ then $[R_{b_1} R_{b_2} R_{b_3}] = R_{[b_1, b_2, b_3]}$ and since $[b_1, b_2, b_3] \in \mathfrak{B}$, it is clear that $R_{\mathfrak{J}}(\mathfrak{B})$ is a Lie triple system of linear transformations in $\mathfrak{J}$. Hence $l_1(\mathfrak{B}) = R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$. If we restrict R_b to $\mathfrak{B}$ we obtain the multiplication by b in $\mathfrak{B}$ and it is clear that we have the restriction homomorphism of $l_1(\mathfrak{B})$ onto $\mathfrak{L}(\mathfrak{B})$ sending R_b into its restriction $R_b \mid \mathfrak{B}$. Since $1 \in \mathfrak{B}$ it is clear that if $b \neq 0$ then R_b is not contained in the kernel of the restriction homomorphism. It follows that the kernel is the set of elements $\sum [R_{b_i} R_{c_i}]$, $b_i, c_i \in \mathfrak{B}$, such that $x(\sum [R_{b_i} R_{c_i}]) = 0$ for all $x \in \mathfrak{B}$. If $b, c \in \mathfrak{B}$ then $b \vartriangle c = R_{b.c} - [R_b R_c] \in l_1(\mathfrak{B})$. Since $1 \in \mathfrak{B}$ the argument used for $\mathfrak{J}$ shows that every element of $l_1(\mathfrak{B})$ is a sum of elements $b \vartriangle c$, $b, c \in \mathfrak{B}$.

Let $L_1(\mathfrak{B})$ be the subalgebra of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ generated by the subalgebra $\mathfrak{B}$ with 1 of $\mathfrak{J}$ considered as a subspace of $\mathfrak{K}(\mathfrak{J}) = \mathfrak{J} + \bar{\mathfrak{J}} + \mathfrak{L}(\mathfrak{J})$. Then $L_1(\mathfrak{B}) \supseteq \mathfrak{B} + \bar{\mathfrak{B}} + [\mathfrak{B}\bar{\mathfrak{B}}]$

and since $[\mathfrak{B}\bar{\mathfrak{B}}]$ is the space spanned by the elements $b \triangle c = [b, \bar{c}]$, $b, c \in \mathfrak{B}$, it is clear that $[\mathfrak{B}\bar{\mathfrak{B}}] = l_1(\mathfrak{B})$ and $\mathfrak{B} + \bar{\mathfrak{B}} + [\mathfrak{B}\bar{\mathfrak{B}}] = \mathfrak{B} + \bar{\mathfrak{B}} + l_1(\mathfrak{B})$. Also, if $b, c, d \in \mathfrak{B}$ then $[d, b \triangle c] = d(b \triangle c) \in \mathfrak{B}$ and $[\bar{d}, b \triangle c] = \overline{d(b \triangle c)} = -\overline{d(c \triangle b)} \in \mathfrak{B}$. Hence it is clear that $\mathfrak{B} + \bar{\mathfrak{B}} + l_1(\mathfrak{B})$ is a subalgebra of $\mathfrak{K}(\mathfrak{J})$ and this is invariant under ε. Consequently, $L_1(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + l_1(\mathfrak{B})$. Since $1 \in \mathfrak{B}$, $\mathfrak{A} = \Phi e + \Phi f + \Phi h \subseteq L_1(\mathfrak{B})$.

We shall define next a larger subalgebra than $L_1(\mathfrak{B})$ determined by $\mathfrak{B}$. For this we let

$$d(\mathfrak{B}) = \{D \in \operatorname{Inder} \mathfrak{J} \mid \mathfrak{B}D \subseteq \mathfrak{B}\}. \tag{40}$$

Then $d(\mathfrak{B})$ is a subalgebra of Inder $\mathfrak{J}$ containing $[R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$. Since $[R_b D] = R_{bD}$ for a derivation D it is clear that $[R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$ is an ideal in $d(\mathfrak{B})$ and that $l_2(\mathfrak{B}) \equiv R_{\mathfrak{J}}(\mathfrak{B}) + d(\mathfrak{B})$ is a subalgebra of $\mathfrak{L}(\mathfrak{J})$ containing $l_1(\mathfrak{B}) = R_{\mathfrak{J}}(\mathfrak{B}) + [R(\mathfrak{B}), R(\mathfrak{B})]$ as ideal. Also since $bL \in \mathfrak{B}$ if $b \in \mathfrak{B}$ and $L \in l_2(\mathfrak{B})$ and $l_2(\mathfrak{B})^{\varepsilon} = l_2(\mathfrak{B})$ it is clear that $L_2(\mathfrak{B}) \equiv \mathfrak{B} + \bar{\mathfrak{B}} + l_2(\mathfrak{B})$ is a subalgebra of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ containing $L_1(\mathfrak{B})$ as ideal. Finally, we note that any subalgebra of $L_2(\mathfrak{B})$ containing $L_1(\mathfrak{B})$ has the form $\mathfrak{B} + \bar{\mathfrak{B}} + R_{\mathfrak{J}}(\mathfrak{B}) + \mathfrak{M}$ where $\mathfrak{M}$ is a subalgebra of $d(\mathfrak{B})$ containing $[R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$ and hence is invariant under ε.

Conversely, let $\mathfrak{M}$ be any subalgebra of $(\mathfrak{K}, \varepsilon)$ containing $\mathfrak{A}$. Let $a + \bar{b} + L \in \mathfrak{M}$. Then

$$\begin{aligned} \tfrac{1}{2}[a + \bar{b} + L, h] &= [a + \bar{b} + L, R_1] \\ &= a - \bar{b} \in \mathfrak{M}. \end{aligned} \tag{41}$$

Then $\frac{1}{2}[a - \bar{b}, h] = a + \bar{b} \in \mathfrak{M}$ and consequently a, $\bar{b}$ and $L \in \mathfrak{M}$. Thus

$$\mathfrak{M} = (\mathfrak{M} \cap \mathfrak{J}) + (\mathfrak{M} \cap \bar{\mathfrak{J}}) + (\mathfrak{M} \cap \mathfrak{L}(\mathfrak{J})). \tag{42}$$

Put $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$. Then $\bar{\mathfrak{M}} = \mathfrak{M}$ implies that $\mathfrak{M} \cap \bar{\mathfrak{J}} = \bar{\mathfrak{B}}$ and $\overline{\mathfrak{M} \cap \mathfrak{L}(\mathfrak{J})} = \mathfrak{M} \cap \mathfrak{L}(\mathfrak{J})$ is a subalgebra of $(\mathfrak{L}(\mathfrak{J}), \varepsilon)$. We have $\mathfrak{M} = \mathfrak{B} + \bar{\mathfrak{B}} + (\mathfrak{M} \cap \mathfrak{L}(\mathfrak{J}))$. Let $b, c \in \mathfrak{B}$. Then $\bar{c} \in \bar{\mathfrak{B}}$ and $[b, \bar{c}] = b \triangle c = R_{b.c} - [R_b R_c] \in \mathfrak{M} \cap \mathfrak{L}(\mathfrak{J})$. Then $[1, b \triangle c] = 1(b \triangle c) = b . c \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a subalgebra of $\mathfrak{J}$ containing 1. Since $[\mathfrak{B}, \bar{\mathfrak{B}}] = l_1(\mathfrak{B})$ we see that $\mathfrak{M} \supseteq L_1(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + l_1(\mathfrak{B})$. Since $\mathfrak{M} = \bar{\mathfrak{M}}$ and $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$ it is immediate that $\mathfrak{M} = \mathfrak{B} + \bar{\mathfrak{B}} + (R_{\mathfrak{J}}(\mathfrak{B}) + \mathfrak{N})$ where $\mathfrak{N}$ is a subalgebra of Inder $\mathfrak{J}$ containing $[R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$. If $D \in \mathfrak{N}$ and $b \in \mathfrak{B}$ then $bD = [b, D] \in \mathfrak{B}$. Hence $\mathfrak{N} \subseteq d(\mathfrak{B})$ defined by (40). Thus $\mathfrak{M} \subseteq L_2(\mathfrak{B})$ and we have proved the following

THEOREM 13. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{J}$ containing* 1. *Then $L_1(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + l_1(\mathfrak{B})$ where $l_1(\mathfrak{B}) = R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R_{\mathfrak{J}}(\mathfrak{B})]$ and $L_2(\mathfrak{B}) = \mathfrak{B} + \mathfrak{B} + l_2(\mathfrak{B})$ where $l_2(\mathfrak{B}) = R_{\mathfrak{J}}(\mathfrak{B}) + d(\mathfrak{B})$ (as in (40)) are subalgebras of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ containing $\mathfrak{A} = \Phi e + \Phi f + \Phi h$. Also $L_1(\mathfrak{B})$ is an ideal in $L_2(\mathfrak{B})$ and any subalgebra of $L_2(\mathfrak{B})$ containing $L_1(\mathfrak{B})$ is a subalgebra of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$. Conversely, if $\mathfrak{M}$ is any* *subalgebra of* $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ *containing* $\mathfrak{A}$ *then* $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$ *is a subalgebra of* $\mathfrak{J}$ *containing* 1 *and* $L_1(\mathfrak{B}) \subseteq \mathfrak{M} \subseteq L_2(\mathfrak{B})$.

Next let $\mathfrak{B}$ be an ideal in $\mathfrak{J}$. Then the ideal in $\mathfrak{L}(\mathfrak{J})$ generated by $R_{\mathfrak{J}}(\mathfrak{B})$ is $i_1(\mathfrak{B}) \equiv R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})]$. Let $I_1(\mathfrak{B})$ be the ideal in $\mathfrak{K}(\mathfrak{J})$ generated by the subspace $\mathfrak{B}$ of $\mathfrak{J}$. Then $I_1(\mathfrak{B})$ contains every $R_b, b \in \mathfrak{B}$, since $R_b = b \bigtriangleup 1 = [b, \bar{1}]$. Hence $I_1(\mathfrak{B})$ contains $i_1(\mathfrak{B})$. Also $I_1(\mathfrak{B})$ contains $\bar{\mathfrak{B}}$ since it contains $\bar{b} = \overline{1R_b} = [R_b, \bar{1}]$, $b \in \mathfrak{B}$. Hence $I_1(\mathfrak{B}) \supseteq \mathfrak{B} + \bar{\mathfrak{B}} + i_1(\mathfrak{B})$. Since it is clear that the right-hand side is an ideal, we have

$$I_1(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + i_1(\mathfrak{B}), \qquad i_1(\mathfrak{B}) = R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})]. \tag{43}$$

Next consider the Lie algebras $\mathfrak{L}(\mathfrak{J}/\mathfrak{B})$ and $\mathfrak{K}(\mathfrak{J}/\mathfrak{B})$ determined by the Jordan algebra $\mathfrak{J}/\mathfrak{B}$. It is clear that we have the homomorphism of $\mathfrak{L}(\mathfrak{J})$ onto $\mathfrak{L}(\mathfrak{J}/\mathfrak{B})$ such that $R_a \to R_{a+\mathfrak{B}}$. An element D in Inder $\mathfrak{J}$ is in the kernel of this homomorphism, if and only if $\mathfrak{J}D \subseteq \mathfrak{B}$. Hence if we put

$$p(\mathfrak{B}) = \{D \in \text{Inder } \mathfrak{J} \mid \mathfrak{J}D \subseteq \mathfrak{B}\} \tag{44}$$

then $p(\mathfrak{B})$ is an ideal in Inder $\mathfrak{J}$. Since $\mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \text{Inder } \mathfrak{J}$ and $\mathfrak{L}(\mathfrak{J}/\mathfrak{B}) = R(\mathfrak{J}/\mathfrak{B}) \oplus \text{Inder } \mathfrak{J}/\mathfrak{B}$ it is clear that the kernel of the homomorphism of $\mathfrak{L}(\mathfrak{J})$ onto $\mathfrak{L}(\mathfrak{J}/\mathfrak{B})$ is $R_{\mathfrak{J}}(\mathfrak{B}) + p(\mathfrak{B})$. The homomorphism of $\mathfrak{L}(\mathfrak{J})$ has a unique extension to a homomorphism of $\mathfrak{K}(\mathfrak{J})$ onto $\mathfrak{K}(\mathfrak{J}/\mathfrak{B})$ such that $a \to a + \mathfrak{B}$, $\bar{a} \to \overline{a + \mathfrak{B}}$, $a \in \mathfrak{J}$. Since $\mathfrak{K}(\mathfrak{J}) = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J}/\mathfrak{B}) = \mathfrak{J}/\mathfrak{B} \oplus \overline{\mathfrak{J}/\mathfrak{B}} \oplus \mathfrak{L}(\mathfrak{J}/\mathfrak{B})$ it is clear that the kernel of the homomorphism of $\mathfrak{K}(\mathfrak{J})$ is

$$I_2(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + i_2(\mathfrak{B}), \qquad i_2(\mathfrak{B}) = R_{\mathfrak{J}}(\mathfrak{B}) + p(\mathfrak{B}). \tag{45}$$

Since $\mathfrak{B}$ is an ideal it is clear that $\mathfrak{J}[R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})] \subseteq \mathfrak{B}$. Hence $p(\mathfrak{B}) \supseteq [R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})]$, $i_2(\mathfrak{B}) \supseteq i_1(\mathfrak{B})$ and $I_2(\mathfrak{B}) \supseteq I_1(\mathfrak{B})$. Also we have $[p(\mathfrak{B}), R(\mathfrak{J})] \subseteq R_{\mathfrak{J}}(\mathfrak{B})$ which implies that $[i_2(\mathfrak{B}), \mathfrak{L}(\mathfrak{J})] \subseteq i_1(\mathfrak{B})$ and $[I_2(\mathfrak{B}), \mathfrak{K}(\mathfrak{J})] \subseteq I_1(\mathfrak{B})$.

Now let $\mathfrak{M}$ be any ideal in $\mathfrak{K}(\mathfrak{J})$. Then (41) holds. Hence we have the decomposition (42). Let $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$ and $b \in \mathfrak{B}$. Then $R_b = [b, \bar{1}] \in \mathfrak{M}$ and $\bar{b} = [R_b, \bar{1}] \in \mathfrak{M} \cap \bar{\mathfrak{J}}$. Similarly, if $\bar{b} \in \mathfrak{M} \cap \bar{\mathfrak{J}}$ then $b \in \mathfrak{B}$. Hence $\mathfrak{M} = \mathfrak{B} + \bar{\mathfrak{B}} + (\mathfrak{M} \cap \mathfrak{L}(\mathfrak{J}))$ and $\mathfrak{M} \cap \mathfrak{L}(\mathfrak{J})$ is an ideal of $\mathfrak{L}(\mathfrak{J})$ containing $R_{\mathfrak{J}}(\mathfrak{B})$ and hence containing $R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})]$. If $b \in \mathfrak{B}$ and $a \in \mathfrak{J}$ then $a \,.\, b = [b, R_a] \in \mathfrak{B}$. Hence $\mathfrak{B}$ is an ideal in $\mathfrak{J}$ and $\mathfrak{M} \supseteq I_1(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + i_1(\mathfrak{B})$. Let $L = R_c + D \in \mathfrak{M} \cap \mathfrak{L}(\mathfrak{J})$ where $D \in \text{Inder } \mathfrak{J}$. Then $[1, L] = 1L = c \in \mathfrak{B}$ so $R_c \in R_{\mathfrak{J}}(\mathfrak{B}) \subseteq \mathfrak{M}$. Then $D \in \mathfrak{M}$ and $aD = [a, D] \in \mathfrak{B}$ for every $a \in \mathfrak{J}$, Thus $D \in p(\mathfrak{B})$ and $\mathfrak{M} \subseteq I_2(\mathfrak{B})$. We note also that since $\mathfrak{M} = \mathfrak{B} + \bar{\mathfrak{B}} + R_{\mathfrak{J}}(\mathfrak{B}) + (\mathfrak{M} \cap \text{Inder } \mathfrak{J})$ it is clear that $\bar{\mathfrak{M}} = \mathfrak{M}$, that is, $\mathfrak{M}$ is an ideal of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$.

We summarize our results in the following

THEOREM 14. *Let* $\mathfrak{B}$ *be an ideal in the Jordan algebra* $\mathfrak{J}$ *with* 1. *Then* $I_1(\mathfrak{B})$ *given in* (43) *is the ideal in* $\mathfrak{K}(\mathfrak{J})$ *generated by the subspace* $\mathfrak{B}$ *of* $\mathfrak{K}(\mathfrak{J})$ $(\subseteq \mathfrak{J})$

and $I_2(\mathfrak{B})$ *defined by* (45) *and* (44) *is the kernel of the homomorphism of* $\mathfrak{K}(\mathfrak{J})$ *onto* $\mathfrak{K}(\mathfrak{J}/\mathfrak{B})$ *such that* $a \to a + \mathfrak{B}$, $\bar{a} \to \overline{a + \mathfrak{B}}$, $R_a \to R_{a+\mathfrak{B}}$, $a \in \mathfrak{J}$. *We have*

$$(46) \qquad I_2(\mathfrak{B}) \supseteq I_1(\mathfrak{B}_1), \qquad [I_2(\mathfrak{B}), \mathfrak{K}(\mathfrak{J})] \subseteq I_1(\mathfrak{B}).$$

If $\mathfrak{M}$ *is any ideal in* $\mathfrak{K}(\mathfrak{J})$ *then* $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$ *is an ideal in* $\mathfrak{J}$ *and* $I_2(\mathfrak{B}) \supseteq \mathfrak{M} \supseteq I_1(\mathfrak{B})$. *Moreover,* $\mathfrak{M}$ *is an ideal of* $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$.

The definitions give $I_2(0) = 0$ and $I_1(\mathfrak{J}) = \mathfrak{K}(\mathfrak{J})$. Hence we have the following

COROLLARY. $\mathfrak{J}$ *is simple if and only if* $\mathfrak{K}(\mathfrak{J})$ *is simple.*

Theorems 13 and 14 give a mapping $\mathfrak{M} \to \mathfrak{J} \cap \mathfrak{M}$ of the set of subalgebras of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ containing $\mathfrak{A}$ and the set of ideals of $\mathfrak{K}(\mathfrak{J})$ onto the set of subalgebras of $\mathfrak{J}$ containing 1 and the set of ideals of $\mathfrak{J}$ respectively. If $\mathfrak{B}$ is an ideal in $\mathfrak{J}$ we have the sequence of ideals $\mathfrak{B} = \mathfrak{B}^{(0)} \supseteq \mathfrak{B}^{(1)} \supseteq \cdots$ where

$$\mathfrak{B}^{(k)} = \mathfrak{B}^{(k-1)} . \mathfrak{B}^{(k-1)} + \mathfrak{B}^{(k-1)} . \mathfrak{B}^{(k-1)} . \mathfrak{J}$$

(cf. §5.2). We shall now call this the *Penico sequence* for $\mathfrak{B}$ and we shall say that $\mathfrak{B}$ is *Penico solvable* if there exists an integer $s \geqq 1$ such that $\mathfrak{B}^{(s)} = 0$. We now have the following result (Koecher [9]).

THEOREM 15. *Let* $\mathfrak{M}$ *be an ideal in* $\mathfrak{K}(\mathfrak{J})$, $\mathfrak{B}$ *the corresponding ideal* $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$ *in* $\mathfrak{J}$. *Then* $\mathfrak{M}$ *is solvable if and only if* $\mathfrak{B}$ *is Penico solvable.*

PROOF. If $\mathfrak{M}$ is an ideal in a Lie algebra we define the derived series $\mathfrak{M} = \mathfrak{M}^{(0)} \supseteq \mathfrak{M}^{(1)} \supseteq \cdots$ inductively by $\mathfrak{M}^{(k)} = [\mathfrak{M}^{(k-1)}, \mathfrak{M}^{(k-1)}]$. These are ideals of the algebra and solvability means that there exists an s such that $\mathfrak{M}^{(s)} = 0$. Now let $\mathfrak{M}$ be an ideal in $\mathfrak{K}(\mathfrak{J})$, $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{J}$. Then $I_1(\mathfrak{B}) \subseteq \mathfrak{M} \subseteq I_2(\mathfrak{B})$, so, by (46), $[\mathfrak{M}, \mathfrak{K}(\mathfrak{J})] \subseteq I_1(\mathfrak{B})$. Hence $\mathfrak{M}^{(1)} = [\mathfrak{M}, \mathfrak{M}] \subseteq I_1(\mathfrak{B})$ and so, by induction, $\mathfrak{M}^{(k+1)} \subseteq I_1(\mathfrak{B})^{(k)}$. This shows that $\mathfrak{M}$ is solvable if and only if $I_1(\mathfrak{B})$ is solvable. For, if $\mathfrak{M}$ is solvable then $I_1(\mathfrak{B})$ is solvable since any subalgebra of a solvable Lie algebra is solvable. Conversely, if $I_1(\mathfrak{B})$ is solvable then $I_1(\mathfrak{B})^{(s)} = 0$ for a suitable s; hence $\mathfrak{M}^{(s+1)} = 0$. Next we note that $(I_1(\mathfrak{B}))^{(1)} \cap \mathfrak{J} = \mathfrak{B}^{(1)}$ since $I_1(\mathfrak{B}) = \mathfrak{B} + \bar{\mathfrak{B}} + R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})]$ which gives $(I_1(\mathfrak{B}))^{(1)} \cap \mathfrak{J} = \mathfrak{B}(R_{\mathfrak{J}}(\mathfrak{B}) + [R_{\mathfrak{J}}(\mathfrak{B}), R(\mathfrak{J})]) = \mathfrak{B} . \mathfrak{B} + \mathfrak{B} . \mathfrak{B} . \mathfrak{J} = \mathfrak{B}^{(1)}$. Now $(I_1(\mathfrak{B}))^{(1)} \cap \mathfrak{J} = \mathfrak{B}^{(1)}$ implies that $I_1(\mathfrak{B}^{(1)}) \subseteq (I_1(\mathfrak{B}))^{(1)} \subseteq I_2(\mathfrak{B}^{(1)})$. By induction, we have $I_1(\mathfrak{B}^{(k)}) \subseteq (I_1(\mathfrak{B}))^{(k)}$. Also $(I_1(\mathfrak{B}))^{(1)} \subseteq I_2(\mathfrak{B}^{(1)})$ implies that $(I_1(\mathfrak{B}))^{(2)} = [(I_1(\mathfrak{B}))^{(1)}, (I_1(\mathfrak{B}))^{(1)}] \subseteq [I_2(\mathfrak{B}^{(1)}), I_2(\mathfrak{B}^{(1)})] \subseteq [I_2(\mathfrak{B}^{(1)}), \mathfrak{K}(\mathfrak{J})] \subseteq I_1(\mathfrak{B}^{(1)})$. One can now establish by induction that

$$(I_1(\mathfrak{B}))^{(2k-1)} \subseteq I_2(\mathfrak{B}^{(k)}), \qquad (I_1(\mathfrak{B}))^{(2k)} \subseteq I_1(\mathfrak{B}^{(k)}).$$

Now suppose $\mathfrak{B}$ is Penico solvable. Then there exists an s such that $\mathfrak{B}^{(s)} = 0$. Then $I_1(\mathfrak{B})^{(2s-1)} = 0$ and $I_1(\mathfrak{B})$ is a solvable Lie algebra. Then $\mathfrak{M}$ is solvable.

Conversely, assume $\mathfrak{M}$ is solvable. Then $I_1(\mathfrak{B})$ is solvable so $I_1(\mathfrak{B}^{(k)}) \subseteq I_1(\mathfrak{B})^{(k)}$ implies that $\mathfrak{B}$ is Penico solvable.

It is clear that if an ideal $\mathfrak{B}$ in a Jordan algebra $\mathfrak{J}$ is Penico solvable then it is solvable. The converse holds if $\mathfrak{J}$ is finite dimensional (p. 192). We recall that the radical of a finite-dimensional Lie algebra $\mathfrak{L}$, rad $\mathfrak{L}$, is defined to be the maximal solvable ideal. $\mathfrak{L}$ is semisimple if and only if rad $\mathfrak{L} = 0$. We can now state the following corollary to Theorem 15.

COROLLARY. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with* 1. *Then* rad $\mathfrak{K}(\mathfrak{J}) = I_2(\text{rad } \mathfrak{J})$ *and $\mathfrak{J}$ is semisimple if and only if $\mathfrak{K}(\mathfrak{J})$ is semisimple.*

PROOF. Since $I_2(\text{rad } \mathfrak{J}) \cap \mathfrak{J} = \text{rad } \mathfrak{J}$ is solvable, $I_2(\text{rad } \mathfrak{J})$ is a solvable ideal in $\mathfrak{K}(\mathfrak{J})$ so $I_2(\text{rad } \mathfrak{J}) \subseteq \text{rad } \mathfrak{K}(\mathfrak{J})$. On the other hand, $\mathfrak{B} = \text{rad } \mathfrak{K}(\mathfrak{J}) \cap \mathfrak{J}$ is a solvable ideal in $\mathfrak{J}$ and rad $\mathfrak{K}(\mathfrak{J}) \subseteq I_2(\mathfrak{B}) \subseteq I_2(\text{rad } \mathfrak{J})$. Hence rad $\mathfrak{K}(\mathfrak{J}) = I_2(\text{rad } \mathfrak{J})$. Clearly this implies that $\mathfrak{J}$ is semisimple if and only if $\mathfrak{K}(\mathfrak{J})$ is semisimple.

EXERCISES

1. Let $\mathfrak{M}_i$, $i = 1, 2$, be ideals in $\mathfrak{K}(\mathfrak{J})$, $\mathfrak{B}_i = \mathfrak{M}_i \cap \mathfrak{J}$. Show that $\mathfrak{B}_1 i_2(\mathfrak{B}_2) + \mathfrak{B}_2 i_2(\mathfrak{B}_1)$, where $i_2(\mathfrak{B}) = \{L \in \mathfrak{L}(\mathfrak{J}) \mid \mathfrak{J}L \subseteq \mathfrak{B}\}$, is an ideal in $\mathfrak{J}$ and

$$\mathfrak{B}_1 . \mathfrak{B}_2 + \mathfrak{B}_1 . \mathfrak{B}_2 . \mathfrak{J} \subseteq [\mathfrak{M}_1, \mathfrak{M}_2] \cap \mathfrak{J} \subseteq \mathfrak{B}_1 i_2(\mathfrak{B}_2) + \mathfrak{B}_2 i_2(\mathfrak{B}_1).$$

Define $\mathfrak{B}^{\langle 0 \rangle} = \mathfrak{B}$, $\mathfrak{B}^{\langle k \rangle} = \mathfrak{B}^{\langle k-1 \rangle} i_2(\mathfrak{B}^{\langle k-1 \rangle})$ and prove that an ideal $\mathfrak{B}$ is Penico solvable if and only if there exists an s such that $\mathfrak{B}^{\langle s \rangle} = 0$.

2. Let $\mathfrak{J}$ be finite-dimensional semisimple, $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ the decomposition of $\mathfrak{J}$ into simple ideals $\mathfrak{J}_i$. Show that $\mathfrak{K}_i \equiv I_1(\mathfrak{J}_i) = I_2(\mathfrak{J}_i)$ and $\mathfrak{K}(\mathfrak{J}) = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \oplus \cdots \oplus \mathfrak{K}_s$ is the decomposition of $\mathfrak{K}(\mathfrak{J})$ into simple ideals.

3. Let η be an automorphism of $\mathfrak{J}$. Show that $L \to \eta^{-1} L \eta$ is an automorphism of $(\mathfrak{L}(\mathfrak{J}), \varepsilon)$ and $\zeta: a + \bar{b} + L \to a\eta + \overline{b\eta} + \eta^{-1} L \eta$ is an automorphism of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ leaving fixed the elements of the subalgebra $\mathfrak{A} = \Phi 1 + \Phi \bar{1} + \Phi R_1$. Show that the converse holds: if ζ is an automorphism of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ leaving the elements of $\mathfrak{A}$ fixed then ζ is obtained from an automorphism η of $\mathfrak{J}$ in the manner indicated.

4. Prove the analogue of exercise 3 for derivations.

7. **Application to a proof of the Albert-Penico Theorem.** Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1 over a field of characteristic 0. We shall now apply the ideal and subalgebra correspondences of the last section and a theorem on Lie algebras to prove the Albert-Penico theorem on $\mathfrak{J}$. We recall that this asserts that there exists a subalgebra $\mathfrak{B}$ containing 1 such that $\mathfrak{J} = \mathfrak{B} \oplus \text{rad } \mathfrak{J}$. This will be proved by proving an extension due to Taft ([4]) of the Levi decomposition theorem for Lie algebras. We have seen that the radical rad $\mathfrak{K}(\mathfrak{J})$ of $\mathfrak{K}(\mathfrak{J})$ is $I_2(\text{rad } \mathfrak{J})$. By Taft's theorem, we shall have the existence of a subalgebra $\mathfrak{S}$ of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ containing $\mathfrak{A} = \Phi 1 + \Phi \bar{1} + \Phi R_1$ such that $\mathfrak{K}(\mathfrak{J}) = \mathfrak{S} \oplus \text{rad } \mathfrak{K}(\mathfrak{J})$.

Since rad $\mathfrak{K}(\mathfrak{J}) = \text{rad } \mathfrak{J} + \overline{\text{rad } \mathfrak{J}} + (\text{rad } \mathfrak{K}(\mathfrak{J}) \cap \mathfrak{L}(\mathfrak{J}))$ and $\mathfrak{S} = \mathfrak{B} \oplus \bar{\mathfrak{B}} \oplus (\mathfrak{S} \cap \mathfrak{L}(\mathfrak{J}))$ where $\mathfrak{B}$ is a subalgebra with 1, and $\mathfrak{K}(\mathfrak{J}) = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$, this will give $\mathfrak{J} = \mathfrak{B} \oplus \text{rad } \mathfrak{J}$.

We proceed to derive Taft's theorem. Let G be a group. Then we shall call an algebra $\mathfrak{E}/\Phi$ a *G-algebra* if $\mathfrak{E}$ is a G-module such that for each $\sigma \in G$ the mapping $x \to x^\sigma$ is either an automorphism or antiautomorphism of $\mathfrak{E}$. We recall that the G-module property is that $x \to x^\sigma$ is a linear transformation in $\mathfrak{E}$, $x^1 = x$ if 1 is the identity of G and $x^{\sigma\tau} = (x^\sigma)^\tau$, $\sigma, \tau \in G$. The notions of G-subalgebra, homomorphism for G-algebras, etc. are clear. As for ordinary algebras, we shall call an exact sequence of G-algebras $0 \to \mathfrak{M} \underset{\alpha}{\to} \mathfrak{E} \underset{\beta}{\to} \mathfrak{B} \to 0$ an *extension* of $\mathfrak{B}$ by $\mathfrak{M}$. We may assume that $\mathfrak{M} \subseteq \mathfrak{E}$ and α is the injection mapping.

We now assume that $\mathfrak{M}^2 = 0$ and that the extension splits both as G-module and as algebra extension (without the G). The first condition is that there exists a G-module homomorphism δ of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta\beta = 1_{\mathfrak{B}}$. Then $\mathfrak{B}^\delta$ is a G-submodule of $\mathfrak{E}$ and $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{B}^\delta$. The second is that there exists an algebra homomorphism δ' of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta'\beta = 1_{\mathfrak{B}}$. Then $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{B}^{\delta'}$ and $\mathfrak{B}^{\delta'}$ is a subalgebra. We shall now fix δ and we consider all the δ' satisfying $\delta'\beta = 1_{\mathfrak{B}}$. As in §2.8, we put $h(a,b) = a^\delta b^\delta - (ab)^\delta$, $a, b \in \mathfrak{B}$. Then $h(a,b) \in \mathfrak{M}$ and $(a,b) \to h(a,b)$ is a bilinear mapping of $\mathfrak{B}$ into $\mathfrak{M}$. Also, if $\mu = \delta' - \delta$ then μ is a linear mapping of $\mathfrak{B}$ into $\mathfrak{M}$. Moreover, since $(ab)^{\delta'} = a^{\delta'}b^{\delta'}$ and $\mathfrak{M}^2 = 0$ we have

$$h(a,b) = (ab)^\mu - a^\delta b^\mu - a^\mu b^\delta. \tag{47}$$

Conversely, if $\mu \in \text{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ satisfies (47) then $\delta' = \delta + \mu$ is an algebra homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta'\beta = 1_{\mathfrak{B}}$. Also $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{B}^{\delta'}$ and $\mathfrak{B}^{\delta'}$ is a subalgebra of $\mathfrak{E}$. Let $\mathfrak{H}$ be the subset of $\mu \in \text{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ satisfying (47). Then if $\mu_1, \mu_2 \in \mathfrak{H}$, $\nu = \mu_1 - \mu_2$ satisfies $(ab)^\nu = a^\nu b^\delta + a^\delta b^\nu$, $a, b \in \mathfrak{B}$. The set $\mathfrak{D}$ of the mappings ν is a subspace of $\text{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ and $\mathfrak{H}$ is a coset $\mu + \mathfrak{D}$ where $\mu \in \mathfrak{H}$.

We can make $\text{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ into a G-module by defining L^σ for $L \in \text{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ and $\sigma \in G$ by $L^\sigma = \sigma_{\mathfrak{B}}{}^{-1} L \sigma_{\mathfrak{M}}$ where $\sigma_{\mathfrak{B}}$($\sigma_{\mathfrak{M}}$, etc.) is the mapping $x \to x^\sigma$ in $\mathfrak{B}$ ($\mathfrak{M}$, etc.). Let $\mu \in \mathfrak{H} = \mu + \mathfrak{D}$. Then $\delta' = \delta + \mu$ is an algebra homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$ and $\sigma_{\mathfrak{B}}{}^{-1}\delta'\sigma_{\mathfrak{E}}$ is also an algebra homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$. Hence $\sigma_{\mathfrak{B}}{}^{-1}\delta'\sigma_{\mathfrak{E}} - \delta \in \mathfrak{H}$. Since δ is a G-homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$, $\delta = \sigma_{\mathfrak{B}}{}^{-1}\delta\sigma_{\mathfrak{E}}$. Hence $\sigma_{\mathfrak{B}}{}^{-1}\delta'\sigma_{\mathfrak{E}} - \delta = \sigma_{\mathfrak{B}}{}^{-1}(\delta' - \delta)\sigma_{\mathfrak{E}} = \sigma_{\mathfrak{B}}{}^{-1}(\delta' - \delta)\sigma_{\mathfrak{M}} \in \mathfrak{H}$. Thus $\mu^\sigma \in \mathfrak{H}$. Hence $\mathfrak{H}^\sigma \subseteq \mathfrak{H}$ and consequently $\mathfrak{D}^\sigma \subseteq \mathfrak{D}$. The argument shows also that $\delta' = \delta + \mu$ is a G-algebra homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$ if and only if $\mu^\sigma = \mu$ for all $\sigma \in G$.

Let $\mathfrak{T}_\mu$ be the subspace of $\mathfrak{D}$ spanned by the elements $\mu^\sigma - \mu^\tau$, $\sigma, \tau \in G$. Then $\mathfrak{T}_\mu{}^\sigma \subseteq \mathfrak{T}_\mu$ and $(\mathfrak{T}_\mu + \mu)^\sigma = \mathfrak{T}_\mu + \mu$, $\sigma \in G$. Then we have the following criterion.

LEMMA. *Suppose there exists a G-submodule* $\mathfrak{C}$ *of* $\text{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ *such that* $\text{Hom}_\Phi(\mathfrak{B},\mathfrak{M}) = \mathfrak{T}_\mu \oplus \mathfrak{C}$. *Then there exists a G-algebra homomorphism* δ'' *of* $\mathfrak{B}$ *into* $\mathfrak{E}$ *such that the restriction* $\delta''|\mathfrak{A} = \delta'|\mathfrak{A}$ $(\delta' = \delta + \mu)$ *for every G-subalgebra* $\mathfrak{A}$ *of* $\mathfrak{B}$ *such that* $\delta'|\mathfrak{A}$ *is a G-algebra homomorphism.*

PROOF. Since $\mathrm{Hom}_\Phi(\mathfrak{B},\mathfrak{M}) = \mathfrak{T}_\mu \oplus \mathfrak{C}$ it is clear that $(\mathfrak{T}_\mu + \mu) \cap \mathfrak{C} = \{\mu'\}$. Since $(\mathfrak{T}_\mu + \mu)^\sigma \subseteq \mathfrak{T}_\mu + \mu$ and $\mathfrak{C}^\sigma \subseteq \mathfrak{C}$ we have $\mu'^\sigma = \mu'$, $\sigma \in G$. Since $\mu' \in \mathfrak{T}_\mu + \mu \subseteq \mathfrak{D} + \mu$ it is clear that $\mu' \in \mathfrak{H}$ and $\delta'' \equiv \delta + \mu'$ is a G-algebra homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$. Now let $\mathfrak{A}$ be a G-subalgebra of $\mathfrak{B}$ such that $\delta' | \mathfrak{A}$ is a G-algebra homomorphism. We have $\delta'' - \delta' = \mu' - \mu = \sum \alpha_{ij}(\mu^{\sigma_i} - \mu^{\sigma_j})$, $\sigma_i, \sigma_j \in G$, $\alpha_{ij} \in \Phi$. Since $\delta' | \mathfrak{A}$ is a G-homomorphism, $\mu | \mathfrak{A}$ is a G-homomorphism. It follows that $\mu^\sigma | \mathfrak{A} = \mu | \mathfrak{A}$, $\sigma \in G$. Then $\mu^{\sigma_i} | \mathfrak{A} = \mu^{\sigma_j} | \mathfrak{A}$ and consequently $\delta'' | \mathfrak{A} = \delta' | \mathfrak{A}$.

The hypothesis of the lemma is satisfied if $\mathrm{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ is completely reducible as G-module. Now we recall that if G is finite and the order $(G: 1)$ of G is not divisible by the characteristic then every G-module is completely reducible. We shall use this to prove

THEOREM 16. *Let G be a finite group whose order is not divisible by the characteristic and let $0 \to \mathfrak{M} \underset{\alpha}{\to} \mathfrak{E} \underset{\beta}{\to} \mathfrak{B} \to 0$ be an exact sequence of G-algebras such that $\mathfrak{M} \subseteq \mathfrak{E}$, α is the injection mapping and $\mathfrak{M}^2 = 0$. Let $\mathfrak{A}$ be a G-subalgebra of $\mathfrak{B}$. Assume there exists an algebra homomorphism δ' of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta'\beta = 1_\mathfrak{B}$ and the restriction $\delta' | \mathfrak{A}$ is a G-homomorphism. Then there exists a G-algebra homomorphism δ'' of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta''\beta = 1_\mathfrak{B}$ and $\delta'' | \mathfrak{A} = \delta' | \mathfrak{A}$.*

PROOF. We have $\mathfrak{E} = \mathfrak{M} \oplus \mathfrak{B}_1$ where $\mathfrak{B}_1$ is a G-submodule of $\mathfrak{E}$. It is clear that the restriction $\beta | \mathfrak{B}_1$ is a G-isomorphism of $\mathfrak{B}_1$ onto $\mathfrak{B}$. Let δ be its inverse. Then δ is a G-homomorphism of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta\beta = 1_\mathfrak{B}$. We apply the foregoing considerations to this δ and to $\mu = \delta' - \delta$. Since $\mathrm{Hom}_\Phi(\mathfrak{B},\mathfrak{M})$ is completely reducible as G-module, the lemma gives a G-algebra homomorphism δ'' of $\mathfrak{B}$ into $\mathfrak{E}$ such that $\delta'' | \mathfrak{A} = \delta' | \mathfrak{A}$.

For our immediate purposes it is useful to have the following alternative formulation of Theorem 16.

COROLLARY. *Let G be a finite group whose order is not divisible by the characteristic, $\mathfrak{E}$ a G-algebra, $\mathfrak{M}$ a G-ideal in $\mathfrak{E}$ such that $\mathfrak{M}^2 = 0$. Assume there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{E}$ such that $\mathfrak{E} = \mathfrak{B} \oplus \mathfrak{M}$ and let $\mathfrak{A}$ be a G-subalgebra of $\mathfrak{B}$. Then there exists a G-subalgebra $\mathfrak{C}$ of $\mathfrak{E}$ containing $\mathfrak{A}$ such that $\mathfrak{E} = \mathfrak{C} \oplus \mathfrak{M}$.*

PROOF. We have the exact sequence of G-algebra $0 \to \mathfrak{M} \underset{\alpha}{\to} \mathfrak{E} \underset{\beta}{\to} \mathfrak{E}/\mathfrak{M} \to 0$ where α is the injection and β is the canonical homomorphism of $\mathfrak{E}$ onto $\mathfrak{E}/\mathfrak{M}$. If $x = b + u$, $b \in \mathfrak{B}$, $u \in \mathfrak{M}$ then $x + \mathfrak{M} \to b$ is an algebra homomorphism δ' of $\mathfrak{E}/\mathfrak{M}$ into $\mathfrak{E}$ such that $\delta'\beta = 1_\mathfrak{B}$. The restriction of δ' to the G-subalgebra $(\mathfrak{A} + \mathfrak{M})/\mathfrak{M}$ is the mapping $a + \mathfrak{M} \to a, a \in \mathfrak{A}$, and this is a G-homomorphism. Hence, by Theorem 16, there exists a G-algebra homomorphism δ'' of $\mathfrak{E}/\mathfrak{M}$ into $\mathfrak{E}$ such that $\delta''\beta = 1_{\mathfrak{E}/\mathfrak{M}}$ and $\delta'' | (\mathfrak{A} + \mathfrak{M})/\mathfrak{M} = \delta' | (\mathfrak{A} + \mathfrak{M})/\mathfrak{M}$. Then $\mathfrak{E} = \mathfrak{M} \oplus (\mathfrak{E}/\mathfrak{M})^{\delta''}$ where $(\mathfrak{E}/\mathfrak{M})$ is a G-subalgebra of $\mathfrak{E}$ containing $((\mathfrak{A} + \mathfrak{M})/\mathfrak{M})^{\delta''} = ((\mathfrak{A} + \mathfrak{M})/\mathfrak{M})^{\delta'} = \mathfrak{A}$.

We shall now apply this to obtain a generalization of the Levi decomposition theorem for Lie algebras. Let G be an arbitrary finite group, $\mathfrak{L}$ a finite-dimensional G-Lie algebra of characteristic 0, $\mathfrak{A}$ a semisimple G-subalgebra of $\mathfrak{L}$. The maximal solvable ideal $\mathfrak{M}$ of $\mathfrak{L}$ is clearly a G-ideal in $\mathfrak{L}$. We now have the following

THEOREM 17 (TAFT). *Let G be a finite group, $\mathfrak{L}$ a finite-dimensional G-Lie algebra of characteristic* 0, *$\mathfrak{M}$ the maximal solvable ideal of $\mathfrak{L}$, $\mathfrak{A}$ a semisimple G-subalgebra of $\mathfrak{L}$. Then there exists a G-subalgebra $\mathfrak{C}$ of $\mathfrak{L}$ containing $\mathfrak{A}$ such that $\mathfrak{L} = \mathfrak{C} \oplus \mathfrak{M}$.*

PROOF. If $[\mathfrak{M},\mathfrak{M}] \neq 0$ the inductive argument on dimensionality given in Reduction II of the proof of the Albert-Penico-Taft theorem (p. 288) is applicable. Hence we may assume $[\mathfrak{M},\mathfrak{M}] = 0$. By the Levi-Malcev-Harish-Chandra theorems (Jacobson, *Lie Algebras*, pp. 91–92) there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{L}$ containing $\mathfrak{A}$ such that $\mathfrak{L} = \mathfrak{B} \oplus \mathfrak{M}$. Then the above corollary gives a G-subalgebra $\mathfrak{C}$ of $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{C} \oplus \mathfrak{M}$ and $\mathfrak{C} \supseteq \mathfrak{A}$.

We now apply this to $\mathfrak{K}(\mathfrak{J})$ considered as a G-Lie algebra for the cyclic group of order two where the action of the element $\neq 1$ is the automorphism ε. If $\mathfrak{A} = \Phi 1 + \Phi \bar{1} + \Phi R_1$ then $\mathfrak{A}$ is simple and $\mathfrak{A}^\varepsilon = \mathfrak{A}$. Hence the foregoing theorem gives a subalgebra $\mathfrak{S}$ of $(\mathfrak{K}(\mathfrak{J}), \varepsilon)$ containing $\mathfrak{A}$ such that $\mathfrak{K}(\mathfrak{J}) = \mathfrak{M} \oplus \mathfrak{S}$. As we saw before, this implies the Albert-Penico theorem.

EXERCISES

1. Let G be a finite group of order not divisible by the characteristic, $\mathfrak{J}$ a finite-dimensional G-Jordan algebra. Prove that there exist a G-subalgebra $\mathfrak{B}$ of $\mathfrak{J}$ such that $\mathfrak{J} = \mathfrak{B} \oplus \operatorname{rad} \mathfrak{J}$.

2. Assume that G and $\mathfrak{J}$ are as in 1. and the characteristic is 0. Let $\mathfrak{A}$ be semisimple G-subalgebra of $\mathfrak{J}$. Prove that the $\mathfrak{B}$ in 1. can be chosen so that $\mathfrak{B} \supseteq \mathfrak{A}$.

8. **Associative bilinear forms.** Let $\mathfrak{J}$ be an arbitrary Jordan algebra with $1, f$ an associative symmetric bilinear form on $\mathfrak{J}$. Thus $f(a,b) = f(b,a)$, $f(a\,.\,b,c) = f(a,b\,.\,c)$, $a,b,c \in \mathfrak{J}$. We shall now show how f can be transferred to invariant forms on Inder $\mathfrak{J}$, $\mathfrak{L}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$ (cf. Braun and Koecher [1]). The associativity condition is equivalent to

$$f(aR_b, c) = f(a, cR_b). \tag{48}$$

This implies

$$f(a[R_bR_d], c) = f(a, c[R_dR_b]). \tag{49}$$

Also we have

$$f(c, d\,.\,a\,.\,b) = f(c\,.\,b,\ d\,.\,a) = f(c\,.\,b\,.\,d,\ a),$$

which gives

$$f(c, d[R_aR_b]) = f(c\,.\,b,\ d\,.\,a) - f(c\,.\,a,\ d\,.\,b) = f(a, b[R_cR_d]). \tag{50}$$

Let $D = \sum_i [R_{a_i} R_{b_i}]$, $E = \sum_j [R_{c_j} R_{d_j}]$. Then (50) implies that

$$(51)\qquad \sum_j f(c_j, d_j D) = \sum_i f(a_i, b_i E) = \sum_{i,j} f(c_j . b_i, d_j . a_i) - \sum_{i,j} f(c_j . a_i, d_j . b_i).$$

The first of these formulas shows that the element in Φ which is given is independent of the particular representation of D as $\sum [R_{a_i} R_{b_i}]$ and the second shows that it is independent of the expression of $E = \sum [R_{c_j} R_{a_j}]$. Hence

$$(52)\qquad f_d(D, E) = \sum_j f(c_j\ d_j D) = \sum_i f(a_i, b_i E)$$

defines a single valued function from Inder $\mathfrak{J} \times$ Inder $\mathfrak{J}$ into Φ. It is clear also that this is bilinear. The last expression in (51) (or (52)) shows that f_d is symmetric:

$$(53)\qquad f_d(D, E) = f_d(E, D).$$

Let $F \in$ Inder $\mathfrak{J}$. Then

$$\begin{aligned} f_d([[R_a R_b], F], [R_c R_d]) &+ f_a([R_a R_b], [[R_c R_d], F]) \\ &= f_d([R_{aF} R_b] + [R_a R_{bF}], [R_c R_d]) + f_d([R_a R_b], [R_{cF} R_d] + [R_c R_{dF}]) \\ &= f(c . b, d . aF) - f(c . aF, d . b) + f(c . bF, d . a) - f(c . a, d . bF) \\ &\quad + f(cF . b, d . a) - f(cF . a, d . b) + f(c . b, dF . a) - f(c . a, dF . b) \\ &= f(c . b, (d . a)F) + f((c . b)F, d . a) - f((c . a)F, d . b) \\ &\qquad - f(c . a, (d . b)F). \end{aligned}$$

Since F has the form $F = \sum [R_{g_k} R_{h_k}]$ this is 0 by (49). It follows that

$$(54)\qquad f_d([DF], E) = f_d(D, [F, E])$$

for $D, E, F \in$ Inder $\mathfrak{J}$, that is, f_d is an associative form on the Lie algebra Inder $\mathfrak{J}$.

We consider next the Lie algebra $\mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) \oplus$ Inder $\mathfrak{J}$ and we extend f_d to a symmetric bilinear form on $\mathfrak{L}(\mathfrak{J})$ by defining

$$(55)\qquad f_l(R_a + D, R_b + E) = f(a, b) + f_d(D, E)$$

for $a, b \in \mathfrak{J}$, $D, E \in$ Inder $\mathfrak{J}$. We have

$$\begin{aligned} f_l([R_a + D, R_c + F], R_b + E) &+ f_l(R_a + D, [R_b + E, R_c + F]) \\ &= f_l([R_a R_c] + R_{aF} - R_{cD} + [DF], R_b + E) \\ &\quad + f_l(R_a + D, [R_b R_c] + R_{bF} - R_{cE} + [EF]) \\ &= f_d([R_a R_c] + [DF], E) + f(aF - cD, b) \\ &\quad + f_d(D, [R_b R_c] + [EF]) + f(a, bF - cE) \\ &= f_d([R_a R_c], E) + f(aF - cD, b) + f_d(D, [R_b R_c]) + f(a, bF - cE) \\ &= f(a, cE) + f(aF, b) - f(cD, b) + f(b, cD) + f(a, bF) - f(a, cE) = 0, \end{aligned}$$

by (49). Hence f_l is a symmetric associative bilinear form on $\mathfrak{L}(\mathfrak{J})$. It is clear from the definition (55) that $R(\mathfrak{J})$ and Inder $\mathfrak{J}$ are orthogonal subspaces relative to f_l and f_l coincides with f_d on Inder $\mathfrak{J}$. We have $f_l(a \triangle b, R_c) = f_l(R_{a.b} - [R_a R_b], R_c)$ $= f(a.b, c) = f(a.c, b) = f(aR_c, b)$. Then $f_l(a \triangle b, [R_c R_d]) = -f_l([(a \triangle b), R_d], R_c)$ $= -f_l(a.d \triangle b, R_c) + f_l(a \triangle b.d,\ R_c)$ (by (25)) $= -f(a.d.c, b) + f(a.c, b.d)$ $= f(a[R_c R_d], b)$. Hence we have

$$f_l(a \triangle b, L) = f(aL, b) \tag{56}$$

for $a, b \in \mathfrak{J}$, $L \in \mathfrak{L}(\mathfrak{J})$.

We consider finally the Lie algebra $\mathfrak{K}(\mathfrak{J}) = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$. We define for $a_i \in \mathfrak{J}$, $\bar{b}_i \in \bar{\mathfrak{J}}$, $L_i \in \mathfrak{L}(\mathfrak{J})$,

$$f_k(a_1 + \bar{b}_1 + L_1,\ a_2 + \bar{b}_2 + L_2) = f(a_1, b_2) + f(a_2, b_1) - f_l(L_1, L_2). \tag{57}$$

This is a symmetric bilinear form on $\mathfrak{K}(\mathfrak{J})$ whose restriction to the subalgebra $\mathfrak{L}(\mathfrak{J})$ coincides with $-f_l$. It is clear also that $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ are totally isotropic and $\mathfrak{J} + \bar{\mathfrak{J}}$ is orthogonal to $\mathfrak{L}(\mathfrak{J})$ relative to this form. We have

$$\begin{aligned} &f_\kappa([a_1 + \bar{b}_1 + L_1,\ a_2 + \bar{b}_2 + L_2],\ a_3 + \bar{b}_3 + L_3) \\ &\quad = f(a_1 L_2 - a_2 L_1,\ b_3) \\ &\qquad + f(a_3, b_1 \bar{L}_2 - b_2 \bar{L}_1) - f_l(a_1 \triangle b_2, L_3) + f_l(a_2 \triangle b_1,\ L_3) \\ &\qquad - f_l([L_1 L_2],\ L_3) \\ &\quad = f(a_1 L_2,\ b_3) - f(a_2 L_1,\ b_3) - f(a_3 L_2,\ b_1) \\ &\qquad + f(a_3 L_1,\ b_2) - f(a_1 L_3,\ b_2) + f(a_2 L_3, b_1) - f_l([L_1 L_2],\ L_3), \end{aligned}$$

by (48), (49) and (56). Since f_l is associative $f_l([L_1 L_2], L_3) = f_l([L_2 L_3], L_1)$. It follows that the right-hand side of the foregoing formula is invariant under cyclic permutation of 1,2,3. Hence the left-hand side is invariant under cyclic permutation of 1,2,3 and this is equivalent to the associativity of f_k on $\mathfrak{K}(\mathfrak{J})$.

Now suppose the form f is nondegenerate and let $L \in \mathfrak{L}(\mathfrak{J})^{\perp}$, the radical of the form f_l on $\mathfrak{L}(\mathfrak{J})$. Then (56) implies that $f(aL, b) = 0$ for all a, b. Hence $L = 0$, which shows that f_l is nondegenerate on $\mathfrak{L}(\mathfrak{J})$. Since $\mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) \oplus$ Inder $\mathfrak{J}$ and $R(\mathfrak{J})$ and Inder $\mathfrak{J}$ are orthogonal relative to f_l it is clear that Inder $\mathfrak{J}$ is a non-isotropic subspace of $\mathfrak{L}(\mathfrak{J})$. Since $f_l = f_d$ on Inder $\mathfrak{J}$, f_d is nondegenerate. We note finally that f_k is nondegenerate since the definition shows that $\mathfrak{J} + \bar{\mathfrak{J}}$ is not isotropic. Moreover, $\mathfrak{J} + \bar{\mathfrak{J}}$ is orthogonal to $\mathfrak{L}(\mathfrak{J})$ and this space is not isotropic.

If $\mathfrak{J}$ is simple then we have seen that $\mathfrak{K}(\mathfrak{J})$ is simple. Also if P is an extension of the base field then $\mathfrak{K}(\mathfrak{J})_P \cong \mathfrak{K}(\mathfrak{J}_P)$. It follows that if $\mathfrak{J}$ is central simple then $\mathfrak{K}(\mathfrak{J})$ is central simple. If, in addition, the dimensionality is finite and the characteristic is 0 then the Killing form of $\mathfrak{K}(\mathfrak{J})$ is nondegenerate and every invariant symmetric bilinear form on $\mathfrak{K}(\mathfrak{J})$ is a multiple of the Killing form (cf. Jacobson, *Lie Algebras*, ex. 9, p. 104). It follows that in this case f_k is a nonzero multiple of the Killing form on $\mathfrak{K}(\mathfrak{J})$.

9. **Examples. Functorial constructions.** We assume first that $\mathfrak{J}$ is finite-dimensional special central simple of degree $m \geqq 3$. Then $\mathfrak{J}$ can be identified with an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a finite-dimensional central simple associative algebra with involution which is perfect (§5.7). Thus $\mathfrak{A}$ and the injection mapping of $\mathfrak{H}(\mathfrak{A}, J)$ into $\mathfrak{A}$ is a unital special universal envelope for $\mathfrak{J} = \mathfrak{H}(\mathfrak{A}, J)$ and J is the main involution. We have shown in §6.9 that Der $\mathfrak{J}$ consists of the set of mappings $x \to [xd]$, $d \in \mathfrak{S}(\mathfrak{A}, J)$, the set of J-skew elements of $\mathfrak{A}$. We showed also that the Lie algebra $\mathfrak{M}(\mathfrak{J})$ of linear transformations in $\mathfrak{J}$ having the generic norm as Lie semi-invariant coincides with Der $\mathfrak{J} + R(\mathfrak{J})$ and this is the set of mappings

$$(58)\qquad S_a\colon x \to a^J x + xa, \qquad x \in \mathfrak{J}, \qquad a \in \mathfrak{A}.$$

If we are not in the exceptional case in which the center of $\mathfrak{A}$ is two dimensional and the characteristic is a prime p dividing m then Der $\mathfrak{J}$ = Inder $\mathfrak{J}$ and $\mathfrak{M}(\mathfrak{J}) = \mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) + [R(\mathfrak{J}), R(\mathfrak{J})]$. If $h \in \mathfrak{J}$ then $S_h = R_{2h}$. Hence if $h, k \in \mathfrak{J}$ then

$$(59)\qquad h \triangle k = R_{h.k} - [R_h R_k] = S_{\frac{1}{2}kh}.$$

We now consider the algebra $\mathfrak{A}_2$ of 2×2 matrices with entries in $\mathfrak{A}$ and the involution J_Q in $\mathfrak{A}_2\colon X \to Q^{-1} X^{Jt} Q$ where $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $X^J = (x_{ij}{}^J)$ for $X = (x_{ij})$. A direct calculation shows that the Lie algebra $\mathfrak{S}(\mathfrak{A}_2, J_Q)$ of J_Q-skew elements of $\mathfrak{A}_2$ is the set of matrices of the form

$$(60)\qquad \begin{pmatrix} a & k \\ h & -a^J \end{pmatrix}, \qquad a \in \mathfrak{A}, \qquad h, k \in \mathfrak{H}(\mathfrak{A}, J).$$

Clearly the subset of $\mathfrak{S}(\mathfrak{A}_2, J_Q)$ of diagonal elements diag $\{a, -a^J\}$ is a subalgebra isomorphic to $\mathfrak{A}^-$ and we have the homomorphism diag$\{a, -a^J\} \to S_a$ of this subalgebra onto $\mathfrak{M}(\mathfrak{J})$. Also we have $[he_{12}, ke_{21}] = hke_{11} - khe_{22}$ $=$ diag$\{hk, -(hk)^J\}$, $[he_{12}, ae_{11} - a^J e_{22}] = -(ha^J + ah)e_{12}$ and $[ke_{21}, ae_{11} - a^J e_{22}]$ $= (ka + a^J k)e_{21}$. It is easily seen from these formulas that we have a homomorphism of the Lie algebra $\mathfrak{S}(\mathfrak{A}_2, J_Q)$ onto the Lie algebra $\mathfrak{J} + \bar{\mathfrak{J}} + R(\mathfrak{J}) +$ Der $\mathfrak{J}$ defined in exercise 1 p. 328. We proceed now to define a homomorphism of a subalgebra of $\mathfrak{S}(\mathfrak{A}_2, J_Q)$ into $\mathfrak{K}(\mathfrak{J})$.

Let $\mathfrak{A}^* = \mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$. Since $[[\mathfrak{J}\mathfrak{J}]\mathfrak{J}] \subseteq \mathfrak{J}$ in $\mathfrak{A}^-$, $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ is a subalgebra of $\mathfrak{A}^-$ and this is the Lie subalgebra generated by $\mathfrak{J}$. Since $1 \in \mathfrak{J}$ and $hk = h.k$ $+ \frac{1}{2}[h, k]$ it is clear that $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}] = \mathfrak{J}^2$ the space spanned by the products hk, $h, k \in \mathfrak{J}$. Let $\mathfrak{S}(\mathfrak{A}_2, J_Q)^*$ denote the subset of $\mathfrak{S}(\mathfrak{A}_2, J_Q)$ of matrices of the form (60) with $a \in \mathfrak{A}^*$. One checks directly that $\mathfrak{S}(\mathfrak{A}_2, J_Q)^*$ is a subalgebra. If $a \in \mathfrak{A}^*$

then $a = \sum h_i k_i$, $h_i, k_i \in \mathfrak{J}$. Hence (59) shows that $S_a \in \mathfrak{L}(\mathfrak{J})$. The foregoing formulas show that the mapping

$$\begin{pmatrix} a & -\frac{1}{2}k \\ h & -a^J \end{pmatrix} \to h + \bar{k} + S_a \tag{61}$$

is a homomorphism of $\mathfrak{S}(\mathfrak{A}_2, J_Q)^*$ onto $\mathfrak{K}(\mathfrak{J})$.

We can now determine the structure of the Lie algebras Der $\mathfrak{J}$, Inder $\mathfrak{J}$, $\mathfrak{M}(\mathfrak{J})$, $\mathfrak{L}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$. We have the homomorphism $d \to S_d$ $(x \to [xd])$ of $\mathfrak{S}(\mathfrak{A}, J)$ onto Der $\mathfrak{J}$. Since $\mathfrak{J}$ generates $\mathfrak{A}$ the kernel of this homomorphism is $\mathfrak{S}(\mathfrak{A}, J) \cap \mathfrak{C}$ where $\mathfrak{C}$ is the center of $\mathfrak{A}$. If $\mathfrak{C} = \Phi$ so $\mathfrak{A}$ is simple and J is of first kind, then $\mathfrak{S}(\mathfrak{A}, J) \cap \mathfrak{C} = 0$ and Der $\mathfrak{J} \cong \mathfrak{S}(\mathfrak{A}, J)$. Also in this case Der $\mathfrak{J}$ = Inder $\mathfrak{J}$ and if Ω is the algebraic closure of the base field then $\mathfrak{J}_\Omega \cong \mathfrak{H}(\mathfrak{D}_m, J_1)$ where either $\mathfrak{D} = \Phi$ or $(\mathfrak{D}, j)$ is a split quaternion algebra with involution. Then $(\text{Der}\ \mathfrak{J})_\Omega = (\text{Inder}\ \mathfrak{J})_\Omega \cong \mathfrak{S}(\mathfrak{D}_m, J_1)$ is a simple Lie algebra of type B if $\mathfrak{D} = \Phi$ and m is odd, of type D if $\mathfrak{D} = \Phi$ and m is even and of type C if $\mathfrak{D}$ is quaternion (Jacobson, *Lie Algebras*, pp. 298–302 and Seligman, *Modular Lie Algebras*, Chapter III). If $\mathfrak{A}$ is not simple then $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is a central simple ideal and $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{A}, J) \cong \mathfrak{B}^+$. Then Der $\mathfrak{J} \cong \mathfrak{S}(\mathfrak{A}, J)/(\mathfrak{C} \cap \mathfrak{S}(\mathfrak{A}, J)) \cong \mathfrak{B}^-/\Phi$. Thus in this case Der $\mathfrak{J}$ is a Lie algebra of type A_I. We have Der $\mathfrak{J}$ = Inder $\mathfrak{J}$ unless $p \mid m$. In the latter case it is easily seen that Inder $\mathfrak{J} \cong [\mathfrak{B}^-, \mathfrak{B}^-]/\Phi$. The last case we have to consider is that in which $\mathfrak{A}$ is simple but not central simple, that is, J is an involution of second kind. Then the center $P = \Phi(i)$, $i^2 = \alpha \in \Phi$ and $i^J = -i$ and Der $\mathfrak{J} \cong \mathfrak{S}(\mathfrak{A}, J)/\Phi i$. Unless the characteristic is a prime p dividing m, Der $\mathfrak{J}$ = Inder $\mathfrak{J}$. If $p \mid m$ then Inder $\mathfrak{J} \cong [\mathfrak{S}(\mathfrak{A}, J), \mathfrak{S}(\mathfrak{A}, J)]/\Phi i$. The Lie algebras $\mathfrak{S}(\mathfrak{A}, J)/\Phi i$ are of type A_{II}. It is well known that the Lie algebras of types A,B,C,D which we obtained here for Inder $\mathfrak{J}$ are simple except for the case in which $\mathfrak{J}_\Omega \cong \mathfrak{H}(\Omega_4, J_1)$ (loc. cit. above).

We consider next the Lie algebras $\mathfrak{M}(\mathfrak{J})$ and $\mathfrak{L}(\mathfrak{J})$. We have the homomorphism $a \to S_a$ of $\mathfrak{A}^-$ onto $\mathfrak{M}(\mathfrak{J})$. Since $S_a = 0$ implies $1\ S_a = a^J + a = 0$, the kernel of the homomorphism is $\mathfrak{S}(\mathfrak{A}, J) \cap \mathfrak{C}$. If $\mathfrak{A}$ is central simple (J of first kind) then $\mathfrak{S}(\mathfrak{A}, J) \cap \mathfrak{C} = 0$ and $\mathfrak{M}(\mathfrak{J}) \cong \mathfrak{A}^-$. If $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ where $\mathfrak{B}$ is an ideal then $\mathfrak{S}(\mathfrak{A}, J) \cap \mathfrak{C} = \Phi(1, -1)$ so $\mathfrak{M}(\mathfrak{J}) \cong \mathfrak{A}^-/\Phi(1, -1)$. If $\mathfrak{A}$ is simple and $\mathfrak{C}$ is a quadratic extension field $\Phi(i)$ with $i^J = -i$ then $\mathfrak{M}(\mathfrak{J}) \cong \mathfrak{A}^-/\Phi i$. We have $\mathfrak{L}(\mathfrak{J}) = \mathfrak{M}(\mathfrak{J})$ unless $\mathfrak{C} \supset \Phi$ and the characteristic is a prime p dividing m. It is easy to see that in this case $\mathfrak{L}(\mathfrak{J}) \cong ([\mathfrak{B}^-, \mathfrak{B}^-] \oplus [(\mathfrak{B}^J)^-, (\mathfrak{B}^J)^-]/\Phi(1, -1)$ if $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ and $\mathfrak{L}(\mathfrak{J}) \cong [\mathfrak{A}^-, \mathfrak{A}^-]/\Phi i$ if $\mathfrak{A}$ is simple and J is of second kind.

We now consider the algebras $\mathfrak{K}(\mathfrak{J})$ and we obtain first an improved description of the Lie algebra $\mathfrak{S}(\mathfrak{A}_2, J_Q)^*$. If $\mathfrak{A}$ is central simple (equivalently, J is of first kind) then the proof of Theorem 6.9 shows that $\mathfrak{S}(\mathfrak{A}, J) = [\mathfrak{J}\mathfrak{J}]$. Since $\mathfrak{A} = \mathfrak{J} + \mathfrak{S}(\mathfrak{A}, J)$ we have $\mathfrak{A}^* = \mathfrak{A}$. Hence in this case

$$\mathfrak{S}(\mathfrak{A}_2, J_Q)^* = \mathfrak{S}(\mathfrak{A}_2, J_Q). \tag{62}$$

We shall show next that if $\mathfrak{A}$ is not central simple then

(63) $$\mathfrak{S}(\mathfrak{A}_2, J_Q)^* = [\mathfrak{S}(\mathfrak{A}_2, J_Q), \mathfrak{S}(\mathfrak{A}_2, J_Q)].$$

We note first that if $a_i \in \mathfrak{A}$, $h_i, k_i \in \mathfrak{J}$ then

(64) $$\left[\begin{pmatrix} a_1 & k_1 \\ h_1 & -a_1^J \end{pmatrix}, \begin{pmatrix} a_2 & k_2 \\ h_2 & -a_2^J \end{pmatrix}\right] = \begin{pmatrix} a & k \\ h & -a^J \end{pmatrix}$$

where

(65) $$\begin{aligned} a &= [a_1 a_2] + k_1 h_2 - k_2 h_1, \\ h &= h_1 a_2 + a_2^J h_1 - h_2 a_1 - a_1^J h_2, \\ k &= a_1 k_2 + k_2 a_1^J - a_2 k_1 - k_1 a_2^J. \end{aligned}$$

These formulas show that $[\mathfrak{S}(\mathfrak{A}_2, J_Q), \mathfrak{S}(\mathfrak{A}_2, J_Q)]$ is the set of matrices of the form (60) in which $h, k \in \mathfrak{J}$ and $a \in [\mathfrak{A}, \mathfrak{A}] + \mathfrak{A}^*$, $\mathfrak{A}^* = \mathfrak{J}^2 = \mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$. Hence $\mathfrak{S}(\mathfrak{A}_2, J_Q)^* \subseteq [\mathfrak{S}(\mathfrak{A}_2, J_Q), \mathfrak{S}(\mathfrak{A}_2, J_Q)]$ and equality holds if and only if $[\mathfrak{A}\mathfrak{A}] \subseteq \mathfrak{A}^*$. Since $\mathfrak{A} = \mathfrak{J} + \mathfrak{S}(\mathfrak{A}, J)$ and $[\mathfrak{S}(\mathfrak{A}, J), \mathfrak{J}] \subseteq \mathfrak{J}$, $[\mathfrak{A}\mathfrak{A}] \subseteq \mathfrak{A}^*$ if and only if $[\mathfrak{S}(\mathfrak{A}, J), \mathfrak{S}(\mathfrak{A}, J)] \subseteq [\mathfrak{J}\mathfrak{J}]$. Now this holds if $\mathfrak{A}$ is not central simple. It is enough to prove this for Φ algebraically closed. Then $\mathfrak{A} = \Phi_m \oplus \Phi_m$ and J can be taken to the involution $(b_1, b_2^t) \to (b_2, b_1^t)$, $b_i \in \Phi_m$. We have

$$[(b, b^t), (c, c^t)] = [(b, -b^t), (c, -c^t)].$$

Hence $[\mathfrak{S}(\mathfrak{A}, J), \mathfrak{S}(\mathfrak{A}, J)] = [\mathfrak{J}\mathfrak{J}]$ and so (63) holds.

The kernel of the homomorphism (61) of $\mathfrak{S}(\mathfrak{A}_2, J_Q)^*$ onto $\mathfrak{K}(\mathfrak{J})$ is the set of matrices $\delta 1 = \text{diag}\{\delta, \delta\}$ where $\delta \in \mathfrak{A}^* \cap \mathfrak{S}(\mathfrak{A}, J) \cap \mathfrak{C}$, $\mathfrak{C}$ the center of $\mathfrak{A}$. If $\mathfrak{A}$ is central simple the kernel is 0 and $\mathfrak{K}(\mathfrak{J}) \cong \mathfrak{S}(\mathfrak{A}_2, J_Q)$, by (62). If $\mathfrak{A}$ is not central simple, then $\mathfrak{A}^* \cap \mathfrak{S}(\mathfrak{A}, J) = (\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]) \cap \mathfrak{S}(\mathfrak{A}, J) = [\mathfrak{J}\mathfrak{J}] = [\mathfrak{S}(\mathfrak{A}, J), \mathfrak{S}(\mathfrak{A}, J)]$. Hence the kernel of (61) in this case is $[\mathfrak{S}(\mathfrak{A}, J), \mathfrak{S}(\mathfrak{A}, J)] \cap \mathfrak{C}$. As in the proof of Theorem 6.9, one sees that this is 0 unless $p \mid m$ in which case the intersection is one dimensional. Accordingly, we have $\mathfrak{K}(\mathfrak{J}) \cong [\mathfrak{S}(\mathfrak{A}_2, J_Q), \mathfrak{S}(\mathfrak{A}_2, J_Q)]$ if $p \nmid m$ and $\mathfrak{K}(\mathfrak{J}) \cong [\mathfrak{S}(\mathfrak{A}_2, J_Q), \mathfrak{S}(\mathfrak{A}_2, J_Q)]/\Phi(1, -1)$ if $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^J$ and $p \mid m$ and $\mathfrak{K}(\mathfrak{J}) \cong [\mathfrak{S}(\mathfrak{A}_2, J_Q), \mathfrak{S}(\mathfrak{A}_2, J_Q)]/\Phi i$ if $\mathfrak{A}$ is simple, J of second kind, $\mathfrak{C} = \Phi(i)$, $i^J = -i$ and $p \mid m$. We remark that if $\mathfrak{A}$ is not simple then the isomorphism indicated can be replaced by $\mathfrak{K}(\mathfrak{J}) \cong [\mathfrak{B}^-, \mathfrak{B}^-]$ or $\mathfrak{K}(\mathfrak{J}) \cong [\mathfrak{B}^-, \mathfrak{B}^-]/\Phi 1$ according as $p \nmid m$ or $p \mid m$. We remark also that if $\mathfrak{A}$ is central simple and J is of orthogonal (symplectic) type in the sense that $(\mathfrak{A}_\Omega, J) \cong (\Omega_m, J_1)$ ($\cong (\mathfrak{Q}_m, J_1)$, $\mathfrak{Q}$ quaternion with standard involution) then the involution J_Q in $\mathfrak{A}_2$ is of symplectic (orthogonal) type. This is easily seen, for example, by calculating $\dim \mathfrak{S}(\mathfrak{A}_2, J_Q)$.

We close our discussion of the Lie algebras Inder $\mathfrak{J}$, Der $\mathfrak{J}$ etc. by noting that the mappings $\mathfrak{J} \to$ Inder $\mathfrak{J}$ etc. are not satisfactory from the category point of view since a homomorphism of $\mathfrak{J}_1$ into $\mathfrak{J}_2$ does not always give rise to one between Inder $\mathfrak{J}_1$ into Inder $\mathfrak{J}_2$ etc. This is the case if the homomorphism is surjective but may not be so otherwise. We shall now indicate some modifications of the Lie algebra constructions which are free of this defect. We restrict the considerations to Jordan algebras with 1. Let $U_1(\mathfrak{J})$ be the unital universal multiplication algebra of $\mathfrak{J}$ and consider $\mathfrak{J}$ as imbedded in $U_1(\mathfrak{J})$ in the usual way. Then $\mathfrak{J}$ is subsystem of the Lie triple system $(U_1(\mathfrak{J})^-)^{(2)}$. Hence $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}] = \mathfrak{J} \oplus [\mathfrak{J}\mathfrak{J}]$ is the subalgebra of $U_1(\mathfrak{J})^-$ generated by $\mathfrak{J}$. Also we have the subalgebra $[\mathfrak{J}\mathfrak{J}]$. We have the canonical homomorphism of $[\mathfrak{J}\mathfrak{J}]$ onto Inder $\mathfrak{J}$ such that $\sum [a_i b_i] \to \sum [R_{a_i} R_{b_i}]$ and of $\mathfrak{J} \oplus [\mathfrak{J}\mathfrak{J}]$ onto $\mathfrak{L}(\mathfrak{J})$ such that $a \to R_a$. We can replace the Lie algebras Inder $\mathfrak{J}$ and $\mathfrak{L}(\mathfrak{J})$ by $[\mathfrak{J}\mathfrak{J}]$ and $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ respectively. It is clear that a homomorphism of the Jordan algebra $\mathfrak{J}_1$ into the Jordan algebra $\mathfrak{J}_2$ sending $1 \to 1$ gives rise to a unique homomorphism of $[\mathfrak{J}_1\mathfrak{J}_1]$ into $[\mathfrak{J}_2\mathfrak{J}_2]$ and of $\mathfrak{J}_1 + [\mathfrak{J}_1\mathfrak{J}_1]$ into $\mathfrak{J}_2 + [\mathfrak{J}_2\mathfrak{J}_2]$. We obtain in this way functors from the category of Jordan algebras with 1, where the morphisms are homomorphisms preserving 1, into the category of Lie algebras.

We can also make a similar modification of the construction of the Lie algebra $\mathfrak{K}(\mathfrak{J})$. Here we let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be vector spaces isomorphic to $\mathfrak{J}$ under the mappings $a \to a_i$, $i = 1, 2$. We consider the space $\mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus (\mathfrak{J} + [\mathfrak{J}\mathfrak{J}])$, where $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ is the subspace of $U_1(\mathfrak{J})$ defined in the last paragraph, and make this into an algebra with composition $[\ ,\ ]$ such that

$$
\begin{aligned}
[a_1 + b_2 + l, c_1 + d_2 + m] &= (aM)_1 - (cL)_1 + (bM^\varepsilon)_2 - (dL^\varepsilon)_2 \\
&\quad + (a \,.\, d - [ad] - b \,.\, c + [bc] + [lm])
\end{aligned}
\tag{66}
$$

where $a, b, c, d \in \mathfrak{J}$, $l, m \in \mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ and $l \to L$, $m \to M$ in the canonical homomorphism of $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ onto $\mathfrak{L}(\mathfrak{J})$. If we have a homomorphism η of $\mathfrak{J}$ into a second Jordan algebra mapping $1 \to 1$ then $a_1 + b_2 + l \to a^{\eta}{}_1, + b^{\eta}{}_2 + l^{\eta}$, where $l \to l^{\eta}$ is the homomorphism of $\mathfrak{J} + [\mathfrak{J}\mathfrak{J}]$ such that $a \to a^{\eta}$, is a Lie algebra homomorphism. In this way we obtain another functor from the category of Jordan algebras with 1 into the category of Lie algebras.

Exercises

1. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{V}$ the Jordan algebra of a nondegenerate symmetric bilinear form f on $\mathfrak{V}$ with dim $\mathfrak{V} > 1$. Let $\mathfrak{Y}$ be a three-dimensional vector space with bilinear form g such that the matrix of g relative to a basis (u, v, w) is

$$
\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
$$

Let $\mathfrak{W} = \mathfrak{V} \oplus \mathfrak{Y}$ and define a symmetric bilinear form h on $\mathfrak{W}$ so that $\mathfrak{V} \perp \mathfrak{Y}$ and $h = f$ on $\mathfrak{V}$ and $h = g$ on $\mathfrak{Y}$. Show that $\mathfrak{K}(\mathfrak{J})$ is isomorphic to the Lie algebra of linear transformations in $\mathfrak{W}$ which are skew relative to h. Use this to prove that the Lie algebra $\mathfrak{S}(\Phi_n, J_1)$ is central simple if $n \geqq 5$.

2. Use the results on $\mathfrak{K}(\mathfrak{J})$ in the text to prove that the symplectic Lie algebra $\mathfrak{S}(\Phi_{2m}, J_S)$ where J_S is the involution $X \to S^{-1}X^t S$, $S = \text{diag}\,\{Q, Q, \cdots, Q\}$, $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is simple if $m \geqq 3$.

3. Use the same method to show that $[\Phi_m^-, \Phi_m^-]$ is simple if $m \geqq 3$ and m is not divisible by the characteristic and that $[\Phi_m^-, \Phi_m^-]/\Phi 1$ is simple if $m \geqq 3$ and is divisible by the characteristic p of Φ.

10. **Associator nilpotent Jordan algebras.** If $\mathfrak{L}$ is a Lie algebra we shall write

$$[a_1 a_2 \cdots a_N]' = [[\cdots[[a_1 a_2]a_3]\cdots]a_N], \qquad a_i \in \mathfrak{L}. \tag{67}$$

We recall that $\mathfrak{L}$ is called nilpotent if there exists a positive integer N such that every $[a_1 a_2 \cdots a_N]' = 0$, $a_i \in \mathfrak{L}$. The minimum N with this property is called the index of nilpotency. We have, for $1 < k+1 < N$,

$$\begin{aligned} [[a_1 \cdots a_k]', [a_{k+1} \cdots a_N]'] &= [[a_1 \cdots a_k]', [[a_{k+1} \cdots a_{N-1}]' a_N]] \\ &= [[[a_1 \cdots a_k]', [a_{k+1} \cdots a_{N-1}]']a_N] \\ &\quad - [[a_1 \cdots a_k a_N]', [a_{k+1} \cdots a_{N-1}]']. \end{aligned} \tag{68}$$

This implies that any product of N $a_i \in \mathfrak{L}$ in any association is a linear combination of the normalized products $[a_1 \cdots a_N]'$. Hence any such product is 0 if $\mathfrak{L}$ is nilpotent of index N. If S is a set of generators for a Lie algebra $\mathfrak{L}$ then (68) implies also that every element of $\mathfrak{L}$ is a linear combination of normalized products $[s_1 \cdots s_r]'$, $s_i \in S$. Hence it is clear (again using (68)) that $\mathfrak{L}$ is nilpotent if there exists an integer N such that every $[s_1 s_2 \cdots s_N]' = 0$, $s_i \in S$.

We shall call a Lie triple system $\mathfrak{T}$ *nilpotent* if there exists a positive odd integer M such that every product

$$[[\cdots[[a_1 a_2 a_3]a_4 a_5]\cdots]a_{M-1} a_M], \qquad a_i \in \mathfrak{T} \tag{69}$$

is 0. The minimal M for this is the *index* of nilpotency. If $\mathfrak{L}$ is a Lie algebra then $\mathfrak{L}$ is Lie triple system $\mathfrak{L}^{(2)}$ relative to the composition $[abc] = [[ca]b] = [cab]'$. It follows that (69) is a normalized product $[a_{i_1} \cdots a_{i_M}]'$ for some permutation $(i_1 \cdots i_M)$ of $(1, \cdots, M)$ and conversely any such normalized product can be expressed in the form (69). Hence it is clear that $\mathfrak{L}$ is nilpotent if and only if $\mathfrak{L}^{(2)}$ is nilpotent. It is clear also that if $\mathfrak{T}$ is a subsystem of $\mathfrak{L}^{(2)}$ which generates $\mathfrak{L}$, so that $\mathfrak{L} = \mathfrak{T} + [\mathfrak{T}\mathfrak{T}]$, then $\mathfrak{L}$ is nilpotent if and only if $\mathfrak{T}$ is nilpotent.

We now define a Jordan algebra $\mathfrak{J}$ to be *associator nilpotent* if the associator

Lie triple system $\mathfrak{J}$ (with $[abc] = [a,b,c]$) is nilpotent. In $\mathfrak{J}$ we have $[x,a,b] = x(R_aR_b - R_{a,b})$ and we define

$$R_{a,b} = R_aR_b - R_{a,b}. \tag{70}$$

Then it is clear that the definition of associator nilpotency of $\mathfrak{J}$ is equivalent to the existence of a positive integer k such that

$$R_{a_1,b_1}R_{a_2,b_2}\cdots R_{a_k\,b_k} = 0 \tag{71}$$

for all $a_i, b_i \in \mathfrak{J}$.

It is clear that any subalgebra and any homomorphic image of an associator nilpotent Lie algebra is associator nilpotent. Also since the condition for associator nilpotency is multilinear, $\mathfrak{J}$ is associator nilpotent if and only if $\mathfrak{J}_{\mathrm{P}}$ is associator nilpotent for any extension field P of the base field of $\mathfrak{J}$. If $\mathfrak{J}$ has an identity element and Γ is a subfield of the center of $\mathfrak{J}$ containing 1 then $\mathfrak{J}$ can be regarded as an algebra over Γ. It is clear also that $\mathfrak{J}$ is associator nilpotent if and only if $\mathfrak{J}/\Gamma$ is associator nilpotent. We shall now determine the structure of finite-dimensional associator nilpotent Jordan algebras with 1. For this purpose we shall say that a Jordan algebra $\mathfrak{J}$ with 1 is *purely inseparable* if $\mathfrak{J}$ contains a nil ideal $\mathfrak{N}$ such that $\mathfrak{J}/\mathfrak{N}$ is an (associative) purely inseparable (algebraic) field over Φ. If the base field Φ is perfect (for example, if the characteristic is 0) then the condition is that $\mathfrak{J}/\mathfrak{N} \cong \Phi$, which is equivalent to: $\mathfrak{J} = \Phi 1 + \mathfrak{N}$. We recall that a finite-dimensional nil algebra is nilpotent. Hence in this case the word "nil" in the definition of a purely inseparable Jordan algebra can be replaced by "nilpotent." We now prove the following

LEMMA 1. *If $\mathfrak{J}$ is a finite-dimensional purely inseparable Jordan algebra with 1 and Ω is the algebraic closure of the base field Φ then $\mathfrak{J}_\Omega = \Omega 1 + \mathfrak{P}$ where $\mathfrak{P}$ is a nil (nilpotent) ideal.*

PROOF. We note first that if P/Φ is a purely inseparable field then $\mathrm{P}_\Omega = \Omega 1 + \mathfrak{M}$ where $\mathfrak{M}$ is the nil radical (= ideal of nilpotent elements) of the commutative associative algebra P_Ω. This is trivial if the characteristic is 0. If the characteristic is $p \neq 0$ and $a \in \mathrm{P}$ then $a^{p^e} = \alpha 1$, $\alpha \in \Phi$, for some positive integer e. Now there exists a $\beta \in \Omega$ such that $\beta^{p^e} = \alpha$. Then in P_Ω we have $a = \beta 1 + z$ where $z = a - \beta 1$ satisfies $z^{p^e} = a^{p^e} - \beta^{p^e} 1 = 0$. It follows that every element of P_Ω has the form $\gamma 1 + w$, $\gamma \in \Omega$, w nilpotent. This means that $\mathrm{P}_\Omega = \Omega 1 + \mathfrak{M}$, $\mathfrak{M}$ the nil radical. Now suppose $\mathfrak{J}$ is a finite-dimensional Jordan algebra with 1 which is purely inseparable over Φ. Then $\mathfrak{J}$ has a nil ideal $\mathfrak{N}$ such that $\mathrm{P} \equiv \mathfrak{J}/\mathfrak{N}$ is a purely inseparable field. Then $\mathfrak{J}_\Omega/\mathfrak{N}_\Omega$, which can be identified with P_Ω, has the form $\Omega 1 + \mathfrak{M}$ where $\mathfrak{M}$ is the nil radical. Then we have an ideal $\mathfrak{P}$ in $\mathfrak{J}_\Omega$ such that $\mathfrak{P}/\mathfrak{N}_\Omega = \mathfrak{M}$ and $\mathfrak{J}_\Omega/\mathfrak{P} \cong \Omega$. Since $\mathfrak{N}$ is nilpotent $\mathfrak{N}_\Omega$ is nilpotent and hence nil and consequently $\mathfrak{P}$ is a nil ideal in $\mathfrak{J}_\Omega$. Then $\mathfrak{J}_\Omega = \Omega 1 + \mathfrak{P}$.

We shall call an element a of a finite-dimensional Jordan algebra with 1 *separable* if the subalgebra $\Phi[a]$ generated by a and 1 is separable. We remark that if e is an idempotent then e is separable and if a is separable then $\Phi[a]_\Omega = \Omega e_1 \oplus \Omega e_2 \oplus \cdots \oplus \Omega e_r$ where the e_i are orthogonal idempotents such that $\sum e_i = 1$.

LEMMA 2. *If $\mathfrak{J}$ is a finite-dimensional associator nilpotent Jordan algebra with* 1 *then every separable element of $\mathfrak{J}$ and in particular every idempotent is contained in the center.*

PROOF. Since $\mathfrak{J}_\Omega$ is associator nilpotent and the center of $\mathfrak{J}_\Omega$ is $\mathfrak{C}_\Omega$, $\mathfrak{C}$ the center of $\mathfrak{J}$, it is sufficient to prove that idempotents are central (= contained in the center). Hence let e be an idempotent in $\mathfrak{J}$ and let $\mathfrak{J} = \mathfrak{J}_0(e) + \mathfrak{J}_{\frac{1}{2}}(e) + \mathfrak{J}_1(e)$ be the Peirce decomposition relative to e. Put $f = 1 - e$. If $a \in \mathfrak{J}_{\frac{1}{2}}$ then $4[a,e,f] = a$. Iteration of this k times gives

$$2^{2k}[\cdots[[a,e,f],e,f]\cdots e,f] = a.$$

Since $\mathfrak{J}$ is associator nilpotent $a = 0$. Thus $\mathfrak{J}_{\frac{1}{2}} = 0$ and $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{J}_1$. This implies that $e \in \mathfrak{C}$.

We can now establish the following criterion.

THEOREM 18. *A necessary and sufficient condition that a finite-dimensional Jordan algebra $\mathfrak{J}$ with* 1 *is associator nilpotent is that $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ where $\mathfrak{J}_i$ is an ideal which contains a separable subfield Γ_i of its center containing the identity 1_i of $\mathfrak{J}_i$ such that $\mathfrak{J}_i/\Gamma_i$ is purely inseparable.*

PROOF. To prove the sufficiency it is enough to show that any finite-dimensional purely inseparable Jordan algebra (with 1) is associator nilpotent. By Lemma 1, we have $\mathfrak{J}_\Omega = \Omega 1 + \mathfrak{P}$, Ω the algebraic closure of Φ, $\mathfrak{P}$ a nil ideal. Then any higher associator of the form (69) with $a_i \in \mathfrak{J}_\Omega$ equals an associator of the same form in elements $z_i \in \mathfrak{P}$. Since $\mathfrak{P}$ is finite-dimensional nil it is nilpotent. Hence for a suitable integer M all the associators in which the $a_i = z_i \in \mathfrak{P}$ are 0. Thus $\mathfrak{J}_\Omega$ and hence $\mathfrak{J}$ is associator nilpotent. Conversely, assume $\mathfrak{J}$ is associator nilpotent and write $1 = \sum_1^s e_i$ where the e_i are primitive idempotents. By Lemma 2, the e_i are central so if $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ is the Peirce decomposition relative to the e_i then the $\mathfrak{J}_{ij} = 0$ for $i \neq j$. Then $\mathfrak{J} = \sum_1^s \mathfrak{J}_i$ where $\mathfrak{J}_i = \mathfrak{J}_{ii}$ is an ideal. It suffices to show that the $\mathfrak{J}_i$ have the property stated in the theorem. We may therefore simplify the notation and assume 1 is primitive. Since we can lift idempotents from $\mathfrak{J}/\text{rad}\,\mathfrak{J}$ to $\mathfrak{J}$ (Lemma 2 of §3.7) it is clear that the identity is a primitive idempotent in $\mathfrak{S} = \mathfrak{J}/\text{rad}\,\mathfrak{J}$. Since $\mathfrak{S}$ is semisimple and 1 is primitive in this algebra it follows from Theorem 4.3 that $\mathfrak{S}$ (which contains no absolute zero divisors by Theorem 5.7) is a division algebra. Now let Γ be the set of separable elements of $\mathfrak{J}$. Then Γ is contained in the center of $\mathfrak{J}$. If $\gamma \in \Gamma$ then $\Phi[\gamma]$ is a separable associative commutative algebra with only one nonzero idempotent. Hence $\Phi[\gamma]$ is a field. If $\gamma_1, \gamma_2 \in \Gamma$ then the subalgebra $\Phi[\gamma_1, \gamma_2]$ generated by γ_1 and γ_2 is a homomorphic image of $\Phi[\gamma_1] \otimes \Phi[\gamma_2]$ which is a direct sum of fields. Since 1 is primitive in $\Phi[\gamma_1, \gamma_2]$ this is a field. Hence Γ is a separable subfield of the center containing 1 and we may regard $\mathfrak{J}$ as an algebra over Γ. Then $\mathfrak{J}/\Gamma$ has the same properties as $\mathfrak{J}$ and in addition every element of $\mathfrak{J}/\Gamma$ which is separable (over Γ, hence over Φ) is contained in Γ. To simplify the notation we assume that $\Gamma = \Phi$. Now let P be the center of $\mathfrak{S} = \mathfrak{J}/\text{rad}\,\mathfrak{J}$ and consider $\mathfrak{S}$ as an algebra over P. Then $(\mathfrak{S}/\mathrm{P})_\Omega$, for Ω the algebraic closure of P, is simple. Since this algebra is also associator nilpotent the result established shows that this is a division algebra. Since Ω is algebraically closed we must have $(\mathfrak{S}/\mathrm{P})_\Omega = \Omega 1$. Hence $\mathfrak{S}/\mathrm{P} = \mathrm{P}$, so $\mathfrak{J}/\text{rad}\,\mathfrak{J}$ is a field. We claim that this is purely inseparable. Otherwise, we have an $a \in \mathfrak{J}$ such that the minimum polynomial $\phi(\lambda)$ of $a + \text{rad}\,\mathfrak{J}$ is separable of degree > 1. Then the minimum polynomial of a is of the form $\phi(\lambda)^r$. It follows that $\Phi[a]$ contains an element b whose minimum polynomial is $\phi(\lambda)$ (Jacobson, *Lie Algebras*, p. 96, where the proof given for characteristic 0 is valid for $\phi(\lambda)$ separable). This contradicts the fact that all the separable elements of $\mathfrak{J}$ are contained in Φ and completes the proof of the theorem.

If Φ is a perfect field then the condition established reduces to the following

COROLLARY 1. *If Φ is perfect then a necessary and sufficient condition that a finite-dimensional Jordan algebra $\mathfrak{J}/\Phi$ with* 1 *is associator nilpotent is that $\mathfrak{J}$ is a direct sum of ideals $\mathfrak{J}_i$ which have the form $\mathfrak{J}_i = \Gamma_i \oplus \mathfrak{N}_i$ where Γ_i is a subfield of the center and $\mathfrak{N}_i$ is a nil ideal.*

We shall also state explicitly the following useful

COROLLARY 2. *If Φ is algebraically closed then a necessary and sufficient condition that a finite-dimensional Jordan algebra $\mathfrak{J}/\Phi$ with* 1 *is associator nilpotent is that $\mathfrak{J}$ is a direct sum of ideals $\mathfrak{J}_i$ of the form $\mathfrak{J}_i = \Phi 1_i + \mathfrak{N}_i$, where 1_i is the identity of $\mathfrak{J}_i$ and $\mathfrak{N}_i$ is a nil ideal.*

We shall derive next an analogue of Engel's theorem in the theory of Lie algebras. We recall that for (abstract) finite-dimensional Lie algebras this asserts that $\mathfrak{L}$ is nilpotent if and only if ad a is nilpotent for every $a \in \mathfrak{L}$. We now note that if $\mathfrak{J}$ is associator nilpotent then, by (71), $R_{a,b}$ is nilpotent for every $a, b \in \mathfrak{J}$. In particular $R_{a,a}$ is nilpotent for every a. We shall now call an element a of a Jordan algebra $\mathfrak{J}$ *associator nilpotent* in $\mathfrak{J}$ if $J_a \equiv R_{a,a}$ is a nilpotent linear transformation in $\mathfrak{J}$. Thus a is associator nilpotent in $\mathfrak{J}$ if and only if there exists an integer N such that $[[\cdots[[x,a,a],a,a]\cdots],a,a] = 0$ for all $x \in \mathfrak{J}$, where the number of a's displayed in our expression is $2N$. We prove first the following

LEMMA 3 (SCHAFER). *If $a \in \mathfrak{J}$ is associator nilpotent then every linear transformation R_{a^i,a^j}, $i,j = 0,1,2,\cdots$ is nilpotent.*

PROOF. Since we are defining $a^{\cdot 0} = 1$ and $R_{a^{\cdot i}, a^{\cdot j}} = R_{a^{\cdot i}} R_{a^{\cdot j}} - R_{a^{\cdot i+j}}$ the result is clear if i or $j = 0$. Hence assume $i, j > 0$. Since the operators $R_{a^{\cdot i}, a^{\cdot j}}$ are contained in the commutative algebra of linear transformations $\mathfrak{A}$ generated by 1, $R_a, R_{a^{\cdot 2}}$ our result will follow by showing that every $R_{a^{\cdot i}, a^{\cdot j}}$, $i, j > 0$, has the form $T_{ij} J_a$ where $T_{ij} \in \mathfrak{A}$. We prove this by induction on $h = i + j$. For this we note that the basic Jordan identity $R_c R_{a.b} - R_c R_b R_a + R_b R_{a.c} - R_{a.c.b} + R_a R_{b.c} - R_a R_b R_c = 0$ can be written in terms of the operators $R_{u,v}$ as $R_{b,a.c} = R_c R_{b,a} + R_a R_{b,c}$. Since $R_{a^{\cdot i}, a^{\cdot j}} = R_{a^{\cdot j}, a^{\cdot i}}$ and $R_{a,a} = J_a$ it is enough to show that $R_{a^{\cdot i}, a^{\cdot j+1}} = T_{i,j+1} J_a$ assuming $R_{a^{\cdot k}, a^{\cdot l}} = T_{k,l} J_a$ for all $k, l > 0$ such that $k, l > 0$ and $k + l \leqq i + j$. Since we have $R_{a^{\cdot i}, a^{\cdot j+1}} = R_{a^{\cdot j}} R_{a^{\cdot i}, a} + R_a R_{a^{\cdot i}, a^{\cdot j}}$ the result is clear.

We can now prove the following analogue of Engel's theorem in the theory of Lie algebras.

THEOREM 19. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1. Then $\mathfrak{J}$ is associator nilpotent if and only if every element of $\mathfrak{J}$ is associator nilpotent.*

PROOF. The necessity of the condition has been noted above. Conversely, assume every element of $\mathfrak{J}$ is associator nilpotent. We show first that this implies the key result, Lemma 2, used in the proof of Theorem 18, namely, that every separable element $a \in \mathfrak{J}$ is in the center. Let Ω be the algebraic closure of Φ. Then, by Schafer's lemma, $R_{a^{\cdot i}, a^{\cdot j}}$ is nilpotent in $\mathfrak{J}$ for all $i, j = 0, 1, 2, \cdots$. Hence every $R_{a^{\cdot i}, a^{\cdot j}}$ is nilpotent in $\mathfrak{J}_\Omega$. In $\mathfrak{J}_\Omega$ we have the orthogonal idempotents $e_k \in \Omega[a]$ such that $a = \sum \alpha_i e_i$, $\alpha_i \in \Omega$, and $\sum e_k = 1$. Then R_{e_k, e_l} is a linear combination of the operators $R_{a^{\cdot i}, a^{\cdot j}}$. Since these are nilpotent and commute, R_{e_k, e_l} is nilpotent. Let $\mathfrak{J}_\Omega = \sum_{k \leq l} \mathfrak{J}_{kl}$ be the Peirce decomposition of $\mathfrak{J}_\Omega$ relative to the e_k. If $k \neq l$, $\mathfrak{J}_{kl} = \mathfrak{J} U_{e_k, e_l} = \mathfrak{J} R_{e_k} R_{e_l} = \mathfrak{J} R_{e_k, e_l}$ since $R_{e_k, e_l} = R_{e_k} R_{e_l}$. Since R_{e_k, e_l} is nilpotent we have $\mathfrak{J}_{kl} = 0$. Then the e_k are in the center of $\mathfrak{J}_\Omega$ and a is in the center of $\mathfrak{J}$. We note next that if $n = \dim \mathfrak{J}$ and Φ is infinite then the hypothesis states that the polynomial mapping $a \to J_a{}^n = R_{a,a}{}^n$ of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ is 0. It follows that the polynomial mapping $a \to J_a{}^n$ of $\mathfrak{J}_\Omega$ is 0. This shows that for infinite base fields the hypothesis carries over to $\mathfrak{J}_\Omega$. This and the result noted permits us to carry over word for word the necessity part of the proof of Theorem 18 to show that $\mathfrak{J}$ has the structure given in Theorem 18. Then $\mathfrak{J}$ is associator nilpotent by this theorem. Now assume Φ finite. Then the necessity proof of Theorem 18 carries over to show that it is sufficient to prove the result with the added conditions that $\mathfrak{J}/\mathrm{rad}\ \mathfrak{J}$ is a division algebra and every element of $\mathfrak{J}$ which is separable is contained in $\Phi 1$. We claim that in this case $\mathfrak{J}/\mathrm{rad}\ \mathfrak{J} \cong \Phi$. Otherwise, let $\bar{a} = a + \mathrm{rad}\ \mathfrak{J} \notin \Phi 1$. Then the minimum polynomial $\mu(\lambda)$ of $\bar{a}$ is of degree $\geqq 1$ and this is irreducible. Moreover, since Φ is perfect $\mu(\lambda)$ is separable. Then the lifting property noted in the last part of the proof of Theorem 18 shows that we may assume that $\mu(a) = 0$. Then a is a separable element of $\mathfrak{J}$ and so this is contained in $\Phi 1$. This contradicts the assumption that $\bar{a} \notin \Phi 1$. Hence $\mathfrak{J}/\mathrm{rad}\ \mathfrak{J} \cong \Phi$ and $\mathfrak{J} = \Phi 1 + \mathrm{rad}\ \mathfrak{J}$ so this is associator nilpotent by the sufficiency part of Theorem 18.

We now consider the unital universal multiplication envelope $U_1(\mathfrak{J})$ for $\mathfrak{J}$ a Jordan algebra with 1 and we imbed $\mathfrak{J}$ as a subspace of $U_1(\mathfrak{J})$ as usual. We shall be interested in the Lie algebra $U_1(\mathfrak{J})^-$, its subalgebra $\mathfrak{J}+[\mathfrak{J}\mathfrak{J}]$ generated by $\mathfrak{J}$ and the subalgebra $\mathfrak{A}(\mathfrak{J})$ of $U_1(\mathfrak{J})^-$ generated by the elements $ab - a.b$, $a, b \in \mathfrak{J}$. Since $[a,b] = (ab - a.b) - (ba - b.a)$ it is clear that $\mathfrak{A}(\mathfrak{J}) \supseteq [\mathfrak{J}\mathfrak{J}]$ which is also a subalgebra of $U_1(\mathfrak{J})^-$. Since $[a,b,c] = [abc] = [[ca]b]$ in $\mathfrak{J}$ it is clear that the Lie triple system structure of $\mathfrak{J}$ given by the associator composition coincides with that given as a subsystem of $(U_1(\mathfrak{J})^-)^{(2)}$. Hence $\mathfrak{J}$ is associator nilpotent if and only if $\mathfrak{J}$ as subsystem of $(U_1(\mathfrak{J})^-)^{(2)}$ is a nilpotent Lie triple system and this is the case if and only if the Lie algebra $\mathfrak{J}+[\mathfrak{J}\mathfrak{J}]$ is nilpotent. Hence we have the equivalence of the first two conditions in the following theorem.

THEOREM 20. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with* 1, *$U_1(\mathfrak{J})$ the unital universal multiplication envelope of $\mathfrak{J}$, $\mathfrak{A}(\mathfrak{J})$ the subalgebra of the Lie algebra $U_1(\mathfrak{J})^-$ generated by the elements $ab - a.b$, $a, b \in \mathfrak{J}$. Then the following conditions are equivalent*: (1) *$\mathfrak{J}$ is associator nilpotent.* (2) *$\mathfrak{J}+[\mathfrak{J}\mathfrak{J}]$ is a nilpotent Lie algebra.* (3) *$U_1(\mathfrak{J})^-$ is nilpotent. These imply*: (4) *$\mathfrak{A}(\mathfrak{J})$ is nilpotent.*

PROOF. Since we have noted the equivalence of (1) and (2) and since a subalgebra of a nilpotent Lie algebra is nilpotent everything will follow if we can show that (1) implies (3). Also we may assume the base field Φ is algebraically closed. Then by Corollary 2 to Theorem 18, $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \cdots \oplus \mathfrak{J}_s$ where the $\mathfrak{J}_i$ are ideals and $\mathfrak{J}_i = \Phi 1_i + \mathfrak{N}_i$ where 1_i is the identity of $\mathfrak{J}_i$ and $\mathfrak{N}_i$ is a nil ideal. By the extension of Theorem 2.16 from two to s components (which is immediate), $U_1(\mathfrak{J})$ is a direct sum of ideals which are isomorphic to the algebras $U_1(\mathfrak{J}_i)$ and the algebras $S_1(\mathfrak{J}_i) \otimes S_1(\mathfrak{J}_j)$ where $i \neq j$ and $S_1(\mathfrak{J}_i)$ is the special unital universal envelope of $\mathfrak{J}_i$. By Theorem 5.2 and the remarks following it, $\mathfrak{N}_i$ generates a nilpotent ideal $\mathfrak{R}_i$ in $U_1(\mathfrak{J}_i)$ and the image of $\mathfrak{N}_i$ in $S_1(\mathfrak{J}_i)$ generates a nilpotent ideal $\mathfrak{S}_i$ in $S_1(\mathfrak{J}_i)$. By Theorem 2.1 (5) and 2.11 (4), $U_1(\mathfrak{J}_i)/\mathfrak{R}_i \cong U_1(\mathfrak{J}_i/\mathfrak{N}_i) \cong U_1(\Phi 1_i) \cong \Phi$ and $S_1(\mathfrak{J}_i)/\mathfrak{S}_i \cong S_1(\mathfrak{J}_i/\mathfrak{N}_i) \cong S_1(\Phi 1_i) \cong \Phi$. Hence $U_1(\mathfrak{J}_i) = \Phi 1 + \mathfrak{R}_i$ and $S_1(\mathfrak{J}_i) = \Phi 1 + \mathfrak{S}_i$ where $\mathfrak{R}_i$ and $\mathfrak{S}_i$ are nilpotent ideals. Then $S_1(\mathfrak{J}_i) \otimes S_1(\mathfrak{J}_j) = \Phi 1 + \mathfrak{S}_{ij}$ where $\mathfrak{S}_{ij}$ is a nilpotent ideal. Now it is clear that if $\mathfrak{E}$ is an associative algebra with 1 such that $\mathfrak{E} = \Phi 1 + \mathfrak{M}$ where $\mathfrak{M}$ is a nilpotent ideal, then $\mathfrak{E}^-$ is nilpotent Lie algebra. Since we have shown that $U_1(\mathfrak{J})$ is a direct sum of ideals having this structure it is clear that $U_1(\mathfrak{J})^-$ is nilpotent.

EXERCISES

1. Let $\mathfrak{J}$ be finite-dimensional associator nilpotent with 1. Show that $U(\mathfrak{J})^-$ is nilpotent.

2. Let $\mathfrak{J}$ be finite-dimensional Jordan with 1. Suppose the subalgebra of

$U(\mathfrak{J})^-$ generated by the elements $ab - a,b$ is nilpotent. Does this imply that $\mathfrak{J}$ is associator nilpotent?

11. **Cartan subalgebras of Jordan algebras. Associator regular elements.** Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1, $\mathfrak{K}$ an associator nilpotent subalgebra containing 1. The mapping $a \to R_a$, $a \in \mathfrak{K}$, $R_a \in \mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$, is a unital multiplication specialization. Hence we have the homomorphism of $U_1(\mathfrak{K})$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ mapping $a \to R_a$, $a \in \mathfrak{K}$. This induces a homomorphism of $U_1(\mathfrak{K})^-$ and of the subalgebra $\mathfrak{A}(\mathfrak{K})$ of $U_1(\mathfrak{K})^-$ generated by the elements $ab - a.b$, $a, b \in \mathfrak{K}$. We denote the image of $\mathfrak{A}(\mathfrak{K})$ under this homomorphism by $\mathfrak{A}_\mathfrak{J}(\mathfrak{K})$. Clearly this is the subalgebra of the Lie algebra $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})^-$ generated by the elements $R_{a,b} = R_a R_b - R_{a.b}$, $a, b \in \mathfrak{K}$. Since $\mathfrak{K}$ is associator nilpotent, $\mathfrak{A}(\mathfrak{K})$ is a nilpotent Lie algebra by Theorem 20. Hence its homomorphic image $\mathfrak{A}_\mathfrak{J}(\mathfrak{K})$ is a nilpotent Lie algebra of linear transformations in the vector space $\mathfrak{J}$.

We recall that if $\mathfrak{L}$ is a nilpotent Lie algebra of linear transformations in the finite-dimensional vector space $\mathfrak{M}$ then we have the Fitting decomposition of $\mathfrak{M}$ relative to $\mathfrak{L}$: $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1$, where $\mathfrak{M}_0 = \{z \in \mathfrak{M} \mid zL^n = 0,\ L \in \mathfrak{L}\}$, $n = \dim \mathfrak{M}$, and $\mathfrak{M}_1 = \bigcap_i \mathfrak{M}(\mathfrak{L}^*)^i$ where $\mathfrak{L}^*$ is the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{M},\mathfrak{M})$ generated by $\mathfrak{L}$ (without 1) (Jacobson, *Lie Algebras*, p. 39). These are invariant subspaces relative to $\mathfrak{L}$ and $\mathfrak{M}_0(\mathfrak{L}^*)^n = 0$. We remark that $\mathfrak{M}_1$ is the only complementary space of $\mathfrak{M}_0$ in $\mathfrak{M}$ which is invariant relative to $\mathfrak{L}$. For, if $\mathfrak{M}'$ is any such space then $\mathfrak{M}'$ has the Fitting decomposition $\mathfrak{M}' = \mathfrak{M}'_0 \oplus \mathfrak{M}'_1$, where $\mathfrak{M}_0' = \{z \in \mathfrak{M}' \mid zL^{n'} = 0,\ L \in \mathfrak{L}\}$, $n' = \dim \mathfrak{M}'$, $\mathfrak{M}_1' = \bigcap_i \mathfrak{M}'(\mathfrak{L}^*)^i$. Then $\mathfrak{M}_0' \subseteq \mathfrak{M}_0$ so $\mathfrak{M}_0' = 0$ so $\mathfrak{M}' = \mathfrak{M}_1' \subseteq \mathfrak{M}_1$. Hence $\mathfrak{M}' = \mathfrak{M}_1$. Let P be an extension field of the base field and consider the extension space $\mathfrak{M}_P$. If $A \in \mathfrak{L}$ then A has a unique extension to a linear transformation in $\mathfrak{M}_P$ and the set $P\mathfrak{L}$ of P-linear combinations of these linear transformations is a Lie algebra which can be identified with $\mathfrak{L}_P$. If $\mathfrak{L}$ is nilpotent then so is $\mathfrak{L}_P$. It is clear also that the Fitting decomposition $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1$ of $\mathfrak{M}$ relative to $\mathfrak{L}$ gives rise to the Fitting decomposition $\mathfrak{M}_P = \mathfrak{M}_{0P} \oplus \mathfrak{M}_{1P}$, that is, $\mathfrak{M}_{iP} = (\mathfrak{M}_P)_i$.

We now consider the Fitting decomposition of the vector space $\mathfrak{J}$ of a finite-dimensional Jordan algebra with 1 relative to the Lie algebra of linear transformations $\mathfrak{A}_\mathfrak{J}(\mathfrak{K})$ where $\mathfrak{K}$ is an associator nilpotent subalgebra containing 1. If $a \in \mathfrak{J}$ then we define

$$\mathfrak{Z}_a = \{z \in \mathfrak{J} \mid zJ_a{}^n = 0\} \tag{72}$$

where $n = \dim \mathfrak{J}$. Thus a is associator nilpotent in $\mathfrak{J}$ if and only if $\mathfrak{Z}_a = \mathfrak{J}$. We remark also that $\Phi[a] \subseteq \mathfrak{Z}_a$. Moreover, the proof of Schafer's lemma shows that for any $i,j = 1\ 2, \cdots$, $R_{a^i, a^j} = T_{ij} J_a = J_a T_{ij}$ where T_{ij} is in the algebra of operators generated by 1, R_a and R_{a^2}. Hence it is clear that $z(R_{a^i, a^j})^n = 0$ for all $z \in \mathfrak{Z}_a$ and $i,j = 0, 1, 2, \cdots$. We now have the following

THEOREM 21. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with* 1, *$\mathfrak{K}$ an* *associator nilpotent subalgebra containing* 1, $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$ *the nilpotent Lie algebra of linear transformations in* $\mathfrak{J}$ *generated by the transformations* $R_{a,b} = R_aR_b - R_{a.b}$, $a, b \in \mathfrak{K}$. *Let* $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{J}_1$ *be the Fitting decomposition of* $\mathfrak{J}$ *relative to* $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$. *Then* $\mathfrak{J}_0$ *is a subalgebra and* $\mathfrak{J}_0 \cdot \mathfrak{J}_1 \subseteq \mathfrak{J}_1$. *Moreover, if the base field is algebraically closed then* $\mathfrak{J}_0 = \bigcap_{b \in \mathfrak{K}} \mathfrak{Z}_b$ *where* $\mathfrak{Z}_a$ *is defined by* (72).

PROOF. It is sufficient to prove the result for algebraically closed base fields. In this case, by Corollary 2 to Theorem 18, $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \oplus \cdots \oplus \mathfrak{K}_s$ where $\mathfrak{K}_i$ is an ideal in $\mathfrak{K}$ which has the form $\mathfrak{K}_i = \Phi e_i + \mathfrak{N}_i$, e_i the identity of $\mathfrak{K}_i$ and $\mathfrak{N}_i$ a nil ideal. Let $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$ be the Peirce decomposition of $\mathfrak{J}$ relative to the e_i. We shall now show that

$$\mathfrak{J}_0 = \sum_1^s \mathfrak{J}_{ii}, \qquad \mathfrak{J}_1 = \sum_{i<j} \mathfrak{J}_{ij}. \tag{73}$$

Let $a, b \in \mathfrak{K}$, $x \in \mathfrak{J}_{ii}$. Then $a = \sum(\alpha_j e_j + z_j)$, $b = \sum(\beta_j e_j + w_j)$ where $\alpha_j, \beta_j \in \Phi$, $z_j, w_j \in \mathfrak{N}_j$. Hence $xR_{a,b} = [x, \alpha_i e_i + z_i, \ \beta_i e_i + w_i] = [x, z_i, w_i] = xR_{z_i,w_i}$. Since $\mathfrak{N}_i$ is a nilpotent Jordan algebra the associative algebra $\mathfrak{N}_i^*$ of linear transformations generated by the $R_{u_i}, u_i \in \mathfrak{N}_i$ is nilpotent (p. 194). The foregoing equation shows that if $x \in \mathfrak{J}_{ii}$ and $a, b \in \mathfrak{K}$ then $xR_{a,b} = xV$ where $V \in \mathfrak{N}_i^*$. It follows that if $L \in \mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$ then $xL = xW$ where $W \in \mathfrak{N}_i^*$. Then $xL^n = 0$ and so $\mathfrak{J}_{ii} \subseteq \mathfrak{J}_0$. Hence $\sum \mathfrak{J}_{ii} \subseteq \mathfrak{J}_0$. It is clear also from the definition of $\mathfrak{J}_0$ that

$$\mathfrak{J}_0 \subseteq \bigcap_{b \in \mathfrak{K}} \mathfrak{Z}_b. \tag{74}$$

We note next that it is clear from the properties of the Peirce decomposition that $\mathfrak{J}_{ij}R_b \subseteq \mathfrak{J}_{ij}$ if $b \in \mathfrak{K} \subseteq \sum \mathfrak{J}_{ii}$. Hence $\mathfrak{J}_{ij}\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K}) \subseteq \mathfrak{J}_{ij}$. Let $b = \sum \beta_i e_i$ where the β_i are distinct elements of Φ. If $x \in \mathfrak{Z}_b$ then $x(R_{b^{\cdot k}, b^{\cdot l}})^n = 0$ for $k, l = 0\ 1, 2, \cdots$. Hence $x(R_{e_i,e_j})^n = 0$ for $i, j = 1, 2, \cdots, s$. This and $\mathfrak{J}_{ij}\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K}) \subseteq \mathfrak{J}_{ij}$ imply that $\mathfrak{Z}_b \cap \mathfrak{J}_{ij} = 0$ if $i \neq j$ and that $\bigcap_{b \in \mathfrak{K}} \mathfrak{Z}_b \subseteq \sum \mathfrak{J}_{ii}$. Hence, by (74), $\sum \mathfrak{J}_{ii} = \mathfrak{J}_0 = \bigcap_{b \in \mathfrak{K}} \mathfrak{Z}_b$. Since $\sum_{i<j} \mathfrak{J}_{ij}$ is invariant under $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$ and is a complement of $\sum \mathfrak{J}_{ii}$ in $\mathfrak{J}$ we have $\mathfrak{J}_1 = \sum_{i<j} \mathfrak{J}_{ij}$. The fact that $\mathfrak{J}_0$ is a subalgebra and $\mathfrak{J}_1 \cdot \mathfrak{J}_0 \subseteq \mathfrak{J}_1$ are clear from the properties of the Peirce decomposition.

If $\mathfrak{K}$ is an associator nilpotent subalgebra containing 1 of the finite-dimensional Jordan algebra $\mathfrak{J}$ with 1 then there exists a k such that $x\, R_{a_1,b_1}R_{a_2,b_2} \cdots R_{a_k,b_k} = 0$ for $x, a_i, b_i \in \mathfrak{K}$. It follows that $\mathfrak{K} \subseteq \mathfrak{J}_0$ the Fitting null component of $\mathfrak{J}$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$. We shall now define a *Cartan subalgebra* $\mathfrak{K}$ of a finite-dimensional Jordan algebra with 1 as a subalgebra containing 1 such that $\mathfrak{K} = \mathfrak{J}_0$. This is equivalent to: $\mathfrak{K}$ is an associator nilpotent subalgebra containing 1 such that if $x \in \mathfrak{J}$ satisfies $xR_{a,b} \in \mathfrak{K}$ for all $a, b \in \mathfrak{K}$ then $x \in \mathfrak{K}$.(*). If x satisfies the condition $xR_{a,b} \in \mathfrak{K}$ for all $a, b \in \mathfrak{K}$ then $xR_{a_1,b_1} \cdots R_{a_k,b_k} = 0$ for all $a_i, b_i \in \mathfrak{K}$. It follows that $x \in \mathfrak{J}_0$, so if $\mathfrak{K}$ is a Cartan subalgebra then $x \in \mathfrak{K}$. On the other hand, suppose $\mathfrak{K}$

* This alternative definition is due to Kimura.

is associator nilpotent with 1 and $\mathfrak{K}$ is not a Cartan subalgebra. Then $\mathfrak{J}_0 \supset \mathfrak{K}$ and $\mathfrak{J}_0(\mathfrak{A}_0(\mathfrak{K})^*)^n = 0$ where $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})^*$ is the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ generated by $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$ and $n = \dim \mathfrak{J}$. Since $\mathfrak{K}$ is an invariant subspace under $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})^*$ there exists a vector $x \in \mathfrak{J}_0$, $\notin \mathfrak{K}$ such that $x\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})^* \subseteq \mathfrak{K}$. Then $xR_{a,b} \in \mathfrak{K}$ for all $a, b \in \mathfrak{K}$ but $x \notin \mathfrak{K}$. Hence $\mathfrak{K}$ does not satisfy the given condition.

If $a \in \mathfrak{J}$ then $\Phi[a]$ is associative and hence is associator nilpotent. It is clear from the remarks following (72) that the subspace $\mathfrak{Z}_a$ defined by (72) coincides with the Fitting null component of $\mathfrak{J}$ relative to the Lie algebra $\mathfrak{A}_{\mathfrak{J}}(\Phi[a])$. Let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{J}/\Phi$ and let $M(a)$ be the matrix of $J_a{}^n$ relative to this basis. Then it is clear that $\dim \mathfrak{Z}_a = n - \mathrm{rank}\, M(a)$. The element a will be called *associator regular* if $\dim \mathfrak{Z}_a$ is minimal. As before, let $x = \sum \xi_i u_i$, ξ_i indeterminates, be a generic element and define $\mathfrak{Z}_x$ and $M(x)$ as for a. Then the entries of $M(x)$ are in $\Phi[\xi_1, \xi_2, \cdots, \xi_n]$. If r is the rank of this matrix then $l = n - r = \dim \mathfrak{Z}_x$. It is clear that rank $M(a) \leqq r$ and a is associator regular if rank $M(a) = r$. Let $\{N_j(x)\}$ be the set of nonzero r-rowed minors in the matrix $M(x)$. Then for infinite Φ there exist $a = \sum \alpha_i u_i$, $\alpha_i \in \Phi$, such that $N_j(a) \neq 0$ for some j. Any such element is associator regular in $\mathfrak{J}$. Hence it is clear that for infinite Φ the set of associator regular elements is a Zariski open subset of $\mathfrak{J}$.

We now prove the following analogue of the theorem of existence of Cartan subalgebras of Lie algebras.

THEOREM 22. *If a is an associator regular element of a finite-dimensional Jordan algebra with* 1 *over an infinite field Φ then $\mathfrak{Z}_a$ is a Cartan subalgebra of $\mathfrak{J}$ containing a.*

PROOF. Let $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{J}_1$ be the Fitting decomposition of $\mathfrak{J}$ relative to the abelian, hence, nilpotent, Lie algebra $\mathfrak{A}_{\mathfrak{J}}(\Phi[a])$. Then we have noted that $\mathfrak{J}_0 = \mathfrak{Z}_a$. Let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{J}$ such that $(u_1, \cdots, u_l)$ is a basis for $\mathfrak{Z}_a$ and $(u_{l+1}, \cdots, u_n)$ is a basis for $\mathfrak{J}_1$. Since $\mathfrak{Z}_a \cdot \mathfrak{Z}_a \subseteq \mathfrak{Z}_a$ and $\mathfrak{J}_1 \cdot \mathfrak{Z}_a \subseteq \mathfrak{J}_1$ the matrix $M(b)$, $b \in \mathfrak{Z}_a$, of $J_b{}^n$ relative to the basis $(u_1, \cdots, u_n)$ has the form

$$\begin{pmatrix} N(b) & 0 \\ 0 & P(b) \end{pmatrix} \tag{75}$$

where $N(b)$ and $P(b)$ are the matrices of the restrictions to $\mathfrak{Z}_a$ and $\mathfrak{J}_1$ respectively. We shall now show that $N(b) = 0$ for all $b \in \mathfrak{Z}_a$. Otherwise, let $b \in \mathfrak{Z}_a$ have a nonzero entry ρ in one of the matrices $N(b)$. The matrix $N(a) = 0$. On the other hand, a is associator regular. Hence det $P(a) \neq 0$. Let ξ, η be indeterminates and consider the element $\xi a + \eta b \in \mathfrak{J}_{\Phi(\xi,\eta)}$. This is contained in $(\mathfrak{Z}_a)_{\Phi(\xi,\eta)}$ and it is clear that the determinant of the $r + 1$ rowed submatrix of $M(\xi a + \eta b)$ which is formed from the one-rowed matrix in the position of ρ and the r-rowed matrix $\xi P(a) + \eta P(b)$ has the form $f(\xi, \eta) = f_1(\xi, \eta) f_2(\xi, \eta)$ where $f_1(1, 0) = \det P(a)$, $f_2(0, 1) = \rho$. Hence $f(\xi, \eta) \neq 0$ and since Φ is infinite we can choose $\xi = \alpha$, $\eta = \beta$ in Φ such that $f(\alpha, \beta) \neq 0$. Then $c = \alpha a + \beta b \in \mathfrak{J}$ and has the property that the rank of

$M(c)$ is $r + 1$. This contradicts the associator regularity of a. Hence we can conclude that $N(b) = 0$, $b \in \mathfrak{Z}_a$. This implies that every element of the subalgebra $\mathfrak{Z}_a$ is an associator nilpotent in $\mathfrak{Z}_a$. Then, by Theorem 19, $\mathfrak{Z}_a$ is an associator nilpotent subalgebra of $\mathfrak{J}$ and $\mathfrak{Z}_a$ contains 1. Since $\mathfrak{Z}_a$ is associator nilpotent the Fitting null component of $\mathfrak{J}$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$ contains $\mathfrak{Z}_a$. Since $\mathfrak{Z}_a$ and $\mathfrak{J}_1$ are invariant under $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$ the null component of $\mathfrak{J}$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$ is $\mathfrak{Z}_a +$ the Fitting null component of $\mathfrak{J}_1$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$. Since $\mathfrak{J}_1$ is the Fitting one component of $\mathfrak{A}_{\mathfrak{J}}(\Phi[a])$ it is contained in the Fitting one component of $\mathfrak{J}$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$. Hence the Fitting null component of $\mathfrak{J}$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$ is $\mathfrak{Z}_a$ and $\mathfrak{Z}_a$ is a Cartan subalgebra.

We remark also that the decomposition $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathfrak{J}_1$ of $\mathfrak{J}$ into Fitting components relative to $\mathfrak{A}_{\mathfrak{J}}(\Phi[a])$ is also the decomposition of $\mathfrak{J}$ into Fitting components relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{Z}_a)$. This implies that an associator regular element cannot be contained in two distinct Cartan subalgebras.

Exercises

1. Let $\mathfrak{J}$ be a finite-dimensional central simple Jordan algebra, a an element of $\mathfrak{J}$ whose minimum polynomial has m distinct roots where $m = \deg \mathfrak{J}$. Show that $\Phi[a]$ is a Cartan subalgebra of $\mathfrak{J}$. Show that the dimensionality of any Cartan subalgebra of $\mathfrak{J}$ is m.
2. Determine a Cartan subalgebra of the Jordan algebra of triangular matrices.

12. **Application to generic traces.** Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1. We have called an element a of $\mathfrak{J}$ of maximum degree if the minimum polynomial of a coincides with its generic minimum polynomial. Moreover, we have seen that if the base field is infinite the set of elements of maximum degree is a Zariski open subset of $\mathfrak{J}$ (p. 224). We have seen in the last section that the set of associator regular elements is Zariski open. Hence the set of associator regular elements of maximum degree is Zariski open. By Theorem 22, if Φ is infinite there exist Cartan subalgebras of $\mathfrak{J}$ containing associator regular elements of maximum degree. We use these to obtain formulas for the generic trace and degree of $\mathfrak{J}$.

THEOREM 23. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with 1 over an algebraically closed field, $\mathfrak{K}$ a Cartan subalgebra of $\mathfrak{J}$ containing an associator regular element of maximum degree. Let $\mathfrak{J} = \mathfrak{K} \oplus \mathfrak{J}_1$ be the Fitting decomposition of $\mathfrak{J}$ relative to $\mathfrak{A}_{\mathfrak{J}}(\mathfrak{K})$, $\mathfrak{K} = \sum_1^s \oplus \mathfrak{K}_i$ where $\mathfrak{K}_i$ is an ideal in $\mathfrak{K}$ and $\mathfrak{K}_i = \Phi e_i \oplus \mathfrak{N}_i$, e_i the identity of $\mathfrak{K}_i$, $\mathfrak{N}_i$ a nil ideal and let n_i be the maximum index of nilpotency of the elements of $\mathfrak{N}_i$. Then if $a \in \mathfrak{J}$ and $a = \sum_1^s(\alpha_i e_i + z_i) + a_1$, $\alpha_i \in \Phi$, $z_i \in \mathfrak{N}_i$, $a_1 \in \mathfrak{J}_1$, then we have the formula*

$$t(a) = \sum n_i \alpha_i \tag{76}$$

for the generic trace and $\sum_1^s n_i$ is the degree of $\mathfrak{J}$.

PROOF. If $b \in \mathfrak{K}$ then the generic minimum polynomial $m_{b,\mathfrak{K}}(\lambda)$ of b relative to $\mathfrak{K}$ is a factor of the generic minimum polynomial $m_b(\lambda)$ of b in $\mathfrak{J}$. Since $\mathfrak{K}$ contains an element of maximum degree it follows that $m_{b,\mathfrak{K}}(\lambda) = m_b(\lambda)$. If $b = \sum(\beta_i e_i + w_i)$ where $\beta_i \in \Phi$ and $w_i \in \mathfrak{N}_i$ then the minimum polynomial $\mu_b(\lambda)$ of b in $\mathfrak{K}$ (and in $\mathfrak{J}$) is the least common multiple L.C.M. $(\lambda - \beta_i)^{m_i}$ where m_i is the index of nilpotency of w_i. It follows from this and the definition of the generic minimum polynomial that $m_b(\lambda) = m_{b,\mathfrak{K}}(\lambda) = \prod_1^s(\lambda - \beta_i)^{n_i}$ where n_i is the maximum index of nilpotency of the elements of $\mathfrak{N}_i$. Then it is clear that the degree of $\mathfrak{J}$ $= \deg m_b(\lambda) = \sum_1^s n_i$ and the generic trace, $t(b) = \sum_1^s n_i\beta_i$ for $b \in \mathfrak{K}$. We have seen in the proof of Theorem 21 that if $\mathfrak{J} = \sum_{i \leq j}\mathfrak{J}_{ij}$ is the Peirce decomposition of $\mathfrak{J}$ relative to the e_i, then $\mathfrak{J}_1 = \sum_{i<j}\mathfrak{J}_{ij}$ and $\mathfrak{K}_i = \mathfrak{J}_{ii}$. Hence if $a = \sum(\alpha_i e_i + z_i)$ $+ a_1$, $\alpha_i \in \Phi$, $z_i \in \mathfrak{N}_i$, $a_1 \in \mathfrak{J}_1$ then $a_1 = \sum_{i<j}a_{ij}$ where $a_{ij} \in \mathfrak{J}_{ij}$. Then $t(a)$ $= t(\sum(\alpha_i e_i + z_i)) + \sum_{i<j}t(a_{ij})$. Since $\sum(\alpha_i e_i + z_i) \in \mathfrak{K}$ we have $t(\sum(\alpha_i e_i + z_i))$ $= \sum n_i\alpha_i$. Since $a_{ij} = 4[a_{ij}, e_i, e_j]$ if $i < j$, $t(a_{ij}) = 0$ since t is associative. Hence we have $t(a_1) = 0$ and $t(a) = \sum_1^s n_i\alpha_i$.

We can use the formula (76) for the generic trace to give a proof of the following result which is independent of the determination of the simple algebras over an algebraically closed field.

THEOREM 24. *If $\mathfrak{J}$ is finite-dimensional separable then the generic trace form $t(a, b) = t(a . b)$ is nondegenerate* (*cf. p.* 240).

PROOF. Since $\mathfrak{J}$ is separable it has an identity element and $\mathfrak{J}_\Omega$ for Ω the algebraic closure is a direct sum of simple ideals $\mathfrak{J}_i$. Then the $\mathfrak{J}_i$ are orthogonal relative to the generic trace form and the restriction of this form to $\mathfrak{J}_i$ is the generic trace form in $\mathfrak{J}_i$ (p. 228). Hence it suffices to prove the nondegeneracy of the generic trace form of a simple Jordan algebra over an algebraically closed field. Then we may choose $\mathfrak{K}$ as in Theorem 23 and we have the formula (76). On the other hand, we have the reduced trace t_E such that $t_E(a) = \sum_1^s\alpha_i$ relative to the set of idempotents $E = \{e_1, e_2, \cdots, e_s\}$. Evidently $t_E(e_i) = t_E(e_i, e_i) \neq 0$ so $t_E \neq 0$. Since $\mathfrak{J}$ is simple and t_E is associative, it follows that t_E is nondegenerate. Since $\mathfrak{N}_i$ defined in Theorem 23 is clearly in the radical of t_E we have $\mathfrak{N}_i = 0$, $i = 1, 2, \cdots, s$. Then (76) gives $t(a) = \sum\alpha_i = t_E(a)$. Hence the reduced trace t_E coincides with the generic trace and consequently the generic trace form t is nondegenerate.

We remark that the converse of the preceding theorem was proved before (Theorem 6.5) and the proof of this did not require the determination of the simple algebras.

13. **Conjugacy of Cartan subalgebras**. Throughout this section we assume the base field is of characteristic 0 and we shall make use of some important concepts and results from the theory of algebraic groups of linear transformations. All of these can be found in Chevalley's two books, *Théorie des Groupes de Lie,* Tome II

and III. We recall first that if $\mathfrak{L}$ is a Lie algebra of linear transformations in a finite-dimensional vector space $\mathfrak{M}$ then the Lie algebra $\bar{\mathfrak{L}}$ of the intersection G of all the algebraic groups of linear transformations whose Lie algebras contain $\mathfrak{L}$ contains $\mathfrak{L}$ (Chevalley, Tome II, pp. 158–169 or Hochschild [2]). $\bar{\mathfrak{L}}$ is called the algebraic hull of $\mathfrak{L}$. The group G is an irreducible algebraic group. We now apply this to the Lie algebra Inder $\mathfrak{J}$ where $\mathfrak{J}$ is a finite-dimensional Jordan algebra and we denote the corresponding group G as I. Since the group of automorphisms of $\mathfrak{J}$ is an algebraic group whose Lie algebra is Der $\mathfrak{J}$ (Chevalley, Tome II, p. 179) it is clear that I is contained in the group of automorphisms. We shall now prove the following analogue of the classical conjugacy theorem for Cartan subalgebras of a Lie algebra.

THEOREM 25. *Let $\mathfrak{J}$ be a finite-dimensional Jordan algebra with* 1 *over an algebraically closed field of characteristic* 0. *Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be Cartan subalgebras of $\mathfrak{J}$. Then there exists an automorphism $s \in I$ such that $\mathfrak{K}_1^s = \mathfrak{K}_2$.*

PROOF. We pattern our proof after Chevalley's proof of the corresponding Lie algebra result (Chevalley, Tome III, pp. 215–219). Let Ω_1 be the orbit of $\mathfrak{K}_1$ under I. Then we wish to show that Ω_1 contains a Zariski open subset of $\mathfrak{J}$. We note first that since $\mathfrak{K}_1$ and I are irreducible, Ω_1 is épais, that is, it is irreducible and contains a nonvacuous open subset of its Zariski closure (Prop. 3, p. 193 of Chevalley, Tome III). Then the desired result that Ω_1 contains an open subset will follow if we can show that the dimensionality of the irreducible set Ω_1 is $n = \dim \mathfrak{J}$. Let $a \in \Omega_1$ and let T be the tangent space to Ω_1 at a. Then it is known that T contains $\mathfrak{K}_1 + a(\text{Inder } \mathfrak{J})$ where $a(\text{Inder } \mathfrak{J}) = \{aD \mid D \in \text{Inder } \mathfrak{J}\}$ (Chevalley, Tome III, p. 192). Since the base field is algebraically closed there exists a set of primitive idempotents e_i such that $\sum_1^s e_i = 1$ and $\mathfrak{K}_i = \sum \mathfrak{J}_{ii}$ where $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ is the Peirce decomposition of $\mathfrak{J}$ relative to the e_i. Also $\mathfrak{J}_{ii} = \Phi e_i + \mathfrak{N}_i$ where $\mathfrak{N}_i$ is a nil ideal in $\mathfrak{J}_{ii}$. Let O_1 denote the open subset of $\mathfrak{K}_1$ consisting of the elements $\sum (\alpha_i e_i + z_i)$, $\alpha_i \in \Phi$, $z_i \in \mathfrak{N}_i$ satisfying $\prod_{i<j}(\alpha_i - \alpha_j) \neq 0$. Let $a \in O_1$ and consider $a[R_{e_i} R_{a_{ij}}]$ where $i \neq j$ and $a_{ij} \in \mathfrak{J}_{ij}$. We have

$$
\begin{aligned}
a[R_{e_i}R_{a_{ij}}] &= (\sum_k (\alpha_k e_k + z_k))\,[R_{e_i}R_{a_{ij}}] \\
&= (\alpha_i e_i + z_i)[R_{e_i}R_{a_{ij}}] + (\alpha_j e_j + z_j)[R_{e_i}R_{a_{ij}}] \\
&= \tfrac{1}{4}(\alpha_i - \alpha_j)a_{ij} + \tfrac{1}{2}(z_i - z_j) \cdot a_{ij} \\
&= a_{ij}S_{ij}
\end{aligned}
\tag{77}
$$

where $S_{ij} = \frac{1}{4}R_{(\alpha_i - \alpha_j)1} + \frac{1}{2}R_{z_i - z_j}$. Since $\alpha_i \neq \alpha_j$ and $R_{z_i - z_j}$, is nilpotent it is clear that S_{ij} is invertible. Since it maps $\mathfrak{J}_{ij}$ into itself we have $\mathfrak{J}_{ij}S_{ij} = \mathfrak{J}_{ij}$. Hence (77) shows that $a(\text{Inder } \mathfrak{J})$ contains $\mathfrak{J}_{ij}$ for every $i \neq j$. It follows that the tangent space to Ω_1 at a contains $\mathfrak{K}_1$ and every $\mathfrak{J}_{ij}$, $i \neq j$. Hence this coincides with $\mathfrak{J}$. Since O_1 is open in $\mathfrak{K}_1$ and Ω_1 is the orbit of $\mathfrak{K}_1$ under I it is clear that O_1 contains a simple point of Ω_1. Hence dim $\Omega_1 = n$ and consequetly Ω_1 contains an open subset of $\mathfrak{J}$. Similarly, the orbit Ω_2 of $\mathfrak{K}_2$ under I contains an open subset of $\mathfrak{J}$. Since the set of associator regular elements of $\mathfrak{J}$ is Zariski open we now see that there exists an associator regular element b in $\mathfrak{J}$ and automorphisms $s_1, s_2 \in I$ such that $b \in \mathfrak{K}_i^{s_i}$. Now $\mathfrak{Z}_b$ and $\mathfrak{K}_i^{s_i}$ are Cartan subalgebras. Hence $\mathfrak{Z}_b = \mathfrak{K}_i^{s_i}$. Then $\mathfrak{K}_1^{s_1} = \mathfrak{K}_2^{s_2}$ and $\mathfrak{K}_2 = \mathfrak{K}_1^{s}$ where $s = s_1 s_2^{-1} \in I$.

CHAPTER IX

EXCEPTIONAL JORDAN ALGEBRAS

This chapter is devoted to the study of the Jordan algebras $\mathfrak{H}(\mathfrak{O}_3, J_1)$ of 3×3 hermitian matrices with entries taken from an octonion algebra $\mathfrak{O}$, and the algebras which become these on extension of the base field. As we have seen in Chapter IV and V, these algebras have a number of important characterizations. First, we recall that the algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ are precisely the Jordan algebras which are reduced, simple and exceptional (that is, not special, see §5.6). Also the algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and the Jordan division algebras which become these on extension of the base field are just the finite-dimensional central simple exceptional Jordan algebras (§5.7). Thus the algebras we are concerned with can be separated into two classes: the reduced ones and the division algebras. It is useful also to separate into two subclasses the reduced ones: namely, the ones which are split in the sense that the octonion algebra $\mathfrak{O}$ is split and the nonsplit ones.

After some preliminary results on identities the first main task we address ourselves to is that of determining conditions for the isomorphism of two reduced algebras $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. These are given in two theorems, the first of which is due to Albert and Jacobson [1] and the second to Springer [4]. For the proofs of both (given in §2 and §4 respectively) the set Π of elements of rank one in $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ plays an important role. These are the elements a of $\mathfrak{J}$ which are nonzero but have the property that adj $a = 0$ where adj a is defined by the generic minimum polynomial as in §6.3. For the proof of Springer's theorem we shall use also some results on groups of automorphisms and norm equivalences of $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. These are given in §3 and have important applications also in other parts of the chapter.

In §5 we collect a number of other important results on Aut $\mathfrak{J}$, mostly, for their own sake. The groups Aut $\mathfrak{J}$ are simple algebraic groups of type F_4 so the results we obtain, particularly, the determination of the involutions of the groups and their centralizers are of interest for the study of this important type of algebraic (or finite) group.

In §6 we consider split algebras and we prove an analogue of the classical invariant factor theorem for these algebras.

In §§7–10 we are concerned with reduced algebras $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ which are not split, so $\mathfrak{O}$ is an octonion division algebra. With any such algebra we can associate a projective plane $\mathfrak{P}(\mathfrak{J})$ which is defined by means of the elements of rank one of $\mathfrak{J}$. The plane $\mathfrak{P}(\mathfrak{J})$ depends only on the octonion coefficient algebra and these are equivalent to some planes which were constructed first by Moufang [1]. We study the main properties of these planes and their projective mappings and relate these to the Jordan algebras.

In §11 we consider some of the main connections between the exceptional Jordan algebras $\mathfrak{J}$ and Lie algebras. We study the Jordan algebras Der $\mathfrak{J}$, $\mathfrak{L}_0(\mathfrak{J})$ and the Tits-Koecher algebras $\mathfrak{K}(\mathfrak{J})$. The dimensionalities of these are respectively 52, 78 and 133 and except for $\mathfrak{L}_0(\mathfrak{J})$ in characteristic three the Lie algebras are all simple.

In the last section of this chapter we give a new construction, due to Tits, of exceptional Jordan division algebras. This construction, which is extremely simple, leads to substantially simpler proofs of Albert's main results on Jordan division algebras ([25] and [32]).

1. **Preliminaries on identities and subalgebras.** We recall that a Jordan algebra $\mathfrak{J}$ is called reduced if $\mathfrak{J}$ has an identity element $1 = \sum_1^n e_i$ where the e_i are orthogonal idempotents which are absolutely primitive in the sense that every element of the Peirce space $\mathfrak{J}_{ii} = \mathfrak{J}U_{e_i}$ has the form $\alpha e_i + z$ where $\alpha \in \Phi$ and z is nilpotent (§5.4). A set of e_i satisfying these conditions is called a reducing set of idempotents for $\mathfrak{J}$. If $\mathfrak{J}$ is reduced and simple then $\mathfrak{J}_{ii} = \Phi e_i$ for every e_i in the reducing set $\{e_i\}$. Moreover, if $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ is the Peirce decomposition relative to the e_i and $a_{ij} \in \mathfrak{J}_{ij}$, $i \neq j$, then $a_{ij}^{\cdot 2} = \alpha(e_i + e_j)$ where $\alpha \in \Phi$ (see §5.6 for this and other results on reduced simple Jordan algebras). There exist a_{ij} such that $a_{ij}^{\cdot 2} \neq 0$. If $\mathfrak{J}$ is reduced and simple and $\{1\}$ is a reducing set of idempotents then clearly $\mathfrak{J} = \Phi 1$ and if $\mathfrak{J}$ has a reducing set of two idempotents then $\mathfrak{J}$ is the Jordan algebra of a nondegenerate symmetric bilinear form. If $\mathfrak{J}$ has a reducing set of $n \geqq 3$ idempotents then we can choose for $j = 2, \cdots, n$ an element $u_{1j} \in \mathfrak{J}_{1j}$ such that $u_{1j}^{\cdot 2} = \gamma_j^{-1}(e_1 + e_j)$, $\gamma_j \in \Phi$. Then there exists an isomorphism η of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{D}_n, J_\gamma)$ where $(\mathfrak{D}, j)$ is a composition algebra and J_γ is the canonical involution determined by the diagonal matrix diag$\{1, \gamma_2, \cdots, \gamma_n\}$. Moreover, we have $e_i^\eta = e_{ii}$ and $u_{1j}^\eta = 1[ij]$, where, in general, $a[ij] = ae_{ij} + \gamma_j^{-1}\gamma_i \bar{a} e_{ji}$, $a \in \mathfrak{D}$ (see §3.2). We shall now call η a *coordinatization* of $\mathfrak{J}$ and we shall say that this is *adapted* to the elements e_i and u_{1j} if the foregoing conditions hold. Any reduced simple Jordan algebra which is not the Jordan algebra of a symmetric bilinear form is finite dimensional. The only exceptional algebras in this class are the algebras isomorphic to $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $(\mathfrak{O}, j)$ is an octonion algebra.

We recall that if $A \in \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, say,

$$A = \begin{bmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3 \bar{b} \\ \gamma_2^{-1}\gamma_1 \bar{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2 \bar{a} & \alpha_3 \end{bmatrix} \tag{1}$$

or, $A = \sum_1^3 \alpha_i e_{ii} + a[23] + b[31] + c[12]$, and

$$A \times A = A^{\cdot 2} - t(A)A + \tfrac{1}{2}(t(A)^2 - t(A^{\cdot 2}))1 \tag{2}$$

where $t(A) = \sum_1^3 \alpha_i$ then

$$(3) \qquad A.(A \times A) = n(A)1$$

where

$$(4) \quad n(A) = \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_3{}^{-1}\gamma_2 n(a) - \alpha_2\gamma_1{}^{-1}\gamma_3 n(b) - \alpha_3\gamma_2{}^{-1}\gamma_1 n(c) + t((ca)b)$$

(see §6.4). Here the n and t on the right-hand side are the generic norm and trace in the octonion algebra $\mathfrak{O}$: $t(a)1 = a + \bar{a}$, $n(a)1 = a\bar{a} = \bar{a}a$. We remark that if the coordinates a, b, c in A are contained in a commutative subalgebra then $n(A) = \det A$ the usual determinant of the matrix A. We have seen in §6.4 that $t(A) = \sum_1^3 \alpha_i$ and $n(A)$ are the generic trace and norm respectively of $A \in \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. The generic minimum polynomial is

$$(5) \qquad m_A(\lambda) = \lambda^3 - t(A)\lambda^2 + \tfrac{1}{2}[t(A)^2 - t(A^{\cdot 2})]\lambda - n(A).$$

We recall also that direct calculation gives $A \times A =$

$$(6) \qquad \begin{bmatrix} \alpha_2\alpha_3 - \gamma_3{}^{-1}\gamma_2 n(a) & \gamma_1{}^{-1}\gamma_2\bar{b}\bar{a} - \alpha_3 c & ca - \gamma_1{}^{-1}\gamma_3\alpha_2\bar{b} \\ ab - \gamma_2{}^{-1}\gamma_1\alpha_3\bar{c} & \alpha_1\alpha_3 - \gamma_1{}^{-1}\gamma_3 n(b) & \gamma_2{}^{-1}\gamma_3\bar{c}\bar{b} - \alpha_1 a \\ \gamma_3{}^{-1}\gamma_1\bar{a}\bar{c} - \alpha_2 b & bc - \gamma_3{}^{-1}\gamma_2\alpha_1\bar{a} & \alpha_1\alpha_2 - \gamma_2{}^{-1}\gamma_1 n(c) \end{bmatrix}.$$

Now let $\mathfrak{J}$ be an arbitrary finite-dimensional exceptional central simple Jordan algebra and let Ω be the algebraic closure of the base field. Since the property of a Jordan algebra of being special is linear (Theorem 2.8) it is clear that $\mathfrak{J}_\Omega$ is an exceptional simple Jordan algebra. Also $\mathfrak{J}_\Omega$ is reduced (Theorem 5.4). Hence our results apply to this algebra and these can be transferred to $\mathfrak{J}$ itself. In particular, we see that the degree of $\mathfrak{J}$ is three and its dimensionality is 27 since this is clearly the dimensionality of $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. Also, for $a \in \mathfrak{J}$ we have

$$(3') \qquad a.(a \times a) = n(a)1$$

where

$$(2') \qquad a \times a = a^{\cdot 2} - t(a)a + \tfrac{1}{2}[t(a)^2 - t(a^{\cdot 2})]1$$

and $t(a)$ and $n(a)$ are the generic trace and norm of a. Thus $a \times a = \operatorname{adj} a$ as defined in §6.3 and the generic minimum polynomial of a is

$$(5') \qquad m_a(\lambda) = \lambda^3 - t(a)\lambda^2 + \tfrac{1}{2}[t(a)^2 - t(a^{\cdot 2})]\lambda - n(a).$$

We remark also that $\mathfrak{J}$ is separable since it is central simple. Hence the generic trace form $t(a, b) = t(a.b)$ is nondegenerate. We recall also that any algebra $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ is unramified in the sense that the discriminant $\delta(x)$ of $m_x(\lambda)$ for the generic element x is not 0 (p. 232). It follows that any finite-dimensional exceptional central simple Jordan algebra $\mathfrak{J}$ is unramified. An element $a \in \mathfrak{J}$ satisfies $\delta(a) \neq 0$

if and only if $m_a(\lambda)$ has distinct roots. Since the minimum polynomial $\mu_a(\lambda)$ is a factor of $m_a(\lambda)$ and has the same roots except for multiplicities as $m_a(\lambda)$, it is clear that $\delta(a) \neq 0$ if and only if $\mu_a(\lambda)$ has three distinct roots. For infinite base fields the set of elements satisfying this condition is Zariski open in $\mathfrak{J}$.

Since the degree of $\mathfrak{J}$ is three it is clear that the cardinality of any set E of non-zero orthogonal idempotents does not exceed three and if $E = \{e_1, e_2, e_3\}$ then $e_1 + e_2 + e_3 = 1$ and the e_i are primitive. Moreover, the e_i are primitive in $\mathfrak{J}_\Omega$. Let $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$ be the Peirce decomposition of $\mathfrak{J}$ relative to the e_i. Then the Peirce decomposition of $\mathfrak{J}_\Omega$ is $\sum \mathfrak{J}_{ij\Omega}$ and since e_i is primitive in $\mathfrak{J}_\Omega$ we have $\mathfrak{J}_{ii\Omega} = \Omega e_i$. Hence $\mathfrak{J}_{ii} = \Phi e_i$. Thus $\mathfrak{J}$ is reduced and E is a reducing set of idempotents for $\mathfrak{J}$. If we use a coordinatization η of $\mathfrak{J}$ adapted to the e_i and elements u_{1j}, $j = 2, 3$ in $\mathfrak{J}_{1j}$ such that $u_{1j}{}^{\cdot 2} \neq 0$ then the generic trace $t(e_i) = t(e_i^\eta) = t(e_{ii}) = 1$.

Since $\mathfrak{J}$ is a finite-dimensional simple Jordan algebra, $1 = \sum_1^r e_i$ where the e_i are completely primitive orthogonal idempotents, that is, the Peirce space $\mathfrak{J}_{ii} = \mathfrak{J}U_{e_i}$ are Jordan division algebras (Theorem 4.3 and Corollary 1 to Theorem 5.7). Moreover, the e_i are connected (by the proof of the Second Structure Theorem, p. 179). We have just seen that $r \leqq 3$ and if $r = 3$ then $\mathfrak{J}$ is reduced. If $r = 1$ then 1 is completely primitive and hence $\mathfrak{J}$ is a division algebra. Now suppose $r = 2$ and let $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{22}$ be the Peirce decomposition relative to e_1, e_2, so $\mathfrak{J}_{12} = \mathfrak{J}_{\frac{1}{2}}(e_i)$. Let $a \in \mathfrak{J}_{12}$. Then $4[a, e_1, e_2] = a$ so $t(a) = 0$. Also since $\mathfrak{J}_{12}{}^{\cdot 2} \subseteq \mathfrak{J}_{11} + \mathfrak{J}_{22}$ and $\mathfrak{J}_{ij} \cdot \mathfrak{J}_{jj} = \mathfrak{J}_{ij} = \mathfrak{J}_{ij} \cdot \mathfrak{J}_{ii}$ we have $a^{\cdot 3} \in \mathfrak{J}_{12}$. The equation $m_a(a) = 0$ reduces to $a^{\cdot 3} = \frac{1}{2}t(a^{\cdot 2})a + n(a)1$ since $m_a(\lambda)$ is given by (5′) and $t(a) = 0$. Since $a^{\cdot 3}$, $a \in \mathfrak{J}_{12}$ and $1 \in \mathfrak{J}_{11} + \mathfrak{J}_{22}$ this gives $a^{\cdot 3} = \frac{1}{2}t(a^{\cdot 2})a$ and $n(a) = 0$. The latter condition implies that a is not invertible so we have shown that there exist no invertible elements in $\mathfrak{J}_{12}$. This contradicts the connectedness of e_1 and e_2 so the case $r = 2$ is ruled out. Thus we see that any finite dimensional exceptional central simple Jordan algebra is either reduced or it is a division algebra.

Now let e be any idempotent of $\mathfrak{J}$ different from 0 and 1. (The existence of such an idempotent implies $\mathfrak{J}$ is reduced.) The proof of Theorem 4.3 shows that e and $1 - e$ can be written as sums of completely primitive orthogonal idempotents. It follows that one and only one of the idempotents is completely primitive and the other is not. We may as well assume it is $e = e_1$. Then $1 - e = e_2 + e_3$ where the e_i are completely primitive. Hence the e_i are absolutely primitive and $t(e_i) = 1$. Thus we see that if e is an idempotent $\neq 0, 1$ in $\mathfrak{J}$ then either $t(e) = 1$ or $t(e) = 2$ and the first alternative holds if and only if e is primitive (completely primitive, absolutely primitive).

We remark finally that every reduced exceptional simple Jordan algebra is central simple and finite dimensional. Hence the two classes of Jordan algebras described by the modifiers: "reduced exceptional simple" and "finite dimensional exceptional central simple nondivision algebra" coincide.

We shall develop next some basic identities on adj $a = a \times a$, $t(a)$ and $n(a)$. Since these are unchanged under extension of the base field we may assume this is infinite and, if we wish, that it is algebraically closed. Then we can apply the concepts and results on polynomial mappings of §6.2.

We wish to linearize the mappings adj and n. Since the characteristic is $\neq 2$ but may be three we introduce

$$a \times b = \tfrac{1}{2}\Delta^b{}_a\text{adj} \tag{7}$$

and

$$(a, b, c) = \tfrac{1}{2}\Delta^c{}_a\Delta^b n. \tag{8}$$

If the characteristic is not three then it is perferable to replace (a, b, c) by

$$n(a, b, c) \equiv \tfrac{1}{3}(a, b, c) = \tfrac{1}{3}!\, \Delta^c{}_a\Delta^b n. \tag{9}$$

By Euler's equation (p. 219), $\Delta^a{}_a\text{adj} = 2$ adj a so $a \times a$ as defined by (7) agrees with the original $a \times a$. Since $\Delta^c{}_a\Delta^b$ adj is symmetric and bilinear in c and b it is clear that $a \times b$ is symmetric and bilinear in a and b. It follows that

$$a \times b = \tfrac{1}{2}[(a + b) \times (a + b) - a \times a - b \times b] \tag{10}$$

so by the definition (2′) of $a \times a$ (or by differentiation) we have

$$\begin{aligned} a \times b = a \,.\, b - \tfrac{1}{2}t(a)b - \tfrac{1}{2}t(b)a \\ + \tfrac{1}{2}[t(a)t(b) - t(a \,.\, b)]1. \end{aligned} \tag{11}$$

In a similar manner one sees that (a, b, c) is symmetric and trilinear and $(a, a, a) = 3n(a)$, so if the characteristic is not three, then $n(a, a, a) = n(a)$. This gives directly:

$$\begin{aligned} n(a, b, c) = \tfrac{1}{3}!\, [n(a + b + c) - n(a + b) - n(b + c) - n(a + c) \\ + n(a) + n(b) + n(c)] \end{aligned} \tag{12}$$

which is equivalent to

$$\begin{aligned} (a, b, c) = \tfrac{1}{2}\, [n(a + b + c) - n(a + b) - n(b + c) - n(a + c) \\ + n(a) + n(b) + n(c)]. \end{aligned} \tag{12′}$$

We shall now show that this form of (12) is valid also in the characteristic three case. Let $F(a, b, c)$ denote the right-hand side of (12′). We claim first that, since n is a homogeneous polynomial function of degree three, $F(a, b, c)$ is symmetric and trilinear in a, b, c. This is equivalent to showing that if $f(\xi) \equiv f(\xi_1, \cdots, \xi\ )$ is a homogeneous cubic polynomial in indeterminates ξ_i, and η_i, ζ_i are other indeterminates then $Ef \equiv f(\xi + \eta + \zeta) - f(\xi + \eta) - f(\xi + \zeta) - f(\eta + \zeta) + f(\xi) + f(\eta) + f(\zeta)$ is homogeneous of first degree in each of the sets of indeterminates $\xi_1, \cdots, \xi_n$; $\eta_1, \cdots, \eta_n$; $\zeta_1, \cdots, \zeta_n$. It is enough to verify this for monomials since E is a linear mapping of the polynomial algebra. For monomials one can either carry through a direct verification or establish the result in the ring $Z[\xi_1, \cdots, \xi_n]$ by using the corresponding functional result on vector spaces over the rationals. We show next that $\Delta^b{}_a n = (a, a, b)$. For this we fix a and consider $(a, a, b) - \Delta^b{}_a n$ as function of a. This is homogeneous of degree two or 0, and applying $\frac{1}{2}\Delta^c$ gives $\frac{1}{2}\Delta^c{}_a(a, a, b) - \frac{1}{2}\Delta^c{}_a\Delta^b\ n = (a, c, b) - (a, c, b) = 0$. Since $(a, a, b) - \Delta^b{}_a$ is homogeneous of degree two or is 0 this implies that $(a, a, b) = \Delta^b{}_a n$. We can now use this result and the chain rule for differentiation to evaluate $\Delta^d F$ for $F(a, b, c)$ given by the right-hand side regarded as a function of a. A simple calculation which we suppress gives $\Delta^d F = (d, b, c)$. On the other hand, since $F(a, b, c)$ is linear in a we have $\Delta^d F = F(d, b, c)$. Hence $F(a, b, c) = (a, b, c)$ and (12′) holds.

As a special case of formula (68″) of Chapter VI (p. 243) we have $\Delta^b{}_a n = t(\text{adj } a, b)$. Then, by (7) and (8) we have

$$(a, b, c) = t(a \times b, c) \tag{13}$$

which implies that the right-hand side is symmetric in a, b and c. Substituting (11) in this we obtain

$$\begin{aligned}(a, b, c) = {}& t(a \,.\, b \,.\, c) - \tfrac{1}{2}t(a)t(b \,.\, c) - \tfrac{1}{2}t(b)t(a \,.\, c) \\ & - \tfrac{1}{2}t(c)t(a \,.\, b) + \tfrac{1}{2}t(a)t(b)t(c).\end{aligned} \tag{14}$$

For characteristic $\neq 3$ this gives formulas for $n(a, b, c)$ and for $n(a)$ in terms of the trace functions.

We have demonstrated here and elsewhere the efficacy of the differential calculus in establishing identities. We shall now develop another method for verifying identities. For this we shall need some additional consequences of the structure theory. Suppose the base field is algebraically closed and $E = \{e_i\}$ is a reducing set of idempotents in the reduced simple exceptional Jordan algebra $\mathfrak{J}$. Let u_{12} and u_{13} be elements of the corresponding Peirce spaces $\mathfrak{J}_{12}$ and $\mathfrak{J}_{13}$ such that $u_{12}{}^{\cdot 2} = \mu^2(e_1 + e_2)$, $u_{13}{}^{\cdot 2} = \nu^2(e_1 + e_3)$ where $\mu\nu \neq 0$. Then $(\mu^{-1}u_{12})^{\cdot 2} = e_1 + e_2$, $(\nu^{-1}u_{13})^{\cdot 2} = e_1 + e_3$. Let η be a coordinatization of $\mathfrak{J}$ adapted to the e_i and $\mu^{-1}u_{12}, \nu^{-1}u_{13}$. Then $\mathfrak{J}^\eta = \mathfrak{H}(\mathfrak{O}_3, J_1)$ where J_1 is the standard involution. Moreover, $e_i{}^\eta = e_{ii}$, $u_{12}{}^\eta = \mu[12] = \mu[21]$, $u_{13}{}^\eta = \nu[13] = \nu[31]$. Let $F = \{f_i\}$ be a second reducing set of idempotents for $\mathfrak{J}$, v_{12}, v_{13} elements of the corresponding Peirce (1,2) and (1,3) spaces such that $v_{12}{}^{\cdot 2} \neq 0$, $v_{13}{}^{\cdot 2} \neq 0$. Then there is an isomorphism ζ of $\mathfrak{J}$ onto an algebra $\mathfrak{H}(\mathfrak{O}_3{}', J_1)$ where $\mathfrak{O}'$ (like $\mathfrak{O}$) is octonion such that $f_i{}^\zeta = e_{ii}$, $v_{12}{}^\zeta = \mu'[12]$, $v_{13}{}^\zeta = \nu'[13]$ (in $\mathfrak{H}(\mathfrak{O}_3{}', J_1)$) where $\mu', \nu' \in \Phi$. Since the base field is algebraically closed the octonion algebras $(\mathfrak{O}, j)$ and $(\mathfrak{O}', j)$ are split and hence are isomorphic (Theorem 4.7, p. 169). Hence there exists an isomorphism of $\mathfrak{H}(\mathfrak{O}_3, J_1)$ onto $\mathfrak{H}(\mathfrak{O}_3{}', J_1)$ sending $1[ij] \rightarrow 1[ij]$ for all i,j. It follows that there exists an automorphism σ of $\mathfrak{J}$ such that $e_i = f_i$, $i = 1, 2, 3$ and $u_{12}{}^\sigma = \alpha v_{12}, u_{13}{}^\sigma = \beta v_{13}$, $\alpha, \beta \in \Phi$.

Let $\mathfrak{J}^{(r)}$ be a vector space direct sum of r copies of $\mathfrak{J}$ and let f be a polynomial function on $\mathfrak{J}^{(r)}$. We denote the elements of $\mathfrak{J}^{(r)}$ as r-tuples $(a_1, a_2, \cdots, a_r)$, $a_i \in \mathfrak{J}$, and let $f(a_1, a_2, \cdots, a_r)$ be the value of f at $(a_1, a_2, \cdots, a_r)$. We can now prove the following

LEMMA 1. *Let $\mathfrak{J}$ be a finite-dimensional exceptional simple Jordan algebra over an algebraically closed field and let f be a polynomial function on $\mathfrak{J}^{(r)}$. Assume that if $f(a_1, a_2, \cdots, a_r) = 0$ then $f(a_1^{\sigma}, a_2^{\sigma}, \cdots, a_r^{\sigma}) = 0$ for every automorphism σ of $\mathfrak{J}$. Assume also that if η is a particular coordinatization of $\mathfrak{J}$ onto $\mathfrak{H}(\mathfrak{O}_3, J_1)$ then $f(b_1^{\eta^{-1}}, b_2^{\eta^{-1}}, a_3, \cdots, a_r) = 0$ for all b_1 of the form $\sum \alpha_i e_{ii}$, b_2 of the form $\sum \beta_i e_{ii} + u[23] + \mu[12] + \nu[13]$ and arbitrary $a_j, j \geqq 3$, where $\alpha_i, \beta_i, \mu, \nu \in \Phi$ and $u \in \mathfrak{O}$. Then $f = 0$.*

PROOF. We note first that since the set of elements a such that the minimum polynomial $\mu_a(\lambda)$ has three distinct roots is Zariski open it is enough to prove that $f(a_1, a_2, \cdots, a_r) = 0$ for all a_1 such that $\mu_{a_1}(\lambda)$ has three distinct roots, a_j arbitrary in $\mathfrak{J}$ if $j \geqq 2$. Then $a_1 = \sum_1^3 \alpha_i f_i$ where the f_i form a reducing set of idempotents. Then there exists an automorphism σ in $\mathfrak{J}$ such that $a_1^{\sigma} = b_1^{\eta^{-1}}$ where $b_1 = \sum \alpha_i e_{ii}$. The first hypothesis implies that if $f(b_1^{\eta^{-1}}, a_2, \cdots, a_r) = 0$ for all $b_1 = \sum \alpha_i e_{ii}$, a_j arbitrary for $j \geqq 2$, then $f = 0$. Write $a_2^{\eta} = \sum \beta_i e_{ii} + u[23] + v[12] + w[13]$, $\beta_i \in \Phi$, $u, v, w \in \mathfrak{O}$. Then the subset of a_2 such that $n(v)n(w) \neq 0$ is Zariski open so if $f(b_1^{\eta^{-1}}, a_2, \cdots, a_r) = 0$ for $b_1 = \sum \alpha_i e_{ii}$ and all a_2 in this subset, a_j arbitrary if $j \geqq 3$ then $f = 0$. The condition $n(v)n(w) \neq 0$ and the result noted above on automorphism implies that there exists an automorphism σ of $\mathfrak{J}$ such that $b_1^{\eta^{-1}\sigma} = b_1^{\eta^{-1}}$ and $a_2^{\sigma} = b_2^{\eta^{-1}}$ where $b_2 = \sum \beta_i e_{ii} + u'[23] + \mu[12] + \nu[13]$ where $\mu, \nu \in \Phi$ and $u' \in \mathfrak{O}$. It follows that $f = 0$ if $f(b_1^{\eta^{-1}}, b_2^{\eta^{-1}}, a_3, \cdots, a_r) = 0$ for arbitrary $a_j \in \mathfrak{J}$ and b_1 and b_2 of the indicated form.

We shall use this lemma to prove the following important identity for $Q(a) = \frac{1}{2}t(a^{\cdot 2}) = \frac{1}{2}t(a, a)$:

$$(15) \qquad 4[Q(a^{\cdot 2}) - Q(a)^2] = t(a)[2t(a^{\cdot 3}) - t(a)t(a^{\cdot 2}) + 2n(a)].$$

It suffices to prove this for algebraically closed base fields. Since $t(a^{\eta}) = t(a)$ and $n(a^{\eta}) = n(a)$ for any isomorphism η it is sufficient in view of the lemma to verify (15) for every a of the form $\sum_1^3 \alpha_i e_{ii}$ in $\mathfrak{H}(\mathfrak{O}_3, J_1)$. Then $Q(a) = \frac{1}{2}\sum_1^3 \alpha_i^2$, $Q(a^{\cdot 2}) = \frac{1}{2}\sum \alpha_i^4$, $t(a^{\cdot k}) = \sum \alpha_i^k$, $n(a) = \alpha_1\alpha_2\alpha_3$. Then the verification of (15) becomes a schoolboy exercise.

In the sequel we shall be concerned from time to time with subalgebras of a finite-dimensional central simple exceptional Jordan algebra $\mathfrak{J}$ generated by two elements. Such subalgebras are special by the Shirshov-Cohn theorem. We shall now establish the following result on dimensionalities of such subalgebras.

LEMMA 2. *Let $\mathfrak{J}$ be a finite-dimensional central simple exceptional Jordan algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{J}$ generated by 1 and two elements a and b. Then* $\dim \mathfrak{B} \leqq 9$.

PROOF. We may assume the base field is algebraically closed. We suppose first that the minimum polynomial of a has three distinct roots. Then $\Phi[a] = \Phi e_1 \oplus \Phi e_2 \oplus \Phi e_3$ where the e_i are orthogonal idempotents $\neq 0$. Using a suitable coordinatization we can identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{O}_3, J_1)$ in such a way that e_i is identified with e_{ii}. Then we have $b = \sum \beta_i e_i + u[23] + v[31] + w[12]$ and $4b . e_1 . e_2 = w[12] \in \mathfrak{B}$. Similarly, $u[23]$ and $v[31] \in \mathfrak{B}$. Evidently $\mathfrak{B}$ is generated by the e_i and the elements $u[23]$, $v[31]$, $w[12]$. Since for $c, d \in \mathfrak{O}$ and i, j, k distinct we have the multiplication formula $2c[ij].d[jk] = cd[ik]$ we see that $\mathfrak{B}$ contains the elements $uv[21] = v\bar{u}[12]$, $vw[32]$ and $wu[13]$. It is immediate from the formulas (18′)–(22′) of §3.2 that the product of any two of the nine elements e_1, e_2, e_3, $u[23]$, $v[31]$, $w[12]$, $uv[21]$, $vw[32]$, $wu[13]$ is a linear combination of these elements. Hence $\dim \mathfrak{B} \leqq 9$ if $\mu_a(\lambda)$ has three distinct roots. Now let $FJ^{(2)}$ be the free Jordan algebra with 1 generated by two elements x and y. If $a, b \in \mathfrak{J}$ let $\eta(a, b)$ be the homomorphism of $FJ^{(2)}$ such that $1 \to 1$, $x \to a$, $y \to b$. Then if $z_1, z_2, \cdots, z_{10}$ are fixed elements of $FJ^{(2)}$ we can define a mapping of $\mathfrak{J}^{(2)}$ into the exterior algebra $E(\mathfrak{J})$ by $(a, b) \to z_1^{\eta(a,b)} \wedge z_2^{\eta(a,b)} \wedge \cdots \wedge z_{10}^{\eta(a\,b)}$ where $\wedge$ denotes the product in $E(\mathfrak{J})$. This is a polynomial mapping. Moreover, since the subalgebra $\mathfrak{B}$ generated by 1 and a, b such that $\mu_a(\lambda)$ has three distinct roots has dimensionality < 10, it is clear that the mapping we have defined is 0 for all a in an open subset and all b in $\mathfrak{J}$. It follows that the mapping is 0. This implies that if $\mathfrak{B}$ is a subalgebra generated by 1 and any a and b then any ten elements $c_i \in \mathfrak{B}$ are linearly dependent since $c_1 \wedge \cdots \wedge c_{10} = 0$. Hence $\dim \mathfrak{B} \leqq 9$.

The bound given in Lemma 2 is exact. For example, in $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_y)$ take $a = \sum_1^3 \alpha_i e_{ii}$ where the α_i are distinct and let $b = 1[23] + 1[31] + w[12]$ where $w \notin \Phi 1$. Then the subalgebra $\mathfrak{B}$ generated by 1, a and b contains every $1[ij]$, $i,j = 1,2,3$ and $w[12]$. Hence, by Theorem 3.2, $\mathfrak{B} = \mathfrak{H}(\mathfrak{D}_3, J_y)$ where $\mathfrak{D}$ is the subalgebra of $(\mathfrak{O}, j)$ generated by 1 and w. Since $\dim \mathfrak{D} = 2$, $\dim \mathfrak{B} = 9$.

EXERCISES

1. (Seligman). Prove the following identity for $\mathfrak{J}$ ($\mathfrak{J}$ any finite-dimensional central simple exceptional Jordan algebra):

$$4\,t(a \times a, b \times (c \times a)) = t(b, c)n(a) + t(a, b)(a, a, c).$$

2. Let $(u_1, u_2, \cdots, u_{27})$ be a basis for $\mathfrak{J}$, $\xi_1, \xi_2, \cdots, \xi_{27}, \eta_1, \eta_2, \cdots, \eta_{27}$ indeterminates $\mathrm{P} = \Phi(\xi_i, \eta_j)$. Show that the subalgebra of $\mathfrak{J}_{\mathrm{P}}$ generated by 1, $x = \sum \xi_i u_i$, and $y = \sum \eta_i u_i$ is special central simple of degree three and dimension nine.

3. Let $\mathfrak{J}$ be as in 1., Φ infinite and let f be an element of the free Jordan algebra $FJ^{(2)}$ with 1 and two generators x, y which is an identity for $\Phi_3{}^+$, that is, f is mapped into 0 by every homomorphism of $FJ^{(2)}$ into $\Phi_3{}^+$ (cf. §1.9). Show that f is an identity for $\mathfrak{J}$.

4. Let $\mathfrak{J}$ be as in exercise 1, and let $p_k = [x^{\cdot k}, y, y]$ in $FJ^{(2)}$. Show that

$$f = [[p_3, p_1, p_1], p_2, p_2] \neq 0$$

and that f is an identity for $\mathfrak{J}$.

5. Let $\mathfrak{J}$ be a Jordan algebra with 1 such that there exists a positive integer n such that every element of $\mathfrak{J}$ is algebraic of degree $\leqq n$, Show that there exists an element $f \neq 0$ in $FJ^{(2)}$ which is an identity for $\mathfrak{J}$.

6. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ the Jordan algebra of a symmetric bilinear form f on the vector space $\mathfrak{B}$ and let $FJ^{(5)}$ be the free Jordan algebra with 1 and generators x_1, x_2, x_3, x_4, x_5. Show that $[[x_1, x_2, x_3]^{\cdot 2}, x_4, x_5]$ defines a nontrivial identity for $\mathfrak{J}$.

2. **Elements of rank one. Uniqueness of the coefficient algebra.** Let $\mathfrak{J}$ be a finite dimensional central simple exceptional Jordan algebra. An element $a \in \mathfrak{J}$ is said to be of *rank one* if $a \neq 0$ and $a \times a = 0$. Let $\Pi = \Pi(\mathfrak{J})$ be the set of elements of rank one in $\mathfrak{J}$. If $(a, b, c) = \frac{1}{2}\Delta^c{}_a\Delta^b{}_a n$ as in (8) then we have seen in (13) that $(a, b, c) = t(a \times b, c)$. Since the trace form is nondegenerate it is clear that $a \in \Pi$ if and only if $a \neq 0$ and

$$(a, a, b) = 0 \tag{16}$$

for all $b \in \mathfrak{J}$. If a is an element of $\mathfrak{J}$ then a defines a symmetric bilinear form F_a on $\mathfrak{J}$ given by $F_a(u, v) = (a, u, v)$. The radical $\mathfrak{N}_a$ of F_a is the set of z such that $(a, z, v) = 0$ for all v. Hence we see that $a \in \Pi$ if and only if $a \neq 0$ and $a \in \mathfrak{N}_a$. We have $a \times a = a^{\cdot 2} - t(a)a + \frac{1}{2}[t(a)^2 - t(a^{\cdot 2})]1$. Hence if $a \times a = 0$ and $t(a) = 0$ then $a^{\cdot 2} = \frac{1}{2}t(a^{\cdot 2})1$ so $t(a^{\cdot 2}) = \frac{3}{2}\, t(a^{\cdot 2})$ and $t(a^{\cdot 2}) = 0$. Thus $a^{\cdot 2} = 0$. If $a \times a = 0$ and $t(a) = 1$ then $a^{\cdot 2} - a = \frac{1}{2}[t(a^{\cdot 2}) - 1]1$ so $t(a^{\cdot 2}) - 1 = \frac{3}{2}(t(a^{\cdot 2}) - 1) = 0$. Then $a^{\cdot 2} = a$. It follows that $a \in \Pi$ if and only if $a \neq 0$ and $a^{\cdot 2} = 0$ or $a = \alpha e$ where $\alpha \neq 0$ is in Φ and e is a primitive idempotent. It is clear from this that $\Pi = \varnothing$ or $\Pi \neq \varnothing$ according as $\mathfrak{J}$ is a division algebra or $\mathfrak{J}$ is reduced.

Let $\mathfrak{J}'$ be a second finite-dimensional central simple exceptional Jordan algebra and let η be a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$, so η is linear and bijective and $n(a^\eta) = \rho n(a)$ for a fixed nonzero ρ in Φ and all $a \in \mathfrak{J}$. Since η is additive it is clear from (12′) that

$$(a^\eta, b^\eta, c^\eta) = \rho(a, b, c),\ a, b, c \in \mathfrak{J}. \tag{17}$$

Since $(a, b, c) = t(a \times b, c)$ we have $t(a^\eta \times b^\eta, c^\eta) = \rho t(a \times b, c)$. Let η^* denote the adjoint of η relative to the trace forms in $\mathfrak{J}$ and $\mathfrak{J}'$: $t(a^\eta, b') = t(a, b'^{\eta^*})$, $a \in \mathfrak{J}$, $b' \in \mathfrak{J}'$. Also put $\hat{\eta} = (\eta^*)^{-1}$. Then we have $t((a^\eta \times b^\eta)^{\eta^*}, c) = \rho t(a \times b, c)$ and since t is nondegenerate, $(a^\eta \times b^\eta)^{\eta^*} = \rho a \times b$, or

$$a^\eta \times b^\eta = \rho(a \times b)^{\hat{\eta}}. \tag{18}$$

This evidently implies that if $a \in \Pi(\mathfrak{J})$ then $a^\eta \in \Pi(\mathfrak{J}')$. Let F_a be the symmetric bilinear form defined above by a: $F_a(b, c) = (a, b, c)$. Then it is clear from (17) that $F_{a^\eta}(b^\eta, c^\eta) = \rho F_a(b,c)$. Hence η is a similarity with ratio ρ of F_a on $\mathfrak{J}$ with F_{a^η} on $\mathfrak{J}'$.

It follows from the results of §6.7 that if $|\Phi| > 3$ then η is a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ if and only if η is an isotopy, η is an isomorphism if and only if it is a norm similarity and $1^\eta = 1'$ the identity of $\mathfrak{J}'$. Also if η is a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ then η^* is a norm similarity of $\mathfrak{J}'$ onto $\mathfrak{J}$ and $\hat{\eta}$ is one from $\mathfrak{J}$ to $\mathfrak{J}'$. These results can be established also by another argument which is valid also in the extreme case in which $|\Phi| = 3$. We shall indicate this method in the exercises. We remark also that any octonion algebra over a finite field is split since any nondegenerate quadratic form on a vector space of dimensionality > 3 over a finite field has positive Witt index. It follows that if $\mathfrak{J}$ is reduced and has a coordinatization such that the associated octonion algebra is a division algebra then the base field is necessarily infinite. This case will be the one of primary interest in this section.

We recall that the norm similarities of $\mathfrak{J}$ form a group $\mathrm{M}(\mathfrak{J})$ which contains the norm preserving group $\mathrm{M}_1(\mathfrak{J})$ consisting of the bijective linear mappings η in $\mathfrak{J}$ such that $n(a^\eta) = n(a)$, $a \in \mathfrak{J}$. Moreover, $\mathrm{M}_1(\mathfrak{J})$ contains the subgroups $\mathrm{M}_1{}^{(1)}(\mathfrak{J})$ and $\mathrm{M}_1{}^{(2)}(\mathfrak{J})$ which are defined to be the sets of mappings in $\mathfrak{J}$ of the form ΠU_{a_i} such that $\Pi n(a_i)^2 = 1$ and $\Pi n(a_i) = 1$ respectively. We now assume that $\mathfrak{J}$ contains subalgebras $\mathfrak{K}$ which are nine-dimensional reduced simple of degree three and nonsplit. It is immediate from the determination of reduced simple Jordan algebras given in §5.6 that such a subalgebra is isomorphic to an algebra $\mathfrak{H}(\mathrm{P}_3, J_\gamma)$ where (P, j) is a quadratic field and conversely. It is clear that if $\mathfrak{J}$ has a coordinatization onto $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $\mathfrak{O}$ is a division algebra then $\mathfrak{J}$ satisfies our condition. If $\mathfrak{K}$ is any subalgebra of $\mathfrak{J}$ we let $\mathrm{M}_1{}^{(1)}{}_{\mathfrak{J}}(\mathfrak{K})$ and $\mathrm{M}_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{K})$ be the subgroups of $\mathrm{M}_1(\mathfrak{J})$ of elements of the form ΠU_{a_i} where the $a_i \in \mathfrak{K}$ and $\Pi n(a_i)^2 = 1$ and $\Pi n(a_i) = 1$ respectively. Also if $\mathfrak{J}$ contains nonsplit nine-dimensional reduced simple subalgebras $\mathfrak{K}$ of degree three then we let $\mathrm{M}_1{}^{(3)}(\mathfrak{J})$ be the subgroup of $\mathrm{M}_1(\mathfrak{J})$ generated by all the $\mathrm{M}_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{K})$ for $\mathfrak{K}$ of the type specified.

Let $\mathfrak{K}$ be as indicated. Then $\mathfrak{K}$ contains a reducing set of three idempotents e_i. We must have $\sum_1{}^3 e_i = 1$, the identity of $\mathfrak{J}$ so $\mathfrak{K}$ contains 1. Also $\mathfrak{K}$ contains an element u_{1j} in the Peirce space $\mathfrak{K}_{1j}$ (hence in $\mathfrak{J}_{1j}$), $j = 2, 3$, such that $u_{1j}{}^{\cdot 2} = \gamma^{-1}(e_1 + e_j)$. Then we have a coordinatization η of $\mathfrak{J}$ adapted to the e_i and u_{1j} and it is clear that $\mathfrak{K}^\eta$ is a subalgebra of the form $\mathfrak{H}(\mathrm{P}_3, J_\gamma)$ of $\mathfrak{J}^\eta = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where (P, j) is a quadratic subfield of $(\mathfrak{O}, j)$. We now consider the group acting in $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ corresponding to $\mathrm{M}_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{K})$. This is the set of mappings in $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ of the form $U_{A_1} U_{A_2} \cdots U_{A_r}$ where the $A_i \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ and $\Pi_1^r\, n(A_i) = 1$, or, since $n(A_i)$ coincides with the usual determinant, $\det(A_1 A_2 \cdots A_r) = 1$. Then we have

LEMMA 1. *Let (P, j) be a quadratic subfield of the octonion algebra $(\mathfrak{O}, j)$. Let $B \in \mathrm{P}_3$ and $X \in \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and put $B^* = B^{J_\gamma}$. Then $(B^*X)B = B^*(XB)$ and $T_B\colon X \to B^*XB \equiv B^*(XB)$ is a mapping of $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ into itself. Moreover, the* *set of* T_B *with* $\det B = 1$ *coincides with the group of mappings of the form* $U_{A_1}U_{A_2}\cdots U_{A_r}$ *where the* $A_i \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ *and* $\Pi n(A_i) = 1$ *and this group is generated by the* $T_{Q_{ij}}$ *where* $Q_{ij} = 1 + qe_{ij}$, $q \in \mathrm{P}$, $i \neq j$. *Finally,* $T_B{}^*$, *the adjoint of* T_B *relative to the trace form in* $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, *coincides with* T_{B^*}. *Hence the group of these elements with* $\det B = 1$ *is closed under the adjoint mapping.*

Proof. Since P is a quadratic subfield of $\mathfrak{O}$ we have $[\rho_1, \rho_2, x] = [\rho_1, x, \rho_2] = [x, \rho_1, \rho_2] = 0$ if $x \in \mathfrak{O}$ and $\rho_i \in \mathrm{P}$ by the alternative identities. It follows that if $X \in \mathfrak{O}_3$ and $B_1, B_2 \in \mathrm{P}_3$ then $[B_1, B_2, X] = [B_1, X, B_2] = [X, B_1, B_2] = 0$. If $B \in \mathrm{P}_3$ then $B^* = B^{J_\gamma} \in \mathrm{P}_3$ so $B^*XB = B^*(XB) = (B^*X)B$. It follows that $T_B\colon X \to B^*XB$ maps $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ into itself and $T_{B_1B_2} = T_{B_1}T_{B_2}$ if $B_i \in \mathrm{P}_3$. If $A \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ then the indicated associativities imply that $XU_A = 2X.A.A - X.A^{\cdot 2} = AXA$. Hence, if $A_i \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ then $XU_{A_1}U_{A_2}\cdots U_{A_r} = A_rA_{r-1}\cdots A_1XA_1A_2\cdots A_r = B^*XB$ where $B = A_1A_2\cdots A_r \in \mathrm{P}_3$. Thus if $\Pi_1^r n(A_i) = 1$ then $\det B = 1$ and $U_{A_1}U_{A_2}\cdots U_{A_r} = T_B$. We note next that $T_{B_1}T_{B_2} = T_{B_1B_2}$ and the fact that the unimodular group $SL(\mathrm{P}, 3)$ is generated by the elementary matrices $Q_{ij} = 1 + qe_{ij}$, $q \in \mathrm{P}$, $i \neq j$, imply that every T_B, $\det B = 1$, is a product of $T_{Q_{ij}}$. Thus to show that every T_B, $\det B = 1$, has the form $U_{A_1}U_{A_2}\cdots U_{A_r}$ where the $A_l \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ and $\Pi n(A_i) = 1$ it is enough to show that every Q_{ij} is a product of $A_l \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$. Let $k \neq i, j$. Then $(1 + qe_{ij})(\bar{q}^{-1}e_{ij} + q^{-1}e_{ji} + e_{kk}) = \bar{q}^{-1}e_{ij} + q^{-1}e_{ji} + e_{ii} + e_{kk}$ and $C_1 = \bar{q}^{-1}e_{ij} + q^{-1}e_{ji} + e_{kk}$ and $C_2 = \bar{q}^{-1}e_{ij} + q^{-1}e_{ji} + e_{ii} + e_{kk} \in \mathfrak{H}(\mathrm{P}_3, J_1)$. Then $A_l = C_l\gamma \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ and $Q_{ij} = A_2A_1{}^{-1}$ is a product of elements of $\mathfrak{H}(\mathrm{P}_3, J_\gamma)$. We note next that if $A_i \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ then $U_{A_i}{}^* = U_{A_i}$ since R_A is selfadjoint for every $A \in \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ (by the associativity of the trace form). If $B = A_1A_2\cdots A_r$ then $T_B = U_{A_1}U_{A_2}\cdots U_{A_r}$ and $T_B{}^* = U_{A_r}U_{A_{r-1}}\cdots U_{A_1} = T_{A_r\ldots A_1} = T_{B^*}$. This completes the proof.

We have denoted the set of elements of rank one of the reduced exceptional Jordan algebra $\mathfrak{J}$ by Π. If $a \in \Pi$ we write $[a]$ for the ray Φ^*a of multiples αa, $\alpha \neq 0$ in Φ. Also let $[\Pi]$ be the collection of these rays. We have seen that the group $\mathrm{M}(\mathfrak{J})$ maps Π into itself. Hence this maps $[\Pi]$ into itself and the same is true of the varioussubgroups of $\mathrm{M}(\mathfrak{J})$ we have defined. We now prove

Lemma 2. *Let* $\mathfrak{J}$ *be a reduced simple exceptional Jordan algebra which has a coordinatization* η *onto an algebra* $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ *such that* $\mathfrak{O}$ *is an octonion division algebra. Let* $\mathrm{M}_1{}^{(3)}(\mathfrak{J})$ *be the subgroup of* $\mathrm{M}(\mathfrak{J})$ *generated by the elements of the form* $\zeta = \Pi U_{a_i}$ *where the elements* a_i *for a given* ζ *are all contained in the same nonsplit nine-dimensional reduced simple subalgebra of degree three, and* $\Pi n(a_i) = 1$. *Then* $\mathrm{M}_1^{(3)}(\mathfrak{J})$ *is transitive on* $[\Pi]$.

Proof. We may identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$. Since $\mathfrak{O}$ is a division algebra any $q \in \mathfrak{O}$ is contained in a quadratic subfield. Hence it is clear from Lemma 1 that the subgroup of $\mathrm{M}(\mathfrak{J})$ generated by the elements $T_{Q_{ij}}$ where $Q_{ij} = 1 + qe_{ij}$, $i \neq j$, $q \in \mathfrak{O}$, is a subgroup of $\mathrm{M}_1{}^{(3)}(\mathfrak{J})$. Hence it is sufficient to show that this subgroup is transitive on $[\Pi]$. If $A \in \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ is given as in (1), so $A = \sum_1^3 \alpha_i e_{ii} + a[23] + b[31] + c[12]$, then it is immediate from the formula (6) for $A \times A$ that if $A \times A = 0$ and $\alpha_1 = 0$ then $n(b) = n(c) = 0$ so $b = c = 0$. Also if $b = 0$ then either $\alpha_1 = 0$ or $\alpha_3 = 0$. It follows from this and similar considerations on the other entries of A that if $A \in \Pi$ then either A is a multiple of an element of one of the following forms:

$$(19)\qquad \begin{gathered} e_{11} + \gamma_2^{-1}\gamma_1 n(q) e_{22} + q[12], \qquad e_{22} + \gamma_3^{-1}\gamma_2 n(q) e_{33} + q[23], \\ e_{33} + \gamma_1^{-1}\gamma_3 n(q) e_{11} + q[31], \qquad q \in \mathfrak{O}, \end{gathered}$$

or every entry of the matrix A is nonzero. A direct calculation shows that the effect of $T_{Q_{ij}}$ is given by the following table:

$$(20)\qquad \begin{aligned} e_{ii}T_{Q_{ij}} &= e_{ii} + \gamma_j^{-1}\gamma_i n(q) e_{jj} + q[ij],\ e_{jj}T_{Q_{ij}} = e_{jj}, \quad e_{kk}T_{Q_{ij}} = e_{kk}, \\ x[ij]T_{Q_{ij}} &= x[ij] + \gamma_j^{-1}\gamma_i t(\bar{q}x) e_{jj}, \qquad x[jk]T_{Q_{ij}} = x[jk], \\ x[ik\ T_{Q_{ij}} &= x[ik] + (\bar{q}x)[jk], \qquad i,j,k \neq. \end{aligned}$$

The first of these shows that the orbit of e_{ii} under $\mathrm{M}_1^{(3)}(\mathfrak{J})$ contains every element of the form $e_{ii} + \gamma_j^{-1}\gamma_i n(q) e_{jj} + q[ij]$, $q \in \mathfrak{O}$. We observe next that Π contains elements all of whose entries are nonzero. For example, $(\sum_{i,j} e_{ij})\gamma$ is such an element. If A is as in (1), then (20) shows that the component in $\mathfrak{J}_{ij}$ (in the Peirce decomposition of $\mathfrak{J}$ relative to the e_{ii}) of the element $AT_{Q_{ij}}$ is $x[ij] + \alpha_{ii} q[ij]$ where $x[ij]$ is the component of A in $\mathfrak{J}_{ij}$. If $\alpha_{ii} \neq 0$, q can be chosen so that this is 0. Hence if $A \in \Pi$ has all of its entries $\neq 0$ then for every $i \neq j$ the orbit of A under $\mathrm{M}_1^{(3)}(\mathfrak{J})$ contains either an element of the form $\rho(e_{jj} + \gamma_k^{-1}\gamma_j n(q) e_{kk} + q[jk])$ or of the form $\rho(e_{ii} + \gamma_k^{-1}\gamma_i n(q) e_{kk} + q[ik])$ where $\rho \neq 0$ is in Φ. It follows that the orbit of $[A]$ contains $[e_i]$, $i = 1,2,3$. Thus the orbit of $[e_i]$ contains every $[A]$ having only nonzero entries. Clearly our results imply that the orbit of $[e_i]$ is $[\Pi]$ so we have the transitivity of $\mathrm{M}_1^{(3)}(\mathfrak{J})$ on $[\Pi]$.

We have seen above that if η is a norm similarity in $\mathfrak{J}$ then the symmetric bilinear forms F_a and F_{a^η} are similar for any $a \in \mathfrak{J}$. Hence the following is an immediate consequence of Lemma 2.

LEMMA 3. *If $\mathfrak{J}$ is as in Lemma 2 then the symmetric bilinear forms F_a, F_b determined by any two elements $a, b \in \Pi$ are similar.*

We shall now determine the form of F_e for a primitive idempotent e. Let $\mathfrak{J} = \mathfrak{J}_0(e) + \mathfrak{J}_1(e) + \mathfrak{J}_{\frac{1}{2}}(e)$ be the Peirce decomposition of $\mathfrak{J}$ relative e. Also put $f = 1 - e$ and let $f = e_2 + e_3$ where the e_i are orthogonal idempotents. Then $E = \{e_1 = e, e_2, e_3\}$ is a reducing set of idempotents for $\mathfrak{J}$. Let $\mathfrak{J} = \sum \mathfrak{J}_{ij}$ be the Peirce decomposition relative to the e_i. Then $\mathfrak{J}_1(e) = \mathfrak{J}_{11} = \Phi e_1, \mathfrak{J}_{\frac{1}{2}}(e) = \mathfrak{J}_{12} + \mathfrak{J}_{13}$, $\mathfrak{J}_0(e) = \Phi e_2 + \Phi e_3 + \mathfrak{J}_{23}$ and $\dim \mathfrak{J}_{\frac{1}{2}} = 16$, $\dim \mathfrak{J}_0(e) = 10$. We have $e \times e = 0$

and if $a \in \mathfrak{J}_{\frac{1}{2}}$ then $a \times e = a . e - \frac{1}{2}t(a)e - \frac{1}{2}t(e)a + \frac{1}{2}[t(a)t(e) - t(a . e)]1 = 0$ since $a . e = \frac{1}{2}a$, $t(e) = 1$ and $t(a) = 0$. Since $F_e(u,v) = (e,u,v) = t(e \times u, v)$ the radical $\mathfrak{R}_e$ of F_e is the set of u such that $e \times u = 0$ and we see that $\mathfrak{R}_e \supseteq \Phi e + \mathfrak{J}_{\frac{1}{2}}$. Now let $b \in \mathfrak{J}_0(e)$ so we can write $b = \alpha e_2 + \beta e_3 + c$ where $\alpha, \beta \in \Phi$ and $c \in \mathfrak{J}_{23}$. Then we have $e \times b = \frac{1}{2}(\beta e_2 + \alpha e_3 - c)$ and so

$$\begin{aligned} F_e(b,b) &= t(e \times b, b) = \tfrac{1}{2}t(\beta e_2 + \alpha e_3 - c, \alpha e_2 + \beta e_3 + c) \\ &= \alpha\beta - \tfrac{1}{2}t(c^{\cdot 2}) = \alpha\beta - Q(c). \end{aligned} \tag{21}$$

Hence $\mathfrak{J}_0$ is nonisotropic relative to F_e and consequently $\mathfrak{R}_e = \mathfrak{J}_1(e) + \mathfrak{J}_{\frac{1}{2}}(e)$ and $\mathfrak{J}/\mathfrak{R}_e \cong \mathfrak{J}_0(e)$ where the isomorphism is one of the vector spaces relative to F_e and the induced form in $\mathfrak{J}/\mathfrak{R}_e$. The restriction of F_e to $\Phi e_2 + \Phi e_3$ has positive Witt index so this space is a hyperbolic plane. Since any two hyperbolic planes are isomorphic it follows from Witt's theorem (Jacobson, *Lectures*, vol. II, p. 162) that the equivalence class of the restriction of Q to $\mathfrak{J}_{23}$ is determined by that of F_e on $\mathfrak{J}$. Thus the equivalence class of Q on $\mathfrak{J}_{23}$ is independent of the choice of the orthogonal idempotents e_2 and e_3 such that $e_2 + e_3 = f = 1 - e$. Now let $u_{1j} \in \mathfrak{J}_{1j}$ so that $u_{1j}^{\cdot 2} = \gamma_j^{-1}(e_1 + e_j)$, $j = 2, 3$, and let η be a coordinatization of $\mathfrak{J}$ adapted to the e_i and u_{1j}. Let $\mathfrak{J}^\eta = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and let $a \in \mathfrak{O}$; then $a[23]^{\cdot 2} = \gamma_2^{-1}\gamma_3 n(a)(e_{22} + e_{33})$ and $Q(a[23]) = \gamma_2^{-1}\gamma_3 n(a)$. It follows that we have a similarity with ratio $\gamma_2^{-1}\gamma_3$ of the restriction of Q to $\mathfrak{J}_{23}$ with the norm form on $\mathfrak{O}$. We can now prove the following

LEMMA 4. *Let $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{H}(\mathfrak{O}'_3, J_{\gamma'})$ be exceptional Jordan matrix algebras, $\mathfrak{O}, \mathfrak{O}'$ octonion algebras. Assume that there exists a norm similarity of $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ onto $\mathfrak{H}(\mathfrak{O}'_3, J_{\gamma'})$ which maps the element e_{11} of $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ onto e_{11} of $\mathfrak{H}(\mathfrak{O}'_3, J_{\gamma'})$. Then the norm forms of $\mathfrak{O}$ and $\mathfrak{O}'$ are similar.*

PROOF. The hypothesis implies that there exists a similarity of the forms Q on the Peirce (2,3)-spaces of $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and $\mathfrak{H}(\mathfrak{O}_3', J_{\gamma'})$ relative to the diagonal idempotents. Then the result noted just before the statement of the lemma implies the norm similarity of $\mathfrak{O}$ and $\mathfrak{O}'$.

We now consider the norm forms of composition algebras and we prove the following

LEMMA 5. *Let $(\mathfrak{D}_i, j)$, $i = 1, 2$, be a composition algebra, $(\mathfrak{B}_i, j)$ a composition subalgebra (= nonisotropic subalgebra containing 1). Assume the norm forms of $(\mathfrak{D}_1, j)$ and $(\mathfrak{D}_2, j)$ are similar and that σ is an isomorphism of $(\mathfrak{B}_1, j)$ onto $(\mathfrak{B}_2, j)$. Then σ can be extended to an isomorphism of $(\mathfrak{D}_1, j)$ onto $(\mathfrak{D}_2, j)$.*

PROOF. Our first assumption is that there exists a bijective linear mapping η of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that for a $\rho \neq 0$ in Φ we have $n_2(a_1^\eta) = \rho n_1(a_1)$, $a_1 \in \mathfrak{D}_1$, n_i the norm form on $\mathfrak{D}_i$. Then $n_2(1^\eta) = \rho$ if 1 is the identity of $\mathfrak{D}_1$. Then 1^η is invertible in $\mathfrak{D}_2$. Let ζ be defined by $a_1^\zeta = a_1^\eta(1^\eta)^{-1}$. Then $n_2(a_1^\zeta) = n_2(a_1^\eta)n_2(1^\eta)^{-1} = n_1(a_1)$.

Hence n_2 and n_1 are equivalent, that is, we may assume $\rho = 1$. Now let σ be an isomorphism of $(\mathfrak{B}_1, j)$ onto $(\mathfrak{B}_2, j)$. Then $n_2(b_1^\sigma) = n_1(b_1)$, $b_1 \in \mathfrak{B}_1$. Hence by Witt's theorem, there exists a bijective linear mapping τ of $\mathfrak{B}_1{}^\perp$ onto $\mathfrak{B}_2{}^\perp$ such that $n_2(c_1^\tau) = n_1(c_1)$, $c_1 \in \mathfrak{B}_1{}^\perp$. This implies that we can choose $q_i \in \mathfrak{B}_i{}^\perp$ such that $q_i^2 = \mu 1 \neq 0$. The proof of the structure theorem on composition algebras (Theorem 4.5, p. 164) shows that $\mathfrak{C}_i = (\mathfrak{B}_i + \mathfrak{B}_i q_i, j)$ is a composition subalgebra of $(\mathfrak{D}_i, j)$ and $a_1 + b_1 q_1 \to a_1{}^\sigma + b_1{}^\sigma q_2$, $a_1, b_1 \in \mathfrak{B}_1$, is an isomorphism of $(\mathfrak{C}_1, j)$ onto $(\mathfrak{C}_2, j)$ which extends σ. Continuing in this way we obtain the result.

If we apply Lemma 5 to $\mathfrak{B}_1 = \mathfrak{B}_2 = \Phi$ we see that if two composition algebras have similar norm forms then the algebras are isomorphic.

We are now ready to prove the uniqueness (up to isomorphism) of the composition algebras specified in the structure theorem for reduced simple exceptional Jordan algebras. The result is the following

THEOREM 1 (ALBERT-JACOBSON [1]). *Let $\mathfrak{J}$ be a reduced simple exceptional Jordan algebra, η_1 and η_2 coordinatizations of $\mathfrak{J}$ as $\mathfrak{H}(\mathfrak{O}, J_\gamma)$ and $\mathfrak{H}(\mathfrak{O}', J_{\gamma'})$, respectively, where $\mathfrak{O}$ and $\mathfrak{O}'$ are octonion algebras. Then $(\mathfrak{O}, j)$ and $(\mathfrak{O}', j)$ are isomorphic.*

PROOF. If the algebras $(\mathfrak{O}, j)$ and $(\mathfrak{O}', j)$ are split then they are isomorphic. Hence we may assume that $(\mathfrak{O}, j)$ is a division algebra. Let e_j, $j = 1, 2, 3$, be elements of $\mathfrak{J}$ such that $e_j{}^{\eta_1} = e_{jj}$ and let $\mathfrak{J} = \sum_{j \leqq k} \mathfrak{J}_{jk}$ be the Peirce decomposition relative to the orthogonal idempotents e_j. Similarly, let g_j, $j = 1, 2, 3$ be elements of $\mathfrak{J}$ such that $g_j{}^{\eta_2} = e_{jj}$ (in $\mathfrak{H}(\mathfrak{O}'_3, J_{\gamma'})$), $\mathfrak{J} = \sum \mathfrak{J}'_{jk}$ the corresponding Peirce decomposition. Since $\mathfrak{O}$ is a division algebra and $\mathrm{M}(\mathfrak{J})$ contains all scalar multiplications by nonzero elements of Φ it is clear from Lemma 2 that there exists a norm similarity in $\mathfrak{J}$ mapping e_1 onto g_1. Hence there exists a norm similarity of $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ onto $\mathfrak{H}(\mathfrak{O}'_3, J_{\gamma'})$ mapping e_{11} onto e_{11}. By Lemma 4, $\mathfrak{O}$ and $\mathfrak{O}'$ are norm similar. Hence $(\mathfrak{O}, j)$ and $(\mathfrak{O}', j)$ are isomorphic by Lemma 5.

We shall now call the octonion coordinate algebra which is determined up to isomorphism by the reduced exceptional Jordan algebra $\mathfrak{J}$ the *coefficient algebra* of $\mathfrak{J}$. $\mathfrak{J}$ is *split* if its coefficient algebra is split. We shall now show that the problem of classifying reduced simple exceptional Jordan algebras with respect to isotopy (or equivalently norm similarity) is equivalent to that of classifying octonion algebras.

THEOREM 2 (JACOBSON [24]). *Let $\mathfrak{J}$ and $\mathfrak{J}'$ be reduced simple exceptional Jordan algebras. Then the following conditions are equivalent:* (1) *$\mathfrak{J}$ and $\mathfrak{J}'$ are isotopic,* (2) *$\mathfrak{J}$ and $\mathfrak{J}'$ are norm similar,* (3) *$\mathfrak{J}$ and $\mathfrak{J}'$ have isomorphic coefficient algebras.*

PROOF. We may assume the base field infinite since for finite fields all the algebras involved are split and hence are isomorphic. Then, as we have noted before, (1) and (2) are equivalent. Now suppose $\mathfrak{J}$ and $\mathfrak{J}'$ are norm similar. We wish to show that $\mathfrak{J}$ and $\mathfrak{J}'$ have isomorphic coefficient algebras, so we may assume that the coefficient algebra of $\mathfrak{J}'$ is a division algebra. Then, by Lemma 2, $\mathrm{M}(\mathfrak{J}')$ acts transitively on Π. Hence if we have a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ then we also have one which maps a given primitive idempotent of $\mathfrak{J}$ into a primitive idempotent of $\mathfrak{J}'$. Then the proof of Theorem 1 shows that $\mathfrak{J}$ and $\mathfrak{J}'$ have isomorphic coefficient algebras. Thus (2) implies (3). Now assume (3). Then $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{O}_3\ J_\gamma)$ and $\mathfrak{J}' \cong \mathfrak{H}(\mathfrak{O}_3, J_\delta)$. We have seen that $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and $\mathfrak{H}(\mathfrak{O}_3, J_\delta)$ are isotopes (p. 60). Hence $\mathfrak{J}$ and $\mathfrak{J}'$ are isotopic so (3) implies (1).

We shall conclude this section with a useful isomorphism extension theorem for reduced exceptional simple Jordan algebras.

THEOREM 3. *Let $\mathfrak{K}$ and $\mathfrak{K}'$ be reduced simple subalgebras of degree three of a reduced exceptional simple Jordan algebra $\mathfrak{J}$, σ an isomorphism of $\mathfrak{K}$ onto $\mathfrak{K}'$. Then σ can be extended to an automorphism in $\mathfrak{J}$.*

PROOF. Let e_i, $i = 1,2,3$, be orthogonal idempotents in $\mathfrak{K}$, u_{12}, u_{13} elements of the Peirce spaces $\mathfrak{K}_{12}$, $\mathfrak{K}_{13}$ of $\mathfrak{K}$ relative to the e_i such that $u_{12}^{\cdot 2} \neq 0$, $u_{13}^{\cdot 2} \neq 0$. Let η be a coordinatization of $\mathfrak{J}$ adapted to the e_i, u_{12}, u_{13}, and ζ a coordinatizaton of $\mathfrak{J}$ adapted to the elements $e_i^\sigma, u_{12}^\sigma, u_{13}^\sigma$. Then $\mathfrak{J}^\eta = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{J}^\zeta = \mathfrak{H}(\mathfrak{O}_3', J_{\gamma'})$ where $(\mathfrak{O},j)$, $(\mathfrak{O}',j)$ are isomorphic octonion algebras. By Theorem 3.2, $\mathfrak{K}^\eta = \mathfrak{H}(\mathfrak{B}_3, J_\gamma)$, $\mathfrak{K}'^\zeta = \mathfrak{H}(\mathfrak{B}_3', J_{\gamma'})$ where $\mathfrak{B}$ and $\mathfrak{B}'$ are composition subalgebras of $(\mathfrak{O},j)$ and $(\mathfrak{O}',j)$ respectively. Now $\eta^{-1}\sigma\zeta \mid \mathfrak{H}(\mathfrak{B}_3, J_\gamma)$ is an isomorphism of $\mathfrak{H}(\mathfrak{B}_3, J_\gamma)$ onto $\mathfrak{H}(\mathfrak{B}_3', J_{\gamma'})$ such that $1[ij] \to 1\{ij\}$ (defined in $\mathfrak{H}(\mathfrak{B}_3, J_\gamma)$ and $\mathfrak{H}(\mathfrak{B}_3', J_{\gamma'})$ respectively). By Theorem 3.3 this has the form $\sum b_{ij}[ij] \to \sum b_{ij}^\beta\{ij\}$ where β is an isomorphism of $(\mathfrak{B},j)$ onto $(\mathfrak{B}',j)$. By Lemma 5, β can be extended to an isomorphism β of $(\mathfrak{O},j)$ onto $(\mathfrak{O}',j)$. Then $\sum a_{ij}[ij] \to \sum a_{ij}^\beta\{ij\}$ is an isomorphism τ of $\mathfrak{J}^\eta$ onto $\mathfrak{J}'^\zeta$ and $\eta\tau\zeta^{-1}$ is an automorphism of $\mathfrak{J}$ extending the given isomorphism of $\mathfrak{K}$ onto $\mathfrak{K}'$.

EXERCISES

1. Let $\mathfrak{J}$ and $\mathfrak{J}'$ be reduced simple exceptional Jordan algebras η a norm similarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ such that $1^\eta = 1$ (the identity of $\mathfrak{J}'$). Note that $n((1 + a)^\eta) = n(1 + a)$ and $n((1 - a)^\eta) = n(1 - a)$ imply that $t(a^\eta) = t(a)$ and $t((a^{\cdot 2})^\eta) = t(a^{\cdot 2})$. Note also that $a \times 1 = \frac{1}{2}t(a)1 - \frac{1}{2}a$, $1 \times 1 = 1$. Show that these relations and (18) imply that $\hat{\eta} = \eta$ so $(a \times b)^\eta = a^\eta \times b^\eta$. Hence show that η is an isomorphism.

2. Show that a linear mapping of $\mathfrak{J}$ into $\mathfrak{J}'$ is a norm similarity if and only if it is an isotopy. Prove that η^* is a norm similarity.

3. **Aut** $\mathfrak{J}/\Phi e$. Let e be a primitive idempotent in a reduced exceptional simple Jordan algebra $\mathfrak{J}$ and let Aut $\mathfrak{J}/\Phi e$ denote the subgroup of the group of automorphisms Aut $\mathfrak{J}$ leaving fixed the element e. In this section we shall determine Aut $\mathfrak{J}/\Phi e$ and a larger subgroup of $\mathrm{M}(\mathfrak{J})$ which we shall denote as $\mathrm{M}(\mathfrak{J}, e)$ whose

definition is given below. The results we shall give can be formulated best in terms of certain Clifford groups and spin groups. We shall therefore begin our discussion with a survey of the concepts and results on Clifford algebras and groups which will be needed. A full account of these results can be found in Chevalley's *The Algebraic Theory of Spinors*, Chapter II. The results we shall derive here, particularly, those on $M(\mathfrak{J}, e)$, will be used in the next section to derive necessary and sufficient conditions for the isomorphism of reduced simple exceptional Jordan algebras.

Let $\mathfrak{B}$ be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form f. Let $\mathfrak{J} = \Phi 1 \oplus \mathfrak{B}$ be the associated Jordan algebra, $C(\mathfrak{B},f)$ the Clifford algebra, which we consider as containing $\mathfrak{J}$ (as a subalgebra of $C(\mathfrak{B},f)^+$). We recall the definition of the *even* (or *second*) Clifford algebra $C^e(\mathfrak{B},f)$ as the subalgebra of $C(\mathfrak{B},f)$ generated by the elements uv, $u, v \in \mathfrak{B}$. If $\dim \mathfrak{B} = n$ then $\dim C(\mathfrak{B},f) = 2^n$ and $\dim C^e(\mathfrak{B},f) = 2^{n-1}$. We have described the structure of $C(\mathfrak{B},f)$ in Theorem 7.2. In a similar manner one can determine the structure of $C^e(\mathfrak{B},f)$. One can show that if n is odd then this algebra is central simple while if n is even then either $C^e(\mathfrak{B},f)$ is simple with center a quadratic field, or it is a direct sum of two simple ideals.

We recall that the *Clifford group* $\Gamma(\mathfrak{B},f)$ is defined to be the subgroup of the group of invertible elements $c \in C(\mathfrak{B},f)$ such that $c^{-1}\mathfrak{J}c \subseteq \mathfrak{J}$. If $c \in \Gamma(\mathfrak{B},f)$ then the mapping $\chi(c)\colon x \to c^{-1}xc$, $x \in \mathfrak{B}$, is contained in the orthogonal group $O(\mathfrak{B},f)$ of the vector space $\mathfrak{B}$ relative to f. This is clear since $x^2 = f(x)1$ $(f(x) = f(x,x))$ and $(c^{-1}xc)^2 = f(x)1$. It is clear also that $c \to \chi(c)$ is a representation of $\Gamma(\mathfrak{B},f)$ acting in $\mathfrak{B}$. This is called the *vector representation* χ of the Clifford group. If $v \in \mathfrak{B}$ and $f(v) \neq 0$ then $v^{\cdot 2} = v^2 = f(v)1$ so v is invertible in the Jordan algebra $\mathfrak{J}$ and in the associative algebra $C(\mathfrak{B},f)$. If $x \in \mathfrak{B}$ we have $v \,.\, x = \frac{1}{2}(vx + xv) = f(x,v)1$ so $v^{-1}xv = -x + 2f(x,v)v^{-1} = -x + 2f(x,v)f(v)^{-1}v$. Thus $v \in \Gamma(\mathfrak{B},f)$ and

$$\chi(v) = -S_v \tag{22}$$

where S_v is the symmetry determined by v, that is, the reflection in the hyperplane orthogonal to v:

$$S_v\colon x \to x - \frac{2f(x,v)}{f(v,v)}v. \tag{23}$$

It is now clear that $\Gamma(\mathfrak{B},f)$ contains all the elements $v_1 v_2 \cdots v_r$ where $v_i \in \mathfrak{B}$ and $f(v_i) \neq 0$. We recall also that the operator $U_v = 2R_v{}^2 - R_{v^{\cdot 2}}$ in $\Gamma(\mathfrak{B},f)^+$ has the form $y \to vyv$. Comparison of this with $x \to v^{-1}xv$ shows that

$$\chi(v) = f(v)^{-1}\bar{U}_v \tag{24}$$

where $\bar{U}_v = U_v \,|\, \mathfrak{B}$, the restriction of U_v to $\mathfrak{B}$. It is clear from (24) that if $f(v_i) \neq 0$ then

$$\chi(v_1 v_2 \cdots v_r) = (\Pi f(v_i))^{-1}\bar{U}_{v_1}\bar{U}_{v_2}\cdots \bar{U}_{v}\,. \tag{25}$$

We define the *even* (or *second*) *Clifford group* $\Gamma^e(\mathfrak{V},f)$ to be $\Gamma(\mathfrak{V},f) \cap C^e(\mathfrak{V},f)$. It is clear that $\Gamma^e = \Gamma^e(\mathfrak{V},f)$ contains all the elements $v_1 v_2 \cdots v_{2r}$ where $v_i \in \mathfrak{V}$ and $f(v_i) \neq 0$. By (22), $\chi(v_1 v_2 \cdots v_{2r}) = S_{v_1} S_{v_2} \cdots S_{v_{2r}}$. By the Cartan-Dieudonné Theorem every rotation is a product of an even number of symmetries. Hence the image $\chi(\Gamma^e)$ contains the rotation group. If $\dim \mathfrak{V}$ is even $C(\mathfrak{V},f)$ is central simple so $\ker \chi = \Phi^*$, the set of nonzero elements of Φ. If $v \in \mathfrak{V}$ and $f(v) \neq 0$ then $v \in \Gamma = \Gamma(\mathfrak{V},f)$ but $v \notin \Gamma^e$. Hence $\chi(v) \notin \chi(\Gamma^e)$. Thus in this case $O(\mathfrak{V},f) \supseteq \chi(\Gamma) \supset \chi(\Gamma^e) \supseteq O^+(\mathfrak{V},f)$, the rotation group. Since O^+ has index two in O it is clear that $O(\mathfrak{V},f) = \chi(\Gamma)$ and $O^+(\mathfrak{V},f) = \chi(\Gamma^e)$ if $\mathfrak{V}$ is even dimensional. If $\dim \mathfrak{V}$ is odd then $C(\mathfrak{V},f)$ has the center $\Phi 1 + \Phi w$ where $w = u_1 u_2 \cdots u_n$ and $(u_1, u_2, \cdots, u_n)$ is an orthogonal basis for $\mathfrak{V}$. Then $w \notin C^e(\mathfrak{V},f)$. An automorphism η of $C(\mathfrak{V},f)$ is inner if and only if $w^\eta = w$. On the other hand if $A \in O(\mathfrak{V},f)$ then $wA = (\det A)w$, as can be seen directly. Hence if $c \in \Gamma$ then $\chi(c) \in O^+(\mathfrak{V},f)$. Hence in the odd-dimensional case $\chi(\Gamma) = \chi(\Gamma^e) = O^+$. Thus in both cases $\chi(\Gamma^e) = O^+$. Also in both cases $\ker \chi \cap \Gamma^e = \Phi^*$. It follows that Γ^e is the set of elements of the form $v_1 v_2 \cdots v_{2r}$ where $\Pi_1^{2r} f(v_i) \neq 0$.

If $c \in \Gamma$ and $x \in \mathfrak{V}$ then $c^{-1}xc = x\chi(c)$. Applying the main involution π we obtain $c^\pi x (c^\pi)^{-1} = x\chi(c)$. Hence $cc^\pi x = xcc^\pi$ and since $\mathfrak{V}$ generates the Clifford algebra, cc^π is in the center of $C(\mathfrak{V},f)$. If $c \in \Gamma^e$ then $c^\pi \in \Gamma^e$ and cc^π is in the intersection of Γ^e with the center of $C(\mathfrak{V},f)$. Hence $cc^\pi = \lambda(c)1$ where $\lambda(c) \in \Phi^*$. If $c_1, c_2 \in \Gamma^e$ then $(c_1 c_2)(c_1 c_2)^\pi = c_1 c_2 c_2^\pi c_1^\pi = \lambda(c_1)\lambda(c_2)1$. Hence $\lambda(c_1 c_2) = \lambda(c_1)\lambda(c_2)$. The subgroup of Γ^e of c such that $\lambda(c) = 1$ is called the *spin group*, Spin $(\mathfrak{V},f)$, and its image $\chi(\text{Spin}(\mathfrak{V},f))$ in the orthogonal group is called the *reduced orthogonal group* $O'(\mathfrak{V},f)$. If $v \in \mathfrak{V}$ then $vv^\pi = v^2 = f(v)1$. Hence if $c = v_1 v_2 \cdots v_{2r}$, $v_i \in \mathfrak{V}$, $f(v_i) \neq 0$ then $\lambda(c) = \Pi_1^{2r} f(v_i)$. Since we have seen that Γ^e is the set of elements $v_1 v_2 \cdots v_{2r}$, $v_i \in \mathfrak{V}$, $f(v_i) \neq 0$ it is clear that Spin$(\mathfrak{V}, Q)$ is the set of elements $v_1 v_2 \cdots v_{2r}$ such that $\Pi f(v_i) = 1$.

The reduced orthogonal group is a subgroup of the rotation group $O^+(\mathfrak{V},f)$ of $\mathfrak{V}$ relative to f. If A is any rotation then there exists a $c \in \Gamma^e$ which is determined up to a factor in Φ^* such that $\chi(c) = A$. Since $\lambda(\gamma c) = \gamma^2 \lambda(c)$ if $\gamma \in \Phi^*$ it is clear that the coset $\lambda(c)$ modulo squares of elements of Φ^* is determined by A (where $\chi(c) = A$). Let Φ^{*2} denote the subgroup of squares of elements of Φ^*. Then we define the *spinorial norm* $\lambda(A)$ of the rotation A to be $\lambda(A) = \lambda(c)\Phi^{*2}$ where $\chi(c) = A$. If $A = S_{v_1} S_{v_2} \cdots S_{v_{2r}}$ is a representation of A as a product of symmetries then $c = v_1 v_2 \cdots v_{2r} \in \Gamma^e$, $\chi(c) = A$ and $\lambda(c) = \Pi_1^{2r} f(v_i)$. Hence $\lambda(A) = \Pi_1^{2r} f(v_i)\Phi^{*2}$. It is clear that λ is a homomorphism of $O^+(\mathfrak{V},f)$ into Φ^*/Φ^{*2}. If $\lambda(A) = 1$ then we have a $c \in \Gamma^e$ such that $\chi(c) = A$ and $\lambda(c) = \gamma^2$, $\gamma \in \Phi^*$. If we replace c by $\gamma^{-1}c$ then we get $\chi(\gamma^{-1}c) = A$, $\lambda(\gamma^{-1}c) = 1$ so $A \in O'(\mathfrak{V},f)$. The converse is clear. Thus the reduced orthogonal group $O'(\mathfrak{V},f)$ is the kernel of the spinorial norm mapping. It is clear also that $O'(\mathfrak{V},f)$ consists of the rotations $S_{v_1} S_{v_2} \cdots S_{v_{2r}}$ such that $\Pi_1^{2r} f(v_i)$ is a square.

We shall now apply these notions and results to the study of the group

Aut $\mathfrak{J}/\Phi e$ where e is a primitive idempotent in the reduced exceptional simple Jordan algebra $\mathfrak{J}$. Let $\mathfrak{J} = \mathfrak{J}_1(e) + \mathfrak{J}_{\frac{1}{2}}(e) + \mathfrak{J}_0(e)$ be the Peirce decomposition of $\mathfrak{J}$ relative to e. Then $\mathfrak{J}_1(e) = \Phi e$. We shall need to derive some formulas for products of elements in the Peirce spaces $\mathfrak{J}_i$ and for Q on these spaces where $Q(a) = \frac{1}{2}t(a^{\cdot 2})$. If we imbed e in a reducing set of idempotents $e_1 = e, e_2, e_3$ and let $\mathfrak{J} = \sum \mathfrak{J}_{ij}$ be the corresponding Peirce decomposition, then $\mathfrak{J}_1(e) = \Phi e_1$, $\mathfrak{J}_0(e) = \Phi e_2 + \Phi e_3 + \mathfrak{J}_{23}$, $\mathfrak{J}_{\frac{1}{2}}(e) = \mathfrak{J}_{12} + \mathfrak{J}_{13}$. Hence, as we saw before, $\dim \mathfrak{J}_0 = 10$ and $\dim \mathfrak{J}_{\frac{1}{2}} = 16$. We can write $\mathfrak{J}_0 = \Phi f \oplus \mathfrak{V}$ where $f = 1 - e = e_2 + e_3$ is the identity element of $\mathfrak{J}_0$ and $\mathfrak{V} = \Phi(e_2 - e_3) + \mathfrak{J}_{23}$. Thus $\dim \mathfrak{V} = 9$ and $\mathfrak{V}$ is the subspace of $\mathfrak{J}_0$ of elements of generic trace 0. If $v = \alpha(e_2 - e_3) + w$, $\alpha \in \Phi$, $w \in \mathfrak{J}_{23}$, then $v^{\cdot 2} = \alpha^2 f + w^{\cdot 2}$ and $w^{\cdot 2} = \beta f$. Hence $v^{\cdot 2} = \delta f$ where $\delta \in \Phi$. Also $Q(v^{\cdot 2}) = \frac{1}{2}t(v^{\cdot 2}) = \delta$ so $v^{\cdot 2} = Q(v)f$. This gives the formula

$$(26) \qquad v_1 . v_2 = Q(v_1, v_2)f, \qquad v_i \in \mathfrak{V}$$

where $Q(v_1, v_2) = \frac{1}{2}t(v_1, v_2)$. Since the restriction of the trace form to the $\mathfrak{J}_{ij}$ is nondegenerate it is clear that Q is nondegenerate on $\mathfrak{J}_{23}$ and hence on $\mathfrak{V}$. Next let $y \in \mathfrak{J}_{\frac{1}{2}}$. Then $y^{\cdot 2} \in \mathfrak{J}_0 + \mathfrak{J}_1$ so $y^{\cdot 2} = \xi e + \eta f + v$ where $\xi, \eta \in \Phi$ and $v \in \mathfrak{V}$. Since $e . f = 0 = e . v$, $t(y^{\cdot 2} . e) = t(\xi e) = \xi$. Hence, by associativity of t, $t(y, y . e) = \xi$ and $Q(y) = \frac{1}{2}t(y, y) = t(y, y . e) = \xi$. Also $t(y^{\cdot 2} . f) = t(\eta f + v) = 2\eta$ which leads in the same way to $\eta = \frac{1}{2}Q(y)$. Hence $y^{\cdot 2} = \frac{1}{2}Q(y)(1 + e) + v$. We now define $y \circ y = v$, the component of $y^{\cdot 2}$ in $\mathfrak{V}$ in the decomposition $\mathfrak{J}_0 + \mathfrak{J}_1 = \Phi e \oplus \Phi f \oplus \mathfrak{V}$. Then we have the formula

$$(27) \qquad y \circ y = v = y^{\cdot 2} - \tfrac{1}{2}Q(y)(1 + e).$$

If we let $y_1 \circ y_2$ be the linearized form of $y \circ y$, $y_i \in \mathfrak{J}_{\frac{1}{2}}$, so

$$y_1 \circ y_2 = \tfrac{1}{2}[(y_1 + y_2) \circ (y_1 + y_2) - y_1 \circ y_1 - y_2 \circ y_2]$$

then we have

$$(28) \qquad y_1 \circ y_2 = y_1 . y_2 - \tfrac{1}{2}Q(y_1, y_2)(1 + e).$$

We have $t(y) = 0$ for $y \in \mathfrak{J}_{\frac{1}{2}}$ and $y^{\cdot 3} \in \mathfrak{J}_{\frac{1}{2}}$ so $t(y^{\cdot 3}) = 0$. Since $y^{\cdot 3} - t(y)y^{\cdot 2} + \frac{1}{2}[t(y)^2 - t(y^{\cdot 2})]y - n(y)1 = 0$, $y^{\cdot 3} = Q(y)y + n(y)1$, which implies

$$(29) \qquad y^{\cdot 3} = Q(y)y, \ n(y) = 0, \quad y \in \mathfrak{J}_{\frac{1}{2}}.$$

This and (27) give directly

$$(30) \qquad (y \circ y) . y = \tfrac{1}{4}Q(y)y.$$

Let $\mathfrak{J}'$ be the subspace of elements of trace 0. We have $\mathfrak{V} + \mathfrak{J}_{\frac{1}{2}} \subseteq \mathfrak{J}'$. If $a \in \mathfrak{J}'$ then $a^{\cdot 3} = \frac{1}{2}t(a^{\cdot 2})a + n(a)1$. Applying Δ^b to this considered as a functional relation we obtain $2b . a . a + a^{\cdot 2} . b = \frac{1}{2}t(a^{\cdot 2})b + t(a . b)a + (a, a, b)1 = \frac{1}{2}t(a^{\cdot 2})b + t(a,b)a + t(a \times a, b)1$ (by (13)). Since $a \times a = a^{\cdot 2} - \frac{1}{2}t(a^{\cdot 2})1$ this gives for $a, b \in \mathfrak{J}'$

(31) $$2b.a.a + a^{\cdot 2}.b = Q(a)b + t(a,b)a + t(a^{\cdot 2},b)1.$$

Put $a = y \in \mathfrak{J}_{\frac{1}{2}}$, $b = v \in \mathfrak{B}$. Then $t(v,y) = 0$ and $2v.y.y + y^{\cdot 2}.v = Q(y)v + t(y^{\cdot 2},v)1$ By (28) this gives

$$Q(v.y,y)(1+e) + 2(v.y) \circ y + \tfrac{1}{2}Q(y)(1+e).v + (y \circ y).v$$
$$= Q(y)v + t(y^{\cdot 2},v)1.$$

Equating components in $\mathfrak{B}$ in the decomposition $\mathfrak{J} = \Phi e \oplus \Phi f \oplus \mathfrak{B} \oplus \mathfrak{J}_{\frac{1}{2}}$ we obtain

(32) $$(v.y) \circ y = \tfrac{1}{4}Q(y)v, \qquad v \in \mathfrak{B}, \qquad y \in \mathfrak{J}_{\frac{1}{2}}.$$

Next we note that (15) gives

(33) $$Q(a^{\cdot 2}) = Q(a)^2, \qquad a \in \mathfrak{J}'.$$

Applying Δ^b, $b \in \mathfrak{J}'$, gives $2t(a^{\cdot 3},b) = t(a^{\cdot 2})t(a.b)$, or $2t(a^{\cdot 2},a.b) = t(a^{\cdot 2})t(a.b)$ Applying Δ^b again to the corresponding functions of a gives $4t(a.b,a.b) + 2t(a^{\cdot 2},b^{\cdot 2}) = 2t(a,b)^2 + t(a^{\cdot 2})t(b^{\cdot 2})$ or

(34) $$4Q(a.b) + t(a^{\cdot 2},\ b^{\cdot 2}) = t(a,b)^2 + 2Q(a)Q(b),$$

$a, b \in \mathfrak{J}'$. If $b = v \in \mathfrak{B}$ and $a = y \in \mathfrak{J}_{\frac{1}{2}}$ then $t(v,y) = 0$ so this and (26) and (28) imply

(35) $$Q(v.y) = \tfrac{1}{4}Q(v)Q(y), \qquad v \in \mathfrak{B}, \qquad y \in \mathfrak{J}_{\frac{1}{2}}.$$

Since $y \circ y \in \mathfrak{B}$ this and (30) give $\frac{1}{4}Q(y \circ y)Q(y) = \frac{1}{16}Q(y)^2 Q(y)$. Hence

(36) $$Q(y \circ y) = \tfrac{1}{4}Q(y)^2$$

holds if $Q(y) \neq 0$. Since this is a polynomial function relation this is valid for all $y \in \mathfrak{J}_{\frac{1}{2}}$.

We are now ready to consider the group Aut $\mathfrak{J}/\Phi e$ and a larger group $\mathrm{M}(\mathfrak{J},e)$ which we shall define. Let $b \in \mathfrak{J}_0(e)$ and put $\theta(b) = e + b \in \mathfrak{J}_1 + \mathfrak{J}_0$. We consider the mapping $U_{\theta(b)}$ in $\mathfrak{J}$. It is clear that $\mathfrak{J}_i U_{\theta(b)} \subseteq \mathfrak{J}_i$ and $eU_{\theta(b)} = e$. The restriction of $U_{\theta(b)}$ to $\mathfrak{J}_0$ coincides with that of U_b so if $b = f$ this is the identity mapping. If $v \in \mathfrak{B}$ and $Q(v) \neq 0$ then $fU_{\theta(v)} = v^{\cdot 2} = Q(v)f$ and if $x \in \mathfrak{B}$ then $xU_{\theta(v)} = xU_v = -Q(v)xS_v$, by (22) and (24), where S_v is the reflection in $\mathfrak{B}$ in the hyperplane orthogonal to v. Thus

(37) $$fU_{\theta(v)} = Q(v)f,$$
$$xU_{\theta(v)} = -Q(v)xS_v, \qquad v, x \in \mathfrak{B}, \qquad Q(v) \neq 0.$$

To obtain the restriction of $U_{\theta(b)}$ to $\mathfrak{J}_{\frac{1}{2}}$ we note that $U_{\theta(b)} = U_{e+b} = U_e + U_b + 2(R_eR_b + R_bR_e - R_{e.b}) = U_e + U_b + 2(R_eR_b + R_bR_e)$. Since $y.e = \frac{1}{2}y$ if $y \in \mathfrak{J}_{\frac{1}{2}}$ and $y.b^{\cdot 2} = 2y.b.b$ if $b \in \mathfrak{J}_0$ we have

(38) $$yU_{\theta(b)} = 2yR_b = 2y.b, \qquad y \in \mathfrak{J}_{\frac{1}{2}}, \qquad b \in \mathfrak{J}_0.$$

We have $\theta(f)=e+f=1$ so $U_{\theta(f)}=1$. Also if $v\in\mathfrak{V}$ then $yU_{\theta(v)}{}^2=4y.v.v=2y.v^{\cdot 2}=2Q(v)y.f=Q(v)y$. This equation shows that if we let $U_{\theta(v)}'$, $v\in\mathfrak{V}$, denote the restriction of $U_{\theta(v)}$ to $\mathfrak{J}_{\frac{1}{2}}$ then $(U_{\theta(v)}')^2=Q(v)1$ (in $\mathfrak{J}_{\frac{1}{2}}$). This implies that we have a representation ρ of the Clifford algebra $C(\mathfrak{V},f)$ acting in $\mathfrak{J}_{\frac{1}{2}}$ such that $1\to 1$, $v\to U_{\theta(v)}'$. By restriction, we obtain representations of $C^e(\mathfrak{V},Q)$, Γ, Γ^e and $\mathrm{Spin}(\mathfrak{V},Q)$. Since $\dim\mathfrak{V}=9$ and Q is nondegenerate, $C^e(\mathfrak{V},Q)$ is central simple and $\dim C^e(\mathfrak{V},Q)=2^8$. Hence the homomorphism ρ of $C^e(\mathfrak{V},Q)$ into $\mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})$ is a monomorphism. Since $\dim(\mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}}))=16^2=2^8$ it is clear that ρ is an isomorphism and $C^e(\mathfrak{V},Q)^\rho=\mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})$ so ρ is an irreducible representation of $C^e(\mathfrak{V},Q)$. By Chevalley (loc.cit. p. 57), it follows that the space spanned by $\mathrm{Spin}(\mathfrak{V},Q)^\rho$ is $\mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})$ so the representation ρ of $\mathrm{Spin}(\mathfrak{V},Q)$ is also irreducible.

Now let $c\in\Gamma^e(\mathfrak{V},Q)$ the even Clifford group. Then c has the form $v_1v_2\cdots v_{2r}$ where the $v_i\in\mathfrak{V}$ and $Q(v_i)\neq 0$. Then $\lambda(c)=\prod_1^{2r}Q(v_i)$ Put

$$\eta=U_{\theta(v_1)}U_{\theta v(2)}\cdots U_{\theta(v_{2r})}. \tag{39}$$

Then the foregoing formulas show that

$$\begin{aligned} e^\eta &= e, \quad f^\eta=\lambda(c)f,\\ x^\eta &= \lambda(c)xc^\chi, \quad x\in\mathfrak{V},\\ y^\eta &= yc^\rho, \quad y\in\mathfrak{J}_{\frac{1}{2}}.\end{aligned} \tag{40}$$

It is clear from these formulas that η is independent of the representation of c as $c=v_1v_2\cdots v_{2r}$. It is clear also that η is a norm similarity of $\mathfrak{J}$ with multiplier $\lambda(c)^2$. We now write $\eta=\eta(c)$ and we have the homomorphism $\eta\colon c\to\eta(c)$ of $\Gamma^e(\mathfrak{V},Q)$. We denote the subgroup of $\mathrm{M}(\mathfrak{J})$ which is the image $\Gamma^e(\mathfrak{V},Q)^\eta$ as $\mathrm{M}(\mathfrak{J},e)$. Since ρ is a monomorphism of the even Clifford algebra the last equation of (40) shows that η is a monomorphism of $\Gamma^e(\mathfrak{V},Q)$.

Now let $c\in\mathrm{Spin}(\mathfrak{V},Q)$. Then $c=v_1v_2\cdots v_{2r}$ where $\lambda(c)=1$. Then (40) shows that $e^\eta=e$, $f^\eta=f$ so $1^\eta=1$. Hence, by the Corollary to Theorem 6.7 (p. 245), η is an automorphism of $\mathfrak{J}$. Since $e^\eta=e$, $\eta\in\mathrm{Aut}\,\mathfrak{J}/\Phi e$. Hence η is a monomorphism of $\mathrm{Spin}(\mathfrak{V},Q)$ into $\mathrm{Aut}\,\mathfrak{J}/\Phi e$.

If A is a similarity in a vector space equipped with a nondegenerate symmetric bilinear form then we write $\mu(A)$ for the coset $\mu\Phi^{*2}$ of the multiplier μ of A in Φ^*/Φ^{*2}. We shall now prove

LEMMA 1. *If B is a rotation in $\mathfrak{V}$ relative to Q then there exists a similarity A in $\mathfrak{J}_{\frac{1}{2}}$ relative to Q satisfying*

$$(x.y)A=xB.yA, \qquad x\in\mathfrak{V}, \qquad y\in\mathfrak{J}_{\frac{1}{2}}. \tag{41}$$

Moreover, A is determined up to a factor in Φ^* by this condition, $\mu(A)=\lambda(B)$ the spinorial norm of B, and

(42) $$yA \circ yA = \mu(y \circ y)B, \qquad y \in \mathfrak{J}_{\frac{1}{2}},$$

where μ is the multiplier of A.

PROOF. Write $B = S_{v_1}S_{v_2} \cdots S_{v_{2r}}$ where the $v_i \in \mathfrak{V}$ and $Q(v_i) \neq 0$. Put $A = 2^{2r}R'_{v_1}R'_{v_2} \cdots R'_{v_{2r}}$ where R'_{v_i} is the restriction of R_{v_i} to $\mathfrak{J}_{\frac{1}{2}}$. If $y \in \mathfrak{J}_{\frac{1}{2}}$ then, by (35), $Q(y \,.\, v_i) = \frac{1}{4}Q(v_i)Q(y)$. Hence A is a similarity of ratio $\mu = \Pi_1^{2r}\, Q(v_i)$. Since the spinorial norm of B is $\Pi_1^{2r}\, Q(v_i)\Phi^{*2}$, it is clear that $\mu(A) = \lambda(B)$. We claim that (41) and (42) hold. Using a field extension argument it is enough to prove this under the additional assumption that the base field is algebraically closed. Let $v_1{}' = \mu^{-\frac{1}{2}}v_1$. Then $S_{v_1{}'}S_{v_2} \cdots S_{v_{2r}} = B$ and $2^{2r}R'_{v_1{}'}R'_{v_2} \cdots R'_{v_{2r}} = \mu^{-} A$. Put $\eta = U_{\theta(v_1{}')}U_{\theta(v_2)} \cdots U_{\theta(v_{2r})}$. Then, since $Q(v_1{}')Q(v_2) \cdots Q(v_{2r}) = 1$, the result proved above shows that η is an automorphism of $\mathfrak{J}$ fixing e and the formulas show that the restrictions of η to $\mathfrak{V}$ and $\mathfrak{J}_{\frac{1}{2}}$ are B and $\mu^{-\frac{1}{2}}A$ respectively. Then we have $(x \,.\, y)\,(\mu^{-\frac{1}{2}}A) = xB \,.\, y(\mu^{-\frac{1}{2}}A)$ if $x \in \mathfrak{V}$, $y \in \mathfrak{J}_{\frac{1}{2}}$. Hence (41) holds. Also we have $y(\mu^{-\frac{1}{2}}A) \circ y(\mu^{-\frac{1}{2}}A) = (y \circ y)B$. Hence (42) holds. We note next that if $A = A_i$, $B = B_i$, $i = 1, 2$, satisfy (41) then so does $A = A_1A_2$, $B = B_1B_2$ and $A = A_1{}^{-1}$, $B = B_1{}^{-1}$. Hence to prove the uniqueness statement of the lemma it is enough to prove this for $B = 1$. Then $(x \,.\, y)A = x \,.\, yA$ so, by (38) and the definition of the representation ρ, A commutes with every element of $C(\mathfrak{V}, Q)^{\rho}$. Since $C^e(\mathfrak{V}, Q)^{\rho} = \mathrm{Hom}_{\Phi}(\mathfrak{J}_{\frac{1}{2}}, \mathfrak{J}_{\frac{1}{2}})$ it follows that A commutes with every element of $\mathrm{Hom}_{\Phi}(\mathfrak{J}_{\frac{1}{2}}, \mathfrak{J}_{\frac{1}{2}})$. Hence A is a scalar multiplication.

We can now prove

THEOREM 4. *Let $c \in \mathrm{Spin}(\mathfrak{V}, Q)$ and write $c = v_1v_2 \cdots v_{2r}$ where the $v_i \in \mathfrak{V}$, $\Pi Q(v_i) = 1$. Put $\eta(c) = U_{\theta(v_1)}U_{\theta(v_2)} \cdots U_{\theta(v_{2r})}$. Then $\eta \colon c \to \eta(c)$ is an isomorphism of* Spin$(\mathfrak{V}, Q)$ *onto* Aut $\mathfrak{J}/\Phi e$.

PROOF. We have seen that η is a monomorphism of Spin$(\mathfrak{V}, Q)$ into Aut $\mathfrak{J}/\Phi e$. It remains to show that it is an epimorphism. Let $\eta \in \mathrm{Aut}\, \mathfrak{J}/\Phi e$. Then $e^{\eta} = e$, $f^{\eta} = f$, $\mathfrak{V}^{\eta} = \mathfrak{V}$, $\mathfrak{J}_{\frac{1}{2}}{}^{\eta} = \mathfrak{J}_{\frac{1}{2}}$ and the restrictions B and A of η to $\mathfrak{V}$ and $\mathfrak{J}_{\frac{1}{2}}$ respectively are Q-orthogonal. Clearly (41) holds. We now claim that B is a rotation. Since the lemma provides a similarity A in $\mathfrak{J}_{\frac{1}{2}}$ satisfying (41) for any given rotation, it is enough to show that no linear mapping $A \neq 0$ exists in $\mathfrak{J}_{\frac{1}{2}}$ such that $(x \,.\, y)A = (xB) \,.\, (yA)$ for $B = -1$. Then $yR_xA = -\, yAR_x$, $y \in \mathfrak{J}_{\frac{1}{2}}$, $x \in \mathfrak{V}$, and A commutes with every linear transformation $R_{v_1}R_{v_2}$, $v_i \in \mathfrak{V}$. Since $C^e(\mathfrak{V}, Q)^{\rho} = \mathrm{Hom}_{\Phi}(\mathfrak{J}_{\frac{1}{2}}, \mathfrak{J}_{\frac{1}{2}})$ it follows that A is a scalar multiplication. Then $yR_xA = -\, yAR_x$ implies $A = 0$. Hence the restriction B of η to $\mathfrak{V}$ is a rotation. Also the restriction A of η to $\mathfrak{J}_{\frac{1}{2}}$ is Q-orthogonal. Hence, by Lemma 1, $\mu(A) = \lambda(B)$ and since $\mu(A) = 1$, $B \in O'(\mathfrak{V}, Q)$. Let $c \in \mathrm{Spin}(\mathfrak{V}, Q)$ satisfy $c^{\chi} = B$. Then it follows from the definition of η that $\eta(c) = \eta$.

COROLLARY 1. Aut $\mathfrak{J}/\Phi e$ *acts irreducibly in* $\mathfrak{V}$ *and in* $\mathfrak{J}_{\frac{1}{2}}$.

PROOF. The proof of the theorem shows that the action of Aut $\mathfrak{J}/\Phi e$ in $\mathfrak{J}_{\frac{1}{2}}$ is the same as that of $\mathrm{Spin}(\mathfrak{V},Q)^{\rho}$ and we have noted that this is irreducible. The action of Aut $\mathfrak{J}/\Phi e$ in $\mathfrak{V}$ is the same as that of $\mathrm{Spin}(\mathfrak{V},Q)^{\chi} = O'(\mathfrak{V},Q)$. Since $O'(\mathfrak{V},Q)$ contains the commutator subgroup of $O(\mathfrak{V},Q)$ and the latter acts irreducibly in $\mathfrak{V}$ (Chevalley, loc. cit. p. 53 and p. 33), Aut $\mathfrak{J}/\Phi e$ is irreducible in $\mathfrak{V}$.

We recall the definition in §2 of the subgroup $\mathrm{M}_1{}^{(3)}(\mathfrak{J})$ of $\mathrm{M}(\mathfrak{J})$ generated by the elements $\zeta = U_{a_1}U_{a_2}\cdots U_{a_r}$ where for given ζ the a_i are contained in a non-split nine-dimensional reduced simple subalgebra of degree three and $\Pi_1^r\, n(a_i) = 1$. Then we have the

Corollary 2. *If the coefficient algebra of* $\mathfrak{J}$ *is a division algebra then* Aut $\mathfrak{J}/\Phi e \subseteq \mathrm{M}_1{}^{(3)}(\mathfrak{J})$.

Proof. We may identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $\mathfrak{O}$ is an octonion division algebra, and we may assume $e = e_{11}$. Then we have seen that any $\eta \in \mathrm{Aut}\,\mathfrak{J}/\Phi e$ has the form $\eta = U_{\theta(v_1)}U_{\theta(v_2)}\cdots U_{\theta(v_{2r})}$ where $v_i \in \mathfrak{V} = \Phi(e_{22} - e_{33}) + \mathfrak{J}_{23}$ and $\Pi_1^{2r}\, Q(v_i) = 1$. We have $v_i = \beta_i(e_{22} - e_{33}) + w_i[23]$ where $\beta_i \in \Phi$, $w_i \in \mathfrak{O}$. Then $\theta(v_i)$ is contained in a subalgebra $\mathfrak{K}_i = \mathfrak{H}(\mathrm{P}_{i3}, J_\gamma)$ where P_i is a quadratic subfield of $\mathfrak{O}$. Also $n(\theta(v_i)) = -Q(v_i)$. If $\delta \in \Phi$ we put $D(\delta) = e_{11} + e_{22} + \delta e_{33}$ so $n(D(\delta)) = \delta$. Now put $\delta_0 = 1$, $\delta_i = \Pi_{j=1}^{i}(-Q(v_j))^{-1}$ for $1 \leqq i \leqq 2r$ and

$$\eta_i = U_{D(\delta_{i-1})^{-1}}U_{\theta(v_i)}\,U_{D(\delta_i)}, \qquad i = 1, 2, \cdots, 2r. \tag{43}$$

Then, since $\delta_0 = 1 = \delta_{2r}$, $\eta_1\eta_2\cdots\eta_{2r} = \eta$. Moreover, $D(\delta_{i-1})^{-1}$, $\theta(v_i)$, $D(\delta_i) \in \mathfrak{K}_i$ and $n(D(\delta_{i-1})^{-1})n(\theta(v_i))n(D(\delta_i)) = 1$. Hence η_i and consequently $\eta \in \mathrm{M}_1{}^{(3)}(\mathfrak{J})$.

We note next the following

Corollary 3. *The center of* Aut $\mathfrak{J}/\Phi e$ *coincides with the kernel of the restriction homomorphism of* Aut $\mathfrak{J}/\Phi e$ *to* $\mathfrak{J}_0(e)$ *(or to* $\mathfrak{V}$*). This consists of* 1 *and the element* ζ_e *such that* $\zeta_e = 1$ *on* $\mathfrak{J}_0(e)$ *and* $\zeta_e = -1$ *on* $\mathfrak{J}_{\frac{1}{2}}(e)$.

This follows easily from the proof of the theorem. We leave this as an exercise for the reader.

We return to the considerations of Lemma 1 and the group $\mathrm{M}(\mathfrak{J}, e)$ of elements of the form $\eta = U_{\theta(v_1)}U_{\theta(v_2)}\cdots U_{\theta(v_{2r})}$ where the $v_i \in \mathfrak{V}$ and $Q(v_i) \neq 0$. It is clear from Lemma 1 that any spinorial norm of a rotation in $\mathfrak{V}$ relative to Q is a coset $\mu(A) = \mu\Phi^{*2}$ where A is a similarity in $\mathfrak{J}_{\frac{1}{2}}$ relative to Q. We shall prove that conversely any $\mu(A)$ for a similarity in $\mathfrak{J}_{\frac{1}{2}}$ relative to Q is a spinorial norm. For this we require

Lemma 2. *Let* y_1 *and* y_2 *be any two elements of* $\mathfrak{J}_{\frac{1}{2}}$ *such that* $Q(y_1)Q(y_2) \neq 0$. *Then there exists a mapping* $\eta \in \mathrm{M}(\mathfrak{J}, e)$ *such that* $y_1{}^{\eta} = y_2$.

Proof. There is no loss in generality in assuming $y_1 = u_{12}$ where $u_{12} \in \mathfrak{J}_{12}$ and $\sum_{i \leqq j}\mathfrak{J}_{ij}$ is the Peirce decompositions of $\mathfrak{J}$ relative to orthogonal idempotents $e_1 = e, e_2, e_3$. We note that (30) (or (36)) imply that $y_i \circ y_i \neq 0$. Assume first that these elements of $\mathfrak{B}$ are linearly dependent so that we have $y_2 \circ y_2 = \beta y_1 \circ y_1 \neq 0$, $\beta \in \Phi$. Also $y_1 \circ y_1 = u_{12} \circ u_{12} = \frac{1}{2}Q(y_1)(e_2 - e_3)$, by (27). By (30), $(e_2 - e_3).y_2 = \delta(y_2 \circ y_2).y_2 = \nu y_2$. Since $\mathfrak{J}_{\frac{1}{2}} = \mathfrak{J}_{12} + \mathfrak{J}_{13}$ is the decomposition of $\mathfrak{J}_{\frac{1}{2}}$ into its characteristic subspaces of the restriction $R'_{e_2-e_3}$ of $R_{e_2-e_3}$ to $\mathfrak{J}_{\frac{1}{2}}$ and since the corresponding characteristic roots are $\frac{1}{2}$ and $-\frac{1}{2}$ we have $\nu = \pm\frac{1}{2}$ and $y_2 \in \mathfrak{J}_{12}$ or $y_2 \in \mathfrak{J}_{13}$. If $y_2 \in \mathfrak{J}_{13}$ then it is clear (for example, from the Coordinatization Theorem) that there exists an element $v_2 = v_{23} \in \mathfrak{J}_{23} \subseteq \mathfrak{B}$ such that $2y_1.v_2 = y_2$. Put $v_1 = e_2 - e_3 \in \mathfrak{B}$. Then $4y_1.v_1.v_2 = y_2$. Then if $\eta = U_{\theta(v_1)}U_{\theta(v_2)}$, $y_1^{\eta} = y_2$, by (38). If $y_2 \in \mathfrak{J}_{12}$ then $y_2' = y_2 U_{\theta(v_1)}U_{\theta(v_2)} \in \mathfrak{J}_{13}$ if $v_1 = e_2 - e_3$ and v_2 is any element of $\mathfrak{J}_{23}$. Also $Q(y_2') \neq 0$ if $Q(v_2) \neq 0$, so we can reduce this case to the preceding one. Now assume that $y_1 \circ y_1$ and $y_2 \circ y_2$ are linearly independent. By (36), $Q(y_2 \circ y_2) = \beta^2 Q(y_1 \circ y_1)$, $\beta \in \Phi$. By Witt's theorem there exists a $B \in O(\mathfrak{B}, Q)$ such that $(y_2 \circ y_2)B = \beta(y_1 \circ y_1)$. Since $\dim \mathfrak{B} = 9$ and β can be replaced by $-\beta$ we may assume that B is a rotation. By Lemma 1 and its proof there exists an $\eta = U_{\theta(v_1)}U_{\theta(v_2)} \cdots U_{\theta(v_{2r})}$ such that $Q(v_i) \neq 0$ and $(y_2 \circ y_2)^{\eta} = \beta(y_1 \circ y_1)$. By (42), $y_2^{\eta} \circ y_2^{\eta} = \delta(y_1 \circ y_1)$ so if we replace y_2 by y_2^{η} we have reduction to the case considered first.

THEOREM 5 (SPRINGER). *The group of multipliers of similarities of $\mathfrak{J}_{\frac{1}{2}}$ relative to Q coincides with the group of spinorial norms of the rotation group in $\mathfrak{B}$ relative to Q.*

PROOF. Lemma 1 shows that every spinorial norm has the form $\mu(A)$ where A is a similarity in $\mathfrak{J}_{\frac{1}{2}}$ relative to Q. Now let C be any similarity in $\mathfrak{J}_{\frac{1}{2}}$ and let $y_1 \in \mathfrak{J}_{\frac{1}{2}}$ satisfy $Q(y_1) \neq 0$. Then $y_2 = y_1 C \in \mathfrak{J}_{\frac{1}{2}}$ and $Q(y_2) = \mu Q(y_1)$ where μ is the multiplier of C. By Lemma 2 there exists an $\eta = U_{\theta(v_1)}U_{\theta(v_2)} \cdots U_{\theta(v_{2r})}$, $Q(v_i) \neq 0$, such that $y_1^{\eta} = y_2$. If we let A and B denote the restrictions of η to $\mathfrak{J}_{\frac{1}{2}}$ and $\mathfrak{B}$ respectively then the proof of Lemma 1 shows that $y_1 A = y_2$ and $\mu(A) = \lambda(B)$. Then $\mu(C) = \mu(A) = \lambda(B)$.

EXERCISES

1. Let $\mathfrak{B}$ be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form, v a vector in $\mathfrak{B}$ with $f(v,v) \neq 0$. Show that the subgroup of $\mathrm{Spin}(\mathfrak{B},f)$ of elements c such that $vc^{\chi} = v$ is isomorphic to $\mathrm{Spin}((\Phi v)^{\perp}, f)$.

2. Show that $\mathrm{Aut}\,\mathfrak{J}/\sum_1^3 \Phi e_i$ the subgroup of $\mathrm{Aut}\,\mathfrak{J}$ ($\mathfrak{J}$ as usual) leaving fixed the three orthogonal idempotents e_i is isomorphic to $\mathrm{Spin}(\mathfrak{D}, n)$ where $\mathfrak{D}$ is a coefficient algebra of $\mathfrak{J}$ and n is its norm form.

4. **Conditions for isomorphism.** In this section we shall prove Springer's theorem that two reduced exceptional simple Jordan algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$ having isomorphic coefficient algebras are isomorphic if and only if the quadratic forms $Q(x) = \frac{1}{2}t(x^{\cdot 2})$ defined by the two algebras are equivalent. Taken together with

Theorem 1 this implies that two reduced exceptional simple Jordan algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are isomorphic if and only if they have isomorphic coefficient algebras and equivalent forms Q. It is clear also that isomorphism of the algebras implies equivalence of the forms. If e_1, e_2, e_3 are othogonal primitive idempotents in $\mathfrak{J} = \mathfrak{J}_1$ and $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ the corresponding Peirce decomposition then any $a \in \mathfrak{J}$ has the form $a = \sum \alpha_i e_i + \sum_{i<j} a_{ij}$ where $\alpha_i \in \Phi$ and $a_{ij} \in \mathfrak{J}_{ij}$. Then $Q(a) = \frac{1}{2} \sum \alpha_i^2 + \sum_{i<j} Q(a_{ij})$. It follows from Witt's theorem that the restriction of Q to the 24 dimensional space $\mathfrak{J}_{12} + \mathfrak{J}_{13} + \mathfrak{J}_{23}$ is determined by Q on $\mathfrak{J}$. Hence the forms Q on $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are equivalent if and only if the corresponding forms on the 24 dimensional spaces are equivalent. We remark also that the theorem we wish to prove is trivial if the coefficient algebras are split since in this case the algebras $\mathfrak{J}_i$ are necessarily isomorphic. We shall therefore assume from now on that the coefficient algebras are octonion division algebras.

Let e be a primitive idempotent in $\mathfrak{J}$. As in §2, we consider the symmetric bilinear form F_e on $\mathfrak{J}$ defined by $F_e(a, b) = (e, a, b) = t(e \times a, b)$. We have seen that if $\mathfrak{R}_e$ is the radical of this form then $\mathfrak{J}/\mathfrak{R}_e$ with the induced form is an orthogonal direct sum of a hyperbolic plane and a subspace equivalent to $\mathfrak{J}_{23}$ equipped with the form $-Q$, where $\mathfrak{J}_{23}$, as usual, is the Peirce (2, 3)-space defined by orthogonal idempotents $e_1 = e$, e_2 and e_3. If $u_{ij} \in \mathfrak{J}_{ij}$, $i \neq j$, then we have $u_{ij}^{\cdot 2} = Q(u_{ij})(e_i + e_j)$. If we choose $u_{12} \in \mathfrak{J}_{12}$, $u_{13} \in \mathfrak{J}_{13}$ so that $Q(u_{12}) = \gamma_2^{-1} \neq 0$, $Q(u_{13}) = \gamma_3^{-1} \neq 0$ and choose a coordinatization of $\mathfrak{J}$ adapted to the e_i and u_{12}, u_{13} then we have $a[12]^{\cdot 2} = \gamma_2^{-1} n(a)(e_{11} + e_{22})$, $a[13]^{\cdot 2} = \gamma_3^{-1} n(a)(e_{11} + e_{33})$, $a[23]^{\cdot 2} = \gamma_3^{-1} \gamma_2 n(a)(e_{22} + e_{33})$, if $a \in \mathfrak{D}$ (where $\mathfrak{J}^\eta = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$). Hence Q on $\mathfrak{J}_{23}$ is equivalent to $\gamma_3^{-1} \gamma_2 n$ where n here is the norm form on the coefficient algebra $\mathfrak{D}$ of $\mathfrak{J}$.

Now let $\mathrm{N}^* = \mathrm{N}^*(\mathfrak{D})$ be the set of elements $n(a) \neq 0$, $a \in \mathfrak{D}$. Then N^* is a subgroup of the multiplicative group Φ^* containing Φ^{*2}. If $\alpha \neq 0$ then we shall call the coset $\alpha \mathrm{N}^*$ in Φ^*/N^* the *norm class* of α. Now let e and everything else be as in the last paragraph. Choose a hyperbolic plane in $\mathfrak{J}/\mathfrak{R}_e$ and let u be any nonisotropic vector such that $u + \mathfrak{R}_e \in \mathfrak{J}/\mathfrak{R}_e$ is orthogonal to the chosen hyperbolic plane. Then the equivalences of forms we have noted imply that the norm class $-(e, u, u)\mathrm{N}^* = \gamma_2^{-1} \gamma_3 \mathrm{N}^*$. Thus it is clear that this norm class is independent of the choice of u and by Witt's theorem, it is also independent of the choice of the hyperbolic plane. Accordingly, we shall call $-(e, u, u)\mathrm{N}^*$ the *norm class* $\kappa(e)$ of the primitive idempotent e. We have $\kappa(e) = \gamma_3^{-1} \gamma_2 \mathrm{N}^*$. It is clear from the Coordinatization Theorem that this can be determined by choosing primitive orthogonal idempotents $e_1 = e, e_2, e_3$ and $u_{23} \in \mathfrak{J}_{23}$, the Peirce (2, 3)-space, so that $Q(u_{23}) \neq 0$. Then $\kappa(e) = Q(u_{23})\mathrm{N}^*$. It follows that $\kappa(e_2) = Q(u_{13})\mathrm{N}^* = \gamma_3^{-1}\mathrm{N}^*$ and $\kappa(e_3) = \gamma_2^{-1}\mathrm{N}^*$ if the γ's are as before.

Let $\zeta \in \mathrm{M}(\mathfrak{J})$ and suppose $n(a^\zeta) = \mu n(a)$, so μ is the multiplier of the similarity ζ. Suppose $f = e^\zeta$ is a primitive idempotent. Then $(e^\zeta, u^\zeta, u^\zeta) = \mu(e, u, u)$ and hence $\kappa(f) = \mu\kappa(e)$. Next let $\zeta \in \mathrm{M}_1(\mathfrak{J})$, so $n(a^\zeta) = n(a)$ and assume $e^\zeta = \mu f$ where f is a primitive idempotent. Then $e^{\mu^{-1}\zeta} = f$ and the multiplier of $\mu^{-1}\zeta$ is μ^{-3}. Hence $\kappa(f) = \mu^{-3}\kappa(e) = \mu\kappa(e)$.

We can now prove

LEMMA 1. *Let e_i, $i = 1,2,3$, be orthogonal primitive idempotents in $\mathfrak{J}$, $\mathfrak{T}$ the subspace of $\mathfrak{J}$ orthogonal to the e_i (relative to Q). Then the set of norm classes of primitive idempotents of $\mathfrak{J}$ coincides with the set of norm classes of the elements $Q(u)$, $u \in \mathfrak{T}$, $Q(u) \neq 0$.*

PROOF. As before, let $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ be the Peirce decomposition of $\mathfrak{J}$ relative to the e_i so $\mathfrak{T} = \sum_{i<j} \mathfrak{J}_{ij}$ and if $u_{1j} \in \mathfrak{J}_{1j}, j = 2,3$ satisfies $Q(u_{1j}) = \gamma_j^{-1}$, then we can identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{O}$, the coefficient algebra. Also in our identification $e_i = e_{ii}$ and $\mathfrak{J}_{ij} = \{a[ij] \mid a \in \mathfrak{O}\}$. If $u = a[12] + b[13] + c[23]$ then $Q(u) = \gamma_2^{-1}n(a) + \gamma_3^{-1}n(b) + \gamma_3^{-1}\gamma_2 n(c)$ and we have seen that $\kappa(e) = \gamma_2^{-1}\gamma_3 \mathrm{N}^*$ which is clearly of the form $Q(u)\mathrm{N}^*$, $u \in \mathfrak{J}_{23}$. Since any primitive idempotent can be imbedded in a set of three orthogonal primitive idempotents and since the spaces $\mathfrak{T}$ orthogonal to any two such sets of idempotents are Q-equivalent, it is clear that if e is any primitive idempotent then $\kappa(e) = Q(u)\mathrm{N}^*$ for some u in the given space $\mathfrak{T}$. We now consider the mapping $T_{Q_{ij}}$ defined, by $q \in Q$ and $i \neq j$ and a similar mapping $T_{S_{ij}}$ defined by $s \in \mathfrak{O}$. Then, by (20), we have

$$g \equiv e_{11} T_{Q_{12}} T_{S_{13}} = e_{11} + \gamma_2^{-1}\gamma_1 n(q) e_{22} + \gamma_3^{-1}\gamma_1 n(s) e_{33} + q[12] + s[13] + \bar{s}q[32] .$$

Then $t(g) = 1 + \gamma_2^{-1}\gamma_1 n(q) + \gamma_3^{-1}\gamma_1 n(s)$ and $g \times g = 0$. Hence if $t(g) \neq 0$ then $t(g)^{-1}g$ is a primitive idempotent. Since $T_{Q_{12}} T_{S_{13}} \in \mathrm{M}_1(\mathfrak{J})$ (Lemma 1 of §2) the result noted above shows that $\kappa(t(g)^{-1}g) = t(g)\kappa(e_{11}) = \gamma_2^{-1}\gamma_3 + \gamma_2^{-2}\gamma_3\gamma_1 n(q) + \gamma_2^{-1}\gamma_1 n(s)$. Since $\gamma_1 = 1$ this implies that if $\beta = \gamma_2^{-1}n(a) + \gamma_3^{-1}n(b) + \gamma_3^{-1}\gamma_2 n(c) \neq 0$ and $n(c) \neq 0$ then $\beta\mathrm{N}^*$ is a norm class of a primitive idempotent of $\mathfrak{J}$. We recall that $\kappa(e_{22}) = \gamma_3^{-1}\mathrm{N}^*$ and $\kappa(e_{33}) = \gamma_2^{-1}\mathrm{N}^*$. Also, by (20), $e_{22} T_{Q_{23}} = e_{22} + \gamma_3^{-1}\gamma_2 n(q) e_{33} + q[23]$ and the trace of this element is $1 + \gamma_3^{-1}\gamma_2 n(q)$ so if this is $\neq 0$ then the argument just used shows that $(\gamma_3^{-1} + \gamma_3^{-2}\gamma_2 n(q))\mathrm{N}^*$ is a norm class of a primitive idempotent in $\mathfrak{J}$. Hence if $\beta = \gamma_2^{-1}n(a) + \gamma_3^{-1}n(b) \neq 0$ and $n(b) \neq 0$ then $\beta\mathrm{N}^*$ is a norm class of a primitive idempotent in $\mathfrak{J}$. It is now clear that every $Q(u)\mathrm{N}^*$, $u \in \mathfrak{T}$, $Q(u) \neq 0$ is a norm class of a primitive idempotent.

The foregoing lemma shows that the norm classes of primitive idempotents of $\mathfrak{J}$ are determined by the form Q. We need to consider next the norm classes of primitive idempotents orthogonal to a given primitive idempotent e. By Witt's theorem it is clear that the equivalence class of the quadratic form Q on $\mathfrak{J}_{\frac{1}{2}}(e)$ is determined by $\kappa(e)$ and Q on $\mathfrak{J}$ (or on a subspace $\mathfrak{T}$ as defined in Lemma 1). We now have

LEMMA 2. *If e is a primitive idempotent then the set of norm classes of* *primitive idempotents e' orthogonal to e coincides with the set of norm classes of the multipliers of similarities of the quadratic form $n(x)+\kappa(e)n(y)$ with the quadratic form Q on $\mathfrak{J}_{\frac{1}{2}}(e)$.*

PROOF. First, let e_1, e_2, e_3 be primitive idempotents with $e_1 = e$ and let the notations be as before. Then we may take $\kappa = \kappa(e) = \gamma_2^{-1}\gamma_3$ and Q on $\mathfrak{J}_{\frac{1}{2}}$ is the quadratic form $\gamma_2^{-1}n(x) + \gamma_3^{-1}n(y)$. Since $\gamma_3^{-1}(n(x) + \gamma_2^{-1}\gamma_3 n(y))$ is equivalent to Q on $\mathfrak{J}_{\frac{1}{2}}$ it is clear that γ_3^{-1} is a multiplier of a similarity of $n(x) + \kappa n(y)$ with Q on $\mathfrak{J}_{\frac{1}{2}}(e)$. Also we have seen that $\kappa(e_2) = \gamma_3^{-1}\mathrm{N}^*$ and since e_2 can be taken to be any primitive idempotent orthogonal to $e = e_1$ we see that the norm class of any primitive idempotent e' orthogonal to e is as specified in the lemma. Next let μ be any multiplier of a similarity of $n(x) + \kappa n(y)$ with Q on $\mathfrak{J}_{\frac{1}{2}}$. Then $\gamma_3\mu$ is the multiplier of a similarity of Q on $\mathfrak{J}_{\frac{1}{2}}$ and, by Theorem 5, $\gamma_3\mu\mathrm{N}^*$ is a spinorial norm of a rotation B in the space $\mathfrak{V}$ of elements of trace 0 in $\mathfrak{J}_0(e)$ relative to Q. By Lemma 1 of §3, we have a similarity A in $\mathfrak{J}_{\frac{1}{2}}(e)$ such that (41) holds and $\mu(A) = \gamma_3\mu\mathrm{N}^*$. Let $v = (e_2 - e_3)B$. Then $v \in \mathfrak{V}$ and $Q(v) = Q(e_2 - e_3) = 1$ so $e_2' = \frac{1}{2}(f + v)$ and $e_3' = \frac{1}{2}(f - v)$ are primitive idempotents and e_1, e'_2, e'_3 are orthogonal. We shall now show that $\kappa(e_2') = \mu\mathrm{N}^*$ which will complete the proof. Let $u \in \mathfrak{J}_{13}$ satisfy $Q(u) \neq 0$. Then $\kappa(e_2) = \gamma_3^{-1}\mathrm{N}^* = Q(u)\mathrm{N}^*$. Also $u\,.\,(e_2 - e_3) = -\frac{1}{2}u$ so, by (42), $(uA)\,.\,v = -\frac{1}{2}(uA)$. Since $(uA)\,.\,f = \frac{1}{2}uA$ we have $uA\,.\,e_3 = \frac{1}{2}uA$. Also $uA\,.\,e_1 = \frac{1}{2}uA$ so uA is in the Peirce (1,3)-space for the idempotents e_1, e_2', e_3'. Hence $\kappa(e_2') = Q(uA)\mathrm{N}^* = \gamma_3\mu Q(u)\mathrm{N}^* = \mu\mathrm{N}^*$ as required.

We can now prove

THEOREM 6 (SPRINGER). *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be reduced simple exceptional Jordan algebras with isomorphic coefficient algebras and equivalent forms Q (or trace forms). Then $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are isomorphic.*

PROOF. Let e_1, e_2, e_3 be orthogonal primitive idempotents in $\mathfrak{J}_1$, u_{1j} an element in the Peirce space $\mathfrak{J}_{11j}$ of $\mathfrak{J}_1$ relative to these idempotents such that $u_{1j}^{\cdot 2} = \gamma_j^{-1}(e_1 + e_j)$. Then $\mathfrak{J}_1 \cong \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $\mathfrak{O}$ is the coefficient algebra and $\kappa(e_1) = \gamma_2^{-1}\gamma_3\mathrm{N}^*$ and $\kappa(e_2) = \gamma_3^{-1}\mathrm{N}^*$. By Lemmas 1 and 2 there exist orthogonal primitive idempotents e_1', e_2' in $\mathfrak{J}_2$ such that $\kappa(e_1') = \gamma_2^{-1}\gamma_3\mathrm{N}^*$ and $\kappa(e_2') = \gamma_3^{-1}\mathrm{N}^*$. It follows readily that $\mathfrak{J}_2 \cong \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ so $\mathfrak{J}_1 \cong \mathfrak{J}_2$.

The method of proof of this theorem can be applied in the same way to a single algebra to give the following

COROLLARY. *Let $\mathfrak{J}$ be a reduced simple exceptional Jordan algebra. Then two primitive idempotents $e, e' \in \mathfrak{J}$ are in the same orbit under* Aut $\mathfrak{J}$ *if and only if the norm classes $\kappa(e) = \kappa(e')$.*

EXERCISES

1. Let $\mathfrak{J}$ be a reduced simple exceptional Jordan algebra, $\mathfrak{X}$ a subspace orthogonal to three primitive orthogonal idempotents. Show that $\mathfrak{J}$ contains nilpotent elements $\neq 0$ if and only if Q has positive Witt index on $\mathfrak{X}$.

2. Assume the coefficient algebra is an octonion division algebra. Show that any element a can be imbedded in a subalgebra of the form $\mathfrak{H}(\mathrm{P}, J_\gamma)$ where (P, j) is a quadratic field.

3. Use exercise 2 to show that if $\mathfrak{J}$ contains a nonzero nilpotent element then $\mathfrak{J}$ has the form $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $\gamma = \mathrm{diag}\{1, -1, 1\}$. Hence prove the following three conditions on a reduced simple exceptional Jordan algebra are equivalent: (1) $\mathfrak{J}$ has nilpotent element $\neq 0$, (2) the quadratic form Q on a subspace $\mathfrak{X}$ orthogonal to three primitive orthogonal idempotents has positive Witt index, (3) Q in $\mathfrak{X}$ is equivalent to $n(x) - n(y) + n(z)$ where n is the norm form on the coefficient algebra. Show that two $\mathfrak{J}$'s containing nonzero nilpotents are isomorphic if and only if their coefficient algebras are isomorphic.

4. Show that if $\mathfrak{J}$ (as in ex. 3) contains a nilpotent element $\neq 0$ then $\mathfrak{J}$ contains a nilpotent element of index three.

5. Call a primitive idempotent e of $\mathfrak{J}$ an *s-idempotent* if $\mathfrak{J}_0(e)$ contains nonzero nilpotent elements. Show that e is an s-idempotent if and only if the norm class $\kappa(e) = (-1)\mathrm{N}^*$. Hence show that any two s-idempotents are in the same orbit under Aut $\mathfrak{J}$.

6. Show that $\mathfrak{J}$ is split if and only if the restriction of Q to the subspace $\mathfrak{X}$ in exercise 1 of maximum Witt index.

7. Let Φ be an algebraic number field with t real conjugate fields. Use Hasse's theorem to show that there are at most 3^t nonisomorphic reduced exceptional simple Jordan algebras over Φ (cf. ex. 7, p. 171). The number 3^t is actually the exact number of these algebras (Albert-Jacobson [1]).

5. **More on Aut $\mathfrak{J}$.** As usual, let $\mathfrak{J}$ be a reduced simple exceptional Jordan algebra. We prove first

THEOREM 7. *The only subspaces of $\mathfrak{J}$ invariant under* Aut $\mathfrak{J}$ *are* $\mathfrak{J}$, $\Phi 1$ *and* $\mathfrak{J}'$, *the subspace of elements of trace* 0.

PROOF. We have seen in §3 (Corollary 1 to Theorem 4) that if e is a primitive idempotent then Aut $\mathfrak{J}/\Phi e$ acts irreducibly on $\mathfrak{J}_{\frac{1}{2}}(e)$ and on $\mathfrak{V}$, the subspace of $\mathfrak{J}_0(e)$ of elements of trace 0. It follows that the only invariant subspaces of $\mathfrak{J}$ relative to Aut $\mathfrak{J}/\Phi e$ are sums of $\mathfrak{J}_{\frac{1}{2}}(e)$, $\mathfrak{V}$ and subspaces of $\Phi e + \Phi(1-e)$. Now let $e_i, i = 1,2,3$, be primitive orthogonal idempotents in $\mathfrak{J}$, $\mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$ the Peirce decomposition determined by the e_i. Then the only invariant subspaces of $\mathfrak{J}$ relative to Aut $\mathfrak{J}/\Phi e_i$ are sums of the subspaces: $\mathfrak{J}_{ij} + \mathfrak{J}_{ik}$, $\mathfrak{J}_{jk} + \Phi(e_j - e_k)$, subspaces of $\Phi e_i + \Phi(e_j + e_k)$ where i, j, k are unequal. Now let $\mathfrak{K}$ be a subspace invariant under Aut $\mathfrak{J}$. Then $\mathfrak{K}$ Aut $\mathfrak{J}/\Phi e_i \subseteq \mathfrak{K}$ for every i, and this implies that either $\mathfrak{K}$ contains every $\mathfrak{J}_{ij}$ and every $\Phi(e_i - e_j)$, $i \neq j$, or $\mathfrak{K}$ is a subspace of $\Phi e_i + \Phi(e_j + e_k)$ for all unequal i, j, k. In the first case, $\mathfrak{K}$ contains $\mathfrak{J}'$ and in the second $\mathfrak{K} = \Phi 1$ and these two alternatives imply the theorem.

If $\mathfrak{B}$ is a subalgebra of an algebra $\mathfrak{A}$ then we denote the subgroup of the automorphism group Aut $\mathfrak{A}$ leaving fixed the elements of $\mathfrak{B}$ as Aut $\mathfrak{A}/\mathfrak{B}$. We now consider Aut $\mathfrak{J}/\mathfrak{K}$ whese $\mathfrak{J}$ is a reduced simple exceptional Jordan algebra and $\mathfrak{K}$ is a reduced simple subalgebra of degree three. Then, as we have seen, we can identify $\mathfrak{J}$ with $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $(\mathfrak{O},j)$ an algebra of octonions and $\mathfrak{K}$ with the subalgebra $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ where $(\mathfrak{D},j)$ is a composition subalgebra of $(\mathfrak{O},j)$. The possible dimensionalities for $\mathfrak{D}$ are $\dim\mathfrak{D} = 1,2,4,8$. In the first case $\mathfrak{D} = \Phi$ and $\dim\mathfrak{K} = 6$. If $\dim\mathfrak{D} = 2$ so $\mathfrak{D} = \mathrm{P}$ a quadratic subfield of $\mathfrak{O}$ or a direct sum of two copies of Φ, then $\dim\mathfrak{K} = 9$. Next we may have $\dim\mathfrak{D} = 4$ in which case $(\mathfrak{D},j)$ is a quaternion algebra and $\dim\mathfrak{K} = 15$. Finally we have the trivial case $\mathfrak{D} = \mathfrak{O}$ and $\dim\mathfrak{K} = \dim\mathfrak{J} = 27$.

Suppose first that $D = \Phi$ so $\mathfrak{K} = \mathfrak{H}(\Phi_3, J_\gamma)$ and $\dim\mathfrak{K} = 6$. Then $\eta \in \mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ if and only if η is an automorphism of $\mathfrak{J}$ and $1[ij]^\eta = 1[ij]$, $i,j = 1,2,3$. By Theorem 3.3, these automorphisms have the form $\sum\alpha_i e_{ii} + a[12] + b[23] + c[31] \to \sum\alpha_i e_{ii} + a^\beta[12] + b^\beta[23] + c^\beta[31]$ where β is an automorphism of $(\mathfrak{O},j)$. Conversely, any $\beta \in \mathrm{Aut}(\mathfrak{O},j)$ defines in the manner indicated an automorphism $\eta \in \mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$. Thus we can identify Aut $\mathfrak{J}/\mathfrak{K}$ for $\mathfrak{K} = \mathfrak{H}(\Phi_3, J_\gamma)$ with $\mathrm{Aut}(\mathfrak{O},j)$.

Now let $\mathfrak{K} = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ where $(\mathfrak{D},j)$ is an arbitrary composition subalgebra of $(\mathfrak{O},j)$. Then $\mathfrak{D} \supseteq \Phi$. Hence it is clear that Aut $\mathfrak{J}/\mathfrak{K}$ is the set of mappings $\sum\alpha_i e_{ii} + a[12] + b[23] + c[31] \to \sum a_i e_{ii} + a^\beta[12] + b^\beta[23] + c^\beta[31]$ where $\beta \in \mathrm{Aut}(\mathfrak{O},j)/\mathfrak{D}$ the subgroup of automorphisms of $(\mathfrak{O},j)$ leaving fixed the elements of $\mathfrak{D}$. In this way we can identify Aut $\mathfrak{J}/\mathfrak{K}$ with $\mathrm{Aut}(\mathfrak{O},j)/\mathfrak{D}$.

We proceed to derive some properties of Aut $\mathfrak{O}$ which we shall need. We note first that if $\beta \in \mathrm{Aut}\,\mathfrak{O}$ then $t(a^\beta) = t(a)$ and $n(a^\beta) = n(a)$ for the generic trace and norm in $(\mathfrak{O},j)$ (Theorem 6.1, p. 224). Since $\mathfrak{O} = \Phi 1 \oplus \mathfrak{O}_0$ where $\mathfrak{O}_0 = \Phi 1^\perp$ is the set of elements of trace 0 we have $(\Phi 1)^\beta = \Phi 1$ and $\mathfrak{O}_0{}^\beta = \mathfrak{O}_0$ for $\beta \in \mathrm{Aut}\,\mathfrak{O}$. It follows that $\bar{a}^\beta = \overline{a^\beta}$ so $\beta \in \mathrm{Aut}(\mathfrak{O},j)$. Thus Aut $\mathfrak{O} = \mathrm{Aut}(\mathfrak{O},j)$. Also the condition $n(a^\beta) = n(a)$ shows that $\beta \in O(\mathfrak{O}, n)$ the orthogonal group in $\mathfrak{O}$ relative to n. We now take up the problem of determining Aut$\mathfrak{O}/\mathfrak{D}$ in the following cases: I. $\mathfrak{D}$ a quaternion subalgebra of $(\mathfrak{O},j)$, II. $\mathfrak{D} = \mathrm{P}$ a quadratic subfield of $(\mathfrak{O},j)$, III. $\mathfrak{O}$ split and $\mathfrak{D} = \Phi u_1 + \Phi u_2$ where the u_i are orthogonal idempotents $\neq 0$ and $\bar{u}_1 = u_2$, $\bar{u}_2 = u_1$.

I. $\mathfrak{D}$ quaternion. We can decompose $\mathfrak{O} = \mathfrak{D} \oplus \mathfrak{D}^\perp$ (relative to the norm form in $\mathfrak{O}$) and if $q \in \mathfrak{D}^\perp$ and $n(q) \neq 0$ then $\mathfrak{D}^\perp = \mathfrak{D}q$ (§4.3, p. 163). If $\beta \in \mathrm{Aut}\mathfrak{O}/\mathfrak{D}$ then $\mathfrak{D}^{\perp\beta} = \mathfrak{D}$ so $q^\beta = uq$, $u \in \mathfrak{D}$. Since $n(q) = n(q^\beta) = n(u)n(q)$, $n(u) = 1$. Conversely, if $u \in \mathfrak{D}$ satisfies $n(u) = 1$ then $q' = uq$ satisfies $q' \in \mathfrak{D}$ and $n(q') = n(q)$. The formulas of §4.3 giving the multiplication in $\mathfrak{O} = \mathfrak{D} + \mathfrak{D}q = \mathfrak{D} + \mathfrak{D}q'$ show that $a + bq \to a + bq'$, $a, b \in \mathfrak{D}$, is an automorphism β in $\mathfrak{D}$. Clearly $\beta \in \mathrm{Aut}\mathfrak{O}/\mathfrak{D}$. It is now clear that the mapping $\beta \to u$ is an isomorphism of Aut$\mathfrak{O}/\mathfrak{D}$ onto the multiplicative group of elements u of norm one in $\mathfrak{D}$.

II. $\mathfrak{D} = \mathrm{P}$ a quadratic subfield of $(\mathfrak{O},j)$. If $x \in \mathfrak{O}$ and $\rho_1, \rho_2 \in \mathrm{P}$ then $(\rho_1\rho_2)x = \rho_1(\rho_2 x)$ since P is generated by a single element and $\mathfrak{O}$ is alternative. Hence $\mathfrak{O}$ is a vector space over P relative to ρx. Since the dimensionality of $\mathfrak{O}$ over Φ is 8 its dimensionality over P is $\dim_P \mathfrak{D} = 4$. Let $i \in P$ satisfy $\bar{i} = -i$ so $P = \Phi(i)$ and $i^2 = \alpha 1$, $\alpha \in \Phi$. If $x, y \in \mathfrak{D}$ we define

$$h(x,y) = n(x,y) + \alpha^{-1} i n(ix,y) \in P. \tag{44}$$

Evidently h is additive in x and y and since $n(iy,x) = -\underline{n(ix,y)}$, by (10) of p. 163, $h(y,x) = n(y,x) + \alpha^{-1} i n(iy,x) = n(x,y) - \alpha^{-1} i n(ix,y) = \overline{h(x,y)}$. Also if $\delta \in \Phi$ then $h(\delta x, y) = \delta h(x,y)$ and $h(ix,y) = n(ix,y) + \alpha^{-1} i n(i^2 x, y) = i n(x,y) + n(ix,y) = ih(x,y)$. Hence h is a hermitian form on $\mathfrak{D}/P$. It is clear that $h(x,y) = 0$ implies $n(x,y) = 0$ so h is nondegenerate. Moreover, if $\mathfrak{M}$ is a P-subspace of $\mathfrak{D}$ then $\mathfrak{M}^{\perp}$ as defined by h is the same as $\mathfrak{M}^{\perp}$ relative to n.

We now define for $x, y, z \in \mathfrak{D}$

$$h(x,y,z) = h(x, y \wedge z) \in P \tag{45}$$

where $y \wedge z = \frac{1}{2}[yz]$. Evidently $h(x,y,z)$ is additive in x, y and z and is linear in x. We now let $\mathfrak{B} = P^{\perp}$, so $\dim_P \mathfrak{B} = 3$. We consider the restriction of $h(x,y,z)$ to $\mathfrak{B}$. We note first that if $x, y, z \in \mathfrak{D}$, then

$$\begin{aligned} n(y, z \wedge x) &= \tfrac{1}{2} n(y, zx - xz) = -\tfrac{1}{2} n(zy - yz, x) \\ &= n(x, y \wedge z) \end{aligned} \tag{46}$$

(by (10) p. 163 and $\bar{z} = -z$, since $z \perp 1$). If $z \in \mathfrak{B}$ then $iz + zi = 0$ which implies that $i_L z_L + z_L i_L = 0$ and $i_R z_R + z_R i_R = 0$ since $a \to a_L$ and $a \to a_R$ are associative specializations of $\mathfrak{D}^+$. Thus $z(ix) = -i(zx)$ and $(zi)x = -(zx)i$. Hence if $x, y, z \in \mathfrak{B}$ then

$$\begin{aligned} n(iy, z \wedge x) &= \tfrac{1}{2} n(iy, zx - xz) \\ &= \tfrac{1}{2} n(iy, zx) + \tfrac{1}{2} n(yi, xz) = -\tfrac{1}{2} n(y, i(zx) + (xz)i) \\ &= \tfrac{1}{2} n(y, z(ix) + (xi)z) = \tfrac{1}{2} n(y, z(ix) - (ix)z) \\ &= -\tfrac{1}{2} n(zy, ix) + \tfrac{1}{2} n(yz, ix) = n(ix, y \wedge z). \end{aligned} \tag{47}$$

Since $h(x,y,z) = h(x, y \wedge z) = n(x, y \wedge z) + \alpha^{-1} i n(ix, y \wedge z)$ it follows from (46) and (47) that $h(y,z,x) = h(x,y,z)$ for $x, y, z \in \mathfrak{B} = P^{\perp}$. It follows that $h(x,y,z)$ is trilinear and alternating on $\mathfrak{B}$.

It is easily seen from the inductive construction of $\mathfrak{D}$ that there exists a quaternion subalgebra $\mathfrak{Q}$ orthogonal to i. Moreover, $\mathfrak{Q}$ has a basis $(1, v_1, v_2, v_3)$ where $\bar{v}_i = -v_i$, $n(v_i) \neq 0$, $v_1 = v_2 v_3 = -v_3 v_2$. Then the $v_i \in \mathfrak{B}$ and $n(v_1, v_2 \wedge v_3) = n(v_1, v_1) \neq 0$. It follows that $h(v_1, v_2, v_3) \neq 0$ and since $h(x,y,z)$ is P-trilinear and alternating on $\mathfrak{B}$, the v_i are P-independent and so form a basis for $\mathfrak{B}/P$. If (w_1, w_2, w_3) is any basis for $\mathfrak{B}/P$ then $w_i = \sum \rho_{ij} v_j$, $\rho_{ij} \in P$ and $h(w_1, w_2, w_3) = \det(\rho_{ij}) h(v_1, v_2, v_3) \neq 0$.

Now let $\beta \in \operatorname{Aut} \mathfrak{D}/P$. Since β is an orthogonal transformation in $\mathfrak{D}$ relative to n and β is P-linear we have $h(x^{\beta}, y^{\beta}) = n(x^{\beta}, y^{\beta}) + \alpha^{-1} i n(ix^{\beta}, y^{\beta}) = n(x,y)$

$+\alpha^{-1}in((ix)^\beta, y^\beta) = n(x,y) + \alpha^{-1}in(ix,y) = h(x,y)$, $x, y \in \mathfrak{O}$, so β is unitary in $\mathfrak{O}/\mathrm{P}$ relative to h. Since $\mathrm{P}^\beta = \mathrm{P}$ we have $\mathfrak{B}^\beta = \mathfrak{B}$. Let B denote the restriction of β to $\mathfrak{B}$ and let (w_1, w_2, w_3) be a basis of $\mathfrak{B}/\mathrm{P}$. Then $w_iB \in \mathfrak{B}$ and $h(w_1B, w_2B, w_3B) = (\det B)h(w_1, w_2, w_3)$. On the other hand, since β is an automorphism, $w_2{}^\beta \wedge w_3{}^\beta = (w_2 \wedge w_3)^\beta$. Hence $h(w_1B, w_2B, w_3B) = h(w_1{}^\beta, w_2{}^\beta, w_3{}^\beta) = h(w_1{}^\beta, w_2{}^\beta \wedge w_3{}^\beta) = h(w_1{}^\beta, (w_2 \wedge w_3)^\beta) = h(w_1, w_2 \wedge w_3) = h(w_1, w_2, w_3)$. Since $h(w_1, w_2, w_3) \neq 0$ we see that $\det B = 1$. Thus $B \in SU(\mathfrak{B}, h)$ the unimodular unitary group in $\mathfrak{B}/\mathrm{P}$ relative to h.

Conversely, let $B \in SU(\mathfrak{B}, h)$. Extend B to a linear transformation β in $\mathfrak{O}/\mathrm{P}$ such that $\rho^\beta = \rho$, $\rho \in \mathrm{P}$. We claim that $\beta \in \mathrm{Aut}\,\mathfrak{O}/\mathrm{P}$. If $x \in \mathfrak{B}$ and $\rho \in \mathrm{P}$ then $(\rho x)^\beta = \rho x^\beta$ is clear. Also $x\rho = \bar{\rho}x$ so $(x\rho)^\beta = x^\beta\rho$ and if $\rho_1, \rho_2 \in \mathrm{P}$ then $(\rho_1\rho_2)^\beta = \rho_1{}^\beta\rho_2{}^\beta$. Now let (w_1, w_2, w_3) be a basis for $\mathfrak{B}/\mathrm{P}$. Then $h(w_1, w_2, w_3) = h(w_1B, w_2B, w_3B) = h(w_1B, w_2B \wedge w_3B)$ and $h(w_1, w_2, w_3) = h(w_1, w_2 \wedge w_3) = h(w_1{}^\beta, (w_2 \wedge w_3)^\beta)$ since β is unitary on $\mathfrak{O}/\mathrm{P}$. This implies that $(w_2 \wedge w_3)^\beta = w_2{}^\beta \wedge w_3{}^\beta$ if w_2, w_3 are P-linearly independent elements of $\mathfrak{B}$. Also $h(v_2{}^\beta, v_3{}^\beta) = h(v_2, v_3)$ implies $n(v_2{}^\beta, v_3{}^\beta) = n(v_2, v_3)$ and this implies that $v_2{}^\beta v_3{}^\beta + v_3{}^\beta v_2{}^\beta = v_2v_3 + v_3v_2$. Hence $(v_2v_3)^\beta = v_2^\beta v_3{}^\beta$ holds for any linearly independent $v_i \in \mathfrak{B}$. It follows that this holds for all $v_i \in \mathfrak{B}$ and that β is an automorphism. Clearly $\beta \in \mathrm{Aut}\,\mathfrak{B}/\mathrm{P}$.

The result we have established is that $\beta \in \mathrm{Aut}\,\mathfrak{O}/\mathrm{P}$ if and only if β is P-linear on $\mathfrak{O}$, $1^\beta = 1$ and the restriction of β to $\mathfrak{B}$ is in $SU(\mathfrak{B}, h)$. Hence we have an isomorphism of $\mathrm{Aut}\,\mathfrak{O}/\mathrm{P}$ with $SU(\mathfrak{B}, h)$. We remark finally that the hermitian form h is independent of the choice of i in P since any other element of P satisfying $\bar{x} = -x$ is of the form δi where $\delta \neq 0$ is in Φ. It is clear from (44) that replacement of i by δi does not change h.

III. $\mathfrak{O}$ split, $\mathfrak{D} = \Phi u_1 + \Phi u_2$, where the u_i are nonzero orthogonal idempotents such that $\bar{u}_1 = u_2, \bar{u}_2 = u_1$. Since the maximum number of nonzero orthogonal idempotents in $\mathfrak{O}$ (and in $\mathfrak{O}^+$) is two we have $u_1 + u_2 = 1$. Let $\mathfrak{O} = \mathfrak{O}_{11} \oplus \mathfrak{O}_{12} \oplus \mathfrak{O}_{21} \oplus \mathfrak{O}_{22}$ be the Peirce decomposition of $\mathfrak{O}$ relative to the u_i (see p.). Then $\mathfrak{O}_{ij} = u_i\mathfrak{O}u_j$. Since the u_i are primitive idempotents in $\mathfrak{O}^+$ which is a Jordan algebra of a nondegenerate symmetric bilinear form on $\mathfrak{O}_0$ we have $\mathfrak{O}_{ii} = u_i\mathfrak{O}u_i = \Phi u_i$. If $i \neq j$ then $\mathfrak{O}_{ij}\mathfrak{O}_{ji} \subseteq \mathfrak{O}_{ii} = \Phi u_i$ and $\mathfrak{O}_{ij}{}^2 \subseteq \mathfrak{O}_{ji}$ (loc. cit.). If $a_{ij} \in \mathfrak{O}_{ij}$ then $a_{ij} = u_ia(1 - u_i)$, $a \in \mathfrak{O}$. Hence $a_{ij}{}^2 = 0$ and $n(a_{ij}) = 0$. It follows that $\mathfrak{O}_{ij}$ is a totally isotropic subspace of $\mathfrak{O}$ relative to n. We have also $t(a_{ij}) = 0$ so $\mathfrak{O}_{ij} \subseteq \mathfrak{O}_0$. Since $n(a,b) = \frac{1}{2}t(a\bar{b})$ we have $n(u_i, a_{ij}) = 0 = n(u_j, a_{ij})$. Since n is nondegenerate it is now clear that n gives a nondegenerate bilinear pairing of $\mathfrak{O}_{ij}$ and $\mathfrak{O}_{ji}$. Hence $\dim \mathfrak{O}_{12} = \dim \mathfrak{O}_{21} = 3$. If $a_{12} \in \mathfrak{O}_{12}$ and $a_{21} \in \mathfrak{O}_{21}$ then $a_{12}a_{21} = \alpha_1 u_1$ and $a_{21}a_{12} = \alpha_2 u_2$, $\alpha_i \in \Phi$. Since $u_i + \bar{u}_i = 1$, $t(u_i) = 1$ and so $n(a_{12}, a_{21}) = \frac{1}{2}t(a_{12}\bar{a}_{21}) = -\frac{1}{2}t(a_{12}a_{21}) = -\frac{1}{2}\alpha_1$. Thus $\alpha_1 = -2n(a_{12}, a_{21})$ and, similarly, $\alpha_2 = -2n(a_{21}, a_{12}) = -2n(a_{12}, a_{21}) = \alpha_1$. Thus

$$a_{12}a_{21} = -2n(a_{12}, a_{21})u \qquad a_{21}a_{12} = -2n(a_{12}, a_{21})u_2 \tag{48}$$

for $a_{ij} \in \mathfrak{O}_{ij}$. We now put

$$n(x, y, z) = n(x, y \wedge z) \in \Phi. \tag{49}$$

It follows from (46) that $n(x, y, z)$ is alternating and Φ-trilinear on $\mathfrak{O}_0$. It is easy to see as in the preceding case that if $i \neq j$ and $a_{ij}, b_{ij}, c_{ij} \in \mathfrak{O}_{ij}$ then (a_{ij}, b_{ij}, c_{ij}) is a basis for $\mathfrak{O}_{ij}$ if and only if $n(a_{ij}, b_{ij}, c_{ij}) \neq 0$.

Now let $\beta \in \mathrm{Aut}\,\mathfrak{O}/\mathfrak{D}$, $\mathfrak{D} = \Phi u_1 + \Phi u_2$. Then $\mathfrak{O}_{ij}{}^{\beta} = \mathfrak{O}_{ij}$. Let B_{ij} be the restriction of β to $\mathfrak{O}_{ij}$. Then (48) implies that $n(a_{12}B_{12}, a_{21}B_{21}) = n(a_{12}, a_{21})$. Now we recall that if $\mathfrak{V}$ and $\mathfrak{V}^*$ is a pair of finite-dimensional vector spaces with a nondegenerate pairing $\langle x, y^* \rangle$, $x \in \mathfrak{V}$, $y^* \in \mathfrak{V}^*$, then any linear transformation B in $\mathfrak{V}$ has a unique adjoint B^* in $\mathfrak{V}^*$ such that $\langle xB, y^* \rangle = \langle x, y^*B^* \rangle$ for all x, y^*. The relation is a symmetric one. We write $\hat{B} = (B^*)^{-1} = (B^{-1})^*$ if B is nonsingular. Then it is clear from the above relation that $B_{21} = \hat{B}_{12}$ (relative to the pairing by n). Also if (a_{ij}, b_{ij}, c_{ij}) is a basis for $\mathfrak{O}_{ij}$ then $n(a_{ij}B_{ij}, b_{ij}B_{ij}, c_{ij}B_{ij}) = n(a_{ij}, b_{ij}, c_{ij})$ and this implies that B_{ij} is unimodular. It is easily seen, as in the discussion of the case II, that, conversely, if B_{12} is any unimodular linear transformation in $\mathfrak{O}_{12}$ and we let B be the linear transformation in $\mathfrak{O}$ which is the identity on the u_i, B_{12} on $\mathfrak{O}_{12}$ and $\hat{B}_{12}$ on $\mathfrak{O}_{21}$, then $\beta \in \mathrm{Aut}\,\mathfrak{O}/\mathfrak{D}$. Thus we have an isomorphism of $\mathrm{Aut}\,\mathfrak{O}/\mathfrak{D}$ onto the unimodular group $SL(\mathfrak{O}_{12})$ in $\mathfrak{O}_{12}$.

If we take into account the results on the groups $\mathrm{Aut}\,\mathfrak{O}/\mathfrak{D}$ and the connection between these and the groups $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ we obtain the following theorem.

THEOREM 8. *Let* $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{K} = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ *where* $(\mathfrak{O}, j)$ *is an octonion algebra and* $(\mathfrak{D}, j)$ *is a composition subalgebra. Assume we have one of the following three cases*: I. $\mathfrak{D}$ *is a quaternion algebra*, II. $\mathfrak{D}$ *is a quadratic field*, III. $\mathfrak{D}$ *is split two-dimensional* ($\cong \Phi \oplus \Phi$). *Then we have the following possibilities for* $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$: I. $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ *is isomorphic to the multiplicative group of elements of norm* 1 *in* $\mathfrak{D}$, II. $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ *is isomorphic to* $SU(\mathfrak{V}, h)$ *where* $\mathfrak{V} = \mathrm{P}^{\perp}$, $\mathrm{P} = \mathfrak{D}$ *in* $\mathfrak{O}$ *and* h *is the hermitian form defined by* (44), III. $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ *is isomorphic to* $SL(\Phi, 3)$.

We remark that, as we indicated at the beginning, this gives a description of $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ for any reduced simple exceptional Jordan algebra $\mathfrak{J}$ and any reduced simple subalgebra of degree three with $\dim \mathfrak{K} = 9$ or 15. Also it should be noted that the results we obtained are considerably sharper than we stated in the theorem since they can be used to describe the action of $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ on $\mathfrak{J}$ in all cases.

We consider next the problem of determining the elements of order two in $\mathrm{Aut}\,\mathfrak{O}$ and $\mathrm{Aut}\,\mathfrak{J}$. In group theory such elements are called the involutions belonging to the group. However, we shall avoid this terminology in the present context for an obvious reason. First, let β be an automorphism of period two in $\mathfrak{O}$. Since β is orthogonal relative to n we have $\mathfrak{O} = \mathfrak{O}_1 \oplus \mathfrak{O}_{-1}$ whese $\mathfrak{O}_i = \{x_i \mid x_i{}^{\beta} = ix_i\}$. Then the $\mathfrak{O}_i$ are nonisotropic subspaces and $\mathfrak{O}_1 \perp \mathfrak{O}_{-1}$. Also $\mathfrak{O}_1$ is a subalgebra.

Choose $q \in \mathfrak{O}_{-1}$ such that $n(q) \neq 0$. Then q_R is a nonsingular linear transformation which interchanges $\mathfrak{O}_1$ and $\mathfrak{O}_{-1}$. Hence $\dim \mathfrak{O}_1 = \dim \mathfrak{O}_{-1} = 4$ and $\mathfrak{D} = \mathfrak{O}_1$ is a quaternion subalgebra. Thus β is the reflection in a quaternion subalgebra in the sense that β is the identity mapping on the quaternion subalgebra and is -1 on its orthogonal complement. Conversely, it is easy to see from the description of $\mathfrak{O}$ as $\mathfrak{D} + \mathfrak{D}q$, $q \in \mathfrak{D}^{\perp}$, that if $\mathfrak{D}$ is any quaternion subalgebra of $(\mathfrak{O}, j)$ then the reflection in $\mathfrak{D}$ is an automorphism of period two in $\mathfrak{O}$.

Next let η be an automorphism of period two in $\mathfrak{J}$, where we assume initially also that the base field is algebraically closed. We have $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{-1}$ where $\mathfrak{J}_i = \{x_i \mid x_i^{\eta} = ix_i\}$ and $\mathfrak{J}_1$ is a subalgebra containing 1. Since η is orthogonal relative to the trace form, $\mathfrak{K} \equiv \mathfrak{J}_1$ and $\mathfrak{J}_{-1}$ are orthogonal and nonisotropic. Thus $\mathfrak{J}_{-1} = \mathfrak{K}^{\perp}$. If $\mathfrak{K} = \Phi 1$ then $\mathfrak{K}^{\perp} = \mathfrak{J}'$ the set of elements of trace 0. Then $a^{\cdot 2} = \alpha 1$, $\alpha \in \Phi$, for every $a \in \mathfrak{J}'$. This is impossible. We note next that if $\mathfrak{B}$ is any subalgebra of $\mathfrak{J}$ which is not isotropic (relative to the generic trace form on $\mathfrak{J}$) then $\mathfrak{B}$ contains no nil ideals $\neq 0$ so $\mathfrak{B}$ is semisimple, and since the base field is algebraically closed, either $\mathfrak{B}$ is one dimensional or it contains an idempotent $\neq 0$, $\neq$ the identity element of $\mathfrak{B}$. In particular, this applies to $\mathfrak{K}$ and since $\mathfrak{K} \neq \Phi 1$ we see that $\mathfrak{K}$ contains an idempotent $e \neq 0, 1$. Since either e or $1 - e$ is primitive we may assume e primitive. Then $\eta \in \mathrm{Aut}\,\mathfrak{J}/\Phi e$ as in §3.

Let $\mathfrak{J} = \mathfrak{J}_0(e) + \mathfrak{J}_1(e) + \mathfrak{J}_{\frac{1}{2}}(e)$ and $\mathfrak{K} = \mathfrak{K}_0(e) + \mathfrak{K}_1(e) + \mathfrak{K}_{\frac{1}{2}}(e)$ be the Peirce decompositions of $\mathfrak{J}$ and $\mathfrak{K}$ relative to e. Then $\mathfrak{K}_i(e) = \mathfrak{K} \cap \mathfrak{J}_i(e)$ and $\mathfrak{J}_0(e) = \Phi f \oplus \mathfrak{V}$ where $f = 1 - e$ and $\mathfrak{V}$ is the subspace of elements of trace 0 in $\mathfrak{J}_0(e)$. Also $\mathfrak{J}_0(e)$ is the Jordan algebra of the nondegenerate symmetric bilinear form Q on $\mathfrak{V}$ (cf. §3). Evidently $\mathfrak{K}_0(e)$ is a subalgebra of $\mathfrak{J}_0(e)$ which is nonisotropic relative to t (or $Q = \frac{1}{2}t$). Hence either $\mathfrak{K}_0(e)$ contains an idempotent $e_2 \neq 0$, $\neq f$ or $\mathfrak{K}_0(e) = \Phi f$. Then $\mathfrak{K} = \Phi e + \Phi f + \mathfrak{K}_{\frac{1}{2}}(e)$ and $\mathfrak{V} \subseteq \mathfrak{K}^{\perp}$. Then $v^{\eta} = -v$ for all v in the nine-dimensional space $\mathfrak{V}$. This contradicts the fact that any element of $\mathrm{Aut}\,\mathfrak{J}/\Phi e$ induces a rotation in $\mathfrak{V}$ relative to Q (§3). Hence this is ruled out and we can conclude that $\mathfrak{K}$ contains three orthogonal primitive idempotents e_i. Let $\mathfrak{J} = \sum_{i \leqq j} \mathfrak{J}_{ij}$, $\mathfrak{K} = \sum_{i \leqq j} \mathfrak{K}_{ij}$ be the corresponding Peirce decompositions of $\mathfrak{J}$ and $\mathfrak{K}$. Then the $\mathfrak{J}_{ij}$ are nonisotropic and the $\mathfrak{K}_{ij}$ are nonisotropic or 0. Moreover, $\mathfrak{J}_{ij} = \mathfrak{K}_{ij} \oplus \mathfrak{K}_{ij}'$ whese $\mathfrak{K}_{ij}' = \mathfrak{J}_{ij} \cap \mathfrak{K}^{\perp}$. We now distinguish two cases: I. $\mathfrak{K}_{ij}$ and $\mathfrak{K}_{ik} \neq 0$ for some choice of $i, j, k = 1, 2, 3$, $i, j, k \neq$. II. $\mathfrak{K}_{ij} = 0 = \mathfrak{K}_{ik}$, i, j, k as in I. We may assume $(i, j, k) = (1, 2, 3)$.

I. In this case we can choose $u_{12} \in \mathfrak{K}_{12}$, $u_{13} \in \mathfrak{K}_{13}$ such that $u_{12}^{\cdot 2} \neq 0$, $u_{13}^{\cdot 2} \neq 0$. Then we have a coordinatization of $\mathfrak{J}$ adapted to the e_i, u_{12}, u_{13} and if we identify $\mathfrak{J}$ with the coordinate algebra $\mathfrak{H}(\mathfrak{O}_3, J_{\gamma})$, we see that the given automorphism η fixes every $1[ij]$. Then η has the form $\sum a[ij] \to \sum a^{\beta}[ij]$ where β is an automorphism of period two in $\mathfrak{O}$. Then we have seen that β is a reflection in the subalgebra $\mathfrak{K} = \mathfrak{H}(\mathfrak{D}_3, J_{\gamma})$ (in the sense that $\eta = 1$ on $\mathfrak{K}$ and -1 on $\mathfrak{K}^{\perp}$). Conversely, it is immediate that any such reflection is an automorphism of period two in $\mathfrak{J}$.

II. Here $\eta = -1$ on $\mathfrak{J}_{12}$ and $\mathfrak{J}_{13}$ so $\eta = 1$ on $\mathfrak{J}_{23} = \mathfrak{J}_{12} \cdot \mathfrak{J}_{13}$. Then η is the element $\zeta_e \neq 1$ in the center of $\mathrm{Aut}\,\mathfrak{J}/\Phi e$ (Corollary 3 to Theorem 4, p. 377).

Our results establish the following theorem in the special case of an algebraically closed base field.

THEOREM 9. *Let $\mathfrak{J}$ be a finite-dimensional exceptional central simple Jordan algebra. Then $\mathfrak{J}$ is reduced if and only if* $\mathrm{Aut}\,\mathfrak{J}$ *contains elements of period two. If the condition holds then any $\eta \in \mathrm{Aut}\,\mathfrak{J}$ of period two is either*: I. *a reflection in a sixteen dimensional central simple subalgebra of degree three, or* II. *the center element $\zeta_e \neq 1$ in a subgroup* $\mathrm{Aut}\,\mathfrak{J}/\Phi e$, *e a primitive idempotent.*

PROOF. Let η be an element of order two in $\mathrm{Aut}\,\mathfrak{J}$ and let $\mathfrak{K}$ be the subalgebra of fixed points, Ω the algebraic closure of the base field Φ. If η is the linear extension of η to $\mathfrak{J}_\Omega$ then $\eta \in \mathrm{Aut}\,\mathfrak{J}_\Omega$, $\eta^2 = 1$, and $\mathfrak{K}_\Omega$ is the subalgebra of $\mathfrak{J}_\Omega$ of fixed points under η. Our determination of η in the algebraically closed case shows that either $\mathfrak{K}_\Omega \cong \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ where $(\mathfrak{D}, j)$ is a (split) quaternion algebra or $\mathfrak{K}_\Omega = \mathfrak{J}_{\Omega 1}(e) \oplus \mathfrak{J}_{\Omega 0}(e)$ a direct sum of the one dimensional ideal $\mathfrak{J}_{\Omega 1}(e) = \Omega e$ and a ten-dimensional ideal which is an algebra of a nondegenerate symmetric bilinear form on a nine-dimensional vector space. In the first case, $\mathfrak{K}$ is central simple and is isomorphic to an algebra $\mathfrak{H}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra with involution (§5.7). Since $\mathfrak{A}_\Omega \cong \mathfrak{D}_3 \cong \Omega_6$ and $\mathfrak{A}$ has an involution, the dimensionality of its division algebra (in the Wedderburn theorem) is a power of two. It follows that either $\mathfrak{A} \cong \Phi_6$ or $\mathfrak{A} \cong \Delta_3$ where Δ is a division algebra. Also if we take into account the dimensionality of $\mathfrak{H}(\mathfrak{A}, J)$ we see that $(\mathfrak{A}, J) \cong (\mathfrak{D}_3, J_\gamma)$ where $(\mathfrak{D}, j)$ is quaternion with standard involution. Thus $\mathfrak{K} \cong \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ in the first case. Evidently this implies that $\mathfrak{J}$ is reduced and η is of the type I specified in the statement of the theorem. Next assume that $\mathfrak{K}_\Omega = \mathfrak{J}_{\Omega 1}(e) \oplus \mathfrak{J}_{\Omega 0}(e)$ as indicated. Then it is easily seen that $\mathfrak{K}$ itself has the same kind of structure so again $\mathfrak{J}$ is reduced. Moreover, it is clear that $\eta = \zeta_e$ for a primitive idempotent e. Conversely, it is clear that if $\mathfrak{J}$ is reduced then it does contain automorphisms of period two of both of the indicated types.

We shall call the elements of order two in $\mathrm{Aut}\,\mathfrak{J}$ of the types indicated automorphism of order two of *type* I and *type* II respectively.

EXERCISES

1. Show that the centralizer $\mathfrak{C}(\eta)$ of an automorphism of period two and type I in $\mathrm{Aut}\,\mathfrak{J}$ is isomorphic to $\mathrm{Aut}\,\mathfrak{D} \times SL(\mathfrak{D})$ where $(\mathfrak{D}, j)$ is a quaternion subalgebra of the coefficient algebra of $\mathfrak{J}$, $\mathrm{Aut}\,\mathfrak{D}$ is its automorphism group and $SL(\mathfrak{D})$ is the subgroup of elements of (generic) norm 1 in $\mathfrak{D}$. (Note: $\mathrm{Aut}\,\mathfrak{D} \cong \mathfrak{D}^*/\Phi^*$ where $\mathfrak{D}^*$ is the multiplicative group of units of $\mathfrak{D}$, by the Skolem-Noether theorem.)

2. Show that the centralizer $\mathfrak{C}(\eta)$ of $\eta \in \mathrm{Aut}\,\mathfrak{J}$ of period two and type II is a subgroup $\mathrm{Aut}\,\mathfrak{J}/\Phi e \cong \mathrm{Spin}(\mathfrak{V}, Q)$ (as in §3).

3. Show that if $\mathfrak{J}$ is split and e_i and f_i, $i = 1,2,3$, are primitive orthogonal idempotents then there exists an automorphism η of $\mathfrak{J}$ such that $e_i^{\eta} = f_i$.

4. Show that if $\mathfrak{J}$ is split then any two automorphisms of $\mathfrak{J}$ of period two and type II are conjugate in Aut$\mathfrak{J}$.

5. Assume Φ is either finite or algebraically closed. Prove that any two automorphisms of $\mathfrak{J}$ of period two and type I are conjugate in Aut$\mathfrak{J}$.

6. Let Φ be finite, $|\Phi| = q$ and let P be the set of primitive idempotents in $\mathfrak{J}$. Show that

$$|P| = q^8(q^8 + q^4 + 1).$$

Show that

$$|\mathrm{Aut}\mathfrak{J}| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1).$$

6. **Invariant factor theorem for split $\mathfrak{J}$.** In this section we shall prove the following analogue of the invariant factor theorem in a matrix algebra Φ_n.

THEOREM 10. *Let $\mathfrak{J}$ be a finite-dimensional split exceptional simple Jordan algebra. Then $a, b \in \mathfrak{J}$ are in the same orbit under* Aut$\mathfrak{J}$ *if and only if a and b have the same minimum polynomials and the same generic minimum polynomials in $\mathfrak{J}$.*

We note first that the classical invariant factor theorem implies that two matrices $a, b \in \Phi_3$ are similar if and only if they have the same minimum polymonials and characteristic polynomials. We recall also that if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are subalgebras of $\mathfrak{J}$ isomorphic to Φ_3^+ then there exists an $\eta \in \mathrm{Aut}\mathfrak{J}$ such that $\mathfrak{K}_1^{\eta} = \mathfrak{K}_2$ (Theorem 3 p. 370). It is clear from these two results that the sufficiency of the condition given in Theorem 10 will follow if we can show that any element $a \in \mathfrak{J}$ can be imbedded in a subalgebra $\mathfrak{K} \cong \Phi_3^+$. Since the necessity of the condition in Theorem 10 is trivial the theorem will follow from the following imbedding theorem.

THEOREM 11. *Let $\mathfrak{J}$ be as in Theorem 10. Then any element $a \in \mathfrak{J}$ can be imbedded in a subalgebra $\mathfrak{K} \cong \Phi_3^+$.*

The proof will be based on several lemmas as follow.

LEMMA 1. *Any element a of a split finite-dimensional exceptional simple Jordan algebra is contained in a subalgebra $\mathfrak{K}$ of the form $\mathfrak{H}(\mathrm{P}_3, \mathfrak{J}_\gamma)$ where* P *is a subalgebra of a split octonion algebra generated by* 1 *and a single element u.*

PROOF (MCCRIMMON). Let $\{e_i\}$ be a reducing set of idempotents for $\mathfrak{J}$, $\mathfrak{J} = \sum \mathfrak{J}_{ij}$ the corresponding Peirce decomposition and write $a = \sum \alpha_i e_i + \sum_{i<j} a_{ij}$ where $\alpha_i \in \Phi$ and $a_{ij} \in \mathfrak{J}_{ij}$. We have the following possibilities for the a_{ij}: I. At least two $Q(a_{ij}) \neq 0$, II. Only one $Q(a_{ij}) \neq 0$, III. All $Q(a_{ij}) = 0$.

In case I we consider the subalgebra $\mathfrak{K}$ generated by the e_i and a. As in the proof of Lemma 2 on p. 362, $\mathfrak{K}$ is generated by the e_i and a_{ij}. If $Q(a_{ij}) \neq 0$ then e_i and e_j are connected idempotents in $\mathfrak{K}$. Since this is the case for two a_{ij} and connectedness for orthogonal idempotents is a transitive relation, all the e_i are connected in $\mathfrak{K}$. Hence by the Coordinatization Theorem and the fact that the e_i are absolutely primitive in $\mathfrak{J}$ (and hence in $\mathfrak{K}$) it follows that $\mathfrak{K} \cong \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ (cf. the proof of Theorem 5.8 on p. 203). Since $\dim \mathfrak{K} \leqq 9$, by Lemma 2 on p. 362 we have $\dim \mathrm{P} \leqq 2$ so the result is clear in this case.

We shall obtain reductions of cases II and III to case I by introducing new reducing sets of idempotents $\{e_i'\}$ relative to which the situation is as in I. We note that if $b_{12} \in \mathfrak{J}_{12}$ satisfies $Q(b_{12}) = 1$ so $b_{12}^{\cdot 2} = e$ where $e = e_1 + e_2$ then

$$(50) \qquad e_1' = \tfrac{1}{2}(e + b_{12}), \qquad e_2' = \tfrac{1}{2}(e - b_{12}), \qquad e_3' = e_3$$

are nonzero orthogonal idempotents and so form a reducing set. If $a = \sum \alpha_i' e_i' + \sum_{i<j} a_{ij}'$ is the Peirce decomposition of a relative to the e_i' then we claim that

$$(51) \qquad a_{12}' = \tfrac{1}{2}(\alpha_1 - \alpha_2)(e_1 - e_2) + (a_{12} - Q(a_{12}, b_{12})\, b_{12})$$

$$(52) \qquad a_{13}' = \tfrac{1}{2} a_{13} + \tfrac{1}{2} a_{23} + b_{12} \cdot a_{13} - b_{12} \cdot a_{23}$$

$$(53) \qquad a_{23}' = \tfrac{1}{2} a_{13} + \tfrac{1}{2} a_{23} - b_{12} \cdot a_{13} - b_{12} \cdot a_{23}.$$

We have

$$\begin{aligned} a_{12}' &= 2\{e_1' a e_2'\} = \tfrac{1}{2}\{e + b_{12}, a, e - b_{12}\} = \tfrac{1}{2}\{eae\} - \tfrac{1}{2}\{b_{12} a b_{12}\} \\ &= \tfrac{1}{2}(\alpha_1 e_1 + \alpha_2 e_2 + a_{12}) - \tfrac{1}{2}(\alpha_2 e_1 + \alpha_1 e_2) - \tfrac{1}{2}\{b_{12} a_{12} b_{12}\} \quad \text{(by PD10)} \\ &= \tfrac{1}{2}(\alpha_1 - \alpha_2)(e_1 - e_2) + (a_{12} - Q(a_{12}, b_{12}) b_{12}), \end{aligned}$$

which is (51). Also

$$\begin{aligned} a_{13}' &= 2\{e_1' a e_3'\} = \{e + b_{12}, a, e_3\} \\ &= \tfrac{1}{2} a_{13} + \tfrac{1}{2} a_{23} + b_{12} \cdot a_{13} + b_{12} \cdot a_{23}, \end{aligned}$$

which is (52). Similarly, we obtain (53).

Case II is trivial if two of the $a_{ij} = 0$ since in this case $a \in \mathfrak{H}(\mathrm{P}_3, J_\gamma)$ where P is generated by 1 and u where $a_{ki} = u[ki]$ and $a_{ij} = a_{jk} = 0$, $i, j, k \neq$. We now assume at least two $a_{ij} \neq 0$. By symmetry, we may suppose $a_{13} \neq 0$, $Q(a_{23}) \neq 0$, $Q(a_{12}) = Q(a_{13}) = 0$. Since $\mathfrak{J}_{12}$ is eight dimensional and the restriction of Q to $\mathfrak{J}_{12}$ is nondegenerate and of maximal Witt index it is easy to see that if $c_{12} \neq 0$ in $\mathfrak{J}_{12}$ and α is any element of Φ then there exists a $b_{12} \in \mathfrak{J}_{12}$ such that $Q(b_{12}) = 1$ and $Q(b_{12}, c_{12}) = \alpha$. We note next that since $a_{13} \neq 0$ and $Q(a_{23}) \neq 0$, $c_{12} = a_{13} \cdot a_{23}$ $\neq 0$. Hence, since $|\Phi| \geqq 3$, there exsits a $b_{12} \in \mathfrak{J}_{12}$ such that $Q(b_{12}) = 1$ and

$2Q(b_{12}, c_{12}) \neq \pm \frac{1}{2} Q(a_{23})$. We use this b_{12} to define the reducing set of idempotents $\{e_i'\}$ as above. We claim that we now have $Q(a_{13}') \neq 0$, $Q(a_{23}') \neq 0$. Now

$$\begin{aligned} Q(a'_{13})e'_3 &= a'_{13}{}^{\cdot 2} . e'_3 = a'_{13}{}^{\cdot 2} . e_3 \\ &= (\tfrac{1}{2}a_{13} + b_{12} \cdot a_{23} + \tfrac{1}{2}a_{23} + b_{12} \cdot a_{13})^{\cdot 2} . e_3 \\ &= (\tfrac{1}{2}a_{13} + b_{12} . a_{23})^{\cdot 2} . e_3 + (\tfrac{1}{2}a_{23} + b_{12} . a_{13})^{\cdot 2} . e_3 \\ &= Q(\tfrac{1}{2}a_{13} + b_{12} . a_{23})e_3 + Q(\tfrac{1}{2}a_{23} + b_{12} . a_{13})e_3 \\ &= [Q(a_{13}, b_{12} . a_{23}) + Q(b_{12} . a_{23}) + \tfrac{1}{2}Q(a_{23}) + Q(a_{23}, b_{12} . a_{13})]e_3 \\ &= [2Q(b_{12}, c_{12}) + \tfrac{1}{2} Q(a_{23})]e_3 \neq 0 \end{aligned}$$

(since $Q(a_{ij} . a_{jk}) = Q(a_{ij})Q(a_{jk})$ if $i, j, k \neq$ and Q is an associative bilinear form). Hence $Q(a_{13}') \neq 0$ and similarly $Q(a_{23}') \neq 0$. Thus we have case I relative to the e_i'.

Case III is trivial if $a_{12} = a_{23} = a_{13} = 0$ so we may assume $a_{12} \neq 0$, $Q(a_{12}) = Q(a_{23}) = Q(a_{13}) = 0$. We can choose $b_{12} \in \mathfrak{J}_{12}$ such that $Q(b_{12}) = 1$ and $Q(a_{12}, b_{12}) \neq \pm \frac{1}{2}(\alpha_1 - \alpha_2)$. We use this to define the e_i' as before. Then

$$\begin{aligned} Q(a'_{12})e &= a'_{12}{}^{\cdot 2} = [\tfrac{1}{2}(\alpha_1 - \alpha_2)(e_1 - e_2) + (a_{12} - Q(a_{12}, b_{12}) b_{12})]^{\cdot 2} \\ &= \tfrac{1}{4}(\alpha_1 - \alpha_2)^2 e + (a_{12} - Q(a_{12} . b_{12}) b_{12})^{\cdot 2} \\ &= \tfrac{1}{4}(\alpha_1 - \alpha_2)^2 e - 2Q(a_{12}, b_{12})^2 e + Q(a_{12}, b_{12})^2 e \\ &= [\tfrac{1}{4}(\alpha_1 - \alpha_2)^2 - Q(a_{12}, b_{12})^2] e \neq 0. \end{aligned}$$

Hence $Q(a_{12}') \neq 0$ and we have either case I or II relative to the e_i'. This completes the proof.

Lemma 2. *Any element u of a split octonion algebra can be imbedded in a split quaternion subalgebra.*

Proof. We may assume $\bar{u} = -u \neq 0$. If $n(u)1 = u\bar{u} \neq 0$ we choose v so that $\bar{v} = -v$, $v \perp u$ and $n(v) = -1$. This can be done since the norm form has maximal Witt index. Then $v^2 = 1$ and the subalgebra generated by $1, u, v$ is split quaternion. Next assume $n(u) = 0$. Then we can choose v so that $\bar{v} = -v$, $n(v) = 0$, $n(u, v) = \frac{1}{2}$. Then if we put $w = u + v$, $z = u - v$ we have $n(w) = 1$, $n(z) = -1$, $w \perp z$. Hence the subalgebra generated by $1, w, z$ is a split quaternion and this contains the element u.

Lemmas 1 and 2 evidently imply that if a is any element of a finite-dimensional split exceptional simple Jordan algebra then a can be imbedded in a subalgebra of the form $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ where $(\mathfrak{D}, j)$ is a split quaternion algebra. Then $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ is isomorphic to $\mathfrak{H}(\Phi_6, J_S)$ where J_S is an involution defined by a skew symmetric matrix. Then $\mathfrak{H}(\Phi_6, J_S)$ is isomorphic to the Jordan algebra of linear transformations in a six dimensional vector space which are selfadjoint relative to a nondegenerate skew bilinear form f. Hence the proof of Theorem 11 can be completed by proving

LEMMA 3. *Let f be a nondegenerate skew bilinear form on a $2n$-dimensional vector space $\mathfrak{V}$, $\mathfrak{H}$ the Jordan algebra of linear transformations in $\mathfrak{V}$ which are selfadjoint relative to f. Then any $A \in \mathfrak{H}$ can be imbedded in a subalgebra $\mathfrak{K}$ containing 1 such that $\mathfrak{K} \cong \Phi_n{}^+$.*

PROOF. We shall obtain the proof by deriving a "rational" canonical form for A. Let $\mu(\lambda) = \lambda^m - \alpha_1\lambda^{m-1} + \cdots + (-1)^m\alpha_m$ be the minimum polynomial of A and let $x \in \mathfrak{V}$ have order polynomial $\mu(\lambda)$. Then $(x, xA, \cdots, xA^{m-1})$ are linearly independent and $f(xA^i, xA^j) = f(xA^{i+j}, x) = -f(x, xA^{i+j}) = -f(xA^i, xA^j) = 0$. Hence the subspace $\sum_0^{m-1}\Phi(xA^i)$ is totally isotropic. Since f is nondegenerate there exists a $y \in \mathfrak{V}$ such that $f(xA^{m-1}, y) = 1$, $f(xA^i, y) = 0$, $0 \leqq i \leqq m-2$. Then $f(xA^{m-2}, yA) = 1$, $f(xA^i, yA) = 0$, $0 \leqq i \leqq m-3$, $f(xA^{m-3}, yA^2) = 1$, $f(xA^i, yA^2) = 0$, $0 \leqq i \leqq m-4$, etc. It follows that $(y, yA, \cdots, yA^{m-1})$ are linearly independent so the order polynomial of y is also $\mu(\lambda)$. Moreover, $\sum_0^{m-1}\Phi(yA^i)$ is totally isotropic and $\mathfrak{V}_1 \equiv \sum\Phi(xA^i) + \sum\Phi(yA^i)$ is not isotropic. Hence $\mathfrak{V} = \mathfrak{V}_1 \oplus \mathfrak{V}_1{}^\perp$. Since A is selfadjoint and maps $\mathfrak{V}_1$ into itself we have $\mathfrak{V}_1{}^\perp A \subseteq \mathfrak{V}_1{}^\perp$. We can repeat the process with the restriction of A to $\mathfrak{V}_1{}^\perp$. Continuing in this way we obtain a decomposition of $\mathfrak{V} = \mathfrak{X} \oplus \mathfrak{Y}$ where $\mathfrak{X}$ and $\mathfrak{Y}$ are totally isotropic n-dimensional subspaces of $\mathfrak{V}$ such that $\mathfrak{X}A \subseteq \mathfrak{X}$ and $\mathfrak{Y}A \subseteq \mathfrak{Y}$. Let $\mathfrak{K}$ be the subalgebra of $\mathfrak{H}$ of linear transformations B such that $\mathfrak{X}B \subseteq \mathfrak{X}$, $\mathfrak{Y}B \subseteq \mathfrak{Y}$. It is clear that f gives a nondegenerate bilinear pairing of $\mathfrak{X}$ and $\mathfrak{Y}$. Hence if C is a linear transformation in $\mathfrak{X}$ there exists a unique linear transformation C^* in $\mathfrak{Y}$ such that $f(xC, y) = f(x, yC^*)$, $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$. Then the linear mapping B in $\mathfrak{V}$ which coincides with C on $\mathfrak{X}$ and with C^* on $\mathfrak{Y}$ is selfadjoint and so $B \in \mathfrak{K}$. It is now clear that $\mathfrak{K}$ is isomorphic to $\mathrm{Hom}_\Phi(\mathfrak{X}, \mathfrak{X})^+$ and so to $\Phi_n{}^+$. Since $A \in \mathfrak{K}$ we have our result.

This completes the proof of Theorem 11 and hence also of the invariant factor theorem (Theorem 10).

EXERCISES

1. (Taussky-Zassenhaus). Let $\mathfrak{V}$ be a finite-dimensional vector space over a field Φ and let $A \in \mathrm{Hom}_\Phi(\mathfrak{V}, \mathfrak{V})$. Show that there exists a nondegenerate symmetric bilinear form f on $\mathfrak{V}$ such that A is selfadjoint relative to f.

2. Let $\mathfrak{K} = \mathfrak{H}(\mathrm{P}_3, J_\gamma)$, P a quadratic field. Show that any element $a \in \mathfrak{K}$ can be imbedded in a subalgebra isomorphic to an algebra $\mathfrak{H}(\Phi_3, J_\gamma)$.

3. Let $\mathfrak{J}$ be a reduced simple exceptional Jordan algebra. Show that any element $a \in \mathfrak{J}$ can be imbedded in a subalgebra $\mathfrak{K}$ which is isomorphic to an algebra $\mathfrak{H}(\Phi_3, J_\gamma)$ and that if a is nilpotent then we may take $\gamma = \mathrm{diag}\{1, -1, 1\}$.

4. Prove that any two nilpotent elements of $\mathfrak{J}$ (as in 3) of index three nilpotency are in the same orbit under Aut$\mathfrak{J}$.

5. Let $\mathfrak{J}$ be a split exceptional simple Jordan algebra, Γ a cubic subfield. Show that every isomorphism of Γ/Φ in $\mathfrak{J}/\Phi$ can be extended to an automorphism. Show that if Φ is finite and $|\Phi| = q$ then the number of isomorphisms of Γ into $\mathfrak{J}$ is $q^{12}(q^8 - 1)(q^4 - 1)$. Use this and exercise 6, p. 389 to prove that the order of the group Aut$\mathfrak{J}/\Gamma$ of automorphisms of $\mathfrak{J}$ leaving fixed the elements of Γ is

$$q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1).$$

6. Show that if $\mathfrak{J}$ is split then $M_1(\mathfrak{J})$ is transitive on the set of elements of generic norm 1 in $\mathfrak{J}$.

7. Assume the base field Φ of $\mathfrak{J}$ is finite, $|\Phi| = q$. Show that if $N_\alpha, \alpha \in \Phi$, is the subset of $a \in \mathfrak{J}$ such that $n(a) = \alpha$ then $|N_\alpha| = |N_\beta|$ if $\alpha, \beta \neq 0$. Prove that

$$|N_0| = (q - 1)(q^{21} + q^{17} - q^{12})$$

and

$$|N_1| = q^{12}(q^9 - 1)(q^5 - 1).$$

Use this and exercise 6 p. 389 to prove that the order of $M_1(\mathfrak{J})$ is

$$q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1).$$

8. Prove that any finite-dimensional simple exceptional Jordan algebra is regular (see p. 179 for definition).

7. **Moufang projective planes.** In this section we shall associate with a reduced exceptional simple Jordan algebra $\mathfrak{J}$ with coefficient algebra $\mathfrak{O}$ a division algebra, a projective plane $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$. The planes $\mathfrak{P}(\mathfrak{J})$ and $\mathfrak{P}(\mathfrak{J}')$ determined by isotopic Jordan algebras $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic. Since $\mathfrak{J}$ and $\mathfrak{J}'$ are isotopic if and only if they have isomorphic coefficient algebras we may regard $\mathfrak{P}$ as determined by the coefficient algebra and, accordingly, we shall write also $\mathfrak{P} = \mathfrak{P}(\mathfrak{O})$. The projective planes $\mathfrak{P}(\mathfrak{O})$ are isomorphic to some which were first defined in an entirely different manner by R. Moufang [1]. The present construction of Moufang planes was first given by P. Jordan in [3] for the case: Φ, the field of reals. This was rediscovered by Freudenthal in [1] and formulated for arbitrary base fields of characteristic $\neq 2, 3$ by Springer in [2]. We shall develop the theory here for arbitrary base fields of characteristic $\neq 2$. Our primary interest will be in the "geometric algebra" of Moufang planes, which is concerned with the algebraic formulations of the basic geometric concepts related to these planes.

We suppose first that $\mathfrak{J}$ is an arbitrary reduced simple exceptional Jordan algebra. As in §2, we let $\Pi = \Pi(\mathfrak{J})$ be the set of elements of rank one in $\mathfrak{J}$ $(a \neq 0, a \times a = 0)$ and we let $[\Pi]$ be the set of rays $[a] = \Phi^* a$, $a \in \Pi$. We can use $[\Pi]$ to define an incidence structure $\mathfrak{P}(\mathfrak{J})$. We recall that such a structure is an ordered triple (π, λ, ι) where π and λ are nonvacuous sets called respectively the set of points and the set of lines of the structure and ι is a correspondence of π to λ, that is, ι is a subset of the product set $\pi \times \lambda$ (cf. Pickert [1, p. 2]). If $P \in \pi$ and $l \in \lambda$ satisfy $(P, l) \in \iota$ then the point P and the line l are said to be *incident* or P is said to lie on l and l is said to pass through P. We now define $\mathfrak{P}(\mathfrak{J})$ by putting $\pi = [\Pi]$, $\lambda = [\Pi]$ and defining *incidence* of $P = [x]$, $l = [u]$ by the condition $t(x, u) = 0$, that is, $x \perp u$ relative to the trace form.

If $\mathfrak{J}/\Phi$ and $\mathfrak{J}'/\Phi'$ are finite-dimensional Jordan algebra with 1 (or more generally strictly power associative algebras) over the fields Φ and Φ' respectively, then a mapping η of $\mathfrak{J}$ into $\mathfrak{J}'$ is called a *norm semisimilarity* if η is a bijective semilinear mapping and there exists a $\rho \neq 0$ in Φ' such that $n(a^\eta) = \rho n(a)^\sigma$ where σ is the isomorphism of Φ onto Φ' associated with η ($(\alpha x)^\eta = \alpha^\sigma x^\eta$, $\alpha \in \Phi$, $x \in \mathfrak{J}$). Suppose now that $\mathfrak{J}$ and $\mathfrak{J}'$ are reduced simple exceptional Jordan algebras. Then we shall show that a norm semisimilarity η of $\mathfrak{J}/\Phi$ onto $\mathfrak{J}'/\Phi'$ determines an isomorphism $[\eta]$ of the incidence structure $\mathfrak{P}(\mathfrak{J})$ onto $\mathfrak{P}(\mathfrak{J}')$, that is, bijective mappings $P \to P'$, $l \to l'$ of the sets of points and the sets of lines such that for any point P and line l, P is incident to l if and only if P' is incident to l'. We note first that if $(a, b, c) = \frac{1}{2}\Delta^c{}_a\, \Delta^b n$ in $\mathfrak{J}$ (and similarly in $\mathfrak{J}'$) then we have the formula (12′) for (a, b, c): $(a, b, c) = \frac{1}{2}[n(a + b + c) - n(a + b) - n(b + c) - n(a + c) + n(a) + n(b) + n(c)]$. Evidently this implies that $(a^\eta, b^\eta, c^\eta) = \rho(a, b, c)^\sigma$ if η is a norm semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ (with multiplier ρ). We recall also that $(a, b, c) = t(a \times b, c)$, which implies that $a \in \Pi$ if and only if $a \neq 0$ and $(a, a, x) = 0$ for all $x \in \mathfrak{J}$. It is now clear that $a \in \Pi(\mathfrak{J})$ if and only if $a^\eta \in \Pi(\mathfrak{J}')$ for the norm semisimilarity η and that $[a] \to [a^\eta]$ is a bijection of $[\Pi(\mathfrak{J})]$ onto $[\Pi(\mathfrak{J}')]$.

We recall next that every semilinear mapping η of $\mathfrak{J}/\Phi$ into $\mathfrak{J}'/\Phi'$ defines a unique mapping η^* of $\mathfrak{J}'$ into $\mathfrak{J}$ by the condition that

$$t(a', b^\eta) = t(a'^{\eta^*}, b)^\sigma \tag{54}$$

for all $a' \in \mathfrak{J}'$, $b \in \mathfrak{J}$, where the t on the left is the generic trace in $\mathfrak{J}'$ that on the right is the generic trace in $\mathfrak{J}$, and σ is the isomorphism of Φ onto Φ' associated with η. Then η^* is a semilinear mapping with associated isomorphism σ^{-1} of Φ' onto Φ. If η is bijective so is η^* and $\hat{\eta} = (\eta^*)^{-1}$ is a bijective semilinear mapping of $\mathfrak{J}$ onto $\mathfrak{J}'$ with associated isomorphism σ.

Now let η be a norm semisimilarity with multiplier ρ. Then $t(a^\eta \times b^\eta, c^\eta) = (a^\eta, b^\eta, c^\eta) = \rho(a, b, c)^\sigma = \rho t(a \times b, c)^\sigma$ gives

$$(a^\eta \times b^\eta)^{\eta^*} = \rho^{\sigma^{-1}} a \times b \tag{55}$$

or

$$a^\eta \times b^\eta = \rho(a \times b)^{\hat{\eta}}. \tag{55′}$$

Suppose further that $1^\eta = 1$, the identity element of $\mathfrak{J}'$. Then the condition $n(a^\eta) = \rho n(a)^\sigma$ for $a = 1$ gives $\rho = 1$. Also $n((1 \pm a)^\eta) = n(1 \pm a)^\sigma$ and the formula $n(\lambda 1 - a) = m_a(\lambda) = \lambda^3 - t(a)\lambda^2 + \frac{1}{2}[t(a)^2 - t(a^2)]\lambda - n(a)$ imply $t(a^\eta) = t(a)^\sigma$,

$t((a^\eta)^{\cdot 2}) = t(a^{\cdot 2})^\sigma$. Also the formula for $a \times b$ gives $a \times 1 = \frac{1}{2}t(a)1 - \frac{1}{2}a$, $1 \times 1 = 1$. Then, by (55′) with $\rho = 1$, we obtain $1^{\hat\eta} = (1 \times 1)^{\hat\eta} = 1^\eta \times 1^\eta = 1$ and $(a \times 1)^{\hat\eta} = a^\eta \times 1$ so $\frac{1}{2}t(a)^\sigma 1^{\hat\eta} - \frac{1}{2}a^{\hat\eta} = \frac{1}{2}t(a^\eta)1 - \frac{1}{2}a^\eta$. Hence $a^{\hat\eta} = a^\eta$ and $(a \times b)^\eta = a^\eta \times b^\eta$ by (55′). It follows that $(a^{\cdot 2})^\eta = (a^\eta)^{\cdot 2}$ and $(a\,.\,b)^\eta = a^\eta\,.\,b^\eta$. Thus η is a semilinear isomorphism of $\mathfrak{J}/\Phi$ onto $\mathfrak{J}'/\Phi'$, that is, η is a bijective semilinear mapping which preserves the product: $(a\,.\,b)^\eta = a^\eta\,.\,b^\eta$. Conversely, if η is a semilinear isomorphism of $\mathfrak{J}/\Phi$ onto $\mathfrak{J}'/\Phi'$ then 1^η is the identity of $\mathfrak{J}'$ and the proof of Theorem 6.1 (VI) (p.) carries over to show that $n(a^\eta) = n(a)^\sigma$, $a \in \mathfrak{J}$, so η is a norm semisimilarity. Also we have $\hat\eta = \eta$ as before.

Now assume η is an arbitrary norm semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$. Then 1^η is invertible in $\mathfrak{J}'$ and this element is the identity element of the isotope $(\mathfrak{J}, u')$ where $u' = (1^\eta)^{\cdot -1}$. The formula relating the generic norm of a Jordan algebra and an isotope (p. 242) shows that the identity mapping is a norm similarity of an algebra onto any isotope. Hence η is a norm semisimilarity of $\mathfrak{J}$ onto the isotope $(\mathfrak{J}', u')$ sending 1 into the identity of the latter algebra. Hence, by the result just proved, η is a semilinear isomorphism of $\mathfrak{J}$ onto $(\mathfrak{J}', u')$ and the mapping $\eta^{*(u')}$ defined as in (54) by the generic trace forms in $\mathfrak{J}$ and $(\mathfrak{J}', u')$ satisfies $\eta^{*(u')} = \eta^{-1}$. Now it is clear that η as mapping of $\mathfrak{J}$ onto $\mathfrak{J}'$ is the product of the semilinear isomorphism η of $\mathfrak{J}$ onto $(\mathfrak{J}', u')$ and the identity mapping considered as norm similarity of $(\mathfrak{J}', u')$ onto $\mathfrak{J}'$. The formula (77) of Chapter VI (p. 246) shows that for the identity mapping $1_{\mathfrak{J}'}$ of $(\mathfrak{J}', u')$ onto $\mathfrak{J}'$ we have $1_{\mathfrak{J}'}{}^{*'} = U_{1^\eta}{}^{(u')}$, where the $*'$ denotes the mapping associated with $1_{\mathfrak{J}'}$ in place of η in (54) using the generic trace forms of $(\mathfrak{J}', u')$ and $\mathfrak{J}'$, and $U_{1^\eta}{}^{(u')}$ is the U-operator in $(\mathfrak{J}', u')$ determined by 1^η. Now we have $\eta^* = (\eta 1_{\mathfrak{J}'})^* = 1_{\mathfrak{J}'}{}^{*'}\eta^{*(u')} = U_{1^\eta}{}^{(u')}\eta^{*(u')} = U_{1^\eta}{}^{(u')}\eta^{-1}$ and $\hat\eta = \eta(U_{1^\eta}{}^{(u')})^{-1}$ which shows that $\hat\eta$ is a norm semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$. Consequently, $[a] \to [a^\eta]$ is a bijection of $[\Pi(\mathfrak{J})]$ onto $[\Pi(\mathfrak{J}')]$.

We can now associate with the given norm semisimilarity η of $\mathfrak{J}/\Phi$ onto $\mathfrak{J}'/\Phi'$ a mapping $[\eta]$ of the incidence structure $\mathfrak{P}(\mathfrak{J})$ onto $\mathfrak{P}(\mathfrak{J}')$ by means of the following formulas:

$$(56)\qquad [\eta]:\quad \begin{aligned} P &= [x] \to P' = [x^\eta], \qquad P \in \pi(\mathfrak{J}),\\ l &= [u] \to l' = [u^{\hat\eta}], \qquad l \in \lambda(\mathfrak{J}). \end{aligned}$$

As we have seen, this is bijective for the points of $\mathfrak{P}(\mathfrak{J})$ and $\mathfrak{P}(\mathfrak{J}')$ and the lines of $\mathfrak{P}(\mathfrak{J})$ and $\mathfrak{P}(\mathfrak{J}')$. Also we have from (54) that $t(a^{\hat\eta}, b^\eta) = t(a, b)^\sigma$ for $a, b \in \mathfrak{J}$ which show that $a \perp b$ if and only if $a^{\hat\eta} \perp b^\eta$. Hence it is clear that $[\eta]$ is an isomorphism of the incidence structures.

A first consequence of this result, is that if $\mathfrak{J}$ and $\mathfrak{J}'$ have isomorphic coefficient algebras (so $\mathfrak{J}$ and $\mathfrak{J}'$ are isotopic) then the incidence structures $\mathfrak{J}$ and $\mathfrak{J}'$ are isomorphic. Thus we see that $\mathfrak{P}(\mathfrak{J})$ depends only on the coefficient algebra $\mathfrak{O}$ of $\mathfrak{J}$, and we may write $\mathfrak{P}(\mathfrak{O})$ for $\mathfrak{P}(\mathfrak{J})$.

We shall now assume that $\mathfrak{O}$ is a division algebra and we shall show that in this case $\mathfrak{P}(\mathfrak{O})$ is a projective plane. We recall that an incidence structure is a *projective plane* if it satisfies the following axioms.

I. Any two distinct points are incident with one and only one line.

II. Any two distinct lines are incident with one and only one point.

III. There exist four points no three of which are incident with the same line.

We can state these conditions in a more relaxed form that two distinct points determine a unique line (or lie on a unique line) and two distinct lines meet (or intersect) in a unique point. Also there exist four points no three of which are collinear.

We shall now verify these axioms for $\mathfrak{P}(\mathfrak{O})$, $\mathfrak{O}$ an octonion division algebra.

LEMMA 1. *Let A, $B \in \Pi(\mathfrak{J})$ where $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{O}$ a division algebra, and assume A, B are linearly independent. Then $A \times B \in \Pi(\mathfrak{J})$.*

PROOF. Let A be as in (1): $A = \sum_1^3 \alpha_i e_{ii} + a[23] + b[31] + c[12]$, $a, b, c \in \mathfrak{O}$. Suppose first that $B = e_{11}$. Then $A \times e_{11} = \frac{1}{2}(\alpha_3 e_{22} + \alpha_2 e_{33} - a[23])$. If $A \times e_{11} = 0$ we have $\alpha_2 = \alpha_3 = a = 0$. Now we have shown in the proof of Lemma 2 of §2 that if $A \in \Pi$ then either A is a nonzero multiple of an element of the form given in (19) or all the entries of A are $\neq 0$. Hence $A \times e_{11} = 0$ implies A is a multiple of e_{11} contrary to hypothesis. Hence $A \times e_{11} \neq 0$. The conditions that $A \times A = 0$ given by (6) include $\alpha_2 \alpha_3 = \gamma_3{}^{-1} \gamma_2 n(a)$ which implies that

$$A \times e_{11} = \tfrac{1}{2}(\alpha_3 e_{22} + \alpha_2 e_{33} - a[23])$$

satisfies $(A \times e_{11}) \times (A \times e_{11}) = 0$. Hence $A \times e_{11} \in \Pi$. Now let A and B be arbitrary linearly independent elements of Π. By Lemma 2 of §2, there exists an $\eta \in \mathrm{M}(\mathfrak{J})$ such that $B^\eta = e_{11}$. If ρ is the multiplier of η then (56′) gives $\rho (A \times B)^{\hat{\eta}} = A^\eta \times e_{11}$ and A^η, e_{11} are linearly independent elements of Π. Then $A^\eta \times e_{11} \in \Pi$ which implies that $A \times B \in \Pi$.

LEMMA 2. *Let $A, B, \mathfrak{J}$ be as in Lemma 1. Then $C \in \Pi(\mathfrak{J})$ satisfies $t(A, C) = 0 = t(B, C)$ if and only if C is a multiple of $A \times B$.*

PROOF. We have $t(A, A \times B) = (A, B, A) = (A, A, B) = t(A \times A, B) = 0$ and similarly $t(B, A \times B) = 0$. Hence any multiple of $A \times B$ satisfies the conditions. Next let C satisfy the conditions. Assume first that $C = e_{11}$. Then $t(A, C) = 0$ implies that the $(1,1)$ entry of A is 0. Then (19) shows that A has the form $\alpha_2 e_{22} + \alpha_3 e_{33} + a[23]$. Similarly, B has the form $\beta_2 e_{22} + \beta_3 e_{33} + b[23]$. Direct calculation shows that $A \times B$ is a multiple of $C = e_{11}$. Now suppose C is any element of Π satisfying $t(A, C) = 0 = t(B, C)$. By Lemma 2 of §2, there exists an $\eta \in \mathrm{M}(\mathfrak{J})$ such that $C^\eta = e_{11}$. Then $t(A^{\hat{\eta}}, e_{11}) = t(A, C) = 0$ and $t(B^{\hat{\eta}}, e_{11}) = 0$. Hence $A^{\hat{\eta}} \times B^{\hat{\eta}}$ is a multiple of e_{11} by the result just proved. By (56′) for $\hat{\eta}$ in place of η we see that this implies that $(A \times B)^\eta$ is a multiple of $e_{11} = C^\eta$. Then $A \times B$ is a multiple of C.

We can now prove

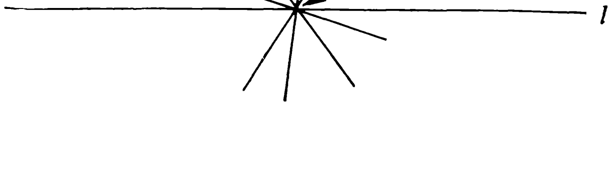
THEOREM 12. *Let $\mathfrak{J}$ be a reduced exceptional simple Jordan algebra whose coefficient algebra $\mathfrak{O}$ is an octonion division algebra. Then the incidence structure $\mathfrak{P}(\mathfrak{J}) = \mathfrak{P}(\mathfrak{O})$ is a projective plane.*

PROOF. Let $P_1 = [x_1]$, $P_2 = [x_2]$ be distinct points in $\mathfrak{P}(\mathfrak{J})$ so $x_1, x_2 \in \Pi(\mathfrak{J})$ and these elements are linearly independent. Put $u = x_1 \times x_2$. Then $u \in \Pi$ and $t(x_i, u) = 0$. Thus $[u]$ is a line incident with P_1 and P_2. Now let $[v]$ be any line incident with P_1 and P_2. Then, by Lemma 2, $[v] = [u]$. Hence $[x_1 \times x_2]$ is the only line incident to P_1 and P_2. Thus axiom I holds. In the same way axiom II holds. To see that axiom III holds we take $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ and take $P_i = [e_{ii}]$, for $i = 1,2,3$ and P_4 to be any element of Π all of whose entries are nonzero. (The proof of Lemma 2 of §2 shows that there exist such elements in Π.) Now if $P = [x]$, $Q = [y]$, $R = [z]$ where $x, y, z \in \Pi$ then it is clear from Lemma 2 that P, Q and R are collinear if and only if $(x, y, z) = 0$. It follows directly that no three of the four points P_1, P_2, P_3, P_4 are collinear.

EXERCISES

1. Show that $\mathfrak{P}(\mathfrak{J})$ is not a projective plane if $\mathfrak{J}$ is split.
2. Verify that if $\mathfrak{J}$ is reduced and not split then $P = [x]$ is collinear with $[e_1]$ and $[e_2]$ where the e_i are primitive orthogonal idempotents if and only if $x \in \Phi e_1 + \Phi e_2 + \mathfrak{J}_{12}$.

8. **Elations.** It is customary to call an isomorphism of a projective plane on a second one a *projective transformation* and an automorphism of a projective plane a *collineation*. If (π, λ, ι) is a projective plane then the dual structure (λ, π, ι') where ι' is the set of pairs (l, P) such that $(P, l) \in \iota$ is a projective plane. An isomorphism of a plane onto its dual is called a *correlation*. It is clear that the collineations form a group of transformations on $\lambda \cup \pi$ and it is easy to see that this group is determined by its action on π or on λ. It is easy to see also that if a collineation has a line l of fixed points (that is, all points incident with l are fixed) and it has two distinct fixed points not on l then the collineation is the identity. Let l be a line, P a point on l. Then we define an *elation* τ with *center* P and *axis* l to be a collineation $\neq 1$ which leaves fixed every point on l and every line through P:



It is clear that the center and axis are uniquely determined and if X is any point not on the axis then the image X^τ is on the line PX (incident with X and P). Also since τ^{-1} is an elation with center P and axis l it is clear that $X^\tau \neq P$. It is easy to see from the definition and the axioms that τ is determined by specifying its center and axis and the image X' of any point X not on the axis. The elations with a fixed center and axis together with the identity mapping form a subgroup of the collineation group. The subgroup of the collineation group generated by all the elations is called the *little projective group* (or *unimodular group*) of the projective plane. If τ is an elation with center P and axis l and ε is a collineation then $\varepsilon^{-1}\tau\varepsilon$ is an elation with center P^ε and axis l^ε. Hence the little projective group is an invariant subgroup of the collineation group.

We now consider the projective plane $\mathfrak{P}(\mathfrak{J})$ defined by the reduced simple exceptional Jordan algebra $\mathfrak{J}$ with coefficient algebra $\mathfrak{O}$, an octonion division algebra. Let $\mathrm{N}(\mathfrak{J})$ denote the group of norm semisimilarities of $\mathfrak{J}$ onto $\mathfrak{J}$. We have seen that every $\eta \in \mathrm{N}(\mathfrak{J})$ determines a collineation $[\eta]$ in $\mathfrak{P}(\mathfrak{J})$ as in (57). The mapping $\eta \to [\eta]$ is a homomorphism of $\mathrm{N}(\mathfrak{J})$ into the collineation group $\Gamma(\mathfrak{P})$ of $\mathfrak{P}$, whose image we denote as $[\mathrm{N}(\mathfrak{J})]$. Similarly, the subgroup $\mathrm{M}(\mathfrak{J})$ of norm similarities (and its various subgroups $\mathrm{M}_1(\mathfrak{J})$, $\mathrm{M}_1^{(1)}(\mathfrak{J})$, $\mathrm{M}_1^{(2)}(\mathfrak{J})$, $\mathrm{M}_1^{(3)}(\mathfrak{J})$ previously defined) defines a subgroup $[\mathrm{M}(\mathfrak{J})]$ ($[\mathrm{M}_1(\mathfrak{J})]$ etc.) of $\Gamma(\mathfrak{P})$. We shall now determine the kernel of the homomorphism $\eta \to [\eta]$ of $\mathrm{N}(\mathfrak{J})$ in the following

LEMMA 1. *If $\eta \in \mathrm{N}(\mathfrak{J})$ then $[\eta] = 1$ if and only if $\eta \in \Phi^*$, that is, η is a multiplication by a nonzero element of Φ.*

PROOF. If ν is a norm similarity of $\mathfrak{J}$ onto a second reduced exceptional simple Jordan algebra then $\eta \to \nu^{-1}\eta\nu$ and $[\eta] = [\nu]^{-1}[\eta][\nu]$ are isomorphisms of $\mathrm{N}(\mathfrak{J})$ onto $\mathrm{N}(\mathfrak{J}')$ and of $[\mathrm{N}(\mathfrak{J})]$ onto $[\mathrm{N}(\mathfrak{J}')]$ respectively. Hence it is enough to prove the lemma for a suitable isotope $\mathfrak{J}'$ of $\mathfrak{J}$. We take this to have the form $\mathfrak{H}(\mathfrak{O}_3, J_1)$. Consider the four points $[e_{ii}]$, $i = 1,2,3$, $[A]$ where $A = \sum_{i,j} e_{ij}$. We have $(e_{11}, e_{22}, e_{33}) = t(e_{11} \times e_{22}, e_{33}) = t(\frac{1}{2}e_{33}, e_{33}) = \frac{1}{2}$ and similarly, $(e_{ii}, e_{jj}, A) = \frac{1}{2}$ if $i \neq j$. Now suppose η is a norm semisimilarity of $\mathfrak{H} = \mathfrak{H}(\mathfrak{O}_3, J_1)$ such that $[\eta] = 1$ and let ρ be the multiplier and σ the automorphism in Φ of η. If $a, b, c \in \mathfrak{H}$ then we have $(a^\eta, b^\eta, c^\eta) = \rho(a,b,c)^\sigma$. We have $e_{ii}{}^\eta = \lambda_i e_{ii}$, $A^\eta = \lambda A$. Since $(\frac{1}{2}1)^\sigma = \frac{1}{2}1$ these conditions give $\lambda_1\lambda_2\lambda_3 = \rho$, $\lambda_i\lambda_j\lambda = \rho$ if $i \neq j$. Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and so replacing η by $\lambda^{-1}\eta$ we may assume $e_{ii}{}^\eta = e_{ii}$, $A^\eta = A$. Then $1^\eta = 1$ so η is a semilinear automorphism of $\mathfrak{H}$ (by the results of §7). Since $e_{ii}{}^\eta = e_{ii}$ we have $\mathfrak{H}_{ij}{}^\eta \subseteq \mathfrak{H}_{ij}$ for the the Peirce spaces $\mathfrak{H}_{ij}$ relative to the e_{ii}. Now let $i \neq j$ and let $q \in \mathfrak{O}$. Then the element $e_{ii} + n(q)e_{jj} + q[ij] \in \Pi$. Hence $(e_{ii} + n(q)e_{jj} + q[ij])^\eta = e_{ii} + n(q)^\sigma e_{jj} + q[ij]^\eta$ is a multiple of $e_{ii} + n(q)e_{jj} + q[ij]$. This gives $q[ij]^\eta = q[ij]$. It follows that η is the identity mapping, which proves the lemma.

It is clear from this result that the kernel of the homomorphism $\eta \to [\eta]$ of $\mathrm{M}_1(\mathfrak{J})$ consists of the multiplications by elements $\omega \in \Phi$ such that $\omega^3 = 1$. Hence the kernel of $M_1(\mathfrak{J})$ is either the identity or it has order three. The main result we shall prove in this section is that $M_1(\mathfrak{J}) = M_1^{(3)}(\mathfrak{J})$ and the image of this group under the indicated homomorphism is the little projective group of $\mathfrak{P}(\mathfrak{J})$.

Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$, $\mathfrak{R} = \mathfrak{H}(P_3, J_\gamma)$, where (P, j) is a quadratic subfield of $(\mathfrak{D}, j)$. As in §2, let $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ be the subgroup of $M_1(\mathfrak{J})$ of elements of the form $U_{A_1}U_{A_2}\cdots U_{A_r}$ whese the $A_i \in \mathfrak{R}$ and $\Pi_1^r\ n(A_i) = 1$. Then we have seen in Lemma 1 of §2 that $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ coincides with the set of mappings $T_B: X \to B^*XB$ where $B \in P_3$. We saw also that this group is generated by the mappings $T_{Q_{ij}}$ where $Q_{ij} = 1 + qe_{ij}$, $q \in P$, $i \neq j$. Moreover, (20) gives the form of $T_{Q_{ij}}$. According to this we see that every element of the form $\alpha e_{jj} + \beta e_{kk} + a[jk]$, $j \neq k$, is fixed under $T_{Q_{ij}}$. These include all the elements $A \in \Pi$ such that $t(e_{ii}, A) = 0$. Hence $[T_{Q_{ij}}]$ leaves fixed every point on the line $[e_{ii}]$. We saw also that $T_{Q_{ij}*} = T_{Q^*_{ij}}$ and $Q_{ij}{}^* = 1 + \gamma_j{}^{-1}\gamma_i\bar{q}e_{ji}$. Hence $\hat{T}_{Q_{ij}} = T_{R_{ji}}$ where $r = -\gamma_j{}^{-1}\gamma_i\bar{q}$ and consequently $[T_{Q_{ij}}]$ leaves fixed every line through the point $[e_{jj}]$. Since $t(e_{jj}, e_{ii}) = 0$, $[e_{jj}]$ is on the line $[e_{ii}]$ so $[T_{Q_{ij}}]$ is an elation with $[e_{jj}]$ as center and $[e_{ii}]$ as axis. Since the $T_{Q_{ij}}$ generate $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ we see that the image $[M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})]$ of $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ in the collineation group is contained in the little projective group $\Lambda(\mathfrak{P})$ of $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$. We note also that $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ contains every $T_{\omega 1}$ where $\omega \in \Phi$ and $\omega^3 = 1$. Since this is the multiplication by ω^2 we see that $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ contains the kernel of the homomorphism of $M_1(\mathfrak{J})$ into the collineation group.

We recall that $M_1{}^{(3)}(\mathfrak{J})$ has been defined essentially as the subgroup of $M_1(\mathfrak{J})$ generated by all the subgroups $M_1{}^{(2)}{}_{\mathfrak{J}}(\mathfrak{R})$ where $\mathfrak{R}$ is a reduced simple subalgebra of degree three and nine dimensions. Clearly, $[M_1{}^{(3)}(\mathfrak{J})] \subseteq \Lambda(\mathfrak{P})$ and $M_1{}^{(3)}(\mathfrak{J})$ contains every mapping $T_{Q_{ij}}$ where $i \neq j$ and $q \in \mathfrak{D}$. By (20), we have $e_{ii}T_{Q_{ij}} = e_{ii} + \gamma_j{}^{-1}\gamma_i n(q)e_{jj} + q[ij]$, which shows that there exists a $T_{Q_{ij}}$ such that $[T_{Q_{ij}}]$ maps the point $[e_{ii}]$ into any point $\neq [e_{jj}]$ on the line $[e_{kk}]$ joining $[e_{ii}]$ and $[e_{jj}]$, $k \neq i, j$. This implies that $[M_1{}^{(3)}(\mathfrak{J})]$ contains every elation with center $[e_{jj}]$ and axis $[e_{ii}]$. We note also that $[M_1{}^{(3)}(\mathfrak{J})]$ is transitive on the set of points of $\mathfrak{P}(\mathfrak{J})$. Since $M_1{}^{(3)}(\mathfrak{J})$ is mapped into itself by the mapping $\eta \to \hat{\eta}$ (by Lemma 1 of §2) it is clear that $[M_1{}^{(3)}(\mathfrak{J})]$ is transitive on the set of lines of $\mathfrak{P}(\mathfrak{J})$.

We wish to show that $[M_1{}^{(3)}(\mathfrak{J})]$ contains every elation of $\mathfrak{P}(\mathfrak{J})$. For this it suffices to show that if l is any line and P is a point on l then there exists an $\varepsilon = [\eta]$, $\eta \in M_1{}^{(3)}(\mathfrak{J})$, such that $[e_{11}]^\varepsilon = l$, $[e_{22}]^\varepsilon = P$ for the line $[e_{11}]$ and the point $[e_{22}]$. Since $[M_1{}^{(3)}(\mathfrak{J})]$ is transitive on the lines we may assume $l = [e_{11}]$ and P is on $[e_{11}]$.

We show first that if $[X]$ is any point not on the line $[e_{11}]$ then there exists a $\zeta \in M_1{}^{(3)}(\mathfrak{J})$ such that $[\zeta]$ leaves fixed every point on the line $[e_{11}]$ and maps $[X]$ into the point $[e_{11}]$. Write $X = \sum_1^3 \xi_i e_{ii} + x[23] + y[31] + z[12]$, $x, y, z \in \mathfrak{D}$. Since X is not on the line $[e_{11}]$, $\xi_1 \neq 0$. We have $e_{22}T_{Q_{12}} = e_{22}$, $e_{33}T_{Q_{13}} = e_{33}$, by (20). The formulas (20) show also that q can be chosen so that $XT_{Q_{12}} = \xi_1 e_{11} + \xi_2{}' e_{22} + \xi_3{}' e_{33} + x'[23] + y'[31]$. Since $X' \in \Pi$ and $\xi_1 \neq 0$, X' is a multiple of an element of the form $e_{11} + \gamma_3{}^{-1}\gamma_2 n(r) e_{33} + r[13]$. Then it is clear that we have an elation with center $[e_{33}]$ and axis $[e_{11}]$ mapping $[X']$ into $[e_{11}]$. Thus we have a product of two elations with axis $[e_{11}]$ mapping $[X]$ into $[e_{11}]$, so this mapping has the required properties.

Now let $P = [X]$ be on the line $[e_{11}]$. We wish to show that there exists an $[\eta]$, $\eta \in \mathrm{M}_1{}^{(3)}(\mathfrak{J})$, such that $[e_{11}]^{[\eta]} = [e_{11}]$ for the line $[e_{11}]$ and $[X]^{[\eta]} = [e_{22}]$ (which is on the line $[e_{11}]$). Assume first that $[X] \neq [e_{33}]$. Then $[X]$ is not on the line $[e_{22}]$ so the result just proved (for the line $[e_{11}]$ and point $[e_{11}]$) shows that there exists a $[\eta]$, $\eta \in \mathrm{M}_1{}^{(3)}(\mathfrak{J})$, such that $[\eta]$ fixes every point on the line $[e_{22}]$ and $[X]^{[\eta]} = [e_{22}]$. Since $[\eta]$ maps $[e_{33}] \to [e_{33}]$ and $[X] \to [e_{22}]$ it is clear that $[\eta]$ maps the line $[e_{11}]$ into itself and the point $[X]$ into $[e_{22}]$, as required. Suppose next that $[X] = [e_{33}]$. Then the required result will follow by showing that there exists an $\eta \in \mathrm{M}_1{}^{(3)}(\mathfrak{J})$ such that $[e_{22}]^{[\eta]} = [e_{33}]$ and $[e_{33}]^{[\eta]} = [e_{22}]$. Put $B = e_{11} + e_{23} - e_{32}$. Then $\eta = T_B \in \mathrm{M}_1{}^{(3)}(\mathfrak{J})$ and a simple calculation shows that $[e_{22}]^{[\eta]} = [e_{33}]$, $[e_{33}]^{[\eta]} = [e_{22}]$. We remark also for later use that $[e_{11}]^{[\eta]} = [e_{11}]$.

The result we have established, that given any line l and point P on l there exists an $[\eta]$, $\eta \in \mathrm{M}_1{}^{(3)}(\mathfrak{J})$, such that $[e_{11}] \to l$, $[e_{22}] \to P$, and the facts that $[\mathrm{M}_1{}^{(3)}(\mathfrak{J})] \subseteq \Lambda(\mathfrak{P})$ and contains all elations with center $[e_{22}]$ and axis $[e_{11}]$ imply the following

LEMMA 2. $[\mathrm{M}_1{}^{(3)}(\mathfrak{J})] = \Lambda(\mathfrak{P})$.

We shall show next that $\mathrm{M}_1(\mathfrak{J}) = \mathrm{M}_1{}^{(3)}(\mathfrak{J})$. Since $\mathrm{M}_1{}^{(3)}(\mathfrak{J})$ contains the kernel of the homomorphism $\eta \to [\eta]$ of $\mathrm{M}_1(\mathfrak{J})$ into the collineation group this will follow if we can show that $[\mathrm{M}_1(\mathfrak{J})] \subseteq \Lambda(\mathfrak{P})$.

An ordered k-tuple $(P_1, P_2, \cdots, P_k)$ of points of a projective plane will be called *independent* if no three of the points P_i, P_j, P_k, $i, j, k \neq$, are collinear. We now prove the following important transitivity property:

LEMMA 3. *The little projective group $\Lambda(\mathfrak{P})$ of $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$ is transitive on independent ordered triples of points.*

PROOF. It is enough to show that if (Q_1, Q_2, Q_3) is an independent ordered triple of points then there exists an element of $\Lambda(\mathfrak{P})$ such that $Q_i \to [e_{ii}]$, $i = 1, 2, 3$. Since we have seen that $\Lambda(\mathfrak{P}) = [\mathrm{M}_1{}^{(3)}(\mathfrak{J})]$ is transitive on points we may assume also that $Q_1 = [e_{11}]$. Suppose first that Q_2 is not on the line $[e_{22}]$. Then, as we saw above, there exists an element of $\Lambda(\mathfrak{P})$ leaving fixed every point on the line $[e_{22}]$, so in particular the points $[e_{11}]$ and $[e_{33}]$, and sending Q_2 into $[e_{22}]$. This reduces the proof to the case $Q_1 = [e_{11}]$, $Q_2 = [e_{22}]$. Then Q_3 is not on the line $[e_{33}]$ so there is an element of $\Lambda(\mathfrak{P})$ leaving the line $[e_{33}]$ pointwise fixed and sending Q_3 into $[e_{33}]$. This proves the result if Q_2 is not on the line $[e_{22}]$ (and $Q_1 = [e_{11}]$). Now assume Q_2 is on the line $[e_{22}]$. Then Q_2 is not on the line $[e_{33}]$ since $Q_2 \neq [e_{11}] = Q_1$. Then the argument shows that we have an element of $\Lambda(\mathfrak{P})$ fixing $[e_{11}]$ and $[e_{22}]$ and sending Q_2 into $[e_{33}]$. Since we have seen that we have an element in $\Lambda(\mathfrak{P})$ fixing the point $[e_{11}]$ and interchanging $[e_{22}]$ and $[e_{33}]$ we have a reduction of this case to the previous one.

We proceed with the proof that $[M_1(\mathfrak{J})] = \Lambda(\mathfrak{P}) = [M_1{}^{(3)}(\mathfrak{J})]$. Since we may replace $\mathfrak{J}$ by an isotope we may assume $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_1)$. Let $\eta \in M_1(\mathfrak{J})$. To prove that $\eta \in M_1{}^{(3)}(\mathfrak{J})$ it is sufficient, in view of Lemma 3, to do this for any η such that $e_{ii}{}^{\eta} = \lambda_i e_{ii}$, $\lambda_i \in \Phi^*$, $i = 1,2,3$. Put $A = \sum e_{ij}$, so $A \in \Pi(\mathfrak{J})$. Assume first that $A^{\eta} \in \mathfrak{H}(\Phi_3, J_1)$. It follows from (18) (or (56)) and $2e_{ii} \times e_{jj} = e_{kk}$, $2(1 \times A) = A$ that $e_{ii}{}^{\hat{\eta}}$, $A^{\hat{\eta}} \in \mathfrak{H}(\Phi_3, J_1)$. Applying this again with the roles of $\hat{\eta}$ and η interchanged we see that $(e_{ii} \times A)^{\eta} \in \mathfrak{L} \doteq \mathfrak{H}(\Phi_3, J_1)$. Since $2e_{ii} \times A = e_{jj} + e_{kk} - e_{jk} - e_{kj}$, $i,j,k \neq$, the six elements e_{ii}, $e_{ii} \times A$ form a basis for $\mathfrak{L}$. Hence $\mathfrak{L}^{\eta} = \mathfrak{L}$ and the restriction of η to $\mathfrak{L}$ is in $M_1(\mathfrak{L})$. Then, by Theorem 6.8 (see also p. 249) there exists a matrix $M \in \Phi_3$ such that $X^{\eta} = \rho M^* X M$ for all $X \in \mathfrak{L}$ where $\rho^3 \det M^2 = 1$. Then $\det M$ is a cube and replacing M by a suitable multiple permits us to assume $\rho = 1$. Then $\det M = \pm 1$ so replacing M by $-M$, if necessary, gives $\det M = 1$. Then $T_M \in M_1^{(3)}(\mathfrak{J})$ and we have $X^{\eta} = XT_M$ for all $X \in \mathfrak{L}$. If we replace η by $\zeta = \eta T_M{}^{-1}$ we obtain $\zeta \in M_1(\mathfrak{J})$, $e_{ii}{}^{\zeta} = e_{ii}$, $A^{\zeta} = A$. Then $1^{\zeta} = 1$ and ζ is an automorphism which fixed every e_{ii}. Then $\zeta \in M_1{}^{(3)}(\mathfrak{J})$ by Corollary 2 to Theorem 4 (p. 377). Hence $\eta \in M_1{}^{(3)}(\mathfrak{J})$ if $A^{\eta} \in \mathfrak{L}$. We now drop this supplementary condition and note that no three of the points $[e_{ii}]$, $[A^{\eta}]$ are collinear since no three of $[e_{ii}]$, $[A]$ are collinear. We shall therefore have a reduction to the case which we have settled by proving

LEMMA 4. *Let C be an element of $\Pi(\mathfrak{J})$, $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_1)$ such that the point $[C]$ does not lie on any of the lines $[e_{ii}]$, $i = 1,2,3$. Then there exists an element $\zeta \in M_1{}^{(3)}(\mathfrak{J})$ such that $e_{ii}{}^{\zeta} = \lambda_i e_{ii}$, $\lambda_i \in \Phi^*$, and $C^{\zeta} \in \mathfrak{H}(\Phi_3, J_1)$.*

PROOF. The conditions on C imply that $C = \sum \alpha_i e_{ii} + a[23] + b[31] + c[12]$ where every α_i, a, b and c are nonzero. Put $Q = e_{11} + c^{-1}e_{22} + ce_{33}$. This is in $\Phi[c]_3$ and has determinant 1, so $T_Q \in M_1{}^{(3)}(\mathfrak{J})$. Then CT_Q has the same form as C and for this element $c = 1$. Hence we may suppose C has this property. Then the conditions that $C \times C = 0$ (see (6)) imply that $a, b, c = 1$ are contained in a quadratic subfield P of $\mathfrak{O}$. Put $R = b^{-1}e_{11} + \bar{b}e_{22} + e_{33} = SU$ where $S = b^{-1}e_{11} + \bar{b}e_{22} + b\bar{b}^{-1}e_{33}$, $U = e_{11} + e_{22} + b\bar{b}^{-1}e_{33}$. Then $CT_R = \sum \alpha_i' e_{ii} + a'[23] + 1[31] + 1[12]$ and $CT_R = CT_S T_U$. Now $\det S = 1$ so $T_S \in M_1{}^{(3)}(\mathfrak{J})$ and the mapping T_U restricted to $\mathfrak{H}(P_3, J_1)$ is an automorphism since $n(b^{-1}\bar{b}) = 1$ and U is unitary. Also $e_1 T_U = e_1$. Hence T_U can be extended to an automorphism in $\mathfrak{J}$ leaving e_1 fixed. By Corollary 2 to Theorem 4, this extension is contained in $M_1{}^{(3)}(\mathfrak{J})$. Thus there exists a $\zeta \in M_1{}^{(3)}(\mathfrak{J})$ such that $C^{\zeta} = CT_R$. Since $CT_R \in \Pi$ the conditions imply that $a' \in \Phi$. Hence $C^{\zeta} = CT_R \in \mathfrak{H}(\Phi_3, J_1)$. It is clear also that $e_{ii}{}^{\zeta} = e_{ii}T_R = \lambda_i e_{ii}$.

This result and Lemma 2 complete the proof of the following

THEOREM 13 (JACOBSON). *Let $\mathfrak{J}$ be a reduced exceptional simple Jordan algebra whose coefficient algebra is a division algebra. Then* $M_1(\mathfrak{J}) = M_1{}^{(3)}(\mathfrak{J})$

and the image of this group under the homomorphism $\eta \to [\eta]$ into the collineation group of $\mathfrak{P}(\mathfrak{J})$ is the little projective group $\Lambda(\mathfrak{P})$.

Since the kernel of the indicated homomorphism is $\Phi^* \cap \mathrm{M}_1(\mathfrak{J})$ we see that $\Lambda(\mathfrak{P}) \cong \mathrm{M}_1(\mathfrak{J})/(\Phi^* \cap \mathrm{M}_1(\mathfrak{J}))$.

EXERCISES

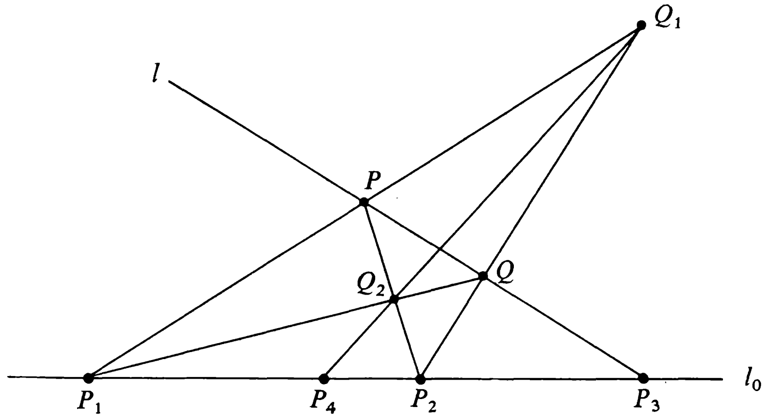
1. Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_\gamma)$, $\mathfrak{O}$ an octonion division algebra. Show that $\mathrm{M}_1(\mathfrak{J})$ is generated by the mappings $T_{Q_{ij}}$, $Q = 1 + qe_{ij}$, $i \neq j$, $q \in \mathfrak{O}$. (Note that this is analogous to the theorem that the unimodular group of matrices over a field is generated by elementary matrices.)

2. Prove that any two elations in $\mathfrak{P}(\mathfrak{J})$ are conjugate in $\Lambda(\mathfrak{P})$.

3. Let $\Lambda_1$be an invariant subgroup $\neq 1$ in $\Lambda = \Lambda(\mathfrak{P})$ and let $\eta \neq 1$ be in Λ_1. Choose P so that $P^\eta \neq P$ and let τ be an elation with center P^η and axis PP^η. Put $\omega = \eta\tau\eta^{-1}\tau^{-1}$. Show that $\omega \neq 1$ is in Λ_1 and leaves the line $l = PP^\eta$ invariant. Show also that if, in addition, η leaves PP^η invariant then ω leaves fixed every point of l. Hence show that Λ_1 contains an element $\eta \neq 1$ which has a line l of fixed points. Show that if this is not an elation then it has a fixed point F not on l. Then if τ is an elation with axis l, $\omega = \eta\tau\eta^{-1}\tau^{-1} \neq 1$ is an elation. Thus show that Λ_1 contains an elation and hence $\Lambda_1 = \Lambda$ and Λ is a simple group.

4. Prove that if l is any line of $\mathfrak{P}$, P a point on l and X and X' points such that P, X, X' are distinct and collinear then there exists an elation with center P, axis l mapping X into X'.

9. **Harmonicity.** A projective plane $\mathfrak{P}$ is said to be *harmonic* if the harmonic conjugate P_4 of a point P_3 relative to the pair of points P_1, P_2 is uniquely determined. Here P_1, P_2, P_3 are distinct points on a line l_0 and P_4 is determined by the following construction: Let l be any line through P_3 distinct from l_0, P, Q points on l such that P, Q, P_3 are distinct. Consider the four lines P_1P, P_1Q, P_2P, P_2Q and let Q_1 be the intersection of P_1P and P_2Q, Q_2 the intersection of P_2P and P_1Q. Then Q_1 and Q_2 are distinct and P_4 is the intersection of Q_1Q_2 with l_0. Thus we have the following figure:



We define an *involution* of $\mathfrak{P}$ to be a collineation of $\mathfrak{P}$ of period two. The harmonic property is a consequence of the existence of certain involutions and elations in $\mathfrak{P}$. For, suppose there exists an involution $\varepsilon(l, P_1, P_2)$ leaving l pointwise fixed and interchanging P_1 and P_2. Then P_1P and P_2P are interchanged by $\varepsilon(l, P_1, P_2)$ and so are P_1Q and P_2Q. Hence the points Q_1 and Q_2 are interchanged and so the line Q_1Q_2 is invariant under $\varepsilon(l, P_1, P_2)$. Then P_4 is fixed under $\varepsilon(l, P_1, P_2)$ since the lines Q_1Q_2 and $P_1P_2 = l_0$ are invariant and P_4 is their intersection. Since a collineation which has a line of fixed points and has two distinct fixed points not on this line in necessarily the identity it is clear that P_4 is the only fixed point of $\varepsilon(l, P_1, P_2)$ not on l and so P_4 is determined by $\varepsilon(l, P_1, P_2)$. Now let l' be another line through P_3 distinct from l_0 and let $\varepsilon(l', P_1, P_2)$ be an involution having l' as line of fixed points and interchanging P_1 and P_2. Assume also that we have an elation τ with axis l_0 and center P_2 mapping l into l'. Then $\varepsilon(l', P_1, P_2) = \tau^{-1}\varepsilon(l, P_1, P_2)\tau$ and so $P_4 = P_4^{\tau}$ is the fixed point of $\varepsilon(l', P_1, P_2)$ which is not on l'. Hence we see that P_4 is independent also of the choice of l.

Now let $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$ be the projective plane associated with the reduced simple exceptional Jordan algebra $\mathfrak{J}$ with coefficient algebra $\mathfrak{D}$ a division algebra. Let the notations be as in the last paragraph and let P' be the intersection of the line P_2P with l'. Then it follows easily from the results of §8 (see ex.4) that there exists an elation τ with center P_2 and axis l_0 mapping P into P'. Then $l^{\tau} = l'$ as required above. Hence the harmonic property of $\mathfrak{P}$ will be established if we can show that given any line l and distinct points P_1, P_2 not on l then there exists an involution $\varepsilon(l, P_1, P_2)$ having l as line of fixed points and interchanging P_1 and P_2.

Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ and let η be the linear mapping in $\mathfrak{J}$ such that $e_{ii}{}^{\eta} = e_{ii}$, $a[12]^{\eta} = -a[12], a[13]^{\eta} = -a[13], a[23]^{\eta} = a[23]$, $a \in \mathfrak{D}$. Then it is clear that η is an automorphism of period two in $\mathfrak{J}$ so $[\eta]$ is an involution in $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$. The points on the line $[e_{11}]$ have the form $[\alpha e_{22} + \beta e_{33} + a[23]]$. Hence every point on the line $[e_{11}]$ is fixed by $[\eta]$. Since $[\eta] \neq 1$ we can choose a point P_1 such that $P_2 = P_1{}^{[\eta]} \neq P_1$. Then P_1 and P_2 are interchanged by the involution $[\eta]$. Hence $[\eta] = \varepsilon([e_{11}], P_1, P_2)$, the involution having $[e_{11}]$ as line of fixed points and interchanging the distinct points P_1, P_2. We prove next the following transitivity property of the subgroup $[\mathrm{M}(\mathfrak{H})]$ of the collineation group.

LEMMA. $[\mathrm{M}(\mathfrak{J})]$ *is transitive on independent ordered quadruples of points.*

PROOF. There is no loss in generality in assuming $\mathfrak{J} = \mathfrak{H}(\mathfrak{D}_3, J_1)$ and we do this to simplify the calculations. In view of Lemmas 3 and 4 of §8 it is sufficient to prove that if $P_i = Q_i = [e_{ii}]$, $i = 1, 2, 3$, $P_4 = A = \sum e_{ij}$ and $Q_4 = B \in \mathfrak{H}(\Phi_3, J_1)$ then there exists a $\zeta \in \mathrm{M}(\mathfrak{J})$ such that $P_j{}^{[\zeta]} = Q_j$, $j = 1, 2, 3, 4$. We may assume that the first row of the matrix B is $(1, \alpha, \beta)$, $\alpha, \beta \in \Phi$. Since B is symmetric and has rank one we have

$$B = \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & \alpha^2 & \alpha\beta \\ \beta & \alpha\beta & \beta^2 \end{bmatrix}.$$

Let $D = \operatorname{diag}\{1,\alpha,\beta\}$. Then $D \in \mathfrak{J}$ and we have $AU_D = B$. Moreover, $U_D \in \mathrm{M}(\mathfrak{J})$ since $n(XU_D) = n(D)^2 n(X)$. Then $\zeta = U_D$ has the required properties since we also have $[e_{ii}]^{[\zeta]} = [e_{ii}]$, $i = 1,2,3$.

Now let l be any line, Q_1 and Q_2 distinct points not on l. We have seen that there exists an involution having $[e_{11}]$as line of fixed points. Let P_1, P_2 be distinct points interchanged by this involution which we denote as $\varepsilon([e_{11}], P_1, P_2)$. Let P_3, P_4 be points on $[e_{11}]$ which are distinct and are distinct from the intersection of $[e_{11}]$ and the line P_1, P_2. In a similar manner choose Q_3, Q_4 on the given line l. Then (P_1, P_2, P_3, P_4) and (Q_1, Q_2, Q_3, Q_4) satisfy the condition of the Lemma so there exists a $\zeta \in \mathrm{M}(\mathfrak{J})$ such that $P_j^{[\zeta]} = Q_j$, $j = 1,2,3,4$. Then $[\zeta]^{-1}\varepsilon([e_{11}], P_1, P_2)[\zeta]$ is an involution having l as line of fixed points and interchanging Q_1, Q_2. As we have seen, the existence of such involutions for any l and points Q_1, Q_2 implies the following

THEOREM 14. *The plane* $\mathfrak{P}(\mathfrak{J}) = \mathfrak{P}(\mathfrak{O})$ *is harmonic.*

We remark that the harmonic property was the primary one singled out by Moufang for the projective planes which she constructed. It is well known that this property holds in any Desarguesian projective plane. Moreover, the property is equivalent to a weaker form of Desargues' theorem called the little Desargues' theorem (Pickert [1, p. 187] or M. Hall [1, p. 17]).

EXERCISES

1. Call an involution ε of $\mathfrak{P}(\mathfrak{J})$ of *first kind* if it has a line of fixed points. Show that any such involution is contained in the little projective group and any two such involutions are conjugate in the full collineation group but not necessarily in the little projective group.

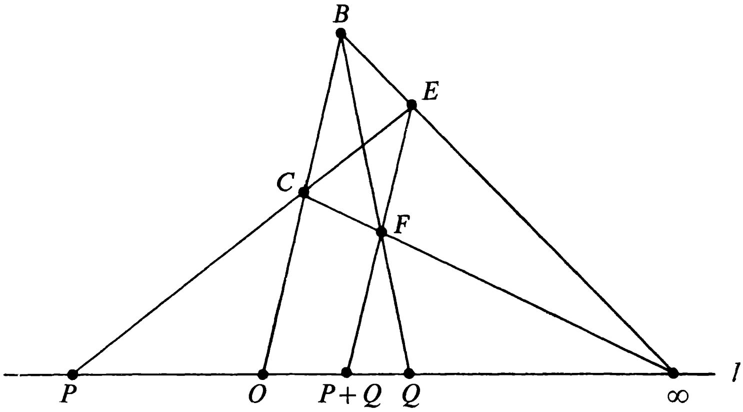
2. Let $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_1)$, $\mathfrak{O}$ an octonion division algebra. Let β be a reflection in a quaternion subalgebra $\mathfrak{Q}$ of $\mathfrak{O}$: β is linear and is 1 on $\mathfrak{Q}$ and -1 on $\mathfrak{Q}^{\perp}$. Let η be the linear mapping in $\mathfrak{J}$ such that $a[ij] \to a^{\beta}[ij]$, $i,j = 1,2,3$. Show that $[\eta]$ is an involution having the four fixed points $[e_{ii}]$ and $[A]$ where $A = \sum e_{ij}$. Show that $[\eta]$ is not of first kind.

3. An involution ε of $\mathfrak{P}(\mathfrak{J})$ will be called of *second kind* if there exists a set of four fixed points of ε no three of which are collinear. Show that every involution of $\mathfrak{P}(\mathfrak{J})$ is either of first kind or second kind.

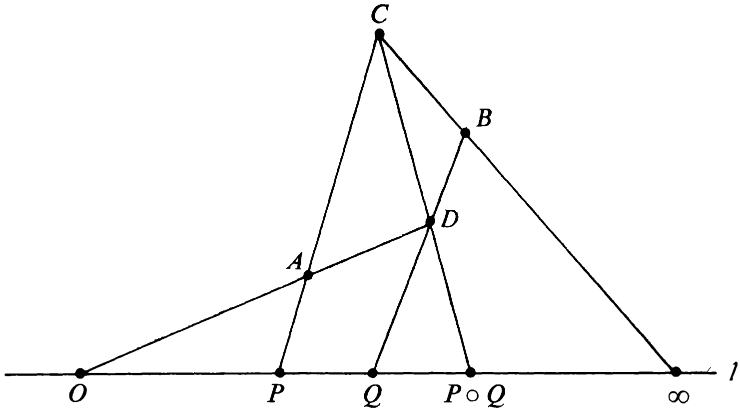
4. (Pickert). Show that $\Lambda(\mathfrak{P})$ for $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$ has the transitivity property of the above Lemma if and only if $\Phi^* = \Phi^{*3} = \{\alpha^3 \mid \alpha \in \Phi^*\}$.

10. **Fundamental theorem of projective geometry for the planes** $\mathfrak{P}(\mathfrak{J})$. Let $\mathfrak{J}/\Phi$ and $\mathfrak{J}'/\Phi'$ be reduced simple exceptional Jordan algebras whose coefficient al-

gebras $\mathfrak{O}/\Phi$ and $\mathfrak{O}'/\Phi'$ are octonion division algebras. We have seen at the very beginning of our discussion (§7) that if η is a norm semisimilarity of $\mathfrak{J}$ onto $\mathfrak{J}'$ then the mapping $[\eta]$ defined by (56) is a projective transformation of the geometry $\mathfrak{P}(\mathfrak{J})$ onto $\mathfrak{P}(\mathfrak{J}')$. We shall now prove the analogue of the classical "fundamental theorem of projective geometry" of Desarguesian geometries (Artin, *Geometric Algebra*, p. 88), namely, that every projective transformation of $\mathfrak{P}(\mathfrak{J})$ onto $\mathfrak{P}(\mathfrak{J}')$ has the form $[\eta]$ for some norm semisimilarity η. We shall base the proof on the classical constructions of addition and multiplication of points on a line. Let l be a line in a projective plane $\mathfrak{P}$ and O, ∞ distinct points on l, B, C points on a line l' through O distinct from l such that B, C and O are distinct. Let P and Q be points on l distinct from ∞. Let E be the intersection of the lines PC and $B\infty$, F the intersection of lines BQ and $C\infty$. Then we define $P + Q$ to be the point of intersection of lines EF and l:



Next let l, O, ∞ be as before and let A, B be two points such that no three points in $\{O, \infty, A, B\}$ are collinear. Let P, Q be points of l distinct from O, ∞ Let C be the point of intersection of ∞B and AP, D the intersection of OA and BQ. Then we define the product $P \circ Q$ to be the point of intersection of CD with l:



Now let $\mathfrak{P} = \mathfrak{P}(\mathfrak{J})$ where $\mathfrak{J} = \mathfrak{H}(\mathfrak{O}_3, J_1)$ and let l be the line $[e_{11}]$, $O = [e_{22}]$, $\infty = [e_{33}]$, $B = [e_{11}]$ and let C be the point of intersection of $A = [\sum e_{ij}]$, ∞ with $l' = OB$. Let $P_q = e_{22} + n(q)e_{33} + q[32]$, $P_r = e_{22} + n(r)e_{33} + r[32]$ where $p, q \in \mathfrak{O}$. Then a direct calculation which we leave to the reader shows that

$$P_p + P_q = P_{p+q}. \tag{57}$$

Similarly, in the second construction if we take $A = \sum e_{ij}$, O, ∞ and B as before then we find that

$$P_p \circ P_q = P_{pq}. \tag{58}$$

We can now prove the following "fundamental theorem" which was proved first by Springer [2] in a different way:

THEOREM 15. *Let $\mathfrak{J}_1/\Phi_1$ and $\mathfrak{J}_2/\Phi_2$ be reduced simple exceptional Jordan algebras whose coefficient algebras $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are octonion division algebras. Then any projective transformation of the plane $\mathfrak{P}(\mathfrak{J}_1)$ onto $\mathfrak{P}(\mathfrak{J}_2)$ has the form $[\eta]$ where η is a norm semisimilarity of $\mathfrak{J}_1/\Phi_1$ onto $\mathfrak{J}_2/\Phi_2$.*

PROOF. Without loss of generality we may assume $\mathfrak{J}_i = \mathfrak{H}(\mathfrak{O}_i, J_1)$. Let τ be projective transformation of $\mathfrak{P}(\mathfrak{J}_1)$ onto $\mathfrak{P}(\mathfrak{J}_2)$. Consider the points $[e_{ii}]$, $A = [\sum e_{ij}]$ in $\mathfrak{P}(\mathfrak{J}_1)$. Since no three of these are collinear the same is true of their images under τ. Then, by the lemma of §9, we can multiply τ on the right by a collineation of $\mathfrak{P}(\mathfrak{J}_2)$ which is of the form $[\zeta]$, $\zeta \in \mathrm{M}(\mathfrak{J}_2)$, to arrange that $[e_{ii}] \to [e_{ii}]$, $A \to A$ (as defined in $\mathfrak{H}(\mathfrak{O}_2, J_1)$. Thus we may assume $[e_{ii}]^\tau = [e_{ii}]$, $A^\tau = A$. Since τ is projective it follows that if we carry out the constructions of addition and multiplication of points on the line $[e_{11}]$ in $\mathfrak{P}_i$ by using the "base" $[e_{11}]$, $[e_{22}]$, $[e_{33}]$, A, then we have $(P_q + P_r)^\tau = P_q^{\,\tau} + P_r^{\,\tau}$, $(P_q \circ P_r)^\tau = P_q^{\,\tau} \circ P_r^{\,\tau}$ if $q, r \neq 0$. Now put $P_q^{\,\tau} = P_q^{\,\beta}$, $q \in \mathfrak{O}$. Then it is clear from (57) and (58) that β is a ring isomorphism of $\mathfrak{O}_1$ onto $\mathfrak{O}_2$. Let $M(\mathfrak{O}_i)$, $C(\mathfrak{O}_i)$ be the multiplication ring and centroid respectively of $\mathfrak{O}_i$ (cf. §5.7). The first of these is the ring of endomorphisms in $\mathfrak{O}_i$ generated by the left and right multiplications and the second is the centralizer of $M(\mathfrak{O}_i)$ in the complete ring of endomorphisms of $(\mathfrak{O}_i, +)$. It is immediate that a ring isomorphism β of $\mathfrak{O}_1$ onto $\mathfrak{O}_2$ determines ring isomorphisms $A \to \beta^{-1}A\beta$, $\gamma \to \beta^{-1}\gamma\beta$ of $M(\mathfrak{O}_1)$ onto $M(\mathfrak{O}_2)$ and of $C(\mathfrak{O}_1)$ onto $C(\mathfrak{O}_2)$. Since $\mathfrak{O}_i$ is central simple over Φ_i its centroid consists of the set of multiplications by elements of $\Phi_i 1$. Hence the isomorphism $\gamma \to \beta^{-1}\gamma\beta$ corresponds to an isomorphism σ of Φ_1 onto Φ_2 and since $(x\gamma)^\beta = x^\beta(\beta^{-1}\gamma\beta)$, $x \in \mathfrak{O}_1$ we have $(\alpha x)^\beta = \alpha^\sigma x^\beta$, $\alpha \in \Phi_1$, $x \in \mathfrak{O}_1$. Thus β is a σ-semilinear isomorphism of $\mathfrak{O}_1/\Phi_1$, onto $\mathfrak{O}_2/\Phi_2$. Then we have the σ-semilinear isomorphism η of $\mathfrak{J}_1$ onto $\mathfrak{J}_2$ such that $q[kl] \to q^\beta[kl]$, $q \in \mathfrak{O}_1$, $k, l = 1,2,3$. Then η is a norm semisimilarity and we have $(e_{22} + n(q)e_{33} + q[32])^\eta = e_{22} + n(q^\beta)e_{33} + q^\beta[32]$ which implies that $P_q^{\,[\eta]} = P_q^{\,\tau}$, $q \in \mathfrak{O}_1$. Hence the restriction of τ and $[\eta]$ to the points of the line $[e_{11}]$ are identical. Also we have

$A^{[\eta]} = A$ and $[e_{11}]^{[\eta]} = [e_{11}]^{\tau}$. Since two projective transformations are identical if they have the same effect on all points of a line and on two distinct points not on the line we have $\tau = [\eta]$ as required.

11. **Connections with exceptional Lie algebras.** In this section we shall specialize the main constructions of Lie algebras from Jordan algebras given in the last chapter to the case of finite-dimensional central simple exceptional Jordan algebras. We consider first the Lie algebra Der $\mathfrak{J}$ for $\mathfrak{J}$ of this type. If $D \in \mathrm{Der}\,\mathfrak{J}$ then $1D = 0$ and D is skew relative to the generic trace form (Theorem 6.1 (vii)), p. 224). Hence D is skew relative to the symmetric bilinear form $Q = \frac{1}{2}t$.

Now let $\mathfrak{J}$ be reduced and let e be a primitive idempotent in $\mathfrak{J}$, $\mathfrak{J} = \mathfrak{J}_0(e) \oplus \mathfrak{J}_{\frac{1}{2}}(e) \oplus \mathfrak{J}_1(e)$ the corresponding Peirce decomposition. Let $\mathrm{Der}\,\mathfrak{J}/\Phi e$ be the subalgebra of $\mathrm{Der}\,\mathfrak{J}$ mapping e into 0. If $D \in \mathrm{Der}\,\mathfrak{J}/\Phi e$ then the condition $x_i . e = ix_i, x_i \in \mathfrak{J}_i = \mathfrak{J}_i(e)$, gives $(x_i D) . e = ix_i D$. Hence $\mathfrak{J}_i D \subseteq \mathfrak{J}_i$. In particular, the restriction D_0 of D to $\mathfrak{J}_0$ is a derivation in this subalgebra of $\mathfrak{J}$. As in §3, we have $\mathfrak{J}_0 = \Phi f \oplus \mathfrak{V}$ where $f = 1 - e$ is the identity of $\mathfrak{J}_0$ and $\mathfrak{V}$ is the subspace of elements of generic trace 0 in $\mathfrak{J}_0$. We have $v^{\cdot 2} = Q(v)f$, if $v \in \mathfrak{V}$, and $\mathfrak{J}_0$ is the Jordan algebra of the nondegenerate symmetric bilinear form Q on $\mathfrak{V}$. It follows from this that $\mathrm{Der}\,\mathfrak{J}_0 = \mathrm{Inder}\,\mathfrak{J}_0$ coincides with the set of mappings of the form $\sum [\bar{R}_{v_i}\bar{R}_{w_i}]$ where $v_i, w_i \in \mathfrak{V}$ and $\bar{R}_b$ for $b \in \mathfrak{J}_0$ is the multiplication in $\mathfrak{J}_0$ determined by b (ex.1, p. 328). It is immediate also that $\mathrm{Der}\,\mathfrak{J}_0$ maps $\mathfrak{V}$ into itself and the set of restrictions of the derivations in $\mathfrak{J}_0$ to $\mathfrak{V}$ is the complete set of linear transformations in $\mathfrak{V}$ which are skew relative to Q. Since $\dim \mathfrak{V} = 9$ this implies that $\mathrm{Der}\,\mathfrak{J}_0$, which is evidently isomorphic to the Lie algebra of linear transformations in $\mathfrak{V}$ which are Q-skew, is a simple Lie algebra (cf. ex. 1, p. 343).

We can now prove the following key result.

THEOREM 16. *Let e be a primitive idempotent in the reduced simple exceptional Jordan algebra. Let* $\mathrm{Der}\,\mathfrak{J}/\Phi e$ *be the subalgebra of* $\mathrm{Der}\,\mathfrak{J}$ *mapping e into* 0. *Then the restriction mapping* $D \to D_0 = D|\mathfrak{J}_0$ *is an isomorphism of* $\mathrm{Der}\,\mathfrak{J}/\Phi e$ *onto* $\mathrm{Der}\,\mathfrak{J}_0$. *Moreover, the set of restrictions to* $\mathfrak{J}_{\frac{1}{2}}$ *of the elements of* $\mathrm{Der}\,\mathfrak{J}/\Phi e$ *is an irreducible set of linear transformations in* $\mathfrak{J}_{\frac{1}{2}}$.

PROOF. Let $D_0 \in \mathrm{Der}\,\mathfrak{J}_0$ so $D_0 = \sum[\bar{R}_{v_i}\bar{R}_{w_i}]$ where $v_i, w_i \in \mathfrak{V}$ and $\bar{R}$ denotes multiplication in $\mathfrak{J}_0$. Put $D = \sum[R_{v_i}R_{w_i}]$ where R is multiplication in $\mathfrak{J}$. Then $D \in \mathrm{Der}\,\mathfrak{J}/\Phi e$ and the restriction of D to $\mathfrak{J}_0$ is D_0. Hence $D \to D|\mathfrak{J}_0$ is surjective on $\mathrm{Der}\,\mathfrak{J}_0$. Now suppose $D \in \mathrm{Der}\,\mathfrak{J}/\Phi e$ satisfies $D|\mathfrak{J}_0 = 0$. Let $D_{\frac{1}{2}}$ be the restriction of D to $\mathfrak{J}_{\frac{1}{2}}$. If $y \in \mathfrak{J}_{\frac{1}{2}}$ and $v \in \mathfrak{V}$ then $y . v \in \mathfrak{J}_{\frac{1}{2}}$ and $(y . v)D_{\frac{1}{2}} = (yD_{\frac{1}{2}}).v$. Thus $D_{\frac{1}{2}}$ commutes with all the multiplications R_v' in $\mathfrak{J}_{\frac{1}{2}}$ determined by the elements $v \in \mathfrak{V}$. In §3 we defined a representation ρ of the Clifford algebra $C(\mathfrak{V},Q)$ by linear transformations in $\mathfrak{J}_{\frac{1}{2}}$ such that $v \to U_{\theta(v)}' = 2R_v'$ (cf. (38)). We saw that $C^e(\mathfrak{V},Q)^\rho = \mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})$ so, a fortiori, $C(\mathfrak{V},Q)^\rho = \mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})$. Since we have just shown that $D_{\frac{1}{2}}$ commutes with every R_v' we see that $D_{\frac{1}{2}}$ is a multiplication by an element of Φ. Since $D_{\frac{1}{2}}$ is skew relative to the symmetric bilinear form Q we have $D_{\frac{1}{2}} = 0$. Since $eD = 0$ and $\mathfrak{J}_0 D = 0$ by hypothesis we have $D = 0$. Thus $D \to D \mid \mathfrak{J}_0$ is an isomorphism of $\mathrm{Der}\,\mathfrak{J} / \Phi e$ onto $\mathrm{Der}\,\mathfrak{J}_0$. Since $C^e(\mathfrak{V},Q)$ is generated by the products $v_1 v_2$, $v_i \in \mathfrak{V}$ it is clear that $\mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})\ C^e(\mathfrak{V},Q)^\rho$ is generated by the products $R_{v_1}{}'R_{v_2}{}'$. Now we have the relation $y.(a.b) = y.a.b + y.b.a$ for $y \in \mathfrak{J}_{\frac{1}{2}}$, $a, b \in \mathfrak{J}_0$ which shows that $2R'_{v_1}.R'_{v_2} = R'_{v_1 \cdot v_2} = Q(v_1, v_2)1$. Since $R_{v_1}{}'R_{v_2}{}' = R_{v_1}{}'.R_{v_2}{}' + \frac{1}{2}[R_{v_1}{}', R_{v_2}{}']$ this implies that $\mathrm{Hom}_\Phi(\mathfrak{J}_{\frac{1}{2}},\mathfrak{J}_{\frac{1}{2}})$ is generated by 1 and the mappings $[R_{v_1}{}'R_{v_2}{}']$. Since $[R_{v_1}{}'R_{v_2}{}']$ is the restriction to $\mathfrak{J}_{\frac{1}{2}}$ of $D = [R_{v_1} R_{v_2}] \in \mathrm{Der}\,\mathfrak{J}/\Phi e$ it is clear that the Lie algebra of restrictions to $\mathfrak{J}_{\frac{1}{2}}$ of the linear transformations in $\mathrm{Der}\,\mathfrak{J}/\Phi e$ is irreducible in $\mathfrak{J}_{\frac{1}{2}}$.

Now let E be any derivation in $\mathfrak{J}$. Then $e^{\cdot 2} = e$ gives $2e.eE = eE$ so $eE \in \mathfrak{J}_{\frac{1}{2}} = \mathfrak{J}_{\frac{1}{2}}(e)$. On the other hand, let $a \in \mathfrak{J}_{\frac{1}{2}}$ and put $E_a = 4[R_e R_a]$. Then E_a is a derivation in $\mathfrak{J}$ and $eE_a = 4(e.e.a - e.a.e) = a$. We now have the linear mapping $\varepsilon: a \to E_a$ of $\mathfrak{J}_{\frac{1}{2}}$ into $\mathrm{Der}\,\mathfrak{J}$ which is clearly a monomorphism. If E is any derivation then $a = eE \in \mathfrak{J}_{\frac{1}{2}}$ and $eE_a = a = eE$. Hence $D = E - E_a \in \mathrm{Der}\,\mathfrak{J}/\Phi e$. Thus $\mathrm{Der}\,\mathfrak{J} = \mathrm{Der}\,\mathfrak{J}/\Phi e + \varepsilon(\mathfrak{J}_{\frac{1}{2}})$. Since $eE_a = a$ it is clear also that $\mathrm{Der}\,\mathfrak{J}/\Phi e \cap \varepsilon(\mathfrak{J}_{\frac{1}{2}}) = 0$. Hence we have

$$\mathrm{Der}\,\mathfrak{J} = \mathrm{Der}\,\mathfrak{J}/\Phi e \oplus \varepsilon(\mathfrak{J}_{\frac{1}{2}}). \tag{59}$$

Now let $\mathfrak{D} \in \mathrm{Der}\,\mathfrak{J}/\Phi e$. Then $[E_a, D] = 4[[R_e R_a], D] = 4[R_e R_{aD}] = E_{aD}$. Hence $[\varepsilon(\mathfrak{J}_{\frac{1}{2}}), \mathrm{Der}\,\mathfrak{J}/\Phi e] \subseteq \varepsilon(\mathfrak{J}_{\frac{1}{2}})$ and $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ as $\mathrm{Der}\,\mathfrak{J}/\Phi e$ Lie module is isomorphic to $\mathfrak{J}_{\frac{1}{2}}$ as $\mathrm{Der}\,\mathfrak{J}/\Phi e$ Lie module (defined by the representation $D \to D_{\frac{1}{2}}$). Since we have seen that $\mathfrak{J}_{\frac{1}{2}}$ is irreducible relative to $\mathrm{Der}\,\mathfrak{J}/\Phi e$ we see that $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ is irreducible relative to the set of mappings $X \to [X, D]$, $D \in \mathrm{Der}\,\mathfrak{J}/\Phi e$. The situation we now have is a special case, as we shall see, of the one described in the following

LEMMA. *Let $\mathfrak{D}$ be a Lie algebra, $\mathfrak{D}_0$ a subalgebra such that $\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}'$ where $\mathfrak{D}'$ is a subspace of $\mathfrak{D}$ such that $[\mathfrak{D}', \mathfrak{D}_0] \subseteq \mathfrak{D}'$. Assume that $\mathfrak{D}_0$ and $\mathfrak{D}'$ are irreducible and not isomorphic as Lie modules for $\mathfrak{D}_0$ (under $[X, D]$, $X \in \mathfrak{D}_0$ or $\mathfrak{D}'$, $D \in \mathfrak{D}_0$) and $\mathfrak{D}'$ is not a subalgebra. Then $\mathfrak{D}$ is a simple Lie algebra.*

PROOF. The hypothesis implies that the only $\mathfrak{D}_0$-submodules of $\mathfrak{D}$ are 0, $\mathfrak{D}_0, \mathfrak{D}'$ and $\mathfrak{D}$. Now let $\mathfrak{B}$ be an ideal $\neq 0$ in $\mathfrak{D}$. Then $\mathfrak{B}$ is a $\mathfrak{D}_0$-submodule of $\mathfrak{D}$ so either $\mathfrak{B} = \mathfrak{D}_0, \mathfrak{D}'$ or $\mathfrak{D}$. If $\mathfrak{B} = \mathfrak{D}_0$ then $\mathfrak{B} \supseteq [\mathfrak{D}_0, \mathfrak{D}'] = \mathfrak{D}'$ (by irreducibility of $\mathfrak{D}'$). This contradicts $\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}'$. If $\mathfrak{B} = \mathfrak{D}'$ then $\mathfrak{D}'$ is a subalgebra contrary to hypothesis. Hence $\mathfrak{B} = \mathfrak{D}$ and $\mathfrak{D}$ is simple.

We can now prove the main result on $\mathrm{Der}\,\mathfrak{J}$.

THEOREM 17. *Let $\mathfrak{J}$ be a finite-dimensional exceptional central simple Jordan algebra. Then every derivation of $\mathfrak{J}$ is inner and* $\mathrm{Der}\,\mathfrak{J}$ *is a central simple Lie algebra of* 52 *dimensions. Moreover, the only invariant subspaces of $\mathfrak{J}$ relative to* $\mathrm{Der}\,\mathfrak{J}$ *are* $0, \Phi 1, \mathfrak{J}'$ *and* $\mathfrak{J}$.

PROOF. It suffices to prove the statements for Φ algebraically closed. Then $\mathfrak{J}$

is reduced and our results are applicable. Since $\varepsilon: a \to E_a$ is a monomorphism, $\dim \operatorname{Der} \mathfrak{J} = \dim \operatorname{Der} \mathfrak{J}/\Phi e + \dim \mathfrak{J}_{\frac{1}{2}}$, by (59). We have seen also that $\operatorname{Der}\mathfrak{J}/\Phi e$ is isomorphic to the Lie algebra of linear transformations in the nine-dimensional vector space $\mathfrak{V}$ which are skew relative to Q. The dimensionality of the latter algebra is $\frac{1}{2}(9 \times 8) = 36$. Since $\dim \mathfrak{J}_{\frac{1}{2}} = 16$ we have $\dim \operatorname{Der} \mathfrak{J} = 36 + 16 = 52$. It is clear from the definition that E_a is an inner derivation. Hence every element of $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ is an inner derivation. Also we have seen that the elements of $\operatorname{Der}\mathfrak{J}/\Phi e$ have the form $\sum [R_{v_i} R_{w_i}]$, $v_i, w_i \in \mathfrak{V}$. Hence these are inner and (59) shows that $\operatorname{Der}\mathfrak{J} = \operatorname{Inder}\mathfrak{J}$. To prove the simplicity of $\operatorname{Der}\mathfrak{J}$ we shall apply the Lemma with $\mathfrak{D}_0 = \operatorname{Der}\mathfrak{J}/\Phi e$ and $\mathfrak{D}' = \varepsilon(\mathfrak{J}_{\frac{1}{2}})$. Then we have $\operatorname{Der}\mathfrak{J} = \mathfrak{D}_0 \oplus \mathfrak{D}'$. Also $\mathfrak{D}_0$ is a simple Lie algebra so $\mathfrak{D}_0$ is irreducible as $\mathfrak{D}_0$-module. Also we have seen that $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ is $\mathfrak{D}_0$-irreducible. Since $\dim \mathfrak{D}_0 = 36$ and $\dim \mathfrak{D}' = 16$ these are not isomorphic as $\mathfrak{D}_0$-modules. Now let $a, b \in \mathfrak{J}_{\frac{1}{2}}$. Then

$$\begin{aligned}[E_a E_b] &= 16[[R_e R_a], [R_e R_b]] \\ &= 16[[[R_e R_a] R_e] R_b] - 16[[[R_e R_a] R_b] R_e] \\ &= 16[R_{[a,e,e]} R_b] - 16[R_{[a\, b,e]} R_e].\end{aligned}$$

Since $[a,e,e] = -\frac{1}{4}a$ and $[a,b,e] = a.b.e - \frac{1}{2}a.b = \frac{1}{2}a.b.(e-f) \in \mathfrak{J}_0 + \mathfrak{J}_1$ this gives

$$[E_a E_b] = -4[R_a R_b].$$

Since $e[R_a R_b] = \frac{1}{2}(a.b - b.a) = 0$, $[E_a E_b] \in \mathfrak{D}_0$. Now we can choose $a, b \in \mathfrak{J}_{\frac{1}{2}}$ so that $[R_a R_b] \neq 0$. Thus let $e = e_1, e_2, e_3$ be orthogonal idempotents and let $a = u_{12} \in \mathfrak{J}_{12}$, $b = u_{13} \in \mathfrak{J}_{13}$ satisfy $u_{12}^{\cdot 2} \neq 0$, $u_{13}^{\cdot 2} \neq 0$. Then we can choose a coordinatization so that u_{12} and u_{13} are mapped into $1[12]$ and $1[13]$. Since $[1[12], 1[23], 1[13]] \neq 0$ we see that $[R_a R_b] \neq 0$. We now see that $\mathfrak{D}' = \varepsilon(\mathfrak{J}_{\frac{1}{2}})$ is not a subalgebra of $\operatorname{Der}\mathfrak{J}$. Hence $\operatorname{Der}\mathfrak{J}$ is simple by the Lemma. To prove the statement on the invariant subspaces we note that the only invariant subspaces of $\mathfrak{J}$ relative to $\operatorname{Der}\mathfrak{J}/\Phi e$ are sums of the spaces $\mathfrak{V}$, $\mathfrak{J}_{\frac{1}{2}}$ and subspaces of $\Phi e + \Phi f$. This is clear since $\mathfrak{V}$ and $\mathfrak{J}_{\frac{1}{2}}$ are irreducible and not isomorphic relative to $\operatorname{Der}\mathfrak{J}/\Phi e$. Hence the proof of the statement on the invariant subspace of $\mathfrak{J}$ relative to $\operatorname{Aut}\mathfrak{J}/\Phi e$ (Theorem 7) carries over to prove the corresponding statement on $\operatorname{Der}\mathfrak{J}/\Phi e$.

We have shown in §7.7 that if $\mathfrak{J}$ is a finite-dimensional separable Jordan algebra such that the degrees of the simple components of $\mathfrak{J}$ over their centers is not divisible by the characteristic of the base field then every derivation of $\mathfrak{J}$ into any finite-dimensional bimodule is inner. In particular this applies to $\mathfrak{J}$ exceptional simple of characteristic $\neq 3$. We shall now give another proof of this special case which is valid also in the characteristic three case. The result is the following

COROLLARY. *Let $\mathfrak{J}$ be a finite-dimensional separable simple exceptional Jordan algebra, $\mathfrak{M}$ a finite-dimensional bimodule for $\mathfrak{J}$. Then every derivation of $\mathfrak{J}$ into $\mathfrak{M}$ is inner.*

PROOF. The reductions given in §7.7 show that it suffices to prove this for algebraically closed base field and $\mathfrak{M}$ unital and irreducible. Then $\mathfrak{J}$ is as in Theorem 17 and $\mathfrak{M}$ as $\mathfrak{J}$-bimodule is isomorphic to $\mathfrak{J}$ by §7.4. Hence the result follows from Theorem 17.

If we take into account the results of §7.7 and the present one we see that we can state the theorem on derivations of separable Jordan algebras into bimodules in the following way. Let $\mathfrak{J}$ be a finite-dimensional separable Jordan algebra. Then the derivations of $\mathfrak{J}$ into finite-dimensional bimodules are inner if and only if the degrees over their centers of the special simple components are not divisible by the characteristic. This is Harris' theorem in its sharpest form.

In Theorem 8 we determined $\mathrm{Aut}\,\mathfrak{J}/\mathfrak{K}$ for $\mathfrak{J}$ reduced simple exceptional and $\mathfrak{K}$ a reduced simple subalgebra of degree three and dimension nine or fifteen. The same method can be used to obtain analogous results for $\mathrm{Der}\,\mathfrak{J}/\mathfrak{K}$, the subalgebra of the derivation algebra mapping $\mathfrak{K}$ into 0. We leave it to the reader to work out the details of this.

We consider next the Lie algebra $\mathfrak{L}(\mathfrak{J}) = \{R_a + \sum[R_{b_i}R_{c_i}] \mid a, b_i, c_i \in \mathfrak{J}\}$ where $\mathfrak{J}$ is finite dimensional central simple exceptional. We have $\mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \mathrm{Der}\,\mathfrak{J}$ since $\mathrm{Der}\,\mathfrak{J} = \mathrm{Inder}\,\mathfrak{J}$ and $a \to R_a$ is a monomorphism. Hence $\dim \mathfrak{L}(\mathfrak{J}) = 27 + 52 = 79$. It is clear that $\mathfrak{L}(\mathfrak{J})$ contains the ideal $\mathfrak{L}_0(\mathfrak{J}) = R(\mathfrak{J}') \oplus \mathrm{Der}\,\mathfrak{J}$ and we have the following result on this algebra.

THEOREM 18. *Let $\mathfrak{J}$ be a finite-dimensional central simple exceptional Jordan algebra, $\mathfrak{L}_0(\mathfrak{J})$ the Lie algebra of linear transformations of the form $R_a + \sum[R_{b_i}R_{c_i}]$ where $a, b_i, c_i \in \mathfrak{J}$ and $t(a) = 0$. Then $\dim \mathfrak{L}_0(\mathfrak{J}) = 78$ and $\mathfrak{L}_0(\mathfrak{J})$ is central simple if the characteristic is $\neq 3$ and $\mathfrak{L}_0(\mathfrak{J})$ contains the ideal $\Phi 1$ and $\mathfrak{L}_0(\mathfrak{J})/\Phi 1$ is central simple if the characteristic is three. Moreover, $\mathfrak{J}$ is irreducible under the action of $\mathfrak{L}_0(\mathfrak{J})$ for any characteristic.*

PROOF. We may assume the base field is algebraically closed. The fact that $\dim \mathfrak{L}_0(\mathfrak{J}) = 78$ is clear since $\dim R(\mathfrak{J}') = \dim \mathfrak{J}' = 26$ and $\dim \mathrm{Der}\,\mathfrak{J} = 52$. If $a \in \mathfrak{J}$ and $D \in \mathrm{Der}\,\mathfrak{J}$ then $[R_a D] = R_{aD}$. Hence the mapping $a \to R_a$ is an isomorphism of $\mathfrak{J}$ and $R(\mathfrak{J})$ regarded as $\mathrm{Der}\,\mathfrak{J}$-modules. Since we have determined the submodules of $\mathfrak{J}$ under $\mathrm{Der}\,\mathfrak{J}$ in Theorem 17, we can deduce from the isomorphism the submodules of $R(\mathfrak{J})$. It is clear from this that if the characteristic is not three then $R(\mathfrak{J}')$ is an irreducible $\mathrm{Der}\,\mathfrak{J}$-module and if the characteristic is three, so $t(1) = 0$, then $R(\mathfrak{J}')/\Phi 1$ is $\mathrm{Der}\,\mathfrak{J}$-irreducible. It is clear also that $[R(\mathfrak{J}'), R(\mathfrak{J}')] = \mathrm{Der}\,\mathfrak{J}$ so $R(\mathfrak{J}')$ is not a subalgebra of $\mathfrak{L}_0(\mathfrak{J})$. Since $\mathrm{Der}\,\mathfrak{J}$ is simple this is an irreducible module for $\mathrm{Der}\,\mathfrak{J}$. Hence if the characteristic is not three, we can apply the Lemma to $\mathfrak{D}_0 = \mathrm{Der}\,\mathfrak{J}$, $\mathfrak{D}' = R(\mathfrak{J}')$ to conclude that $\mathfrak{L}_0(\mathfrak{J}) = \mathrm{Der}\,\mathfrak{J} \oplus R(\mathfrak{J}')$ is simple. If the characteristic is three we use the same argument with $\mathfrak{D}_0 = \mathrm{Der}\,\mathfrak{J}$ and $\mathfrak{D}' = R(\mathfrak{J}')/\Phi 1$ to prove $\mathfrak{L}_0(\mathfrak{J})/\Phi 1$ is simple. Now let $\mathfrak{B}$ be a subspace $\neq 0$ invariant under $\mathfrak{L}_0(\mathfrak{J})$. Then $\mathfrak{B}$ is invariant under $\mathrm{Der}\,\mathfrak{J}$ so either $\mathfrak{B} = \Phi 1, \mathfrak{B} = \mathfrak{J}$

or $\mathfrak{B} = \mathfrak{J}$, by Theorem 17. Now $\mathfrak{B} = \Phi 1$ is ruled out since $\mathfrak{B}R(\mathfrak{J}') \subseteq \mathfrak{B}$. Also $\mathfrak{B} = \mathfrak{J}'$ is impossible since this and $\mathfrak{B}R(\mathfrak{J}') \subseteq \mathfrak{B}$ would imply that $\mathfrak{J}'$ is a subalgebra, which is clearly not so. Hence $\mathfrak{B} = \mathfrak{J}$ and $\mathfrak{J}$ is irreducible under $\mathfrak{L}_0(\mathfrak{J})$.

We note that by Theorem 6.11 (p. 257) if the base field is infinite then $\mathfrak{L}_0(\mathfrak{J})$ and $\mathfrak{B}(\mathfrak{J})$ coincide respectively with $\mathfrak{M}_0(\mathfrak{J})$ and $\mathfrak{M}(\mathfrak{J})$ the Lie algebras of linear transformations having the generic norm as Lie invariant and Lie semi-invariant.

Finally, we consider the Lie algebra $\mathfrak{K}(\mathfrak{J})$ defined by the Tits-Koecher construction from the finite-dimensional central simple exceptional Jordan algebra $\mathfrak{J}$. We have seen that $\mathfrak{J}$ central simple implies that $\mathfrak{K}(\mathfrak{J})$ is central simple. Also the definition $\mathfrak{K}(\mathfrak{J}) = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$ shows that $\dim \mathfrak{K}(\mathfrak{J}) = 27 + 27 + 79 = 133$. It can be seen that the simple Lie algebras $\mathrm{Der}\,\mathfrak{J}$, $\mathfrak{L}_0(\mathfrak{J})$ (for characteristic $\neq 3$) and $\mathfrak{K}(\mathfrak{J})$ are of types F_4, E_6 and E_7 respectively. This is easily seen, from the Cartan decompositions relative to suitable Cartan subalgebras. We shall indicate this in the exercises.

Exercises

1. Let $\mathfrak{J}$ be a reduced, simple exceptional Jordan algebra, $\{e_i\}$ a reducing set of idempotents for $\mathfrak{J}$. Let $\mathrm{Der}\,\mathfrak{J}/\sum\Phi e_i$ be subalgebra of $\mathrm{Der}\,\mathfrak{J}$ such that $e_i D = 0$. Show that $\mathrm{Der}\,\mathfrak{J}/\sum\Phi e_i$ is isomorphic to the Lie algebra $\mathfrak{S}(\mathfrak{O}, n)$ of linear transformations in the octonion coefficient algebra $\mathfrak{O}$ which are skew relative to the norm form n in $\mathfrak{O}$.

2. Let $\mathfrak{J}$ as in exercise 1 be split and let $\mathfrak{H}$ be a splitting Cartan subalgebra of the subalgebra $\mathrm{Der}\,\mathfrak{J}/\sum\Phi e_i$. Show that this is a splitting Cartan subalgebra for $\mathrm{Der}\,\mathfrak{J}$. Determine the corresponding roots and show that $\mathrm{Der}\,\mathfrak{J}$ is of type F_4 (cf. the author's *Lie Algebras*, pp. 144–145).

3. Let $\mathfrak{J}$ and $\mathfrak{H}$ be as in exercise 2 and let $\tilde{\mathfrak{H}} = \mathfrak{H} + R(\mathfrak{J}' \cap \sum\Phi e_i)$ where $R(\mathfrak{J}' \cap \sum\Phi e_i)$ denotes the set R_b, $b \in \mathfrak{J}' \cap \sum\Phi e_i$. Show that if the characteristic is $\neq 3$ then $\tilde{\mathfrak{H}}$ is a splitting Cartan subalgebra whose corresponding root system is of type E_6.

4. Let $\mathfrak{J}$ be as in exercise 2. Determine a Cartan subalgebra for $\mathfrak{K}(\mathfrak{J})$ and show that $\mathfrak{K}(\mathfrak{J})$ is split of type E_7.

5. Let $\mathfrak{J}$ be a finite-dimensional central simple exceptional Jordan algebra, $\mathfrak{K}$ a separable cubic subfield of $\mathfrak{J}$ and let $\mathrm{Der}\,\mathfrak{J}/\mathfrak{K}$ be the subalgebra of $\mathrm{Der}\,\mathfrak{J}$ mapping $\mathfrak{K}$ into 0. Show that $\mathrm{Der}\,\mathfrak{J}/\mathfrak{K}$ is a central simple Lie algebra of type D_4 in the sense that $(\mathrm{Der}\,\mathfrak{J}/\mathfrak{K})_\Omega$ for Ω the algebraic closure of Φ is the simple Lie algebra D_4. Prove that $\mathrm{Der}\,\mathfrak{J}/\mathfrak{K}$ is not isomorphic to a Lie algebra $\mathfrak{S}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra with involution.

6. Let $\mathfrak{J}$ be as in exercise 1, e a primitive idempotent and let $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ be defined by $\mathfrak{J}$ and e as in the text. Show that $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ is a subsystem of the Lie triple system $(\mathrm{Der}\,\mathfrak{J})^{(2)}$. Use Theorem 8.2 (p. 313) to show that $\mathrm{Der}\,\mathfrak{J}$ is the universal Lie envelope for $\varepsilon(\mathfrak{J}_{\frac{1}{2}})$ if the characteristic is 0. Does this hold also for prime characteristic?

7. Let $\mathfrak{J}$ be as in exercise 5 and let $\mathfrak{T} = R(\mathfrak{J}')$. Let α be a nonzero element of Φ

and define the Lie triple system $(\mathfrak{T}, \alpha)$ as in exercise 4, (p. 312). Assume the characteristic is 0 and show that the universal Lie envelope of $(\mathfrak{T}, \alpha)$ is a simple Lie algebra of type E_6.

8. (Tomber). Let $\mathfrak{J}$ be as in exercise 1, Φ algebraically closed of characteristic 0. Prove that every automorphism of Der$\mathfrak{J}$ has the form $D \to \eta^{-1} D \eta$ where η is a uniquely determined automorphism of $\mathfrak{J}$. Use this and "Galois cohomology" (see Chapter X of the author's *Lie algebras*) to prove that if $\mathfrak{L}$ is a Lie algebra over a field Φ of characteristic 0 such that $\mathfrak{L}_\Omega$ for Ω the algebraic closure of Φ is isomorphic to Der $\tilde{\mathfrak{J}}/\Omega$, $\tilde{\mathfrak{J}}$ the reduced exceptional simple Jordan algebra over Ω then $\mathfrak{L}$ is isomorphic to a Lie algebra Der$\mathfrak{J}$ where $\mathfrak{J}$ is a finite-dimensional central simple exceptional Jordan algebra.

12. **Exceptional Jordan division algebras.** The first examples of finite-dimensional exceptional Jordan division algebras were given by Albert in [25]. In this paper Albert studied the finite-dimensional central simple exceptional Jordan algebras which contain a given cyclic cubic field. This study was continued in [32] which gave a construction also of the finite-dimensional central simple exceptional Jordan algebras containing a given cubic noncyclic subfield. In both of Albert's constructions the algebras were given as fixed point sets of semilinear automorphisms of an extension algebra. In this section we shall give two constructions of finite-dimensional central simple exceptional Jordan algebras which have been communicated to the author by Tits and which are considerably simpler than Albert's. The first construction, as we shall see, gives all the finite-dimensional exceptional central simple Jordan algebras containing a given subalgebra of the form $\mathfrak{A}^+$ where $\mathfrak{A}$ is a nine-dimensional central simple associative algebra. It is easy to give the conditions that such an algebra be a division algebra and to construct division algebras of this type over appropriate base fields.

Tits' second construction gives all the finite-dimensional central simple exceptional Jordan algebras containing a given subalgebra of the form $\mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{A}$ is nine-dimensional simple associative with center a quadratic field and J is an involution of second kind. Since, as is easily seen, every finite-dimensional central simple exceptional Jordan algebra contains subalgebras of the form $\mathfrak{A}^+$ or of the form $\mathfrak{H}(\mathfrak{A}, J)$ as indicated, Tits' constructions give all the finite-dimensional central simple exceptional Jordan algebras. Both of Tits' constructions are rational in the sense that they do not require an extension of the base field for their definition.

In this section we shall study in some detail the algebras obtained from Tits' first construction. Since an algebra of the second type becomes one of the first type if one makes a suitable quadratic field extension, one can use the first construction alone to derive the main results on exceptional Jordan division algebras. Hence the second construction will not be essential for our purposes and so we shall be content to indicate this only in the exercises at the end of the section.

We proceed to give Tits' first construction. Let $\mathfrak{A}$ be a nine-dimensional central simple associative algebra, $\mathfrak{A}^+$ the corresponding Jordan algebra. Then $\mathfrak{A}$ and $\mathfrak{A}^+$ are of degree three. If $a, b \in \mathfrak{A}$ we define $a \times b = a.b - \frac{1}{2}t(a)b - \frac{1}{2}t(b)a + \frac{1}{2}(t(a)t(b) - t(a.b))1$ (as in §1) so $a \times a = \operatorname{adj} a$. Also we define $\bar{a} = a \times 1 = \frac{1}{2}t(a)1 - \frac{1}{2}a$. Let μ be a nonzero element of Φ and let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ be three copies of $\mathfrak{A}$: $\mathfrak{A}_i$ is a vector space and we have a linear isomorphism $a \to a_i$ of $\mathfrak{A}$ onto $\mathfrak{A}_i$. We now put $(\mathfrak{A}, \mu) = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ and we define a product in $(\mathfrak{A}, \mu)$ so that $\mathfrak{A}_i . \mathfrak{A}_j \subseteq \mathfrak{A}_{i+j}$ where the subscripts are reduced modulo three and where multiplication of $a_i, a_i', b_i, b_i', c_i, c_i' \in \mathfrak{A}_i$ is given by the following table:

(60)

	a'_0	b'_1	c'_2
a_0	$a.a'$	$\bar{a}b'$	$c'\bar{a}$
b_1	$\overline{a'}b$	$\mu b \times b'$	$\overline{bc'}$
c_2	$c\overline{a'}$	$\overline{b'c}$	$\mu^{-1}c \times c'$

Here the subscripts in the body of the table have been omitted since these can be immediately supplied by using $\mathfrak{A}_i\mathfrak{A}_j \subseteq \mathfrak{A}_{i+j}$. Thus we have the complete product formula:

$$(a_0 + b_1 + c_2).(a_0' + b_1' + c_2') \tag{61}$$
$$= (a.a' + \overline{bc'} + \overline{b'c})_0 + (\bar{a}b' + \overline{a'}b + \mu^{-1}c \times c')_1$$
$$+ (c'\bar{a} + c\overline{a'} + \mu b \times b')_2.$$

It is clear that $(\mathfrak{A}, \mu)$ is a commutative algebra.

We now note that if u is an invertible element in the associative algebra $\mathfrak{A}$ then $(\mathfrak{A}, \mu) \cong (\mathfrak{A}, \mu n(u))$ where $n(u)$ is the generic norm of u as defined in $\mathfrak{A}$. Observe first that if a and b are elements of a finite-dimensional associative algebra with 1 then

$$\operatorname{adj}(ab) = (\operatorname{adj} b)(\operatorname{adj} a). \tag{62}$$

It is sufficient to prove this for invertible elements and for these we have $(ab)\operatorname{adj}(ab) = n(ab)$ and $(ab)(\operatorname{adj} b)(\operatorname{adj} a) = n(b)n(a) = n(ab)$ so (62) is valid. The formula (62) in $\mathfrak{A}$ gives the following formulas:

$$u^{-1}a \times u^{-1}b = n(u)^{-1}(a \times b)u, \tag{63}$$
$$au \times bu = n(u)u^{-1}(a \times b)$$

for arbitrary a, b and invertible u. These and the relation $\overline{u^{-1}au} = u^{-1}\bar{a}u$ imply that we have the following multiplication table in $(\mathfrak{A}, \mu)$:

(64)

	$(u^{-1}a'u)_0$	$(u^{-1}b')_1$	$(c'u)_2$
$(u^{-1}au)_0$	$u^{-1}(a\,.\,a')u$	$u^{-1}\bar{a}b'$	$c'\bar{a}u$
$(u^{-1}b)_1$	$u^{-1}\overline{a'}b$	$\mu n(u)^{-1}(b \times b')u$	$\overline{u^{-1}bc'u}$
$(cu)_2$	$c\overline{a'}u$	$\overline{u^{-1}b'cu}$	$\mu^{-1}n(u)u^{-1}(c \times c')$

It is clear from this table that we have an isomorphism of $(\mathfrak{A},\mu)$ onto $(\mathfrak{A},\mu n(u)^{-1})$ such that $(u^{-1}au)_0 + (u^{-1}b)_1 + (cu)_2 \to a_0 + b_1 + c_2$. In particular, if δ is any nonzero element of Φ then $(\mathfrak{A},\mu)$ and $(\mathfrak{A},\mu\delta^{-3})$ are isomorphic. Hence if μ is a cube in Φ then $(\mathfrak{A},\mu) \cong (\mathfrak{A},1)$.

We shall now show that if Ω is the algebraic closure of Φ then $(\mathfrak{A},\mu)_\Omega$ is the split exceptional simple Jordan algebra $\mathfrak{H}(\mathfrak{O}_3,J_1)$. This will imply that $(\mathfrak{A},\mu)$ is a finite-dimensional central simple exceptional Jordan algebra. We shall do this by beginning at the other end with $\mathfrak{H}(\mathfrak{O}_3,J_1)$ and showing that if we take $\mathfrak{A}_0 = \mathfrak{H}(\mathrm{P}_3,J_1)$ where $\mathrm{P} \cong \Phi \oplus \Phi$ then we can choose subspaces $\mathfrak{A}_1, \mathfrak{A}_2$ in $\mathfrak{H}(\mathfrak{O}_3,J_1)$ such that we have the multiplication table (60) with $\mu = 1$ for these subspaces.

We need first a multiplication table for a split octonion algebra which is a little more explicit than the one we encountered in §5. For this purpose we let $(\mathfrak{Q},j)$ be any quaternion algebra and we make the standard construction of the octonion algebra $\mathfrak{O} = \mathfrak{Q} \oplus \mathfrak{Q}l$ as on p. , where we now suppose that $l^2 = 1$. Then it is clear that $\mathfrak{O}$ is split. Put $u_1 = \frac{1}{2}(1 + l)$, $u_2 = \frac{1}{2}(1 - l)$. Then the u_i are orthogonal idempotents, $u_1 + u_2 = 1$, $\bar{u}_1 = u_2$, $\bar{u}_2 = u_1$. Moreover, if $\mathfrak{Q}_0$ denotes the set of elements of trace 0 in $\mathfrak{Q}$ then we have

(65) $$O = \Phi u_1 \oplus \Phi u_2 \oplus \mathfrak{Q}_0 u_1 \oplus \mathfrak{Q}_0 u_2.$$

If $b \in \mathfrak{Q}_0$ then $u_1 b = \frac{1}{2}(1 + l)b = \frac{1}{2}b - \frac{1}{2}bl = \frac{1}{2}b(1 - l) = bu_2$ and similarly $u_2 b = bu_1$. It follows that $\mathfrak{Q}_0 u_1 = u_2\mathfrak{O}u_1$, $\mathfrak{Q}_0 u_2 = u_1\mathfrak{O}u_2$ so (64) is the Peirce decomposition of $\mathfrak{O}$ relative to the u_i. If $b_i \in \mathfrak{Q}_0$ then the multiplication formulas in $\mathfrak{O}$ give

(66) $$(b_1u_i)(b_2u_i) = (b_1 \wedge b_2)u_j,$$

(67) $$(b_1u_i)(b_2u_j) = -\,n(b_1, b_2)u_j$$

where $i \neq j$ and $b_1 \wedge b_2 = \frac{1}{2}[b_1, b_2]$ (cf. (48)). Now suppose $\mathfrak{Q}$ is the usual quaternion algebra with the basis $(1,x_1,x_2,x_3)$ where $\bar{x}_i = -\,x_i$, $x_i^2 = -\,1$, $x_ix_j = -\,x_jx_i = x_k$, (i,j,k) a cyclic permutation of $(1,2,3)$. Then in $\mathfrak{B} = \mathfrak{Q}_0$ we have the usual metric $n(\sum_1^3 \xi_i x_i) = \sum \xi_i^2$ and $b_1 \wedge b_2$ is the usual cross product of vectors in three-dimensional (Euclidean) space since $x_i \wedge x_j = x_k$.

We now consider the Jordan algebra $\mathfrak{H}(\mathfrak{O}_3, J_1)$ where $\mathfrak{O}$ is the split octonion algebra so $\mathfrak{O} = \Phi u_1 \oplus \Phi u_2 \oplus \mathfrak{V}u_1 \oplus \mathfrak{V}u_2$ where $\mathfrak{V}$ is the three-dimensional space as just indicated. Put $\mathrm{P} = \Phi u_1 \oplus \Phi u_2$, $\mathfrak{A}_0 = \mathfrak{H}(\mathrm{P}_3, J_1)$, $\mathfrak{A}_1 = (\mathfrak{V}u_1)[12] + (\mathfrak{V}u_1)[23] + (\mathfrak{V}u_1)[31]$, $\mathfrak{A}_2 = (\mathfrak{V}u_2)[12] + (\mathfrak{V}u_2)[23] + (\mathfrak{V}u_2)[31]$ where $(\mathfrak{V}u_i)[jk] = \{w[jk] \mid w \in \mathfrak{V}u_i\}$. We now define linear mappings $X = \sum \xi_{ij} e_{ij} \to X_i$ of $\mathfrak{A} = \Phi_3$ onto $\mathfrak{A}_i$ by the following formulas:

$$\begin{aligned} X_0 &= \sum \xi_{ii} e_{ii} + (\xi_{12}u_1 + \xi_{21}u_2)[12] + (\xi_{23}u_1 + \xi_{32}u_2)[23] \\ &\quad + (\xi_{31}u_1 + \xi_{13}u_2)[31] \\ X_1 &= -v_3u_1[12] - v_1u_1[23] - v_2u_1[31] \end{aligned}$$

where $v_i = \sum_j \xi_{ij} x_j$, (x_1, x_2, x_3) the above basis for $\mathfrak{V}$,

$$X_2 = -v_3'u_2[12] - v_1'u_2[23] - v_2'u_2[31], \quad v_i' = \sum_j \xi_{ji} x_j.$$

We have seen on p. 127 that $X \to X_0$ is an isomorphism of $\Phi_3{}^+$ onto $\mathfrak{A}_0$. By (66) we have $X_1{}^{\cdot 2} = -(v_1 \wedge v_2)u_2[12] - (v_2 \wedge v_3)u_2[23] - [v_3 \wedge v_1]u_2[31]$. Since $X \times X \equiv \operatorname{adj} X = \sum \eta_{ij} e_{ij}$ where $\eta_{ij} =$ co-factor of ξ_{ji} in X we have $(X \times X)_2 = X_1{}^{\cdot 2}$. Similarly, $(X \times X)_1 = X_2{}^{\cdot 2}$. Now for $i \neq j$ and $a, b \in \mathfrak{O}$, $a[ij] . b[ij] = n(a,b)(e_{ii} + e_{jj})$ and $n(a,b) = 0$ if $a \in \Phi u_1 + \Phi u_2$ and $b \in \mathfrak{V}u_i$. Hence if $A = \sum \alpha_{ij} e_{ij}$ and $B = \sum \beta_{ij} e_{ij}$ then

$$\begin{aligned} 2A_0 . B_1 &= -\ [(\alpha_{11} + \alpha_{22})b_3 - \alpha_{31}b_1 - \alpha_{32}b_2]u[12] \\ &\quad -\ [(\alpha_{22} + \alpha_{33})b_1 - \alpha_{12}b_2 - \alpha_{13}b_3]u_1[23] \\ &\quad -\ [(\alpha_{33} + \alpha_{11})b_2 - \alpha_{23}b_3 - \alpha_{21}b_1]u_1[31] \end{aligned}$$

where $b_i = \sum_j \beta_{ij} x_j$. On the other hand,

$$2\bar{A} = t(A)1 - A = \begin{bmatrix} \alpha_{22} + \alpha_{33} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{11} + \alpha_{33} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{11} + \alpha_{22} \end{bmatrix}$$

which implies that the row vectors of the matrix $2\bar{A}B$ are respectively $(\alpha_{22} + \alpha_{33})\beta_1 - \alpha_{13}\beta_2 - \alpha_{13}\beta_3$, $-\alpha_{21}\beta_1 + (\alpha_{11} + \alpha_{33})\beta_2 - \alpha_{23}\beta_3$, $-\alpha_{31}\beta_1 - \alpha_{32}\beta_2 + (\alpha_{11} + \alpha_{22})\beta_3$ where $\beta_1, \beta_2, \beta_3$ are the row vectors of B. It follows that $A_0 . B_1 = (\bar{A}B)_1$. Similarly, we have $A_0 . B_2 = (B\bar{A})_2$. Next let $a_i = \sum_j \alpha_{ij} v_j$, $b_i' = \sum_j \beta_{ji} v_j$. Then

$$\begin{aligned} 2A_1 . B_2 &= 2(a_3u_1[12] + a_1u_1[23] + a_2u_1[31]).(b_3'u_2[12] \\ &\qquad\qquad + b_1'u_2[23] + b_2'u_2[31]) \\ &= (n(a_3, b_3') + n(a_2, b_2'))e_{11} + (n(a_1, b_1') + n(a_3, b_3'))e_{22} \\ &\quad + (n(a_2, b_2') + n(a_1, b_1'))e_{33} - (n(a_1, b_2')u_2 + n(a_2, b_1')u_1)[21] \\ &\quad - (n(a_2, b_3')u_2 + n(a_3, b_2')u_1)[32] - (n(a_3, b_1')u_2 + n(a_1, b_3')u_1)[13]. \end{aligned}$$

Since $n(a_i, b_j') = \sum_k \alpha_{ik}\beta_{kj}$ one sees easily that $A_1 . B_2 = (\overline{AB})_0$. It is clear from the formulas we have just derived and (60) that if $\mathfrak{O}$ is the split octonion algebra then $\mathfrak{H}(\mathfrak{O}_3, J_1) \cong (\Phi_3, 1)$. This implies

THEOREM 19. *The algebra* $(\mathfrak{A}, \mu)$ *where* $\mathfrak{A}$ *is a nine-dimensional central simple associative algebra and* $\mu \neq 0$ *is in* Φ *is a finite-dimensional central simple exceptional Jordan algebra.*

PROOF. If Ω is the algebraic closure of Φ then $(\mathfrak{A}, \mu)_\Omega = (\mathfrak{A}_\Omega, \mu) \cong (\mathfrak{A}_\Omega, 1) = (\Omega_3, 1) \cong \mathfrak{H}(\mathfrak{O}_3, J_1)$. Hence the result is clear.

We determine next the conditions that $(\mathfrak{A}, \mu)$ is a division algebra and the structure of $(\mathfrak{A}, \mu)$ if this is not a division algebra. The result is the following

THEOREM 20. *Necessary and sufficient conditions that* $(\mathfrak{A}, \mu)$ *is a division algebra are*: $\mathfrak{A}$ *is a division algebra and* μ *is not a (generic) norm of an element of* $\mathfrak{A}$. *If* $(\mathfrak{A}, \mu)$ *is not a division algebra then this algebra is split.*

PROOF. Suppose first that $\mathfrak{A}$ is a division algebra but $(\mathfrak{A}, \mu)$ is not. Then $(\mathfrak{A}, \mu)$ contains an idempotent $e \neq 0, 1$ and $e = a_0 + b_1 + c_2$ where either $b \neq 0$ or $c \neq 0$ since $\mathfrak{A}_0 \cong \mathfrak{A}^+$ has no idempotents $\neq 0, 1$. By (60), we have $a^{\cdot 2} + 2\,\overline{bc} = a$, $2\,\bar{a}b + \mu^{-1}c \times c = b$, $2c\bar{a} + \mu b \times b = c$. If we multiply the second of these equations on the left by c and the third on the right by b we obtain $2c\bar{a}b + \mu^{-1}n(c) = cb$, $2c\bar{a}b + \mu n(b) = cb$. Then $\mu n(b) = \mu^{-1}n(c)$. Since $\mathfrak{A}$ is a division algebra $n(b) = 0$ implies $b = 0$. Hence $n(b) \neq 0$ and $n(c) \neq 0$. Then $\mu^2 = n(c)n(b)^{-1} = n(cb^{-1})$. This implies that $\mu = n(\mu bc^{-1})$. Hence if $\mathfrak{A}$ is a division algebra and μ is not a norm of an element of $\mathfrak{A}$ then $(\mathfrak{A}, \mu)$ contains no idempotents $\neq 0, 1$ and so $(\mathfrak{A}, \mu)$ is a division algebra. It is clear also that if $\mathfrak{A}$ is not a division algebra then $(\mathfrak{A}, \mu)$ is not since this contains a subalgebra isomorphic to $\mathfrak{A}^+$. Now assume $\mathfrak{A}$ is a division algebra and $\mu = n(c)$ for some $c \in \mathfrak{A}$. Then, as we have seen above, $(\mathfrak{A}, \mu) \cong (\mathfrak{A}, 1)$. Now in $(\mathfrak{A}, 1)$ consider the element $f = 1_0 + 1_1 + 1_2$. Then it is immediate from the multiplication table (60) that $f^{\cdot 2} = 3f$. This shows that $(\mathfrak{A}, 1)$ and, consequently, $(\mathfrak{A}, \mu)$ is not a division algebra. It remains to prove the last statement that if $(\mathfrak{A}, \mu)$ is not a division algebra then this Jordan algebra is split. Since a finite-dimensional exceptional central simple Jordan algebra which contains a subalgebra isomorphic to $\Phi_3{}^+$ is split we may assume that $\mathfrak{A}$ is a division algebra. Let Γ be a cubic subfield of $\mathfrak{A}$. Then Γ is a splitting field for $\mathfrak{A}$: $\mathfrak{A}_\Gamma \cong \Gamma_3$. Then Γ is a splitting field for $(\mathfrak{A}, \mu)$, that is, $(\mathfrak{A}, \mu)_\Gamma$ is split. On the other hand, $(\mathfrak{A}, \mu)$ is reduced and the condition that such an algebra is split is that the quadratic form $Q = \frac{1}{2}t$ has maximal Witt index on the subspace $\mathfrak{T}$ orthogonal to a reducing set of idempotents (ex. 6, p. 832). Then the extension of Q on $\mathfrak{T}_\Gamma$ has maximal Witt index. Since Γ is odd dimensional it follows from a theorem due to Springer [3] that Q has maximal Witt index on $\mathfrak{T}$ and so $(\mathfrak{A}, \mu)$ is split.

To actually construct a division algebra $(\mathfrak{A}, \mu)$ we shall make use of the following

LEMMA 1. *Let $\mathfrak{A}$ be a finite-dimensional strictly power associative division algebra over Φ (see p. 227), ξ an indeterminate. Then $\mathfrak{A}_{\mathrm{P}}$ for $\mathrm{P} = \Phi(\xi)$ is a division algebra and ξ is not a (generic) norm in $\mathfrak{A}_{\mathrm{P}}$ if* $\dim \mathfrak{A} > 1$.

PROOF. Let $(u_1, u_2, \cdots, u_n)$ be a basis for $\mathfrak{A}/\Phi$, m the degree of $\mathfrak{A}$. Any element of $\tilde{\mathfrak{A}} = \mathfrak{A}_{\mathrm{P}}$ has the form $\tilde{a} = \sum \rho_i(\xi)u_i$ where $\rho_i(\xi)$ is a rational expression in ξ with coefficients in Φ. If the generic norm $n(\tilde{a}) = 0$ for $\tilde{a} \neq 0$ we may assume the $\rho_i(\xi)$ are polynomials in ξ and not every $\rho_i(\xi)$ is divisible by ξ. Then $a = \sum \rho_i(0)u_i \neq 0$ in $\mathfrak{A}$ and $n(a) = 0$ contrary to the assumption that $\mathfrak{A}$ is a division algebra. Hence $\tilde{a} \neq 0$ in $\tilde{\mathfrak{A}}$ implies $n(\tilde{a}) \neq 0$ so $\tilde{\mathfrak{A}}$ is a division algebra. Now assume $\xi = n(\tilde{a})$, $\tilde{a} = \sum \rho_i(\xi)u_i$. Let $v(\xi)$ be a least common multiple of the denominators of the $\rho_i(\xi)$. Then we have $\xi v(\xi)^m = n(\sum \sigma_i(\xi)u_i)$ where the $\sigma_i(\xi)$ are polynomials in ξ. If ξ is a common factor of the $\sigma_i(\xi)$ then $n(\sum \sigma_i(\xi)u_i) = \xi^m n(\sum \sigma_i'(\xi)u_i)$ where the $\sigma_i'(\xi)$ are polynomials. If we assume $\dim \mathfrak{A} > 1$ then $m > 1$ and so the relation $\xi v(\xi)^m = \xi^m n(\sum \sigma_i'(\xi)u_i)$ implies that $\xi \mid v(\xi)$. Then $\xi^m \mid v(\xi)^m$ and we can cancel the ξ^m. Continuing in this way we may assume that $\xi v(\xi)^m = n(\sum \sigma_i(\xi)u_i)$ where the $\sigma_i(\xi)$ are polynomials in ξ and not every $\sigma_i(\xi)$ is divisible by ξ. Then $a = \sum \sigma_i(0)u_i \neq 0$ and $n(a) = 0$ contrary to the assumption that $\mathfrak{A}$ is a division algebra.

We can now prove the following theorem which is an immediate consequence of Lemma 1 and the criterion given in Theorem 20.

THEOREM 21 (ALBERT). *Let Φ be a field such that there exists a nine-dimensional central associative division algebra over Φ. Then there exist finite-dimensional central exceptional Jordan division algebras over $\mathrm{P} = \Phi(\xi)$, ξ an indeterminate.*

PROOF. Let $\mathfrak{A}$ be a nine-dimensional associative central division algebra over Φ. Then the algebra $(\mathfrak{A}_{\mathrm{P}}, \xi)$ is an exceptional central Jordan division algebra.

Examples of fields Φ satisfying the conditions of the theorem are: Φ, the rationals and Φ, any p-adic field.

Now let $\mathfrak{J}$ be any finite-dimensional exceptional central simple Jordan algebra. Assume $\mathfrak{J}$ contains a subalgebra $\mathfrak{A}_0$ isomorphic to an algebra $\mathfrak{A}^+$ where $\mathfrak{A}$ is a central simple associative algebra of degree three. We shall show that $\mathfrak{J}$ is isomorphic to an algebra of the form $(\mathfrak{A}, \mu)$. For the proof of this we shall need to apply the theory of bimodules for the Jordan algebra $\mathfrak{A}^+$.

We recall that any unital bimodule for $\mathfrak{A}^+$ is a unital right module for the unital universal multiplication envelope $U_1(\mathfrak{A}^+)$. By Theorems 7.8 and 7.10, $U_1(\mathfrak{A}^+) \cong S_1''(\mathfrak{A}^+)$ is a direct sum of five central simple ideals of dimensionalilties 9^2, 6^2, 6^2, 3^2, 3^2 respectively. Corresponding to these we have five isomorphism classes of irreducible unital bimodules for the Jordan algebra $\mathfrak{A}^+$. We recall also that the component of $U_1(\mathfrak{A}^+)$ of dimensionality 9^2 is Φ_9 and a representative of the corresponding isomorphism class of irreducible unital bimodules is the regular bimodule $\mathfrak{A}^+$.

We now construct the Jordan algebra $(\mathfrak{A}, 1) = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ as before. It is clear from the multiplication table $\mathfrak{A}_i . \mathfrak{A}_j \subseteq \mathfrak{A}_{i+j}$ that the $\mathfrak{A}_i$ are sub-bimodules of $(\mathfrak{A}, 1)$ considered as unital bimodule for $\mathfrak{A}_0$ under the usual action. The table (60) shows that these bimodules for $\mathfrak{A}^+ \cong \mathfrak{A}_0$ can be identified as vector spaces with $\mathfrak{A}$ where the actions in the three cases are respectively $x . a$, $\bar{a}x$ and $x\bar{a}$, x in the bimodule, a in $\mathfrak{A}^+$. In this way we obtain three bimodules, the first of which is the regular bimodule for $\mathfrak{A}^+$. We shall call the second and third the *left* and *right* bimodule for $\mathfrak{A}^+$. The sub-bimodules of the left (right) bimodule for $\mathfrak{A}^+$ are the left (right) ideals so this bimodule is irreducible or a direct sum of three isomorphic irreducible bimodules according as $\mathfrak{A}$ is a division algebra or $\mathfrak{A} \cong \Phi_3$. Since the regular bimodule for $\mathfrak{A}^+$ is nine dimensional and remains irreducible on extension of the base field to its algebraic closure while the left and right bimodules decompose as direct sums of three-dimensional submodules on extension of the base field to its algebraic closure, it is clear that the regular bimodule is not isomorphic to an irreducible component of the left or right bimodules for $\mathfrak{A}^+$. We note next that the irreducible bimodules provided by the left and right bimodules are not isomorphic. To prove this it is sufficient to show that the left and right bimodules are not isomorphic. Suppose they are. Then we have a bijective linear mapping S in $\mathfrak{A}$ such that $(\bar{a}x)S = (xS)\bar{a}$, $a, x \in \mathfrak{A}$. Since $a \to \bar{a}$ is surjective this gives $(ax)S = (xS)a$, $a, x \in \mathfrak{A}$. Putting $x = 1$ gives $aS = (1S)a$ or $xS = (1S)x$. Then $(ax)S = (1S)(ax)$ and $(xS)a = (1S)(xa)$, so $(1S)(ax) = (1S)(xa)$. Since $xS = (1S)x$ and S is bijective, $1S$ is invertible. Hence we have $ax = xa$, $a, x \in \mathfrak{A}$, which contradicts the fact that $\mathfrak{A}$ is central simple and $\dim \mathfrak{A} > 1$.

It is clear that the subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ generated by the mappings $x \to \bar{a}x$ ($x \to x\bar{a}$) is the set $\mathfrak{A}_L$ ($\mathfrak{A}_R$) of left (right) multiplications. Since $\mathfrak{A}_L \cong \mathfrak{A}_R^0$ and $\mathfrak{A}_R \cong \mathfrak{A}$ it is clear that the simple component of $U_1(\mathfrak{A}^+)$ corresponding to the irreducible bimodules of the left and the right bimodule for $\mathfrak{A}$ are isomorphic to $\mathfrak{A}^0$ and $\mathfrak{A}$ respectively. Since $\dim \mathfrak{A} = 3^2$ this identifies the components of dimensionality 3^2.

Now suppose that $\mathfrak{A}^+$ is imbedded as a subalgebra $\mathfrak{A}_0$ of a finite-dimensional central simple exceptional Jordan algebra. Then the isomorphism of $\mathfrak{A}^+$ and $\mathfrak{A}_0$ permits us to regard $\mathfrak{J}$ as unital $\mathfrak{A}^+$-bimodule in the obvious way. We claim that this bimodule has the structure $\mathfrak{J} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ where $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are sub-bimodules isomorphic to the regular, left and right bimodules respectively for $\mathfrak{A}^+$. If $\mathfrak{J}$ is split this is clear since in this case $\mathfrak{J} \cong (\mathfrak{A}, 1) = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ and the identity isomorphism on the common subalgebra $\mathfrak{A}_0$ can be extended to an isomorphism of $(\mathfrak{A}, 1)$ onto $\mathfrak{J}$. This gives an $\mathfrak{A}^+$-bimodule isomorphism of $(\mathfrak{A}, 1) = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ onto $\mathfrak{J}$ and establishes our assertion on $\mathfrak{J}$. Also, we see that in the split case the subalgebra $\mathfrak{E}$ of $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ generated by the mappings corresponding to the elements of $\mathfrak{A}^+$ is isomorphic to $\Phi_9 \oplus \mathfrak{A} \oplus \mathfrak{A}^0$ Now let $\mathfrak{J}$ be arbitrary and consider the algebra of linear transformations $\mathfrak{E}$ just defined. Since $\mathfrak{E}$ is a homomorphic image of $U_1(\mathfrak{A}^+)$, $\mathfrak{E}$ is a direct sum of certain of the five simple components of $U_1(\mathfrak{A}^+)$. On the other hand, if Ω is the algebraic closure of Φ then $\mathfrak{E}_\Omega \cong \Omega_9 \oplus \Omega_3 \oplus \Omega_3$. It follows that $\mathfrak{E} \cong \Phi_9 \oplus \mathfrak{A} \oplus \mathfrak{A}^0$. This in turn implies that $\mathfrak{J}$ as $\mathfrak{A}^+$-bimodule is a direct sum of irreducible sub-bimodules isomorphic to the regular, left and right bimodules for $\mathfrak{A}^+$. If $\mathfrak{A}$ is split, $\mathfrak{J}$ is split and the desired decomposition holds for $\mathfrak{J}$. If $\mathfrak{A}$ is a division algebra then the left and right bimodules are irreducible so again we have $\mathfrak{J} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ where these are isomorphic to the regular, left and right bimodules for $\mathfrak{A}^+$ respectively. We note also that in all cases the sub-bimodules $\mathfrak{A}_i$ of $\mathfrak{J}$ are uniquely determined as the homogeneous components of $\mathfrak{J}$ as completely reducible right $U_1(\mathfrak{A}^+)$-module (cf. Jacobson, *Structure of Rings*, p. 63).

The decomposition $\mathfrak{J} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ gives a linear isomorphism η_i, $i = 0,1,2$, of $\mathfrak{A}$ onto $\mathfrak{A}_i$ such that η_0 is an algebra isomorphism of $\mathfrak{A}^+$ onto $\mathfrak{A}_0$ and we have $x^{\eta_1}.a^{\eta_0} = (\bar{a}x)^{\eta_1}$, $x^{\eta_2}.a^{\eta_0} = (x\bar{a})^{\eta_2}$, $a, x \in \mathfrak{A}$. This shows that we have the first row and column of the multiplication table (60). We shall now show that we can modify η_1 so that we have the complete table (60) (for the η_i in place of the mappings $a \to a_i$). We now consider the extension algebra $\tilde{\mathfrak{J}} \equiv \mathfrak{J}_\Omega = \tilde{\mathfrak{A}}_0 \oplus \tilde{\mathfrak{A}}_1 \oplus \tilde{\mathfrak{A}}_2$ where $\tilde{\mathfrak{A}}_i = \mathfrak{A}_{i\Omega}$ and we let η_i denote the Ω-linear extension of η_i to $\tilde{\mathfrak{A}} = \mathfrak{A}_\Omega$ onto $\tilde{\mathfrak{A}}_i$. Then η_0 is an algebra isomorphism and the foregoing formulas hold for $x, a \in \tilde{\mathfrak{A}}$. Now form the Jordan algebra $(\tilde{\mathfrak{A}}, 1) = \tilde{\mathfrak{A}}_0 \oplus \tilde{\mathfrak{A}}_1 \oplus \tilde{\mathfrak{A}}_2$ using the mapping η_i for the mappings $a \to a_i$ given initially and the multiplication table (60). Then $\tilde{\mathfrak{J}} \cong (\tilde{\mathfrak{A}}, 1)$ and so the identity mapping on $\tilde{\mathfrak{A}}_0$ can be extended to an isomorphism λ of $(\tilde{\mathfrak{A}}, 1)$ onto $\tilde{\mathfrak{J}}$. Since the spaces $\tilde{\mathfrak{A}}_i$ in the two algebras are the homogeneous components of the algebras as bimodules for $\tilde{\mathfrak{A}}^+$ and these correspond respectively to the regular, left and right bimodules for $\tilde{\mathfrak{A}}^+$ it is clear that λ maps $\tilde{\mathfrak{A}}_i$ onto itself. Put $\zeta_i = \eta_i\lambda$, so $\zeta_0 = \eta_0$, and ζ_i is a linear isomorphism of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}_i$. If $a, b \in \mathfrak{A}$ we have $a^{\eta_1}.b^{\eta_1} = (a \times b)^{\eta_2}$ in $(\mathfrak{A}, 1)$. Applying λ gives $a^{\zeta_1}.b^{\zeta_1} = (a \times b)^{\zeta_2}$ in $\tilde{\mathfrak{J}}$. Similarly, we have $a^{\zeta_2}.b^{\zeta_2} = (a \times b)^{\zeta_1}$, $a^{\zeta_1}.b^{\zeta_2} = (\overline{ab})^{\zeta_0}$.

We now compare the mappings η_i, ζ_i as mappings into the algebra $\tilde{\mathfrak{J}}$. We have $a^{\eta_0}.b^{\eta_1} = (\bar{a}b)^{\eta_1}$, $a^{\eta_0}.b^{\zeta_1} = (\bar{a}b)^{\zeta_1}$. Hence $(\bar{a}b)^{\eta_1\zeta_1^{-1}} = (a^{\eta_0}.b^{\eta_1})^{\zeta_1^{-1}} = (a^{\eta_0}.b^{\eta_1\zeta_1^{-1}\zeta_1})^{\zeta_1^{-1}} = (\bar{a}b^{\eta_1\zeta_1^{-1}})$. Since this holds for all $a, b \in \tilde{\mathfrak{A}}$, it follows, as is well known, that there exists an invertible element $u \in \tilde{\mathfrak{A}}$ such that $b^{\eta_1\zeta_1^{-1}} = bu$, $b \in \tilde{\mathfrak{A}}$. Then $b^{\eta_1} = (bu)^{\zeta_1}$ and $b^{\zeta_1} = (bu^{-1})^{\eta_1}$. Similarly, there exists an invertible v in $\tilde{\mathfrak{A}}$ such that $b^{\eta_2} = (vb)^{\zeta_2}$, $b^{\zeta_2} = (v^{-1}b)^{\eta_2}$, $b \in \tilde{\mathfrak{A}}$.

Now let $a, b \in \mathfrak{A}$. Then $a^{\eta_i}, b^{\eta_i} \in \mathfrak{J}$ so $a^{\eta_i}.b^{\eta_j} \in \mathfrak{J}$. Then $a^{\eta_1}.b^{\eta_1} = (au)^{\zeta_1}.(bu)^{\zeta_1} = (au \times bu)^{\zeta_2} = (n(u)u^{-1}(a \times b))^{\zeta_2} = (n(u)v^{-1}u^{-1}(a \times b))^{\eta_2}$ which implies that $n(u)v^{-1}u^{-1} \in \mathfrak{A}$. Similarly, the formula for $a^{\eta_2}.b^{\eta_2}$ shows that $n(v)^{-1}v^{-1}u^{-1} \in \mathfrak{A}$ and that for $a^{\eta_1}.b^{\eta_2}$ shows that $uv \in \mathfrak{A}$. Then $n(u), n(v) \in \Phi$ and $v^{-1}u^{-1} \in \mathfrak{A}$. If $w \in \mathfrak{A}$ the mapping $x \to xw$ is an automorphism of $\mathfrak{A}$ considered as left $\mathfrak{A}^+$-

bimodule. Hence we may replace the mapping η_1 by η_1' given by $x^{\eta_1'} = (xw)^{\eta_1}$. Taking $w = v^{-1}u^{-1}$ we obtain $b^{\eta_1'} = (bv^{-1})^{\zeta_1}$. Then the element u in the foregoing calculations is replaced by v^{-1}. Hence we obtain $a^{\eta_1'} . b^{\eta_1'} = n(v)^{-1}(a \times b)^{\eta_2}$, $a^{\eta_2} . b^{\eta_2} = n(v)(a \times b)^{\eta_1'}$, $a^{\eta_1'} . b^{\eta_2'} = (\overline{ab})^{\eta_0}$. These relations and the others involving η_0 implb that $\mathfrak{A} \cong (\mathfrak{A}_0, n(v)^{-1})$. We have therefore proved the following

THEOREM 22. *Let $\mathfrak{J}$ be a finite-dimensional exceptional central simple Jordan algebra. Assume $\mathfrak{J}$ contains a subalgebra $\mathfrak{A}_0$ isomorphic to $\mathfrak{A}^+$ where $\mathfrak{A}$ is a central simple associative algebra of degree three. Then $\mathfrak{J} \cong (\mathfrak{A}, \mu)$ for some $\mu \neq 0$ in Φ.*

We shall now seek to clarify the position of the algebras of the form $(\mathfrak{A}, \mu)$ in the general theory of exceptional Jordan division algebras. For the sake of simplicity (to avoid discussing inseparable elements) we assume the characteristic is not three. Let $\mathfrak{J}$ be a finite-dimensional central exceptional Jordan division algebra. If $a \in \mathfrak{J}$ the minimum polynomial $\mu_a(\lambda)$ of a is irreducible. Hence the generic minimum polynomial $m_a(\lambda)$ is a power of $\mu_a(\lambda)$. Since the degree of $m_a(\lambda)$ is three we see that either $\mu_a(\lambda) = \lambda - \alpha$, which is the case if and only if $a = \alpha 1 \in \Phi 1$, or $m_a(\lambda) = \mu_a(\lambda)$ is of degree three. Thus any $a \notin \Phi 1$ generates a cubic subfield of $\mathfrak{J}$ and this is separable, since the characteristic is not three. We consider next the subalgebra generated by two elements in the following

LEMMA 2. *Let $\mathfrak{J}$ be a finite-dimensional central exceptional Jordan division algebra over a field of characteristic not three, a and b elements of $\mathfrak{J}$. Then the subalgebra $\mathfrak{K}$ generated by a and b is of one of the following types*: (1) $\mathfrak{K} = \Phi 1$, (2) *$\mathfrak{K}$ a separable cubic field*, (3) *$\mathfrak{K} \cong \mathfrak{A}^+$ where $\mathfrak{A}$ is an associative division algebra of degree three*, (4) *$\mathfrak{K} \cong \mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{A}$ is an associative division algebra of degree three over its center, which is a quadratic field, and J is an involution of second kind.*

PROOF. Suppose neither (1) nor (2) holds. Then $a \notin \Phi 1$ and $b \notin \Phi[a]$. Hence $\dim \mathfrak{K} > 3$. Clearly $\mathfrak{K}$ is a special Jordan division algebra. We claim that this is central. Otherwise, we have an element c in the center of $\mathfrak{K}$ such that $\Phi[c]$ is a separable cubic field. If $x \in \mathfrak{K}$ then $[x, c^{\cdot i}, c^{\cdot j}] = 0$ for $i, j = 0, 1, 2, \cdots$ and in $\mathfrak{K}_\Omega$ we have $\Omega[c] = \Omega e_1 + \Omega e_2 + \Omega e_3$ where the e_i are nonzero orthogonal idempotents. Then $[x, e_i, e_j] = 0$ which implies (via the Peirce decomposition relative to the e_i) that $x \in \Omega[c]$. Then $x \in \Phi[c]$, contrary to $\dim \mathfrak{K} > 3$. Hence $\mathfrak{K}$ is central and clearly the degree of $\mathfrak{K}$ is three. We can now apply the classification of special central simple algebras given in §5.7 (cf. ex.1 on p. 211) to conclude that $\mathfrak{K}$ is of one of the types (3) or (4).[1]

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[1] We remark that the case $\mathfrak{K} = \mathfrak{H}(\mathfrak{A}, J)$, $(\mathfrak{A}, J)$ central simple of degree three, is ruled out since these algebras are reduced.

LEMMA 3. *If $\mathfrak{K} = \mathfrak{H}(\mathfrak{A}, J)$ where $\mathfrak{A}$ is simple with center a quadratic field* P *and J is an involution of second kind then $\mathfrak{K}_P \cong (\mathfrak{A}/P)^+$. If $\mathfrak{K}$ is as in Lemma* 2 *then $\mathfrak{K}_\Sigma$ is a division algebra for any quadratic extension field Σ of Φ.*

PROOF. Let $P = \Phi(u)$ where $u^2 \in \Phi$ and $u^J = -u$. Then $\mathfrak{A} = \mathfrak{H}(\mathfrak{A}, J) \oplus \mathfrak{H}(\mathfrak{A}, J)u$. It follows from this that $\mathfrak{H}(\mathfrak{A}, J)_P \cong (\mathfrak{A}/P)^+$. Now let $\mathfrak{K}$ be as in Lemma 2 and let Σ be a quadratic field. If $\mathfrak{K}$ is as in (1) or (2) then it is clear that $\mathfrak{K}_\Sigma$ is a field. If $\mathfrak{K}$ is as in case (3), so $\mathfrak{K} = \mathfrak{A}^+$, where $\mathfrak{A}$ is an associative central division algebra of degree three, then $\mathfrak{A}_\Sigma$ is either a division algebra or $\mathfrak{A}_\Sigma \cong \Sigma_3$. The latter possibility is ruled out since the dimensionality of any splitting field for $\mathfrak{A}$ is $\geqq 3$. Hence $\mathfrak{A}_\Sigma$ is a division algebra and $\mathfrak{K}_\Sigma = (\mathfrak{A}^+)_\Sigma$ is a division algebra. Now let $\mathfrak{K}$ be as in (4) where P is the center of $\mathfrak{A}$. Then $(\mathfrak{K}_\Sigma)_P = (\mathfrak{K}_P)_\Sigma \cong (\mathfrak{A}/P)^+{}_\Sigma$ is a division algebra, by case (3). This implies that $\mathfrak{K}_\Sigma$ is a division algebra.

We can now prove

THEOREM 23. *Let $\mathfrak{J}$ be a finite-dimensional central exceptional division algebra. Then $\mathfrak{J}_P$ is a division algebra for any quadratic extension field* P *of Φ and there exist quadratic extensions* P *such that $\mathfrak{J}_P \cong (\mathfrak{A}, \mu)$ for an associative central division algebra $\mathfrak{A}$ of degree three over* P.

PROOF. Let $(1, u)$ be a basis for P/Φ. Then every element of $\mathfrak{J}_P$ has the form $a + bu$ where $a, b \in \mathfrak{J}$. Let $\mathfrak{K}$ be the subalgebra of $\mathfrak{J}$ generated by a and b. Then $a + bu \in \mathfrak{K}_P$ which is a Jordan division algebra by Lemma 3. Hence if $a + bu \neq 0$ then $a + bu$ has an inverse in $\mathfrak{K}_P$. Hence $\mathfrak{J}_P$ is a division algebra. Now let $a, b \in \mathfrak{J}$ such that $a \notin \Phi 1$ and $b \notin \Phi[a]$. Then the subalgebra $\mathfrak{K}$ generated by a and b is either as in (3) or (4) of Lemma 2. If we have case (3), $\mathfrak{J}$ is of the form $(\mathfrak{A}, \mu)$ by Theorem 22 and it remains of this form under any quadratic extension of the base field. If $\mathfrak{K}$ is as in (4) then we let P be the center of $\mathfrak{A}$. Then $\mathfrak{J}_P$ has the form $(\mathfrak{A}, \mu)$ by the first part of Lemma 3 and Theorem 22.

We shall now apply Theorem 23 and an arithmetic result due to Hasse and Schilling to prove the following

THEOREM 24 (ALBERT). *If Φ is an algebraic number field then every finite-dimensional central simple exceptional Jordan algebra over Φ is reduced.*

PROOF. Otherwise, we have a finite-dimensional central exceptional Jordan division algebra $\mathfrak{J}$ over Φ. By Theorem 23, we may assume $\mathfrak{J}$ is of the form $(\mathfrak{A}, \mu)$. On the other hand, it follows from a theorem of Hasse and Schilling ([1], cf. also Albert [33]) that μ is a generic norm of an element of $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is not a division algebra by Theorem 20. This contradiction proves the theorem.

It is interesting to study also directly the finite-dimensional central simple exceptional Jordan algebras which contain subalgebras of the form $\mathfrak{H}(\mathfrak{A}, J)$ where J is of second kind as in (4) of Lemma 3. The construction of these algebras will be indicated in exercises 5 and 6 below.

Exercises

1. Show that there exists an isomorphism of $(\mathfrak{A}, \mu)$ onto $(\mathfrak{A}, \nu)$ which is the identity on $\mathfrak{A}$ if and only if $\mu\nu^{-1}$ is a generic norm of an element of $\mathfrak{A}$.

2. Show that $(\mathfrak{A}, \mu) \cong (\mathfrak{A}^0, \mu^{-1})$.

3. Show that if $d \in \mathfrak{A}$ and the generic trace $t(d) = 0$ then the mapping $a_0 + b_1 + c_2 \to (bd)_1 - (dc)_2$ in $(\mathfrak{A}, \mu) = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ is a derivation mapping the subalgebra $\mathfrak{A}_0$ into 0. Prove that any derivation D in $(\mathfrak{A}, \mu)$ such that $\mathfrak{A}_0 D = 0$ has this form.

4. Determine the automorphisms of $(\mathfrak{A}, \mu)$ which map $\mathfrak{A}_0$ into itself.

5. (Tits). Let $(\mathfrak{A}, J)$ be a simple associative algebra of degree three with center a quadratic extension P of the base field, J an involution of second kind in $\mathfrak{A}$. Let $\mu \neq 0$ be in P and a an element of $\mathfrak{H}(\mathfrak{A}, J)$ such that $n(a) = \mu\mu^J$. Let $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A}$ and define the Φ-linear mapping $\mathfrak{A}$ into $\mathfrak{B}$ by $z \to z^* = (z, az^J)$. Let $\mathfrak{A}^*$ be the image in $\mathfrak{B}$ of this mapping. Let $(\mathfrak{A}, J, \mu, a) = \mathfrak{H} \oplus \mathfrak{A}^*$ where $\mathfrak{H} = \mathfrak{H}(\mathfrak{A}, J)$ and define a product in $(\mathfrak{A}, J, \mu, a)$ by linearizing the squaring operator defined by

$$(h + z^*)^{\cdot 2} = (h^{\cdot 2} + \overline{2zaz^J}) + (2\bar{h}z + \mu^J(z^J \times z^J)a^{-1})^*.$$

Show that $(\mathfrak{A}, J, \mu, a)_{\mathrm{P}} \cong (\mathfrak{A}/\mathrm{P}, \mu)$ and hence show that $(\mathfrak{A}, J, \mu, a)$ is an exceptional simple Jordan algebra containing $\mathfrak{H}(\mathfrak{A}, J)$ as a subalgebra.

6. (Tits). Show that any finite-dimensional central simple exceptional Jordan algebra which contains a subalgebra of the form $\mathfrak{H}(\mathfrak{A}, J)$, $\mathfrak{A}$ simple associative with involution J of second kind is an algebra $(\mathfrak{A}, J, \mu, a)$ as in exercise 5.

FURTHER RESULTS AND OPEN QUESTIONS

CHAPTER I

There are a number of important open questions concerning identities and inverses in Jordan algebras. We proceed to indicate some of these.

First, as in the text, let $FJ^{(r)}$ denote the free Jordan algebra with 1 and r (free) generators $x_1, x_2, \cdots, x_r$ (p. 40) and let $FSJ^{(r)}$ be the free special Jordan algebra with 1 and r (free) generators $u_1, u_2, \cdots, u_r$ (p. 7). Let $\mathfrak{K}^{(r)}$ be the kernel of the homomorphism of $FJ^{(r)}$ onto $FSJ^{(r)}$ sending $1 \to 1$, $x_i \to u_i$, $i = 1, 2, \cdots, r$. It is immediate that $\mathfrak{K}^{(r)}$ is a T-ideal in $FJ^{(r)}$, that is, an ideal mapped into itself by every homomorphism of $FJ^{(r)}$ into itself. We know that $\mathfrak{K}^{(r)} \neq 0$ if $r \geqq 3$. A Jordan algebra with 1 and r generators is a homomorphic image of a special Jordan algebra if and only if it satisfies all the identities in $\mathfrak{K}^{(r)}$. The results of Glennie indicated in the text (p. 51) provide nonzero elements of $\mathfrak{K}^{(3)}$. However, a complete determination of $\mathfrak{K}^{(r)}$ for $r \geqq 3$ has not yet been given. It is not known either if $\mathfrak{K}^{(r)}$, $r \geqq 3$, is finitely generated as a T-ideal.

Let $\Phi\{\{x_1, \cdots, x_r\}\}'$ be the free nonassociative algebra generated by $x_1, \cdots, x_r$, $\mathfrak{T}$, the ideal in this algebra generated by all the elements of the form $ab - ba$, $(a^2b)a - a^2(ba)$, a, $b \in \Phi\{\{x_i\}\}$. Assume that Φ is infinite and let $f \in \Phi\{\{x_i\}\}$ be homogeneous of degree n. We claim that if f is an identity for all finite-dimensional Jordan algebras, then f is an identity for all Jordan algebras. (This result has been communicated to us by Koecher.) We have to show that the hypothesis implies that $f \in \mathfrak{T}$. Since Φ is infinite, $\mathfrak{T}$ is a homogeneous ideal, that is, if $f_1 + f_2 + \cdots + f_k \in \mathfrak{T}$ where $\deg f_j = j$, then every $f_j \in \mathfrak{T}$. Let $\mathfrak{T}_{n+1}$ be the ideal generated by $\mathfrak{T}$ and all monomials in the x's of degree $\geqq n + 1$. Then $\mathfrak{T}_{n+1}$ is also a homogeneous ideal and the homogeneous part of degree n in this ideal is contained in $\mathfrak{T}$. It is clear that $\Phi\{\{x_i\}\}/\mathfrak{T}_{n+1}$ is a Jordan algebra and since there are only a finite number of distinct monomials in the x_i of degree $\leqq n$, $\Phi\{\{x_i\}\}/\mathfrak{T}_{n+1}$ is finite dimensional. Hence f is an identity for this Jordan algebra. Then $f \in \mathfrak{T}_{n+1}$, which implies that $f \in \mathfrak{T}$.

Let $\{\mathfrak{J}_\alpha \mid \alpha \in \mathrm{A}\}$ be a family of Jordan algebras with 1. We define a *free composition* of the $\mathfrak{J}_\alpha$ to be a Jordan algebra $\mathfrak{P}$ with 1 together with a family $\{\eta_\alpha \mid \alpha \in \mathrm{A}\}$ where η_α is a homomorphism of $\mathfrak{J}_\alpha$ into $\mathfrak{P}$ sending $1 \to 1$ such that, if $\mathfrak{J}$ is any Jordan algebra with 1 and ζ_α is a homomorphism of $\mathfrak{J}_\alpha$ into $\mathfrak{J}$ with $1 \to 1$, then there exists a unique homomorphism λ of $\mathfrak{P}$ into $\mathfrak{J}$ satisfying $1^\lambda = 1$ and $\zeta_\alpha = \eta_\alpha \lambda$, $\alpha \in \mathrm{A}$. It is immediate that any two free compositions are equivalent in the obvious categorical sense. In particular, the algebra $\mathfrak{P}$ is determined up to isomorphism. Also $\mathfrak{P}$ is generated by the subalgebras $\mathfrak{J}_\alpha^{\eta_\alpha}$. It is easy to prove the existence of a free composition for any family of Jordan algebras. For example, let $\mathfrak{J}_1 = FJ^{(r)}/\mathfrak{K}$, $\mathfrak{J}_2 = FJ^{(s)}/\mathfrak{L}$ where $\mathfrak{K}$ and $\mathfrak{L}$ are ideals. Identify $FJ^{(r)}$ and $FJ^{(s)}$ with the subalgebras of $FJ^{(r+s)}$ generated by $x_1, \cdots, x_r$ and $x_{r+1}, \cdots, x_{r+s}$ respectively and let $\mathfrak{M}$ be the ideal in $FJ^{(r+s)}$ generated by $\mathfrak{K}$ and $\mathfrak{L}$. Then it is easily seen that $\mathfrak{P} = FJ^{(r+s)}/\mathfrak{M}$ and the homomorphisms $x + \mathfrak{K} \rightarrow x + \mathfrak{M}$, $x \in FJ^{(r)}$ and $y + \mathfrak{L} \rightarrow y + \mathfrak{M}$, $y \in FJ^{(s)}$ constitute a free composition of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. This procedure is easily generalized (see Cohn's *Universal Algebra*, p. 142 for a general argument). We shall call the free composition $\{\mathfrak{P}, \eta_\alpha\}$ the *free product* of the $\mathfrak{J}_\alpha$ if η_α is a monomorphism for every $\alpha \in \mathrm{A}$. In this case, if $\mathrm{A} = \{1, 2, \cdots, n\}$, then we write $\mathfrak{P} = \mathfrak{J}_1 * \mathfrak{J}_2 * \cdots * \mathfrak{J}_n$. It is easy to see that the free product of associative algebras (same definitions as for Jordan algebras) exists. This implies that the free product of special Jordan algebras exist. On the other hand, free products may not exist for exceptional Jordan algebras. For example, let $\mathfrak{J}_1$ be a finite-dimensional simple exceptional Jordan algebra. Then it is known that if $\mathfrak{J}$ is any Jordan algebra with 1 containing $\mathfrak{J}_1$ as subalgebra with 1, then $\mathfrak{J} \cong \mathfrak{J}_1 \otimes \mathfrak{C}$, where $\mathfrak{C}$ is an associative Jordan algebra with 1 (Jacobson [18]). It follows that $\mathfrak{J}$ satisfies every multilinear identity which holds for $\mathfrak{J}_1$. Now let $\mathfrak{J}_2 = FJ^{(2)}$ and let $\mathfrak{P}$ be a free composition of $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Then $\mathfrak{P}$ satisfies all the multilinear identities satisfied by $\mathfrak{J}_1$. Hence $\mathfrak{P}$ cannot contain a subalgebra isomorphic to $\mathfrak{J}_2$ (cf. ex. 4, p. 363). Thus $\mathfrak{P}$ is not the free product of $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be associative Jordan algebras (with 1) and $\mathfrak{J}_3 = FJ^{(1)}$ with generator z. Since the $\mathfrak{J}_i$ are special the free product $\mathfrak{J}_1 * \mathfrak{J}_2 * \mathfrak{J}_3$ exists. Also the free associative product $\mathfrak{P}'$ of the associative algebras $\mathfrak{J}_i$ exists. Let $\mathfrak{J}_1 *' \mathfrak{J}_2 *' \mathfrak{J}_3$ be the subalgebra of the special Jordan algebra $\mathfrak{P}'^+$ generated by the images of the $\mathfrak{J}_i$ in $\mathfrak{P}'$. We have the canonical homomorphism σ of $\mathfrak{J}_1 * \mathfrak{J}_2 * \mathfrak{J}_3$ onto $\mathfrak{J}_1 *' \mathfrak{J}_2 *' \mathfrak{J}_3$. Let $\mathfrak{K}$ be the kernel of σ and let $\mathfrak{Z}$ be the subspace of $\mathfrak{J}_1 * \mathfrak{J}_2 * \mathfrak{J}_3$ spanned by products of elements of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ and a single z (that is, degree 1 in z). If $\mathfrak{J}_1 = \mathfrak{J}_2 = FJ^{(1)}$, then Macdonald's theorem states that $\mathfrak{K} \cap \mathfrak{Z} = 0$. The same result has been proved by McCrimmon in [9] if $\mathfrak{J}_1 = \mathfrak{J}_2$ is isomorphic to the group algebra of an infinite cyclic group. In both of these cases the universal multiplication envelope of $\mathfrak{J}_i$ is commutative (cf. §2.3 and §2.13). We conjecture that this condition on the $\mathfrak{J}_i$, $i = 1, 2$, is sufficient to insure that $\mathfrak{K} \cap \mathfrak{Z} = 0$.* This would constitute a generalization of Macdonald's and McCrimmon's theorems. Another consequence of the conjectured result would be that the free product of two associative Jordan algebras having commutative universal envelopes is special. This would constitute a generalization of Shirshov's theorem.

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* This conjecture was arrived at in a conversation with McCrimmon.

McCrimmon's analogue of Macdonald's theorem can be used to establish identities in x, y, $x^{\cdot -1}$, $y^{\cdot -1}$, z which are of degree one or zero in z by verifying these for special Jordan algebras. Examples of this sort are the identities $\{xyx\}.\{x^{\cdot -1}y^{\cdot -1}x^{\cdot -1}\} = 1$, $\{xyx\}^{\cdot 2}.\{x^{\cdot -1}y^{\cdot -1}x^{\cdot -1}\} = \{xyx\}$, which follow directly from Theorem 1.13(6), p. 52. Another example is

$$\{y\{(x^{\cdot -1} + y^{\cdot -1})\{xzx\}(x^{\cdot -1} + y^{\cdot -1})\}y\} = \{(x + y)z(x + y)\}.$$

On the other hand, McCrimmon's theorem is not adequate for establishing more general rational identities in x, y which involve inverses of polynomials in x and y and more general rational expressions, e.g. $x^{\cdot -1} + (y^{\cdot -1} - x)^{\cdot -1}$ (which occurs in Hua's identity, ex. 3, p. 54). To handle such identities we consider the free Jordan algebra $FJ^{(n+2)}$ generated by $x, y, x_1, \cdots, x_n$. Let $P_i = P_i(x, y, x_1, \cdots, x_i)$ be an element of the subalgebra generated by $1, x, y, x_1, \cdots, x_i$, $i = 0, 1, \cdots, n$. Let $I = I(P_0, \cdots, P_n)$ be the ideal in $FJ^{(n+2)}$ generated by the $2n$ elements $P_{i-1}.x_i - 1$, $P_{i-1}.x_i^{\cdot 2} - x_i$. Now let $\mathfrak{J}$ be a Jordan algebra with 1, a, b elements of $\mathfrak{J}$. Assume $P_0(a,b)$ is invertible in $\mathfrak{J}$ with a_1 as its inverse, $P_1(a, b, a_1)$ is invertible with inverse a_2, and, in general, $P_{i-1}(a, b, a_1, \cdots, a_{i-1})$ is invertible with inverse a_i. Then the homomorphism of $FJ^{(n+2)}$ into $\mathfrak{J}$ such that $1 \to 1$, $x \to a$, $y \to b$, $x_i \to a_i$ maps I into 0, so we have the homomorphism of $FJ^{(n+2)}/I$ into $\mathfrak{J}$ sending $1 \to 1$, $x + I \to a$, $y + I \to b$, $x_i + I \to a_i$. It is clear also that $P_{i-1} + I$ is invertible in $FJ^{(n+2)}/I$ with inverse $x_i + I$. The elements of I may be regarded as rational identities in the two elements x, y. As an example, let $n = 3$, $P_0 = x$, $P_1 = y$, $P_2 = x - x_2$. Then Hua's identity is equivalent to the statement that the following elements are contained in I:

$$(x_1 - x_3).(x - \{xyx\}) - 1, \qquad (x_1 - x_3)^{\cdot 2}.(x - \{xyx\}) - (x_1 - x_3).$$

We conjecture that for any choice of the P_i the subalgebra of $\mathfrak{J}$ generated by 1, a, b, $a_1, \cdots, a_n$ (as indicated) is special (assuming that $P_{i-1}(a, b, a_1, \cdots, a_{i-1})$ is invertible with inverse a_i). This would constitute a generalization of the theorem of Shirshov-Cohn. If $\mathfrak{J}$ is algebraic (that is, every element of $\mathfrak{J}$ is algebraic) then the subalgebra generated by 1, a, b, $a_1, \cdots, a_n$ is generated by $1, a, b$. Hence this is special by the Shirshov-Cohn theorem. In general, if the subalgebra generated by $1, a, b, a_1, \cdots, a_n$ is special, so that this can be identified with a subalgebra of $\mathfrak{A}^+$, $\mathfrak{A}$ associative, then a_i and $P_{i-1}(a, b_1, a_1, \cdots, a_{i-1})$ are inverses in the associative algebra $\mathfrak{A}$. Then the verification of the fact that certain elements of $\mathfrak{J}$ are mapped into 0 under the homomorphism we defined can be reduced to the verification of certain rational identities in associative algebras. In particular, it is clear that the validity of Hua's identity for algebraic Jordan algebras, hence for finite-dimensional Jordan algebras, is a consequence of Hua's identity for associative algebras. In general, if the conjecture we have made is true, then any rational identity in two elements of a Jordan algebra could be established by proving corresponding rational identities in two elements of an associative algebra.

One can also attempt to reduce identities which are rational in two elements and integral and of the first degree in a third element to associative rational identities in a similar manner. Here let $FJ^{(n+3)}$ be the free Jordan algebra generated by $1, x, y, z, x_1, \cdots, x_n$, $P_i(x, y, z, x_1, \cdots, x_i)$ an element of the subalgebra generated by $1, x, y, z, x_1, \cdots, x_i$, I the ideal generated by $P_{i-1} . x_i - 1$, $P_{i-1} . x_i^{.2} - x_i$. Let $\Phi\{u, v, w, u_1, \cdots, u_n\}$ be the free associative algebra generated by 1 and the indicated elements, I_s the ideal generated by $P_{i-1}u_i - 1$, $u_iP_{i-1} - 1$, where $P_i = P_i(u, v, w, u_1, \cdots, u_n)$. Then we have a homomorphism of $FJ^{(n+3)}$ into the Jordan algebra $(\Phi\{\cdots\}/I_s)^+$ such that $1 \to 1$, $x \to u + I_s$, $y \to v + I_s$, $z \to w + I_s$, $x_i \to u_i + I_s$. We conjecture that if $f \in FJ^{(n+3)}$ is of degree 1 in z and is mapped into 0 under the homomorphism of $FJ^{(n+3)}$ into $(\Phi\{\cdots\}/I_s)^+$, then $f \in I$. If this were the case, then we would have a reduction of identities rational in two elements and integral of the first degree in a third element to associative rational identities.

There are a number of interesting questions on the embeddability of Jordan integral domains in division rings. We recall first a well-known result of Ore's that any associative ring without zero-divisors $\neq 0$ which has the (left) common multiple property can be embedded in a division ring (Jacobson, *Theory of Rings*, p. 118 or Cohn's *Universal Algebra*, p. 275). The common multiple property for an associative ring $\mathfrak{A}$ is that if a and b are nonzero elements of $\mathfrak{A}$ then $\mathfrak{A}a \cap \mathfrak{A}b \neq 0$ for the left ideals $\mathfrak{A}a$, $\mathfrak{A}b$. We have called an element of a Jordan algebra $\mathfrak{J}$ a zero divisor in $\mathfrak{J}$ if the mapping U_a is not injective in $\mathfrak{J}$, that is, there exists a $b \neq 0$ such that $bU_a = 0$. We shall now say that the elements $a, b \in \mathfrak{J}$ *have a common multiple* if $\mathfrak{J}U_a \cap \mathfrak{J}U_b \neq 0$ for the quadratic ideals $\mathfrak{J}U_a$, $\mathfrak{J}U_b$. In view of Ore's result one is tempted to conjecture that if $\mathfrak{J}$ is a Jordan algebra in which any two nonzero elements have a common multiple then $\mathfrak{J}$ can be embedded in a Jordan division algebra.

It is a well-known result, due independently to Malcev and B. H. Neumann, that the free associative algebra $\Phi\{x_1, x_2, \cdots, x_n\}$ can be embedded in a division algebra (Cohn's *Universal Algebra*, p. 276). It follows from this that the free Jordan algebra $FJ^{(2)}$ can be embedded in a division algebra. Hence this is an integral domain. Is the free Jordan algebra $FJ^{(n)}$, $n \geqq 3$, an integral domain and can this be embedded in a division algebra? If the latter statement is true then it would provide examples of Jordan division algebras which are infinite dimensional over their centers. On the other hand, it is also possible that $FJ^{(n)}$ contains nonzero zero-divisors and possibly nonzero nilpotent elements if $n \geqq 3$. This is the situation for free alternative algebras (see Humm and Kleinfeld [7]).

CHAPTER II

The notion of bimodule and universal multiplication envelope for associative and Lie algebras are the starting points of the cohomology theories for these classes of algebras. The theory in the first case was initiated by Hochschild and in the second case by Chevalley and Eilenberg. We recall the basic foundational results in these theories (see Cartan-Eilenberg [1] for the details). Let $\mathfrak{A}$ be an associative algebra with 1, $\mathfrak{M}$ a unital bimodule for $\mathfrak{A}$. The unital universal multiplication envelope for $\mathfrak{A}$ is $U_1(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{A}^0$ and $\mathfrak{A}$ is a bimodule (the regular bimodule) for $\mathfrak{A}$. One defines the *n*th *cohomology group of* $\mathfrak{A}$ *with coefficients in* $\mathfrak{M}$ for $n = 0, 1, 2$, by $H^n(\mathfrak{A},\mathfrak{M}) = \mathrm{Ext}^n_{U_1(\mathfrak{A})}(\mathfrak{A},\mathfrak{M})$. These are vector spaces over the base field Φ, $H^0(\mathfrak{A},\mathfrak{M})$ is isomorphic to the subspace of $\mathfrak{M}$ of elements u such that $ua = au$, $a \in \mathfrak{A}$, $H^1(\mathfrak{A},\mathfrak{M}) \cong \mathrm{Der}(\mathfrak{A},\mathfrak{M})/\mathrm{Inder}(\mathfrak{A},\mathfrak{M})$, where $\mathrm{Inder}(\mathfrak{A},\mathfrak{M})$ is the subspace of derivations of $\mathfrak{A}$ into $\mathfrak{M}$ of the form $a \to [a,u]$, $a \in \mathfrak{A}$, $u \in \mathfrak{M}$. $H^2(\mathfrak{A},\mathfrak{M})$ is isomorphic to the vector space of equivalence classes of null (= singular) extensions $0 \to \mathfrak{M} \to \mathfrak{E} \to \mathfrak{A} \to 0$ where the vector space compositions on such equivalence classes are the Baer compositions. Next let $\mathfrak{L}$ be a Lie algebra, $U(\mathfrak{L})$ the universal multiplication envelope (= Birkhoff-Witt algebra) and regard Φ as a trivial bimodule for $\mathfrak{L}$ so that $\alpha l = 0 = l\alpha$, $\alpha \in \Phi$, $l \in \mathfrak{L}$. If $\mathfrak{M}$ is an $\mathfrak{L}$-bimodule, one defines $H^n(\mathfrak{L},\mathfrak{M}) = \mathrm{Ext}^n_{U(\mathfrak{L})}(\Phi,\mathfrak{M})$, $n = 0, 1, 2, \cdots$. Then $H^0(\mathfrak{L},\mathfrak{M})$ is isomorphic to the subspace of $\mathfrak{M}$ of u such that $ul = 0$ and $H^1(\mathfrak{L},\mathfrak{M}) \cong \mathrm{Der}(\mathfrak{L},\mathfrak{M})/\mathrm{Inder}(\mathfrak{L},\mathfrak{M})$ where $\mathrm{Inder}(\mathfrak{L},\mathfrak{M})$ is the set of derivations of the form $l \to lu$ $(= -ul)$, $l \in \mathfrak{L}$, $u \in \mathfrak{M}$. Also, as in the associative case $H^2(\mathfrak{L},\mathfrak{M})$ is isomorphic to the vector space of equivalence classes of null extensions of $\mathfrak{L}$ by $\mathfrak{M}$.

In both the associative and Lie cases, one has simple standard resolutions of the bimodules $\mathfrak{A}$ and Φ respectively which lead to determinations of $H^n(-,\mathfrak{M})$ by cochains, cocycles and coboundaries. Also in both cases one has important interpretations of the vanishing of H^n for $n = 1, 2, 3$.

A definition of cohomology spaces for $n \geqq 2$ in any variety of algebras has been given by Gerstenhaber in [2] as follows. Let $\mathfrak{A} \in \mathscr{V}(I)$, the variety of algebras defined by a set of identities I, $\mathfrak{M}$ an I-bimodule for $\mathfrak{A}$. Define a *singular extension of length two* of $\mathfrak{A}$ by $\mathfrak{M}$ to be a null extension of $\mathfrak{A}$ by $\mathfrak{M}$ as defined on p. 91. If $n > 2$, define a *singular extension of length n* to be an exact sequence of bimodules $0 \to \mathfrak{M} \to \mathfrak{M}_{n-1} \to \cdots \to \mathfrak{M}_2 \to \mathfrak{C} \to 0$ together with a singular extension $0 \to \mathfrak{C} \to \mathfrak{E} \to \mathfrak{A} \to 0$. These give the exact sequence

$$0 \to \mathfrak{M} \to \mathfrak{M}_{n-1} \to \cdots \to \mathfrak{M}_2 \to \mathfrak{E} \to \mathfrak{A} \to 0.$$

Morphisms, equivalences, addition and scalar multiplication of equivalence classes of singular extensions can be defined. Then one defines the *n*th *cohomology group* $H^n(\mathfrak{A},\mathfrak{M})$ *for* $n \geqq 2$ of $\mathfrak{A} \in \mathscr{V}(I)$ with coefficients in $\mathfrak{M}$ as the vector space of equivalence classes of singular extensions of length n of $\mathfrak{A}$ by $\mathfrak{M}$. These definitions are equivalent to the classical ones in the associative and Lie cases.

Gerstenhaber's definitions of $H^n(\mathfrak{A},\mathfrak{M})$ for $n \geqq 2$ have been recently supplemented by Glassman in [1] to give definitions of $H^0(\mathfrak{A},\mathfrak{M})$ and $H^1(\mathfrak{A},\mathfrak{M})$. We shall now indicate these. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be I-bimodules for $\mathfrak{A} \in \mathscr{V}(I)$, η a homomorphism of $\mathfrak{M}_1$ into $\mathfrak{M}_2$. Then any derivation D of $\mathfrak{A}$ into $\mathfrak{M}_1$ determines the derivation $D\eta$ of $\mathfrak{A}$ into $\mathfrak{M}_2$ so we have the linear mapping $\tilde{\eta}: D \to D\eta$ of $\mathrm{Der}(\mathfrak{A},\mathfrak{M}_1)$ into $\mathrm{Der}(\mathfrak{A},\mathfrak{M}_2)$. In this way one obtains a functor from the category of $\mathfrak{A}$-bimodules to the category of vector spaces so that $\mathfrak{M} \to \mathrm{Der}(\mathfrak{A},\mathfrak{M})$, $\eta \to \tilde{\eta}$. An *inner derivation functor* J: $\mathfrak{M} \to J(\mathfrak{A},\mathfrak{M})$, $\eta \to \tilde{\eta} \mid J(\mathfrak{A},\mathfrak{M})$ is defined to be a subfunctor of the preceding which respects epimorphisms. It can be shown that the inner derivation functors are in 1-1 correspondence with submodules of $\mathrm{Der}(\mathfrak{A}, U(\mathfrak{A}))$ considered in a natural way as left $U(\mathfrak{A})$-module. Relative to the choice of J one defines $H_J^1(\mathfrak{A},\mathfrak{M}) = \mathrm{Der}(\mathfrak{A},\mathfrak{M})/J(\mathfrak{A},\mathfrak{M})$. Now assume the inner derivation functor J is finitely generated in the sense that the left $U(\mathfrak{A})$-module $J(U(\mathfrak{A}))$ is finitely generated. Let $\{d_1, d_2, \cdots, d_k\}$ be a set of generators for this module and let X_i, $1 \leqq i \leqq k$, be a free $\mathfrak{A}$-bimodule with generator x_i corresponding to the generator 1 of $U(\mathfrak{A})$. Let $\bar{d}_i$ be the derivation of $\mathfrak{A}$ into X_i corresponding to d_i and let Y be the submodule of $\sum_1^k \oplus X_i$ generated by the elements $a \sum_1^k \bar{d}_i$, $a \in \mathfrak{A}$. Put $\mathfrak{C}_{\{d_i\}} = (\sum \oplus X_i)/Y$ and define $H^0_{J_1\{d_i\}}(\mathfrak{A},\mathfrak{M}) = \mathrm{Hom}_{U(\mathfrak{A})}(\mathfrak{C}_{\{d_i\}},\mathfrak{M})$ for any bimodule $\mathfrak{M}$ of $\mathfrak{A}$. For suitable choices of J and $\{d_i\}$ in the associative and Lie cases the definitions of H^0 and H^1 given by Glassman are equivalent to the usual ones. Also if $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is a short exact sequence of $\mathfrak{A}$-bimodules, then one can define connecting homomorphisms to obtain the usual long exact sequence of cohomology spaces.

All the results just indicated can be developed for algebras over commutative rings. The notion of bimodules, universal multiplication envelopes and extensions have been considered by Knopfmacher in [1] and [2] in this generality.

CHAPTER III

The Corollary (p. 138) to the Coordinatization Theorem has been generalized recently by McCrimmon in [11]. If e_1 and e_2 are nonzero orthogonal idempotents in a Jordan algebra $\mathfrak{J}$, then these are said to be *interconnected* if $e_i \in \mathfrak{J}_{ij}^2 . e_i$, $i \neq j = 1, 2$, $\mathfrak{J}_{ij} = \mathfrak{J}U_{e_i,e_j}$. This is equivalent to: $\mathfrak{J}_{ij}^2 . e_i = \mathfrak{J}_{ii}$, $\mathfrak{J}_{ii} = \mathfrak{J}U_{e_i}$. It is clear that connectedness implies interconnectedness. McCrimmon has proved that if $\mathfrak{J}$ is a Jordan algebra with $1 = \sum_1^n e_i$, where the e_i are nonzero interconnected orthogonal idempotents and $n \geqq 4$, then $\mathfrak{J}$ is special (and hence any Jordan algebra containing $\mathfrak{J}$ as subalgebra with the same identity is special). This theorem had been conjectured by the author and was suggested by Martindale's theorem. It is easy to see that any two nonzero idempotents of a simple Jordan algebra are interconnected. As has been shown by McCrimmon, a consequence of this and a generalization of the foregoing theorem to algebras not necessarily containing 1 is the following striking result: If $\mathfrak{J}$ is a simple Jordan algebra (not necessarily containing 1) which contains three nonzero orthogonal idempotents whose sum is not an identity element for $\mathfrak{J}$, then $\mathfrak{J}$ is special.

CHAPTER IV

The structure theory of this chapter reduces the study of the Jordan algebras which are nondegenerate and satisfy the minimum conditions on quadratic ideals to that of Jordan division algebras. Thus the situation is quite similar to that which obtains for Artinian semisimple algebras. At the present time the relation between Jordan and associative division algebras remains to be clarified. A natural question here is the following: Is every special Jordan division algebra of one of the following types: (1) a Jordan algebra of a symmetric bilinear form, (2) Δ^+ where Δ is an associative division algebra, (3) $\mathfrak{H}(\Delta, J)$ where Δ is a division algebra with involution J? At the present no examples of exceptional Jordan division algebras which are infinite dimensional over their centers are known.

There is no satisfactory analysis as yet of degenerate Jordan algebras satisfying appropriate minimum conditions on quadratic ideals. In particular, it would be interesting to have a definition of a radical for a Jordan algebra satisfying the minimum conditions which is analogous to one of the definitions of the radical of an Artinian associative algebras.

It would be interesting to extend to Jordan algebras other aspects of the associative structure theory. One interesting direction would be the development of a theory of Jordan *PI*-algebras, that is, Jordan algebras satisfying a polynomial identity. Since there exist nontrivial identities which are satisfied by all special Jordan algebras, the so-called *s*-identities (p. 49) it is natural to define a *PI*-Jordan algebra to be one which satisfies an identity p which is not an *s*-identity. For example, the Jordan algebra of a nondegenerate symmetric bilinear form satisfies the identity $p = [[x_1, x_2, x_3]^{\cdot 2}, x_4, x_5]$ (ex. 6, p. 364), and it is easily seen that this is not an *s*-identity. Are these and simple Jordan algebras which are finite dimensional over their centers the only simple *PI*-Jordan algebras? The same question can be asked for Jordan division algebras.

The structure theory of this chapter has been extended by McCrimmon ([6] and [14]) to algebras over an arbitrary commutative ring with 1. As we indicated in Chapter I, in considering special Jordan algebras, to obtain a general theory one has to replace the bilinear composition $a.b$ by the composition bU_a which is linear in b and quadratic in a, and, if the existence of 1 is not assumed, then one must consider also the unary composition $a^{\cdot 2}$. To simplify the discussion we stick to the case of algebras with 1. It is convenient to formulate the axioms in terms of the mappings U_a. For the sake of comparison with the Jordan algebras considered in this book, we give first an analogous definition of the usual ones in operator form. Accordingly, we *define a Jordan algebra with* 1 *over a field* Φ *of characteristic not two* to be a triple $(\mathfrak{J}, R, 1)$, where $\mathfrak{J}$ is a vector space over Φ, 1 a particular element of $\mathfrak{J}$, and R a mapping of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ satisfying the following conditions:

1. R is Φ-linear.
2. $R_1 = 1$.
3. $[R_a, R_{aR_a}] = 0$.
4. $R_a = L_a$ if $xL_a \equiv aR_x$.

It is clear that this definition is equivalent to the usual one and that if P is an extension field of Φ then $\mathfrak{J}_P$, 1 and the extension R of R to a linear mapping of $\mathfrak{J}_P$ satisfies conditions 1–4. Hence we obtain the Jordan algebra $(\mathfrak{J}_P, R, 1)$. The foregoing definition can also be given for left unital Φ-modules over an arbitrary commutative ring with 1, provided that Φ contains an element $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$. In this case also $\mathfrak{J}_P$, R and 1 would satisfy the same axioms for any commutative ring extension P of Φ (with the same 1). A *homomorphism* η of a Jordan algebra $(\mathfrak{J}, R, 1)$ into a Jordan algebra $(\mathfrak{J}', R', 1')$ is a Φ-homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$ such that $1^\eta = 1'$ and $(bR_a)^\eta = b^\eta R'_{a^\eta}$.

We now give McCrimmon's definition of a *quadratic Jordan algebra with* 1 *over an arbitrary commutative ring* Φ *with* 1 as a triple $(\mathfrak{J}, U, 1)$ where $\mathfrak{J}$ is a unital left Φ-module, 1 a distinguished element of $\mathfrak{J}$, and U is a mapping $a \to U_a$ of $\mathfrak{J}$ into $\mathrm{Hom}_\Phi(\mathfrak{J},\mathfrak{J})$ satisfying the following axioms:

1. U is Φ-quadratic, that is, $U_{\alpha a} = \alpha^2 U_a$, $\alpha \in \Phi$, $a \in \mathfrak{J}$ and $U_{a,b} \equiv U_{a+b} - U_a - U_b$ is Φ-bilinear in a and b.
2. $U_1 = 1$.
3. $U_a U_b U_a = U_{bU_a}$.
4. $V_{a,b} U_b = U_b V_{b,a}$ where $xV_{a,b} = bU_{a,x}$.

If P is a commutative ring extension with 1 of Φ, then U has a unique extension to a quadratic mapping of $\mathfrak{J}_P = P \otimes_\Phi \mathfrak{J}$ into $\mathrm{Hom}_P(\mathfrak{J}_P, \mathfrak{J}_P)$. Here if $\rho_i \in P$, $a_i \in \mathfrak{J}$ then $U_{\Sigma \rho_i a_i} = \sum \rho_i^2 U_{a_i} + \sum_{i<j} \rho_i \rho_j U_{a_i, a_j}$. We require also

5. Conditions 3 and 4 hold in every $\mathfrak{J}_P$.

Axiom 5 is equivalent to intrinsic conditions on $\mathfrak{J}$, namely, the validity of certain linearizations of 3 and 4. It is easy to see that if Φ is a field with more than three elements, then these linearizations are consequences of 3 and 4. Hence 5 is superfluous in this case.

We put $a^{\cdot 2} = 1U_a$, $a \circ b = (a+b)^{\cdot 2} - a^{\cdot 2} - b^{\cdot 2} = 1U_{a,b}$, $V_a = U_{a,1}$. Then $a \circ b = b \circ a$ and $a \circ a = 2a^{\cdot 2}$. Also $V_1 = U_{1,1} = 2$. We write $\{abc\} = bU_{a,c}$, so $\{aba\} = 2bU_{a,a}$ since $U_{a,a} = 2U_a$. Also, $\{abc\} = cV_{a,b}$.

As a special case of the notion of homomorphism for general algebras with finitary compositions (cf. Cohn's *Universal Algebra*, p. 49), one defines a *homomorphism* η of $(\mathfrak{J}, U, 1)$ into the quadratic Jordan algebra $(\mathfrak{J}', U', 1')$ to be a Φ-homomorphism of $\mathfrak{J}$ into $\mathfrak{J}'$ such that $1^\eta = 1'$ and $(bU_a)^\eta = b^\eta U_{a^\eta}$. The class of quadratic Jordan algebras with 1 over Φ is a category whose morphisms are homomorphisms. If Φ is a field of characteristic not two and $\mathfrak{J}$ is a Jordan algebra with 1 over Φ then $\mathfrak{J}$ defines a quadratic Jordan algebra with 1, $(\mathfrak{J}, U, 1)$

in which $1 = 1$, $U_a = 2R_a^2 - R_{a^{\cdot 2}}$. All the conditions except 4 are clear and this has been proved on p. 328. Conversely, let $(\mathfrak{J}, U, 1)$ be a quadratic Jordan algebra with 1 over a field of characteristic not two. Then it can be shown that, if we put $R_a = \frac{1}{2}V_a$, we obtain a Jordan algebra with 1, $(\mathfrak{J}, R, 1)$. One can show also that the two constructions are inverses and homomorphisms of $(\mathfrak{J}, U, 1)$ coincide with homomorphisms of $(\mathfrak{J}, R, 1)$. One obtains in this way a category isomorphism of the category of quadratic Jordan algebras with 1 over Φ with the category of Jordan algebras with 1 over Φ. The same result holds for any ring Φ which contains an element $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$, if one defines a Jordan algebra over Φ as indicated above (that is, as in the field case).

Let $\mathfrak{A}$ be an associative algebra with 1 over a commutative ring Φ (with 1). Define $xU_a = axa$. Then $xU_{a,b} = axb + bxa$, $xV_{a,b} = abx + xba$. It is immediate that 1–5 hold, so $\mathfrak{A}$ with 1 and the indicated U is a quadratic Jordan algebra with 1, $\mathfrak{A}^{(q)} = (\mathfrak{A}, U, 1)$. If $(\mathfrak{J}, U, 1)$ is a quadratic Jordan algebra with 1, a *subalgebra* $\mathfrak{K}$ is a Φ-submodule containing 1 and closed under bU_a. Subalgebras of algebras $\mathfrak{A}^{(q)}$, $\mathfrak{A}$ associative with 1, are called *special quadratic Jordan algebras with* 1. If $(\mathfrak{A}, J)$ is an associative algebra with involution and 1 then $\mathfrak{H}(\mathfrak{A}, J)$, the set of J-symmetric elements, is a subalgebra of $\mathfrak{A}^{(q)}$. Let $\mathfrak{B}$ be a vector space over a field of any characteristic, f a quadratic form. Then it is readily verified that the subspace $\mathfrak{J} = \Phi 1 + \mathfrak{B}$ of the Clifford algebra $C(\mathfrak{B}, f)$ is a subalgebra of $C(\mathfrak{B}, f)^{(q)}$. We call this the *quadratic Jordan algebra* of f.

We shall see in a moment that the standard exceptional Jordan algebras also have analogues in quadratic Jordan algebras (see also the notes on Chapter IX).

It seems likely that much of the usual Jordan theory can be carried over to quadratic Jordan algebras. We proceed to indicate that this is indeed the case with the structure theory as developed in this chapter. In the notes on the remaining chapters, extensions of some other parts of the theory will be indicated.

We consider first the basic concepts of the structure theory. One defines an *inner* (= quadratic) *ideal* $\mathfrak{B}$ of $(\mathfrak{J}, U, 1)$ to be a Φ-submodule of $\mathfrak{J}$ such that $\mathfrak{J}U_b \subseteq \mathfrak{B}$, $b \in \mathfrak{B}$. An *outer ideal* $\mathfrak{B}$ is a Φ-submodule such that $bU_a \in \mathfrak{B}$, $b \in \mathfrak{B}$, $a \in \mathfrak{J}$. An *ideal* is a subset which is both an inner and an outer ideal. If $\mathfrak{B}$ is an ideal it is easily seen that if $b \in \mathfrak{B}$ and $a, c \in \mathfrak{J}$, then $bU_{a,c}$, $b^{\cdot 2}$, $b \circ c$, $bV_{a,c} \in \mathfrak{B}$. It follows that the Φ-module $\mathfrak{J}/\mathfrak{B}$ is a quadratic Jordan algebra with 1 in the obvious way. The kernel of a homomorphism is an ideal. If $b \in \mathfrak{J}$ then $\mathfrak{J}U_b$ is an inner ideal called the *principal inner ideal* generated by b. If Φ contains $\frac{1}{2}$, then any outer ideal is an ideal. Otherwise, this may not be the case, as the following examples show: (1) Let $\mathfrak{J}$ be the special quadratic Jordan algebra $\mathfrak{H}(Z_n)$ $n \times n$ symmetric matrices over the ring Z of integers, and let $\mathfrak{B}$ be the subset of integral matrices with even diagonal elements. Then $\mathfrak{B}$ is an outer ideal which is not an ideal. (2) Let Φ be a field of characteristic two, P an extension field such that the subfield $\Phi(\mathrm{P}^2)$ over Φ generated by the squares of the elements of P does not coincide with P. Let $\mathfrak{H}(\mathrm{P}_n)$ be the special quadratic Φ-algebra o

$n \times n$ symmetric matrices over P. Let $\mathfrak{B}$ be the subset consisting of the matrices with diagonal entries in $\Phi(\mathrm{P}^2)$. Then $\mathfrak{B}$ is an outer ideal which is not an ideal.

The outer ideal $\mathfrak{J}'$ of $(\mathfrak{J}, U, 1)$ generated by 1 is a subalgebra which we shall call the *core* of $\mathfrak{J}$. Any subalgebra of $\mathfrak{J}$ containing $\mathfrak{J}'$ will be called an *ample* subalgebra of $\mathfrak{J}$.

In an associative algebra with 1 the relations $aba = a$ and $ab^2a = 1$ imply $ab = 1 = ba$. This suggests the definition: $a \in (\mathfrak{J}, U, 1)$ is *invertible* with b as an *inverse* if $bU_a = a$, $b^{\cdot 2}U_a = 1$. This is equivalent to the usual notions in a Jordan algebra and one has a close analogue of Theorem 1.13 (on inverses). For example, the inverse is unique and $U_aU_b = 1 = U_bU_a$ for b the inverse $a^{\cdot -1}$. A quadratic Jordan algebra with 1 is a *division algebra* if every nonzero element of the algebra is invertible.

If u is an invertible element of $(\mathfrak{J}, U, 1)$, then we put $1^{(u)} = u^{\cdot -1}$, $U_a^{(u)} = U_uU_a$. Then $(\mathfrak{J}, U^{(u)}, 1^{(u)})$ is a quadratic Jordan algebra called the *u-isotope* of $(\mathfrak{J}, U, 1)$. As in the case of Jordan algebra, isotopy is an equivalence relation.

An element e of a quadratic Jordan algebra $(\mathfrak{J}, U, 1)$ is *idempotent* if $e^{\cdot 2} = e$ and the idempotents e, f are *orthogonal* if $fU_e = e \circ f = eU_f = 0$. Let $\{e_i \mid i = 1, \cdots, n\}$ be orthogonal idempotents such that $\sum_1^n e_i = 1$. Put $E_{ii} = U_{e_i}$, $E_{ij} = U_{e_i, e_j}$ for $i \neq j$. Then $1 = \sum_{i \leq j} E_{ij}$ and the E_{ij} are orthogonal projections. Hence $\mathfrak{J} = \sum \oplus \mathfrak{J}_{ij}$. This is called the *Peirce decomposition* of $\mathfrak{J}$ relative to the e_i. The $\mathfrak{J}_{ii}$ are inner ideals. There are many important relations connecting the Peirce components $\mathfrak{J}_{ij}$ analogous to those we derived for Jordan algebras. One defines connectedness and strong connectedness for orthogonal idempotents in a quadratic Jordan algebra with 1 as for Jordan algebras. The principal results carry over: transitivity holds and one can pass from connected idempotents to strongly connected idempotents in an isotope as in Lemma 5 of §3.1 (p. 123).

We consider next the definition of quadratic Jordan matrix algebras. For the sake of simplicity we stick to the case of standard involutions; the more general case of canonical involutions is reducible to this via isotopy by a diagonal matrix. First, let $(\mathfrak{D}, j)$ be an associative algebra with involution and 1. For any $n = 1, 2, 3, \cdots$ we define the standard quadratic Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ to be the set of $n \times n$ matrices over $\mathfrak{D}$ which are symmetric under the standard involution $X \to \bar{X}^t$ $(\bar{d} = d^j)$ where $AU_B = BAB$ and 1 and the Φ-module structure are as usual. In studying these algebras it is convenient to modify our usual definition of $a[ij]$, $a \in \mathfrak{D}$ by putting $a[ii] = ae_{ii}$, $a \in \mathfrak{H}(\mathfrak{D}, j)$, $a[ij] = ae_{ij} + \bar{a}e_{ji}$, $a \in \mathfrak{D}$. Using these one can develop formulas for the U operator analogous to (18′)–(23′) of p. 126. These and the formulas which one obtains from the Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_3)$, where $(\mathfrak{D}, j)$ is alternative over a field of characteristic $\neq 2$, lead to the definition of $\mathfrak{H}(\mathfrak{D}_3)$ for $(\mathfrak{D}, j)$ alternative. Here one begins with an alternative algebra with involution $(\mathfrak{D}, j)$ and 1 satisfying the condition that $d\bar{d} \in N(\mathfrak{D})$, the nucleus of $\mathfrak{D}$, for every $d \in \mathfrak{D}$. Then it can be shown that $\mathfrak{D}' = \mathfrak{H}(\mathfrak{D}, j) \cap N(\mathfrak{D})$ is a Φ-submodule of $\mathfrak{D}$ such that $dd'\bar{d} \in \mathfrak{D}'$ for all $d \in \mathfrak{D}$, $d' \in \mathfrak{D}'$. One defines $\mathfrak{H}(\mathfrak{D}_3)$

to be the set of matrices which are symmetric under the standard involution in $\mathfrak{D}_3$ and *which have diagonal elements in* $\mathfrak{D}'$. (Note that if $\mathfrak{D}$ is associative this is the same as our former definition.) Equivalently, we let $\mathfrak{H}(\mathfrak{D}_3)$ be the Φ-module direct sum of three copies of $\mathfrak{D}$ and three copies of $\mathfrak{D}'$ denoted respectively as $\mathfrak{D}[ij]$, $i<j$, $\mathfrak{D}'[ii]$, $i,j = 1,2,3$. If $i>j$, we put $d[ij] = \bar{d}[ji]$, where $\bar{d}[ji]$ is the image in $\mathfrak{D}[ji]$ of $\bar{d}\in\mathfrak{D}$. It is easy to see that there is a unique quadratic mapping $a \to U_a$ of $\mathfrak{H}(\mathfrak{D}_3) = \mathfrak{D}[12] \oplus \mathfrak{D}[23] \oplus \mathfrak{D}[13] \oplus \mathfrak{D}'[11] \oplus \mathfrak{D}'[22] \oplus \mathfrak{D}'[33]$ into $\mathrm{Hom}_\Phi(\mathfrak{H}(\mathfrak{D}_3),\mathfrak{H}(\mathfrak{D}_3))$ such that the following formulas hold:

$$b'[ii]\,U_{a'[ii]} = a'b'a'[ii], \tag{1}$$

$$a'[ii]\,U_{a[ij]} = \bar{a}a'a[jj], \tag{2}$$

$$b[ij]\,U_{a[ij]} = a\bar{b}a[ij], \tag{3}$$

$$\{a'[ii]b[ij]c'[jj]\} = a'bc'[ij], \tag{4}$$

$$\{a'[ii]b[ij]c[ji]\} = (a'bc + \overline{a'bc})[ii], \tag{5}$$

$$\{a'[ii]b[ij]c[jk]\} = a'bc[ik], \tag{6}$$

$$\{a'[ii]b'[ii]c[ij]\} = a'b'c[ij], \tag{7}$$

$$\{a[ij]b'[jj]c[jk]\} = ab'c[ik], \tag{8}$$

$$\{a[ij]b[ji]c[ik]\} = a(bc)[ik], \tag{9}$$

$$\{a[ij]b[jk]c[ki]\} = (a(bc) + \overline{a(bc)})[ii], \tag{10}$$

where $\{abc\} = bU_{a,c}$ and it is understood that all the U formulas not covered by these and $a[ji] = \bar{a}[ij]$ are 0 (e.g. $a'[ii]U_{b'[jj]} = 0$ if $i \neq j$). We remark that the condition that $d\bar{d}\in N(\mathfrak{D})$ implies that $d\bar{d}$ and $d+\bar{d}\in\mathfrak{D}'$ for $d\in\mathfrak{D}$. Hence the right-hand sides of (1)–(10) are contained in $\mathfrak{H}(\mathfrak{D}_3)$. It is a formidable task which has been carried out by McCrimmon to show that $(\mathfrak{H}(\mathfrak{D}_3), U, 1)$, where $1 = \sum_1^3 1[ii]$ satisfies the axioms for a quadratic Jordan. We shall now define a *standard quadratic Jordan matrix algebra* to be either an algebra of the form $(\mathfrak{H}(\mathfrak{D}_n), U, 1)$, where $(\mathfrak{D}, j)$ is associative with involution and 1, or one of the algebras $(\mathfrak{H}(\mathfrak{D}_3), U, 1)$ just defined.

Let $\mathfrak{J}$ be such a standard quadratic Jordan matrix algebra and let $\mathfrak{D}_0$ be the Φ-submodule of $\mathfrak{D}$ generated by the elements $d\bar{d}$, $d\in\mathfrak{D}$. Then it is easily seen that if we identify $d'[ii]$ with $d'e_{ii}$, $d[ij]$ with $de_{ij}+\bar{d}e_{ji}$, $i<j$, then the core $\mathfrak{J}' = \mathfrak{H}(\mathfrak{D}_n)'$, $n \geqq 3$ is the subset of matrices having diagonal elements in $\mathfrak{D}_0$. The elements $e_i = 1[ii]$ and $1[ij]$, $i<j$, are contained in $\mathfrak{J}'$. The e_i are orthogonal idempotents such that $\sum e_i = 1$ and e_i and e_j, $i \neq j$, are strongly connected by $1[ij]$. As for ordinary Jordan algebras, one has a Strong Coordinatization Theorem which is a structure theorem for quadratic Jordan algebras $(\mathfrak{J}, U, 1)$ such that $1 = \sum_1^n e_i$, where the e_i are nonzero strongly connected orthogonal idempotents and $n \geqq 3$

The result states that under these conditions one has a homomorphism η of $(\mathfrak{J}, U, 1)$ into a standard quadratic Jordan matrix algebra $\mathfrak{H}(\mathfrak{D}_n)$ such that $\mathfrak{J}^\eta$ is an ample subalgebra of $\mathfrak{H}(\mathfrak{D}_n)$ and the kernel of η is an ideal consisting of absolute zero divisors z (that is, $U_z = 0$) such that $2z = 0$.

We can now state McCrimmon's extensions of our structure theorems. The class of algebras to which these apply are the quadratic Jordan algebras with 1, $(\mathfrak{J}, U, 1)$ which contain no absolute zero-divisors $\neq 0$, and which satisfy the minimum conditions for inner (= quadratic) ideals (p. 157). Such an algebra is a direct sum of simple quadratic Jordan algebras satisfying the same conditions. A determination of these can be given by the following theorem: A quadratic Jordan algebra $(\mathfrak{J}, U, 1)$ is simple with minimum conditions for inner ideals if and only if it is of one of the following types: I. a quadratic Jordan division algebra, II. an ample subalgebra of a quadratic Jordan algebra of a nondegenerate quadratic form, III. an ample subalgebra of a quadratic Jordan matrix algebra $\mathfrak{H}(\mathfrak{O}_3, J_\gamma)$ where $(\mathfrak{O}, j)$ is an octonion algebra with standard involution, IV. an ample subalgebra of a quadratic Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$, where $(\mathfrak{A}, J)$ is a simple Artinian associative algebra with involution.

We note finally that McCrimmon has succeeded also in extending the theory to quadratic Jordan algebras without 1 by giving axioms in terms of the U operator and a unary composition $a \rightarrow a^{\cdot 2}$, which insure that such a system can be imbedded in a quadratic Jordan algebra with 1.

Chapter V

Braun and Koecher [1] have defined the radical of finite-dimensional Jordan algebra $\mathfrak{J}/\Phi$ with 1 to be the intersection of the radicals of all the symmetric bilinear forms f on $\mathfrak{J}$ into extension fields of Φ such that $f(z, 1) = 0$ for all nilpotent z. It is clear that $\mathrm{rad}\,\mathfrak{J}$ as defined in this chapter is contained in the Braun-Koecher radical $\mathfrak{R}$. Conversely, let $a \notin \mathrm{rad}\,\mathfrak{J}$, so a has a nonzero image $\bar{a}$ in some simple homomorphic image $\bar{\mathfrak{J}}$ of $\mathfrak{J}$. Let $\bar{\mathrm{P}}$ be the center of $\bar{\mathfrak{J}}$ so $\bar{\mathrm{P}}$ is a finite-dimensional extension field of Φ. Let $\bar{t}$ be the generic trace bilinear form on $\bar{\mathfrak{J}}/\bar{\mathrm{P}}$ so $\bar{t}$ is nondegenerate (by Chapter VI, p. 240 or Chapter VIII, p. 353), $\bar{t}$ has values in $\bar{\mathrm{P}}$ and $\bar{t}(\bar{z}, \bar{1}) = 0$ for $\bar{z}$ nilpotent. Since $\bar{a} \neq 0$, $\bar{a}$ is not contained in the radical of $\bar{t}$. Define f by $f(x, y) = \bar{t}(\bar{x}, \bar{y})$. Then it is immediate that f is a form of the type considered by Braun and Koecher and a is not contained in the radical of f. Hence $a \notin \mathfrak{R}$. Thus $\mathfrak{R} \subseteq \mathrm{rad}\,\mathfrak{J}$ and $\mathfrak{R} = \mathrm{rad}\,\mathfrak{J}$.

Chapter VI

The author has proved in [32] that the generic norm $n(x)$, $x = \sum_1^n \xi_i u_i$, of any finite-dimensional simple Jordan algebra is an irreducible polynomial in $\Phi[\xi_1, \cdots, \xi_n]$. The method of proof of this result is an extension of one used by Dieudonné in [2] to prove the same result for simple associative algebras. Since any simple associative algebra $\mathfrak{A}$ over a field of characteristic not two determines the simple Jordan algebra $\mathfrak{A}^+$ with the same generic norm, the theorem for associative algebras of characteristic not two follows from the Jordan algebra result. The Jordan theorem gives the following consequence of Theorem 3 (iv): If $\mathfrak{A}$ is a simple Jordan algebra and the remaining hypotheses are as in Theorem 3 (iv) then $Q(\xi_1, \cdots, \xi_n)$ is a power of the generic norm. It is easy to extend this result to the semisimple case.

The classical theorem on composition of quadratic forms (see Theorem 4.5, Jacobson [22], and the references given there) has been generalized by Schafer in [14]. His results are that if $\mathfrak{A}$ is an algebra which possesses a nondegenerate homogeneous form $Q(x)$ of degree n permitting the composition $Q(xy) = Q(x)Q(y)$ and the characteristic is 0 or $p > n$, then $\mathfrak{A}$ is alternative. Moreover, if $\mathfrak{A}$ is finite dimensional, then $\mathfrak{A}$ is separable. Nondegeneracy of Q in Schafer's sense means that if $Q(x_1, x_2, \cdots, x_n)$ is the symmetric multilinear form associated with Q as usual so that $Q(x, \cdots, x) = Q(x)$, then the only z such that $Q(z, x_2, \cdots, x_n) = 0$ for all x_i is $z = 0$. Schafer's result, along with Theorem 3, can be used to give a complete determination of the nondegenerate Q which permit composition on a finite-dimensional algebra. Schafer conjectured that his conclusions are valid without the assumption of finiteness of dimensionality. In another paper [13] Schafer considered the problem of characterizing Jordan algebras by commutativity and the existence of a nondegenerate form Q satisfying the Jordan composition $Q(\{xyx\}) = Q(x)^2Q(y)$. He was able to treat this problem only for $\deg Q = 2$ and 3.

An extensive generalization of these results of Schafer's has been given by McCrimmon in [2]. He considers finite-dimensional algebras $\mathfrak{A}/\Phi$ and defines a form (homogeneous polynomial function of degree > 0) Q on $\mathfrak{A}$ to *admit composition* if there exists two rational mappings E annd F of $\mathfrak{A}$ into $\mathrm{Hom}_\Phi(\mathfrak{A}, \mathfrak{A})$ such that: 1. $E(1) = 1 = F(1)$, 2. $\Delta_1^a\ E = \alpha a_L$, $\Delta_1^a\ F = \beta a_R$ where α, β are nonzero elements of Φ and a_L and a_R are respectively the left and right multiplications by a, 3. $Q(aE_b) = Q(a)\,e(b)$, $Q(aF_b) = Q(a)f(b)$ for some rational functions e and f in $\mathfrak{A}$ (whenever the various functions are defined). A form Q is called *nondegenerate* if the corresponding *trace form* $\tau(a,b) = -\Delta_1^a\Delta^b \log Q$ is a nondegenerate symmetric bilinear form in a and b. If Q admits composition and the characteristic is 0 or a prime p exceeding the degree of Q, then this condition is equivalent to Schafer's conditions for nondegeneracy. With these definitions, McCrimmon has proved that if $\mathfrak{A}$ is a finite-dimensional algebra with 1 over an infinite field which possesses a nondegenerate form Q admitting composition, then $\mathfrak{A}$ is a separable noncommutative Jordan algebra (definition in ex. 8, p. 33) and $\mathfrak{A}^+$ is a separable (commutative) Jordan algebra. The converse is also valid as is shown by McCrimmon in [2] and Osborn in [7]. McCrimmon's theorem implies Schafer's theorem on composition in the usual sense with the improvement that the condition on the characteristic can be replaced by the assumption that the number of elements in the base field exceeds the degree of Q. McCrimmon's theorem yields also the generalization of Schafer's results on Jordan composition to forms of arbitrary degree.

In [8] McCrimmon generalized the foregoing results to algebras for which finiteness of dimensionality is not assumed and has obtained an affirmative answer to Schafer's conjecture noted above. In [10] the same author has given an extension of the theory of the generic minimum polynomial to strictly power associative algebras with 1 which may be infinite dimensional but are "generically algebraic" in a certain sense.

The basic tool in these papers of McCrimmon's is that of the differential calculus of rational mappings. This method was used systematically first by Koecher in applying the structure theory of Jordan algebras over the real field to determine the homogeneous domains of positivity and more general "ω-domains" (see Koecher [4] and also Vinberg [1]).

The following theorem has been proved by Meyberg in [3]: Let $\mathfrak{J}$ and $\mathfrak{J}'$ be finite-dimensional central simple Jordan algebras with the same underlying vector space of characteristic $\neq 3$ (and $\neq 2$) and assume that $\mathfrak{J}$ and $\mathfrak{J}'$ have the same structure group (equivalently, in view of Theorems 6.6 and 6.7, if $|\Phi|$ is large enough, the same groups of norm similarities), then $\mathfrak{J}$ and $\mathfrak{J}'$ are isotopes.

Chapter VII

It would be interesting to obtain proof of the main results of this chapter (the separability of $U(\mathfrak{J})$ for $\mathfrak{J}$ separable, the Albert-Penico-Taft theorem, Harris' theorem) without using the classification of simple Jordan algebras. For characteristic 0, such proofs are given in Chapter VIII. Perhaps it may be possible to transfer the results of this case to characteristic p by a method of reduction modulo p.

It is unknown at the present time whether or not anything like Theorem 6.16 is valid in the characteristic p case. In particular, the following question appears to be unsettled: Let $\mathfrak{J} = \mathfrak{K} \oplus \mathfrak{N}$ where $\mathfrak{K}$ is separable and $\mathfrak{N}$ is the radical. Let $\mathfrak{B}$ be a separable subalgebra. Then does there exist an automorphism η such that $\mathfrak{B}^{\eta} \subseteq \mathfrak{K}$? (Theorem 6.15 gives an affirmative answer in a special case.)

Chapter VIII

We shall indicate the group theoretic background for some of the Lie algebras considered in this chapter and the extensions of these notions to quadratic Jordan algebras (cf. Koecher [8] and McCrimmon [6]).

We consider first the structure group $\Gamma(\mathfrak{J})$. We defined this to be the group of isotopies of $\mathfrak{J}$ onto $\mathfrak{J}$ and we saw in §1.12 that $\Gamma(\mathfrak{J})$ is the set of bijective linear mappings α in $\mathfrak{J}$ for which there exists a linear mapping β such that $U_x\alpha = \beta U_{x^\alpha}$, $x \in \mathfrak{J}$. Then 1^α is invertible and $\beta = \alpha(U_{1^\alpha})^{-1}$. Hence the condition is:

$U_{x^\alpha} = U_{1^\alpha}\alpha^{-1} U_{x^\alpha}$ or, with a slight change of notation, we can define $\Gamma(\mathfrak{J})$ to be the set of bijective linear mappings W of $\mathfrak{J}$ such that

$$(1) \qquad U_{xW} = W^*U_xW, \qquad W^* = U_{1W}W^{-1}, \quad x \in \mathfrak{J}.$$

It follows that $W \in \Gamma(\mathfrak{J})$ if and only if W is a bijective linear mapping in $\mathfrak{J}$ such that there exists a bijective linear mapping W^* in $\mathfrak{J}$ satisfying $U_{xW} = W^*U_xW$. Then necessarily $W^* = U_{1W}W^{-1}$.

These considerations carry over to quadratic Jordan algebras $(\mathfrak{J}, U, 1)$ with 1 over an arbitrary commutative ring Φ with 1. We define the *structure group* $\Gamma(\mathfrak{J})$ to be the group of bijective Φ-endomorphisms W of $\mathfrak{J}$ for which there exists a bijective Φ-endormorphism W^* such that $U_{xW} = W^*U_xW$, $x \in \mathfrak{J}$. It is clear from the axiom 3 for quadratic Jordan algebras with 1 that if a is invertible, then $U_a \in \Gamma(\mathfrak{J})$. Also, $U_{1W} = W^*W$ is bijective, so $1W$ is invertible and $W^* = U_{1W}W^{-1} \in \Gamma(\mathfrak{J})$. It is immediate that $W \to W^*$ is an anti-automorphism in $\Gamma(\mathfrak{J})$ and $W^{**} = W$. If a is invertible and $W \in \Gamma(\mathfrak{J})$, then $U_{aW} = W^*U_aW$ is invertible. Hence aW is invertible in $\mathfrak{J}$ and $(aW)^{\cdot -1} = (aW)U_{aW}^{-1} = a^{\cdot -1}(W^*)^{-1}$. If $W \in \Gamma(\mathfrak{J})$ and $1W = 1$ then $W^* = W^{-1}$ and $U_xW = WU_{xW}$. This states that W is a homomorphism of $(\mathfrak{J}, U, 1)$ so W is an automorphism. Hence the group of automorphisms $\mathrm{Aut}\,\mathfrak{J} \subseteq \Gamma(\mathfrak{J})$ and $\mathrm{Aut}\,\mathfrak{J}$ is the subgroup of $\Gamma(\mathfrak{J})$ fixing the element 1. The subgroup $\Gamma_1(\mathfrak{J})$ of $\Gamma(\mathfrak{J})$ generated by the invertible U_a is called the *inner structure group*. The elements of $\Gamma_1(\mathfrak{J}) \cap \mathrm{Aut}\,\mathfrak{J}$ are called *inner automorphisms*.

Let $\mathfrak{D}$ be the algebra over Φ with basis $(1, t)$, where $t^2 = 1$ (cf. the proof of Theorem 2.1(7) on p. 67). Consider $(\mathfrak{J}_{\mathfrak{D}}, U, 1)$ and let $\mathfrak{G}(\mathfrak{J})$ be the set Φ of endomorphisms A of $\mathfrak{J}$ such that $1 + At \in \Gamma(\mathfrak{J}_{\mathfrak{D}})$. Here A is the $\mathfrak{D}$-endomorphism in $\mathfrak{J}_{\mathfrak{D}}$ extending the given A. The condition that $W = 1 + At \in \Gamma(\mathfrak{J}_{\mathfrak{D}})$ is that (1) holds for $x \in \mathfrak{J}$. Since $W^{-1} = 1 - At$, this gives the condition that $A \in \mathfrak{G}(\mathfrak{J})$ if and only if

$$(2) \qquad U_{x,xA} = U_{1,1A}U_x + [U_x, A], \qquad x \in \mathfrak{J}.$$

It follows from this, or directly from the definition, that $\mathfrak{G}(\mathfrak{J})$ is a Lie algebra (subalgebra of $\mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})^-$) which is restricted if $p\Phi = 0$ for a prime p (cf. Serre, *Lie Algebras and Lie Groups*, New York, 1965, p. L.A. 1.3). We call $\mathfrak{G}(\mathfrak{J})$ the *structure Lie algebra* of $(\mathfrak{J}, U, 1)$. It is easily seen that $A \in \mathfrak{G}(\mathfrak{J})$ if and only if there exists an $A' \in \mathrm{Hom}_\Phi(\mathfrak{J}, \mathfrak{J})$ such that

$$(3) \qquad U_{x,xA} = A'U_x + U_xA, \qquad x \in \mathfrak{J}.$$

Then A' is uniquely determined and in fact, $A' = U_{1A} - A$. One can deduce from the axioms for quadratic Jordan algebras with 1 the identity $U_xV_{a,b} + V_{b,a}U_x = U_{x,xV_{a,b}}$ which shows that $V_{a,b} \in \mathfrak{G}(\mathfrak{J})$. Since $U_{1,1A} = V_{1,1A}$ we see that $A' \in \mathfrak{G}(\mathfrak{J})$. Hence $A \to A'$ is an anti-automorphism of the Lie algebra $\mathfrak{G}(\mathfrak{J})$ and $\varepsilon: A \to \bar{A} \equiv -A'$ is an automorphism satisfying $\varepsilon^2 = 1$. Also we have $\bar{V}_{a,b} = -V_{b,a}$.

If $A \in \mathfrak{G}(\mathfrak{J})$ and $W \in \Gamma(\mathfrak{J})$, then $W^{-1}(1 + At)W \in \Gamma(\mathfrak{J}_{\mathfrak{D}})$. Hence $W^{-1}AW \in \mathfrak{G}(\mathfrak{J})$. Thus $\Gamma(\mathfrak{J})$ acts on $\mathfrak{G}(\mathfrak{J})$ by means of $A \to W^{-1}AW$ and this is an automorphism of $\mathfrak{G}(\mathfrak{J})$. One can establish easily from the definitions that $(W^{-1}AW)' = W^*A'(W^{-1})^*$.

The defining condition (3) for $A \in \mathfrak{G}(\mathfrak{J})$ gives $U_{x,yA} + U_{y,xA} = A'U_{x,y} + U_{x,y}A$ which implies that $AV_{x,z} + V_{xA,z} = V_{x,zA'} + V_{x,z}A$. Hence $[V_{a,b}, A] = V_{aA,b} + V_{a,b\bar{A}}$. This implies that the subset $\mathfrak{L}(\mathfrak{J})$ of sums of the operators $V_{a,b}$ is an ideal in $\mathfrak{G}(\mathfrak{J})$. We shall call this ideal the *inner structure Lie algebra* of $(\mathfrak{J}, U, 1)$.

We define a *derivation* D of $(\mathfrak{J}, U, 1)$ to be a Φ-endomorphism such that $1D = 0$. and $[U_x, D] = U_{x,xD}$, $x \in \mathfrak{J}$. It is clear from (3) that $D \in \mathfrak{G}(\mathfrak{J})$ and $D' = -D$ or $\bar{D} = D$. The set of derivations Der $\mathfrak{J}$ is the subalgebra of $\mathfrak{G}(\mathfrak{J})$ of mappings satisfying $1D = 0$. This is clear from (2). The elements of $\mathfrak{L}(\mathfrak{J}) \cap \operatorname{Der}\mathfrak{J}$ are called *inner derivations*. Among these one has the *strictly inner derivations* $V_{a,b} - V_{b,a}$. The inner derivations and strictly inner derivations constitute ideals Inder $\mathfrak{J}$ and Strinder $\mathfrak{J}$ in Der $\mathfrak{J}$.

We consider next the extension of the Tits-Koecher construction of the Lie algebra $\mathfrak{K}(\mathfrak{J})$ to quadratic Jordan algebras. Since $V_{a,b} = 2a \triangle b$ if $\mathfrak{J}$ is a Jordan algebra and $V_{a,b}$ is defined as in $(\mathfrak{J}, U, 1)$ with $U_a = 2R_a^2 - R_{a^2}$, one is lead to define $\mathfrak{K}(\mathfrak{J}) = \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{L}(\mathfrak{J})$, where $\bar{\mathfrak{J}}$ is a copy of $\mathfrak{J}$ and $\mathfrak{L}(\mathfrak{J})$ is the inner structure Lie algebra. We define a product [,] in $\mathfrak{K}(\mathfrak{J})$ as in (26) on p. 325 with the modification that $a \triangle b$ is replaced by $V_{a,b}$ (cf. Koecher [8]). As in ex. 1, p. 329, we can introduce the same bilinear product [,] in $\mathfrak{F}(\mathfrak{J}) \equiv \mathfrak{J} \oplus \bar{\mathfrak{J}} \oplus \mathfrak{G}(\mathfrak{J})$, $\mathfrak{G}(\mathfrak{J})$ the structure Lie algebra. The verification given in the text carries over to show that $\mathfrak{F}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$ are Lie algebras. We have the automorphism ε in $\mathfrak{F}(\mathfrak{J})$: $a + \bar{b} + A \to b + \bar{a} + \bar{A}$ (cf. p. 327) and $\varepsilon^2 = 1$.

We now return to Jordan algebras with 1 (over a field of characteristic $\neq 2$, though everything is valid for a commutative ring Φ containing $\frac{1}{2}$). Along with the Jordan algebra $\mathfrak{J}$ we consider the associated quadratic Jordan algebra $(\mathfrak{J}, U, 1)$, where $U_a = 2R_a^2 - R_{a^2}$. Then $V_{a,b} = 2a \triangle b = 2(R_{a.b} - [R_aR_b])$. A linear mapping D is a derivation of the Jordan algebra if and only if it is a derivation of $(\mathfrak{J}, U, 1)$. Since $\mathfrak{G}(\mathfrak{J})$ includes the $R_a = \frac{1}{2}V_{a,1}$, it is immediate that $\mathfrak{G}(\mathfrak{J}) = R(\mathfrak{J}) \oplus \operatorname{Der}\mathfrak{J}$ and $\mathfrak{L}(\mathfrak{J}) = R(\mathfrak{J}) \oplus [R(\mathfrak{J}), R(\mathfrak{J})]$.

We shall now prove a recent theorem of Koecher's which is an important addition to the theory of $\mathfrak{K}(\mathfrak{J})$ given in the text. This states that any derivation D of $\mathfrak{K}(\mathfrak{J})$ into $\mathfrak{F}(\mathfrak{J})$ can be extended to an inner derivation D of $\mathfrak{F}(\mathfrak{J})$ and every derivation of $\mathfrak{F}(\mathfrak{J})$ is inner. To be consistent with the text, we use the definitions given in the text rather than the modifications just indicated for quadratic Jordan algebras. We note first that if A is a linear transformation in $\mathfrak{J}$ such that there exists a linear transformation $\tilde{A}$ such that $[a \triangle b, A] = aA \triangle b + a \triangle b\tilde{A}$, then $A \in \mathfrak{G}(\mathfrak{J})$ and $\tilde{A} = \bar{A}$. This follows by retracing the steps of the argument given above to show that $\mathfrak{L}(\mathfrak{J})$ is an ideal in $\mathfrak{G}(\mathfrak{J})$. Now let D be a derivation of the ideal $\mathfrak{K}(\mathfrak{J})$ into $\mathfrak{F}(\mathfrak{J})$. Since $[a + \bar{b} + L, R_1] = a - \bar{b}$ we can subtract an inner derivation from D to obtain a derivation mapping R_1 into $\mathfrak{G}(\mathfrak{J})$. Hence we may assume $R_1 D \in \mathfrak{G}(\mathfrak{J})$. Since $[a, R_1] = a$ for $a \in \mathfrak{J}$, we have $[aD, R_1] + [a, R_1 D] = aD$, and since $[aD, R_1] \in \mathfrak{J} + \bar{\mathfrak{J}}$ and $R_1 D \in \mathfrak{G}(\mathfrak{J})$, so $[a, R_1 D] \in \mathfrak{J}$, we have $aD \in \mathfrak{J} + \bar{\mathfrak{J}}$. Hence we can write $aD = aD_1 + aD_2$, where D_1 and D_2 are linear mappings of $\mathfrak{J}$ into $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ respectively. Then $[a, R_1] = a$ gives $aD_1 + aD_2 = aD_1 - aD_2 + a(R_1 D)$. Since $aR_1 D \in \mathfrak{J}$, we obtain $D_2 = 0$, so $\mathfrak{J}D \subseteq \mathfrak{J}$. Similarly, $\bar{\mathfrak{J}}D \subseteq \bar{\mathfrak{J}}$. Then $[\mathfrak{J}\bar{\mathfrak{J}}]D \subseteq [\mathfrak{J}\bar{\mathfrak{J}}]$, and since $[\mathfrak{J}\bar{\mathfrak{J}}] = \mathfrak{L}$, we see that also $\mathfrak{L}D \subseteq \mathfrak{L}$. Let D_1 be the restriction of D to $\mathfrak{J}$ and write the restriction of D to $\bar{\mathfrak{J}}$ as $\bar{x} \to \overline{xD_2}$, $x \in \mathfrak{J}$. Let D_3 be the restriction of D to $\mathfrak{L}$. If $a \in \mathfrak{J}$ and $L \in \mathfrak{L}$, then $[a, L] = aL$ and $aLD_1 = [aD_1, L] + [a, LD_3] = aD_1 L + aLD_3$. Hence $LD_3 = [L, D_1]$. If $b \in \mathfrak{J}$, then $[a \triangle b, D_1] = (a \triangle b)D_3 = [a, \bar{b}]D = [aD_1, \bar{b}] + [a, \overline{bD_2}] = aD_1 \triangle b + a \triangle bD_2$. By the result noted at the beginning, we have $D_1 \in \mathfrak{G}(\mathfrak{J})$ and $D_2 = \bar{D}_1$. It follows that the derivation D coincides with the restrictions to $\mathfrak{K}(\mathfrak{J})$ of the inner derivation $X \to [X, D_1]$ determined by $D_1 \in \mathfrak{G}(\mathfrak{J})$. It follows easily that every derivation of $\mathfrak{F}(\mathfrak{J})$ is inner and that $\operatorname{Der} \mathfrak{F}(\mathfrak{J}) \cong \mathfrak{F}(\mathfrak{J}) \cong \operatorname{Der} \mathfrak{K}(\mathfrak{J})$.

We consider next the automorphisms of $\mathfrak{F}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$. We have already defined the automorphism ε: $a + \bar{b} + L \to b + \bar{a} + \bar{L}$ of $\mathfrak{F}(\mathfrak{J})$ and $\mathfrak{K}(\mathfrak{J})$. Let $W \in \Gamma(\mathfrak{J})$ the structure group and $A \in \mathfrak{G}(\mathfrak{J})$. Then $W^{-1}AW \in \mathfrak{G}(\mathfrak{J})$. The definition of $\Gamma(\mathfrak{J})$ implies that $W^{-1}(a \triangle b)W = aW \triangle b(W^*)^{-1}$. Hence $A \to W^{-1}AW$ is an automorphism of $\mathfrak{G}(\mathfrak{J})$ which maps $\mathfrak{L}(\mathfrak{J})$ into itself. Also, $a + \bar{b} + A \to aW + \overline{b(W^*)^{-1}} + W^{-1}AW$ is an automorphism α_W of $\mathfrak{F}(\mathfrak{J})$ which maps $\mathfrak{K}(\mathfrak{J})$ into itself. It can be shown that the α_W can be characterized as the automorphisms of $\mathfrak{F}(\mathfrak{J})$ ($\mathfrak{K}(\mathfrak{J})$) which map $\mathfrak{J}$ into itself. If $a \in \mathfrak{J}$, then $(\operatorname{ad} a)^3 = 0$ in $\mathfrak{F}(\mathfrak{J})$ (cf. ex. 4, p. 329). It follows that $\tau_a = \exp(\operatorname{ad} a)$ where $X \exp(\operatorname{ad} a)) = 1 + [Xa] + \frac{1}{2}[[Xa]a]$ is an automorphism in $\mathfrak{F}(\mathfrak{J})$ mapping $\mathfrak{K}(\mathfrak{J})$ into itself. The automorphism ε, α_W for $W \in \Gamma(\mathfrak{J})$, τ_a for $a \in \mathfrak{J}$ generate a subgroup $\operatorname{Aut}' \mathfrak{F}(\mathfrak{J})$ of the group of automorphisms of $\mathfrak{F}(\mathfrak{J})$. Relations between these generators have been considered by Koecher. There exist Jordan algebras $\mathfrak{J}$ such that $\operatorname{Aut}' \mathfrak{F}(\mathfrak{J}) \neq \operatorname{Aut} \mathfrak{F}(\mathfrak{J})$.

For finite-dimensional Jordan algebras, Koecher has defined in a recent paper [10] a group of rational mappings which is closely related to the foregoing groups. This is the group $\Xi(\mathfrak{J})$ generated by the structure group $\Gamma(\mathfrak{J})$, the translations $t_a: x \to x + a$ and the mapping $j: x \to -x^{\cdot -1}$ defined on the set of invertible elements. In general, the mappings of Ξ are rational defined on Zariski open subsets of $\mathfrak{J}$ and the composition is the usual composition $\circ$ (cf. §6.2). The Hua identity can be written in operator form as $j \circ t_a \circ j \circ t_{a^{\cdot -1}} \circ j \circ t_a = U_a$. This shows that the inner structure group $\Gamma_1(\mathfrak{J})$ is contained in the subgroup $\Xi_1(\mathfrak{J})$ generated by j and the translations. If T denotes the subgroup of translations, then it follows from Hua's identity that $\Xi = \Gamma \circ T \circ j \circ T \circ j \circ T$ (with obvious meaning). Koecher has given a beautiful characterization of the group Ξ by a differential equation such that a rational mapping is contained in Ξ if and only if it satisfies the equation. Using this, he has proved that Ξ is an algebraic group (not linear). Also he has studied the question of the uniqueness of the representation of an element according to the decomposition of Ξ as $\Gamma \circ T \circ j \circ T \circ j \circ T$. One obtains a linear representation of Ξ onto $\mathrm{Aut}'\mathfrak{F}(j)$ sending $W \to \alpha_W$, $t_a \to \tau_a$, $j \to \varepsilon$.

Chapter IX

A number of characterizations of finite-dimensional exceptional central simple Jordan algebras based on forms have been given. We have already noted Schafer's characterization in [13] by means of cubic forms satisfying the Jordan composition (see the notes on Chapter VI). Another important characterization is one due to Springer in [1]. In this, one assumes that $\mathfrak{J}$ is a finite-dimensional algebra with 1 over a field of characteristic $\neq 2, 3$ equipped with a quadratic form Q with non-degenerate associated bilinear form (x, y) such that (1) $Q(x^2) = Q(x)^2$ if $(x, 1) = 0$, (2) $(xy, z) = (x, yz)$, (3) $Q(1) = 3/2$. Then Springer has shown that $\mathfrak{J}$ is a central simple Jordan algebra of degree three with $Q(x) = \frac{1}{2}t(x^{\cdot 2})(x^{\cdot 2} = x^2)$ and conversely.

Another characterization based on cubic forms is one which was first suggested by Freudenthal (cf. [6]) and was established by Springer in [5] for finite-dimensional algebras over fields of characteristic $\neq 2, 3$. Recently, this has been extended by McCrimmon in [15] to quadratic Jordan algebras over arbitrary fields. The Springer-McCrimmon axioms are the following. Let $\mathfrak{J}$ be a finite-dimensional vector space over an arbitrary field Φ equipped with a cubic form n and a distinguished element $l \in \mathfrak{J}$ satisfying the following conditions:

1. $n(1) = 1$.
2. $t(x, y) = -\Delta_x^1 \Delta_y \log n$ is a nondegenerate symmetric bilinear form.
3. If $x^{\#}$ is defined by $t(x^{\#}, y) = \Delta_x^y n$, then $x^{\#\#} = n(x)x$.

Now define $x \times y = (x + y)^{\#} - x^{\#} - y^{\#}$ and

$$yU_x = t(x, y)x - x^{\#} \times y. \tag{1}$$

Then McCrimmon has shown, using the techniques of the differential calculus of rational mappings, that $(\mathfrak{J}, U, 1)$ is a quadratic Jordan algebra with 1. (Actually, the hypothesis of finiteness of dimensionality is not essential. One requires only a formulation which permits the application of the differential calculus.) Let $(\mathfrak{D}, j)$ be a composition algebra over an arbitrary field Φ, $\mathfrak{H}(\mathfrak{D}_3, J_\gamma)$ the space of 3×3 matrices over $\mathfrak{D}$ which are symmetric under the canonical involution $X \to \gamma^{-1}\bar{X}^t\gamma$, $\gamma = \mathrm{diag}\,\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_i \neq 0$ in Φ. Let $n(X)$ be defined as usual (equation (50), p. 232) and let 1 be the usual identity matrix. Then it can be shown that the conditions 1–3 are satisfied, so if U is defined by (1), then $(\mathfrak{H}(\mathfrak{D}_3, J_\gamma), U, 1)$ is a quadratic Jordan algebra with 1. The structure then defined is an isotope of a standard quadratic Jordan matrix algebra as defined in the notes on Chapter IV by an octonion algebra.

Similarly, one can extend Tits' constructions to quadratic Jordan algebras. We indicate only the first of these. Hence let $\mathfrak{A}$ be a central simple associative algebra (with 1) of degree three. Then the generic norm n on $\mathfrak{A}$ is a cubic form and

$t(a, b) = -\Delta_a^1 \Delta_b \log n$ is a nondegenerate symmetric bilinear form on $\mathfrak{A}$. Let $\mathfrak{J} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$ a direct sum of three copies of $\mathfrak{A}$, $a \to a_i$ a linear isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_i$. Let μ be a nonzero element of Φ and define

$$n(x) = n(a) + \mu n(b) + \mu^{-1} n(c)$$

for $x = a_0 + b_1 + c_2$, $a,b,c \in \mathfrak{A}$. Then it can be shown that this and $1 = 1_0$ satisfy the condition 1–3 so (1) defines a quadratic Jordan algebra with 1. If the characteristic is not two, this coincides with the quadratic Jordan algebra defined by Tits.

There is an extensive literature on exceptional algebraic groups, Lie algebras and geometries connected with exceptional Jordan algebras. We indicate this briefly. Tae-Il Suh in [1] has proved that any isomorphism between the little projective groups of Moufang projective planes is induced by an isomorphism or correlation between the planes. His proof is based on a classification of involutions contained in the little projective group. Another method of obtaining Suh's result and an extension of this to the middle projective group has been given by Veldkamp in [2]. An extensive study of elliptic and hyperbolic Moufang planes, that is, the geometry of Moufang planes relative to certain types of polarities, has been made by Springer and Veldkamp in [1]. Recently, Springer and Veldkamp in [2] and Veldkamp in [3], [4] and [5] have developed a geometry of split Jordan algebras. This is again based on the elements of rank one.

Additional results on the groups of automorphisms and of norm preserving transformations of reduced exceptional simple Jordan algebras are given in Jacobson [26] and [27]. These are algebraic groups of types F_4 and E_6 respectively. Soda in [1] has studied the groups $\mathrm{Aut}\,\mathfrak{J} \,|\, \mathfrak{K}$, where $\mathfrak{J}$ is reduced simple exceptional and $\mathfrak{K}$ is a cubic subfield. These are exceptional simple algebraic groups of type D_4, which had been studied previously in a geometric fashion by Tits [3]. Soda's results are based in part on some given by Springer in [9].

In a series of papers [6]–[14], Freudenthal has studied certain real forms of the Lie algebras E_7 and E_8 and related these to symplectic and metasymplectic geometries. Certain forms of Lie algebras of types D_4 and E_6 have been studied respectively by Allen in [1] and by Ferrar in [1].

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