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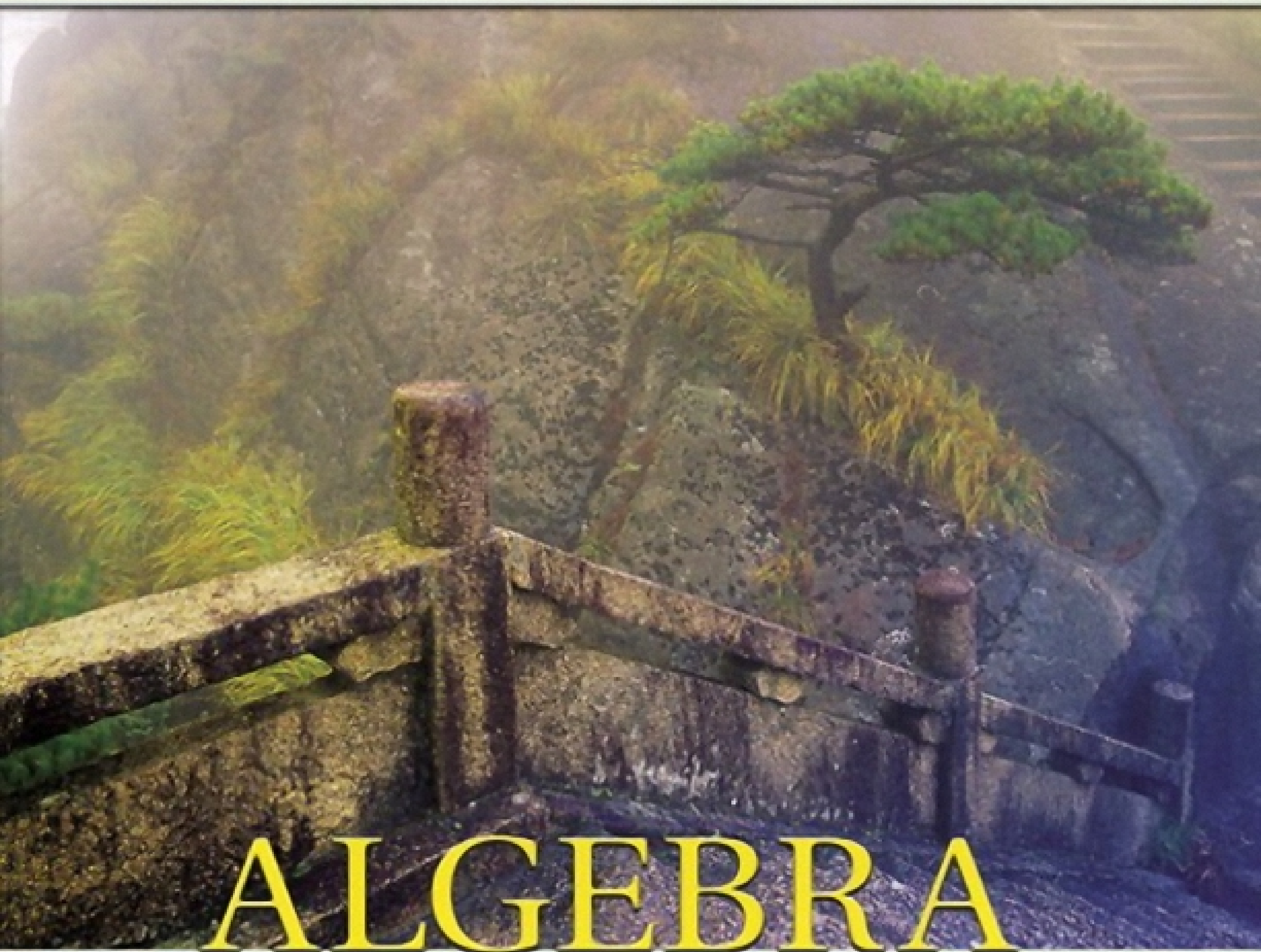


ALGEBRA FOR COLLEGE STUDENTS

Fifth Edition

1.1 Basic Concepts

OBJECTIVES

1. Write sets using set notation.
2. Use number lines.
3. Know the common sets of numbers.
4. Find additive inverses.
5. Use absolute value.
6. Use inequality symbols.
7. Graph sets of real numbers.

In this chapter we review some of the basic symbols and rules of algebra.

OBJECTIVE 1 Write sets using set notation. A **set** is a collection of objects called the **elements** or **members** of the set. In algebra, the elements of a set are usually numbers. Set braces, { }, are used to enclose the elements. For example, 2 is an element of the set {1, 2, 3}. Since we can count the number of elements in the set {1, 2, 3}, it is a *finite set*.

In our study of algebra, we refer to certain sets of numbers by name. The set

$$N = \{1, 2, 3, 4, 5, 6, \ldots\}$$

is called the **natural numbers** or the **counting numbers.** The three dots show that the list continues in the same pattern indefinitely. We cannot list all of the elements of the set of natural numbers, so it is an *infinite set.*

When 0 is included with the set of natural numbers, we have the set of **whole numbers,** written

$$W = \{0, 1, 2, 3, 4, 5, 6, \ldots\}.$$

A set containing no elements, such as the set of whole numbers less than 0, is called the **empty set,** or **null set,** usually written $\emptyset$ or { }.

CAUTION Do not write $\{\emptyset\}$ for the empty set; $\{\emptyset\}$ is a set with one element, $\emptyset$. Use the notation $\emptyset$ or { } for the empty set.

To write the fact that 2 is an element of the set {1, 2, 3}, we use the symbol $\in$ (read "is an element of").

$$2 \in \{1, 2, 3\}$$

The number 2 is also an element of the set of natural numbers N, so we may write

$$2 \in N.$$

To show that 0 is *not* an element of set N, we draw a slash through the symbol $\in$.

$$0 \notin N$$

Two sets are equal if they contain exactly the same elements. For example, $\{1, 2\} = \{2, 1\}$, because the sets contain the same elements. (Order doesn't matter.) On the other hand, $\{1, 2\} \neq \{0, 1, 2\}$ ($\neq$ means "is not equal to") since one set contains the element 0 while the other does not.

In algebra, letters called **variables** are often used to represent numbers or to define sets of numbers. For example,

$$\{x \mid x \text{ is a natural number between 3 and 15}\}$$

(read "the set of all elements x such that x is a natural number between 3 and 15") defines the set

$$\{4, 5, 6, 7, \ldots, 14\}.$$

The notation $\{x \mid x$ is a natural number between 3 and 15$\}$ is an example of **set-builder notation.**

$$\underbrace{\{}_{\text{the set of}}\ \underbrace{x}_{\text{all elements } x}\ \underbrace{\mid}_{\text{such that}}\ \underbrace{x \text{ has property } P}_{x \text{ has a given property } P}\}$$

EXAMPLE 1 Listing the Elements in Sets

List the elements in each set.

(a) $\{x \mid x$ is a natural number less than 4$\}$
The natural numbers less than 4 are 1, 2, and 3. This set is $\{1, 2, 3\}$.

(b) $\{y \mid y$ is one of the first five even natural numbers$\} = \{2, 4, 6, 8, 10\}$

(c) $\{z \mid z$ is a natural number greater than or equal to 7$\}$
The set of natural numbers greater than or equal to 7 is an infinite set, written with three dots as $\{7, 8, 9, 10, \ldots\}$.

Now Try Exercise 1.

EXAMPLE 2 Using Set-Builder Notation to Describe Sets

Use set-builder notation to describe each set.

(a) $\{1, 3, 5, 7, 9\}$
There are often several ways to describe a set with set-builder notation. One way to describe this set is

$$\{y \mid y \text{ is one of the first five odd natural numbers}\}.$$

(b) $\{5, 10, 15, \ldots\}$
This set can be described as $\{x \mid x$ is a multiple of 5 greater than 0$\}$.

Now Try Exercises 13 and 15.

OBJECTIVE 2 Use number lines. A good way to get a picture of a set of numbers is to use a **number line.** To construct a number line, choose any point on a horizontal line and label it 0. Next, choose a point to the right of 0 and label it 1. The distance from 0 to 1 establishes a scale that can be used to locate more points, with positive numbers to the right of 0 and negative numbers to the left of 0. The number 0 is neither positive nor negative. A number line is shown in Figure 1.

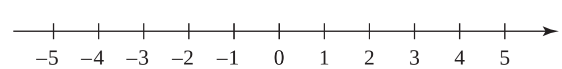


FIGURE 1

The set of numbers identified on the number line in Figure 1, including positive and negative numbers and 0, is part of the set of **integers,** written

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Each number on a number line is called the **coordinate** of the point that it labels, while the point is the **graph** of the number. Figure 2 shows a number line with several selected points graphed on it.

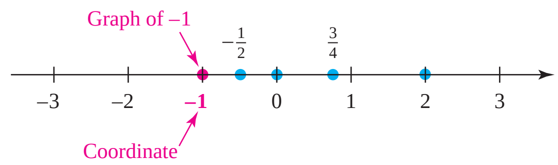
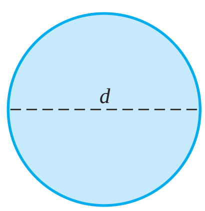


FIGURE 2

The fractions $-\frac{1}{2}$ and $\frac{3}{4}$, graphed on the number line in Figure 2, are examples of rational numbers. A **rational number** can be expressed as the quotient of two integers, with denominator not 0. Rational numbers can also be written in decimal form, either as terminating decimals such as $\frac{3}{5} = .6$, $\frac{1}{8} = .125$, or $\frac{11}{4} = 2.75$, or as repeating decimals such as $\frac{1}{3} = .33333\ldots$ or $\frac{3}{11} = .272727\ldots$. A repeating decimal is often written with a bar over the repeating digit(s). Using this notation, $.2727\ldots$ is written $.\overline{27}$.



$\pi = \frac{C}{d}$

Decimal numbers that neither terminate nor repeat are *not* rational, and thus are called **irrational numbers.** Many square roots are irrational numbers; for example, $\sqrt{2} = 1.4142136\ldots$ and $-\sqrt{7} = -2.6457513\ldots$ repeat indefinitely without pattern. (Some square roots *are* rational: $\sqrt{16} = 4$, $\sqrt{100} = 10$, and so on.) Another irrational number is π, the ratio of the distance around or circumference of a circle to its diameter.

Some of the rational and irrational numbers just discussed are graphed on the number line in Figure 3. The rational numbers together with the irrational numbers make up the set of **real numbers.** Every point on a number line corresponds to a real number, and every real number corresponds to a point on the number line.

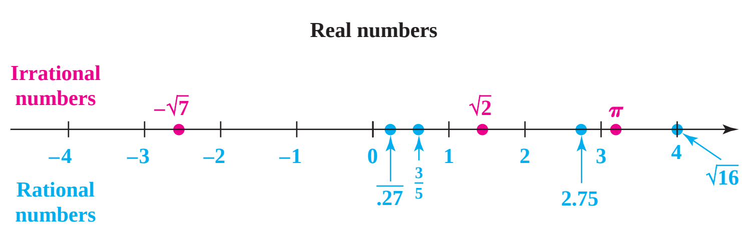


FIGURE 3

OBJECTIVE 3 Know the common sets of numbers. The following sets of numbers will be used throughout the rest of this text.

Sets of Numbers

Natural numbers or counting numbers	$\{1, 2, 3, 4, 5, 6, \ldots\}$
Whole numbers	$\{0, 1, 2, 3, 4, 5, 6, \ldots\}$
Integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers	$\left\{\frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers, } q \neq 0\right\}$ *Examples:* $\frac{4}{1}$ or 4, 1.3, $-\frac{9}{2}$ or $-4\frac{1}{2}$, $\frac{16}{8}$ or 2, $\sqrt{9}$ or 3, $.\overline{6}$
Irrational numbers	$\{x \mid x$ is a real number that is not rational$\}$ *Examples:* $\sqrt{3}$, $-\sqrt{2}$, π
Real numbers	$\{x \mid x$ is represented by a point on a number line$\}$*

The relationships among these various sets of numbers are shown in Figure 4; in particular, the figure shows that the set of real numbers includes both the rational and irrational numbers. Every real number is either rational or irrational. Also, notice that the integers are elements of the set of rational numbers and that whole numbers and natural numbers are elements of the set of integers.

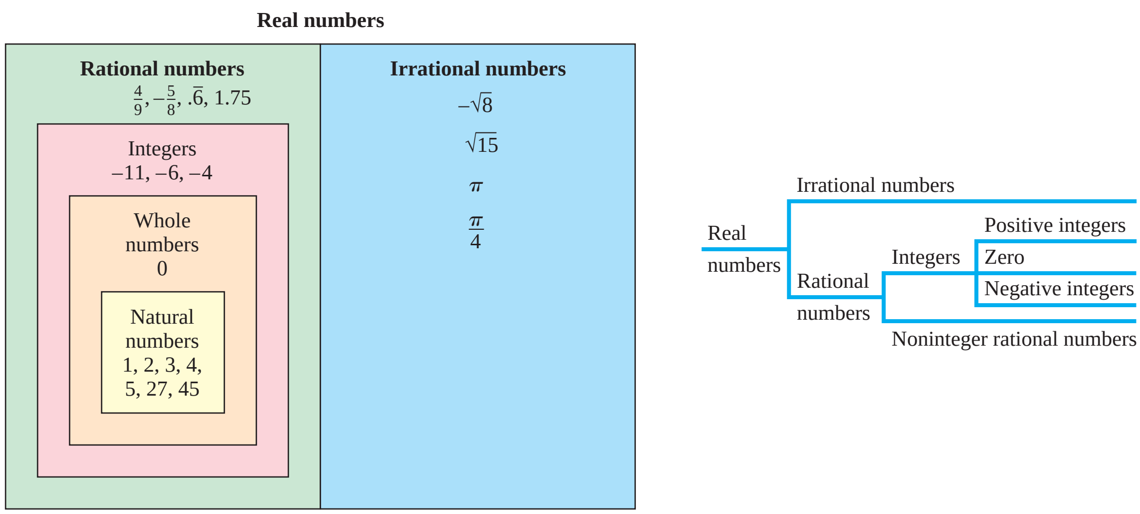


FIGURE 4 The Real Numbers

EXAMPLE 3 Identifying Examples of Number Sets

Which numbers in

$$\left\{-8, -\sqrt{6}, -\frac{9}{64}, 0, .5, \frac{2}{3}, 1.\overline{12}, \sqrt{3}, 2\right\}$$

are elements of each set?

(a) Integers
-8, 0, and 2 are integers.

(b) Rational numbers
-8, $-\frac{9}{64}$, 0, .5, $\frac{2}{3}$, $1.\overline{12}$, and 2 are rational numbers.

(c) Irrational numbers
$-\sqrt{6}$ and $\sqrt{3}$ are irrational numbers.

(d) Real numbers
All the numbers in the given set are real numbers.

Now Try Exercise 25.

*An example of a number that is not a coordinate of a point on a number line is $\sqrt{-1}$. This number, called an *imaginary number,* is discussed in Chapter 8.

EXAMPLE 4 Determining Relationships between Sets of Numbers

Decide whether each statement is *true* or *false.*

(a) All irrational numbers are real numbers.

This is true. As shown in Figure 4, the set of real numbers includes all irrational numbers.

(b) Every rational number is an integer.

This statement is false. Although some rational numbers are integers, other rational numbers, such as $\frac{2}{3}$ and $-\frac{1}{4}$, are not.

Now Try Exercise 27.

OBJECTIVE 4 Find additive inverses. Look again at the number line in Figure 1. For each positive number, there is a negative number on the opposite side of 0 that lies the same distance from 0. These pairs of numbers are called *additive inverses*, *negatives*, or *opposites* of each other. For example, 5 is the additive inverse of -5, and -5 is the additive inverse of 5.

Additive Inverse

For any real number a, the number $-a$ is the **additive inverse** of a.

Change the sign of a number to get its additive inverse. The sum of a number and its additive inverse is always 0.

The symbol "$-$" can be used to indicate any of the following:

1. a negative number, such as -9 or -15;
2. the additive inverse of a number, as in "-4 is the additive inverse of 4";
3. subtraction, as in $12 - 3$.

In the expression $-(-5)$, the symbol "$-$" is being used in two ways: the first $-$ indicates the additive inverse of -5, and the second indicates a negative number, -5. Since the additive inverse of -5 is 5, then $-(-5) = 5$. This example suggests the following property.

For any real number a, $\quad \boldsymbol{-(-a) = a.}$

Number	*Additive Inverse*
6	-6
-4	4
$\frac{2}{3}$	$-\frac{2}{3}$
-8.7	8.7
0	0

Numbers written with positive or negative signs, such as $+4$, $+8$, -9, and -5, are called **signed numbers.** A positive number can be called a signed number even though the positive sign is usually left off. The table in the margin shows the additive inverses of several signed numbers. The number 0 is its own additive inverse.

OBJECTIVE 5 Use absolute value. Geometrically, the **absolute value** of a number a, written $|a|$, is the distance on the number line from 0 to a. For example, the absolute value of 5 is the same as the absolute value of -5 because each number lies five units from 0. See Figure 5. That is,

$$|5| = 5 \qquad \text{and} \qquad |-5| = 5.$$

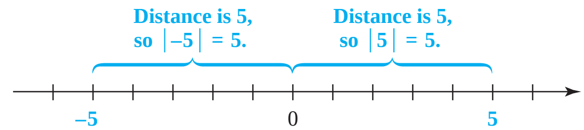


FIGURE 5

CAUTION Because absolute value represents distance, and distance is always positive (or 0), ***the absolute value of a number is always positive (or 0).***

The formal definition of absolute value follows.

Absolute Value

$$|a| = \begin{cases} a & \text{if } a \text{ is positive or } 0 \\ -a & \text{if } a \text{ is negative} \end{cases}$$

The second part of this definition, $|a| = -a$ if a is negative, requires careful thought. If a is a *negative* number, then $-a$, the additive inverse or opposite of a, is a positive number, so $|a|$ is positive. For example, if $a = -3$, then

$$|a| = |-3| = -(-3) = 3. \qquad |a| = -a \text{ if } a \text{ is negative.}$$

EXAMPLE 5 Evaluating Absolute Value Expressions

Find the value of each expression.

(a) $|13| = 13$ **(b)** $|-2| = -(-2) = 2$ **(c)** $|0| = 0$

(d) $-|8|$

Evaluate the absolute value first. Then find the additive inverse.

$$-|8| = -(8) = -8$$

(e) $-|-8|$

Work as in part (d): $|-8| = 8$, so

$$-|-8| = -(8) = -8.$$

(f) $|-2| + |5|$

Evaluate each absolute value first, then add.

$$|-2| + |5| = 2 + 5 = 7$$

Now Try Exercises 43, 47, 49, and 53.

Absolute value is useful in applications comparing size without regard to sign.

EXAMPLE 6 Comparing Rates of Change in Industries

The projected annual rates of employment change (in percent) in some of the fastest growing and most rapidly declining industries from 1994 through 2005 are shown in the table on the next page.

Industry (1994–2005)	*Percent Rate of Change*
Health services	5.7
Computer and data processing services	4.9
Child day care services	4.3
Footware, except rubber and plastic	−6.7
Household audio and video equipment	−4.2
Luggage, handbags, and leather products	−3.3

Source: U.S. Bureau of Labor Statistics.

Photo not available

What industry in the list is expected to see the greatest change? the least change?

We want the greatest *change*, without regard to whether the change is an increase or a decrease. Look for the number in the list with the largest absolute value. That number is found in footware, since $|-6.7| = 6.7$. Similarly, the least change is in the luggage, handbags, and leather products industry: $|-3.3| = 3.3$.

Now Try Exercise 59.

OBJECTIVE 6 Use inequality symbols. The statement $4 + 2 = 6$ is an **equation;** it states that two quantities are equal. The statement $4 \neq 6$ (read "4 is not equal to 6") is an **inequality,** a statement that two quantities are *not* equal. When two numbers are not equal, one must be less than the other. The symbol $<$ means "is less than." For example,

$$8 < 9, \quad -6 < 15, \quad -6 < -1, \quad \text{and} \quad 0 < \frac{4}{3}.$$

The symbol $>$ means "is greater than." For example,

$$12 > 5, \quad 9 > -2, \quad -4 > -6, \quad \text{and} \quad \frac{6}{5} > 0.$$

Notice that in each case, the symbol "points" toward the smaller number.

The number line in Figure 6 shows the graphs of the numbers 4 and 9. We know that $4 < 9$. On the graph, 4 is to the left of 9. The smaller of two numbers is always to the left of the other on a number line.

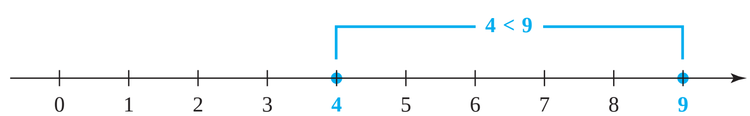


FIGURE 6

Inequalities on a Number Line

On a number line,

$\boldsymbol{a < b}$ if a is to the left of b; $\quad$ $\boldsymbol{a > b}$ if a is to the right of b.

We can use a number line to determine order. As shown on the number line in Figure 7, -6 is located to the left of 1. For this reason, $-6 < 1$. Also, $1 > -6$. From the same number line, $-5 < -2$, or $-2 > -5$.

FIGURE 7

CAUTION Be careful when ordering negative numbers. Since -5 is to the left of -2 on the number line in Figure 7, $-5 < -2$, or $-2 > -5$. In each case, the symbol points to -5, the smaller number.

The following table summarizes results about positive and negative numbers in both words and symbols.

Words	*Symbols*
Every negative number is less than 0. Every positive number is greater than 0.	If a is negative, then $a < 0$. If a is positive, then $a > 0$.
0 is neither positive nor negative.	

In addition to the symbols $\neq$, $<$, and $>$, the symbols $\leq$ and $\geq$ are often used.

INEQUALITY SYMBOLS

Symbol	*Meaning*	*Example*
$\neq$	is not equal to	$3 \neq 7$
$<$	is less than	$-4 < -1$
$>$	is greater than	$3 > -2$
$\leq$	is less than or equal to	$6 \leq 6$
$\geq$	is greater than or equal to	$-8 \geq -10$

The following table shows several inequalities and why each is true.

Inequality	*Why It Is True*
$6 \leq 8$	$6 < 8$
$-2 \leq -2$	$-2 = -2$
$-9 \geq -12$	$-9 > -12$
$-3 \geq -3$	$-3 = -3$
$6 \cdot 4 \leq 5(5)$	$24 < 25$

Notice the reason why $-2 \leq -2$ is true. With the symbol $\leq$, if *either* the $<$ part *or* the $=$ part is true, then the inequality is true. This is also the case with the $\geq$ symbol.

In the last row of the table, recall that the dot in $6 \cdot 4$ indicates the product 6×4, or 24, and $5(5)$ means 5×5, or 25. Thus, the inequality $6 \cdot 4 \leq 5(5)$ becomes $24 \leq 25$, which is true.

OBJECTIVE 7 Graph sets of real numbers. Inequality symbols and variables are used to write sets of real numbers. For example, the set $\{x \mid x > -2\}$ consists of all the real numbers greater than -2. On a number line, we show the elements of this set (the set of all real numbers to the right of -2) by drawing an arrow from -2 to the right. We use a parenthesis at -2 to indicate that -2 is *not* an element of the given set. The result, shown in Figure 8, is the graph of the set $\{x \mid x > -2\}$.

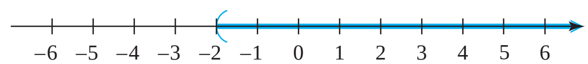


FIGURE 8

The set of numbers greater than -2 is an example of an **interval** on the number line. To write intervals, we use **interval notation.** Using this notation, we write the interval of all numbers greater than -2 as $(-2, \infty)$. The infinity symbol ∞ does not indicate a number; it shows that the interval includes all real numbers greater than -2. The left parenthesis indicates that -2 is not included. A parenthesis is *always* used next to the infinity symbol in interval notation. The set of all real numbers is written in interval notation as $(-\infty, \infty)$.

VIDEO

EXAMPLE 7 Graphing an Inequality Written in Interval Notation

Write $\{x \mid x < 4\}$ in interval notation and graph the interval.

The interval is written $(-\infty, 4)$. The graph is shown in Figure 9. Since the elements of the set are all real numbers *less than* 4, the graph extends to the left.

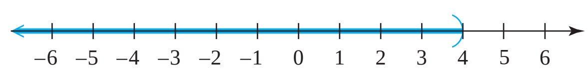


FIGURE 9

Now Try Exercise 101.

The set $\{x \mid x \leq -6\}$ includes all real numbers less than or equal to -6. To show that -6 is part of the set, a square bracket is used at -6, as shown in Figure 10. In interval notation, this set is written $(-\infty, -6]$.

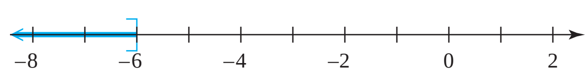


FIGURE 10

EXAMPLE 8 Graphing an Inequality Written in Interval Notation

Write $\{x \mid x \geq -4\}$ in interval notation and graph the interval.

This set is written in interval notation as $[-4, \infty)$. The graph is shown in Figure 11. We use a square bracket at -4 since -4 is part of the set.

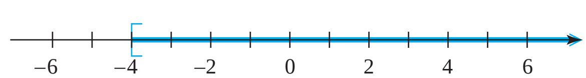


FIGURE 11

Now Try Exercise 103.

NOTE In a previous course you may have graphed { using an open circle instead of a parenthesis at -2. Also, you may have graphed $\{x \mid x \geq -4\}$ using a solid dot instead of a bracket at -4.

It is common to graph sets of numbers that are *between* two given numbers. For example, the set $\{x \mid -2 < x < 4\}$ includes all real numbers between -2 and 4, but not the numbers -2 and 4 themselves. This set is written in interval notation as $(-2, 4)$. The graph has a heavy line between -2 and 4 with parentheses at -2 and 4. See Figure 12. The inequality $-2 < x < 4$ is read "-2 is less than x and x is less than 4," or "x is between -2 and 4."

FIGURE 12

EXAMPLE 9 Graphing a Three-Part Inequality

Write $\{x \mid 3 < x \leq 10\}$ in interval notation and graph the interval.

Use a parenthesis at 3 and a square bracket at 10 to get $(3, 10]$ in interval notation. The graph is shown in Figure 13. Read the inequality $3 < x \leq 10$ as "3 is less than x and x is less than or equal to 10," or "x is between 3 and 10, excluding 3 and including 10."

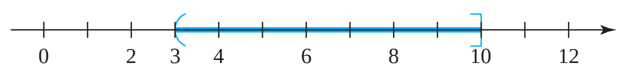


FIGURE 13

Now Try Exercise 109.

1.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 1/Videotape 1

Write each set by listing its elements. See Example 1.

1. $\{x \mid x$ is a natural number less than $6\}$

2. $\{m \mid m$ is a natural number less than $9\}$

3. $\{z \mid z$ is an integer greater than $4\}$

4. $\{y \mid y$ is an integer greater than $8\}$

5. $\{z \mid z$ is an integer less than or equal to $4\}$

6. $\{p \mid p$ is an integer less than $3\}$

7. $\{a \mid a$ is an even integer greater than $8\}$

8. $\{k \mid k$ is an odd integer less than $1\}$

9. $\{x \mid x$ is an irrational number that is also rational$\}$

10. $\{r \mid r$ is a number that is both positive and negative$\}$

11. $\{p \mid p$ is a number whose absolute value is $4\}$

12. $\{w \mid w$ is a number whose absolute value is $7\}$

Write each set using set-builder notation. See Example 2. (More than one description is possible.)

13. $\{2, 4, 6, 8\}$

14. $\{11, 12, 13, 14\}$

15. $\{4, 8, 12, 16, \ldots\}$

16. $\{\ldots, -6, -3, 0, 3, 6, \ldots\}$

17. A student claimed that $\{x \mid x$ is a natural number greater than $3\}$ and $\{y \mid y$ is a natural number greater than $3\}$ actually name the same set, even though different variables are used. Was this student correct?

18. A student claimed that $\{\emptyset\}$ and $\emptyset$ name the same set. Was this student correct?

Graph the elements of each set on a number line.

19. $\{-3, -1, 0, 4, 6\}$

20. $\{-4, -2, 0, 3, 5\}$

21. $\left\{-\frac{2}{3}, 0, \frac{4}{5}, \frac{12}{5}, \frac{9}{2}, 4.8\right\}$

22. $\left\{-\frac{6}{5}, -\frac{1}{4}, 0, \frac{5}{6}, \frac{13}{4}, 5.2, \frac{11}{2}\right\}$

23. Explain the difference between the graph of a number and the coordinate of a point.

24. Explain why the real numbers $.36$ and $.\overline{36}$ have different points as graphs on a number line.

*Which elements of each set are **(a)** natural numbers, **(b)** whole numbers, **(c)** integers, **(d)** rational numbers, **(e)** irrational numbers, **(f)** real numbers? See Example 3.*

25. $\left\{-8, -\sqrt{5}, -.6, 0, \frac{3}{4}, \sqrt{3}, \pi, 5, \frac{13}{2}, 17, \frac{40}{2}\right\}$

26. $\left\{-9, -\sqrt{6}, -.7, 0, \frac{6}{7}, \sqrt{7}, 4.\overline{6}, 8, \frac{21}{2}, 13, \frac{75}{5}\right\}$

Decide whether each statement is true *or* false. *If false, tell why. See Example 4.*

27. Every integer is a whole number.

28. Every natural number is an integer.

29. Every irrational number is an integer.

30. Every integer is a rational number.

31. Every natural number is a whole number.

32. Some rational numbers are irrational.

33. Some rational numbers are whole numbers.

34. Some real numbers are integers.

35. The absolute value of any number is the same as the absolute value of its additive inverse.

36. The absolute value of any nonzero number is positive.

*Give **(a)** the additive inverse and **(b)** the absolute value of each number. See the discussion of additive inverses and Example 5.*

37. 6 **38.** 8 **39.** -12 **40.** -15 **41.** $\frac{6}{5}$ **42.** $.13$

Find the value of each expression. See Example 5.

43. $|-8|$

44. $|-11|$

45. $\left|\frac{3}{2}\right|$

46. $\left|\frac{7}{4}\right|$

47. $-|5|$

48. $-|17|$

49. $-|-2|$

50. $-|-8|$

51. $-|4.5|$

52. $-|12.6|$

53. $|-2| + |3|$

54. $|-16| + |12|$

55. $|-9| - |-3|$

56. $|-10| - |-5|$

57. $|-1| + |-2| - |-3|$

58. $|-6| + |-4| - |-10|$

Solve each problem. See Example 6.

EXERCISES

59. The table shows the percent change in population from 1990 through 1999 for some of the largest cities in the United States.

City	*Percent Change*
New York	1.4
Los Angeles	4.2
Chicago	.6
Philadelphia	−10.6
Houston	8.7
Detroit	−6.1

Source: U.S. Bureau of the Census.

(a) Which city had the greatest change in population? What was this change? Was it an increase or a decline?

(b) Which city had the smallest change in population? What was this change? Was it an increase or a decline?

60. The table gives the net trade balance, in millions of dollars, for selected U.S. trade partners for April 2002.

Country	*Trade Balance (in millions of dollars)*
Germany	−2815
China	−7552
Netherlands	823
France	−951
Turkey	96
Australia	373

Source: U.S. Bureau of the Census.

A negative balance means that imports exceeded exports, while a positive balance means that exports exceeded imports.

(a) Which country had the greatest discrepancy between exports and imports? Explain.

(b) Which country had the smallest discrepancy between exports and imports? Explain.

Sea level refers to the surface of the ocean. The depth of a body of water such as an ocean or sea can be expressed as a negative number, representing average depth in feet below sea level. On the other hand, the altitude of a mountain can be expressed as a positive number, indicating its height in feet above sea level. The table gives selected depths and heights.

Body of Water	*Average Depth in Feet (as a negative number)*	*Mountain*	*Altitude in Feet (as a positive number)*
Pacific Ocean	−12,925	McKinley	20,320
South China Sea	−4,802	Point Success	14,158
Gulf of California	−2,375	Matlalcueyetl	14,636
Caribbean Sea	−8,448	Rainier	14,410
Indian Ocean	−12,598	Steele	16,644

Source: World Almanac and Book of Facts, 2002.

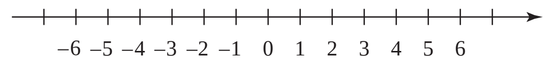
61. List the bodies of water in order, starting with the deepest and ending with the shallowest.

62. List the mountains in order, starting with the shortest and ending with the tallest.

63. *True* or *false:* The absolute value of the depth of the Pacific Ocean is greater than the absolute value of the depth of the Indian Ocean.

64. *True* or *false:* The absolute value of the depth of the Gulf of California is greater than the absolute value of the depth of the Caribbean Sea.

Use the number line to answer true *or* false *to each statement.*



65. $-6 < -2$ **66.** $-4 < -3$ **67.** $-4 > -3$ **68.** $-2 > -1$

69. $3 > -2$ **70.** $5 > -3$ **71.** $-3 \geq -3$ **72.** $-4 \leq -4$

Rewrite each statement with $>$ *so that it uses* $<$ *instead; rewrite each statement with* $<$ *so that it uses* $>$.

73. $6 > 2$ **74.** $4 > 1$ **75.** $-9 < 4$ **76.** $-5 < 1$

77. $-5 > -10$ **78.** $-8 > -12$ **79.** $0 < x$ **80.** $-2 < x$

Use an inequality symbol to write each statement.

81. 7 is greater than y.

82. -4 is less than 12.

83. 5 is greater than or equal to 5.

84. -3 is less than or equal to -3.

85. $3t - 4$ is less than or equal to 10.

86. $5x + 4$ is greater than or equal to 19.

87. $5x + 3$ is not equal to 0.

88. $6x + 7$ is not equal to -3.

89. t is between -3 and 5.

90. r is between -4 and 12.

91. $3x$ is between -3 and 4, including -3 and excluding 4.

92. $5y$ is between -2 and 6, excluding -2 and including 6.

First simplify each side of the inequality. Then tell whether the resulting statement is true *or* false.

93. $-6 < 7 + 3$ **94.** $-7 < 4 + 2$ **95.** $2 \cdot 5 \geq 4 + 6$

96. $8 + 7 \leq 3 \cdot 5$ **97.** $-|-3| \geq -3$ **98.** $-|-5| \leq -5$

99. $-8 > -|-6|$ **100.** $-9 > -|-4|$

Write each set using interval notation and graph the interval. See Examples 7–9.

101. $\{x \mid x > -1\}$ **102.** $\{x \mid x < 5\}$ **103.** $\{x \mid x \leq 6\}$

104. $\{x \mid x \geq -3\}$ **105.** $\{x \mid 0 < x < 3.5\}$ **106.** $\{x \mid -4 < x < 6.1\}$

107. $\{x \mid 2 \leq x \leq 7\}$ **108.** $\{x \mid -3 \leq x \leq -2\}$ **109.** $\{x \mid -4 < x \leq 3\}$

110. $\{x \mid 3 \leq x < 6\}$ **111.** $\{x \mid 0 < x \leq 3\}$ **112.** $\{x \mid -1 \leq x < 6\}$

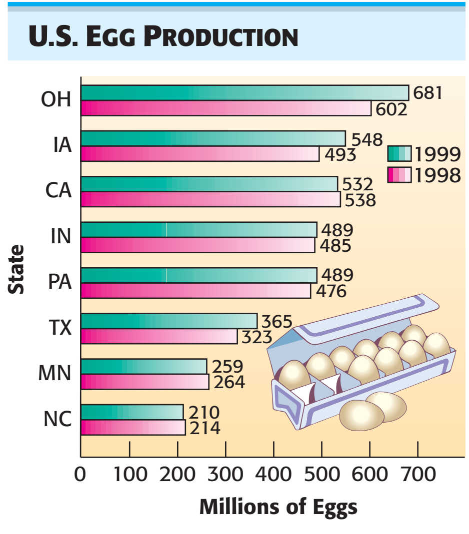
The graph on the next page shows egg production in millions of eggs in selected states for 1998 and 1999. Use this graph to work Exercises 113–116.

113. In 1999, which states had production greater than 500 million eggs?

114. In which states was 1999 egg production less than 1998 egg production?

115. If x represents 1999 egg production for Texas (TX) and y represents 1999 egg production for Ohio (OH), which is true: $x < y$ or $x > y$?

116. If x represents 1999 egg production for Indiana (IN) and y represents 1999 egg production for Pennsylvania (PA), write an equation or inequality that compares the production in these two states.



Source: Iowa Agricultural Statistics.

117. List the sets of numbers introduced in this section. Give a short explanation, including three examples, for each set.

118. List at least five symbols introduced in this section, and give a true statement involving each one.

1.2 Operations on Real Numbers

OBJECTIVES

1 Add real numbers.

2 Subtract real numbers.

3 Find the distance between two points on a number line.

4 Multiply real numbers.

5 Divide real numbers.

In this section we review the rules for adding, subtracting, multiplying, and dividing real numbers.

OBJECTIVE 1 Add real numbers. Recall that the answer to an addition problem is called the **sum.** The rules for adding real numbers follow.

Adding Real Numbers

Like signs To add two numbers with the *same* sign, add their absolute values. The sign of the answer (either + or −) is the same as the sign of the two numbers.

Unlike signs To add two numbers with *different* signs, subtract the smaller absolute value from the larger. The sign of the answer is the same as the sign of the number with the larger absolute value.

EXAMPLE 1 Adding Two Negative Numbers

Find each sum.

(a) $-12 + (-8)$

First find the absolute values.

$$|-12| = 12 \qquad \text{and} \qquad |-8| = 8$$

Because -12 and -8 have the *same* sign, add their absolute values. Both numbers are negative, so the answer is negative.

$$-12 + (-8) = -(12 + 8) = -(20) = -20$$

(b) $-6 + (-3) = -(|-6| + |-3|) = -(6 + 3) = -9$

(c) $-1.2 + (-.4) = -(1.2 + .4) = -1.6$

(d) $-\frac{5}{6} + \left(-\frac{1}{3}\right) = -\left(\frac{5}{6} + \frac{1}{3}\right) = -\left(\frac{5}{6} + \frac{2}{6}\right) = -\frac{7}{6}$

Now Try Exercise 11.

EXAMPLE 2 Adding Numbers with Different Signs

VIDEO

Find each sum.

(a) $-17 + 11$

First find the absolute values.

$$|-17| = 17 \quad \text{and} \quad |11| = 11$$

Because -17 and 11 have *different* signs, subtract their absolute values.

$$17 - 11 = 6$$

The number -17 has a larger absolute value than 11, so the answer is negative.

$$-17 + 11 = -6$$

Negative because $|-17| > |11|$

(b) $4 + (-1)$

Subtract the absolute values, 4 and 1. Because 4 has the larger absolute value, the sum must be positive.

$$4 + (-1) = 4 - 1 = 3$$

Positive because $|4| > |-1|$

(c) $-9 + 17 = 17 - 9 = 8$

(d) $-16 + 12$

The absolute values are 16 and 12. Subtract the absolute values. The negative number has the larger absolute value, so the answer is negative.

$$-16 + 12 = -(16 - 12) = -4$$

(e) $-\frac{4}{5} + \frac{2}{3}$

Write each fraction with a common denominator.

$$\frac{4}{5} = \frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15} \quad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$$

$$-\frac{4}{5} + \frac{2}{3} = -\frac{12}{15} + \frac{10}{15}$$

$$= -\left(\frac{12}{15} - \frac{10}{15}\right) \qquad -\frac{12}{15} \text{ has the larger absolute value.}$$

$$= -\frac{2}{15} \qquad \text{Subtract.}$$

(f) $-2.3 + 5.6 = 3.3$

Now Try Exercises 13, 15, and 17.

OBJECTIVE 2 Subtract real numbers. Recall that the answer to a subtraction problem is called the **difference.** Thus, the difference between 6 and 4 is 2. To see how subtraction should be defined, compare the following two statements.

$$6 - 4 = 2$$
$$6 + (-4) = 2$$

Similarly, $9 - 3 = 6$ and $9 + (-3) = 6$ so that $9 - 3 = 9 + (-3)$. To subtract 3 from 9, we add the additive inverse of 3 to 9. These examples suggest the following rule for subtraction.

Subtraction

For all real numbers a and b,

$$\boldsymbol{a - b = a + (-b).}$$

That is, change the sign of the second number and add.

EXAMPLE 3 Subtracting Real Numbers

Find each difference.

Change to addition.
Change sign of second number.

(a) $6 - 8 = 6 + (-8) = -2$

Changed
Sign changed

(b) $-12 - 4 = -12 + (-4) = -16$

(c) $-10 - (-7) = -10 + [-(-7)]$ This step is often omitted.
$= -10 + 7$
$= -3$

(d) $-2.4 - (-8.1) = -2.4 + 8.1 = 5.7$

(e) $\dfrac{8}{3} - \left(-\dfrac{5}{3}\right) = \dfrac{8}{3} + \dfrac{5}{3} = \dfrac{13}{3}$

Now Try Exercises 19, 23, 25, and 27.

When working a problem that involves both addition and subtraction, add and subtract in order from left to right. Work inside brackets or parentheses first.

EXAMPLE 4 Adding and Subtracting Real Numbers

Perform the indicated operations.

(a) $$\begin{aligned} 15 - (-3) - 5 - 12 &= (15 + 3) - 5 - 12 && \text{Work from left to right.} \\ &= 18 - 5 - 12 \\ &= 13 - 12 \\ &= 1 \end{aligned}$$

(b) $$\begin{aligned} -9 - [-8 - (-4)] + 6 &= -9 - [-8 + 4] + 6 && \text{Work inside brackets.} \\ &= -9 - [-4] + 6 \\ &= -9 + 4 + 6 \\ &= -5 + 6 \\ &= 1 \end{aligned}$$

Now Try Exercises 39 and 41.

OBJECTIVE 3 Find the distance between two points on a number line. The number line in Figure 14 shows several points. To find the distance between the points 4 and 7, we subtract: $7 - 4 = 3$. Since distance is always positive (or 0), we must be careful to subtract in such a way that the answer is positive (or 0). Or, to avoid this problem altogether, we can find the absolute value of the difference. Then the distance between 4 and 7 is either

$$|7 - 4| = |3| = 3 \qquad \text{or} \qquad |4 - 7| = |-3| = 3.$$

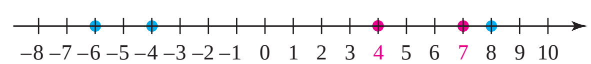


FIGURE 14

Distance

The **distance** between two points on a number line is the absolute value of the difference between the numbers.

EXAMPLE 5 Finding Distance between Points on the Number Line

Find the distance between each pair of points from Figure 14.

(a) 8 and -4

Find the absolute value of the difference of the numbers, taken in either order.

$$|8 - (-4)| = 12 \qquad \text{or} \qquad |-4 - 8| = 12$$

(b) -4 and -6

$$|-4 - (-6)| = 2 \qquad \text{or} \qquad |-6 - (-4)| = 2$$

Now Try Exercise 51.

OBJECTIVE 4 Multiply real numbers. The answer to a multiplication problem is called the **product.** For example, 24 is the product of 8 and 3. The rules for finding signs of products of real numbers are given next.

Multiplying Real Numbers

Like signs The product of two numbers with the *same* sign is positive.

Unlike signs The product of two numbers with *different* signs is negative.

EXAMPLE 6 Multiplying Real Numbers

Find each product.

(a) $-3(-9) = 27$ Same sign; product is positive.

(b) $-.5(-.4) = .2$ **(c)** $-\frac{3}{4}\left(-\frac{5}{3}\right) = \frac{5}{4}$

(d) $6(-9) = -54$ Different signs; product is negative.

(e) $-.05(.3) = -.015$ **(f)** $\frac{2}{3}(-3) = -2$ **(g)** $-\frac{5}{8}\left(\frac{12}{13}\right) = -\frac{15}{26}$

Now Try Exercises 61, 65, 67, and 73.

OBJECTIVE 5 Divide real numbers. Earlier, we defined subtraction in terms of addition. Now we define division in terms of multiplication. The result of dividing one number by another is called the **quotient.** The quotient of two real numbers $a \div b$ $(b \neq 0)$ is the real number q such that $q \cdot b = a$. That is,

$$a \div b = q \quad \text{only if} \quad q \cdot b = a.$$

For example, $36 \div 9 = 4$ since $4 \cdot 9 = 36$. Similarly, $35 \div (-5) = -7$ since $-7(-5) = 35$. The quotient $a \div b$ can also be denoted $\frac{a}{b}$. Thus, $35 \div (-5)$ can be written $\frac{35}{-5}$. As above, $\frac{35}{-5} = -7$ since -7 answers the question, "What number multiplied by -5 gives the product 35?" Now consider $\frac{5}{0}$. There is *no* number whose product with 0 gives 5. On the other hand, $\frac{0}{0}$ would be satisfied by *every* real number, because any number multiplied by 0 gives 0. When dividing, we always want a *unique* quotient, and therefore ***division by 0 is undefined.*** Thus,

$$\frac{15}{0} \text{ is undefined} \qquad \text{and} \qquad -\frac{1}{0} \text{ is undefined.}$$

CAUTION Division by 0 is undefined. However, dividing 0 by a nonzero number gives the quotient 0. For example,

$$\frac{6}{0} \text{ is undefined,} \quad \text{but} \quad \frac{0}{6} = 0 \quad (\text{since } 0 \cdot 6 = 0).$$

Be careful when 0 is involved in a division problem.

Recall that $\frac{a}{b} = a \cdot \frac{1}{b}$. Thus, dividing by b is the same as multiplying by $\frac{1}{b}$. If $b \neq 0$, then $\frac{1}{b}$ is the **reciprocal** (or *multiplicative inverse*) of b. When multiplied, reciprocals have a product of 1. The table on the next page gives several numbers and their reciprocals. There is no reciprocal for 0 because there is no number that can be multiplied by 0 to give a product of 1.

Number	*Reciprocal*	
$-\frac{2}{5}$	$-\frac{5}{2}$	$-\frac{2}{5}\left(-\frac{5}{2}\right) = 1$
-6	$-\frac{1}{6}$	$-6\left(-\frac{1}{6}\right) = 1$
$\frac{7}{11}$	$\frac{11}{7}$	$\frac{7}{11}\left(\frac{11}{7}\right) = 1$
.05	20	$.05(20) = 1$
0	None	

CAUTION A number and its additive inverse have *opposite* signs; however, a number and its reciprocal always have the *same* sign.

The preceding discussion suggests the following definition of division.

Division

For all real numbers a and b (where $b \neq 0$),

$$a \div b = \frac{a}{b} = a \cdot \frac{1}{b}.$$

That is, multiply the first number by the reciprocal of the second number.

Since division is defined as multiplication by the reciprocal, the rules for signs of quotients are the same as those for signs of products.

Dividing Real Numbers

Like signs The quotient of two nonzero real numbers with the *same* sign is positive.

Unlike signs The quotient of two nonzero real numbers with *different* signs is negative.

EXAMPLE 7 Dividing Real Numbers

Find each quotient.

(a) $\frac{-12}{4} = -12 \cdot \frac{1}{4} = -3$ $\quad \frac{a}{b} = a \cdot \frac{1}{b}$

(b) $\frac{6}{-3} = 6\left(-\frac{1}{3}\right) = -2$ $\quad$ The reciprocal of -3 is $-\frac{1}{3}$.

(c) $\frac{-\frac{2}{3}}{-\frac{5}{9}} = -\frac{2}{3} \cdot \left(-\frac{9}{5}\right) = \frac{6}{5}$ $\quad$ The reciprocal of $-\frac{5}{9}$ is $-\frac{9}{5}$.

Now Try Exercises 75, 77, and 87.

The rules for multiplication and division suggest the following results.

Equivalent Forms of a Fraction

The fractions $\dfrac{-x}{y}$, $\dfrac{x}{-y}$, and $-\dfrac{x}{y}$ are equivalent. (Assume $y \neq 0$.)

Example: $\dfrac{-4}{7} = \dfrac{4}{-7} = -\dfrac{4}{7}$.

The fractions $\dfrac{x}{y}$ and $\dfrac{-x}{-y}$ are equivalent.

Example: $\dfrac{4}{7} = \dfrac{-4}{-7}$.

The forms $\frac{x}{-y}$ and $\frac{-x}{-y}$ are not used very often.

Every fraction has three signs: the sign of the numerator, the sign of the denominator, and the sign of the fraction itself. Changing any two of these three signs does not change the value of the fraction. Changing only one sign, or changing all three, *does* change the value.

1.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 1/Videotape 1

Complete each statement and give an example.

1. The sum of a positive number and a negative number is 0 if __________.

2. The sum of two positive numbers is a __________ number.

3. The sum of two negative numbers is a __________ number.

4. The sum of a positive number and a negative number is negative if __________.

5. The sum of a positive number and a negative number is positive if __________.

6. The difference between two positive numbers is negative if __________.

7. The difference between two negative numbers is negative if __________.

8. The product of two numbers with like signs is __________.

9. The product of two numbers with unlike signs is __________.

10. The quotient formed by any nonzero number divided by 0 is __________, and the quotient formed by 0 divided by any nonzero number is __________.

Add or subtract as indicated. See Examples 1–3.

11. $-6 + (-13)$	**12.** $-8 + (-15)$	**13.** $13 + (-4)$
14. $19 + (-13)$	**15.** $-\frac{7}{3} + \frac{3}{4}$	**16.** $-\frac{5}{6} + \frac{3}{8}$
17. $-2.3 + .45$	**18.** $-.238 + 4.55$	**19.** $-6 - 5$
20. $-8 - 13$	**21.** $8 - (-13)$	**22.** $13 - (-22)$
23. $-16 - (-3)$	**24.** $-21 - (-8)$	**25.** $-12.31 - (-2.13)$

26. $-15.88 - (-9.22)$ **27.** $\frac{9}{10} - \left(-\frac{4}{3}\right)$ **28.** $\frac{3}{14} - \left(-\frac{1}{4}\right)$

29. $|-8 - 6|$ **30.** $|-7 - 9|$ **31.** $-|-4 + 9|$

32. $-|-5 + 7|$ **33.** $-2 - |-4|$ **34.** $9 - |-13|$

Perform the indicated operations. See Example 4.

35. $-7 + 5 - 9$ **36.** $-12 + 13 - 19$ **37.** $6 - (-2) + 8$

38. $7 - (-3) + 12$ **39.** $-9 - 4 - (-3) + 6$ **40.** $-10 - 5 - (-12) + 8$

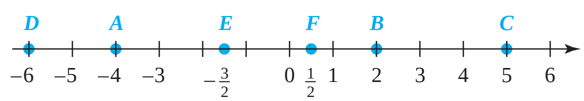
41. $-8 - (-12) - (2 - 6)$ **42.** $-3 + (-14) + (-5 + 3)$ **43.** $-.382 + 4 - .6$

44. $3 - 2.94 - (-.63)$ **45.** $\left(-\frac{5}{4} - \frac{2}{3}\right) + \frac{1}{6}$ **46.** $\left(-\frac{5}{8} + \frac{1}{4}\right) - \left(-\frac{1}{4}\right)$

47. $-\frac{3}{4} - \left(\frac{1}{2} - \frac{3}{8}\right)$ **48.** $\frac{7}{5} - \left(\frac{9}{10} - \frac{3}{2}\right)$

49. $|-11| - |-5| - |7| + |-2|$ **50.** $|-6| + |-3| - |4| - |-8|$

The number line has several points labeled. Find the distance between each pair of points. See Example 5.



51. *A* and *B* **52.** *A* and *C* **53.** *D* and *F* **54.** *E* and *C*

55. Give an example of a difference between two negative numbers that is equal to 5. State the rule for determining the sign of the answer after subtraction has been changed to addition.

56. Give an example of a sum of a positive number and a negative number that is equal to 4. State the rule for determining the sign of the answer when adding two numbers with different signs.

57. A statement that is often heard is “Two negatives give a positive.” When is this true? When is it not true? Give a more precise statement that conveys this message.

58. Explain why the reciprocal of a nonzero number must have the same sign as the number.

Multiply. See Example 6.

59. $5(-7)$ **60.** $6(-6)$ **61.** $-8(-5)$ **62.** $-10(-4)$

63. $-10\left(-\frac{1}{5}\right)$ **64.** $-\frac{1}{2}(-12)$ **65.** $\frac{3}{4}(-16)$ **66.** $\frac{4}{5}(-35)$

67. $-\frac{5}{2}\left(-\frac{12}{25}\right)$ **68.** $-\frac{9}{7}\left(-\frac{35}{36}\right)$ **69.** $-\frac{3}{8}\left(-\frac{24}{9}\right)$ **70.** $-\frac{2}{11}\left(-\frac{99}{4}\right)$

71. $-2.4(-2.45)$ **72.** $-3.45(-2.14)$ **73.** $3.4(-3.14)$ **74.** $5.66(-2.1)$

Divide where possible. See Example 7.

75. $\frac{-14}{2}$ **76.** $\frac{-26}{13}$ **77.** $\frac{-24}{-4}$ **78.** $\frac{-36}{-9}$ **79.** $\frac{100}{-25}$

80. $\frac{300}{-60}$ **81.** $\frac{0}{-8}$ **82.** $\frac{0}{-10}$ **83.** $\frac{5}{0}$ **84.** $\frac{12}{0}$

85. $-\frac{10}{17} \div \left(-\frac{12}{5}\right)$ **86.** $-\frac{22}{23} \div \left(-\frac{33}{4}\right)$ **87.** $\dfrac{\frac{12}{13}}{-\frac{4}{3}}$ **88.** $\dfrac{\frac{5}{6}}{-\frac{1}{30}}$

89. $-\frac{27.72}{13.2}$ **90.** $\frac{-126.7}{36.2}$ **91.** $\frac{-100}{-.01}$ **92.** $\frac{-50}{-.05}$

Solve each problem.

93. The highest temperature ever recorded in Juneau, Alaska, was 90°F. The lowest temperature ever recorded there was −22°F. What is the difference between these two temperatures? (*Source: World Almanac and Book of Facts*, 2002.)

94. On August 10, 1936, a temperature of 120°F was recorded in Arkansas. On February 13, 1905, Arkansas recorded a temperature of −29°F. What is the difference between these two temperatures? (*Source: World Almanac and Book of Facts*, 2002.)

95. The Standard and Poor's 500, an index measuring the performance of 500 leading stocks, had an annual return of 37.58% in 1995. For 2000, its annual return was −9.10%. Find the difference between these two percents. (*Source:* Legg Mason Wood Walker, Inc.)

96. When George W. Bush took office in January 2001, the U.S. federal budget was at a record surplus of \$236 billion. It was at a record deficit of −\$255 billion when his father, George H. W. Bush, left office in January 1993. Find the difference between these two amounts. (*Source: Economic Report of the President*, 2001.)

The table shows Social Security finances (in billions of dollars). Use this table to work Exercises 97 and 98.

Year	Tax Revenue	Cost of Benefits
2000	538	409
2010*	916	710
2020*	1479	1405
2030*	2041	2542

*Projected
Source: Social Security Board of Trustees.

Photo not available

97. Find the difference between Social Security tax revenue and cost of benefits for each year shown in the table.

98. Interpret your answer for 2030.

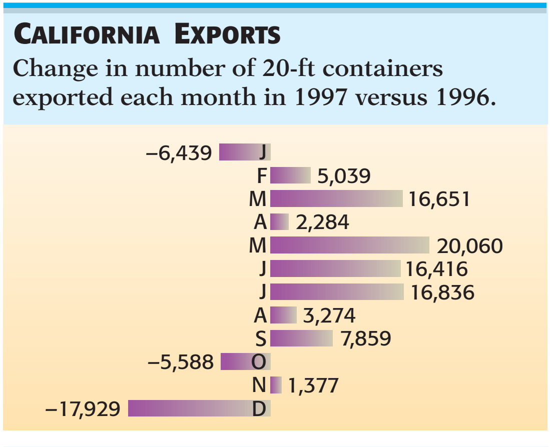
Use the graph of California exports to work Exercises 99–102.

99. What is the difference between the January and February changes?

100. What is the difference between the changes in April and May?

101. Which of the following is the best estimate of the difference between the October and November changes?

A. -6000 **B.** -6500
C. -7000 **D.** -7500



Source: *USA Today*, June 23, 1998.

102. Which of the following is the best estimate of the difference between the November and December changes?

A. 17,000 **B.** 18,000 **C.** 19,000 **D.** 20,000

RELATING CONCEPTS (EXERCISES 103–106)

For Individual or Group Work

In Section 1.1 we discussed the meanings of $a < b$, $a = b$, and $a > b$. Choose two numbers a and b such that $a < b$. ***Work Exercises 103–106 in order.***

103. Find the difference $a - b$.

104. How does the answer in Exercise 103 compare to 0? (Is it greater than, less than, or equal to 0?)

105. Repeat Exercise 103 with different values for a and b.

106. How does the answer in Exercise 105 compare to 0? Based on your observations in these exercises, complete the following statement: If $a < b$, then $a - b$ ____ 0.

1.3 Exponents, Roots, and Order of Operations

OBJECTIVES

1. Use exponents.
2. Find square roots.
3. Use the order of operations.
4. Evaluate algebraic expressions for given values of variables.

Two or more numbers whose product is a third number are **factors** of that third number. For example, 2 and 6 are factors of 12 since $2 \cdot 6 = 12$. Other integer factors of 12 are 1, 3, 4, 12, -1, -2, -3, -4, -6, and -12.

OBJECTIVE 1 Use exponents. In algebra, we use *exponents* as a way of writing products of repeated factors. For example, the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ is written

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{5 factors of 2}} = 2^5.$$

The number 5 shows that 2 is used as a factor 5 times. The number 5 is the *exponent*, and 2 is the *base*.

$$\overset{\text{Base}}{2}{}^{5\ \leftarrow\ \text{Exponent}}$$

Read 2^5 as "2 to the fifth power" or simply "2 to the fifth." Multiplying the five 2s gives

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.$$

Exponential Expression

If a is a real number and n is a natural number, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a},$$

where n is the **exponent,** a is the **base,** and a^n is an **exponential expression.** Exponents are also called **powers.**

EXAMPLE 1 Using Exponential Notation

Write each expression using exponents.

(a) $4 \cdot 4 \cdot 4$

Here, 4 is used as a factor 3 times.

$$\underbrace{4 \cdot 4 \cdot 4}_{3 \text{ factors of } 4} = 4^3$$

Read 4^3 as "4 cubed."

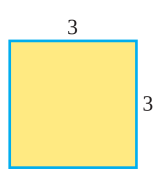
(b) $\frac{3}{5} \cdot \frac{3}{5} = \left(\frac{3}{5}\right)^2$ 2 factors of $\frac{3}{5}$

Read $\left(\frac{3}{5}\right)^2$ as "$\frac{3}{5}$ squared."

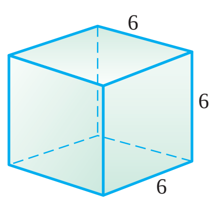
(c) $(-6)(-6)(-6)(-6) = (-6)^4$

(d) $x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$

Now Try Exercises 13, 15, 17, and 19.



(a) $3 \cdot 3 = 3$ squared, or 3^2



(b) $6 \cdot 6 \cdot 6 = 6$ cubed, or 6^3

FIGURE 15

In parts (a) and (b) of Example 1, we used the terms *squared* and *cubed* to refer to powers of 2 and 3, respectively. The term *squared* comes from the figure of a square, which has the same measure for both length and width, as shown in Figure 15(a). Similarly, the term *cubed* comes from the figure of a cube. As shown in Figure 15(b), the length, width, and height of a cube have the same measure.

EXAMPLE 2 Evaluating Exponential Expressions

Write each expression without exponents and evaluate.

(a) $5^2 = 5 \cdot 5 = 25$ 5 is used as a factor 2 times.

(b) $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$ $\frac{2}{3}$ is used as a factor 3 times.

(c) $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$

Now Try Exercises 21 and 27.

Be careful when evaluating an exponential expression with a negative sign.

EXAMPLE 3 Evaluating Exponential Expressions with Negative Signs

Evaluate each exponential expression.

(a) $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$

(b) $(-2)^6$

The exponent 6 applies to the number -2.

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = 64 \qquad \text{The base is } -2.$$

(c) -2^6

Since there are no parentheses, the exponent 6 applies *only* to the number 2, not to -2.

$$-2^6 = -(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = -64 \qquad \text{The base is 2.}$$

Now Try Exercises 29, 31, and 33.

Example 3 suggests the following generalizations.

The product of an *odd* number of negative factors is negative.
The product of an *even* number of negative factors is positive.

CAUTION As shown in Examples 3(b) and (c), it is important to distinguish between $-a^n$ and $(-a)^n$.

$$-a^n = -1\underbrace{(a \cdot a \cdot a \cdot \ldots \cdot a)}_{n \text{ factors of } a} \qquad \text{The base is } a.$$

$$(-a)^n = \underbrace{(-a)(-a) \cdot \ldots \cdot (-a)}_{n \text{ factors of } -a} \qquad \text{The base is } -a.$$

OBJECTIVE 2 Find square roots. As we saw in Example 2(a), $5^2 = 5 \cdot 5 = 25$, so 5 squared is 25. The opposite (inverse) of squaring a number is called taking its **square root.** For example, a square root of 25 is 5. Another square root of 25 is -5 since $(-5)^2 = 25$; thus, 25 has two square roots, 5 and -5.

We write the positive or *principal* square root of a number with the symbol $\sqrt{}$, called a **radical sign.** For example, the positive or principal square root of 25 is written $\sqrt{25} = 5$. The negative square root of 25 is written $-\sqrt{25} = -5$. Since the square of any nonzero real number is positive, *the square root of a negative number, such as $\sqrt{-25}$, is not a real number.*

EXAMPLE 4 Finding Square Roots

Find each square root that is a real number.

(a) $\sqrt{36} = 6$ since 6 is positive and $6^2 = 36$.

(b) $\sqrt{0} = 0$ since $0^2 = 0$.

(c) $\sqrt{\dfrac{9}{16}} = \dfrac{3}{4}$ since $\left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$.

(d) $\sqrt{.16} = .4$ since $(.4)^2 = .16$. **(e)** $\sqrt{100} = 10$ since $10^2 = 100$.

(f) $-\sqrt{100} = -10$ since the negative sign is outside the radical sign.

(g) $\sqrt{-100}$ is not a real number because the negative sign is inside the radical sign. No *real number* squared equals -100.

Notice the difference among the expressions in parts (e), (f), and (g). Part (e) is the positive or principal square root of 100, part (f) is the negative square root of 100, and part (g) is the square root of -100, which is not a real number.

Now Try Exercises 37, 41, 43, and 47.

CAUTION The symbol $\sqrt{}$ is used only for the *positive* square root, except that $\sqrt{0} = 0$. The symbol $-\sqrt{}$ is used for the negative square root.

OBJECTIVE 3 Use the order of operations. To simplify an expression such as $5 + 2 \cdot 3$, what should we do first—add 5 and 2, or multiply 2 and 3? When an expression involves more than one operation symbol, we use the following **order of operations.**

AUDIO

Order of Operations

1. Work separately above and below any **fraction bar.**
2. If **grouping symbols** such as **parentheses ( ), square brackets [ ],** or **absolute value bars | |** are present, start with the innermost set and work outward.
3. Evaluate all **powers, roots,** and **absolute values.**
4. Do any **multiplications** or **divisions** in order, working from left to right.
5. Do any **additions** or **subtractions** in order, working from left to right.

EXAMPLE 5 Using the Order of Operations

Simplify.

(a) $5 + 2 \cdot 3$

First multiply and then add.

$$\begin{aligned} 5 + 2 \cdot 3 &= 5 + 6 && \text{Multiply.} \\ &= 11 && \text{Add.} \end{aligned}$$

(b) $24 \div 3 \cdot 2 + 6$

Multiplications and divisions are done *in the order in which they appear from left to right*, so divide first.

$$\begin{aligned} 24 \div 3 \cdot 2 + 6 &= 8 \cdot 2 + 6 && \text{Divide.} \\ &= 16 + 6 && \text{Multiply.} \\ &= 22 && \text{Add.} \end{aligned}$$

Now Try Exercises 53 and 57.

VIDEO

EXAMPLE 6 Using the Order of Operations

Simplify.

(a) $10 \div 5 + 2|3 - 4|$

Evaluate the absolute value first.

$$\begin{aligned} 10 \div 5 + 2|3 - 4| &= 10 \div 5 + 2|-1| && \text{Subtract inside the absolute value bars.} \\ &= 10 \div 5 + 2 \cdot 1 && \text{Take the absolute value.} \\ &= 2 + 2 && \text{Divide and multiply.} \\ &= 4 && \text{Add.} \end{aligned}$$

(b) $4 \cdot 3^2 + 7 - (2 + 8)$

Work inside the parentheses first.

$$\begin{aligned} 4 \cdot 3^2 + 7 - (2 + 8) &= 4 \cdot 3^2 + 7 - 10 && \text{Add inside parentheses.} \\ &= 4 \cdot 9 + 7 - 10 && \text{Evaluate powers.} \\ &= 36 + 7 - 10 && \text{Multiply.} \\ &= 43 - 10 && \text{Add.} \\ &= 33 && \text{Subtract.} \end{aligned}$$

(c) $\frac{1}{2} \cdot 4 + (6 \div 3 - 7)$

Work inside the parentheses, dividing before subtracting.

$$\begin{aligned} \frac{1}{2} \cdot 4 + (6 \div 3 - 7) &= \frac{1}{2} \cdot 4 + (2 - 7) && \text{Divide inside parentheses.} \\ &= \frac{1}{2} \cdot 4 + (-5) && \text{Subtract inside parentheses.} \\ &= 2 + (-5) && \text{Multiply.} \\ &= -3 && \text{Add.} \end{aligned}$$

Now Try Exercises 65 and 71.

YOU TRY IT

EXAMPLE 7 Using the Order of Operations

Simplify $\dfrac{5 + (-2^3)(2)}{6 \cdot \sqrt{9} - 9 \cdot 2}$.

Work separately above and below the fraction bar.

$$\begin{aligned} \frac{5 + (-2^3)(2)}{6 \cdot \sqrt{9} - 9 \cdot 2} &= \frac{5 + (-8)(2)}{6 \cdot 3 - 9 \cdot 2} && \text{Evaluate powers and roots.} \\ &= \frac{5 - 16}{18 - 18} && \text{Multiply.} \\ &= \frac{-11}{0} && \text{Subtract.} \end{aligned}$$

Since division by 0 is undefined, the given expression is undefined.

Now Try Exercise 75.

Photo not available

OBJECTIVE 4 Evaluate algebraic expressions for given values of variables. Any collection of numbers, variables, operation symbols, and grouping symbols, such as

$$6ab, \qquad 5m - 9n, \qquad \text{and} \qquad -2(x^2 + 4y),$$

is called an **algebraic expression.** Algebraic expressions have different numerical values for different values of the variables. We can evaluate such expressions by *substituting* given values for the variables.

Algebraic expressions are used in problem solving. For example, if movie tickets cost \$7 each, the amount in dollars you pay for x tickets can be represented by the algebraic expression $7x$. We can substitute different numbers of tickets to get the costs to purchase those tickets.

EXAMPLE 8 Evaluating Algebraic Expressions

Evaluate each expression if $m = -4$, $n = 5$, $p = -6$, and $q = 25$.

(a) $5m - 9n$

Replace m with -4 and n with 5.

$$5m - 9n = 5(-4) - 9(5) = -20 - 45 = -65$$

(b) $$\frac{m + 2n}{4p} = \frac{-4 + 2(5)}{4(-6)} = \frac{-4 + 10}{-24} = \frac{6}{-24} = -\frac{1}{4}$$

(c) $-3m^3 - n^2(\sqrt{q}) = -3(-4)^3 - (5)^2(\sqrt{25})$ Substitute; $m = -4$, $n = 5$, and $q = 25$.

$= -3(-64) - 25(5)$ Evaluate powers and roots.

$= 192 - 125$ Multiply.

$= 67$ Subtract.

Now Try Exercises 79 and 85.

CAUTION To avoid errors when evaluating expressions, use parentheses around any negative numbers that are substituted for variables, as shown in Example 8.

1.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 1/Videotape 1

Decide whether each statement is true *or* false. *If false, correct the statement so it is true.*

1. $-4^6 = (-4)^6$

2. $-4^7 = (-4)^7$

3. $\sqrt{16}$ is a positive number.

4. $3 + 5 \cdot 6 = 3 + (5 \cdot 6)$

5. $(-2)^7$ is a negative number.

6. $(-2)^8$ is a positive number.

7. The product of 8 positive factors and 8 negative factors is positive.

8. The product of 3 positive factors and 3 negative factors is positive.

9. In the exponential expression -3^5, -3 is the base.

10. $\sqrt{a}$ is positive for all positive numbers a.

11. Evaluate each exponential expression.

(a) 8^2 **(b)** -8^2
(c) $(-8)^2$ **(d)** $-(-8)^2$

12. Evaluate each exponential expression.

(a) 4^3 **(b)** -4^3
(c) $(-4)^3$ **(d)** $-(-4)^3$

Write each expression using exponents. See Example 1.

13. $10 \cdot 10 \cdot 10 \cdot 10$ **14.** $8 \cdot 8 \cdot 8$ **15.** $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$

16. $\frac{1}{2} \cdot \frac{1}{2}$ **17.** $(-9)(-9)(-9)$ **18.** $(-4)(-4)(-4)(-4)$

19. $z \cdot z \cdot z \cdot z \cdot z \cdot z \cdot z$ **20.** $a \cdot a \cdot a \cdot a \cdot a$

Evaluate each expression. See Examples 2 and 3.

21. 4^2 **22.** 2^4 **23.** $.28^3$ **24.** $.91^3$

25. $\left(\frac{1}{5}\right)^3$ **26.** $\left(\frac{1}{6}\right)^4$ **27.** $\left(\frac{4}{5}\right)^4$ **28.** $\left(\frac{7}{10}\right)^3$

29. $(-5)^3$ **30.** $(-2)^5$ **31.** $(-2)^8$ **32.** $(-3)^6$

33. -3^6 **34.** -4^6 **35.** -8^4 **36.** -10^3

Find each square root. If it is not a real number, say so. See Example 4.

37. $\sqrt{81}$ **38.** $\sqrt{64}$ **39.** $\sqrt{169}$ **40.** $\sqrt{225}$

41. $-\sqrt{400}$ **42.** $-\sqrt{900}$ **43.** $\sqrt{\frac{100}{121}}$ **44.** $\sqrt{\frac{225}{169}}$

45. $-\sqrt{.49}$ **46.** $-\sqrt{.64}$ **47.** $\sqrt{-36}$ **48.** $\sqrt{-121}$

49. Match each square root with the appropriate value or description.

(a) $\sqrt{144}$ **(b)** $\sqrt{-144}$ **(c)** $-\sqrt{144}$

A. -12 **B.** 12 **C.** Not a real number

50. Explain why $\sqrt{-900}$ is not a real number.

51. If a is a positive number, is $-\sqrt{-a}$ positive, negative, or not a real number?

52. If a is a positive number, is $-\sqrt{a}$ positive, negative, or not a real number?

Simplify each expression. Use the order of operations. See Examples 5–7.

53. $12 + 3 \cdot 4$ **54.** $15 + 5 \cdot 2$ **55.** $6 \cdot 3 - 12 \div 4$

56. $9 \cdot 4 - 8 \div 2$ **57.** $10 + 30 \div 2 \cdot 3$ **58.** $12 + 24 \div 3 \cdot 2$

59. $-3(5)^2 - (-2)(-8)$ **60.** $-9(2)^2 - (-3)(-2)$ **61.** $5 - 7 \cdot 3 - (-2)^3$

62. $-4 - 3 \cdot 5 + 6^2$ **63.** $-7(\sqrt{36}) - (-2)(-3)$ **64.** $-8(\sqrt{64}) - (-3)(-7)$

65. $6|4 - 5| - 24 \div 3$ **66.** $-4|2 - 4| + 8 \cdot 2$ **67.** $|-6 - 5|(-8) + 3^2$

68. $(-6 - 3)|-2 - 3| \div 9$ **69.** $6 + \frac{2}{3}(-9) - \frac{5}{8} \cdot 16$ **70.** $7 - \frac{3}{4}(-8) + 12 \cdot \frac{5}{6}$

71. $-14\left(-\frac{2}{7}\right) \div (2 \cdot 6 - 10)$ **72.** $-12\left(-\frac{3}{4}\right) - (6 \cdot 5 \div 3)$

73. $\dfrac{(-5 + \sqrt{4})(-2^2)}{-5 - 1}$ **74.** $\dfrac{(-9 + \sqrt{16})(-3^2)}{-4 - 1}$

75. $\dfrac{2(-5) + (-3)(-2)}{-8 + 3^2 - 1}$ **76.** $\dfrac{3(-4) + (-5)(-8)}{2^3 - 2 - 6}$

77. $\dfrac{5 - 3\left(\dfrac{-5 - 9}{-7}\right) - 6}{-9 - 11 + 3 \cdot 7}$ **78.** $\dfrac{-4\left(\dfrac{12 - (-8)}{3 \cdot 2 + 4}\right) - 5(-1 - 7)}{-9 - (-7) - [-5 - (-8)]}$

Evaluate each expression if $a = -3$, $b = 64$, and $c = 6$. See Example 8.

79. $3a + \sqrt{b}$ **80.** $-2a - \sqrt{b}$ **81.** $\sqrt{b} + c - a$ **82.** $\sqrt{b} - c + a$

83. $4a^3 + 2c$ **84.** $-3a^4 - 3c$ **85.** $\dfrac{2c + a^3}{4b + 6a}$ **86.** $\dfrac{3c + a^2}{2b - 6c}$

Solve each problem.

Residents of Linn County, Iowa in the Cedar Rapids Community School District can use the expression

$$(v \times .5485 - 4850) \div 1000 \times 31.44$$

to determine their property taxes, where v is home value. (*Source: The Gazette*, August 19, 2000.) Use the expression to calculate the amount of property taxes to the nearest dollar that the owner of a home with each of the following values would pay. Follow the order of operations.

87. \$100,000 **88.** \$150,000 **89.** \$200,000

The Blood Alcohol Concentration (BAC) of a person who has been drinking is given by the expression

$$\text{number of oz} \times \%\text{ alcohol} \times .075 \div \text{body weight in lb} - \text{hr of drinking} \times .015.$$

(*Source:* Lawlor, J., *Auto Math Handbook: Mathematical Calculations, Theory, and Formulas for Automotive Enthusiasts*, HP Books, 1991.)

90. Suppose a policeman stops a 190-lb man who, in 2 hr, has ingested four 12-oz beers (48 oz), each having a 3.2% alcohol content.

(a) Substitute the values in the formula, and write the expression for the man's BAC.
(b) Calculate the man's BAC to the nearest thousandth. Follow the order of operations.

91. Find the BAC to the nearest thousandth for a 135-lb woman who, in 3 hr, has drunk three 12-oz beers (36 oz), each having a 4.0% alcohol content.

92. (a) Calculate the BACs in Exercises 90 and 91 if each person weighs 25 lb more and the rest of the variables stay the same. How does increased weight affect a person's BAC?
(b) Predict how decreased weight would affect the BAC of each person in Exercises 90 and 91. Calculate the BACs if each person weighs 25 lb less and the rest of the variables stay the same.

93. An approximation of the percent of U.S. households investing in mutual funds during the years 1980 through 2000 can be obtained by substituting a given year for x in the expression

$$2.2023x - 4356.6$$

and then evaluating. (*Source:* Investment Company Institute.) Approximate the percent of U.S. households investing in mutual funds in each year. Round answers to the nearest tenth of a percent.

Photo not available

(a) 1980 **(b)** 1990 **(c)** 2000
(d) How has the percent of households investing in mutual funds changed from 1980 to 2000?

94. An approximation of federal spending on education in billions of dollars from 1997 through 2001 can be obtained using the expression

$$3.31714x - 6597.86,$$

where x represents the year. (*Source:* U.S. Department of Education.)

(a) Use this expression to complete the table. Round answers to the nearest tenth.

Year	*Education Spending (in billions of dollars)*
1997	26.5
1998	29.8
1999	______
2000	______
2001	______

(b) Describe the trend in the amount of federal spending on education during these years.

1.4 Properties of Real Numbers

OBJECTIVES

1. Use the distributive property.
2. Use the inverse properties.
3. Use the identity properties.
4. Use the commutative and associative properties.
5. Use the multiplication property of 0.

The study of any object is simplified when we know the properties of the object. For example, a property of water is that it freezes when cooled to 0°C. Knowing this helps us to predict the behavior of water.

The study of numbers is no different. The basic properties of real numbers studied in this section reflect results that occur consistently in work with numbers, so they have been generalized to apply to expressions with variables as well.

OBJECTIVE 1 Use the distributive property. Notice that

$$2(3 + 5) = 2 \cdot 8 = 16$$

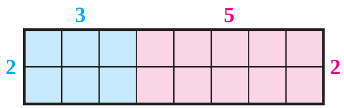
and

$$2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16,$$

so

$$2(3 + 5) = 2 \cdot 3 + 2 \cdot 5.$$

This idea is illustrated by the divided rectangle in Figure 16. Similarly,



Area of left part is $2 \cdot 3 = 6$.
Area of right part is $2 \cdot 5 = 10$.
Area of total rectangle is $2(3 + 5) = 16$.
FIGURE 16

$$-4[5 + (-3)] = -4(2) = -8$$

and

$$-4(5) + (-4)(-3) = -20 + 12 = -8,$$

so

$$-4[5 + (-3)] = -4(5) + (-4)(-3).$$

These arithmetic examples are generalized to *all* real numbers as the **distributive property of multiplication with respect to addition,** or simply the **distributive property.**

AUDIO

Distributive Property

For any real numbers a, b, and c,

$$\boldsymbol{a(b + c) = ab + ac} \quad \text{and} \quad \boldsymbol{(b + c)a = ba + ca}.$$

The distributive property can also be written

$$ab + ac = a(b + c) \quad \text{and} \quad ba + ca = (b + c)a.$$

It can be extended to more than two numbers as well.

$$a(b + c + d) = ab + ac + ad$$

This property is important because it provides a way to rewrite a *product* $a(b + c)$ as a sum $ab + ac$, or a *sum* as a product.

NOTE When we rewrite $a(b + c)$ as $ab + ac$, we sometimes refer to the process as "removing parentheses."

VIDEO EXPLORE

EXAMPLE 1 Using the Distributive Property

Use the distributive property to rewrite each expression.

(a) $3(x + y)$

Use the first form of the property to rewrite this product as a sum.

$$3(x + y) = 3x + 3y$$

(b) $-2(5 + k) = -2(5) + (-2)(k)$

$= -10 - 2k$

(c) $4x + 8x$

Use the second form of the property to rewrite this sum as a product.

$$4x + 8x = (4 + 8)x = 12x$$

(d) $3r - 7r = 3r + (-7r)$ Definition of subtraction

$= [3 + (-7)]r$ Distributive property

$= -4r$

(e) $5p + 7q$

Because there is no common number or variable here, we cannot use the distributive property to rewrite the expression.

(f) $6(x + 2y - 3z) = 6x + 6(2y) + 6(-3z)$

$= 6x + 12y - 18z$

Now Try Exercises 11, 13, 15, and 19.

As illustrated in Example 1(d), the distributive property can also be used for subtraction, so

$$a(b - c) = ab - ac.$$

OBJECTIVE 2 Use the inverse properties. In Section 1.1 we saw that the additive inverse of a number a is $-a$ and that the sum of a number and its additive inverse is 0. For example, 3 and -3 are additive inverses, as are -8 and 8. The number 0 is its own additive inverse. In Section 1.2, we saw that two numbers with a product of 1 are reciprocals. As mentioned there, another name for reciprocal is *multiplicative inverse*. This is similar to the idea of an additive inverse. Thus, 4 and $\frac{1}{4}$ are multiplicative inverses, as are $-\frac{2}{3}$ and $-\frac{3}{2}$. (Recall that reciprocals have the same sign.) We can extend these properties of arithmetic, the **inverse properties** of addition and multiplication, to the real numbers of algebra.

Inverse Properties

For any real number a, there is a single real number $-a$, such that

$$\boldsymbol{a + (-a) = 0} \qquad \text{and} \qquad \boldsymbol{-a + a = 0.}$$

The inverse "undoes" addition with the result 0.

For any *nonzero* real number a, there is a single real number $\frac{1}{a}$ such that

$$\boldsymbol{a \cdot \frac{1}{a} = 1} \qquad \text{and} \qquad \boldsymbol{\frac{1}{a} \cdot a = 1.}$$

The inverse "undoes" multiplication with the result 1.

OBJECTIVE 3 Use the identity properties. The numbers 0 and 1 each have a special property. Zero is the only number that can be added to any number to get that number. That is, adding 0 leaves the identity of a number unchanged. For this reason, 0 is called the **identity element for addition** or the **additive identity.** In a similar way, multiplying by 1 leaves the identity of any number unchanged, so 1 is the **identity element for multiplication** or the **multiplicative identity.** The following **identity properties** summarize this discussion and extend these properties from arithmetic to algebra.

Identity Properties

For any real number a,

$$\boldsymbol{a + 0 = 0 + a = a.}$$

Start with a number a; add 0. The answer is "identical" to a.

Also, $$\boldsymbol{a \cdot 1 = 1 \cdot a = a.}$$

Start with a number a; multiply by 1. The answer is "identical" to a.

EXAMPLE 2 Using the Identity Property $1 \cdot a = a$

Simplify each expression.

(a) $12m + m = 12m + 1m$ Identity property

$= (12 + 1)m$ Distributive property

$= 13m$ Add inside parentheses.

(b) $y + y = 1y + 1y$ Identity property
$= (1 + 1)y$ Distributive property
$= 2y$ Add inside parentheses.

(c) $-(m - 5n) = -1(m - 5n)$ Identity property
$= -1(m) + (-1)(-5n)$ Distributive property
$= -m + 5n$ Multiply.

Now Try Exercises 21 and 23.

Expressions such as $12m$ and $5n$ from Example 2 are examples of *terms.* A **term** is a number or the product of a number and one or more variables. Terms with exactly the same variables raised to exactly the same powers are called **like terms.** Some examples of like terms are

$$5p \text{ and } -21p \qquad -6x^2 \text{ and } 9x^2. \qquad \text{Like terms}$$

Some examples of unlike terms are

$$3m \text{ and } 16x \qquad 7y^3 \text{ and } -3y^2. \qquad \text{Unlike terms}$$

The numerical factor in a term is called the **numerical coefficient,** or just the **coefficient.** For example, in the term $9x^2$, the coefficient is 9.

OBJECTIVE 4 Use the commutative and associative properties. Simplifying expressions as in parts (a) and (b) of Example 2 is called **combining like terms.** Only like terms may be combined. To combine like terms in an expression such as

$$-2m + 5m + 3 - 6m + 8,$$

we need two more properties. From arithmetic, we know that

$$3 + 9 = 12 \qquad \text{and} \qquad 9 + 3 = 12.$$

Also,

$$3 \cdot 9 = 27 \qquad \text{and} \qquad 9 \cdot 3 = 27.$$

Furthermore, notice that

$$(5 + 7) + (-2) = 12 + (-2) = 10$$

and

$$5 + [7 + (-2)] = 5 + 5 = 10.$$

Also,

$$(5 \cdot 7)(-2) = 35(-2) = -70$$

and

$$(5)[7 \cdot (-2)] = 5(-14) = -70.$$

These arithmetic examples can be extended to algebra.

Commutative and Associative Properties

For any real numbers a, b, and c,

$$\left.\begin{aligned} a + b &= b + a \\ \text{and} \qquad ab &= ba. \end{aligned}\right\} \quad \textbf{Commutative properties}$$

Reverse the order of two terms or factors.

Also,

$$\left.\begin{aligned} a + (b + c) &= (a + b) + c \\ \text{and} \qquad a(bc) &= (ab)c. \end{aligned}\right\} \quad \textbf{Associative properties}$$

Shift parentheses among three terms or factors; order stays the same.

The commutative properties are used to change the *order* of the terms or factors in an expression. Think of commuting from home to work and then from work to home. The associative properties are used to *regroup* the terms or factors of an expression. Remember, to *associate* is to be part of a group.

EXAMPLE 3 Using the Commutative and Associative Properties

Simplify $-2m + 5m + 3 - 6m + 8$.

$$\begin{aligned} &-2m + 5m + 3 - 6m + 8 \\ &= (-2m + 5m) + 3 - 6m + 8 && \text{Order of operations} \\ &= (-2 + 5)m + 3 - 6m + 8 && \text{Distributive property} \\ &= 3m + 3 - 6m + 8 \end{aligned}$$

By the order of operations, the next step would be to add $3m$ and 3, but they are unlike terms. To get $3m$ and $-6m$ together, use the associative and commutative properties. Begin by inserting parentheses and brackets according to the order of operations.

$$\begin{aligned} &[(3m + 3) - 6m] + 8 \\ &= [3m + (3 - 6m)] + 8 && \text{Associative property} \\ &= [3m + (-6m + 3)] + 8 && \text{Commutative property} \\ &= [(3m + [-6m]) + 3] + 8 && \text{Associative property} \\ &= (-3m + 3) + 8 && \text{Combine like terms.} \\ &= -3m + (3 + 8) && \text{Associative property} \\ &= -3m + 11 && \text{Add.} \end{aligned}$$

In practice, many of these steps are not written down, but you should realize that the commutative and associative properties are used whenever the terms in an expression are rearranged to combine like terms.

Now Try Exercise 27.

EXAMPLE 4 Using the Properties of Real Numbers

Simplify each expression.

(a) $5y - 8y - 6y + 11y$

$$\begin{aligned} &= (5 - 8 - 6 + 11)y && \text{Distributive property} \\ &= 2y && \text{Combine like terms.} \end{aligned}$$

(b) $3x + 4 - 5(x + 1) - 8$

$$
\begin{aligned}
&= 3x + 4 - 5x - 5 - 8 && \text{Distributive property} \\
&= 3x - 5x + 4 - 5 - 8 && \text{Commutative property} \\
&= -2x - 9 && \text{Combine like terms.}
\end{aligned}
$$

(c)
$$
\begin{aligned}
8 - (3m + 2) &= 8 - 1(3m + 2) && \text{Identity property} \\
&= 8 - 3m - 2 && \text{Distributive property} \\
&= 6 - 3m && \text{Combine like terms.}
\end{aligned}
$$

(d)
$$
\begin{aligned}
3x(5)(y) &= [3x(5)]y && \text{Order of operations} \\
&= [3(x \cdot 5)]y && \text{Associative property} \\
&= [3(5x)]y && \text{Commutative property} \\
&= [(3 \cdot 5)x]y && \text{Associative property} \\
&= (15x)y && \text{Multiply.} \\
&= 15(xy) && \text{Associative property} \\
&= 15xy
\end{aligned}
$$

As previously mentioned, many of these steps are not usually written out.

Now Try Exercises 29 and 31.

OBJECTIVE 5 Use the multiplication property of 0. The additive identity property gives a special property of 0, namely that $a + 0 = a$ for any real number a. The **multiplication property of 0** gives a special property of 0 that involves multiplication: The product of any real number and 0 is 0.

Multiplication Property of 0

For any real number a,

$$a \cdot 0 = 0 \quad \text{and} \quad 0 \cdot a = 0.$$

1.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 1/Videotape 1

Choose the correct response in Exercises 1–4.

1. The identity element for addition is

A. $-a$ **B.** 0 **C.** 1 **D.** $\frac{1}{a}$.

2. The identity element for multiplication is

A. $-a$ **B.** 0 **C.** 1 **D.** $\frac{1}{a}$.

3. The additive inverse of a is

A. $-a$ **B.** 0 **C.** 1 **D.** $\frac{1}{a}$.

4. The multiplicative inverse of a, where $a \neq 0$, is

A. $-a$ **B.** 0 **C.** 1 **D.** $\frac{1}{a}$.

Complete each statement.

5. The multiplication property of 0 says that the ______ of 0 and any real number is ______.

6. The commutative property is used to change the ______ of two terms or factors.
7. The associative property is used to change the ______ of three terms or factors.
8. Like terms are terms with the ______ variables raised to the ______ powers.
9. When simplifying an expression, only ______ terms can be combined.
10. The coefficient in the term $-8yz^2$ is ______.

Simplify each expression. See Examples 1 and 2.

11. $2(m + p)$
12. $3(a + b)$
13. $-12(x - y)$
14. $-10(p - q)$
15. $5k + 3k$
16. $6a + 5a$
17. $7r - 9r$
18. $4n - 6n$
19. $-8z + 4w$
20. $-12k + 3r$
21. $a + 7a$
22. $s + 9s$
23. $-(2d - f)$
24. $-(3m - n)$

Simplify each expression. See Examples 1–4.

25. $-12y + 4y + 3 + 2y$
26. $-5r - 9r + 8r - 5$
27. $-6p + 5 - 4p + 6 + 11p$
28. $-8x - 12 + 3x - 5x + 9$
29. $3(k + 2) - 5k + 6 + 3$
30. $5(r - 3) + 6r - 2r + 4$
31. $-2(m + 1) - (m - 4)$
32. $6(a - 5) - (a + 6)$
33. $.25(8 + 4p) - .5(6 + 2p)$
34. $.4(10 - 5x) - .8(5 + 10x)$
35. $-(2p + 5) + 3(2p + 4) - 2p$
36. $-(7m - 12) - 2(4m + 7) - 8m$
37. $2 + 3(2z - 5) - 3(4z + 6) - 8$
38. $-4 + 4(4k - 3) - 6(2k + 8) + 7$

Complete each statement so that the indicated property is illustrated. Simplify each answer, if possible.

39. $5x + 8x =$ ______ (distributive property)
40. $9y - 6y =$ ______ (distributive property)
41. $5(9r) =$ ______ (associative property)
42. $-4 + (12 + 8) =$ ______ (associative property)
43. $5x + 9y =$ ______ (commutative property)
44. $-5 \cdot 7 =$ ______ (commutative property)
45. $1 \cdot 7 =$ ______ (identity property)
46. $-12x + 0 =$ ______ (identity property)
47. $-\frac{1}{4}ty + \frac{1}{4}ty =$ ______ (inverse property)
48. $-\frac{9}{8}\left(-\frac{8}{9}\right) =$ ______ (inverse property)
49. $8(-4 + x) =$ ______ (distributive property)
50. $3(x - y + z) =$ ______ (distributive property)
51. $0(.875x + 9y - 88z) =$ ______ (multiplication property of 0)
52. $0(35t^2 - 8t + 12) =$ ______ (multiplication property of 0)
53. Give an "everyday" example of a commutative operation and of one that is not commutative.
54. Give an "everyday" example of inverse operations.

The distributive property can be used to mentally perform calculations. For example, calculate $38 \cdot 17 + 38 \cdot 3$ as follows.

$$\begin{aligned} 38 \cdot 17 + 38 \cdot 3 &= 38(17 + 3) \quad \text{Distributive property} \\ &= 38(20) \\ &= 760 \end{aligned}$$

Use the distributive property to calculate each value mentally.

55. $96 \cdot 19 + 4 \cdot 19$

56. $27 \cdot 60 + 27 \cdot 40$

57. $58 \cdot \frac{3}{2} - 8 \cdot \frac{3}{2}$

58. $8.75(15) - 8.75(5)$

59. $4.31(69) + 4.31(31)$

60. $\frac{8}{5}(17) + \frac{8}{5}(13)$

RELATING CONCEPTS (EXERCISES 61–66)

For Individual or Group Work

When simplifying the expression $3x + 4 + 2x + 7$ *to* $5x + 11$, *several important steps are usually done mentally.* ***Work Exercises 61–66 in order,*** *providing the property that justifies each statement in the given simplification. (These steps could be done in other orders.)*

61. $3x + 4 + 2x + 7 = (3x + 4) + (2x + 7)$

62. $= 3x + (4 + 2x) + 7$

63. $= 3x + (2x + 4) + 7$

64. $= (3x + 2x) + (4 + 7)$

65. $= (3 + 2)x + (4 + 7)$

66. $= 5x + 11$

67. Write a paragraph explaining the properties introduced in this section. Give examples.

68. Explain how the distributive property is used to combine like terms. Give an example.

69. By the distributive property, $a(b + c) = ab + ac$. This property is more completely named the distributive property of multiplication with respect to addition. Is there a distributive property of addition with respect to multiplication? That is, does

$$a + (b \cdot c) = (a + b)(a + c)$$

for all real numbers, a, b, and c? To find out, try some sample values of a, b, and c.

70. Are there *any* different numbers that satisfy the statement $a - b = b - a$? Give an example if your answer is yes.

2.1 Linear Equations in One Variable

OBJECTIVES

1. Decide whether a number is a solution of a linear equation.
2. Solve linear equations using the addition and multiplication properties of equality.
3. Solve linear equations using the distributive property.
4. Solve linear equations with fractions or decimals.
5. Identify conditional equations, contradictions, and identities.

In the previous chapter we began to use *algebraic expressions.* Some examples of algebraic expressions are

$$8x + 9, \qquad y - 4, \qquad \text{and} \qquad \frac{x^3y^8}{z}. \qquad \text{Algebraic expressions}$$

Equations and inequalities compare algebraic expressions, just as a balance scale compares the weights of two quantities. Many applications of mathematics lead to *equations,* statements that two algebraic expressions are equal. A *linear equation in one variable* involves only real numbers and one variable raised to the first power. Examples are

$$x + 1 = -2, \qquad x - 3 = 5, \qquad \text{and} \qquad 2k + 5 = 10. \qquad \text{Linear equations}$$

It is important to be able to distinguish between algebraic expressions and equations. *An equation always contains an equals sign, while an expression does not.*

Linear Equation in One Variable

A **linear equation in one variable** can be written in the form

$$Ax + B = C,$$

where A, B, and C are real numbers, with $A \neq 0$.

A linear equation is also called a **first-degree equation** since the highest power on the variable is one. Some examples of equations that are not linear (that is, *nonlinear*) are

$$x^2 + 3y = 5, \qquad \frac{8}{x} = -22, \qquad \text{and} \qquad \sqrt{x} = 6. \qquad \text{Nonlinear equations}$$

OBJECTIVE 1 Decide whether a number is a solution of a linear equation. If the variable in an equation can be replaced by a real number that makes the statement true, then that number is a **solution** of the equation. For example, 8 is a solution of the equation $x - 3 = 5$, since replacing x with 8 gives a true statement. An equation is *solved* by finding its **solution set,** the set of all solutions. The solution set of the equation $x - 3 = 5$ is $\{8\}$.

Equivalent equations are equations that have the same solution set. To solve an equation, we usually start with the given equation and replace it with a series of simpler equivalent equations. For example,

$$5x + 2 = 17, \qquad 5x = 15, \qquad \text{and} \qquad x = 3$$

are all equivalent since each has the solution set $\{3\}$.

OBJECTIVE 2 Solve linear equations using the addition and multiplication properties of equality. Two important properties that are used in producing equivalent equations are the **addition** and **multiplication properties of equality.**

AUDIO

Addition and Multiplication Properties of Equality

Addition Property of Equality

For all real numbers A, B, and C, the equations

$$A = B \qquad \text{and} \qquad A + C = B + C$$

are equivalent.

That is, the same number may be added to each side of an equation without changing the solution set.

Multiplication Property of Equality

For all real numbers A and B, and for $C \neq 0$, the equations

$$A = B \qquad \text{and} \qquad AC = BC$$

are equivalent.

That is, each side of an equation may be multiplied by the same nonzero number without changing the solution set.

Because subtraction and division are defined in terms of addition and multiplication, respectively, these properties can be extended: The same number may be subtracted from each side of an equation, and each side of an equation may be divided by the same nonzero number, without changing the solution set.

YOU TRY IT

VIDEO

EXAMPLE 1 Using the Addition and Multiplication Properties to Solve a Linear Equation

Solve $4x - 2x - 5 = 4 + 6x + 3$.

The goal is to use the addition and multiplication properties to get x alone on one side of the equation. First, combine like terms on each side of the equation to obtain

$$2x - 5 = 7 + 6x.$$

Next, use the addition property to get the terms with x on the same side of the equation and the remaining terms (the numbers) on the other side. One way to do this is to first add 5 to each side.

$$\begin{aligned}
2x - 5 + 5 &= 7 + 6x + 5 && \text{Add 5.}\\
2x &= 12 + 6x && \text{Combine like terms.}\\
2x - 6x &= 12 + 6x - 6x && \text{Subtract } 6x.\\
-4x &= 12 && \text{Combine like terms.}\\
\frac{-4x}{-4} &= \frac{12}{-4} && \text{Divide by } -4.\\
x &= -3
\end{aligned}$$

To be sure that -3 is the solution, check by substituting for x in the *original* equation.

Check:

$$\begin{aligned}
4x - 2x - 5 &= 4 + 6x + 3 && && \text{Original equation}\\
4(-3) - 2(-3) - 5 &= 4 + 6(-3) + 3 && ? && \text{Let } x = -3.\\
-12 + 6 - 5 &= 4 - 18 + 3 && ? && \text{Multiply.}\\
-11 &= -11 && && \text{True}
\end{aligned}$$

The true statement indicates that $\{-3\}$ is the solution set.

Now Try Exercise 13.

NOTE Notice that in Example 1 the equality symbols are aligned in a column. Do not use more than one equality symbol in a horizontal line of work when solving an equation.

We use the following steps to solve a linear equation in one variable. (Some equations may not require all these steps.)

Solving a Linear Equation in One Variable

Step 1 **Clear fractions.** Eliminate any fractions by multiplying each side by the least common denominator.

Step 2 **Simplify each side separately.** Use the distributive property to clear parentheses and combine like terms as needed.

Step 3 **Isolate the variable terms on one side.** Use the addition property to get all terms with variables on one side of the equation and all numbers on the other.

Step 4 **Isolate the variable.** Use the multiplication property to get an equation with just the variable (with coefficient 1) on one side.

Step 5 **Check.** Substitute the proposed solution into the original equation.

OBJECTIVE 3 Solve linear equations using the distributive property. In Example 1 we did not use Step 1 or the distributive property in Step 2 as given in the box. Many equations, however, will require one or both of these steps.

EXAMPLE 2 Using the Distributive Property to Solve a Linear Equation

Solve $2(k - 5) + 3k = k + 6$.

Step 1 Since there are no fractions in this equation, Step 1 does not apply.

Step 2 Use the distributive property to simplify and combine terms on the left side of the equation.

$$2(k - 5) + 3k = k + 6$$

$$2k - 10 + 3k = k + 6 \qquad \text{Distributive property}$$

$$5k - 10 = k + 6 \qquad \text{Combine like terms.}$$

Step 3 Next, use the addition property of equality.

$$5k - 10 + 10 = k + 6 + 10 \qquad \text{Add 10.}$$

$$5k = k + 16 \qquad \text{Combine like terms.}$$

$$5k - k = k + 16 - k \qquad \text{Subtract } k.$$

$$4k = 16 \qquad \text{Combine like terms.}$$

Step 4 Use the multiplication property of equality to get just k on the left.

$$\frac{4k}{4} = \frac{16}{4} \qquad \text{Divide by 4.}$$

$$k = 4$$

Step 5 Check that the solution set is $\{4\}$ by substituting 4 for k in the original equation.

Now Try Exercise 15.

NOTE Because of space limitations, we will not always show the check when solving an equation. To be sure that your solution is correct, you should *always* check your work.

OBJECTIVE 4 Solve linear equations with fractions or decimals. When fractions or decimals appear as coefficients in equations, our work can be made easier if we multiply each side of the equation by the least common denominator (LCD) of all the fractions. This is an application of the multiplication property of equality, and it produces an equivalent equation with integer coefficients.

EXAMPLE 3 Solving a Linear Equation with Fractions

Solve $\dfrac{x+7}{6} + \dfrac{2x-8}{2} = -4.$

Start by eliminating the fractions. Multiply both sides by the LCD, 6.

Step 1 $$6\left(\frac{x+7}{6} + \frac{2x-8}{2}\right) = 6(-4)$$

Step 2 $$6\left(\frac{x+7}{6}\right) + 6\left(\frac{2x-8}{2}\right) = 6(-4) \qquad \text{Distributive property}$$

$$x + 7 + 3(2x - 8) = -24 \qquad \text{Multiply.}$$

$$x + 7 + 6x - 24 = -24 \qquad \text{Distributive property}$$

$$7x - 17 = -24 \qquad \text{Combine like terms.}$$

Step 3 $$7x - 17 + 17 = -24 + 17 \qquad \text{Add 17.}$$

$$7x = -7 \qquad \text{Combine like terms.}$$

Step 4 $$\frac{7x}{7} = \frac{-7}{7} \qquad \text{Divide by 7.}$$

$$x = -1$$

Step 5 Check by substituting -1 for x in the original equation.

$$\frac{x+7}{6} + \frac{2x-8}{2} = -4$$

$$\frac{-1+7}{6} + \frac{2(-1)-8}{2} \stackrel{?}{=} -4 \qquad \text{Let } x = -1.$$

$$\frac{6}{6} + \frac{-10}{2} \stackrel{?}{=} -4$$

$$1 - 5 \stackrel{?}{=} -4$$

$$-4 = -4 \qquad \text{True}$$

The solution checks, so the solution set is $\{-1\}$.

Now Try Exercise 41.

In later sections we solve problems involving interest rates and concentrations of solutions. These problems involve percents that are converted to decimals. The equations that are used to solve such problems involve decimal coefficients. We can clear these decimals by multiplying by a power of 10 that allows us to obtain integer coefficients.

EXAMPLE 4 Solving a Linear Equation with Decimals

Solve $.06x + .09(15 - x) = .07(15)$.

Since each decimal number is given in hundredths, multiply both sides of the equation by 100. (This is done by moving the decimal points two places to the right.) To multiply the second term, $.09(15 - x)$, by 100, remember the associative property: To multiply three terms, first multiply any two of them. Here we will multiply $100(.09)$ first to get 9, so the product $100(.09)(15 - x)$ becomes $9(15 - x)$.

$$.06x + .09(15 - x) = .07(15)$$

$$.06x + .09(15 - x) = .07(15) \qquad \text{Multiply each term by 100.}$$

$$6x + 9(15 - x) = 7(15)$$

$$6x + 9(15) - 9x = 105 \qquad \text{Distributive property; multiply.}$$

$$-3x + 135 = 105 \qquad \text{Combine like terms.}$$

$$-3x + 135 - 135 = 105 - 135 \qquad \text{Subtract 135.}$$

$$-3x = -30$$

$$\frac{-3x}{-3} = \frac{-30}{-3} \qquad \text{Divide by } -3.$$

$$x = 10$$

Check to verify that the solution set is $\{10\}$.

Now Try Exercise 45.

OBJECTIVE 5 Identify conditional equations, contradictions, and identities. All the preceding equations had solution sets containing one element; for example, $2(k - 5) + 3k = k + 6$ has solution set $\{4\}$. Some equations that appear to be linear have no solutions, while others have an infinite number of solutions. The table gives the names of these types of equations.

Type of Equation	*Number of Solutions*	*Indication When Solving*
Conditional	One	Final line is $x =$ a number. (See Example 5(a).)
Contradiction	None; solution set $\emptyset$	Final line is false, such as $0 = 1$. (See Example 5(c).)
Identity	Infinite; solution set {all real numbers}	Final line is true, such as $0 = 0$. (See Example 5(b).)

EXAMPLE 5 Recognizing Conditional Equations, Identities, and Contradictions

Solve each equation. Decide whether it is a *conditional equation*, an *identity*, or a *contradiction*.

(a) $5x - 9 = 4(x - 3)$

$$\begin{aligned} 5x - 9 &= 4x - 12 && \text{Distributive property} \\ 5x - 9 - 4x &= 4x - 12 - 4x && \text{Subtract } 4x. \\ x - 9 &= -12 && \text{Combine like terms.} \\ x - 9 + 9 &= -12 + 9 && \text{Add 9.} \\ x &= -3 \end{aligned}$$

The solution set, $\{-3\}$, has only one element, so $5x - 9 = 4(x - 3)$ is a conditional equation.

(b) $5x - 15 = 5(x - 3)$

Use the distributive property to clear parentheses on the right side.

$$\begin{aligned} 5x - 15 &= 5x - 15 \\ 0 &= 0 && \text{Subtract } 5x \text{ and add 15.} \end{aligned}$$

The final line, $0 = 0$, indicates that the solution set is {all real numbers}, and the equation $5x - 15 = 5(x - 3)$ is an identity. (*Note:* The first step yielded $5x - 15 = 5x - 15$, which is true for all values of x. We could have identified the equation as an identity at that point.)

(c)
$$\begin{aligned} 5x - 15 &= 5(x - 4) \\ 5x - 15 &= 5x - 20 && \text{Distributive property} \\ 5x - 15 - 5x &= 5x - 20 - 5x && \text{Subtract } 5x. \\ -15 &= -20 && \text{False} \end{aligned}$$

Since the result, $-15 = -20$, is *false*, the equation has no solution. The solution set is $\emptyset$, so the equation $5x - 15 = 5(x - 4)$ is a contradiction.

Now Try Exercises 53, 57, and 59.

2.1 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 2/Videotape 2

1. Which equations are linear equations in x?

A. $3x + x - 1 = 0$ **B.** $8 = x^2$ **C.** $6x + 2 = 9$ **D.** $\frac{1}{2}x - \frac{1}{x} = 0$

2. Which of the equations in Exercise 1 are nonlinear equations in x? Explain why.

3. Decide whether 6 is a solution of $3(x + 4) = 5x$ by substituting 6 for x. If it is not a solution, explain why.

4. Use substitution to decide whether -2 is a solution of $5(x + 4) - 3(x + 6) = 9(x + 1)$. If it is not a solution, explain why.

5. The equation $4[x + (2 - 3x)] = 2(4 - 4x)$ is an identity. Let x represent the number of letters in your last name. Is this number a solution of this equation? Check your answer.

6. The expression $.06(10 - x)(100)$ is equivalent to which of the following?

A. $.06 - .06x$ **B.** $60 - 6x$ **C.** $6 - 6x$ **D.** $6 - .06x$

7. Identify each as an *expression* or an *equation.*

(a) $3x = 6$ **(b)** $3x + 6$

(c) $5x + 6(x - 3) = 12x + 6$ **(d)** $5x + 6(x - 3) - (12x + 6)$

8. Explain why $6x + 9 = 6x + 8$ cannot have a solution. (No work is necessary.)

Solve and check each equation. See Examples 1 and 2.

9. $7x + 8 = 1$

10. $5x - 4 = 21$

11. $7x - 5x + 15 = x + 8$

12. $2x + 4 - x = 4x - 5$

13. $12w + 15w - 9 + 5 = -3w + 5 - 9$

14. $-4t + 5t - 8 + 4 = 6t - 4$

15. $2(x + 3) = -4(x + 1)$

16. $4(t - 9) = 8(t + 3)$

17. $3(2w + 1) - 2(w - 2) = 5$

18. $4(x - 2) + 2(x + 3) = 6$

19. $2x + 3(x - 4) = 2(x - 3)$

20. $6x - 3(5x + 2) = 4(1 - x)$

21. $6p - 4(3 - 2p) = 5(p - 4) - 10$

22. $-2k - 3(4 - 2k) = 2(k - 3) + 2$

23. $-[2z - (5z + 2)] = 2 + (2z + 7)$

24. $-[6x - (4x + 8)] = 9 + (6x + 3)$

25. $-3m + 6 - 5(m - 1) = 4m - (2m - 4) - 9m + 5$

26. $4(k + 2) - 8k - 5 = -3k + 9 - 2(k + 6)$

27. $-[3x - (2x + 5)] = -4 - [3(2x - 4) - 3x]$

28. $2[-(x - 1) + 4] = 5 + [-(6x - 7) + 9x]$

29. $-(9 - 3a) - (4 + 2a) - 3 = -(2 - 5a) + (-a) + 1$

30. $-(-2 + 4x) - (3 - 4x) + 5 = -(-3 + 6x) + x + 1$

31. To solve the linear equation

$$\frac{8x}{3} - \frac{2x}{4} = -13,$$

we are allowed to multiply each side by the least common denominator of all the fractions in the equation. What is this least common denominator?

32. To solve the linear equation

$$.05x + .12(x + 5000) = 940,$$

we multiply each side by a power of 10 so that all coefficients are integers. What is the smallest power of 10 that will accomplish this goal?

33. Suppose that in solving the equation

$$\frac{1}{3}x + \frac{1}{2}x = \frac{1}{6}x,$$

you begin by multiplying each side by 12, rather than the *least* common denominator, 6. Would you get the correct solution anyway? Explain.

34. What is the final line in the check for the solution of the equation in Example 4?

Solve each equation. See Examples 3 and 4.

35. $-\dfrac{5}{9}k = 2$

36. $\dfrac{3}{11}z = -5$

37. $\dfrac{6}{5}x = -1$

38. $-\dfrac{7}{8}r = 6$

39. $\dfrac{m}{2} + \dfrac{m}{3} = 5$

40. $\dfrac{x}{5} - \dfrac{x}{4} = 1$

41. $\dfrac{x - 10}{5} + \dfrac{2}{5} = -\dfrac{x}{3}$

42. $\dfrac{2r - 3}{7} + \dfrac{3}{7} = -\dfrac{r}{3}$

43. $\dfrac{4t + 1}{3} = \dfrac{t + 5}{6} + \dfrac{t - 3}{6}$

44. $\dfrac{2x + 5}{5} = \dfrac{3x + 1}{2} + \dfrac{-x + 7}{2}$

45. $.05x + .12(x + 5000) = 940$

46. $.09k + .13(k + 300) = 61$

47. $.02(50) + .08r = .04(50 + r)$

48. $.20(14{,}000) + .14t = .18(14{,}000 + t)$

49. $.05x + .10(200 - x) = .45x$

50. $.08x + .12(260 - x) = .48x$

51. A student tried to solve the equation $8x = 7x$ by dividing each side by x, obtaining $8 = 7$. He gave the solution set as $\emptyset$. Why is this incorrect?

52. Explain the distinction between a conditional equation, an identity, and a contradiction.

Decide whether each equation is conditional, *an* identity, *or a* contradiction. *Give the solution set. See Example 5.*

53. $-2p + 5p - 9 = 3(p - 4) - 5$

54. $-6k + 2k - 11 = -2(2k - 3) + 4$

55. $6x + 2(x - 2) = 9x + 4$

56. $-4(x + 2) = -3(x + 5) - x$

57. $-11m + 4(m - 3) + 6m = 4m - 12$

58. $3p - 5(p + 4) + 9 = -11 + 15p$

59. $7[2 - (3 + 4r)] - 2r = -9 + 2(1 - 15r)$

60. $4[6 - (1 + 2m)] + 10m = 2(10 - 3m) + 8m$

Decide whether each pair of equations is equivalent. If not equivalent, explain why.

61. $5x = 10$ and $\dfrac{5x}{x + 2} = \dfrac{10}{x + 2}$

62. $x + 1 = 9$ and $\dfrac{x + 1}{8} = \dfrac{9}{8}$

63. $x = -3$ and $\dfrac{x}{x + 3} = \dfrac{-3}{x + 3}$

64. $m = 1$ and $\dfrac{m + 1}{m - 1} = \dfrac{2}{m - 1}$

65. $k = 4$ and $k^2 = 16$

66. $p^2 = 36$ and $p = 6$

2.2 Formulas

OBJECTIVES

1. Solve a formula for a specified variable.
2. Solve applied problems using formulas.
3. Solve percent problems.

Photo not available

EXPLORE

A **mathematical model** is an equation or inequality that describes a real situation. Models for many applied problems already exist; they are called *formulas*. A **formula** is a mathematical equation in which variables are used to describe a relationship. Some formulas that we will be using are

$$d = rt, \qquad I = prt, \qquad \text{and} \qquad P = 2L + 2W.$$

A list of some common formulas used in algebra is given inside the covers of the book.

OBJECTIVE 1 Solve a formula for a specified variable. In some applications, the appropriate formula may be solved for a different variable than the one to be found. For example, the formula $I = prt$ says that interest on a loan or investment equals principal (amount borrowed or invested) times rate (annually in percent) times time at interest (in years). To determine how long it will take for an investment at a stated interest rate to earn a predetermined amount of interest, it would help to first solve the formula for t. This process is called **solving for a specified variable.**

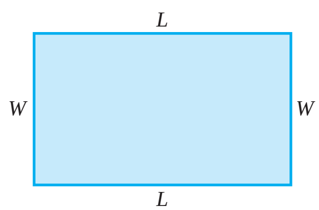
The steps used in the following examples are very similar to those used in solving linear equations. When you are solving for a specified variable, the key is to treat that variable as if it were the only one; treat all other variables like numbers (constants). The following additional suggestions may be helpful.

Solving for a Specified Variable

Step 1 Get all terms containing the specified variable on one side of the equation and all terms without that variable on the other side.

Step 2 If necessary, use the distributive property to combine the terms with the specified variable.* The result should be the product of a sum or difference and the variable.

Step 3 Divide both sides by the factor that is the coefficient of the specified variable.



Perimeter, P, distance around a rectangle, is given by
$P = 2L + 2W$.

FIGURE 1

EXAMPLE 1 Solving for a Specified Variable

Solve the formula $P = 2L + 2W$ for W.

This formula gives the relationship between the perimeter (distance around) a rectangle, P, the length of the rectangle, L, and the width of the rectangle, W. See Figure 1.

Solve the formula for W by getting W alone on one side of the equals sign. To begin, subtract $2L$ from both sides.

$$P = 2L + 2W$$

Step 1 $$P - 2L = 2L + 2W - 2L \quad \text{Subtract } 2L.$$

$$P - 2L = 2W$$

Step 2 is not needed here.

Step 3 $$\frac{P - 2L}{2} = \frac{2W}{2} \quad \text{Divide both sides by 2.}$$

$$\frac{P - 2L}{2} = W \quad \text{or} \quad W = \frac{P - 2L}{2}$$

Now Try Exercise 9.

CAUTION In Step 3 of Example 1, you cannot simplify the fraction by dividing 2 into the term $2L$. The subtraction in the numerator must be done before the division.

$$\frac{P - 2L}{2} \neq P - L$$

EXAMPLE 2 Solving a Formula with Parentheses

The formula for the perimeter of a rectangle is sometimes written in the equivalent form $P = 2(L + W)$. Solve this form for W.

*Using the distributive property to write $ab + ac$ as $a(b + c)$ is called *factoring.* See Chapter 6.

One way to begin is to use the distributive property on the right side of the equation to get $P = 2L + 2W$, which we would then solve as in Example 1. Another way to begin is to divide by the coefficient 2.

$$P = 2(L + W)$$

$$\frac{P}{2} = L + W \qquad \text{Divide by 2.}$$

$$\frac{P}{2} - L = W \quad \text{or} \quad W = \frac{P}{2} - L \qquad \text{Subtract } L.$$

We can show that this result is equivalent to our result in Example 1 by multiplying L by $\frac{2}{2}$.

$$\frac{P}{2} - \frac{2}{2}(L) = W \qquad \frac{2}{2} = 1, \text{ so } L = \frac{2}{2}(L).$$

$$\frac{P}{2} - \frac{2L}{2} = W$$

$$\frac{P - 2L}{2} = W \qquad \text{Subtract fractions.}$$

The final line agrees with the result in Example 1.

Now Try Exercise 15.

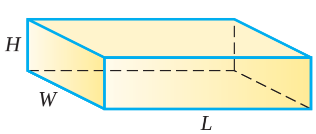


FIGURE 2

A rectangular solid has the shape of a box, but is solid. See Figure 2. The labels H, W, and L represent the height, width, and length of the figure, respectively. The surface area of any solid three-dimensional figure is the total area of its surface. For a rectangular solid, the surface area A is

$$A = 2HW + 2LW + 2LH.$$

YOU TRY IT

EXAMPLE 3 Using the Distributive Property to Solve for a Specified Variable

Given the surface area, height, and width of a rectangular solid, write a formula for the length.

To solve for the length L, treat L as the only variable and treat all other variables as constants.

$$A = 2HW + 2LW + 2LH$$

$$A - 2HW = 2LW + 2LH \qquad \text{Subtract } 2HW.$$

$$A - 2HW = L(2W + 2H) \qquad \text{Use the distributive property on the right side.}$$

$$\frac{A - 2HW}{2W + 2H} = L \qquad \text{Divide by } 2W + 2H.$$

or

$$L = \frac{A - 2HW}{2W + 2H}$$

Now Try Exercise 21.

CAUTION The most common error in working a problem like Example 3 is not using the distributive property correctly. We must write the expression so that the specified variable is a *factor;* then we can divide by its coefficient in the final step.

OBJECTIVE 2 Solve applied problems using formulas. The next example uses the distance formula, $d = rt$, which relates d, the distance traveled, r, the rate or speed, and t, the travel time.

EXAMPLE 4 Finding Average Speed

Janet Branson found that usually it took her $\frac{3}{4}$ hr each day to drive a distance of 15 mi to work. What was her speed?

Find the formula for speed (rate) r by solving $d = rt$ for r.

$$d = rt$$

$$\frac{d}{t} = \frac{rt}{t} \qquad \text{Divide by } t.$$

$$\frac{d}{t} = r \quad \text{or} \quad r = \frac{d}{t}$$

Notice that only Step 3 was needed to solve for r in this example. Now find Janet's speed by substituting the given values of d and t into this formula.

$$r = \frac{d}{t}$$

$$r = \frac{15}{\frac{3}{4}} \qquad \text{Let } d = 15,\ t = \tfrac{3}{4}.$$

$$r = 15 \cdot \frac{4}{3} \qquad \text{Multiply by the reciprocal of } \tfrac{3}{4}.$$

$$r = 20$$

Her speed averaged 20 mph. (That is, at times she may have traveled a little faster or a little slower than 20 mph, but overall her speed was 20 mph.)

Now Try Exercise 25.

PROBLEM SOLVING

As seen in Example 4, it may be convenient to first solve for a specific unknown variable before substituting the given values. This is particularly useful when we wish to substitute several different values for the same variable. For example, an economics class might need to solve the equation $I = prt$ for r to find rates that produce specific amounts of interest for various principals and times.

OBJECTIVE 3 Solve percent problems. An important everyday use of mathematics involves the concept of percent. Percent is written with the symbol %. The word

percent means "per one hundred." One percent means "one per one hundred" or "one one-hundredth."

$$1\% = .01 \qquad \text{or} \qquad 1\% = \frac{1}{100}$$

Solving a Percent Problem

Let a represent a partial amount of b, the base, or whole amount. Then the following formula can be used to solve a percent problem.

$$\frac{\textbf{amount}}{\textbf{base}} = \frac{a}{b} = \textbf{percent (represented as a decimal)}$$

For example, if a class consists of 50 students and 32 are males, then the percent of males in the class is

$$\begin{aligned} \frac{\text{amount}}{\text{base}} &= \frac{a}{b} \\ &= \frac{32}{50} && \text{Let } a = 32,\ b = 50. \\ &= .64 \quad \text{or} \quad 64\%. \end{aligned}$$

EXAMPLE 5 Solving Percent Problems

(a) A 50-L mixture of acid and water contains 10 L of acid. What is the percent of acid in the mixture?

The given amount of the mixture is 50 L, and the part that is acid (percentage) is 10 L. Let x represent the percent of acid. Then, the percent of acid in the mixture is

$$\begin{aligned} x &= \frac{10}{50} \\ x &= .20 \quad \text{or} \quad 20\%. \end{aligned}$$

(b) If a savings account balance of \$3550 earns 8% interest in one year, how much interest is earned?

Let x represent the amount of interest earned (that is, the part of the whole amount invested). Since $8\% = .08$, the equation is

$$\begin{aligned} \frac{x}{3550} &= .08 && \tfrac{a}{b} = \text{percent} \\ x &= .08(3550) && \text{Multiply by 3550.} \\ x &= 284. \end{aligned}$$

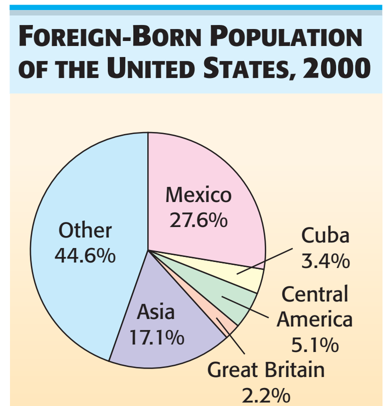
The interest earned is \$284.

Now Try Exercises 37 and 39.

Graphs sometimes represent the percents of a whole amount that satisfy certain conditions.

EXAMPLE 6 Interpreting Percents from a Graph

The country of origin of immigrants to the United States is shifting from mainly European in the 19th century to Hispanic (from Mexico and Central America) and Asian. In 2000, the foreign-born population of the United States from all countries was 28,379,000 to the nearest thousand. Use the graph in Figure 3 to determine how many came from Asia.



Source: U.S. Bureau of the Census.

FIGURE 3

According to the graph, 17.1% of the foreign-born population came from Asia. Let x represent the required number of immigrants from Asia.

$$\frac{x}{28{,}379{,}000} = .171$$
$$x = .171(28{,}379{,}000)$$
$$x = 4{,}852{,}809$$

Therefore, 4,853,000 to the nearest thousand immigrants came from Asia in 2000.

Now Try Exercise 47.

2.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 2/Videotape 2

RELATING CONCEPTS (EXERCISES 1–6)

For Individual or Group Work

Consider the following equations:

First Equation	**Second Equation**
$x = \dfrac{5x + 8}{3}$	$t = \dfrac{bt + k}{c} \quad (c \neq 0).$

Solving the second equation for t requires the same logic as solving the first equation for x. When solving for t, we treat all other variables as though they were constants. ***Work Exercises 1–6 in order,*** *to see the "parallel logic" of solving for x and solving for t.*

1. (a) Clear the first equation of fractions by multiplying each side by 3.

(b) Clear the second equation of fractions by multiplying each side by c.

2. (a) Get the terms involving x on the left side of the first equation by subtracting $5x$ from each side.
   (b) Get the terms involving t on the left side of the second equation by subtracting bt from each side.

3. (a) Combine like terms on the left side of the first equation. What property allows us to write $3x - 5x$ as $(3 - 5)x = -2x$?
   (b) Write the expression on the left side of the second equation so that t is a factor. What property allows us to do this?

4. (a) Divide each side of the first equation by the coefficient of x.
   (b) Divide each side of the second equation by the coefficient of t.

5. Look at your answer for the second equation. What restriction must be placed on the variables? Why is this necessary?

6. Write a short paragraph summarizing what you have learned in this group of exercises.

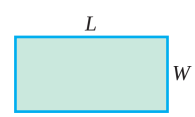
Solve each formula for the specified variable. See Examples 1 and 2.

EXERCISES

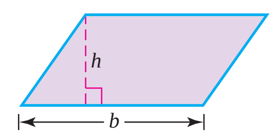
7. $I = prt$ for r (simple interest)

8. $d = rt$ for t (distance)

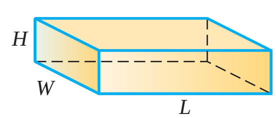
9. $P = 2L + 2W$ for L
(perimeter of a rectangle)



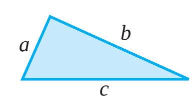
10. $A = bh$ for b
(area of a parallelogram)



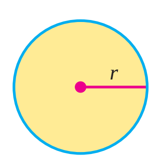
11. $V = LWH$ for W
(volume of a rectangular solid)



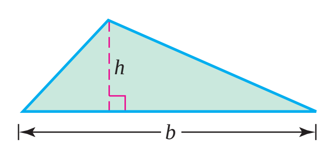
12. $P = a + b + c$ for b
(perimeter of a triangle)



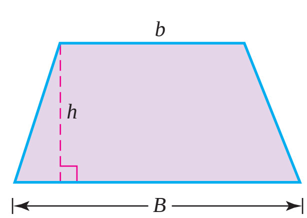
13. $C = 2\pi r$ for r
(circumference of a circle)



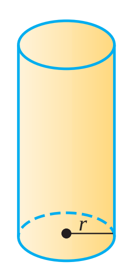
14. $A = \frac{1}{2}bh$ for h (area of a triangle)



15. $A = \frac{1}{2}h(B + b)$ for B
(area of a trapezoid)



16. $S = 2\pi rh + 2\pi r^2$ for h
(surface area of a right circular cylinder)



17. $F = \frac{9}{5}C + 32$ for C
(Celsius to Fahrenheit)

18. $C = \frac{5}{9}(F - 32)$ for F
(Fahrenheit to Celsius)

19. When a formula is solved for a particular variable, several different equivalent forms may be possible. If we solve $A = \frac{1}{2}bh$ for h, one possible correct answer is

$$h = \frac{2A}{b}.$$

Which one of the following is *not* equivalent to this?

A. $h = 2\left(\frac{A}{b}\right)$ **B.** $h = 2A\left(\frac{1}{b}\right)$ **C.** $h = \frac{A}{\frac{1}{2}b}$ **D.** $h = \frac{\frac{1}{2}A}{b}$

20. Suppose the formula

$$A = 2HW + 2LW + 2LH$$

is solved for L as follows.

$$A = 2HW + 2LW + 2LH$$
$$A - 2LW - 2HW = 2LH$$
$$\frac{A - 2LW - 2HW}{2H} = L$$

While there are no algebraic errors here, what is wrong with the final equation, if we are interested in solving for L?

Solve each equation for the specified variable. Use the distributive property to factor as necessary. See Example 3.

21. $2k + ar = r - 3y$ for r

22. $4s + 7p = tp - 7$ for p

23. $w = \frac{3y - x}{y}$ for y

24. $c = \frac{-2t + 4}{t}$ for t

Solve each problem. See Example 4.

25. In 1998 Jeff Gordon won the World 600 (mile) race with a speed of 136.424 mph. Find his time to the nearest thousandth. (*Source: Sports Illustrated 1999 Sports Almanac.*)

photo not available

26. In 1975, rain shortened the Indianapolis 500 race to 435 mi. It was won by Bobby Unser, who averaged 149.213 mph. What was his time to the nearest thousandth? (*Source: Sports Illustrated 1998 Sports Almanac.*)

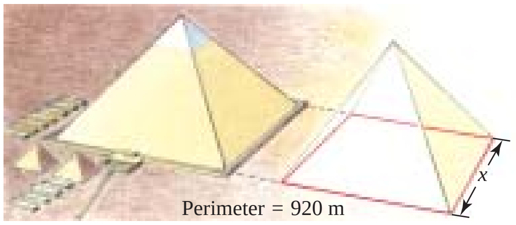
27. Faye Korn traveled from Kansas City to Louisville, a distance of 520 mi, in 10 hr. Find her rate in miles per hour.

28. The distance from Melbourne to London is 10,500 mi. If a jet averages 500 mph between the two cities, what is its travel time in hours?

29. As of 2001, the highest temperature ever recorded in Chicago was 40°C. Find the corresponding Fahrenheit temperature. (*Source: World Almanac and Book of Facts*, 2002.)

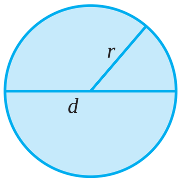
30. The lowest temperature recorded in Salt Lake City in 1997 was 8°F. Find the corresponding Celsius temperature. (*Source: World Almanac and Book of Facts*, 1999.)

31. The base of the Great Pyramid of Cheops is a square whose perimeter is 920 m. What is the length of each side of this square? (*Source: Atlas of Ancient Archaeology.*)

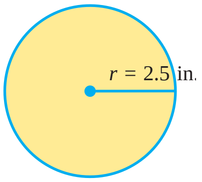


32. The Peachtree Plaza Hotel in Atlanta is in the shape of a cylinder with radius 46 m and height 220 m. Find its volume to the nearest tenth. (*Hint:* Use the π key on your calculator.)

33. The circumference of a circle is 480π in. What is its radius? What is its diameter?



34. The radius of a circle is 2.5 in. What is the diameter of the circle? What is its circumference?



35. A cord of wood contains 128 ft^3 (cubic feet) of wood. If a stack of wood is 4 ft wide and 4 ft high, how long must it be if it contains exactly 1 cord?

36. Give one set of possible dimensions for a stack of wood that contains 1.5 cords. (See Exercise 35.)

Solve each problem. See Example 5.

EXERCISES

37. A mixture of alcohol and water contains a total of 36 oz of liquid. There are 9 oz of pure alcohol in the mixture. What percent of the mixture is water? What percent is alcohol?

38. A mixture of acid and water is 35% acid. If the mixture contains a total of 40 L, how many liters of pure acid are in the mixture? How many liters of pure water are in the mixture?

39. A real estate agent earned \$6300 commission on a property sale of \$210,000. What is her rate of commission?

40. A certificate of deposit for 1 yr pays \$221 simple interest on a principal of \$3400. What is the interest rate being paid on this deposit?

When a consumer loan is paid off ahead of schedule, the finance charge is smaller than if the loan were paid off over its scheduled life. By one method, called the rule of 78, *the amount of* unearned interest *(finance charge that need not be paid) is given by*

$$u = f \cdot \frac{k(k + 1)}{n(n + 1)},$$

where u is the amount of unearned interest (money saved) when a loan scheduled to run n payments is paid off k payments ahead of schedule. The total scheduled finance charge is f. Use this formula to solve Exercises 41–44.

41. Rhonda Alessi bought a new Ford and agreed to pay it off in 36 monthly payments. The total finance charge was $700. Find the unearned interest if she paid the loan off 4 payments ahead of schedule.

42. Charles Vosburg bought a car and agreed to pay it off in 36 monthly payments. The total finance charge on the loan was $600. With 12 payments remaining, Charles decided to pay the loan in full. Find the amount of unearned interest.

43. The finance charge on a loan taken out by Vic Denicola is $380.50. If there were 24 equal monthly installments needed to repay the loan, and the loan is paid in full with 8 months remaining, find the amount of unearned interest.

44. Adrian Ortega is scheduled to repay a loan in 24 equal monthly installments. The total finance charge on the loan is $450. With 9 payments remaining, he decides to repay the loan in full. Find the amount of unearned interest.

Exercises 45 and 46 deal with winning percentage in the standings of sports teams. Winning percentage (Pct.) is commonly expressed as a decimal rounded to the nearest thousandth. To find the winning percentage of a team, divide the number of wins (W) *by the total number of games played* (W + L).

45. At the start of play on September 15, 2000, the standings of the Central Division of the American League were as shown. Find the winning percentage of each team.

(a) Chicago **(b)** Cleveland
(c) Detroit

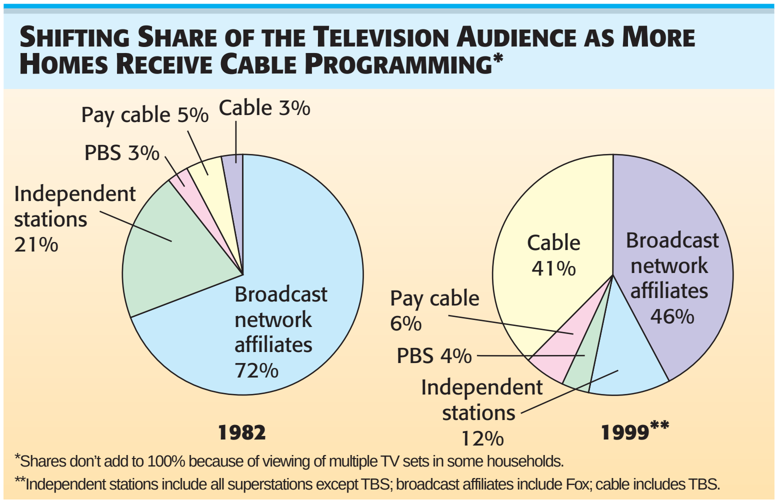
	W	L	Pct.
Chicago	87	58	
Cleveland	77	65	
Detroit	71	74	
Kansas City	68	78	.466
Minnesota	63	82	.434

46. Repeat Exercise 45 for the following standings for the Eastern Division of the National League.

(a) Atlanta **(b)** New York
(c) Florida

	W	L	Pct.
Atlanta	86	60	
New York	84	62	
Florida	69	76	
Montreal	61	84	.421
Philadelphia	60	85	.414

Television networks have been losing viewers to cable programming since 1982, as the two graphs show. Use these graphs to answer Exercises 47–50. See Example 6.



Source: Nielsen Media Research, National Cable Television Association Report, Spring, 2000.

47. In a typical group of 50,000 television viewers, how many would have watched cable in 1982?

48. In 1982, how many of a typical group of 110,000 viewers watched independent stations?

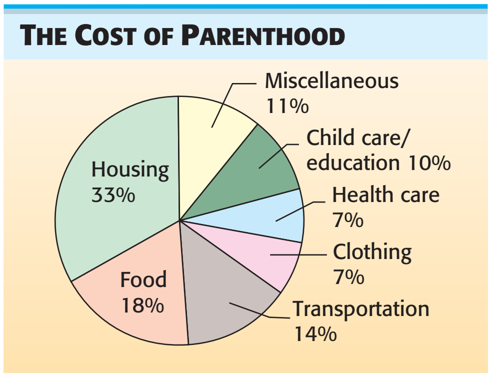
49. How many of a typical group of 35,000 viewers watched cable in 1999?

50. In a typical group of 65,000 viewers, how many watched independent stations in 1999?

An average middle-income family will spend $160,140 *to raise a child born in 1999 from birth to age 17. The graph shows the percent spent for various categories. Use the graph to answer Exercises 51 and 52. See Example 6.*

51. To the nearest dollar, how much will be spent to provide housing for the child?

52. **(a)** To the nearest dollar, how much will be spent for health care?
(b) Use your answer from part (a) to decide how much will be spent for transportation.



Source: U.S. Department of Agriculture.

2.3 Applications of Linear Equations

OBJECTIVES

1. Translate from words to mathematical expressions.
2. Write equations from given information.
3. Distinguish between expressions and equations.
4. Use the six steps in solving an applied problem.
5. Solve further percent problems.
6. Solve investment problems.
7. Solve mixture problems.

OBJECTIVE 1 Translate from words to mathematical expressions. Producing a mathematical model of a real situation often involves translating verbal statements into mathematical statements.

PROBLEM SOLVING

Usually there are key words and phrases in a verbal problem that translate into mathematical expressions involving addition, subtraction, multiplication, and division. Translations of some commonly used expressions follow.

Translating from Words to Mathematical Expressions

Verbal Expression	*Mathematical Expression (where x and y are numbers)*
Addition	
The **sum** of a number and 7	$x + 7$
6 **more than** a number	$x + 6$
3 **plus** a number	$3 + x$
24 **added to** a number	$x + 24$
A number **increased by** 5	$x + 5$
The **sum** of two numbers	$x + y$
Subtraction	
2 **less than** a number	$x - 2$
12 **minus** a number	$12 - x$
A number **decreased by** 12	$x - 12$
A number **subtracted from** 10	$10 - x$
The **difference between** two numbers	$x - y$
Multiplication	
16 **times** a number	$16x$
A number **multiplied by** 6	$6x$
$\frac{2}{3}$ **of** a number (used with fractions and percent)	$\frac{2}{3}x$
Twice (2 times) a number	$2x$
The **product** of two numbers	xy
Division	
The **quotient** of 8 and a number	$\frac{8}{x}$ $(x \neq 0)$
A number **divided by** 13	$\frac{x}{13}$
The **ratio** of two numbers or the **quotient** of two numbers	$\frac{x}{y}$ $(y \neq 0)$

CAUTION Because subtraction and division are not commutative operations, it is important to correctly translate expressions involving them. For example, "2 less than a number" is translated as $x - 2$, *not* $2 - x$. "A number subtracted from 10" is expressed as $10 - x$, *not* $x - 10$.

For division, it is understood that the number by which we are dividing is the denominator, and the number that is divided is the numerator. For example, "a number divided by 13" and "13 divided into x" both translate as $\frac{x}{13}$. Similarly, "the quotient of x and y" is translated as $\frac{x}{y}$.

OBJECTIVE 2 Write equations from given information. The symbol for equality, $=$, is often indicated by the word *is*. In fact, because equal mathematical expressions represent names for the same number, any words that indicate the idea of "sameness" translate to $=$.

EXAMPLE 1 Translating Words into Equations

Translate each verbal sentence into an equation.

Verbal Sentence	*Equation*
Twice a number, decreased by 3, is 42.	$2x - 3 = 42$
The product of a number and 12, decreased by 7, is 105.	$12x - 7 = 105$
The quotient of a number and the number plus 4 is 28.	$\frac{x}{x + 4} = 28$
The quotient of a number and 4, plus the number, is 10.	$\frac{x}{4} + x = 10$

Now Try Exercises 7 and 17.

OBJECTIVE 3 Distinguish between expressions and equations. To distinguish between algebraic expressions and equations, remember that an expression translates as a phrase. An equation includes the $=$ symbol and translates as a sentence.

EXAMPLE 2 Distinguishing between Expressions and Equations

Decide whether each is an *expression* or an *equation.*

(a) $2(3 + x) - 4x + 7$

There is no equals sign, so this is an expression.

(b) $2(3 + x) - 4x + 7 = -1$

Because of the equals sign, this is an equation. Note that the expression in part (a) simplifies to the expression $-2x + 13$, and the equation in part (b) has solution 7.

Now Try Exercises 21 and 23.

OBJECTIVE 4 Use the six steps in solving an applied problem. Throughout this book we will be solving different types of applications. While there is no one method that will allow us to solve all types of applied problems, the following six steps are helpful.

Solving an Applied Problem

Step 1 **Read** the problem carefully until you understand what is given and what is to be found.

Step 2 **Assign a variable** to represent the unknown value, using diagrams or tables as needed. Write down what the variable represents. If necessary, express any other unknown values in terms of the variable.

(continued)

Step 3 **Write an equation** using the variable expression(s).
Step 4 **Solve** the equation.
Step 5 **State the answer** to the problem. Does it seem reasonable?
Step 6 **Check** the answer in the words of the original problem.

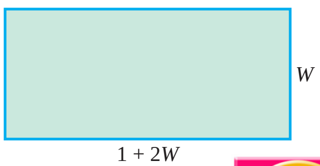


FIGURE 4

VIDEO

EXAMPLE 3 Solving a Geometry Problem

The length of a rectangle is 1 cm more than twice the width. The perimeter of the rectangle is 110 cm. Find the length and the width of the rectangle.

Step 1 **Read** the problem. What must be found? The length and width of the rectangle. What is given? The length is 1 cm more than twice the width; the perimeter is 110 cm.

Step 2 **Assign a variable.** Make a sketch, as in Figure 4. Choose a variable: let $W =$ the width; then $1 + 2W =$ the length.

Step 3 **Write an equation.** The perimeter of a rectangle is given by the formula $P = 2L + 2W$.

$$P = 2L + 2W$$
$$110 = 2(1 + 2W) + 2W \qquad \text{Let } L = 1 + 2W \text{ and } P = 110.$$

Step 4 **Solve** the equation obtained in Step 3.

$$110 = 2(1 + 2W) + 2W$$
$$110 = 2 + 4W + 2W \qquad \text{Distributive property}$$
$$110 = 2 + 6W \qquad \text{Combine like terms.}$$
$$110 - 2 = 2 + 6W - 2 \qquad \text{Subtract 2.}$$
$$108 = 6W$$
$$\frac{108}{6} = \frac{6W}{6} \qquad \text{Divide by 6.}$$
$$18 = W$$

Step 5 **State the answer.** The width of the rectangle is 18 cm and the length is $1 + 2(18) = 37$ cm.

Step 6 **Check** the answer by substituting these dimensions into the words of the original problem.

Now Try Exercise 29.

YOU TRY IT

EXAMPLE 4 Finding Unknown Numerical Quantities

Two outstanding major league pitchers in recent years are Roger Clemens and Greg Maddux. Between 1984 and 1999, they pitched a total of 916 games. Clemens pitched 44 more games than Maddux. How many games did each player pitch? (*Source: Who's Who in Baseball*, 2000.)

Photo not available

Step 1 **Read** the problem. We are asked to find the number of games each player pitched.

Step 2 **Assign a variable** to represent the number of games of one of the men.

Let $m =$ the number of games for Maddux.

We must also find the number of games for Clemens. Since he pitched 44 more games than Maddux,

$$m + 44 = \text{Clemens' number of games.}$$

Step 3 **Write an equation.** The sum of the numbers of games is 916, so

$$\underset{m}{\text{Maddux's games}} + \underset{(m + 44)}{\text{Clemens' games}} = \underset{916.}{\text{Total}}$$

Step 4 **Solve** the equation.

$$\begin{aligned} m + (m + 44) &= 916 & \\ 2m + 44 &= 916 & \text{Combine like terms.} \\ 2m &= 872 & \text{Subtract 44.} \\ m &= 436 & \text{Divide by 2.} \end{aligned}$$

Step 5 **State the answer.** Since m represents the number of Maddux's games, Maddux pitched 436 games. Also, $m + 44 = 436 + 44 = 480$ is the number of games pitched by Clemens.

Step 6 **Check.** 480 is 44 more than 436, and the sum of 436 and 480 is 916. The conditions of the problem are satisfied, and our solution checks.

Now Try Exercise 33.

CAUTION A common error in solving applied problems is forgetting to answer all the questions asked in the problem. In Example 4, we were asked for the number of games for *each* player, so there was extra work in Step 5 in order to find Clemens' number.

OBJECTIVE 5 Solve further percent problems. Recall from the previous section that percent means "per one hundred," so 5% means .05, 14% means .14, and so on.

EXAMPLE 5 Solving a Percent Problem

In 2002 there were 301 long-distance area codes in the United States. This was an increase of 250% over the number when the area code plan originated in 1947. How many area codes were there in 1947? (*Source:* SBC Telephone Directory.)

Step 1 **Read** the problem. We are given that the number of area codes increased by 250% from 1947 to 2002, and there were 301 area codes in 2002. We must find the original number of area codes.

Step 2 **Assign a variable.** Let x represent the number of area codes in 1947. Since $250\% = 250(.01) = 2.5$, $2.5x$ represents the number of codes added since then.

Step 3 **Write an equation** from the given information.

$$\underset{x}{\text{the number in 1947}} + \underset{2.5x}{\text{the increase}} = \underset{301}{301}$$

Step 4 **Solve** the equation.

$$1x + 2.5x = 301 \quad \text{Identity property}$$
$$3.5x = 301 \quad \text{Combine like terms.}$$
$$x = 86 \quad \text{Divide by 3.5.}$$

Step 5 **State the answer.** There were 86 area codes in 1947.

Step 6 **Check** that the increase, $301 - 86 = 215$, is 250% of 86.

Now Try Exercise 39.

CAUTION Watch for two common errors that occur in solving problems like the one in Example 5.

1. Do not try to find 250% of 301 and subtract that amount from 301. The 250% should be applied to *the amount in 1947, not the amount in 2002.*
2. Do not write the equation as

$$x + 2.5 = 301.$$

The percent must be multiplied by some amount; in this case, the amount is the number of area codes in 1947, giving $2.5x$.

OBJECTIVE 6 Solve investment problems. We use linear equations to solve certain types of investment problems. The investment problems in this chapter deal with *simple interest.* In most real-world applications, *compound interest* (covered in a later chapter) is used.

EXAMPLE 6 Solving an Investment Problem

After winning the state lottery, Mark LeBeau has \$40,000 to invest. He will put part of the money in an account paying 4% interest and the remainder into stocks paying 6% interest. His accountant tells him that the total annual income from these investments should be \$2040. How much should he invest at each rate?

Step 1 **Read** the problem again.

Step 2 **Assign a variable.**

Let $x =$ the amount to invest at 4%;

then $40{,}000 - x =$ the amount to invest at 6%.

The formula for interest is $I = prt$. Here the time, t, is 1 yr. Use a table to organize the given information.

Principal	*Rate (as a decimal)*	*Interest*	
x	.04	$.04x$	
$40{,}000 - x$	.06	$.06(40{,}000 - x)$	
40,000		2040	← **Total**

Step 3 **Write an equation.** The last column of the table gives the equation.

$$\underset{.04x}{\text{interest at 4\%}} + \underset{.06(40{,}000 - x)}{\text{interest at 6\%}} = \underset{2040}{\text{total interest}}$$

Step 4 **Solve** the equation. We do so without clearing decimals.

$$\begin{aligned} .04x + .06(40{,}000) - .06x &= 2040 && \text{Distributive property} \\ -.02x + 2400 &= 2040 && \text{Combine like terms; multiply.} \\ -.02x &= -360 && \text{Subtract 2400.} \\ x &= 18{,}000 && \text{Divide by } -.02. \end{aligned}$$

Step 5 **State the answer.** Mark should invest \$18,000 of the money at 4% and \$40,000 − \$18,000 = \$22,000 at 6%.

Step 6 **Check** by finding the annual interest at each rate; they should total \$2040.

Now Try Exercise 43.

NOTE In Example 6, we chose to let the variable represent the amount invested at 4%. Students often ask, "Can I let the variable represent the other unknown?" The answer is yes. The equation will be different, but in the end the answers will be the same.

OBJECTIVE 7 Solve mixture problems. Mixture problems involving rates of concentration can be solved with linear equations.

EXAMPLE 7 Solving a Mixture Problem

A chemist must mix 8 L of a 40% acid solution with some 70% solution to get a 50% solution. How much of the 70% solution should be used?

Step 1 **Read** the problem. The problem asks for the amount of 70% solution to be used.

Step 2 **Assign a variable.** Let x = the number of liters of 70% solution to be used. The information in the problem is illustrated in Figure 5.

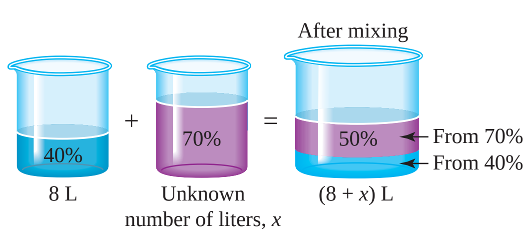


FIGURE 5

Use the given information to complete a table as shown on the next page.

Number of Liters	Percent (as a decimal)	Liters of Pure Acid
8	.40	$.40(8) = 3.2$
x	.70	$.70x$
$8 + x$	.50	$.50(8 + x)$

Sum must equal

The numbers in the last column were found by multiplying the strengths and the numbers of liters. The number of liters of pure acid in the 40% solution plus the number of liters in the 70% solution must equal the number of liters in the 50% solution.

Step 3 **Write an equation.**

$$3.2 + .70x = .50(8 + x)$$

Step 4 **Solve.**

$$
\begin{aligned}
3.2 + .70x &= 4 + .50x && \text{Distributive property} \\
.20x &= .8 && \text{Subtract 3.2 and } .50x. \\
x &= 4 && \text{Divide by .20.}
\end{aligned}
$$

Step 5 **State the answer.** The chemist should use 4 L of the 70% solution.

Step 6 **Check.** 8 L of 40% solution plus 4 L of 70% solution is $8(.40) + 4(.70) =$ 6 L of acid. Similarly, $8 + 4$ or 12 L of 50% solution has $12(.50) = 6$ L of acid in the mixture. The total amount of pure acid is 6 L both before and after mixing, so the answer checks.

Now Try Exercise 51.

In some mixture problems, you will need to use the fact that the percent of acid in pure water is 0%. Similarly, pure acid is 100% acid.

YOU TRY IT

EXAMPLE 8 Solving a Mixture Problem When One Ingredient Is Pure

The octane rating of gasoline is a measure of its antiknock qualities. For a standard fuel, the octane rating is the percent of isooctane. How many liters of pure isooctane should be mixed with 200 L of 94% isooctane, referred to as 94 octane, to get a mixture that is 98% isooctane?

Step 1 **Read** the problem. The problem asks for the amount of pure isooctane.

Step 2 **Assign a variable.** Let $x =$ the number of liters of pure (100%) isooctane. Fill in a table with the given information. Recall that $100\% = 100(.01) = 1$.

Number of Liters	Percent (as a decimal)	Liters of Pure Isooctane
x	1	x
200	.94	$.94(200)$
$x + 200$	.98	$.98(x + 200)$

Step 3 **Write an equation.** The equation comes from the last column of the table, as in Example 7.

$$x + .94(200) = .98(x + 200)$$

Step 4 **Solve.**

$$x + .94(200) = .98x + .98(200) \quad \text{Distributive property}$$
$$x + 188 = .98x + 196 \quad \text{Multiply.}$$
$$.02x = 8 \quad \text{Subtract } .98x \text{ and } 188.$$
$$x = 400 \quad \text{Divide by } .02.$$

Step 5 **State the answer.** 400 L of isooctane are needed.

Step 6 **Check** by showing that $400 + .94(200) = .98(400 + 200)$.

Now Try Exercise 53.

CONNECTIONS

Probably the most famous study of problem-solving techniques was developed by George Polya (1888–1985). Among his many publications was the modern classic *How to Solve It.* In this book, Polya proposed a four-step process for problem solving.

Polya's Four-Step Process for Problem Solving

1. **Understand the problem.** You must first decide what you are to find.
2. **Devise a plan.** Here are some strategies that may prove useful.

   Problem-Solving Strategies
   If a formula applies, use it.
   Write an equation and solve it.
   Draw a sketch.
   Make a table or a chart.
   Look for a pattern.
   Use trial and error.
   Work backward.

   We used the first of these strategies in the previous section. In this section we used the next three strategies.
3. **Carry out the plan.** This is where the algebraic techniques you are learning in this book can be helpful.
4. **Look back and check.** Is your answer reasonable? Does it answer the question that was asked?

For Discussion or Writing

Compare Polya's four steps with the six steps for problem solving given earlier. Which of our steps correspond with each of Polya's steps?

2.3 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 2/Videotape 2

In each of the following, **(a)** *translate as an expression and* **(b)** *translate as an equation or inequality. Use x to represent the number.*

1. **(a)** 12 more than a number ______
(b) 12 is more than a number. ______

2. **(a)** 3 less than a number ______
(b) 3 is less than a number. ______

3. **(a)** 4 smaller than a number ______
(b) 4 is smaller than a number. ______

4. **(a)** 6 greater than a number ______
(b) 6 is greater than a number. ______

5. Which one of the following is *not* a valid translation of "20% of a number"?

A. $.20x$ **B.** $.2x$

C. $\frac{x}{5}$ **D.** $20x$

6. Explain why $13 - x$ is *not* a correct translation of "13 less than a number."

Translate each verbal phrase into a mathematical expression. Use x to represent the unknown number. See Example 1.

7. Twice a number, decreased by 13

8. The product of 6 and a number, decreased by 12

9. 12 increased by three times a number

10. 12 more than one-half of a number

11. The product of 8 and 12 less than a number

12. The product of 9 more than a number and 6 less than the number

13. The quotient of three times a number and 7

14. The quotient of 6 and five times a nonzero number

Use the variable x for the unknown, and write an equation representing the verbal sentence. Then solve the problem. See Example 1.

15. The sum of a number and 6 is -31. Find the number.

16. The sum of a number and -4 is 12. Find the number.

17. If the product of a number and -4 is subtracted from the number, the result is 9 more than the number. Find the number.

18. If the quotient of a number and 6 is added to twice the number, the result is 8 less than the number. Find the number.

19. When $\frac{2}{3}$ of a number is subtracted from 12, the result is 10. Find the number.

20. When 75% of a number is added to 6, the result is 3 more than the number. Find the number.

Decide whether each is an expression *or an* equation. *See Example 2.*

21. $5(x + 3) - 8(2x - 6)$

22. $-7(z + 4) + 13(z - 6)$

23. $5(x + 3) - 8(2x - 6) = 12$

24. $-7(z + 4) + 13(z - 6) = 18$

25. $\frac{r}{2} - \frac{r + 9}{6} - 8$

26. $\frac{r}{2} - \frac{r + 9}{6} = 8$

27. In your own words, list the six steps suggested for solving an applied problem.

28. In a recent year the two most popular places where book buyers shopped were large chain bookstores and small chain/independent bookstores. In a sample of book buyers, 70 more shopped at large chain bookstores than at small chain/independent bookstores. A total of 442 book buyers shopped at these two types of stores. Complete the problem-solving steps to find how many buyers shopped at each type of bookstore. (*Source:* Book Industry Study Group.)

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Step 1 We are asked to find ______________________________.

Step 2 Let $x =$ the number of book buyers at large chain bookstores.
Then $x - 70 =$ ______________________________.

Step 3 ______ + ______ = 442

Step 4 $x =$ ______

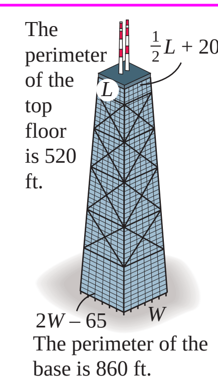
Step 5 There were ______ large chain bookstore shoppers and ______ small chain/independent shoppers.

Step 6 The number of ______ was ______ more than the number of ______ and the total number of these shoppers was ______.

Use the six-step problem-solving method to solve each problem. See Examples 3 and 4.

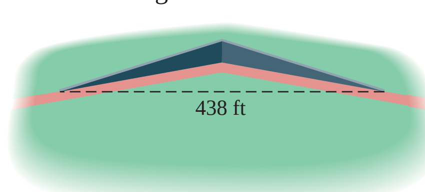
29. The John Hancock Center in Chicago has a rectangular base. The length of the base measures 65 ft less than twice the width. The perimeter of this base is 860 ft. What are the dimensions of the base?

30. The John Hancock Center (Exercise 29) tapers as it rises. The top floor is rectangular and has perimeter 520 ft. The width of the top floor measures 20 ft more than one-half its length. What are the dimensions of the top floor?



31. The Bermuda Triangle supposedly causes trouble for aircraft pilots. It has a perimeter of 3075 mi. The shortest side measures 75 mi less than the middle side, and the longest side measures 375 mi more than the middle side. Find the lengths of the three sides.

32. The Vietnam Veterans Memorial in Washington, D.C., is in the shape of two sides of an isosceles triangle. If the two walls of equal length were joined by a straight line of 438 ft, the perimeter of the resulting triangle would be 931.5 ft. Find the lengths of the two walls. (*Source:* Pamphlet obtained at Vietnam Veterans Memorial.)



33. The two companies with top revenues in the Fortune 500 list for 2002 were Wal-Mart and Exxon Mobil. Their revenues together totaled $412 million. Exxon Mobil revenues were $28 million less than Wal-Mart revenues. What were the revenues for each corporation? (*Source:* www.fortune.com/lists/F500/)

34. In a recent year, video rental revenue was $.27 billion more than twice video sales revenue. Together, these revenues amounted to $9.81 billion. What was the revenue from each of these sources? (*Source:* Paul Kagan Associates, Inc.)

35. In the 1996 presidential election, Bill Clinton and Bob Dole together received 538 electoral votes. Clinton received 220 more votes than Dole. How many votes did each candidate receive? (*Source:* Congressional Quarterly, Inc.)

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36. Ted Williams and Rogers Hornsby were two great hitters. Together they got 5584 hits in their careers. Hornsby got 276 more hits than Williams. How many base hits did each get? (*Source:* Neft, D. S. and R. M. Cohen, *The Sports Encyclopedia: Baseball*, St. Martins Griffin; New York, 1997.)

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Solve each percent problem. See Example 5.

EXERCISES

37. Composite scores on the ACT exam rose from 20.6 in 1990 to 21.0 in 1998. What percent increase was this? (*Source:* The American College Testing Program.)

38. In 1998, the number of participants in the ACT exam was 995,000. Earlier, in 1990, a total of 817,000 took the exam. What percent increase was this? (*Source:* The American College Testing Program.)

39. In 1990, the average tuition for public 4-year universities in the United States was $2035 for full-time students. By 2000, it had risen approximately 85%. To the nearest dollar, what was the approximate cost in 2000? (*Source:* National Center for Education Statistics, U.S. Department of Education.)

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40. In 1990, the average tuition for private 4-year universities in the United States was $10,348 for full-time students. By 2000, it had risen approximately 86.6%. To the nearest dollar, what was the approximate cost in 2000? (*Source:* National Center for Education Statistics, U.S. Department of Education.)

41. At the end of a day, Jeff Hornsby found that the total cash register receipts at the motel where he works amounted to $2725. This included the 9% sales tax charged. Find the amount of the tax.

42. Fino Roverato sold his house for $159,000. He got this amount knowing that he would have to pay a 6% commission to his agent. What amount did he have after the agent was paid?

Solve each investment problem. See Example 6.

43. Carter Fenton earned \$12,000 last year by giving tennis lessons. He invested part at 3% simple interest and the rest at 4%. In one year, he earned a total of \$440 in interest. How much did he invest at each rate?

Principal	Rate (as a decimal)	Interest
x	.03	
	.04	

44. Melissa Wright won \$60,000 on a slot machine in Las Vegas. She invested part at 2% simple interest and the rest at 3%. In one year, she earned a total of \$1600 in interest. How much was invested at each rate?

Principal	Rate (as a decimal)	Interest
x	.02	

45. Michael Pellissier invested some money at 4.5% simple interest and \$1000 less than twice this amount at 3%. His total annual income from the interest was \$1020. How much was invested at each rate?

46. Holly Rioux invested some money at 3.5% simple interest, and \$5000 more than 3 times this amount at 4%. In one year, she earned \$1440 in interest. How much did she invest at each rate?

47. Jerry and Lucy Keefe have \$29,000 invested in stocks paying 5%. How much additional money should they invest in certificates of deposit paying 2% so that the total return on the two investments is 3%?

48. Ron Hampton placed \$15,000 in an account paying 6%. How much additional money should he deposit at 4% so that the total return on the two investments is 5.5%?

Solve each problem involving rates of concentration and mixtures. See Examples 7 and 8.

49. Ten liters of a 4% acid solution must be mixed with a 10% solution to get a 6% solution. How many liters of the 10% solution are needed?

Liters of Solution	Percent (as a decimal)	Liters of Pure Acid
10	.04	
x	.10	
	.06	

50. How many liters of a 14% alcohol solution must be mixed with 20 L of a 50% solution to get a 30% solution?

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Liters of Solution	Percent (as a decimal)	Liters of Pure Alcohol
x	.14	
	.50	

51. In a chemistry class, 12 L of a 12% alcohol solution must be mixed with a 20% solution to get a 14% solution. How many liters of the 20% solution are needed?

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52. How many liters of a 10% alcohol solution must be mixed with 40 L of a 50% solution to get a 40% solution?

53. How much pure dye must be added to 4 gal of a 25% dye solution to increase the solution to 40%? (*Hint:* Pure dye is 100% dye.)

54. How much water must be added to 6 gal of a 4% insecticide solution to reduce the concentration to 3%? (*Hint:* Water is 0% insecticide.)

55. Randall Albritton wants to mix 50 lb of nuts worth \$2 per lb with some nuts worth \$6 per lb to make a mixture worth \$5 per lb. How many pounds of \$6 nuts must he use?

56. Lee Ann Spahr wants to mix tea worth 2¢ per oz with 100 oz of tea worth 5¢ per oz to make a mixture worth 3¢ per oz. How much 2¢ tea should be used?

57. Why is it impossible to mix candy worth \$4 per lb and candy worth \$5 per lb to obtain a final mixture worth \$6 per lb?

58. Write an equation based on the following problem, solve the equation, and explain why the problem has no solution.

How much 30% acid should be mixed with 15 L of 50% acid to obtain a mixture that is 60% acid?

RELATING CONCEPTS (EXERCISES 59–63)

For Individual or Group Work

Consider each problem.

Problem A
Jack has \$800 invested in two accounts. One pays 5% interest per year and the other pays 10% interest per year. The amount of yearly interest is the same as he would get if the entire \$800 was invested at 8.75%. How much does he have invested at each rate?

Problem B
Jill has 800 L of acid solution. She obtained it by mixing some 5% acid with some 10% acid. Her final mixture of 800 L is 8.75% acid. How much of each of the 5% and 10% solutions did she use to get her final mixture?

In Problem A, let x represent the amount invested at 5% *interest, and in Problem B, let y represent the amount of* 5% *acid used.* ***Work Exercises 59–63 in order.***

59. (a) Write an expression in x that represents the amount of money Jack invested at 10% in Problem A.

(b) Write an expression in y that represents the amount of 10% acid solution Jill used in Problem B.

60. (a) Write expressions that represent the amount of interest Jack earns per year at 5% and at 10%.

(b) Write expressions that represent the amount of pure acid in Jill's 5% and 10% acid solutions.

61. (a) The sum of the two expressions in part (a) of Exercise 60 must equal the total amount of interest earned in one year. Write an equation representing this fact.

(b) The sum of the two expressions in part (b) of Exercise 60 must equal the amount of pure acid in the final mixture. Write an equation representing this fact.

2.4 Further Applications of Linear Equations

OBJECTIVES

1. Solve problems about different denominations of money.
2. Solve problems about uniform motion.
3. Solve problems about angles.

There are three common applications of linear equations that we did not discuss in Section 2.3: money problems, uniform motion problems, and problems involving the angles of a triangle.

OBJECTIVE 1 Solve problems about different denominations of money. These problems are very similar to the simple interest problems in Section 2.3.

PROBLEM SOLVING

In problems involving money, use the basic fact that

$$\begin{bmatrix}\text{number of monetary} \\ \text{units of the same kind}\end{bmatrix} \times [\text{denomination}] = \begin{bmatrix}\text{total monetary} \\ \text{value}\end{bmatrix}.$$

For example, 30 dimes have a monetary value of $30(.10) = 3.00$ dollars. Fifteen 5-dollar bills have a value of $15(5) = 75$ dollars.

EXAMPLE 1 Solving a Money Denomination Problem

For a bill totaling $5.65, a cashier received 25 coins consisting of nickels and quarters. How many of each type of coin did the cashier receive?

Step 1 **Read** the problem. The problem asks that we find the number of nickels and the number of quarters the cashier received.

Step 2 **Assign a variable.**

Let x represent the number of nickels;
then $25 - x$ represents the number of quarters.

We can organize the information in a table as we did with investment problems.

	Number of Coins	Denomination	Value
Nickels	x	.05	$.05x$
Quarters	$25 - x$	.25	$.25(25 - x)$
			5.65 ← Total

Step 3 **Write an equation.** From the last column of the table,

$$.05x + .25(25 - x) = 5.65.$$

Step 4 **Solve.**

$$\begin{aligned} .05x + .25(25 - x) &= 5.65 \\ 5x + 25(25 - x) &= 565 && \text{Multiply by 100.} \\ 5x + 625 - 25x &= 565 && \text{Distributive property} \\ -20x &= -60 && \text{Subtract 625; combine terms.} \\ x &= 3 && \text{Divide by } -20. \end{aligned}$$

Step 5 **State the answer.** The cashier has 3 nickels and $25 - 3 = 22$ quarters.

Step 6 **Check.** The cashier has $3 + 22 = 25$ coins, and the value of the coins is $\$.05(3) + \$.25(22) = \$5.65$, as required.

Now Try Exercise 11.

CAUTION Be sure that your answer is reasonable when working problems like Example 1. Because you are dealing with a number of coins, the correct answer can neither be negative nor a fraction.

OBJECTIVE 2 Solve problems about uniform motion.

PROBLEM SOLVING

Uniform motion problems use the distance formula, $d = rt$. In this formula, **when rate (or speed) is given in miles per hour, time must be given in hours.** To solve such problems, **draw a sketch** to illustrate what is happening in the problem, and make a table to summarize the given information.

EXAMPLE 2 Solving a Motion Problem (Motion in Opposite Directions)

VIDEO EXPLORE

Two cars leave the same place at the same time, one going east and the other west. The eastbound car averages 40 mph, while the westbound car averages 50 mph. In how many hours will they be 300 mi apart?

Step 1 **Read** the problem. We are looking for the time it takes for the two cars to be 300 mi apart.

Step 2 **Assign a variable.** A sketch shows what is happening in the problem: The cars are going in *opposite* directions. See Figure 6.

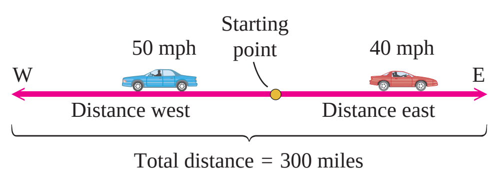


FIGURE 6

Let x represent the time traveled by each car. Summarize the information of the problem in a table. When the expressions for rate and time are entered, *fill in each distance by multiplying rate by time* using the formula $d = rt$.

	Rate	Time	Distance
Eastbound Car	40	x	$40x$
Westbound Car	50	x	$50x$
			300 ← Total

Step 3 **Write an equation.** From the sketch in Figure 6, the sum of the two distances is 300.

$$40x + 50x = 300$$

Step 4 **Solve.**

$$90x = 300 \qquad \text{Combine like terms.}$$

$$x = \frac{300}{90} \qquad \text{Divide by 90.}$$

$$x = \frac{10}{3} \qquad \text{Lowest terms}$$

Step 5 **State the answer.** The cars travel $\frac{10}{3} = 3\frac{1}{3}$ hr, or 3 hr 20 min.

Step 6 **Check.** The eastbound car traveled $40\left(\frac{10}{3}\right) = \frac{400}{3}$ mi, and the westbound car traveled $50\left(\frac{10}{3}\right) = \frac{500}{3}$ mi, for a total of $\frac{400}{3} + \frac{500}{3} = \frac{900}{3} = 300$ mi, as required.

Now Try Exercise 21.

CAUTION It is a common error to write 300 as the distance for each car in Example 2. Three hundred miles is the *total* distance traveled.

As in Example 2, in general, the equation for a problem involving motion in opposite directions is of the form

$$\text{partial distance} + \text{partial distance} = \text{total distance.}$$

EXAMPLE 3 Solving a Motion Problem (Motion in the Same Direction)

VIDEO

YOU TRY IT

Jeff Bezzone can bike to work in $\frac{3}{4}$ hr. When he takes the bus, the trip takes $\frac{1}{4}$ hr. If the bus travels 20 mph faster than Jeff rides his bike, how far is it to his workplace?

Step 1 **Read** the problem. We must find the distance between Jeff's home and his workplace.

Step 2 **Assign a variable.** Although the problem asks for a distance, it is easier here to let x be Jeff's speed when he rides his bike to work. Then the speed of the bus is $x + 20$. For the trip by bike,

$$d = rt = x \cdot \frac{3}{4} = \frac{3}{4}x,$$

and by bus,

$$d = rt = (x + 20) \cdot \frac{1}{4} = \frac{1}{4}(x + 20).$$

Summarize this information in a table.

	Rate	*Time*	*Distance*	
Bike	x	$\frac{3}{4}$	$\frac{3}{4}x$	Same
Bus	$x + 20$	$\frac{1}{4}$	$\frac{1}{4}(x + 20)$	

Step 3 **Write an equation.** The key to setting up the correct equation is to understand that the distance in each case is the same. See Figure 7.

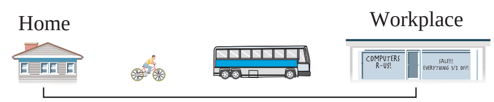


FIGURE 7

Since the distance is the same in each case,

$$\frac{3}{4}x = \frac{1}{4}(x + 20).$$

Step 4 **Solve** the equation. First multiply each side by 4.

$$4\left(\frac{3}{4}x\right) = 4\left(\frac{1}{4}\right)(x + 20) \qquad \text{Multiply by 4.}$$

$$3x = x + 20 \qquad \text{Multiply; identity property}$$

$$2x = 20 \qquad \text{Subtract } x.$$

$$x = 10 \qquad \text{Divide by 2.}$$

Step 5 **State the answer.** The required distance is

$$d = \frac{3}{4}x = \frac{3}{4}(10) = \frac{30}{4} = 7.5 \text{ mi.}$$

Step 6 **Check** by finding the distance using

$$d = \frac{1}{4}(x + 20) = \frac{1}{4}(10 + 20) = \frac{30}{4} = 7.5 \text{ mi,}$$

the same result.

Now Try Exercise 25.

As in Example 3, the equation for a problem involving motion in the same direction is often of the form

one distance = other distance.

NOTE In Example 3 it was easier to let the variable represent a quantity other than the one that we were asked to find. This is the case in some problems. It takes practice to learn when this approach is the best, and practice means working lots of problems!

OBJECTIVE 3 Solve problems about angles. An important result of Euclidean geometry (the geometry of the Greek mathematician Euclid) is that the sum of the angle measures of any triangle is 180°. This property is used in the next example.

EXPLORE

YOU TRY IT

VIDEO

EXAMPLE 4 Finding Angle Measures

Find the value of x, and determine the measure of each angle in Figure 8.

Step 1 **Read** the problem. We are asked to find the measure of each angle.

Step 2 **Assign a variable.** Let x represent the measure of one angle.

Step 3 **Write an equation.** The sum of the three measures shown in the figure must be 180°.

$$x + (x + 20) + (210 - 3x) = 180$$

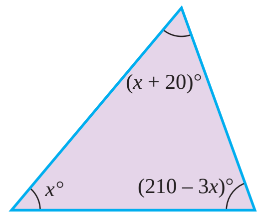


FIGURE 8

Step 4 **Solve.**

$$
\begin{aligned}
-x + 230 &= 180 && \text{Combine like terms.}\\
-x &= -50 && \text{Subtract 230.}\\
x &= 50 && \text{Divide by } -1.
\end{aligned}
$$

Step 5 **State the answer.** One angle measures 50°, another measures $x + 20 = 50 + 20 = 70°$, and the third measures $210 - 3x = 210 - 3(50) = 60°$.

Step 6 **Check.** Since $50° + 70° + 60° = 180°$, the answers are correct.

Now Try Exercise 31.

We discuss another way to solve the problems in this section in Chapter 4.

2.4 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 2/Videotape 2

Solve each problem.

1. What amount of money is found in a coin hoard containing 38 nickels and 26 dimes?

2. The distance between Cape Town, South Africa, and Miami is 7700 mi. If a jet averages 480 mph between the two cities, what is its travel time in hours?

3. Tri Phong traveled from Chicago to Des Moines, a distance of 300 mi, in 5 hr. What was his rate in miles per hour?

4. A square has perimeter 40 in. What would be the perimeter of an equilateral triangle whose sides each measure the same length as the side of the square?

Write a short explanation in Exercises 5–8.

5. Read over Example 3 in this section. The solution of the equation is 10. Why is *10 mph* not the answer to the problem?

6. Suppose that you know that two angles of a triangle have equal measures, and the third angle measures 36°. Explain in a few words the strategy you would use to find the measures of the equal angles without actually writing an equation.

7. In a problem about the number of coins of different denominations, would an answer that is a fraction be reasonable? What about a negative number?

8. In a motion problem the rate is given as x mph and the time is given as 30 min. What variable expression represents the distance in miles?

Solve each problem. See Example 1.

9. Otis Taylor has a box of coins that he uses when playing poker with his friends. The box currently contains 44 coins, consisting of pennies, dimes, and quarters. The number of pennies is equal to the number of dimes, and the total value is \$4.37. How many of each denomination of coin does he have in the box?

Number of Coins	Denomination	Value
x	.01	$.01x$
x		
	.25	
		4.37 ← Total

10. Nana Nantambu found some coins while looking under her sofa pillows. There were equal numbers of nickels and quarters, and twice as many half-dollars as quarters. If she found \$2.60 in all, how many of each denomination of coin did she find?

Number of Coins	Denomination	Value
x	.05	$.05x$
x		
$2x$	.50	
		2.60 ← Total

EXERCISES

11. Kim Falgout's daughter, Madeline, has a piggy bank with 47 coins. Some are quarters, and the rest are half-dollars. If the total value of the coins is \$17.00, how many of each denomination does she have?

12. John Joslyn has a jar in his office that contains 39 coins. Some are pennies, and the rest are dimes. If the total value of the coins is \$2.64, how many of each denomination does he have?

13. Dave Bowers collects U.S. gold coins. He has a collection of 41 coins. Some are \$10 coins, and the rest are \$20 coins. If the face value of the coins is \$540, how many of each denomination does he have?

14. In the nineteenth century, the United States minted two-cent and three-cent pieces. Frances Steib has three times as many three-cent pieces as two-cent pieces, and the face value of these coins is \$2.42. How many of each denomination does she have?

15. A total of 550 people attended a Kenny Loggins concert. Floor tickets cost \$40 each, while balcony tickets cost \$28 each. If a total of \$20,800 was collected, how many of each type of ticket were sold?

16. The Delgado Community College production of *The Music Man* was a big success. For opening night, 410 tickets were sold. Students paid \$3 each, while nonstudents paid \$7 each. If a total of \$1650 was collected, how many students and how many nonstudents attended?

In Exercises 17–20, find the rate based on the information provided. Use a calculator and round your answers to the nearest hundredth. All events were at the 2000 Summer Olympics in Sidney, Australia. (*Source:* http://espn.go.com/oly/summer00)

photo not available

	Event	Participant	Distance	Time
17.	100-m hurdles, Women	Olga Shishigina, Kazakhstan	100 m	12.65 sec
18.	400-m hurdles, Women	Irina Privalova, Russia	400 m	53.02 sec
19.	400-m hurdles, Men	Angelo Taylor, USA	400 m	47.50 sec
20.	400-m dash, Men	Michael Johnson, USA	400 m	43.84 sec

Solve each problem. See Examples 2 and 3.

21. Two steamers leave a port on a river at the same time, traveling in opposite directions. Each is traveling 22 mph. How long will it take for them to be 110 mi apart?

	Rate	Time	Distance
First Steamer		t	
Second Steamer	22		
			110

22. A train leaves Kansas City, Kansas, and travels north at 85 km per hr. Another train leaves at the same time and travels south at 95 km per hr. How long will it take before they are 315 km apart?

	Rate	Time	Distance
First Train	85	t	
Second Train			
			315

23. Agents Mulder and Scully are driving to Georgia to investigate "Big Blue," a giant aquatic reptile reported to inhabit one of the local lakes. Mulder leaves Washington at 8:30 A.M. and averages 65 mph. His partner, Scully, leaves at 9:00 A.M., following the same path and averaging 68 mph. At what time will Scully catch up with Mulder?

	Rate	Time	Distance
Mulder			
Scully			

24. Lois and Clark are covering separate stories and have to travel in opposite directions. Lois leaves the *Daily Planet* at 8:00 A.M. and travels at 35 mph. Clark leaves at 8:15 A.M. and travels at 40 mph. At what time will they be 140 mi apart?

	Rate	Time	Distance
Lois			
Clark			

25. Latrella can get to school in 15 min if she rides her bike. It takes her 45 min if she walks. Her speed when walking is 10 mph slower than her speed when riding. What is her speed when she rides?

	Rate	Time	Distance
Riding			
Walking			

26. When Dewayne drives his car to work, the trip takes 30 min. When he rides the bus, it takes 45 min. The average speed of the bus is 12 mph less than his speed when driving. Find the distance he travels to work.

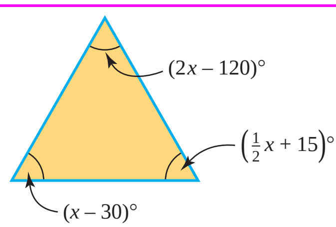
	Rate	*Time*	*Distance*
Car			
Bus			

27. Johnny leaves Memphis to visit his cousin, Anne Hoffman, who lives in the town of Hornsby, TN, 80 mi away. He travels at an average speed of 50 mph. One-half hour later, Anne leaves to visit Johnny, traveling at an average speed of 60 mph. How long after Anne leaves will it be before they meet?

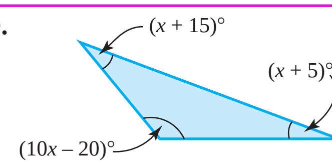
28. On an automobile trip, Aimee Cardella maintained a steady speed for the first two hours. Rush-hour traffic slowed her speed by 25 mph for the last part of the trip. The entire trip, a distance of 125 mi, took $2\frac{1}{2}$ hr. What was her speed during the first part of the trip?

Find the measure of each angle in the triangles shown. (Be sure to substitute your value of x into each angle expression.) See Example 4.

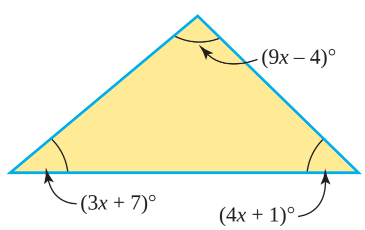
29.



30.



31.



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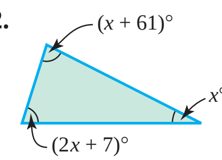


RELATING CONCEPTS (EXERCISES 33–36)

For Individual or Group Work

Consider the following two figures. ***Work Exercises 33–36 in order.***

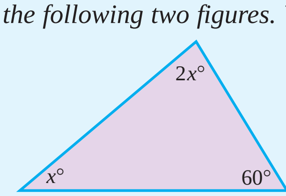


FIGURE A

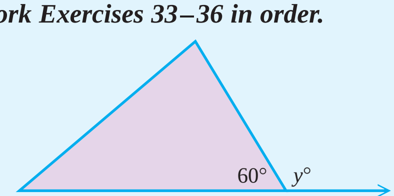


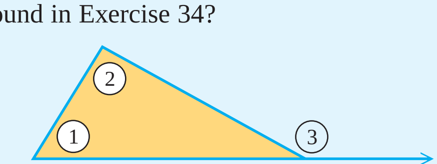
FIGURE B

33. Solve for the measures of the unknown angles in Figure A.

34. Solve for the measure of the unknown angle marked $y°$ in Figure B.

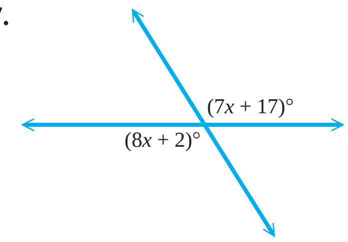
35. Add the measures of the two angles you found in Exercise 33. How does the sum compare to the measure of the angle you found in Exercise 34?

36. From Exercises 33–35, make a conjecture (an educated guess) about the relationship among the angles marked 1, 2, and 3 in the figure shown here.

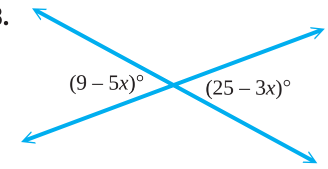


In Exercises 37 and 38, the angles marked with variable expressions are called ***vertical angles.*** *It is shown in geometry that vertical angles have equal measures. Find the measure of each angle.*

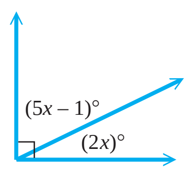
37.



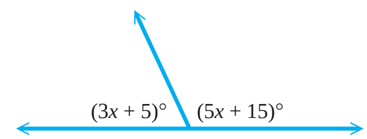
38.



39. Two angles whose sum is 90° are called **complementary angles.** Find the measures of the complementary angles shown in the figure.



40. Two angles whose sum is 180° are called **supplementary angles.** Find the measures of the supplementary angles shown in the figure.



Another type of application often studied in introductory and intermediate algebra courses involves consecutive integers. Consecutive integers are integers that follow each other in counting order, such as 8, 9, and 10. Suppose we wish to solve the following problem:

> Find three consecutive integers such that the sum of the first and third, increased by 3, is 50 more than the second.

Let x represent the first of the unknown integers. Then $x + 1$ will be the second, and $x + 2$ will be the third. The equation we need can be found by going back to the words of the original problem.

$$\underbrace{x + (x + 2)}_{\text{Sum of the first and third}} \quad \underbrace{+\,3}_{\text{increased by 3}} \quad \underbrace{=}_{\text{is}} \quad \underbrace{(x + 1) + 50}_{\text{50 more than the second.}}$$

The solution of this equation is 46, meaning that the first integer is $x = 46$, the second is $x + 1 = 47$, and the third is $x + 2 = 48$. The three integers are 46, 47, and 48. Check by substituting these numbers back into the words of the original problem.

Solve each problem involving consecutive integers.

EXERCISES

41. Find three consecutive integers such that the sum of the first and twice the second is 17 more than twice the third.

42. Find four consecutive integers such that the sum of the first three is 54 more than the fourth.

43. If I add my current age to the age I will be next year on this date, the sum is 103 years. How old will I be 10 years from today?

44. Two pages facing each other in this book have 189 as the sum of their page numbers. What are the two page numbers?

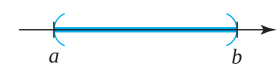
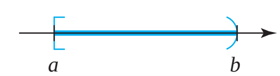
2.5 Linear Inequalities in One Variable

OBJECTIVES

1. Solve linear inequalities using the addition property.
2. Solve linear inequalities using the multiplication property.
3. Solve linear inequalities with three parts.
4. Solve applied problems using linear inequalities.

In Section 1.1, we used interval notation to write solution sets of inequalities, using a parenthesis to indicate that an endpoint is not included and a square bracket to indicate that an endpoint is included. We summarize the various types of intervals here.

Interval Notation

Type of Interval	*Set*	*Interval Notation*	*Graph*
Open interval	$\{x \mid a < x\}$	(a, ∞)	
	$\{x \mid a < x < b\}$	(a, b)	
	$\{x \mid x < b\}$	$(-\infty, b)$	
	$\{x \mid x \text{ is a real number}\}$	$(-\infty, \infty)$	
Half-open interval	$\{x \mid a \le x\}$	$[a, \infty)$	
	$\{x \mid a < x \le b\}$	$(a, b]$	
	$\{x \mid a \le x < b\}$	$[a, b)$	
	$\{x \mid x \le b\}$	$(-\infty, b]$	
Closed interval	$\{x \mid a \le x \le b\}$	$[a, b]$	

An inequality says that two expressions are *not* equal. Solving inequalities is similar to solving equations.

AUDIO

Linear Inequality in One Variable

A **linear inequality in one variable** can be written in the form

$$Ax + B < C,$$

where A, B, and C are real numbers, with $A \neq 0$.

(Throughout this section we give definitions and rules only for $<$, but they are also valid for $>$, $\le$, and $\ge$.) Examples of linear inequalities include

$$x + 5 < 2, \qquad x - 3 \ge 5, \qquad \text{and} \qquad 2k + 5 \le 10. \qquad \text{Linear inequalities}$$

OBJECTIVE 1 Solve linear inequalities using the addition property. We solve an inequality by finding all numbers that make the inequality true. Usually, an inequality has an infinite number of solutions. These solutions, like solutions of equations, are found by producing a series of simpler equivalent inequalities. **Equivalent inequalities** are inequalities with the same solution set. We use the addition and multiplication properties of inequality to produce equivalent inequalities.

Addition Property of Inequality

For all real numbers A, B, and C, the inequalities

$$A < B \qquad \text{and} \qquad A + C < B + C$$

are equivalent.

That is, adding the same number to each side of an inequality does not change the solution set.

As with equations, the addition property can be used to *subtract* the same number from each side of an inequality.

EXAMPLE 1 Using the Addition Property of Inequality

Solve $x - 7 < -12$.

Add 7 to each side.

$$x - 7 + 7 < -12 + 7 \qquad \text{Add 7.}$$
$$x < -5$$

Check: Substitute -5 for x in the equation $x - 7 = -12$. The result should be a true statement.

$$x - 7 = -12$$
$$-5 - 7 = -12 \quad ? \qquad \text{Let } x = -5.$$
$$-12 = -12 \qquad \text{True}$$

This shows that -5 is the boundary point. Now test a number on each side of -5 to verify that numbers *less than* -5 make the inequality true. Choose -4 and -6.

$$x - 7 < -12$$

$-4 - 7 < -12$? Let $x = -4$.	$-6 - 7 < -12$? Let $x = -6$.
$-11 < -12$ False	$-13 < -12$ True
-4 is not in the solution set.	-6 is in the solution set.

The check confirms that $(-\infty, -5)$, graphed in Figure 9, is the correct solution set.

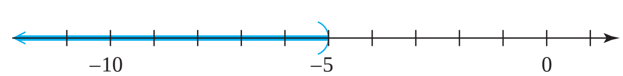


FIGURE 9

Now Try Exercise 9.

EXAMPLE 2 Using the Addition Property of Inequality

Solve $14 + 2m \leq 3m$ and graph the solution set.

First, subtract $2m$ from each side.

$$14 + 2m \leq 3m$$

$$14 + 2m - 2m \leq 3m - 2m \qquad \text{Subtract } 2m.$$

$$14 \leq m \qquad \text{Combine like terms.}$$

The inequality $14 \leq m$ (14 is less than or equal to m) can also be written $m \geq 14$ (m is greater than or equal to 14). Notice that in each case, the inequality symbol points to the smaller number, 14.

Check:

$$14 + 2m = 3m$$

$$14 + 2(14) \stackrel{?}{=} 3(14) \qquad \text{Let } m = 14.$$

$$42 = 42 \qquad \text{True}$$

So 14 satisfies the equality part of $\leq$. Choose 10 and 15 as test points.

$$14 + 2m < 3m$$

$14 + 2(10) \stackrel{?}{<} 3(10)$ Let $m = 10$.	$14 + 2(15) \stackrel{?}{<} 3(15)$ Let $m = 15$.
$34 < 30$ False	$44 < 45$ True
10 is not in the solution set.	15 is in the solution set.

The check confirms that $[14, \infty)$ is the correct solution set. See Figure 10.

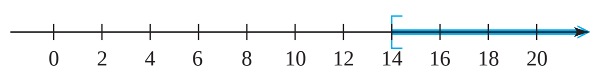


FIGURE 10

Now Try Exercise 21.

CAUTION Errors often occur in graphing inequalities when the variable term is on the right side. (This is probably due to the fact that we read from left to right.) To guard against such errors, it is a good idea to rewrite these inequalities so that the variable is on the left, as discussed in Example 2.

OBJECTIVE 2 Solve linear inequalities using the multiplication property. Solving an inequality such as $3x \leq 15$ requires dividing each side by 3, using the *multiplication property of inequality,* which is a little more involved than the multiplication property of *equality*. To see how the multiplication property of inequality works, start with the true statement

$$-2 < 5.$$

Multiply each side by, say, 8.

$$-2(8) < 5(8) \qquad \text{Multiply by 8.}$$

$$-16 < 40 \qquad \text{True}$$

This gives a true statement. Start again with $-2 < 5$, and multiply each side by -8.

$$-2(-8) < 5(-8) \qquad \text{Multiply by } -8.$$

$$16 < -40 \qquad \text{False}$$

The result, $16 < -40$, is false. To make it true, we must change the direction of the inequality symbol to get

$$16 > -40. \qquad \text{True}$$

As these examples suggest, multiplying each side of an inequality by a *negative* number reverses the direction of the inequality symbol. The same is true for dividing by a negative number since division is defined in terms of multiplication.

AUDIO

Multiplication Property of Inequality

For all real numbers A, B, and C, with $C \neq 0$,

(a) the inequalities

$$A < B \quad \text{and} \quad AC < BC$$

are equivalent **if $C > 0$;**

(b) the inequalities

$$A < B \quad \text{and} \quad AC > BC$$

are equivalent **if $C < 0$.**

That is, each side of an inequality may be multiplied (or divided) by a *positive* number without changing the direction of the inequality symbol. ***Multiplying (or dividing) by a* negative *number requires that we reverse the inequality symbol.***

CAUTION Remember to reverse the direction of the inequality symbol when multiplying or dividing by a *negative* number.

EXAMPLE 3 Using the Multiplication Property of Inequality

Solve each inequality and graph the solution set.

(a) $5m \leq -30$

Use the multiplication property to divide each side by 5. Since $5 > 0$, do *not* reverse the inequality symbol.

$$5m \leq -30$$

$$\frac{5m}{5} \leq \frac{-30}{5} \qquad \text{Divide by 5.}$$

$$m \leq -6$$

Check that the solution set is the interval $(-\infty, -6]$, graphed in Figure 11.

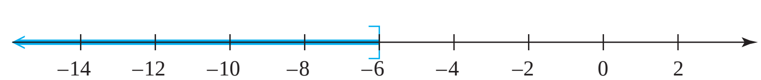


FIGURE 11

(b) $-4k \le 32$

Divide each side by -4. Since $-4 < 0$, reverse the inequality symbol.

$$-4k \le 32$$

$$\frac{-4k}{-4} \ge \frac{32}{-4} \qquad \text{Divide by } -4 \text{ and reverse the symbol.}$$

$$k \ge -8$$

Check the solution set. Figure 12 shows the graph of the solution set, $[-8, \infty)$.

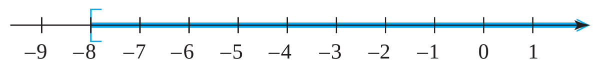


FIGURE 12

Now Try Exercises 13 and 17.

The steps used in solving a linear inequality are given here.

Solving a Linear Inequality

Step 1 **Simplify each side separately.** Use the distributive property to clear parentheses and combine like terms as needed.

Step 2 **Isolate the variable terms on one side.** Use the addition property of inequality to get all terms with variables on one side of the inequality and all numbers on the other side.

Step 3 **Isolate the variable.** Use the multiplication property of inequality to change the inequality to the form $x < k$ or $x > k$.

Remember: Reverse the direction of the inequality symbol ***only*** when ***multiplying or dividing each side of an inequality by a negative number.***

EXAMPLE 4 **Solving a Linear Inequality Using the Distributive Property**

Solve $-3(x + 4) + 2 \ge 7 - x$ and graph the solution set.

Step 1 $-3x - 12 + 2 \ge 7 - x$ Distributive property

$$-3x - 10 \ge 7 - x$$

Step 2 $-3x - 10 + x \ge 7 - x + x$ Add x.

$$-2x - 10 \ge 7$$

$$-2x - 10 + 10 \ge 7 + 10 \qquad \text{Add 10.}$$

$$-2x \ge 17$$

Step 3 $\dfrac{-2x}{-2} \le \dfrac{17}{-2}$ Divide by -2; change $\ge$ to $\le$.

$$x \le -\frac{17}{2}$$

Figure 13 shows the graph of the solution set, $\left(-\infty, -\frac{17}{2}\right]$.

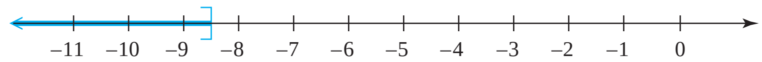


FIGURE 13

Now Try Exercise 23.

NOTE In Step 2 of Example 4, if we add $3x$ to both sides of the inequality, we have

$$
\begin{aligned}
-3x - 10 + 3x &\geq 7 - x + 3x && \text{Add } 3x.\\
-10 &\geq 2x + 7\\
-10 - 7 &\geq 2x + 7 - 7 && \text{Subtract 7.}\\
-17 &\geq 2x\\
-\frac{17}{2} &\geq x. && \text{Divide by 2.}
\end{aligned}
$$

The final line is read "$-\frac{17}{2}$ is greater than or equal to x," which means the same thing as "x is less than or equal to $-\frac{17}{2}$." Thus, the solution set is the same.

EXAMPLE 5 Solving a Linear Inequality with Fractions

Solve $-\frac{2}{3}(r-3) - \frac{1}{2} < \frac{1}{2}(5-r)$ and graph the solution set.

To clear fractions, multiply each side by the least common denominator, 6.

$$
\begin{aligned}
-\frac{2}{3}(r-3) - \frac{1}{2} &< \frac{1}{2}(5-r)\\
6\left[-\frac{2}{3}(r-3) - \frac{1}{2}\right] &< 6\left[\frac{1}{2}(5-r)\right] && \text{Multiply by 6.}\\
6\left[-\frac{2}{3}(r-3)\right] - 6\left(\frac{1}{2}\right) &< 6\left[\frac{1}{2}(5-r)\right] && \text{Distributive property}\\
-4(r-3) - 3 &< 3(5-r)
\end{aligned}
$$

Step 1

$$
\begin{aligned}
-4r + 12 - 3 &< 15 - 3r && \text{Distributive property}\\
-4r + 9 &< 15 - 3r
\end{aligned}
$$

Step 2

$$
\begin{aligned}
-4r + 9 + 3r &< 15 - 3r + 3r && \text{Add } 3r.\\
-r + 9 &< 15\\
-r + 9 - 9 &< 15 - 9 && \text{Subtract 9.}\\
-r &< 6
\end{aligned}
$$

Step 3 To solve for r, multiply each side of the inequality by -1. Since -1 is negative, change the direction of the inequality symbol.

$$
\begin{aligned}
-1(-r) &> -1(6) && \text{Multiply by } -1\text{, change } < \text{ to } >.\\
r &> -6
\end{aligned}
$$

Check that the solution set is $(-6, \infty)$. See the graph in Figure 14 on the next page.

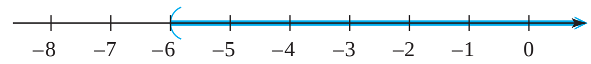


FIGURE 14

Now Try Exercise 29.

OBJECTIVE 3 Solve linear inequalities with three parts. For some applications, it is necessary to work with an inequality such as

$$3 < x + 2 < 8,$$

where $x + 2$ is *between* 3 and 8. To solve this inequality, we subtract 2 from each of the three parts of the inequality, giving

$$3 - 2 < x + 2 - 2 < 8 - 2$$
$$1 < x < 6.$$

Thus, x must be between 1 and 6, so $x + 2$ will be between 3 and 8. The solution set, $(1, 6)$, is graphed in Figure 15.

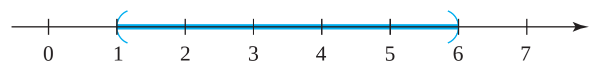


FIGURE 15

CAUTION When inequalities have three parts, the order of the parts is important. It would be *wrong* to write an inequality as $8 < x + 2 < 3$, since this would imply that $8 < 3$, a false statement. In general, three-part inequalities are written so that the symbols point in the same direction and both point toward the smaller number.

EXPLORE
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EXAMPLE 6 Solving a Three-Part Inequality

Solve $-2 \leq -3k - 1 \leq 5$ and graph the solution set.

Begin by adding 1 to each of the three parts to isolate the variable term in the middle.

$$-2 + 1 \leq -3k - 1 + 1 \leq 5 + 1 \qquad \text{Add 1 to each part.}$$
$$-1 \leq -3k \leq 6$$
$$\frac{-1}{-3} \geq \frac{-3k}{-3} \geq \frac{6}{-3} \qquad \text{Divide each part by } -3\text{; reverse the inequality symbols.}$$
$$\frac{1}{3} \geq k \geq -2$$
$$-2 \leq k \leq \frac{1}{3} \qquad \text{Rewrite in the order on the number line.}$$

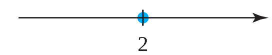
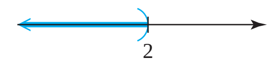
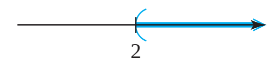
Check that the solution set is $\left[-2, \frac{1}{3}\right]$, as shown in Figure 16.

FIGURE 16

Now Try Exercise 51.

Examples of the types of solution sets to be expected from solving linear equations or linear inequalities are shown below.

Solution Sets of Linear Equations and Inequalities

Equation or Inequality	*Typical Solution Set*	*Graph of Solution Set*
Linear equation $5x + 4 = 14$	$\{2\}$	
Linear inequality $5x + 4 < 14$	$(-\infty, 2)$	
$5x + 4 > 14$	$(2, \infty)$	
Three-part inequality $-1 \le 5x + 4 \le 14$	$[-1, 2]$	

OBJECTIVE 4 Solve applied problems using linear inequalities. In addition to the familiar "is less than" and "is greater than," the expressions "is no more than" and "is at least" also indicate inequalities. Expressions for inequalities sometimes appear in applied problems. The table shows how to interpret these expressions.

Word Expression	*Interpretation*
a is at least b	$a \ge b$
a is no less than b	$a \ge b$
a is at most b	$a \le b$
a is no more than b	$a \le b$

In Examples 7 and 8, we show how to solve applied problems with inequalities. We use the six problem-solving steps, changing Step 3 from "Write an equation" to "Write an inequality."

EXAMPLE 7 Using a Linear Inequality to Solve a Rental Problem

A rental company charges \$15 to rent a chain saw, plus \$2 per hr. Al Ghandi can spend no more than \$35 to clear some logs from his yard. What is the *maximum* amount of time he can use the rented saw?

Step 1 **Read** the problem again.

Step 2 **Assign a variable.** Let $h =$ the number of hours he can rent the saw.

Step 3 **Write an inequality.** He must pay \$15, plus \$$2h$, to rent the saw for h hours, and this amount must be *no more than* \$35.

$$\overbrace{15 + 2h}^{\text{Cost of renting}} \quad \overbrace{\le}^{\text{is no more than}} \quad \overbrace{35}^{\text{35 dollars.}}$$

Step 4 **Solve.**

$$2h \le 20 \qquad \text{Subtract 15.}$$

$$h \le 10 \qquad \text{Divide by 2.}$$

Step 5 **State the answer.** He can use the saw for a maximum of 10 hr. (Of course, he may use it for less time, as indicated by the inequality $h \leq 10$.)

Step 6 **Check.** If Al uses the saw for 10 hr, he will spend $15 + 2(10) = 35$ dollars, the maximum amount.

Now Try Exercise 65.

EXAMPLE 8 Finding an Average Test Score

Martha has scores of 88, 86, and 90 on her first three algebra tests. An average score of at least 90 will earn an A in the class. What possible scores on her fourth test will earn her an A average?

Let x represent the score on the fourth test. Her average score must be at least 90. To find the average of four numbers, add them and then divide by 4.

$$\overbrace{\frac{88 + 86 + 90 + x}{4}}^{\text{Average}} \quad \overbrace{\geq}^{\text{is at least}} \quad \overbrace{90}^{90.}$$

$$\frac{264 + x}{4} \geq 90 \qquad \text{Add the scores.}$$

$$264 + x \geq 360 \qquad \text{Multiply by 4.}$$

$$x \geq 96 \qquad \text{Subtract 264.}$$

Photo not available

She must score 96 or more on her fourth test.

Check: $$\frac{88 + 86 + 90 + 96}{4} = \frac{360}{4} = 90$$

A score of 96 or more will give an average of at least 90, as required.

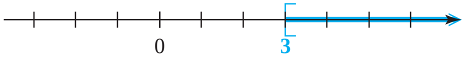
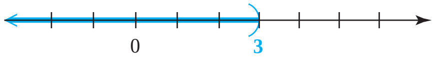
Now Try Exercise 63.

2.5 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 3/Videotape 3

Match each inequality in Column I with the correct graph or interval notation in Column II.

I	II
1. $x \leq 3$	**A.** 
2. $x > 3$	**B.** 
3. $x < 3$	**C.** $(3, \infty)$
4. $x \geq 3$	**D.** $(-\infty, 3]$
5. $-3 \leq x \leq 3$	**E.** $(-3, 3)$
6. $-3 < x < 3$	**F.** $[-3, 3]$

7. Explain how to determine whether to use parentheses or brackets when graphing the solution set of an inequality.

8. Describe the steps used to solve a linear inequality. Explain when it is necessary to reverse the inequality symbol.

Solve each inequality. Give the solution set in both interval and graph forms. See Examples 1–5.

9. $x + 4 \geq 20$

10. $t + 40 \geq 50$

11. $3k - 1 > 20$

12. $5z - 6 < 64$

13. $4x < 16$

14. $2m > 10$

15. $-\frac{3}{4}r \geq 30$

16. $-1.5y \leq -\frac{9}{2}$

17. $-1.3m \geq -5.2$

18. $-2.5y \leq -1.25$

19. $\frac{2k - 5}{-4} > 5$

20. $\frac{3z - 2}{-5} < 6$

21. $6x - 4 \geq -2x$

22. $-2m + 8 \leq 2m$

23. $-(4 + r) + 2 - 3r < -14$

24. $-(9 + k) - 5 + 4k \geq 4$

25. $-3(z - 6) > 2z - 2$

26. $-2(x + 4) \leq 6x + 16$

27. $\frac{2}{3}(3k - 1) \geq \frac{3}{2}(2k - 3)$

28. $\frac{7}{5}(10m - 1) < \frac{2}{3}(6m + 5)$

29. $-\frac{1}{4}(p + 6) + \frac{3}{2}(2p - 5) < 10$

30. $\frac{3}{5}(k - 2) - \frac{1}{4}(2k - 7) \leq 3$

31. $3(2x - 4) - 4x < 2x + 3$

32. $7(4 - x) + 5x < 2(16 - x)$

33. $8\left(\frac{1}{2}x + 3\right) < 8\left(\frac{1}{2}x - 1\right)$

34. $10x + 2(x - 4) < 12x - 10$

RELATING CONCEPTS (EXERCISES 35–39)

For Individual or Group Work

Work Exercises 35–39 in order.

35. Solve the linear equation $5(x + 3) - 2(x - 4) = 2(x + 7)$ and graph the solution set on a number line.

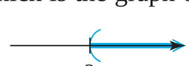
36. Solve the linear inequality $5(x + 3) - 2(x - 4) > 2(x + 7)$ and graph the solution set on a number line.

37. Solve the linear inequality $5(x + 3) - 2(x - 4) < 2(x + 7)$ and graph the solution set on a number line.

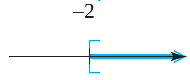
38. Graph all the solution sets of the equation and inequalities in Exercises 35–37 on the same number line. What set do you obtain?

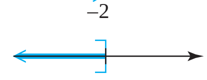
39. Based on the results of Exercises 35–37, complete the following using a conjecture (educated guess): The solution set of $-3(x + 2) = 3x + 12$ is $\{-3\}$, and the solution set of $-3(x + 2) < 3x + 12$ is $(-3, \infty)$. Therefore the solution set of $-3(x + 2) > 3x + 12$ is ______.

40. Which is the graph of $-2 < x$?

A. 

B. 

C. 

D. 

Solve each inequality. Give the solution set in both interval and graph forms. See Example 6.

41. $-4 < x - 5 < 6$ **42.** $-1 < x + 1 < 8$ **43.** $-9 \leq k + 5 \leq 15$

44. $-4 \leq m + 3 \leq 10$ **45.** $-6 \leq 2z + 4 \leq 16$ **46.** $-15 < 3p + 6 < -12$

47. $-19 \leq 3x - 5 \leq 1$ **48.** $-16 < 3t + 2 < -10$ **49.** $-1 \leq \dfrac{2x - 5}{6} \leq 5$

50. $-3 \leq \dfrac{3m + 1}{4} \leq 3$ **51.** $4 \leq 5 - 9x < 8$ **52.** $4 \leq 3 - 2x < 8$

Find the unknown numbers in each description.

53. Six times a number is between -12 and 12.

54. Half a number is between -3 and 2.

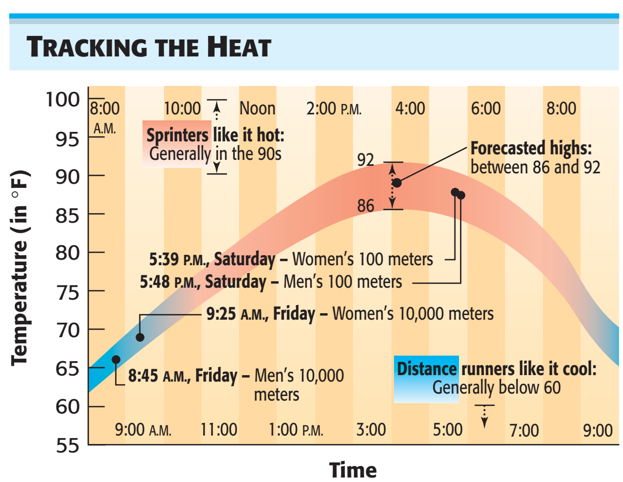
55. When 1 is added to twice a number, the result is greater than or equal to 7.

56. If 8 is subtracted from a number, then the result is at least 5.

57. One third of a number is added to 6, giving a result of at least 3.

58. Three times a number, minus 5, is no more than 7.

The July 14th weather forecast by time of day for the 2000 U.S. Olympic Track and Field Trials, held July 14–23, 2000, in Sacramento, California, is shown in the figure. Use this graph to work Exercises 59–62.



Source: Accuweather, Bee research.

photo not available

59. Sprinters prefer Fahrenheit temperatures in the 90s. Using the upper boundary of the forecast, in what time period is the temperature expected to be at least 90°F?

60. Distance runners prefer cool temperatures. During what time period are temperatures predicted to be no more than 70°F? Use the lower forecast boundary.

61. What range of temperatures is predicted for the Women's 100-m event?

62. What range of temperatures is forecast for the Men's 10,000-m event?

Solve each problem. See Examples 7 and 8.

63. Margaret Westmoreland earned scores of 90 and 82 on her first two tests in English Literature. What score must she make on her third test to keep an average of 84 or greater?

64. Jacques d'Hemecourt scored 92 and 96 on his first two tests in Methods in Teaching Mathematics. What score must he make on his third test to keep an average of 90 or greater?

65. A couple wishes to rent a car for one day while on vacation. Avis automobile rental wants \$35.00 per day and 14¢ per mile, while Enterprise wants \$34.00 per day and 16¢ per mile. After how many miles would the price to rent from Enterprise exceed the price to rent from Avis?

photo not available

66. Jane and Terry Brandsma went to Long Island for a week. They needed to rent a car, so they checked out two rental firms. Avis wanted \$28 per day, with no mileage fee. Downtown Toyota wanted \$108 per week and 14¢ per mile. How many miles would they have to drive before the Avis price is less than the Toyota price?

A product will produce a profit only when the revenue R from selling the product exceeds the cost C of producing it. In Exercises 67 and 68, find the smallest whole number of units x that must be sold for the business to show a profit for the item described.

67. Peripheral Visions, Inc. finds that the cost to produce x studio-quality videotapes is $C = 20x + 100$, while the revenue produced from them is $R = 24x$ (C and R in dollars).

68. Speedy Delivery finds that the cost to make x deliveries is $C = 3x + 2300$, while the revenue produced from them is $R = 5.50x$ (C and R in dollars).

69. Suppose that $4 < x < 1$. What can you say about x?

70. What is wrong with writing a statement of inequality as $10 > 5 < 8$? What three-part inequality would avoid this problem?

71. A BMI (body mass index) between 19 and 25 is considered healthy. Use the formula

$$\text{BMI} = \frac{704 \times (\text{weight in pounds})}{(\text{height in inches})^2}$$

to find the weight range w, to the nearest pound, that gives a healthy BMI for each height. (*Source: Washington Post.*)

(a) 72 in. **(b)** Your height in inches

72. To achieve the maximum benefit from exercising, the heart rate in beats per minute should be in the target heart rate zone (THR). For a person aged A, the formula is

$$.7(220 - A) \leq THR \leq .85(220 - A).$$

Find the THR to the nearest whole number for each age. (*Source:* Hockey, Robert V., *Physical Fitness: The Pathway to Healthful Living,* Times Mirror/Mosby College Publishing, 1989.)

(a) 35 **(b)** Your age

2.6 Set Operations and Compound Inequalities

OBJECTIVES

1 Find the intersection of two sets.

2 Solve compound inequalities with the word *and*.

3 Find the union of two sets.

4 Solve compound inequalities with the word *or*.

The table shows symptoms of an underactive thyroid and an overactive thyroid.

Underactive Thyroid	*Overactive Thyroid*
Sleepiness, *s*	Insomnia, *i*
Dry hands, *d*	Moist hands, *m*
Intolerance of cold, *c*	Intolerance of heat, *h*
Goiter, *g*	Goiter, *g*

Source: The Merck Manual of Diagnosis and Therapy, 16th Edition, Merck Research Laboratories, 1992.

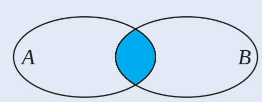
Let N be the set of symptoms for an underactive thyroid, and let O be the set of symptoms for an overactive thyroid. Suppose we are interested in the set of symptoms that are found in *both* sets N *and* O. In this section we discuss the use of the words *and* and *or* as they relate to sets and inequalities.

OBJECTIVE 1 Find the intersection of two sets. The intersection of two sets is defined using the word *and*.

Intersection of Sets

For any two sets A and B, the **intersection** of A and B, symbolized $A \cap B$, is defined as follows:

$$A \cap B = \{x \mid x \text{ is an element of } A \textbf{ and } x \text{ is an element of } B\}.$$



EXAMPLE 1 Finding the Intersection of Two Sets

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$. Find $A \cap B$.

The set $A \cap B$ contains those elements that belong to both A *and* B: the numbers 2 and 4. Therefore,

$$\begin{aligned} A \cap B &= \{1, 2, 3, 4\} \cap \{2, 4, 6\} \\ &= \{2, 4\}. \end{aligned}$$

Now Try Exercise 7.

A **compound inequality** consists of two inequalities linked by a connective word such as *and* or *or*. Examples of compound inequalities are

$$x + 1 \le 9 \quad \text{and} \quad x - 2 \ge 3$$

and

$$2x > 4 \quad \text{or} \quad 3x - 6 < 5.$$

OBJECTIVE 2 Solve compound inequalities with the word *and*. Use the following steps.

Solving a Compound Inequality with *and*

Step 1 Solve each inequality in the compound inequality individually.

Step 2 Since the inequalities are joined with *and*, the solution set of the compound inequality will include all numbers that satisfy both inequalities in Step 1 (the intersection of the solution sets).

EXAMPLE 2 Solving a Compound Inequality with *and*

Solve the compound inequality

$$x + 1 \le 9 \quad \text{and} \quad x - 2 \ge 3.$$

Step 1 Solve each inequality in the compound inequality individually.

$$\begin{aligned} x + 1 &\le 9 & &\text{and} & x - 2 &\ge 3 \\ x + 1 - 1 &\le 9 - 1 & &\text{and} & x - 2 + 2 &\ge 3 + 2 \\ x &\le 8 & &\text{and} & x &\ge 5 \end{aligned}$$

Step 2 Because the inequalities are joined with the word *and*, the solution set will include all numbers that satisfy both inequalities in Step 1 at the same time. Thus, the compound inequality is true whenever $x \le 8$ and $x \ge 5$ are both true. The top graph in Figure 17 shows $x \le 8$, and the bottom graph shows $x \ge 5$.

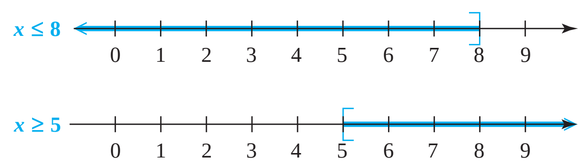


FIGURE 17

Find the intersection of the two graphs in Figure 17 to get the solution set of the compound inequality. The intersection of the two graphs in Figure 18 shows that the solution set in interval notation is [5, 8].

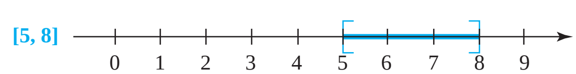


FIGURE 18

Now Try Exercise 27.

EXAMPLE 3 Solving a Compound Inequality with *and*

Solve the compound inequality

$$-3x - 2 > 5 \quad \text{and} \quad 5x - 1 \le -21.$$

Step 1 Solve each inequality separately.

$$\begin{aligned} -3x - 2 &> 5 & &\text{and} & 5x - 1 &\le -21 \\ -3x &> 7 & &\text{and} & 5x &\le -20 \\ x &< -\frac{7}{3} & &\text{and} & x &\le -4 \end{aligned}$$

The graphs of $x < -\frac{7}{3}$ and $x \le -4$ are shown in Figure 19.

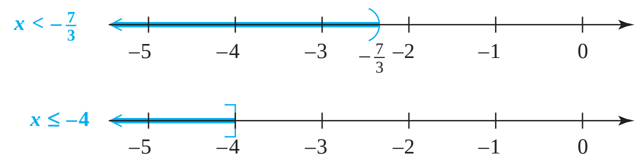


FIGURE 19

Step 2 Now find all values of x that satisfy both conditions; that is, the real numbers that are less than $-\frac{7}{3}$ and also less than or equal to -4. As shown by the graph in Figure 20, the solution set is $(-\infty, -4]$.

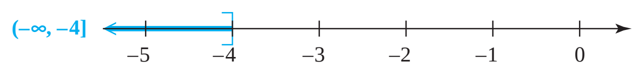


FIGURE 20

Now Try Exercise 31.

VIDEO

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EXAMPLE 4 Solving a Compound Inequality with *and*

Solve $x + 2 < 5$ and $x - 10 > 2$.

First solve each inequality separately.

$$x + 2 < 5 \quad \text{and} \quad x - 10 > 2$$
$$x < 3 \quad \text{and} \quad x > 12$$

The graphs of $x < 3$ and $x > 12$ are shown in Figure 21.

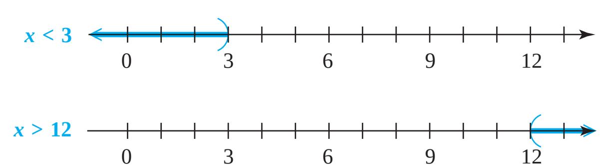


FIGURE 21

There is no number that is both less than 3 *and* greater than 12, so the given compound inequality has no solution. The solution set is $\emptyset$. See Figure 22.

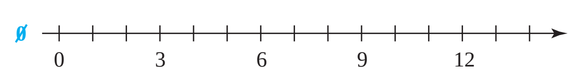


FIGURE 22

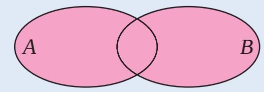
Now Try Exercise 25.

OBJECTIVE 3 Find the union of two sets. The union of two sets is defined using the word *or*.

Union of Sets

For any two sets A and B, the **union** of A and B, symbolized $A \cup B$, is defined as follows:

$$A \cup B = \{x \mid x \text{ is an element of } A \textbf{ or } x \text{ is an element of } B\}.$$



EXAMPLE 5 Finding the Union of Two Sets

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$. Find $A \cup B$.

Begin by listing all the elements of set A: 1, 2, 3, 4. Then list any additional elements from set B. In this case the elements 2 and 4 are already listed, so the only additional element is 6. Therefore,

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4\} \cup \{2, 4, 6\} \\ &= \{1, 2, 3, 4, 6\}. \end{aligned}$$

The union consists of all elements in either A *or* B (or both).

Now Try Exercise 13.

In Example 5, notice that although the elements 2 and 4 appeared in both sets A and B, they are written only once in $A \cup B$.

OBJECTIVE 4 Solve compound inequalities with the word *or*. Use the following steps.

Solving a Compound Inequality with *or*

Step 1 Solve each inequality in the compound inequality individually.

Step 2 Since the inequalities are joined with *or*, the solution set of the compound inequality includes all numbers that satisfy either one of the two inequalities in Step 1 (the union of the solution sets).

EXAMPLE 6 Solving a Compound Inequality with *or*

Solve $6x - 4 < 2x$ or $-3x \leq -9$.

Step 1 Solve each inequality separately.

$$\begin{array}{rcr} 6x - 4 < 2x & \text{or} & -3x \leq -9 \\ 4x < 4 & & \\ x < 1 & \text{or} & x \geq 3 \end{array}$$

The graphs of these two inequalities are shown in Figure 23 on the next page.

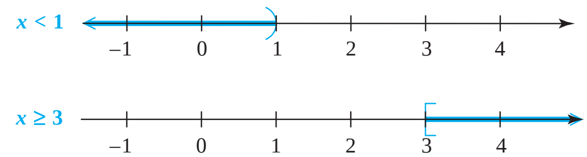


FIGURE 23

Step 2 Since the inequalities are joined with *or,* find the union of the two solution sets. The union is shown in Figure 24 and is written

$$(-\infty, 1) \cup [3, \infty).$$

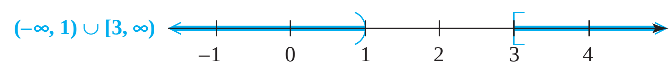


FIGURE 24

Now Try Exercise 43.

CAUTION When inequalities are used to write the solution set in Example 6, it *should* be written as

$$x < 1 \quad \text{or} \quad x \geq 3,$$

which keeps the numbers 1 and 3 in their order on the number line. Writing $3 \leq x < 1$ would imply that $3 \leq 1$, which is ***FALSE.*** There is no other way to write the solution set of such a union.

EXPLORE YOU TRY IT VIDEO

EXAMPLE 7 Solving a Compound Inequality with *or*

Solve $-4x + 1 \geq 9$ or $5x + 3 \leq -12$.

First we solve each inequality separately.

$$\begin{aligned} -4x + 1 &\geq 9 && \text{or} & 5x + 3 &\leq -12 \\ -4x &\geq 8 && \text{or} & 5x &\leq -15 \\ x &\leq -2 && \text{or} & x &\leq -3 \end{aligned}$$

The graphs of these two inequalities are shown in Figure 25.

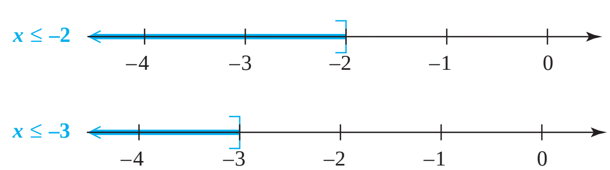


FIGURE 25

By taking the union, we obtain the interval $(-\infty, -2]$. It is graphed in Figure 26.

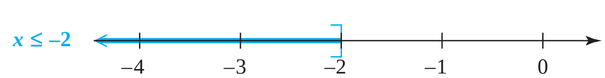


FIGURE 26

Now Try Exercise 37.

EXAMPLE 8 Applying Intersection and Union

The five highest domestic grossing films (adjusted for inflation) are listed in the table.

Five All-Time Highest Grossing Films

Film	Admissions	Gross Income
Gone with the Wind	200,605,313	$972,900,000
Star Wars	178,119,595	$863,900,000
The Sound of Music	142,415,376	$690,700,000
E.T.	135,987,938	$659,500,000
The Ten Commandments	131,000,000	$635,400,000

Source: New York Times Almanac, 2001.

List the elements of the following sets.

(a) The set of top five films with admissions greater than 180,000,000 *and* gross greater than $800,000,000

The only film that satisfies both conditions is *Gone with the Wind,* so the set is

{*Gone with the Wind*}.

(b) The set of top five films with admissions less than 170,000,000 *or* gross greater than $700,000,000

Here, a film that satisfies at least one of the conditions is in the set. This set includes all five films:

{*Gone with the Wind*, *Star Wars*, *The Sound of Music*, *E.T.*, *The Ten Commandments*}.

Now Try Exercise 69.

2.6 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 3/Videotape 3

Decide whether each statement is true *or* false. *If it is false, explain why.*

1. The union of the solution sets of $x + 1 = 5$, $x + 1 < 5$, and $x + 1 > 5$ is $(-\infty, \infty)$.

2. The intersection of the sets $\{x \mid x \geq 7\}$ and $\{x \mid x \leq 7\}$ is $\emptyset$.

3. The union of the sets $(-\infty, 8)$ and $(8, \infty)$ is $\{8\}$.

4. The intersection of the sets $(-\infty, 8]$ and $[8, \infty)$ is $\{8\}$.

5. The intersection of the set of rational numbers and the set of irrational numbers is $\{0\}$.

6. The union of the set of rational numbers and the set of irrational numbers is the set of real numbers.

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5\}$, $C = \{1, 6\}$, *and* $D = \{4\}$. *Specify each set. See Examples 1 and 5.*

EXERCISES

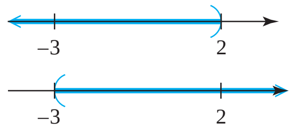
7. $B \cap A$ **8.** $A \cap B$ **9.** $A \cap D$ **10.** $B \cap C$

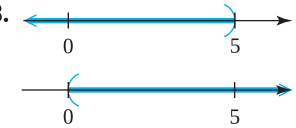
11. $B \cap \emptyset$ **12.** $A \cap \emptyset$ **13.** $A \cup B$ **14.** $B \cup D$

15. Give an example of intersection applied to a real-life situation.

16. A compound inequality uses one of the words *and* or *or*. Explain how you will determine whether to use *intersection* or *union* when graphing the solution set.

Two sets are specified by graphs. Graph the intersection of the two sets.

17.


18.


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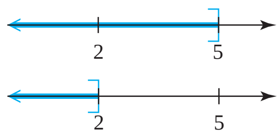
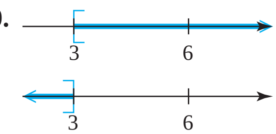


20.


For each compound inequality, give the solution set in both interval and graph forms. See Examples 2–4.

21. $x < 2$ and $x > -3$

22. $x < 5$ and $x > 0$

23. $x \le 2$ and $x \le 5$

24. $x \ge 3$ and $x \ge 6$

25. $x \le 3$ and $x \ge 6$

26. $x \le -1$ and $x \ge 3$

27. $x - 3 \le 6$ and $x + 2 \ge 7$

28. $x + 5 \le 11$ and $x - 3 \ge -1$

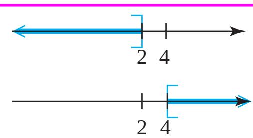
29. $-3x > 3$ and $x + 3 > 0$

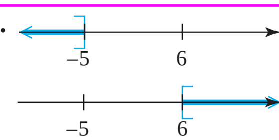
30. $-3x < 3$ and $x + 2 < 6$

31. $3x - 4 \le 8$ and $-4x + 1 \ge -15$

32. $7x + 6 \le 48$ and $-4x \ge -24$

Two sets are specified by graphs. Graph the union of the two sets.

33.


34.


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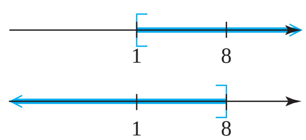

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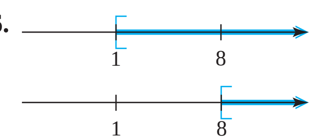


For each compound inequality, give the solution set in both interval and graph forms. See Examples 6 and 7.

37. $x \le 1$ or $x \le 8$

38. $x \ge 1$ or $x \ge 8$

39. $x \ge -2$ or $x \ge 5$

40. $x \le -2$ or $x \le 6$

41. $x \ge -2$ or $x \le 4$

42. $x \ge 5$ or $x \le 7$

43. $x + 2 > 7$ or $1 - x > 6$

44. $x + 1 > 3$ or $x + 4 < 2$

45. $x + 1 > 3$ or $-4x + 1 > 5$

46. $3x < x + 12$ or $x + 1 > 10$

Express each set in the simplest interval form. (Hint: Graph each set and look for the intersection or union.)

47. $(-\infty, -1] \cap [-4, \infty)$

48. $[-1, \infty) \cap (-\infty, 9]$

49. $(-\infty, -6] \cap [-9, \infty)$

50. $(5, 11] \cap [6, \infty)$

51. $(-\infty, 3) \cup (-\infty, -2)$

52. $[-9, 1] \cup (-\infty, -3)$

53. $[3, 6] \cup (4, 9)$

54. $[-1, 2] \cup (0, 5)$

For each compound inequality, decide whether intersection *or* union *should be used. Then give the solution set in both interval and graph forms. See Examples 2, 3, 4, 6, and 7.*

EXERCISES

55. $x < -1$ and $x > -5$

56. $x > -1$ and $x < 7$

57. $x < 4$ or $x < -2$

58. $x < 5$ or $x < -3$

59. $-3x \leq -6$ or $-3x \geq 0$

60. $2x - 6 \leq -18$ and $2x \geq -18$

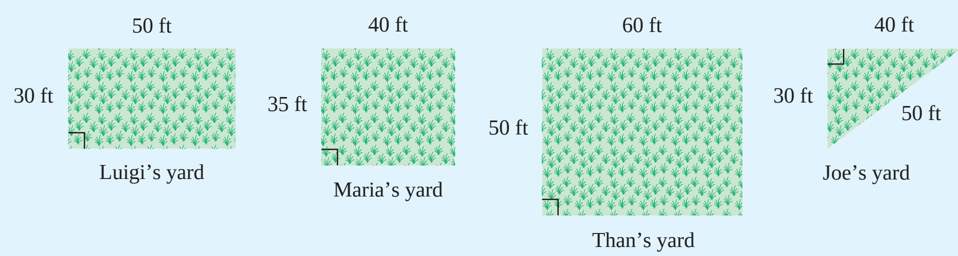
61. $x + 1 \geq 5$ and $x - 2 \leq 10$

62. $-8x \leq -24$ or $-5x \geq 15$

RELATING CONCEPTS (EXERCISES 63–68)

For Individual or Group Work

The figures represent the backyards of neighbors Luigi, Maria, Than, and Joe. Find the area and the perimeter of each yard. Suppose that each resident has 150 ft *of fencing and enough sod to cover* 1400 ft^2 *of lawn. Give the name or names of the residents whose yards satisfy each description.* ***Work Exercises 63–68 in order.***



63. The yard can be fenced *and* the yard can be sodded.

64. The yard can be fenced *and* the yard cannot be sodded.

65. The yard cannot be fenced *and* the yard can be sodded.

66. The yard cannot be fenced *and* the yard cannot be sodded.

67. The yard can be fenced *or* the yard can be sodded.

68. The yard cannot be fenced *or* the yard can be sodded.

EXERCISES

Use the graphs to answer Exercises 69 and 70. See Example 8.

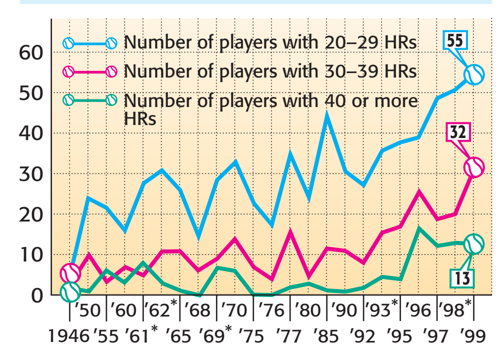
69. In which years did the number of players with 30–39 home runs exceed 20 *and* the number of players with 40 or more home runs exceed 45?

70. In which years were the number of players with 20–29 home runs less than 20 *or* the number of players with 30–39 home runs at least 20?

photo not available

GOING, GOING, GONE

Home runs have been flying out of major-league ballparks at an increasing rate. A breakdown:



*Indicates expansion year

Source: Bee research.

2.7 Absolute Value Equations and Inequalities

OBJECTIVES

1. Use the distance definition of absolute value.
2. Solve equations of the form $|ax + b| = k$, for $k > 0$.
3. Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.
4. Solve absolute value equations that involve rewriting.
5. Solve equations of the form $|ax + b| = |cx + d|$.
6. Solve special cases of absolute value equations and inequalities.

In a production line, quality is controlled by randomly choosing items from the line and checking to see how selected measurements vary from the optimum measure. These differences are sometimes positive and sometimes negative, so they are expressed with absolute value. For example, a machine that fills quart milk cartons might be set to release 1 qt (32 oz) plus or minus 2 oz per carton. Then the number of ounces in each carton should satisfy the *absolute value inequality* $|x - 32| \le 2$, where x is the number of ounces.

OBJECTIVE 1 Use the distance definition of absolute value. In Chapter 1 we saw that the absolute value of a number x, written $|x|$, represents the distance from x to 0 on the number line. For example, the solutions of $|x| = 4$ are 4 and -4, as shown in Figure 27.

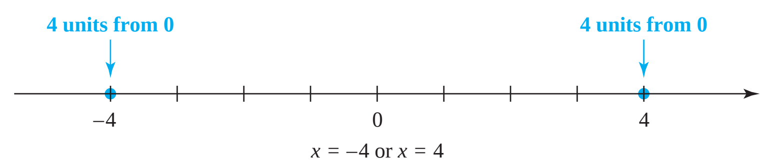


FIGURE 27

Because absolute value represents distance from 0, it is reasonable to interpret the solutions of $|x| > 4$ to be all numbers that are *more* than 4 units from 0. The set $(-\infty, -4) \cup (4, \infty)$ fits this description. Figure 28 shows the graph of the solution set of $|x| > 4$. Because the graph consists of two separate intervals, the solution set is described using *or* as

$$x < -4 \quad \text{or} \quad x > 4.$$

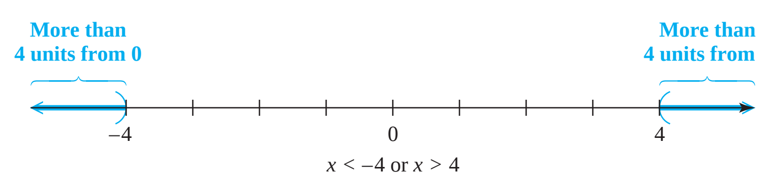


FIGURE 28

The solution set of $|x| < 4$ consists of all numbers that are *less* than 4 units from 0 on the number line. Another way of thinking of this is to think of all numbers *between* -4 and 4. This set of numbers is given by $(-4, 4)$, as shown in Figure 29. Here, the graph shows that $-4 < x < 4$, which means $x > -4$ *and* $x < 4$.

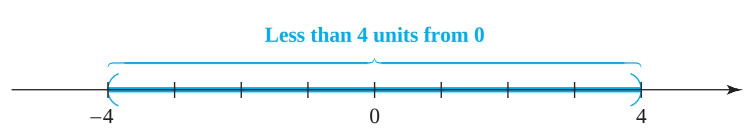


FIGURE 29

The equation and inequalities just described are examples of **absolute value equations and inequalities.** They involve the absolute value of a variable expression and generally take the form

$$|ax + b| = k, \qquad |ax + b| > k, \qquad \text{or} \qquad |ax + b| < k,$$

where k is a positive number. From Figures 27–29, we see that

$|x| = 4$ has the same solution set as $x = -4$ or $x = 4$,
$|x| > 4$ has the same solution set as $x < -4$ or $x > 4$,
$|x| < 4$ has the same solution set as $x > -4$ and $x < 4$.

Thus, we solve an absolute value equation or inequality by first rewriting it as an equivalent statement without absolute value bars. Notice that, except for special cases, the solution set of an absolute value *equation* includes exactly *two points*. However, the solution set of an absolute value *inequality* includes one or more *intervals*. We summarize these facts in the next box.

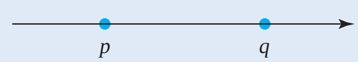
Solving Absolute Value Equations and Inequalities

Let k be a positive real number, and p and q be real numbers.

1. To solve $|ax + b| = k$, solve the compound equation

$$ax + b = k \quad \text{or} \quad ax + b = -k.$$

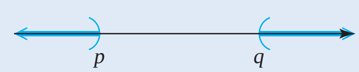
The solution set is usually of the form $\{p, q\}$, which includes two numbers.



2. To solve $|ax + b| > k$, solve the compound inequality

$$ax + b > k \quad \text{or} \quad ax + b < -k.$$

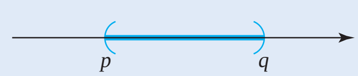
The solution set is of the form $(-\infty, p) \cup (q, \infty)$, which consists of two separate intervals.



3. To solve $|ax + b| < k$, solve the compound inequality

$$-k < ax + b < k.$$

The solution set is of the form (p, q), a single interval.



NOTE Some people prefer to write the compound statements in parts 1 and 2 of the preceding summary as

$$ax + b = k \quad \text{or} \quad -(ax + b) = k$$

and

$$ax + b > k \quad \text{or} \quad -(ax + b) > k.$$

These forms are equivalent to those we give in the summary and produce the same results.

OBJECTIVE 2 Solve equations of the form $|ax + b| = k$, for $k > 0$. The next example shows how we use a compound equation to solve a typical absolute value equation. Remember that because absolute value refers to distance from the origin, each absolute value equation will have two parts.

EXAMPLE 1 Solving an Absolute Value Equation

Solve $|2x + 1| = 7$.

For $|2x + 1|$ to equal 7, $2x + 1$ must be 7 units from 0 on the number line. This can happen only when $2x + 1 = 7$ or $2x + 1 = -7$. This is the first case in the preceding summary. Solve this compound equation as follows.

$$\begin{aligned} 2x + 1 &= 7 \quad &\text{or} \quad 2x + 1 &= -7 \\ 2x &= 6 \quad &\text{or} \quad 2x &= -8 \\ x &= 3 \quad &\text{or} \quad x &= -4 \end{aligned}$$

Check by substitution in the original absolute value equation to verify that the solution set is $\{-4, 3\}$. The graph is shown in Figure 30.

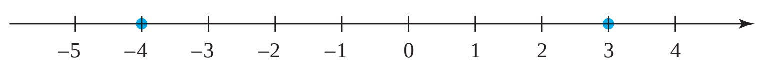


FIGURE 30

Now Try Exercise 11.

OBJECTIVE 3 Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.

EXAMPLE 2 Solving an Absolute Value Inequality with >

Solve $|2x + 1| > 7$.

By part 2 of the summary, this absolute value inequality is rewritten as

$$2x + 1 > 7 \quad \text{or} \quad 2x + 1 < -7,$$

because $2x + 1$ must represent a number that is *more* than 7 units from 0 on either side of the number line. Now, solve the compound inequality.

$$\begin{aligned} 2x + 1 &> 7 \quad &\text{or} \quad 2x + 1 &< -7 \\ 2x &> 6 \quad &\text{or} \quad 2x &< -8 \\ x &> 3 \quad &\text{or} \quad x &< -4 \end{aligned}$$

Check these solutions. The solution set is $(-\infty, -4) \cup (3, \infty)$. See Figure 31. Notice that the graph consists of two intervals.

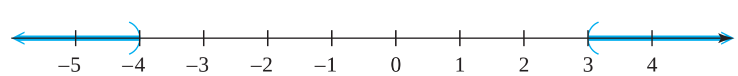


FIGURE 31

Now Try Exercise 25.

VIDEO EXPLORE

EXAMPLE 3 Solving an Absolute Value Inequality with <

Solve $|2x + 1| < 7$.

The expression $2x + 1$ must represent a number that is less than 7 units from 0 on either side of the number line. Another way of thinking of this is to realize that $2x + 1$ must be between -7 and 7. As part 3 of the summary shows, this is written as the three-part inequality

$$-7 < 2x + 1 < 7.$$

We solved such inequalities in Section 2.5 by working with all three parts at the same time.

$$-7 < 2x + 1 < 7$$
$$-8 < 2x < 6 \qquad \text{Subtract 1 from each part.}$$
$$-4 < x < 3 \qquad \text{Divide each part by 2.}$$

Check that the solution set is $(-4, 3)$, so the graph consists of the single interval shown in Figure 32.

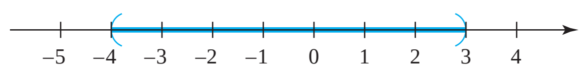


FIGURE 32

Now Try Exercise 39.

Look back at Figures 30, 31, and 32, with the graphs of $|2x + 1| = 7$, $|2x + 1| > 7$, and $|2x + 1| < 7$. If we find the union of the three sets, we get the set of all real numbers. This is because for any value of x, $|2x + 1|$ will satisfy one and only one of the following: it is equal to 7, greater than 7, or less than 7.

CAUTION When solving absolute value equations and inequalities of the types in Examples 1, 2, and 3, remember the following.

1. The methods described apply when the constant is alone on one side of the equation or inequality and is *positive.*
2. Absolute value equations and absolute value inequalities in the form $|ax + b| > k$ translate into "or" compound statements.
3. Absolute value inequalities in the form $|ax + b| < k$ translate into "and" compound statements, which may be written as three-part inequalities.
4. An "or" statement *cannot* be written in three parts. It would be incorrect to use

$$-7 > 2x + 1 > 7$$

in Example 2, because this would imply that $-7 > 7$, which is *false.*

OBJECTIVE 4 Solve absolute value equations that involve rewriting. Sometimes an absolute value equation or inequality requires some rewriting before it can be set up as a compound statement, as shown in the next example.

EXAMPLE 4 Solving an Absolute Value Equation That Requires Rewriting

Solve $|x + 3| + 5 = 12$.

First get the absolute value alone on one side of the equals sign by subtracting 5 from each side.

$$|x + 3| + 5 - 5 = 12 - 5 \qquad \text{Subtract 5.}$$
$$|x + 3| = 7$$

Now use the method shown in Example 1.

$$x + 3 = 7 \quad \text{or} \quad x + 3 = -7$$
$$x = 4 \quad \text{or} \quad x = -10$$

Check that the solution set is $\{4, -10\}$ by substituting into the original equation.

Now Try Exercise 63.

We use a similar method to solve an absolute value *inequality* that requires rewriting.

OBJECTIVE 5 Solve equations of the form $|ax + b| = |cx + d|$. By definition, for two expressions to have the same absolute value, they must either be equal or be negatives of each other.

Solving $|ax + b| = |cx + d|$

To solve an absolute value equation of the form

$$|ax + b| = |cx + d|,$$

solve the compound equation

$$ax + b = cx + d \quad \text{or} \quad ax + b = -(cx + d).$$

EXAMPLE 5 Solving an Equation with Two Absolute Values

Solve $|z + 6| = |2z - 3|$.

This equation is satisfied either if $z + 6$ and $2z - 3$ are equal to each other or if $z + 6$ and $2z - 3$ are negatives of each other.

$$z + 6 = 2z - 3 \quad \text{or} \quad z + 6 = -(2z - 3)$$

Solve each equation.

$$z + 6 = 2z - 3 \quad \text{or} \quad z + 6 = -2z + 3$$
$$6 + 3 = 2z - z \qquad\qquad 3z = -3$$
$$9 = z \quad \text{or} \quad z = -1$$

The solution set is $\{9, -1\}$.

Now Try Exercise 71.

OBJECTIVE 6 Solve special cases of absolute value equations and inequalities. When a typical absolute value equation or inequality involves a *negative constant or 0* alone on one side, use the properties of absolute value to solve. Keep in mind the following.

1. The absolute value of an expression can never be negative: $|a| \geq 0$ for all real numbers a.
2. The absolute value of an expression equals 0 only when the expression is equal to 0.

The next two examples illustrate these special cases.

EXAMPLE 6 Solving Special Cases of Absolute Value Equations

Solve each equation.

(a) $|5r - 3| = -4$

Since the absolute value of an expression can never be negative, there are no solutions for this equation. The solution set is $\emptyset$.

(b) $|7x - 3| = 0$

The expression $7x - 3$ will equal 0 *only* if

$$7x - 3 = 0.$$

The solution of this equation is $\frac{3}{7}$. The solution set is $\left\{\frac{3}{7}\right\}$. It consists of only one element that checks by substitution in the original equation.

Now Try Exercises 79 and 81.

EXAMPLE 7 Solving Special Cases of Absolute Value Inequalities

Solve each inequality.

(a) $|x| \geq -4$

The absolute value of a number is never negative. For this reason, $|x| \geq -4$ is true for *all* real numbers. The solution set is $(-\infty, \infty)$.

(b) $|k + 6| - 3 < -5$

Add 3 to both sides to get the absolute value expression alone on one side.

$$|k + 6| < -2$$

There is no number whose absolute value is less than -2, so this inequality has no solution. The solution set is $\emptyset$.

(c) $|m - 7| + 4 \leq 4$

Adding -4 to both sides gives

$$|m - 7| \leq 0.$$

The value of $|m - 7|$ will never be less than 0. However, $|m - 7|$ will equal 0 when $m = 7$. Therefore, the solution set is $\{7\}$.

Now Try Exercises 85 and 91.

CONNECTIONS

Absolute value is used to find the relative error of a measurement in science, engineering, manufacturing, and other fields. If x_t represents the expected value of a measurement and x represents the actual measurement, then the *relative error in x* equals the absolute value of the difference between x_t and x divided by x_t. That is,

$$\text{relative error in } x = \left|\frac{x_t - x}{x_t}\right|.$$

In many situations in the work world, the relative error must be less than some predetermined amount. For example, suppose a machine filling *quart* milk cartons is set for a relative error no greater than .05. Here $x_t = 32$ oz, the relative error $= .05$ oz, and we must find x, given

$$\left|\frac{32 - x}{32}\right| = \left|1 - \frac{x}{32}\right| \leq .05.$$

For Discussion or Writing

With this tolerance level, how many ounces may a carton contain?

2.7 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

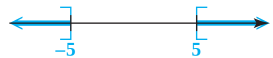
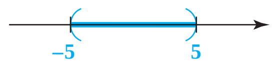
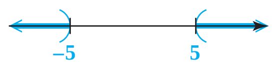
InterAct Math Tutorial Software

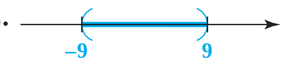
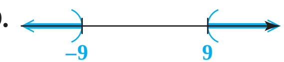
AW Math Tutor Center

MathXL

Digital Video Tutor CD 3/Videotape 3

Match each absolute value equation or inequality in Column I with the graph of its solution set in Column II.

	I		II
1.	$\lvert x \rvert = 5$	**A.**	
	$\lvert x \rvert < 5$	**B.**	
	$\lvert x \rvert > 5$	**C.**	
	$\lvert x \rvert \leq 5$	**D.**	
	$\lvert x \rvert \geq 5$	**E.**	

	I		II
2.	$\lvert x \rvert = 9$	**A.**	
	$\lvert x \rvert > 9$	**B.**	
	$\lvert x \rvert \geq 9$	**C.**	
	$\lvert x \rvert < 9$	**D.**	
	$\lvert x \rvert \leq 9$	**E.**	

3. Explain when to use *and* and when to use *or* if you are solving an absolute value equation or inequality of the form $\lvert ax + b \rvert = k$, $\lvert ax + b \rvert < k$, or $\lvert ax + b \rvert > k$, where k is a positive number.

4. How many solutions will $\lvert ax + b \rvert = k$ have if

(a) $k = 0$; **(b)** $k > 0$; **(c)** $k < 0$?

Solve each equation. See Example 1.

5. $|x| = 12$ 6. $|k| = 14$ 7. $|4x| = 20$ 8. $|5x| = 30$

9. $|y - 3| = 9$ 10. $|p - 5| = 13$ 11. $|2x - 1| = 11$ 12. $|2y + 3| = 19$

13. $|4r - 5| = 17$ 14. $|5t - 1| = 21$ 15. $|2y + 5| = 14$ 16. $|2x - 9| = 18$

17. $\left|\frac{1}{2}x + 3\right| = 2$ 18. $\left|\frac{2}{3}q - 1\right| = 5$

19. $\left|1 + \frac{3}{4}k\right| = 7$ 20. $\left|2 - \frac{5}{2}m\right| = 14$

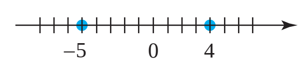
Solve each inequality and graph the solution set. See Example 2.

21. $|x| > 3$ 22. $|y| > 5$ 23. $|k| \geq 4$ 24. $|r| \geq 6$

25. $|r + 5| \geq 20$ 26. $|3x - 1| \geq 8$ 27. $|t + 2| > 10$ 28. $|4x + 1| \geq 21$

29. $|3 - x| > 5$ 30. $|5 - x| > 3$ 31. $|-5x + 3| \geq 12$ 32. $|-2x - 4| \geq 5$

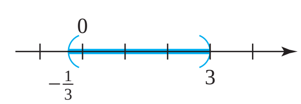
33. The graph of the solution set of $|2x + 1| = 9$ is given here.



Without actually doing the algebraic work, graph the solution set of each inequality, referring to the graph above.

(a) $|2x + 1| < 9$ **(b)** $|2x + 1| > 9$

34. The graph of the solution set of $|3x - 4| < 5$ is given here.



Without actually doing the algebraic work, graph the solution set of the following, referring to the graph above.

(a) $|3x - 4| = 5$ **(b)** $|3x - 4| > 5$

Solve each inequality and graph the solution set. See Example 3. (Hint: Compare your answers to those in Exercises 21–32.)

35. $|x| \leq 3$ 36. $|y| \leq 5$ 37. $|k| < 4$ 38. $|r| < 6$

39. $|r + 5| \leq 20$ 40. $|3x - 1| < 8$ 41. $|t + 2| \leq 10$ 42. $|4x + 1| < 21$

43. $|3 - x| \leq 5$ 44. $|5 - x| \leq 3$ 45. $|-5x + 3| \leq 12$ 46. $|-2x - 4| \leq 5$

Decide which method you should use to solve each absolute value equation or inequality. Find the solution set and on Exercises 47–58, graph the solution set. See Examples 1–3.

47. $|-4 + k| > 9$ 48. $|-3 + t| > 8$ 49. $|r + 5| > 20$ 50. $|2x - 1| < 7$

51. $|7 + 2z| = 5$ 52. $|9 - 3p| = 3$ 53. $|3r - 1| \leq 11$ 54. $|2s - 6| \leq 6$

55. $|-6x - 6| \leq 1$ 56. $|-2x - 6| \leq 5$ 57. $|2x - 1| \geq 7$ 58. $|-4 + k| \leq 9$

59. $|x| - 1 = 4$ 60. $|x + 3| = 10$ 61. $|x + 2| = 3$ 62. $|x - 4| = 1$

Solve each equation or inequality. Give the solution set in set notation for equations and in interval notation for inequalities. See Example 4.

EXERCISES

63. $|x + 4| + 1 = 2$ **64.** $|x + 5| - 2 = 12$ **65.** $|2x + 1| + 3 > 8$

66. $|6x - 1| - 2 > 6$ **67.** $|x + 5| - 6 \le -1$ **68.** $|r - 2| - 3 \le 4$

69. $|2 - x| > 3$ **70.** $|4 - x| < 1$

Solve each equation. See Example 5.

EXERCISES

71. $|3x + 1| = |2x + 4|$ **72.** $|7x + 12| = |x - 8|$

73. $\left|m - \dfrac{1}{2}\right| = \left|\dfrac{1}{2}m - 2\right|$ **74.** $\left|\dfrac{2}{3}r - 2\right| = \left|\dfrac{1}{3}r + 3\right|$

75. $|6x| = |9x + 1|$ **76.** $|13x| = |2x + 1|$

77. $|2p - 6| = |2p + 11|$ **78.** $|3x - 1| = |3x + 9|$

Solve each equation or inequality. See Examples 6 and 7.

EXERCISES

79. $|12t - 3| = -8$ **80.** $|13w + 1| = -3$ **81.** $|4x + 1| = 0$

82. $|6r - 2| = 0$ **83.** $|2q - 1| = -6$ **84.** $|8n + 4| = -4$

85. $|x + 5| > -9$ **86.** $|x + 9| > -3$ **87.** $|7x + 3| \le 0$

88. $|4x - 1| \le 0$ **89.** $|5x - 2| = 0$ **90.** $|4 + 7x| = 0$

91. $|10z + 7| + 3 < 1$ **92.** $|4x + 1| - 2 < -5$

93. The 1998 recommended daily intake (RDI) of calcium for females aged 19–50 is 1000 mg/day. (*Source: World Almanac and Book of Facts*, 2000.) Actual vitamin needs vary from person to person. Write an absolute value inequality, with x representing the RDI, to express the RDI plus or minus 100 mg/day and solve it.

94. The average clotting time of blood is 7.45 sec with a variation of plus or minus 3.6 sec. Write this statement as an absolute value inequality with x representing the time and solve it.

RELATING CONCEPTS (EXERCISES 95–98)

For Individual or Group Work

The ten tallest buildings in Kansas City, Missouri, are listed along with their heights.

Building	Height (in feet)
One Kansas City Place	632
AT&T Town Pavilion	590
Hyatt Regency	504
Kansas City Power and Light	476
City Hall	443
Fidelity Bank and Trust Building	433
1201 Walnut	427
Federal Office Building	413
Commerce Tower	407
City Center Square	404

Source: World Almanac and Book of Facts, 2001.

Use this information to ***work Exercises 95–98 in order.***

95. To find the average of a group of numbers, we add the numbers and then divide by the number of items added. Use a calculator to find the average of the heights.

96. Let k represent the average height of these buildings. If a height x satisfies the inequality

$$|x - k| < t,$$

then the height is said to be within t ft of the average. Using your result from Exercise 95, list the buildings that are within 50 ft of the average.

97. Repeat Exercise 96, but list the buildings that are within 75 ft of the average.

98. **(a)** Write an absolute value inequality that describes the height of a building that is *not* within 75 ft of the average.

(b) Solve the inequality you wrote in part (a).

(c) Use the result of part (b) to list the buildings that are not within 75 ft of the average.

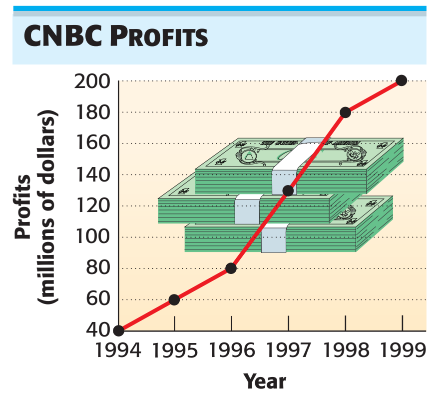
(d) Confirm that your answer to part (c) makes sense by comparing it with your answer to Exercise 97.

3.1 The Rectangular Coordinate System

OBJECTIVES

1. Plot ordered pairs.
2. Find ordered pairs that satisfy a given equation.
3. Graph lines.
4. Find *x*- and *y*-intercepts.
5. Recognize equations of vertical and horizontal lines.
6. Use a graphing calculator to graph an equation.

The line graph in Figure 1 shows profits for cable television station CNBC from 1994 through 1999.



Source: *Fortune*, May 24, 1999, p. 142.

FIGURE 1

The line graph represents information based on a method for locating a point in a plane developed by René Descartes, a 17th-century French mathematician. It is said that Descartes, who was lying in bed ill, was watching a fly crawl about on the ceiling near a corner of the room. It occurred to him that the location of the fly on the ceiling could be described by determining its distances from the two adjacent walls. In this chapter we use this insight to plot points and graph linear equations in two variables whose graphs are straight lines.

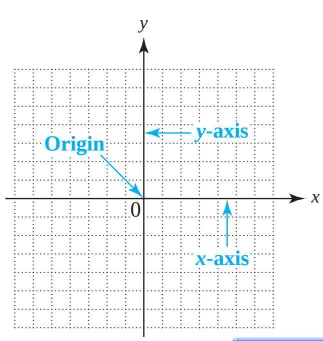


FIGURE 2

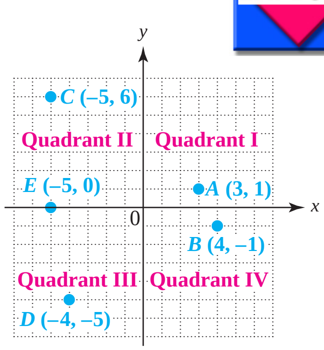


FIGURE 3

OBJECTIVE 1 Plot ordered pairs. Each of the pairs of numbers (3, 1), (−5, 6), and (4, −1) is an example of an **ordered pair;** that is, a pair of numbers written within parentheses in which the order of the numbers is important. We graph an ordered pair using two perpendicular number lines that intersect at their 0 points, as shown in Figure 2. The common 0 point is called the **origin.** The position of any point in this plane is determined by referring to the horizontal number line, the ***x*-axis,** and the vertical number line, the ***y*-axis.** The first number in the ordered pair indicates the position relative to the *x*-axis, and the second number indicates the position relative to the *y*-axis. The *x*-axis and the *y*-axis make up a **rectangular** (or **Cartesian,** for Descartes) **coordinate system.**

To locate, or **plot,** the point on the graph that corresponds to the ordered pair (3, 1), we move three units from 0 to the right along the *x*-axis, and then one unit up parallel to the *y*-axis. The point corresponding to the ordered pair (3, 1) is labeled *A* in Figure 3. Additional points are labeled *B*–*E*. The phrase "the point corresponding to the ordered pair (3, 1)" is often abbreviated as "the point (3, 1)." The numbers in an ordered pair are called the **coordinates** of the corresponding point.

We can relate this method of locating ordered pairs to the line graph in Figure 1. We move along the horizontal axis to a year, then up parallel to the vertical axis to find profit for that year. Thus, we can write the ordered pair (1998, 180) to indicate that in 1998, profit was $180 million.

CAUTION The parentheses used to represent an ordered pair are also used to represent an open interval (introduced in Chapter 1). The context of the discussion tells whether ordered pairs or open intervals are being represented.

The four regions of the graph, shown in Figure 3, are called **quadrants I, II, III,** and **IV,** reading counterclockwise from the upper right quadrant. The points on the x-axis and y-axis do not belong to any quadrant. For example, point E in Figure 3 belongs to no quadrant.

OBJECTIVE 2 Find ordered pairs that satisfy a given equation. Each solution to an equation with two variables, such as $2x + 3y = 6$, includes two numbers, one for each variable. To keep track of which number goes with which variable, we write the solutions as ordered pairs. (If x and y are used as the variables, the x-value is given first.) For example, we can show that $(6, -2)$ is a solution of $2x + 3y = 6$ by substitution.

$$\begin{aligned} 2x + 3y &= 6 \\ 2(6) + 3(-2) &\stackrel{?}{=} 6 \qquad \text{Let } x = 6,\ y = -2. \\ 12 - 6 &\stackrel{?}{=} 6 \\ 6 &= 6 \qquad \text{True} \end{aligned}$$

Because the ordered pair $(6, -2)$ makes the equation true, it is a solution. On the other hand, $(5, 1)$ is *not* a solution of the equation $2x + 3y = 6$ because

$$2(5) + 3(1) = 10 + 3 = 13 \neq 6.$$

To find ordered pairs that satisfy an equation, select any number for one of the variables, substitute it into the equation for that variable, and then solve for the other variable. Two other ordered pairs satisfying $2x + 3y = 6$ are $(0, 2)$ and $(3, 0)$. Since any real number could be selected for one variable and would lead to a real number for the other variable, linear equations in two variables have an infinite number of solutions.

EXAMPLE 1 Completing Ordered Pairs

Complete the table of ordered pairs for $2x + 3y = 6$.

x	y	
-3		← Represents the ordered pair $(-3, \_\_)$
	-4	← Represents the ordered pair $(\_\_, -4)$

First let $x = -3$ and substitute into the equation to find y. Then let $y = -4$ and substitute to find x.

$$\begin{aligned} 2x + 3y &= 6 \\ 2(-3) + 3y &= 6 \qquad \text{Let } x = -3. \\ -6 + 3y &= 6 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

The ordered pair is $(-3, 4)$.

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + 3(-4) &= 6 \qquad \text{Let } y = -4. \\ 2x - 12 &= 6 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

The ordered pair is $(9, -4)$.

These pairs lead to the following completed table.

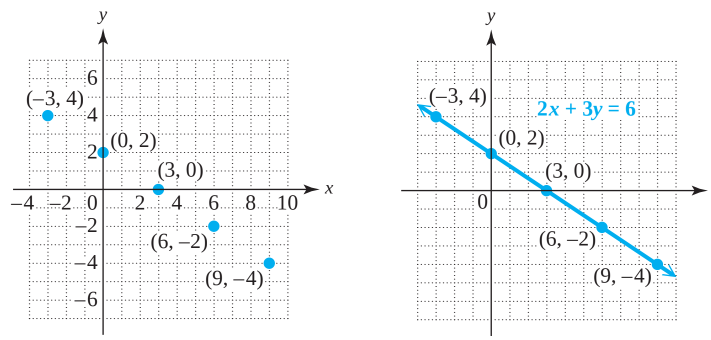
x	y
-3	4
9	-4

Now Try Exercise 23(a).

OBJECTIVE 3 Graph lines. Since an equation in two variables like $2x + 3y = 6$ is satisfied by an infinite number of ordered pairs, how might we express the solution set of such an equation? The **graph of an equation** is the set of points corresponding to *all* ordered pairs that satisfy the equation. It gives a "picture" of the equation.

To graph an equation, we plot a number of ordered pairs that satisfy the equation until we have enough points to suggest the shape of the graph. For example, to graph $2x + 3y = 6$, we plot all ordered pairs found in Objective 2 and Example 1. These points, shown in a table of values and plotted in Figure 4(a), appear to lie on a straight line. If all the ordered pairs that satisfy the equation $2x + 3y = 6$ were graphed, they would form the straight line shown in Figure 4(b).

x	y
-3	4
0	2
3	0
6	-2
9	-4



(a) (b)

FIGURE 4

Now Try Exercise 23(b).

The equation $2x + 3y = 6$ is called a **first-degree equation** because it has no term with a variable to a power greater than 1.

The graph of any first-degree equation in two variables is a straight line.

Since first-degree equations with two variables have straight-line graphs, they are called *linear equations in two variables.* (We discussed linear equations in one variable in Chapter 2.)

Linear Equation in Two Variables

A **linear equation in two variables** can be written in the form

$$Ax + By = C,$$

where A, B, and C are real numbers (A and B not both 0). This form is called **standard form.**

OBJECTIVE 4 Find *x*- and *y*-intercepts. A straight line is determined if any two different points on the line are known, so finding two different points is enough to graph the line. Two useful points for graphing are the x- and y-intercepts. The ***x*-intercept** is the point (if any) where the line intersects the x-axis; likewise, the ***y*-intercept** is the point (if any) where the line intersects the y-axis.* In Figure 4(b), the y-value of the point where the line intersects the x-axis is 0. Similarly, the x-value of the point where the line intersects the y-axis is 0. This suggests a method for finding the x- and y-intercepts.

Finding Intercepts

In the equation of a line, let $y = 0$ to find the x-intercept; let $x = 0$ to find the y-intercept.

EXAMPLE 2 Finding Intercepts

Find the x- and y-intercepts of $4x - y = -3$ and graph the equation.

We find the x-intercept by letting $y = 0$.

$$4x - 0 = -3 \qquad \text{Let } y = 0.$$

$$4x = -3$$

$$x = -\frac{3}{4} \qquad x\text{-intercept is } \left(-\tfrac{3}{4}, 0\right).$$

For the y-intercept, we let $x = 0$.

$$4(0) - y = -3 \qquad \text{Let } x = 0.$$

$$-y = -3$$

$$y = 3 \qquad y\text{-intercept is } (0, 3).$$

The intercepts are the two points $\left(-\frac{3}{4}, 0\right)$ and $(0, 3)$. We show these ordered pairs in the table next to Figure 5 and use them to draw the graph.

x	y
$-\frac{3}{4}$	0
0	3

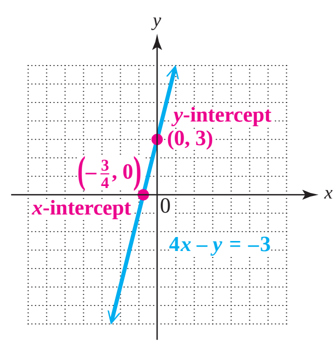


FIGURE 5

Now Try Exercise 33.

NOTE While two points, such as the two intercepts in Figure 5, are sufficient to graph a straight line, it is a good idea to use a third point to guard against errors. Verify by substitution that $(-2, -5)$ also lies on the graph of $4x - y = -3$.

*Some texts define an intercept as a number, not a point. For example, "y-intercept (0, 4)" would be given as "y-intercept 4."

OBJECTIVE 5 Recognize equations of vertical and horizontal lines. A graph can fail to have an x-intercept or a y-intercept, which is why the phrase "if any" was added when discussing intercepts.

EXAMPLE 3 Graphing a Horizontal Line

Graph $y = 2$.

Writing $y = 2$ as $0x + 1y = 2$ shows that any value of x, including $x = 0$, gives $y = 2$, making the y-intercept $(0, 2)$. Since y is always 2, there is no value of x corresponding to $y = 0$, so the graph has no x-intercept. The graph, shown with a table of ordered pairs in Figure 6, is a horizontal line.

x	y
-1	2
0	2
3	2

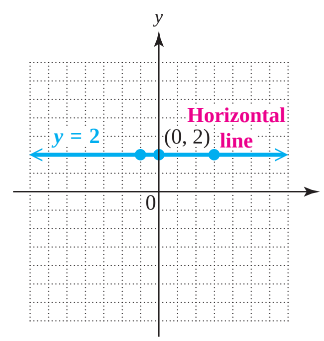


FIGURE 6

Now Try Exercise 39.

EXAMPLE 4 Graphing a Vertical Line

Graph $x + 1 = 0$.

The form $1x + 0y = -1$ shows that every value of y leads to $x = -1$, making the x-intercept $(-1, 0)$. No value of y makes $x = 0$, so the graph has no y-intercept. The only way a straight line can have no y-intercept is to be vertical, as shown in Figure 7.

x	y
-1	-4
-1	0
-1	5

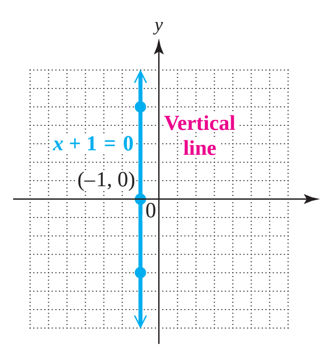


FIGURE 7

Now Try Exercise 43.

CAUTION To avoid confusing equations of horizontal and vertical lines remember that

1. An equation with only the variable x will always intersect the *x-axis* and thus will be *vertical*.
2. An equation with only the variable y will always intersect the *y-axis* and thus will be *horizontal*.

Some lines have both the x- and y-intercepts at the origin.

EXAMPLE 5 Graphing a Line That Passes through the Origin

Graph $x + 2y = 0$.

Find the intercepts.

$$\begin{aligned} x + 2y &= 0 \\ x + 2(0) &= 0 && \text{Let } y = 0. \\ x + 0 &= 0 \\ x &= 0 && x\text{-intercept is } (0, 0). \end{aligned}$$

$$\begin{aligned} x + 2y &= 0 \\ 0 + 2y &= 0 && \text{Let } x = 0. \\ y &= 0 && y\text{-intercept is } (0, 0). \end{aligned}$$

Both intercepts are the same point, (0, 0), which means that the graph passes through the origin. To find another point to graph the line, choose any nonzero number for x, say $x = 4$, and solve for y.

$$\begin{aligned} x + 2y &= 0 \\ 4 + 2y &= 0 && \text{Let } x = 4. \\ 2y &= -4 \\ y &= -2 \end{aligned}$$

This gives the ordered pair $(4, -2)$. These two points lead to the graph shown in Figure 8. As a check, verify that $(-2, 1)$ also lies on the line.

x	y
-2	1
0	0
4	-2

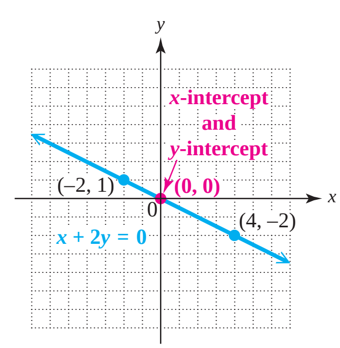
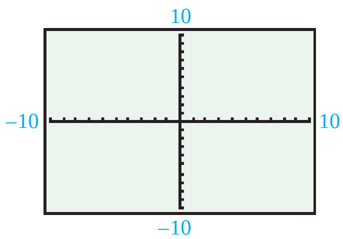


FIGURE 8

To find the additional point, we could have chosen any number (except 0) for y instead of x.

Now Try Exercise 45.



Standard viewing window

FIGURE 9

OBJECTIVE 6 Use a graphing calculator to graph an equation. When graphing by hand, we first set up a rectangular coordinate system, then plot points and draw the graph. Similarly, when graphing with a graphing calculator, we first tell the calculator how to set up a rectangular coordinate system. This involves choosing the minimum and maximum x- and y-values that will determine the viewing screen. In the screen shown in Figure 9, we chose minimum x- and y-values of -10 and maximum x- and y-values of 10. The *scale* on each axis determines the distance between the tick marks; in the screen shown, the scale is 1 for both axes. We refer to this as the *standard viewing window.*

To graph an equation, we usually need to solve the equation for y in order to enter it into the calculator. Once the equation is graphed, we can use the calculator to find the intercepts or any other point on the graph easily.

EXAMPLE 6 Graphing a Linear Equation with a Graphing Calculator and Finding the Intercepts

Use a graphing calculator to graph $4x - y = 3$.

Because we want to be able to see the intercepts on the screen, we use them to determine an appropriate window. Here, the x-intercept is $(.75, 0)$ and the y-intercept is $(0, -3)$. Although many choices are possible, we choose the standard viewing window. We must solve the equation for y to enter it into the calculator.

$$\begin{aligned} 4x - y &= 3 \\ -y &= -4x + 3 && \text{Subtract } 4x. \\ y &= 4x - 3 && \text{Multiply by } -1. \end{aligned}$$

The graph is shown in Figures 10 and 11, which also give the intercepts at the bottoms of the screens.

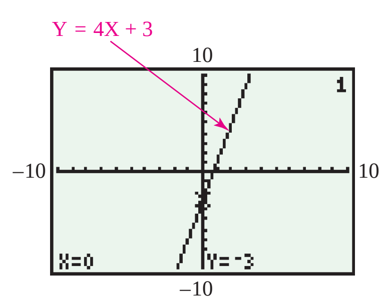


FIGURE 10

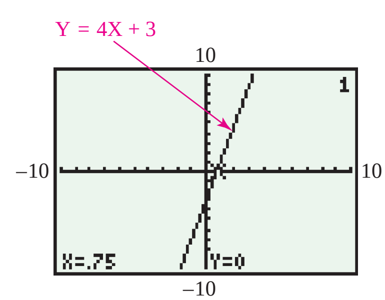


FIGURE 11

Some calculators have the capability of locating the x-intercept (called "Root" or "Zero"). Consult your owner's manual.

Now Try Exercise 59.

3.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 4/Videotape 4

In Exercises 1 and 2, answer each question by locating ordered pairs on the graphs.

1. The graph shows the percent of women in math or computer science professions.

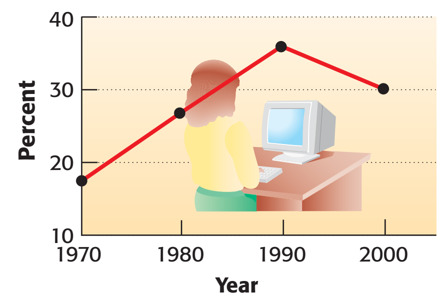
(a) If (x, y) represents a point on the graph, what does x represent? What does y represent?

(b) In what decade (10-yr period) did the percent of women in math or computer science professions decrease?

(c) Write an ordered pair (x, y) that gives the approximate percent of women in math or computer science professions in 1990.

(d) What does the ordered pair (2000, 30) mean in the context of this graph?

WOMEN IN MATH OR COMPUTER SCIENCE PROFESSIONS



Source: U.S. Bureau of the Census and Bureau of Labor Statistics.

2. The graph indicates federal government tax revenues in billions of dollars.

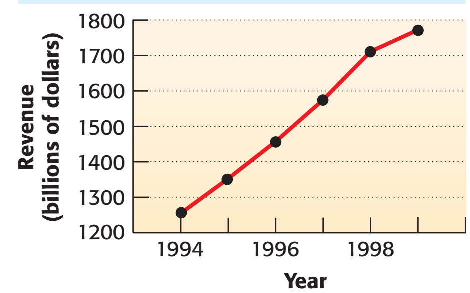
(a) If (x, y) represents a point on the graph, what does x represent? What does y represent?

(b) Estimate revenue in 1996.

(c) Write an ordered pair (x, y) that gives approximate federal tax revenues in 1995.

(d) What does the ordered pair (1998, 1720) mean in the context of this graph?

FEDERAL TAX REVENUES



Source: U.S. Office of Management and Budget.

Fill in each blank with the correct response.

3. The point with coordinates (0, 0) is called the _____ of a rectangular coordinate system.

4. For any value of x, the point $(x, 0)$ lies on the _____-axis.

5. To find the x-intercept of a line, we let _____ equal 0 and solve for _____; to find the y-intercept, we let _____ equal 0 and solve for _____.

6. The equation _____ $= 4$ has a horizontal line as its graph.
(x or y)

7. To graph a straight line, we must find a minimum of _____ points.

8. The point (_____, 4) is on the graph of $2x - 3y = 0$.

Name the quadrant, if any, in which each point is located.

9. **(a)** (1, 6) **(b)** $(-4, -2)$ **(c)** $(-3, 6)$ **(d)** $(7, -5)$ **(e)** $(-3, 0)$

10. **(a)** $(-2, -10)$ **(b)** (4, 8) **(c)** $(-9, 12)$ **(d)** $(3, -9)$ **(e)** $(0, -8)$

11. Use the given information to determine the possible quadrants in which the point (x, y) must lie.

(a) $xy > 0$ **(b)** $xy < 0$ **(c)** $\frac{x}{y} < 0$ **(d)** $\frac{x}{y} > 0$

12. What must be true about the coordinates of any point that lies along an axis?

Plot each point on a rectangular coordinate system.

13. $(2, 3)$ **14.** $(-1, 2)$ **15.** $(-3, -2)$ **16.** $(1, -4)$ **17.** $(0, 5)$
18. $(-2, -4)$ **19.** $(-2, 4)$ **20.** $(3, 0)$ **21.** $(-2, 0)$ **22.** $(3, -3)$

In Exercises 23–28, ***(a)*** *complete the given table for each equation, and then* ***(b)*** *graph the equation. See Example 1 and Figure 4.*

23. $x - y = 3$

x	y
0	
	0
5	
2	

24. $x - y = 5$

x	y
0	
	0
1	
3	

25. $x + 2y = 5$

x	y
0	
	0
2	
	2

26. $x + 3y = -5$

x	y
0	
	0
1	
	−1

27. $4x - 5y = 20$

x	y
0	
	0
2	
	−3

28. $6x - 5y = 30$

x	y
0	
	0
3	
	−2

29. Explain why the graph of $x + y = k$ cannot pass through quadrant III if $k > 0$.

30. Explain how to determine the intercepts and graph of the linear equation $4x - 3y = 12$.

31. A student attempted to graph $4x + 5y = 0$ by finding intercepts. She first let $x = 0$ and found y; then she let $y = 0$ and found x. In both cases, the resulting point was $(0, 0)$. She knew that she needed at least two points to graph the line, but was unsure what to do next because finding intercepts gave her only one point. Explain to her what to do next.

32. What is the equation of the x-axis? What is the equation of the y-axis?

Find the x- and y-intercepts. Then graph each equation. See Examples 2–5.

33. $2x + 3y = 12$ **34.** $5x + 2y = 10$ **35.** $x - 3y = 6$

36. $x - 2y = -4$ **37.** $\frac{2}{3}x - 3y = 7$ **38.** $\frac{5}{7}x + \frac{6}{7}y = -2$

39. $y = 5$ **40.** $y = -3$ **41.** $x = 2$

42. $x = -3$ **43.** $x + 4 = 0$ **44.** $y + 2 = 0$

45. $x + 5y = 0$ **46.** $x - 3y = 0$ **47.** $2x = 3y$

48. $4y = 3x$ **49.** $-\frac{2}{3}y = x$ **50.** $3y = -\frac{4}{3}x$

A linear equation can be used as a model to describe real data in some cases. Exercises 51 and 52 are based on this idea.

51. Track qualifying records at North Carolina Motor Speedway from 1965–1998 are approximated by the linear equation $y = 1.22x + 118$, where y is the speed (in miles per hour) in year x. In the equation, $x = 0$ corresponds to 1965, $x = 10$ corresponds to 1975, and so on. Use the equation to approximate the speed of the 1995 winner, Hut Stricklin. (*Source:* NASCAR.)

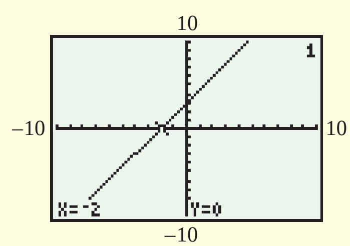
photo not available

52. According to Families USA Foundation, the national average family health care cost in dollars between 1980 and 2000 can be approximated by the linear equation $y = 382.75x + 1742$, where $x = 0$ corresponds to 1980 and $x = 20$ corresponds to 2000. Based on this equation, find the national average health care cost in 2000.

TECHNOLOGY INSIGHTS (EXERCISES 53–57)

53. The screens show the graph of one of the equations in A–D. Which equation is it?

A. $3x + 2y = 6$ **B.** $-3x + 2y = 6$ **C.** $-3x - 2y = 6$ **D.** $3x - 2y = 6$



54. The table of ordered pairs was generated by a graphing calculator with a TABLE feature.

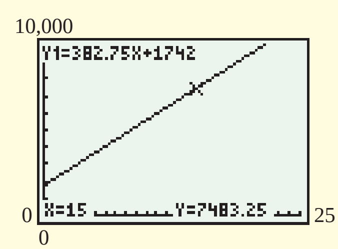
(a) What is the x-intercept?
(b) What is the y-intercept?
(c) Which equation corresponds to this table of values?

A. $Y_1 = 2X - 3$ **B.** $Y_1 = -2X - 3$
C. $Y_1 = 2X + 3$ **D.** $Y_1 = -2X + 3$

X	Y_1
0	3
.5	2
1	1
1.5	0
2	-1
2.5	-2
3	-3

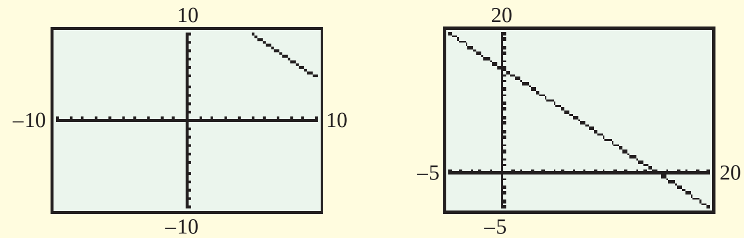
X=0

55. Refer to the equation in Exercise 52. A portion of its graph is shown on the accompanying screen, along with the coordinates of a point on the line displayed at the bottom. How is this point interpreted in the context of the model?

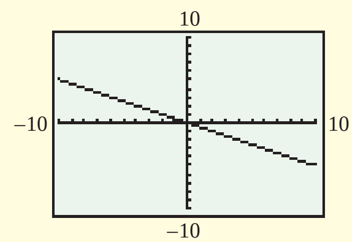


(continued)

56. The screens each show the graph of $x + y = 15$ (which was entered as $y = -x + 15$). However, different viewing windows are used. Which window would be more useful for this graph? Why?



57. The screen shows the graph of $x + 2y = 0$ from Example 5. In what form should you enter the equation into the calculator?



Graph each equation using a graphing calculator. Use a standard viewing window. See Example 6.

58. $4x - y = 10$

59. $5x + 2y = -10$

60. $3x + 4y = -6$

61. $3.6x - y = -5.8$

62. $y - 4.2 = 1.5x$

RELATING CONCEPTS (EXERCISES 63–68)

For Individual or Group Work

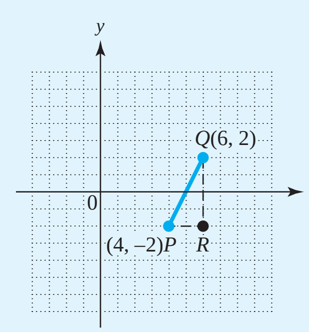
If the endpoints of a line segment are known, then the coordinates of the midpoint of the segment can be found. The figure shows the coordinates of the points P and Q. Let $\overline{PQ}$ represent the line segment with endpoints at P and Q. To derive a formula for the midpoint of $\overline{PQ}$, ***work Exercises 63–68 in order.***

63. In the figure, R is the point with the same x-coordinate as Q and the same y-coordinate as P. Write the ordered pair that corresponds to R.

64. From the graph, determine the coordinates of the midpoint of $\overline{PR}$.

65. From the graph, determine the coordinates of the midpoint of $\overline{QR}$.

66. The x-coordinate of the midpoint M of $\overline{PQ}$ is the x-coordinate of the midpoint of $\overline{PR}$ and the y-coordinate is the y-coordinate of the midpoint of $\overline{QR}$. Write the ordered pair that corresponds to M.



3.2 The Slope of a Line

OBJECTIVES

1. Find the slope of a line given two points on the line.
2. Find the slope of a line given an equation of the line.
3. Graph a line given its slope and a point on the line.
4. Use slopes to determine whether two lines are parallel, perpendicular, or neither.
5. Solve problems involving average rate of change.

Slope (steepness) is used in many practical ways. The slope of a highway (sometimes called the *grade*) is often given as a percent. For example, a 10% $\left(\text{or } \frac{10}{100} = \frac{1}{10}\right)$ slope means the highway rises 1 unit for every 10 horizontal units. Stairs and roofs have slopes too, as shown in Figure 12.

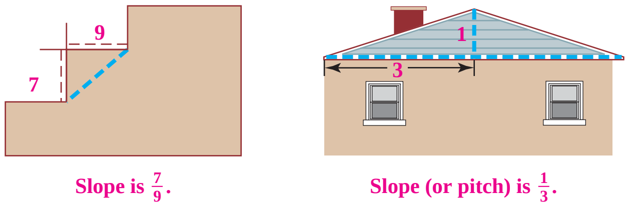


FIGURE 12

In each example mentioned, slope is the ratio of vertical change, or **rise,** to horizontal change, or **run.** A simple way to remember this is to think "slope is rise over run."

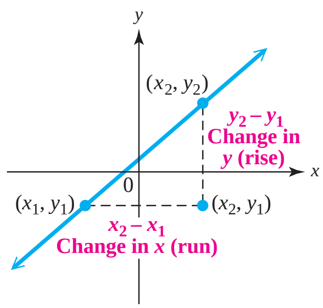


FIGURE 13

OBJECTIVE 1 Find the slope of a line given two points on the line. To get a formal definition of the slope of a line, we designate two different points on the line. To differentiate between the points, we write them as (x_1, y_1) and (x_2, y_2). See Figure 13. (The small numbers 1 and 2 in these ordered pairs are called *subscripts.* Read (x_1, y_1) as "x-sub-one, y-sub-one.")

As we move along the line in Figure 13 from (x_1, y_1) to (x_2, y_2), the y-value changes (vertically) from y_1 to y_2, an amount equal to $y_2 - y_1$. As y changes from y_1 to y_2, the value of x changes (horizontally) from x_1 to x_2 by the amount $x_2 - x_1$. The ratio of the change in y to the change in x (the rise over the run) is called the *slope* of the line, with the letter m traditionally used for slope.

AUDIO

Slope Formula

The **slope** of the line through the distinct points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2).$$

VIDEO EXPLORE

EXAMPLE 1 Finding the Slope of a Line

Find the slope of the line through the points $(2, -1)$ and $(-5, 3)$.

If $(2, -1) = (x_1, y_1)$ and $(-5, 3) = (x_2, y_2)$, then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{-5 - 2} = \frac{4}{-7} = -\frac{4}{7}.$$

See Figure 14. If the pairs are reversed so that $(2, -1) = (x_2, y_2)$ and $(-5, 3) = (x_1, y_1)$, the slope is the same.

$$m = \frac{-1 - 3}{2 - (-5)} = \frac{-4}{7} = -\frac{4}{7}$$

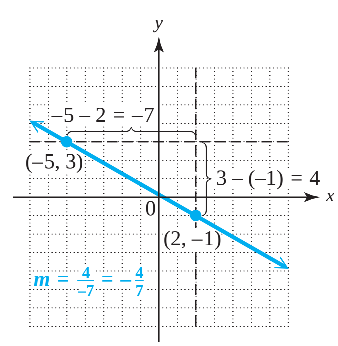


FIGURE 14

Now Try Exercise 19.

Example 1 suggests that the slope is the same no matter which point we consider first. Also, using similar triangles from geometry, we can show that the slope is the same no matter which two different points on the line we choose.

CAUTION In calculating slope, be careful to subtract the y-values and the x-values in the *same order.*

Correct	**Incorrect**
$\dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$	$\cancel{\dfrac{y_2 - y_1}{x_1 - x_2}}$ or $\cancel{\dfrac{y_1 - y_2}{x_2 - x_1}}$

Also, remember that the change in y is the *numerator* and the change in x is the *denominator.*

OBJECTIVE 2 Find the slope of a line given an equation of the line. When an equation of a line is given, one way to find the slope is to use the definition of slope by first finding two different points on the line.

EXAMPLE 2 Finding the Slope of a Line

Find the slope of the line $4x - y = -8$.

The intercepts can be used as the two different points needed to find the slope. Let $y = 0$ to find that the x-intercept is $(-2, 0)$. Then let $x = 0$ to find that the y-intercept is $(0, 8)$. Use these two points in the slope formula. The slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{8 - 0}{0 - (-2)} = \frac{8}{2} = 4.$$

Now Try Exercise 29.

EXAMPLE 3 Finding Slopes of Horizontal and Vertical Lines

Find the slope of each line.

(a) $y = 2$

Figure 6 in Section 3.1 shows that the graph of $y = 2$ is a horizontal line. To find the slope, select two different points on the line, such as $(3, 2)$ and $(-1, 2)$, and use the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{2 - 2}{3 - (-1)} = \frac{0}{4} = 0$$

In this case, the *rise* is 0, so the slope is 0.

(b) $x = -1$

As shown in Figure 7 (Section 3.1), the graph of $x = -1$ or $x + 1 = 0$ is a vertical line. Two points that satisfy the equation $x = -1$ are $(-1, 5)$ and $(-1, -4)$. If we use these two points to try to find the slope, we obtain

$$m = \frac{\text{rise}}{\text{run}} = \frac{-4 - 5}{-1 - (-1)} = \frac{-9}{0}.$$

Since division by 0 is undefined, the slope is undefined. This is why the definition of slope includes the restriction $x_1 \neq x_2$.

Now Try Exercises 35 and 37.

Generalizing from Example 3, we can make the following statements about slopes of horizontal and vertical lines.

AUDIO

Slopes of Horizontal and Vertical Lines

The slope of a horizontal line is 0; the slope of a vertical line is undefined.

The slope of a line can also be found directly from its equation. Look again at the equation $4x - y = -8$ from Example 2. Solve this equation for y.

$$\begin{aligned} 4x - y &= -8 && \text{Equation from Example 2} \\ -y &= -4x - 8 && \text{Subtract } 4x. \\ y &= 4x + 8 && \text{Multiply by } -1. \end{aligned}$$

Notice that the slope, 4, found using the slope formula in Example 2 is the same number as the coefficient of x in the equation $y = 4x + 8$. We will see in the next section that this always happens, *as long as the equation is solved for y.*

VIDEO

EXAMPLE 4 Finding the Slope from an Equation

Find the slope of the graph of $3x - 5y = 8$.

Solve the equation for y.

$$\begin{aligned} 3x - 5y &= 8 && \\ -5y &= -3x + 8 && \text{Subtract } 3x. \\ y &= \frac{3}{5}x - \frac{8}{5} && \text{Divide by } -5. \end{aligned}$$

The slope is given by the coefficient of x, so the slope is $\frac{3}{5}$.

Now Try Exercise 31.

OBJECTIVE 3 Graph a line given its slope and a point on the line. Example 5 shows how to graph a straight line by using the slope and one point on the line.

YOU TRY IT

EXAMPLE 5 Using the Slope and a Point to Graph Lines

Graph each line.

(a) With slope $\frac{2}{3}$ passing through the point $(-1, 4)$

First locate the point $P(-1, 4)$ on a graph as shown in Figure 15. Then use the slope to find a second point. From the slope formula,

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{2}{3},$$

so move *up* 2 units and then 3 units to the *right* to locate another point on the graph (labeled R). The line through $(-1, 4)$ and R is the required graph.

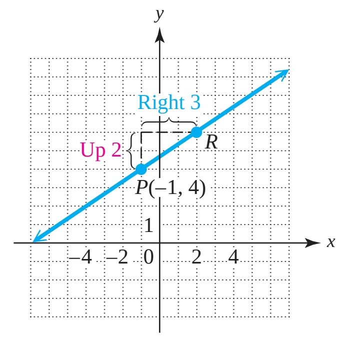


FIGURE 15

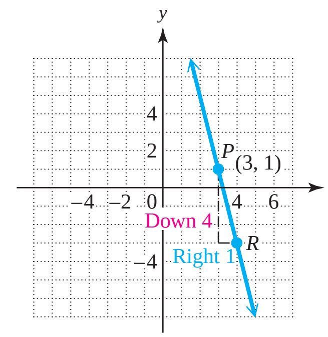


FIGURE 16

(b) Through (3, 1) with slope -4

Start by locating the point $P(3, 1)$ on a graph. Find a second point R on the line by writing -4 as $\frac{-4}{1}$ and using the slope formula.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-4}{1}$$

Move *down* 4 units from (3, 1), and then move 1 unit to the *right*. Draw a line through this second point R and (3, 1), as shown in Figure 16.

The slope also could be written as

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{4}{-1}.$$

In this case the second point R is located *up* 4 units and 1 unit to the *left*. Verify that this approach also produces the line in Figure 16.

Now Try Exercises 41 and 43.

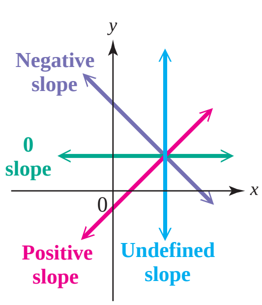


FIGURE 17

In Example 5(a), the slope of the line is the *positive* number $\frac{2}{3}$. The graph of the line in Figure 15 goes up (rises) from left to right. The line in Example 5(b) has *negative* slope, -4. As Figure 16 shows, its graph goes down (falls) from left to right. These facts suggest the following generalization.

A positive slope indicates that the line goes *up* from left to right;
a negative slope indicates that the line goes *down* from left to right.

AUDIO

Figure 17 shows lines of positive, 0, negative, and undefined slopes.

OBJECTIVE 4 Use slopes to determine whether two lines are parallel, perpendicular, or neither. The slopes of a pair of parallel or perpendicular lines are related in a special way. Recall that the slope of a line measures the steepness of the line. Since parallel lines have equal steepness, their slopes must be equal; also, lines with the same slope are parallel.

Slopes of Parallel Lines

Two nonvertical lines with the same slope are parallel; two nonvertical parallel lines have the same slope.

EXAMPLE 6 Determining Whether Two Lines Are Parallel

Are the lines L_1, through $(-2, 1)$ and $(4, 5)$, and L_2, through $(3, 0)$ and $(0, -2)$, parallel?

The slope of L_1 is

$$m_1 = \frac{5-1}{4-(-2)} = \frac{4}{6} = \frac{2}{3}.$$

The slope of L_2 is

$$m_2 = \frac{-2-0}{0-3} = \frac{-2}{-3} = \frac{2}{3}.$$

Because the slopes are equal, the two lines are parallel.

Now Try Exercise 49.

To see how the slopes of perpendicular lines are related, consider a nonvertical line with slope $\frac{a}{b}$. If this line is rotated 90°, the vertical change and the horizontal change are reversed and the slope is $-\frac{b}{a}$, since the horizontal change is now negative. See Figure 18. Thus, the slopes of perpendicular lines have product -1 and are negative reciprocals of each other. For example, if the slopes of two lines are $\frac{3}{4}$ and $-\frac{4}{3}$, then the lines are perpendicular because $\frac{3}{4}\left(-\frac{4}{3}\right) = -1$.

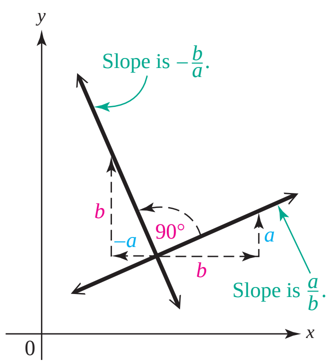


FIGURE 18

Slopes of Perpendicular Lines

If neither is vertical, perpendicular lines have slopes that are negative reciprocals; that is, their product is -1. Also, lines with slopes that are negative reciprocals are perpendicular.

EXAMPLE 7 Determining Whether Two Lines Are Perpendicular

Are the lines with equations $2y = 3x - 6$ and $2x + 3y = -6$ perpendicular?

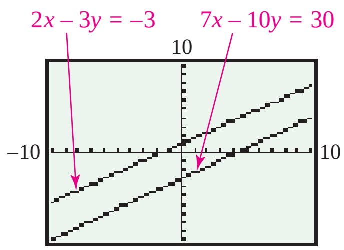
Find the slope of each line by first solving each equation for y.

$2y = 3x - 6$	$2x + 3y = -6$
$y = \underbrace{\frac{3}{2}}_{\text{Slope}}x - 3$	$3y = -2x - 6$
	$y = \underbrace{-\frac{2}{3}}_{\text{Slope}}x - 2$

Since the product of the slopes of the two lines is $\frac{3}{2}\left(-\frac{2}{3}\right) = -1$, the lines are perpendicular.

Now Try Exercise 51.

NOTE In Example 7, alternatively, we could have found the slope of each line by using intercepts and the slope formula.



The graphs are *not* parallel, though they may appear to be.

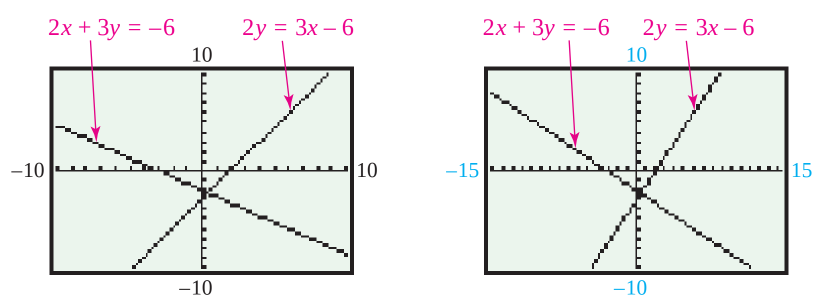
FIGURE 19

We must be careful when interpreting calculator graphs of parallel and perpendicular lines. For example, the graphs of the equations in Figure 19 appear to be parallel. However, checking their slopes algebraically, we find that

$2x - 3y = -3$	$7x - 10y = 30$
$-3y = -2x - 3$	$-10y = -7x + 30$
$y = \frac{2}{3}x + 1$	$y = \frac{7}{10}x - 3.$

Since the slopes $\frac{2}{3}$ and $\frac{7}{10}$ are not equal, the lines are *not* parallel.

Figure 20(a) shows graphs of the perpendicular lines from Example 7. As graphed in the standard viewing window, the lines do not appear to be perpendicular. However, if we use a *square viewing window* as in Figure 20(b), we get a more realistic view. (Many graphing calculators can set a square window automatically. See your owner's manual.) These two cases indicate that we cannot rely completely on what we see on a calculator screen—we must understand the mathematical concepts as well.



In the standard window, the lines *do not* appear to be perpendicular.

(a)

In the square window, the lines *do* appear to be perpendicular.

(b)

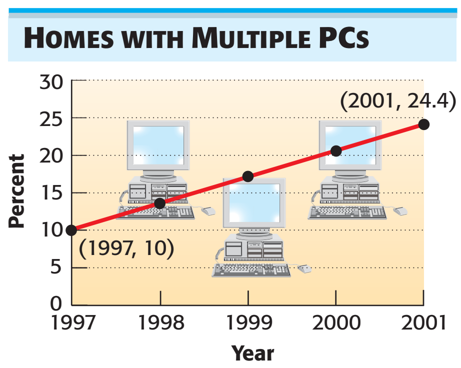
FIGURE 20

OBJECTIVE 5 Solve problems involving average rate of change. We know that the slope of a line is the ratio of the vertical change in y to the horizontal change in x. Thus, slope gives the *average rate of change* in y per unit of change in x, where the value of y depends on the value of x. The next examples illustrate this idea. We assume a linear relationship between x and y.

EXPLORE

EXAMPLE 8 Interpreting Slope as Average Rate of Change

The graph in Figure 21 approximates the percent of U.S. households owning multiple personal computers in the years 1997 through 2001. Find the average rate of change in percent per year.



Source: The Yankee Group.

FIGURE 21

To determine the average rate of change, we need two pairs of data. From the graph, if $x = 1997$, then $y = 10$ and if $x = 2001$, then $y = 24.4$, so we have the ordered pairs (1997, 10) and (2001, 24.4). By the slope formula,

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{24.4 - 10}{2001 - 1997} = \frac{14.4}{4} = 3.6.$$

This means that the number of U.S. households owning multiple computers *increased* by 3.6% each year from 1997 to 2001.

Now Try Exercise 69.

EXAMPLE 9 Interpreting Slope as Average Rate of Change

In 1997, sales of VCRs numbered 16.7 million. In 2002, estimated sales of VCRs were 13.3 million. Find the average rate of change, in millions, per year. (*Source: The Gazette*, June 22, 2002.)

To use the slope formula, we need two ordered pairs. Here, if $x = 1997$, then $y = 16.7$ and if $x = 2002$, then $y = 13.3$, which gives the ordered pairs (1997, 16.7) and (2002, 13.3). (Note that y is in millions.)

$$\text{average rate of change} = \frac{13.3 - 16.7}{2002 - 1997} = \frac{-3.4}{5} = -.68$$

The graph in Figure 22 confirms that the line through the ordered pairs falls from left to right and therefore has negative slope. Thus, sales of VCRs *decreased* by .68 million each year from 1997 to 2002.

Now Try Exercise 71.

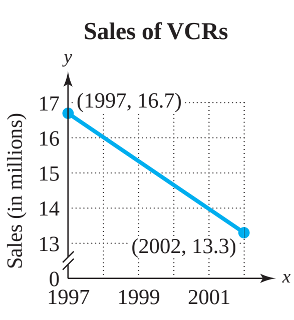


FIGURE 22

3.2 EXERCISES

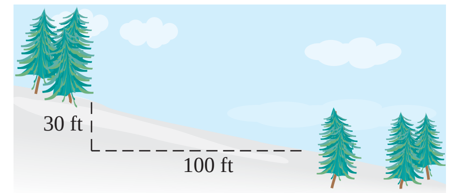
For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 4/Videotape 4

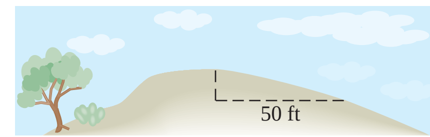
1. A ski slope drops 30 ft for every horizontal 100 ft. Which of the following express its slope? (There are several correct choices.)

A. $-.3$ **B.** $-\frac{3}{10}$ **C.** $-3\frac{1}{3}$

D. $-\frac{30}{100}$ **E.** $-\frac{10}{3}$



2. A hill has slope $-.05$. How many feet in the vertical direction correspond to a run of 50 ft?



3. Match each situation in (a)–(d) with the most appropriate graph in A–D.

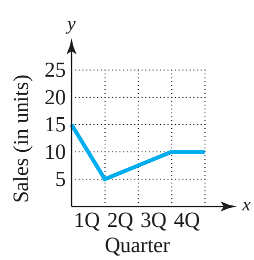
(a) Sales rose sharply during the first quarter, leveled off during the second quarter, and then rose slowly for the rest of the year.

(b) Sales fell sharply during the first quarter, and then rose slowly during the second and third quarters before leveling off for the rest of the year.

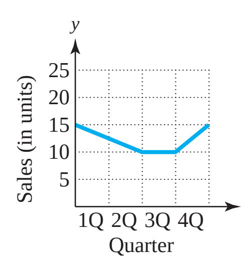
(c) Sales rose sharply during the first quarter, and then fell to the original level during the second quarter before rising steadily for the rest of the year.

(d) Sales fell during the first two quarters of the year, leveled off during the third quarter, and rose during the fourth quarter.

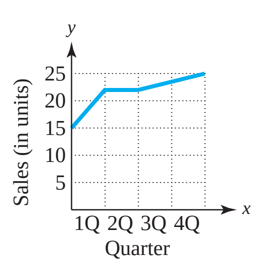
A.



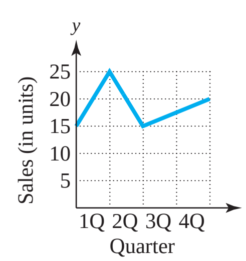
B.



C.



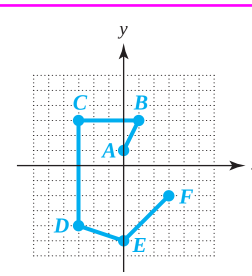
D.



Determine the slope of each line segment in the given figure.

4. AB **5.** BC **6.** CD

7. DE **8.** EF



Calculate the value of each slope m using the slope formula. See Example 1.

9. $m = \dfrac{6 - 2}{5 - 3}$ **10.** $m = \dfrac{5 - 7}{-4 - 2}$ **11.** $m = \dfrac{4 - (-1)}{-3 - (-5)}$

12. $m = \dfrac{-6 - 0}{0 - (-3)}$ **13.** $m = \dfrac{-5 - (-5)}{3 - 2}$ **14.** $m = \dfrac{7 - (-2)}{-3 - (-3)}$

15. Which of the following forms of the slope formula are correct? Explain.

A. $\dfrac{y_1 - y_2}{x_2 - x_1}$ **B.** $\dfrac{y_1 - y_2}{x_1 - x_2}$ **C.** $\dfrac{x_2 - x_1}{y_2 - y_1}$ **D.** $\dfrac{y_2 - y_1}{x_2 - x_1}$

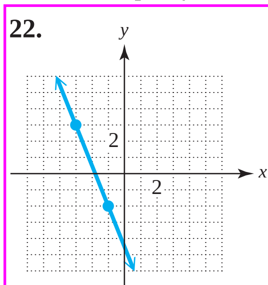
Find the slope of the line through each pair of points. See Example 1.

16. $(-2, -3)$ and $(-1, 5)$ **17.** $(-4, 3)$ and $(-3, 4)$ **18.** $(-4, 1)$ and $(2, 6)$

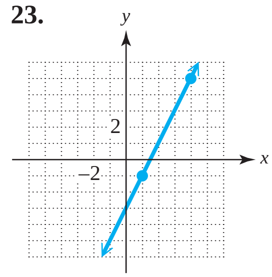
19. $(-3, -3)$ and $(5, 6)$ **20.** $(2, 4)$ and $(-4, 4)$ **21.** $(-6, 3)$ and $(2, 3)$

Find the slope of each line.

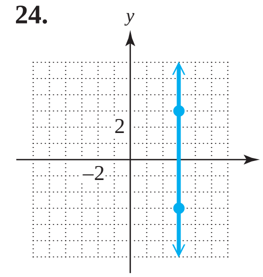
22.



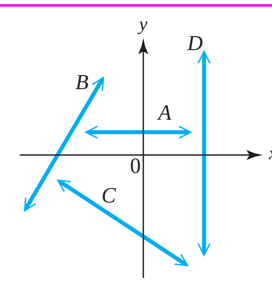
23.



24.



Based on the figure shown here, determine which line satisfies the given description.



25. The line has positive slope.

26. The line has negative slope.

27. The line has slope 0.

28. The line has undefined slope.

Find the slope of the line and sketch the graph. See Examples 1–4.

29. $x + 2y = 4$ **30.** $x + 3y = -6$ **31.** $5x - 2y = 10$

32. $4x - y = 4$ **33.** $y = 4x$ **34.** $y = -3x$

35. $x - 3 = 0$ **36.** $y + 5 = 0$ **37.** $y = -4$

Graph the line described. See Example 5.

38. Through $(-4, 2)$; $m = \dfrac{1}{2}$ **39.** Through $(-2, -3)$; $m = \dfrac{5}{4}$

40. Through $(0, -2)$; $m = -\dfrac{2}{3}$ **41.** Through $(0, -4)$; $m = -\dfrac{3}{2}$

42. Through $(-1, -2)$; $m = 3$ **43.** Through $(-2, -4)$; $m = 4$

44. $m = 0$; through $(2, -5)$ **45.** Undefined slope; through $(-3, 1)$

46. Undefined slope; through $(-4, 1)$

47. $m = 0$; through $(5, 3)$

48. If a line has slope $-\frac{4}{9}$, then any line parallel to it has slope ____, and any line perpendicular to it has slope ____.

Decide whether each pair of lines is parallel, perpendicular, *or* neither. *See Examples 6 and 7.*

49. The line through $(4, 6)$ and $(-8, 7)$ and the line through $(-5, 5)$ and $(7, 4)$

50. The line through $(15, 9)$ and $(12, -7)$ and the line through $(8, -4)$ and $(5, -20)$

51. $2x + 5y = -7$ and $5x - 2y = 1$

52. $x + 4y = 7$ and $4x - y = 3$

53. $2x + y = 6$ and $x - y = 4$

54. $4x - 3y = 6$ and $3x - 4y = 2$

55. $3x = y$ and $2y - 6x = 5$

56. $x = 6$ and $6 - x = 8$

57. $2x + 5y = -8$ and $6 + 2x = 5y$

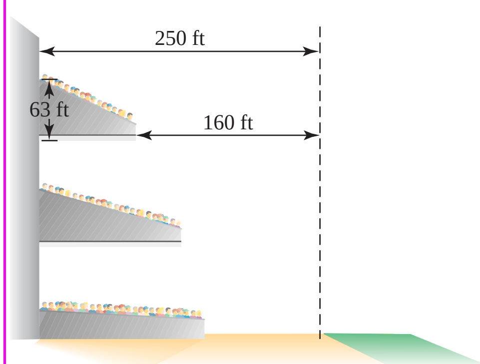
58. $4x + y = 0$ and $5x - 8 = 2y$

59. $4x - 3y = 8$ and $4y + 3x = 12$

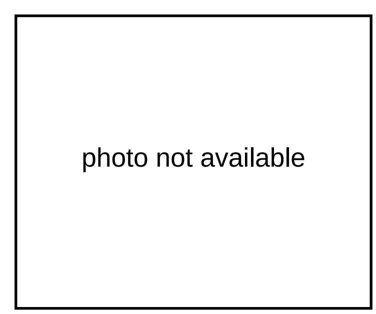
60. $2x = y + 3$ and $2y + x = 3$

Solve each problem.

61. The upper deck at Comiskey Park in Chicago has produced, among other complaints, displeasure with its steepness. It is 160 ft from home plate to the front of the upper deck and 250 ft from home plate to the back. The top of the upper deck is 63 ft above the bottom. What is its slope? (Consider the slope as a positive number here.)

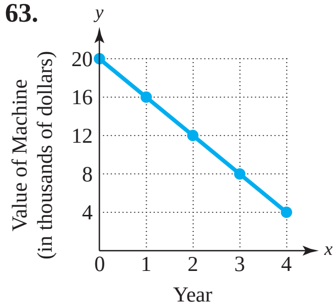


62. When designing the FleetCenter arena in Boston, architects designed the ramps leading up to the entrances so that circus elephants would be able to walk up the ramps. The maximum grade (or slope) that an elephant will walk on is 13%. Suppose that such a ramp was constructed with a horizontal run of 150 ft. What would be the maximum vertical rise the architects could use?

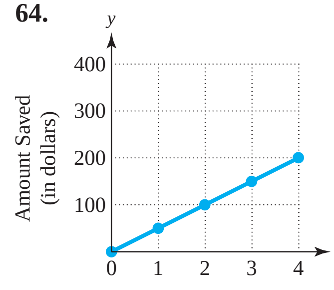


Find and interpret the average rate of change illustrated in each graph.

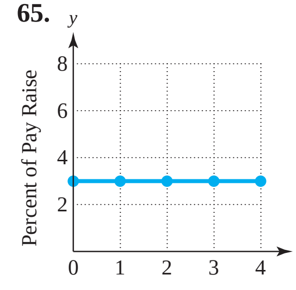
63.



64.



65.



66. If the graph of a linear equation rises from left to right, then the average rate of change is ______________ (positive/negative). If the graph of a linear equation falls from left to right, then the average rate of change is ______________ (positive/negative).

Solve each problem. See Examples 8 and 9.

67. The table gives book publishers' approximate net dollar sales (in millions) from 1995 through 2000.

Book Publishers' Sales

Year	Sales (in millions)
1995	19,000
1996	20,000
1997	21,000
1998	22,000
1999	23,000
2000	24,000

Source: Book Industry Study Group.

(a) Find the average rate of change for 1995–1996, 1995–1999, and 1998–2000.

(b) What do you notice about your answers in part (a)? What does this tell you?

68. The table gives the number of cellular telephone subscribers (in thousands) from 1994 through 1999.

Cellular Telephone Subscribers

Year	Subscribers (in thousands)
1994	24,134
1995	33,786
1996	44,043
1997	55,312
1998	69,209
1999	86,047

Source: Cellular Telecommunications Industry Association, Washington, D.C., *State of the Cellular Industry* (Annual).

(a) Find the average rate of change in subscribers for 1994–1995, 1995–1996, and so on.

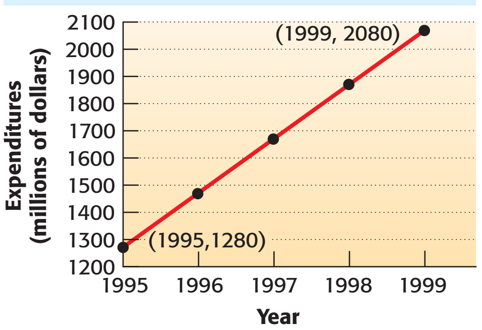
(b) Is the average rate of change in successive years approximately the same? If the ordered pairs in the table were plotted, could an approximately straight line be drawn through them?

69. Merck pharmaceutical company research and development expenditures (in millions of dollars) in recent years are closely approximated by the graph.

(a) Use the given ordered pairs to determine the average rate of change in these expenditures per year.

(b) Explain how a positive rate of change is interpreted in this situation.

RESEARCH AND DEVELOPMENT EXPENDITURES

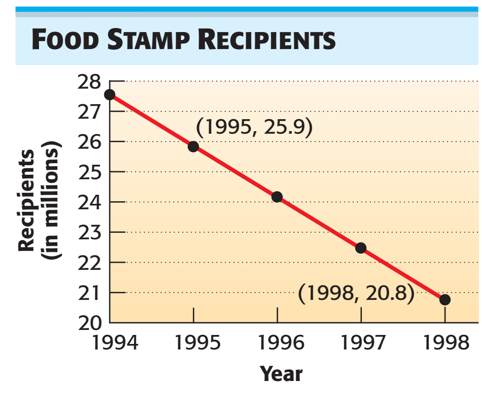


Source: Merck & Co., Inc. 1999 Annual Report.

70. The graph provides a good approximation of the number of food stamp recipients (in millions) from 1994 through 1998.

(a) Use the given ordered pairs to find the average rate of change in food stamp recipients per year during this period.

(b) Interpret what a negative slope means in this situation.



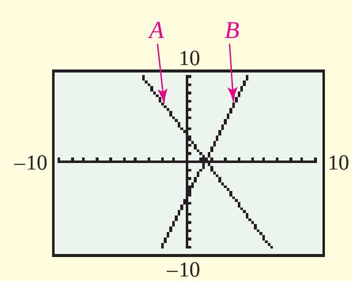
Source: U.S. Bureau of the Census.

71. When introduced in 1997, a DVD player sold for about \$500. In 2002, the average price was \$155. Find and interpret the average rate of change in price per year. (*Source: The Gazette*, June 22, 2002.)

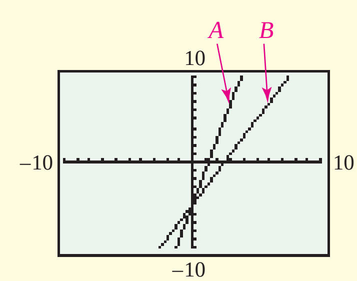
72. In 1997 when DVD players entered the market, .349 million (that is, 349,000) were sold. In 2002, sales of DVD players reached 15.5 million (estimated). Find and interpret the average rate of change in sales, in millions, per year. Round your answer to the nearest hundredth. (*Source: The Gazette*, June 22, 2002.)

TECHNOLOGY INSIGHTS (EXERCISES 73 AND 74)

73. The graphing calculator screen shows two lines. One is the graph of $y_1 = -2x + 3$ and the other is the graph of $y_2 = 3x - 4$. Which is which?



74. The graphing calculator screen shows two lines. One is the graph of $y_1 = 2x - 5$ and the other is the graph of $y_2 = 4x - 5$. Which is which?



Solve each problem, using your knowledge of the slopes of parallel and perpendicular lines.

EXERCISES

75. Show that $(-13, -9)$, $(-11, -1)$, $(2, -2)$, and $(4, 6)$ are the vertices of a parallelogram. (*Hint:* A parallelogram is a four-sided figure with opposite sides parallel.)

76. Is the figure with vertices at $(-11, -5)$, $(-2, -19)$, $(12, -10)$, and $(3, 4)$ a parallelogram? Is it a rectangle? (*Hint:* A rectangle is a parallelogram with a right angle.)

3.3 Linear Equations in Two Variables

OBJECTIVES

1. Write an equation of a line given its slope and y-intercept.
2. Graph a line using its slope and y-intercept.
3. Write an equation of a line given its slope and a point on the line.
4. Write an equation of a line given two points on the line.
5. Write an equation of a line parallel or perpendicular to a given line.
6. Write an equation of a line that models real data.
7. Use a graphing calculator to solve linear equations in one variable.

OBJECTIVE 1 Write an equation of a line given its slope and y-intercept. In the previous section we found the slope of a line from the equation of the line by solving the equation for y. For example, we found that the slope of the line with equation $y = 4x + 8$ is 4, the coefficient of x. What does the number 8 represent?

To find out, suppose a line has slope m and y-intercept $(0, b)$. We can find an equation of this line by choosing another point (x, y) on the line, as shown in Figure 23. Using the slope formula,

$$m = \frac{y - b}{x - 0}$$

$$m = \frac{y - b}{x}$$

$$mx = y - b \qquad \text{Multiply by } x.$$

$$mx + b = y \qquad \text{Add } b.$$

$$y = mx + b. \qquad \text{Rewrite.}$$

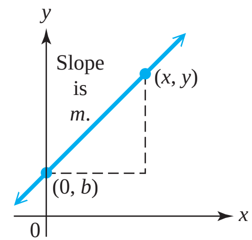


FIGURE 23

This last equation is called the *slope-intercept form* of the equation of a line, because we can identify the slope and y-intercept at a glance. Thus, in the line with equation $y = 4x + 8$, the number 8 indicates that the y-intercept is $(0, 8)$.

Slope-Intercept Form

The **slope-intercept form** of the equation of a line with slope m and y-intercept $(0, b)$ is

$$\mathbf{y = \underbrace{m}_{\text{Slope}}x + \underbrace{b}_{y\text{-intercept is }(0,\, b).}}$$

AUDIO

EXAMPLE 1 Using the Slope-Intercept Form to Find an Equation of a Line

Find an equation of the line with slope $-\frac{4}{5}$ and y-intercept $(0, -2)$.

Here $m = -\frac{4}{5}$ and $b = -2$. Substitute these values into the slope-intercept form.

$$y = mx + b \qquad \text{Slope-intercept form}$$

$$y = -\frac{4}{5}x - 2 \qquad m = -\tfrac{4}{5};\ b = -2$$

Now Try Exercise 19.

OBJECTIVE 2 Graph a line using its slope and *y*-intercept. If the equation of a line is written in slope-intercept form, we can use the slope and y-intercept to obtain its graph.

EXAMPLE 2 Graphing Lines Using Slope and *y*-Intercept

Graph each line using the slope and y-intercept.

(a) $y = 3x - 6$

Here $m = 3$ and $b = -6$. Plot the y-intercept $(0, -6)$. The slope 3 can be interpreted as

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{3}{1}.$$

From $(0, -6)$, move *up* 3 units and to the *right* 1 unit, and plot a second point at $(1, -3)$. Join the two points with a straight line to obtain the graph in Figure 24.

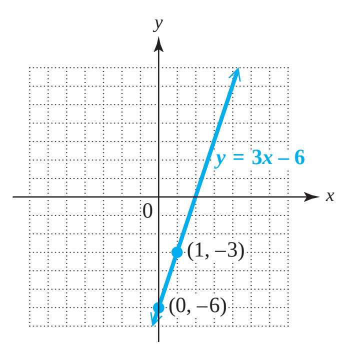


FIGURE 24

(b) $3y + 2x = 9$

Write the equation in slope-intercept form by solving for y.

$$3y + 2x = 9$$

$$3y = -2x + 9 \qquad \text{Subtract } 2x.$$

$$y = \underbrace{-\frac{2}{3}}_{\text{Slope}}x + \underbrace{3}_{y\text{-intercept is } (0,\,3).} \qquad \text{Slope-intercept form}$$

To graph this equation, plot the y-intercept $(0, 3)$. The slope can be interpreted as either $\frac{-2}{3}$ or $\frac{2}{-3}$. Using $\frac{-2}{3}$, move from $(0, 3)$ *down* 2 units and to the *right* 3 units to locate the point (3, 1). The line through these two points is the required graph. See Figure 25. (Verify that the point obtained using $\frac{2}{-3}$ as the slope is also on this line.)

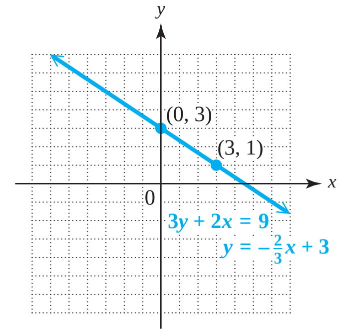


FIGURE 25

Now Try Exercise 25.

NOTE The slope-intercept form of a linear equation is the most useful for several reasons. Every linear equation (of a nonvertical line) has a *unique* (one and only one) slope-intercept form. In Section 3.5 we study *linear functions,* which are defined using slope-intercept form. Also, this is the form we use when graphing a line with a graphing calculator. (See Section 3.1, Example 6.)

OBJECTIVE 3 Write an equation of a line given its slope and a point on the line. Let m represent the slope of a line and (x_1, y_1) represent a given point on the line. Let (x, y) represent any other point on the line. See Figure 26. Then by the slope formula,

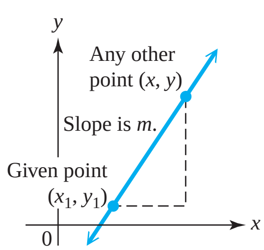


FIGURE 26

$$m = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = y - y_1 \qquad \text{Multiply each side by } x - x_1.$$

$$y - y_1 = m(x - x_1). \qquad \text{Rewrite.}$$

This last equation is the *point-slope form* of the equation of a line.

Point-Slope Form

The **point-slope form** of the equation of a line with slope m passing through the point (x_1, y_1) is

Slope

$$\mathbf{y - y_1 = m(x - x_1).}$$

Given point

To use this form to write the equation of a line, we need to know the coordinates of a point (x_1, y_1) and the slope m of the line.

EXAMPLE 3 Using the Point-Slope Form

Find an equation of the line with slope $\frac{1}{3}$ passing through the point $(-2, 5)$.

Use the point-slope form of the equation of a line, with $(x_1, y_1) = (-2, 5)$ and $m = \frac{1}{3}$.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 5 = \frac{1}{3}[x - (-2)] \qquad y_1 = 5,\ m = \tfrac{1}{3},\ x_1 = -2$$

$$y - 5 = \frac{1}{3}(x + 2)$$

$$3y - 15 = x + 2 \qquad \text{Multiply by 3.}$$

$$-x + 3y = 17 \qquad \text{Subtract } x\text{; add 15.}$$

In Section 3.1, we defined *standard form* for a linear equation as

$$Ax + By = C,$$

where A, B, and C are real numbers. Most often, however, A, B, and C are integers. In this case, let us agree that integers A, B, and C have no common factor (except 1) and $A \geq 0$. For example, the final equation in Example 3, $-x + 3y = 17$, is written in standard form as $x - 3y = -17$.

NOTE The definition of "standard form" is not standard from one text to another. Any linear equation can be written in many different (all equally correct) forms. For example, the equation $2x + 3y = 8$ can be written as

$$2x = 8 - 3y, \qquad 3y = 8 - 2x, \qquad x + \frac{3}{2}y = 4, \qquad 4x + 6y = 16,$$

and so on. In addition to writing it in the form $Ax + By = C$ with $A \geq 0$, let us agree that the form $2x + 3y = 8$ is preferred over any multiples of each side, such as $4x + 6y = 16$. (To write $4x + 6y = 16$ in standard form, divide each side by 2.)

Now Try Exercise 31.

OBJECTIVE 4 Write an equation of a line given two points on the line. To find an equation of a line when two points on the line are known, first use the slope formula to find the slope of the line. Then use the slope with either of the given points and the point-slope form of the equation of a line.

EXAMPLE 4 Finding an Equation of a Line Given Two Points

Find an equation of the line passing through the points $(-4, 3)$ and $(5, -7)$. Write the equation in standard form.

First find the slope by using the slope formula.

$$m = \frac{-7 - 3}{5 - (-4)} = -\frac{10}{9}$$

Use either $(-4, 3)$ or $(5, -7)$ as (x_1, y_1) in the point-slope form of the equation of a line. If you choose $(-4, 3)$, then $-4 = x_1$ and $3 = y_1$.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point-slope form} \\
y - 3 &= -\frac{10}{9}[x - (-4)] && y_1 = 3,\ m = -\tfrac{10}{9},\ x_1 = -4 \\
y - 3 &= -\frac{10}{9}(x + 4) && \\
9y - 27 &= -10x - 40 && \text{Multiply by 9; distributive property.} \\
10x + 9y &= -13 && \text{Standard form}
\end{aligned}
$$

Verify that if $(5, -7)$ were used, the same equation would result.

Now Try Exercise 49.

A horizontal line has slope 0. Using point-slope form, the equation of a horizontal line through the point (a, b) is

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \\
y - b &= 0(x - a) && y_1 = b,\ m = 0,\ x_1 = a \\
y - b &= 0 && \\
y &= b. &&
\end{aligned}
$$

Notice that point-slope form does not apply to a vertical line, since the slope of a vertical line is undefined. A vertical line through the point (a, b) has equation $x = a$.

In summary, horizontal and vertical lines have the following special equations.

Equations of Horizontal and Vertical Lines

The horizontal line through the point (a, b) has equation $\boldsymbol{y = b}$.
The vertical line through the point (a, b) has equation $\boldsymbol{x = a}$.

Now Try Exercises 41 and 43.

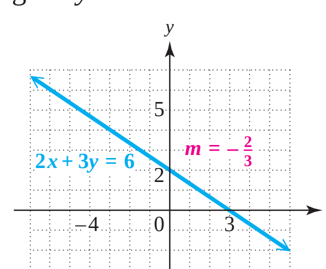
OBJECTIVE 5 Write an equation of a line parallel or perpendicular to a given line. As mentioned in the previous section, parallel lines have the same slope and perpendicular lines have slopes that are negative reciprocals of each other.

EXAMPLE 5 Finding Equations of Parallel or Perpendicular Lines

Find an equation of the line passing through the point $(-4, 5)$ and **(a)** parallel to the line $2x + 3y = 6$; **(b)** perpendicular to the line $2x + 3y = 6$. Write each equation in slope-intercept form.

(a) We find the slope of the line $2x + 3y = 6$ by solving for y.

$$
\begin{aligned}
2x + 3y &= 6 && \\
3y &= -2x + 6 && \text{Subtract } 2x. \\
y &= \underbrace{-\frac{2}{3}}_{\text{Slope}}x + 2 && \text{Divide by 3.}
\end{aligned}
$$



The slope is given by the coefficient of x, so $m = -\frac{2}{3}$. See the figure.

The required equation of the line through $(-4, 5)$ and parallel to $2x + 3y = 6$ must also have slope $-\frac{2}{3}$. To find this equation, we use the point-slope form, with $(x_1, y_1) = (-4, 5)$ and $m = -\frac{2}{3}$.

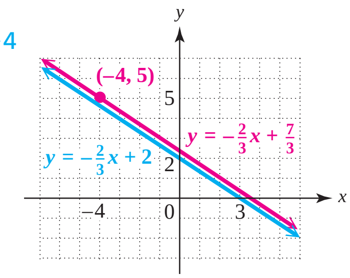
$$y - 5 = -\frac{2}{3}[x - (-4)] \qquad y_1 = 5,\ m = -\tfrac{2}{3},\ x_1 = -4$$

$$y - 5 = -\frac{2}{3}(x + 4)$$

$$y - 5 = -\frac{2}{3}x - \frac{8}{3} \qquad \text{Distributive property}$$

$$y = -\frac{2}{3}x - \frac{8}{3} + \frac{15}{3} \qquad \text{Add } 5 = \tfrac{15}{3}.$$

$$y = -\frac{2}{3}x + \frac{7}{3} \qquad \text{Combine like terms.}$$



We did not clear fractions after the substitution step here because we want the equation in slope-intercept form—that is, solved for y. Both lines are shown in the figure.

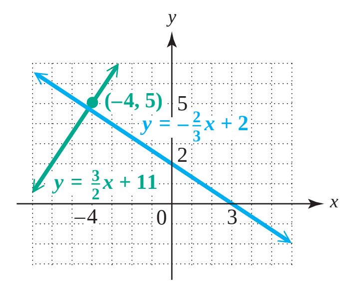
(b) To be perpendicular to the line $2x + 3y = 6$, a line must have a slope that is the negative reciprocal of $-\frac{2}{3}$, which is $\frac{3}{2}$. We use $(-4, 5)$ and slope $\frac{3}{2}$ in the point-slope form to get the equation of the perpendicular line shown in the figure.

$$y - 5 = \frac{3}{2}[x - (-4)] \qquad y_1 = 5,\ m = \tfrac{3}{2},\ x_1 = -4$$

$$y - 5 = \frac{3}{2}(x + 4)$$

$$y - 5 = \frac{3}{2}x + 6 \qquad \text{Distributive property}$$

$$y = \frac{3}{2}x + 11 \qquad \text{Add 5.}$$



Now Try Exercises 61 and 65.

A summary of the various forms of linear equations follows.

AUDIO

Forms of Linear Equations

Equation	*Description*	*When to Use*
$\boldsymbol{y = mx + b}$	**Slope-Intercept Form** Slope is m. y-intercept is $(0, b)$.	The slope and y-intercept can be easily identified and used to quickly graph the equation.
$\boldsymbol{y - y_1 = m(x - x_1)}$	**Point-Slope Form** Slope is m. Line passes through (x_1, y_1).	This form is ideal for finding the equation of a line if the slope and a point on the line or two points on the line are known.

(continued)

Equation	*Description*	*When to Use*
$Ax + By = C$	**Standard Form** (A, B, and C integers, $A \geq 0$) Slope is $-\frac{A}{B}$ $(B \neq 0)$. x-intercept is $\left(\frac{C}{A}, 0\right)$ $(A \neq 0)$. y-intercept is $\left(0, \frac{C}{B}\right)$ $(B \neq 0)$.	The x- and y-intercepts can be found quickly and used to graph the equation. Slope must be calculated.
$y = b$	**Horizontal Line** Slope is 0. y-intercept is $(0, b)$.	If the graph intersects only the y-axis, then y is the only variable in the equation.
$x = a$	**Vertical Line** Slope is undefined. x-intercept is $(a, 0)$.	If the graph intersects only the x-axis, then x is the only variable in the equation.

OBJECTIVE 6 Write an equation of a line that models real data. We can use the information presented in this section to write equations of lines that mathematically describe, or *model*, real data if the given set of data changes at a fairly constant rate. In this case, the data fit a linear pattern, and the rate of change is the slope of the line.

VIDEO

EXAMPLE 6 Determining a Linear Equation to Describe Real Data

Suppose it is time to fill your car with gasoline. At your local station, 89-octane gas is selling for \$1.60 per gal.

(a) Write an equation that describes the cost y to buy x gal of gas.

Experience has taught you that the total price you pay is determined by the number of gallons you buy multiplied by the price per gallon (in this case, \$1.60). As you pump the gas, two sets of numbers spin by: the number of gallons pumped and the price for that number of gallons.

The table uses ordered pairs to illustrate this situation.

Number of Gallons Pumped	*Price of This Number of Gallons*
0	$0(\$1.60) = \$0.00$
1	$1(\$1.60) = \$1.60$
2	$2(\$1.60) = \$3.20$
3	$3(\$1.60) = \$4.80$
4	$4(\$1.60) = \$6.40$

If we let x denote the number of gallons pumped, then the total price y in dollars can be found by the linear equation

$$\overset{\text{Total price}}{y} = 1.60\overset{\text{Number of gallons}}{x}.$$

Theoretically, there are infinitely many ordered pairs (x, y) that satisfy this equation, but here we are limited to nonnegative values for x, since we cannot have a negative number of gallons. There is also a practical maximum value for x in this situation, which varies from one car to another. What determines this maximum value?

(b) You can also get a car wash at the gas station if you pay an additional \$3.00. Write an equation that defines the price for gas and a car wash.

Since an additional \$3.00 will be charged, you pay $1.60x + 3.00$ dollars for x gallons of gas and a car wash, or

$$y = 1.6x + 3. \qquad \text{Delete unnecessary 0s.}$$

(c) Interpret the ordered pairs (5, 11) and (10, 19) in relation to the equation from part (b).

The ordered pair (5, 11) indicates that the price of 5 gal of gas and a car wash is \$11.00. Similarly, (10, 19) indicates that the price of 10 gal of gas and a car wash is \$19.00.

Now Try Exercises 69 and 73.

NOTE In Example 6(a), the ordered pair (0, 0) satisfied the equation, so the linear equation has the form $y = mx$, where $b = 0$. If a realistic situation involves an initial charge plus a charge per unit as in Example 6(b), the equation has the form $y = mx + b$, where $b \neq 0$.

EXAMPLE 7 Finding an Equation of a Line That Models Data

Average annual tuition and fees for in-state students at public 4-year colleges are shown in the table for selected years and graphed as ordered pairs of points in the *scatter diagram* in Figure 27, where $x = 0$ represents 1990, $x = 4$ represents 1994, and so on, and y represents the cost in dollars.

Year	Cost (in dollars)
1990	2035
1994	2820
1996	3151
1998	3486
2000	3774

Source: U.S. National Center for Education Statistics.

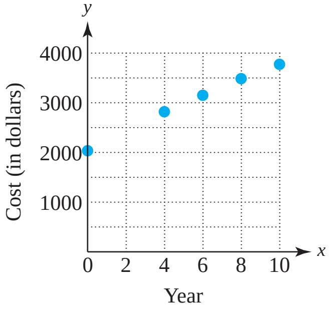


FIGURE 27

(a) Find an equation that models the data.

Since the points in Figure 27 lie approximately on a straight line, we can write a linear equation that models the relationship between year x and cost y. We choose two data points, (0, 2035) and (10, 3774), to find the slope of the line.

$$m = \frac{3774 - 2035}{10 - 0} = \frac{1739}{10} = 173.9$$

The slope 173.9 indicates that the cost of tuition and fees for in-state students at public 4-year colleges increased by about \$174 per year from 1990 to 2000. We use this slope, the y-intercept (0, 2035), and the slope-intercept form to write an equation of the line. Thus,

$$y = 173.9x + 2035.$$

(b) Use the equation from part (a) to approximate the cost of tuition and fees at public 4-year colleges in 2002.

The value $x = 12$ corresponds to the year 2002, so we substitute 12 for x in the equation.

$$y = 173.9x + 2035$$
$$y = 173.9(12) + 2035$$
$$y = 4121.8$$

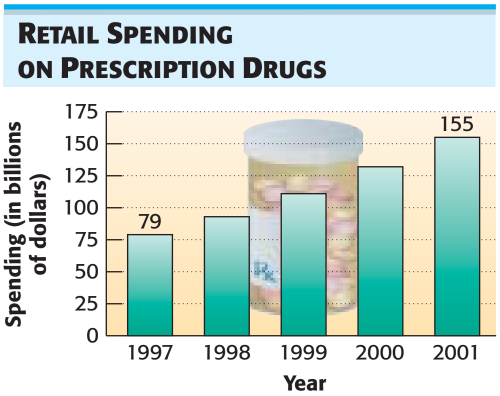
According to the model, average tuition and fees for in-state students at public 4-year colleges in 2002 were about \$4122.

Now Try Exercise 79.

NOTE In Example 7, if we had chosen different data points, we would have gotten a slightly different equation. However, all such equations should be similar.

EXAMPLE 8 Finding an Equation of a Line That Models Data

Retail spending (in billions of dollars) on prescription drugs in the United States is shown in the graph in Figure 28.



Source: American Institute for Research analysis of Scott-Levin data.

FIGURE 28

(a) Write an equation that models the data.

The data shown in the bar graph increase linearly; that is, we could draw a straight line through the tops of any two bars that would be close to the top of each bar. We can use the data and the point-slope form of the equation of a line to get an equation that models the relationship between year x and spending on prescription drugs y. If we let $x = 7$ represent 1997, $x = 8$ represent 1998, and so on, the given data for 1997 and 2001 can be written as the ordered pairs (7, 79) and (11, 155). The slope of the line through these two points is

$$m = \frac{155 - 79}{11 - 7} = \frac{76}{4} = 19.$$

Thus, retail spending on prescription drugs increased by about \$19 billion per year. Using this slope, one of the points, say (7, 79), and the point-slope form, we obtain

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 79 &= 19(x - 7) && (x_1, y_1) = (7, 79);\ m = 19 \\ y - 79 &= 19x - 133 && \text{Distributive property} \\ y &= 19x - 54. && \text{Slope-intercept form} \end{aligned}$$

Thus, retail spending y (in billions of dollars) on prescription drugs in the United States in year x can be approximated by the equation $y = 19x - 54$.

(b) Use the equation from part (a) to predict retail spending on prescription drugs in the United States in 2004. (Assume a constant rate of change.)

Since $x = 7$ represents 1997 and 2004 is 7 yr after 1997, $x = 14$ represents 2004. We substitute 14 for x in the equation.

$$y = 19x - 54 = 19(14) - 54 = 212$$

According to the model, \$212 billion will be spent on prescription drugs in 2004.

Now Try Exercise 81.

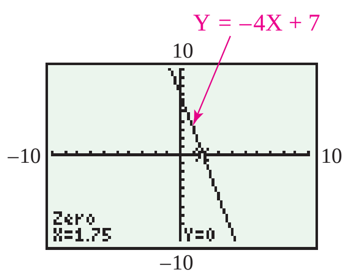


FIGURE 29

OBJECTIVE 7 Use a graphing calculator to solve linear equations in one variable. Figure 29 shows the graph of $Y = -4X + 7$. From the values at the bottom of the screen, we see that when $X = 1.75$, $Y = 0$. This means that $X = 1.75$ satisfies the equation $-4X + 7 = 0$, a linear equation in one variable. Therefore, the solution set of $-4X + 7 = 0$ is $\{1.75\}$. We can verify this algebraically by substitution. (The word "Zero" indicates that the x-intercept has been located.)

EXAMPLE 9 Solving an Equation with a Graphing Calculator

Use a graphing calculator to solve $-2x - 4(2 - x) = 3x + 4$.

We must write the equation as an equivalent equation with 0 on one side.

$$-2x - 4(2 - x) - 3x - 4 = 0 \qquad \text{Subtract } 3x \text{ and } 4.$$

Then we graph $Y = -2X - 4(2 - X) - 3X - 4$ to find the x-intercept. The standard viewing window cannot be used because the x-intercept does not lie in the interval $[-10, 10]$. As seen in Figure 30, the x-intercept of the graph is $(-12, 0)$, and thus the solution (or zero) of the equation is -12. The solution set is $\{-12\}$.

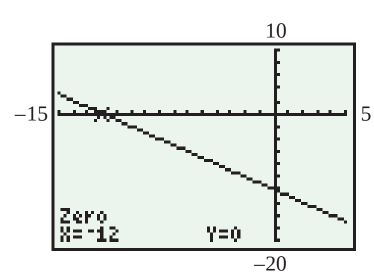


FIGURE 30

Now Try Exercise 83.

3.3 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 4/Videotape 4

1. The following equations all represent the same line. Which one is in standard form as defined in the text?

A. $3x - 2y = 5$ **B.** $2y = 3x - 5$ **C.** $\frac{3}{5}x - \frac{2}{5}y = 1$ **D.** $3x = 2y + 5$

2. Which equation is in point-slope form?

A. $y = 6x + 2$ **B.** $4x + y = 9$ **C.** $y - 3 = 2(x - 1)$ **D.** $2y = 3x - 7$

3. Which equation in Exercise 2 is in slope-intercept form?

4. Write the equation $y + 2 = -3(x - 4)$ in slope-intercept form.

5. Write the equation from Exercise 4 in standard form.

6. Write the equation $10x - 7y = 70$ in slope-intercept form.

Match each equation with the graph that it most closely resembles. (Hint: Determine the signs of m and b to help you make your decision.)

EXERCISES

7. $y = 2x + 3$

8. $y = -2x + 3$

9. $y = -2x - 3$

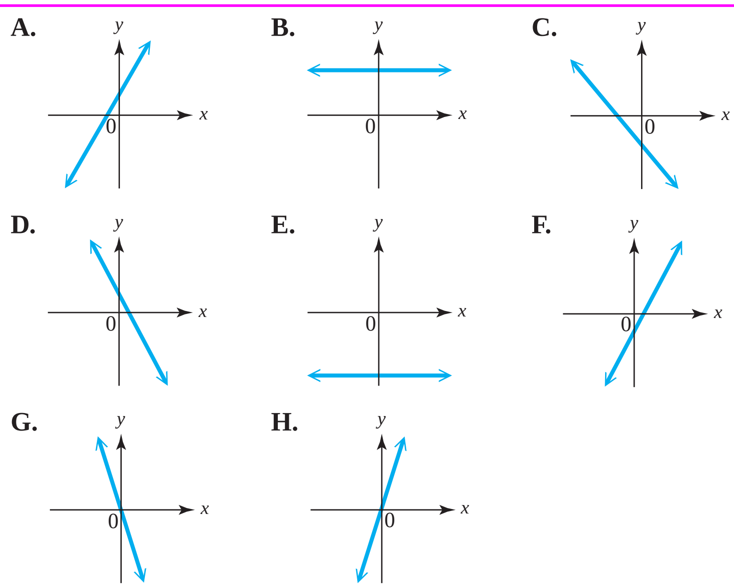
10. $y = 2x - 3$

11. $y = 2x$

12. $y = -2x$

13. $y = 3$

14. $y = -3$



Find the equation in slope-intercept form of the line satisfying the given conditions. See Example 1.

EXERCISES

15. $m = 5$; $b = 15$

16. $m = -2$; $b = 12$

17. $m = -\frac{2}{3}$; $b = \frac{4}{5}$

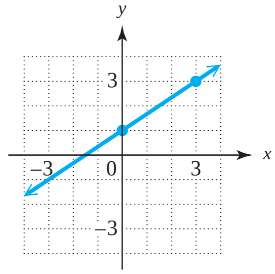
18. $m = -\frac{5}{8}$; $b = -\frac{1}{3}$

19. Slope $\frac{2}{5}$; y-intercept $(0, 5)$

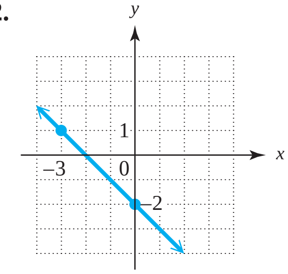
20. Slope $-\frac{3}{4}$; y-intercept $(0, 7)$

Write an equation in slope-intercept form of the line shown in each graph. (Hint: Use the indicated points to find the slope.)

21.



22.



For each equation, ***(a)*** *write it in slope-intercept form,* ***(b)*** *give the slope of the line,* ***(c)*** *give the y-intercept, and* ***(d)*** *graph the line. See Example 2.*

23. $-x + y = 4$

24. $-x + y = 6$

25. $6x + 5y = 30$

26. $3x + 4y = 12$

27. $4x - 5y = 20$

28. $7x - 3y = 3$

29. $x + 2y = -4$

30. $x + 3y = -9$

Find an equation of the line that satisfies the given conditions. Write the equation in standard form. See Example 3.

31. Through $(-2, 4)$; slope $-\frac{3}{4}$

32. Through $(-1, 6)$; slope $-\frac{5}{6}$

33. Through $(5, 8)$; slope -2

34. Through $(12, 10)$; slope 1

35. Through $(-5, 4)$; slope $\frac{1}{2}$

36. Through $(7, -2)$; slope $\frac{1}{4}$

37. x-intercept $(3, 0)$; slope 4

38. x-intercept $(-2, 0)$; slope -5

39. In your own words, list all the forms of linear equations in two variables and describe when each form should be used.

40. Explain why the point-slope form of an equation cannot be used to find the equation of a vertical line.

Write an equation of the line that satisfies the given conditions.

41. Through $(9, 5)$; slope 0

42. Through $(-4, -2)$; slope 0

43. Through $(9, 10)$; undefined slope

44. Through $(-2, 8)$; undefined slope

45. Through $(.5, .2)$; vertical

46. Through $\left(\frac{5}{8}, \frac{2}{9}\right)$; vertical

47. Through $(-7, 8)$; horizontal

48. Through $(2, 7)$; horizontal

Find an equation of the line passing through the given points. Write the equation in standard form. See Example 4.

49. $(3, 4)$ and $(5, 8)$

50. $(5, -2)$ and $(-3, 14)$

51. $(6, 1)$ and $(-2, 5)$

52. $(-2, 5)$ and $(-8, 1)$

53. $\left(-\frac{2}{5}, \frac{2}{5}\right)$ and $\left(\frac{4}{3}, \frac{2}{3}\right)$

54. $\left(\frac{3}{4}, \frac{8}{3}\right)$ and $\left(\frac{2}{5}, \frac{2}{3}\right)$

55. $(2, 5)$ and $(1, 5)$

56. $(-2, 2)$ and $(4, 2)$

57. $(7, 6)$ and $(7, -8)$

58. $(13, 5)$ and $(13, -1)$

59. $(1, -3)$ and $(-1, -3)$

60. $(-4, -6)$ and $(5, -6)$

Find an equation of the line satisfying the given conditions. Write the equation in slope-intercept form. See Example 5.

61. Through $(7, 2)$; parallel to $3x - y = 8$

62. Through $(4, 1)$; parallel to $2x + 5y = 10$

63. Through $(-2, -2)$; parallel to $-x + 2y = 10$

64. Through $(-1, 3)$; parallel to $-x + 3y = 12$

65. Through $(8, 5)$; perpendicular to $2x - y = 7$

66. Through $(2, -7)$; perpendicular to $5x + 2y = 18$

67. Through $(-2, 7)$; perpendicular to $x = 9$

68. Through $(8, 4)$; perpendicular to $x = -3$

Write an equation in the form $y = mx$ for each situation. Then give the three ordered pairs associated with the equation for x-values 0, 5, and 10. See Example 6(a).

69. x represents the number of hours traveling at 45 mph, and y represents the distance traveled (in miles).

70. x represents the number of compact discs sold at \$16 each, and y represents the total cost of the discs (in dollars).

71. x represents the number of gallons of gas sold at \$1.50 per gal, and y represents the total cost of the gasoline (in dollars).

72. x represents the number of days a videocassette is rented at \$3.50 per day, and y represents the total charge for the rental (in dollars).

*For each situation, **(a)** write an equation in the form $y = mx + b$; **(b)** find and interpret the ordered pair associated with the equation for $x = 5$; and **(c)** answer the question. See Examples 6(b) and 6(c).*

73. A membership to the Midwest Athletic Club costs \$99 plus \$39 per month. (*Source:* Midwest Athletic Club.) Let x represent the number of months selected. How much does the first year's membership cost?

photo not available

74. For a family membership, the athletic club in Exercise 73 charges a membership fee of \$159 plus \$60 for each additional family member after the first. Let x represent the number of additional family members. What is the membership fee for a four-person family?

75. A cell phone plan includes 900 anytime minutes for \$50 per month, plus a one-time activation fee of \$25. A Nokia 5165 cell phone is included at no additional charge. (*Source:* U.S. Cellular.) Let x represent the number of months of service. If you sign a 2-yr contract, how much will this cell phone plan cost? (Assume that you never use more than the allotted number of minutes.)

76. Another cell phone plan includes 450 anytime minutes for $35 per month, plus $19.95 for a Nokia 5165 cell phone and $25 for a one-time activation fee. (*Source:* U.S. Cellular.) Let x represent the number of months of service. If you sign a 1-yr contract, how much will this cell phone package cost? (Assume that you never use more than the allotted number of minutes.)

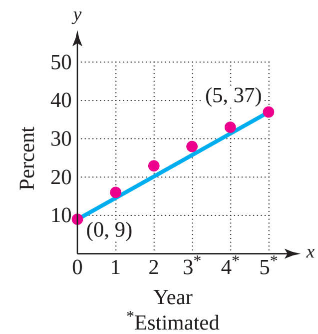
77. A rental car costs $50 plus $.20 per mile. Let x represent the number of miles driven, and y represent the total charge to the renter. How many miles was the car driven if the renter paid $84.60?

78. There is a $30 fee to rent a chain saw, plus $6 per day. Let x represent the number of days the saw is rented and y represent the charge to the user in dollars. If the total charge is $138, for how many days is the saw rented?

Solve each problem. In part (a), give equations in slope-intercept form. See Examples 7 and 8. (*Source for Exercises 79 and 80:* Jupiter Media Metrix.)

EXERCISES

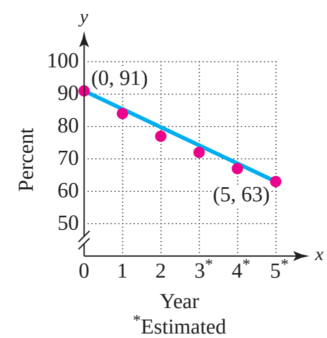
79. The percent of households that access the Internet by high-speed broadband is shown in the graph, where the year 2000 corresponds to $x = 0$.



(a) Use the ordered pairs from the graph to write an equation that models the data. What does the slope tell us in the context of this problem?

(b) Use the equation from part (a) to predict the percent of U.S. households that will access the Internet by broadband in 2006. Round your answer to the nearest percent.

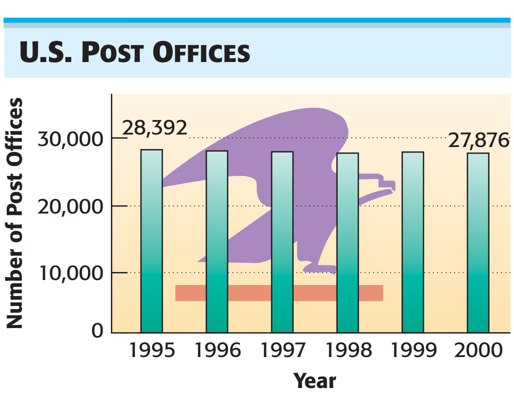
80. The percent of U.S. households that access the Internet by dial-up is shown in the graph, where the year 2000 corresponds to $x = 0$.



(a) Use the ordered pairs from the graph to write an equation that models the data. What does the slope tell us in the context of this problem?

(b) Use the equation from part (a) to predict the percent of U.S. households that will access the Internet by dial-up in 2006. Round your answer to the nearest percent.

81. The number of post offices in the United States is shown in the bar graph.



Source: U.S. Postal Service, *Annual Report of the Postmaster General.*

(a) Use the information given for the years 1995 and 2000, letting $x = 5$ represent 1995, $x = 10$ represent 2000, and y represent the number of post offices, to write an equation that models the data.

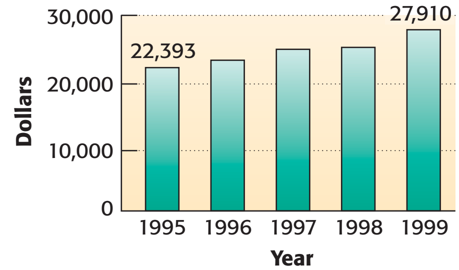
(b) Use the equation to approximate the number of post offices in 1998. How does this result compare to the actual value, 27,952?

82. Median household income of African-Americans is shown in the bar graph.

(a) Use the information given for the years 1995 and 1999, letting $x = 5$ represent 1995, $x = 9$ represent 1999, and y represent the median income, to write an equation that models median household income.

(b) Use the equation to approximate the median income for 1997. How does your result compare to the actual value, \$25,050?

MEDIAN HOUSEHOLD INCOME FOR AFRICAN-AMERICANS



Source: U.S. Bureau of the Census.

TECHNOLOGY INSIGHTS (EXERCISES 83–88)

In Exercises 83–88, do the following.

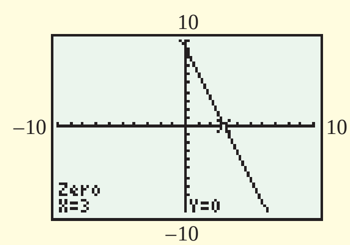
(a) *Simplify and rewrite the equation so that the right side is* 0. *Then replace* 0 *with y.*

(b) *The graph of the equation for y is shown with each exercise. Use the graph to determine the solution of the given equation. See Example 9.*

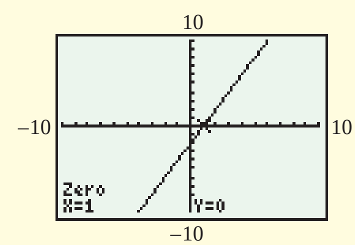
(c) *Solve the equation using the methods of Chapter 2.*

EXERCISES

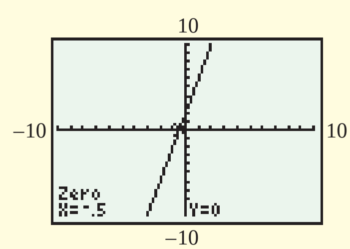
83. $2x + 7 - x = 4x - 2$



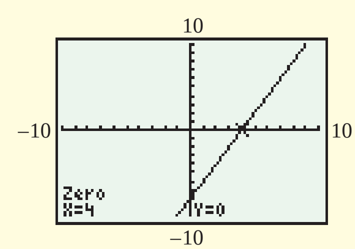
84. $7x - 2x + 4 - 5 = 3x + 1$



85. $3(2x + 1) - 2(x - 2) = 5$

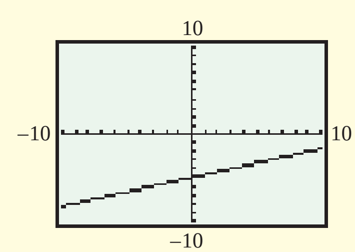


86. $4x - 3(4 - 2x) = 2(x - 3) + 6x + 2$



87. The graph of y_1 is shown in the standard viewing window. Which is the only choice that could possibly be the solution of the equation $y_1 = 0$?

A. -15 **B.** 0 **C.** 5 **D.** 15



88. **(a)** Solve $-2(x - 5) = -x - 2$ using the methods of Chapter 2.

(b) Explain why the standard viewing window of a graphing calculator cannot graphically support the solution found in part (a). What minimum and maximum x-values would make it possible for the solution to be seen?

RELATING CONCEPTS (EXERCISES 89–94)

For Individual or Group Work

In Section 2.2 we learned how formulas can be applied to problem solving. ***Work Exercises 89–94 in order,*** *to see how the formula that relates Celsius and Fahrenheit temperatures is derived.*

89. There is a linear relationship between Celsius and Fahrenheit temperatures. When $C = 0°$, $F =$ _____°, and when $C = 100°$, $F =$ _____°.

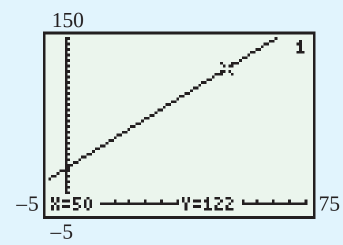
90. Think of ordered pairs of temperatures (C, F), where C and F represent corresponding Celsius and Fahrenheit temperatures. The equation that relates the two scales has a straight-line graph that contains the two points determined in Exercise 89. What are these two points?

91. Find the slope of the line described in Exercise 90.

92. Now think of the point-slope form of the equation in terms of C and F, where C replaces x and F replaces y. Use the slope you found in Exercise 91 and one of the two points determined earlier, and find the equation that gives F in terms of C.

93. To obtain another form of the formula, use the equation you found in Exercise 92 and solve for C in terms of F.

94. The equation found in Exercise 92 is graphed on the graphing calculator screen shown here. Interpret the display at the bottom, in the context of this group of exercises.



3.4 Linear Inequalities in Two Variables

OBJECTIVES

1. Graph linear inequalities in two variables.
2. Graph the intersection of two linear inequalities.
3. Graph the union of two linear inequalities.
4. Use a graphing calculator to solve linear inequalities in one variable.

OBJECTIVE 1 Graph linear inequalities in two variables. In Chapter 2 we graphed linear inequalities in one variable on the number line. In this section we graph linear inequalities in two variables on a rectangular coordinate system.

Linear Inequality in Two Variables

An inequality that can be written as

$$Ax + By < C \qquad \text{or} \qquad Ax + By > C,$$

where A, B, and C are real numbers and A and B are not both 0, is a **linear inequality in two variables.**

The symbols $\leq$ and $\geq$ may replace $<$ and $>$ in the definition.

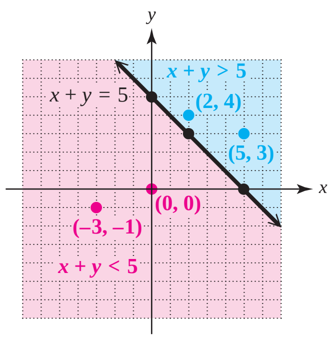


FIGURE 31

Consider the graph in Figure 31. The graph of the line $x + y = 5$ divides the points in the rectangular coordinate system into three sets: those points that lie on the line itself and satisfy the equation $x + y = 5$ [like (0, 5), (2, 3), and (5, 0)], those that lie in the half-plane above the line and satisfy the inequality $x + y > 5$ [like (5, 3) and (2, 4)], and those that lie in the half-plane below the line and satisfy the inequality $x + y < 5$ [like (0, 0) and $(-3, -1)$]. The graph of the line $x + y = 5$ is called the **boundary line** for the inequalities $x + y > 5$ and $x + y < 5$. Graphs of linear inequalities in two variables are *regions* in the real number plane that may or may not include boundary lines.

To graph a linear inequality in two variables, follow these steps.

Graphing a Linear Inequality

Step 1 **Draw the graph of the straight line that is the boundary.** Make the line solid if the inequality involves $\leq$ or $\geq$; make the line dashed if the inequality involves $<$ or $>$.

Step 2 **Choose a test point.** Choose any point not on the line and substitute the coordinates of this point in the inequality.

Step 3 **Shade the appropriate region.** Shade the region that includes the test point if it satisfies the original inequality; otherwise, shade the region on the other side of the boundary line.

EXAMPLE 1 Graphing a Linear Inequality

Graph $3x + 2y \geq 6$.

Step 1 First graph the line $3x + 2y = 6$. The graph of this line, the boundary of the graph of the inequality, is shown in Figure 32.

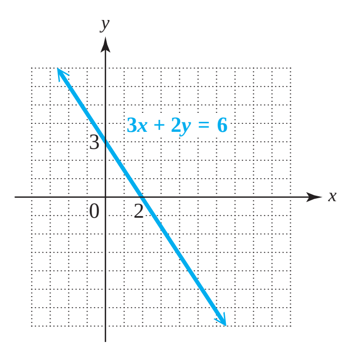


FIGURE 32

Step 2 The graph of the inequality $3x + 2y \geq 6$ includes the points of the line $3x + 2y = 6$ and either the points *above* the line $3x + 2y = 6$ or the points *below* that line. To decide which, select any point not on the boundary line $3x + 2y = 6$ as a test point. The origin, (0, 0), is often a good choice because the substitution is easy. Substitute the values from the test point (0, 0) for x and y in the inequality $3x + 2y > 6$.

$$3(0) + 2(0) > 6 \quad ?$$

$$0 > 6 \quad \text{False}$$

Step 3 Because the result is false, (0, 0) does *not* satisfy the inequality, and so the solution set includes all points on the other side of the line. This region is shaded in Figure 33.

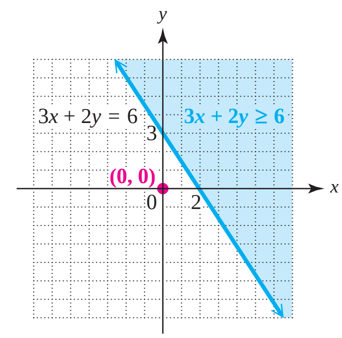


FIGURE 33

Now Try Exercise 7.

If the inequality is written in the form $y > mx + b$ or $y < mx + b$, then the inequality symbol indicates which half-plane to shade.

If $y > mx + b$, then shade above the boundary line;
if $y < mx + b$, then shade below the boundary line.

This method works *only* if the inequality is solved for y.

EXAMPLE 2 Graphing a Linear Inequality

Graph $x - 3y < 4$.

First graph the boundary line, shown in Figure 34. The points of the boundary line do not belong to the inequality $x - 3y < 4$ (because the inequality symbol is $<$, not $\leq$). For this reason, the line is dashed. Now solve the inequality for y.

$$x - 3y < 4$$
$$-3y < -x + 4 \qquad \text{Subtract } x.$$
$$y > \frac{1}{3}x - \frac{4}{3} \qquad \text{Multiply by } -\tfrac{1}{3}\text{; change } < \text{ to } >.$$

Because of the *is greater than* symbol, shade *above* the line. As a check, choose a test point not on the line, say (1, 2), and substitute for x and y in the original inequality.

$$1 - 3(2) < 4 \quad ?$$
$$-5 < 4 \quad \text{True}$$

This result agrees with the decision to shade above the line. The solution set, graphed in Figure 34, includes only those points in the shaded half-plane (not those on the line).

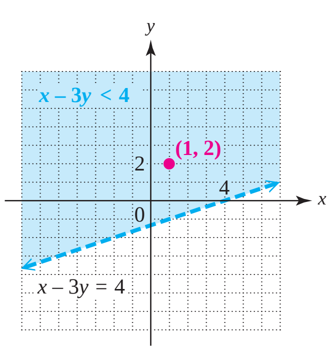


FIGURE 34

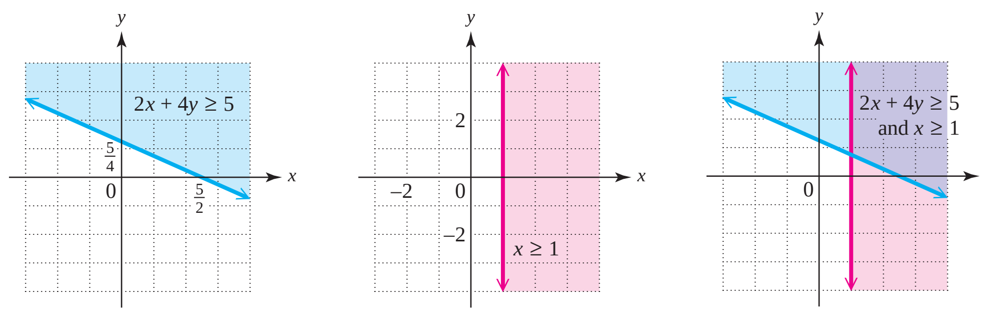
Now Try Exercise 9.

OBJECTIVE 2 Graph the intersection of two linear inequalities. In Section 2.6, we used the words *and* and *or* to solve compound inequalities. In that section, the inequalities had one variable. We can extend those ideas to include inequalities in two variables. A pair of inequalities joined with the word *and* is interpreted as the intersection of the solution sets of the inequalities. The graph of the intersection of two or more inequalities is the region of the plane where all points satisfy all of the inequalities at the same time.

EXAMPLE 3 Graphing the Intersection of Two Inequalities

Graph $2x + 4y \geq 5$ and $x \geq 1$.

To begin, we graph each of the two inequalities $2x + 4y \geq 5$ and $x \geq 1$ separately. The graph of $2x + 4y \geq 5$ is shown in Figure 35(a), and the graph of $x \geq 1$ is shown in Figure 35(b).



(a) (b) (c)

FIGURE 35

In practice, the graphs in Figures 35(a) and (b) are graphed on the same axes. Then we use heavy shading to identify the intersection of the graphs, as shown in Figure 35(c).

To check, we use a test point from each of the four regions formed by the intersection of the boundary lines. Verify that only ordered pairs in the heavily shaded region satisfy both inequalities.

Now Try Exercise 19.

OBJECTIVE 3 Graph the union of two linear inequalities. When two inequalities are joined by the word *or*, we must find the union of the graphs of the inequalities. The graph of the union of two inequalities includes all of the points that satisfy either inequality.

EXAMPLE 4 Graphing the Union of Two Inequalities

Graph $2x + 4y \geq 5$ or $x \geq 1$.

The graphs of the two inequalities are shown in Figures 35(a) and (b) in Example 3. The graph of the union is shown in Figure 36.

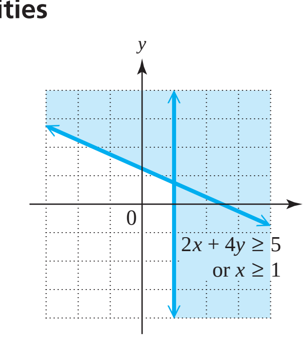


FIGURE 36

Now Try Exercise 29.

CONNECTIONS

Suppose a factory can have *no more than* 200 workers on a shift, but must have *at least* 100 and must manufacture *at least* 3000 units at minimum cost. The managers need to know how many workers should be on a shift in order to produce the required units at minimal cost. *Linear programming* is a method for finding the optimal (best possible) solution that meets all the conditions for such problems. The first step in solving linear programming problems with two variables is to express the conditions (constraints) as inequalities, graph the system of inequalities, and identify the region that satisfies all the inequalities at once.

For Discussion or Writing

Let x represent the number of workers and y represent the number of units manufactured.

1. Write three inequalities expressing the conditions given in the problem.
2. Graph the inequalities from Item 1 and shade the intersection.
3. The cost per worker is \$50 per day and the cost to manufacture 1 unit is \$100. Write an expression representing the total daily cost, C.
4. Find values of x and y for several points in or on the boundary of the shaded region. Include any "corner points."
5. Of the values of x and y that you chose in Item 4, which gives the least cost when substituted in the cost equation from Item 3? What does your answer mean in terms of the given problem? Is your answer reasonable? Explain.

OBJECTIVE 4 Use a graphing calculator to solve linear inequalities in one variable. Recall from Section 3.3 that the x-intercept of the graph of the line $y = mx + b$ indicates the solution of the equation $mx + b = 0$. We can extend this observation to find solutions of the associated inequalities $mx + b > 0$ and $mx + b < 0$. The solution set of $mx + b > 0$ is the set of all x-values for which the graph of $y = mx + b$ is *above* the x-axis. (We consider points above because the symbol is $>$.) On the other hand, the solution set of $mx + b < 0$ is the set of all x-values for which the graph of $y = mx + b$ is *below* the x-axis. (We consider points below because the symbol is $<$.)

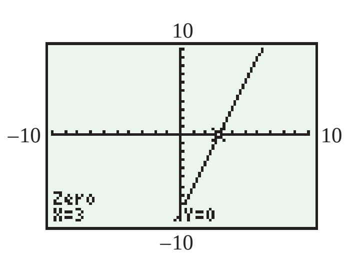


FIGURE 37

For example, in Figure 37 the x-intercept of $y = 3x - 9$ is $(3, 0)$. Therefore,

the solution set of $3x - 9 = 0$ is $\{3\}$.

Because the graph of y lies above the x-axis for x-values greater than 3,

the solution set of $3x - 9 > 0$ is $(3, \infty)$.

Because the graph lies below the x-axis for x-values less than 3,

the solution set of $3x - 9 < 0$ is $(-\infty, 3)$.

To solve the equation $-2(3x + 1) = -2x + 18$ and the associated inequalities $-2(3x + 1) > -2x + 18$ and $-2(3x + 1) < -2x + 18$, we must rewrite the equation so that the right side equals 0:

$$-2(3x + 1) + 2x - 18 = 0.$$

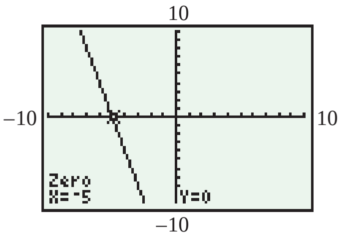


FIGURE 38

Graphing

$$y = -2(3x + 1) + 2x - 18$$

yields the x-intercept $(-5, 0)$, as shown in Figure 38. Because the graph of y lies *above* the x-axis for x-values less than -5,

the solution set of $-2(3x + 1) > -2x + 18$ is $(-\infty, -5)$.

Because the graph of y lies *below* the x-axis for x-values greater than -5,

the solution set of $-2(3x + 1) < -2x + 18$ is $(-5, \infty)$.

3.4 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 4/Videotape 5

In Exercises 1–4, fill in the first blank with either solid *or* dashed. *Fill in the second blank with* either above *or* below.

1. The boundary of the graph of $y \le -x + 2$ will be a ______ line, and the shading will be ______ the line.

2. The boundary of the graph of $y < -x + 2$ will be a ______ line, and the shading will be ______ the line.

3. The boundary of the graph of $y > -x + 2$ will be a ______ line, and the shading will be ______ the line.

4. The boundary of the graph of $y \ge -x + 2$ will be a ______ line, and the shading will be ______ the line.

5. How is the boundary line $Ax + By = C$ used in graphing either $Ax + By < C$ or $Ax + By > C$?

6. Describe the two methods discussed in the text for deciding which region is the solution set of a linear inequality in two variables.

Graph each linear inequality in two variables. See Examples 1 and 2.

EXERCISES

7. $x + y \le 2$ **8.** $x + y \le -3$ **9.** $4x - y < 4$

10. $3x - y < 3$ **11.** $x + 3y \ge -2$ **12.** $x + 4y \ge -3$

13. $x + y > 0$ **14.** $x + 2y > 0$ **15.** $x - 3y \le 0$

16. $x - 5y \le 0$ **17.** $y < x$ **18.** $y \le 4x$

Graph each compound inequality. See Example 3.

EXERCISES

19. $x + y \le 1$ and $x \ge 1$ **20.** $x - y \ge 2$ and $x \ge 3$

21. $2x - y \ge 2$ and $y < 4$ **22.** $3x - y \ge 3$ and $y < 3$

23. $x + y > -5$ and $y < -2$ **24.** $6x - 4y < 10$ and $y > 2$

Use the method described in Section 2.7 to write each inequality as a compound inequality, and graph its solution set in the rectangular coordinate plane.

EXERCISES

25. $|x| < 3$ **26.** $|y| < 5$ **27.** $|x + 1| < 2$ **28.** $|y - 3| < 2$

Graph each compound inequality. See Example 4.

29. $x - y \geq 1$ or $y \geq 2$

30. $x + y \leq 2$ or $y \geq 3$

31. $x - 2 > y$ or $x < 1$

32. $x + 3 < y$ or $x > 3$

33. $3x + 2y < 6$ or $x - 2y > 2$

34. $x - y \geq 1$ or $x + y \leq 4$

TECHNOLOGY INSIGHTS (EXERCISES 35–42)

Match each inequality with its calculator graph. (Hint: Use the slope, y-intercept, and inequality symbol in making your choice.)

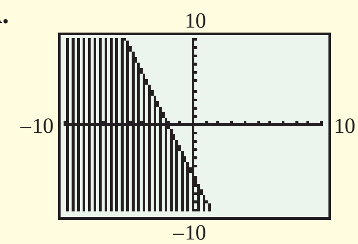
35. $y \leq 3x - 6$

36. $y \geq 3x - 6$

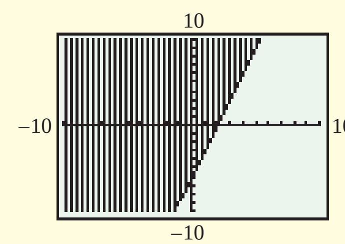
37. $y \leq -3x - 6$

38. $y \geq -3x - 6$

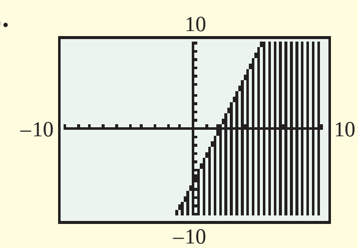
A.



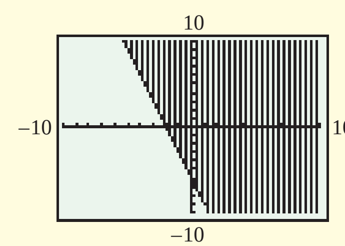
B.



C.

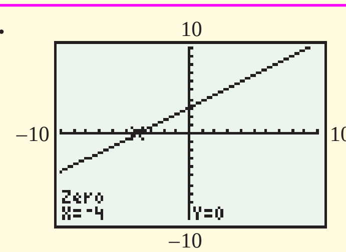


D.

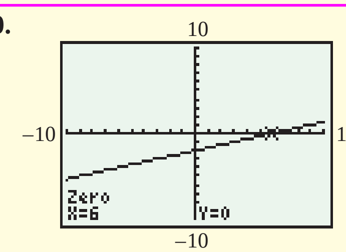


The graph of a linear equation $y = mx + b$ is shown on a graphing calculator screen, along with the x-value of the x-intercept of the line. Use the screen to solve ***(a)*** $y = 0$, ***(b)*** $y < 0$, *and* ***(c)*** $y > 0$. *See Objective 4.*

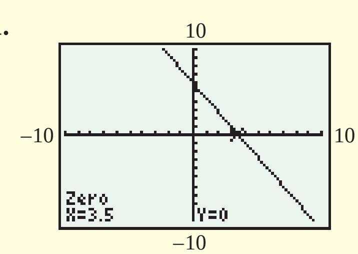
39.



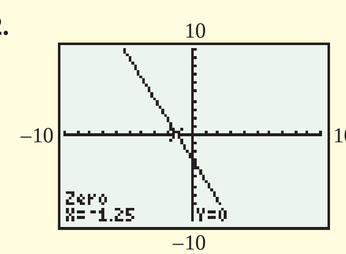
40.



41.



42.



Solve the equation in part (a) and the associated inequalities in parts (b) and (c) using the methods of Chapter 2. Then graph the left side as y in the standard viewing window of a graphing calculator and explain how the graph supports your answers in parts (a)–(c).

43. **(a)** $5x + 3 = 0$
(b) $5x + 3 > 0$
(c) $5x + 3 < 0$

44. **(a)** $6x + 3 = 0$
(b) $6x + 3 > 0$
(c) $6x + 3 < 0$

45. **(a)** $-8x - (2x + 12) = 0$
(b) $-8x - (2x + 12) \geq 0$
(c) $-8x - (2x + 12) \leq 0$

46. **(a)** $-4x - (2x + 18) = 0$
(b) $-4x - (2x + 18) \geq 0$
(c) $-4x - (2x + 18) \leq 0$

3.5 Introduction to Functions

OBJECTIVES

1. Define and identify relations and functions.
2. Find domain and range.
3. Identify functions defined by graphs and equations.
4. Use function notation.
5. Identify linear functions.

Photo not available

We often describe one quantity in terms of another. Consider the following.

- The amount of your paycheck if you are paid hourly depends on the number of hours you worked.
- The cost at the gas station depends on the number of gallons of gas you pumped into your car.
- The distance traveled by a car moving at a constant speed depends on the time traveled.

We can use ordered pairs to represent these corresponding quantities. For example, we indicate the relationship between the amount of your paycheck and hours worked by writing ordered pairs in which the first number represents hours worked and the second number represents paycheck amount in dollars. Then the ordered pair (5, 40) indicates that when you work 5 hr, your paycheck is \$40. Similarly, the ordered pairs (10, 80) and (20, 160) show that working 10 hr results in an \$80 paycheck and working 20 hr results in a \$160 paycheck. In this example, what would the ordered pair (40, 320) indicate?

Since the amount of your paycheck *depends* on the number of hours worked, your paycheck amount is called the *dependent variable*, and the number of hours worked is called the *independent variable.* Generalizing, if the value of the variable y depends on the value of the variable x, then y is the **dependent variable** and x is the **independent variable.**

Independent variable ↓ ↓ Dependent variable

(x, y)

OBJECTIVE 1 Define and identify relations and functions. Since we can write related quantities using ordered pairs, a set of ordered pairs such as

$$\{(5, 40), (10, 80), (20, 160), (40, 320)\}$$

is called a *relation.*

Relation

A **relation** is a set of ordered pairs.

A special kind of relation, called a *function,* is very important in mathematics and its applications.

Function

A **function** is a relation in which, for each value of the first component of the ordered pairs, there is *exactly one value* of the second component.

EXAMPLE 1 Determining Whether Relations Are Functions

Tell whether each relation defines a function.

$$F = \{(1, 2), (-2, 4), (3, -1)\}$$
$$G = \{(-2, -1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$$
$$H = \{(-4, 1), (-2, 1), (-2, 0)\}$$

Relations F and G are functions, because for each different x-value there is exactly one y-value. Notice that in G, the last two ordered pairs have the same y-value (1 is paired with 2, and 2 is paired with 2). This does not violate the definition of function, since the first components (x-values) are different and each is paired with only one second component (y-value).

In relation H, however, the last two ordered pairs have the *same* x-value paired with *two different* y-values (-2 is paired with both 1 and 0), so H is a relation but not a function. ***In a function, no two ordered pairs can have the same first component and different second components.***

Different y-values

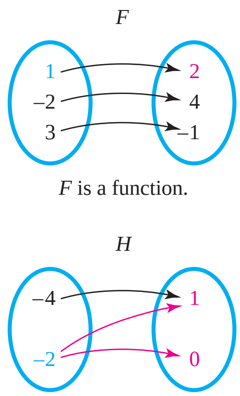
$$H = \{(-4, 1), (-2, 1), (-2, 0)\} \qquad \text{Not a function}$$

Same x-value

Now Try Exercises 5 and 7.

In a function, there is *exactly one* value of the dependent variable, the second component, for each value of the independent variable, the first component. This is what makes functions so important in applications.

NOTE The relation from the beginning of this section representing hours worked and corresponding paycheck amount is a function since each x-value is paired with exactly one y-value. You would not be happy, for example, if you and a coworker each worked 20 hr at the same hourly rate and your paycheck was \$160 while his was \$200.



F is a function.

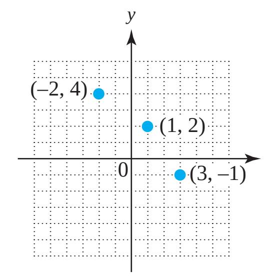
H is not a function.

FIGURE 39

Relations and functions can also be expressed as a correspondence or *mapping* from one set to another, as shown in Figure 39 for function F and relation H from Example 1. The arrow from 1 to 2 indicates that the ordered pair (1, 2) belongs to F—each first component is paired with exactly one second component. In the mapping for set H, which is not a function, the first component -2 is paired with two different second components, 1 and 0.

Since relations and functions are sets of ordered pairs, we can represent them using tables and graphs. A table and graph for function F is shown in Figure 40.

x	y
1	2
-2	4
3	-1



Graph of F

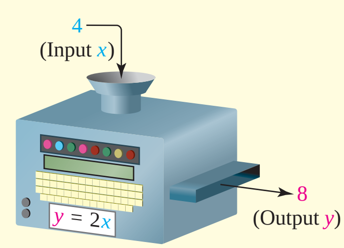
FIGURE 40

Finally, we can describe a relation or function using a rule that tells how to determine the dependent variable for a specific value of the independent variable. The rule may be given in words: the dependent variable is twice the independent variable. Usually the rule is an equation:

$$\underset{\text{Dependent variable}}{y} = 2\underset{\text{Independent variable}}{x}.$$

This is the most efficient way to define a relation or function.

NOTE Another way to think of a function relationship is to think of the independent variable as an input and the dependent variable as an output. This is illustrated by the input-output (function) machine for the function defined by $y = 2x$.



Function machine

OBJECTIVE 2 Find domain and range.

Domain and Range

In a relation, the set of all values of the independent variable (x) is the **domain;** the set of all values of the dependent variable (y) is the **range.**

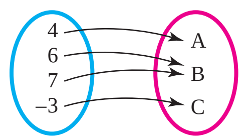
EXAMPLE 2 Finding Domains and Ranges of Relations

Give the domain and range of each relation. Tell whether the relation defines a function.

(a) $\{(3, -1), (4, 2), (4, 5), (6, 8)\}$

The domain, the set of x-values, is $\{3, 4, 6\}$; the range, the set of y-values, is $\{-1, 2, 5, 8\}$. This relation is not a function because the same x-value 4 is paired with two different y-values, 2 and 5.

(b)



The domain of this relation is

$$\{4, 6, 7, -3\};$$

the range is

$$\{A, B, C\}.$$

This mapping defines a function—each x-value corresponds to exactly one y-value.

(c)

x	y
-5	2
0	2
5	2

This is a table of ordered pairs, so the domain is the set of x-values $\{-5, 0, 5\}$ and the range is the set of y-values $\{2\}$. The table defines a function because each different x-value corresponds to exactly one y-value (even though it is the same y-value).

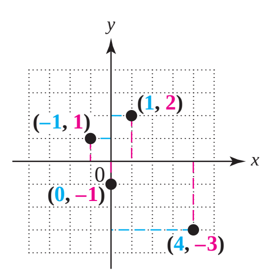
Now Try Exercises 11, 13, and 15.

As mentioned previously, the graph of a relation is the graph of its ordered pairs. The graph gives a picture of the relation, which can be used to determine its domain and range.

EXAMPLE 3 Finding Domains and Ranges from Graphs

Give the domain and range of each relation.

(a)



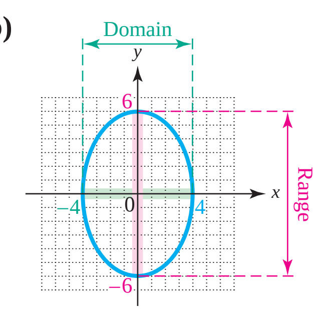
The domain is the set of x-values,

$$\{-1, 0, 1, 4\}.$$

The range is the set of y-values,

$$\{-3, -1, 1, 2\}.$$

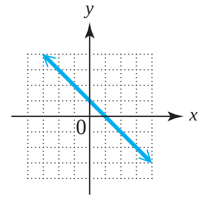
(b)



The x-values of the points on the graph include all numbers between -4 and 4, inclusive. The y-values include all numbers between -6 and 6, inclusive. Using interval notation,

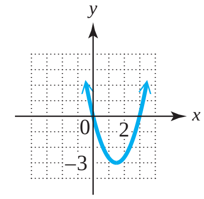
the domain is $[-4, 4]$;
the range is $[-6, 6]$.

(c)



The arrowheads indicate that the line extends indefinitely left and right, as well as up and down. Therefore, both the domain and the range include all real numbers, written $(-\infty, \infty)$.

(d)



The arrowheads indicate that the graph extends indefinitely left and right, as well as upward. The domain is $(-\infty, \infty)$. Because there is a least y-value, -3, the range includes all numbers greater than or equal to -3, written $[-3, \infty)$.

Now Try Exercises 17 and 19.

Since relations are often defined by equations, such as $y = 2x + 3$ and $y^2 = x$, we must sometimes determine the domain of a relation from its equation. In this book, we assume the following agreement on the domain of a relation.

Agreement on Domain

Unless specified otherwise, the domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.

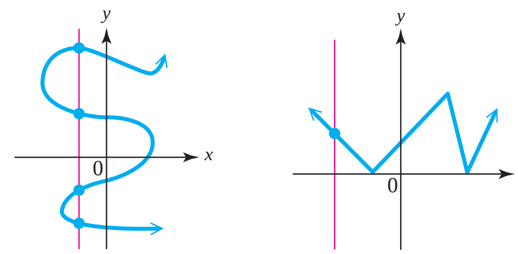
To illustrate this agreement, since any real number can be used as a replacement for x in $y = 2x + 3$, the domain of this function is the set of all real numbers. As another example, the function defined by $y = \frac{1}{x}$ has all real numbers except 0 as domain, since y is undefined if $x = 0$. In general, the domain of a function defined by an algebraic expression is all real numbers, except those numbers that lead to division by 0 or an even root of a negative number.

OBJECTIVE 3 Identify functions defined by graphs and equations. Most of the relations we have seen in the examples are functions—that is, each x-value corresponds to exactly one y-value. Since each value of x leads to only one value of y in a function, any vertical line drawn through the graph of a function must intersect the graph in at most one point. This is the *vertical line test* for a function.

Vertical Line Test

If every vertical line intersects the graph of a relation in no more than one point, then the relation represents a function.

For example, the graph shown in Figure 41(a) is not the graph of a function since a vertical line intersects the graph in more than one point. The graph in Figure 41(b) does represent a function.



Not a function–the same x-value corresponds to four different y-values.

(a)

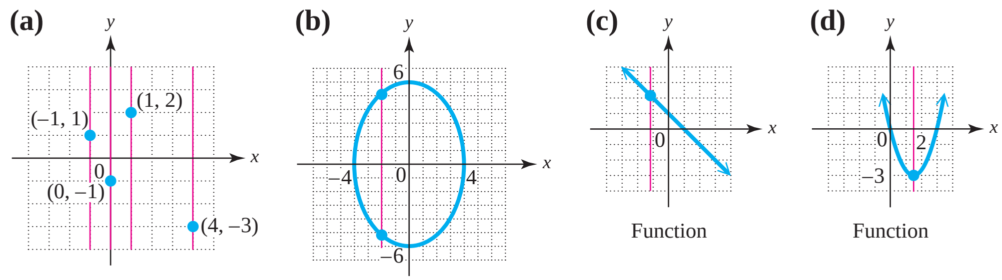
Function–each x-value corresponds to only one y-value.

(b)

FIGURE 41

EXAMPLE 4 Using the Vertical Line Test

Use the vertical line test to determine whether each relation graphed in Example 3 is a function.



(a) Function (b) Not a function (c) Function (d) Function

The graphs in (a), (c), and (d) represent functions. The graph of the relation in (b) fails the vertical line test, since the same x-value corresponds to two different y-values; therefore, it is not the graph of a function.

Now Try Exercise 21.

NOTE Graphs that do not represent functions are still relations. Remember that all equations and graphs represent relations and that all relations have a domain and range.

The vertical line test is a simple method for identifying a function defined by a graph. It is more difficult to decide whether a relation defined by an equation is a function. The next example gives some hints that may help.

EXAMPLE 5 Identifying Functions from Their Equations

Decide whether each relation defines a function and give the domain.

(a) $y = x + 4$

In the defining equation (or rule), $y = x + 4$, y is always found by adding 4 to x. Thus, each value of x corresponds to just one value of y and the relation defines a function; x can be any real number, so the domain is $\{x \mid x \text{ is a real number}\}$ or $(-\infty, \infty)$.

(b) $y = \sqrt{2x - 1}$

For any choice of x in the domain, there is exactly one corresponding value for y (the radical is a nonnegative number), so this equation defines a function. Refer to the agreement on domain stated previously. Since the equation involves a square root, the quantity under the radical sign cannot be negative. Thus,

$$\begin{aligned} 2x - 1 &\geq 0 \\ 2x &\geq 1 \\ x &\geq \frac{1}{2}, \end{aligned}$$

and the domain of the function is $\left[\frac{1}{2}, \infty\right)$.

(c) $y^2 = x$

The ordered pairs (16, 4) and (16, -4) both satisfy this equation. Since one value of x, 16, corresponds to two values of y, 4 and -4, this equation does not define a function. Because x is equal to the square of y, the values of x must always be nonnegative. The domain of the relation is $[0, \infty)$.

(d) $y \leq x - 1$

By definition, y is a function of x if every value of x leads to exactly one value of y. In this example, a particular value of x, say 1, corresponds to many values of y. The ordered pairs (1, 0), (1, -1), (1, -2), (1, -3), and so on, all satisfy the inequality. For this reason, an inequality never defines a function. Any number can be used for x, so the domain is the set of real numbers, $(-\infty, \infty)$.

(e) $y = \dfrac{5}{x - 1}$

Given any value of x in the domain, we find y by subtracting 1, then dividing the result into 5. This process produces exactly one value of y for each value in the domain, so this equation defines a function. The domain includes all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving for x.

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned}$$

Thus, the domain includes all real numbers except 1. In interval notation this is written as $(-\infty, 1) \cup (1, \infty)$.

Now Try Exercises 27, 29, and 35.

In summary, three variations of the definition of function are given here.

AUDIO

Variations of the Definition of Function

1. A **function** is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A **function** is a set of ordered pairs in which no first component is repeated.
3. A **function** is a rule or correspondence that assigns exactly one range value to each domain value.

OBJECTIVE 4 Use function notation. When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say "y is a function of x" to emphasize that y *depends on* x. We use the notation

$$y = f(x),$$

called **function notation,** to express this and read $f(x)$ as "f of x." (In this special notation the parentheses do not indicate multiplication.) The letter f stands for *function.* For example, if $y = 9x - 5$, we can name this function f and write

$$f(x) = 9x - 5.$$

Note that ***f*(*x*) *is just another name for the dependent variable y.*** For example, if $y = f(x) = 9x - 5$ and $x = 2$, then we find y, or $f(2)$, by replacing x with 2.

$$\begin{aligned} y &= f(2) \\ &= 9 \cdot 2 - 5 \\ &= 18 - 5 \\ &= 13. \end{aligned}$$

The statement "if $x = 2$, then $y = 13$" represents the ordered pair $(2, 13)$ and is abbreviated with function notation as

$$f(2) = 13.$$

Read $f(2)$ as "f of 2" or "f at 2." Also,

$$f(0) = 9 \cdot 0 - 5 = -5 \qquad \text{and} \qquad f(-3) = 9(-3) - 5 = -32.$$

These ideas and the symbols used to represent them can be illustrated as follows.

Name of the function

$$y = f(x) = \overbrace{9x - 5}^{\text{Defining expression}}$$

Value of the function Name of the independent variable

CAUTION The symbol $f(x)$ *does not* indicate "f times x," but represents the y-value for the indicated x-value. As just shown, $f(2)$ is the y-value that corresponds to the x-value 2.

EXAMPLE 6 Using Function Notation

Let $f(x) = -x^2 + 5x - 3$. Find the following.

(a) $f(2)$

Replace x with 2.

$$\begin{aligned} f(x) &= -x^2 + 5x - 3 \\ f(2) &= -2^2 + 5 \cdot 2 - 3 \\ &= -4 + 10 - 3 \\ &= 3 \end{aligned}$$

Thus, $f(2) = 3$; the ordered pair $(2, 3)$ belongs to f.

(b) $f(q)$

Replace x with q.

$$f(x) = -x^2 + 5x - 3$$
$$f(q) = -q^2 + 5q - 3$$

The replacement of one variable with another is important in later courses.

Now Try Exercises 41 and 45.

Sometimes letters other than f, such as g, h, or capital letters F, G, and H are used to name functions.

EXPLORE

EXAMPLE 7 Using Function Notation

Let $g(x) = 2x + 3$. Find and simplify $g(a + 1)$.

$$g(x) = 2x + 3$$
$$g(a + 1) = 2(a + 1) + 3 \quad \text{Replace } x \text{ with } a + 1.$$
$$= 2a + 2 + 3$$
$$= 2a + 5$$

Now Try Exercise 49.

Functions can be evaluated in a variety of ways, as shown in Example 8.

EXAMPLE 8 Using Function Notation

For each function, find $f(3)$.

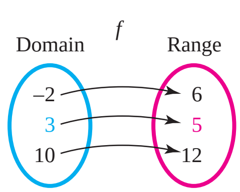
(a) $f(x) = 3x - 7$

$$f(3) = 3(3) - 7 \quad \text{Replace } x \text{ with } 3.$$
$$f(3) = 2$$

(b) $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$

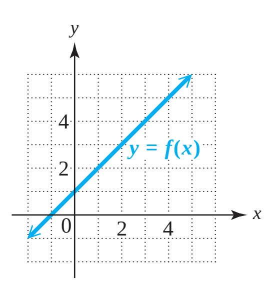
We want $f(3)$, the y-value of the ordered pair where $x = 3$. As indicated by the ordered pair $(3, 1)$, when $x = 3$, $y = 1$, so $f(3) = 1$.

(c)



The domain element 3 is paired with 5 in the range, so $f(3) = 5$.

(d)



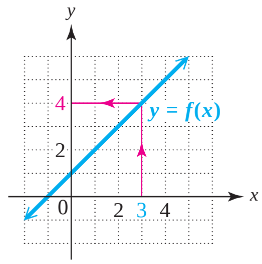


FIGURE 42

To evaluate $f(3)$, find 3 on the x-axis. See Figure 42. Then move up until the graph of f is reached. Moving horizontally to the y-axis gives 4 for the corresponding y-value. Thus, $f(3) = 4$.

Now Try Exercises 53 and 57.

If a function f is defined by an equation with x and y, not with function notation, use the following steps to find $f(x)$.

Finding an Expression for $f(x)$

Step 1 Solve the equation for y.

Step 2 Replace y with $f(x)$.

EXAMPLE 9 Writing Equations Using Function Notation

Rewrite each equation using function notation. Then find $f(-2)$ and $f(a)$.

(a) $y = x^2 + 1$

This equation is already solved for y. Since $y = f(x)$,

$$f(x) = x^2 + 1.$$

To find $f(-2)$, let $x = -2$.

$$\begin{aligned} f(-2) &= (-2)^2 + 1 \\ &= 4 + 1 \\ &= 5 \end{aligned}$$

Find $f(a)$ by letting $x = a$: $\quad f(a) = a^2 + 1.$

(b) $x - 4y = 5$

First solve $x - 4y = 5$ for y. Then replace y with $f(x)$.

$$\begin{aligned} x - 4y &= 5 \\ x - 5 &= 4y \\ y &= \frac{x-5}{4} \quad \text{so} \quad f(x) = \frac{1}{4}x - \frac{5}{4} \end{aligned}$$

Now find $f(-2)$ and $f(a)$.

$$f(-2) = \frac{1}{4}(-2) - \frac{5}{4} = -\frac{7}{4} \qquad \text{Let } x = -2.$$

$$f(a) = \frac{1}{4}a - \frac{5}{4} \qquad \text{Let } x = a.$$

Now Try Exercise 59.

OBJECTIVE 5 Identify linear functions. Our first two-dimensional graphing was of straight lines. Linear equations (except for vertical lines with equations $x = a$) define *linear functions*.

AUDIO

> **Linear Function**
>
> A function that can be defined by
>
> $$\boldsymbol{f(x) = mx + b}$$
>
> for real numbers m and b is a **linear function.**

Recall from Section 3.3 that m is the slope of the line and $(0, b)$ is the y-intercept. In Example 9(b), we wrote the equation $x - 4y = 5$ as the linear function defined by

$$f(x) = \underset{\text{Slope}}{\frac{1}{4}}x - \underset{y\text{-intercept is } \left(0, -\frac{5}{4}\right).}{\frac{5}{4}}.$$

To graph this function, plot the y-intercept and use the definition of slope as $\frac{\text{rise}}{\text{run}}$ to find a second point on the line. Draw the straight line through the points to obtain the graph. (See Section 3.3, Example 2.)

A linear function defined by $\boldsymbol{f(x) = b}$ (whose graph is a horizontal line) is sometimes called a **constant function.** The domain of any linear function is $(-\infty, \infty)$. The range of a nonconstant linear function is $(-\infty, \infty)$, while the range of the constant function defined by $f(x) = b$ is $\{b\}$.

Now Try Exercise 67.

3.5 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 5/Videotape 5

1. In your own words, define a function and give an example.

2. In your own words, define the domain of a function and give an example.

3. In an ordered pair of a relation, is the first element the independent or the dependent variable?

4. Give an example of a relation that is not a function, having domain $\{-3, 2, 6\}$ and range $\{4, 6\}$. (There are many possible correct answers.)

Tell whether each relation defines a function. See Example 1.

5. $\{(5, 1), (3, 2), (4, 9), (7, 6)\}$

6. $\{(8, 0), (5, 4), (9, 3), (3, 8)\}$

7. $\{(2, 4), (0, 2), (2, 5)\}$

8. $\{(9, -2), (-3, 5), (9, 2)\}$

9. $\{(-3, 1), (4, 1), (-2, 7)\}$

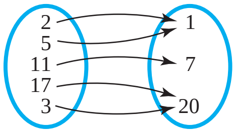
10. $\{(-12, 5), (-10, 3), (8, 3)\}$

Decide whether each relation defines a function and give the domain and range. See Examples 1–4.

11. $\{(1, 1), (1, -1), (0, 0), (2, 4), (2, -4)\}$

12. $\{(2, 5), (3, 7), (4, 9), (5, 11)\}$

13.



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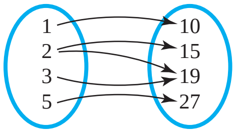


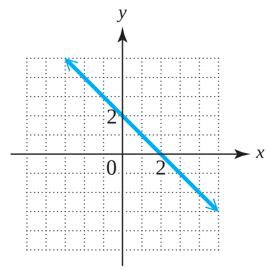
15.

x	y
1	5
1	2
1	-1
1	-4

16.

x	y
4	-3
2	-3
0	-3
-2	-3

17.



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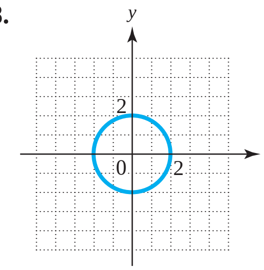

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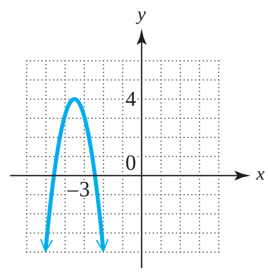

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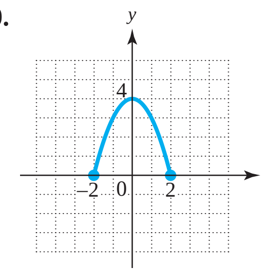

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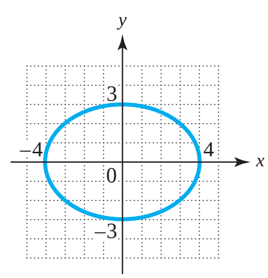

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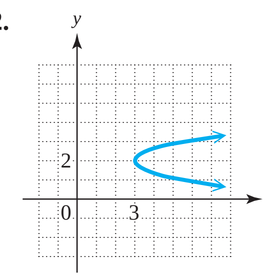


Decide whether each relation defines y as a function of x. Give the domain. See Example 5.

EXERCISES

23. $y = x^2$ **24.** $y = x^3$ **25.** $x = y^6$ **26.** $x = y^4$

27. $y = 2x - 6$ **28.** $y = -6x + 8$ **29.** $x + y < 4$ **30.** $x - y < 3$

31. $y = \sqrt{x}$ **32.** $y = -\sqrt{x}$ **33.** $xy = 1$ **34.** $xy = -3$

35. $y = \sqrt{4x + 2}$ **36.** $y = \sqrt{9 - 2x}$ **37.** $y = \dfrac{2}{x - 9}$ **38.** $y = \dfrac{-7}{x - 16}$

39. Choose the correct response: The notation $f(3)$ means

A. the variable f times 3 or $3f$.
B. the value of the dependent variable when the independent variable is 3.
C. the value of the independent variable when the dependent variable is 3.
D. f equals 3.

40. Give an example of a function from everyday life. (*Hint:* Fill in the blanks: _____ depends on _____, so _____ is a function of _____.)

Let $f(x) = -3x + 4$ and $g(x) = -x^2 + 4x + 1$. Find the following. See Examples 6 and 7.

41. $f(0)$ **42.** $f(-3)$ **43.** $g(-2)$ **44.** $g(10)$

45. $f(p)$ **46.** $g(k)$ **47.** $f(-x)$ **48.** $g(-x)$

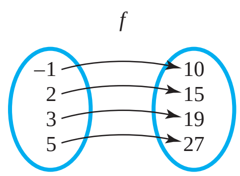
49. $f(x + 2)$ **50.** $f(a + 4)$ **51.** $f(2m - 3)$ **52.** $f(3t - 2)$

For each function, find ***(a)*** *$f(2)$ and* ***(b)*** *$f(-1)$. See Example 8.*

53. $f = \{(-1, 3), (4, 7), (0, 6), (2, 2)\}$

54. $f = \{(2, 5), (3, 9), (-1, 11), (5, 3)\}$

55.



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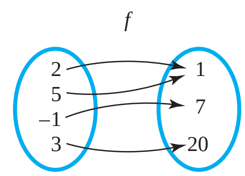

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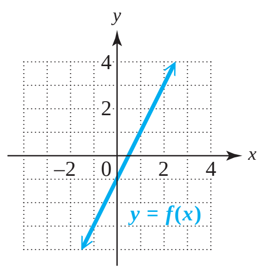

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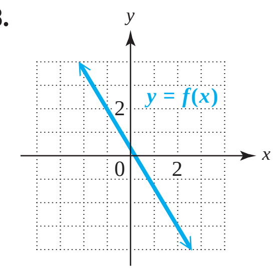


An equation that defines y as a function of x is given. ***(a)*** *Solve for y in terms of x and replace y with the function notation $f(x)$.* ***(b)*** *Find $f(3)$. See Example 9.*

59. $x + 3y = 12$ **60.** $x - 4y = 8$ **61.** $y + 2x^2 = 3$

62. $y - 3x^2 = 2$ **63.** $4x - 3y = 8$ **64.** $-2x + 5y = 9$

65. Fill in each blank with the correct response.

The equation $2x + y = 4$ has a straight _____ as its graph. One point that lies on the graph is (3, _____). If we solve the equation for y and use function notation, we obtain $f(x) =$ _____. For this function, $f(3) =$ _____, meaning that the point (_____, _____) lies on the graph of the function.

66. Which of the following defines a linear function?

A. $y = \dfrac{x - 5}{4}$ **B.** $y = \dfrac{1}{x}$ **C.** $y = x^2$ **D.** $y = \sqrt{x}$

Graph each linear function. Give the domain and range. See Objective 5.

67. $f(x) = -2x + 5$ **68.** $g(x) = 4x - 1$ **69.** $h(x) = \dfrac{1}{2}x + 2$

70. $F(x) = -\dfrac{1}{4}x + 1$ **71.** $G(x) = 2x$ **72.** $H(x) = -3x$

73. $g(x) = -4$ **74.** $f(x) = 5$

75. Suppose that a package weighing x lb costs $f(x)$ dollars to mail to a given location, where

$$f(x) = 2.75x.$$

(a) What is the value of $f(3)$?

(b) In your own words, describe what 3 and the value $f(3)$ mean in part (a), using the terminology *independent variable* and *dependent variable.*

(c) How much would it cost to mail a 5-lb package? Interpret this question and its answer using function notation.

EXERCISES

76. Suppose that a Yellow Cab driver charges \$1.50 per mile.

(a) Fill in the table with the correct response for the price $f(x)$ he charges for a trip of x mi.

x	$f(x)$
0	
1	
2	
3	

(b) The linear function that gives a rule for the amount charged is $f(x) =$ ______.

(c) Graph this function for the domain {0, 1, 2, 3}.

photo not available

Forensic scientists use the lengths of certain bones to calculate the height of a person. Two bones often used are the tibia (t), the bone from the ankle to the knee, and the femur (r), the bone from the knee to the hip socket. A person's height (h) is determined from the lengths of these bones using functions defined by the following formulas. All measurements are in centimeters.

For men: $h(r) = 69.09 + 2.24r$ or $h(t) = 81.69 + 2.39t$

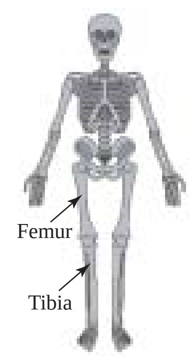
For women: $h(r) = 61.41 + 2.32r$ or $h(t) = 72.57 + 2.53t$

77. Find the height of a man with a femur measuring 56 cm.

78. Find the height of a man with a tibia measuring 40 cm.

79. Find the height of a woman with a femur measuring 50 cm.

80. Find the height of a woman with a tibia measuring 36 cm.



Federal regulations set standards for the size of the quarters of marine mammals. A pool to house sea otters must have a volume of "the square of the sea otter's average adult length (in meters) multiplied by 3.14 *and by* .91 *meter." If x represents the sea otter's average adult length and* $f(x)$ *represents the volume (in cubic meters) of the corresponding pool size, this formula can be written as*

$$f(x) = .91(3.14)x^2.$$

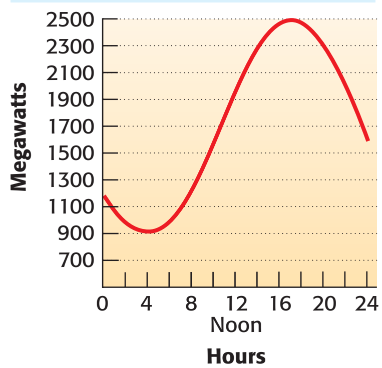
photo not available

Find the volume of the pool for each adult sea otter length (in meters). Round answers to the nearest hundredth.

81. .8 **82.** 1.0 **83.** 1.2 **84.** 1.5

85. The graph shows the daily megawatts of electricity used on a record-breaking summer day in Sacramento, California.

ELECTRICITY USE

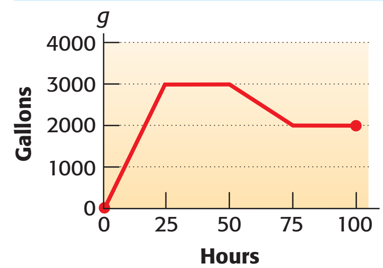


Source: Sacramento Municipal Utility District.

(a) Is this the graph of a function?
(b) What is the domain?
(c) Estimate the number of megawatts used at 8 A.M.
(d) At what time was the most electricity used? the least electricity?
(e) Call this function f. What is $f(12)$? What does it mean?

86. Refer to the graph to answer the questions.

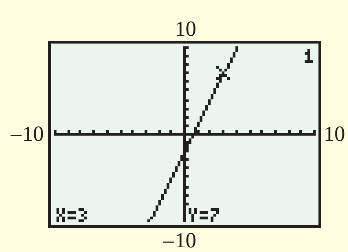
GALLONS OF WATER IN A POOL AT TIME t



(a) What numbers are possible values of the independent variable? the dependent variable?
(b) For how long is the water level increasing? decreasing?
(c) How many gallons of water are in the pool after 90 hr?
(d) Call this function f. What is $f(0)$? What does it mean?
(e) What is $f(25)$? What does it mean?

TECHNOLOGY INSIGHTS (EXERCISES 87 AND 88)

87. The calculator screen shows the graph of a linear function $y = f(x)$, along with the display of coordinates of a point on the graph. Use function notation to write what the display indicates.



88. The table was generated by a graphing calculator for a linear function $Y_1 = f(X)$. Use the table to work parts (a)–(e).

(a) What is $f(2)$?
(b) If $f(X) = -3.7$, what is the value of X?
(c) What is the slope of the line?
(d) What is the y-intercept of the line?
(e) Find the expression for $f(X)$.

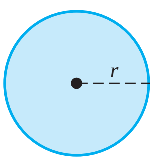
X	Y1
0	3.5
1	2.3
2	1.1
3	-.1
4	-1.3
5	-2.5
6	-3.7

X=0

3.6 Variation

OBJECTIVES

1 Write an equation expressing direct variation.
2 Find the constant of variation, and solve direct variation problems.
3 Solve inverse variation problems.
4 Solve joint variation problems.
5 Solve combined variation problems.



$C = 2\pi r$

Certain types of functions are very common, especially in business and the physical sciences. These are functions where y depends on a multiple of x, or y depends on a number divided by x. In such situations, y is said to *vary directly as x* (in the first case) or *vary inversely as x* (in the second case). For example, by the distance formula, the distance traveled varies directly as the rate (or speed) and the time. Formulas for area and volume are other familiar examples of *direct variation*.

On the other hand, the force required to keep a car from skidding on a curve varies inversely as the radius of the curve. Another example of *inverse variation* is how travel time is inversely proportional to rate or speed.

OBJECTIVE 1 Write an equation expressing direct variation. The circumference of a circle is given by the formula $C = 2\pi r$, where r is the radius of the circle. See the figure. Circumference is always a constant multiple of the radius. (C is always found by multiplying r by the constant 2π.) Thus,

As the *radius increases*, the *circumference increases.*

The reverse is also true.

As the *radius decreases*, the *circumference decreases.*

Because of this, the circumference is said to *vary directly* as the radius.

Direct Variation

y* varies directly as *x if there exists a real number k such that

$$\mathbf{y = kx.}$$

Also, y is said to be **proportional to** x. The number k is called the **constant of variation.** In direct variation, for $k > 0$, as the value of x increases, the value of y also increases. Similarly, as x decreases, y decreases.

OBJECTIVE 2 Find the constant of variation, and solve direct variation problems. The direct variation equation $y = kx$ defines a linear function, where the constant of variation k is the slope of the line. For example, we wrote the equation

$$y = 1.60x$$

to describe the cost y to buy x gal of gas in Example 6 of Section 3.3. The cost varies directly as, or is proportional to, the number of gallons of gas purchased. That is, as the number of gallons of gas increases, cost increases; also, as the number of gallons of gas decreases, cost decreases. The constant of variation k is 1.60, the cost of 1 gal of gas.

EXAMPLE 1 Finding the Constant of Variation and the Variation Equation

Steven Pusztai is paid an hourly wage. One week he worked 43 hr and was paid \$795.50. How much does he earn per hour?

Let h represent the number of hours he works and P represent his corresponding pay. Then, P varies directly as h, so

$$P = kh.$$

Here, k represents Steven's hourly wage. Since $P = 795.50$ when $h = 43$,

$$795.50 = 43k$$
$$k = 18.50. \qquad \text{Use a calculator.}$$

His hourly wage is \$18.50, and P and h are related by

$$P = 18.50h.$$

Now Try Exercise 31.

EXAMPLE 2 Solving a Direct Variation Problem

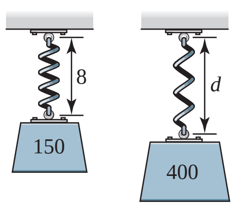


FIGURE 43

Hooke's law for an elastic spring states that the distance a spring stretches is proportional to the force applied. If a force of 150 newtons* stretches a certain spring 8 cm, how much will a force of 400 newtons stretch the spring? See Figure 43.

If d is the distance the spring stretches and f is the force applied, then $d = kf$ for some constant k. Since a force of 150 newtons stretches the spring 8 cm, use these values to find k.

$$d = kf \qquad \text{Variation equation}$$
$$8 = k \cdot 150 \qquad \text{Let } d = 8 \text{ and } f = 150.$$
$$k = \frac{8}{150} \qquad \text{Find } k.$$
$$k = \frac{4}{75}$$

Substitute $\frac{4}{75}$ for k in the variation equation $d = kf$ to get

$$d = \frac{4}{75}f.$$

For a force of 400 newtons,

$$d = \frac{4}{75}(400) \qquad \text{Let } f = 400.$$
$$= \frac{64}{3}.$$

The spring will stretch $\frac{64}{3}$ cm if a force of 400 newtons is applied.

Now Try Exercise 35.

*A newton is a unit of measure of force used in physics.

In summary, use the following steps to solve a variation problem.

Solving a Variation Problem

Step 1 Write the variation equation.

Step 2 Substitute the initial values and solve for k.

Step 3 Rewrite the variation equation with the value of k from Step 2.

Step 4 Substitute the remaining values, solve for the unknown, and find the required answer.

The direct variation equation $y = kx$ is a linear equation. However, other kinds of variation involve other types of equations. For example, one variable can be proportional to a power of another variable.

Direct Variation as a Power

y* varies directly as the *n*th power of *x if there exists a real number k such that

$$y = kx^n.$$

An example of direct variation as a power is the formula for the area of a circle, $A = \pi r^2$. Here, π is the constant of variation, and the area varies directly as the *square* of the radius.

EXAMPLE 3 Solving a Direct Variation Problem

The distance a body falls from rest varies directly as the square of the time it falls (disregarding air resistance). If a skydiver falls 64 ft in 2 sec, how far will she fall in 8 sec?

Step 1 If d represents the distance the skydiver falls and t the time it takes to fall, then d is a function of t, and, for some constant k,

$$d = kt^2.$$

Step 2 To find the value of k, use the fact that the skydiver falls 64 ft in 2 sec.

$$\begin{aligned} d &= kt^2 && \text{Variation equation} \\ 64 &= k(2)^2 && \text{Let } d = 64 \text{ and } t = 2. \\ k &= 16 && \text{Find } k. \end{aligned}$$

Step 3 Using 16 for k, the variation equation becomes

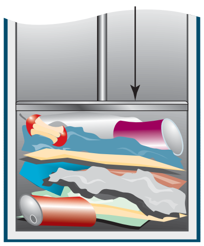
$$d = 16t^2.$$

Step 4 Now let $t = 8$ to find the number of feet the skydiver will fall in 8 sec.

$$\begin{aligned} d &= 16(8)^2 && \text{Let } t = 8. \\ &= 1024 \end{aligned}$$

The skydiver will fall 1024 ft in 8 sec.

Now Try Exercise 37.



As pressure on trash increases, volume of trash decreases.

FIGURE 44

OBJECTIVE 3 Solve inverse variation problems. In direct variation, where $k > 0$, as x increases, y increases. Similarly, as x decreases, y decreases. Another type of variation is *inverse variation*. With inverse variation, where $k > 0$, as one variable increases, the other variable decreases. For example, in a closed space, volume decreases as pressure increases, as illustrated by a trash compactor. See Figure 44. As the compactor presses down, the pressure on the trash increases; in turn, the trash occupies a smaller space.

AUDIO

Inverse Variation

y* varies inversely as *x if there exists a real number k such that

$$y = \frac{k}{x}.$$

Also, ***y* varies inversely as the *n*th power of *x*** if there exists a real number k such that

$$y = \frac{k}{x^n}.$$

The inverse variation equation also defines a function. Since x is in the denominator, these functions are *rational functions*. (See Chapter 7.) Another example of inverse variation comes from the distance formula. In its usual form, the formula is

$$d = rt$$

Dividing each side by r gives

$$t = \frac{d}{r}.$$

Here, t (time) varies inversely as r (rate or speed), with d (distance) serving as the constant of variation. For example, if the distance between Chicago and Des Moines is 300 mi, then

$$t = \frac{300}{r}$$

and the values of r and t might be any of the following.

$$\left.\begin{aligned} r &= 50, t = 6 \\ r &= 60, t = 5 \\ r &= 75, t = 4 \end{aligned}\right\} \text{As } r \text{ increases, } t \text{ decreases.} \qquad \left.\begin{aligned} r &= 30, t = 10 \\ r &= 25, t = 12 \\ r &= 20, t = 15 \end{aligned}\right\} \text{As } r \text{ decreases, } t \text{ increases.}$$

If we *increase* the rate (speed) we drive, time *decreases*. If we *decrease* the rate (speed) we drive, what happens to time?

EXAMPLE 4 Solving an Inverse Variation Problem

In the manufacturing of a certain medical syringe, the cost of producing the syringe varies inversely as the number produced. If 10,000 syringes are produced, the cost is \$2 per syringe. Find the cost per syringe to produce 25,000 syringes.

Let $x =$ the number of syringes produced,

and $c =$ the cost per syringe.

Photo not available

Here, as production increases, cost decreases and as production decreases, cost increases. Since c varies inversely as x, there is a constant k such that

$$c = \frac{k}{x}.$$

Find k by replacing c with 2 and x with 10,000.

$$2 = \frac{k}{10{,}000}$$

$$20{,}000 = k \qquad \text{Multiply by 10,000.}$$

Since $c = \frac{k}{x}$,

$$c = \frac{20{,}000}{25{,}000} = .80. \qquad \text{Let } k = 20{,}000 \text{ and } x = 25{,}000.$$

The cost per syringe to make 25,000 syringes is \$.80.

Now Try Exercise 39.

EXAMPLE 5 Solving an Inverse Variation Problem

The weight of an object above Earth varies inversely as the square of its distance from the center of Earth. A space shuttle in an elliptical orbit has a maximum distance from the center of Earth (*apogee*) of 6700 mi. Its minimum distance from the center of Earth (*perigee*) is 4090 mi. See Figure 45. If an astronaut in the shuttle weighs 57 lb at its apogee, what does the astronaut weigh at its perigee?

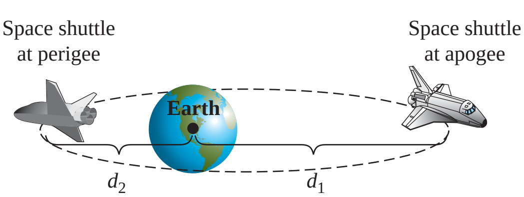


FIGURE 45

If w is the weight and d is the distance from the center of Earth, then

$$w = \frac{k}{d^2}$$

for some constant k. At the apogee the astronaut weighs 57 lb, and the distance from the center of Earth is 6700 mi. Use these values to find k.

$$57 = \frac{k}{(6700)^2} \qquad \text{Let } w = 57 \text{ and } d = 6700.$$

$$k = 57(6700)^2$$

Then the weight at the perigee with $d = 4090$ mi is

$$w = \frac{57(6700)^2}{(4090)^2} \approx 153 \text{ lb.} \qquad \text{Use a calculator.}$$

Now Try Exercise 43.

OBJECTIVE 4 Solve joint variation problems. It is common for one variable to depend on several others. If one variable varies directly as the *product* of several other variables (perhaps raised to powers), the first variable is said to *vary jointly* as the others.

Joint Variation

y* varies jointly as *x* and *z if there exists a real number k such that

$$y = kxz.$$

CAUTION Note that *and* in the expression "y varies directly as x *and* z" translates as the product $y = kxz$. The word *and* does not indicate addition here.

EXAMPLE 6 Solving a Joint Variation Problem

The interest on a loan or an investment is given by the formula $I = prt$. Here, for a given principal p, the interest earned I varies jointly as the interest rate r and the time t the principal is left at interest. If an investment earns \$100 interest at 5% for 2 yr, how much interest will the same principal earn at 4.5% for 3 yr?

We use the formula $I = prt$, where p is the constant of variation because it is the same for both investments. For the first investment, we have $I = 100$, $r = .05$, and $t = 2$, so

$$\begin{aligned} I &= prt \\ 100 &= p(.05)(2) && \text{Let } I = 100,\ r = .05, \text{ and } t = 2. \\ 100 &= .1p \\ \frac{100}{.1} &= p \\ p &= 1000. \end{aligned}$$

Now we find I when $p = 1000$, $r = .045$, and $t = 3$.

$$\begin{aligned} I &= 1000(.045)(3) && \text{Let } p = 1000,\ r = .045, \text{ and } t = 3. \\ I &= 135 \end{aligned}$$

The interest will be \$135.

Now Try Exercise 45.

OBJECTIVE 5 Solve combined variation problems. There are many combinations of direct and inverse variation. Example 7 shows a typical **combined variation** problem.

EXAMPLE 7 Solving a Combined Variation Problem

Body mass index, or BMI, is used by physicians to assess a person's level of fatness. A BMI from 19 through 25 is considered desirable. BMI varies directly as an individual's weight in pounds and inversely as the square of the individual's height in inches. A person who weighs 118 lb and is 64 in. tall has a BMI of 20. (The BMI is rounded to the nearest whole number.) Find the BMI of a person who weighs 165 lb with a height of 70 in. (*Source: Washington Post.*)

Let B represent the BMI, w the weight, and h the height. Then

$$B = \frac{kw}{h^2}.$$

BMI varies directly as the weight.

BMI varies inversely as the square of the height.

To find k, let $B = 20$, $w = 118$, and $h = 64$.

$$20 = \frac{k(118)}{64^2}$$

$$k = \frac{20(64^2)}{118} \qquad \text{Multiply by } 64^2\text{; divide by 118.}$$

$$k \approx 694 \qquad \text{Use a calculator.}$$

Now find B when $k = 694$, $w = 165$, and $h = 70$.

$$B = \frac{694(165)}{70^2} \approx 23 \qquad \text{Nearest whole number}$$

The person's BMI is 23.

Photo not available

Now Try Exercise 47.

3.6 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 5/Videotape 5

Use personal experience or intuition to determine whether the situation suggests direct *or* inverse *variation.*

1. The number of different lottery tickets you buy and your probability of winning that lottery

2. The rate and the distance traveled by a pickup truck in 3 hr

3. The amount of pressure put on the accelerator of a car and the speed of the car

4. The number of days from now until December 25 and the magnitude of the frenzy of Christmas shopping

5. Your age and the probability that you believe in Santa Claus

6. The surface area of a balloon and its diameter

7. The number of days until the end of the baseball season and the number of home runs that Barry Bonds has

8. The amount of gasoline you pump and the amount you pay

Determine whether each equation represents direct, inverse, joint, *or* combined *variation.*

9. $y = \frac{3}{x}$

10. $y = \frac{8}{x}$

11. $y = 10x^2$

12. $y = 2x^3$

13. $y = 3xz^4$

14. $y = 6x^3z^2$

15. $y = \frac{4x}{wz}$

16. $y = \frac{6x}{st}$

17. For $k > 0$, if y varies directly as x, when x increases, y __________, and when x decreases, y __________.

18. For $k > 0$, if y varies inversely as x, when x increases, y __________, and when x decreases, y __________.

Solve each problem.

EXERCISES

19. If x varies directly as y, and $x = 9$ when $y = 3$, find x when $y = 12$.

20. If x varies directly as y, and $x = 10$ when $y = 7$, find y when $x = 50$.

21. If a varies directly as the square of b, and $a = 4$ when $b = 3$, find a when $b = 2$.

22. If h varies directly as the square of m, and $h = 15$ when $m = 5$, find h when $m = 7$.

23. If z varies inversely as w, and $z = 10$ when $w = .5$, find z when $w = 8$.

24. If t varies inversely as s, and $t = 3$ when $s = 5$, find s when $t = 5$.

25. If m varies inversely as p^2, and $m = 20$ when $p = 2$, find m when $p = 5$.

26. If a varies inversely as b^2, and $a = 48$ when $b = 4$, find a when $b = 7$.

27. p varies jointly as q and r^2, and $p = 200$ when $q = 2$ and $r = 3$. Find p when $q = 5$ and $r = 2$.

28. f varies jointly as g^2 and h, and $f = 50$ when $g = 4$ and $h = 2$. Find f when $g = 3$ and $h = 6$.

29. Explain the difference between inverse variation and direct variation.

30. What is meant by the constant of variation in a direct variation problem? If you were to graph the linear equation $y = kx$ for some nonnegative constant k, what role would the value of k play in the graph?

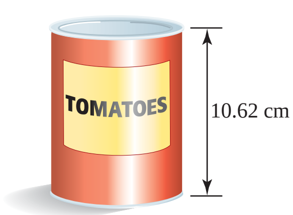
Solve each problem involving variation. See Examples 1–7.

EXERCISES

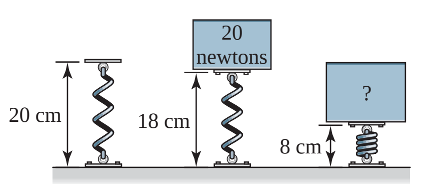
31. Todd bought 8 gal of gasoline and paid \$13.59. To the nearest tenth of a cent, what is the price of gasoline per gallon?

32. Melissa gives horseback rides at Shadow Mountain Ranch. A 2.5-hr ride costs \$50.00. What is the price per hour?

33. The volume of a can of tomatoes is proportional to the height of the can. If the volume of the can is 300 cm^3 when its height is 10.62 cm, find the volume of a can with height 15.92 cm.



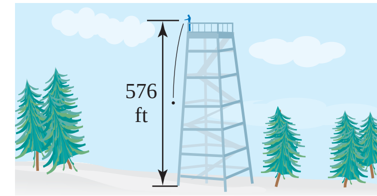
34. The force required to compress a spring is proportional to the change in length of the spring. If a force of 20 newtons is required to compress a certain spring 2 cm, how much force is required to compress the spring from 20 cm to 8 cm?



35. The weight of an object on Earth is directly proportional to the weight of that same object on the moon. A 200-lb astronaut would weigh 32 lb on the moon. How much would a 50-lb dog weigh on the moon?

36. The pressure exerted by a certain liquid at a given point varies directly as the depth of the point beneath the surface of the liquid. The pressure at 30 m is 80 newtons per m^2. What pressure is exerted at 50 m?

37. For a body falling freely from rest (disregarding air resistance), the distance the body falls varies directly as the square of the time. If an object is dropped from the top of a tower 576 ft high and hits the ground in 6 sec, how far did it fall in the first 4 sec?



38. The amount of water emptied by a pipe varies directly as the square of the diameter of the pipe. For a certain constant water flow, a pipe emptying into a canal will allow 200 gal of water to escape in an hour. The diameter of the pipe is 6 in. How much water would a 12-in. pipe empty into the canal in an hour, assuming the same water flow?

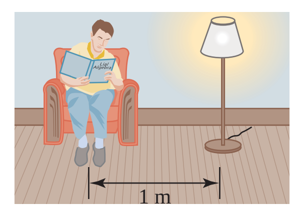
39. Over a specified distance, speed varies inversely with time. If a Dodge Viper on a test track goes a certain distance in one-half minute at 160 mph, what speed is needed to go the same distance in three-fourths minute?

40. For a constant area, the length of a rectangle varies inversely as the width. The length of a rectangle is 27 ft when the width is 10 ft. Find the width of a rectangle with the same area if the length is 18 ft.

41. The frequency of a vibrating string varies inversely as its length. That is, a longer string vibrates fewer times in a second than a shorter string. Suppose a piano string 2 ft long vibrates 250 cycles per sec. What frequency would a string 5 ft long have?

42. The current in a simple electrical circuit varies inversely as the resistance. If the current is 20 amps when the resistance is 5 ohms, find the current when the resistance is 7.5 ohms.

43. The amount of light (measured in foot-candles) produced by a light source varies inversely as the square of the distance from the source. If the illumination produced 1 m from a light source is 768 foot-candles, find the illumination produced 6 m from the same source.



44. The force with which Earth attracts an object above Earth's surface varies inversely with the square of the distance of the object from the center of Earth. If an object 4000 mi from the center of Earth is attracted with a force of 160 lb, find the force of attraction if the object were 6000 mi from the center of Earth.

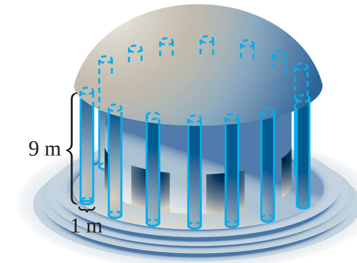
45. For a given interest rate, simple interest varies jointly as principal and time. If \$2000 left in an account for 4 yr earned interest of \$280, how much interest would be earned in 6 yr?

46. The collision impact of an automobile varies jointly as its mass and the square of its speed. Suppose a 2000-lb car traveling at 55 mph has a collision impact of 6.1. What is the collision impact of the same car at 65 mph?

47. The force needed to keep a car from skidding on a curve varies inversely as the radius of the curve and jointly as the weight of the car and the square of the speed. If 242 lb of force keep a 2000-lb car from skidding on a curve of radius 500 ft at 30 mph, what force would keep the same car from skidding on a curve of radius 750 ft at 50 mph?

photo not available

48. The maximum load that a cylindrical column with a circular cross section can hold varies directly as the fourth power of the diameter of the cross section and inversely as the square of the height. A 9-m column 1 m in diameter will support 8 metric tons. How many metric tons can be supported by a column 12 m high and $\frac{2}{3}$ m in diameter?



Load = 8 metric tons

49. The number of long-distance phone calls between two cities in a certain time period varies jointly as the populations of the cities, p_1 and p_2, and inversely as the distance between them. If 80,000 calls are made between two cities 400 mi apart, with populations of 70,000 and 100,000, how many calls are made between cities with populations of 50,000 and 75,000 that are 250 mi apart?

50. A body mass index from 27 through 29 carries a slight risk of weight-related health problems, while one of 30 or more indicates a great increase in risk. Use your own height and weight and the information in Example 7 to determine your BMI and whether you are at risk.

51. Natural gas provides 35.8% of U.S. energy. (*Source:* U.S. Energy Department.) The volume of gas varies inversely as the pressure and directly as the temperature. (Temperature must be measured in *Kelvin* (K), a unit of measurement used in physics.) If a certain gas occupies a volume of 1.3 L at 300 K and a pressure of 18 newtons per cm^2, find the volume at 340 K and a pressure of 24 newtons per cm^2.

52. The maximum load of a horizontal beam that is supported at both ends varies directly as the width and the square of the height and inversely as the length between the supports. A beam 6 m long, .1 m wide, and .06 m high supports a load of 360 kg. What is the maximum load supported by a beam 16 m long, .2 m wide, and .08 m high?

Exercises 53 and 54 describe weight-estimation formulas that fishermen have used over the years. Girth *is the distance around the body of the fish.* (*Source: Sacramento Bee,* November 9, 2000.)

EXERCISES

53. The weight of a bass varies jointly as its girth and the square of its length. A prize-winning bass weighed in at 22.7 lb and measured 36 in. long with 21 in. girth. How much would a bass 28 in. long with 18 in. girth weigh?

54. The weight of a trout varies jointly as its length and the square of its girth. One angler caught a trout that weighed 10.5 lb and measured 26 in. long with 18 in. girth. Find the weight of a trout that is 22 in. long with 15 in. girth.

RELATING CONCEPTS (EXERCISES 55–62)

For Individual or Group Work

A routine activity such as pumping gasoline can be related to many of the concepts studied in this chapter. Suppose that premium unleaded costs \$1.25 *per gallon.* ***Work Exercises 55–62 in order.***

55. Zero gallons of gasoline cost \$0.00, while 1 gallon costs \$1.25. Represent these two pieces of information as ordered pairs of the form (gallons, price).

56. Use the information from Exercise 55 to find the slope of the line on which the two points lie.

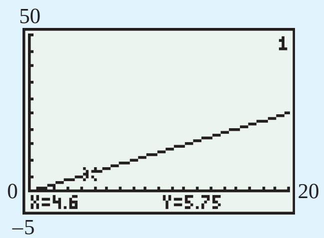
57. Write the slope-intercept form of the equation of the line on which the two points lie.

58. Using function notation, if $f(x) = ax + b$ represents the line from Exercise 57, what are the values of a and b?

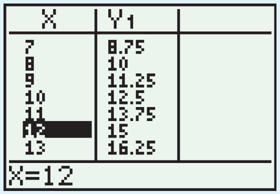
59. How does the value of a from Exercise 58 relate to gasoline in this situation? With relationship to the line, what do we call this number?

60. Why does the equation from Exercise 57 satisfy the conditions for direct variation? In the context of variation, what do we call the value of a?

61. The graph of the equation from Exercise 57 is shown in the calculator screen. How is the display at the bottom of the screen interpreted in the context of these exercises?



62. The table was generated by a graphing calculator, with Y_1 entered as the equation from Exercise 57. Interpret the entry for $X = 12$ in the context of these exercises.

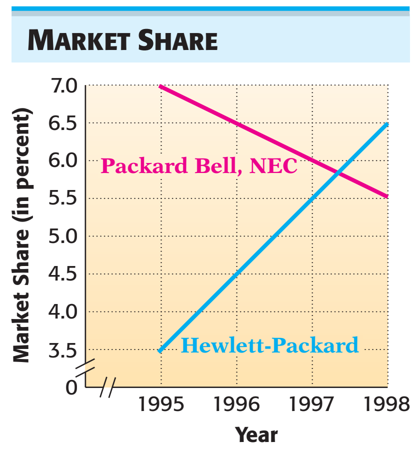


4.1 Systems of Linear Equations in Two Variables

OBJECTIVES

1. Decide whether an ordered pair is a solution of a linear system.
2. Solve linear systems by graphing.
3. Solve linear systems (with two equations and two variables) by elimination.
4. Solve special systems.
5. Solve linear systems (with two equations and two variables) by substitution.
6. Recognize how a graphing calculator is used to solve a linear system.

The worldwide personal computer market share for different manufacturers has varied, with first one, then another obtaining a larger share. As shown in Figure 1, Hewlett-Packard's share rose from 1995 through 1998, while Packard Bell, NEC saw its share decline. The graphs intersect at the point when the two companies had the same market share.



Source: Intelliquest; IDC.

FIGURE 1

We could use a linear equation to model the graph of Hewlett-Packard's market share and another linear equation to model the graph of Packard Bell, NEC's market share. Such a set of equations is called a **system of equations,** in this case a **system of linear equations.** The point where the graphs in Figure 1 intersect is a solution of each of the individual equations. It is also the solution of the system of linear equations.

OBJECTIVE 1 Decide whether an ordered pair is a solution of a linear system. The **solution set of a linear system** of equations contains all ordered pairs that satisfy all the equations of the system at the same time.

EXAMPLE 1 Deciding Whether an Ordered Pair Is a Solution

Decide whether the given ordered pair is a solution of the system.

(a) $\begin{aligned} x + y &= 6 \\ 4x - y &= 14 \end{aligned}$; $(4, 2)$

Replace x with 4 and y with 2 in each equation of the system.

$x + y = 6$	$4x - y = 14$
$4 + 2 = 6$?	$4(4) - 2 = 14$?
$6 = 6$ True	$14 = 14$ True

Since $(4, 2)$ makes both equations true, $(4, 2)$ is a solution of the system.

(b) $\begin{array}{r} 3x + 2y = 11 \\ x + 5y = 36 \end{array}$; $(-1, 7)$

$3x + 2y = 11$	$x + 5y = 36$
$3(-1) + 2(7) = 11$?	$-1 + 5(7) = 36$?
$-3 + 14 = 11$	$-1 + 35 = 36$
$11 = 11$ True	$34 = 36$ False

The ordered pair $(-1, 7)$ is not a solution of the system, since it does not make *both* equations true.

Now Try Exercises 11 and 13.

OBJECTIVE 2 Solve linear systems by graphing. One way to find the solution set of a linear system of equations is to graph each equation and find the point where the graphs intersect.

VIDEO

EXAMPLE 2 Solving a System by Graphing

Solve the system of equations by graphing.

$$x + y = 5 \qquad (1)$$
$$2x - y = 4 \qquad (2)$$

When we graph these linear equations as shown in Figure 2, the graph suggests that the point of intersection is the ordered pair (3, 2).

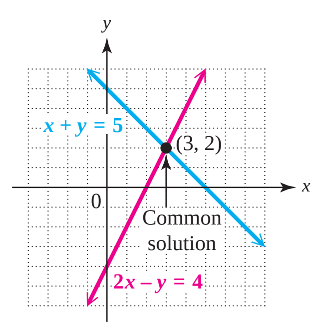


FIGURE 2

To be sure that (3, 2) is a solution of *both* equations, we check by substituting 3 for x and 2 for y in each equation.

$x + y = 5$ (1)	$2x - y = 4$ (2)
$3 + 2 = 5$?	$2(3) - 2 = 4$?
$5 = 5$ True	$6 - 2 = 4$?
	$4 = 4$ True

Since (3, 2) makes both equations true, $\{(3, 2)\}$ is the solution set of the system.

Now Try Exercise 15.

There are three possibilities for the solution set of a linear system in two variables.

AUDIO

Graphs of Linear Systems in Two Variables

1. The two graphs intersect in a single point. The coordinates of this point give the only solution of the system. In this case the system is **consistent,** and the equations are **independent.** This is the most common case. See Figure 3(a).
2. The graphs are parallel lines. In this case the system is **inconsistent;** that is, there is no solution common to both equations of the system, and the solution set is $\emptyset$. See Figure 3(b).
3. The graphs are the same line. In this case the equations are **dependent,** since any solution of one equation of the system is also a solution of the other. The solution set is an infinite set of ordered pairs representing the points on the line. See Figure 3(c).

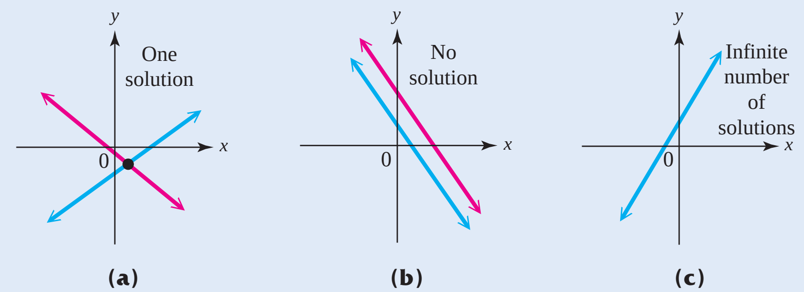


FIGURE 3

OBJECTIVE 3 Solve linear systems (with two equations and two variables) by elimination. While it is possible to find the solution of a system of equations by graphing, it can be difficult to read exact coordinates, especially if they are not integers, from a graph. Because of this, we usually use algebraic methods to solve systems. One such method, called the **elimination method,** involves combining the two equations of the system so that one variable is *eliminated*. This is done using the following fact.

$$\text{If} \quad a = b \text{ and } c = d, \quad \text{then} \quad a + c = b + d.$$

The general method of solving a system by the elimination method is summarized as follows.

AUDIO

Solving a Linear System by Elimination

Step 1 **Write both equations in standard form $Ax + By = C$.**

Step 2 **Make the coefficients of one pair of variable terms opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the x- or y-terms is 0.

Step 3 **Add** the new equations to eliminate a variable. The sum should be an equation with just one variable.

Step 4 **Solve** the equation from Step 3 for the remaining variable.

Step 5 **Find the other value.** Substitute the result of Step 4 into either of the original equations and solve for the other variable.

Step 6 **Check** the solution in both of the original equations. Then write the solution set.

EXAMPLE 3 Solving a System by Elimination

Solve the system

$$5x - 2y = 4 \qquad (1)$$
$$2x + 3y = 13. \qquad (2)$$

Step 1 Both equations are in standard form.

Step 2 Suppose that you wish to eliminate the variable x. One way to do this is to multiply equation (1) by 2 and equation (2) by -5.

$$\begin{aligned} 10x - 4y &= 8 && \text{2 times each side of equation (1)} \\ -10x - 15y &= -65 && \text{−5 times each side of equation (2)} \end{aligned}$$

Step 3 Now add.

$$\begin{aligned} 10x - 4y &= 8 \\ -10x - 15y &= -65 \\ \hline -19y &= -57 \end{aligned}$$

Step 4 Solve for y. $\quad y = 3 \quad$ Divide by -19.

Step 5 To find x, substitute 3 for y in either equation (1) or (2). Substituting in equation (2) gives

$$\begin{aligned} 2x + 3y &= 13 && (2) \\ 2x + 3(3) &= 13 && \text{Let } y = 3. \\ 2x + 9 &= 13 \\ 2x &= 4 && \text{Subtract 9.} \\ x &= 2. && \text{Divide by 2.} \end{aligned}$$

Step 6 The solution is (2, 3). To check, substitute 2 for x and 3 for y in both equations (1) and (2).

$$\begin{aligned} 5x - 2y &= 4 && (1) \\ 5(2) - 2(3) &= 4 && ? \\ 10 - 6 &= 4 && ? \\ 4 &= 4 && \text{True} \end{aligned}
\qquad\Bigg|\qquad
\begin{aligned} 2x + 3y &= 13 && (2) \\ 2(2) + 3(3) &= 13 && ? \\ 4 + 9 &= 13 && ? \\ 13 &= 13 && \text{True} \end{aligned}$$

The solution set is $\{(2, 3)\}$.

Now Try Exercise 19.

EXAMPLE 4 Solving a System with Fractional Coefficients

Solve the system

$$5x - 2y = 4 \qquad (1)$$
$$\frac{1}{2}x + \frac{3}{4}y = \frac{13}{4}. \qquad (2)$$

If an equation in a system has fractional coefficients, as in equation (2), first multiply by the least common denominator to clear the fractions.

$$4\left(\frac{1}{2}x + \frac{3}{4}y\right) = 4 \cdot \frac{13}{4} \qquad \text{Multiply equation (2) by the LCD, 4.}$$

$$4 \cdot \frac{1}{2}x + 4 \cdot \frac{3}{4}y = 4 \cdot \frac{13}{4} \qquad \text{Distributive property}$$

$$2x + 3y = 13 \qquad \text{Equivalent to equation (2)}$$

The system of equations becomes

$$\begin{aligned} 5x - 2y &= 4 && (1) \\ 2x + 3y &= 13, && \text{Equation (2) with fractions cleared} \end{aligned}$$

which is identical to the system we solved in Example 3. The solution set is $\{(2, 3)\}$. To confirm this, check the solution in both equations (1) and (2).

Now Try Exercise 31.

NOTE If an equation in a system contains decimal coefficients, it is best to first clear the decimals by multiplying by an appropriate power of ten, depending on the number of decimal places. Then solve the system. For example, we multiply *each side* of the equation

$$.5x + .75y = 3.25$$

by 100 to get the equivalent equation

$$50x + 75y = 325.$$

OBJECTIVE 4 Solve special systems. As we saw in Figures 3(b) and (c), some systems of linear equations have no solution or an infinite number of solutions. Examples 5 and 6 show how to recognize these systems when solving algebraically.

EXAMPLE 5 Solving a System of Dependent Equations

Solve the system

$$\begin{aligned} 2x - y &= 3 && (1) \\ 6x - 3y &= 9. && (2) \end{aligned}$$

We multiply equation (1) by -3, and then add the result to equation (2).

$$\begin{aligned} -6x + 3y &= -9 && -3 \text{ times each side of equation (1)} \\ 6x - 3y &= 9 && (2) \\ \hline 0 &= 0 && \text{True} \end{aligned}$$

Adding these equations gives the true statement $0 = 0$. In the original system, we could get equation (2) from equation (1) by multiplying equation (1) by 3. Because of this, equations (1) and (2) are equivalent and have the same graph, as shown in Figure 4. The equations are dependent. The solution set is the set of all points on the line with equation $2x - y = 3$, written

$$\{(x, y) \mid 2x - y = 3\}$$

and read "the set of all ordered pairs (x, y), such that $2x - y = 3$."

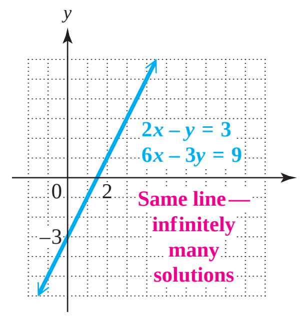


FIGURE 4

Now Try Exercise 23.

NOTE When a system has dependent equations and an infinite number of solutions, as in Example 5, either equation of the system could be used to write the solution set. We prefer to use an equation (in standard form) with coefficients that are integers having no common factor (except 1). Other texts may express such solutions differently.

EXAMPLE 6 Solving an Inconsistent System

Solve the system

$$x + 3y = 4 \qquad (1)$$
$$-2x - 6y = 3. \qquad (2)$$

Multiply equation (1) by 2, and then add the result to equation (2).

$$\begin{aligned} 2x + 6y &= 8 && \text{Equation (1) multiplied by 2} \\ -2x - 6y &= 3 && (2) \\ \hline 0 &= 11 && \text{False} \end{aligned}$$

The result of the addition step is a false statement, which indicates that the system is inconsistent. As shown in Figure 5, the graphs of the equations of the system are parallel lines. There are no ordered pairs that satisfy both equations, so there is no solution for the system; the solution set is $\emptyset$.

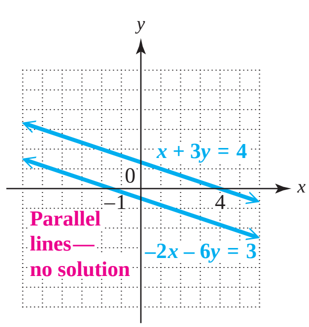


FIGURE 5

Now Try Exercise 25.

The results of Examples 5 and 6 are generalized as follows.

Special Cases of Linear Systems

If both variables are eliminated when a system of linear equations is solved,

1. there are infinitely many solutions if the resulting statement is *true;*
2. there is no solution if the resulting statement is *false.*

Slopes and y-intercepts can be used to decide if the graphs of a system of equations are parallel lines or if they coincide. In Example 5, writing each equation in slope-intercept form shows that both lines have slope 2 and y-intercept $(0, -3)$, so the graphs are the same line and the system has an infinite solution set.

In Example 6, both equations have slope $-\frac{1}{3}$ but y-intercepts $\left(0, \frac{4}{3}\right)$ and $\left(0, -\frac{1}{2}\right)$, showing that the graphs are two distinct parallel lines. Thus, the system has no solution.

OBJECTIVE 5 Solve linear systems (with two equations and two variables) by substitution. Linear systems can also be solved algebraically by the **substitution method.** This method is most useful for solving linear systems in which one variable has coefficient 1 or -1.

The substitution method is summarized as follows.

Solving a Linear System by Substitution

Step 1 **Solve one of the equations for either variable.** If one of the variable terms has coefficient 1 or -1, choose it since the substitution method is usually easier this way.

Step 2 **Substitute** for that variable in the other equation. The result should be an equation with just one variable.

Step 3 **Solve** the equation from Step 2.

Step 4 **Find the other value.** Substitute the result from Step 3 into the equation from Step 1 to find the value of the other variable.

Step 5 **Check** the solution in both of the original equations. Then write the solution set.

EXAMPLE 7 Solving a System by Substitution

Solve the system

$$\begin{aligned} 3x + 2y &= 13 \qquad (1)\\ 4x - y &= -1. \qquad (2) \end{aligned}$$

Step 1 To use the substitution method, first solve one of the equations for either x or y. Since the coefficient of y in equation (2) is -1, it is easiest to solve for y in equation (2).

$$\begin{aligned} 4x - y &= -1 && (2)\\ -y &= -1 - 4x && \text{Subtract } 4x.\\ y &= 1 + 4x && \text{Multiply by } -1. \end{aligned}$$

Step 2 Substitute $1 + 4x$ for y in equation (1).

$$\begin{aligned} 3x + 2y &= 13 && (1)\\ 3x + 2(1 + 4x) &= 13 && \text{Let } y = 1 + 4x. \end{aligned}$$

Step 3 Solve for x.

$$\begin{aligned} 3x + 2 + 8x &= 13 && \text{Distributive property}\\ 11x &= 11 && \text{Combine terms; subtract 2.}\\ x &= 1 && \text{Divide by 11.} \end{aligned}$$

Step 4 Now solve for y. Since $y = 1 + 4x$,

$$y = 1 + 4(1) = 5. \qquad \text{Let } x = 1.$$

Step 5 Check the solution (1, 5) in both equations (1) and (2).

$$\begin{aligned} 3x + 2y &= 13 && (1) \\ 3(1) + 2(5) &= 13 && ? \\ 3 + 10 &= 13 && ? \\ 13 &= 13 && \text{True} \end{aligned} \qquad \begin{aligned} 4x - y &= -1 && (2) \\ 4(1) - 5 &= -1 && ? \\ 4 - 5 &= -1 && ? \\ -1 &= -1 && \text{True} \end{aligned}$$

The solution set is $\{(1, 5)\}$.

Now Try Exercise 43.

EXAMPLE 8 Solving a System by Substitution

Solve the system

$$\begin{aligned} \frac{2}{3}x - \frac{1}{2}y &= \frac{7}{6} && (1) \\ 3x - 2y &= 6. && (2) \end{aligned}$$

This system will be easier to solve if you clear the fractions in equation (1). Multiply by the LCD, 6.

$$\begin{aligned} 6 \cdot \frac{2}{3}x - 6 \cdot \frac{1}{2}y &= 6 \cdot \frac{7}{6} \\ 4x - 3y &= 7 && (3) \end{aligned}$$

Now the system consists of equations (2) and (3). To use the substitution method, one equation must be solved for one of the two variables. Solve equation (2) for x.

$$\begin{aligned} 3x &= 2y + 6 \\ x &= \frac{2y + 6}{3} \end{aligned}$$

Substitute $\frac{2y + 6}{3}$ for x in equation (3).

$$\begin{aligned} 4x - 3y &= 7 && (3) \\ 4\left(\frac{2y + 6}{3}\right) - 3y &= 7 && \text{Let } x = \tfrac{2y+6}{3}. \\ 3\left[4\left(\frac{2y + 6}{3}\right)\right] - 3(3y) &= 3(7) && \text{Multiply by 3 to clear the fraction.} \\ 4(2y + 6) - 9y &= 21 \\ 8y + 24 - 9y &= 21 && \text{Distributive property} \\ 24 - y &= 21 && \text{Combine terms.} \\ -y &= -3 && \text{Subtract 24.} \\ y &= 3 && \text{Multiply by } -1. \end{aligned}$$

Since $x = \frac{2y + 6}{3}$ and $y = 3$,

$$x = \frac{2(3) + 6}{3} = \frac{6 + 6}{3} = 4.$$

A check verifies that the solution set is $\{(4, 3)\}$.

Now Try Exercise 49.

NOTE While the substitution method is not usually the best choice for solving a system like the one in Example 8, it is sometimes necessary to use it to solve a system of *nonlinear* equations.

OBJECTIVE 6 Recognize how a graphing calculator is used to solve a linear system. In Example 2 we showed how to solve the system

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

by graphing the two lines and finding their point of intersection. We can also do this with a graphing calculator.

EXAMPLE 9 Finding the Solution Set of a System from a Graphing Calculator Screen

The solution set of the system

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

can be found from the calculator screen in Figure 6. The two lines were graphed by solving the first equation to get $y = 5 - x$ and the second to get $y = 2x - 4$. The coordinates of their point of intersection are displayed at the bottom of the screen, indicating that the solution set is $\{(3, 2)\}$. (Compare this graph to the one found in Figure 2.)

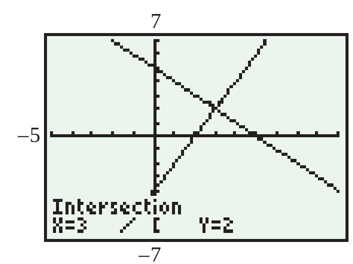


FIGURE 6

Now Try Exercise 59.

4.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 5/Videotape 6

Fill in the blanks with the correct responses.

1. If $(3, -6)$ is a solution of a linear system in two variables, then substituting ________ for x and ________ for y leads to true statements in *both* equations.

2. A solution of a system of independent linear equations in two variables is a(n) ________.

3. If the solution process leads to a false statement such as $0 = 5$ when solving a system, the solution set is ________.

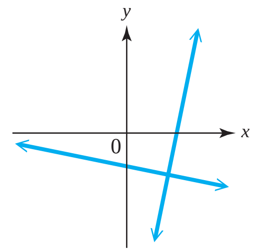
4. If the solution process leads to a true statement such as $0 = 0$ when solving a system, the system has ________ equations.

5. If the two lines forming a system have the same slope and different y-intercepts, the system has ________ solution(s).
(how many?)

6. If the two lines forming a system have different slopes, the system has ________ solution(s).
(how many?)

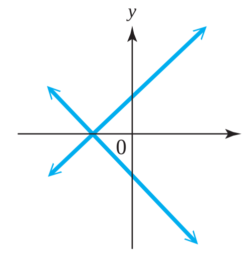
7. Which ordered pair could possibly be a solution of the graphed system of equations? Why?

A. $(3, 3)$
B. $(-3, 3)$
C. $(-3, -3)$
D. $(3, -3)$



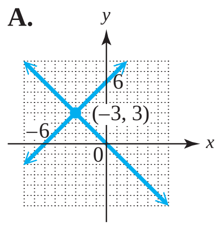
8. Which ordered pair could possibly be a solution of the graphed system of equations? Why?

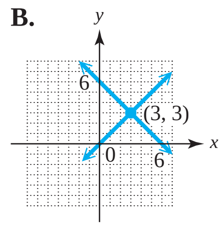
A. $(3, 0)$
B. $(-3, 0)$
C. $(0, 3)$
D. $(0, -3)$

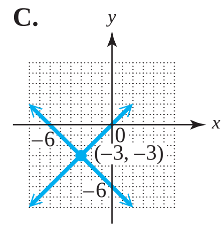


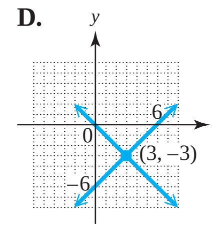
9. Match each system with the correct graph.

(a) $x + y = 6$, $x - y = 0$ **(b)** $x + y = -6$, $x - y = 0$ **(c)** $x + y = 0$, $x - y = -6$ **(d)** $x + y = 0$, $x - y = 6$

A.


B.


C.


D.


Decide whether the given ordered pair is a solution of the given system. See Example 1.

10. $\begin{aligned} x + y &= 6 \\ x - y &= 4 \end{aligned}$; $(5, 1)$

11. $\begin{aligned} x - y &= 17 \\ x + y &= -1 \end{aligned}$; $(8, -9)$

12. $\begin{aligned} 2x - y &= 8 \\ 3x + 2y &= 20 \end{aligned}$; $(5, 2)$

13. $\begin{aligned} 3x - 5y &= -12 \\ x - y &= 1 \end{aligned}$; $(-1, 2)$

Solve each system by graphing. See Example 2.

14. $\begin{aligned} x + y &= 4 \\ 2x - y &= 2 \end{aligned}$

15. $\begin{aligned} x + y &= -5 \\ -2x + y &= 1 \end{aligned}$

16. $\begin{aligned} x - 4y &= -4 \\ 3x + y &= 1 \end{aligned}$

Solve each system by elimination. If the system is inconsistent or has dependent equations, say so. See Examples 3–6.

17. $\begin{aligned} 2x - 5y &= 11 \\ 3x + y &= 8 \end{aligned}$

18. $\begin{aligned} -2x + 3y &= 1 \\ -4x + y &= -3 \end{aligned}$

19. $\begin{aligned} 3x + 4y &= -6 \\ 5x + 3y &= 1 \end{aligned}$

20. $\begin{aligned} 4x + 3y &= 1 \\ 3x + 2y &= 2 \end{aligned}$

21. $\begin{aligned} 3x + 3y &= 0 \\ 4x + 2y &= 3 \end{aligned}$

22. $\begin{aligned} 8x + 4y &= 0 \\ 4x - 2y &= 2 \end{aligned}$

23. $\begin{aligned} 7x + 2y &= 6 \\ -14x - 4y &= -12 \end{aligned}$

24. $\begin{aligned} x - 4y &= 2 \\ 4x - 16y &= 8 \end{aligned}$

25. $\begin{aligned} 5x - 5y &= 3 \\ x - y &= 12 \end{aligned}$

26. $\begin{aligned} 2x - 3y &= 7 \\ -4x + 6y &= 14 \end{aligned}$

27. $\begin{aligned} x - \frac{1}{2}y &= 2 \\ -x + \frac{2}{5}y &= -\frac{8}{5} \end{aligned}$

28. $\begin{aligned} \frac{3}{2}x + y &= 3 \\ \frac{2}{3}x + \frac{1}{3}y &= 1 \end{aligned}$

29. $\begin{aligned} x + y &= 0 \\ 2x - 2y &= 0 \end{aligned}$

30. $\begin{aligned} 3x + 3y &= 0 \\ -2x - y &= 0 \end{aligned}$

31. $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= -\frac{1}{3} \\ \frac{1}{2}x + 2y &= -7 \end{aligned}$

32. $\begin{aligned} \frac{1}{5}x + y &= \frac{6}{5} \\ \frac{1}{10}x + \frac{1}{3}y &= \frac{5}{6} \end{aligned}$

Write each equation in slope-intercept form and then tell how many solutions the system has. Do not actually solve.

EXERCISES

33. $\begin{aligned} 3x + 7y &= 4 \\ 6x + 14y &= 3 \end{aligned}$

34. $\begin{aligned} -x + 2y &= 8 \\ 4x - 8y &= 1 \end{aligned}$

35. $\begin{aligned} 2x &= -3y + 1 \\ 6x &= -9y + 3 \end{aligned}$

36. $\begin{aligned} 5x &= -2y + 1 \\ 10x &= -4y + 2 \end{aligned}$

37. Suppose that two linear equations are graphed on the same set of coordinate axes. Sketch what the graph might look like if the system has the given description.

(a) The system has a single solution. **(b)** The system has no solution.
(c) The system has infinitely many solutions.

38. Assuming you want to minimize the amount of work required, tell whether you would use the substitution or elimination method to solve each system. Explain your answers. Then solve the system.

(a) $\begin{aligned} 6x - y &= 5 \\ y &= 11x \end{aligned}$ **(b)** $\begin{aligned} 3x + y &= -7 \\ x - y &= -5 \end{aligned}$ **(c)** $\begin{aligned} 3x - 2y &= 0 \\ 9x + 8y &= 7 \end{aligned}$

Solve each system by substitution. If the system is inconsistent or has dependent equations, say so. See Examples 5–8.

EXERCISES

39. $\begin{aligned} 4x + y &= 6 \\ y &= 2x \end{aligned}$

40. $\begin{aligned} 2x - y &= 6 \\ y &= 5x \end{aligned}$

41. $\begin{aligned} 3x - 4y &= -22 \\ -3x + y &= 0 \end{aligned}$

42. $\begin{aligned} -3x + y &= -5 \\ x + 2y &= 0 \end{aligned}$

43. $\begin{aligned} -x - 4y &= -14 \\ 2x - y &= 1 \end{aligned}$

44. $\begin{aligned} -3x - 5y &= -17 \\ 4x &= y - 8 \end{aligned}$

45. $\begin{aligned} 5x - 4y &= 9 \\ 3 - 2y &= -x \end{aligned}$

46. $\begin{aligned} 6x - y &= -9 \\ 4 + 7x &= -y \end{aligned}$

47. $\begin{aligned} x &= 3y + 5 \\ x &= \frac{3}{2}y \end{aligned}$

48. $\begin{aligned} x &= 6y - 2 \\ x &= \frac{3}{4}y \end{aligned}$

49. $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= 3 \\ -3x + y &= 0 \end{aligned}$

50. $\begin{aligned} \frac{1}{4}x - \frac{1}{5}y &= 9 \\ 5x - y &= 0 \end{aligned}$

51. $\begin{aligned} y &= 2x \\ 4x - 2y &= 0 \end{aligned}$

52. $\begin{aligned} x &= 3y \\ 3x - 9y &= 0 \end{aligned}$

53. $\begin{aligned} x &= 5y \\ 5x - 25y &= 5 \end{aligned}$

54. $\begin{aligned} y &= -4x \\ 8x + 2y &= 4 \end{aligned}$

TECHNOLOGY INSIGHTS (EXERCISES 55–58)

55. The table shown was generated by a graphing calculator. The functions defined by Y_1 and Y_2 are linear. Based on the table, what are the coordinates of the point of intersection of the graphs?

X	Y_1	Y_2
0	-7	-1
1	-6	-2
2	-5	-3
3	-4	-4
4	-3	-5
5	-2	-6
6	-1	-7

X=0

56. The functions defined by Y_1 and Y_2 in the table are linear.

(a) Use the methods of Chapter 3 to find the equation for Y_1.

(b) Use the methods of Chapter 3 to find the equation for Y_2.

(c) Solve the system of equations formed by Y_1 and Y_2.

X	Y_1	Y_2
0	4	7
1	8	5
2	12	3
3	16	1
4	20	-1
5	24	-3
6	28	-5

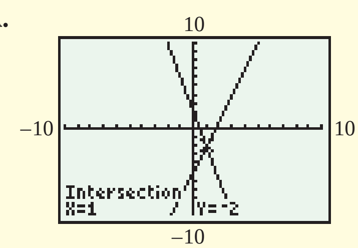
X=0

57. The solution set of the system

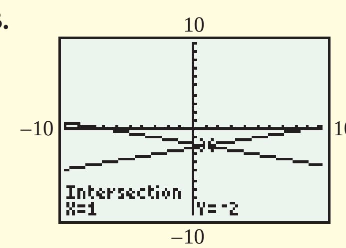
$$y_1 = 3x - 5$$
$$y_2 = -4x + 2$$

is $\{(1, -2)\}$. Using slopes and y-intercepts, determine which one of the two calculator graphs is the appropriate one for this system.

A.

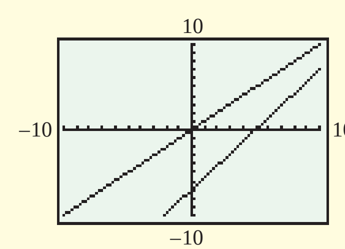


B.



58. Which one of the ordered pairs listed could be the only possible solution of the system whose graphs are shown in the standard viewing window of a graphing calculator?

A. $(15, -15)$ **B.** $(15, 15)$
C. $(-15, 15)$ **D.** $(-15, -15)$



For each system ***(a)*** *solve by elimination or substitution and* ***(b)*** *use a graphing calculator to support your result. In part (b), be sure to solve each equation for y first. See Example 9.*

59. $x + y = 10$
$2x - y = 5$

60. $6x + y = 5$
$-x + y = -9$

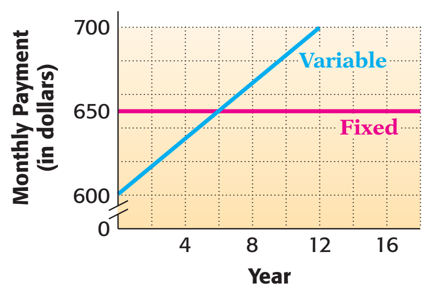
61. $3x - 2y = 4$
$3x + y = -2$

62. $2x - 3y = 3$
$2x + 2y = 8$

Answer the questions in Exercises 63–66 by observing the graphs provided.

63. Annette Hebert compared the monthly payments she would incur for two types of mortgages: fixed-rate and variable-rate. Her observations led to the following graphs.

MORTGAGE SHOPPING

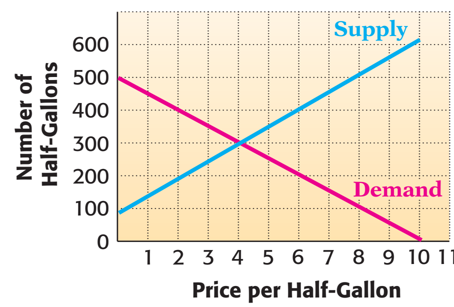


(a) For which years would the monthly payment be more for the fixed-rate mortgage than for the variable-rate mortgage?

(b) In what year would the payments be the same, and what would those payments be?

64. The figure shows graphs that represent supply and demand for a certain brand of low-fat frozen yogurt at various prices per half-gallon (in dollars).

THE FORTUNES OF FROZEN YOGURT



(a) At what price does supply equal demand?

(b) For how many half-gallons does supply equal demand?

(c) What are the supply and demand at a price of $2 per half-gallon?

65. The graph shows network share (the percentage of TV sets in use) for the early evening news programs for the three major broadcast networks from 1986 through 2000.

(a) Between what years did the ABC early evening news dominate?

(b) During what year did ABC's dominance end? Which network equaled ABC's share that year? What was that share?

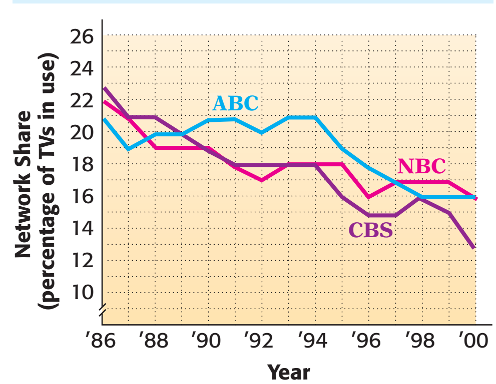
(c) During what years did ABC and CBS have equal network share? What was the share for each of these years?

(d) Which networks most recently had equal share? Write their share as an ordered pair of the form

(year, share).

(e) Describe the general trend in viewership for the three major networks during these years.

WHO'S WATCHING THE EVENING NEWS?



Source: Nielsen Media Research.

66. The graph at the top of the next page shows how the production of vinyl LPs, audiocassettes, and compact discs (CDs) changed over the years from 1986 through 1998.

(a) In what year did cassette production and CD production reach equal levels? What was that level?

(b) Express the point of intersection of the graphs of LP production and CD production as an ordered pair of the form

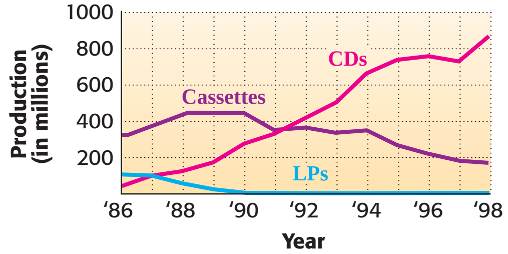
(year, production level).

(c) Between what years did cassette production first stabilize and remain fairly constant?

(d) Describe the trend in CD production from 1986 through 1998. If a straight line were used to approximate its graph, would the line have positive, negative, or 0 slope?

(e) If a straight line were used to approximate the graph of cassette production from 1990 through 1998, would the line have positive, negative, or 0 slope? Explain.

THE SOUNDS OF MUSIC



Source: Recording Industry Association of America.

Use the graph given at the beginning of this section (repeated here) to work Exercises 67–70.

67. For which years was Hewlett-Packard's share less than Packard Bell, NEC's share?

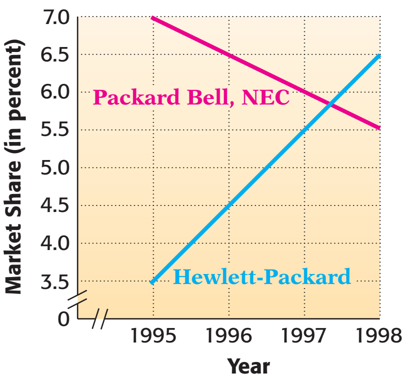
68. Estimate the year in which market share for Hewlett-Packard and Packard Bell, NEC was the same. About what was this share?

69. If $x = 0$ represents 1995 and $x = 3$ represents 1998, the market shares y (in percent) of these companies are closely modeled by the linear equations in the following system.

$$y = -.5x + 7 \qquad \text{Packard Bell, NEC}$$
$$y = x + 3.5 \qquad \text{Hewlett-Packard}$$

Solve this system. Round values to the nearest tenth as necessary.

MARKET SHARE



Source: Intelliquest; IDC.

70. Using your solution from Exercise 69, in what month and year did these companies have the same market share? What was that share?

A system such as

$$\frac{3}{x} + \frac{4}{y} = \frac{5}{2}$$
$$\frac{5}{x} - \frac{3}{y} = \frac{7}{4}$$

can be solved by elimination. One way to do this is to let $p = \frac{1}{x}$ and $q = \frac{1}{y}$. Substitute, solve for p and q, and then find x and y. $\left(\text{Hint: } \frac{3}{x} = 3 \cdot \frac{1}{x} = 3p.\right)$ Use this method to solve each system.

71. $$\frac{3}{x} + \frac{4}{y} = \frac{5}{2}$$
$$\frac{5}{x} - \frac{3}{y} = \frac{7}{4}$$

72. $$\frac{4}{x} - \frac{9}{y} = -1$$
$$-\frac{7}{x} + \frac{6}{y} = -\frac{3}{2}$$

4.2 Systems of Linear Equations in Three Variables

OBJECTIVES

1 Understand the geometry of systems of three equations in three variables.

A solution of an equation in three variables, such as

$$2x + 3y - z = 4,$$

is called an **ordered triple** and is written (x, y, z). For example, the ordered triple $(0, 1, -1)$ is a solution of the equation, because

$$2(0) + 3(1) - (-1) = 0 + 3 + 1 = 4.$$

Verify that another solution of this equation is $(10, -3, 7)$.

2 Solve linear systems (with three equations and three variables) by elimination.

3 Solve linear systems (with three equations and three variables) where some of the equations have missing terms.

4 Solve special systems (with three equations and three variables).

In the rest of this chapter, the term *linear equation* is extended to equations of the form

$$Ax + By + Cz + \cdots + Dw = K,$$

where not all the coefficients $A, B, C, \ldots, D$ equal 0. For example,

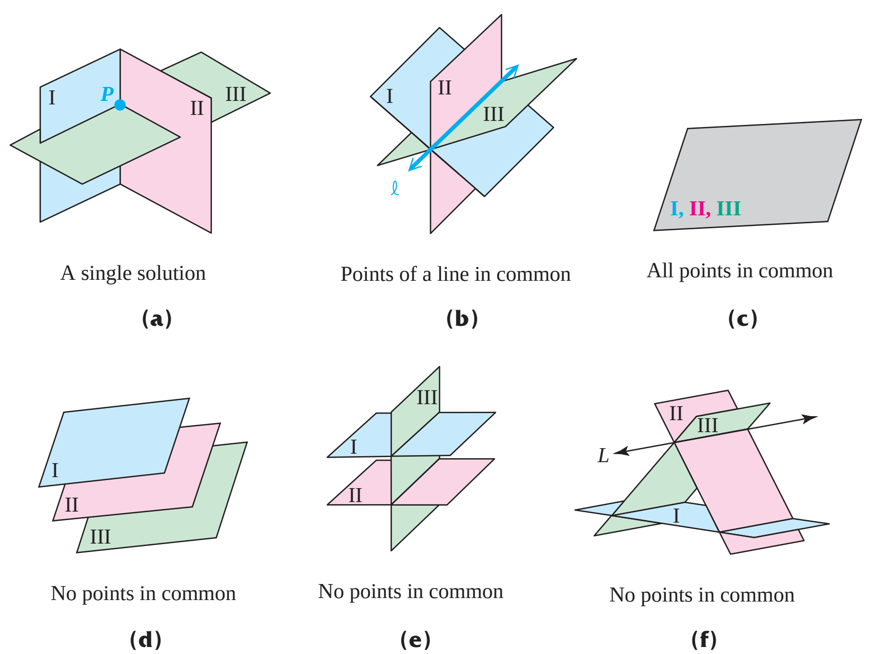
$$2x + 3y - 5z = 7 \qquad \text{and} \qquad x - 2y - z + 3u - 2w = 8$$

are linear equations, the first with three variables and the second with five variables.

OBJECTIVE 1 Understand the geometry of systems of three equations in three variables. In this section, we discuss the solution of a system of linear equations in three variables, such as

$$\begin{aligned} 4x + 8y + z &= 2 \\ x + 7y - 3z &= -14 \\ 2x - 3y + 2z &= 3. \end{aligned}$$

Theoretically, a system of this type can be solved by graphing. However, the graph of a linear equation with three variables is a *plane*, not a line. Since the graph of each equation of the system is a plane, which requires three-dimensional graphing, this method is not practical. However, it does illustrate the number of solutions possible for such systems, as shown in Figure 7.



A single solution **(a)** Points of a line in common **(b)** All points in common **(c)**

No points in common **(d)** No points in common **(e)** No points in common **(f)**

FIGURE 7

Figure 7 illustrates the following cases.

AUDIO

Graphs of Linear Systems in Three Variables

1. The three planes may meet at a single, common point that is the solution of the system. See Figure 7(a).

(continued)

2. The three planes may have the points of a line in common so that the infinite set of points that satisfy the equation of the line is the solution of the system. See Figure 7(b).
3. The three planes may coincide so that the solution of the system is the set of all points on a plane. See Figure 7(c).
4. The planes may have no points common to all three so that there is no solution of the system. See Figures 7(d), (e), and (f).

OBJECTIVE 2 Solve linear systems (with three equations and three variables) by elimination. Since graphing to find the solution set of a system of three equations in three variables is impractical, these systems are solved with an extension of the elimination method, summarized as follows.

AUDIO

Solving a Linear System in Three Variables

Step 1 **Eliminate a variable.** Use the elimination method to eliminate any variable from any two of the original equations. The result is an equation in two variables.

Step 2 **Eliminate the same variable again.** Eliminate the *same* variable from any *other* two equations. The result is an equation in the same two variables as in Step 1.

Step 3 **Eliminate a different variable and solve.** Use the elimination method to eliminate a second variable from the two equations in two variables that result from Steps 1 and 2. The result is an equation in one variable that gives the value of that variable.

Step 4 **Find a second value.** Substitute the value of the variable found in Step 3 into either of the equations in two variables to find the value of the second variable.

Step 5 **Find a third value.** Use the values of the two variables from Steps 3 and 4 to find the value of the third variable by substituting into an appropriate equation.

Step 6 **Check** the solution in all of the original equations. Then write the solution set.

EXPLORE YOU TRY IT VIDEO

EXAMPLE 1 Solving a System in Three Variables

Solve the system

$$\begin{aligned} 4x + 8y + z &= 2 && (1)\\ x + 7y - 3z &= -14 && (2)\\ 2x - 3y + 2z &= 3. && (3) \end{aligned}$$

Step 1 As before, the elimination method involves eliminating a variable from the sum of two equations. The choice of which variable to eliminate is arbitrary.

Suppose we decide to begin by eliminating z. We multiply equation (1) by 3 and then add the result to equation (2).

$$\begin{array}{rcll} 12x + 24y + 3z &=& 6 & \text{Multiply each side of (1) by 3.} \\ x + 7y - 3z &=& -14 & \text{(2)} \\ \hline 13x + 31y \quad &=& -8 & \text{Add.} \quad \text{(4)} \end{array}$$

Step 2 Equation (4) has only two variables. To get another equation without z, we multiply equation (1) by -2 and add the result to equation (3). It is essential at this point to *eliminate the same variable, z*.

$$\begin{array}{rcll} -8x - 16y - 2z &=& -4 & \text{Multiply each side of (1) by } -2. \\ 2x - 3y + 2z &=& 3 & \text{(3)} \\ \hline -6x - 19y \quad &=& -1 & \text{Add.} \quad \text{(5)} \end{array}$$

Step 3 Now we solve the system of equations (4) and (5) for x and y. This step is possible only if the *same* variable is eliminated in Steps 1 and 2.

$$\begin{array}{rcll} 78x + 186y &=& -48 & \text{Multiply each side of (4) by 6.} \\ -78x - 247y &=& -13 & \text{Multiply each side of (5) by 13.} \\ \hline -61y &=& -61 & \text{Add.} \\ y &=& 1 & \end{array}$$

Step 4 Now we substitute 1 for y in either equation (4) or (5). Choosing (5) gives

$$\begin{array}{rcll} -6x - 19y &=& -1 & \text{(5)} \\ -6x - 19(1) &=& -1 & \text{Let } y = 1. \\ -6x - 19 &=& -1 & \\ -6x &=& 18 & \\ x &=& -3. & \end{array}$$

Step 5 We substitute -3 for x and 1 for y in any one of the three original equations to find z. Choosing (1) gives

$$\begin{array}{rcll} 4x + 8y + z &=& 2 & \text{(1)} \\ 4(-3) + 8(1) + z &=& 2 & \text{Let } x = -3 \text{ and } y = 1. \\ -4 + z &=& 2 & \\ z &=& 6. & \end{array}$$

Step 6 It appears that the ordered triple $(-3, 1, 6)$ is the only solution of the system. We must check that the solution satisfies all three equations of the system. For equation (1),

$$\begin{array}{rcll} 4x + 8y + z &=& 2 & \text{(1)} \\ 4(-3) + 8(1) + 6 &=& 2 & ? \\ -12 + 8 + 6 &=& 2 & ? \\ 2 &=& 2. & \text{True} \end{array}$$

Because $(-3, 1, 6)$ also satisfies equations (2) and (3), the solution set is $\{(-3, 1, 6)\}$.

Now Try Exercise 3.

OBJECTIVE 3 Solve linear systems (with three equations and three variables) where some of the equations have missing terms. When this happens, one elimination step can be omitted.

EXAMPLE 2 Solving a System of Equations with Missing Terms

Solve the system

$$\begin{aligned} 6x - 12y &= -5 & (1)\\ 8y + z &= 0 & (2)\\ 9x - z &= 12. & (3) \end{aligned}$$

Since equation (3) is missing the variable y, eliminate y using equations (1) and (2).

$$\begin{aligned} 12x - 24y \quad\quad &= -10 && \text{Multiply each side of (1) by 2.}\\ 24y + 3z &= 0 && \text{Multiply each side of (2) by 3.}\\ \hline 12x \quad\quad + 3z &= -10 && \text{Add.} \quad (4) \end{aligned}$$

Use this result, together with equation (3), to eliminate z. Multiply equation (3) by 3.

$$\begin{aligned} 27x - 3z &= 36 && \text{Multiply each side of (3) by 3.}\\ 12x + 3z &= -10 && (4)\\ \hline 39x \quad\quad &= 26 && \text{Add.}\\ x = \frac{26}{39} &= \frac{2}{3} \end{aligned}$$

Substituting into equation (3) gives

$$\begin{aligned} 9x - z &= 12 && (3)\\ 9\left(\frac{2}{3}\right) - z &= 12 && \text{Let } x = \tfrac{2}{3}.\\ 6 - z &= 12\\ z &= -6. \end{aligned}$$

Substituting -6 for z in equation (2) gives

$$\begin{aligned} 8y + z &= 0 && (2)\\ 8y - 6 &= 0 && \text{Let } z = -6.\\ 8y &= 6\\ y &= \frac{3}{4}. \end{aligned}$$

Check in each of the original equations of the system to verify that the solution set of the system is $\left\{\left(\frac{2}{3}, \frac{3}{4}, -6\right)\right\}$.

Now Try Exercise 21.

OBJECTIVE 4 Solve special systems (with three equations and three variables). Linear systems with three variables may be inconsistent or may include dependent equations. The next examples illustrate these cases.

VIDEO

EXAMPLE 3 Solving an Inconsistent System with Three Variables

Solve the system

$$\begin{aligned} 2x - 4y + 6z &= 5 && (1)\\ -x + 3y - 2z &= -1 && (2)\\ x - 2y + 3z &= 1. && (3) \end{aligned}$$

Eliminate x by adding equations (2) and (3) to get the equation

$$y + z = 0.$$

Now, *eliminate x again*, using equations (1) and (3).

$$\begin{aligned} -2x + 4y - 6z &= -2 && \text{Multiply each side of (3) by } -2.\\ 2x - 4y + 6z &= 5 && (1)\\ \hline 0 &= 3 && \text{False} \end{aligned}$$

The resulting false statement indicates that equations (1) and (3) have no common solution. Thus, the system is inconsistent and the solution set is $\emptyset$. The graph of this system would show these two planes parallel to one another.

Now Try Exercise 29.

NOTE If you get a false statement when adding as in Example 3, you do not need to go any further with the solution. Since two of the three planes are parallel, it is not possible for the three planes to have any common points.

EXAMPLE 4 Solving a System of Dependent Equations with Three Variables

Solve the system

$$\begin{aligned} 2x - 3y + 4z &= 8 && (1)\\ -x + \frac{3}{2}y - 2z &= -4 && (2)\\ 6x - 9y + 12z &= 24. && (3) \end{aligned}$$

Multiplying each side of equation (1) by 3 gives equation (3). Multiplying each side of equation (2) by -6 also gives equation (3). Because of this, the equations are dependent. All three equations have the same graph, as illustrated in Figure 7(c). The solution set is written

$$\{(x, y, z) \mid 2x - 3y + 4z = 8\}.$$

Although any one of the three equations could be used to write the solution set, we use the equation in standard form with coefficients that are integers with no common factor (except 1), as we did in Section 4.1.

Now Try Exercise 33.

We can extend the method discussed in this section to solve larger systems. For example, to solve a system of four equations in four variables, eliminate a variable from three pairs of equations to get a system of three equations in three unknowns. Then proceed as shown above.

4.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 6/Videotape 6

1. Explain what the following statement means: The solution set of the system

$$\begin{aligned} 2x + y + z &= 3 \\ 3x - y + z &= -2 \\ 4x - y + 2z &= 0 \end{aligned}$$

is $\{(-1, 2, 3)\}$.

2. The two equations

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \end{aligned}$$

have a common solution of (1, 2, 3). Which equation would complete a system of three linear equations in three variables having solution set $\{(1, 2, 3)\}$?

A. $3x + 2y - z = 1$ **B.** $3x + 2y - z = 4$
C. $3x + 2y - z = 5$ **D.** $3x + 2y - z = 6$

Solve each system of equations. See Example 1.

3. $\begin{aligned} 2x - 5y + 3z &= -1 \\ x + 4y - 2z &= 9 \\ x - 2y - 4z &= -5 \end{aligned}$

4. $\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + z &= 1 \\ x + 2y + 2z &= -1 \end{aligned}$

5. $\begin{aligned} 3x + 2y + z &= 8 \\ 2x - 3y + 2z &= -16 \\ x + 4y - z &= 20 \end{aligned}$

6. $\begin{aligned} -3x + y - z &= -10 \\ -4x + 2y + 3z &= -1 \\ 2x + 3y - 2z &= -5 \end{aligned}$

7. $\begin{aligned} 2x + 5y + 2z &= 0 \\ 4x - 7y - 3z &= 1 \\ 3x - 8y - 2z &= -6 \end{aligned}$

8. $\begin{aligned} 5x - 2y + 3z &= -9 \\ 4x + 3y + 5z &= 4 \\ 2x + 4y - 2z &= 14 \end{aligned}$

9. $\begin{aligned} x + 2y + z &= 4 \\ 2x + y - z &= -1 \\ x - y - z &= -2 \end{aligned}$

10. $\begin{aligned} x - 2y + 5z &= -7 \\ -2x - 3y + 4z &= -14 \\ -3x + 5y - z &= -7 \end{aligned}$

11. $\begin{aligned} \frac{1}{3}x + \frac{1}{6}y - \frac{2}{3}z &= -1 \\ -\frac{3}{4}x - \frac{1}{3}y - \frac{1}{4}z &= 3 \\ \frac{1}{2}x + \frac{3}{2}y + \frac{3}{4}z &= 21 \end{aligned}$

12. $\begin{aligned} \frac{2}{3}x - \frac{1}{4}y + \frac{5}{8}z &= 0 \\ \frac{1}{5}x + \frac{2}{3}y - \frac{1}{4}z &= -7 \\ -\frac{3}{5}x + \frac{4}{3}y - \frac{7}{8}z &= -5 \end{aligned}$

13. $\begin{aligned} -x + 2y + 6z &= 2 \\ 3x + 2y + 6z &= 6 \\ x + 4y - 3z &= 1 \end{aligned}$

14. $\begin{aligned} 2x + y + 2z &= 1 \\ x + 2y + z &= 2 \\ x - y - z &= 0 \end{aligned}$

15. $\begin{aligned} x + y - z &= -2 \\ 2x - y + z &= -5 \\ -x + 2y - 3z &= -4 \end{aligned}$

16. $\begin{aligned} x + 2y + 3z &= 1 \\ -x - y + 3z &= 2 \\ -6x + y + z &= -2 \end{aligned}$

Solve each system of equations. See Example 2.

17. $\begin{aligned} 2x - 3y + 2z &= -1 \\ x + 2y + z &= 17 \\ 2y - z &= 7 \end{aligned}$

18. $\begin{aligned} 2x - y + 3z &= 6 \\ x + 2y - z &= 8 \\ 2y + z &= 1 \end{aligned}$

19. $\begin{aligned} 4x + 2y - 3z &= 6 \\ x - 4y + z &= -4 \\ -x + 2z &= 2 \end{aligned}$

20. $\begin{aligned} 2x + 3y - 4z &= 4 \\ x - 6y + z &= -16 \\ -x + 3z &= 8 \end{aligned}$

21. $\begin{aligned} 2x + y &= 6 \\ 3y - 2z &= -4 \\ 3x - 5z &= -7 \end{aligned}$

22. $\begin{aligned} 4x - 8y &= -7 \\ 4y + z &= 7 \\ -8x + z &= -4 \end{aligned}$

23. $\begin{aligned} -5x + 2y + z &= 5 \\ -3x - 2y - z &= 3 \\ -x + 6y &= 1 \end{aligned}$

24. $\begin{aligned} x + y - z &= 0 \\ 2y - z &= 1 \\ 2x + 3y - 4z &= -4 \end{aligned}$

25. $\begin{aligned} 4x - z &= -6 \\ \frac{3}{5}y + \frac{1}{2}z &= 0 \\ \frac{1}{3}x + \frac{2}{3}z &= -5 \end{aligned}$

26. $\begin{aligned} 5x - 2z &= 8 \\ 4y + 3z &= -9 \\ \frac{1}{2}x + \frac{2}{3}y &= -1 \end{aligned}$

27. Using your immediate surroundings, give an example of three planes that

(a) intersect in a single point;
(b) do not intersect;
(c) intersect in infinitely many points.

28. Suppose that a system has infinitely many ordered triple solutions of the form (x, y, z) such that

$$x + y + 2z = 1.$$

Give three specific ordered triples that are solutions of the system.

Solve each system of equations. If the system is inconsistent or has dependent equations, say so. See Examples 1, 3, and 4.

29. $\begin{aligned} 2x + 2y - 6z &= 5 \\ -3x + y - z &= -2 \\ -x - y + 3z &= 4 \end{aligned}$

30. $\begin{aligned} -2x + 5y + z &= -3 \\ 5x + 14y - z &= -11 \\ 7x + 9y - 2z &= -5 \end{aligned}$

31. $\begin{aligned} -5x + 5y - 20z &= -40 \\ x - y + 4z &= 8 \\ 3x - 3y + 12z &= 24 \end{aligned}$

32. $\begin{aligned} x + 4y - z &= 3 \\ -2x - 8y + 2z &= -6 \\ 3x + 12y - 3z &= 9 \end{aligned}$

33. $\begin{aligned} 2x + y - z &= 6 \\ 4x + 2y - 2z &= 12 \\ -x - \frac{1}{2}y + \frac{1}{2}z &= -3 \end{aligned}$

34. $\begin{aligned} 2x - 8y + 2z &= -10 \\ -x + 4y - z &= 5 \\ \frac{1}{8}x - \frac{1}{2}y + \frac{1}{8}z &= -\frac{5}{8} \end{aligned}$

35. $\begin{aligned} x + y - 2z &= 0 \\ 3x - y + z &= 0 \\ 4x + 2y - z &= 0 \end{aligned}$

36. $\begin{aligned} 2x + 3y - z &= 0 \\ x - 4y + 2z &= 0 \\ 3x - 5y - z &= 0 \end{aligned}$

Extend the method of this section to solve each system. Express the solution in the form (x, y, z, w).

37. $\begin{aligned} x + y + z - w &= 5 \\ 2x + y - z + w &= 3 \\ x - 2y + 3z + w &= 18 \\ -x - y + z + 2w &= 8 \end{aligned}$

38. $\begin{aligned} 3x + y - z + 2w &= 9 \\ x + y + 2z - w &= 10 \\ x - y - z + 3w &= -2 \\ -x + y - z + w &= -6 \end{aligned}$

39. $$\begin{aligned} 3x + y - z + w &= -3 \\ 2x + 4y + z - w &= -7 \\ -2x + 3y - 5z + w &= 3 \\ 5x + 4y - 5z + 2w &= -7 \end{aligned}$$

40. $$\begin{aligned} x - 3y + 7z + w &= 11 \\ 2x + 4y + 6z - 3w &= -3 \\ 3x + 2y + z + 2w &= 19 \\ 4x + y - 3z + w &= 22 \end{aligned}$$

RELATING CONCEPTS (EXERCISES 41–50)

For Individual or Group Work

Suppose that on a distant planet a function of the form

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

describes the height in feet of a projectile x sec after it has been projected upward. ***Work Exercises 41–50 in order,*** *to see how this can be related to a system of three equations in three variables a, b, and c.*

photo not available

41. After 1 sec, the height of a certain projectile is 128 ft. Thus, $f(1) = 128$. Use this information to find one equation in the variables a, b, and c. (*Hint:* Substitute 1 for x and 128 for $f(x)$.)

42. After 1.5 sec, the height is 140 ft. Find a second equation in a, b, and c.

43. After 3 sec, the height is 80 ft. Find a third equation in a, b, and c.

44. Write a system of three equations in a, b, and c, based on your answers in Exercises 41–43. Solve the system.

45. What is the function f for this particular projectile?

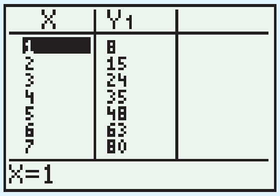
46. In the function f written in Exercise 45, the _____ of the projectile is a function of the _____ elapsed since it was projected.

47. What was the initial height of the projectile? (*Hint:* Find $f(0)$.)

48. The projectile reaches its maximum height in 1.625 sec. Find its maximum height.

49. In Chapter 9 we discuss graphs of functions of the form $f(x) = ax^2 + bx + c$ $(a \neq 0)$. Use a system of equations to find the values of a, b, and c for the function of this form that satisfies $f(1) = 2$, $f(-1) = 0$, and $f(-2) = 8$. Then write the expression for $f(x)$.

50. The accompanying table was generated by a graphing calculator for a function $Y_1 = aX^2 + bX + c$. Use any three points shown to find the values of a, b, and c. Then write the expression for Y_1.



51. Discuss why it is necessary to eliminate the same variable in the first two steps of the elimination method with three equations and three variables.

52. In Step 3 of the elimination method for solving systems in three variables, does it matter which variable is eliminated? Explain.

4.3 Applications of Systems of Linear Equations

OBJECTIVES

1. Solve geometry problems using two variables.
2. Solve money problems using two variables.
3. Solve mixture problems using two variables.
4. Solve distance-rate-time problems using two variables.
5. Solve problems with three variables using a system of three equations.

Many applied problems involve more than one unknown quantity. Although some problems with two unknowns can be solved using just one variable, it is often easier to use two variables. To solve a problem with two unknowns, we must write two equations that relate the unknown quantities. The system formed by the pair of equations can then be solved using the methods of this chapter.

The following steps, based on the six-step problem-solving method first introduced in Chapter 2, give a strategy for solving applied problems using more than one variable.

Solving an Applied Problem by Writing a System of Equations

Step 1 **Read** the problem carefully until you understand what is given and what is to be found.

Step 2 **Assign variables** to represent the unknown values, using diagrams or tables as needed. *Write down* what each variable represents.

Step 3 **Write a system of equations** that relates the unknowns.

Step 4 **Solve** the system of equations.

Step 5 **State the answer** to the problem. Does it seem reasonable?

Step 6 **Check** the answer in the words of the original problem.

OBJECTIVE 1 Solve geometry problems using two variables. Problems about the perimeter of a geometric figure often involve two unknowns and can be solved using systems of equations.

EXAMPLE 1 Finding the Dimensions of a Soccer Field

Unlike football, where the dimensions of a playing field cannot vary, a rectangular soccer field may have a width between 50 and 100 yd and a length between 50 and 100 yd. Suppose that one particular field has a perimeter of 320 yd. Its length measures 40 yd more than its width. What are the dimensions of this field? (*Source: Microsoft Encarta Encyclopedia 2000.*)

Step 1 **Read** the problem again. We are asked to find the dimensions of the field.

Step 2 **Assign variables.** Let L = the length and W = the width. Figure 8 shows a soccer field with the length labeled L and the width labeled W.

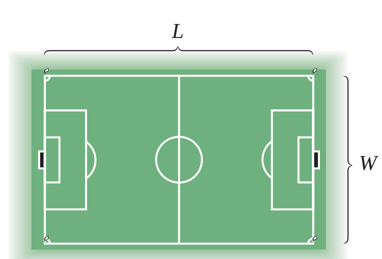


FIGURE 8

Step 3 **Write a system of equations.** Because the perimeter is 320 yd, we find one equation by using the perimeter formula:

$$2L + 2W = 320.$$

Because the length is 40 yd more than the width, we have

$$L = W + 40.$$

The system is

$$\begin{aligned} 2L + 2W &= 320 && (1) \\ L &= W + 40. && (2) \end{aligned}$$

Step 4 **Solve** the system of equations. Since equation (2) is solved for L, we can use the substitution method. We substitute $W + 40$ for L in equation (1), and solve for W.

$$\begin{aligned} 2L + 2W &= 320 && (1) \\ 2(W + 40) + 2W &= 320 && \text{Let } L = W + 40. \\ 2W + 80 + 2W &= 320 && \text{Distributive property} \\ 4W + 80 &= 320 && \text{Combine terms.} \\ 4W &= 240 && \text{Subtract 80.} \\ W &= 60 && \text{Divide by 4.} \end{aligned}$$

Let $W = 60$ in the equation $L = W + 40$ to find L.

$$L = 60 + 40 = 100$$

Step 5 **State the answer.** The length is 100 yd, and the width is 60 yd. The answer is reasonable, since both dimensions are within the ranges given in the problem.

Step 6 **Check.** The perimeter of this soccer field is

$$2(100) + 2(60) = 320 \text{ yd},$$

and the length, 100 yd, is 40 yd more than the width, since

$$100 - 40 = 60.$$

The answer is correct.

Now Try Exercise 3.

OBJECTIVE 2 Solve money problems using two variables. Professional sport ticket prices increase annually. Average per-ticket prices in three of the four major sports (football, basketball, and hockey) now exceed \$30.00.

EXAMPLE 2 Solving a Problem about Ticket Prices

During recent National Hockey League and National Basketball Association seasons, two hockey tickets and one basketball ticket purchased at their average prices would have cost \$110.40. One hockey ticket and two basketball tickets would have cost \$106.32. What were the average ticket prices for the two sports? (*Source:* Team Marketing Report, Chicago.)

Photo not available

Step 1 **Read** the problem again. There are two unknowns.

Step 2 **Assign variables.** Let h represent the average price for a hockey ticket and b represent the average price for a basketball ticket.

Step 3 **Write a system of equations.** Because two hockey tickets and one basketball ticket cost a total of \$110.40, one equation for the system is

$$2h + b = 110.40.$$

By similar reasoning, the second equation is

$$h + 2b = 106.32.$$

Therefore, the system is

$$\begin{aligned} 2h + b &= 110.40 && \text{(1)} \\ h + 2b &= 106.32. && \text{(2)} \end{aligned}$$

Step 4 **Solve** the system of equations. To eliminate h, multiply equation (2) by -2 and add.

$$\begin{aligned} 2h + b &= 110.40 && \text{(1)} \\ \underline{-2h - 4b} &\underline{= -212.64} && \text{Multiply each side of (2) by } -2. \\ -3b &= -102.24 && \text{Add.} \\ b &= 34.08 && \text{Divide by } -3. \end{aligned}$$

To find the value of h, let $b = 34.08$ in equation (2).

$$\begin{aligned} h + 2b &= 106.32 && \text{(2)} \\ h + 2(34.08) &= 106.32 && \text{Let } b = 34.08. \\ h + 68.16 &= 106.32 && \text{Multiply.} \\ h &= 38.16 && \text{Subtract 68.16.} \end{aligned}$$

Step 5 **State the answer.** The average price for one basketball ticket was \$34.08. For one hockey ticket, the average price was \$38.16.

Step 6 **Check** that these values satisfy the conditions stated in the problem.

Now Try Exercise 11.

OBJECTIVE 3 Solve mixture problems using two variables. We solved mixture problems earlier using one variable. For many mixture problems it seems more natural to use more than one variable and a system of equations.

EXAMPLE 3 Solving a Mixture Problem

How many ounces each of 5% hydrochloric acid and 20% hydrochloric acid must be combined to get 10 oz of solution that is 12.5% hydrochloric acid?

Step 1 **Read** the problem. Two solutions of different strengths are being mixed together to get a specific amount of a solution with an "in-between" strength.

Step 2 **Assign variables.** Let x represent the number of ounces of 5% solution and y represent the number of ounces of 20% solution. Use a table to summarize the information from the problem.

Ounces of Solution	Percent (as a decimal)	Ounces of Pure Acid
x	5% = .05	$.05x$
y	20% = .20	$.20y$
10	12.5% = .125	(.125)10

Figure 9 illustrates what is happening in the problem.

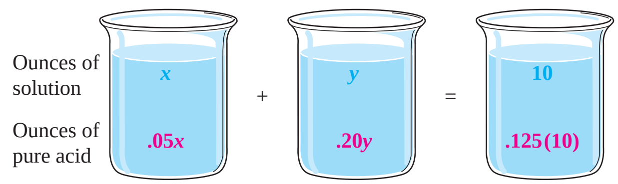


FIGURE 9

Step 3 **Write a system of equations.** When the x oz of 5% solution and the y oz of 20% solution are combined, the total number of ounces is 10, so

$$x + y = 10. \qquad (1)$$

The ounces of acid in the 5% solution ($.05x$) plus the ounces of acid in the 20% solution ($.20y$) should equal the total ounces of acid in the mixture, which is $(.125)10$, or 1.25. That is,

$$.05x + .20y = 1.25. \qquad (2)$$

Notice that these equations can be quickly determined by reading down in the table or using the labels in Figure 9.

Step 4 **Solve** the system of equations (1) and (2). Eliminate x by first multiplying equation (2) by 100 to clear it of decimals and then multiplying equation (1) by -5.

$$\begin{aligned} 5x + 20y &= 125 && \text{Multiply each side of (2) by 100.} \\ -5x - 5y &= -50 && \text{Multiply each side of (1) by } -5. \\ \hline 15y &= 75 && \text{Add.} \\ y &= 5 \end{aligned}$$

Because $y = 5$ and $x + y = 10$, x is also 5.

Step 5 **State the answer.** The desired mixture will require 5 oz of the 5% solution and 5 oz of the 20% solution.

Step 6 **Check** that these values satisfy both equations of the system.

Now Try Exercise 17.

Photo not available

CONNECTIONS

Problems that can be solved by writing a system of equations have been of interest historically. The following problem appeared in a Hindu work that dates back to about 850 A.D.

> The mixed price of 9 citrons (a lemonlike fruit shown in the photo) and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and the price of a wood apple here, having distinctly separated those prices well.

For Discussion or Writing

What do you think is meant by "the mixed price" in the problem quoted above? Write a system of equations for this problem. (You will be asked to solve it in Exercise 35.)

OBJECTIVE 4 Solve distance-rate-time problems using two variables. Motion problems require the distance formula, $d = rt$, where d is distance, r is rate (or speed), and t is time. These applications often lead to systems of equations, as in the next example.

EXAMPLE 4 Solving a Motion Problem

A car travels 250 km in the same time that a truck travels 225 km. If the speed of the car is 8 km per hr faster than the speed of the truck, find both speeds.

Step 1 **Read** the problem again. Given the distances traveled, we need to find the speed of each vehicle.

Step 2 **Assign variables.**

Let $x =$ the speed of the car,
and $y =$ the speed of the truck.

As in Example 3, a table helps organize the information. Fill in the given information for each vehicle (in this case, distance) and use the assigned variables for the unknown speeds (rates).

	d	r	t
Car	250	x	
Truck	225	y	

The table shows nothing about time. To get an expression for time, solve the distance formula, $d = rt$, for t.

$$\frac{d}{r} = t$$

The two times can be written as $\frac{250}{x}$ and $\frac{225}{y}$.

Step 3 **Write a system of equations.** The problem states that the car travels 8 km per hr faster than the truck. Since the two speeds are x and y,

$$x = y + 8. \qquad (1)$$

Both vehicles travel for the same time, so from the table,

$$\frac{250}{x} = \frac{225}{y}.$$

This is not a linear equation. However, multiplying each side by xy gives

$$250y = 225x,$$

which is linear. The system is

$$\begin{aligned} x &= y + 8 \\ 250y &= 225x. \qquad (2) \end{aligned}$$

Step 4 **Solve** the system of equations by substitution. Replace x with $y + 8$ in equation (2).

$$\begin{aligned} 250y &= 225x && (2) \\ 250y &= 225(y + 8) && \text{Let } x = y + 8. \\ 250y &= 225y + 1800 && \text{Distributive property} \\ 25y &= 1800 && \text{Subtract } 225y. \\ y &= 72 && \text{Divide by 25.} \end{aligned}$$

Because $x = y + 8$, the value of x is $72 + 8 = 80$.

Step 5 **State the answer.** The car's speed is 80 km per hr, and the truck's speed is 72 km per hr.

Step 6 **Check.** This is especially important since one of the equations had variable denominators.

$$\text{Car:}\quad t = \frac{d}{r} = \frac{250}{80} = 3.125$$

$$\text{Truck:}\quad t = \frac{d}{r} = \frac{225}{72} = 3.125$$

Times are equal.

Since $80 - 72 = 8$, the conditions of the problem are satisfied.

Now Try Exercise 27.

OBJECTIVE 5 Solve problems with three variables using a system of three equations. To solve such problems, we extend the method used for two unknowns. Since three variables are used, three equations are necessary to find a solution.

Photo not available

EXAMPLE 5 Solving a Problem Involving Prices

At Panera Bread, a loaf of honey wheat bread costs \$2.40, a loaf of pumpernickel bread costs \$3.35, and a loaf of French bread costs \$2.10. On a recent day, three times as many loaves of honey wheat were sold as pumpernickel. The number of loaves of French bread sold was 5 less than the number of loaves of honey wheat sold. Total receipts for these breads were \$56.90. How many loaves of each type of bread were sold? (*Source:* Panera Bread menu.)

Step 1 **Read** the problem again. There are three unknowns in this problem.

Step 2 **Assign variables** to represent the three unknowns.

Let x = the number of loaves of honey wheat,
y = the number of loaves of pumpernickel,
and z = the number of loaves of French bread.

Step 3 **Write a system of three equations** using the information in the problem. Since three times as many loaves of honey wheat were sold as pumpernickel,

$$x = 3y, \quad \text{or} \quad x - 3y = 0. \qquad (1)$$

Also,

Number of loaves of French bread	equals	5 less than the number of loaves of honey wheat.
z	$=$	$x - 5$,

so

$$x - z = 5. \qquad (2)$$

Multiplying the cost of a loaf of each kind of bread by the number of loaves of that kind sold and adding gives the total receipts.

$$2.40x + 3.35y + 2.10z = 56.90$$

Multiply each side of this equation by 100 to clear it of decimals.

$$240x + 335y + 210z = 5690 \qquad (3)$$

Step 4 **Solve** the system of three equations using the method shown in Section 4.2. Solving the system

$$
\begin{aligned}
x - 3y &= 0 && (1)\\
x - z &= 5 && (2)\\
240x + 335y + 210z &= 5690 && (3)
\end{aligned}
$$

leads to

$$x = 12, \quad y = 4, \quad \text{and} \quad z = 7.$$

Step 5 **State the answer.** The solution is (12, 4, 7), so 12 loaves of honey wheat, 4 loaves of pumpernickel, and 7 loaves of French bread were sold.

Step 6 **Check.** Since $12 = 3 \cdot 4$, the number of loaves of honey wheat is three times the number of loaves of pumpernickel. Also, $12 - 7 = 5$, so the number of loaves of French bread is 5 less than the number of loaves of honey wheat. Multiply the appropriate cost per loaf by the number of loaves sold and add the results to check that total receipts were \$56.90.

Now Try Exercise 45.

YOU TRY IT

EXAMPLE 6 Solving a Business Production Problem

A company produces three color television sets, models X, Y, and Z. Each model X set requires 2 hr of electronics work, 2 hr of assembly time, and 1 hr of finishing time. Each model Y requires 1, 3, and 1 hr of electronics, assembly, and finishing time, respectively. Each model Z requires 3, 2, and 2 hr of the same work, respectively. There are 100 hr available for electronics, 100 hr available for assembly, and 65 hr available for finishing per week. How many of each model should be produced each week if all available time must be used?

Step 1 **Read** the problem again. There are three unknowns.

Step 2 **Assign variables.**

Let x = the number of model X produced per week,
y = the number of model Y produced per week,
and z = the number of model Z produced per week.

We organize the information in a table.

	Each Model X	Each Model Y	Each Model Z	Totals
Hours of Electronics Work	2	1	3	100
Hours of Assembly Time	2	3	2	100
Hours of Finishing Time	1	1	2	65

Step 3 **Write a system of three equations.** The x model X sets require $2x$ hr of electronics, the y model Y sets require $1y$ (or y) hr of electronics, and the

z model Z sets require $3z$ hr of electronics. Since 100 hr are available for electronics,

$$2x + y + 3z = 100. \qquad (1)$$

Similarly, from the fact that 100 hr are available for assembly,

$$2x + 3y + 2z = 100, \qquad (2)$$

and the fact that 65 hr are available for finishing leads to the equation

$$x + y + 2z = 65. \qquad (3)$$

Again, notice the advantage of setting up a table. By reading across, we can easily determine the coefficients and constants in the equations of the system.

Step 4 **Solve** the system

$$\begin{aligned} 2x + y + 3z &= 100 \\ 2x + 3y + 2z &= 100 \\ x + y + 2z &= 65 \end{aligned}$$

to find $x = 15$, $y = 10$, and $z = 20$.

Step 5 **State the answer.** The company should produce 15 model X, 10 model Y, and 20 model Z sets per week.

Step 6 **Check** that these values satisfy the conditions of the problem.

Now Try Exercise 47.

4.3 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 6/Videotape 6

Solve each problem. See Example 1.

1. During the 2000 Major League Baseball season, the St. Louis Cardinals played 162 games. They won 28 more games than they lost. What was their win–loss record that year?

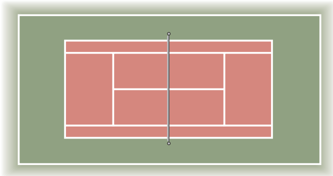
2. Refer to Exercise 1. During the same 162-game season, the Chicago Cubs lost 32 more games than they won. What was the team's win–loss record?

2000 MLB FINAL STANDINGS NATIONAL LEAGUE CENTRAL

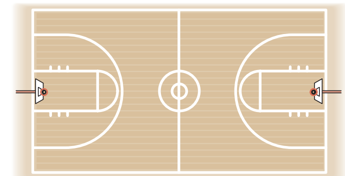
Team	W	L
St. Louis	___	___
Cincinnati	85	77
Milwaukee	73	89
Houston	72	90
Pittsburgh	69	93
Chicago	___	___

Source: www.mlb.com

3. Venus and Serena measured a tennis court and found that it was 42 ft longer than it was wide and had a perimeter of 228 ft. What were the length and the width of the tennis court?



4. Shaq and Kobe found that the width of their basketball court was 44 ft less than the length. If the perimeter was 288 ft, what were the length and the width of their court?

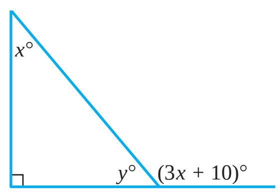


5. The two biggest U.S. companies in terms of revenue in 2000 were ExxonMobil and General Motors. ExxonMobil's revenue was \$29 billion more than that of General Motors. Total revenue for the two companies was \$399 billion. What was the revenue for each company? (*Source:* Bridge News, MarketGuide.com)

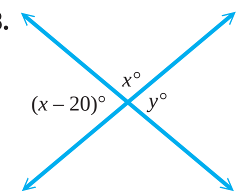
6. The top two U.S. trading partners during the first four months of 2000 were Canada and Mexico. Exports and imports with Mexico were \$57 billion less than those with Canada. Total exports and imports involving these two countries were \$211 billion. How much were U.S. exports and imports with each country? (*Source:* U.S. Bureau of the Census.)

In Exercises 7 and 8, find the measures of the angles marked x and y. Remember that (1) *the sum of the measures of the angles of a triangle is* 180°, (2) *supplementary angles have a sum of* 180°, *and* (3) *vertical angles have equal measures.*

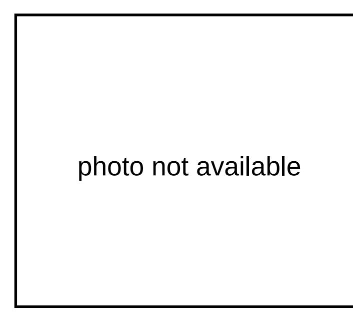
7.



8.



The Fan Cost Index (FCI) represents the cost of four average-price tickets, four small soft drinks, two small beers, four hot dogs, parking for one car, two game programs, and two souvenir caps to a sporting event. For example, in a recent year, the FCI for Major League Baseball was \$105.63. *This was by far the least for the four major professional sports.* (*Source:* Team Marketing Report, Chicago.)



Use the concept of FCI in Exercises 9 and 10. See Example 2.

9. The FCI prices for the National Hockey League and the National Basketball Association totaled \$423.12. The hockey FCI was \$16.36 more than that of basketball. What were the FCIs for these sports?

10. The FCI prices for Major League Baseball and the National Football League totaled \$311.03. The football FCI was \$105.87 more than that of baseball. What were the FCIs for these sports?

Solve each problem. See Example 2.

11. Andrew McGinnis works at Wendy's Old Fashioned Hamburgers. During one particular lunch hour, he sold 15 single hamburgers and 10 double hamburgers, totaling \$63.25. Another lunch hour, he sold 30 singles and 5 doubles, totaling \$78.65. How much did each type of burger cost? (*Source:* Wendy's Old Fashioned Hamburgers menu.)

12. Tokyo and New York are among the most expensive cities worldwide for business travelers. Using average costs per day for each city (which includes room, meals, laundry, and two taxi fares), 2 days in Tokyo and 3 days in New York cost \$2015. Four days in Tokyo and 2 days in New York cost \$2490. What is the average cost per day for each city? (*Source:* ECA International.)

The formulas $p = br$ (percentage = base × rate) and $I = prt$ (simple interest = principal × rate × time) are used in the applications in Exercises 17–24. In general, we are using

$$\text{portion} = \text{whole} \times \text{percent}.$$

To prepare to use these formulas, answer the questions in Exercises 13 and 14.

EXERCISES

13. If a container of liquid contains 60 oz of solution, what is the number of ounces of pure acid if the given solution contains the following acid concentrations?

(a) 10% **(b)** 25% **(c)** 40% **(d)** 50%

14. If \$5000 is invested in an account paying simple annual interest, how much interest will be earned during the first year at the following rates?

(a) 2% **(b)** 3% **(c)** 4% **(d)** 3.5%

15. If a pound of turkey costs \$.99, how much will x pounds cost?

16. If a ticket to the movie *Eight Legged Freaks* costs \$8 and y tickets are sold, how much is collected from the sale?

Solve each problem. See Example 3.

17. How many gallons each of 25% alcohol and 35% alcohol should be mixed to get 20 gal of 32% alcohol?

Gallons of Solution	Percent (as a decimal)	Gallons of Pure Alcohol
x	25% = .25	
y	35% = .35	
20	32% =	

18. How many liters each of 15% acid and 33% acid should be mixed to get 120 L of 21% acid?

Liters of Solution	Percent (as a decimal)	Liters of Pure Acid
x	15% = .15	
y	33% =	
120	21% =	

EXERCISES

19. Pure acid is to be added to a 10% acid solution to obtain 54 L of a 20% acid solution. What amounts of each should be used?

20. A truck radiator holds 36 L of fluid. How much pure antifreeze must be added to a mixture that is 4% antifreeze to fill the radiator with a mixture that is 20% antifreeze?

21. A party mix is made by adding nuts that sell for $2.50 per kg to a cereal mixture that sells for $1 per kg. How much of each should be added to get 30 kg of a mix that will sell for $1.70 per kg?

	Number of Kilograms	*Price per Kilogram*	*Value*
Nuts	x	2.50	
Cereal	y	1.00	
Mixture		1.70	

22. A popular fruit drink is made by mixing fruit juices. Such a drink with 50% juice is to be mixed with another drink that is 30% juice to get 200 L of a drink that is 45% juice. How much of each should be used?

	Liters of Drink	*Percent (as a decimal)*	*Liters of Pure Juice*
50% Juice	x	.50	
30% Juice	y	.30	
Mixture		.45	

23. A total of $3000 is invested, part at 2% simple interest and part at 4%. If the total annual return from the two investments is $100, how much is invested at each rate?

Principal	*Rate (as a decimal)*	*Interest*
x	.02	$.02x$
y	.04	$.04y$
3000		100

24. An investor will invest a total of $15,000 in two accounts, one paying 4% annual simple interest, and the other 3%. If he wants to earn $550 annual interest, how much should he invest at each rate?

Principal	*Rate (as a decimal)*	*Interest*
x	.04	
y	.03	
15,000		

The formula $d = rt$ (distance = rate × time) is used in the applications in Exercises 27–30. To prepare to use this formula, answer the questions in Exercises 25 and 26.

EXERCISES

25. If the speed of a killer whale is 25 mph and the whale swims for y hr, how many miles does the whale travel?

26. If the speed of a boat in still water is 10 mph, and the speed of the current of a river is x mph, what is the speed of the boat

(a) going upstream (that is, against the current, which slows the boat down);

(b) going downstream (that is, with the current, which speeds the boat up)?

Solve each problem. See Example 4.

27. A train travels 150 km in the same time that a plane covers 400 km. If the speed of the plane is 20 km per hr less than 3 times the speed of the train, find both speeds.

	r	*t*	*d*
Train	x		150
Plane	y		400

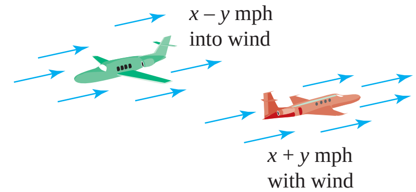
28. A freight train and an express train leave towns 390 km apart, traveling toward one another. The freight train travels 30 km per hr slower than the express train. They pass one another 3 hr later. What are their speeds?

	r	*t*	*d*
Freight Train	x	3	
Express Train	y	3	

29. In his motorboat, Bill Ruhberg travels upstream at top speed to his favorite fishing spot, a distance of 36 mi, in 2 hr. Returning, he finds that the trip downstream, still at top speed, takes only 1.5 hr. Find the speed of Bill's boat and the speed of the current.

	r	t	d
Upstream	$x - y$	2	
Downstream	$x + y$		

30. Traveling for 3 hr into a steady headwind, a plane flies 1650 mi. The pilot determines that flying *with* the same wind for 2 hr, he could make a trip of 1300 mi. Find the speed of the plane and the speed of the wind.



Use the problem-solving techniques of this section to solve each problem with two variables. See Examples 1–4.

31. At age 61, rock icon Tina Turner generated the most revenue on the concert circuit in 2000. Turner and second-place 'N Sync together took in \$157 million from ticket sales. If 'N Sync took in \$3.8 million less than Turner, how much did each generate? (*Source:* Pollstar.)

photo not available

32. Carol Britz plans to mix pecan clusters that sell for \$3.60 per lb with chocolate truffles that sell for \$7.20 per lb to get a mixture that she can sell in Valentine boxes for \$4.95 per lb. How much of the \$3.60 clusters and the \$7.20 truffles should she use to create 80 lb of the mix?

	Number of Pounds	Price per Pound	Value
Pecan Clusters	x		
Chocolate Truffles	y		
Valentine Mixture	80		

33. Tickets to a production of *King Lear* at Broward Community College cost \$5 for general admission or \$4 with a student ID. If 184 people paid to see a performance and \$812 was collected, how many of each type of ticket were sold?

34. At a business meeting at Panera Bread, the bill for two cappuccinos and three house lattes was \$10.95. At another table, the bill for one cappuccino and two house lattes was \$6.65. How much did each type of beverage cost? (*Source:* Panera Bread menu.)

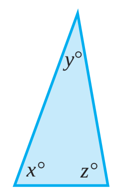
35. The mixed price of 9 citrons and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and the price of a wood apple here, having distinctly separated those prices well. (*Source:* Hindu work, A.D. 850.)

36. Braving blizzard conditions on the planet Hoth, Luke Skywalker sets out at top speed in his snow speeder for a rebel base 4800 mi away. He travels into a steady headwind and makes the trip in 3 hr. Returning, he finds that the trip back, still at top speed but now with a tailwind, takes only 2 hr. Find the top speed of Luke's snow speeder and the speed of the wind.

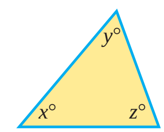
	r	t	d
Into Headwind			
With Tailwind			

Solve each problem involving three variables. See Examples 5 and 6. (In Exercises 37–40, remember that the sum of the measures of the angles of a triangle is 180°.*)*

37. In the figure, $z = x + 10$ and $x + y = 100$. Determine a third equation involving x, y, and z, and then find the measures of the three angles.



38. In the figure, x is 10 less than y and 20 less than z. Write a system of equations and find the measures of the three angles.



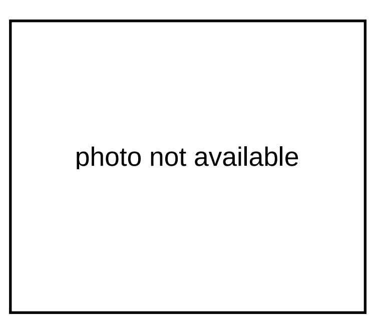
39. In a certain triangle, the measure of the second angle is 10° more than three times the first. The third angle measure is equal to the sum of the measures of the other two. Find the measures of the three angles.

40. The measure of the largest angle of a triangle is 12° less than the sum of the measures of the other two. The smallest angle measures 58° less than the largest. Find the measures of the angles.

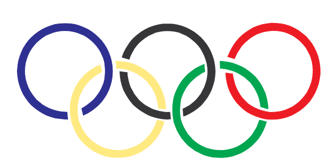
41. The perimeter of a triangle is 70 cm. The longest side is 4 cm less than the sum of the other two sides. Twice the shortest side is 9 cm less than the longest side. Find the length of each side of the triangle.

42. The perimeter of a triangle is 56 in. The longest side measures 4 in. less than the sum of the other two sides. Three times the shortest side is 4 in. more than the longest side. Find the lengths of the three sides.

43. In a random sample of 100 Americans of voting age, 10 more Americans identify themselves as Independents than Republicans. Six fewer Americans identify themselves as Republicans than Democrats. Assuming that all of those sampled are Republican, Democrat, or Independent, how many of those in the sample identify themselves with each political affiliation? (*Source:* The Gallup Organization.)



44. In the 2000 Summer Olympics in Sydney, Australia, the United States earned 14 more gold medals than silver. The number of bronze medals earned was 17 less than twice the number of silver medals. The United States earned a total of 97 medals. How many of each kind of medal did the United States earn? (*Source: The Gazette*, October 2, 2000.)



45. Tickets for one show on the Harlem Globetrotters' 75th Anniversary Tour cost \$10, \$18, or, for VIP seats, \$30. So far, five times as many \$18 tickets have been sold as VIP tickets. The number of \$10 tickets equals the number of \$18 tickets plus twice the number of VIP tickets. Sales of these tickets total $9500. How many of each kind of ticket have been sold? (*Source:* www.ticketmaster.com)

46. Three kinds of tickets are available for a *Prosthetic Forehead* concert: “up close,” “in the middle,” and “far out.” “Up close” tickets cost $10 more than “in the middle” tickets, while “in the middle” tickets cost $10 more than “far out” tickets. Twice the cost of an “up close” ticket is $20 more than 3 times the cost of a “far out” ticket. Find the price of each kind of ticket.

47. A hardware supplier manufactures three kinds of clamps, types A, B, and C. Production restrictions require it to make 10 units more type C clamps than the total of the other types and twice as many type B clamps as type A. The shop must produce a total of 490 units of clamps per day. How many units of each type can be made per day?

48. A Mardi Gras trinket manufacturer supplies three wholesalers, A, B, and C. The output from a day’s production is 320 cases of trinkets. She must send wholesaler A three times as many cases as she sends B, and she must send wholesaler C 160 cases less than she provides A and B together. How many cases should she send to each wholesaler to distribute the entire day’s production to them?

49. A plant food is to be made from three chemicals. The mix must include 60% of the first and second chemicals. The second and third chemicals must be in a ratio of 4 to 3 by weight. How much of each chemical is needed to make 750 kg of the plant food?

50. How many ounces of 5% hydrochloric acid, 20% hydrochloric acid, and water must be combined to get 10 oz of solution that is 8.5% hydrochloric acid, if the amount of water used must equal the total amount of the other two solutions?

51. During a recent National Hockey League regular season, the Dallas Stars played 82 games. Together, their wins and losses totaled 74. They tied 18 fewer games than they lost. How many wins, losses, and ties did they have that year?

Team	GP	W	L	T	GF	GA	Pts
Dallas	82	___	___	___	252	198	104
Detroit	82	38	26	18	253	197	94
Phoenix	82	38	37	7	240	243	83
St. Louis	82	36	35	11	236	239	83
Chicago	82	34	35	13	223	210	81
Toronto	82	30	44	8	230	273	68

Source: *Sports Illustrated Sports Almanac.*

52. During a recent National Hockey League season, the Boston Bruins played 82 games. Their losses and ties totaled 56, and they had 21 fewer wins than losses. How many wins, losses, and ties did they have that year?

Team	GP	W	L	T	GF	GA	Pts
Buffalo	82	40	30	12	237	208	92
Pittsburgh	82	38	36	8	285	280	84
Ottawa	82	31	36	15	226	234	77
Montreal	82	31	36	15	249	276	77
Hartford	82	32	39	11	226	256	75
Boston	82	___	___	___	234	300	61

Source: *Sports Illustrated Sports Almanac.*

4.4 Solving Systems of Linear Equations by Matrix Methods

OBJECTIVES

1. Define a matrix.
2. Write the augmented matrix for a system.
3. Use row operations to solve a system with two equations.
4. Use row operations to solve a system with three equations.
5. Use row operations to solve special systems.

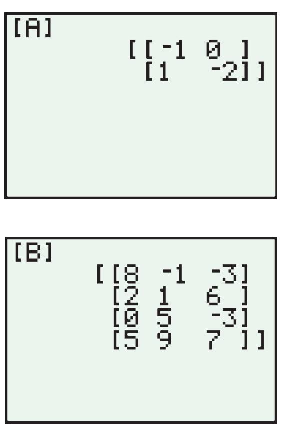


FIGURE 10

OBJECTIVE 1 Define a matrix. An ordered array of numbers such as

$$\begin{array}{c c} & \text{Columns} \\ \text{Rows} & \begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 2 \end{bmatrix} \end{array}$$

is called a **matrix.** The numbers are called **elements** of the matrix. Matrices (the plural of *matrix*) are named according to the number of **rows** and **columns** they contain. The rows are read horizontally, and the columns are read vertically. For example, the first row in the preceding matrix is 2 3 5 and the first column is $\begin{matrix} 2 \\ 7 \end{matrix}$. This matrix is a 2×3 (read "two by three") matrix because it has 2 rows and 3 columns. The number of rows is given first, and then the number of columns. Two other examples follow.

$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \quad 2 \times 2 \text{ matrix} \qquad \begin{bmatrix} 8 & -1 & -3 \\ 2 & 1 & 6 \\ 0 & 5 & -3 \\ 5 & 9 & 7 \end{bmatrix} \quad 4 \times 3 \text{ matrix}$$

A **square matrix** is one that has the same number of rows as columns. The 2×2 matrix is a square matrix.

Figure 10 shows how a graphing calculator displays the preceding two matrices. Work with matrices is made much easier by using technology when available. Consult your owner's manual for details.

In this section, we discuss a matrix method of solving linear systems that is really just a very structured way of using the elimination method. The advantage of this new method is that it can be done by a graphing calculator or a computer, allowing large systems of equations to be solved easily.

OBJECTIVE 2 Write the augmented matrix for a system. To begin, we write an *augmented matrix* for the system. An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups. For example, to solve the system

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5, \end{aligned}$$

start with the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right].$$

Place the coefficients of the variables to the left of the bar, and the constants to the right. The bar separates the coefficients from the constants. The matrix is just a shorthand way of writing the system of equations, so the rows of the augmented matrix can be treated the same as the equations of a system of equations.

We know that exchanging the position of two equations in a system does not change the system. Also, multiplying any equation in a system by a nonzero number does not change the system. Comparable changes to the augmented matrix of a system of equations produce new matrices that correspond to systems with the same solutions as the original system.

The following **row operations** produce new matrices that lead to systems having the same solutions as the original system.

Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The elements in any row may be multiplied by any nonzero real number.
3. Any row may be changed by adding to the elements of the row the product of a real number and the corresponding elements of another row.

Examples of these row operations follow.

Row operation 1:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 1 & 0 & 7 \\ 4 & 8 & -3 \\ 2 & 3 & 9 \end{bmatrix}.$$

Interchange row 1 and row 3.

Row operation 2:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 6 & 9 & 27 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Multiply the numbers in row 1 by 3.

Row operation 3:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 0 & 3 & -5 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Multiply the numbers in row 3 by −2; add them to the corresponding numbers in row 1.

The third row operation corresponds to the way we eliminated a variable from a pair of equations in the previous sections.

OBJECTIVE 3 Use row operations to solve a system with two equations. Row operations can be used to rewrite a matrix until it is the matrix of a system where the solution is easy to find. The goal is a matrix in the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

for systems with two or three equations, respectively. Notice that there are 1s down the diagonal from upper left to lower right and 0s below the 1s. A matrix written this way is said to be in **row echelon form.** When these matrices are rewritten as systems of equations, the value of one variable is known, and the rest can be found by substitution. The following examples illustrate this method.

VIDEO

YOU TRY IT

EXAMPLE 1 Using Row Operations to Solve a System with Two Variables

Use row operations to solve the system

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5. \end{aligned}$$

We start with the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Now we use the various row operations to change this matrix into one that leads to a system that is easier to solve.

It is best to work by columns. We start with the first column and make sure that there is a 1 in the first row, first column position. There is already a 1 in this position. Next, we get 0 in every position below the first. To get a 0 in row two, column one, we use the third row operation and add to the numbers in row two the result of multiplying each number in row one by -2. (We abbreviate this as $-2R_1 + R_2$.) Row one remains unchanged.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 + 1(-2) & 1 + -3(-2) & -5 + 1(-2) \end{array}\right]$$

Original number from row two

-2 times number from row one

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right] \quad -2R_1 + R_2$$

The matrix now has a 1 in the first position of column one, with 0 in every position below the first.

Now we go to column two. A 1 is needed in row two, column two. We get this 1 by using the second row operation, multiplying each number of row two by $\frac{1}{7}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -1 \end{array}\right] \quad \tfrac{1}{7}R_2$$

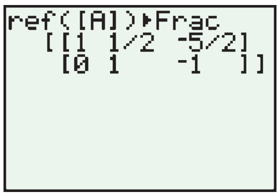
This augmented matrix leads to the system of equations

$$\begin{aligned} 1x - 3y &= 1 \\ 0x + 1y &= -1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x - 3y &= 1 \\ y &= -1. \end{aligned}$$

From the second equation, $y = -1$. We substitute -1 for y in the first equation to get

$$\begin{aligned} x - 3y &= 1 \\ x - 3(-1) &= 1 \\ x + 3 &= 1 \\ x &= -2. \end{aligned}$$

(a)



(b)

FIGURE 11

The solution set of the system is $\{(-2, -1)\}$. Check this solution by substitution in both equations of the system.

Now Try Exercise 3.

If the augmented matrix of the system in Example 1 is entered as matrix [A] in a graphing calculator (Figure 11(a)) and the row echelon form of the matrix is found (Figure 11(b)), then the system becomes

$$\begin{aligned} x + \frac{1}{2}y &= -\frac{5}{2} \\ y &= -1. \end{aligned}$$

While this system looks different from the one we obtained in Example 1, it is equivalent, since its solution set is also $\{(-2, -1)\}$.

OBJECTIVE 4 Use row operations to solve a system with three equations. A linear system with three equations is solved in a similar way. We use row operations to get 1s down the diagonal from left to right and all 0s below each 1.

VIDEO

YOU TRY IT

EXAMPLE 2 Using Row Operations to Solve a System with Three Variables

Use row operations to solve the system

$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5. \end{aligned}$$

Start by writing the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

This matrix already has 1 in row one, column one. Next get 0s in the rest of column one. First, add to row two the results of multiplying each number of row one by -3. This gives the matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 1 & 3 & 2 & 5 \end{array}\right]. \qquad -3R_1 + R_2$$

Now add to the numbers in row three the results of multiplying each number of row one by -1.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \qquad -1R_1 + R_3$$

Introduce 1 in row two, column two by multiplying each number in row two by $\frac{1}{6}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \quad \frac{1}{6}R_2$$

To obtain 0 in row three, column two, add to row three the results of multiplying each number in row two by -4.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \quad -4R_2 + R_3$$

Finally, obtain 1 in row three, column three by multiplying each number in row three by $\frac{3}{23}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \frac{3}{23}R_3$$

This final matrix gives the system of equations

$$\begin{aligned} x - y + 5z &= -6 \\ y - \frac{8}{3}z &= \frac{14}{3} \\ z &= -1. \end{aligned}$$

Substitute -1 for z in the second equation, $y - \frac{8}{3}z = \frac{14}{3}$, to find that $y = 2$. Finally, substitute 2 for y and -1 for z in the first equation, $x - y + 5z = -6$, to determine that $x = 1$. The solution set of the original system is $\{(1, 2, -1)\}$. Check by substitution.

Now Try Exercise 15.

OBJECTIVE 5 Use row operations to solve special systems. In the final example we show how to recognize inconsistent systems or systems with dependent equations when solving these systems with row operations.

EXAMPLE 3 Recognizing Inconsistent Systems or Dependent Equations

Use row operations to solve each system.

(a)
$$\begin{aligned} 2x - 3y &= 8 \\ -6x + 9y &= 4 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -6 & 9 & 4 \end{array}\right] \quad \frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 28 \end{array}\right] \quad 6R_1 + R_2$$

The corresponding system of equations is

$$x - \frac{3}{2}y = 4$$
$$0 = 28, \quad \text{False}$$

which has no solution and is inconsistent. The solution set is $\emptyset$.

(b) $-10x + 12y = 30$
$5x - 6y = -15$

$$\left[\begin{array}{cc|c} -10 & 12 & 30 \\ 5 & -6 & -15 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{6}{5} & -3 \\ 5 & -6 & -15 \end{array}\right] \quad -\tfrac{1}{10}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{6}{5} & -3 \\ 0 & 0 & 0 \end{array}\right] \quad -5R_1 + R_2$$

The corresponding system is

$$x - \frac{6}{5}y = -3$$
$$0 = 0, \quad \text{True}$$

which has dependent equations. Using the second equation of the original system, we write the solution set as

$$\{(x, y) \,|\, 5x - 6y = -15\}.$$

Now Try Exercises 11 and 13.

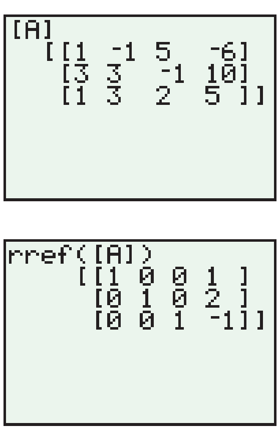


FIGURE 12

CONNECTIONS

An extension of the matrix method described in this section involves transforming an augmented matrix into **reduced row echelon form.** This form has 1s down the main diagonal and 0s above and below this diagonal. For example, the matrix for the system in Example 2 could be transformed into the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \text{which gives the equivalent system} \quad \begin{aligned} x &= 1 \\ y &= 2 \\ z &= -1. \end{aligned}$$

The calculator screens in Figure 12 indicate how easily this transformation can be obtained using technology.

For Discussion or Writing

1. Write the reduced row echelon form for the matrix of the system in Example 1.
2. If transforming to reduced row echelon form leads to all 0s in the final row, what kind of system is represented?

4.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 6/Videotape 6

1. Consider the matrix $\begin{bmatrix} -2 & 3 & 1 \\ 0 & 5 & -3 \\ 1 & 4 & 8 \end{bmatrix}$ and answer the following.

(a) What are the elements of the second row?
(b) What are the elements of the third column?
(c) Is this a square matrix? Explain why or why not.
(d) Give the matrix obtained by interchanging the first and third rows.
(e) Give the matrix obtained by multiplying the first row by $-\frac{1}{2}$.
(f) Give the matrix obtained by multiplying the third row by 3 and adding to the first row.

2. Give the dimensions of each matrix.

(a) $\begin{bmatrix} 3 & -7 \\ 4 & 5 \\ -1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 4 & 9 & 0 \\ -1 & 2 & -4 \end{bmatrix}$

(c)
```
[A]
      [[6   3 ]
       [-2  5 ]
       [4   10]
       [1   -1]]
```

(d)
```
[B]
      [[-1  7  1]
       [0   2  1]
       [-3  5  4]
       [8  -9  1]]
```

Complete the steps in the matrix solution of each system by filling in the boxes. Give the final system and the solution set. See Example 1.

3. $4x + 8y = 44$
$2x - y = -3$

$\left[\begin{array}{cc|c} 4 & 8 & 44 \\ 2 & -1 & -3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & \square & \square \\ 2 & -1 & -3 \end{array}\right]$ $\frac{1}{4}R_1$

$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & \square & \square \end{array}\right]$ $-2R_1 + R_2$

$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 1 & \square \end{array}\right]$ $-\frac{1}{5}R_2$

4. $2x - 5y = -1$
$3x + y = 7$

$\left[\begin{array}{cc|c} 2 & -5 & -1 \\ 3 & 1 & 7 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -\frac{5}{2} & \square \\ 3 & 1 & 7 \end{array}\right]$ $\frac{1}{2}R_1$

$\left[\begin{array}{cc|c} 1 & -\frac{5}{2} & -\frac{1}{2} \\ 0 & \square & \square \end{array}\right]$ $-3R_1 + R_2$

$\left[\begin{array}{cc|c} 1 & -\frac{5}{2} & -\frac{1}{2} \\ 0 & 1 & \square \end{array}\right]$ $\frac{2}{17}R_2$

Use row operations to solve each system. See Examples 1 and 3.

5. $x + y = 5$
$x - y = 3$

6. $x + 2y = 7$
$x - y = -2$

7. $2x + 4y = 6$
$3x - y = 2$

8. $4x + 5y = -7$
$x - y = 5$

9. $\begin{aligned} 3x + 4y &= 13 \\ 2x - 3y &= -14 \end{aligned}$

10. $\begin{aligned} 5x + 2y &= 8 \\ 3x - y &= 7 \end{aligned}$

11. $\begin{aligned} -4x + 12y &= 36 \\ x - 3y &= 9 \end{aligned}$

12. $\begin{aligned} 2x - 4y &= 8 \\ -3x + 6y &= 5 \end{aligned}$

13. $\begin{aligned} 2x + y &= 4 \\ 4x + 2y &= 8 \end{aligned}$

14. $\begin{aligned} -3x - 4y &= 1 \\ 6x + 8y &= -2 \end{aligned}$

Complete the steps in the matrix solution of each system by filling in the boxes. Give the final system and the solution set. See Example 2.

15. $\begin{aligned} x + y - z &= -3 \\ 2x + y + z &= 4 \\ 5x - y + 2z &= 23 \end{aligned}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 2 & 1 & 1 & 4 \\ 5 & -1 & 2 & 23 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & \square & \square & \square \\ 0 & \square & \square & \square \end{array}\right] \quad \begin{array}{l} \\ -2R_1 + R_2 \\ -5R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & \square & \square \\ 0 & -6 & 7 & 38 \end{array}\right] \quad -1R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & \square & \square \end{array}\right] \quad 6R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & \square \end{array}\right] \quad -\tfrac{1}{11}R_3$$

16. $\begin{aligned} 2x + y + 2z &= 11 \\ 2x - y - z &= -3 \\ 3x + 2y + z &= 9 \end{aligned}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 11 \\ 2 & -1 & -1 & -3 \\ 3 & 2 & 1 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \square & \square & \square \\ 2 & -1 & -1 & -3 \\ 3 & 2 & 1 & 9 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & \frac{11}{2} \\ 0 & \square & \square & \square \\ 0 & \square & \square & \square \end{array}\right] \quad \begin{array}{l} \\ -2R_1 + R_2 \\ -3R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & \frac{11}{2} \\ 0 & 1 & \square & \square \\ 0 & \frac{1}{2} & -2 & -\frac{15}{2} \end{array}\right] \quad -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & \frac{11}{2} \\ 0 & 1 & \frac{3}{2} & 7 \\ 0 & 0 & \square & \square \end{array}\right] \quad -\tfrac{1}{2}R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & \frac{11}{2} \\ 0 & 1 & \frac{3}{2} & 7 \\ 0 & 0 & 1 & \square \end{array}\right] \quad -\tfrac{4}{11}R_3$$

Use row operations to solve each system. See Examples 2 and 3.

17. $\begin{aligned} x + y - 3z &= 1 \\ 2x - y + z &= 9 \\ 3x + y - 4z &= 8 \end{aligned}$

18. $\begin{aligned} 2x + 4y - 3z &= -18 \\ 3x + y - z &= -5 \\ x - 2y + 4z &= 14 \end{aligned}$

19. $\begin{aligned} x + y - z &= 6 \\ 2x - y + z &= -9 \\ x - 2y + 3z &= 1 \end{aligned}$

20. $\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + 2z &= 0 \\ x + y + 2z &= -1 \end{aligned}$

21. $$\begin{aligned} x - y &= 1 \\ y - z &= 6 \\ x + z &= -1 \end{aligned}$$

22. $$\begin{aligned} x + y &= 1 \\ 2x - z &= 0 \\ y + 2z &= -2 \end{aligned}$$

23. $$\begin{aligned} x - 2y + z &= 4 \\ 3x - 6y + 3z &= 12 \\ -2x + 4y - 2z &= -8 \end{aligned}$$

24. $$\begin{aligned} 4x + 8y + 4z &= 9 \\ x + 3y + 4z &= 10 \\ 5x + 10y + 5z &= 12 \end{aligned}$$

25. $$\begin{aligned} x + 2y + 3z &= -2 \\ 2x + 4y + 6z &= -5 \\ x - y + 2z &= 6 \end{aligned}$$

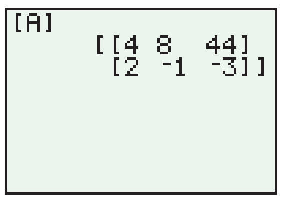
26. $$\begin{aligned} x + 3y + z &= 1 \\ 2x + 6y + 2z &= 2 \\ 3x + 9y + 3z &= 3 \end{aligned}$$

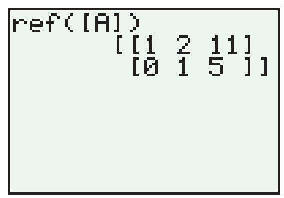
27. Write a short explanation of each term. Include examples.

(a) Matrix **(b)** Row of a matrix
(c) Column of a matrix **(d)** Square matrix
(e) Augmented matrix **(f)** Row operations on a matrix

28. Compare the use of the third row operation on a matrix and the elimination method of solving a system of linear equations. Give examples.

The augmented matrix for the system in Exercise 3 is shown in the graphing calculator screen on the left as matrix [A]. *The screen in the middle shows the row echelon form for* [A]. *Compare it to the matrix shown in the answer section for Exercise 3. The screen on the right shows the reduced row echelon form, and from this it can be determined by inspection that the solution set of the system is* {(1, 5)}.





Use a graphing calculator and either one of the two matrix methods illustrated to solve each system.

29. $$\begin{aligned} 4x + y &= 5 \\ 2x + y &= 3 \end{aligned}$$

30. $$\begin{aligned} 5x + 3y &= 7 \\ 7x - 3y &= -19 \end{aligned}$$

31. $$\begin{aligned} 5x + y - 3z &= -6 \\ 2x + 3y + z &= 5 \\ -3x - 2y + 4z &= 3 \end{aligned}$$

32. $$\begin{aligned} x + y + z &= 3 \\ 3x - 3y - 4z &= -1 \\ x + y + 3z &= 11 \end{aligned}$$

33. $$\begin{aligned} x + z &= -3 \\ y + z &= 3 \\ x + y &= 8 \end{aligned}$$

34. $$\begin{aligned} x - y &= -1 \\ -y + z &= -2 \\ x + z &= -2 \end{aligned}$$

5.1 Integer Exponents and Scientific Notation

OBJECTIVES

1. Use the product rule for exponents.
2. Define 0 and negative exponents.
3. Use the quotient rule for exponents.
4. Use the power rules for exponents.
5. Simplify exponential expressions.
6. Use the rules for exponents with scientific notation.

Recall that we use exponents to write products of repeated factors. For example,

$$2^5 \quad \text{is defined as} \quad 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.$$

The number 5, the *exponent*, shows that the *base* 2 appears as a factor 5 times. The quantity 2^5 is called an *exponential* or a *power*. We read 2^5 as "2 to the fifth power" or "2 to the fifth."

OBJECTIVE 1 Use the product rule for exponents. There are several useful rules that simplify work with exponents. For example, the product $2^5 \cdot 2^3$ can be simplified as follows.

$$5 + 3 = 8$$

$$2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^8$$

This result, that products of exponential expressions with the *same base* are found by adding exponents, is generalized as the **product rule for exponents.**

Product Rule for Exponents

If m and n are natural numbers and a is any real number, then

$$\boldsymbol{a^m \cdot a^n = a^{m+n}}.$$

That is, when multiplying powers of like bases, keep the same base and add the exponents.

To see that the product rule is true, use the definition of an exponent as follows.

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{a \text{ appears as a factor } m \text{ times.}} \qquad a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{a \text{ appears as a factor } n \text{ times.}}$$

From this,

$$\begin{aligned} a^m \cdot a^n &= \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} \\ &= \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{(m+n) \text{ factors}} \\ a^m \cdot a^n &= a^{m+n}. \end{aligned}$$

EXAMPLE 1 Using the Product Rule for Exponents

Apply the product rule for exponents, if possible, in each case.

(a) $3^4 \cdot 3^7 = 3^{4+7} = 3^{11}$

(b) $5^3 \cdot 5 = 5^3 \cdot 5^1 = 5^{3+1} = 5^4$

(c) $y^3 \cdot y^8 \cdot y^2 = y^{3+8+2} = y^{13}$

(d)
$$\begin{aligned} (5y^2)(-3y^4) &= 5(-3)y^2y^4 && \text{Associative and commutative properties} \\ &= -15y^{2+4} && \text{Product rule} \\ &= -15y^6 \end{aligned}$$

(e) $(7p^3q)(2p^5q^2) = 7(2)p^3p^5qq^2 = 14p^8q^3$

(f) $x^2 \cdot y^4$

The product rule does not apply because the bases are not the same.

Now Try Exercises 7, 11, 13, and 15.

CAUTION Be careful in problems like Example 1(a) not to multiply the bases. Notice that $3^4 \cdot 3^7 \neq 9^{11}$. Remember to keep the *same* base and add the exponents.

OBJECTIVE 2 Define 0 and negative exponents. So far we have discussed only positive exponents. How should we define a 0 exponent? Suppose we multiply 4^2 by 4^0. Extending the product rule, we should have

$$4^2 \cdot 4^0 = 4^{2+0} = 4^2.$$

For the product rule to hold true, 4^0 must equal 1, and so we define a^0 this way for any nonzero real number a.

Zero Exponent

If a is any nonzero real number, then

$$\boldsymbol{a^0 = 1.}$$

The expression **0^0 is undefined.***

EXAMPLE 2 Using 0 as an Exponent

Evaluate each expression.

(a) $6^0 = 1$

(b) $(-6)^0 = 1$ Base is -6.

(c) $-6^0 = -(6^0) = -(1) = -1$ Base is 6.

(d) $-(-6)^0 = -1$

(e) $5^0 + 12^0 = 1 + 1 = 2$

(f) $(8k)^0 = 1, \quad k \neq 0$

Now Try Exercises 17 and 25.

How should we define a negative exponent? Extending the product rule again,

$$8^2 \cdot 8^{-2} = 8^{2+(-2)} = 8^0 = 1.$$

This indicates that 8^{-2} is the reciprocal of 8^2. But $\frac{1}{8^2}$ is the reciprocal of 8^2, and a number can have only one reciprocal. Therefore, it is reasonable to conclude that $8^{-2} = \frac{1}{8^2}$. We can generalize and make the following definition.

Negative Exponent

For any natural number n and any nonzero real number a,

$$\boldsymbol{a^{-n} = \frac{1}{a^n}.}$$

*In advanced treatments, 0^0 is called an *indeterminant form.*

With this definition, the expression a^n is meaningful for any integer exponent n and any nonzero real number a. The product rule is valid for integers m and n.

CAUTION *A negative exponent does not indicate a negative number;* negative exponents lead to reciprocals. For example,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}. \qquad \text{Not negative}$$

On the other hand,

$$-3^{-2} = -\frac{1}{3^2} = -\frac{1}{9} \qquad \text{Negative}$$

because of the negative sign preceding the base.

VIDEO

EXAMPLE 3 Using Negative Exponents

In parts (a)–(f), write each expression with only positive exponents.

(a) $2^{-3} = \frac{1}{2^3}$

(b) $6^{-1} = \frac{1}{6^1} = \frac{1}{6}$

(c) $(5z)^{-3} = \frac{1}{(5z)^3}, \quad z \neq 0$

Base is $5z$.

(d) $5z^{-3} = 5\left(\frac{1}{z^3}\right) = \frac{5}{z^3}, \quad z \neq 0$

Base is z.

(e) $-m^{-2} = -\frac{1}{m^2}, \quad m \neq 0$

(What is the base here?)

(f) $(-m)^{-2} = \frac{1}{(-m)^2}, \quad m \neq 0$

(What is the base here?)

In parts (g) and (h), evaluate each expression.

(g) $3^{-1} + 4^{-1} = \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$ $\quad \frac{1}{3} \cdot \frac{4}{4} = \frac{4}{12}; \frac{1}{4} \cdot \frac{3}{3} = \frac{3}{12}$

(h) $5^{-1} - 2^{-1} = \frac{1}{5} - \frac{1}{2} = \frac{2}{10} - \frac{5}{10} = -\frac{3}{10}$

Now Try Exercises 33, 37, 39, 41, and 45.

CAUTION In Example 3(g), note that $3^{-1} + 4^{-1} \neq (3 + 4)^{-1}$. The expression on the left is equal to $\frac{7}{12}$, as shown in the example, while the expression on the right is $7^{-1} = \frac{1}{7}$. Similar reasoning can be applied to part (h).

YOU TRY IT

EXAMPLE 4 Using Negative Exponents

Evaluate each expression.

(a) $$\frac{1}{2^{-3}} = \frac{1}{\frac{1}{2^3}} = 1 \div \frac{1}{2^3} = 1 \cdot \frac{2^3}{1} = 2^3 = 8$$

Multiply by the reciprocal.

(b) $$\frac{2^{-3}}{3^{-2}} = \frac{\frac{1}{2^3}}{\frac{1}{3^2}} = \frac{1}{2^3} \div \frac{1}{3^2} = \frac{1}{2^3} \cdot \frac{3^2}{1} = \frac{3^2}{2^3} = \frac{9}{8}$$

Now Try Exercises 51 and 53.

Example 4 suggests the following generalizations.

Special Rules for Negative Exponents

If $a \neq 0$ and $b \neq 0$, then $$\frac{1}{a^{-n}} = a^n \quad \text{and} \quad \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}.$$

OBJECTIVE 3 Use the quotient rule for exponents. A quotient, such as $\frac{a^8}{a^3}$, can be simplified in much the same way as a product. (In all quotients of this type, assume that the denominator is not 0.) Using the definition of an exponent,

$$\frac{a^8}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a \cdot a \cdot a \cdot a = a^5.$$

Notice that $8 - 3 = 5$. In the same way,

$$\frac{a^3}{a^8} = \frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a^5} = a^{-5}.$$

Here $3 - 8 = -5$. These examples suggest the following **quotient rule for exponents.**

Quotient Rule for Exponents

If a is any nonzero real number and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}.$$

That is, when dividing powers of like bases, keep the same base and subtract the exponent of the denominator from the exponent of the numerator.

EXAMPLE 5 Using the Quotient Rule for Exponents

Apply the quotient rule for exponents, if possible, and write each result using only positive exponents.

Numerator exponent

Denominator exponent

(a) $\frac{3^7}{3^2} = 3^{7-2} = 3^5$

Minus sign

(b) $\frac{p^6}{p^2} = p^{6-2} = p^4, \quad p \neq 0$

(c) $\frac{k^7}{k^{12}} = k^{7-12} = k^{-5} = \frac{1}{k^5}, \quad k \neq 0$

(d) $\frac{2^7}{2^{-3}} = 2^{7-(-3)} = 2^{7+3} = 2^{10}$

(e) $\dfrac{8^{-2}}{8^{5}} = 8^{-2-5} = 8^{-7} = \dfrac{1}{8^{7}}$

(f) $\dfrac{6}{6^{-1}} = \dfrac{6^{1}}{6^{-1}} = 6^{1-(-1)} = 6^{2}$

(g) $\dfrac{z^{-5}}{z^{-8}} = z^{-5-(-8)} = z^{3}, \quad z \neq 0$

(h) $\dfrac{a^{3}}{b^{4}}, \quad b \neq 0$

The quotient rule does not apply because the bases are different.

Now Try Exercises 63, 67, 75, and 77.

CAUTION As seen in Example 5, be very careful when working with quotients that involve negative exponents in the denominator. Always be sure to write the numerator exponent, then a minus sign, and then the denominator exponent.

OBJECTIVE 4 Use the power rules for exponents. The expression $(3^4)^2$ can be simplified as

$$(3^4)^2 = 3^4 \cdot 3^4 = 3^{4+4} = 3^8,$$

where $4 \cdot 2 = 8$. This example suggests the first **power rule for exponents.** The other two power rules can be demonstrated with similar examples.

Power Rules for Exponents

If a and b are real numbers and m and n are integers, then

(a) $(a^m)^n = a^{mn}$, **(b)** $(ab)^m = a^m b^m$, and **(c)** $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m} \quad (b \neq 0)$.

That is,

(a) to raise a power to a power, multiply exponents;

(b) to raise a product to a power, raise each factor to that power; and

(c) to raise a quotient to a power, raise the numerator and the denominator to that power.

EXAMPLE 6 Using the Power Rules for Exponents

Use one or more power rules to simplify each expression.

(a) $(p^8)^3 = p^{8 \cdot 3} = p^{24}$

(b) $\left(\dfrac{2}{3}\right)^4 = \dfrac{2^4}{3^4} = \dfrac{16}{81}$

(c) $(3y)^4 = 3^4 y^4 = 81y^4$

(d) $(6p^7)^2 = 6^2 p^{7 \cdot 2} = 6^2 p^{14} = 36p^{14}$

(e) $\left(\dfrac{-2m^5}{z}\right)^3 = \dfrac{(-2)^3 m^{5 \cdot 3}}{z^3} = \dfrac{(-2)^3 m^{15}}{z^3} = \dfrac{-8m^{15}}{z^3}, \quad z \neq 0$

Now Try Exercises 79, 81, 83, and 87.

The reciprocal of a^n is $\frac{1}{a^n} = \left(\frac{1}{a}\right)^n$. Also, by definition, a^n and a^{-n} are reciprocals since

$$a^n \cdot a^{-n} = a^n \cdot \frac{1}{a^n} = 1.$$

Thus, since both are reciprocals of a^n,

$$a^{-n} = \left(\frac{1}{a}\right)^n.$$

Some examples of this result are

$$6^{-3} = \left(\frac{1}{6}\right)^3 \quad \text{and} \quad \left(\frac{1}{3}\right)^{-2} = 3^2.$$

This discussion can be generalized as follows.

More Special Rules for Negative Exponents

If $a \neq 0$ and $b \neq 0$ and n is an integer, then

$$\boldsymbol{a^{-n} = \left(\frac{1}{a}\right)^n} \quad \text{and} \quad \boldsymbol{\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n}.$$

That is, any nonzero number raised to the negative *n*th power is equal to the reciprocal of that number raised to the *n*th power.

EXAMPLE 7 Using Negative Exponents with Fractions

Write each expression with only positive exponents and then evaluate.

(a) $\left(\frac{3}{7}\right)^{-2} = \left(\frac{7}{3}\right)^2 = \frac{49}{9}$ **(b)** $\left(\frac{4}{5}\right)^{-3} = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$

Now Try Exercise 55.

The definitions and rules of this section are summarized here.

Definitions and Rules for Exponents

For all integers m and n and all real numbers a and b, the following rules apply.

Product Rule	$a^m \cdot a^n = a^{m+n}$
Quotient Rule	$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$
Zero Exponent	$a^0 = 1 \quad (a \neq 0)$
Negative Exponent	$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$

(continued)

Power Rules	$(a^m)^n = a^{mn}$	$(ab)^m = a^m b^m$
	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$	
Special Rules	$\frac{1}{a^{-n}} = a^n \quad (a \neq 0)$	$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n} \quad (a, b \neq 0)$
	$a^{-n} = \left(\frac{1}{a}\right)^n \quad (a \neq 0)$	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad (a, b \neq 0)$

OBJECTIVE 5 Simplify exponential expressions. With the rules of exponents developed so far in this section, we can simplify expressions that involve one or more rules.

EXAMPLE 8 Using the Definitions and Rules for Exponents

Simplify each expression so that no negative exponents appear in the final result. Assume all variables represent nonzero real numbers.

(a) $3^2 \cdot 3^{-5} = 3^{2+(-5)} = 3^{-3} = \frac{1}{3^3}$ or $\frac{1}{27}$

(b) $x^{-3}x^{-4}x^2 = x^{-3+(-4)+2} = x^{-5} = \frac{1}{x^5}$

(c) $(4^{-2})^{-5} = 4^{(-2)(-5)} = 4^{10}$

(d) $(x^{-4})^6 = x^{(-4)6} = x^{-24} = \frac{1}{x^{24}}$

(e)
$$\begin{aligned}\frac{x^{-4}y^2}{x^2y^{-5}} &= \frac{x^{-4}}{x^2} \cdot \frac{y^2}{y^{-5}} \\ &= x^{-4-2} \cdot y^{2-(-5)} \\ &= x^{-6}y^7 \\ &= \frac{y^7}{x^6}\end{aligned}$$

(f)
$$\begin{aligned}(2^3x^{-2})^{-2} &= (2^3)^{-2} \cdot (x^{-2})^{-2} \\ &= 2^{-6}x^4 \\ &= \frac{x^4}{2^6} \quad \text{or} \quad \frac{x^4}{64}\end{aligned}$$

(g)
$$\begin{aligned}\left(\frac{3x^2}{y}\right)^2\left(\frac{4x^3}{y^{-2}}\right)^{-1} &= \frac{3^2(x^2)^2}{y^2} \cdot \frac{y^{-2}}{4x^3} && \text{Combination of rules} \\ &= \frac{9x^4}{y^2} \cdot \frac{y^{-2}}{4x^3} && \text{Power rule} \\ &= \frac{9}{4}x^{4-3}y^{-2-2} && \text{Quotient rule} \\ &= \frac{9x}{4y^4}\end{aligned}$$

Now Try Exercises 91, 93, 107, and 117.

NOTE There is often more than one way to simplify expressions like those in Example 8. For instance, we could simplify Example 8(e) as follows.

$$\frac{x^{-4}y^2}{x^2y^{-5}} = \frac{y^5y^2}{x^4x^2} \qquad \text{Use } \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}.$$

$$= \frac{y^7}{x^6} \qquad \text{Product rule}$$

OBJECTIVE 6 Use the rules for exponents with scientific notation. Scientists often use numbers that are very large or very small. For example, the number of one-celled organisms that will sustain a whale for a few hours is 400,000,000,000,000, and the shortest wavelength of visible light is approximately .0000004 m. It is simpler to write these numbers using *scientific notation.*

In scientific notation, a number is written with the decimal point after the first nonzero digit and multiplied by a power of 10, as indicated in the following definition.

Scientific Notation

A number is written in **scientific notation** when it is expressed in the form

$$a \times 10^n$$

where $1 \le |a| < 10$, and n is an integer.

For example, in scientific notation,

$$8000 = 8 \times 1000 = 8 \times 10^3.$$

When using scientific notation, it is customary to use a times sign $\times$ instead of a multiplication dot. The following numbers are not in scientific notation.

$.230 \times 10^4$	46.5×10^{-3}
.230 is less than 1.	46.5 is greater than 10.

To write a number in scientific notation, use the following steps. (If the number is negative, ignore the negative sign, go through these steps, and then attach a negative sign to the result.)

Converting to Scientific Notation

Step 1 **Position the decimal point.** Place a caret, ^, to the right of the first nonzero digit, where the decimal point will be placed.

Step 2 **Determine the numeral for the exponent.** Count the number of digits from the decimal point to the caret. This number gives the absolute value of the exponent on 10.

Step 3 **Determine the sign for the exponent.** Decide whether multiplying by 10^n should make the result of Step 1 larger or smaller. The exponent should be positive to make the result larger; it should be negative to make the result smaller.

It is helpful to remember that for $n \ge 1$, $10^{-n} < 1$ and $10^n \ge 10$.

EXAMPLE 9 Writing Numbers in Scientific Notation

Write each number in scientific notation.

(a) 820,000

Step 1 Place a caret to the right of the 8 (the first nonzero digit) to mark the new location of the decimal point.

$$8_{\wedge}20{,}000$$

Step 2 Count from the decimal point, which is understood to be after the last 0, to the caret.

$$8.\underbrace{20{,}000}_{\text{Count 5 places.}}. \leftarrow \text{Decimal point}$$

Step 3 Since the number 8.2 is to be made larger, the exponent on 10 is positive.

$$820{,}000 = 8.2 \times 10^{5}$$

(b) .0000072

Count from left to right.

$$.\underbrace{000007}_{\text{6 places}}.2$$

Since the number 7.2 is to be made smaller, the exponent on 10 is negative.

$$.0000072 = 7.2 \times 10^{-6}$$

(c) $-.\underbrace{00004}_{\text{Count 5 places.}}62 = -4.62 \times 10^{-5}$

Now Try Exercises 127, 131, and 133.

To convert a number written in scientific notation to standard notation, just work in reverse.

Converting from Scientific Notation

Multiplying a number by a positive power of 10 makes the number larger, so move the decimal point to the right if n is positive in 10^n.

Multiplying by a negative power of 10 makes a number smaller, so move the decimal point to the left if n is negative.

If n is 0, leave the decimal point where it is.

EXAMPLE 10 Converting from Scientific Notation to Standard Notation

Write each number in standard notation.

(a) 6.93×10^{7}

$$6.\underbrace{9300000}_{\text{7 places}} \qquad \text{Attach 0s as necessary.}$$

We moved the decimal point 7 places to the right. (It was necessary to attach five 0s.)

$$6.93 \times 10^{7} = 69{,}300{,}000$$

(b) 4.7×10^{-6}

$$\underbrace{000004}_{\text{6 places}}.7 \qquad \text{Attach 0s as necessary.}$$

We moved the decimal point 6 places to the left.

$$4.7 \times 10^{-6} = .0000047$$

(c) $-1.083 \times 10^{0} = -1.083 \times 1 = -1.083$

Now Try Exercises 135, 137, and 139.

When problems require operations with numbers that are very large and/or very small, it is often advantageous to write the numbers in scientific notation first, and then perform the calculations using the rules for exponents.

EXAMPLE 11 Using Scientific Notation in Computation

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Evaluate $\dfrac{1{,}920{,}000 \times .0015}{.000032 \times 45{,}000}$.

First, express all numbers in scientific notation.

$$\frac{1{,}920{,}000 \times .0015}{.000032 \times 45{,}000} = \frac{1.92 \times 10^{6} \times 1.5 \times 10^{-3}}{3.2 \times 10^{-5} \times 4.5 \times 10^{4}}$$

Next, use the commutative and associative properties and the rules for exponents to simplify the expression.

$$\begin{aligned}\frac{1{,}920{,}000 \times .0015}{.000032 \times 45{,}000} &= \frac{1.92 \times 1.5 \times 10^{6} \times 10^{-3}}{3.2 \times 4.5 \times 10^{-5} \times 10^{4}}\\ &= \frac{1.92 \times 1.5 \times 10^{3}}{3.2 \times 4.5 \times 10^{-1}}\\ &= \frac{1.92 \times 1.5}{3.2 \times 4.5} \times 10^{4}\\ &= .2 \times 10^{4}\\ &= (2 \times 10^{-1}) \times 10^{4}\\ &= 2 \times 10^{3}\\ &= 2000\end{aligned}$$

Now Try Exercise 149.

EXAMPLE 12 Using Scientific Notation to Solve Problems

In 1990, the national health care expenditure was \$695.6 billion. By 2000, this figure had risen by a factor of 1.9; that is, it almost doubled in only 10 yr. (*Source:* U.S. Centers for Medicare & Medicaid Services.)

(a) Write the 1990 health care expenditure using scientific notation.

$$\begin{aligned}695.6 \text{ billion} = 695.6 \times 10^{9} &= (6.956 \times 10^{2}) \times 10^{9}\\ &= 6.956 \times 10^{11} \qquad \text{Product rule}\end{aligned}$$

In 1990, the expenditure was $\$6.956 \times 10^{11}$.

(b) What was the expenditure in 2000?
Multiply the result in part (a) by 1.9.

$$(6.956 \times 10^{11}) \times 1.9 = (1.9 \times 6.956) \times 10^{11}$$ Commutative and associative properties

$$= 13.216 \times 10^{11}$$ Round to three decimal places.

The 2000 expenditure was about $1,321,600,000,000, over $1 trillion.

Now Try Exercise 153.

5.1 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 7/Videotape 7

Decide whether each expression has been simplified correctly. If not, correct it.

1. $(ab)^2 = ab^2$

2. $y^2 \cdot y^6 = y^{12}$

3. $\left(\frac{4}{a}\right)^3 = \frac{4^3}{a} \quad (a \neq 0)$

4. $xy^0 = 0 \quad (y \neq 0)$

5. State the product rule for exponents in your own words. Give an example.

6. Your friend evaluated $4^5 \cdot 4^2$ as 16^7. Explain to him why his answer is incorrect.

Apply the product rule for exponents, if possible, in each case. See Example 1.

7. $13^4 \cdot 13^8$

8. $9^6 \cdot 9^4$

9. $x^3 \cdot x^5 \cdot x^9$

10. $y^4 \cdot y^5 \cdot y^6$

11. $(-3w^5)(9w^3)$

12. $(-5x^2)(3x^4)$

13. $(2x^2y^5)(9xy^3)$

14. $(8s^4t)(3s^3t^5)$

15. $r^2 \cdot s^4$

16. $p^3 \cdot q^2$

In Exercises 17 and 18, match the expression in Column I with its equivalent expression in Column II. Choices may be used once, more than once, or not at all. See Example 2.*

	I	II
17. (a)	9^0	**A.** 0
(b)	-9^0	**B.** 1
(c)	$(-9)^0$	**C.** -1
(d)	$-(-9)^0$	**D.** 9
		E. -9

	I	II
18. (a)	$2x^0$	**A.** 0
(b)	$-2x^0$	**B.** 1
(c)	$(2x)^0$	**C.** -1
(d)	$(-2x)^0$	**D.** 2
		E. -2

Evaluate. Assume all variables represent nonzero real numbers. See Example 2.*

19. 25^0

20. 14^0

21. -7^0

22. -10^0

23. $(-15)^0$

24. $(-20)^0$

25. $3^0 + (-3)^0$

26. $5^0 + (-5)^0$

27. $-3^0 + 3^0$

28. $-5^0 + 5^0$

29. $-4^0 - m^0$

30. $-8^0 - k^0$

*The authors thank Mitchel Levy of Broward Community College for his suggestions for these exercises.

In Exercises 31 and 32, match the expression in Column I with its equivalent expression in Column II. Choices may be used once, more than once, or not at all. See Example 3.*

	I	II
31.	**(a)** 4^{-2}	**A.** 16
	(b) -4^{-2}	**B.** $\frac{1}{16}$
	(c) $(-4)^{-2}$	**C.** -16
	(d) $-(-4)^{-2}$	**D.** $-\frac{1}{16}$

	I	II
32.	**(a)** 5^{-3}	**A.** 125
	(b) -5^{-3}	**B.** -125
	(c) $(-5)^{-3}$	**C.** $\frac{1}{125}$
	(d) $-(-5)^{-3}$	**D.** $-\frac{1}{125}$

Write each expression with only positive exponents. Assume all variables represent nonzero real numbers. In Exercises 45–48, simplify each expression. See Example 3.

33. 5^{-4} **34.** 7^{-2} **35.** 8^{-1} **36.** 12^{-1}

37. $(4x)^{-2}$ **38.** $(5t)^{-3}$ **39.** $4x^{-2}$ **40.** $5t^{-3}$

41. $-a^{-3}$ **42.** $-b^{-4}$ **43.** $(-a)^{-4}$ **44.** $(-b)^{-6}$

45. $5^{-1} + 6^{-1}$ **46.** $2^{-1} + 8^{-1}$ **47.** $8^{-1} - 3^{-1}$ **48.** $6^{-1} - 4^{-1}$

49. Consider the expressions $-a^n$ and $(-a)^n$. In some cases they are equal and in some cases they are not. Using $n = 2, 3, 4, 5$, and 6 and $a = 2$, draw a conclusion as to when they are equal and when they are opposites.

50. Your friend thinks that $(-3)^{-2}$ is a negative number. Why is she incorrect?

Evaluate each expression. See Examples 4 and 7.

51. $\dfrac{1}{4^{-2}}$ **52.** $\dfrac{1}{3^{-3}}$ **53.** $\dfrac{2^{-2}}{3^{-3}}$ **54.** $\dfrac{3^{-3}}{2^{-2}}$

55. $\left(\dfrac{2}{3}\right)^{-3}$ **56.** $\left(\dfrac{3}{2}\right)^{-3}$ **57.** $\left(\dfrac{4}{5}\right)^{-2}$ **58.** $\left(\dfrac{5}{4}\right)^{-2}$

*In Exercises 59 and 60, match the expression in Column I with its equivalent expression in Column II. Choices may be used once, more than once, or not at all.**

	I	II
59.	**(a)** $\left(\frac{1}{3}\right)^{-1}$	**A.** $\frac{1}{3}$
	(b) $\left(-\frac{1}{3}\right)^{-1}$	**B.** 3
	(c) $-\left(\frac{1}{3}\right)^{-1}$	**C.** $-\frac{1}{3}$
	(d) $-\left(-\frac{1}{3}\right)^{-1}$	**D.** -3

	I	II
60.	**(a)** $\left(\frac{2}{5}\right)^{-2}$	**A.** $\frac{25}{4}$
	(b) $\left(-\frac{2}{5}\right)^{-2}$	**B.** $-\frac{25}{4}$
	(c) $-\left(\frac{2}{5}\right)^{-2}$	**C.** $\frac{4}{25}$
	(d) $-\left(-\frac{2}{5}\right)^{-2}$	**D.** $-\frac{4}{25}$

61. State the quotient rule for exponents in your own words. Give an example.

62. State the three power rules for exponents in your own words. Give examples.

*The authors thank Mitchel Levy of Broward Community College for his suggestions for these exercises.

Apply the quotient rule for exponents, if applicable, and write each result using only positive exponents. Assume all variables represent nonzero real numbers. See Example 5.

63. $\dfrac{4^8}{4^6}$ **64.** $\dfrac{5^9}{5^7}$ **65.** $\dfrac{x^{12}}{x^8}$ **66.** $\dfrac{y^{14}}{y^{10}}$

67. $\dfrac{r^7}{r^{10}}$ **68.** $\dfrac{y^8}{y^{12}}$ **69.** $\dfrac{6^4}{6^{-2}}$ **70.** $\dfrac{7^5}{7^{-3}}$

71. $\dfrac{6^{-3}}{6^7}$ **72.** $\dfrac{5^{-4}}{5^2}$ **73.** $\dfrac{7}{7^{-1}}$ **74.** $\dfrac{8}{8^{-1}}$

75. $\dfrac{r^{-3}}{r^{-6}}$ **76.** $\dfrac{s^{-4}}{s^{-8}}$ **77.** $\dfrac{x^3}{y^2}$ **78.** $\dfrac{y^5}{t^3}$

Use one or more power rules to simplify each expression. Assume all variables represent nonzero real numbers. See Example 6.

79. $(x^3)^6$ **80.** $(y^5)^4$ **81.** $\left(\dfrac{3}{5}\right)^3$ **82.** $\left(\dfrac{4}{3}\right)^2$

83. $(4t)^3$ **84.** $(5t)^4$ **85.** $(-6x^2)^3$ **86.** $(-2x^5)^5$

87. $\left(\dfrac{-4m^2}{t}\right)^3$ **88.** $\left(\dfrac{-5n^4}{r^2}\right)^3$

Simplify each expression so that no negative exponents appear in the final result. Assume all variables represent nonzero real numbers. See Examples 1–8.

89. $3^5 \cdot 3^{-6}$ **90.** $4^4 \cdot 4^{-6}$ **91.** $a^{-3}a^2a^{-4}$ **92.** $k^{-5}k^{-3}k^4$

93. $(k^2)^{-3}k^4$ **94.** $(x^3)^{-4}x^5$ **95.** $-4r^{-2}(r^4)^2$ **96.** $-2m^{-1}(m^3)^2$

97. $(5a^{-1})^4(a^2)^{-3}$ **98.** $(3p^{-4})^2(p^3)^{-1}$ **99.** $(z^{-4}x^3)^{-1}$ **100.** $(y^{-2}z^4)^{-3}$

101. $7k^2(-2k)(4k^{-5})^0$ **102.** $3a^2(-5a^{-6})(-2a)^0$ **103.** $\dfrac{(p^{-2})^0}{5p^{-4}}$

104. $\dfrac{(m^4)^0}{9m^{-3}}$ **105.** $\dfrac{(3pq)q^2}{6p^2q^4}$ **106.** $\dfrac{(-8xy)y^3}{4x^5y^4}$

107. $\dfrac{4a^5(a^{-1})^3}{(a^{-2})^{-2}}$ **108.** $\dfrac{12k^{-2}(k^{-3})^{-4}}{6k^5}$ **109.** $\dfrac{(-y^{-4})^2}{6(y^{-5})^{-1}}$

110. $\dfrac{2(-m^{-1})^{-4}}{9(m^{-3})^2}$ **111.** $\dfrac{(2k)^2m^{-5}}{(km)^{-3}}$ **112.** $\dfrac{(3rs)^{-2}}{3^2r^2s^{-4}}$

113. $\dfrac{(2k)^2k^3}{k^{-1}k^{-5}}(5k^{-2})^{-3}$ **114.** $\dfrac{(3r^2)^2r^{-5}}{r^{-2}r^3}(2r^{-6})^2$ **115.** $\left(\dfrac{3k^{-2}}{k^4}\right)^{-1} \cdot \dfrac{2}{k}$

116. $\left(\dfrac{7m^{-2}}{m^{-3}}\right)^{-2} \cdot \dfrac{m^3}{4}$ **117.** $\left(\dfrac{2p}{q^2}\right)^3\left(\dfrac{3p^4}{q^{-4}}\right)^{-1}$ **118.** $\left(\dfrac{5z^3}{2a^2}\right)^{-3}\left(\dfrac{8a^{-1}}{15z^{-2}}\right)^{-3}$

119. $\dfrac{2^2y^4(y^{-3})^{-1}}{2^5y^{-2}}$ **120.** $\dfrac{3^{-1}m^4(m^2)^{-1}}{3^2m^{-2}}$

121. $\dfrac{(2m^2p^3)^2(4m^2p)^{-2}}{(-3mp^4)^{-1}(2m^3p^4)^3}$ **122.** $\dfrac{(-5y^3z^4)^2(2yz^5)^{-2}}{10(y^4z)^3(3y^3z^2)^{-1}}$

123. $\dfrac{(-3y^3x^3)(-4y^4x^2)(x^2)^{-4}}{18x^3y^2(y^3)^3(x^3)^{-2}}$ **124.** $\dfrac{(2m^3x^2)^{-1}(3m^4x)^{-3}}{(m^2x^3)^3(m^2x)^{-5}}$

125. $\left(\frac{p^2q^{-1}}{2p^{-2}}\right)^2 \cdot \left(\frac{p^3 \cdot 4q^{-2}}{3q^{-5}}\right)^{-1} \cdot \left(\frac{pq^{-5}}{q^{-2}}\right)^3$ **126.** $\left(\frac{a^6b^{-2}}{2a^{-2}}\right)^{-1} \cdot \left(\frac{6a^{-2}}{5b^{-4}}\right)^2 \cdot \left(\frac{2b^{-1}a^2}{3b^{-2}}\right)^{-1}$

Write each number in scientific notation. See Example 9.

127. 530 **128.** 1600 **129.** .830 **130.** .0072

131. .00000692 **132.** .875 **133.** $-38{,}500$ **134.** $-976{,}000{,}000$

Write each number in standard notation. See Example 10.

135. 7.2×10^4 **136.** 8.91×10^2 **137.** 2.54×10^{-3}

138. 5.42×10^{-4} **139.** -6×10^4 **140.** -9×10^3

141. 1.2×10^{-5} **142.** 2.7×10^{-6}

Find each value. See Example 11.

143. $\frac{12 \times 10^4}{2 \times 10^6}$ **144.** $\frac{16 \times 10^5}{4 \times 10^8}$ **145.** $\frac{3 \times 10^{-2}}{12 \times 10^3}$

146. $\frac{5 \times 10^{-3}}{25 \times 10^2}$ **147.** $\frac{.05 \times 1600}{.0004}$ **148.** $\frac{.003 \times 40{,}000}{.00012}$

149. $\frac{20{,}000 \times .018}{300 \times .0004}$ **150.** $\frac{840{,}000 \times .03}{.00021 \times 600}$

Solve each problem. See Example 12.

151. The U.S. budget didn't pass **\$1,000,000,000** until 1917. Seventy years later (in 1987) it exceeded **\$1,000,000,000,000** for the first time. President George W. Bush's budget request for fiscal 2003 was **\$2,128,000,000,000.** If stacked in dollar bills, this amount would stretch **144,419** mi, almost two-thirds of the distance to the moon. Write the four boldfaced numbers in scientific notation. (*Source: The Gazette*, February 5, 2002.)

152. In 1970, Wal-Mart had **1500** employees. In 1997, Wal-Mart became the largest private employer in the United States, with **680,000** employees. In 1999, Wal-Mart became the largest private employer in the world, with **1,100,000** employees. By 2007, the company is expected to have **2,200,000** employees. Write these four numbers in scientific notation. (*Source:* Wal-Mart.)

Photo not available

153. On October 28, 1998, IBM announced a computer capable of 3.9×10^8 operations per second. This was 15,000 times faster than the normal desktop computer at that time. What was the number of operations that the normal desktop could do? (*Source:* IBM.)

154. In the early years of the Powerball Lottery, a player would choose five numbers from 1 through 49 and one number from 1 through 42. It can be shown that there are about 8.009×10^7 different ways to do this. Suppose that a group of 2000 persons decided to purchase tickets for all these numbers and each ticket cost \$1.00. How much should each person have expected to pay? (*Source:* www.powerball.com)

155. The speed of light is approximately 3×10^{10} cm per sec. How long will it take light to travel 9×10^{12} cm?

156. The average distance from Earth to the sun is 9.3×10^7 mi. How long would it take a rocket, traveling at 2.9×10^3 mph, to reach the sun?

157. A *light-year* is the distance that light travels in one year. Find the number of miles in a light-year if light travels 1.86×10^5 mi per sec.

158. Use the information given in the previous two exercises to find the number of minutes necessary for light from the sun to reach Earth.

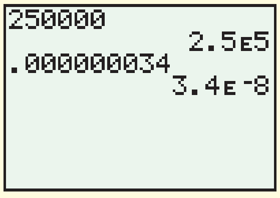
159. **(a)** The planet Mercury has an average distance from the sun of 3.6×10^7 mi, while the average distance of Venus to the sun is 6.7×10^7 mi. How long would it take a spacecraft traveling at 1.55×10^3 mph to travel from Venus to Mercury? (Give your answer in hours, in standard notation.)

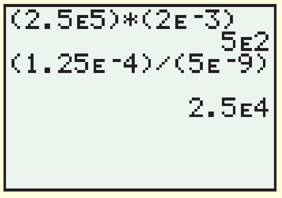
(b) Use the information from part (a) to find the number of days it would take the spacecraft to travel from Venus to Mercury. Round your answer to the nearest whole number of days.

160. When the distance between the centers of the moon and Earth is 4.60×10^8 m, an object on the line joining the centers of the moon and Earth exerts the same gravitational force on each when it is 4.14×10^8 m from the center of Earth. How far is the object from the center of the moon at that point?

TECHNOLOGY INSIGHTS (EXERCISES 161–164)

The screen on the left shows how a graphing calculator displays 250,000 *and* .000000034 *in scientific notation. When put in scientific mode, it will calculate and display results as shown in the screen on the right.*





Predict the result the calculator will give for each screen. (Use the usual scientific notation to write your answers.)

161.
```
(1.5E12)*(5E-3)
```

162.
```
(3.2E-5)*(3E12)
```

163.
```
(8.4E14)/(2.1E-3
)
```

164.
```
(2.5E10)/(2E-3)
```

5.2 Adding and Subtracting Polynomials

OBJECTIVES

1. Know the basic definitions for polynomials.
2. Find the degree of a polynomial.
3. Add and subtract polynomials.

OBJECTIVE 1 Know the basic definitions for polynomials. Just as whole numbers are the basis of arithmetic, *polynomials* are fundamental in algebra. To understand polynomials, we review several words from Chapter 1. A *term* is a number, a variable, or the product or quotient of a number and one or more variables raised to powers. Examples of terms include

$$4x, \quad \frac{1}{2}m^5 \left(\text{or } \frac{m^5}{2}\right), \quad -7z^9, \quad 6x^2z, \quad \frac{5}{3x^2}, \quad \text{and} \quad 9.$$

Terms

The number in the product is called the *numerical coefficient*, or just the *coefficient*. In the term $8x^3$, the coefficient is 8. In the term $-4p^5$, it is -4. The coefficient of the term k is understood to be 1. The coefficient of $-r$ is -1. In the term $\frac{x}{3}$, the coefficient is $\frac{1}{3}$ since $\frac{x}{3} = \frac{1x}{3} = \frac{1}{3}x$. More generally, any factor in a term is the coefficient of the product of the remaining factors. For example, $3x^2$ is the coefficient of y in the term $3x^2y$, and $3y$ is the coefficient of x^2 in $3x^2y$.

Recall that any combination of variables or constants (numerical values) joined by the basic operations of addition, subtraction, multiplication, and division (except by 0), or raising to powers or taking roots is called an *algebraic expression*. The simplest kind of algebraic expression is a *polynomial*.

AUDIO

Polynomial

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents and no variables appear in denominators.

Examples of polynomials include

$$3x - 5, \quad 4m^3 - 5m^2p + 8, \quad \text{and} \quad -5t^2s^3.$$

Polynomials

Even though the expression $3x - 5$ involves subtraction, it is a sum of terms since it could be written as $3x + (-5)$.

Some examples of expressions that are not polynomials are

$$x^{-1} + 3x^{-2}, \quad \sqrt{9 - x}, \quad \text{and} \quad \frac{1}{x}. \qquad \text{Not polynomials}$$

The first of these is not a polynomial because it has negative integer exponents, the second because it involves a variable under a radical, and the third because it contains a variable in the denominator.

Most of the polynomials used in this book contain only one variable. A polynomial containing only the variable x is called a **polynomial in x.** A polynomial in one variable is written in **descending powers** of the variable if the exponents on the variable decrease from left to right. For example,

$$x^5 - 6x^2 + 12x - 5$$

is a polynomial in descending powers of x. The term -5 in this polynomial can be thought of as $-5x^0$, since $-5x^0 = -5(1) = -5$.

EXAMPLE 1 Writing Polynomials in Descending Powers

Write each polynomial in descending powers of the variable.

(a) $y - 6y^3 + 8y^5 - 9y^4 + 12$

Write the polynomial as

$$8y^5 - 9y^4 - 6y^3 + y + 12.$$

(b) $-2 + m + 6m^2 - 4m^3$ is written as $-4m^3 + 6m^2 + m - 2$.

Now Try Exercise 1.

Some polynomials with a specific number of terms are so common that they are given special names. A polynomial with exactly three terms is a **trinomial,** and a polynomial with exactly two terms is a **binomial.** A single-term polynomial is a **monomial.** The table gives examples.

Type of Polynomial	*Examples*
Monomial	$5x, \quad 7m^9, \quad -8, \quad x^2y^2$
Binomial	$3x^2 - 6, \quad 11y + 8, \quad 5a^2b + 3a$
Trinomial	$y^2 + 11y + 6, \quad 8p^3 - 7p + 2m, \quad -3 + 2k^5 + 9z^4$
None of these	$p^3 - 5p^2 + 2p - 5, \quad -9z^3 + 5c^3 + 2m^5 + 11r^2 - 7r$

EXPLORE

OBJECTIVE 2 Find the degree of a polynomial. The **degree of a term** with one variable is the exponent on the variable. For example, the degree of $2x^3$ is 3, the degree of $-x^4$ is 4, and the degree of $17x$ (that is, $17x^1$) is 1. The degree of a term with more than one variable is defined to be the sum of the exponents on the variables. For example, the degree of $5x^3y^7$ is 10, because $3 + 7 = 10$.

The greatest degree of any term in a polynomial is called the **degree of the polynomial.** In most cases, we will be interested in finding the degree of a polynomial in one variable. For example, $4x^3 - 2x^2 - 3x + 7$ has degree 3, because the greatest degree of any term is 3 (the degree of $4x^3$).

The table shows several polynomials and their degrees.

Polynomial	*Degree*
$9x^2 - 5x + 8$	2
$17m^9 + 18m^{14} - 9m^3$	14
$5x$	1, because $5x = 5x^1$
-2	0, because $-2 = -2x^0$ (Any nonzero constant has degree 0.)
$5a^2b^5$	7, because $2 + 5 = 7$
$x^3y^9 + 12xy^4 + 7xy$	12, because the degrees of the terms are 12, 5, and 2; 12 is the greatest.

NOTE The number 0 has no degree, since 0 times a variable to any power is 0.

Now Try Exercises 21, 25, and 27.

OBJECTIVE 3 Add and subtract polynomials. We use the distributive property to simplify polynomials by combining terms. For example,

$$\begin{aligned} x^3 + 4x^2 + 5x^2 - 1 &= x^3 + (4 + 5)x^2 - 1 && \text{Distributive property} \\ &= x^3 + 9x^2 - 1. \end{aligned}$$

On the other hand, the terms in the polynomial $4x + 5x^2$ cannot be combined. As these examples suggest, only terms containing exactly the same variables to the same powers may be combined. Recall that such terms are called *like terms*.

CAUTION Remember that only *like terms* can be combined.

EXAMPLE 2 Combining Like Terms

Combine like terms.

(a) $$\begin{aligned} -5y^3 + 8y^3 - y^3 &= (-5 + 8 - 1)y^3 && \text{Distributive property} \\ &= 2y^3 \end{aligned}$$

(b) $$\begin{aligned} 6x + 5y - 9x + 2y &= 6x - 9x + 5y + 2y && \text{Associative and commutative properties} \\ &= -3x + 7y && \text{Combine like terms.} \end{aligned}$$

Since $-3x$ and $7y$ are unlike terms, no further simplification is possible.

(c) $$\begin{aligned} 5x^2y - 6xy^2 + 9x^2y + 13xy^2 &= 5x^2y + 9x^2y - 6xy^2 + 13xy^2 \\ &= 14x^2y + 7xy^2 \end{aligned}$$

Now Try Exercises 31, 37, and 43.

We use the following rule to add two polynomials.

Adding Polynomials

To add two polynomials, combine like terms.

Polynomials can be added horizontally or vertically, as seen in the next example.

EXAMPLE 3 Adding Polynomials

Add: $(3a^5 - 9a^3 + 4a^2) + (-8a^5 + 8a^3 + 2)$.

Use the commutative and associative properties to rearrange the polynomials so that like terms are together. Then use the distributive property to combine like terms.

$$\begin{aligned}
&(3a^5 - 9a^3 + 4a^2) + (-8a^5 + 8a^3 + 2) \\
&\quad = 3a^5 - 8a^5 - 9a^3 + 8a^3 + 4a^2 + 2 \\
&\quad = -5a^5 - a^3 + 4a^2 + 2 \qquad \text{Combine like terms.}
\end{aligned}$$

Add these same two polynomials vertically by placing like terms in columns.

$$\begin{array}{rrrr}
3a^5 & -\ 9a^3 & +\ 4a^2 & \\
-8a^5 & +\ 8a^3 & & +\ 2 \\
\hline
-5a^5 & -\ \ a^3 & +\ 4a^2 & +\ 2
\end{array}$$

Now Try Exercises 51 and 65.

In Chapter 1, we defined subtraction of real numbers as

$$a - b = a + (-b).$$

That is, we add the first number and the negative (or opposite) of the second. We can give a similar definition for subtraction of polynomials by defining the **negative of a polynomial** as that polynomial with the sign of every coefficient changed.

Subtracting Polynomials

To subtract two polynomials, add the first polynomial and the negative of the *second* polynomial.

EXAMPLE 4 Subtracting Polynomials

Subtract: $(-6m^2 - 8m + 5) - (-5m^2 + 7m - 8)$.

Change every sign in the second polynomial and add.

$$\begin{aligned}
&(-6m^2 - 8m + 5) - (-5m^2 + 7m - 8) \\
&\quad = -6m^2 - 8m + 5 + 5m^2 - 7m + 8 \\
&\quad = -6m^2 + 5m^2 - 8m - 7m + 5 + 8 \qquad \text{Rearrange terms.} \\
&\quad = -m^2 - 15m + 13 \qquad \text{Combine like terms.}
\end{aligned}$$

Check by adding the sum, $-m^2 - 15m + 13$, to the second polynomial. The result should be the first polynomial.

To subtract these two polynomials vertically, write the first polynomial above the second, lining up like terms in columns.

$$\begin{array}{rrr}
-6m^2 & -\ 8m & +\ 5 \\
-5m^2 & +\ 7m & -\ 8 \\
\hline
\end{array}$$

Change all the signs in the second polynomial and add.

$$\begin{array}{rrrl}
-6m^2 & -\ \ 8m & +\ \ 5 & \\
+\ 5m^2 & -\ \ 7m & +\ \ 8 & \quad \text{Change all signs.} \\
\hline
-m^2 & -\ 15m & +\ 13 & \quad \text{Add in columns.}
\end{array}$$

Now Try Exercises 61 and 69.

5.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 7/Videotape 7

Write each polynomial in descending powers of the variable. See Example 1.

1. $2x^3 + x - 3x^2 + 4$ **2.** $3y^2 + y^4 - 2y^3 + y$ **3.** $4p^3 - 8p^5 + p^7$

4. $q^2 + 3q^4 - 2q + 1$ **5.** $-m^3 + 5m^2 + 3m^4 + 10$ **6.** $4 - x + 3x^2$

Give the numerical coefficient and the degree of each term.

7. $7z$ **8.** $3r$ **9.** $-15p^2$ **10.** $-27k^3$ **11.** x^4

12. y^6 **13.** $\frac{t}{6}$ **14.** $\frac{m}{4}$ **15.** $-mn^5$ **16.** $-a^5b$

Identify each polynomial as a monomial, binomial, trinomial, *or* none of these. *Also give the degree.*

17. 25 **18.** 5 **19.** $7m - 22$

20. $6x + 15$ **21.** $-7y^6 + 11y^8$ **22.** $12k^2 - 9k^5$

23. $-5m^3 + 6m - 9m^2$ **24.** $4z^2 - 11z + 2$

25. $-6p^4q - 3p^3q^2 + 2pq^3 - q^4$ **26.** $8s^3t - 4s^2t^2 + 2st^3 + 9$

27. Which one of the following is a trinomial in descending powers, having degree 6?

A. $5x^6 - 4x^5 + 12$ **B.** $6x^5 - x^6 + 4$
C. $2x + 4x^2 - x^6$ **D.** $4x^6 - 6x^4 + 9x^2 - 8$

28. Give an example of a polynomial of four terms in the variable x, having degree 5, written in descending powers, lacking a fourth-degree term.

Combine terms. See Example 2.

29. $5z^4 + 3z^4$ **30.** $8r^5 - 2r^5$ **31.** $-m^3 + 2m^3 + 6m^3$

32. $3p^4 + 5p^4 - 2p^4$ **33.** $x + x + x + x + x$ **34.** $z - z - z + z$

35. $m^4 - 3m^2 + m$ **36.** $5a^5 + 2a^4 - 9a^3$ **37.** $y^2 + 7y - 4y^2$

38. $2c^2 - 4 + 8 - c^2$ **39.** $2k + 3k^2 + 5k^2 - 7$ **40.** $4x^2 + 2x - 6x^2 - 6$

41. $n^4 - 2n^3 + n^2 - 3n^4 + n^3$ **42.** $2q^3 + 3q^2 - 4q - q^3 + 5q^2$

43. $3ab^2 + 7a^2b - 5ab^2 + 13a^2b$ **44.** $6m^2n - 8mn^2 + 3mn^2 - 7m^2n$

45. $4 - (2 + 3m) + 6m + 9$ **46.** $8a - (3a + 4) - (5a - 3)$

47. $6 + 3p - (2p + 1) - (2p + 9)$ **48.** $4x - 8 - (-1 + x) - (11x + 5)$

49. Define *polynomial* in your own words. Give examples. Include the words *term*, *monomial*, *binomial*, and *trinomial* in your explanation.

50. Write a paragraph explaining how to add and subtract polynomials. Give examples.

Add or subtract as indicated. See Examples 3 and 4.

51. $(5x^2 + 7x - 4) + (3x^2 - 6x + 2)$ **52.** $(4k^3 + k^2 + k) + (2k^3 - 4k^2 - 3k)$

53. $(6t^2 - 4t^4 - t) + (3t^4 - 4t^2 + 5)$ **54.** $(3p^2 + 2p - 5) + (7p^2 - 4p^3 + 3p)$

55. $(y^3 + 3y + 2) + (4y^3 - 3y^2 + 2y - 1)$ **56.** $(2x^5 - 2x^4 + x^3 - 1) + (x^4 - 3x^3 + 2)$

57. $(3r + 8) - (2r - 5)$ **58.** $(2d + 7) - (3d - 1)$

59. $(2a^2 + 3a - 1) - (4a^2 + 5a + 6)$ **60.** $(q^4 - 2q^2 + 10) - (3q^4 + 5q^2 - 5)$

61. $(z^5 + 3z^2 + 2z) - (4z^5 + 2z^2 - 5z)$

62. $(5t^3 - 3t^2 + 2t) - (4t^3 + 2t^2 + 3t)$

63. Add.
$$\begin{array}{r} 21p - 8 \\ -9p + 4 \\ \hline \end{array}$$

64. Add.
$$\begin{array}{r} 15m - 9 \\ 4m + 12 \\ \hline \end{array}$$

65. Add.
$$\begin{array}{r} -12p^2 + 4p - 1 \\ 3p^2 + 7p - 8 \\ \hline \end{array}$$

66. Add.
$$\begin{array}{r} -6y^3 + 8y + 5 \\ 9y^3 + 4y - 6 \\ \hline \end{array}$$

67. Subtract.
$$\begin{array}{r} 12a + 15 \\ 7a - 3 \\ \hline \end{array}$$

68. Subtract.
$$\begin{array}{r} -3b + 6 \\ 2b - 8 \\ \hline \end{array}$$

69. Subtract.
$$\begin{array}{r} 6m^2 - 11m + 5 \\ -8m^2 + 2m - 1 \\ \hline \end{array}$$

70. Subtract.
$$\begin{array}{r} -4z^2 + 2z - 1 \\ 3z^2 - 5z + 2 \\ \hline \end{array}$$

71. Add.
$$\begin{array}{r} 12z^2 - 11z + 8 \\ 5z^2 + 16z - 2 \\ -4z^2 + 5z - 9 \\ \hline \end{array}$$

72. Add.
$$\begin{array}{r} -6m^3 + 2m^2 + 5m \\ 8m^3 + 4m^2 - 6m \\ -3m^3 + 2m^2 - 7m \\ \hline \end{array}$$

73. Add.
$$\begin{array}{rrrr} 6y^3 & - 9y^2 & & + 8 \\ 4y^3 & + 2y^2 & + 5y & \\ \hline \end{array}$$

74. Add.
$$\begin{array}{rrrr} -7r^8 & + 2r^6 & - r^5 & \\ & 3r^6 & & + 5 \\ \hline \end{array}$$

75. Subtract.
$$\begin{array}{rrrr} -5a^4 & & + 8a^2 & - 9 \\ & 6a^3 & - a^2 & + 2 \\ \hline \end{array}$$

76. Subtract.
$$\begin{array}{rrrr} & - 2m^3 & + 8m^2 & \\ m^4 & - m^3 & & + 2m \\ \hline \end{array}$$

Perform the indicated operations. See Examples 2–4.

77. Subtract $4y^2 - 2y + 3$ from $7y^2 - 6y + 5$.

78. Subtract $-(-4x + 2z^2 + 3m)$ from $[(2z^2 - 3x + m) + (z^2 - 2m)]$.

79. $(-4m^2 + 3n^2 - 5n) - [(3m^2 - 5n^2 + 2n) + (-3m^2) + 4n^2]$

80. $[-(4m^2 - 8m + 4m^3) - (3m^2 + 2m + 5m^3)] + m^2$

81. $[-(y^4 - y^2 + 1) - (y^4 + 2y^2 + 1)] + (3y^4 - 3y^2 - 2)$

82. $[2p - (3p - 6)] - [(5p - (8 - 9p)) + 4p]$

83. $-[3z^2 + 5z - (2z^2 - 6z)] + [(8z^2 - [5z - z^2]) + 2z^2]$

84. $5k - (5k - [2k - (4k - 8k)]) + 11k - (9k - 12k)$

5.3 Polynomial Functions

OBJECTIVES

1. Recognize and evaluate polynomial functions.
2. Use a polynomial function to model data.
3. Add and subtract polynomial functions.
4. Graph basic polynomial functions.

OBJECTIVE 1 Recognize and evaluate polynomial functions. In Chapter 3 we studied linear (first-degree polynomial) functions, defined as $f(x) = mx + b$. Now we consider more general polynomial functions.

Polynomial Function

A **polynomial function of degree *n*** is defined by

$$\boldsymbol{f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,}$$

for real numbers $a_n, a_{n-1}, \ldots, a_1$, and a_0, where $a_n \neq 0$ and n is a whole number.

Another way of describing a polynomial function is to say that it is a function defined by a polynomial in one variable, consisting of one or more terms. It is usually written in descending powers of the variable, and its degree is the degree of the polynomial that defines it.

Suppose that we consider the polynomial $3x^2 - 5x + 7$, so

$$f(x) = 3x^2 - 5x + 7.$$

If $x = -2$, then $f(x) = 3x^2 - 5x + 7$ takes on the value

$$\begin{aligned} f(-2) &= 3(-2)^2 - 5(-2) + 7 && \text{Let } x = -2. \\ &= 3(4) + 10 + 7 \\ &= 29. \end{aligned}$$

Thus, $f(-2) = 29$ and the ordered pair $(-2, 29)$ belongs to f.

EXAMPLE 1 Evaluating Polynomial Functions

Let $f(x) = 4x^3 - x^2 + 5$. Find each value.

(a) $f(3)$

$$\begin{aligned} f(x) &= 4x^3 - x^2 + 5 \\ f(3) &= 4(3)^3 - 3^2 + 5 && \text{Substitute 3 for } x. \\ &= 4(27) - 9 + 5 && \text{Order of operations} \\ &= 108 - 9 + 5 \\ &= 104 \end{aligned}$$

(b) $$\begin{aligned} f(-4) &= 4(-4)^3 - (-4)^2 + 5 && \text{Let } x = -4. \\ &= 4(-64) - 16 + 5 \\ &= -267 \end{aligned}$$

Now Try Exercise 3.

While f is the most common letter used to represent functions, recall that other letters such as g and h are also used. The capital letter P is often used for polynomial functions. Note that the function defined as $P(x) = 4x^3 - x^2 + 5$ yields the same ordered pairs as the function f in Example 1.

OBJECTIVE 2 Use a polynomial function to model data. Polynomial functions can be used to approximate data. They are usually valid for small intervals, and they allow us to predict (with caution) what might happen for values just outside the intervals. These intervals are often periods of years, as shown in Example 2.

EXAMPLE 2 Using a Polynomial Model to Approximate Data

The number of U.S. households estimated to see and pay at least one bill on-line each month during the years 2000 through 2006 can be modeled by the polynomial function defined by

$$P(x) = .808x^2 + 2.625x + .502,$$

where $x = 0$ corresponds to the year 2000, $x = 1$ corresponds to 2001, and so on, and $P(x)$ is in millions. Use this function to approximate the number of households expected to pay at least one bill on-line each month in 2004.

Since $x = 4$ corresponds to 2004, we must find $P(4)$.

$$P(x) = .808x^2 + 2.625x + .502$$

$$P(4) = .808(4)^2 + 2.625(4) + .502 \qquad \text{Let } x = 4.$$

$$= 23.93 \qquad \text{Evaluate.}$$

Thus, in 2004 about 23.93 million households are expected to pay at least one bill on-line each month.

Now Try Exercise 9.

OBJECTIVE 3 Add and subtract polynomial functions. The operations of addition, subtraction, multiplication, and division are also defined for functions. For example, businesses use the equation "profit equals revenue minus cost," written using function notation as

$$P(x) = R(x) - C(x),$$

where x is the number of items produced and sold. Thus, the profit function is found by subtracting the cost function from the revenue function.

We define the following **operations on functions.**

Adding and Subtracting Functions

If $f(x)$ and $g(x)$ define functions, then

$$(f + g)(x) = f(x) + g(x) \qquad \text{Sum}$$

and

$$(f - g)(x) = f(x) - g(x). \qquad \text{Difference}$$

In each case, the domain of the new function is the intersection of the domains of $f(x)$ and $g(x)$.

EXAMPLE 3 Adding and Subtracting Functions

For the polynomial functions defined by

$$f(x) = x^2 - 3x + 7 \quad \text{and} \quad g(x) = -3x^2 - 7x + 7,$$

find **(a)** the sum and **(b)** the difference.

(a) $(f + g)(x) = f(x) + g(x)$ — Use the definition.

$= (x^2 - 3x + 7) + (-3x^2 - 7x + 7)$ — Substitute.

$= -2x^2 - 10x + 14$ — Add the polynomials.

(b) $(f - g)(x) = f(x) - g(x)$ — Use the definition.

$= (x^2 - 3x + 7) - (-3x^2 - 7x + 7)$ — Substitute.

$= (x^2 - 3x + 7) + (3x^2 + 7x - 7)$ — Change subtraction to addition.

$= 4x^2 + 4x$ — Add.

Now Try Exercise 15.

YOU TRY IT

EXAMPLE 4 Adding and Subtracting Functions

For the functions defined by

$$f(x) = 10x^2 - 2x \quad \text{and} \quad g(x) = 2x,$$

find each of the following.

(a) $(f + g)(2)$

$$\begin{aligned} (f + g)(2) &= f(2) + g(2) && \text{Use the definition.} \\ &= [10(2)^2 - 2(2)] + 2(2) && \text{Substitute.} \\ &= 40 \end{aligned}$$

Alternatively, we could first find $(f + g)(x)$.

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) && \text{Use the definition.} \\ &= (10x^2 - 2x) + 2x && \text{Substitute.} \\ &= 10x^2 && \text{Combine like terms.} \end{aligned}$$

Then,

$$(f + g)(2) = 10(2)^2 = 40. \qquad \text{The result is the same.}$$

(b) $(f - g)(x)$ and $(f - g)(1)$

$$\begin{aligned} (f - g)(x) &= f(x) - g(x) && \text{Use the definition.} \\ &= (10x^2 - 2x) - 2x && \text{Substitute.} \\ &= 10x^2 - 4x && \text{Combine like terms.} \end{aligned}$$

Then,

$$(f - g)(1) = 10(1)^2 - 4(1) = 6. \qquad \text{Substitute.}$$

Confirm that $f(1) - g(1)$ gives the same result.

Now Try Exercises 17 and 19.

OBJECTIVE 4 Graph basic polynomial functions. Functions were introduced in Section 3.5. Recall that each input (or x-value) of a function results in one output (or y-value). The simplest polynomial function is the **identity function,** defined by $f(x) = x$. The domain (set of x-values) of this function is all real numbers, $(-\infty, \infty)$, and it pairs each real number with itself. Therefore, the range (set of y-values) is also $(-\infty, \infty)$. Its graph is a straight line, as first seen in Chapter 3. (Notice that a *linear function* is a specific kind of polynomial function.) Figure 1 shows the graph of $f(x) = x$ and a table of selected ordered pairs.

x	$f(x) = x$
-2	-2
-1	-1
0	0
1	1
2	2

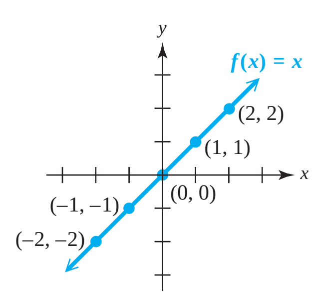


FIGURE 1

Another polynomial function, defined by $f(x) = x^2$, is the **squaring function.** For this function, every real number is paired with its square. The input can be any real number, so the domain is $(-\infty, \infty)$. Since the square of any real number is nonnegative, the range is $[0, \infty)$. Its graph is a *parabola*. Figure 2 shows the graph and a table of selected ordered pairs.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

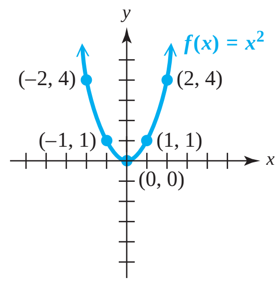


FIGURE 2

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8

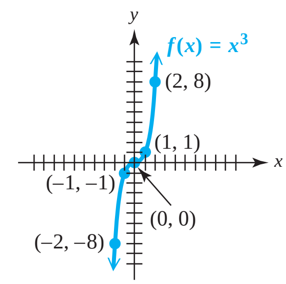


FIGURE 3

The **cubing function** is defined by $f(x) = x^3$. Every real number is paired with its cube. The domain and the range are both $(-\infty, \infty)$. Its graph is neither a line nor a parabola. See Figure 3 and the table of ordered pairs.

EXAMPLE 5 Graphing Variations of the Identity, Squaring, and Cubing Functions

Graph each function by creating a table of ordered pairs. Give the domain and range of each function by observing the graphs.

(a) $f(x) = 2x$

To find each range value, multiply the domain value by 2. Plot the points and join them with a straight line. See Figure 4. Both the domain and the range are $(-\infty, \infty)$.

x	$f(x) = 2x$
-2	-4
-1	-2
0	0
1	2
2	4

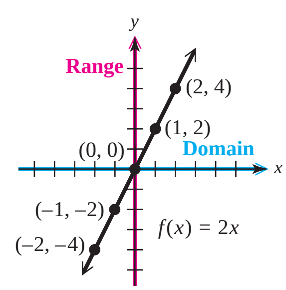


FIGURE 4

(b) $f(x) = -x^2$

For each input x, square it and then take its opposite. Plotting and joining the points gives a parabola that opens down. It is a *reflection* of the graph of the squaring function. See the table and Figure 5 on the next page. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0]$.

x	$f(x) = -x^2$
-2	-4
-1	-1
0	0
1	-1
2	-4

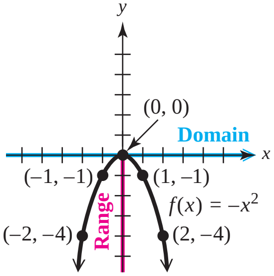


FIGURE 5

x	$f(x) = x^3 - 2$
-2	-10
-1	-3
0	-2
1	-1
2	6

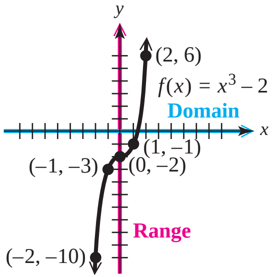


FIGURE 6

(c) $f(x) = x^3 - 2$

For this function, cube the input and then subtract 2 from the result. The graph is that of the cubing function *shifted* 2 units down. See the table and Figure 6. The domain and range are both $(-\infty, \infty)$.

Now Try Exercises 31, 33, and 35.

5.3 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 7/Videotape 7

For each polynomial function, find **(a)** $f(-1)$ *and* **(b)** $f(2)$. *See Example 1.*

1. $f(x) = 6x - 4$

2. $f(x) = -2x + 5$

3. $f(x) = x^2 - 3x + 4$

4. $f(x) = 3x^2 + x - 5$

5. $f(x) = 5x^4 - 3x^2 + 6$

6. $f(x) = -4x^4 + 2x^2 - 1$

7. $f(x) = -x^2 + 2x^3 - 8$

8. $f(x) = -x^2 - x^3 + 11x$

Solve each problem. See Example 2.

9. The number of airports in the United States during the period from 1970 through 1997 can be approximated by the polynomial function defined by

$$f(x) = -6.77x^2 + 445.34x + 11{,}279.82,$$

where $x = 0$ represents 1970, $x = 1$ represents 1971, and so on. Use this function to approximate the number of airports in each given year. (*Source:* U.S. Federal Aviation Administration.)

(a) 1970 **(b)** 1985 **(c)** 1997

Photo not available

10. The number of cases commenced by U.S. Courts of Appeals during the period from 1990 through 1998 can be approximated by the polynomial function defined by

$$f(x) = -145.32x^2 + 2610.84x + 41{,}341.13,$$

where $x = 0$ represents 1990, $x = 1$ represents 1991, and so on. Use this function to approximate the number of cases commenced in each given year. (*Source:* Administrative Office of the U.S. Courts, *Statistical Tables for the Federal Judiciary,* annual.)

(a) 1993 **(b)** 1995 **(c)** 1997

Photo not available

11. The number of bank debit cards issued during the period from 1990 through 2000 can be modeled by the polynomial function defined by

$$P(x) = -.31x^3 + 5.8x^2 - 15x + 9,$$

where $x = 0$ corresponds to the year 1990, $x = 1$ corresponds to 1991, and so on, and $P(x)$ is in millions. Use this function to approximate the number of bank debit cards issued in each given year. Round answers to the nearest million. (*Source: Statistical Abstract of the United States*, 2000.)

(a) 1990 **(b)** 1996 **(c)** 1999

12. The number of military personnel on active duty in the United States during the period from 1990 through 1998 is approximated by the polynomial function defined by

$$P(x) = .0045x^3 - .072x^2 + .19x + 3.7,$$

where $x = 0$ corresponds to 1990, $x = 1$ corresponds to 1991, and so on, and $P(x)$ is in millions. Based on this model, how many military personnel were on active duty in each given year? Round answers to the nearest tenth of a million. (*Source:* U.S. Department of Defense.)

(a) 1990 **(b)** 1994 **(c)** 1998

For each pair of functions, find ***(a)*** $(f + g)(x)$ *and* ***(b)*** $(f - g)(x)$. *See Example 3.*

13. $f(x) = 5x - 10$, $g(x) = 3x + 7$

14. $f(x) = -4x + 1$, $g(x) = 6x + 2$

15. $f(x) = 4x^2 + 8x - 3$, $g(x) = -5x^2 + 4x - 9$

16. $f(x) = 3x^2 - 9x + 10$, $g(x) = -4x^2 + 2x + 12$

Let $f(x) = x^2 - 9$, $g(x) = 2x$, *and* $h(x) = x - 3$. *Find each of the following. See Example 4.*

17. $(f + g)(x)$ **18.** $(f - g)(x)$ **19.** $(f + g)(3)$

20. $(f - g)(-3)$ **21.** $(f - h)(x)$ **22.** $(f + h)(x)$

23. $(f - h)(-3)$ **24.** $(f + h)(-2)$ **25.** $(g + h)(-10)$

26. $(g - h)(10)$ **27.** $(g - h)(-3)$ **28.** $(g + h)\left(\frac{1}{3}\right)$

29. Construct two polynomial functions defined by $f(x)$, a polynomial of degree 3, and $g(x)$, a polynomial of degree 4. Find $(f - g)(x)$ and $(g - f)(x)$. Use your answers to decide whether subtraction of polynomial functions is a commutative operation. Explain.

30. Find two polynomial functions defined by $f(x)$ and $g(x)$ such that

$$(f + g)(x) = 3x^3 - x + 3.$$

Graph each function. Give the domain and range. See Example 5.

31. $f(x) = -2x + 1$ **32.** $f(x) = 3x + 2$ **33.** $f(x) = -3x^2$

34. $f(x) = \frac{1}{2}x^2$ **35.** $f(x) = x^3 + 1$ **36.** $f(x) = -x^3 + 2$

5.4 Multiplying Polynomials

OBJECTIVES

1. Multiply terms.
2. Multiply any two polynomials.
3. Multiply binomials.
4. Find the product of the sum and difference of two terms.
5. Find the square of a binomial.
6. Multiply polynomial functions.

OBJECTIVE 1 Multiply terms. Recall that the product of the two terms $3x^4$ and $5x^3$ is found by using the commutative and associative properties, along with the rules for exponents.

$$\begin{aligned}(3x^4)(5x^3) &= 3 \cdot 5 \cdot x^4 \cdot x^3\\ &= 15x^{4+3}\\ &= 15x^7\end{aligned}$$

EXAMPLE 1 Multiplying Monomials

Find each product.

(a) $-4a^3(3a^5) = -4(3)a^3 \cdot a^5 = -12a^8$

(b) $2m^2z^4(8m^3z^2) = 2(8)m^2 \cdot m^3 \cdot z^4 \cdot z^2 = 16m^5z^6$

Now Try Exercises 5 and 7.

OBJECTIVE 2 Multiply any two polynomials. We use the distributive property to extend this process to find the product of any two polynomials.

EXAMPLE 2 Multiplying Polynomials

Find each product.

(a) $-2(8x^3 - 9x^2)$

$$\begin{aligned}-2(8x^3 - 9x^2) &= -2(8x^3) - 2(-9x^2) \quad \text{Distributive property}\\ &= -16x^3 + 18x^2\end{aligned}$$

(b) $$\begin{aligned}5x^2(-4x^2 + 3x - 2) &= 5x^2(-4x^2) + 5x^2(3x) + 5x^2(-2)\\ &= -20x^4 + 15x^3 - 10x^2\end{aligned}$$

(c) $(3x - 4)(2x^2 + x)$

Use the distributive property to multiply each term of $2x^2 + x$ by $3x - 4$.

$$(3x - 4)(2x^2 + x) = (3x - 4)(2x^2) + (3x - 4)(x)$$

Here $3x - 4$ has been treated as a single expression so that the distributive property could be used. Now use the distributive property two more times.

$$\begin{aligned}&= 3x(2x^2) + (-4)(2x^2) + (3x)(x) + (-4)(x)\\ &= 6x^3 - 8x^2 + 3x^2 - 4x\\ &= 6x^3 - 5x^2 - 4x\end{aligned}$$

(d) $$\begin{aligned}2x^2(x + 1)(x - 3) &= 2x^2[(x + 1)(x) + (x + 1)(-3)]\\ &= 2x^2[x^2 + x - 3x - 3]\\ &= 2x^2(x^2 - 2x - 3)\\ &= 2x^4 - 4x^3 - 6x^2\end{aligned}$$

Now Try Exercises 11, 13, and 19.

It is often easier to multiply polynomials by writing them vertically. To find the product $(3x - 4)(2x^2 + x)$ from Example 2(c) vertically, proceed as follows. (Notice how this process is similar to that of finding the product of two numbers, such as 24×78.)

1. Multiply x and $3x - 4$.

$$\begin{array}{rr} & 3x - 4 \\ & 2x^2 + x \\ \hline x(3x-4) \rightarrow & 3x^2 - 4x \end{array}$$

2. Multiply $2x^2$ and $3x - 4$. Line up like terms of the products in columns.

$$\begin{array}{rrrr} & & 3x & -\ 4 \\ & & 2x^2 & +\ x \\ \hline & & 3x^2 & -\ 4x \\ 2x^2(3x-4) \rightarrow & 6x^3 & -\ 8x^2 & \end{array}$$

3. Combine like terms.

$$6x^3 - 5x^2 - 4x$$

EXAMPLE 3 Multiplying Polynomials Vertically

Find each product.

(a) $(5a - 2b)(3a + b)$

$$\begin{array}{rrrl} & 5a & -\ 2b & \\ & 3a & +\ b & \\ \hline & 5ab & -\ 2b^2 & \leftarrow b(5a - 2b) \\ 15a^2 & -\ 6ab & & \leftarrow 3a(5a - 2b) \\ \hline 15a^2 & -\ ab & -\ 2b^2 & \text{Combine like terms.} \end{array}$$

(b) $(3m^3 - 2m^2 + 4)(3m - 5)$

$$\begin{array}{rrrrrl} & 3m^3 & -\ 2m^2 & +\ 4 & & \\ & & 3m & -\ 5 & & \\ \hline & -15m^3 & +\ 10m^2 & & -\ 20 & -5(3m^3 - 2m^2 + 4) \\ 9m^4 & -\ 6m^3 & & +\ 12m & & 3m(3m^3 - 2m^2 + 4) \\ \hline 9m^4 & -\ 21m^3 & +\ 10m^2 & +\ 12m & -\ 20 & \text{Combine like terms.} \end{array}$$

Now Try Exercise 25.

NOTE We can also use a rectangle to model polynomial multiplication. For example, to find the product

$$(5a - 2b)(3a + b),$$

from Example 3(a), label a rectangle with each term as shown here. Then put the product of each pair of monomials in the appropriate box.

	$3a$	b
$5a$		
$-2b$		

	$3a$	b
$5a$	$15a^2$	$5ab$
$-2b$	$-6ab$	$-2b^2$

The product of the original binomials is the sum of these four monomial products.

$$\begin{aligned}(5a - 2b)(3a + b) &= 15a^2 + 5ab - 6ab - 2b^2 \\ &= 15a^2 - ab - 2b^2\end{aligned}$$

OBJECTIVE 3 Multiply binomials. When working with polynomials, the product of two binomials occurs repeatedly. There is a shortcut method for finding these products. Recall that a binomial has just two terms, such as $3x - 4$ or $2x + 3$. We can find the product of these binomials using the distributive property as follows.

$$\begin{aligned}(3x - 4)(2x + 3) &= 3x(2x + 3) - 4(2x + 3)\\ &= 3x(2x) + 3x(3) - 4(2x) - 4(3)\\ &= 6x^2 + 9x - 8x - 12\end{aligned}$$

Before combining like terms to find the simplest form of the answer, we check the origin of each of the four terms in the sum. First, $6x^2$ is the product of the two *first* terms.

$$(3x - 4)(2x + 3) \qquad 3x(2x) = 6x^2 \qquad \text{First terms}$$

To get $9x$, the *outer* terms are multiplied.

$$(3x - 4)(2x + 3) \qquad 3x(3) = 9x \qquad \text{Outer terms}$$

The term $-8x$ comes from the *inner* terms.

$$(3x - 4)(2x + 3) \qquad -4(2x) = -8x \qquad \text{Inner terms}$$

Finally, -12 comes from the *last* terms.

$$(3x - 4)(2x + 3) \qquad -4(3) = -12 \qquad \text{Last terms}$$

The product is found by combining these four results.

$$\begin{aligned}(3x - 4)(2x + 3) &= 6x^2 + 9x + (-8x) + (-12)\\ &= 6x^2 + x - 12\end{aligned}$$

To keep track of the order of multiplying these terms, we use the initials $\boxed{\text{FOIL}}$ (First, Outer, Inner, Last). All the steps of the FOIL method can be done as follows. Try to do as many of these steps as possible mentally.

First Last
$(3x - 4)(2x + 3)$
Inner
Outer

$6x^2$ -12
$(3x - 4)(2x + 3)$
$-8x$
$9x$
x Add.

EXAMPLE 4 Using the FOIL Method

Use the FOIL method to find each product.

(a) $(4m - 5)(3m + 1)$

First terms	$(4m - 5)(3m + 1)$	$4m(3m) = 12m^2$
Outer terms	$(4m - 5)(3m + 1)$	$4m(1) = 4m$
Inner terms	$(4m - 5)(3m + 1)$	$-5(3m) = -15m$
Last terms	$(4m - 5)(3m + 1)$	$-5(1) = -5$

Simplify by combining the four terms.

$$\begin{aligned}(4m-5)(3m+1) &= \overset{\text{F}}{12m^2} + \overset{\text{O}}{4m} - \overset{\text{I}}{15m} - \overset{\text{L}}{5}\\ &= 12m^2 - 11m - 5\end{aligned}$$

The procedure can be written in compact form as follows.

$12m^2$ $\quad -5$

$$(4m-5)(3m+1)$$

$-15m$

$4m$

$-11m$ Add.

Combine these four results to get $12m^2 - 11m - 5$.

$$\textbf{(b)}\ (6a-5b)(3a+4b) = \overset{\text{First}}{18a^2} + \overset{\text{Outer}}{24ab} - \overset{\text{Inner}}{15ab} - \overset{\text{Last}}{20b^2}$$
$$= 18a^2 + 9ab - 20b^2$$

$$\textbf{(c)}\ (2k+3z)(5k-3z) = 10k^2 - 6kz + 15kz - 9z^2 \qquad \text{FOIL}$$
$$= 10k^2 + 9kz - 9z^2$$

Now Try Exercises 35 and 39.

OBJECTIVE 4 Find the product of the sum and difference of two terms. Some types of binomial products occur frequently. The product of the sum and difference of the same two terms, x and y, is

$$\begin{aligned}(x+y)(x-y) &= x^2 - xy + xy - y^2 \qquad \text{FOIL}\\ &= x^2 - y^2.\end{aligned}$$

Product of the Sum and Difference of Two Terms

The **product of the sum and difference of the two terms** x **and** y is the difference of the squares of the terms.

$$(x+y)(x-y) = x^2 - y^2$$

EXAMPLE 5 Multiplying the Sum and Difference of Two Terms

Find each product.

$$\textbf{(a)}\ (p+7)(p-7) = p^2 - 7^2$$
$$= p^2 - 49$$

$$\textbf{(b)}\ (2r+5)(2r-5) = (2r)^2 - 5^2$$
$$= 2^2r^2 - 25$$
$$= 4r^2 - 25$$

$$\textbf{(c)}\ (6m+5n)(6m-5n) = (6m)^2 - (5n)^2$$
$$= 36m^2 - 25n^2$$

(d) $2x^3(x + 3)(x - 3) = 2x^3(x^2 - 9)$
$$= 2x^5 - 18x^3$$

Now Try Exercises 47 and 51.

OBJECTIVE 5 Find the square of a binomial. Another special binomial product is the *square of a binomial*. To find the square of $x + y$, or $(x + y)^2$, multiply $x + y$ by itself.

$$(x + y)(x + y) = x^2 + xy + xy + y^2$$
$$= x^2 + 2xy + y^2$$

A similar result is true for the square of a difference.

Square of a Binomial

The **square of a binomial** is the sum of the square of the first term, twice the product of the two terms, and the square of the last term.

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

EXAMPLE 6 Squaring Binomials

Find each product.

(a) $(m + 7)^2 = m^2 + 2 \cdot m \cdot 7 + 7^2$
$$= m^2 + 14m + 49$$

(b) $(p - 5)^2 = p^2 - 2 \cdot p \cdot 5 + 5^2$
$$= p^2 - 10p + 25$$

(c) $(2p + 3v)^2 = (2p)^2 + 2(2p)(3v) + (3v)^2$
$$= 4p^2 + 12pv + 9v^2$$

(d) $(3r - 5s)^2 = (3r)^2 - 2(3r)(5s) + (5s)^2$
$$= 9r^2 - 30rs + 25s^2$$

Now Try Exercises 59 and 63.

CAUTION As the products in the formula for the square of a binomial show,

$$(x + y)^2 \neq x^2 + y^2.$$

More generally,

$$(x + y)^n \neq x^n + y^n \quad (n \neq 1).$$

We can use the patterns for the special products with more complicated products, as the following example shows.

EXAMPLE 7 Multiplying More Complicated Binomials

Find each product.

(a) $[(3p - 2) + 5q][(3p - 2) - 5q]$

$= (3p - 2)^2 - (5q)^2$ Product of sum and difference of terms

$= 9p^2 - 12p + 4 - 25q^2$ Square both quantities.

(b) $[(2z + r) + 1]^2 = (2z + r)^2 + 2(2z + r)(1) + 1^2$ Square of a binomial

$= 4z^2 + 4zr + r^2 + 4z + 2r + 1$ Square again; use the distributive property.

(c) $(x + y)^3 = (x + y)^2(x + y)$

$= (x^2 + 2xy + y^2)(x + y)$ Square $x + y$.

$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

(d) $(2a + b)^4 = (2a + b)^2(2a + b)^2$

$= (4a^2 + 4ab + b^2)(4a^2 + 4ab + b^2)$ Square $2a + b$.

$= 16a^4 + 16a^3b + 4a^2b^2 + 16a^3b + 16a^2b^2$
$\quad + 4ab^3 + 4a^2b^2 + 4ab^3 + b^4$

$= 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$

Now Try Exercises 69, 73, and 77.

OBJECTIVE 6 Multiply polynomial functions. In the previous section we introduced operations on functions and saw how functions can be added and subtracted. Functions can also be multiplied.

Multiplying Functions

If $f(x)$ and $g(x)$ define functions, then

$$(fg)(x) = f(x) \cdot g(x). \quad \text{Product}$$

The domain of the product function is the intersection of the domains of $f(x)$ and $g(x)$.

EXAMPLE 8 Multiplying Polynomial Functions

For $f(x) = 3x + 4$ and $g(x) = 2x^2 + x$, find $(fg)(x)$ and $(fg)(-1)$.

$(fg)(x) = f(x) \cdot g(x)$ Use the definition.

$= (3x + 4)(2x^2 + x)$

$= 6x^3 + 3x^2 + 8x^2 + 4x$ FOIL

$= 6x^3 + 11x^2 + 4x$ Combine like terms.

Then

$(fg)(-1) = 6(-1)^3 + 11(-1)^2 + 4(-1)$ Let $x = -1$.

$= -6 + 11 - 4$

$= 1.$

(What does $f(-1) \cdot g(-1)$ equal?)

Now Try Exercises 113 and 115.

5.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 8/Videotape 7

Match each product in Column I with the correct polynomial in Column II.

I	II
1. $(2x-5)(3x+4)$	**A.** $6x^2+23x+20$
2. $(2x+5)(3x+4)$	**B.** $6x^2+7x-20$
3. $(2x-5)(3x-4)$	**C.** $6x^2-7x-20$
4. $(2x+5)(3x-4)$	**D.** $6x^2-23x+20$

Find each product. See Examples 1–3.

EXERCISES

5. $-8m^3(3m^2)$ **6.** $4p^2(-5p^4)$ **7.** $14x^2y^3(-2x^5y)$

8. $-5m^3n^4(4m^2n^5)$ **9.** $3x(-2x+5)$ **10.** $5y(-6y-1)$

11. $-q^3(2+3q)$ **12.** $-3a^4(4-a)$ **13.** $6k^2(3k^2+2k+1)$

14. $5r^3(2r^2-3r-4)$ **15.** $(2m+3)(3m^2-4m-1)$

16. $(4z-2)(z^2+3z+5)$ **17.** $m(m+5)(m-8)$

18. $p(p-6)(p+4)$ **19.** $4z(2z+1)(3z-4)$ **20.** $2y(8y-3)(2y+1)$

21. $4x^3(x-3)(x+2)$ **22.** $2y^5(y-8)(y+2)$ **23.** $(2y+3)(3y-4)$

24. $(5m-3)(2m+6)$

25. $\begin{array}{r} -b^2+3b+3 \\ \underline{2b+4} \end{array}$

26. $\begin{array}{r} -r^2-4r+8 \\ \underline{3r-2} \end{array}$

27. $\begin{array}{r} 5m-3n \\ \underline{5m+3n} \end{array}$

28. $\begin{array}{r} 2k+6q \\ \underline{2k-6q} \end{array}$

29. $\begin{array}{r} 2z^3-5z^2+8z-1 \\ \underline{4z+3} \end{array}$

30. $\begin{array}{r} 3z^4-2z^3+\;z-5 \\ \underline{2z-5} \end{array}$

31. $\begin{array}{r} 2p^2+3p+6 \\ \underline{3p^2-4p-1} \end{array}$

32. $\begin{array}{r} 5y^2-2y+4 \\ \underline{2y^2+\;y+3} \end{array}$

Use the FOIL method to find each product. See Example 4.

EXERCISES

33. $(m+5)(m-8)$ **34.** $(p-6)(p+4)$ **35.** $(4k+3)(3k-2)$

36. $(5w+2)(2w+5)$ **37.** $(z-w)(3z+4w)$ **38.** $(s+t)(2s-5t)$

39. $(6c-d)(2c+3d)$ **40.** $(2m-n)(3m+5n)$ **41.** $(.2x+1.3)(.5x-.1)$

42. $(.5y-.4)(.1y+2.1)$ **43.** $\left(3w+\frac{1}{4}z\right)(w-2z)$ **44.** $\left(5r-\frac{2}{3}y\right)(r+5y)$

45. Describe the FOIL method in your own words.

46. Explain why the product of the sum and difference of two terms is not a trinomial.

Find each product. See Example 5.

EXERCISES

47. $(2p-3)(2p+3)$ **48.** $(3x-8)(3x+8)$ **49.** $(5m-1)(5m+1)$

50. $(6y+3)(6y-3)$ **51.** $(3a+2c)(3a-2c)$ **52.** $(5r-4s)(5r+4s)$

53. $\left(4x-\frac{2}{3}\right)\left(4x+\frac{2}{3}\right)$ **54.** $\left(3t+\frac{5}{4}\right)\left(3t-\frac{5}{4}\right)$ **55.** $(4m+7n^2)(4m-7n^2)$

56. $(2k^2+6h)(2k^2-6h)$ **57.** $(5y^3+2)(5y^3-2)$ **58.** $(3x^3+4)(3x^3-4)$

Find each square. See Example 6.

59. $(y - 5)^2$ **60.** $(a - 3)^2$ **61.** $(2p + 7)^2$

62. $(3z + 8)^2$ **63.** $(4n + 3m)^2$ **64.** $(5r + 7s)^2$

65. $\left(k - \frac{5}{7}p\right)^2$ **66.** $\left(q - \frac{3}{4}r\right)^2$

67. How do the expressions $(x + y)^2$ and $x^2 + y^2$ differ?

68. Find the product $101 \cdot 99$ using the special product rule $(x + y)(x - y) = x^2 - y^2$.

Find each product. See Example 7.

69. $[(5x + 1) + 6y]^2$ **70.** $[(3m - 2) + p]^2$

71. $[(2a + b) - 3]^2$ **72.** $[(4k + h) - 4]^2$

73. $[(2a + b) - 3][(2a + b) + 3]$ **74.** $[(m + p) + 5][(m + p) - 5]$

75. $[(2h - k) + j][(2h - k) - j]$ **76.** $[(3m - y) + z][(3m - y) - z]$

77. $(y + 2)^3$ **78.** $(z - 3)^3$ **79.** $(5r - s)^3$

80. $(x + 3y)^3$ **81.** $(q - 2)^4$ **82.** $(r + 3)^4$

Find each product.

83. $(2a + b)(3a^2 + 2ab + b^2)$ **84.** $(m - 5p)(m^2 - 2mp + 3p^2)$

85. $(4z - x)(z^3 - 4z^2x + 2zx^2 - x^3)$ **86.** $(3r + 2s)(r^3 + 2r^2s - rs^2 + 2s^3)$

87. $(m^2 - 2mp + p^2)(m^2 + 2mp - p^2)$ **88.** $(3 + x + y)(-3 + x - y)$

89. $ab(a + b)(a + 2b)(a - 3b)$ **90.** $mp(m - p)(m - 2p)(2m + p)$

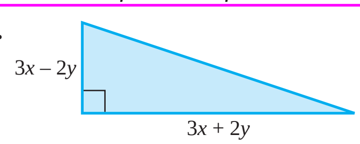
In Exercises 91–94, two expressions are given. Replace x with 3 and y with 4 to show that, in general, the two expressions do not equal each other.

91. $(x + y)^2$; $x^2 + y^2$ **92.** $(x + y)^3$; $x^3 + y^3$

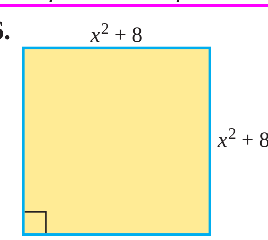
93. $(x + y)^4$; $x^4 + y^4$ **94.** $(x + y)^5$; $x^5 + y^5$

Find the area of each figure. Express it as a polynomial in descending powers of the variable x. Refer to the formulas on the inside covers of this book if necessary.

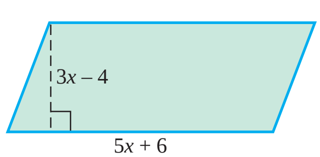
95.



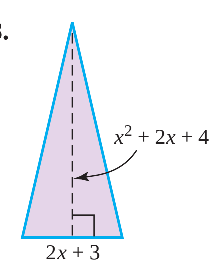
96.



97.



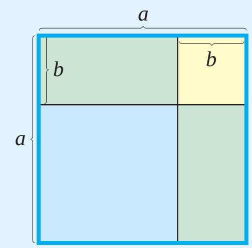
98.



RELATING CONCEPTS (EXERCISES 99–106)

For Individual or Group Work

Consider the figure. ***Work Exercises 99–106 in order.***



99. What is the length of each side of the blue square in terms of a and b?

100. What is the formula for the area of a square? Use the formula to write an expression, in the form of a product, for the area of the blue square.

101. Each green rectangle has an area of _____. Therefore, the total area in green is represented by the polynomial _____.

102. The yellow square has an area of _____.

103. The area of the entire colored region is represented by _____, because each side of the entire colored region has length _____.

104. The area of the blue square is equal to the area of the entire colored region minus the total area of the green squares minus the area of the yellow square. Write this as a simplified polynomial in a and b.

105. (a) What must be true about the expressions for the area of the blue square you found in Exercises 100 and 104?

(b) Write an equation based on your answer in part (a). How does this reinforce one of the main ideas of this section?

106. Draw a figure and give a similar proof for $(a + b)^2 = a^2 + 2ab + b^2$.

For each pair of functions, find the product $(fg)(x)$. *See Example 8.*

107. $f(x) = 2x$, $g(x) = 5x - 1$

108. $f(x) = 3x$, $g(x) = 6x - 8$

109. $f(x) = x + 1$, $g(x) = 2x - 3$

110. $f(x) = x - 7$, $g(x) = 4x + 5$

111. $f(x) = 2x - 3$, $g(x) = 4x^2 + 6x + 9$

112. $f(x) = 3x + 4$, $g(x) = 9x^2 - 12x + 16$

Let $f(x) = x^2 - 9$, $g(x) = 2x$, *and* $h(x) = x - 3$. *Find each of the following. See Example 8.*

113. $(fg)(x)$

114. $(fh)(x)$

115. $(fg)(2)$

116. $(fh)(1)$

117. $(gh)(x)$

118. $(fh)(-1)$

119. $(gh)(-3)$

120. $(fg)(-2)$

5.5 Dividing Polynomials

OBJECTIVES

1. Divide a polynomial by a monomial.
2. Divide a polynomial by a polynomial of two or more terms.
3. Divide polynomial functions.

OBJECTIVE 1 Divide a polynomial by a monomial. We now discuss polynomial division, beginning with division by a monomial. (Recall that a monomial is a single term, such as $8x$, $-9m^4$, or $11y^2$.)

Dividing by a Monomial

To divide a polynomial by a monomial, divide each term in the polynomial by the monomial, and then write each quotient in lowest terms.

EXAMPLE 1 Dividing a Polynomial by a Monomial

Divide.

(a) $$\frac{15x^2 - 12x + 6}{3} = \frac{15x^2}{3} - \frac{12x}{3} + \frac{6}{3} \qquad \text{Divide each term by 3.}$$
$$= 5x^2 - 4x + 2 \qquad \text{Write in lowest terms.}$$

Check this answer by multiplying it by the divisor, 3. You should get $15x^2 - 12x + 6$ as the result.

$$\underset{\text{Divisor}}{3}\underbrace{(5x^2 - 4x + 2)}_{\text{Quotient}} = \underbrace{15x^2 - 12x + 6}_{\text{Original polynomial}}$$

(b) $$\frac{5m^3 - 9m^2 + 10m}{5m^2} = \frac{5m^3}{5m^2} - \frac{9m^2}{5m^2} + \frac{10m}{5m^2} \qquad \text{Divide each term by } 5m^2.$$
$$= m - \frac{9}{5} + \frac{2}{m} \qquad \text{Write in lowest terms.}$$

This result is not a polynomial. (Why?) The quotient of two polynomials need not be a polynomial.

(c) $$\frac{8xy^2 - 9x^2y + 6x^2y^2}{x^2y^2} = \frac{8xy^2}{x^2y^2} - \frac{9x^2y}{x^2y^2} + \frac{6x^2y^2}{x^2y^2}$$
$$= \frac{8}{x} - \frac{9}{y} + 6$$

Now Try Exercises 5, 9, and 11.

OBJECTIVE 2 Divide a polynomial by a polynomial of two or more terms. The process for dividing one polynomial by another polynomial that is not a monomial is similar to that for dividing whole numbers.

EXAMPLE 2 Dividing a Polynomial by a Polynomial

Divide $\dfrac{2m^2 + m - 10}{m - 2}$.

Write the problem as if dividing whole numbers, making sure that both polynomials are written in descending powers of the variables.

$$m - 2\overline{)2m^2 + m - 10}$$

Divide the first term of $2m^2 + m - 10$ by the first term of $m - 2$. Since $\dfrac{2m^2}{m} = 2m$, place this result above the division line.

$$\begin{array}{r} 2m \phantom{{}+m-10} \\ m - 2\overline{)2m^2 + m - 10} \end{array} \quad \leftarrow \text{Result of } \frac{2m^2}{m}$$

Multiply $m - 2$ and $2m$, and write the result below $2m^2 + m - 10$.

$$\begin{array}{r} 2m \phantom{{}+m-10} \\ m - 2\overline{)2m^2 + m - 10} \\ \underline{2m^2 - 4m} \phantom{{}-10} \end{array} \quad \leftarrow 2m(m-2) = 2m^2 - 4m$$

Now subtract $2m^2 - 4m$ from $2m^2 + m$. Do this by mentally changing the signs on $2m^2 - 4m$ and *adding*.

$$\begin{array}{r@{}l}
 & 2m \\
m - 2 & \overline{)2m^2 + m - 10} \\
 & \underline{2m^2 - 4m} \\
 & \phantom{)2m^2 +{}} 5m \quad \leftarrow \text{Subtract. The difference is } 5m.
\end{array}$$

Bring down -10 and continue by dividing $5m$ by m.

$$\begin{array}{r@{}l}
 & 2m + 5 \quad \leftarrow \frac{5m}{m} = 5 \\
m - 2 & \overline{)2m^2 + m - 10} \\
 & \underline{2m^2 - 4m} \\
 & \phantom{)2m^2 +{}} 5m - 10 \quad \leftarrow \text{Bring down } -10. \\
 & \phantom{)2m^2 +{}} \underline{5m - 10} \quad \leftarrow 5(m - 2) = 5m - 10 \\
 & \phantom{)2m^2 + 5m - 1{}} 0 \quad \leftarrow \text{Subtract. The difference is 0.}
\end{array}$$

Finally, $(2m^2 + m - 10) \div (m - 2) = 2m + 5$. Check by multiplying $m - 2$ and $2m + 5$. The result should be $2m^2 + m - 10$.

Now Try Exercise 17.

EXAMPLE 3 Dividing a Polynomial with a Missing Term

VIDEO EXPLORE

Divide $3x^3 - 2x + 5$ by $x - 3$.

Make sure that $3x^3 - 2x + 5$ is in descending powers of the variable. Add a term with 0 coefficient as a placeholder for the missing x^2-term.

$$x - 3\overline{)3x^3 + \underbrace{0x^2}_{\text{Missing term}} - 2x + 5}$$

Start with $\dfrac{3x^3}{x} = 3x^2$.

$$\begin{array}{r@{}l}
 & 3x^2 \quad \leftarrow \frac{3x^3}{x} = 3x^2 \\
x - 3 & \overline{)3x^3 + 0x^2 - 2x + 5} \\
 & \underline{3x^3 - 9x^2} \quad \leftarrow 3x^2(x - 3)
\end{array}$$

Subtract by mentally changing the signs on $3x^3 - 9x^2$ and adding.

$$\begin{array}{r@{}l}
 & 3x^2 \\
x - 3 & \overline{)3x^3 + 0x^2 - 2x + 5} \\
 & \underline{3x^3 - 9x^2} \\
 & \phantom{)3x^3 +{}} 9x^2 \quad \leftarrow \text{Subtract.}
\end{array}$$

Bring down the next term.

$$\begin{array}{r@{}l}
 & 3x^2 \\
x - 3 & \overline{)3x^3 + 0x^2 - 2x + 5} \\
 & \underline{3x^3 - 9x^2} \\
 & \phantom{)3x^3 +{}} 9x^2 - 2x \quad \leftarrow \text{Bring down } -2x.
\end{array}$$

In the next step, $\frac{9x^2}{x} = 9x$.

$$\begin{array}{rll}
 & 3x^2 + 9x \phantom{{}+5} & \leftarrow \frac{9x^2}{x} = 9x \\
x - 3 \big) & \overline{3x^3 + 0x^2 - 2x + 5} & \\
 & \underline{3x^3 - 9x^2} \phantom{{}- 2x + 5} & \\
 & 9x^2 - 2x \phantom{{}+5} & \\
 & \underline{9x^2 - 27x} \phantom{{}+5} & \leftarrow 9x(x-3) \\
 & 25x + 5 & \leftarrow \text{Subtract; bring down 5.}
\end{array}$$

Finally, $\frac{25x}{x} = 25$.

$$\begin{array}{rll}
 & 3x^2 + 9x + 25 & \leftarrow \frac{25x}{x} = 25 \\
x - 3 \big) & \overline{3x^3 + 0x^2 - 2x + 5} & \\
 & \underline{3x^3 - 9x^2} \phantom{{}- 2x + 5} & \\
 & 9x^2 - 2x \phantom{{}+5} & \\
 & \underline{9x^2 - 27x} \phantom{{}+5} & \\
 & 25x + 5 & \\
 & \underline{25x - 75} & \leftarrow 25(x-3) \\
 & 80 & \leftarrow \text{Remainder}
\end{array}$$

Write the remainder, 80, as the numerator of the fraction $\frac{80}{x-3}$. In summary,

$$\frac{3x^3 - 2x + 5}{x - 3} = 3x^2 + 9x + 25 + \frac{80}{x - 3}.$$

Check by multiplying $x - 3$ and $3x^2 + 9x + 25$ and adding 80 to the result. You should get $3x^3 - 2x + 5$.

Now Try Exercise 33.

CAUTION Remember to include $\frac{\text{remainder}}{\text{divisor}}$ as part of the answer.

YOU TRY IT

EXAMPLE 4 Dividing by a Polynomial with a Missing Term

Divide $6r^4 + 9r^3 + 2r^2 - 8r + 7$ by $3r^2 - 2$.

The polynomial $3r^2 - 2$ has a missing term. Write it as $3r^2 + 0r - 2$ and divide as usual.

$$\begin{array}{rl}
 & 2r^2 + 3r + 2 \\
3r^2 + \underbrace{0r}_{\text{Missing term}} - 2 \big) & \overline{6r^4 + 9r^3 + 2r^2 - 8r + 7} \\
 & \underline{6r^4 + 0r^3 - 4r^2} \phantom{{}- 8r + 7} \\
 & 9r^3 + 6r^2 - 8r \phantom{{}+7} \\
 & \underline{9r^3 + 0r^2 - 6r} \phantom{{}+7} \\
 & 6r^2 - 2r + 7 \\
 & \underline{6r^2 + 0r - 4} \\
 & -2r + 11
\end{array}$$

Since the degree of the remainder, $-2r + 11$, is less than the degree of the divisor, $3r^2 - 2$, the division process is now finished. The result is written

$$2r^2 + 3r + 2 + \frac{-2r + 11}{3r^2 - 2}.$$

Now Try Exercise 37.

CAUTION Remember the following steps when dividing a polynomial by a polynomial of two or more terms.

1. Be sure the terms in both polynomials are in descending powers.
2. Write any missing terms with 0 placeholders.

EXAMPLE 5 Performing a Division with a Fractional Coefficient in the Quotient

Divide $2p^3 + 5p^2 + p - 2$ by $2p + 2$.

$$\frac{3p^2}{2p} = \frac{3}{2}p$$

$$
\begin{array}{r@{}l}
 & p^2 + \frac{3}{2}p - 1 \\
2p + 2 & \overline{)\,2p^3 + 5p^2 + p - 2} \\
 & \underline{2p^3 + 2p^2} \\
 & \phantom{2p^3 + {}} 3p^2 + p \\
 & \phantom{2p^3 + {}} \underline{3p^2 + 3p} \\
 & \phantom{2p^3 + 5p^2 {}} -2p - 2 \\
 & \phantom{2p^3 + 5p^2 {}} \underline{-2p - 2} \\
 & \phantom{2p^3 + 5p^2 + p - {}} 0
\end{array}
$$

Since the remainder is 0, the quotient is $p^2 + \dfrac{3}{2}p - 1$.

Now Try Exercise 39.

OBJECTIVE 3 Divide polynomial functions. In the preceding sections, we used operations on functions to add, subtract, and multiply polynomial functions. We now define the quotient of two functions.

Dividing Functions

If $f(x)$ and $g(x)$ define functions, then

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}. \qquad \text{Quotient}$$

The domain of the quotient function is the intersection of the domains of $f(x)$ and $g(x)$, excluding any values of x where $g(x) = 0$.

EXAMPLE 6 Dividing Polynomial Functions

For $f(x) = 2x^2 + x - 10$ and $g(x) = x - 2$, find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(-3)$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2 + x - 10}{x - 2}$$

This quotient was found in Example 2, with m replacing x. The result here is $2x + 5$, so

$$\left(\frac{f}{g}\right)(x) = 2x + 5, \quad x \neq 2.$$

Then

$$\left(\frac{f}{g}\right)(-3) = 2(-3) + 5 = -1. \qquad \text{Let } x = -3.$$

(Which is easier to find here—$\left(\frac{f}{g}\right)(-3)$ or $\frac{f(-3)}{g(-3)}$?)

Now Try Exercises 57 and 59.

5.5 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 8/Videotape 7

Complete each statement with the correct word or words.

1. We find the quotient of two monomials by using the _____ rule for _____.

2. To divide a polynomial by a monomial, divide _____ of the polynomial by the _____.

3. When dividing polynomials that are not monomials, first write them in _____.

4. If a polynomial in a division problem has a missing term, insert a term with _____ as a placeholder.

Divide. See Example 1.

5. $\dfrac{15x^3 - 10x^2 + 5}{5}$

6. $\dfrac{27m^4 - 18m^3 + 9m}{9}$

7. $\dfrac{9y^2 + 12y - 15}{3y}$

8. $\dfrac{80r^2 - 40r + 10}{10r}$

9. $\dfrac{15m^3 + 25m^2 + 30m}{5m^2}$

10. $\dfrac{64x^3 - 72x^2 + 12x}{8x^3}$

11. $\dfrac{14m^2n^2 - 21mn^3 + 28m^2n}{14m^2n}$

12. $\dfrac{24h^2k + 56hk^2 - 28hk}{16h^2k^2}$

13. $\dfrac{8wxy^2 + 3wx^2y + 12w^2xy}{4wx^2y}$

14. $\dfrac{12ab^2c + 10a^2bc + 18abc^2}{6a^2bc}$

Complete the division.

15.
$$\begin{array}{r}
r^2 \phantom{{}+25r-6} \\
3r-1\,\overline{)\,3r^3 - 22r^2 + 25r - 6} \\
\underline{3r^3 - r^2} \phantom{{}+25r-6} \\
-21r^2 \phantom{{}+25r-6}
\end{array}$$

16.
$$\begin{array}{r}
3b^2 \phantom{{}-4b-40} \\
2b-5\,\overline{)\,6b^3 - 7b^2 - 4b - 40} \\
\underline{6b^3 - 15b^2} \phantom{{}-4b-40} \\
8b^2 \phantom{{}-4b-40}
\end{array}$$

Divide. See Examples 2–5.

17. $\dfrac{y^2 + 3y - 18}{y + 6}$

18. $\dfrac{q^2 + 4q - 32}{q - 4}$

19. $\dfrac{3t^2 + 17t + 10}{3t + 2}$

20. $\dfrac{2k^2 - 3k - 20}{2k + 5}$

21. $\dfrac{p^2 + 2p + 20}{p + 6}$

22. $\dfrac{x^2 + 11x + 16}{x + 8}$

23. $\dfrac{3m^3 + 5m^2 - 5m + 1}{3m - 1}$

24. $\dfrac{8z^3 - 6z^2 - 5z + 3}{4z + 3}$

25. $(2z^3 - 5z^2 + 6z - 15) \div (2z - 5)$

26. $(3p^3 + p^2 + 18p + 6) \div (3p + 1)$

27. $(4x^3 + 9x^2 - 10x + 3) \div (4x + 1)$

28. $(10z^3 - 26z^2 + 17z - 13) \div (5z - 3)$

29. $\dfrac{6x^3 - 19x^2 + 14x - 15}{3x^2 - 2x + 4}$

30. $\dfrac{8m^3 - 18m^2 + 37m - 13}{2m^2 - 3m + 6}$

31. $(x^3 + 2x - 3) \div (x - 1)$

32. $(2x^3 - 11x^2 + 28) \div (x - 5)$

33. $(3x^3 - x + 4) \div (x - 2)$

34. $(3k^3 + 9k - 14) \div (k - 2)$

35. $\dfrac{4k^4 + 6k^3 + 3k - 1}{2k^2 + 1}$

36. $\dfrac{6y^4 + 4y^3 + 4y - 6}{3y^2 + 2y - 3}$

37. $(x^4 - 4x^3 + 5x^2 - 3x + 2) \div (x^2 + 3)$

38. $(3t^4 + 5t^3 - 8t^2 - 13t + 2) \div (t^2 - 5)$

39. $(2p^3 + 7p^2 + 9p + 2) \div (2p + 2)$

40. $(3a^2 - 11a + 17) \div (2a + 6)$

41. $\dfrac{p^3 - 1}{p - 1}$

42. $\dfrac{8a^3 + 1}{2a + 1}$

Divide.

43. $\left(2x^2 - \dfrac{7}{3}x - 1\right) \div (3x + 1)$

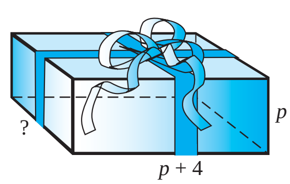
44. $\left(m^2 + \dfrac{7}{2}m + 3\right) \div (2m + 3)$

45. $\left(3a^2 - \dfrac{23}{4}a - 5\right) \div (4a + 3)$

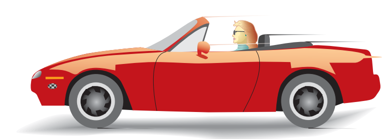
46. $\left(3q^2 + \dfrac{19}{5}q - 3\right) \div (5q - 2)$

Solve each problem.

47. The volume of a box is $2p^3 + 15p^2 + 28p$. The height is p and the length is $p + 4$; find the width.



48. Suppose a car travels a distance of $(2m^3 + 15m^2 + 13m - 63)$ km in $(2m + 9)$ hr. Find its rate of speed.



49. For $P(x) = x^3 - 4x^2 + 3x - 5$, find $P(-1)$. Then divide $P(x)$ by $D(x) = x + 1$. Compare the remainder to $P(-1)$. What do these results suggest?

50. Let $P(x) = 4x^3 - 8x^2 + 13x - 2$, and $D(x) = 2x - 1$. Use division to find polynomials $Q(x)$ and $R(x)$ so that $P(x) = Q(x) \cdot D(x) + R(x)$.

For each pair of functions, find the quotient $\left(\frac{f}{g}\right)(x)$ and give any x-values that are not in the domain of the quotient function. See Example 6.

51. $f(x) = 10x^2 - 2x$, $g(x) = 2x$

52. $f(x) = 18x^2 - 24x$, $g(x) = 3x$

53. $f(x) = 2x^2 - x - 3$, $g(x) = x + 1$

54. $f(x) = 4x^2 - 23x - 35$, $g(x) = x - 7$

55. $f(x) = 8x^3 - 27$, $g(x) = 2x - 3$

56. $f(x) = 27x^3 + 64$, $g(x) = 3x + 4$

Let $f(x) = x^2 - 9$, $g(x) = 2x$, and $h(x) = x - 3$. Find each of the following. See Example 6.

57. $\left(\dfrac{f}{g}\right)(x)$

58. $\left(\dfrac{f}{h}\right)(x)$

59. $\left(\dfrac{f}{g}\right)(2)$

60. $\left(\dfrac{f}{h}\right)(1)$

61. $\left(\dfrac{h}{g}\right)(x)$

62. $\left(\dfrac{f}{h}\right)(-3)$

63. $\left(\dfrac{h}{g}\right)(3)$

64. $\left(\dfrac{f}{g}\right)(-1)$

6.1 Greatest Common Factors; Factoring by Grouping

OBJECTIVES

1. Factor out the greatest common factor.
2. Factor by grouping.

Writing a polynomial as the product of two or more simpler polynomials is called **factoring** the polynomial. For example, the product of $3x$ and $5x - 2$ is $15x^2 - 6x$, and $15x^2 - 6x$ can be factored as the product $3x(5x - 2)$.

$$3x(5x - 2) = 15x^2 - 6x \qquad \text{Multiplying}$$

$$15x^2 - 6x = 3x(5x - 2) \qquad \text{Factoring}$$

Notice that both multiplying and factoring use the distributive property, but in opposite directions. Factoring "undoes" or reverses multiplying.

OBJECTIVE 1 Factor out the greatest common factor. The first step in factoring a polynomial is to find the *greatest common factor* for the terms of the polynomial. The **greatest common factor (GCF)** is the largest term that is a factor of all terms in the polynomial. For example, the greatest common factor for $8x + 12$ is 4, since 4 is the largest term that is a factor of (divides into) both $8x$ and 12. Using the distributive property,

$$8x + 12 = 4(2x) + 4(3) = 4(2x + 3).$$

As a check, multiply 4 and $2x + 3$. The result should be $8x + 12$. Using the distributive property this way is called *factoring out the greatest common factor.*

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

(a) $9z - 18$
Since 9 is the GCF, factor 9 from each term.

$$9z - 18 = 9 \cdot z - 9 \cdot 2 = 9(z - 2)$$

(b) $56m + 35p = 7(8m + 5p)$

(c) $2y + 5$ There is no common factor other than 1.

(d) $12 + 24z = 12 \cdot 1 + 12 \cdot 2z$
$= 12(1 + 2z)$ 12 is the GCF.

Now Try Exercise 5.

CAUTION In Example 1(d), remember to write the factor 1. Always check answers by multiplying.

VIDEO

EXAMPLE 2 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

(a) $9x^2 + 12x^3$
The numerical part of the greatest common factor is 3, the largest number that divides into both 9 and 12. For the variable parts, x^2 and x^3, use the least exponent that appears on x; here the least exponent is 2. The GCF is $3x^2$.

$$9x^2 + 12x^3 = 3x^2(3) + 3x^2(4x) \qquad \text{GCF} = 3x^2$$
$$= 3x^2(3 + 4x)$$

(b) $32p^4 - 24p^3 + 40p^5 = 8p^3(4p) + 8p^3(-3) + 8p^3(5p^2) \qquad \text{GCF} = 8p^3$

$$= 8p^3(4p - 3 + 5p^2)$$

(c) $3k^4 - 15k^7 + 24k^9 = 3k^4(1 - 5k^3 + 8k^5)$

(d) $24m^3n^2 - 18m^2n + 6m^4n^3$

The numerical part of the GCF is 6. Here 2 is the least exponent that appears on m, while 1 is the least exponent on n. The GCF is $6m^2n$.

$$24m^3n^2 - 18m^2n + 6m^4n^3 = 6m^2n(4mn) + 6m^2n(-3) + 6m^2n(m^2n^2)$$
$$= 6m^2n(4mn - 3 + m^2n^2)$$

(e) $25x^2y^3 + 30y^5 - 15x^4y^7 = 5y^3(5x^2 + 6y^2 - 3x^4y^4)$

Now Try Exercises 9 and 13.

A greatest common factor need not be a monomial. The next example shows a binomial greatest common factor.

EXAMPLE 3 Factoring Out a Binomial Factor

Factor out the greatest common factor.

(a) $(x - 5)(x + 6) + (x - 5)(2x + 5)$

The greatest common factor here is $x - 5$.

$$(x - 5)(x + 6) + (x - 5)(2x + 5) = (x - 5)[(x + 6) + (2x + 5)]$$
$$= (x - 5)(x + 6 + 2x + 5)$$
$$= (x - 5)(3x + 11)$$

(b) $z^2(m + n)^2 + x^2(m + n)^2 = (m + n)^2(z^2 + x^2)$

(c) $p(r + 2s)^2 - q(r + 2s)^3 = (r + 2s)^2[p - q(r + 2s)]$

$$= (r + 2s)^2(p - qr - 2qs)$$

(d) $(p - 5)(p + 2) - (p - 5)(3p + 4)$

$$= (p - 5)[(p + 2) - (3p + 4)]$$
$$= (p - 5)[p + 2 - 3p - 4] \qquad \text{Be careful with signs.}$$
$$= (p - 5)[-2p - 2]$$
$$= (p - 5)[-2(p + 1)] \text{ or } -2(p - 5)(p + 1) \qquad \text{Look for a common factor.}$$

Now Try Exercises 21 and 25.

When the coefficient of the term of greatest degree is negative, it is sometimes preferable to factor out the -1 that is understood along with the GCF.

EXAMPLE 4 Factoring Out a Negative Common Factor

VIDEO

YOU TRY IT

Factor $-a^3 + 3a^2 - 5a$ in two ways.

First, a could be used as the common factor, giving

$$-a^3 + 3a^2 - 5a = a(-a^2) + a(3a) + a(-5)$$
$$= a(-a^2 + 3a - 5).$$

Because of the leading negative sign, $-a$ could be used as the common factor.

$$-a^3 + 3a^2 - 5a = -a(a^2) + (-a)(-3a) + (-a)(5)$$
$$= -a(a^2 - 3a + 5)$$

Either answer is correct.

Now Try Exercise 33.

NOTE Example 4 showed two ways of factoring a polynomial. Sometimes there may be a reason to prefer one of these forms over the other, but both are correct.

OBJECTIVE 2 Factor by grouping. Sometimes the *individual terms* of a polynomial have a greatest common factor of 1, but it still may be possible to factor the polynomial by using a process called *factoring by grouping.* We usually factor by grouping when a polynomial has more than three terms. For example, to factor the polynomial

$$ax - ay + bx - by,$$

group the terms as follows.

Terms with common factor a Terms with common factor b

$$(ax - ay) + (bx - by)$$

Then factor $ax - ay$ as $a(x - y)$ and factor $bx - by$ as $b(x - y)$ to get

$$(ax - ay) + (bx - by) = a(x - y) + b(x - y).$$

On the right, the common factor is $x - y$. The final factored form is

$$ax - ay + bx - by = (x - y)(a + b).$$

EXAMPLE 5 Factoring by Grouping

Factor $3x - 3y - ax + ay$.

Grouping terms gives

$$(3x - 3y) + (-ax + ay) = 3(x - y) + a(-x + y).$$

There is no simple common factor here. However, if we factor out $-a$ instead of a in the second group of terms, we get

$$3(x - y) - a(x - y).$$

Now factor out the common binomial factor $(x - y)$ to obtain

$$(x - y)(3 - a).$$

Check by multiplying: $(x - y)(3 - a) = 3x - ax - 3y + ay$

$$= 3x - 3y - ax + ay$$

Now Try Exercise 43.

NOTE In Example 5, different grouping would lead to the product

$$(a - 3)(y - x).$$

Verify by multiplying that this is also correct.

Use the following steps to factor by grouping.

Factoring by Grouping

Step 1 **Group terms.** Collect the terms into groups so that each group has a common factor.

Step 2 **Factor within the groups.** Factor out the common factor in each group.

Step 3 **Factor the entire polynomial.** If each group now has a common factor, factor it out. If not, try a different grouping.

EXAMPLE 6 Factoring by Grouping

Factor $6ax + 12bx + a + 2b$ by grouping.

$$6ax + 12bx + a + 2b = (6ax + 12bx) + (a + 2b) \qquad \text{Group terms.}$$

Now factor $6x$ from the first group, and use the identity property of multiplication to introduce the factor 1 in the second group.

$$\begin{aligned}(6ax + 12bx) + (a + 2b) &= 6x(a + 2b) + 1(a + 2b) \\ &= (a + 2b)(6x + 1) \qquad \text{Factor out } a + 2b.\end{aligned}$$

Again, as in Example 1(d), remember to write the 1. Check by multiplying.

Now Try Exercise 41.

EXAMPLE 7 Rearranging Terms before Factoring by Grouping

Factor $p^2q^2 - 10 - 2q^2 + 5p^2$.

Neither the first two terms nor the last two terms have a common factor except the identity element 1. Rearrange and group the terms as follows.

$$\begin{aligned}&(p^2q^2 - 2q^2) + (5p^2 - 10) && \text{Rearrange and group the terms.} \\ &= q^2(p^2 - 2) + 5(p^2 - 2) && \text{Factor out the common factors.} \\ &= (p^2 - 2)(q^2 + 5) && \text{Factor out } p^2 - 2.\end{aligned}$$

Check by multiplying.

Now Try Exercise 53.

CAUTION In Example 7, do not stop at the step

$$q^2(p^2 - 2) + 5(p^2 - 2).$$

This expression is *not in factored form* because it is a *sum* of two terms, $q^2(p^2 - 2)$ and $5(p^2 - 2)$, not a *product.*

6.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 8/Videotape 8

Factor out the greatest common factor. Simplify the factors, if possible. See Examples 1–4.

1. $12m + 60$ **2.** $15r - 27$ **3.** $8k^3 + 24k$

4. $9z^4 + 81z$ **5.** $xy - 5xy^2$ **6.** $5h^2j + hj$

7. $-4p^3q^4 - 2p^2q^5$ **8.** $-3z^5w^2 - 18z^3w^4$

9. $21x^5 + 35x^4 + 14x^3$ **10.** $6k^3 - 36k^4 + 48k^5$

11. $10t^5 - 8t^4 - 2t^3$ **12.** $6p^3 - 3p^2 + 9p^4$

13. $15a^2c^3 - 25ac^2 + 5ac$ **14.** $15y^3z^3 + 27y^2z^4 - 36yz^5$

15. $16z^2n^6 + 64zn^7 - 32z^3n^3$ **16.** $5r^3s^5 + 10r^2s^2 - 15r^4s^2$

17. $-27m^3p^5 + 36m^4p^3 - 72m^5p^4$ **18.** $-50r^4t^2 + 80r^3t^3 - 90r^2t^4$

19. $14a^3b^2 + 7a^2b - 21a^5b^3 + 42ab^4$ **20.** $12km^3 - 24k^3m^2 + 36k^2m^4 - 60k^4m^3$

21. $(m - 4)(m + 2) + (m - 4)(m + 3)$ **22.** $(z - 5)(z + 7) + (z - 5)(z + 9)$

23. $(2z - 1)(z + 6) - (2z - 1)(z - 5)$ **24.** $(3x + 2)(x - 4) - (3x + 2)(x + 8)$

25. $5(2 - x)^2 - (2 - x)^3 + 4(2 - x)$ **26.** $3(5 - x)^4 + 2(5 - x)^3 - (5 - x)^2$

27. $4(3 - x)^2 - (3 - x)^3 + 3(3 - x)$ **28.** $2(t - s) + 4(t - s)^2 - (t - s)^3$

29. $15(2z + 1)^3 + 10(2z + 1)^2 - 25(2z + 1)$

30. $6(a + 2b)^2 - 4(a + 2b)^3 + 12(a + 2b)^4$

31. $5(m + p)^3 - 10(m + p)^2 - 15(m + p)^4$

32. $-9a^2(p + q) - 3a^3(p + q)^2 + 6a(p + q)^3$

Factor each polynomial twice. First use a common factor with a positive coefficient, and then use a common factor with a negative coefficient. See Example 4.

33. $-r^3 + 3r^2 + 5r$ **34.** $-t^4 + 8t^3 - 12t$ **35.** $-12s^5 + 48s^4$

36. $-16y^4 + 64y^3$ **37.** $-2x^5 + 6x^3 + 4x^2$ **38.** $-5a^3 + 10a^4 - 15a^5$

Factor by grouping. See Examples 5–7.

39. $mx + 3qx + my + 3qy$ **40.** $2k + 2h + jk + jh$ **41.** $10m + 2n + 5mk + nk$

42. $3ma + 3mb + 2ab + 2b^2$ **43.** $4 - 2q - 6p + 3pq$ **44.** $20 + 5m + 12n + 3mn$

45. $p^2 - 4zq + pq - 4pz$ **46.** $r^2 - 9tw + 3rw - 3rt$ **47.** $2xy - 8y + 3x - 12$

48. $6y^2 + 9y + 4xy + 6x$ **49.** $m^3 + 4m^2 - 6m - 24$ **50.** $2a^3 + a^2 - 14a - 7$

51. $-3a^3 - 3ab^2 + 2a^2b + 2b^3$ **52.** $-16m^3 + 4m^2p^2 - 4mp + p^3$

53. $4 + xy - 2y - 2x$ **54.** $2ab^2 - 4 - 8b^2 + a$

55. $8 + 9y^4 - 6y^3 - 12y$ **56.** $x^3y^2 - 3 - 3y^2 + x^3$

57. $1 - a + ab - b$ **58.** $2ab^2 - 8b^2 + a - 4$

Factor out the variable that is raised to the smaller exponent. (For example, in Exercise 59, factor out m^{-5}.)

59. $3m^{-5} + m^{-3}$ **60.** $k^{-2} + 2k^{-4}$

61. $3p^{-3} + 2p^{-2}$ **62.** $-5q^{-3} + 8q^{-2}$

63. When directed to factor the polynomial $4x^2y^5 - 8xy^3$ completely, a student wrote $2xy^3(2xy^2 - 4)$. When the teacher did not give him full credit, he complained because when his answer is multiplied out, the result is the original polynomial. Was the teacher justified in her grading? Why or why not?

64. Refer to Exercise 58. One form of the answer is $(2b^2 + 1)(a - 4)$. Give two other acceptable factored forms of $2ab^2 - 8b^2 + a - 4$.

65. Which one of the following is an example of a polynomial in factored form?

A. $3x^2y^3 + 6x^2(2x + y)$ **B.** $5(x + y)^2 - 10(x + y)^3$

C. $(-2 + 3x)(5y^2 + 4y + 3)$ **D.** $(3x + 4)(5x - y) - (3x + 4)(2x - 1)$

6.2 Factoring Trinomials

OBJECTIVES

1. Factor trinomials when the coefficient of the squared term is 1.
2. Factor trinomials when the coefficient of the squared term is not 1.
3. Use an alternative method for factoring trinomials.
4. Factor by substitution.

OBJECTIVE 1 Factor trinomials when the coefficient of the squared term is 1. We begin by finding the product of $x + 3$ and $x - 5$.

$$\begin{aligned}(x + 3)(x - 5) &= x^2 - 5x + 3x - 15\\ &= x^2 - 2x - 15\end{aligned}$$

We see by this result that the factored form of $x^2 - 2x - 15$ is $(x + 3)(x - 5)$.

Multiplication

Factored form $\rightarrow$ $(x + 3)(x - 5) = x^2 - 2x - 15$ $\leftarrow$ Product

Factoring

Since multiplying and factoring are operations that "undo" each other, factoring trinomials involves using FOIL backwards. As shown here, the x^2-term came from multiplying x and x, and -15 came from multiplying 3 and -5.

Product of x and x is x^2.

$$(\mathbf{x} + \mathbf{3})(\mathbf{x} - \mathbf{5}) = \mathbf{x^2} - 2x - \mathbf{15}$$

Product of 3 and -5 is -15.

We found the $-2x$ in $x^2 - 2x - 15$ by multiplying the outer terms, then the inner terms, and adding.

Outer terms: $x(-5) = -5x$

$$(x + 3)(x - 5)$$

Inner terms: $3 \cdot x = 3x$

Add to get $-2x$.

Based on this example, use the following steps to factor a trinomial $x^2 + bx + c$, with 1 as the coefficient of the squared term.

Factoring $x^2 + bx + c$

Step 1 **Find pairs whose product is *c*.** Find all pairs of integers whose product is the third term of the trinomial, c.

Step 2 **Find pairs whose sum is *b*.** Choose the pair whose sum is the coefficient of the middle term, b.

If there are no such integers, the polynomial cannot be factored. A polynomial that cannot be factored with integer coefficients is **prime.**

EXAMPLE 1 Factoring Trinomials in $x^2 + bx + c$ Form

Factor each polynomial.

(a) $y^2 + 2y - 35$

Step 1 Find pairs of numbers whose product is -35.	*Step 2* Write sums of those numbers.	
$-35(1)$	$-35 + 1 = -34$	
$35(-1)$	$35 + (-1) = 34$	
$7(-5)$	$7 + (-5) = 2$	← Coefficient of the middle term
$5(-7)$	$5 + (-7) = -2$	

The required numbers are 7 and -5, so

$$y^2 + 2y - 35 = (y + 7)(y - 5).$$

Check by finding the product of $y + 7$ and $y - 5$.

(b) $r^2 + 8r + 12$

Look for two numbers with a product of 12 and a sum of 8. Of all pairs of numbers having a product of 12, only the pair 6 and 2 has a sum of 8. Therefore,

$$r^2 + 8r + 12 = (r + 6)(r + 2).$$

Because of the commutative property, it would be equally correct to write $(r + 2)(r + 6)$. Check by multiplying.

Now Try Exercises 5 and 7.

EXAMPLE 2 Recognizing a Prime Polynomial

Factor $m^2 + 6m + 7$.

Look for two numbers whose product is 7 and whose sum is 6. Only two pairs of integers, 7 and 1 and -7 and -1, give a product of 7. Neither of these pairs has a sum of 6, so $m^2 + 6m + 7$ cannot be factored with integer coefficients and is prime.

Now Try Exercise 9.

We use a similar process to factor a trinomial that has more than one variable.

EXAMPLE 3 Factoring a Trinomial in Two Variables

Factor $p^2 + 6ap - 16a^2$.

Look for two expressions whose product is $-16a^2$ and whose sum is $6a$. The quantities $8a$ and $-2a$ have the necessary product and sum, so

$$p^2 + 6ap - 16a^2 = (p + 8a)(p - 2a).$$

Now Try Exercise 11.

Sometimes a trinomial will have a common factor that should be factored out first.

EXAMPLE 4 Factoring a Trinomial with a Common Factor

Factor $16y^3 - 32y^2 - 48y$.

Start by factoring out the greatest common factor, $16y$.

$$16y^3 - 32y^2 - 48y = 16y(y^2 - 2y - 3)$$

To factor $y^2 - 2y - 3$, look for two integers whose product is -3 and whose sum is -2. The necessary integers are -3 and 1, so

$$16y^3 - 32y^2 - 48y = 16y(y - 3)(y + 1).$$

Now Try Exercise 31.

CAUTION When factoring, always look for a common factor first. Remember to write the common factor as part of the answer.

OBJECTIVE 2 Factor trinomials when the coefficient of the squared term is not 1. We can use a generalization of the method shown in Objective 1 to factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$. To factor $3x^2 + 7x + 2$, for example, we first identify the values of a, b, and c.

$$ax^2 + bx + c$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$3x^2 + 7x + 2$$
$$a = 3, \quad b = 7, \quad c = 2$$

The product ac is $3 \cdot 2 = 6$, so we must find integers having a product of 6 and a sum of 7 (since the middle term has coefficient 7). The necessary integers are 1 and 6, so we write $7x$ as $1x + 6x$, or $x + 6x$, giving

$$3x^2 + 7x + 2 = 3x^2 + \underbrace{x + 6x}_{x + 6x = 7x} + 2.$$

Now we factor by grouping.

$$\begin{aligned} 3x^2 + x + 6x + 2 &= (3x^2 + x) + (6x + 2) \\ &= x(3x + 1) + 2(3x + 1) \\ 3x^2 + 7x + 2 &= (3x + 1)(x + 2) \end{aligned}$$

VIDEO

EXAMPLE 5 Factoring a Trinomial in $ax^2 + bx + c$ Form

Factor $12r^2 - 5r - 2$.

Since $a = 12$, $b = -5$, and $c = -2$, the product ac is -24. The two integers whose product is -24 and whose sum is -5 are -8 and 3.

$$12r^2 - 5r - 2 = 12r^2 + 3r - 8r - 2 \qquad \text{Write } -5r \text{ as } 3r - 8r.$$
$$= 3r(4r + 1) - 2(4r + 1) \qquad \text{Factor by grouping.}$$
$$= (4r + 1)(3r - 2) \qquad \text{Factor out the common factor.}$$

Now Try Exercise 19.

OBJECTIVE 3 Use an alternative method for factoring trinomials. Alternatively, trying repeated combinations and using FOIL is helpful when the product ac is large. This method is shown using the two polynomials we just factored.

EXAMPLE 6 Factoring Trinomials in $ax^2 + bx + c$ Form

Factor each trinomial.

(a) $3x^2 + 7x + 2$

To factor this polynomial, we must find the correct numbers to put in the blanks.

$$3x^2 + 7x + 2 = (\_\_x + \_\_)(\_\_x + \_\_)$$

Addition signs are used since all the signs in the polynomial indicate addition. The first two expressions have a product of $3x^2$, so they must be $3x$ and $1x$ or x.

$$3x^2 + 7x + 2 = (3x + \_\_)(x + \_\_)$$

The product of the two last terms must be 2, so the numbers must be 2 and 1. There is a choice. The 2 could be used with the $3x$ or with the x. Only one of these choices can give the correct middle term, $7x$. We use FOIL to try each one.

$3x$	$6x$
$(3x + 2)(x + 1)$	$(3x + 1)(x + 2)$
$2x$	x
$3x + 2x = 5x$	$6x + x = 7x$
Wrong middle term	Correct middle term

Therefore, $3x^2 + 7x + 2 = (3x + 1)(x + 2)$.

(b) $12r^2 - 5r - 2$

To reduce the number of trials, we note that the trinomial has no common factor. This means that neither of its factors can have a common factor. We should keep this in mind as we choose factors. We try 4 and 3 for the two first terms.

$$12r^2 - 5r - 2 = (4r\_\_)(3r\_\_)$$

We do not know what signs to use yet. The factors of -2 are -2 and 1 or -1 and 2. We try both possibilities to see if we obtain the correct middle term, $-5r$.

	$8r$
$(4r - 2)(3r + 1)$	$(4r - 1)(3r + 2)$
Wrong: $4r - 2$ has a common factor of 2.	$-3r$
	$8r - 3r = 5r$
	Wrong middle term

The middle term on the right is $5r$, instead of the $-5r$ that is needed. We get $-5r$ by interchanging the signs in the factors.

$$-8r$$

$$(4r + 1)(3r - 2)$$

$$3r$$

$$-8r + 3r = -5r$$

Correct middle term

Thus, $12r^2 - 5r - 2 = (4r + 1)(3r - 2)$.

Now Try Exercise 21.

NOTE As shown in Example 6(b), if the terms of a polynomial have no common factor (except 1), then none of the terms of its factors can have a common factor. Remembering this will eliminate some potential factors.

This alternative method of factoring a trinomial $ax^2 + bx + c$, $a \neq 1$, is summarized here.

AUDIO

Factoring $ax^2 + bx + c$

Step 1 **Find pairs whose product is *a*.** Write all pairs of integer factors of the coefficient of the squared term, a.

Step 2 **Find pairs whose product is *c*.** Write all pairs of integer factors of the last term, c.

Step 3 **Choose inner and outer terms.** Use FOIL and various combinations of the factors from Steps 1 and 2 until the necessary middle term is found.

If no such combinations exist, the trinomial is prime.

EXAMPLE 7 Factoring a Trinomial in Two Variables

Factor $18m^2 - 19mx - 12x^2$.

There is no common factor (except 1). Follow the steps to factor the trinomial. There are many possible factors of both 18 and -12. Try 6 and 3 for 18 and -3 and 4 for -12.

$(6m - 3x)(3m + 4x)$	$(6m + 4x)(3m - 3x)$
Wrong: common factor	Wrong: common factors

Since 6 and 3 do not work as factors of 18, try 9 and 2 instead, with 3 and -4 as factors of -12.

	$27mx$
$(9m + 3x)(2m - 4x)$	$(9m - 4x)(2m + 3x)$
Wrong: common factors	$-8mx$
	$27mx + (-8mx) = 19mx$
	Wrong middle term

The result on the right differs from the correct middle term only in sign, so interchange the signs in the factors. Check by multiplying.

$$18m^2 - 19mx - 12x^2 = (9m + 4x)(2m - 3x)$$

Now Try Exercise 23.

EXAMPLE 8 Factoring $ax^2 + bx + c$, $a < 0$

Factor $-3x^2 + 16x + 12$.

While it is possible to factor this polynomial directly, it is helpful to first factor out -1. Then proceed as in the earlier examples.

$$\begin{aligned} -3x^2 + 16x + 12 &= -1(3x^2 - 16x - 12) \\ &= -1(3x + 2)(x - 6) \\ &= -(3x + 2)(x - 6) \end{aligned}$$

This factored form can be written in other ways. Two of them are

$$(-3x - 2)(x - 6) \quad \text{and} \quad (3x + 2)(-x + 6).$$

Verify that these both give the original trinomial when multiplied.

Now Try Exercise 33.

EXAMPLE 9 Factoring a Trinomial with a Common Factor

Factor $16y^3 + 24y^2 - 16y$.

$$\begin{aligned} 16y^3 + 24y^2 - 16y &= 8y(2y^2 + 3y - 2) && \text{GCF} = 8y \\ &= 8y(2y - 1)(y + 2) && \text{Remember the common factor.} \end{aligned}$$

Now Try Exercise 39.

OBJECTIVE 4 Factor by substitution. Sometimes we can factor a more complicated polynomial by making a substitution of a variable for an expression.

EXAMPLE 10 Factoring a Polynomial Using Substitution

Factor $2(x + 3)^2 + 5(x + 3) - 12$.

Since the binomial $x + 3$ appears to powers 2 and 1, we let the substitution variable represent $x + 3$. We may choose any letter we wish except x. We choose y to equal $x + 3$.

$$\begin{aligned} 2(x + 3)^2 + 5(x + 3) - 12 &= 2y^2 + 5y - 12 && \text{Let } y = x + 3. \\ &= (2y - 3)(y + 4) && \text{Factor.} \end{aligned}$$

Now we replace y with $x + 3$ and simplify to get

$$\begin{aligned} 2(x + 3)^2 + 5(x + 3) - 12 &= [2(x + 3) - 3][(x + 3) + 4] \\ &= (2x + 6 - 3)(x + 7) \\ &= (2x + 3)(x + 7). \end{aligned}$$

Now Try Exercise 49.

CAUTION Remember to make the final substitution of $x + 3$ for y in Example 10.

EXAMPLE 11 Factoring a Trinomial in $ax^4 + bx^2 + c$ Form

Factor $6y^4 + 7y^2 - 20$.

The variable y appears to powers in which the larger exponent is twice the smaller exponent. In a case such as this, let the substitution variable equal the smaller power. Here, let $m = y^2$. Then $m^2 = (y^2)^2 = y^4$, and the given trinomial becomes

$$6m^2 + 7m - 20,$$

which is factored as

$$6m^2 + 7m - 20 = (3m - 4)(2m + 5).$$

Since $m = y^2$,

$$6y^4 + 7y^2 - 20 = (3y^2 - 4)(2y^2 + 5).$$

Now Try Exercise 59.

NOTE Some students feel comfortable enough about factoring to factor polynomials like the one in Example 11 directly, without using the substitution method.

6.2 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 9/Videotape 8

1. Which one of the following is *not* a valid way of starting the process of factoring $12x^2 + 29x + 10$?

A. $(12x \quad\quad)(x \quad\quad)$ **B.** $(4x \quad\quad)(3x \quad\quad)$
C. $(6x \quad\quad)(2x \quad\quad)$ **D.** $(8x \quad\quad)(4x \quad\quad)$

2. Which one of the following is the completely factored form of $2x^6 - 5x^5 - 3x^4$?

A. $x^4(2x + 1)(x - 3)$ **B.** $x^4(2x - 1)(x + 3)$
C. $(2x^5 + x^4)(x - 3)$ **D.** $x^3(2x^2 + x)(x - 3)$

3. Which one of the following is *not* a factored form of $-x^2 + 16x - 60$?

A. $(x - 10)(-x + 6)$ **B.** $(-x - 10)(x + 6)$
C. $(-x + 10)(x - 6)$ **D.** $-(x - 10)(x - 6)$

4. Which one of the following is the completely factored form of $4x^2 - 4x - 24$?

A. $4(x - 2)(x + 3)$ **B.** $4(x + 2)(x + 3)$
C. $4(x + 2)(x - 3)$ **D.** $4(x - 2)(x - 3)$

Factor each trinomial. See Examples 1–9.

EXERCISES

5. $y^2 + 7y - 30$ **6.** $z^2 + 2z - 24$ **7.** $p^2 + 15p + 56$ **8.** $k^2 - 11k + 30$

9. $-m^2 + 11m - 60$ **10.** $-p^2 + 12p + 27$ **11.** $a^2 - 2ab - 35b^2$

12. $z^2 + 8zw + 15w^2$ **13.** $y^2 - 3yq - 15q^2$ **14.** $k^2 - 11hk + 28h^2$

15. $x^2y^2 + 11xy + 18$ **16.** $p^2q^2 - 5pq - 18$ **17.** $-6m^2 - 13m + 15$

18. $-15y^2 + 17y + 18$ **19.** $10x^2 + 3x - 18$ **20.** $8k^2 + 34k + 35$

21. $20k^2 + 47k + 24$ **22.** $27z^2 + 42z - 5$ **23.** $15a^2 - 22ab + 8b^2$

EXERCISES

24. $15p^2 + 24pq + 8q^2$ **25.** $36m^2 - 60m + 25$ **26.** $25r^2 - 90r + 81$

27. $40x^2 + xy + 6y^2$ **28.** $14c^2 - 17cd - 6d^2$ **29.** $6x^2z^2 + 5xz - 4$

30. $8m^2n^2 - 10mn + 3$ **31.** $24x^2 + 42x + 15$ **32.** $36x^2 + 18x - 4$

33. $-15a^2 - 70a + 120$ **34.** $-12a^2 - 10a + 42$

35. $-11x^3 + 110x^2 - 264x$ **36.** $-9k^3 - 36k^2 + 189k$

37. $2x^3y^3 - 48x^2y^4 + 288xy^5$ **38.** $6m^3n^2 - 24m^2n^3 - 30mn^4$

39. $6a^3 + 12a^2 - 90a$ **40.** $3m^4 + 6m^3 - 72m^2$

41. $13y^3 + 39y^2 - 52y$ **42.** $4p^3 + 24p^2 - 64p$

43. $12p^3 - 12p^2 + 3p$ **44.** $45t^3 + 60t^2 + 20t$

45. When a student was given the polynomial $4x^2 + 2x - 20$ to factor completely on a test, the student lost some credit when her answer was $(4x + 10)(x - 2)$. She complained to her teacher that when we multiply $(4x + 10)(x - 2)$, we get the original polynomial. Write a short explanation of why she lost some credit for her answer, even though the product is indeed $4x^2 + 2x - 20$.

46. When factoring the polynomial $-4x^2 - 29x + 24$, Margo obtained $(-4x + 3)(x + 8)$, while Steve got $(4x - 3)(-x - 8)$. Who is correct? Explain your answer.

Factor each trinomial. See Example 10.

EXERCISES

47. $12p^6 - 32p^3r + 5r^2$ **48.** $2y^6 + 7xy^3 + 6x^2$

49. $10(k + 1)^2 - 7(k + 1) + 1$ **50.** $4(m - 5)^2 - 4(m - 5) - 15$

51. $3(m + p)^2 - 7(m + p) - 20$ **52.** $4(x - y)^2 - 23(x - y) - 6$

53. $a^2(a + b)^2 - ab(a + b)^2 - 6b^2(a + b)^2$

54. $m^2(m - p) + mp(m - p) - 2p^2(m - p)$

55. $p^2(p + q) + 4pq(p + q) + 3q^2(p + q)$

56. $2k^2(5 - y) - 7k(5 - y) + 5(5 - y)$

57. $z^2(z - x) - zx(x - z) - 2x^2(z - x)$

58. $r^2(r - s) - 5rs(s - r) - 6s^2(r - s)$

Factor each trinomial. See Example 11.

59. $p^4 - 10p^2 + 16$ **60.** $k^4 + 10k^2 + 9$ **61.** $2x^4 - 9x^2 - 18$

62. $6z^4 + z^2 - 1$ **63.** $16x^4 + 16x^2 + 3$ **64.** $9r^4 + 9r^2 + 2$

6.3 Special Factoring

OBJECTIVES

1. Factor a difference of squares.
2. Factor a perfect square trinomial.
3. Factor a difference of cubes.
4. Factor a sum of cubes.

OBJECTIVE 1 Factor a difference of squares. The special products introduced in Section 5.4 are used in reverse when factoring. Recall that the product of the sum and difference of two terms leads to a **difference of squares,** a pattern that occurs often when factoring.

Difference of Squares

$$x^2 - y^2 = (x + y)(x - y)$$

EXAMPLE 1 Factoring Differences of Squares

Factor each polynomial.

(a) $4a^2 - 64$

There is a common factor of 4.

$$\begin{aligned} 4a^2 - 64 &= 4(a^2 - 16) && \text{Factor out the common factor.} \\ &= 4(a + 4)(a - 4) && \text{Factor the difference of squares.} \end{aligned}$$

(b) $$16m^2 - 49p^2 = \overset{A^2}{(4m)^2} - \overset{B^2}{(7p)^2} = (\overset{A}{4m} + \overset{B}{7p})(\overset{A}{4m} - \overset{B}{7p})$$

(c) $$\begin{aligned} 81k^2 - (a + 2)^2 &= \overset{A^2}{(9k)^2} - \overset{B^2}{(a + 2)^2} = (\overset{A}{9k} + \overbrace{a + 2}^{B})(\overset{A}{9k} - \overbrace{(a + 2)}^{B}) \\ &= (9k + a + 2)(9k - a - 2) \end{aligned}$$

We could have used the method of substitution here.

(d) $$\begin{aligned} x^4 - 81 &= (x^2 + 9)(x^2 - 9) && \text{Factor the difference of squares.} \\ &= (x^2 + 9)(x + 3)(x - 3) && \text{Factor } x^2 - 9. \end{aligned}$$

Now Try Exercises 9, 13, and 15.

CAUTION Assuming no greatest common factor except 1, it is not possible to factor (with real numbers) a *sum* of squares such as $x^2 + 25$. In particular, $x^2 + y^2 \neq (x + y)^2$.

OBJECTIVE 2 Factor a perfect square trinomial. Two other special products from Section 5.4 lead to the following rules for factoring.

Perfect Square Trinomial

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

Because the trinomial $x^2 + 2xy + y^2$ is the square of $x + y$, it is called a **perfect square trinomial.** In this pattern, both the first and the last terms of the trinomial must be perfect squares. In the factored form, twice the product of the first and the last terms must give the middle term of the trinomial. It is important to understand these patterns in terms of words, since they occur with many different symbols (other than x and y).

$4m^2 + 20m + 25$	$p^2 - 8p + 64$
Perfect square trinomial	Not a perfect square trinomial; middle term should be $\pm 16p$.

EXAMPLE 2 Factoring Perfect Square Trinomials

YOU TRY IT

Factor each polynomial.

(a) $144p^2 - 120p + 25$

Here $144p^2 = (12p)^2$ and $25 = 5^2$. The sign on the middle term is $-$, so if $144p^2 - 120p + 25$ is a perfect square trinomial, the factored form will have to be

$$(12p - 5)^2.$$

Take twice the product of the two terms to see if this is correct.

$$2(12p)(-5) = -120p$$

This is the middle term of the given trinomial, so

$$144p^2 - 120p + 25 = (12p - 5)^2.$$

(b) $4m^2 + 20mn + 49n^2$

If this is a perfect square trinomial, it will equal $(2m + 7n)^2$. By the pattern in the box, if multiplied out, this squared binomial has a middle term of $2(2m)(7n) = 28mn$, which *does not equal* $20mn$. Verify that this trinomial cannot be factored by the methods of the previous section either. It is prime.

VIDEO

(c) $(r + 5)^2 + 6(r + 5) + 9 = [(r + 5) + 3]^2$

$$= (r + 8)^2,$$

since $2(r + 5)(3) = 6(r + 5)$, the middle term.

(d) $m^2 - 8m + 16 - p^2$

Since there are four terms, use factoring by grouping. The first three terms here form a perfect square trinomial. Group them together, and factor as follows.

$$(m^2 - 8m + 16) - p^2 = (m - 4)^2 - p^2$$

The result is the difference of squares. Factor again to get

$$(m - 4)^2 - p^2 = (m - 4 + p)(m - 4 - p).$$

Now Try Exercises 23, 25, and 33.

Perfect square trinomials, of course, can be factored using the general methods shown earlier for other trinomials. The patterns given here provide "shortcuts."

OBJECTIVE 3 Factor a difference of cubes. A **difference of cubes,** $x^3 - y^3$, can be factored as follows.

Difference of Cubes

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

We could check this pattern by finding the product of $x - y$ and $x^2 + xy + y^2$.

EXAMPLE 3 Factoring Differences of Cubes

Factor each polynomial.

(a) $m^3 - 8 = m^3 - 2^3 = (m - 2)(m^2 + 2m + 2^2) = (m - 2)(m^2 + 2m + 4)$

Check:

$$m^3 \qquad -8$$

$$(m - 2)(m^2 + 2m + 4)$$

$$-2m$$

Opposite of the product of the cube roots gives the middle term.

(b) $27x^3 - 8y^3 = (3x)^3 - (2y)^3$

$$= (3x - 2y)[(3x)^2 + (3x)(2y) + (2y)^2]$$

$$= (3x - 2y)(9x^2 + 6xy + 4y^2)$$

(c) $1000k^3 - 27n^3 = (10k)^3 - (3n)^3$

$$= (10k - 3n)[(10k)^2 + (10k)(3n) + (3n)^2]$$

$$= (10k - 3n)(100k^2 + 30kn + 9n^2)$$

Now Try Exercises 37 and 39.

OBJECTIVE 4 Factor a sum of cubes. While an expression of the form $x^2 + y^2$ (a sum of squares) cannot be factored with real numbers, a **sum of cubes** is factored as follows.

Sum of Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

To verify this result, find the product of $x + y$ and $x^2 - xy + y^2$. Compare this pattern with the pattern for a difference of cubes.

NOTE The sign of the second term in the binomial factor of a sum or difference of cubes is *always the same* as the sign in the original polynomial. In the trinomial factor, the first and last terms are *always positive;* the sign of the middle term is *the opposite of* the sign of the second term in the binomial factor.

EXAMPLE 4 Factoring Sums of Cubes

Factor each polynomial.

(a) $r^3 + 27 = r^3 + 3^3 = (r + 3)(r^2 - 3r + 3^2)$

$= (r + 3)(r^2 - 3r + 9)$

(b) $27z^3 + 125 = (3z)^3 + 5^3 = (3z + 5)[(3z)^2 - (3z)(5) + 5^2]$

$= (3z + 5)(9z^2 - 15z + 25)$

(c) $(x + 2)^3 + t^3 = [(x + 2) + t][(x + 2)^2 - (x + 2)t + t^2]$

$= (x + 2 + t)(x^2 + 4x + 4 - xt - 2t + t^2)$

Now Try Exercises 41 and 43.

CAUTION A common error is to think that the xy-term has a coefficient of 2 when factoring the sum or difference of cubes. Since there is no coefficient of 2, expressions of the form $x^2 + xy + y^2$ and $x^2 - xy + y^2$ usually cannot be factored further.

The special types of factoring in this section are summarized here. *These should be memorized.*

Special Types of Factoring

Difference of Squares	$x^2 - y^2 = (x + y)(x - y)$
Perfect Square Trinomial	$x^2 + 2xy + y^2 = (x + y)^2$
	$x^2 - 2xy + y^2 = (x - y)^2$
Difference of Cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of Cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

6.4 A General Approach to Factoring

OBJECTIVES

1. Factor out any common factor.
2. Factor binomials.
3. Factor trinomials.
4. Factor polynomials of more than three terms.

In this section, we summarize and apply the factoring methods presented in the preceding sections. A polynomial is completely factored when the polynomial is in the following form.

1. The polynomial is written as a product of prime polynomials with integer coefficients.

2. None of the polynomial factors can be factored further, except that a monomial factor need not be factored completely.

Factoring a Polynomial

Step 1 **Factor out any common factor.**

Step 2 **If the polynomial is a binomial,** check to see if it is the difference of squares, the difference of cubes, or the sum of cubes.

If the polynomial is a trinomial, check to see if it is a perfect square trinomial. If it is not, factor as in Section 6.2.

If the polynomial has more than three terms, try to factor by grouping.

Step 3 **Check** the factored form by multiplying.

OBJECTIVE 1 Factor out any common factor. This step is always the same, regardless of the number of terms in the polynomial.

EXAMPLE 1 Factoring Out a Common Factor

Factor each polynomial.

(a) $9p + 45 = 9(p + 5)$ **(b)** $8m^2p^2 + 4mp = 4mp(2mp + 1)$

(c) $5x(a + b) - y(a + b) = (a + b)(5x - y)$

Now Try Exercises 13 and 23.

OBJECTIVE 2 Factor binomials. Use one of the following rules.

For a **binomial** (two terms), check for the following.

Difference of squares	$x^2 - y^2 = (x + y)(x - y)$
Difference of cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

EXAMPLE 2 Factoring Binomials

Factor each polynomial, if possible.

(a) $64m^2 - 9n^2 = (8m)^2 - (3n)^2$ Difference of squares

$= (8m + 3n)(8m - 3n)$

(b) $8p^3 - 27 = (2p)^3 - 3^3$ Difference of cubes

$= (2p - 3)[(2p)^2 + (2p)(3) + 3^2]$

$= (2p - 3)(4p^2 + 6p + 9)$

(c) $100m^3 + 1 = (10m)^3 + 1^3$ Sum of cubes

$= (10m + 1)[(10m)^2 - (10m)(1) + 1^2]$

$= (10m + 1)(100m^2 - 10m + 1)$

(d) $25m^2 + 121$ is prime. It is the sum of squares.

Now Try Exercises 7, 11, and 29.

OBJECTIVE 3 Factor trinomials.

For a **trinomial** (three terms), decide if it is a perfect square trinomial of the form

$$x^2 + 2xy + y^2 = (x + y)^2, \quad \text{or} \quad x^2 - 2xy + y^2 = (x - y)^2,$$

or, if not, use the methods of Section 6.2.

EXAMPLE 3 Factoring Trinomials

Factor each trinomial.

(a) $p^2 + 10p + 25 = (p + 5)^2$ Perfect square trinomial

(b) $49z^2 - 42z + 9 = (7z - 3)^2$ Perfect square trinomial

(c) $y^2 - 5y - 6 = (y - 6)(y + 1)$
The numbers -6 and 1 have a product of -6 and a sum of -5.

(d) $r^2 + 18r + 72 = (r + 6)(r + 12)$

(e) $2k^2 - k - 6 = (2k + 3)(k - 2)$
Use either method from Section 6.2.

(f)
$$\begin{aligned} 28z^2 + 6z - 10 &= 2(14z^2 + 3z - 5) && \text{Factor out the common factor.} \\ &= 2(7z + 5)(2z - 1) \end{aligned}$$

Now Try Exercises 9, 19, and 41.

OBJECTIVE 4 Factor polynomials of more than three terms. Try factoring by grouping.

EXAMPLE 4 Factoring Polynomials with More Than Three Terms

Factor each polynomial.

(a)
$$\begin{aligned} xy^2 - y^3 + x^3 - x^2y &= (xy^2 - y^3) + (x^3 - x^2y) \\ &= y^2(x - y) + x^2(x - y) \\ &= (x - y)(y^2 + x^2) \end{aligned}$$

(b)
$$\begin{aligned} 20k^3 + 4k^2 - 45k - 9 &= (20k^3 + 4k^2) - (45k + 9) && \text{Be careful with signs.} \\ &= 4k^2(5k + 1) - 9(5k + 1) \\ &= (5k + 1)(4k^2 - 9) && 5k + 1 \text{ is a common factor.} \\ &= (5k + 1)(2k + 3)(2k - 3) && \text{Difference of squares} \end{aligned}$$

(c)
$$\begin{aligned} 4a^2 + 4a + 1 - b^2 &= (4a^2 + 4a + 1) - b^2 && \text{Associative property} \\ &= (2a + 1)^2 - b^2 && \text{Perfect square trinomial} \\ &= (2a + 1 + b)(2a + 1 - b) && \text{Difference of squares} \end{aligned}$$

(d) $8m^3 + 4m^2 - n^3 - n^2$
First, notice that the terms must be rearranged before grouping because

$$(8m^3 + 4m^2) - (n^3 + n^2) = 4m^2(2m + 1) - n^2(n + 1),$$

which cannot be factored further. Write the polynomial as follows.

$$\begin{aligned} 8m^3 + 4m^2 - n^3 - n^2 &= (8m^3 - n^3) + (4m^2 - n^2) && \text{Group the cubes and squares.} \\ &= (2m - n)(4m^2 + 2mn + n^2) + (2m - n)(2m + n) && \text{Factor each group.} \\ &= (2m - n)(4m^2 + 2mn + n^2 + 2m + n) && \text{Factor out the common factor } 2m - n. \end{aligned}$$

Now Try Exercises 21 and 45.

6.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 9/Videotape 9

Factor each polynomial. See Examples 1–4.

1. $100a^2 - 9b^2$
2. $10r^2 + 13r - 3$
3. $3p^4 - 3p^3 - 90p^2$
4. $k^4 - 16$
5. $3a^2pq + 3abpq - 90b^2pq$
6. $49z^2 - 16$
7. $225p^2 + 256$
8. $18m^3n + 3m^2n^2 - 6mn^3$
9. $6b^2 - 17b - 3$
10. $k^2 - 6k - 16$
11. $x^3 - 1000$
12. $6t^2 + 19tu - 77u^2$
13. $4(p + 2) + m(p + 2)$
14. $40p - 32r$
15. $9m^2 - 45m + 18m^3$
16. $4k^2 + 28kr + 49r^2$
17. $54m^3 - 2000$
18. $mn - 2n + 5m - 10$
19. $9m^2 - 30mn + 25n^2$
20. $2a^2 - 7a - 4$
21. $kq - 9q + kr - 9r$
22. $56k^3 - 875$
23. $16z^3x^2 - 32z^2x$
24. $9r^2 + 100$
25. $x^2 + 2x - 35$
26. $9 - a^2 + 2ab - b^2$
27. $x^4 - 625$
28. $2m^2 - mn - 15n^2$
29. $p^3 + 64$
30. $48y^2z^3 - 28y^3z^4$
31. $64m^2 - 625$
32. $14z^2 - 3zk - 2k^2$
33. $12z^3 - 6z^2 + 18z$
34. $225k^2 - 36r^2$
35. $256b^2 - 400c^2$
36. $z^2 - zp - 20p^2$
37. $1000z^3 + 512$
38. $64m^2 - 25n^2$
39. $10r^2 + 23rs - 5s^2$
40. $12k^2 - 17kq - 5q^2$
41. $24p^3q + 52p^2q^2 + 20pq^3$
42. $32x^2 + 16x^3 - 24x^5$
43. $48k^4 - 243$
44. $14x^2 - 25xq - 25q^2$
45. $m^3 + m^2 - n^3 - n^2$
46. $64x^3 + y^3 - 16x^2 + y^2$
47. $x^2 - 4m^2 - 4mn - n^2$
48. $4r^2 - s^2 - 2st - t^2$
49. $18p^5 - 24p^3 + 12p^6$
50. $k^2 - 6k + 16$
51. $2x^2 - 2x - 40$
52. $27x^3 - 3y^3$
53. $(2m + n)^2 - (2m - n)^2$
54. $(3k + 5)^2 - 4(3k + 5) + 4$
55. $50p^2 - 162$
56. $y^2 + 3y - 10$
57. $12m^2rx + 4mnrx + 40n^2rx$
58. $18p^2 + 53pr - 35r^2$
59. $21a^2 - 5ab - 4b^2$
60. $x^2 - 2xy + y^2 - 4$
61. $x^2 - y^2 - 4$
62. $(5r + 2s)^2 - 6(5r + 2s) + 9$
63. $(p + 8q)^2 - 10(p + 8q) + 25$
64. $z^4 - 9z^2 + 20$
65. $21m^4 - 32m^2 - 5$
66. $(x - y)^3 - (27 - y)^3$
67. $(r + 2t)^3 + (r - 3t)^3$
68. $16x^3 + 32x^2 - 9x - 18$
69. $x^5 + 3x^4 - x - 3$
70. $x^{16} - 1$
71. $m^2 - 4m + 4 - n^2 + 6n - 9$
72. $x^2 + 4 + x^2y + 4y$

6.5 Solving Equations by Factoring

OBJECTIVES

1. Learn and use the zero-factor property.
2. Solve applied problems that require the zero-factor property.

In the previous four sections, we factored polynomial *expressions*. Now we can use factoring to solve polynomial *equations*. In Chapter 2, we developed methods for solving linear, or first-degree, equations. Solving higher-degree polynomial equations requires other methods.

OBJECTIVE 1 Learn and use the zero-factor property. Some polynomial equations can be solved by factoring. Solving equations by factoring depends on a special property of the number 0, called the **zero-factor property.**

Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if $\boldsymbol{ab = 0}$, then either $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$.

To prove the zero-factor property, we first assume $a \neq 0$. (If a does equal 0, then the property is proved already.) If $a \neq 0$, then $\frac{1}{a}$ exists, and both sides of $ab = 0$ can be multiplied by $\frac{1}{a}$ to get

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$
$$b = 0.$$

Thus, if $a \neq 0$, then $b = 0$, and the property is proved.

CAUTION If $ab = 0$, then $a = 0$ or $b = 0$. However, if $ab = 6$, for example, it is not necessarily true that $a = 6$ or $b = 6$; in fact, it is very likely that *neither* $a = 6$ *nor* $b = 6$. *The zero-factor property works only for a product equal to 0.*

EXAMPLE 1 Using the Zero-Factor Property to Solve an Equation

Solve $(x + 6)(2x - 3) = 0$.

Here the product of $x + 6$ and $2x - 3$ is 0. By the zero-factor property, this can be true only if

$$x + 6 = 0 \quad \text{or} \quad 2x - 3 = 0.$$

Solve these two equations.

$$\begin{aligned} x + 6 &= 0 & \text{or} \quad 2x - 3 &= 0 \\ x &= -6 & \text{or} \quad 2x &= 3 \\ & & x &= \frac{3}{2} \end{aligned}$$

Check these two solutions by substitution in the original equation.

If $x = -6$, then

$$\begin{aligned} (x + 6)(2x - 3) &= 0 \\ (-6 + 6)[2(-6) - 3] &= 0 \quad ? \\ 0(-15) &= 0 \quad ? \\ 0 &= 0. \quad \text{True} \end{aligned}$$

If $x = \frac{3}{2}$, then

$$\begin{aligned} (x + 6)(2x - 3) &= 0 \\ \left(\frac{3}{2} + 6\right)\left(2 \cdot \frac{3}{2} - 3\right) &= 0 \quad ? \\ \frac{15}{2}(0) &= 0 \quad ? \\ 0 &= 0. \quad \text{True} \end{aligned}$$

Both solutions check; the solution set is $\left\{-6, \frac{3}{2}\right\}$.

Now Try Exercise 5.

Since the product $(x + 6)(2x - 3)$ equals $2x^2 + 9x - 18$, the equation of Example 1 has a squared term and is an example of a *quadratic equation.* A quadratic equation has degree 2.

Quadratic Equation

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, with $a \neq 0$, is a **quadratic equation.** This form is called **standard form.**

Quadratic equations are discussed in more detail in Chapter 9.

The steps involved in solving a quadratic equation by factoring are summarized here.

Solving a Quadratic Equation by Factoring

Step 1 **Write in standard form.** Rewrite the equation if necessary so that one side is 0.

Step 2 **Factor** the polynomial.

Step 3 **Use the zero-factor property.** Set each variable factor equal to 0.

Step 4 **Find the solution(s).** Solve each equation formed in Step 3.

Step 5 **Check** each solution in the *original* equation.

EXAMPLE 2 Solving Quadratic Equations by Factoring

Solve each equation.

(a) $2x^2 + 3x = 2$

Step 1
$$\begin{aligned} 2x^2 + 3x &= 2 \\ 2x^2 + 3x - 2 &= 0 \quad \text{Standard form} \end{aligned}$$

Step 2
$$(2x - 1)(x + 2) = 0 \quad \text{Factor.}$$

Step 3 $2x - 1 = 0$ or $x + 2 = 0$ Zero-factor property

Step 4 $2x = 1$ or $x = -2$ Solve each equation.

$$x = \frac{1}{2}$$

Step 5 Check each solution in the original equation.

If $x = \frac{1}{2}$, then

$$2x^2 + 3x = 2$$
$$2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) = 2 \quad ?$$
$$2\left(\frac{1}{4}\right) + \frac{3}{2} = 2 \quad ?$$
$$\frac{1}{2} + \frac{3}{2} = 2 \quad ?$$
$$2 = 2. \quad \text{True}$$

If $x = -2$, then

$$2x^2 + 3x = 2$$
$$2(-2)^2 + 3(-2) = 2 \quad ?$$
$$2(4) - 6 = 2 \quad ?$$
$$8 - 6 = 2 \quad ?$$
$$2 = 2. \quad \text{True}$$

Because both solutions check, the solution set is $\left\{\frac{1}{2}, -2\right\}$.

(b) $4x^2 = 4x - 1$

$$4x^2 - 4x + 1 = 0 \quad \text{Standard form}$$
$$(2x - 1)^2 = 0 \quad \text{Factor.}$$
$$2x - 1 = 0 \quad \text{Zero-factor property}$$
$$2x = 1$$
$$x = \frac{1}{2}$$

There is only one solution because the trinomial is a perfect square. The solution set is $\left\{\frac{1}{2}\right\}$.

Now Try Exercises 11 and 29.

EXAMPLE 3 Solving a Quadratic Equation with a Missing Term

Solve $5z^2 - 25z = 0$.

This quadratic equation has a missing term. Comparing it with the standard form $ax^2 + bx + c = 0$ shows that $c = 0$. The zero-factor property can still be used.

$$5z^2 - 25z = 0$$
$$5z(z - 5) = 0 \quad \text{Factor.}$$
$$5z = 0 \quad \text{or} \quad z - 5 = 0 \quad \text{Zero-factor property}$$
$$z = 0 \quad \text{or} \quad z = 5$$

The solutions are 0 and 5, as can be verified by substituting in the original equation. The solution set is $\{0, 5\}$.

Now Try Exercise 19.

CAUTION Remember to include the solution 0 when writing the solution set of the equation in Example 3.

EXAMPLE 4 Solving an Equation That Requires Rewriting

Solve $(2q + 1)(q + 1) = 2(1 - q) + 6$.

Write the equation in standard form $ax^2 + bx + c = 0$ by first multiplying on each side.

$$(2q + 1)(q + 1) = 2(1 - q) + 6$$
$$2q^2 + 3q + 1 = 2 - 2q + 6$$
$$2q^2 + 5q - 7 = 0 \qquad \text{Standard form}$$
$$(2q + 7)(q - 1) = 0 \qquad \text{Factor.}$$
$$2q + 7 = 0 \quad \text{or} \quad q - 1 = 0 \qquad \text{Zero-factor property}$$
$$2q = -7 \quad \text{or} \quad q = 1$$
$$q = -\frac{7}{2}$$

Check that the solution set is $\left\{-\frac{7}{2}, 1\right\}$.

Now Try Exercise 35.

The zero-factor property can be extended to solve certain polynomial equations of degree 3 or higher, as shown in the next example.

EXAMPLE 5 Solving an Equation of Degree 3

Solve $-x^3 + x^2 = -6x$.

Start by adding $6x$ to each side to get 0 on the right side.

$$-x^3 + x^2 + 6x = 0$$

To make the factoring step easier, multiply each side by -1.

$$x^3 - x^2 - 6x = 0$$
$$x(x^2 - x - 6) = 0 \qquad \text{Factor out } x.$$
$$x(x - 3)(x + 2) = 0 \qquad \text{Factor the trinomial.}$$

Use the zero-factor property, extended to include the three variable factors.

$$x = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 3 \quad \text{or} \quad x = -2$$

Check that the solution set is $\{0, 3, -2\}$.

Now Try Exercise 37.

OBJECTIVE 2 Solve applied problems that require the zero-factor property. The next example shows an application that leads to a quadratic equation. We continue to use the six-step problem-solving method introduced in Chapter 2.

EXAMPLE 6 Using a Quadratic Equation in an Application

Some surveyors are surveying a lot that is in the shape of a parallelogram. They find that the longer sides of the parallelogram are each 8 m longer than the distance

between them. The area of the lot is 48 m^2. Find the length of the longer sides and the distance between them.

Step 1 **Read** the problem again. There will be two answers.

Step 2 **Assign a variable.** Let x represent the distance between the longer sides. Then $x + 8$ is the length of each longer side. See Figure 1.

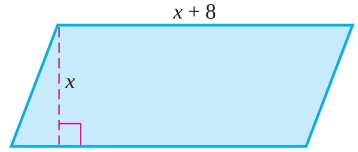


FIGURE 1

Step 3 **Write an equation.** The area of a parallelogram is given by $A = bh$, where b is the length of the longer side and h is the distance between the longer sides. Here $b = x + 8$ and $h = x$.

$$A = bh$$
$$48 = (x + 8)x \qquad \text{Let } A = 48,\ b = x + 8,\ h = x.$$

Step 4 **Solve.**

$$48 = x^2 + 8x \qquad \text{Distributive property}$$
$$0 = x^2 + 8x - 48 \qquad \text{Standard form}$$
$$0 = (x + 12)(x - 4) \qquad \text{Factor.}$$
$$x + 12 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{Zero-factor property}$$
$$x = -12 \quad \text{or} \quad x = 4$$

Step 5 **State the answer.** A distance cannot be negative, so reject -12 as a solution. The only possible solution is 4, so the distance between the longer sides is 4 m. The length of the longer sides is $4 + 8 = 12$ m.

Step 6 **Check.** The length of the longer sides is 8 m more than the distance between them, and the area is $4 \cdot 12 = 48$ m^2 as required, so the answer checks.

Now Try Exercise 57.

CAUTION When applications lead to quadratic equations, a solution of the equation may not satisfy the physical requirements of the problem, as in Example 6. Reject such solutions.

A function defined by a quadratic polynomial is called a *quadratic function.* In Chapter 9 we investigate quadratic functions in detail. The next example uses such a function.

EXAMPLE 7 Using a Quadratic Function in an Application

Quadratic functions are used to describe the height a falling object or a propelled object reaches in a specific time. For example, if a toy rocket is launched vertically

Photo not available

upward from ground level with an initial velocity of 128 ft per sec, then its height in feet after t sec is a function defined by

$$h(t) = -16t^2 + 128t,$$

if air resistance is neglected. After how many seconds will the rocket be 220 ft above the ground?

We must let $h(t) = 220$ and solve for t.

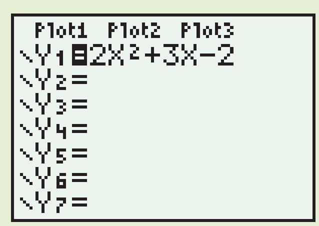
$$\begin{aligned}
220 &= -16t^2 + 128t && \text{Let } h(t) = 220.\\
16t^2 - 128t + 220 &= 0 && \text{Standard form}\\
4t^2 - 32t + 55 &= 0 && \text{Divide by 4.}\\
(2t - 11)(2t - 5) &= 0 && \text{Factor.}\\
2t - 11 = 0 \quad \text{or} \quad 2t - 5 &= 0 && \text{Zero-factor property}\\
t = 5.5 \quad \text{or} \quad t &= 2.5
\end{aligned}$$

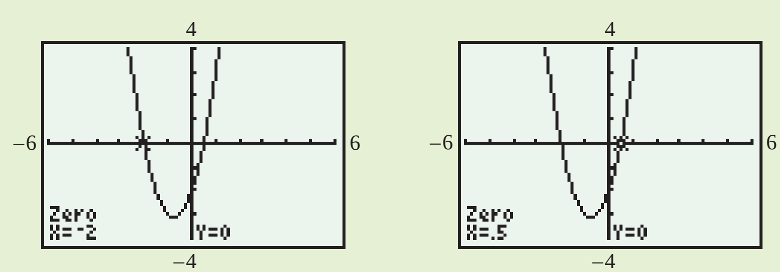
The rocket will reach a height of 220 ft twice: on its way up at 2.5 sec and again on its way down at 5.5 sec.

Now Try Exercise 65.

CONNECTIONS

In Section 5.3 we saw that the graph of $f(x) = x^2$ is a parabola. In general, the graph of $f(x) = ax^2 + bx + c$, $a \neq 0$, is a parabola, and the x-intercepts of its graph give the real number solutions of the equation $ax^2 + bx + c = 0$. In the screens, we show how a graphing calculator can locate these x-intercepts (called *zeros* of the function) for $Y_1 = f(X) = 2X^2 + 3X - 2$. Notice that this quadratic expression was found on the left side of the equation in Example 2(a) earlier in this section, where the equation was written in standard form.





The x-intercepts (zeros) given with the graphs are the same as the solutions found in Example 2(a). This method of graphical solution can be used for any type of equation.

6.5 EXERCISES

For Extra Help

Student's Solutions Manual

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AW Math Tutor Center

MathXL

Digital Video Tutor CD 10/Videotape 9

1. Explain in your own words how the zero-factor property is used in solving a quadratic equation.

2. One of the following equations is *not* in proper form for using the zero-factor property. Which one is it? Explain why it is not in proper form.

A. $(x + 2)(x - 6) = 0$ **B.** $x(3x - 7) = 0$
C. $3t(t + 8)(t - 9) = 0$ **D.** $y(y - 3) + 6(y - 3) = 0$

Solve each equation using the zero-factor property. See Example 1.

3. $(x - 5)(x + 10) = 0$ **4.** $(x + 3)(x + 7) = 0$
5. $(2k - 5)(3k + 8) = 0$ **6.** $(3q - 4)(2q + 5) = 0$

Solve each equation. See Examples 2–4.

7. $m^2 - 3m - 10 = 0$ **8.** $x^2 + x - 12 = 0$ **9.** $z^2 + 9z + 18 = 0$
10. $x^2 - 18x + 80 = 0$ **11.** $2x^2 = 7x + 4$ **12.** $2x^2 = 3 - x$
13. $15k^2 - 7k = 4$ **14.** $3c^2 + 3 = -10c$
15. $2x^2 - 12 - 4x = x^2 - 3x$ **16.** $3p^2 + 9p + 30 = 2p^2 - 2p$
17. $(5z + 1)(z + 3) = -2(5z + 1)$ **18.** $(3x + 1)(x - 3) = 2 + 3(x + 5)$
19. $4p^2 + 16p = 0$ **20.** $2a^2 - 8a = 0$
21. $6m^2 - 36m = 0$ **22.** $-3m^2 + 27m = 0$
23. $-3m^2 + 27 = 0$ **24.** $-2a^2 + 8 = 0$
25. $4p^2 - 16 = 0$ **26.** $9x^2 - 81 = 0$
27. $-x^2 = 9 - 6x$ **28.** $-m^2 - 8m = 16$
29. $9k^2 + 24k + 16 = 0$ **30.** $4m^2 - 20m + 25 = 0$
31. $(x - 3)(x + 5) = -7$ **32.** $(x + 8)(x - 2) = -21$
33. $(2x + 1)(x - 3) = 6x + 3$ **34.** $(3x + 2)(x - 3) = 7x - 1$
35. $(x + 3)(x - 6) = (2x + 2)(x - 6)$ **36.** $(2x + 1)(x + 5) = (x + 11)(x + 3)$

Solve each equation. See Example 5.

37. $2x^3 - 9x^2 - 5x = 0$ **38.** $6x^3 - 13x^2 - 5x = 0$
39. $9t^3 = 16t$ **40.** $25x^3 = 64x$
41. $2r^3 + 5r^2 - 2r - 5 = 0$ **42.** $2p^3 + p^2 - 98p - 49 = 0$
43. $-x^3 + 6x^2 + 9x - 54 = 0$ **44.** $6t^3 + 5t^2 - 6t - 5 = 0$
45. $x^3 - 3x^2 - 4x + 12 = 0$

46. A student tried to solve the equation in Exercise 39 by first dividing each side by t, obtaining $9t^2 = 16$. She then solved the resulting equation by the zero-factor property to get the solution set $\left\{-\frac{4}{3}, \frac{4}{3}\right\}$. What was incorrect about her procedure?

47. Without actually solving each equation, determine which one of the following has 0 in its solution set.

A. $4x^2 - 25 = 0$ **B.** $x^2 + 2x - 3 = 0$ **C.** $6x^2 + 9x + 1 = 0$ **D.** $x^3 + 4x^2 = 3x$

Solve each equation.

48. $2(x - 1)^2 - 7(x - 1) - 15 = 0$

49. $4(2k + 3)^2 - (2k + 3) - 3 = 0$

50. $5(3a - 1)^2 + 3 = -16(3a - 1)$

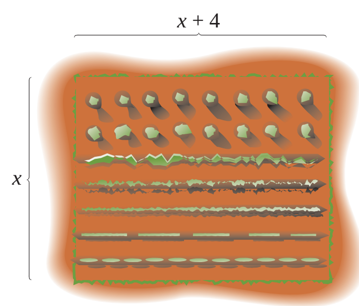
51. $2(m + 3)^2 = 5(m + 3) - 2$

52. $(2k - 3)^2 = 16k^2$

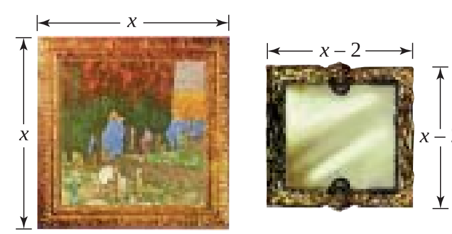
53. $9p^2 = (5p + 2)^2$

Solve each problem. See Examples 6 and 7.

54. A garden has an area of 320 ft^2. Its length is 4 ft more than its width. What are the dimensions of the garden?



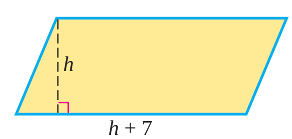
55. A square mirror has sides measuring 2 ft less than the sides of a square painting. If the difference between their areas is 32 ft^2, find the lengths of the sides of the mirror and the painting.



EXERCISES

56. A sign has the shape of a triangle. The length of the base is 3 m less than the height. What are the measures of the base and the height, if the area is 44 m^2?

57. The base of a parallelogram is 7 ft more than the height. If the area of the parallelogram is 60 ft^2, what are the measures of the base and the height?



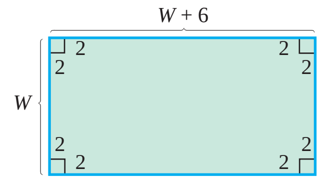
58. A farmer has 300 ft of fencing and wants to enclose a rectangular area of 5000 ft^2. What dimensions should she use?

59. A rectangular landfill has an area of 30,000 ft^2. Its length is 200 ft more than its width. What are the dimensions of the landfill?

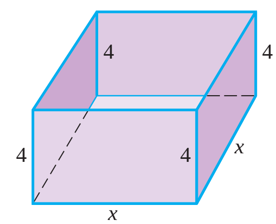
60. Find two consecutive integers such that the sum of their squares is 61.

61. Find two consecutive integers such that their product is 72.

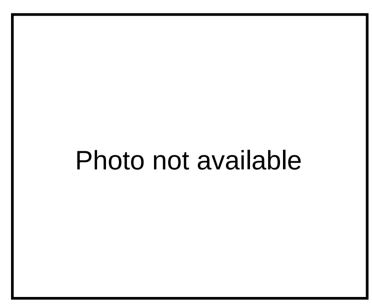
62. A box with no top is to be constructed from a piece of cardboard whose length measures 6 in. more than its width. The box is to be formed by cutting squares that measure 2 in. on each side from the four corners, and then folding up the sides. If the volume of the box will be 110 in.^{3}, what are the dimensions of the piece of cardboard?



63. The surface area of the box with open top shown in the figure is 161 in.^{2}. Find the dimensions of the base. (*Hint:* The surface area of the box is a function defined by $S(x) = x^2 + 16x$.)



64. Refer to Example 7. After how many seconds will the rocket be 240 ft above the ground? 112 ft above the ground?



65. If an object is propelled upward with an initial velocity of 64 ft per sec from a height of 80 ft, then its height in feet t sec after it is propelled is a function defined by

$$f(t) = -16t^2 + 64t + 80.$$

How long after it is propelled will it hit the ground? (*Hint:* When it hits the ground, its height is 0 ft.)

66. If a baseball is dropped from a helicopter 625 ft above the ground, then its distance in feet from the ground t sec later is a function defined by

$$f(t) = -16t^2 + 625.$$

How long after it is dropped will it hit the ground?

67. If a rock is dropped from a building 576 ft high, then its distance in feet from the ground t sec later is a function defined by

$$f(t) = -16t^2 + 576.$$

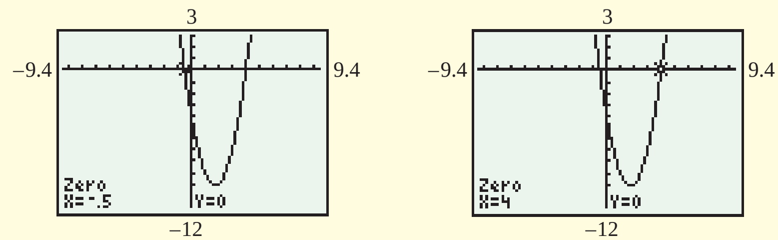
How long after it is dropped will it hit the ground?

TECHNOLOGY INSIGHTS (EXERCISES 68–71)

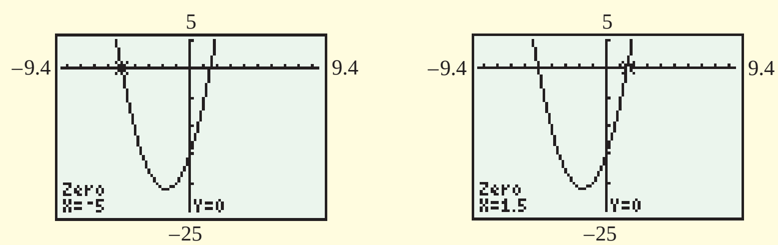
As shown in the Connections box following Example 7, the solutions of the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are represented on the graph of the quadratic function $f(x) = ax^2 + bx + c$ by the x-intercepts.

Solve each equation using the zero-factor property, and confirm that your solutions correspond to the x-intercepts (zeros) shown on the accompanying graphing calculator screens.

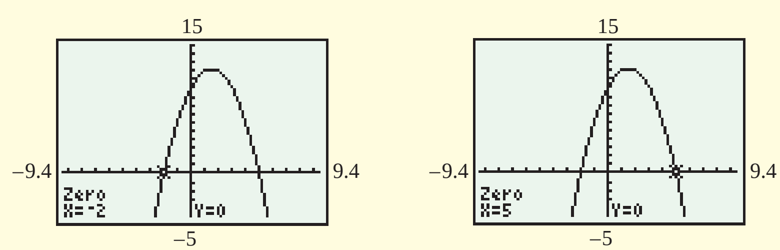
68. $2x^2 - 7x - 4 = 0$



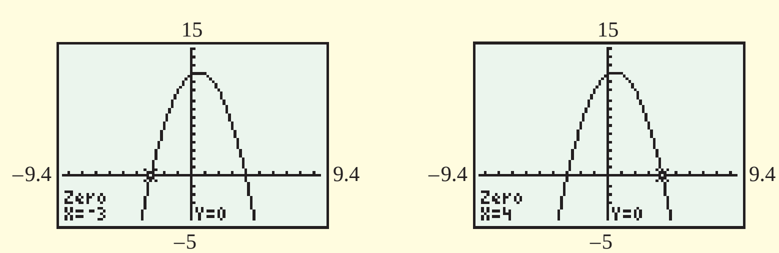
69. $2x^2 + 7x - 15 = 0$



70. $-x^2 + 3x = -10$



71. $-x^2 + x = -12$



7.1 Rational Expressions and Functions; Multiplying and Dividing

OBJECTIVES

1. Define rational expressions.
2. Define rational functions and describe their domains.
3. Write rational expressions in lowest terms.
4. Multiply rational expressions.
5. Find reciprocals for rational expressions.
6. Divide rational expressions.

OBJECTIVE 1 Define rational expressions. In arithmetic, a rational number is the quotient of two integers, with the denominator not 0. In algebra, a **rational expression** or *algebraic fraction* is the quotient of two polynomials, again with the denominator not 0. For example,

$$\frac{x}{y}, \quad \frac{-a}{4}, \quad \frac{m+4}{m-2}, \quad \frac{8x^2-2x+5}{4x^2+5x}, \quad \text{and} \quad x^5 \left(\text{or } \frac{x^5}{1}\right)$$

are all rational expressions. In other words, rational expressions are the elements of the set

$$\left\{ \frac{P}{Q} \,\middle|\, P, Q \text{ polynomials, with } Q \neq 0 \right\}.$$

OBJECTIVE 2 Define rational functions and describe their domains. A function that is defined by a rational expression is called a **rational function** and has the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $Q(x) \neq 0$.

The domain of a rational function includes all real numbers except those that make $Q(x)$—that is, the denominator—equal to 0. For example, the domain of

$$f(x) = \frac{2}{x-5}$$

includes all real numbers except 5, because 5 would make the denominator equal to 0.

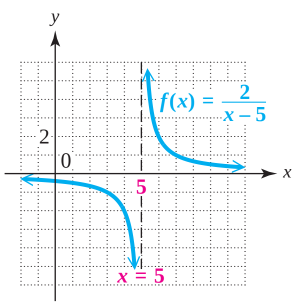


FIGURE 1

Figure 1 shows a graph of the function defined by $f(x) = \frac{2}{x-5}$. Notice that the graph does not exist when $x = 5$. It does not intersect the dashed vertical line whose equation is $x = 5$. This line is an *asymptote*. We discuss graphs of rational functions in more detail in Section 7.4.

YOU TRY IT

EXAMPLE 1 Finding Numbers That Are Not in the Domains of Rational Functions

Find all numbers that are not in the domain of each rational function, and then state the domain.

(a) $f(x) = \frac{3}{7x - 14}$

The only values that cannot be used are those that make the denominator 0. To find these values, set the denominator equal to 0 and solve the resulting equation.

$$7x - 14 = 0$$
$$7x = 14 \qquad \text{Add 14.}$$
$$x = 2 \qquad \text{Divide by 7.}$$

The number 2 cannot be used as a replacement for x; the domain of f includes all real numbers except 2.

(b) $g(x) = \dfrac{3 + x}{x^2 - 4x + 3}$

Set the denominator equal to 0, and solve the equation.

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x - 3)(x - 1) &= 0 \qquad \text{Factor.} \\ x - 3 = 0 \quad \text{or} \quad x - 1 &= 0 \qquad \text{Zero-factor property} \\ x = 3 \quad \text{or} \quad x &= 1 \end{aligned}$$

The domain of g includes all real numbers except 3 and 1.

(c) $h(x) = \dfrac{8x + 2}{3}$

The denominator, 3, can never be 0, so the domain includes all real numbers.

(d) $f(x) = \dfrac{2}{x^2 + 4}$

Setting $x^2 + 4$ equal to 0 leads to $x^2 = -4$. There is no real number whose square is -4. Therefore, any real number can be used, and as in part (c), the domain includes all real numbers.

Now Try Exercises 11, 15, 17, and 19.

OBJECTIVE 3 Write rational expressions in lowest terms. In arithmetic, we write the fraction $\frac{15}{20}$ in lowest terms by dividing the numerator and denominator by 5 to get $\frac{3}{4}$. We write rational expressions in lowest terms in a similar way, using the **fundamental property of rational numbers.**

Fundamental Property of Rational Numbers

If $\frac{a}{b}$ is a rational number and if c is any nonzero real number, then

$$\frac{a}{b} = \frac{ac}{bc}.$$

That is, the numerator and denominator of a rational number may either be multiplied or divided by the same nonzero number without changing the value of the rational number.

Since $\frac{c}{c}$ is equivalent to 1, the fundamental property is based on the identity property of multiplication.

A rational expression is a quotient of two polynomials. Since the value of a polynomial is a real number for every value of the variable for which it is defined, any statement that applies to rational numbers will also apply to rational expressions. We use the following steps to write rational expressions in lowest terms.

Writing a Rational Expression in Lowest Terms

Step 1 **Factor** both numerator and denominator to find their greatest common factor (GCF).

Step 2 **Apply the fundamental property.**

EXAMPLE 2 Writing Rational Expressions in Lowest Terms

Write each rational expression in lowest terms.

(a) $\dfrac{8k}{16} = \dfrac{k \cdot 8}{2 \cdot 8} = \dfrac{k}{2} \cdot 1 = \dfrac{k}{2}$

Here, the GCF of the numerator and denominator is 8. We then applied the fundamental property.

(b) $\dfrac{8 + k}{16}$

The numerator cannot be factored, so this expression cannot be simplified further and is in lowest terms.

(c)
$$\frac{a^2 - a - 6}{a^2 + 5a + 6} = \frac{(a - 3)(a + 2)}{(a + 3)(a + 2)}$$
Factor the numerator and the denominator.
$$= \frac{a - 3}{a + 3} \cdot 1$$
$\dfrac{a + 2}{a + 2} = 1$
$$= \frac{a - 3}{a + 3}$$
Lowest terms

(d) $\dfrac{y^2 - 4}{2y + 4} = \dfrac{(y + 2)(y - 2)}{2(y + 2)} = \dfrac{y - 2}{2}$

(e)
$$\frac{x^3 - 27}{x - 3} = \frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$$
Factor the difference of cubes.
$$= x^2 + 3x + 9$$
Lowest terms

(f)
$$\frac{pr + qr + ps + qs}{pr + qr - ps - qs} = \frac{(pr + qr) + (ps + qs)}{(pr + qr) - (ps + qs)}$$
Group terms.
$$= \frac{r(p + q) + s(p + q)}{r(p + q) - s(p + q)}$$
Factor within groups.
$$= \frac{(p + q)(r + s)}{(p + q)(r - s)}$$
Factor by grouping.
$$= \frac{r + s}{r - s}$$
Lowest terms

Now Try Exercises 27, 31, 35, 43, and 47.

CAUTION Be careful! When using the fundamental property of rational numbers, only common *factors* may be divided. For example,

$$\frac{y - 2}{2} \neq y \quad \text{and} \quad \frac{y - 2}{2} \neq y - 1$$

because the 2 in $y - 2$ is not a *factor* of the numerator. To see this, replace y with a number and evaluate the fraction. For example, if $y = 5$, then

$$\frac{y - 2}{2} = \frac{5 - 2}{2} = \frac{3}{2}.$$

This does not equal y or $y - 1 = 5 - 1 = 4$. *Remember to factor before writing a fraction in lowest terms.*

In the rational expression from Example 2(c),

$$\frac{a^2 - a - 6}{a^2 + 5a + 6}, \quad \text{or} \quad \frac{(a - 3)(a + 2)}{(a + 3)(a + 2)},$$

a can take any value except -3 or -2 since these values make the denominator 0. In the simplified rational expression

$$\frac{a - 3}{a + 3},$$

a cannot equal -3. Because of this,

$$\frac{a^2 - a - 6}{a^2 + 5a + 6} = \frac{a - 3}{a + 3}$$

for all values of a except -3 or -2. From now on such statements of equality will be made with the understanding that they apply only for those real numbers that make neither denominator equal 0. We will no longer state such restrictions.

EXAMPLE 3 Writing Rational Expressions in Lowest Terms

Write each rational expression in lowest terms.

(a) $\dfrac{m - 3}{3 - m}$

In this rational expression, the numerator and denominator are opposites. The given expression can be written in lowest terms by writing the denominator as $-1(m - 3)$, giving

$$\frac{m - 3}{3 - m} = \frac{m - 3}{-1(m - 3)} = \frac{1}{-1} = -1.$$

The numerator could have been rewritten instead to get the same result.

(b)
$$\begin{aligned} \frac{r^2 - 16}{4 - r} &= \frac{(r + 4)(r - 4)}{4 - r} & \\ &= \frac{(r + 4)(r - 4)}{-1(r - 4)} & \text{Write } 4 - r \text{ as } -1(r - 4). \\ &= \frac{r + 4}{-1} & \text{Fundamental property} \\ &= -(r + 4) \quad \text{or} \quad -r - 4 & \text{Lowest terms} \end{aligned}$$

Now Try Exercises 49 and 51.

As shown in Examples 3(a) and (b), the quotient

$$\frac{a}{-a} \quad (a \neq 0)$$

can be simplified as

$$\frac{a}{-a} = \frac{a}{-1(a)} = \frac{1}{-1} = -1.$$

The following statement summarizes this result.

In general, if the numerator and the denominator of a rational expression are opposites, then the expression equals -1.

Based on this result,

$$\frac{q-7}{7-q}=-1 \qquad \text{and} \qquad \frac{-5a+2b}{5a-2b}=-1.$$

However,

$$\frac{r-2}{r+2}$$

cannot be simplified further since the numerator and the denominator are *not* opposites.

OBJECTIVE 4 Multiply rational expressions. To multiply rational expressions, follow these steps. (In practice, we usually simplify before multiplying.)

Multiplying Rational Expressions

Step 1 **Factor** all numerators and denominators as completely as possible.

Step 2 **Apply the fundamental property.**

Step 3 **Multiply** remaining factors in the numerator and remaining factors in the denominator. Leave the denominator in factored form.

Step 4 **Check** to be sure the product is in lowest terms.

EXAMPLE 4 Multiplying Rational Expressions

Multiply.

(a)
$$\frac{5p-5}{p}\cdot\frac{3p^2}{10p-10}=\frac{5(p-1)}{p}\cdot\frac{3p\cdot p}{2\cdot 5(p-1)} \qquad \text{Factor.}$$
$$=\frac{1}{1}\cdot\frac{3p}{2} \qquad \text{Lowest terms}$$
$$=\frac{3p}{2} \qquad \text{Multiply.}$$

(b)
$$\frac{k^2+2k-15}{k^2-4k+3}\cdot\frac{k^2-k}{k^2+k-20}=\frac{(k+5)(k-3)}{(k-3)(k-1)}\cdot\frac{k(k-1)}{(k+5)(k-4)}$$
$$=\frac{k}{k-4}$$

(c)
$$(p-4)\cdot\frac{3}{5p-20}=\frac{p-4}{1}\cdot\frac{3}{5p-20} \qquad \text{Write } p-4 \text{ as } \frac{p-4}{1}.$$
$$=\frac{p-4}{1}\cdot\frac{3}{5(p-4)} \qquad \text{Factor.}$$
$$=\frac{3}{5}$$

(d) $$\frac{x^2+2x}{x+1}\cdot\frac{x^2-1}{x^3+x^2}=\frac{x(x+2)}{x+1}\cdot\frac{(x+1)(x-1)}{x^2(x+1)}$$ Factor.

$$=\frac{(x+2)(x-1)}{x(x+1)}$$ Multiply; lowest terms

(e) $$\frac{x-6}{x^2-12x+36}\cdot\frac{x^2-3x-18}{x^2+7x+12}=\frac{x-6}{(x-6)^2}\cdot\frac{(x+3)(x-6)}{(x+3)(x+4)}$$ Factor.

$$=\frac{1}{x+4}$$ Lowest terms

Remember to include 1 in the numerator when all other factors are eliminated using the fundamental property.

Now Try Exercises 71, 73, and 77.

Rational Expression	*Reciprocal*
$\frac{5}{k}$	$\frac{k}{5}$
$\frac{m^2-9m}{2}$	$\frac{2}{m^2-9m}$
$\frac{0}{4}$	undefined

OBJECTIVE 5 Find reciprocals for rational expressions. The rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are reciprocals of each other if they have a product of 1. The **reciprocal** of a rational expression is defined in the same way: Two rational expressions are reciprocals of each other if they have a product of 1. Recall that 0 has no reciprocal. The table shows several rational expressions and their reciprocals. In the first two cases, check that the product of the rational expression and its reciprocal is 1.

The examples in the table suggest the following procedure.

Finding the Reciprocal

To find the reciprocal of a nonzero rational expression, invert the rational expression.

OBJECTIVE 6 Divide rational expressions. Dividing rational expressions is like dividing rational numbers.

Dividing Rational Expressions

To divide two rational expressions, *multiply* the first by the reciprocal of the second.

EXAMPLE 5 Dividing Rational Expressions

Divide.

(a) $$\frac{2z}{9}\div\frac{5z^2}{18}=\frac{2z}{9}\cdot\frac{18}{5z^2}$$ Multiply by the reciprocal of the divisor.

$$=\frac{2z}{9}\cdot\frac{2\cdot 9}{5z^2}$$ Factor.

$$=\frac{4}{5z}$$ Multiply; lowest terms

(b) $\dfrac{m^2pq^3}{mp^4} \div \dfrac{m^5p^2q}{mpq^2}$

Use the definitions of division and multiplication and the properties of exponents.

$$\begin{aligned}
\frac{m^2pq^3}{mp^4} \div \frac{m^5p^2q}{mpq^2} &= \frac{m^2pq^3}{mp^4} \cdot \frac{mpq^2}{m^5p^2q} && \text{Multiply by the reciprocal.}\\
&= \frac{m^3p^2q^5}{m^6p^6q} && \text{Properties of exponents}\\
&= \frac{q^4}{m^3p^4} && \text{Properties of exponents}
\end{aligned}$$

(c)
$$\begin{aligned}
\frac{8k-16}{3k} \div \frac{3k-6}{4k^2} &= \frac{8k-16}{3k} \cdot \frac{4k^2}{3k-6} && \text{Multiply by the reciprocal.}\\
&= \frac{8(k-2)}{3k} \cdot \frac{4k^2}{3(k-2)} && \text{Factor.}\\
&= \frac{32k}{9} && \text{Multiply; lowest terms}
\end{aligned}$$

VIDEO

(d) $\dfrac{5m^2+17m-12}{3m^2+7m-20} \div \dfrac{5m^2+2m-3}{15m^2-34m+15}$

$$\begin{aligned}
&= \frac{5m^2+17m-12}{3m^2+7m-20} \cdot \frac{15m^2-34m+15}{5m^2+2m-3} && \text{Definition of division}\\
&= \frac{(5m-3)(m+4)}{(m+4)(3m-5)} \cdot \frac{(3m-5)(5m-3)}{(5m-3)(m+1)} && \text{Factor.}\\
&= \frac{5m-3}{m+1} && \text{Lowest terms}
\end{aligned}$$

Now Try Exercises 63, 79, and 85.

7.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 10/Videotape 10

Rational expressions often can be written in lowest terms in seemingly different ways. For example,

$$\frac{y-3}{-5} \quad \textit{and} \quad \frac{-y+3}{5}$$

look different, but we get the second quotient by multiplying the first by -1 *in both the numerator and denominator. To practice recognizing equivalent rational expressions, match the expressions in Exercises 1–6 with their equivalents in choices A–F.*

1. $\dfrac{x-3}{x+4}$ **2.** $\dfrac{x+3}{x-4}$ **3.** $\dfrac{x-3}{x-4}$ **4.** $\dfrac{x+3}{x+4}$ **5.** $\dfrac{3-x}{x+4}$ **6.** $\dfrac{x+3}{4-x}$

A. $\dfrac{-x-3}{4-x}$ **B.** $\dfrac{-x-3}{-x-4}$ **C.** $\dfrac{3-x}{-x-4}$ **D.** $\dfrac{-x+3}{-x+4}$ **E.** $\dfrac{x-3}{-x-4}$ **F.** $\dfrac{-x-3}{x-4}$

7. In Example 1(a), we showed that the domain of the rational function defined by $f(x) = \frac{3}{7x - 14}$ does not include 2. Explain in your own words why this is so. In general, how do we find the value or values excluded from the domain of a rational function?

8. The domain of the rational function defined by $g(x) = \dfrac{x + 1}{x^2 + 3}$ includes all real numbers. Explain.

Find all numbers that are not in the domain of each function. See Example 1.

EXERCISES

9. $f(x) = \dfrac{x}{x - 7}$ **10.** $f(x) = \dfrac{x}{x + 3}$ **11.** $f(x) = \dfrac{6x - 5}{7x + 1}$

12. $f(x) = \dfrac{8x - 3}{2x + 7}$ **13.** $f(x) = \dfrac{12x + 3}{x}$ **14.** $f(x) = \dfrac{9x + 8}{x}$

15. $f(x) = \dfrac{3x + 1}{2x^2 + x - 6}$ **16.** $f(x) = \dfrac{2x + 4}{3x^2 + 11x - 42}$ **17.** $f(x) = \dfrac{x + 2}{14}$

18. $f(x) = \dfrac{x - 9}{26}$ **19.** $f(x) = \dfrac{2x^2 - 3x + 4}{3x^2 + 8}$ **20.** $f(x) = \dfrac{9x^2 - 8x + 3}{4x^2 + 1}$

21. (a) Identify the two *terms* in the numerator and the two *terms* in the denominator of the rational expression $\dfrac{x^2 + 4x}{x + 4}$.

(b) Describe the steps you would use to write this rational expression in lowest terms. (*Hint:* It simplifies to x.)

22. Only one of the following rational expressions can be simplified. Which one is it?

A. $\dfrac{x^2 + 2}{x^2}$ **B.** $\dfrac{x^2 + 2}{2}$ **C.** $\dfrac{x^2 + y^2}{y^2}$ **D.** $\dfrac{x^2 - 5x}{x}$

23. Only one of the following rational expressions is *not* equivalent to $\frac{x - 3}{4 - x}$. Which one is it?

A. $\dfrac{3 - x}{x - 4}$ **B.** $\dfrac{x + 3}{4 + x}$ **C.** $-\dfrac{3 - x}{4 - x}$ **D.** $-\dfrac{x - 3}{x - 4}$

24. Which two of the following rational expressions equal -1?

A. $\dfrac{2x + 3}{2x - 3}$ **B.** $\dfrac{2x - 3}{3 - 2x}$ **C.** $\dfrac{2x + 3}{3 + 2x}$ **D.** $\dfrac{2x + 3}{-2x - 3}$

Write each rational expression in lowest terms. See Example 2.

EXERCISES

25. $\dfrac{x^2(x + 1)}{x(x + 1)}$ **26.** $\dfrac{y^3(y - 4)}{y^2(y - 4)}$ **27.** $\dfrac{(x + 4)(x - 3)}{(x + 5)(x + 4)}$

28. $\dfrac{(2x + 7)(x - 1)}{(2x + 3)(2x + 7)}$ **29.** $\dfrac{4x(x + 3)}{8x^2(x - 3)}$ **30.** $\dfrac{5y^2(y + 8)}{15y(y - 8)}$

31. $\dfrac{3x + 7}{3}$ **32.** $\dfrac{4x - 9}{4}$ **33.** $\dfrac{6m + 18}{7m + 21}$

34. $\dfrac{5r - 20}{3r - 12}$ **35.** $\dfrac{3z^2 + z}{18z + 6}$ **36.** $\dfrac{2x^2 - 5x}{16x - 40}$

37. $\dfrac{2t+6}{t^2-9}$ **38.** $\dfrac{5s-25}{s^2-25}$ **39.** $\dfrac{x^2+2x-15}{x^2+6x+5}$

40. $\dfrac{y^2-5y-14}{y^2+y-2}$ **41.** $\dfrac{8x^2-10x-3}{8x^2-6x-9}$ **42.** $\dfrac{12x^2-4x-5}{8x^2-6x-5}$

43. $\dfrac{a^3+b^3}{a+b}$ **44.** $\dfrac{r^3-s^3}{r-s}$ **45.** $\dfrac{2c^2+2cd-60d^2}{2c^2-12cd+10d^2}$

46. $\dfrac{3s^2-9st-54t^2}{3s^2-6st-72t^2}$ **47.** $\dfrac{ac-ad+bc-bd}{ac-ad-bc+bd}$ **48.** $\dfrac{2xy+2xw+y+w}{2xy+y-2xw-w}$

Write each rational expression in lowest terms. See Example 3.

49. $\dfrac{7-b}{b-7}$ **50.** $\dfrac{r-13}{13-r}$ **51.** $\dfrac{x^2-y^2}{y-x}$ **52.** $\dfrac{m^2-n^2}{n-m}$

53. $\dfrac{(a-3)(x+y)}{(3-a)(x-y)}$ **54.** $\dfrac{(8-p)(x+2)}{(p-8)(x-2)}$ **55.** $\dfrac{5k-10}{20-10k}$

56. $\dfrac{7x-21}{63-21x}$ **57.** $\dfrac{a^2-b^2}{a^2+b^2}$ **58.** $\dfrac{p^2+q^2}{p^2-q^2}$

59. Explain in a few words how to multiply rational expressions. Give an example.

60. Explain in a few words how to divide rational expressions. Give an example.

Multiply or divide as indicated. See Examples 4 and 5.

61. $\dfrac{x^3}{3y}\cdot\dfrac{9y^2}{x^5}$ **62.** $\dfrac{a^4}{5b^2}\cdot\dfrac{25b^4}{a^3}$

63. $\dfrac{5a^4b^2}{16a^2b}\div\dfrac{25a^2b}{60a^3b^2}$ **64.** $\dfrac{s^3t^2}{10s^2t^4}\div\dfrac{8s^4t^2}{5t^6}$

65. $\dfrac{(-3mn)^2\cdot(4m^2n)^3}{16m^2n^4(mn^2)^3}\div\dfrac{24(m^2n^2)^4}{(3m^2n^3)^2}$ **66.** $\dfrac{(-4a^2b^3)^2\cdot(3a^2b^4)^2}{(2a^2b^3)^4\cdot(3a^3b)^2}\div\dfrac{(ab)^4}{(a^2b^3)^2}$

67. $\dfrac{(x+2)(x+1)}{(x+3)(x-2)}\cdot\dfrac{(x+3)(x+4)}{(x+2)(x+1)}$ **68.** $\dfrac{(x+3)(x-4)}{(x-4)(x+2)}\cdot\dfrac{(x+5)(x-6)}{(x+3)(x-6)}$

69. $\dfrac{(2x+3)(x-4)}{(x+8)(x-4)}\div\dfrac{(x-4)(x+2)}{(x-4)(x+8)}$ **70.** $\dfrac{(6x+5)(x-3)}{(x+9)(x-1)}\div\dfrac{(x-3)(2x+7)}{(x-1)(x+9)}$

71. $\dfrac{4x}{8x+4}\cdot\dfrac{14x+7}{6}$ **72.** $\dfrac{12x-20}{5x}\cdot\dfrac{6}{9x-15}$

73. $\dfrac{p^2-25}{4p}\cdot\dfrac{2}{5-p}$ **74.** $\dfrac{a^2-1}{4a}\cdot\dfrac{2}{1-a}$

75. $(7k+7)\div\dfrac{4k+4}{5}$ **76.** $(8y-16)\div\dfrac{3y-6}{10}$

77. $(z^2-1)\cdot\dfrac{1}{1-z}$ **78.** $(y^2-4)\div\dfrac{2-y}{8y}$

79. $\dfrac{m^2-49}{m+1}\div\dfrac{7-m}{m}$ **80.** $\dfrac{k^2-4}{3k^2}\div\dfrac{2-k}{11k}$

81. $\dfrac{12x-10y}{3x+2y}\cdot\dfrac{6x+4y}{10y-12x}$ **82.** $\dfrac{9s-12t}{2s+2t}\cdot\dfrac{3s+3t}{4t-3s}$

83. $\dfrac{x^2 - 25}{x^2 + x - 20} \cdot \dfrac{x^2 + 7x + 12}{x^2 - 2x - 15}$

84. $\dfrac{t^2 - 49}{t^2 + 4t - 21} \cdot \dfrac{t^2 + 8t + 15}{t^2 - 2t - 35}$

85. $\dfrac{6x^2 + 5xy - 6y^2}{12x^2 - 11xy + 2y^2} \div \dfrac{4x^2 - 12xy + 9y^2}{8x^2 - 14xy + 3y^2}$

86. $\dfrac{8a^2 - 6ab - 9b^2}{6a^2 - 5ab - 6b^2} \div \dfrac{4a^2 + 11ab + 6b^2}{9a^2 + 12ab + 4b^2}$

87. $\dfrac{3k^2 + 17kp + 10p^2}{6k^2 + 13kp - 5p^2} \div \dfrac{6k^2 + kp - 2p^2}{6k^2 - 5kp + p^2}$

88. $\dfrac{16c^2 + 24cd + 9d^2}{16c^2 - 16cd + 3d^2} \div \dfrac{16c^2 - 9d^2}{16c^2 - 24cd + 9d^2}$

89. $\left(\dfrac{6k^2 - 13k - 5}{k^2 + 7k} \div \dfrac{2k - 5}{k^3 + 6k^2 - 7k}\right) \cdot \dfrac{k^2 - 5k + 6}{3k^2 - 8k - 3}$

90. $\left(\dfrac{2x^3 + 3x^2 - 2x}{3x - 15} \div \dfrac{2x^3 - x^2}{x^2 - 3x - 10}\right) \cdot \dfrac{5x^2 - 10x}{3x^2 + 12x + 12}$

91. $\dfrac{a^2(2a + b) + 6a(2a + b) + 5(2a + b)}{3a^2(a + 2b) - 2a(a + 2b) - (a + 2b)} \div \dfrac{a + 1}{a - 1}$

92. $\dfrac{2x^2(x - 3z) - 5x(x - 3z) + 2(x - 3z)}{4x^2(3z - x) - 11x(3z - x) + 6(3z - x)} \div \dfrac{4x + 1}{4x - 3}$

7.2 Adding and Subtracting Rational Expressions

OBJECTIVES

1. Add and subtract rational expressions with the same denominator.
2. Find a least common denominator.
3. Add and subtract rational expressions with different denominators.

OBJECTIVE 1 Add and subtract rational expressions with the same denominator. The following steps, used to add or subtract rational numbers, are also used to add or subtract rational expressions.

Adding or Subtracting Rational Expressions

Step 1 **If the denominators are the same,** add or subtract the numerators. Place the result over the common denominator.

If the denominators are different, first find the least common denominator. Write all rational expressions with this least common denominator, and then add or subtract the numerators. Place the result over the common denominator.

Step 2 **Simplify.** Write all answers in lowest terms.

EXAMPLE 1 Adding and Subtracting Rational Expressions with the Same Denominators

Add or subtract as indicated.

(a) $\dfrac{3y}{5} + \dfrac{x}{5} = \dfrac{3y + x}{5}$ ← Add numerators. ← Keep the common denominator.

(b) $$\frac{7}{2r^2} - \frac{11}{2r^2} = \frac{7 - 11}{2r^2}$$ Subtract numerators; keep the common denominator.

$$= \frac{-4}{2r^2}$$

$$= -\frac{2}{r^2}$$ Lowest terms

(c) $$\frac{m}{m^2 - p^2} + \frac{p}{m^2 - p^2} = \frac{m + p}{m^2 - p^2}$$ Add numerators; keep the common denominator.

$$= \frac{m + p}{(m + p)(m - p)}$$ Factor.

$$= \frac{1}{m - p}$$ Lowest terms

(d) $$\frac{4}{x^2 + 2x - 8} + \frac{x}{x^2 + 2x - 8} = \frac{4 + x}{x^2 + 2x - 8}$$

$$= \frac{4 + x}{(x - 2)(x + 4)}$$

$$= \frac{1}{x - 2}$$

Now Try Exercises 7, 9, and 15.

OBJECTIVE 2 Find a least common denominator. We add or subtract rational expressions with different denominators by first writing them with a common denominator, usually the **least common denominator (LCD).**

Finding the Least Common Denominator

Step 1 **Factor** each denominator.

Step 2 **Find the least common denominator.** The LCD is the product of all different factors from each denominator, with each factor raised to the *greatest* power that occurs in any denominator.

EXAMPLE 2 Finding Least Common Denominators

Assume that the given expressions are denominators of two fractions. Find the LCD for each group.

(a) $5xy^2, \quad 2x^3y$

Each denominator is already factored.

$$5xy^2 = 5 \cdot x \cdot y^2$$
$$2x^3y = 2 \cdot x^3 \cdot y$$

Greatest exponent on x is 3.

$$\text{LCD} = 5 \cdot 2 \cdot x^3 \cdot y^2$$ Greatest exponent on y is 2.

$$= 10x^3y^2$$

(b) $k - 3, \quad k$

Each denominator is already factored. The LCD, an expression divisible by *both* $k - 3$ and k, is

$$k(k - 3).$$

It is usually best to leave a least common denominator in factored form.

(c) $y^2 - 2y - 8, \quad y^2 + 3y + 2$

Factor the denominators.

$$\left.\begin{aligned} y^2 - 2y - 8 &= (y - 4)(y + 2) \\ y^2 + 3y + 2 &= (y + 2)(y + 1) \end{aligned}\right\} \text{Factor.}$$

The LCD, divisible by both polynomials, is

$$(y - 4)(y + 2)(y + 1).$$

(d) $8z - 24, \quad 5z^2 - 15z$

$$\left.\begin{aligned} 8z - 24 &= 8(z - 3) \\ 5z^2 - 15z &= 5z(z - 3) \end{aligned}\right\} \text{Factor.}$$

The LCD is $8 \cdot 5z \cdot (z - 3) = 40z(z - 3)$.

(e) $m^2 + 5m + 6, \quad m^2 + 4m + 4, \quad 2(m^2 + 2m - 3)$

$$\left.\begin{aligned} m^2 + 5m + 6 &= (m + 3)(m + 2) \\ m^2 + 4m + 4 &= (m + 2)^2 \\ 2(m^2 + 2m - 3) &= 2(m + 3)(m - 1) \end{aligned}\right\} \text{Factor.}$$

The LCD is $2(m + 3)(m + 2)^2(m - 1)$.

Now Try Exercises 21, 23, 27, and 35.

OBJECTIVE 3 Add and subtract rational expressions with different denominators. Before adding or subtracting two rational expressions, we write each expression with the least common denominator by multiplying its numerator and denominator by the factors needed to get the LCD. This procedure is valid because we are multiplying each rational expression by a form of 1, the identity element for multiplication.

Adding or subtracting rational expressions follows the same procedure as that used for rational numbers. Consider the sum $\frac{7}{15} + \frac{5}{12}$. The LCD for 15 and 12 is 60. Multiply $\frac{7}{15}$ by $\frac{4}{4}$ (a form of 1) and multiply $\frac{5}{12}$ by $\frac{5}{5}$ so that each fraction has denominator 60, and then add the numerators.

$$\begin{aligned} \frac{7}{15} + \frac{5}{12} &= \frac{7 \cdot 4}{15 \cdot 4} + \frac{5 \cdot 5}{12 \cdot 5} && \text{Fundamental property} \\ &= \frac{28}{60} + \frac{25}{60} \\ &= \frac{28 + 25}{60} && \text{Add the numerators.} \\ &= \frac{53}{60} \end{aligned}$$

EXAMPLE 3 Adding and Subtracting Rational Expressions with Different Denominators

Add or subtract as indicated.

(a) $\dfrac{5}{2p} + \dfrac{3}{8p}$

The LCD for $2p$ and $8p$ is $8p$. To write the first rational expression with a denominator of $8p$, multiply by $\frac{4}{4}$.

$$\frac{5}{2p} + \frac{3}{8p} = \frac{5 \cdot 4}{2p \cdot 4} + \frac{3}{8p} \qquad \text{Fundamental principle}$$

$$= \frac{20}{8p} + \frac{3}{8p}$$

$$= \frac{20 + 3}{8p} \qquad \text{Add numerators.}$$

$$= \frac{23}{8p}$$

(b) $\dfrac{6}{r} - \dfrac{5}{r - 3}$

The LCD is $r(r - 3)$. Rewrite each rational expression with this denominator.

$$\frac{6}{r} - \frac{5}{r-3} = \frac{6(r-3)}{r(r-3)} - \frac{r \cdot 5}{r(r-3)} \qquad \text{Fundamental principle}$$

$$= \frac{6r - 18}{r(r-3)} - \frac{5r}{r(r-3)} \qquad \text{Distributive and commutative properties}$$

$$= \frac{6r - 18 - 5r}{r(r-3)} \qquad \text{Subtract numerators.}$$

$$= \frac{r - 18}{r(r-3)} \qquad \text{Combine terms in the numerator.}$$

Now Try Exercises 39 and 43.

CAUTION One of the most common sign errors in algebra occurs when subtracting a rational expression with two or more terms in the numerator. Remember that in this situation, the subtraction sign must be distributed to *every* term in the numerator of the fraction that follows it. Study Example 4 carefully to see how this is done.

EXAMPLE 4 Using the Distributive Property When Subtracting Rational Expressions

Subtract.

(a) $\dfrac{7x}{3x + 1} - \dfrac{x - 2}{3x + 1}$

The denominators are the same for both rational expressions. The subtraction sign must be applied to *both* terms in the numerator of the second rational expression.

Notice the careful use of the distributive property here.

$$\begin{aligned}
\frac{7x}{3x+1} - \frac{x-2}{3x+1} &= \frac{7x - (x-2)}{3x+1} && \text{Write as a single rational expression.}\\
&= \frac{7x - x + 2}{3x+1} && \text{Distributive property; be careful with signs.}\\
&= \frac{6x+2}{3x+1} && \text{Combine terms in the numerator.}\\
&= \frac{2(3x+1)}{3x+1} && \text{Factor the numerator.}\\
&= 2 && \text{Lowest terms}
\end{aligned}$$

(b) $\dfrac{1}{q-1} - \dfrac{1}{q+1}$

$$\begin{aligned}
&= \frac{1(q+1)}{(q-1)(q+1)} - \frac{1(q-1)}{(q+1)(q-1)} && \text{Fundamental property}\\
&= \frac{(q+1) - (q-1)}{(q-1)(q+1)} && \text{Subtract.}\\
&= \frac{q+1-q+1}{(q-1)(q+1)} && \text{Distributive property}\\
&= \frac{2}{(q-1)(q+1)} && \text{Combine terms in the numerator.}
\end{aligned}$$

Now Try Exercises 47 and 53.

In some problems, rational expressions to be added or subtracted have denominators that are opposites of each other. The next example illustrates how to proceed in such a problem.

YOU TRY IT

EXAMPLE 5 Adding Rational Expressions with Denominators That Are Opposites

Add.

$$\frac{y}{y-2} + \frac{8}{2-y}$$

To get a common denominator of $y - 2$, multiply the second expression by -1 in both the numerator and the denominator.

$$\begin{aligned}
\frac{y}{y-2} + \frac{8}{2-y} &= \frac{y}{y-2} + \frac{8(-1)}{(2-y)(-1)}\\
&= \frac{y}{y-2} + \frac{-8}{y-2}\\
&= \frac{y-8}{y-2} && \text{Add the numerators.}
\end{aligned}$$

Now Try Exercise 49.

The next example illustrates addition and subtraction involving more than two rational expressions.

VIDEO

EXAMPLE 6 Adding and Subtracting Three Rational Expressions

Add and subtract as indicated.

$$\frac{3}{x-2}+\frac{5}{x}-\frac{6}{x^2-2x}$$

The denominator of the third rational expression factors as $x(x-2)$, which is the LCD for the three rational expressions.

$$\frac{3}{x-2}+\frac{5}{x}-\frac{6}{x^2-2x}$$

$$=\frac{3x}{x(x-2)}+\frac{5(x-2)}{x(x-2)}-\frac{6}{x(x-2)}$$ Fundamental property

$$=\frac{3x+5(x-2)-6}{x(x-2)}$$ Add and subtract the numerators.

$$=\frac{3x+5x-10-6}{x(x-2)}$$ Distributive property

$$=\frac{8x-16}{x(x-2)}$$ Combine terms in the numerator.

$$=\frac{8(x-2)}{x(x-2)}$$ Factor the numerator.

$$=\frac{8}{x}$$ Lowest terms

Now Try Exercise 55.

EXPLORE

EXAMPLE 7 Subtracting Rational Expressions

Subtract.

$$\frac{m+4}{m^2-2m-3}-\frac{2m-3}{m^2-5m+6}$$

$$=\frac{m+4}{(m-3)(m+1)}-\frac{2m-3}{(m-3)(m-2)}$$ Factor each denominator.

The LCD is $(m-3)(m+1)(m-2)$.

$$=\frac{(m+4)(m-2)}{(m-3)(m+1)(m-2)}-\frac{(2m-3)(m+1)}{(m-3)(m-2)(m+1)}$$ Fundamental property

$$=\frac{(m+4)(m-2)-(2m-3)(m+1)}{(m-3)(m+1)(m-2)}$$ Subtract.

$$=\frac{m^2+2m-8-(2m^2-m-3)}{(m-3)(m+1)(m-2)}$$ Multiply in the numerator.

$$= \frac{m^2 + 2m - 8 - 2m^2 + m + 3}{(m-3)(m+1)(m-2)}$$ Distributive property; be careful with signs.

$$= \frac{-m^2 + 3m - 5}{(m-3)(m+1)(m-2)}$$ Combine terms in the numerator.

If we try to factor the numerator, we find that this rational expression is in lowest terms.

Now Try Exercise 69.

EXAMPLE 8 Adding Rational Expressions

Add.

$$\frac{5}{x^2 + 10x + 25} + \frac{2}{x^2 + 7x + 10}$$

$$= \frac{5}{(x+5)^2} + \frac{2}{(x+5)(x+2)}$$ Factor each denominator.

The LCD is $(x+5)^2(x+2)$.

$$= \frac{5(x+2)}{(x+5)^2(x+2)} + \frac{2(x+5)}{(x+5)^2(x+2)}$$ Fundamental property

$$= \frac{5(x+2) + 2(x+5)}{(x+5)^2(x+2)}$$ Add.

$$= \frac{5x + 10 + 2x + 10}{(x+5)^2(x+2)}$$ Distributive property

$$= \frac{7x + 20}{(x+5)^2(x+2)}$$ Combine terms in the numerator.

Now Try Exercise 77.

7.2 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 10/Videotape 10

RELATING CONCEPTS (EXERCISES 1–6)

For Individual or Group Work

Work Exercises 1–6 in order.

1. Let $x = 4$ and $y = 2$. Evaluate $\frac{1}{x} + \frac{1}{y}$.

2. Let $x = 4$ and $y = 2$. Evaluate $\frac{1}{x + y}$.

3. Are the answers for Exercises 1 and 2 the same? What can you conclude?

4. Let $x = 3$ and $y = 5$. Evaluate $\frac{1}{x} - \frac{1}{y}$.

5. Let $x = 3$ and $y = 5$. Evaluate $\frac{1}{x - y}$.

6. Are the answers for Exercises 4 and 5 the same? What can you conclude?

Add or subtract as indicated. Write all answers in lowest terms. See Example 1.

7. $\dfrac{7}{t}+\dfrac{2}{t}$ **8.** $\dfrac{5}{r}+\dfrac{9}{r}$ **9.** $\dfrac{11}{5x}-\dfrac{1}{5x}$

10. $\dfrac{7}{4y}-\dfrac{3}{4y}$ **11.** $\dfrac{5x+4}{6x+5}+\dfrac{x+1}{6x+5}$ **12.** $\dfrac{6y+12}{4y+3}+\dfrac{2y-6}{4y+3}$

13. $\dfrac{x^2}{x+5}-\dfrac{25}{x+5}$ **14.** $\dfrac{y^2}{y+6}-\dfrac{36}{y+6}$

15. $\dfrac{-3p+7}{p^2+7p+12}+\dfrac{8p+13}{p^2+7p+12}$ **16.** $\dfrac{5x+6}{x^2+x-20}+\dfrac{4-3x}{x^2+x-20}$

17. $\dfrac{a^3}{a^2+ab+b^2}-\dfrac{b^3}{a^2+ab+b^2}$ **18.** $\dfrac{p^3}{p^2-pq+q^2}+\dfrac{q^3}{p^2-pq+q^2}$

19. Write a step-by-step method for adding or subtracting rational expressions that have a common denominator. Illustrate with an example.

20. Write a step-by-step method for adding or subtracting rational expressions that have different denominators. Give an example.

Assume that the expressions given are denominators of fractions. Find the least common denominator (LCD) for each group. See Example 2.

21. $18x^2y^3$, $24x^4y^5$ **22.** $24a^3b^4$, $18a^5b^2$ **23.** $z-2$, z

24. $k+3$, k **25.** $2y+8$, $y+4$ **26.** $3r-21$, $r-7$

27. x^2-81, $x^2+18x+81$ **28.** y^2-16, $y^2-8y+16$

29. $m+n$, $m-n$, m^2-n^2 **30.** $r+s$, $r-s$, r^2-s^2

31. x^2-3x-4, $x+x^2$ **32.** $y^2-8y+12$, y^2-6y

33. $2t^2+7t-15$, $t^2+3t-10$ **34.** s^2-3s-4, $3s^2+s-2$

35. $2y+6$, y^2-9, y **36.** $9x+18$, x^2-4, x

37. One student added two rational expressions and obtained the answer $\frac{3}{5-y}$. Another student obtained the answer $\frac{-3}{y-5}$ for the same problem. Is it possible that both answers are correct? Explain.

38. What is *wrong* with the following work?

$$\frac{x}{x+2}-\frac{4x-1}{x+2}=\frac{x-4x-1}{x+2}=\frac{-3x-1}{x+2}$$

Add or subtract as indicated. Write all answers in lowest terms. See Examples 3–6.

39. $\dfrac{8}{t}+\dfrac{7}{3t}$ **40.** $\dfrac{5}{x}+\dfrac{9}{4x}$ **41.** $\dfrac{5}{12x^2y}-\dfrac{11}{6xy}$

42. $\dfrac{7}{18a^3b^2}-\dfrac{2}{9ab}$ **43.** $\dfrac{1}{x-1}-\dfrac{1}{x}$ **44.** $\dfrac{3}{x-3}-\dfrac{1}{x}$

45. $\dfrac{3a}{a+1}+\dfrac{2a}{a-3}$ **46.** $\dfrac{2x}{x+4}+\dfrac{3x}{x-7}$ **47.** $\dfrac{17y+3}{9y+7}-\dfrac{-10y-18}{9y+7}$

48. $\dfrac{7x+8}{3x+2}-\dfrac{x+4}{3x+2}$ **49.** $\dfrac{2}{4-x}+\dfrac{5}{x-4}$ **50.** $\dfrac{3}{2-t}+\dfrac{1}{t-2}$

51. $\dfrac{w}{w-z}-\dfrac{z}{z-w}$ **52.** $\dfrac{a}{a-b}-\dfrac{b}{b-a}$

53. $\dfrac{5}{12 + 4x} - \dfrac{7}{9 + 3x}$

54. $\dfrac{3}{10x + 15} - \dfrac{8}{12x + 18}$

55. $\dfrac{4x}{x - 1} - \dfrac{2}{x + 1} - \dfrac{4}{x^2 - 1}$

56. $\dfrac{4}{x + 3} - \dfrac{x}{x - 3} - \dfrac{18}{x^2 - 9}$

57. $\dfrac{15}{y^2 + 3y} + \dfrac{2}{y} + \dfrac{5}{y + 3}$

58. $\dfrac{7}{t - 2} - \dfrac{6}{t^2 - 2t} - \dfrac{3}{t}$

59. $\dfrac{5}{x - 2} + \dfrac{1}{x} + \dfrac{2}{x^2 - 2x}$

60. $\dfrac{5x}{x - 3} + \dfrac{2}{x} + \dfrac{6}{x^2 - 3x}$

61. $\dfrac{3x}{x + 1} + \dfrac{4}{x - 1} - \dfrac{6}{x^2 - 1}$

62. $\dfrac{5x}{x + 3} + \dfrac{x + 2}{x} - \dfrac{6}{x^2 + 3x}$

63. $\dfrac{4}{x + 1} + \dfrac{1}{x^2 - x + 1} - \dfrac{12}{x^3 + 1}$

64. $\dfrac{5}{x + 2} + \dfrac{2}{x^2 - 2x + 4} - \dfrac{60}{x^3 + 8}$

65. $\dfrac{2x + 4}{x + 3} + \dfrac{3}{x} - \dfrac{6}{x^2 + 3x}$

66. $\dfrac{4x + 1}{x + 5} - \dfrac{2}{x} + \dfrac{10}{x^2 + 5x}$

67. $\dfrac{3}{(p - 2)^2} - \dfrac{5}{p - 2} + 4$

68. $\dfrac{8}{(3r - 1)^2} + \dfrac{2}{3r - 1} - 6$

*Add or subtract as indicated. Write all answers in lowest terms. See Examples 7 and 8.**

69. $\dfrac{3}{x^2 - 5x + 6} - \dfrac{2}{x^2 - 4x + 4}$

70. $\dfrac{2}{m^2 - 4m + 4} + \dfrac{3}{m^2 + m - 6}$

71. $\dfrac{5x}{x^2 + xy - 2y^2} - \dfrac{3x}{x^2 + 5xy - 6y^2}$

72. $\dfrac{6x}{6x^2 + 5xy - 4y^2} - \dfrac{2y}{9x^2 - 16y^2}$

73. $\dfrac{5x - y}{x^2 + xy - 2y^2} - \dfrac{3x + 2y}{x^2 + 5xy - 6y^2}$

74. $\dfrac{6x + 5y}{6x^2 + 5xy - 4y^2} - \dfrac{x + 2y}{9x^2 - 16y^2}$

75. $\dfrac{r + s}{3r^2 + 2rs - s^2} - \dfrac{s - r}{6r^2 - 5rs + s^2}$

76. $\dfrac{3y}{y^2 + yz - 2z^2} + \dfrac{4y - 1}{y^2 - z^2}$

77. $\dfrac{3}{x^2 + 4x + 4} + \dfrac{7}{x^2 + 5x + 6}$

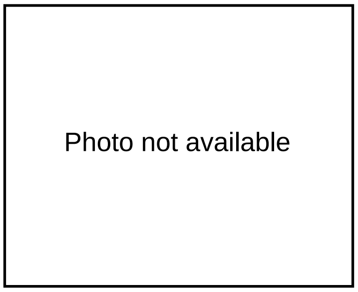
78. $\dfrac{5}{x^2 + 6x + 9} - \dfrac{2}{x^2 + 4x + 3}$

Work each problem.

79. A *concours d'elegance* is a competition in which a maximum of 100 points is awarded to a car based on its general attractiveness. The function defined by the rational expression

$$c(x) = \frac{1010}{49(101 - x)} - \frac{10}{49}$$

approximates the cost, in thousands of dollars, of restoring a car so that it will win x points.



(a) Simplify the expression for $c(x)$ by performing the indicated subtraction.

(b) Use the simplified expression to determine how much it would cost to win 95 points.

* The authors wish to thank Joyce Nemeth of Broward Community College for her suggestions regarding some of these exercises.

80. A *cost-benefit model* expresses the cost of an undertaking in terms of the benefits received. One cost-benefit model gives the cost in thousands of dollars to remove x percent of a certain pollutant as

$$c(x) = \frac{6.7x}{100 - x}.$$

Another model produces the relationship

$$c(x) = \frac{6.5x}{102 - x}.$$

(a) What is the cost found by averaging the two models? (*Hint:* The average of two quantities is half their sum.)
(b) Using the two given models and your answer to part (a), find the cost to the nearest dollar to remove 95% ($x = 95$) of the pollutant.
(c) Average the two costs in part (b) from the given models. What do you notice about this result compared to the cost using the average of the two models?

RELATING CONCEPTS (EXERCISES 81–86)

For Individual or Group Work

In Example 6 we showed that

$$\frac{3}{x-2} + \frac{5}{x} - \frac{6}{x^2 - 2x}$$

is equal to $\frac{8}{x}$*. Algebra is, in a sense, a generalized form of arithmetic.* ***Work Exercises 81–86 in order,*** *to see how the algebra in this example is related to the arithmetic of common fractions.*

81. Perform the following operations, and express your answer in lowest terms.

$$\frac{3}{7} + \frac{5}{9} - \frac{6}{63}$$

82. Substitute 9 for x in the given problem from Example 6. Compare this problem to the one given in Exercise 81. What do you notice?

83. Now substitute 9 for x in the answer given in Example 6. Do your results agree with the result you obtained in Exercise 81?

84. Replace x in the problem from Example 6 with the number of letters in your last name, assuming that this number is not 2. If your last name has two letters, let $x = 3$. Now predict the answer to your problem. Verify that your prediction is correct.

85. Why will $x = 2$ not work for the problem from Example 6?

86. What other value of x is not allowed in the problem from Example 6?

7.3 Complex Fractions

OBJECTIVES

1. Simplify complex fractions by simplifying the numerator and denominator (Method 1).
2. Simplify complex fractions by multiplying by a common denominator (Method 2).
3. Compare the two methods of simplifying complex fractions.
4. Simplify rational expressions with negative exponents.

A **complex fraction** is an expression having a fraction in the numerator, denominator, or both. Examples of complex fractions include

$$\frac{1+\frac{1}{x}}{2}, \qquad \frac{\frac{4}{y}}{6-\frac{3}{y}}, \qquad \text{and} \qquad \frac{\frac{m^2-9}{m+1}}{\frac{m+3}{m^2-1}}.$$

OBJECTIVE 1 Simplify complex fractions by simplifying the numerator and denominator (Method 1). There are two different methods for simplifying complex fractions.

Simplifying a Complex Fraction: Method 1

Step 1 Simplify the numerator and denominator separately.

Step 2 Divide by multiplying the numerator by the reciprocal of the denominator.

Step 3 Simplify the resulting fraction, if possible.

In Step 2, we are treating the complex fraction as a quotient of two rational expressions and dividing. Before performing this step, be sure that both the numerator and denominator are single fractions.

EXAMPLE 1 Simplifying Complex Fractions by Method 1

Use Method 1 to simplify each complex fraction.

(a) $\dfrac{\frac{x+1}{x}}{\frac{x-1}{2x}}$

Both the numerator and the denominator are already simplified, so divide by multiplying the numerator by the reciprocal of the denominator.

$$\frac{\frac{x+1}{x}}{\frac{x-1}{2x}} = \frac{x+1}{x} \div \frac{x-1}{2x} \qquad \text{Write as a division problem.}$$

$$= \frac{x+1}{x} \cdot \frac{2x}{x-1} \qquad \text{Multiply by the reciprocal of } \frac{x-1}{2x}.$$

$$= \frac{2(x+1)}{x-1} \qquad \text{Multiply and simplify.}$$

(b)
$$\frac{2+\frac{1}{y}}{3-\frac{2}{y}}=\frac{\frac{2y}{y}+\frac{1}{y}}{\frac{3y}{y}-\frac{2}{y}}$$

$$=\frac{\frac{2y+1}{y}}{\frac{3y-2}{y}}$$ Simplify the numerator and denominator.

$$=\frac{2y+1}{y}\div\frac{3y-2}{y}$$ Write as a division problem.

$$=\frac{2y+1}{y}\cdot\frac{y}{3y-2}$$ Multiply by the reciprocal of $\frac{3y-2}{y}$.

$$=\frac{2y+1}{3y-2}$$ Multiply and simplify.

Now Try Exercises 5 and 9.

OBJECTIVE 2 Simplify complex fractions by multiplying by a common denominator (Method 2). The second method for simplifying complex fractions uses the identity property for multiplication.

Simplifying a Complex Fraction: Method 2

Step 1 Multiply the numerator and denominator of the complex fraction by the least common denominator of the fractions in the numerator and the fractions in the denominator of the complex fraction.

Step 2 Simplify the resulting fraction, if possible.

EXAMPLE 2 Simplifying Complex Fractions by Method 2

Use Method 2 to simplify each complex fraction.

(a) $\dfrac{2+\frac{1}{y}}{3-\frac{2}{y}}$

Multiply the numerator and denominator by the LCD of all the fractions in the numerator and the denominator of the complex fraction. (This is the same as multiplying by 1.) Here the LCD is y.

$$\frac{2+\frac{1}{y}}{3-\frac{2}{y}}=\frac{2+\frac{1}{y}}{3-\frac{2}{y}}\cdot 1$$

$$= \frac{\left(2 + \frac{1}{y}\right) \cdot y}{\left(3 - \frac{2}{y}\right) \cdot y}$$ Multiply the numerator and denominator by y, since $\frac{y}{y} = 1$.

$$= \frac{2 \cdot y + \frac{1}{y} \cdot y}{3 \cdot y - \frac{2}{y} \cdot y}$$ Distributive property

$$= \frac{2y + 1}{3y - 2}$$

Compare this method of solution with that used in Example 1(b).

(b) $\dfrac{2p + \frac{5}{p-1}}{3p - \frac{2}{p}}$

The LCD is $p(p - 1)$.

$$\frac{2p + \frac{5}{p-1}}{3p - \frac{2}{p}} = \frac{\left(2p + \frac{5}{p-1}\right) \cdot p(p-1)}{\left(3p - \frac{2}{p}\right) \cdot p(p-1)}$$ Multiply the numerator and denominator by the LCD.

$$= \frac{2p[p(p-1)] + \frac{5}{p-1} \cdot p(p-1)}{3p[p(p-1)] - \frac{2}{p} \cdot p(p-1)}$$ Distributive property

$$= \frac{2p[p(p-1)] + 5p}{3p[p(p-1)] - 2(p-1)}$$

$$= \frac{2p^3 - 2p^2 + 5p}{3p^3 - 3p^2 - 2p + 2}$$

This rational expression is in lowest terms.

Now Try Exercises 9 (using Method 2) and 11.

OBJECTIVE 3 Compare the two methods of simplifying complex fractions. Choosing whether to use Method 1 or Method 2 to simplify a complex fraction is usually a matter of preference. Some students prefer one method over the other, while other students feel comfortable with both methods and rely on practice with many examples to determine which method they will use on a particular problem. In the next example, we illustrate how to simplify a complex fraction using both methods so that you can observe the processes and decide for yourself the pros and cons of each method.

YOU TRY IT

EXAMPLE 3 Simplifying Complex Fractions Using Both Methods

Use both Method 1 and Method 2 to simplify each complex fraction.

Method 1

(a) $$\frac{\frac{2}{x-3}}{\frac{5}{x^2-9}}$$

$$= \frac{\frac{2}{x-3}}{\frac{5}{(x-3)(x+3)}}$$

$$= \frac{2}{x-3} \div \frac{5}{(x-3)(x+3)}$$

$$= \frac{2}{x-3} \cdot \frac{(x-3)(x+3)}{5}$$

$$= \frac{2(x+3)}{5}$$

(b) $$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$$

$$= \frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{y^2}{x^2y^2}-\frac{x^2}{x^2y^2}}$$

$$= \frac{\frac{y+x}{xy}}{\frac{y^2-x^2}{x^2y^2}}$$

$$= \frac{y+x}{xy} \div \frac{y^2-x^2}{x^2y^2}$$

$$= \frac{y+x}{xy} \cdot \frac{x^2y^2}{(y-x)(y+x)}$$

$$= \frac{xy}{y-x}$$

Method 2

(a) $$\frac{\frac{2}{x-3}}{\frac{5}{x^2-9}}$$

$$= \frac{\frac{2}{x-3} \cdot (x-3)(x+3)}{\frac{5}{(x-3)(x+3)} \cdot (x-3)(x+3)}$$

$$= \frac{2(x+3)}{5}$$

(b) $$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$$

$$= \frac{\left(\frac{1}{x}+\frac{1}{y}\right) \cdot x^2y^2}{\left(\frac{1}{x^2}-\frac{1}{y^2}\right) \cdot x^2y^2}$$

$$= \frac{xy^2+x^2y}{y^2-x^2}$$

$$= \frac{xy(y+x)}{(y+x)(y-x)}$$

$$= \frac{xy}{y-x}$$

Now Try Exercises 13 and 17.

OBJECTIVE 4 Simplify rational expressions with negative exponents. Rational expressions and complex fractions sometimes involve negative exponents. To simplify such expressions, we begin by rewriting the expressions with only positive exponents.

VIDEO

YOU TRY IT

EXAMPLE 4 Simplifying Rational Expressions with Negative Exponents

Simplify each expression, using only positive exponents in the answer.

(a) $\dfrac{m^{-1} + p^{-2}}{2m^{-2} - p^{-1}}$

First write the expression with only positive exponents using the definition of a negative exponent.

$$\frac{m^{-1} + p^{-2}}{2m^{-2} - p^{-1}} = \frac{\frac{1}{m} + \frac{1}{p^2}}{\frac{2}{m^2} - \frac{1}{p}}$$

Note that the 2 in $2m^{-2}$ is *not* raised to the -2 power (since m is the base for the exponent -2), so $2m^{-2} = \dfrac{2}{m^2}$. Simplify the complex fraction using Method 2, multiplying the numerator and denominator by the LCD, m^2p^2.

$$\frac{\frac{1}{m} + \frac{1}{p^2}}{\frac{2}{m^2} - \frac{1}{p}} = \frac{m^2p^2\left(\frac{1}{m} + \frac{1}{p^2}\right)}{m^2p^2\left(\frac{2}{m^2} - \frac{1}{p}\right)}$$

$$= \frac{m^2p^2 \cdot \frac{1}{m} + m^2p^2 \cdot \frac{1}{p^2}}{m^2p^2 \cdot \frac{2}{m^2} - m^2p^2 \cdot \frac{1}{p}} \qquad \text{Distributive property}$$

$$= \frac{mp^2 + m^2}{2p^2 - m^2p} \qquad \text{Lowest terms}$$

(b) $$\frac{k^{-1}}{k^{-1} + 1} = \frac{\frac{1}{k}}{\frac{1}{k} + 1} \qquad \text{Write with positive exponents.}$$

$$= \frac{k \cdot \frac{1}{k}}{k\left(\frac{1}{k} + 1\right)} \qquad \text{Use Method 2.}$$

$$= \frac{k \cdot \frac{1}{k}}{k \cdot \frac{1}{k} + k \cdot 1}$$

$$= \frac{1}{1 + k}$$

Now Try Exercises 29 and 31.

7.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 11/Videotape 10

1. Explain in your own words the two methods of simplifying complex fractions.

2. Method 2 of simplifying complex fractions says that we can multiply both the numerator and the denominator of the complex fraction by the same nonzero expression. What property of real numbers from Section 1.4 justifies this method?

Use either method to simplify each complex fraction. See Examples 1–3.

3. $\dfrac{\frac{12}{x-1}}{\frac{6}{x}}$

4. $\dfrac{\frac{24}{t+4}}{\frac{6}{t}}$

5. $\dfrac{\frac{k+1}{2k}}{\frac{3k-1}{4k}}$

6. $\dfrac{\frac{1-r}{4r}}{\frac{-1-r}{8r}}$

7. $\dfrac{\frac{4z^2x^4}{9}}{\frac{12x^2z^5}{15}}$

8. $\dfrac{\frac{3y^2x^3}{8}}{\frac{9y^3x^4}{16}}$

9. $\dfrac{\frac{1}{x}+1}{-\frac{1}{x}+1}$

10. $\dfrac{\frac{2}{k}-1}{\frac{2}{k}+1}$

11. $\dfrac{\frac{3}{x}+\frac{3}{y}}{\frac{3}{x}-\frac{3}{y}}$

12. $\dfrac{\frac{4}{t}-\frac{4}{s}}{\frac{4}{t}+\frac{4}{s}}$

13. $\dfrac{\frac{8x-24y}{10}}{\frac{x-3y}{5x}}$

14. $\dfrac{\frac{10x-5y}{12}}{\frac{2x-y}{6y}}$

15. $\dfrac{\frac{x^2-16y^2}{xy}}{\frac{1}{y}-\frac{4}{x}}$

16. $\dfrac{\frac{2}{s}-\frac{3}{t}}{\frac{4t^2-9s^2}{st}}$

17. $\dfrac{y-\frac{y-3}{3}}{\frac{4}{9}+\frac{2}{3y}}$

18. $\dfrac{p-\frac{p+2}{4}}{\frac{3}{4}-\frac{5}{2p}}$

19. $\dfrac{\frac{x+2}{x}+\frac{1}{x+2}}{\frac{5}{x}+\frac{x}{x+2}}$

20. $\dfrac{\frac{y+3}{y}-\frac{4}{y-1}}{\frac{y}{y-1}+\frac{1}{y}}$

RELATING CONCEPTS (EXERCISES 21–26)

For Individual or Group Work

Simplifying a complex fraction by Method 1 is a good way to review the methods of adding, subtracting, multiplying, and dividing rational expressions. Method 2 gives a good review of the fundamental principle of rational expressions. Refer to the following complex fraction and ***work Exercises 21–26 in order.***

$$\dfrac{\frac{4}{m}+\frac{m+2}{m-1}}{\frac{m+2}{m}-\frac{2}{m-1}}$$

21. Add the fractions in the numerator.

22. Subtract as indicated in the denominator.

23. Divide your answer from Exercise 21 by your answer from Exercise 22.

24. Go back to the original complex fraction and find the LCD of all denominators.

25. Multiply the numerator and denominator of the complex fraction by your answer from Exercise 24.

26. Your answers for Exercises 23 and 25 should be the same. Write a paragraph comparing the two methods. Which method do you prefer? Explain why.

Simplify each expression, using only positive exponents in your answer. See Example 4.

EXERCISES

27. $\dfrac{1}{x^{-2}+y^{-2}}$ **28.** $\dfrac{1}{p^{-2}-q^{-2}}$ **29.** $\dfrac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$

30. $\dfrac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$ **31.** $\dfrac{x^{-1}+2y^{-1}}{2y+4x}$ **32.** $\dfrac{a^{-2}-4b^{-2}}{3b-6a}$

33. **(a)** Start with the complex fraction $\dfrac{\frac{3}{mp}-\frac{4}{p}+\frac{8}{m}}{2m^{-1}-3p^{-1}}$ and write it so that there are no negative exponents in your expression.

(b) Explain why $\dfrac{\frac{3}{mp}-\frac{4}{p}+\frac{8}{m}}{\frac{1}{2m}-\frac{1}{3p}}$ would *not* be a correct response in part (a).

(c) Simplify the complex fraction in part (a).

34. Is $\dfrac{m^{-1}+n^{-1}}{m^{-2}+n^{-2}}=\dfrac{m^2+n^2}{m+n}$ a true statement? Explain why or why not.

7.4 Equations with Rational Expressions and Graphs

OBJECTIVES

1 Determine the domain of a rational equation.
2 Solve rational equations.
3 Recognize the graph of a rational function.
4 Solve rational equations using a graphing calculator.

At the beginning of this chapter, we defined the domain of a rational expression as the set of all possible values of the variable. Any value that makes the denominator 0 is excluded.

OBJECTIVE 1 Determine the domain of a rational equation. The **domain of a rational equation** is the intersection (overlap) of the domains of the rational expressions in the equation.

EXAMPLE 1 Determining the Domains of Rational Equations

Find the domain of each equation.

(a) $\dfrac{2}{x} - \dfrac{3}{2} = \dfrac{7}{2x}$

The domains of the three rational terms of the equation $\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$ are, in order, $\{x \mid x \neq 0\}$, $(-\infty, \infty)$, and $\{x \mid x \neq 0\}$. The intersection of these three domains is all real numbers except 0, which may be written $\{x \mid x \neq 0\}$.

(b) $\dfrac{2}{x-3} - \dfrac{3}{x+3} = \dfrac{12}{x^2-9}$

The domains of the three terms are, respectively, $\{x \mid x \neq 3\}$, $\{x \mid x \neq -3\}$, and $\{x \mid x \neq \pm 3\}$. The domain of the equation is the intersection of the three domains, all real numbers except 3 and -3, written $\{x \mid x \neq \pm 3\}$.

Now Try Exercises 5 and 9.

OBJECTIVE 2 Solve rational equations. The easiest way to solve most equations with rational expressions is to multiply all terms in the equation by the least common denominator. This step will clear the equation of all denominators, as the next examples show. *We can do this only with equations, not expressions.*

Because the first step in solving a rational equation is to multiply both sides of the equation by a common denominator, it is *necessary* to either check the potential solutions or verify that they are in the domain.

CAUTION When both sides of an equation are multiplied by a *variable* expression, the resulting "solutions" may not satisfy the original equation. You *must* either determine and observe the domain or check all potential solutions in the original equation. *It is wise to do both.*

VIDEO

EXAMPLE 2 Solving an Equation with Rational Expressions

Solve $\dfrac{2}{x} - \dfrac{3}{2} = \dfrac{7}{2x}$.

The domain, which excludes 0, was found in Example 1(a).

$$2x\left(\frac{2}{x} - \frac{3}{2}\right) = 2x\left(\frac{7}{2x}\right) \qquad \text{Multiply by the LCD, } 2x.$$

$$2x\left(\frac{2}{x}\right) - 2x\left(\frac{3}{2}\right) = 2x\left(\frac{7}{2x}\right) \qquad \text{Distributive property}$$

$$4 - 3x = 7 \qquad \text{Multiply.}$$

$$-3x = 3 \qquad \text{Subtract 4.}$$

$$x = -1 \qquad \text{Divide by } -3.$$

Check:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x} \qquad \text{Original equation}$$

$$\frac{2}{-1} - \frac{3}{2} \stackrel{?}{=} \frac{7}{2(-1)} \qquad \text{Let } x = -1.$$

$$-\frac{7}{2} = -\frac{7}{2} \qquad \text{True}$$

The solution set is $\{-1\}$.

Now Try Exercise 11.

VIDEO

EXAMPLE 3 Solving an Equation with No Solution

Solve $\dfrac{2}{x-3} - \dfrac{3}{x+3} = \dfrac{12}{x^2-9}$.

Using the result from Example 1(b), we know that the domain excludes 3 and -3. We multiply each side by the LCD, $(x+3)(x-3)$.

$$(x+3)(x-3)\left(\frac{2}{x-3} - \frac{3}{x+3}\right) = (x+3)(x-3)\left(\frac{12}{x^2-9}\right)$$

$$\begin{aligned} 2(x+3) - 3(x-3) &= 12 && \text{Distributive property} \\ 2x + 6 - 3x + 9 &= 12 && \text{Distributive property} \\ -x + 15 &= 12 && \text{Combine terms.} \\ -x &= -3 && \text{Subtract 15.} \\ x &= 3 && \text{Divide by } -1. \end{aligned}$$

Since 3 is not in the domain, it cannot be a solution of the equation. Substituting 3 in the original equation shows why.

$$\begin{aligned} \frac{2}{x-3} - \frac{3}{x+3} &= \frac{12}{x^2-9} \\ \frac{2}{3-3} - \frac{3}{3+3} &\stackrel{?}{=} \frac{12}{3^2-9} && \text{Let } x = 3. \\ \frac{2}{0} - \frac{3}{6} &\stackrel{?}{=} \frac{12}{0} \end{aligned}$$

Since division by 0 is undefined, the given equation has no solution and the solution set is $\emptyset$.

Now Try Exercise 25.

EXPLORE

YOU TRY IT

EXAMPLE 4 Solving an Equation with Rational Expressions

Solve $\dfrac{3}{p^2+p-2} - \dfrac{1}{p^2-1} = \dfrac{7}{2(p^2+3p+2)}$.

Factor each denominator to find the LCD, $2(p-1)(p+2)(p+1)$. The domain excludes 1, -2, and -1. Multiply each side by the LCD.

$$2(p-1)(p+2)(p+1)\left[\frac{3}{(p+2)(p-1)} - \frac{1}{(p+1)(p-1)}\right]$$

$$= 2(p-1)(p+2)(p+1)\left[\frac{7}{2(p+2)(p+1)}\right]$$

$$\begin{aligned} 2 \cdot 3(p+1) - 2(p+2) &= 7(p-1) && \text{Distributive property} \\ 6p + 6 - 2p - 4 &= 7p - 7 && \text{Distributive property} \\ 4p + 2 &= 7p - 7 && \text{Combine terms.} \\ 9 &= 3p \\ 3 &= p \end{aligned}$$

Note that 3 is in the domain; substitute 3 for p in the original equation to check that the solution set is $\{3\}$.

Now Try Exercise 37.

VIDEO

EXAMPLE 5 Solving an Equation That Leads to a Quadratic Equation

Solve $\dfrac{2}{3x+1} = \dfrac{1}{x} - \dfrac{6x}{3x+1}$.

Since the denominator $3x + 1$ cannot equal 0, $-\frac{1}{3}$ is excluded from the domain, as is 0. Multiply each side by the LCD, $x(3x + 1)$.

$$x(3x+1)\left(\frac{2}{3x+1}\right) = x(3x+1)\left(\frac{1}{x} - \frac{6x}{3x+1}\right)$$

$$2x = 3x + 1 - 6x^2$$

Since this equation is quadratic, write it in standard form with 0 on the right side.

$$6x^2 - 3x + 2x - 1 = 0$$

$$6x^2 - x - 1 = 0 \quad \text{Standard form}$$

$$(3x+1)(2x-1) = 0 \quad \text{Factor.}$$

$$3x + 1 = 0 \quad \text{or} \quad 2x - 1 = 0 \quad \text{Zero-factor property}$$

$$x = -\frac{1}{3} \quad \text{or} \quad x = \frac{1}{2}$$

Because $-\frac{1}{3}$ is not in the domain of the equation, it is not a solution. Check that the solution set is $\left\{\frac{1}{2}\right\}$.

Now Try Exercise 31.

OBJECTIVE 3 Recognize the graph of a rational function. As mentioned in Section 7.1, a function defined by a rational expression is a *rational function.* Because one or more values of x are excluded from the domain of most rational functions, their graphs are usually *discontinuous.* That is, there will be one or more breaks in the graph. For example, we use point plotting and observing the domain to graph the simple rational function defined by

$$f(x) = \frac{1}{x}.$$

The domain of this function includes all real numbers except 0. Thus, there will be no point on the graph with $x = 0$. The vertical line with equation $x = 0$ is called a **vertical asymptote** of the graph. We show some typical ordered pairs in the table for both negative and positive x-values.

x	-3	-2	-1	$-.5$	$-.25$	$-.1$	$.1$	$.25$	$.5$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	-10	10	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

Notice that the closer positive values of x are to 0, the larger y is. Similarly, the closer negative values of x are to 0, the smaller (more negative) y is. Using this observation, the fact that the domain excludes 0, and plotting the points found above produces the graph in Figure 2.

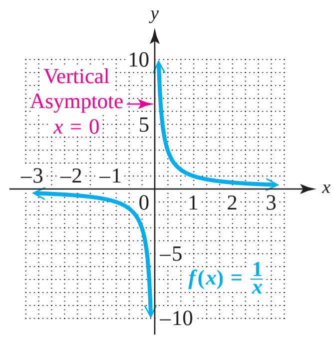


FIGURE 2

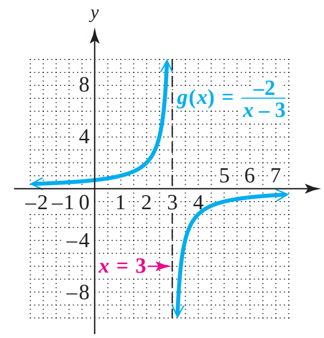


FIGURE 3

The graph of

$$g(x) = \frac{-2}{x-3},$$

is shown in Figure 3. Some ordered pairs that belong to the function are shown in the table.

x	-2	-1	0	1	2	2.5	2.75	3.25	3.5	4	5	6	7
y	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{2}{3}$	1	2	4	8	-8	-4	-2	-1	$-\frac{2}{3}$	$-\frac{1}{2}$

There is no point on the graph for $x = 3$, because 3 is excluded from the domain. The dashed line $x = 3$ represents the asymptote and is not part of the graph. As suggested by the points from the table, the graph gets closer to the vertical asymptote $x = 3$ as the x-values get closer to 3.

OBJECTIVE 4 Solve rational equations using a graphing calculator. Earlier, we solved linear and quadratic equations using a graphing calculator. The procedure is similar with rational equations. Because rational functions usually have values of x that are excluded from the domain, a calculator in *connected mode* may show a vertical line on the screen where an asymptote occurs. Using *dot mode* will usually give a more realistic picture.

In Figure 4, we show the graph of

$$g(x) = \frac{x}{x+3}$$

generated in connected mode and dot mode. If dot mode is used, we must remember that, theoretically, the function is continuous (unbroken) on its domain, in this case, $(-\infty, -3) \cup (-3, \infty)$. As before, the x-intercepts of the graph give the solutions of the equation

$$\frac{x}{x+3} = 0.$$

From the graph, since the x-intercept is $(0, 0)$, we see that the only solution of this equation is $\{0\}$.

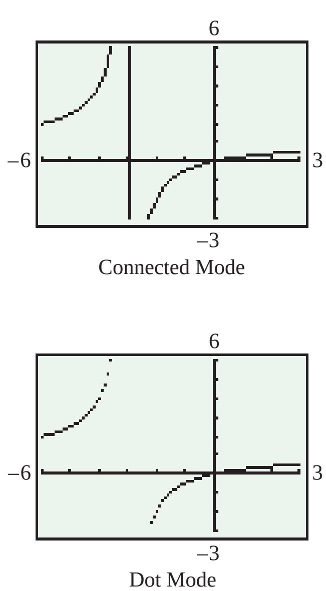


FIGURE 4

EXAMPLE 6 Finding Solutions of a Rational Equation from a Graphing Calculator Screen

Two views of the graph of $f(x) = x^{-2} + x^{-1} - 1.5$ are shown in Figure 5. Use the graph to determine the solution set of $x^{-2} + x^{-1} - 1.5 = 0$.

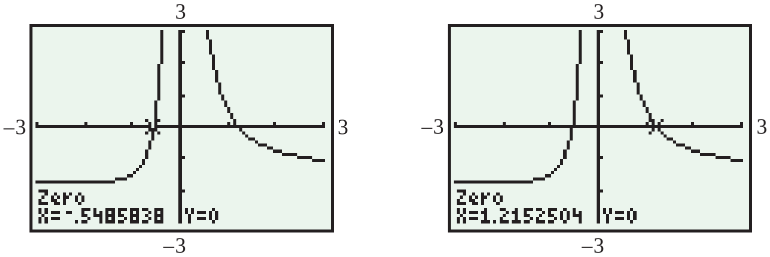


FIGURE 5

Look at the bottom of each calculator screen for the x-values for which $y = 0$, $-.5485838$ and 1.2152504. These are the solutions of the equation. The solution set is $\{-.5485838, 1.2152504\}$.

Now Try Exercise 49.

7.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 11/Videotape 11

As explained in this section, any values that would cause a denominator to equal 0 must be excluded from the domain and consequently as solutions of an equation that has variable expressions in the denominators. ***(a)*** *Without actually solving the equation, list all possible numbers that would have to be rejected if they appeared as potential solutions.* ***(b)*** *Then give the domain using set notation. See Example 1.*

1. $\frac{1}{x+1}-\frac{1}{x-2}=0$

2. $\frac{3}{x+4}-\frac{2}{x-9}=0$

3. $\frac{5}{3x+5}-\frac{1}{x}=\frac{1}{2x+3}$

4. $\frac{6}{4x+7}-\frac{3}{x}=\frac{5}{6x-13}$

5. $\frac{1}{3x}+\frac{1}{2x}=\frac{x}{3}$

6. $\frac{5}{6x}-\frac{8}{2x}=\frac{x}{4}$

7. $\frac{3x+1}{x-4}=\frac{6x+5}{2x-7}$

8. $\frac{4x-1}{2x+3}=\frac{12x-25}{6x-2}$

9. $\frac{2}{x^2-x}+\frac{1}{x+3}=\frac{4}{x-2}$

10. Is it possible that any potential solutions to the equation

$$\frac{x+7}{4}-\frac{x+3}{3}=\frac{x}{12}$$

would have to be rejected? Explain.

Solve each equation. See Examples 2–5.

11. $\frac{-5}{2x}+\frac{3}{4x}=\frac{-7}{4}$

12. $\frac{6}{5x}-\frac{2}{3x}=\frac{-8}{45}$

13. $x-\frac{24}{x}=-2$

14. $p+\frac{15}{p}=-8$

15. $\frac{x-4}{x+6}=\frac{2x+3}{2x-1}$

16. $\frac{5x-8}{x+2}=\frac{5x-1}{x+3}$

17. $\dfrac{3x + 1}{x - 4} = \dfrac{6x + 5}{2x - 7}$

18. $\dfrac{4x - 1}{2x + 3} = \dfrac{12x - 25}{6x - 2}$

19. $\dfrac{1}{y - 1} + \dfrac{5}{12} = \dfrac{-2}{3y - 3}$

20. $\dfrac{4}{m + 2} - \dfrac{11}{9} = \dfrac{1}{3m + 6}$

21. $\dfrac{-2}{3t - 6} - \dfrac{1}{36} = \dfrac{-3}{4t - 8}$

22. $\dfrac{3}{4m + 2} = \dfrac{17}{2} - \dfrac{7}{2m + 1}$

23. $\dfrac{3}{k + 2} - \dfrac{2}{k^2 - 4} = \dfrac{1}{k - 2}$

24. $\dfrac{3}{x - 2} + \dfrac{21}{x^2 - 4} = \dfrac{14}{x + 2}$

25. $\dfrac{1}{y + 2} + \dfrac{3}{y + 7} = \dfrac{5}{y^2 + 9y + 14}$

26. $\dfrac{1}{t + 3} + \dfrac{4}{t + 5} = \dfrac{2}{t^2 + 8t + 15}$

27. $\dfrac{9}{x} + \dfrac{4}{6x - 3} = \dfrac{2}{6x - 3}$

28. $\dfrac{5}{n} + \dfrac{4}{6 - 3n} = \dfrac{2n}{6 - 3n}$

29. $\dfrac{6}{w + 3} + \dfrac{-7}{w - 5} = \dfrac{-48}{w^2 - 2w - 15}$

30. $\dfrac{2}{r - 5} + \dfrac{3}{2r + 1} = \dfrac{22}{2r^2 - 9r - 5}$

31. $\dfrac{x}{x - 3} + \dfrac{4}{x + 3} = \dfrac{18}{x^2 - 9}$

32. $\dfrac{2x}{x - 3} + \dfrac{4}{x + 3} = \dfrac{-24}{x^2 - 9}$

33. $\dfrac{1}{x + 4} + \dfrac{x}{x - 4} = \dfrac{-8}{x^2 - 16}$

34. $\dfrac{5}{x - 4} - \dfrac{3}{x - 1} = \dfrac{x^2 - 1}{x^2 - 5x + 4}$

35. $\dfrac{2}{4x + 7} + \dfrac{x}{3} = \dfrac{6}{12x + 21}$

36. $\dfrac{2}{k^2 + k - 6} + \dfrac{1}{k^2 - k - 2} = \dfrac{4}{k^2 + 4k + 3}$

37. $\dfrac{5}{p^2 + 3p + 2} - \dfrac{3}{p^2 - 4} = \dfrac{1}{p^2 - p - 2}$

38. $\dfrac{5x + 14}{x^2 - 9} = \dfrac{-2x^2 - 5x + 2}{x^2 - 9} + \dfrac{2x + 4}{x - 3}$

39. $\dfrac{4x - 7}{4x^2 - 9} = \dfrac{-2x^2 + 5x - 4}{4x^2 - 9} + \dfrac{x + 1}{2x + 3}$

40. Professor Dan Abbey asked the following question on a test: What is the solution set of $\frac{x + 3}{x + 3} = 1$? Only one student answered it correctly.

(a) What is the solution set?

(b) Many students answered {all real numbers}. Why is this not correct?

Graph each rational function. Give the equation of the vertical asymptote. See Figures 2 and 3.

41. $f(x) = \dfrac{2}{x}$

42. $f(x) = \dfrac{3}{x}$

43. $f(x) = \dfrac{1}{x - 2}$

44. $f(x) = \dfrac{1}{x + 2}$

Solve each problem.

45. The average number of vehicles waiting in line to enter a sports arena parking area is modeled by the function defined by

$$w(x) = \frac{x^2}{2(1 - x)},$$

where x is a quantity between 0 and 1 known as the *traffic intensity.* (*Source:* Mannering, F., and W. Kilareski, *Principles of Highway Engineering and Traffic Control*, John Wiley and Sons, 1990.) To the nearest tenth, find the average number of vehicles waiting for each traffic intensity.

(a) .1 **(b)** .8 **(c)** .9

(d) What happens to waiting time as traffic intensity increases?

46. The percent of deaths caused by smoking is modeled by the rational function defined by

$$p(x) = \frac{x - 1}{x},$$

where x is the number of times a smoker is more likely to die of lung cancer than a nonsmoker. This is called the *incidence rate.* (*Source:* Walker, A., *Observation and Inference: An Introduction to the Methods of Epidemiology,* Epidemiology Resources Inc., 1991.) For example, $x = 10$ means that a smoker is 10 times more likely than a nonsmoker to die of lung cancer.

(a) Find $p(x)$ if x is 10.

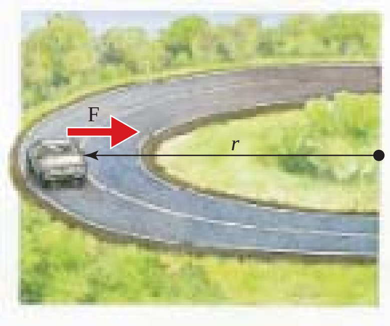
(b) For what values of x is $p(x) = 80\%$? (*Hint:* Change 80% to a decimal.)

(c) Can the incidence rate equal 0? Explain.

47. The force required to keep a 2000-lb car, going 30 mph, from skidding on a curve is given by

$$F(r) = \frac{225{,}000}{r},$$

where r is the radius of the curve in feet.



(a) What radius must a curve have if a force of 450 lb is needed to keep the car from skidding?

(b) As the radius of the curve is lengthened, how is the force affected?

EXERCISES

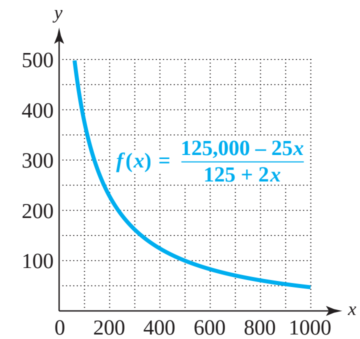
48. The amount of heating oil produced (in gallons per day) by an oil refinery is modeled by the rational function defined by

$$f(x) = \frac{125{,}000 - 25x}{125 + 2x},$$

where x is the amount of gasoline produced (in hundreds of gallons per day). Suppose the refinery must produce 300 gal of heating oil per day to meet the needs of its customers.

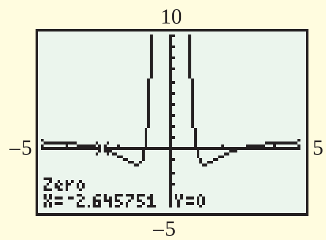
(a) How much gasoline will be produced per day?

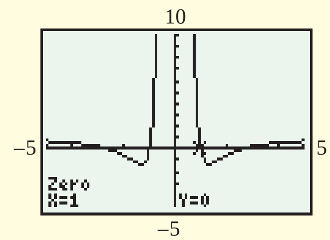
(b) The graph of f is shown in the figure. Use it to decide what happens to the amount of gasoline (x) produced as the amount of heating oil (y) produced increases.



TECHNOLOGY INSIGHTS (EXERCISES 49–52)

Two views of the graph of $f(x) = 7x^{-4} - 8x^{-2} + 1$ are shown. Use the graphs to respond to Exercises 49 and 50. See Example 6.





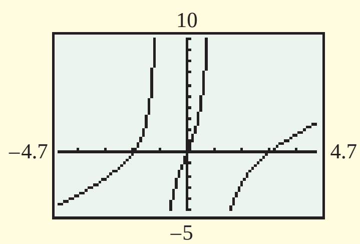
49. How many solutions does the equation $f(x) = 0$ have?

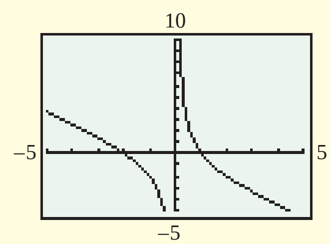
50. Give the solutions of $f(x) = 0$ (to the nearest hundredth if necessary).

In Exercises 51 and 52, use the graph to determine the solution set of the equation $f(x) = 0$. All solutions are integers.

51. $f(x) = \dfrac{x^3 - x^2 - 6x}{x^2 - 1}$

52. $f(x) = \dfrac{-x^3 - x^2 + 2x}{x^2}$





SUMMARY EXERCISES ON OPERATIONS AND EQUATIONS WITH RATIONAL EXPRESSIONS

A common student error is to confuse an equation, such as $\frac{x}{2} + \frac{x}{3} = -5$, with an operation, such as $\frac{x}{2} + \frac{x}{3}$. Look for the equals sign to distinguish between them. Equations are solved for a numerical answer, while problems involving operations result in simplified expressions.

Solving an Equation

Solve: $\frac{x}{2} + \frac{x}{3} = -5.$

Multiply each side by the LCD, 6.

$$6\left(\frac{x}{2} + \frac{x}{3}\right) = 6(-5)$$
$$6\left(\frac{x}{2}\right) + 6\left(\frac{x}{3}\right) = 6(-5)$$
$$3x + 2x = -30$$
$$5x = -30$$
$$x = -6$$

Check that the solution set is $\{-6\}$.

Performing an Operation

Add: $\frac{x}{2} + \frac{x}{3}.$

Write both fractions with the LCD, 6.

$$\frac{x}{2} + \frac{x}{3} = \frac{x \cdot 3}{2 \cdot 3} + \frac{x \cdot 2}{3 \cdot 2}$$
$$= \frac{3x}{6} + \frac{2x}{6}$$
$$= \frac{3x + 2x}{6}$$
$$= \frac{5x}{6}$$

In each exercise, identify as an equation *or an* operation. *Then perform the indicated operation or solve the given equation, as appropriate.*

1. $\frac{x}{2} - \frac{x}{4} = 5$

2. $\frac{4x - 20}{x^2 - 25} \cdot \frac{(x + 5)^2}{10}$

3. $\frac{6}{7x} - \frac{4}{x}$

4. $\dfrac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}}$

5. $\frac{5}{7t} = \frac{52}{7} - \frac{3}{t}$

6. $\frac{x - 5}{3} + \frac{1}{3} = \frac{x - 2}{5}$

7. $\frac{7}{6x} + \frac{5}{8x}$

8. $\frac{4}{x} - \frac{8}{x + 1} = 0$

9. $\dfrac{\frac{6}{x + 1} - \frac{1}{x}}{\frac{2}{x} - \frac{4}{x + 1}}$

10. $\frac{8}{r + 2} - \frac{7}{4r + 8}$

11. $\frac{x}{x + y} + \frac{2y}{x - y}$

12. $\frac{3p^2 - 6p}{p + 5} \div \frac{p^2 - 4}{8p + 40}$

13. $\frac{x - 2}{9} \cdot \frac{5}{8 - 4x}$

14. $\frac{a - 4}{3} + \frac{11}{6} = \frac{a + 1}{2}$

15. $\frac{b^2 + b - 6}{b^2 + 2b - 8} \cdot \frac{b^2 + 8b + 16}{3b + 12}$

16. $\frac{10z^2 - 5z}{3z^3 - 6z^2} \div \frac{2z^2 + 5z - 3}{z^2 + z - 6}$

17. $\frac{5}{x^2 - 2x} - \frac{3}{x^2 - 4}$

18. $\frac{6}{t + 1} + \frac{4}{5t + 5} = \frac{34}{15}$

19. $\dfrac{\frac{5}{x} - \frac{3}{y}}{\frac{9x^2 - 25y^2}{x^2y}}$

20. $\frac{-2}{a^2 + 2a - 3} - \frac{5}{3 - 3a} = \frac{4}{3a + 9}$

21. $\frac{4y^2 - 13y + 3}{2y^2 - 9y + 9} \div \frac{4y^2 + 11y - 3}{6y^2 - 5y - 6}$

22. $\frac{8}{3k + 9} - \frac{8}{15} = \frac{2}{5k + 15}$

23. $\dfrac{3r}{r-2} = 1 + \dfrac{6}{r-2}$

24. $\dfrac{6z^2 - 5z - 6}{6z^2 + 5z - 6} \cdot \dfrac{12z^2 - 17z + 6}{12z^2 - z - 6}$

25. $\dfrac{-1}{3-x} - \dfrac{2}{x-3}$

26. $\dfrac{\dfrac{t}{4} - \dfrac{1}{t}}{1 + \dfrac{t+4}{t}}$

27. $\dfrac{2}{y+1} - \dfrac{3}{y^2 - y - 2} = \dfrac{3}{y-2}$

28. $\dfrac{7}{2x^2 - 8x} + \dfrac{3}{x^2 - 16}$

29. $\dfrac{3}{y-3} - \dfrac{3}{y^2 - 5y + 6} = \dfrac{2}{y-2}$

30. $\dfrac{2k + \dfrac{5}{k-1}}{3k - \dfrac{2}{k}}$

7.5 Applications of Rational Expressions

OBJECTIVES

1. Find the value of an unknown variable in a formula.
2. Solve a formula for a specified variable.
3. Solve applications using proportions.
4. Solve applications about distance, rate, and time.
5. Solve applications about work rates.

VIDEO

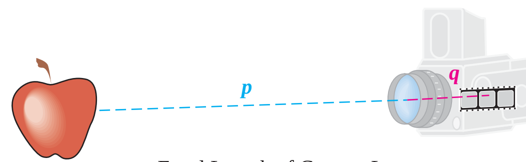
OBJECTIVE 1 Find the value of an unknown variable in a formula. Formulas may contain rational expressions, as does $t = \frac{d}{r}$. We now show how to work with formulas of this type.

EXAMPLE 1 Finding the Value of a Variable in a Formula

In physics the focal length, f, of a lens is given by the formula

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

In the formula, p is the distance from the object to the lens and q is the distance from the lens to the image. See Figure 6. Find q if $p = 20$ cm and $f = 10$ cm.



Focal Length of Camera Lens

FIGURE 6

Replace f with 10 and p with 20.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{10} = \frac{1}{20} + \frac{1}{q} \qquad \text{Let } f = 10,\ p = 20.$$

$$20q \cdot \frac{1}{10} = 20q\left(\frac{1}{20} + \frac{1}{q}\right) \qquad \text{Multiply by the LCD, } 20q.$$

$$2q = q + 20$$

$$q = 20$$

The distance from the lens to the image is 20 cm.

Now Try Exercise 5.

OBJECTIVE 2 Solve a formula for a specified variable. Recall that the goal in solving for a specified variable is to isolate it on one side of the equals sign.

EXAMPLE 2 Solving a Formula for a Specified Variable

Solve $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ for p.

$$fpq \cdot \frac{1}{f} = fpq\left(\frac{1}{p} + \frac{1}{q}\right) \qquad \text{Multiply by the LCD, } fpq.$$

$$pq = fq + fp \qquad \text{Distributive property}$$

Transform the equation so that the terms with p (the specified variable) are on the same side. One way to do this is to subtract fp from each side.

$$pq - fp = fq \qquad \text{Subtract } fp.$$

$$p(q - f) = fq \qquad \text{Factor out } p.$$

$$p = \frac{fq}{q - f} \qquad \text{Divide by } q - f.$$

Now Try Exercise 11.

EXAMPLE 3 Solving a Formula for a Specified Variable

Solve $I = \frac{nE}{R + nr}$ for n.

$$(R + nr)I = (R + nr)\frac{nE}{R + nr} \qquad \text{Multiply by } R + nr.$$

$$RI + nrI = nE$$

$$RI = nE - nrI \qquad \text{Subtract } nrI.$$

$$RI = n(E - rI) \qquad \text{Factor out } n.$$

$$\frac{RI}{E - rI} = n \qquad \text{Divide by } E - rI.$$

Now Try Exercise 15.

CAUTION Refer to the steps in Examples 2 and 3 that factor out the desired variable. This is a step that often gives students difficulty. Remember that the variable for which you are solving *must* be a factor on only one side of the equation, so each side can be divided by the remaining factor in the last step.

We now solve problems that translate into equations with rational expressions, using the six-step problem-solving method from Chapter 2.

OBJECTIVE 3 Solve applications using proportions. A **ratio** is a comparison of two quantities. The ratio of a to b may be written in any of the following ways:

$$a \text{ to } b, \qquad a:b, \qquad \text{or} \qquad \frac{a}{b}.$$

Ratios are usually written as quotients in algebra. A **proportion** is a statement that two ratios are equal. Proportions are a useful and important type of rational equation.

EXAMPLE 4 Solving a Proportion

In 1999, 16 of every 100 Americans had no health insurance coverage. The population at that time was about 274 million. How many million had no health insurance? (*Source:* U.S. Bureau of the Census.)

Step 1 **Read** the problem.

Step 2 **Assign a variable.** Let $x =$ the number (in millions) who had no health insurance.

Step 3 **Write an equation.** To get an equation, set up a proportion. The ratio x to 274 should equal the ratio 16 to 100. Write the proportion and solve the equation.

$$\frac{16}{100} = \frac{x}{274}$$

Step 4 **Solve.**

$$27{,}400\left(\frac{16}{100}\right) = 27{,}400\left(\frac{x}{274}\right) \qquad \text{Multiply by a common denominator.}$$

$$4384 = 100x \qquad \text{Simplify.}$$

$$x = 43.84$$

Step 5 **State the answer.** There were 43.84 million Americans with no health insurance in 1999.

Step 6 **Check** that the ratio of this number to 274 million is equivalent to $\frac{16}{100}$.

Now Try Exercise 31.

EXAMPLE 5 Solving a Proportion Involving Rates

Marissa's car uses 10 gal of gas to travel 210 mi. She has 5 gal of gas in the car, and she wants to know how much more gas she will need to drive 640 mi. If we assume the car continues to use gas at the same rate, how many more gallons will she need?

Step 1 **Read** the problem.

Step 2 **Assign a variable.** Let $x =$ the additional number of gallons of gas needed.

Step 3 **Write an equation.** To get an equation, set up a proportion.

$$\frac{\text{gallons}}{\text{miles}} \rightarrow \frac{10}{210} = \frac{5 + x}{640} \leftarrow \frac{\text{gallons}}{\text{miles}}$$

Step 4 **Solve.** The LCD is $10 \cdot 21 \cdot 64$.

$$10 \cdot 21 \cdot 64\left(\frac{10}{210}\right) = 10 \cdot 21 \cdot 64\left(\frac{5 + x}{640}\right)$$

$$64 \cdot 10 = 21(5 + x)$$

$$640 = 105 + 21x \qquad \text{Distributive property}$$

$$535 = 21x \qquad \text{Subtract 105.}$$

$$25.5 \approx x \qquad \text{Divide by 21; round to the nearest tenth.}$$

Step 5 **State the answer.** Marissa will need about 25.5 more gallons of gas.

Step 6 **Check** the answer in the words of the problem. The 25.5 gal plus the 5 gal equals 30.5 gal.

$$\frac{30.5}{640} \approx .048 \qquad \text{and} \qquad \frac{10}{210} \approx .048$$

Since the rates are equal, the solution is correct.

Now Try Exercise 35.

OBJECTIVE 4 Solve applications about distance, rate, and time. A familiar example of a rate is speed, which is the ratio of distance to time. The next examples use the distance formula $d = rt$ introduced in Chapter 2.

Photo not available

EXAMPLE 6 Solving a Problem about Distance, Rate, and Time

A tour boat goes 10 mi against the current in a small river in the same time that it goes 15 mi with the current. If the speed of the current is 3 mph, find the speed of the boat in still water.

Step 1 **Read** the problem. We must find the speed of the boat in still water.

Step 2 **Assign a variable.**

Let $x =$ the speed of the boat in still water.

When the boat is traveling *against* the current, the current slows the boat down, and the speed of the boat is the difference between its speed in still water and the speed of the current, that is, $x - 3$ mph.

When the boat is traveling *with* the current, the current speeds the boat up, and the speed of the boat is the sum of its speed in still water and the speed of the current, that is, $x + 3$ mph.

Thus, $x - 3 =$ the speed of the boat *against* the current,

and $x + 3 =$ the speed of the boat *with* the current.

Because the time is the same going against the current as with the current, find time in terms of distance and rate (speed) for each situation. Start with the distance formula, $d = rt$, and divide each side by r to get

$$t = \frac{d}{r}.$$

Going against the current, the distance is 10 mi and the rate is $x - 3$ mph, giving

$$t = \frac{d}{r} = \frac{10}{x - 3}.$$

Going with the current, the distance is 15 mi and the rate is $x + 3$ mph, so

$$t = \frac{d}{r} = \frac{15}{x + 3}.$$

This information is summarized in the following table.

	Distance	Rate	Time
Against Current	10	$x - 3$	$\frac{10}{x-3}$
With Current	15	$x + 3$	$\frac{15}{x+3}$

Times are equal.

Step 3 **Write an equation.** Because the times are equal,

$$\frac{10}{x-3} = \frac{15}{x+3}.$$

Step 4 **Solve** this equation. The LCD is $(x + 3)(x - 3)$.

$$(x+3)(x-3)\left(\frac{10}{x-3}\right) = (x+3)(x-3)\left(\frac{15}{x+3}\right) \quad \text{Multiply by the LCD.}$$

$$10(x + 3) = 15(x - 3)$$

$$10x + 30 = 15x - 45 \quad \text{Distributive property}$$

$$30 = 5x - 45 \quad \text{Subtract } 10x.$$

$$75 = 5x \quad \text{Add 45.}$$

$$15 = x \quad \text{Divide by 5.}$$

Step 5 **State the answer.** The speed of the boat in still water is 15 mph.

Step 6 **Check** the answer: $\frac{10}{15-3} = \frac{15}{15+3}$ is true.

Now Try Exercise 39.

EXAMPLE 7 Solving a Problem about Distance, Rate, and Time

At O'Hare International Airport in Chicago, Cheryl and Bill are walking to the gate (at the same speed) to catch their flight to Akron, Ohio. Since Bill wants a window seat, he steps onto the moving sidewalk and continues to walk while Cheryl uses the stationary sidewalk. If the sidewalk moves at 1 m per sec and Bill saves 50 sec covering the 300-m distance, what is their walking speed?

Step 1 **Read** the problem. We must find their walking speed.

Step 2 **Assign a variable.** Let x represent their walking speed in meters per second. Thus Cheryl travels at x m per sec and Bill travels at $x + 1$ m per sec. Since Bill's time is 50 sec less than Cheryl's time, express their times in terms of the known distances and the variable rates. As in Example 6, start with $d = rt$ and divide each side by r to get $t = \frac{d}{r}$. For Cheryl, the distance is 300 m and the rate is x mph. Cheryl's time is

$$t = \frac{d}{r} = \frac{300}{x}.$$

Bill travels 300 m at a rate of $x + 1$ mph, so his time is

$$t = \frac{d}{r} = \frac{300}{x+1}.$$

This information is summarized in the following table.

	Distance	Rate	Time
Cheryl	300	x	$\frac{300}{x}$
Bill	300	$x + 1$	$\frac{300}{x+1}$

Step 3 **Write an equation** using the times from the table.

Bill's time	is	Cheryl's time		less 50 seconds.
$\frac{300}{x+1}$	$=$	$\frac{300}{x}$	$-$	50

Step 4 **Solve.**

$$x(x+1)\left(\frac{300}{x+1}\right) = x(x+1)\left(\frac{300}{x} - 50\right) \qquad \text{Multiply by the LCD, } x(x+1).$$

$$300x = 300(x+1) - 50x(x+1)$$

$$300x = 300x + 300 - 50x^2 - 50x \qquad \text{Distributive property}$$

$$0 = 50x^2 + 50x - 300 \qquad \text{Standard form}$$

$$0 = x^2 + x - 6 \qquad \text{Divide by 50.}$$

$$0 = (x+3)(x-2) \qquad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Zero-factor property}$$

$$x = -3 \quad \text{or} \quad x = 2$$

Discard the negative answer, since speed cannot be negative.

Step 5 **State the answer.** Their walking speed is 2 m per sec.

Step 6 **Check** the solution in the words of the original problem.

Now Try Exercise 45.

OBJECTIVE 5 Solve applications about work rates. Problems about work are closely related to distance problems.

PROBLEM SOLVING

People work at different rates. If the letters r, t, and A represent the rate at which the work is done, the time required, and the amount of work accomplished, respectively, then $A = rt$. Notice the similarity to the distance formula, $d = rt$.

Amount of work can be measured in terms of jobs accomplished. Thus, if 1 job is completed, $A = 1$, and the formula gives the rate as

$$1 = rt$$

$$r = \frac{1}{t}.$$

Rate of Work

If a job can be accomplished in t units of time, then the rate of work is

$$\frac{1}{t} \textbf{ job per unit of time.}$$

To solve a work problem, we begin by using this fact to express all rates of work. See if you can identify the six steps used in the following example.

EXAMPLE 8 Solving a Problem about Work

Letitia and Kareem are working on a neighborhood cleanup. Kareem can clean up all the trash in the area in 7 hr, while Letitia can do the same job in 5 hr. How long will it take them if they work together?

Let $x =$ the number of hours it will take the two people working together. Just as we made a table for the distance formula, $d = rt$, make a table here for $A = rt$, with $A = 1$. Since $A = 1$, the rate for each person will be $\frac{1}{t}$, where t is the time it takes the person to complete the job alone. For example, since Kareem can clean up all the trash in 7 hr, his rate is $\frac{1}{7}$ of the job per hour. Similarly, Letitia's rate is $\frac{1}{5}$ of the job per hour. Fill in the table as shown.

	Rate	*Time Working Together*	*Fractional Part of the Job Done*
Kareem	$\frac{1}{7}$	x	$\frac{1}{7}x$
Letitia	$\frac{1}{5}$	x	$\frac{1}{5}x$

Since together they complete 1 job, the sum of the fractional parts accomplished by them should equal 1.

$$\begin{array}{ccccc} \text{Part done by Kareem} & + & \text{part done by Letitia} & \text{is} & \text{1 whole job.} \\ \frac{1}{7}x & + & \frac{1}{5}x & = & 1 \end{array}$$

Solve this equation. The LCD is 35.

$$\begin{aligned} 35\left(\frac{1}{7}x + \frac{1}{5}x\right) &= 35 \cdot 1 \\ 5x + 7x &= 35 \\ 12x &= 35 \\ x &= \frac{35}{12} \end{aligned}$$

Working together, Kareem and Letitia can do the job in $\frac{35}{12}$ hr, or 2 hr 55 min. Check this result in the original problem.

Now Try Exercise 47.

There is another way to approach problems about work. For instance, in Example 8, x represents the number of hours it will take the two people working together to complete the entire job. In one hour, $\frac{1}{x}$ of the entire job will be completed. In one hour, Kareem completes $\frac{1}{7}$ of the job and Letitia completes $\frac{1}{5}$ of the job. The sum of their rates should equal $\frac{1}{x}$. This gives the equation

$$\frac{1}{7} + \frac{1}{5} = \frac{1}{x}.$$

When each side of this equation is multiplied by $35x$, the result is $5x + 7x = 35$. Notice that this is the same equation we got in Example 8 in the third line from the bottom. Thus the solution of the equation is the same using either approach.

7.5 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

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In Exercises 1–4, a familiar formula is given. Give the letter of the choice that is an equivalent form of the given formula.

EXERCISES

1. $p = br$ (percent)

A. $b = \dfrac{p}{r}$ **B.** $r = \dfrac{b}{p}$ **C.** $b = \dfrac{r}{p}$ **D.** $p = \dfrac{r}{b}$

2. $V = LWH$ (geometry)

A. $H = \dfrac{LW}{V}$ **B.** $L = \dfrac{V}{WH}$ **C.** $L = \dfrac{WH}{V}$ **D.** $W = \dfrac{H}{VL}$

3. $m = \dfrac{F}{a}$ (physics)

A. $a = mF$ **B.** $F = \dfrac{m}{a}$ **C.** $F = \dfrac{a}{m}$ **D.** $F = ma$

4. $I = \dfrac{E}{R}$ (electricity)

A. $R = \dfrac{I}{E}$ **B.** $R = IE$ **C.** $E = \dfrac{I}{R}$ **D.** $E = RI$

Solve each problem. See Example 1.

EXERCISES

5. A gas law in chemistry says that

$$\frac{PV}{T} = \frac{pv}{t}.$$

Suppose that $T = 300$, $t = 350$, $V = 9$, $P = 50$, and $v = 8$. Find p.

6. In work with electric circuits, the formula

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$$

occurs. Find b if $a = 8$ and $c = 12$.

7. A formula from anthropology says that

$$c = \frac{100b}{L}.$$

Find L if $c = 80$ and $b = 5$.

8. The gravitational force between two masses is given by

$$F = \frac{GMm}{d^2}.$$

Find M if $F = 10$, $G = 6.67 \times 10^{-11}$, $m = 1$, and $d = 3 \times 10^{-6}$.

Solve each formula for the specified variable. See Examples 2 and 3.

9. $F = \dfrac{GMm}{d^2}$ for G (physics)

10. $F = \dfrac{GMm}{d^2}$ for M (physics)

11. $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$ for a (electricity)

12. $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$ for b (electricity)

13. $\dfrac{PV}{T} = \dfrac{pv}{t}$ for v (chemistry)

14. $\dfrac{PV}{T} = \dfrac{pv}{t}$ for T (chemistry)

15. $I = \dfrac{nE}{R + nr}$ for r (engineering)

16. $a = \dfrac{V - v}{t}$ for V (physics)

17. $A = \dfrac{1}{2}h(B + b)$ for b (mathematics)

18. $S = \dfrac{n}{2}(a + \ell)d$ for n (mathematics)

19. $\dfrac{E}{e} = \dfrac{R + r}{r}$ for r (engineering)

20. $y = \dfrac{x + z}{a - x}$ for x

21. To solve the equation $m = \frac{ab}{a - b}$ for a, what is the first step?

22. Suppose you are asked to solve the equation

$$rp - rq = p + q$$

for r. What is the first step?

Solve each problem mentally. Use proportions in Exercises 23 and 24.

23. In a mathematics class, 3 of every 4 students are girls. If there are 20 students in the class, how many are girls? How many are boys?

24. In a certain southern state, sales tax on a purchase of \$1.50 is \$.12. What is the sales tax on a purchase of \$6.00?

25. If Marin can mow her yard in 2 hr, what is her rate (in job per hour)?

26. A van traveling from Atlanta to Detroit averages 50 mph and takes 14 hr to make the trip. How far is it from Atlanta to Detroit?

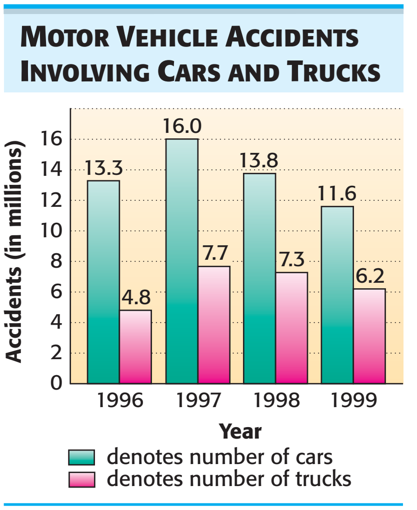
Use the bar graph to answer the questions in Exercises 27–30.

27. In which year was the ratio of truck accidents to car accidents the least?

28. In which year was the ratio of truck accidents to car accidents the greatest?

29. In which year was the ratio of car accidents to truck accidents closest to 3 to 1?

30. In which year was the ratio of car accidents to truck accidents closest to 2 to 1?



Source: National Safety Council.

Use a proportion to solve each problem. See Examples 4 and 5.

31. During the 1997–1998 academic year, the ratio of teachers to students in private high schools was approximately 1 to 24. If a private high school had 554 students, how many teachers would be at the school if this ratio was valid for that school? Round your answer to the nearest whole number. (*Source:* U.S. National Center for Education Statistics, *Private School Universe Survey*, 1997–98.)

Photo not available

32. During the 1998–1999 National Basketball Association season, Shaquille O'Neal of the Los Angeles Lakers played in 49 games for a total of 1705 min. If he had played in all 50 of the team's games, how many minutes would he have played, assuming that the ratio of games to minutes stayed the same? Round your answer to the nearest whole number. (*Source: Sports Illustrated 2000 Sports Almanac.*)

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33. Biologists tagged 500 fish in a lake on January 1. On February 1 they returned and collected a random sample of 400 fish, 8 of which had been previously tagged. Approximately how many fish does the lake have based on this experiment?

34. Suppose that in the experiment of Exercise 33, 10 of the previously tagged fish were collected on February 1. What would be the estimate of the fish population?

Nurses use proportions to determine the amount of a drug to administer when the dose of the drug is measured in milligrams but the drug is packaged in a diluted form in milliliters. (*Source:* Hoyles, Celia, Richard Noss, and Stefano Pozzi, "Proportional Reasoning in Nursing Practice," *Journal for Research in Mathematics Education*, January 2001.) *For example, to find the number of milliliters of fluid needed to administer* 300 mg *of a drug that comes packaged as* 120 mg *in* 2 mL *of fluid, a nurse sets up the proportion*

$$\frac{120 \text{ mg}}{2 \text{ mL}} = \frac{300 \text{ mg}}{x \text{ mL}},$$

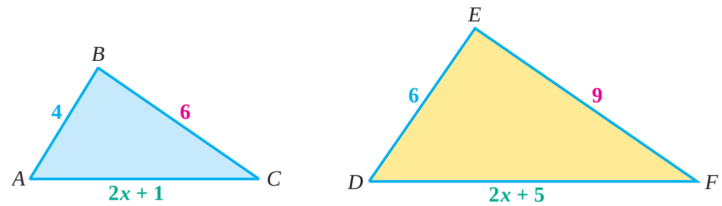
where x represents the amount to administer in milliliters. Use this method to find the correct dose for each prescription.

35. 120 mg of Amakacine packaged as 100 mg in 2-mL vials

36. 1.5 mg of morphine packaged as 20 mg ampules diluted in 10 mL of fluid

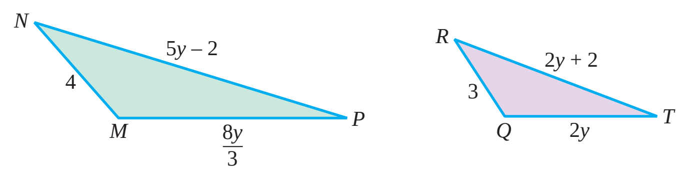
In geometry, it is shown that two triangles with corresponding angle measures equal, called **similar triangles,** *have corresponding sides proportional. For example, in the figure at the top of the next page, angle A = angle D, angle B = angle E, and angle C = angle F, so the triangles are similar. Then the following ratios of corresponding sides are equal.*

$$\frac{4}{6} = \frac{6}{9} = \frac{2x + 1}{2x + 5}$$



37. Solve for x using the given proportion to find the lengths of the third sides of the triangles.

38. Suppose the following triangles are similar. Find y and the lengths of the two longest sides of each triangle.



Solve each problem. See Examples 6 and 7.

39. Kellen's boat goes 12 mph. Find the rate of the current of the river if she can go 6 mi upstream in the same amount of time she can go 10 mi downstream.

	Distance	Rate	Time
Downstream	10	$12 + x$	
Upstream	6	$12 - x$	

40. Kasey can travel 8 mi upstream in the same time it takes her to go 12 mi downstream. Her boat goes 15 mph in still water. What is the rate of the current?

	Distance	Rate	Time
Downstream			
Upstream			

41. Driving from Tulsa to Detroit, Jeff averaged 50 mph. He figured that if he had averaged 60 mph, his driving time would have decreased 3 hr. How far is it from Tulsa to Detroit?

42. If Dr. Dawson rides his bike to his office, he averages 12 mph. If he drives his car, he averages 36 mph. His time driving is $\frac{1}{4}$ hr less than his time riding his bike. How far is his office from home?

43. A private plane traveled from San Francisco to a secret rendezvous. It averaged 200 mph. On the return trip, the average speed was 300 mph. If the total traveling time was 4 hr, how far from San Francisco was the secret rendezvous?

44. Johnny averages 30 mph when he drives on the old highway to his favorite fishing hole, and he averages 50 mph when most of his route is on the interstate. If both routes are the same length, and he saves 2 hr by traveling on the interstate, how far away is the fishing hole?

45. On the first part of a trip to Carmel traveling on the freeway, Marge averaged 60 mph. On the rest of the trip, which was 10 mi longer than the first part, she averaged 50 mph. Find the total distance to Carmel if the second part of the trip took 30 min more than the first part.

46. While on vacation, Jim and Annie decided to drive all day. During the first part of their trip on the highway, they averaged 60 mph. When they got to Houston, traffic caused them to average only 30 mph. The distance they drove in Houston was 100 mi less than their distance on the highway. What was their total driving distance if they spent 50 min more on the highway than they did in Houston?

Solve each problem. See Example 8.

EXERCISES

47. Butch and Peggy want to pick up the mess that their grandson, Grant, has made in his playroom. Butch could do it in 15 min working alone. Peggy, working alone, could clean it in 12 min. How long will it take them if they work together?

	Rate	Time Working Together	Fractional Part of the Job Done
Butch	$\frac{1}{15}$	x	
Peggy	$\frac{1}{12}$	x	

48. Lou can groom Jay Beckenstein's dogs in 8 hr, but it takes his business partner, Janet, only 5 hr to groom the same dogs. How long will it take them to groom Jay's dogs if they work together?

	Rate	Time Working Together	Fractional Part of the Job Done
Lou	$\frac{1}{8}$	x	
Janet	$\frac{1}{5}$	x	

49. Ron Wood can paint a room in 6 hr working alone. If his son Jason helps him, the job takes 4 hr. How long would it take Jason to do the job if he worked alone?

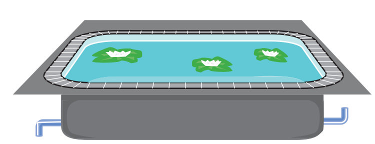
50. Sandi and Cary Goldstein are refinishing a table. Working alone, Cary could do the job in 7 hr. If the two work together, the job takes 5 hr. How long will it take Sandi to refinish the table working alone?

51. If a vat of acid can be filled by an inlet pipe in 10 hr and emptied by an outlet pipe in 20 hr, how long will it take to fill the vat if both pipes are open?

52. A winery has a vat to hold chardonnay. An inlet pipe can fill the vat in 9 hr, while an outlet pipe can empty it in 12 hr. How long will it take to fill the vat if both the outlet and the inlet pipes are open?

53. Suppose that Hortense and Mort can clean their entire house in 7 hr, while their toddler, Mimi, just by being around, can completely mess it up in only 2 hr. If Hortense and Mort clean the house while Mimi is at her grandma's, and then start cleaning up after Mimi the minute she gets home, how long does it take from the time Mimi gets home until the whole place is a shambles?

54. An inlet pipe can fill an artificial lily pond in 60 min, while an outlet pipe can empty it in 80 min. Through an error, both pipes are left open. How long will it take for the pond to fill?



8.1 Radical Expressions and Graphs

OBJECTIVES

1. Find roots of numbers.
2. Find principal roots.
3. Graph functions defined by radical expressions.
4. Find *n*th roots of *n*th powers.
5. Use a calculator to find roots.

OBJECTIVE 1 Find roots of numbers. In Chapter 1 we found square roots of positive numbers such as

$$\sqrt{36} = 6 \text{ because } 6 \cdot 6 = 36 \quad \text{and} \quad \sqrt{144} = 12 \text{ because } 12 \cdot 12 = 144.$$

In this section we extend our discussion of roots to cube roots, fourth roots, and higher roots. In general, $\sqrt[n]{a}$ is a number whose nth power equals a. That is,

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a.$$

The number a is the **radicand,** n is the **index** or **order,** and the expression $\sqrt[n]{a}$ is a **radical.**

EXAMPLE 1 Simplifying Higher Roots

Simplify.

(a) $\sqrt[3]{27} = 3$ because $3^3 = 27$.

(b) $\sqrt[3]{125} = 5$ because $5^3 = 125$.

(c) $\sqrt[4]{16} = 2$ because $2^4 = 16$.

(d) $\sqrt[5]{32} = 2$ because $2^5 = 32$.

Now Try Exercises 5, 19, 21, and 35.

OBJECTIVE 2 Find principal roots. If n is even, positive numbers have two nth roots. For example, both 4 and -4 are square roots of 16, and 2 and -2 are fourth roots of 16. In such cases, the notation $\sqrt[n]{a}$ represents the positive root, called the **principal root.**

nth Root

If n is *even* and a is *positive* or 0, then

$$\sqrt[n]{a} \text{ represents the principal } n\text{th root of } a,$$

and

$$-\sqrt[n]{a} \text{ represents the negative } n\text{th root of } a.$$

If n is *even* and a is *negative*, then

$$\sqrt[n]{a} \text{ is not a real number.}$$

If n is *odd*, then

$$\text{there is exactly one } n\text{th root of } a\text{, written } \sqrt[n]{a}.$$

If n is even, the two nth roots of a are often written together as $\pm\sqrt[n]{a}$, with $\pm$ read "positive or negative."

VIDEO

EXAMPLE 2 Finding Roots

Find each root.

(a) $\sqrt{100} = 10$

Because the radicand is positive, there are two square roots, 10 and -10. We want the principal root, which is 10.

(b) $-\sqrt{100} = -10$

Here, we want the negative square root, -10.

(c) $\sqrt[4]{81} = 3$

(d) $\sqrt[6]{-64}$

The index is even and the radicand is negative, so this is not a real number.

(e) $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$.

Now Try Exercises 1, 15, 17, and 25.

OBJECTIVE 3 Graph functions defined by radical expressions. A **radical expression** is an algebraic expression that contains radicals. For example,

$$3 - \sqrt{x}, \qquad \sqrt[3]{x}, \qquad \text{and} \qquad \sqrt{2x - 1}$$

are radical expressions.

In earlier chapters we graphed functions defined by polynomial and rational expressions. Now we examine the graphs of functions defined by the basic radical expressions $f(x) = \sqrt{x}$ and $f(x) = \sqrt[3]{x}$.

Figure 1 shows the graph of the **square root function** with a table of selected points. Only nonnegative values can be used for x, so the domain is $[0, \infty)$. Because $\sqrt{x}$ is the principal square root of x, it always has a nonnegative value, so the range is also $[0, \infty)$.

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3

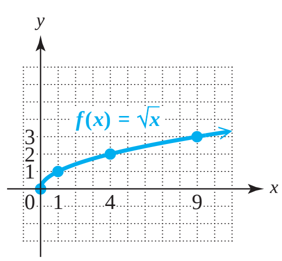


FIGURE 1

x	$f(x) = \sqrt[3]{x}$
-8	-2
-1	-1
0	0
1	1
8	2

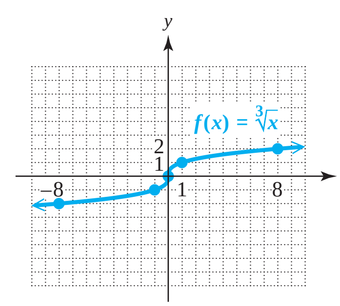


FIGURE 2

Figure 2 shows the graph of the **cube root function** and a table of selected points. Since any real number (positive, negative, or 0) can be used for x in the cube root function, $\sqrt[3]{x}$ can be positive, negative, or 0. Thus both the domain and the range of the cube root function are $(-\infty, \infty)$.

EXAMPLE 3 Graphing Functions Defined with Radicals

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Graph each function by creating a table of values. Give the domain and range.

(a) $f(x) = \sqrt{x - 3}$

A table of values is given on the next page. The x-values were chosen in such a way that the function values are all integers. For the radicand to be nonnegative, we must have $x - 3 \geq 0$, or $x \geq 3$. Therefore, the domain is $[3, \infty)$. Function values are positive or 0, so the range is $[0, \infty)$. The graph is shown in Figure 3 on the next page.

x	$f(x) = \sqrt{x-3}$
3	$\sqrt{3-3} = 0$
4	$\sqrt{4-3} = 1$
7	$\sqrt{7-3} = 2$

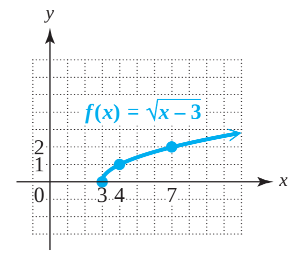


FIGURE 3

(b) $f(x) = \sqrt[3]{x} + 2$

See the table and Figure 4. Both the domain and range are $(-\infty, \infty)$.

x	$f(x) = \sqrt[3]{x} + 2$
-8	$\sqrt[3]{-8} + 2 = 0$
-1	$\sqrt[3]{-1} + 2 = 1$
0	$\sqrt[3]{0} + 2 = 2$
1	$\sqrt[3]{1} + 2 = 3$
8	$\sqrt[3]{8} + 2 = 4$

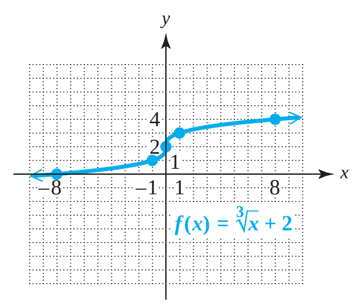


FIGURE 4

Now Try Exercises 37 and 41.

OBJECTIVE 4 Find *n*th roots of *n*th powers. A square root of a^2 (where $a \neq 0$) is a number that can be squared to give a^2. This number is either a or $-a$. Since the symbol $\sqrt{a^2}$ represents the *nonnegative* square root, we express $\sqrt{a^2}$ with absolute value bars as $|a|$, because a may be a negative number.

$\sqrt{a^2}$

For any real number a,

$$\sqrt{a^2} = |a|.$$

EXAMPLE 4 Simplifying Square Roots Using Absolute Value

Find each square root.

(a) $\sqrt{7^2} = |7| = 7$

(b) $\sqrt{(-7)^2} = |-7| = 7$

(c) $\sqrt{k^2} = |k|$

(d) $\sqrt{(-k)^2} = |-k| = |k|$

Now Try Exercises 45, 47, and 53.

We can generalize this idea to any nth root.

$\sqrt[n]{a^n}$

If n is an *even* positive integer, then $\quad \sqrt[n]{a^n} = |a|$,

and if n is an *odd* positive integer, then $\quad \sqrt[n]{a^n} = a$.

That is, use absolute value when n is even; absolute value is not necessary when n is odd.

EXAMPLE 5 Simplifying Higher Roots Using Absolute Value

Simplify each root.

(a) $\sqrt[6]{(-3)^6} = |-3| = 3$ $\quad$ n is even; use absolute value.

(b) $\sqrt[5]{(-4)^5} = -4$ $\quad$ n is odd.

(c) $-\sqrt[4]{(-9)^4} = -|-9| = -9$

(d) $\sqrt[3]{\frac{8}{27}} = \sqrt[3]{\left(\frac{2}{3}\right)^3} = \frac{2}{3}$

(e) $-\sqrt{m^4} = -|m^2| = -m^2$

No absolute value bars are needed here because m^2 is nonnegative for any real number value of m.

(f) $\sqrt[3]{a^{12}} = a^4$ because $a^{12} = (a^4)^3$.

(g) $\sqrt[4]{x^{12}} = |x^3|$

We use absolute value bars to guarantee that the result is not negative (because x^3 can be either positive or negative, depending on x). Also, $|x^3|$ can be written as $x^2 \cdot |x|$.

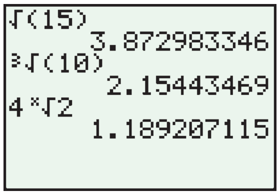
Now Try Exercises 49, 51, 55, and 57.

OBJECTIVE 5 Use a calculator to find roots. While numbers such as $\sqrt{9}$ and $\sqrt[3]{-8}$ are rational, radicals are often irrational numbers. To find approximations of roots such as $\sqrt{15}$, $\sqrt[3]{10}$, and $\sqrt[4]{2}$, we usually use scientific or graphing calculators. Using a calculator, we find

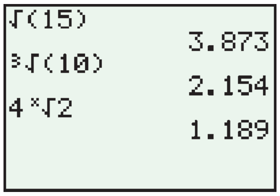
$$\sqrt{15} \approx 3.872983346, \quad \sqrt[3]{10} \approx 2.15443469, \quad \text{and} \quad \sqrt[4]{2} \approx 1.189207115,$$

where the symbol $\approx$ means "is approximately equal to." In this book we usually show approximations rounded to three decimal places. Thus, we would write

$$\sqrt{15} \approx 3.873, \quad \sqrt[3]{10} \approx 2.154, \quad \text{and} \quad \sqrt[4]{2} \approx 1.189.$$



(a)



(b)

FIGURE 5

Figure 5 shows how the preceding approximations are displayed on a TI-83 Plus graphing calculator. In Figure 5(a), eight or nine decimal places are shown, while in Figure 5(b), the number of decimal places is fixed at three.

There is a simple way to check that a calculator approximation is "in the ballpark." Because 16 is a little larger than 15, $\sqrt{16} = 4$ should be a little larger than $\sqrt{15}$. Thus, 3.873 is a reasonable approximation for $\sqrt{15}$.

NOTE The methods for finding approximations differ among makes and models of calculators. You should always consult your owner's manual for keystroke instructions. Be aware that graphing calculators often differ from scientific calculators in the order in which keystrokes are made.

EXAMPLE 6 Finding Approximations for Roots

Use a calculator to verify that each approximation is correct.

(a) $\sqrt{39} \approx 6.245$

(b) $-\sqrt{72} \approx -8.485$

(c) $\sqrt[3]{93} \approx 4.531$

(d) $\sqrt[4]{39} \approx 2.499$

Now Try Exercises 59, 65, 67, and 69.

EXAMPLE 7 Using Roots to Calculate Resonant Frequency

In electronics, the resonant frequency f of a circuit may be found by the formula

$$f = \frac{1}{2\pi\sqrt{LC}},$$

where f is in cycles per second, L is in henrys, and C is in farads.* Find the resonant frequency f if $L = 5 \times 10^{-4}$ henrys and $C = 3 \times 10^{-10}$ farads. Give your answer to the nearest thousand.

Find the value of f when $L = 5 \times 10^{-4}$ and $C = 3 \times 10^{-10}$.

$$f = \frac{1}{2\pi\sqrt{LC}} \qquad \text{Given formula}$$

$$= \frac{1}{2\pi\sqrt{(5 \times 10^{-4})(3 \times 10^{-10})}} \qquad \text{Substitute for } L \text{ and } C.$$

$$\approx 411{,}000 \qquad \text{Use a calculator.}$$

The resonant frequency f is approximately 411,000 cycles per sec.

Now Try Exercise 73.

8.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 12/Videotape 12

Match each expression from Column I with the equivalent choice from Column II. Answers may be used more than once. See Example 2.

I	II
1. $-\sqrt{16}$	**A.** 3
2. $\sqrt{-16}$	**B.** -2
3. $\sqrt[3]{-27}$	**C.** 2
4. $\sqrt[5]{-32}$	**D.** -3
5. $\sqrt[4]{81}$	**E.** -4
6. $\sqrt[3]{8}$	**F.** Not a real number

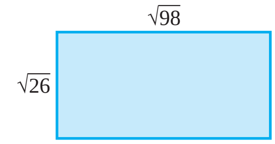
*Henrys and farads are units of measure in electronics.

Choose the closest approximation of each square root.

7. $\sqrt{123.5}$
A. 9 **B.** 10 **C.** 11 **D.** 12

8. $\sqrt{67.8}$
A. 7 **B.** 8 **C.** 9 **D.** 10

Refer to the rectangle to answer the questions in Exercises 9 and 10.



9. Which one of the following is the best estimate of its area?
A. 2500 **B.** 250 **C.** 50 **D.** 100

10. Which one of the following is the best estimate of its perimeter?
A. 15 **B.** 250 **C.** 100 **D.** 30

11. Consider the expression $-\sqrt{-a}$. Decide whether it is *positive, negative,* 0, or *not a real number* if
(a) $a > 0$, **(b)** $a < 0$, **(c)** $a = 0$.

12. If n is odd, under what conditions is $\sqrt[n]{a}$
(a) positive, **(b)** negative, **(c)** 0?

Find each root that is a real number. Use a calculator as necessary. See Examples 1 and 2.

13. $\sqrt{81}$ **14.** $\sqrt{121}$ **15.** $-\sqrt[3]{216}$ **16.** $-\sqrt[3]{343}$
17. $\sqrt[3]{-64}$ **18.** $\sqrt[3]{-125}$ **19.** $\sqrt[3]{512}$ **20.** $\sqrt[3]{1000}$
21. $\sqrt[4]{1296}$ **22.** $\sqrt[4]{625}$ **23.** $-\sqrt[4]{81}$ **24.** $-\sqrt[4]{256}$
25. $\sqrt[4]{-16}$ **26.** $\sqrt[4]{-81}$ **27.** $\sqrt[6]{64}$ **28.** $\sqrt[4]{256}$
29. $\sqrt[6]{-32}$ **30.** $\sqrt[8]{-1}$ **31.** $\sqrt{\frac{64}{81}}$ **32.** $\sqrt{\frac{100}{9}}$
33. $\sqrt[3]{\frac{8}{27}}$ **34.** $\sqrt[4]{\frac{81}{16}}$ **35.** $\sqrt[6]{\frac{1}{64}}$ **36.** $\sqrt[5]{\frac{1}{32}}$

Graph each function and give its domain and range. See Example 3.

37. $f(x) = \sqrt{x + 3}$ **38.** $f(x) = \sqrt{x - 5}$ **39.** $f(x) = \sqrt{x} - 2$
40. $f(x) = \sqrt{x} + 4$ **41.** $f(x) = \sqrt[3]{x} - 3$ **42.** $f(x) = \sqrt[3]{x} + 1$
43. $f(x) = \sqrt[3]{x - 3}$ **44.** $f(x) = \sqrt[3]{x + 1}$

Simplify each root. See Examples 4 and 5.

45. $\sqrt{12^2}$ **46.** $\sqrt{19^2}$ **47.** $\sqrt{(-10)^2}$ **48.** $-\sqrt{13^2}$
49. $\sqrt[6]{(-2)^6}$ **50.** $\sqrt[6]{(-4)^6}$ **51.** $\sqrt[5]{(-9)^5}$ **52.** $\sqrt[5]{(-8)^5}$
53. $\sqrt{x^2}$ **54.** $-\sqrt{x^2}$ **55.** $\sqrt[3]{x^3}$ **56.** $-\sqrt[3]{x^3}$
57. $\sqrt[3]{x^{15}}$ **58.** $\sqrt[4]{k^{20}}$

Find a decimal approximation for each radical. Round the answer to three decimal places. See Example 6.

59. $\sqrt{9483}$ **60.** $\sqrt{6825}$ **61.** $\sqrt{284.361}$ **62.** $\sqrt{846.104}$
63. $\sqrt{7}$ **64.** $\sqrt{11}$ **65.** $-\sqrt{82}$ **66.** $-\sqrt{91}$

67. $\sqrt[3]{423}$ **68.** $\sqrt[3]{555}$ **69.** $\sqrt[4]{100}$ **70.** $\sqrt[4]{250}$

71. $\sqrt[5]{23.8}$ **72.** $\sqrt[5]{98.4}$

Solve each problem. See Example 7.

EXERCISES

73. Use the formula in Example 7 to calculate the resonant frequency of a circuit to the nearest thousand if $L = 7.237 \times 10^{-5}$ henrys and $C = 2.5 \times 10^{-10}$ farads.

74. The threshold weight T for a person is the weight above which the risk of death increases greatly. The threshold weight in pounds for men aged 40–49 is related to height in inches by the formula

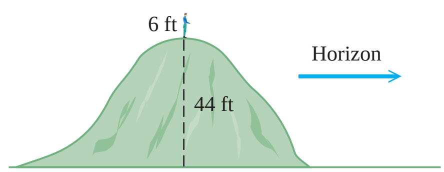
$$h = 12.3\sqrt[3]{T}.$$

What height corresponds to a threshold weight of 216 lb for a 43-year-old man? Round your answer to the nearest inch, and then to the nearest tenth of a foot.

75. According to an article in *The World Scanner Report* (August 1991), the distance D, in miles, to the horizon from an observer's point of view over water or "flat" earth is given by

$$D = \sqrt{2H},$$

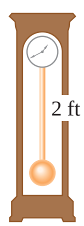
where H is the height of the point of view, in feet. If a person whose eyes are 6 ft above ground level is standing at the top of a hill 44 ft above "flat" earth, approximately how far to the horizon will she be able to see?



76. The time for one complete swing of a simple pendulum is

$$t = 2\pi\sqrt{\frac{L}{g}},$$

where t is time in seconds, L is the length of the pendulum in feet, and g, the force due to gravity, is about 32 ft per sec^2. Find the time of a complete swing of a 2-ft pendulum to the nearest tenth of a second.



77. **Heron's formula** gives a method of finding the area of a triangle if the lengths of its sides are known. Suppose that a, b, and c are the lengths of the sides. Let s denote one-half of the perimeter of the triangle (called the *semiperimeter*); that is,

$$s = \frac{1}{2}(a + b + c).$$

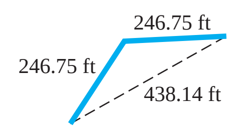
Then the area of the triangle is

$$A = \sqrt{s(s - a)(s - b)(s - c)}.$$

Find the area of the Bermuda Triangle, if the "sides" of this triangle measure approximately 850 mi, 925 mi, and 1300 mi. Give your answer to the nearest thousand square miles.

78. The Vietnam Veterans' Memorial in Washington, D.C., is in the shape of an unenclosed isosceles triangle with equal sides of length 246.75 ft. If the triangle were enclosed, the third side would have length 438.14 ft. Use Heron's formula from the previous exercise to find the area of this enclosure to the nearest hundred square feet. (*Source:* Information pamphlet obtained at the Vietnam Veterans' Memorial.)

Photo not available



Not to scale

79. The formula

$$I = \sqrt{\frac{2P}{L}}$$

relates the coefficient of self-induction L (in henrys), the energy P stored in an electronic circuit (in joules), and the current I (in amps). Find I if $P = 120$ and $L = 80$.

Photo not available

80. When the wind blows, the air feels much colder than the actual temperature. The **windchill factor** measures the cooling effect that the wind has on one's skin. Until recently, the formula that the National Weather Service used to compute windchill was

$$T_{\text{wc}} = .0817\left(3.71\sqrt{V} + 5.81 - .25V\right)(T - 91.4) + 91.4,$$

where T_{wc} is windchill, V is wind speed in miles per hour (mph), and T is air temperature in degrees Fahrenheit. The windchill for various wind speeds and temperatures is shown in the table.

Windchill Factor

Wind Speed (mph)	Air Temperature (°Fahrenheit)														
	35	**30**	**25**	**20**	**15**	**10**	**5**	**0**	**−5**	**−10**	**−15**	**−20**	**−25**	**−30**	**−35**
4	35	30	25	20	15	10	5	0	−5	−10	−15	−20	−25	−30	−35
5	32	27	22	16	11	6	0	−5	−10	−15	−21	−26	−31	−36	−42
10	22	16	10	3	−3	−9	−15	−22	−27	−34	−40	−46	−52	−58	−64
15	16	9	2	−5	−11	−18	−25	−31	−38	−45	−51	−58	−65	−72	−78
20	12	4	−3	−10	−17	−24	−31	−39	−46	−53	−60	−67	−74	−81	−88
25	8	1	−7	−15	−22	−29	−36	−44	−51	−59	−66	−74	−81	−88	−96
30	6	−2	−10	−18	−25	−33	−41	−49	−56	−64	−71	−79	−86	−93	−101
35	4	−4	−12	−20	−27	−35	−43	−52	−58	−67	−74	−82	−89	−97	−105
40	3	−5	−13	−21	−29	−37	−45	−53	−60	−69	−76	−84	−92	−100	−107
45	2	−6	−14	−22	−30	−38	−46	−54	−62	−70	−78	−85	−93	−102	−109

Source: USA Today.

Choose a temperature of 10°F. Use the formula to calculate the windchill for wind speeds of 4, 10, 25, and 40 mph. Round the results to the nearest degree. Do your results match those in the tables?

8.2 Rational Exponents

OBJECTIVES

1. Use exponential notation for nth roots.
2. Define and use expressions of the form $a^{m/n}$.
3. Convert between radicals and rational exponents.
4. Use the rules for exponents with rational exponents.

OBJECTIVE 1 Use exponential notation for *n*th roots. In mathematics we often formulate definitions so that previous rules remain valid. In Chapter 5 we defined 0 as an exponent in such a way that the rules for products, quotients, and powers would still be valid. Now we look at exponents that are rational numbers of the form $\frac{1}{n}$, where n is a natural number.

For the rules of exponents to remain valid, the product $(3^{1/2})^2 = 3^{1/2} \cdot 3^{1/2}$ should be found by adding exponents.

$$\begin{aligned}(3^{1/2})^2 &= 3^{1/2} \cdot 3^{1/2} \\ &= 3^{1/2+1/2} \\ &= 3^1 \\ &= 3\end{aligned}$$

However, by definition $\left(\sqrt{3}\right)^2 = \sqrt{3} \cdot \sqrt{3} = 3$. Since both $(3^{1/2})^2$ and $\left(\sqrt{3}\right)^2$ are equal to 3, it is reasonable to have

$$3^{1/2} = \sqrt{3}.$$

This suggests the following generalization.

> ***a*$^{1/n}$**
>
> If $\sqrt[n]{a}$ is a real number, then $\qquad \boldsymbol{a^{1/n} = \sqrt[n]{a}.}$

EXAMPLE 1 Evaluating Exponentials of the Form $a^{1/n}$

Evaluate each expression.

(a) $64^{1/3} = \sqrt[3]{64} = 4$

(b) $100^{1/2} = \sqrt{100} = 10$

(c) $-256^{1/4} = -\sqrt[4]{256} = -4$

(d) $(-256)^{1/4} = \sqrt[4]{-256}$ is not a real number because the radicand, -256, is negative and the index is even.

(e) $(-32)^{1/5} = \sqrt[5]{-32} = -2$

(f) $\left(\dfrac{1}{8}\right)^{1/3} = \sqrt[3]{\dfrac{1}{8}} = \dfrac{1}{2}$

Now Try Exercises 11, 13, 19, and 25.

> **CAUTION** Notice the difference between parts (c) and (d) in Example 1. The radical in part (c) is the *negative fourth root* of a positive number, while the radical in part (d) is the *principal fourth root of a negative number,* which is not a real number.

OBJECTIVE 2 Define and use expressions of the form $a^{m/n}$. We know that $8^{1/3} = \sqrt[3]{8}$. How should we define a number like $8^{2/3}$? For past rules of exponents to be valid,

$$8^{2/3} = 8^{(1/3)2} = (8^{1/3})^2.$$

Since $8^{1/3} = \sqrt[3]{8}$,

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$$

Generalizing from this example, we define $a^{m/n}$ as follows.

$a^{m/n}$

If m and n are positive integers with m/n in lowest terms, then

$$a^{m/n} = (a^{1/n})^m,$$

provided that $a^{1/n}$ is a real number. If $a^{1/n}$ is not a real number, then $a^{m/n}$ is not a real number.

EXAMPLE 2 Evaluating Exponentials of the Form $a^{m/n}$

Evaluate each exponential.

(a) $36^{3/2} = (36^{1/2})^3 = 6^3 = 216$

(b) $125^{2/3} = (125^{1/3})^2 = 5^2 = 25$

(c) $-4^{5/2} = -(4^{5/2}) = -(4^{1/2})^5 = -(2)^5 = -32$

(d) $(-27)^{2/3} = [(-27)^{1/3}]^2 = (-3)^2 = 9$

Notice how the $-$ sign is used in parts (c) and (d). In part (c), we first evaluate the exponential and then find its negative. In part (d), the $-$ sign is part of the base, -27.

(e) $(-100)^{3/2}$ is not a real number since $(-100)^{1/2}$ is not a real number.

Now Try Exercises 21 and 23.

When a rational exponent is negative, the earlier interpretation of negative exponents is applied.

$a^{-m/n}$

If $a^{m/n}$ is a real number, then

$$a^{-m/n} = \frac{1}{a^{m/n}} \quad (a \neq 0).$$

EXAMPLE 3 Evaluating Exponentials with Negative Rational Exponents

Evaluate each exponential.

(a) $16^{-3/4}$

By the definition of a negative exponent,

$$16^{-3/4} = \frac{1}{16^{3/4}}.$$

Since $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$,

$$16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{8}.$$

(b) $25^{-3/2} = \dfrac{1}{25^{3/2}} = \dfrac{1}{(\sqrt{25})^3} = \dfrac{1}{5^3} = \dfrac{1}{125}$

(c) $\left(\dfrac{8}{27}\right)^{-2/3} = \dfrac{1}{\left(\dfrac{8}{27}\right)^{2/3}} = \dfrac{1}{\left(\sqrt[3]{\dfrac{8}{27}}\right)^2} = \dfrac{1}{\left(\dfrac{2}{3}\right)^2} = \dfrac{1}{\dfrac{4}{9}} = \dfrac{9}{4}$

We could also use the rule $\left(\dfrac{b}{a}\right)^{-m} = \left(\dfrac{a}{b}\right)^m$ here, as follows.

$$\left(\frac{8}{27}\right)^{-2/3} = \left(\frac{27}{8}\right)^{2/3} = \left(\sqrt[3]{\frac{27}{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Now Try Exercises 27 and 29.

CAUTION When using the rule in Example 3(c), we take the reciprocal only of the base, *not* the exponent. Also, be careful to distinguish between exponential expressions like $-16^{1/4}$, $16^{-1/4}$, and $-16^{-1/4}$.

$$-16^{1/4} = -2, \qquad 16^{-1/4} = \frac{1}{2}, \qquad \text{and} \qquad -16^{-1/4} = -\frac{1}{2}.$$

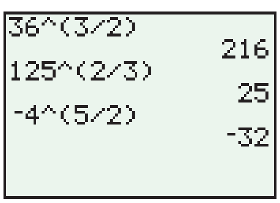


FIGURE 6

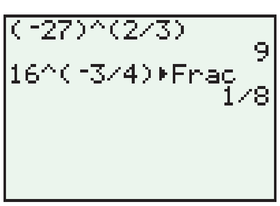


FIGURE 7

The screens in Figures 6 and 7 illustrate how a graphing calculator performs some of the evaluations seen in Examples 2 and 3. (All results on the screens are rational numbers.)

We obtain an alternative definition of $a^{m/n}$ by using the power rule for exponents a little differently than in the earlier definition. If all indicated roots are real numbers, then

$$a^{m/n} = a^{m(1/n)} = (a^m)^{1/n},$$

so

$$a^{m/n} = (a^m)^{1/n}.$$

$a^{m/n}$

If all indicated roots are real numbers, then

$$\boldsymbol{a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}.}$$

We can now evaluate an expression such as $27^{2/3}$ in two ways:

$$27^{2/3} = (27^{1/3})^2 = 3^2 = 9$$

or

$$27^{2/3} = (27^2)^{1/3} = 729^{1/3} = 9.$$

In most cases, it is easier to use $(a^{1/n})^m$.

This rule can also be expressed with radicals as follows.

Radical Form of $a^{m/n}$

If all indicated roots are real numbers, then

$$\boldsymbol{a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.}$$

That is, raise a to the power and then take the root, or take the root and then raise a to the power.

For example,

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4, \qquad \text{and} \qquad 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4,$$

so

$$8^{2/3} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2.$$

OBJECTIVE 3 Convert between radicals and rational exponents. Using the definition of rational exponents, we can simplify many problems involving radicals by converting the radicals to numbers with rational exponents. After simplifying, we convert the answer back to radical form.

EXAMPLE 4 Converting between Rational Exponents and Radicals

Write each exponential as a radical. Assume all variables represent positive real numbers. Use the definition that takes the root first.

(a) $13^{1/2} = \sqrt{13}$ **(b)** $6^{3/4} = \left(\sqrt[4]{6}\right)^3$

(c) $9m^{5/8} = 9\left(\sqrt[8]{m}\right)^5$ **(d)** $6x^{2/3} - (4x)^{3/5} = 6\left(\sqrt[3]{x}\right)^2 - \left(\sqrt[5]{4x}\right)^3$

(e) $r^{-2/3} = \dfrac{1}{r^{2/3}} = \dfrac{1}{\left(\sqrt[3]{r}\right)^2}$

(f) $(a^2 + b^2)^{1/2} = \sqrt{a^2 + b^2}$ Note that $\sqrt{a^2 + b^2} \neq a + b$.

In (g)–(i), write each radical as an exponential. Simplify. Assume all variables represent positive real numbers.

(g) $\sqrt{10} = 10^{1/2}$ **(h)** $\sqrt[4]{3^8} = 3^{8/4} = 3^2 = 9$

(i) $\sqrt[6]{z^6} = z$ since z is positive.

Now Try Exercises 33, 35, 37, 49, and 51.

NOTE In Example 4(i), it was not necessary to use absolute value bars since the directions specifically stated that the variable represents a positive real number. Since the absolute value of the positive real number z is z itself, the answer is simply z. When working exercises with radicals, we often assume that variables represent positive real numbers, which will eliminate the need for absolute value.

OBJECTIVE 4 Use the rules for exponents with rational exponents. The definition of rational exponents allows us to apply the rules for exponents first introduced in Chapter 5.

AUDIO

Rules for Rational Exponents

Let r and s be rational numbers. For all real numbers a and b for which the indicated expressions exist:

$$a^r \cdot a^s = a^{r+s} \qquad a^{-r} = \frac{1}{a^r} \qquad \frac{a^r}{a^s} = a^{r-s} \qquad \left(\frac{a}{b}\right)^{-r} = \frac{b^r}{a^r}$$

$$(a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad a^{-r} = \left(\frac{1}{a}\right)^r.$$

EXAMPLE 5 Applying Rules for Rational Exponents

Write with only positive exponents. Assume all variables represent positive real numbers.

(a) $2^{1/2} \cdot 2^{1/4} = 2^{1/2+1/4} = 2^{3/4}$ Product rule

(b) $\dfrac{5^{2/3}}{5^{7/3}} = 5^{2/3-7/3} = 5^{-5/3} = \dfrac{1}{5^{5/3}}$ Quotient rule

(c)
$$\begin{aligned}
\frac{(x^{1/2}y^{2/3})^4}{y} &= \frac{(x^{1/2})^4(y^{2/3})^4}{y} && \text{Power rule}\\
&= \frac{x^2y^{8/3}}{y^1} && \text{Power rule}\\
&= x^2y^{8/3-1} && \text{Quotient rule}\\
&= x^2y^{5/3}
\end{aligned}$$

(d)
$$\begin{aligned}
\left(\frac{x^4y^{-6}}{x^{-2}y^{1/3}}\right)^{-2/3} &= \frac{(x^4)^{-2/3}(y^{-6})^{-2/3}}{(x^{-2})^{-2/3}(y^{1/3})^{-2/3}}\\
&= \frac{x^{-8/3}y^4}{x^{4/3}y^{-2/9}} && \text{Power rule}\\
&= x^{-8/3-4/3}y^{4-(-2/9)} && \text{Quotient rule}\\
&= x^{-4}y^{38/9}\\
&= \frac{y^{38/9}}{x^4} && \text{Definition of negative exponent}
\end{aligned}$$

The same result is obtained if we simplify within the parentheses first, leading to

$$(x^6y^{-19/3})^{-2/3}.$$

Then, apply the power rule. (Show that the result is the same.)

(e)
$$\begin{aligned}
m^{3/4}(m^{5/4} - m^{1/4}) &= m^{3/4} \cdot m^{5/4} - m^{3/4} \cdot m^{1/4} && \text{Distributive property}\\
&= m^{3/4+5/4} - m^{3/4+1/4} && \text{Product rule}\\
&= m^{8/4} - m^{4/4}\\
&= m^2 - m
\end{aligned}$$

Do not make the common mistake of multiplying exponents in the first step.

Now Try Exercises 57, 59, 65, 75, and 77.

CAUTION Use the rules of exponents in problems like those in Example 5. Do not convert the expressions to radical form.

EXAMPLE 6 Applying Rules for Rational Exponents

Rewrite all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume all variables represent positive real numbers.

(a) $\sqrt[3]{x^2} \cdot \sqrt[4]{x} = x^{2/3} \cdot x^{1/4}$ Convert to rational exponents.

$= x^{2/3+1/4}$ Product rule

$= x^{8/12+3/12}$ Write exponents with a common denominator.

$= x^{11/12}$

(b) $\dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}} = \dfrac{x^{3/2}}{x^{2/3}} = x^{3/2-2/3} = x^{5/6}$

(c) $\sqrt{\sqrt[4]{z}} = \sqrt{z^{1/4}} = (z^{1/4})^{1/2} = z^{1/8}$

Now Try Exercises 83, 85, and 89.

8.2 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 12/Videotape 12

Match each expression from Column I with the equivalent choice from Column II.

I	II
1. $2^{1/2}$	**A.** -4
2. $(-27)^{1/3}$	**B.** 8
3. $-16^{1/2}$	**C.** $\sqrt{2}$
4. $(-16)^{1/2}$	**D.** $-\sqrt{6}$
5. $(-32)^{1/5}$	**E.** -3
6. $(-32)^{2/5}$	**F.** $\sqrt{6}$
7. $4^{3/2}$	**G.** 4
8. $6^{2/4}$	**H.** -2
9. $-6^{2/4}$	**I.** 6
10. $36^{.5}$	**J.** Not a real number

Simplify each expression involving rational exponents. See Examples 1–3.

11. $169^{1/2}$ **12.** $121^{1/2}$ **13.** $729^{1/3}$ **14.** $512^{1/3}$

15. $16^{1/4}$ **16.** $625^{1/4}$ **17.** $\left(\frac{64}{81}\right)^{1/2}$ **18.** $\left(\frac{8}{27}\right)^{1/3}$

19. $(-27)^{1/3}$ **20.** $(-32)^{1/5}$ **21.** $100^{3/2}$ **22.** $64^{3/2}$

23. $-16^{5/2}$ **24.** $-32^{3/5}$ **25.** $(-144)^{1/2}$ **26.** $(-36)^{1/2}$

27. $64^{-3/2}$ **28.** $81^{-3/2}$ **29.** $\left(-\frac{8}{27}\right)^{-2/3}$ **30.** $\left(-\frac{64}{125}\right)^{-2/3}$

31. Explain why $(-64)^{1/2}$ is not a real number, while $-64^{1/2}$ is a real number.

32. Explain why $a^{1/n}$ is defined to be equal to $\sqrt[n]{a}$ when $\sqrt[n]{a}$ is real.

Write with radicals. Assume all variables represent positive real numbers. See Example 4.

33. $12^{1/2}$ **34.** $3^{1/2}$ **35.** $8^{3/4}$

36. $7^{2/3}$ **37.** $(9q)^{5/8} - (2x)^{2/3}$ **38.** $(3p)^{3/4} + (4x)^{1/3}$

39. $(2m)^{-3/2}$ **40.** $(5y)^{-3/5}$ **41.** $(2y + x)^{2/3}$

42. $(r + 2z)^{3/2}$ **43.** $(3m^4 + 2k^2)^{-2/3}$ **44.** $(5x^2 + 3z^3)^{-5/6}$

45. Show that, in general, $\sqrt{a^2 + b^2} \neq a + b$ by replacing a with 3 and b with 4.

46. Suppose someone claims that $\sqrt[n]{a^n + b^n}$ must equal $a + b$, since when $a = 1$ and $b = 0$, a true statement results:

$$\sqrt[n]{a^n + b^n} = \sqrt[n]{1^n + 0^n} = \sqrt[n]{1^n} = 1 = 1 + 0 = a + b.$$

Explain why this is faulty reasoning.

Simplify by first converting to rational exponents. Assume all variables represent positive real numbers. See Example 4.

47. $\sqrt{2^{12}}$ **48.** $\sqrt{5^{10}}$ **49.** $\sqrt[3]{4^9}$ **50.** $\sqrt[4]{6^8}$ **51.** $\sqrt{x^{20}}$

52. $\sqrt{r^{50}}$ **53.** $\sqrt[3]{x} \cdot \sqrt{x}$ **54.** $\sqrt[4]{y} \cdot \sqrt[5]{y^2}$ **55.** $\dfrac{\sqrt[3]{t^4}}{\sqrt[5]{t^4}}$ **56.** $\dfrac{\sqrt[4]{w^3}}{\sqrt[6]{w}}$

Use the rules of exponents to simplify each expression. Write all answers with positive exponents. Assume all variables represent positive real numbers. See Example 5.

57. $3^{1/2} \cdot 3^{3/2}$ **58.** $6^{4/3} \cdot 6^{2/3}$ **59.** $\dfrac{64^{5/3}}{64^{4/3}}$

60. $\dfrac{125^{7/3}}{125^{5/3}}$ **61.** $y^{7/3} \cdot y^{-4/3}$ **62.** $r^{-8/9} \cdot r^{17/9}$

63. $\dfrac{k^{1/3}}{k^{2/3} \cdot k^{-1}}$ **64.** $\dfrac{z^{3/4}}{z^{5/4} \cdot z^{-2}}$ **65.** $\dfrac{(x^{1/4}y^{2/5})^{20}}{x^2}$

66. $\dfrac{(r^{1/5}s^{2/3})^{15}}{r^2}$ **67.** $\dfrac{(x^{2/3})^2}{(x^2)^{7/3}}$ **68.** $\dfrac{(p^3)^{1/4}}{(p^{5/4})^2}$

69. $\dfrac{m^{3/4}n^{-1/4}}{(m^2n)^{1/2}}$ **70.** $\dfrac{(a^2b^5)^{-1/4}}{(a^{-3}b^2)^{1/6}}$ **71.** $\dfrac{p^{1/5}p^{7/10}p^{1/2}}{(p^3)^{-1/5}}$

72. $\dfrac{z^{1/3}z^{-2/3}z^{1/6}}{(z^{-1/6})^3}$ **73.** $\left(\dfrac{b^{-3/2}}{c^{-5/3}}\right)^2 (b^{-1/4}c^{-1/3})^{-1}$ **74.** $\left(\dfrac{m^{-2/3}}{a^{-3/4}}\right)^4 (m^{-3/8}a^{1/4})^{-2}$

75. $\left(\dfrac{p^{-1/4}q^{-3/2}}{3^{-1}p^{-2}q^{-2/3}}\right)^{-2}$ **76.** $\left(\dfrac{2^{-2}w^{-3/4}x^{-5/8}}{w^{3/4}x^{-1/2}}\right)^{-3}$ **77.** $p^{2/3}(p^{1/3} + 2p^{4/3})$

78. $z^{5/8}(3z^{5/8} + 5z^{11/8})$ **79.** $k^{1/4}(k^{3/2} - k^{1/2})$ **80.** $r^{3/5}(r^{1/2} + r^{3/4})$

81. $6a^{7/4}(a^{-7/4} + 3a^{-3/4})$ **82.** $4m^{5/3}(m^{-2/3} - 4m^{-5/3})$

Write with rational exponents, and then apply the properties of exponents. Assume all radicands represent positive real numbers. Give answers in exponential form. See Example 6.

83. $\sqrt[5]{x^3} \cdot \sqrt[4]{x}$ **84.** $\sqrt[6]{y^5} \cdot \sqrt[3]{y^2}$ **85.** $\dfrac{\sqrt{x^5}}{\sqrt{x^8}}$ **86.** $\dfrac{\sqrt[3]{k^5}}{\sqrt[3]{k^7}}$

87. $\sqrt{y} \cdot \sqrt[3]{yz}$ **88.** $\sqrt[3]{xz} \cdot \sqrt{z}$ **89.** $\sqrt[4]{\sqrt[3]{m}}$ **90.** $\sqrt[3]{\sqrt{k}}$

91. $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$ **92.** $\sqrt[3]{\sqrt[5]{\sqrt{y}}}$

Solve each problem.

93. Meteorologists can determine the duration of a storm by using the function defined by

$$T(D) = .07D^{3/2},$$

where D is the diameter of the storm in miles and T is the time in hours. Find the duration of a storm with a diameter of 16 mi. Round your answer to the nearest tenth of an hour.

Photo not available

94. The threshold weight T, in pounds, for a person is the weight above which the risk of death increases greatly. The threshold weight in pounds for men aged 40–49 is related to height in inches by the function defined by

$$h(T) = (1860.867T)^{1/3}.$$

What height corresponds to a threshold weight of 200 lb for a 46-yr-old man? Round your answer to the nearest inch, and then to the nearest tenth of a foot.

Photo not available

RELATING CONCEPTS (EXERCISES 95–102)

For Individual or Group Work

Earlier, we factored expressions like $x^4 - x^5$ by factoring out the greatest common factor to get $x^4 - x^5 = x^4(1 - x)$. We can adapt this approach to factor expressions with rational exponents. When one or more of the exponents is negative or a fraction, we use order on the number line discussed in Chapter 1 to decide on the common factor. In this type of factoring, we want the binomial factor to have only positive exponents, so we always factor out the variable with the least *exponent. A positive exponent is greater than a negative exponent, so in $7z^{5/8} + z^{-3/4}$, we factor out $z^{-3/4}$, because $-\frac{3}{4}$ is less than $\frac{5}{8}$.*

Factor out the given common factor from each expression. Assume all variables represent positive real numbers.

95. $3x^{-1/2} - 4x^{1/2}$; $x^{-1/2}$

96. $m^3 - 3m^{5/2}$; $m^{5/2}$

97. $4t^{-1/2} + 7t^{3/2}$; $t^{-1/2}$

98. $8x^{2/3} + 5x^{-1/3}$; $x^{-1/3}$

99. $4p - p^{3/4}$; $p^{3/4}$

100. $2m^{1/8} - m^{5/8}$; $m^{1/8}$

101. $9k^{-3/4} - 2k^{-1/4}$; $k^{-3/4}$

102. $7z^{-5/8} - z^{-3/4}$; $z^{-3/4}$

8.3 Simplifying Radical Expressions

OBJECTIVES

1. Use the product rule for radicals.
2. Use the quotient rule for radicals.
3. Simplify radicals.
4. Simplify products and quotients of radicals with different indexes.
5. Use the Pythagorean formula.
6. Use the distance formula.

OBJECTIVE 1 Use the product rule for radicals. We now develop rules for multiplying and dividing radicals that have the same index. For example, is the product of two nth-root radicals equal to the nth root of the product of the radicands? Are the expressions $\sqrt{36 \cdot 4}$ and $\sqrt{36} \cdot \sqrt{4}$ equal? To find out, we do the computations:

$$\sqrt{36 \cdot 4} = \sqrt{144} = 12$$
$$\sqrt{36} \cdot \sqrt{4} = 6 \cdot 2 = 12.$$

Notice that in both cases the result is the same. This is an example of the **product rule for radicals.**

Product Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

That is, the product of two radicals is the radical of the product.

AUDIO

We justify the product rule using the rules for rational exponents. Since $\sqrt[n]{a} = a^{1/n}$ and $\sqrt[n]{b} = b^{1/n}$,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}.$$

CAUTION Use the product rule only when the radicals have the *same* index.

EXAMPLE 1 Using the Product Rule

Multiply. Assume all variables represent positive real numbers.

(a) $\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$

(b) $\sqrt{2} \cdot \sqrt{19} = \sqrt{2 \cdot 19} = \sqrt{38}$

(c) $\sqrt{11} \cdot \sqrt{p} = \sqrt{11p}$

(d) $\sqrt{7} \cdot \sqrt{11xyz} = \sqrt{77xyz}$

Now Try Exercises 7, 9, and 11.

EXPLORE

EXAMPLE 2 Using the Product Rule

Multiply. Assume all variables represent positive real numbers.

(a) $\sqrt[3]{3} \cdot \sqrt[3]{12} = \sqrt[3]{3 \cdot 12} = \sqrt[3]{36}$

(b) $\sqrt[4]{8y} \cdot \sqrt[4]{3r^2} = \sqrt[4]{24yr^2}$

(c) $\sqrt[6]{10m^4} \cdot \sqrt[6]{5m} = \sqrt[6]{50m^5}$

(d) $\sqrt[4]{2} \cdot \sqrt[5]{2}$ cannot be simplified using the product rule for radicals because the indexes (4 and 5) are different.

Now Try Exercises 13, 15, 17, and 19.

OBJECTIVE 2 Use the quotient rule for radicals. The **quotient rule for radicals** is similar to the product rule.

Quotient Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and n is a natural number, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

That is, the radical of a quotient is the quotient of the radicals.

EXAMPLE 3 Using the Quotient Rule

Simplify. Assume all variables represent positive real numbers.

(a) $\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$

(b) $\sqrt{\frac{7}{36}} = \frac{\sqrt{7}}{\sqrt{36}} = \frac{\sqrt{7}}{6}$

(c) $\sqrt[3]{-\frac{8}{125}} = \sqrt[3]{\frac{-8}{125}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{125}} = \frac{-2}{5} = -\frac{2}{5}$

(d) $\sqrt[3]{\frac{7}{216}} = \frac{\sqrt[3]{7}}{\sqrt[3]{216}} = \frac{\sqrt[3]{7}}{6}$

(e) $\sqrt[5]{\frac{x}{32}} = \frac{\sqrt[5]{x}}{\sqrt[5]{32}} = \frac{\sqrt[5]{x}}{2}$

(f) $-\sqrt[3]{\frac{m^6}{125}} = -\frac{\sqrt[3]{m^6}}{\sqrt[3]{125}} = -\frac{m^2}{5}$

Now Try Exercises 23, 25, 31, 33, and 35.

OBJECTIVE 3 Simplify radicals. We use the product and quotient rules to simplify radicals. A radical is **simplified** if the following four conditions are met.

Simplified Radical

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have no common factor (except 1).

EXAMPLE 4 Simplifying Roots of Numbers

Simplify.

(a) $\sqrt{24}$

Check to see whether 24 is divisible by a perfect square (the square of a natural number) such as 4, 9, ... Choose the largest perfect square that divides into 24. The largest such number is 4. Write 24 as the product of 4 and 6, and then use the product rule.

$$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4} \cdot \sqrt{6} = 2\sqrt{6}$$

(b) $\sqrt{108}$

The number 108 is divisible by the perfect square 36: $\sqrt{108} = \sqrt{36 \cdot 3}$. If this is not obvious, try factoring 108 into its prime factors.

$$\begin{aligned}\sqrt{108} &= \sqrt{2^2 \cdot 3^3} \\ &= \sqrt{2^2 \cdot 3^2 \cdot 3} \\ &= 2 \cdot 3 \cdot \sqrt{3} && \text{Product rule} \\ &= 6\sqrt{3}\end{aligned}$$

(c) $\sqrt{10}$

No perfect square (other than 1) divides into 10, so $\sqrt{10}$ cannot be simplified further.

(d) $\sqrt[3]{16}$

Look for the largest perfect *cube* that divides into 16. The number 8 satisfies this condition, so write 16 as $8 \cdot 2$ (or factor 16 into prime factors).

$$\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$$

(e)
$$\begin{aligned}-\sqrt[4]{162} &= -\sqrt[4]{81 \cdot 2} && \text{81 is a perfect 4th power.} \\ &= -\sqrt[4]{81} \cdot \sqrt[4]{2} && \text{Product rule} \\ &= -3\sqrt[4]{2}\end{aligned}$$

Now Try Exercises 39, 41, 49, and 55.

CAUTION In simplifying an expression like that in Example 4(b), be careful with which factors belong *outside* the radical sign and which belong *inside.* Note how $2 \cdot 3$ is written outside because $\sqrt{2^2} = 2$ and $\sqrt{3^2} = 3$, while the remaining 3 is left inside the radical.

EXAMPLE 5 Simplifying Radicals Involving Variables

Simplify. Assume all variables represent positive real numbers.

(a)
$$\begin{aligned}\sqrt{16m^3} &= \sqrt{16m^2 \cdot m} \\ &= \sqrt{16m^2} \cdot \sqrt{m} \\ &= 4m\sqrt{m}\end{aligned}$$

No absolute value bars are needed around the m in color because of the assumption that all the variables represent *positive* real numbers.

(b)
$$\begin{aligned}\sqrt{200k^7q^8} &= \sqrt{10^2 \cdot 2 \cdot (k^3)^2 \cdot k \cdot (q^4)^2} && \text{Factor.} \\ &= 10k^3q^4\sqrt{2k} && \text{Remove perfect square factors.}\end{aligned}$$

(c)
$$\begin{aligned}\sqrt[3]{8x^4y^5} &= \sqrt[3]{(8x^3y^3)(xy^2)} && 8x^3y^3 \text{ is the largest perfect cube that divides } 8x^4y^5. \\ &= \sqrt[3]{8x^3y^3} \cdot \sqrt[3]{xy^2} \\ &= 2xy\sqrt[3]{xy^2}\end{aligned}$$

(d) $-\sqrt[4]{32y^9} = -\sqrt[4]{(16y^8)(2y)}$ $\quad$ $16y^8$ is the largest 4th power that divides $32y^9$.

$$= -\sqrt[4]{16y^8} \cdot \sqrt[4]{2y}$$
$$= -2y^2\sqrt[4]{2y}$$

Now Try Exercises 75, 79, 83, and 87.

NOTE From Example 5 we see that if a variable is raised to a power with an exponent divisible by 2, it is a perfect square. If it is raised to a power with an exponent divisible by 3, it is a perfect cube. In general, if it is raised to a power with an exponent divisible by n, it is a perfect nth power.

The conditions for a simplified radical given earlier state that an exponent in the radicand and the index of the radical should have no common factor (except 1). The next example shows how to simplify radicals with such common factors.

YOU TRY IT

EXAMPLE 6 Simplifying Radicals by Using Smaller Indexes

Simplify. Assume all variables represent positive real numbers.

(a) $\sqrt[9]{5^6}$

We can write this radical using rational exponents and then write the exponent in lowest terms. We then express the answer as a radical.

$$\sqrt[9]{5^6} = 5^{6/9} = 5^{2/3} = \sqrt[3]{5^2} \quad \text{or} \quad \sqrt[3]{25}$$

(b) $\sqrt[4]{p^2} = p^{2/4} = p^{1/2} = \sqrt{p}$ (Recall the assumption that $p > 0$.)

Now Try Exercises 93 and 97.

These examples suggest the following rule.

AUDIO

If m is an integer, n and k are natural numbers, and all indicated roots exist, then

$$\sqrt[kn]{a^{km}} = \sqrt[n]{a^m}.$$

OBJECTIVE 4 Simplify products and quotients of radicals with different indexes. Since the product and quotient rules for radicals apply only when they have the same index, we multiply and divide radicals with different indexes by using rational exponents.

VIDEO

EXAMPLE 7 Multiplying Radicals with Different Indexes

Simplify $\sqrt{7} \cdot \sqrt[3]{2}$.

Because the different indexes, 2 and 3, have a least common index of 6, use rational exponents to write each radical as a sixth root.

$$\sqrt{7} = 7^{1/2} = 7^{3/6} = \sqrt[6]{7^3} = \sqrt[6]{343}$$
$$\sqrt[3]{2} = 2^{1/3} = 2^{2/6} = \sqrt[6]{2^2} = \sqrt[6]{4}$$

Therefore,

$$\sqrt{7} \cdot \sqrt[3]{2} = \sqrt[6]{343} \cdot \sqrt[6]{4} = \sqrt[6]{1372}. \qquad \text{Product rule}$$

Now Try Exercise 99.

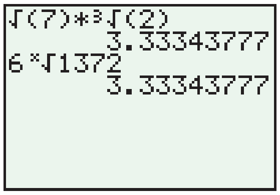


FIGURE 8

Results such as the one in Example 7 can be supported using a calculator, as shown in Figure 8. Notice that the calculator gives the same approximation for the initial product and the final radical that we obtained.

CAUTION The computation in Figure 8 is not *proof* that the two expressions are equal. The algebra in Example 7, however, is valid proof of their equality.

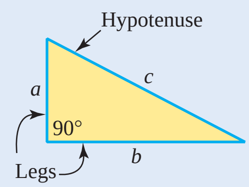
OBJECTIVE 5 Use the Pythagorean formula. The **Pythagorean formula** relates the lengths of the three sides of a right triangle.

Pythagorean Formula

If c is the length of the longest side of a right triangle and a and b are the lengths of the shorter sides, then

AUDIO

$$c^2 = a^2 + b^2.$$



The longest side is the **hypotenuse** and the two shorter sides are the **legs** of the triangle. The hypotenuse is the side opposite the right angle.

EXPLORE

YOU TRY IT

VIDEO

EXAMPLE 8 Using the Pythagorean Formula

Use the Pythagorean formula to find the length of the hypotenuse in the triangle in Figure 9.

To find the length of the hypotenuse c, let $a = 4$ and $b = 6$. Then use the formula.

$$c^2 = a^2 + b^2$$

$$c^2 = 4^2 + 6^2 \qquad \text{Let } a = 4 \text{ and } b = 6.$$

$$c^2 = 52$$

$$c = \sqrt{52} \qquad \text{Choose the principal root.}$$

$$c = \sqrt{4 \cdot 13} \qquad \text{Factor.}$$

$$c = \sqrt{4} \cdot \sqrt{13} \qquad \text{Product rule}$$

$$c = 2\sqrt{13}$$

The length of the hypotenuse is $2\sqrt{13}$.

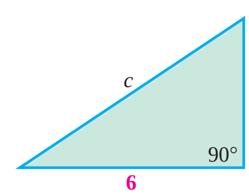


FIGURE 9

Now Try Exercise 109.

CAUTION When using the equation $c^2 = a^2 + b^2$, be sure that the length of the hypotenuse is substituted for c, and that the lengths of the legs are substituted for a and b. Errors often occur because values are substituted incorrectly.

Photo not available

CONNECTIONS

The Pythagorean formula is undoubtedly one of the most widely used and oldest formulas we have. It is very important in trigonometry, which is used in surveying, drafting, engineering, navigation, and many other fields. There is evidence that the Babylonians knew the concept quite well. Although attributed to Pythagoras, it was known to every surveyor from Egypt to China for a thousand years before Pythagoras. In the 1939 movie *The Wizard of Oz,* the Scarecrow asks the Wizard for a brain. When the Wizard presents him with a diploma granting him a Th.D. (Doctor of Thinkology), the Scarecrow recites the following:

> The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. . . .
> Oh joy! Rapture! I've got a brain.

For Discussion or Writing

Did the Scarecrow recite the Pythagorean formula? (An *isosceles triangle* is a triangle with two equal sides.) Is his statement true? Explain.

EXAMPLE 9 Using a Formula from Electronics

The impedance Z of an alternating series circuit is given by the formula

$$Z = \sqrt{R^2 + X^2},$$

where R is the resistance and X is the reactance, both in ohms. Find the value of the impedance if $R = 40$ ohms and $X = 30$ ohms.

Substitute 40 for R and 30 for X in the formula.

$$\begin{aligned} Z &= \sqrt{R^2 + X^2} && \text{Given formula} \\ &= \sqrt{40^2 + 30^2} && \text{Let } R = 40 \text{ and } X = 30. \\ &= \sqrt{1600 + 900} \\ &= \sqrt{2500} \\ &= 50 \end{aligned}$$

The impedance is 50 ohms.

Now Try Exercise 113.

OBJECTIVE 6 Use the distance formula. An important result in algebra is derived by using the Pythagorean formula. The *distance formula* allows us to find the distance between two points in the coordinate plane, or the length of the line segment joining those two points. Figure 10 shows the points $(3, -4)$ and $(-5, 3)$. The vertical line through $(-5, 3)$ and the horizontal line through $(3, -4)$ intersect at the point $(-5, -4)$. Thus, the point $(-5, -4)$ becomes the vertex of the right angle in a right triangle. By the Pythagorean formula, the square of the length of the hypotenuse, d, of the right triangle in Figure 10 is equal to the sum of the squares of the lengths of the two legs a and b:

$$d^2 = a^2 + b^2.$$

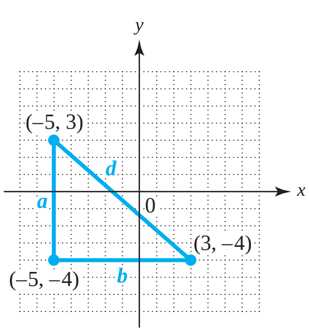


FIGURE 10

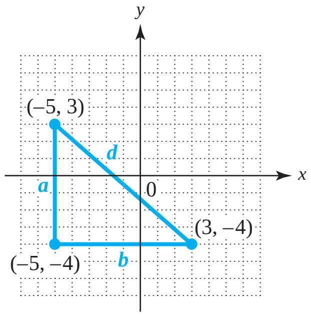


FIGURE 10
(repeated)

The length a is the difference between the y-coordinates of the endpoints. Since the x-coordinate of both points in Figure 10 is -5, the side is vertical, and we can find a by finding the difference between the y-coordinates. We subtract -4 from 3 to get a positive value for a.

$$a = 3 - (-4) = 7$$

Similarly, we find b by subtracting -5 from 3.

$$b = 3 - (-5) = 8$$

Substituting these values into the formula, we obtain

$$\begin{aligned} d^2 &= a^2 + b^2 \\ d^2 &= 7^2 + 8^2 && \text{Let } a = 7 \text{ and } b = 8. \\ d^2 &= 49 + 64 \\ d^2 &= 113 \\ d &= \sqrt{113}. \end{aligned}$$

We choose the principal root since distance cannot be negative. Therefore, the distance between $(-5, 3)$ and $(3, -4)$ is $\sqrt{113}$.

NOTE It is customary to leave the distance in radical form. Do not use a calculator to get an approximation unless you are specifically directed to do so.

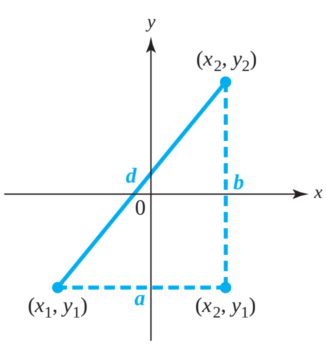


FIGURE 11

This result can be generalized. Figure 11 shows the two points (x_1, y_1) and (x_2, y_2). To find a formula for the distance d between these two points, notice that the distance between (x_1, y_1) and (x_2, y_1) is given by

$$a = |x_2 - x_1|,$$

and the distance between (x_2, y_2) and (x_2, y_1) is given by

$$b = |y_2 - y_1|.$$

From the Pythagorean formula,

$$\begin{aligned} d^2 &= a^2 + b^2 \\ d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2. && |p - q|^2 = (p - q)^2 \end{aligned}$$

Choosing the principal square root gives the **distance formula.**

AUDIO

Distance Formula

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\mathbf{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

VIDEO

EXAMPLE 10 Using the Distance Formula

Find the distance between the points $(-3, 5)$ and $(6, 4)$.

When using the distance formula to find the distance between two points, designating the points as (x_1, y_1) and (x_2, y_2) is arbitrary. We choose $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (6, 4)$.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[6 - (-3)]^2 + (4 - 5)^2} \qquad x_2 = 6,\ y_2 = 4,\ x_1 = -3,\ y_1 = 5 \\ &= \sqrt{9^2 + (-1)^2} \\ &= \sqrt{82} \end{aligned}$$

Now Try Exercise 121.

8.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 12/Videotape 12

Decide whether each statement is true *or* false *by using the product rule explained in this section. Then support your answer by finding a calculator approximation for each expression.*

1. $2\sqrt{12} = \sqrt{48}$ **2.** $\sqrt{72} = 2\sqrt{18}$

3. $3\sqrt{8} = 2\sqrt{18}$ **4.** $5\sqrt{72} = 6\sqrt{50}$

5. Which one of the following is *not* equal to $\sqrt{\frac{1}{2}}$? (Do not use calculator approximations.)

A. $\sqrt{.5}$ **B.** $\sqrt{\dfrac{2}{4}}$ **C.** $\sqrt{\dfrac{3}{6}}$ **D.** $\dfrac{\sqrt{4}}{\sqrt{16}}$

6. Use the π key on your calculator to get a value for π. Now find an approximation for $\sqrt[4]{\dfrac{2143}{22}}$. Does the result mean that π is actually equal to $\sqrt[4]{\dfrac{2143}{22}}$? Why or why not?

Multiply using the product rule. Assume all variables represent positive real numbers. See Examples 1 and 2.

7. $\sqrt{5} \cdot \sqrt{6}$ **8.** $\sqrt{10} \cdot \sqrt{3}$ **9.** $\sqrt{14} \cdot \sqrt{x}$ **10.** $\sqrt{23} \cdot \sqrt{t}$

11. $\sqrt{14} \cdot \sqrt{3pqr}$ **12.** $\sqrt{7} \cdot \sqrt{5xt}$ **13.** $\sqrt[3]{7x} \cdot \sqrt[3]{2y}$ **14.** $\sqrt[3]{9x} \cdot \sqrt[3]{4y}$

15. $\sqrt[4]{11} \cdot \sqrt[4]{3}$ **16.** $\sqrt[4]{6} \cdot \sqrt[4]{9}$ **17.** $\sqrt[4]{2x} \cdot \sqrt[4]{3y^2}$ **18.** $\sqrt[4]{3y^2} \cdot \sqrt[4]{6yz}$

19. $\sqrt[3]{7} \cdot \sqrt[4]{3}$ **20.** $\sqrt[5]{8} \cdot \sqrt[6]{12}$

21. Explain the product rule for radicals in your own words. Give examples.

22. Explain the quotient rule for radicals in your own words. Give examples.

Simplify each radical. Assume all variables represent positive real numbers. See Example 3.

23. $\sqrt{\dfrac{64}{121}}$ **24.** $\sqrt{\dfrac{16}{49}}$ **25.** $\sqrt{\dfrac{3}{25}}$ **26.** $\sqrt{\dfrac{13}{49}}$

27. $\sqrt{\dfrac{x}{25}}$ **28.** $\sqrt{\dfrac{k}{100}}$ **29.** $\sqrt{\dfrac{p^6}{81}}$ **30.** $\sqrt{\dfrac{w^{10}}{36}}$

31. $\sqrt[3]{\dfrac{27}{64}}$ **32.** $\sqrt[3]{\dfrac{216}{125}}$ **33.** $\sqrt[3]{-\dfrac{r^2}{8}}$ **34.** $\sqrt[3]{-\dfrac{t}{125}}$

35. $-\sqrt[4]{\dfrac{81}{x^4}}$ **36.** $-\sqrt[4]{\dfrac{625}{y^4}}$ **37.** $\sqrt[5]{\dfrac{1}{x^{15}}}$ **38.** $\sqrt[5]{\dfrac{32}{y^{20}}}$

Express each radical in simplified form. See Example 4.

EXERCISES

39. $\sqrt{12}$ **40.** $\sqrt{18}$ **41.** $\sqrt{288}$ **42.** $\sqrt{72}$ **43.** $-\sqrt{32}$

44. $-\sqrt{48}$ **45.** $-\sqrt{28}$ **46.** $-\sqrt{24}$ **47.** $\sqrt{-300}$ **48.** $\sqrt{-150}$

49. $\sqrt[3]{128}$ **50.** $\sqrt[3]{24}$ **51.** $\sqrt[3]{-16}$ **52.** $\sqrt[3]{-250}$ **53.** $\sqrt[3]{40}$

54. $\sqrt[3]{375}$ **55.** $-\sqrt[4]{512}$ **56.** $-\sqrt[4]{1250}$ **57.** $\sqrt[5]{64}$ **58.** $\sqrt[5]{128}$

59. A student claimed that $\sqrt[3]{14}$ is not in simplified form, since $14 = 8 + 6$, and 8 is a perfect cube. Was his reasoning correct? Why or why not?

60. Explain in your own words why $\sqrt[3]{k^4}$ is not a simplified radical.

Express each radical in simplified form. Assume all variables represent positive real numbers. See Example 5.

EXERCISES

61. $\sqrt{72k^2}$ **62.** $\sqrt{18m^2}$ **63.** $\sqrt{144x^3y^9}$

64. $\sqrt{169s^5t^{10}}$ **65.** $\sqrt{121x^6}$ **66.** $\sqrt{256z^{12}}$

67. $-\sqrt[3]{27t^{12}}$ **68.** $-\sqrt[3]{64y^{18}}$ **69.** $-\sqrt{100m^8z^4}$

70. $-\sqrt{25t^6s^{20}}$ **71.** $-\sqrt[3]{-125a^6b^9c^{12}}$ **72.** $-\sqrt[3]{-216y^{15}x^6z^3}$

73. $\sqrt[4]{\frac{1}{16}r^8t^{20}}$ **74.** $\sqrt[4]{\frac{81}{256}t^{12}u^8}$ **75.** $\sqrt{50x^3}$ **76.** $\sqrt{300z^3}$

77. $-\sqrt{500r^{11}}$ **78.** $-\sqrt{200p^{13}}$ **79.** $\sqrt{13x^7y^8}$ **80.** $\sqrt{23k^9p^{14}}$

81. $\sqrt[3]{8z^6w^9}$ **82.** $\sqrt[3]{64a^{15}b^{12}}$ **83.** $\sqrt[3]{-16z^5t^7}$ **84.** $\sqrt[3]{-81m^4n^{10}}$

85. $\sqrt[4]{81x^{12}y^{16}}$ **86.** $\sqrt[4]{81t^8u^{28}}$ **87.** $-\sqrt[4]{162r^{15}s^{10}}$ **88.** $-\sqrt[4]{32k^5m^{10}}$

89. $\sqrt{\frac{y^{11}}{36}}$ **90.** $\sqrt{\frac{v^{13}}{49}}$ **91.** $\sqrt[3]{\frac{x^{16}}{27}}$ **92.** $\sqrt[3]{\frac{y^{17}}{125}}$

Simplify each radical. Assume that $x \geq 0$. See Example 6.

EXERCISES

93. $\sqrt[4]{48^2}$ **94.** $\sqrt[4]{50^2}$ **95.** $\sqrt[4]{25}$

96. $\sqrt[6]{8}$ **97.** $\sqrt[10]{x^{25}}$ **98.** $\sqrt[12]{x^{44}}$

Simplify by first writing the radicals as radicals with the same index. Then multiply. Assume all variables represent positive real numbers. See Example 7.

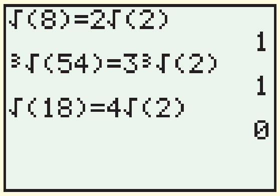
EXERCISES

99. $\sqrt[3]{4} \cdot \sqrt{3}$ **100.** $\sqrt[3]{5} \cdot \sqrt{6}$ **101.** $\sqrt[4]{3} \cdot \sqrt[3]{4}$

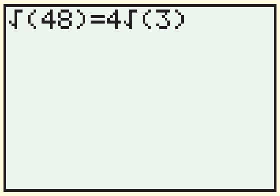
102. $\sqrt[5]{7} \cdot \sqrt[7]{5}$ **103.** $\sqrt{x} \cdot \sqrt[3]{x}$ **104.** $\sqrt[3]{y} \cdot \sqrt[4]{y}$

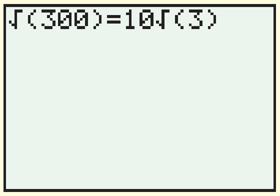
TECHNOLOGY INSIGHTS (EXERCISES 105–108)

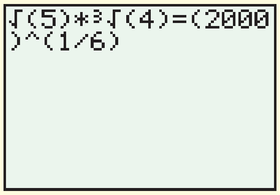
A graphing calculator can be used to test whether two quantities are equal. In the screen shown here, the first two lines of entries both represent true statements, and thus the calculator returns a 1 *to indicate* true. *The third entry is* false, *and the calculator returns a* 0. *These can be verified algebraically using the rules for radicals found in this section.*

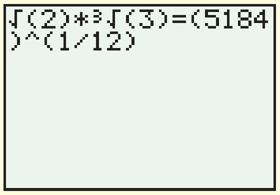


Determine whether the calculator should return a 1 *or a* 0 *for each screen.*

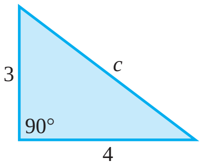
105.


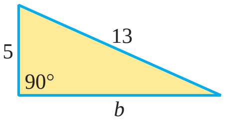
106.


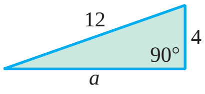
107.


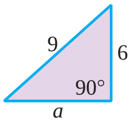
108.


Find the unknown length in each right triangle. Simplify the answer if necessary. See Example 8.

109.


110.


111.


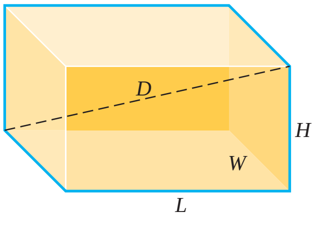
112.


Solve each problem. See Example 9.

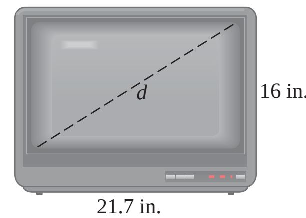
113. The length of the diagonal of a box is given by

$$D = \sqrt{L^2 + W^2 + H^2},$$

where L, W, and H are the length, width, and height of the box. Find the length of the diagonal, D, of a box that is 4 ft long, 3 ft high, and 2 ft wide. Give the exact value, then round to the nearest tenth of a foot.



114. A Sanyo color television, model AVM-2755, has a rectangular screen with a 21.7-in. width. Its height is 16 in. What is the diameter of the screen to the nearest tenth of an inch? (*Source:* Actual measurements of the author's television.)



115. A formula from electronics dealing with impedance of parallel resonant circuits is

$$I = \frac{E}{\sqrt{R^2 + \omega^2 L^2}},$$

where the variables are in appropriate units. Find I if $E = 282$, $R = 100$, $L = 264$, and $\omega = 120\pi$. Give your answer to the nearest thousandth.

116. In the study of sound, one version of the law of tensions is

$$f_1 = f_2 \sqrt{\frac{F_1}{F_2}}.$$

If $F_1 = 300$, $F_2 = 60$, and $f_2 = 260$, find f_1 to the nearest unit.

117. The illumination I, in foot-candles, produced by a light source is related to the distance d, in feet, from the light source by the equation

$$d = \sqrt{\frac{k}{I}},$$

where k is a constant. If $k = 640$, how far from the light source will the illumination be 2 foot-candles? Give the exact value, and then round to the nearest tenth of a foot.

118. The following letter appeared in the column "Ask Tom Why," written by Tom Skilling of the *Chicago Tribune.*

Dear Tom,
I cannot remember the formula to calculate the distance to the horizon. I have a stunning view from my 14th floor condo, 150 feet above the ground. How far can I see?
Ted Fleischaker; Indianapolis, Ind.

Photo not available

Skilling's answer was as follows.

To find the distance to the horizon in miles, take the square root of the height of your view in feet and multiply that result by 1.224. Your answer will be the number of miles to the horizon. (*Source: Chicago Tribune*, August 17, 2002.)

Assuming Ted's eyes are 6 ft above the ground, the total height from the ground is $150 + 6 = 156$ ft. To the nearest tenth of a mile, how far can he see to the horizon?

Find the distance between each pair of points. See Example 10.

119. (6, 13) and (1, 1)

120. (8, 13) and (2, 5)

121. (−6, 5) and (3, −4)

122. (−1, 5) and (−7, 7)

123. (−8, 2) and (−4, 1)

124. (−1, 2) and (5, 3)

125. $(4.7, 2.3)$ and $(1.7, -1.7)$

126. $(-2.9, 18.2)$ and $(2.1, 6.2)$

127. $\left(\sqrt{2}, \sqrt{6}\right)$ and $\left(-2\sqrt{2}, 4\sqrt{6}\right)$

128. $\left(\sqrt{7}, 9\sqrt{3}\right)$ and $\left(-\sqrt{7}, 4\sqrt{3}\right)$

129. $(x + y, y)$ and $(x - y, x)$

130. $(c, c - d)$ and $(d, c + d)$

131. As given in the text, the distance formula is expressed with a radical. Write the distance formula using rational exponents.

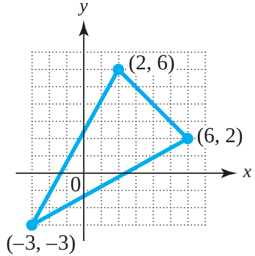
132. An alternative form of the distance formula is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

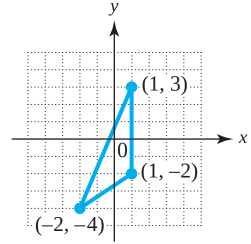
Compare this to the form given in this section, and explain why the two forms are equivalent.

Find the perimeter of each triangle. (Hint: *For Exercise 133,* $\sqrt{k} + \sqrt{k} = 2\sqrt{k}$.)

133.



134.



8.4 Adding and Subtracting Radical Expressions

OBJECTIVES

1. Define a radical expression.
2. Simplify radical expressions involving addition and subtraction.

OBJECTIVE 1 Define a radical expression. Recall from Section 8.1 that a **radical expression** is an algebraic expression that contains radicals. For example,

$$\sqrt[4]{3} + \sqrt{6}, \qquad \sqrt{x + 2y} - 1, \qquad \text{and} \qquad \sqrt{8} - \sqrt{2r}$$

are radical expressions. The examples in the previous section discussed simplifying radical expressions that involve multiplication and division. Now we show how to simplify radical expressions that involve addition and subtraction.

OBJECTIVE 2 Simplify radical expressions involving addition and subtraction. An expression such as $4\sqrt{2} + 3\sqrt{2}$ can be simplified by using the distributive property.

$$4\sqrt{2} + 3\sqrt{2} = (4 + 3)\sqrt{2} = 7\sqrt{2}$$

As another example, $2\sqrt{3} - 5\sqrt{3} = (2 - 5)\sqrt{3} = -3\sqrt{3}$. This is similar to simplifying $4x + 3x$ to $7x$ or $2y - 5y$ to $-3y$.

CAUTION Only radical expressions with the *same index* and the *same radicand* may be combined. Expressions such as $5\sqrt{3} + 2\sqrt{2}$ or $3\sqrt{3} + 2\sqrt[3]{3}$ cannot be simplified by combining terms.

EXAMPLE 1 Adding and Subtracting Radicals

Add or subtract to simplify each radical expression.

(a) $3\sqrt{24} + \sqrt{54}$

Begin by simplifying each radical; then use the distributive property to combine terms.

$$\begin{aligned} 3\sqrt{24} + \sqrt{54} &= 3\sqrt{4} \cdot \sqrt{6} + \sqrt{9} \cdot \sqrt{6} && \text{Product rule} \\ &= 3 \cdot 2\sqrt{6} + 3\sqrt{6} \\ &= 6\sqrt{6} + 3\sqrt{6} \\ &= 9\sqrt{6} && \text{Combine terms.} \end{aligned}$$

(b) $$\begin{aligned} 2\sqrt{20x} - \sqrt{45x} &= 2\sqrt{4} \cdot \sqrt{5x} - \sqrt{9} \cdot \sqrt{5x} && \text{Product rule} \\ &= 2 \cdot 2\sqrt{5x} - 3\sqrt{5x} \\ &= 4\sqrt{5x} - 3\sqrt{5x} \\ &= \sqrt{5x}, \quad x \geq 0 && \text{Combine terms.} \end{aligned}$$

(c) $2\sqrt{3} - 4\sqrt{5}$

Here the radicands differ and are already simplified, so $2\sqrt{3} - 4\sqrt{5}$ cannot be simplified further.

Now Try Exercises 7, 15, and 19.

CAUTION Do not confuse the product rule with combining like terms. The root of a sum ***does not equal*** the sum of the roots. For example,

$$\sqrt{9 + 16} \neq \sqrt{9} + \sqrt{16}$$

since $\quad \sqrt{9 + 16} = \sqrt{25} = 5, \quad$ but $\quad \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$

EXAMPLE 2 Adding and Subtracting Radicals with Higher Indexes

Add or subtract to simplify each radical expression. Assume all variables represent positive real numbers.

(a) $$\begin{aligned} 2\sqrt[3]{16} - 5\sqrt[3]{54} &= 2\sqrt[3]{8 \cdot 2} - 5\sqrt[3]{27 \cdot 2} && \text{Factor.} \\ &= 2\sqrt[3]{8} \cdot \sqrt[3]{2} - 5\sqrt[3]{27} \cdot \sqrt[3]{2} && \text{Product rule} \\ &= 2 \cdot 2 \cdot \sqrt[3]{2} - 5 \cdot 3 \cdot \sqrt[3]{2} \\ &= 4\sqrt[3]{2} - 15\sqrt[3]{2} \\ &= -11\sqrt[3]{2} && \text{Combine terms.} \end{aligned}$$

(b) $$\begin{aligned} 2\sqrt[3]{x^2y} + \sqrt[3]{8x^5y^4} &= 2\sqrt[3]{x^2y} + \sqrt[3]{(8x^3y^3)x^2y} && \text{Factor.} \\ &= 2\sqrt[3]{x^2y} + 2xy\sqrt[3]{x^2y} && \text{Product rule} \\ &= (2 + 2xy)\sqrt[3]{x^2y} && \text{Distributive property} \end{aligned}$$

Now Try Exercises 23 and 29.

CAUTION Remember to write the index when working with cube roots, fourth roots, and so on.

EXAMPLE 3 Adding and Subtracting Radicals with Fractions

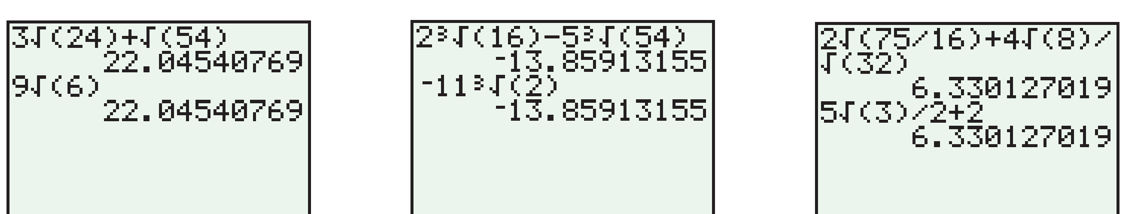
Perform the indicated operations. Assume all variables represent positive real numbers.

(a)
$$2\sqrt{\frac{75}{16}} + 4\frac{\sqrt{8}}{\sqrt{32}} = 2\frac{\sqrt{25 \cdot 3}}{\sqrt{16}} + 4\frac{\sqrt{4 \cdot 2}}{\sqrt{16 \cdot 2}} \quad \text{Quotient rule}$$
$$= 2\left(\frac{5\sqrt{3}}{4}\right) + 4\left(\frac{2\sqrt{2}}{4\sqrt{2}}\right) \quad \text{Product rule}$$
$$= \frac{5\sqrt{3}}{2} + 2 \quad \text{Multiply; } \frac{\sqrt{2}}{\sqrt{2}} = 1.$$
$$= \frac{5\sqrt{3}}{2} + \frac{4}{2} \quad \text{Write with a common denominator.}$$
$$= \frac{5\sqrt{3} + 4}{2}$$

(b)
$$10\sqrt[3]{\frac{5}{x^6}} - 3\sqrt[3]{\frac{4}{x^9}} = 10\frac{\sqrt[3]{5}}{\sqrt[3]{x^6}} - 3\frac{\sqrt[3]{4}}{\sqrt[3]{x^9}} \quad \text{Quotient rule}$$
$$= \frac{10\sqrt[3]{5}}{x^2} - \frac{3\sqrt[3]{4}}{x^3}$$
$$= \frac{10x\sqrt[3]{5}}{x^3} - \frac{3\sqrt[3]{4}}{x^3} \quad \text{Write with a common denominator.}$$
$$= \frac{10x\sqrt[3]{5} - 3\sqrt[3]{4}}{x^3}$$

Now Try Exercises 47 and 53.

A calculator can support some of the results obtained in the examples of this section. In Example 1(a), we simplified $3\sqrt{24} + \sqrt{54}$ to obtain $9\sqrt{6}$. The screen in Figure 12(a) shows that the approximations are the same, suggesting that our simplification was correct. Figure 12(b) shows support for the result of Example 2(a): $2\sqrt[3]{16} - 5\sqrt[3]{54} = -11\sqrt[3]{2}$. Figure 12(c) supports the result of Example 3(a).

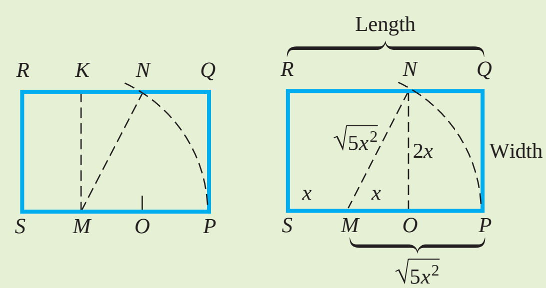


(a) (b) (c)

FIGURE 12

CONNECTIONS

A triangle that has whole number measures for the lengths of two sides may have an irrational number as the measure of the third side. For example, a right triangle with the two shorter sides measuring 1 and 2 units will have a longest side measuring $\sqrt{5}$ units. The ratio of the dimensions of the *golden rectangle*, considered to have the most pleasing dimensions of any rectangle, is irrational. To sketch a golden rectangle, begin with the square *ONRS*. Divide it into two equal parts by segment *MK*, as shown in the figure. Let *M* be the center of a circle with radius *MN*. Sketch the rectangle *PQRS*. This is a golden rectangle, with the property that if the original square is taken away, *PQNO* is still a golden rectangle. If the square with side *OP* is taken away, another golden rectangle results, and so on.



If the sides of the generating square have measure $2x$, then by the Pythagorean formula,

$$MN = \sqrt{x^2 + (2x)^2} = \sqrt{x^2 + 4x^2} = \sqrt{5x^2}.$$

Since *NP* is an arc of the circle with radius *MN*,

$$MP = MN = \sqrt{5x^2}.$$

The ratio of length to width is

$$\frac{\text{length}}{\text{width}} = \frac{x + \sqrt{5x^2}}{2x} = \frac{x + x\sqrt{5}}{2x} = \frac{x\left(1 + \sqrt{5}\right)}{2x} = \frac{1 + \sqrt{5}}{2},$$

which is an irrational number.

For Discussion or Writing

1. The golden rectangle has been widely used in art and architecture. See whether you can find some examples of its use. Use a calculator to approximate the ratio found above, called the *golden ratio*.
2. The sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . . is called the *Fibonacci sequence*. After the first two terms, both 1, every term is found by adding the two preceding terms. Form a sequence of ratios of the successive terms:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ldots.$$

Now use a calculator to find approximations of these ratios. What seems to be happening?

8.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 13/Videotape 13

EXERCISES

1. Which one of the following sums could be simplified without first simplifying the individual radical expressions?

A. $\sqrt{50} + \sqrt{32}$ **B.** $3\sqrt{6} + 9\sqrt{6}$ **C.** $\sqrt[3]{32} - \sqrt[3]{108}$ **D.** $\sqrt[5]{6} - \sqrt[5]{192}$

2. Let $a = 1$ and let $b = 64$.

(a) Evaluate $\sqrt{a} + \sqrt{b}$. Then find $\sqrt{a+b}$. Are they equal?
(b) Evaluate $\sqrt[3]{a} + \sqrt[3]{b}$. Then find $\sqrt[3]{a+b}$. Are they equal?
(c) Complete the following: In general, $\sqrt[n]{a} + \sqrt[n]{b} \neq$ ____________________, based on the observations in parts (a) and (b) of this exercise.

3. Even though the root indexes of the terms are not equal, the sum $\sqrt{64} + \sqrt[3]{125} + \sqrt[4]{16}$ can be simplified quite easily. What is this sum? Why can we add these terms so easily?

4. Explain why $28 - 4\sqrt{2}$ is not equal to $24\sqrt{2}$. (This is a common error among algebra students.)

Simplify. Assume all variables represent positive real numbers. See Examples 1 and 2.

EXERCISES

5. $\sqrt{36} - \sqrt{100}$
6. $\sqrt{25} - \sqrt{81}$
7. $-2\sqrt{48} + 3\sqrt{75}$
8. $4\sqrt{32} - 2\sqrt{8}$
9. $\sqrt[3]{16} + 4\sqrt[3]{54}$
10. $3\sqrt[3]{24} - 2\sqrt[3]{192}$
11. $\sqrt[4]{32} + 3\sqrt[4]{2}$
12. $\sqrt[4]{405} - 2\sqrt[4]{5}$
13. $6\sqrt{18} - \sqrt{32} + 2\sqrt{50}$
14. $5\sqrt{8} + 3\sqrt{72} - 3\sqrt{50}$
15. $5\sqrt{6} + 2\sqrt{10}$
16. $3\sqrt{11} - 5\sqrt{13}$
17. $2\sqrt{5} + 3\sqrt{20} + 4\sqrt{45}$
18. $5\sqrt{54} - 2\sqrt{24} - 2\sqrt{96}$
19. $8\sqrt{2x} - \sqrt{8x} + \sqrt{72x}$
20. $4\sqrt{18k} - \sqrt{72k} + \sqrt{50k}$
21. $3\sqrt{72m^2} - 5\sqrt{32m^2} - 3\sqrt{18m^2}$
22. $9\sqrt{27p^2} - 14\sqrt{108p^2} + 2\sqrt{48p^2}$
23. $-\sqrt[3]{54} + 2\sqrt[3]{16}$
24. $15\sqrt[3]{81} - 4\sqrt[3]{24}$
25. $2\sqrt[3]{27x} - 2\sqrt[3]{8x}$
26. $6\sqrt[3]{128m} + 3\sqrt[3]{16m}$
27. $\sqrt[3]{x^2y} - \sqrt[3]{8x^2y}$
28. $3\sqrt[3]{x^2y^2} - 2\sqrt[3]{64x^2y^2}$
29. $3x\sqrt[3]{xy^2} - 2\sqrt[3]{8x^4y^2}$
30. $6q^2\sqrt[3]{5q} - 2q\sqrt[3]{40q^4}$
31. $5\sqrt[4]{32} + 3\sqrt[4]{162}$
32. $2\sqrt[4]{512} + 4\sqrt[4]{32}$
33. $3\sqrt[4]{x^5y} - 2x\sqrt[4]{xy}$
34. $2\sqrt[4]{m^9p^6} - 3m^2p\sqrt[4]{mp^2}$
35. $2\sqrt[4]{32a^3} + 5\sqrt[4]{2a^3}$
36. $-\sqrt[4]{16r} + 5\sqrt[4]{r}$
37. $\sqrt[3]{64xy^2} + \sqrt[3]{27x^4y^5}$
38. $\sqrt[4]{625s^3t} - \sqrt[4]{81s^7t^5}$

Simplify. Assume all variables represent positive real numbers. See Example 3.

EXERCISES

39. $\sqrt{8} - \dfrac{\sqrt{64}}{\sqrt{16}}$
40. $\sqrt{48} - \dfrac{\sqrt{81}}{\sqrt{9}}$
41. $\dfrac{2\sqrt{5}}{3} + \dfrac{\sqrt{5}}{6}$

42. $\dfrac{4\sqrt{3}}{3} + \dfrac{2\sqrt{3}}{9}$

43. $\sqrt{\dfrac{8}{9}} + \sqrt{\dfrac{18}{36}}$

44. $\sqrt{\dfrac{12}{16}} + \sqrt{\dfrac{48}{64}}$

45. $\dfrac{\sqrt{32}}{3} + \dfrac{2\sqrt{2}}{3} - \dfrac{\sqrt{2}}{\sqrt{9}}$

46. $\dfrac{\sqrt{27}}{2} - \dfrac{3\sqrt{3}}{2} + \dfrac{\sqrt{3}}{\sqrt{4}}$

47. $3\sqrt{\dfrac{50}{9}} + 8\dfrac{\sqrt{2}}{\sqrt{8}}$

48. $9\sqrt{\dfrac{48}{25}} - 2\dfrac{\sqrt{2}}{\sqrt{98}}$

49. $\sqrt{\dfrac{25}{x^8}} - \sqrt{\dfrac{9}{x^6}}$

50. $\sqrt{\dfrac{100}{y^4}} + \sqrt{\dfrac{81}{y^{10}}}$

51. $3\sqrt[3]{\dfrac{m^5}{27}} - 2m\sqrt[3]{\dfrac{m^2}{64}}$

52. $2a\sqrt[4]{\dfrac{a}{16}} - 5a\sqrt[4]{\dfrac{a}{81}}$

53. $3\sqrt[3]{\dfrac{2}{x^6}} - 4\sqrt[3]{\dfrac{5}{x^9}}$

54. $-4\sqrt[3]{\dfrac{4}{t^9}} + 3\sqrt[3]{\dfrac{9}{t^{12}}}$

In Example 1(a) we showed that $3\sqrt{24} + \sqrt{54} = 9\sqrt{6}$. *To support this result, we can find a calculator approximation of* $3\sqrt{24}$, *then find a calculator approximation of* $\sqrt{54}$, *and add these two approximations. Then, we find a calculator approximation of* $9\sqrt{6}$. *It should correspond to the sum that we just found. (For this example, both approximations are* 22.04540769. *Due to rounding procedures, there may be a discrepancy in the final digit if you try to duplicate this work.) Follow this procedure to support the statements in Exercises 55 and 56.*

55. $3\sqrt{32} - 2\sqrt{8} = 8\sqrt{2}$

56. $2\sqrt{40} + 6\sqrt{90} - 3\sqrt{160} = 10\sqrt{10}$

57. A rectangular yard has a length of $\sqrt{192}$ m and a width of $\sqrt{48}$ m. Choose the best estimate of its dimensions. Then estimate the perimeter.

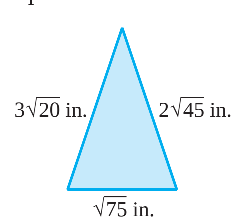
A. 14 m by 7 m **B.** 5 m by 7 m **C.** 14 m by 8 m **D.** 15 m by 8 m

58. If the sides of a triangle are $\sqrt{65}$ in., $\sqrt{35}$ in., and $\sqrt{26}$ in., which one of the following is the best estimate of its perimeter?

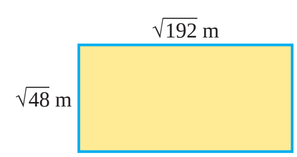
A. 20 in. **B.** 26 in. **C.** 19 in. **D.** 24 in.

Solve each problem. Give answers as simplified radical expressions.

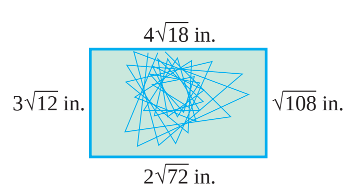
59. Find the perimeter of the triangle.



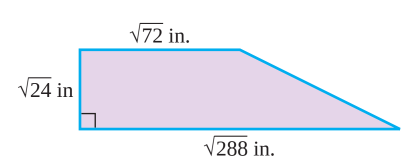
60. Find the perimeter of the rectangle.



61. What is the perimeter of the computer graphic?



62. Find the area of the trapezoid.



8.5 Multiplying and Dividing Radical Expressions

OBJECTIVES

1. Multiply radical expressions.
2. Rationalize denominators with one radical term.
3. Rationalize denominators with binomials involving radicals.
4. Write radical quotients in lowest terms.

YOU TRY IT

OBJECTIVE 1 Multiply radical expressions. We multiply binomial expressions involving radicals by using the FOIL (First, Outer, Inner, Last) method. For example, we find the product of the binomials $\sqrt{5} + 3$ and $\sqrt{6} + 1$ as follows.

$$\begin{aligned}(\sqrt{5} + 3)(\sqrt{6} + 1) &= \overbrace{\sqrt{5} \cdot \sqrt{6}}^{\text{First}} + \overbrace{\sqrt{5} \cdot 1}^{\text{Outer}} + \overbrace{3 \cdot \sqrt{6}}^{\text{Inner}} + \overbrace{3 \cdot 1}^{\text{Last}} \\ &= \sqrt{30} + \sqrt{5} + 3\sqrt{6} + 3\end{aligned}$$

This result cannot be simplified further.

EXAMPLE 1 Multiplying Binomials Involving Radical Expressions

Multiply using FOIL.

(a) $$\begin{aligned}(7 - \sqrt{3})(\sqrt{5} + \sqrt{2}) &= \overset{\text{F}}{7\sqrt{5}} + \overset{\text{O}}{7\sqrt{2}} - \overset{\text{I}}{\sqrt{3} \cdot \sqrt{5}} - \overset{\text{L}}{\sqrt{3} \cdot \sqrt{2}} \\ &= 7\sqrt{5} + 7\sqrt{2} - \sqrt{15} - \sqrt{6}\end{aligned}$$

(b) $$\begin{aligned}&(\sqrt{10} + \sqrt{3})(\sqrt{10} - \sqrt{3}) \\ &= \sqrt{10} \cdot \sqrt{10} - \sqrt{10} \cdot \sqrt{3} + \sqrt{10} \cdot \sqrt{3} - \sqrt{3} \cdot \sqrt{3} \\ &= 10 - 3 \\ &= 7\end{aligned}$$

Notice that this is the kind of product that results in the difference of squares:

$$(x + y)(x - y) = x^2 - y^2.$$

Here, $x = \sqrt{10}$ and $y = \sqrt{3}$.

(c) $$\begin{aligned}(\sqrt{7} - 3)^2 &= (\sqrt{7} - 3)(\sqrt{7} - 3) \\ &= \sqrt{7} \cdot \sqrt{7} - 3\sqrt{7} - 3\sqrt{7} + 3 \cdot 3 \\ &= 7 - 6\sqrt{7} + 9 \\ &= 16 - 6\sqrt{7}\end{aligned}$$

(d) $$\begin{aligned}(5 - \sqrt[3]{3})(5 + \sqrt[3]{3}) &= 5 \cdot 5 + 5\sqrt[3]{3} - 5\sqrt[3]{3} - \sqrt[3]{3} \cdot \sqrt[3]{3} \\ &= 25 - \sqrt[3]{3^2} \\ &= 25 - \sqrt[3]{9}\end{aligned}$$

(e) $$\begin{aligned}(\sqrt{k} + \sqrt{y})(\sqrt{k} - \sqrt{y}) &= (\sqrt{k})^2 - (\sqrt{y})^2 \\ &= k - y, \quad k \geq 0 \text{ and } y \geq 0\end{aligned}$$

Now Try Exercises 13, 17, 23, 27, and 39.

NOTE In Example 1(c) we could have used the formula for the square of a binomial,

$$(x - y)^2 = x^2 - 2xy + y^2,$$

to obtain the same result:

$$\begin{aligned}(\sqrt{7} - 3)^2 &= (\sqrt{7})^2 - 2(\sqrt{7})(3) + 3^2\\ &= 7 - 6\sqrt{7} + 9\\ &= 16 - 6\sqrt{7}.\end{aligned}$$

OBJECTIVE 2 Rationalize denominators with one radical term. As defined earlier, a simplified radical expression will have no radical in the denominator. The origin of this agreement no doubt occurred before the days of high-speed calculation, when computation was a tedious process performed by hand. To see this, consider the radical expression $\frac{1}{\sqrt{2}}$. To find a decimal approximation by hand, it would be necessary to divide 1 by a decimal approximation for $\sqrt{2}$, such as 1.414. It would be much easier if the divisor were a whole number. This can be accomplished by multiplying $\frac{1}{\sqrt{2}}$ by 1 in the form $\frac{\sqrt{2}}{\sqrt{2}}$:

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Now the computation would require dividing 1.414 by 2 to obtain .707, a much easier task.

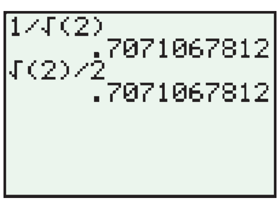


FIGURE 13

With current technology, either form of this fraction can be approximated with the same number of keystrokes. See Figure 13, which shows how a calculator gives the same approximation for both forms of the expression.

A common way of "standardizing" the form of a radical expression is to have the denominator contain no radicals. The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalizing the denominator.**

EXAMPLE 2 Rationalizing Denominators with Square Roots

Rationalize each denominator.

(a) $\frac{3}{\sqrt{7}}$

Multiply the numerator and denominator by $\sqrt{7}$. This is, in effect, multiplying by 1.

$$\frac{3}{\sqrt{7}} = \frac{3 \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}}$$

In the denominator, since $\sqrt{7} \cdot \sqrt{7} = \sqrt{7 \cdot 7} = \sqrt{49} = 7$,

$$\frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}.$$

The denominator is now a rational number.

(b) $$\frac{5\sqrt{2}}{\sqrt{5}} = \frac{5\sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{5\sqrt{10}}{5} = \sqrt{10}$$

(c) $$\frac{6}{\sqrt{12}}$$

Less work is involved if we simplify the radical in the denominator first.

$$\frac{6}{\sqrt{12}} = \frac{6}{\sqrt{4 \cdot 3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}}$$

Now we rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

$$\frac{3 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Now Try Exercises 43, 49, and 51.

EXAMPLE 3 Rationalizing Denominators in Roots of Fractions

Simplify each radical. In part (b), $p > 0$.

(a)
$$\begin{aligned}
\sqrt{\frac{18}{125}} &= \frac{\sqrt{18}}{\sqrt{125}} && \text{Quotient rule} \\
&= \frac{\sqrt{9 \cdot 2}}{\sqrt{25 \cdot 5}} && \text{Factor.} \\
&= \frac{3\sqrt{2}}{5\sqrt{5}} && \text{Product rule} \\
&= \frac{3\sqrt{2} \cdot \sqrt{5}}{5\sqrt{5} \cdot \sqrt{5}} && \text{Multiply by } \frac{\sqrt{5}}{\sqrt{5}}. \\
&= \frac{3\sqrt{10}}{5 \cdot 5} && \text{Product rule} \\
&= \frac{3\sqrt{10}}{25}
\end{aligned}$$

(b)
$$\begin{aligned}
\sqrt{\frac{50m^4}{p^5}} &= \frac{\sqrt{50m^4}}{\sqrt{p^5}} && \text{Quotient rule} \\
&= \frac{5m^2\sqrt{2}}{p^2\sqrt{p}} && \text{Product rule} \\
&= \frac{5m^2\sqrt{2} \cdot \sqrt{p}}{p^2\sqrt{p} \cdot \sqrt{p}} && \text{Multiply by } \frac{\sqrt{p}}{\sqrt{p}}. \\
&= \frac{5m^2\sqrt{2p}}{p^2 \cdot p} && \text{Product rule} \\
&= \frac{5m^2\sqrt{2p}}{p^3}
\end{aligned}$$

Now Try Exercises 55 and 63.

EXAMPLE 4 Rationalizing Denominators with Cube and Fourth Roots

Simplify.

(a) $\sqrt[3]{\dfrac{27}{16}}$

Use the quotient rule and simplify the numerator and denominator.

$$\sqrt[3]{\frac{27}{16}} = \frac{\sqrt[3]{27}}{\sqrt[3]{16}} = \frac{3}{\sqrt[3]{8} \cdot \sqrt[3]{2}} = \frac{3}{2\sqrt[3]{2}}$$

To get a rational denominator, multiply the numerator and denominator by a number that will result in a perfect cube in the radicand in the denominator. Since $2 \cdot 4 = 8$, a perfect cube, multiply the numerator and denominator by $\sqrt[3]{4}$.

$$\sqrt[3]{\frac{27}{16}} = \frac{3}{2\sqrt[3]{2}} = \frac{3 \cdot \sqrt[3]{4}}{2\sqrt[3]{2} \cdot \sqrt[3]{4}} = \frac{3\sqrt[3]{4}}{2\sqrt[3]{8}} = \frac{3\sqrt[3]{4}}{2 \cdot 2} = \frac{3\sqrt[3]{4}}{4}$$

(b) $\sqrt[4]{\dfrac{5x}{z}} = \dfrac{\sqrt[4]{5x}}{\sqrt[4]{z}} \cdot \dfrac{\sqrt[4]{z^3}}{\sqrt[4]{z^3}} = \dfrac{\sqrt[4]{5xz^3}}{\sqrt[4]{z^4}} = \dfrac{\sqrt[4]{5xz^3}}{z}, \quad x \geq 0, z > 0$

Now Try Exercises 71 and 81.

CAUTION It is easy to make mistakes in problems like the one in Example 4(a). A typical error is to multiply the numerator and denominator by $\sqrt[3]{2}$, forgetting that

$$\sqrt[3]{2} \cdot \sqrt[3]{2} \neq 2.$$

You need *three* factors of 2 to obtain 2^3 under the radical. As implied in Example 4(a),

$$\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = 2.$$

OBJECTIVE 3 Rationalize denominators with binomials involving radicals. Recall the special product

$$(x + y)(x - y) = x^2 - y^2.$$

To rationalize a denominator that contains a binomial expression (one that contains exactly two terms) involving radicals, such as

$$\frac{3}{1 + \sqrt{2}},$$

we must use *conjugates*. The conjugate of $1 + \sqrt{2}$ is $1 - \sqrt{2}$. In general, $x + y$ and $x - y$ are **conjugates.**

Rationalizing a Binomial Denominator

Whenever a radical expression has a sum or difference with square root radicals in the denominator, rationalize the denominator by multiplying both the numerator and denominator by the conjugate of the denominator.

For the expression $\frac{3}{1+\sqrt{2}}$, we rationalize the denominator by multiplying both the numerator and denominator by $1-\sqrt{2}$, the conjugate of the denominator.

$$\frac{3}{1+\sqrt{2}} = \frac{3(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$$

Then $(1+\sqrt{2})(1-\sqrt{2}) = 1^2 - (\sqrt{2})^2 = 1 - 2 = -1$. Placing -1 in the denominator gives

$$= \frac{3(1-\sqrt{2})}{-1}$$

$$= \frac{3}{-1}(1-\sqrt{2})$$

$$= -3(1-\sqrt{2}) \quad \text{or} \quad -3 + 3\sqrt{2}.$$

EXPLORE

YOU TRY IT

EXAMPLE 5 Rationalizing Binomial Denominators

Rationalize each denominator.

(a) $\frac{5}{4-\sqrt{3}}$

To rationalize the denominator, multiply both the numerator and denominator by the conjugate of the denominator, $4+\sqrt{3}$.

$$\frac{5}{4-\sqrt{3}} = \frac{5(4+\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})}$$

$$= \frac{5(4+\sqrt{3})}{16-3}$$

$$= \frac{5(4+\sqrt{3})}{13}$$

Notice that the numerator is left in factored form. This makes it easier to determine whether the expression is written in lowest terms.

(b) $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

Multiply the numerator and denominator by $\sqrt{5}-\sqrt{3}$ to rationalize the denominator.

$$\frac{\sqrt{2}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{(\sqrt{2}-\sqrt{3})(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$$

$$= \frac{\sqrt{10}-\sqrt{6}-\sqrt{15}+3}{5-3}$$

$$= \frac{\sqrt{10}-\sqrt{6}-\sqrt{15}+3}{2}$$

(c) $$\frac{3}{\sqrt{5m}-\sqrt{p}} = \frac{3\left(\sqrt{5m}+\sqrt{p}\right)}{\left(\sqrt{5m}-\sqrt{p}\right)\left(\sqrt{5m}+\sqrt{p}\right)}$$

$$= \frac{3\left(\sqrt{5m}+\sqrt{p}\right)}{5m-p}, \quad 5m \neq p,\ m > 0,\ p > 0$$

Now Try Exercises 85, 91, and 95.

OBJECTIVE 4 Write radical quotients in lowest terms.

EXAMPLE 6 Writing Radical Quotients in Lowest Terms

Write each quotient in lowest terms.

(a) $\frac{6+2\sqrt{5}}{4}$

Factor the numerator and denominator, then write in lowest terms.

$$\frac{6+2\sqrt{5}}{4} = \frac{2\left(3+\sqrt{5}\right)}{2\cdot 2} = \frac{3+\sqrt{5}}{2}$$

Here is an alternative method for writing this expression in lowest terms.

$$\frac{6+2\sqrt{5}}{4} = \frac{6}{4} + \frac{2\sqrt{5}}{4} = \frac{3}{2} + \frac{\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$$

(b) $$\frac{5y-\sqrt{8y^2}}{6y} = \frac{5y-2y\sqrt{2}}{6y}, \quad y > 0 \qquad \text{Product rule}$$

$$= \frac{y\left(5-2\sqrt{2}\right)}{6y} \qquad \text{Factor the numerator.}$$

$$= \frac{5-2\sqrt{2}}{6}$$

Note that the final fraction cannot be simplified further because there is no common factor of 2 in the numerator.

Now Try Exercises 107 and 109.

CAUTION Be careful to factor *before* writing a quotient in lowest terms.

CONNECTIONS

In calculus, it is sometimes desirable to rationalize the *numerator* in an expression. The procedure is similar to rationalizing the denominator. For example, to rationalize the numerator of

$$\frac{6-\sqrt{2}}{4},$$

we multiply the numerator and the denominator by the conjugate of the numerator.

$$\frac{6-\sqrt{2}}{4}=\frac{(6-\sqrt{2})(6+\sqrt{2})}{4(6+\sqrt{2})}=\frac{36-2}{4(6+\sqrt{2})}=\frac{34}{4(6+\sqrt{2})}=\frac{17}{2(6+\sqrt{2})}$$

In the final expression, the numerator is rationalized and is in lowest terms.

For Discussion or Writing

Rationalize the numerators of the following expressions, assuming a and b are nonnegative real numbers.

1. $\dfrac{8\sqrt{5}-1}{6}$ **2.** $\dfrac{3\sqrt{a}+\sqrt{b}}{b}$ **3.** $\dfrac{3\sqrt{a}+\sqrt{b}}{\sqrt{b}-\sqrt{a}}$

4. Rationalize the denominator of the expression in Exercise 3, and then describe the difference in the procedure you used from what you did in Exercise 3.

8.5 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 13/Videotape 13

Match each part of a rule for a special product in Column I with the other part in Column II.

I	II
1. $(x + \sqrt{y})(x - \sqrt{y})$	**A.** $x - y$
2. $(\sqrt{x} + y)(\sqrt{x} - y)$	**B.** $x + 2y\sqrt{x} + y^2$
3. $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$	**C.** $x - y^2$
4. $(\sqrt{x} + \sqrt{y})^2$	**D.** $x - 2\sqrt{xy} + y$
5. $(\sqrt{x} - \sqrt{y})^2$	**E.** $x^2 - y$
6. $(\sqrt{x} + y)^2$	**F.** $x + 2\sqrt{xy} + y$

Multiply, then simplify each product. Assume all variables represent positive real numbers. See Example 1.

EXERCISES

7. $\sqrt{6}(3 + \sqrt{2})$

8. $\sqrt{2}(\sqrt{32} - \sqrt{9})$

9. $5(\sqrt{72} - \sqrt{8})$

10. $\sqrt{3}(\sqrt{12} + 2)$

11. $(\sqrt{7} + 3)(\sqrt{7} - 3)$

12. $(\sqrt{3} - 5)(\sqrt{3} + 5)$

13. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})$

14. $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$

15. $(\sqrt{8} - \sqrt{2})(\sqrt{8} + \sqrt{2})$

16. $(\sqrt{20} - \sqrt{5})(\sqrt{20} + \sqrt{5})$

17. $(\sqrt{2} + 1)(\sqrt{3} - 1)$

18. $(\sqrt{3} + 3)(\sqrt{5} - 2)$

19. $(\sqrt{11} - \sqrt{7})(\sqrt{2} + \sqrt{5})$

20. $(\sqrt{6} + \sqrt{2})(\sqrt{3} + \sqrt{2})$

21. $(2\sqrt{3} + \sqrt{5})(3\sqrt{3} - 2\sqrt{5})$

22. $(\sqrt{7} - \sqrt{11})(2\sqrt{7} + 3\sqrt{11})$

23. $(\sqrt{5} + 2)^2$

24. $(\sqrt{11} - 1)^2$

25. $(\sqrt{21} - \sqrt{5})^2$

26. $(\sqrt{6} - \sqrt{2})^2$

27. $(2 + \sqrt[3]{6})(2 - \sqrt[3]{6})$

28. $(\sqrt[3]{3} + 6)(\sqrt[3]{3} - 6)$

29. $\left(2+\sqrt[3]{2}\right)\left(4-2\sqrt[3]{2}+\sqrt[3]{4}\right)$

30. $\left(\sqrt[3]{3}-1\right)\left(\sqrt[3]{9}+\sqrt[3]{3}+1\right)$

31. $\left(3\sqrt{x}-\sqrt{5}\right)\left(2\sqrt{x}+1\right)$

32. $\left(4\sqrt{p}+\sqrt{7}\right)\left(\sqrt{p}-9\right)$

33. $\left(3\sqrt{r}-\sqrt{s}\right)\left(3\sqrt{r}+\sqrt{s}\right)$

34. $\left(\sqrt{k}+4\sqrt{m}\right)\left(\sqrt{k}-4\sqrt{m}\right)$

35. $\left(\sqrt[3]{2y}-5\right)\left(4\sqrt[3]{2y}+1\right)$

36. $\left(\sqrt[3]{9z}-2\right)\left(5\sqrt[3]{9z}+7\right)$

37. $\left(\sqrt{3x}+2\right)\left(\sqrt{3x}-2\right)$

38. $\left(\sqrt{6y}-4\right)\left(\sqrt{6y}+4\right)$

39. $\left(2\sqrt{x}+\sqrt{y}\right)\left(2\sqrt{x}-\sqrt{y}\right)$

40. $\left(\sqrt{p}+5\sqrt{s}\right)\left(\sqrt{p}-5\sqrt{s}\right)$

41. $\left[\left(\sqrt{2}+\sqrt{3}\right)-\sqrt{6}\right]\left[\left(\sqrt{2}+\sqrt{3}\right)+\sqrt{6}\right]$

42. $\left[\left(\sqrt{5}-\sqrt{2}\right)-\sqrt{3}\right]\left[\left(\sqrt{5}-\sqrt{2}\right)+\sqrt{3}\right]$

Rationalize the denominator in each expression. Assume all variables represent positive real numbers. See Examples 2 and 3.

43. $\frac{7}{\sqrt{7}}$ **44.** $\frac{11}{\sqrt{11}}$ **45.** $\frac{15}{\sqrt{3}}$ **46.** $\frac{12}{\sqrt{6}}$ **47.** $\frac{\sqrt{3}}{\sqrt{2}}$

48. $\frac{\sqrt{7}}{\sqrt{6}}$ **49.** $\frac{9\sqrt{3}}{\sqrt{5}}$ **50.** $\frac{3\sqrt{2}}{\sqrt{11}}$ **51.** $\frac{-6}{\sqrt{18}}$ **52.** $\frac{-5}{\sqrt{24}}$

53. $\sqrt{\frac{7}{2}}$ **54.** $\sqrt{\frac{10}{3}}$ **55.** $-\sqrt{\frac{7}{50}}$ **56.** $-\sqrt{\frac{13}{75}}$ **57.** $\sqrt{\frac{24}{x}}$

58. $\sqrt{\frac{52}{y}}$ **59.** $\frac{-8\sqrt{3}}{\sqrt{k}}$ **60.** $\frac{-4\sqrt{13}}{\sqrt{m}}$ **61.** $-\sqrt{\frac{150m^5}{n^3}}$ **62.** $-\sqrt{\frac{98r^3}{s^5}}$

63. $\sqrt{\frac{288x^7}{y^9}}$ **64.** $\sqrt{\frac{242t^9}{u^{11}}}$ **65.** $\frac{5\sqrt{2m}}{\sqrt{y^3}}$

66. $\frac{2\sqrt{5r}}{\sqrt{m^3}}$ **67.** $-\sqrt{\frac{48k^2}{z}}$ **68.** $-\sqrt{\frac{75m^3}{p}}$

Simplify. Assume all variables represent positive real numbers. See Example 4.

69. $\sqrt[3]{\frac{2}{3}}$ **70.** $\sqrt[3]{\frac{4}{5}}$ **71.** $\sqrt[3]{\frac{4}{9}}$ **72.** $\sqrt[3]{\frac{5}{16}}$ **73.** $\sqrt[3]{\frac{9}{32}}$

74. $\sqrt[3]{\frac{10}{9}}$ **75.** $-\sqrt[3]{\frac{2p}{r^2}}$ **76.** $-\sqrt[3]{\frac{6x}{y^2}}$ **77.** $\sqrt[3]{\frac{x^6}{y}}$ **78.** $\sqrt[3]{\frac{m^9}{q}}$

79. $\sqrt[4]{\frac{16}{x}}$ **80.** $\sqrt[4]{\frac{81}{y}}$ **81.** $\sqrt[4]{\frac{2y}{z}}$ **82.** $\sqrt[4]{\frac{7t}{s^2}}$

83. Explain the procedure you will use to rationalize the denominator of the expression in Exercise 85: $\frac{3}{4+\sqrt{5}}$. Would multiplying both the numerator and the denominator of this fraction by $4+\sqrt{5}$ lead to a rationalized denominator? Why or why not?

84. Show, in two ways, that the reciprocal of $\sqrt{6}-\sqrt{5}$ is $\sqrt{6}+\sqrt{5}$. (In general, however, the conjugate is not equal to the reciprocal.)

Rationalize the denominator in each expression. Assume all variables represent positive real numbers and no denominators are 0. *See Example 5.*

EXERCISES

85. $\dfrac{3}{4+\sqrt{5}}$

86. $\dfrac{4}{3-\sqrt{7}}$

87. $\dfrac{\sqrt{8}}{3-\sqrt{2}}$

88. $\dfrac{\sqrt{27}}{2+\sqrt{3}}$

89. $\dfrac{2}{3\sqrt{5}+2\sqrt{3}}$

90. $\dfrac{-1}{3\sqrt{2}-2\sqrt{7}}$

91. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{6}-\sqrt{5}}$

92. $\dfrac{\sqrt{5}+\sqrt{6}}{\sqrt{3}-\sqrt{2}}$

93. $\dfrac{m-4}{\sqrt{m}+2}$

94. $\dfrac{r-9}{\sqrt{r}-3}$

95. $\dfrac{4\sqrt{x}}{\sqrt{x}-2\sqrt{y}}$

96. $\dfrac{5\sqrt{r}}{3\sqrt{r}+\sqrt{s}}$

97. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

98. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

99. $\dfrac{3\sqrt{x}}{\sqrt{x}-2\sqrt{y}}$

100. $\dfrac{5\sqrt{k}}{2\sqrt{k}+\sqrt{q}}$

101. If a and b are both positive numbers and $a^2 = b^2$, then $a = b$. Use this fact to show that $\dfrac{\sqrt{6}-\sqrt{2}}{4} = \sqrt{\dfrac{2-\sqrt{3}}{2}}$.

102. Use a calculator approximation to support your result in Exercise 101.

Write each expression in lowest terms. Assume all variables represent positive real numbers. See Example 6.

EXERCISES

103. $\dfrac{30-20\sqrt{6}}{10}$

104. $\dfrac{24+12\sqrt{5}}{12}$

105. $\dfrac{3-3\sqrt{5}}{3}$

106. $\dfrac{-5+5\sqrt{2}}{5}$

107. $\dfrac{16-4\sqrt{8}}{12}$

108. $\dfrac{12-9\sqrt{72}}{18}$

109. $\dfrac{6p+\sqrt{24p^3}}{3p}$

110. $\dfrac{11y-\sqrt{242y^5}}{22y}$

Rationalize each denominator. Assume all radicals represent real numbers and no denominators are 0.

111. $\dfrac{1}{\sqrt{x+y}}$

112. $\dfrac{5}{\sqrt{m-n}}$

113. $\dfrac{p}{\sqrt{p+2}}$

114. $\dfrac{3q}{\sqrt{5+q}}$

115. The following expression occurs in a certain standard problem in trigonometry:

$$\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}.$$

Show that it simplifies to $\frac{\sqrt{6}-\sqrt{2}}{4}$. Then verify using a calculator approximation.

116. The following expression occurs in a certain standard problem in trigonometry:

$$\frac{\sqrt{3}+1}{1-\sqrt{3}}.$$

Show that it simplifies to $-2-\sqrt{3}$. Then verify using a calculator approximation.

RELATING CONCEPTS (EXERCISES 117–124)

For Individual or Group Work

In Chapter 6 we presented methods of factoring, where the terms in the factors were integers. For example, the binomial $x^2 - 9$ *is a difference of squares and factors as* $(x+3)(x-3)$*. However, we can also use this pattern to factor any binomial if we allow square root radicals in the terms of the factors. For example,* $t - 5$ *can be factored as* $(\sqrt{t} + \sqrt{5})(\sqrt{t} - \sqrt{5})$.

Similarly, we can factor any binomial as the sum or difference of cubes, using the patterns $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ *and* $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$*. For example, we can factor* $y + 2$ *and* $y - 2$ *as follows:*

$$y + 2 = \left(\sqrt[3]{y} + \sqrt[3]{2}\right)\left(\sqrt[3]{y^2} - \sqrt[3]{2y} + \sqrt[3]{4}\right)$$

$$y - 2 = \left(\sqrt[3]{y} - \sqrt[3]{2}\right)\left(\sqrt[3]{y^2} + \sqrt[3]{2y} + \sqrt[3]{4}\right).$$

Use these ideas to ***work Exercises 117–124 in order.***

117. Factor $x - 7$ as the difference of squares.

118. Factor $x - 7$ as the difference of cubes.

119. Factor $x + 7$ as the sum of cubes.

120. Use the result of Exercise 117 to rationalize the denominator of $\dfrac{x+3}{\sqrt{x} - \sqrt{7}}$.

121. Use the result of Exercise 118 to rationalize the denominator of $\dfrac{x+3}{\sqrt[3]{x} - \sqrt[3]{7}}$.

122. Use the result of Exercise 119 to rationalize the denominator of $\dfrac{x+3}{\sqrt[3]{x^2} - \sqrt[3]{7x} + \sqrt[3]{49}}$.

123. Factor the integer 2 as a difference of cubes by first writing it as $5 - 3$.

124. Use the result of Exercise 123 to rationalize the denominator of $\dfrac{2}{\sqrt[3]{5} - \sqrt[3]{3}}$.

Rationalize the numerator in each expression. Assume all variables represent positive real numbers. (Hint: See the Connections box following Example 6.)

125. $\dfrac{6 - \sqrt{2}}{4}$ **126.** $\dfrac{8\sqrt{5} - 1}{6}$ **127.** $\dfrac{3\sqrt{a} + \sqrt{b}}{b}$ **128.** $\dfrac{\sqrt{p} - 3\sqrt{q}}{4q}$

8.6 Solving Equations with Radicals

OBJECTIVES

1. Solve radical equations using the power rule.
2. Solve radical equations that require additional steps.
3. Solve radical equations with indexes greater than 2.
4. Solve radical equations using a graphing calculator.
5. Use the power rule to solve a formula for a specified variable.

OBJECTIVE 1 Solve radical equations using the power rule. An equation that includes one or more radical expressions with a variable is called a **radical equation.** Some examples of radical equations are

$$\sqrt{x-4}=8, \qquad \sqrt{5x+12}=3\sqrt{2x-1}, \qquad \text{and} \qquad \sqrt[3]{6+x}=27.$$

The equation $x = 1$ has only one solution. Its solution set is $\{1\}$. If we square both sides of this equation, we get $x^2 = 1$. This new equation has two solutions: -1 and 1. Notice that the solution of the original equation is also a solution of the squared equation. However, the squared equation has another solution, -1, that is *not* a solution of the original equation. When solving equations with radicals, we use this idea of raising both sides to a power. It is an application of the **power rule.**

Power Rule for Solving an Equation with Radicals

If both sides of an equation are raised to the same power, all solutions of the original equation are also solutions of the new equation.

AUDIO

Read the power rule carefully; it does *not* say that all solutions of the new equation are solutions of the original equation. They may or may not be. Solutions that do not satisfy the original equation are called **extraneous solutions;** they must be discarded.

CAUTION When the power rule is used to solve an equation, ***every solution of the new equation*** **must** ***be checked in the original equation.***

EXAMPLE 1 Using the Power Rule

Solve $\sqrt{3x+4} = 8$.

Use the power rule and square both sides to obtain

$$\begin{aligned} \left(\sqrt{3x+4}\right)^2 &= 8^2 \\ 3x+4 &= 64 \\ 3x &= 60 \\ x &= 20. \end{aligned}$$

To check, substitute the potential solution in the *original* equation.

Check:

$$\begin{aligned} \sqrt{3x+4} &= 8 \\ \sqrt{3 \cdot 20 + 4} &\stackrel{?}{=} 8 \qquad \text{Let } x = 20. \\ \sqrt{64} &\stackrel{?}{=} 8 \\ 8 &= 8 \qquad \text{True} \end{aligned}$$

Since 20 satisfies the *original* equation, the solution set is $\{20\}$.

Now Try Exercise 9.

The solution of the equation in Example 1 can be generalized to give a method for solving equations with radicals.

Solving an Equation with Radicals

Step 1 **Isolate the radical.** Make sure that one radical term is alone on one side of the equation.

Step 2 **Apply the power rule.** Raise both sides of the equation to a power that is the same as the index of the radical.

Step 3 **Solve** the resulting equation; if it still contains a radical, repeat Steps 1 and 2.

Step 4 **Check** all potential solutions in the original equation.

AUDIO

CAUTION Remember Step 4 or you may get an incorrect solution set.

VIDEO YOU TRY IT

EXAMPLE 2 Using the Power Rule

Solve $\sqrt{5q-1} + 3 = 0$.

Step 1 To get the radical alone on one side, subtract 3 from each side.

$$\sqrt{5q-1} = -3$$

Step 2 Now square both sides.

$$\left(\sqrt{5q-1}\right)^2 = (-3)^2$$

Step 3

$$5q - 1 = 9$$
$$5q = 10$$
$$q = 2$$

Step 4 Check the potential solution, 2, by substituting it in the original equation.

Check:

$$\sqrt{5q-1} + 3 = 0$$
$$\sqrt{5 \cdot 2 - 1} + 3 = 0 \quad ? \qquad \text{Let } q = 2.$$
$$3 + 3 = 0 \qquad \text{False}$$

This false result shows that 2 is *not* a solution of the original equation; it is extraneous. The solution set is $\emptyset$.

Now Try Exercise 11.

NOTE We could have determined after Step 1 that the equation in Example 2 has no solution because the expression on the left cannot be negative.

OBJECTIVE 2 Solve radical equations that require additional steps. The next examples involve finding the square of a binomial. Recall that

$$(x + y)^2 = x^2 + 2xy + y^2.$$

EXPLORE

EXAMPLE 3 Using the Power Rule; Squaring a Binomial

Solve $\sqrt{4-x} = x + 2$.

Step 1 The radical is alone on the left side of the equation.

Step 2 Square both sides; the square of $x + 2$ is $(x + 2)^2 = x^2 + 4x + 4$.

$$\left(\sqrt{4-x}\right)^2 = (x+2)^2$$
$$4 - x = x^2 + \underbrace{4x}_{\text{Twice the product of 2 and } x} + 4$$

Step 3 The new equation is quadratic, so get 0 on one side.

$$0 = x^2 + 5x \qquad \text{Subtract 4 and add } x.$$
$$0 = x(x + 5) \qquad \text{Factor.}$$
$$x = 0 \quad \text{or} \quad x + 5 = 0 \qquad \text{Zero-factor property}$$
$$x = -5$$

Step 4 Check each potential solution in the original equation.

Check:

If $x = 0$, then	If $x = -5$, then
$\sqrt{4-x} = x + 2$	$\sqrt{4-x} = x + 2$
$\sqrt{4-0} = 0 + 2 \quad ?$	$\sqrt{4-(-5)} = -5 + 2 \quad ?$
$\sqrt{4} = 2 \quad ?$	$\sqrt{9} = -3 \quad ?$
$2 = 2.$ True	$3 = -3.$ False

The solution set is $\{0\}$. The other potential solution, -5, is extraneous.

Now Try Exercise 27.

CAUTION When a radical equation requires squaring a binomial as in Example 3, remember to include the middle term.

$(x+2)^2 \neq x^2 + 4$	$(x+2)^2 = x^2 + 4x + 4$
INCORRECT	**CORRECT**

EXAMPLE 4 Using the Power Rule; Squaring a Binomial

Solve $\sqrt{x^2 - 4x + 9} = x - 1.$

Squaring both sides gives $(x-1)^2 = x^2 - 2(x)(1) + 1^2$ on the right.

$$\left(\sqrt{x^2 - 4x + 9}\right)^2 = (x-1)^2$$

$$x^2 - 4x + 9 = x^2 - 2x + 1$$

Twice the product of x and -1

Subtract x^2 and 1 from each side; then add $4x$ to each side to obtain

$$8 = 2x$$

$$4 = x.$$

Check this potential solution in the original equation.

Check:

$$\sqrt{x^2 - 4x + 9} = x - 1$$

$$\sqrt{4^2 - 4 \cdot 4 + 9} = 4 - 1 \quad ? \quad \text{Let } x = 4.$$

$$3 = 3 \quad \text{True}$$

The solution set of the original equation is $\{4\}$.

Now Try Exercise 29.

EXAMPLE 5 Using the Power Rule; Squaring Twice

Solve $\sqrt{5x + 6} + \sqrt{3x + 4} = 2.$

Start by getting one radical alone on one side of the equation by subtracting $\sqrt{3x + 4}$ from each side.

$$\sqrt{5x + 6} = 2 - \sqrt{3x + 4}$$

$$\left(\sqrt{5x + 6}\right)^2 = \left(2 - \sqrt{3x + 4}\right)^2 \qquad \text{Square both sides.}$$

$$5x + 6 = 4 - 4\sqrt{3x + 4} + (3x + 4)$$

Twice the product of 2 and $-\sqrt{3x + 4}$

This equation still contains a radical, so square both sides again. Before doing this, isolate the radical term on the right.

$$5x + 6 = 8 + 3x - 4\sqrt{3x + 4}$$

$$2x - 2 = -4\sqrt{3x + 4} \qquad \text{Subtract 8 and } 3x.$$

$$x - 1 = -2\sqrt{3x + 4} \qquad \text{Divide by 2.}$$

$$(x-1)^2 = \left(-2\sqrt{3x + 4}\right)^2 \qquad \text{Square both sides again.}$$

$$x^2 - 2x + 1 = (-2)^2\left(\sqrt{3x + 4}\right)^2 \qquad (ab)^2 = a^2b^2$$

$$x^2 - 2x + 1 = 4(3x + 4)$$

$$x^2 - 2x + 1 = 12x + 16 \qquad \text{Distributive property}$$

$$\begin{aligned} x^2 - 14x - 15 &= 0 && \text{Standard form} \\ (x - 15)(x + 1) &= 0 && \text{Factor.} \\ x - 15 = 0 \quad \text{or} \quad x + 1 &= 0 && \text{Zero-factor property} \\ x = 15 \quad \text{or} \quad x &= -1 \end{aligned}$$

Check each of these potential solutions in the original equation. Only -1 checks, so the solution set, $\{-1\}$, has only one element.

Now Try Exercise 51.

OBJECTIVE 3 Solve radical equations with indexes greater than 2. The power rule also works for powers greater than 2.

VIDEO

EXAMPLE 6 Using the Power Rule for a Power Greater than 2

Solve $\sqrt[3]{z + 5} = \sqrt[3]{2z - 6}$.

Raise both sides to the third power.

$$\begin{aligned} \left(\sqrt[3]{z + 5}\right)^3 &= \left(\sqrt[3]{2z - 6}\right)^3 \\ z + 5 &= 2z - 6 \\ 11 &= z \end{aligned}$$

Check:

$$\begin{aligned} \sqrt[3]{z + 5} &= \sqrt[3]{2z - 6} && \text{Original equation} \\ \sqrt[3]{11 + 5} &\stackrel{?}{=} \sqrt[3]{2 \cdot 11 - 6} && \text{Let } z = 11. \\ \sqrt[3]{16} &= \sqrt[3]{16} && \text{True} \end{aligned}$$

The solution set is $\{11\}$.

Now Try Exercise 37.

OBJECTIVE 4 Solve radical equations using a graphing calculator. In Example 4 we solved the equation $\sqrt{x^2 - 4x + 9} = x - 1$ using algebraic methods. If we write this equation with one side equal to 0, we get

$$\sqrt{x^2 - 4x + 9} - x + 1 = 0.$$

Using a graphing calculator to graph the function defined by

$$f(x) = \sqrt{x^2 - 4x + 9} - x + 1,$$

we obtain the graph shown in Figure 14. Notice that its zero (x-value of the x-intercept) is 4, which is the solution we found in Example 4.

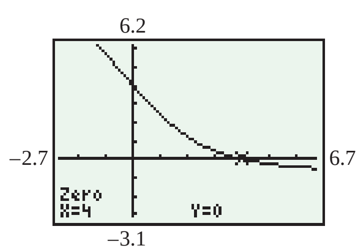


FIGURE 14

In Example 3, we found that the single solution of $\sqrt{4 - x} = x + 2$ is 0, with an extraneous value of -5. If we graph $f(x) = \sqrt{4 - x}$ and $g(x) = x + 2$ in the same window, we find that the x-coordinate of the point of intersection of the two graphs is 0, which is the solution of the equation. See Figure 15.

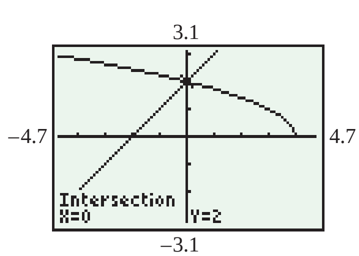
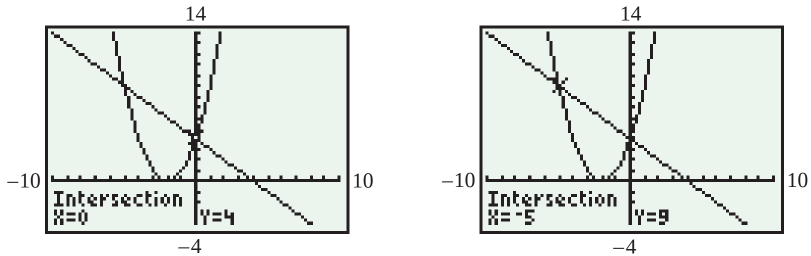


FIGURE 15

We solved the equation in Example 3 by squaring both sides, obtaining $4 - x = x^2 + 4x + 4$. In Figure 16 on the next page, we show that the two functions defined by $f(x) = 4 - x$ and $g(x) = x^2 + 4x + 4$ have two points of intersection. The extraneous value -5 that we found in Example 3 shows up as an x-value of one of these points of intersection. However, our check showed that -5 was not a solution of the *original* equation (before the squaring step). Here we see a graphical interpretation of the extraneous value.



0 is a solution of the original equation.

−5 is *not* a solution of the original equation.

FIGURE 16

OBJECTIVE 5 Use the power rule to solve a formula for a specified variable.

Photo not available

VIDEO

EXAMPLE 7 Solving a Formula from Electronics for a Variable

An important property of a radio frequency transmission line is its *characteristic impedance*, represented by Z and measured in ohms. If L and C are the inductance and capacitance, respectively, per unit of length of the line, then these quantities are related by the formula $Z = \sqrt{\frac{L}{C}}$. Solve this formula for C.

$$Z = \sqrt{\frac{L}{C}} \qquad \text{Given formula}$$

$$Z^2 = \frac{L}{C} \qquad \text{Square both sides.}$$

$$CZ^2 = L \qquad \text{Multiply by } C.$$

$$C = \frac{L}{Z^2} \qquad \text{Divide by } Z^2.$$

Now Try Exercise 67.

8.6 EXERCISES

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Check each equation to see if the given value for x is a solution.

1. $\sqrt{3x+18}=x$

(a) 6 **(b)** -3

2. $\sqrt{3x-3}=x-1$

(a) 1 **(b)** 4

3. $\sqrt{x+2}=\sqrt{9x-2}-2\sqrt{x-1}$

(a) 2 **(b)** 7

4. $\sqrt{8x-3}=2x$

(a) $\frac{3}{2}$ **(b)** $\frac{1}{2}$

5. Is 9 a solution of the equation $\sqrt{x}=-3$? If not, what is the solution of this equation? Explain.

6. Before even attempting to solve $\sqrt{3x+18}=x$, how can you be sure that the equation cannot have a negative solution?

Solve each equation. See Examples 1–4.

EXERCISES

7. $\sqrt{r-2}=3$
8. $\sqrt{q+1}=7$
9. $\sqrt{6k-1}=1$
10. $\sqrt{7m-3}=5$
11. $\sqrt{4r+3}+1=0$
12. $\sqrt{5k-3}+2=0$
13. $\sqrt{3k+1}-4=0$
14. $\sqrt{5z+1}-11=0$
15. $4-\sqrt{x-2}=0$
16. $9-\sqrt{4k+1}=0$
17. $\sqrt{9a-4}=\sqrt{8a+1}$
18. $\sqrt{4p-2}=\sqrt{3p+5}$
19. $2\sqrt{x}=\sqrt{3x+4}$
20. $2\sqrt{m}=\sqrt{5m-16}$
21. $3\sqrt{z-1}=2\sqrt{2z+2}$
22. $5\sqrt{4a+1}=3\sqrt{10a+25}$
23. $k=\sqrt{k^2+4k-20}$
24. $p=\sqrt{p^2-3p+18}$
25. $a=\sqrt{a^2+3a+9}$
26. $z=\sqrt{z^2-4z-8}$
27. $\sqrt{9-x}=x+3$
28. $\sqrt{5-x}=x+1$
29. $\sqrt{k^2+2k+9}=k+3$
30. $\sqrt{a^2-3a+3}=a-1$
31. $\sqrt{r^2+9r+3}=-r$
32. $\sqrt{p^2-15p+15}=p-5$
33. $\sqrt{z^2+12z-4}+4-z=0$
34. $\sqrt{m^2+3m+12}-m-2=0$

35. What is *wrong* with this first step in the solution process for $\sqrt{3x+4}=8-x$. Solve it correctly.

$$3x+4=64+x^2$$

36. Explain what is *wrong* with this first step in the solution process for the equation $\sqrt{5x+6}-\sqrt{x+3}=3$. Then solve it correctly.

$$(5x+6)+(x+3)=9$$

Solve each equation. See Examples 5 and 6.

EXERCISES

37. $\sqrt[3]{2x+5}=\sqrt[3]{6x+1}$
38. $\sqrt[3]{p-1}=2$
39. $\sqrt[3]{a^2+5a+1}=\sqrt[3]{a^2+4a}$
40. $\sqrt[3]{r^2+2r+8}=\sqrt[3]{r^2}$
41. $\sqrt[3]{2m-1}=\sqrt[3]{m+13}$
42. $\sqrt[3]{2k-11}-\sqrt[3]{5k+1}=0$
43. $\sqrt[4]{a+8}=\sqrt[4]{2a}$
44. $\sqrt[4]{z+11}=\sqrt[4]{2z+6}$
45. $\sqrt[3]{x-8}+2=0$
46. $\sqrt[3]{r+1}+1=0$
47. $\sqrt[4]{2k-5}+4=0$
48. $\sqrt[4]{8z-3}+2=0$
49. $\sqrt{k+2}-\sqrt{k-3}=1$
50. $\sqrt{r+6}-\sqrt{r-2}=2$
51. $\sqrt{2r+11}-\sqrt{5r+1}=-1$
52. $\sqrt{3x-2}-\sqrt{x+3}=1$
53. $\sqrt{3p+4}-\sqrt{2p-4}=2$
54. $\sqrt{4x+5}-\sqrt{2x+2}=1$
55. $\sqrt{3-3p}-3=\sqrt{3p+2}$
56. $\sqrt{4x+7}-4=\sqrt{4x-1}$
57. $\sqrt{2\sqrt{x+11}}=\sqrt{4x+2}$
58. $\sqrt{1+\sqrt{24-10x}}=\sqrt{3x+5}$
59. What is the smallest power to which you can raise both sides of the radical equation $\sqrt{x+3}=\sqrt[3]{5+4x}$ so that the radicals are eliminated?
60. What is the smallest power to which you can raise both sides of the radical equation $\sqrt[4]{x+3}=\sqrt[3]{10x+14}$ so that the radicals are eliminated?

61. Use a graphing calculator to solve $\sqrt{3-3x} = 3 + \sqrt{3x+2}$. What is the domain of $y = \sqrt{3-3x} - 3 - \sqrt{3x+2}$?

62. Use a graphing calculator with a window of $[-1, 4]$ by $[-1, 3]$ to solve $\sqrt{2\sqrt{7x+2}} = \sqrt{3x+2}$. What is the domain of $f(x) = \sqrt{2\sqrt{7x+2}} - \sqrt{3x+2}$?

For each equation, rewrite the expressions with rational exponents as radical expressions, and then solve using the procedures explained in this section.

63. $(2x-9)^{1/2} = 2 + (x-8)^{1/2}$

64. $(3w+7)^{1/2} = 1 + (w+2)^{1/2}$

65. $(2w-1)^{2/3} - w^{1/3} = 0$

66. $(x^2-2x)^{1/3} - x^{1/3} = 0$

Solve each formula from electricity and radio for the indicated variable. See Example 7. (Source: Cooke, Nelson M., and Joseph B. Orleans, Mathematics Essential to Electricity and Radio, *McGraw-Hill, 1943.)*

67. $V = \sqrt{\dfrac{2K}{m}}$ for K

68. $V = \sqrt{\dfrac{2K}{m}}$ for m

69. $f = \dfrac{1}{2\pi\sqrt{LC}}$ for L

70. $r = \sqrt{\dfrac{Mm}{F}}$ for F

71. A number of useful formulas involve radicals or radical expressions. Many occur in the mathematics needed for working with objects in space. The formula

$$N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$$

is used to find the rotational rate N of a space station. Here a is the acceleration and r represents the radius of the space station in meters. To find the value of r that will make N simulate the effect of gravity on Earth, the equation must be solved for r, using the required value of N. (*Source:* Kastner, Bernice, *Space Mathematics,* NASA, 1972.)

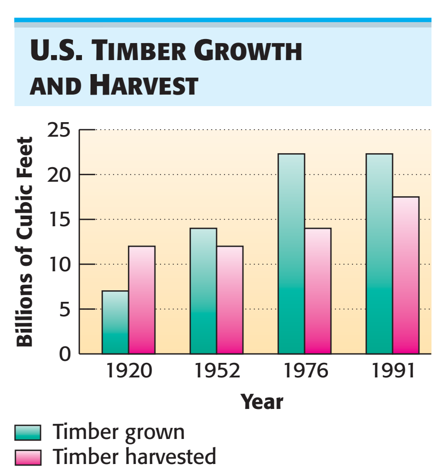
(a) Solve the equation for r.
(b) Find the value of r that makes $N = .063$ rotation per sec, if $a = 9.8$ m per sec^2.
(c) Find the value of r that makes $N = .04$ rotation per sec, if $a = 9.8$ m per sec^2.

If x is the number of years since 1900, the equation $y = x^{.7}$ approximates the timber grown in the United States in billions of cubic feet. Let $x = 20$ represent 1920, $x = 52$ represent 1952, and so on.

72. Replace x in the equation for each year shown in the graph and use a calculator to find the value of y. (Round answers to the nearest billion.)

73. Use the graph to estimate the amount of timber grown for each year shown.

74. Compare the values found from the equation with your estimates from the graph. Does the equation give a good approximation of the data from the graph? In which year is the approximation best?



Source: Figures from U.S. Forest Service. Adapted from *"The Truth about America's Forests,"* Evergreen, 4025 Crater Lake Hwy., Medford, Ore. 97504.

75. From the graph, estimate the amount of timber harvested in each year shown.

76. Use the equation $y = x^{.62}$ and a calculator to approximate the amount of timber harvested in each of the given years. (Round answers to the nearest billion.)

77. Compare your answers from Exercises 75 and 76. Does the equation give a good approximation? For which year is it poorest?

Photo not available

8.7 EXERCISES

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Decide whether each expression is equal to 1, −1, i, or −i.

1. $\sqrt{-1}$ **2.** $-\sqrt{-1}$ **3.** i^2 **4.** $-i^2$ **5.** $\dfrac{1}{i}$ **6.** $(-i)^2$

Write each number as a product of a real number and i. Simplify all radical expressions. See Example 1.

7. $\sqrt{-169}$ **8.** $\sqrt{-225}$ **9.** $-\sqrt{-144}$ **10.** $-\sqrt{-196}$

11. $\sqrt{-5}$ **12.** $\sqrt{-21}$ **13.** $\sqrt{-48}$ **14.** $\sqrt{-96}$

Multiply or divide as indicated. See Examples 2 and 3.

15. $\sqrt{-7} \cdot \sqrt{-15}$ **16.** $\sqrt{-3} \cdot \sqrt{-19}$ **17.** $\sqrt{-4} \cdot \sqrt{-25}$ **18.** $\sqrt{-9} \cdot \sqrt{-81}$

19. $\sqrt{-3} \cdot \sqrt{11}$ **20.** $\sqrt{-10} \cdot \sqrt{2}$ **21.** $\dfrac{\sqrt{-300}}{\sqrt{-100}}$ **22.** $\dfrac{\sqrt{-40}}{\sqrt{-10}}$

23. $\dfrac{\sqrt{-75}}{\sqrt{3}}$ **24.** $\dfrac{\sqrt{-160}}{\sqrt{10}}$

25. **(a)** Every real number is a complex number. Explain why this is so.
(b) Not every complex number is a real number. Give an example of this and explain why this statement is true.

26. Explain how to add, subtract, multiply, and divide complex numbers. Give examples.

Add or subtract as indicated. Write your answers in the form a + bi. See Examples 4 and 5.

27. $(3 + 2i) + (-4 + 5i)$ **28.** $(7 + 15i) + (-11 + 14i)$

29. $(5 - i) + (-5 + i)$ **30.** $(-2 + 6i) + (2 - 6i)$

31. $(4 + i) - (-3 - 2i)$ **32.** $(9 + i) - (3 + 2i)$

33. $(-3 - 4i) - (-1 - 4i)$ **34.** $(-2 - 3i) - (-5 - 3i)$

35. $(-4 + 11i) + (-2 - 4i) + (7 + 6i)$ **36.** $(-1 + i) + (2 + 5i) + (3 + 2i)$

37. $[(7 + 3i) - (4 - 2i)] + (3 + i)$ **38.** $[(7 + 2i) + (-4 - i)] - (2 + 5i)$

39. Fill in the blank with the correct response:

Because $(4 + 2i) - (3 + i) = 1 + i$, using the definition of subtraction, we can check this to find that $(1 + i) + (3 + i) =$ ______________.

40. Fill in the blank with the correct response:

Because $\dfrac{-5}{2 - i} = -2 - i$, using the definition of division, we can check this to find that $(-2 - i)(2 - i) =$ ______________.

Multiply. See Example 6.

41. $(3i)(27i)$ **42.** $(5i)(125i)$ **43.** $(-8i)(-2i)$

44. $(-32i)(-2i)$ **45.** $5i(-6 + 2i)$ **46.** $3i(4 + 9i)$

47. $(4 + 3i)(1 - 2i)$ **48.** $(7 - 2i)(3 + i)$ **49.** $(4 + 5i)^2$

50. $(3 + 2i)^2$ **51.** $2i(-4 - i)^2$ **52.** $3i(-3 - i)^2$

53. $(12 + 3i)(12 - 3i)$ **54.** $(6 + 7i)(6 - 7i)$ **55.** $(4 + 9i)(4 - 9i)$

56. $(7 + 2i)(7 - 2i)$

57. What is the conjugate of $a + bi$?

58. If we multiply $a + bi$ by its conjugate, we get ______________, which is always a real number.

Write each expression in standard form $a + bi$. See Example 7.

59. $\dfrac{2}{1 - i}$ **60.** $\dfrac{29}{5 + 2i}$ **61.** $\dfrac{-7 + 4i}{3 + 2i}$ **62.** $\dfrac{-38 - 8i}{7 + 3i}$

63. $\dfrac{8i}{2 + 2i}$ **64.** $\dfrac{-8i}{1 + i}$ **65.** $\dfrac{2 - 3i}{2 + 3i}$ **66.** $\dfrac{-1 + 5i}{3 + 2i}$

67. $\dfrac{3 + i}{i}$ **68.** $\dfrac{5 - i}{-i}$

TECHNOLOGY INSIGHTS (EXERCISES 69–70)

Predict the answer that the calculator screen will provide for the given complex number operation entry.

69.
```
(8-5i)+2(11+6i)-
(1+i)²
```

70.
```
(4+3i)(2-6i)-2i(
5+7i)
```

71. Recall that if $a \neq 0$, then $\frac{1}{a}$ is called the reciprocal of a. Use this definition to express the reciprocal of $5 - 4i$ in the form $a + bi$.

72. Recall that if $a \neq 0$, then a^{-1} is defined to be $\frac{1}{a}$. Use this definition to express $(4 - 3i)^{-1}$ in the form $a + bi$.

Find each power of i. See Example 8.

73. i^{18} **74.** i^{26} **75.** i^{89} **76.** i^{48} **77.** i^{38}

78. i^{102} **79.** i^{43} **80.** i^{83} **81.** i^{-5} **82.** i^{-17}

83. A student simplified i^{-18} as follows:

$$i^{-18} = i^{-18} \cdot i^{20} = i^{-18+20} = i^2 = -1.$$

Explain the mathematical justification for this correct work.

84. Explain why

$$(46 + 25i)(3 - 6i) \quad \text{and} \quad (46 + 25i)(3 - 6i)i^{12}$$

must be equal. (Do not actually perform the computation.)

Ohm's law for the current I in a circuit with voltage E, resistance R, capacitance reactance X_c, and inductive reactance X_L is

$$I = \frac{E}{R + (X_L - X_c)i}.$$

Use this law to work Exercises 85 and 86.

85. Find I if $E = 2 + 3i$, $R = 5$, $X_L = 4$, and $X_c = 3$.

86. Find E if $I = 1 - i$, $R = 2$, $X_L = 3$, and $X_c = 1$.

Complex numbers will appear again in this book in Chapter 9, when we study quadratic equations. The following exercises examine how a complex number can be a solution of a quadratic equation.

87. Show that $1 + 5i$ is a solution of $x^2 - 2x + 26 = 0$. Then show that its conjugate is also a solution.

88. Show that $3 + 2i$ is a solution of $x^2 - 6x + 13 = 0$. Then show that its conjugate is also a solution.

RELATING CONCEPTS (EXERCISES 89–94)

For Individual or Group Work

Consider the following expressions:

Binomials	**Complex Numbers**
$x + 2$, $3x - 1$	$1 + 2i$, $3 - i$

When we add, subtract, or multiply complex numbers in standard form, the rules are the same as those for the corresponding operations on binomials. That is, we add or subtract like terms, and we use FOIL to multiply. Division, however, is comparable to division by the sum or difference of radicals, where we multiply by the conjugate to get a rational denominator. To express the quotient of two complex numbers in standard form, we also multiply by the conjugate of the denominator. ***Work Exercises 89–94 in order,*** *to better understand these ideas.*

89. (a) Add the two binomials. **(b)** Add the two complex numbers.

(continued)

90. **(a)** Subtract the second binomial from the first.
(b) Subtract the second complex number from the first.

91. **(a)** Multiply the two binomials.
(b) Multiply the two complex numbers.

92. **(a)** Rationalize the denominator: $\frac{\sqrt{3}-1}{1+\sqrt{2}}$.
(b) Write in standard form: $\frac{3-i}{1+2i}$.

93. Explain why the answers for (a) and (b) in Exercise 91 do not correspond as the answers in Exercises 89–90 do.

94. Explain why the answers for (a) and (b) in Exercise 92 do not correspond as the answers in Exercises 89–90 do.

Perform the indicated operations. Give answers in standard form.

95. $\frac{3}{2-i}+\frac{5}{1+i}$

96. $\frac{2}{3+4i}+\frac{4}{1-i}$

97. $\left(\frac{2+i}{2-i}+\frac{i}{1+i}\right)i$

98. $\left(\frac{4-i}{1+i}-\frac{2i}{2+i}\right)4i$

8.7 EXERCISES

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Decide whether each expression is equal to 1, −1, i, or −i.

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Write each number as a product of a real number and i. Simplify all radical expressions. See Example 1.

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Multiply or divide as indicated. See Examples 2 and 3.

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23. $\frac{\sqrt{-75}}{\sqrt{3}}$ **24.** $\frac{\sqrt{-160}}{\sqrt{10}}$

25. **(a)** Every real number is a complex number. Explain why this is so.
(b) Not every complex number is a real number. Give an example of this and explain why this statement is true.

26. Explain how to add, subtract, multiply, and divide complex numbers. Give examples.

Add or subtract as indicated. Write your answers in the form a + bi. See Examples 4 and 5.

27. $(3 + 2i) + (-4 + 5i)$

28. $(7 + 15i) + (-11 + 14i)$

29. $(5 - i) + (-5 + i)$

30. $(-2 + 6i) + (2 - 6i)$

31. $(4 + i) - (-3 - 2i)$

32. $(9 + i) - (3 + 2i)$

33. $(-3 - 4i) - (-1 - 4i)$

34. $(-2 - 3i) - (-5 - 3i)$

35. $(-4 + 11i) + (-2 - 4i) + (7 + 6i)$

36. $(-1 + i) + (2 + 5i) + (3 + 2i)$

37. $[(7 + 3i) - (4 - 2i)] + (3 + i)$

38. $[(7 + 2i) + (-4 - i)] - (2 + 5i)$

39. Fill in the blank with the correct response:

Because $(4 + 2i) - (3 + i) = 1 + i$, using the definition of subtraction, we can check this to find that $(1 + i) + (3 + i) =$ ____________.

40. Fill in the blank with the correct response:

Because $\frac{-5}{2 - i} = -2 - i$, using the definition of division, we can check this to find that $(-2 - i)(2 - i) =$ ____________.

Multiply. See Example 6.

EXERCISES

41. $(3i)(27i)$ **42.** $(5i)(125i)$ **43.** $(-8i)(-2i)$

44. $(-32i)(-2i)$ **45.** $5i(-6 + 2i)$ **46.** $3i(4 + 9i)$

47. $(4 + 3i)(1 - 2i)$ **48.** $(7 - 2i)(3 + i)$ **49.** $(4 + 5i)^2$

50. $(3 + 2i)^2$ **51.** $2i(-4 - i)^2$ **52.** $3i(-3 - i)^2$

53. $(12 + 3i)(12 - 3i)$ **54.** $(6 + 7i)(6 - 7i)$ **55.** $(4 + 9i)(4 - 9i)$

56. $(7 + 2i)(7 - 2i)$

57. What is the conjugate of $a + bi$?

58. If we multiply $a + bi$ by its conjugate, we get ____________, which is always a real number.

Write each expression in standard form $a + bi$. See Example 7.

EXERCISES

59. $\frac{2}{1 - i}$ **60.** $\frac{29}{5 + 2i}$ **61.** $\frac{-7 + 4i}{3 + 2i}$ **62.** $\frac{-38 - 8i}{7 + 3i}$

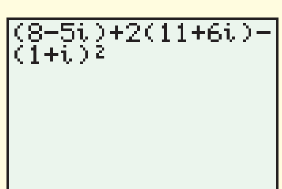
63. $\frac{8i}{2 + 2i}$ **64.** $\frac{-8i}{1 + i}$ **65.** $\frac{2 - 3i}{2 + 3i}$ **66.** $\frac{-1 + 5i}{3 + 2i}$

67. $\frac{3 + i}{i}$ **68.** $\frac{5 - i}{-i}$

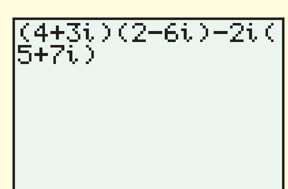
TECHNOLOGY INSIGHTS (EXERCISES 69–70)

Predict the answer that the calculator screen will provide for the given complex number operation entry.

69.



70.



71. Recall that if $a \neq 0$, then $\frac{1}{a}$ is called the reciprocal of a. Use this definition to express the reciprocal of $5 - 4i$ in the form $a + bi$.

72. Recall that if $a \neq 0$, then a^{-1} is defined to be $\frac{1}{a}$. Use this definition to express $(4 - 3i)^{-1}$ in the form $a + bi$.

Find each power of i. See Example 8.

73. i^{18}	**74.** i^{26}	**75.** i^{89}	**76.** i^{48}	**77.** i^{38}
78. i^{102}	**79.** i^{43}	**80.** i^{83}	**81.** i^{-5}	**82.** i^{-17}

83. A student simplified i^{-18} as follows:

$$i^{-18} = i^{-18} \cdot i^{20} = i^{-18+20} = i^2 = -1.$$

Explain the mathematical justification for this correct work.

84. Explain why

$$(46 + 25i)(3 - 6i) \quad \text{and} \quad (46 + 25i)(3 - 6i)i^{12}$$

must be equal. (Do not actually perform the computation.)

Ohm's law for the current I in a circuit with voltage E, resistance R, capacitance reactance X_c, and inductive reactance X_L is

$$I = \frac{E}{R + (X_L - X_c)i}.$$

Use this law to work Exercises 85 and 86.

85. Find I if $E = 2 + 3i$, $R = 5$, $X_L = 4$, and $X_c = 3$.

86. Find E if $I = 1 - i$, $R = 2$, $X_L = 3$, and $X_c = 1$.

Complex numbers will appear again in this book in Chapter 9, when we study quadratic equations. The following exercises examine how a complex number can be a solution of a quadratic equation.

87. Show that $1 + 5i$ is a solution of $x^2 - 2x + 26 = 0$. Then show that its conjugate is also a solution.

88. Show that $3 + 2i$ is a solution of $x^2 - 6x + 13 = 0$. Then show that its conjugate is also a solution.

RELATING CONCEPTS (EXERCISES 89–94)

For Individual or Group Work

Consider the following expressions:

Binomials	**Complex Numbers**
$x + 2, \quad 3x - 1$	$1 + 2i, \quad 3 - i$

When we add, subtract, or multiply complex numbers in standard form, the rules are the same as those for the corresponding operations on binomials. That is, we add or subtract like terms, and we use FOIL to multiply. Division, however, is comparable to division by the sum or difference of radicals, where we multiply by the conjugate to get a rational denominator. To express the quotient of two complex numbers in standard form, we also multiply by the conjugate of the denominator. ***Work Exercises 89–94 in order,*** *to better understand these ideas.*

89. (a) Add the two binomials. **(b)** Add the two complex numbers.

(continued)

90. **(a)** Subtract the second binomial from the first.
(b) Subtract the second complex number from the first.

91. **(a)** Multiply the two binomials.
(b) Multiply the two complex numbers.

92. **(a)** Rationalize the denominator: $\frac{\sqrt{3}-1}{1+\sqrt{2}}$.
(b) Write in standard form: $\frac{3-i}{1+2i}$.

93. Explain why the answers for (a) and (b) in Exercise 91 do not correspond as the answers in Exercises 89–90 do.

94. Explain why the answers for (a) and (b) in Exercise 92 do not correspond as the answers in Exercises 89–90 do.

Perform the indicated operations. Give answers in standard form.

95. $\frac{3}{2-i}+\frac{5}{1+i}$

96. $\frac{2}{3+4i}+\frac{4}{1-i}$

97. $\left(\frac{2+i}{2-i}+\frac{i}{1+i}\right)i$

98. $\left(\frac{4-i}{1+i}-\frac{2i}{2+i}\right)4i$

9.1 The Square Root Property and Completing the Square

OBJECTIVES

1. Review the zero-factor property.
2. Learn the square root property.
3. Solve quadratic equations of the form $(ax + b)^2 = c$ by using the square root property.
4. Solve quadratic equations by completing the square.
5. Solve quadratic equations with solutions that are not real numbers.

We introduced quadratic equations in Section 6.5. Recall that a *quadratic equation* is defined as follows.

Quadratic Equation

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, with $a \neq 0$, is a **quadratic equation.** The given form is called **standard form.**

A quadratic equation is a *second-degree equation,* that is, an equation with a squared term and no terms of higher degree. For example,

$$4m^2 + 4m - 5 = 0 \qquad \text{and} \qquad 3x^2 = 4x - 8$$

are quadratic equations, with the first equation in standard form.

OBJECTIVE 1 Review the zero-factor property. In Section 6.5 we used factoring and the zero-factor property to solve quadratic equations.

Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if $ab = 0$, then $a = 0$ or $b = 0$.

We solved a quadratic equation such as $3x^2 - 5x - 28 = 0$ using the zero-factor property as follows.

$$\begin{aligned}
3x^2 - 5x - 28 &= 0 \\
(3x + 7)(x - 4) &= 0 && \text{Factor.} \\
3x + 7 = 0 \quad \text{or} \quad x - 4 &= 0 && \text{Zero-factor property} \\
3x = -7 \quad \text{or} \quad x &= 4 && \text{Solve each equation.} \\
x = -\frac{7}{3} &
\end{aligned}$$

The solution set is $\left\{-\frac{7}{3}, 4\right\}$.

OBJECTIVE 2 Learn the square root property. Although factoring is the simplest way to solve quadratic equations, not every quadratic equation can be solved easily by factoring. In this section and the next, we develop three other methods of solving quadratic equations based on the following property.

Square Root Property

If x and k are complex numbers and $x^2 = k$, then

$$\boldsymbol{x = \sqrt{k}} \quad \text{or} \quad \boldsymbol{x = -\sqrt{k}.}$$

The following steps justify the square root property.

$$\begin{aligned}
x^2 &= k && \\
x^2 - k &= 0 && \text{Subtract } k. \\
(x - \sqrt{k})(x + \sqrt{k}) &= 0 && \text{Factor.} \\
x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} &= 0 && \text{Zero-factor property} \\
\boldsymbol{x = \sqrt{k}} \quad \text{or} \quad \boldsymbol{x} &\boldsymbol{= -\sqrt{k}} && \text{Solve each equation.}
\end{aligned}$$

CAUTION Remember that if $k \neq 0$, using the square root property always produces *two* square roots, one positive and one negative.

EXAMPLE 1 Using the Square Root Property

Solve each equation.

(a) $x^2 = 5$

By the square root property,

$$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5},$$

and the solution set is $\{\sqrt{5}, -\sqrt{5}\}$.

(b) $4x^2 - 48 = 0$

Solve for x^2.

$$\begin{aligned}
4x^2 - 48 &= 0 && \\
4x^2 &= 48 && \text{Add 48.} \\
x^2 &= 12 && \text{Divide by 4.} \\
x = \sqrt{12} \quad \text{or} \quad x &= -\sqrt{12} && \text{Square root property} \\
x = 2\sqrt{3} \quad \text{or} \quad x &= -2\sqrt{3} && \sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}
\end{aligned}$$

The solutions are $2\sqrt{3}$ and $-2\sqrt{3}$. Check each in the original equation.

Check: $4x^2 - 48 = 0$ Original equation

$$\begin{aligned}
4(2\sqrt{3})^2 - 48 &= 0 \quad ? \\
4(12) - 48 &= 0 \quad ? \\
48 - 48 &= 0 \quad ? \\
0 &= 0 \quad \text{True}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
4(-2\sqrt{3})^2 - 48 &= 0 \quad ? \\
4(12) - 48 &= 0 \quad ? \\
48 - 48 &= 0 \quad ? \\
0 &= 0 \quad \text{True}
\end{aligned}$$

The solution set is $\{2\sqrt{3}, -2\sqrt{3}\}$.

Now Try Exercises 7 and 13.

NOTE Recall that solutions such as those in Example 1 are sometimes abbreviated with the symbol $\pm$ (read "positive or negative"); with this symbol the solutions in Example 1 would be written $\pm\sqrt{5}$ and $\pm 2\sqrt{3}$.

EXAMPLE 2 Using the Square Root Property in an Application

Galileo Galilei (1564–1642) developed a formula for freely falling objects described by

$$d = 16t^2,$$

where d is the distance in feet that an object falls (disregarding air resistance) in t sec, regardless of weight. Galileo dropped objects from the Leaning Tower of Pisa to develop this formula. If the Leaning Tower is about 180 ft tall, use Galileo's formula to determine how long it would take an object dropped from the tower to fall to the ground. (*Source: Microsoft Encarta Encyclopedia 2002.*)

We substitute 180 for d in Galileo's formula.

$$\begin{aligned} d &= 16t^2 && \\ 180 &= 16t^2 && \text{Let } d = 180. \\ 11.25 &= t^2 && \text{Divide by 16.} \\ t = \sqrt{11.25} \quad \text{or} \quad t &= -\sqrt{11.25} && \text{Square root property} \end{aligned}$$

Since time cannot be negative, we discard the negative solution. In applied problems, we usually prefer approximations to exact values. Using a calculator, $\sqrt{11.25} \approx 3.4$ so $t \approx 3.4$. The object would fall to the ground in about 3.4 sec.

Now Try Exercise 25.

OBJECTIVE 3 Solve quadratic equations of the form $(ax + b)^2 = c$ by using the square root property. To solve more complicated equations using the square root property, such as

$$(x - 5)^2 = 36,$$

substitute $(x - 5)^2$ for x^2 and 36 for k in the square root property to obtain

$$\begin{aligned} x - 5 &= \sqrt{36} \quad &\text{or}\quad x - 5 &= -\sqrt{36} \\ x - 5 &= 6 \quad &\text{or}\quad x - 5 &= -6 \\ x &= 11 \quad &\text{or}\quad x &= -1. \end{aligned}$$

Check: $(x - 5)^2 = 36$ Original equation

$$\begin{aligned} (11 - 5)^2 &= 36 \quad ? \\ 6^2 &= 36 \quad ? \\ 36 &= 36 \quad \text{True} \end{aligned}
\qquad\Bigg|\qquad
\begin{aligned} (-1 - 5)^2 &= 36 \quad ? \\ (-6)^2 &= 36 \quad ? \\ 36 &= 36 \quad \text{True} \end{aligned}$$

Since both solutions satisfy the original equation, the solution set is $\{-1, 11\}$.

EXAMPLE 3 Using the Square Root Property

Solve $(2x - 3)^2 = 18$.

$$\begin{aligned} 2x - 3 &= \sqrt{18} \quad &\text{or}\quad 2x - 3 &= -\sqrt{18} && \text{Square root property} \\ 2x &= 3 + \sqrt{18} \quad &\text{or}\quad 2x &= 3 - \sqrt{18} && \\ x &= \frac{3 + \sqrt{18}}{2} \quad &\text{or}\quad x &= \frac{3 - \sqrt{18}}{2} && \\ x &= \frac{3 + 3\sqrt{2}}{2} \quad &\text{or}\quad x &= \frac{3 - 3\sqrt{2}}{2} && \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2} \end{aligned}$$

We show the check for the first solution. The check for the second solution is similar.

Check:

$$(2x - 3)^2 = 18 \quad \text{Original equation}$$

$$\left[2\left(\frac{3 + 3\sqrt{2}}{2}\right) - 3\right]^2 = 18 \quad ?$$

$$\left(3 + 3\sqrt{2} - 3\right)^2 = 18 \quad ?$$

$$\left(3\sqrt{2}\right)^2 = 18 \quad ?$$

$$18 = 18 \quad \text{True}$$

The solution set is $\left\{\frac{3 + 3\sqrt{2}}{2}, \frac{3 - 3\sqrt{2}}{2}\right\}$.

Now Try Exercise 21.

OBJECTIVE 4 Solve quadratic equations by completing the square. We can use the square root property to solve *any* quadratic equation by writing it in the form $(x + k)^2 = n$. That is, we must write the left side of the equation as a perfect square trinomial that can be factored as $(x + k)^2$, the square of a binomial, and the right side must be a constant. Rewriting a quadratic equation in this form is called **completing the square.**

Recall that the perfect square trinomial

$$x^2 + 10x + 25$$

can be factored as $(x + 5)^2$. In the trinomial, the coefficient of x (the first-degree term) is 10 and the constant term is 25. Notice that if we take half of 10 and square it, we get the constant term, 25.

$$\left[\frac{1}{2}(\underset{\text{Coefficient of } x}{10})\right]^2 = 5^2 = \underset{\text{Constant}}{25}$$

Similarly, in

$$x^2 + 12x + 36, \qquad \left[\frac{1}{2}(12)\right]^2 = 6^2 = 36,$$

and in

$$m^2 - 6m + 9, \qquad \left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9.$$

This relationship is true in general and is the idea behind completing the square.

VIDEO

EXAMPLE 4 Solving a Quadratic Equation by Completing the Square

Solve $x^2 + 8x + 10 = 0$.

This quadratic equation cannot be solved by factoring, and it is not in the correct form to solve using the square root property. To solve it by completing the square, we need a perfect square trinomial on the left side of the equation. To get this form, we first subtract 10 from each side.

$$x^2 + 8x + 10 = 0$$

$$x^2 + 8x = -10 \qquad \text{Subtract 10.}$$

We must add a constant to get a perfect square trinomial on the left.

$$\underbrace{x^2 + 8x + \underline{\ ?\ }}_{\text{Needs to be a perfect square trinomial}}$$

To find this constant, we apply the ideas preceding this example—we take half the coefficient of the first-degree term and square the result.

$$\left[\frac{1}{2}(8)\right]^2 = 4^2 = 16 \leftarrow \text{Desired constant}$$

We add this constant, 16, to *each* side of the equation. (Why?)

$$x^2 + 8x + 16 = -10 + 16$$

Now we factor the perfect square trinomial on the left and add on the right.

$$(x + 4)^2 = 6$$

We can solve this equation using the square root property.

$$x + 4 = \sqrt{6} \quad \text{or} \quad x + 4 = -\sqrt{6}$$

$$x = -4 + \sqrt{6} \quad \text{or} \quad x = -4 - \sqrt{6}$$

Check:

$$x^2 + 8x + 10 = 0 \qquad \text{Original equation}$$

$$\left(-4 + \sqrt{6}\right)^2 + 8\left(-4 + \sqrt{6}\right) + 10 \stackrel{?}{=} 0 \qquad \text{Let } x = -4 + \sqrt{6}.$$

$$16 - 8\sqrt{6} + 6 - 32 + 8\sqrt{6} + 10 \stackrel{?}{=} 0$$

$$0 = 0 \qquad \text{True}$$

The check for the second solution is similar. The solution set is

$$\left\{-4 + \sqrt{6}, -4 - \sqrt{6}\right\}.$$

Now Try Exercise 39.

The procedure from Example 4 can be generalized.

EXPLORE

Completing the Square

To solve $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square, use these steps.

Step 1 **Be sure the squared term has coefficient 1.** If the coefficient of the squared term is 1, proceed to Step 2. If the coefficient of the squared term is not 1 but some other nonzero number a, divide each side of the equation by a.

Step 2 **Put the equation in correct form.** Rewrite so that terms with variables are on one side of the equals sign and the constant is on the other side.

Step 3 **Square half the coefficient of the first-degree term.**

Step 4 **Add the square to each side.**

Step 5 **Factor the perfect square trinomial.** One side should now be a perfect square trinomial. Factor it as the square of a binomial. Simplify the other side.

Step 6 **Solve the equation.** Apply the square root property to complete the solution.

NOTE Steps 1 and 2 can be done in either order. With some equations, it is more convenient to do Step 2 first.

YOU TRY IT

EXAMPLE 5 Solving a Quadratic Equation with $a = 1$ by Completing the Square

Solve $k^2 + 5k - 1 = 0$.

Follow the steps in the box. Since the coefficient of the squared term is 1, begin with Step 2.

Step 2 $$k^2 + 5k = 1 \qquad \text{Add 1 to each side.}$$

Step 3 Take half the coefficient of the first-degree term and square the result.

$$\left[\frac{1}{2}(5)\right]^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

Step 4 Add the square to each side of the equation to get

$$k^2 + 5k + \frac{25}{4} = 1 + \frac{25}{4}.$$

Step 5 $$\left(k + \frac{5}{2}\right)^2 = \frac{29}{4} \qquad \text{Factor on the left; add on the right.}$$

Step 6 $$k + \frac{5}{2} = \sqrt{\frac{29}{4}} \quad \text{or} \quad k + \frac{5}{2} = -\sqrt{\frac{29}{4}} \qquad \text{Square root property}$$

$$k + \frac{5}{2} = \frac{\sqrt{29}}{2} \quad \text{or} \quad k + \frac{5}{2} = -\frac{\sqrt{29}}{2}$$

$$k = -\frac{5}{2} + \frac{\sqrt{29}}{2} \quad \text{or} \quad k = -\frac{5}{2} - \frac{\sqrt{29}}{2}$$

$$k = \frac{-5 + \sqrt{29}}{2} \quad \text{or} \quad k = \frac{-5 - \sqrt{29}}{2}$$

Check that the solution set is $\left\{\frac{-5 + \sqrt{29}}{2}, \frac{-5 - \sqrt{29}}{2}\right\}$.

Now Try Exercise 41.

EXAMPLE 6 Solving a Quadratic Equation with $a \neq 1$ by Completing the Square

Solve $2x^2 - 4x - 5 = 0$.

First divide each side of the equation by 2 to get 1 as the coefficient of the squared term.

$$x^2 - 2x - \frac{5}{2} = 0 \qquad \text{Step 1}$$

$$x^2 - 2x = \frac{5}{2} \qquad \text{Step 2}$$

$$\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1 \qquad \text{Step 3}$$

$$x^2 - 2x + 1 = \frac{5}{2} + 1 \qquad \text{Step 4}$$

$$(x - 1)^2 = \frac{7}{2} \qquad \text{Step 5}$$

$$x - 1 = \sqrt{\frac{7}{2}} \quad \text{or} \quad x - 1 = -\sqrt{\frac{7}{2}} \qquad \text{Step 6}$$

$$x = 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}}$$

$$x = 1 + \frac{\sqrt{14}}{2} \quad \text{or} \quad x = 1 - \frac{\sqrt{14}}{2} \qquad \text{Rationalize denominators.}$$

Add the two terms in each solution as follows.

$$1 + \frac{\sqrt{14}}{2} = \frac{2}{2} + \frac{\sqrt{14}}{2} = \frac{2 + \sqrt{14}}{2}$$

$$1 - \frac{\sqrt{14}}{2} = \frac{2}{2} - \frac{\sqrt{14}}{2} = \frac{2 - \sqrt{14}}{2}.$$

Check that the solution set is $\left\{\frac{2 + \sqrt{14}}{2}, \frac{2 - \sqrt{14}}{2}\right\}$.

Now Try Exercise 47.

OBJECTIVE 5 Solve quadratic equations with solutions that are not real numbers. So far, all the equations we have solved using the square root property have had two real solutions. In the equation $x^2 = k$, if $k < 0$, there will be two imaginary solutions.

EXAMPLE 7 Solving Quadratic Equations with Imaginary Solutions

Solve each equation.

(a) $x^2 = -15$

$$x = \sqrt{-15} \quad \text{or} \quad x = -\sqrt{-15} \qquad \text{Square root property}$$

$$x = i\sqrt{15} \quad \text{or} \quad x = -i\sqrt{15} \qquad \sqrt{-1} = i$$

The solution set is $\{i\sqrt{15}, -i\sqrt{15}\}$.

(b) $(t + 2)^2 = -16$

$$t + 2 = \sqrt{-16} \quad \text{or} \quad t + 2 = -\sqrt{-16} \qquad \text{Square root property}$$
$$t + 2 = 4i \quad \text{or} \quad t + 2 = -4i \qquad \sqrt{-16} = 4i$$
$$t = -2 + 4i \quad \text{or} \quad t = -2 - 4i$$

The solution set is $\{-2 + 4i, -2 - 4i\}$.

(c) $x^2 + 2x + 7 = 0$

Solve by completing the square.

$$x^2 + 2x = -7 \qquad \text{Subtract 7.}$$
$$x^2 + 2x + 1 = -7 + 1 \qquad \left[\tfrac{1}{2}(2)\right]^2 = 1\text{; add 1 to each side.}$$
$$(x + 1)^2 = -6 \qquad \text{Factor on the left; add on the right.}$$
$$x + 1 = \pm i\sqrt{6} \qquad \text{Square root property}$$
$$x = -1 \pm i\sqrt{6} \qquad \text{Subtract 1.}$$

The solution set is $\left\{-1 + i\sqrt{6}, -1 - i\sqrt{6}\right\}$.

Now Try Exercises 55, 57, and 61.

NOTE The procedure for completing the square is also used in other areas of mathematics. For example, we use it in Section 9.6 when we graph quadratic equations and again in Chapter 11 when we work with circles.

9.1 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 14/Videotape 14

1. A student was asked to solve the quadratic equation $x^2 = 16$ and did not get full credit for the solution set $\{4\}$. Why?

2. Why can't the zero-factor property be used to solve every quadratic equation?

3. Give a one-sentence description or explanation of each phrase.

(a) Quadratic equation in standard form **(b)** Zero-factor property
(c) Square root property

4. What is wrong with the following "solution"?

$$
\begin{aligned}
x^2 - x - 2 &= 5 \\
(x - 2)(x + 1) &= 5 \\
x - 2 = 5 \quad \text{or} \quad x + 1 &= 5 \qquad \text{Zero-factor property} \\
x = 7 \quad \text{or} \quad x &= 4
\end{aligned}
$$

Use the square root property to solve each equation. See Examples 1 and 3.

EXERCISES

5. $x^2 = 81$	**6.** $z^2 = 225$	**7.** $t^2 = 17$
8. $x^2 = 19$	**9.** $m^2 = 32$	**10.** $x^2 = 54$
11. $t^2 - 20 = 0$	**12.** $p^2 - 50 = 0$	**13.** $3n^2 - 72 = 0$
14. $5z^2 - 200 = 0$	**15.** $(x + 2)^2 = 25$	**16.** $(t + 8)^2 = 9$
17. $(x - 4)^2 = 3$	**18.** $(x + 3)^2 = 11$	**19.** $(t + 5)^2 = 48$

20. $(m - 6)^2 = 27$ **21.** $(3x - 1)^2 = 7$ **22.** $(2x + 4)^2 = 10$

23. $(4p + 1)^2 = 24$ **24.** $(5t - 2)^2 = 12$

Solve Exercises 25 and 26 using Galileo's formula, $d = 16t^2$. Round answers to the nearest tenth. See Example 2.

25. The Gateway Arch in St. Louis, Missouri, is 630 ft tall. How long would it take an object dropped from the top of it to fall to the ground? (*Source: Home & Away,* November/December 2000.)

26. Mount Rushmore National Memorial in South Dakota features a sculpture of four of America's favorite presidents carved into the rim of the mountain, 500 ft above the valley floor. How long would it take a rock dropped from the top of the sculpture to fall to the ground? (*Source: Microsoft Encarta Encyclopedia 2002.*)

27. Of the two equations

$$(2x + 1)^2 = 5 \qquad \text{and} \qquad x^2 + 4x = 12,$$

one is more suitable for solving by the square root property, and the other is more suitable for solving by completing the square. Which method do you think most students would use for each equation?

28. Why would most students find the equation $x^2 + 4x = 20$ easier to solve by completing the square than the equation $5x^2 + 2x = 3$?

29. Decide what number must be added to make each expression a perfect square trinomial.

(a) $x^2 + 6x +$ _____ **(b)** $x^2 + 14x +$ _____ **(c)** $p^2 - 12p +$ _____

(d) $x^2 + 3x +$ _____ **(e)** $q^2 - 9q +$ _____ **(f)** $t^2 - \frac{1}{2}t +$ _____

30. What would be the first step in solving $2x^2 + 8x = 9$ by completing the square?

Determine the number that will complete the square to solve each equation after the constant term has been written on the right side. Do not actually solve. See Examples 4–6.

31. $x^2 + 4x - 2 = 0$ **32.** $t^2 + 2t - 1 = 0$ **33.** $x^2 + 10x + 18 = 0$

34. $x^2 + 8x + 11 = 0$ **35.** $3w^2 - w - 24 = 0$ **36.** $4z^2 - z - 39 = 0$

Solve each equation by completing the square. Use the results of Exercises 31–36 to solve Exercises 39–44. See Examples 4–6.

37. $x^2 - 2x - 24 = 0$ **38.** $m^2 - 4m - 32 = 0$ **39.** $x^2 + 4x - 2 = 0$

40. $t^2 + 2t - 1 = 0$ **41.** $x^2 + 10x + 18 = 0$ **42.** $x^2 + 8x + 11 = 0$

43. $3w^2 - w = 24$ **44.** $4z^2 - z = 39$ **45.** $2k^2 + 5k - 2 = 0$

46. $3r^2 + 2r - 2 = 0$ **47.** $5x^2 - 10x + 2 = 0$ **48.** $2x^2 - 16x + 25 = 0$

49. $9x^2 - 24x = -13$ **50.** $25n^2 - 20n = 1$ **51.** $z^2 - \frac{4}{3}z = -\frac{1}{9}$

52. $p^2 - \frac{8}{3}p = -1$ **53.** $.1x^2 - .2x - .1 = 0$ **54.** $.1p^2 - .4p + .1 = 0$

Find the imaginary solutions of each equation. See Example 7.

55. $x^2 = -12$ **56.** $x^2 = -18$ **57.** $(r - 5)^2 = -3$

58. $(t + 6)^2 = -5$ **59.** $(6k - 1)^2 = -8$ **60.** $(4m - 7)^2 = -27$

61. $m^2 + 4m + 13 = 0$ **62.** $t^2 + 6t + 10 = 0$ **63.** $3r^2 + 4r + 4 = 0$

64. $4x^2 + 5x + 5 = 0$ **65.** $-m^2 - 6m - 12 = 0$ **66.** $-k^2 - 5k - 10 = 0$

Solve for x. Assume that a and b represent positive real numbers.

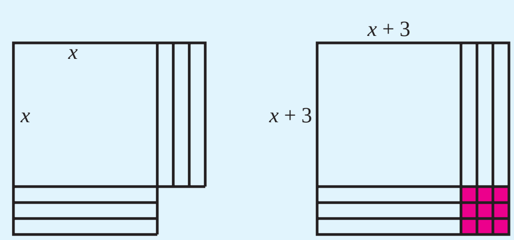
67. $x^2 - b = 0$ **68.** $x^2 = 4b$ **69.** $4x^2 = b^2 + 16$

70. $9x^2 - 25a = 0$ **71.** $(5x - 2b)^2 = 3a$ **72.** $x^2 - a^2 - 36 = 0$

RELATING CONCEPTS (EXERCISES 73–78)

For Individual or Group Work

The Greeks had a method of completing the square geometrically in which they literally changed a figure into a square. For example, to complete the square for $x^2 + 6x$, we begin with a square of side x, as in the figure on the left. We add three rectangles of width 1 to the right side and the bottom to get a region with area $x^2 + 6x$. To fill in the corner (complete the square), we must add 9 1-by-1 squares as shown in the figure on the right.



Work Exercises 73–78 in order.

73. What is the area of the original square?

74. What is the area of each strip?

75. What is the total area of the six strips?

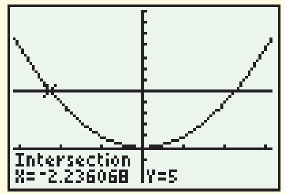
76. What is the area of each small square in the corner of the second figure?

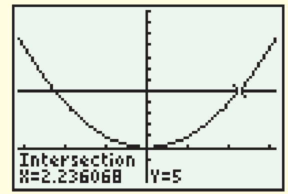
77. What is the total area of the small squares?

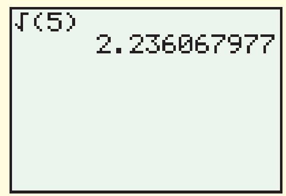
78. What is the area of the new, larger square?

TECHNOLOGY INSIGHTS (EXERCISES 79–80)

Two of the following calculator screens show the intersection points of the graph of $y = x^2$ *and the graph of the horizontal line* $y = 5$. *The other screen shows that* $\sqrt{5} \approx 2.236068$, *so the graphs intersect at* $x = -\sqrt{5}$ *and* $x = \sqrt{5}$. *This supports our solution* $\pm\sqrt{5}$ *for* $x^2 = 5$ *using the square root property in Example 1(a).*

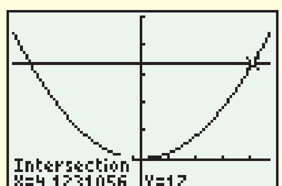


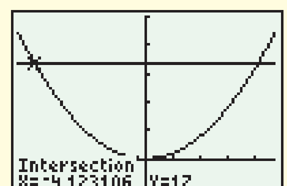




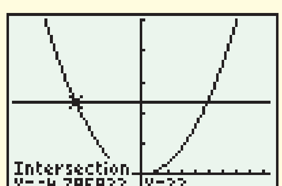
Use the screens in Exercises 79 and 80 to give the exact value of the solutions of $x^2 = k$.

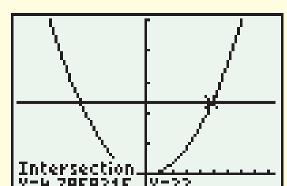
79.





80.





9.2 The Quadratic Formula

OBJECTIVES

1. Derive the quadratic formula.
2. Solve quadratic equations using the quadratic formula.
3. Use the discriminant to determine the number and type of solutions.

The examples in the previous section showed that any quadratic equation can be solved by completing the square; however, completing the square can be tedious and time consuming. In this section, we complete the square to solve the general quadratic equation

$$ax^2 + bx + c = 0,$$

where a, b, and c are complex numbers and $a \neq 0$. The solution of this general equation gives a formula for finding the solution of any specific quadratic equation.

OBJECTIVE 1 Derive the quadratic formula. To solve $ax^2 + bx + c = 0$ by completing the square (assuming $a > 0$), we follow the steps given in Section 9.1.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Divide by } a. \text{ (Step 1)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Subtract } \tfrac{c}{a}. \text{ (Step 2)}$$

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \qquad \text{(Step 3)}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{Add } \tfrac{b^2}{4a^2} \text{ to each side. (Step 4)}$$

Write the left side as a perfect square and rearrange the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a} \qquad \text{(Step 5)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2} \qquad \text{Write with a common denominator.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Add fractions.}$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Square root property (Step 6)}$$

Since

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a},$$

the right side of each equation can be expressed as

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $a < 0$, the same two solutions are obtained. The result is the **quadratic formula,** which is abbreviated as follows.

Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

CAUTION In the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the square root is added to or subtracted from the value of $-b$ *before* dividing by $2a$.

OBJECTIVE 2 Solve quadratic equations using the quadratic formula. To use the quadratic formula, first write the given equation in standard form $ax^2 + bx + c = 0$; then identify the values of a, b, and c and substitute them into the formula.

EXAMPLE 1 Using the Quadratic Formula (Rational Solutions)

Solve $6x^2 - 5x - 4 = 0$.

First, identify the values of a, b, and c of the general quadratic equation, $ax^2 + bx + c = 0$. Here a, the coefficient of the second-degree term, is 6, while b, the coefficient of the first-degree term, is -5, and the constant c is -4. Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-4)}}{2(6)} \qquad a = 6,\ b = -5,\ c = -4$$

$$x = \frac{5 \pm \sqrt{25 + 96}}{12}$$

$$x = \frac{5 \pm \sqrt{121}}{12}$$

$$x = \frac{5 \pm 11}{12}$$

This last statement leads to two solutions, one from $+$ and one from $-$.

$$x = \frac{5 + 11}{12} = \frac{16}{12} = \frac{4}{3} \quad \text{or} \quad x = \frac{5 - 11}{12} = \frac{-6}{12} = -\frac{1}{2}$$

Check each solution by substituting it in the original equation. The solution set is $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$.

Now Try Exercise 5.

We could have used factoring to solve the equation in Example 1.

$$6x^2 - 5x - 4 = 0$$

$$(3x - 4)(2x + 1) = 0 \qquad \text{Factor.}$$

$$3x - 4 = 0 \quad \text{or} \quad 2x + 1 = 0 \qquad \text{Zero-factor property}$$

$$3x = 4 \quad \text{or} \quad 2x = -1 \qquad \text{Solve each equation.}$$

$$x = \frac{4}{3} \quad \text{or} \quad x = -\frac{1}{2} \qquad \text{Same solutions as in Example 1}$$

When solving quadratic equations, it is a good idea to try factoring first. If the equation cannot be factored or if factoring is difficult, then use the quadratic formula.

Later in this section, we will show a way to determine whether factoring can be used to solve a quadratic equation.

EXAMPLE 2 Using the Quadratic Formula (Irrational Solutions)

Solve $4r^2 = 8r - 1$.

Write the equation in standard form as

$$4r^2 - 8r + 1 = 0,$$

and identify $a = 4$, $b = -8$, and $c = 1$. Now use the quadratic formula.

$$\begin{aligned}
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
r &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)} && a = 4,\ b = -8,\ c = 1 \\
&= \frac{8 \pm \sqrt{64 - 16}}{8} \\
&= \frac{8 \pm \sqrt{48}}{8} \\
&= \frac{8 \pm 4\sqrt{3}}{8} && \sqrt{48} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3} \\
&= \frac{4\left(2 \pm \sqrt{3}\right)}{4(2)} && \text{Factor.} \\
&= \frac{2 \pm \sqrt{3}}{2} && \text{Lowest terms}
\end{aligned}$$

The solution set is $\left\{\frac{2 + \sqrt{3}}{2}, \frac{2 - \sqrt{3}}{2}\right\}$.

Now Try Exercise 9.

CAUTION *Every* quadratic equation must be expressed in standard form $ax^2 + bx + c = 0$ before we begin to solve it, whether we use factoring or the quadratic formula. Also, when writing solutions in lowest terms, be sure to *factor first;* then divide out the common factor, as shown in the last two steps in Example 2.

EXAMPLE 3 Using the Quadratic Formula (Imaginary Solutions)

Solve $(9q + 3)(q - 1) = -8$.

To write this equation in standard form, we first multiply and collect all nonzero terms on the left.

$$\begin{aligned}
(9q + 3)(q - 1) &= -8 \\
9q^2 - 6q - 3 &= -8 \\
9q^2 - 6q + 5 &= 0 && \text{Standard form}
\end{aligned}$$

From the equation $9q^2 - 6q + 5 = 0$, we identify $a = 9$, $b = -6$, and $c = 5$, and use the quadratic formula.

$$\begin{aligned}
q &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)} \\
&= \frac{6 \pm \sqrt{-144}}{18} \\
&= \frac{6 \pm 12i}{18} && \sqrt{-144} = 12i \\
&= \frac{6(1 \pm 2i)}{6(3)} && \text{Factor.} \\
&= \frac{1 \pm 2i}{3} && \text{Lowest terms}
\end{aligned}$$

The solution set, written in standard form $a + bi$ for complex numbers, is $\left\{\frac{1}{3} + \frac{2}{3}i, \frac{1}{3} - \frac{2}{3}i\right\}$.

Now Try Exercise 33.

OBJECTIVE 3 Use the discriminant to determine the number and type of solutions. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad \longleftarrow \text{Discriminant}$$

If a, b, and c are integers, the type of solutions of a quadratic equation—that is, rational, irrational, or imaginary—is determined by the expression under the radical sign, $b^2 - 4ac$. Because it distinguishes among the three types of solutions, $b^2 - 4ac$ is called the *discriminant*. By calculating the discriminant before solving a quadratic equation, we can predict whether the solutions will be rational numbers, irrational numbers, or imaginary numbers. (This can be useful in an applied problem, for example, where irrational or imaginary solutions are not acceptable.)

Discriminant

The **discriminant** of $ax^2 + bx + c = 0$ is $\boldsymbol{b^2 - 4ac}$. If a, b, and c are integers, then the number and type of solutions are determined as follows.

Discriminant	*Number and Type of Solutions*
Positive, and the square of an integer	Two rational solutions
Positive, but not the square of an integer	Two irrational solutions
Zero	One rational solution
Negative	Two imaginary solutions

Calculating the discriminant can also help you decide whether to solve a quadratic equation by factoring or by using the quadratic formula. If the discriminant is a perfect square (including 0), then the equation can be solved by factoring. Otherwise, the quadratic formula should be used.

EXAMPLE 4 Using the Discriminant

Find the discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

(a) $6x^2 - x - 15 = 0$

We find the discriminant by evaluating $b^2 - 4ac$.

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(6)(-15) \qquad a = 6,\ b = -1,\ c = -15 \\ &= 1 + 360 \\ &= 361 \end{aligned}$$

A calculator shows that $361 = 19^2$, a perfect square. Since a, b, and c are integers and the discriminant is a perfect square, there will be two rational solutions and the equation can be solved by factoring.

(b) $3m^2 - 4m = 5$

Write the equation in standard form as $3m^2 - 4m - 5 = 0$ to find that $a = 3$, $b = -4$, and $c = -5$.

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(3)(-5) \\ &= 16 + 60 \\ &= 76 \end{aligned}$$

Because 76 is positive but not the square of an integer and a, b, and c are integers, the equation will have two irrational solutions and is best solved using the quadratic formula.

(c) $4x^2 + x + 1 = 0$

Since $a = 4$, $b = 1$, and $c = 1$, the discriminant is

$$1^2 - 4(4)(1) = -15.$$

Since the discriminant is negative and a, b, and c are integers, this quadratic equation will have two imaginary solutions. The quadratic formula should be used to solve it.

(d) $4t^2 + 9 = 12t$

Write the equation as $4t^2 - 12t + 9 = 0$ to find $a = 4$, $b = -12$, and $c = 9$. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-12)^2 - 4(4)(9) \\ &= 144 - 144 \\ &= 0. \end{aligned}$$

Because the discriminant is 0, the quantity under the radical in the quadratic formula is 0, and there is only one rational solution. Again, the equation can be solved by factoring.

Now Try Exercises 37 and 39.

EXAMPLE 5 Using the Discriminant

Find k so that $9x^2 + kx + 4 = 0$ will have only one rational solution.

The equation will have only one rational solution if the discriminant is 0. Since $a = 9$, $b = k$, and $c = 4$, the discriminant is

$$b^2 - 4ac = k^2 - 4(9)(4) = k^2 - 144.$$

Set the discriminant equal to 0 and solve for k.

$$k^2 - 144 = 0$$

$$k^2 = 144 \qquad \text{Subtract 144.}$$

$$k = 12 \quad \text{or} \quad k = -12 \qquad \text{Square root property}$$

The equation will have only one rational solution if $k = 12$ or $k = -12$.

Now Try Exercise 53.

9.2 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 14/Videotape 14

Answer each question in Exercises 1–4.

1. An early version of Microsoft *Word* for Windows included the 1.0 edition of *Equation Editor*. The documentation used the following for the quadratic formula.

$$x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Was this correct? Explain.

2. The Cadillac Bar in Houston, Texas, encourages patrons to write (tasteful) messages on the walls. One person attempted to write the quadratic formula, as shown here.

$$x = \frac{-b\sqrt{b^2 - 4ac}}{2a}$$

Was this correct? Explain.

3. What is wrong with the following "solution" of $5x^2 - 5x + 1 = 0$?

$$x = \frac{5 \pm \sqrt{25 - 4(5)(1)}}{2(5)} \qquad a = 5,\ b = -5,\ c = 1$$

$$x = \frac{5 \pm \sqrt{5}}{10}$$

$$x = \frac{1}{2} \pm \sqrt{5}$$

4. A student claimed that the equation $2x^2 - 5 = 0$ cannot be solved using the quadratic formula because there is no first-degree x-term. Was the student correct? Explain.

Use the quadratic formula to solve each equation. (All solutions for these equations are real numbers.) See Examples 1 and 2.

EXERCISES

5. $m^2 - 8m + 15 = 0$	**6.** $x^2 + 3x - 28 = 0$	**7.** $2k^2 + 4k + 1 = 0$
8. $2w^2 + 3w - 1 = 0$	**9.** $2x^2 - 2x = 1$	**10.** $9t^2 + 6t = 1$
11. $x^2 + 18 = 10x$	**12.** $x^2 - 4 = 2x$	**13.** $4k^2 + 4k - 1 = 0$
14. $4r^2 - 4r - 19 = 0$	**15.** $2 - 2x = 3x^2$	**16.** $26r - 2 = 3r^2$
17. $\frac{x^2}{4} - \frac{x}{2} = 1$	**18.** $p^2 + \frac{p}{3} = \frac{1}{6}$	**19.** $-2t(t + 2) = -3$
20. $-3x(x + 2) = -4$	**21.** $(r - 3)(r + 5) = 2$	**22.** $(k + 1)(k - 7) = 1$

23. $(g + 2)(g - 3) = 1$ **24.** $(x - 5)(x + 2) = 6$ **25.** $p = \dfrac{5(5 - p)}{3(p + 1)}$

26. $k = \dfrac{k + 15}{3(k - 1)}$

Use the quadratic formula to solve each equation. (All solutions for these equations are imaginary numbers.) See Example 3.

27. $x^2 - 3x + 6 = 0$ **28.** $x^2 - 5x + 20 = 0$ **29.** $r^2 - 6r + 14 = 0$

30. $t^2 + 4t + 11 = 0$ **31.** $4x^2 - 4x = -7$ **32.** $9x^2 - 6x = -7$

33. $x(3x + 4) = -2$ **34.** $z(2z + 3) = -2$

35. $(x + 5)(x - 6) = (2x - 1)(x - 4)$ **36.** $(3x - 4)(x + 2) = (2x - 5)(x + 5)$

Use the discriminant to determine whether the solutions for each equation are

A. *two rational numbers;* **B.** *one rational number;*
C. *two irrational numbers;* **D.** *two imaginary numbers.*

Do not actually solve. See Example 4.

37. $25x^2 + 70x + 49 = 0$ **38.** $4k^2 - 28k + 49 = 0$ **39.** $x^2 + 4x + 2 = 0$

40. $9x^2 - 12x - 1 = 0$ **41.** $3x^2 = 5x + 2$ **42.** $4x^2 = 4x + 3$

43. $3m^2 - 10m + 15 = 0$ **44.** $18x^2 + 60x + 82 = 0$

45. Using the discriminant, which equations in Exercises 37–44 can be solved by factoring?

Based on your answer in Exercise 45, solve the equation given in each exercise.

46. Exercise 37 **47.** Exercise 38 **48.** Exercise 41 **49.** Exercise 42

50. Find the discriminant for each quadratic equation. Use it to tell whether the equation can be solved by factoring or whether the quadratic formula should be used. Then solve each equation.

(a) $3k^2 + 13k = -12$ **(b)** $2x^2 + 19 = 14x$

51. Is it possible for the solution of a quadratic equation with integer coefficients to include just one irrational number? Why or why not?

52. Can the solution of a quadratic equation with integer coefficients include one real and one imaginary number? Why or why not?

Find the value of a, b, or c so that each equation will have exactly one rational solution. See Example 5.

53. $p^2 + bp + 25 = 0$ **54.** $r^2 - br + 49 = 0$ **55.** $am^2 + 8m + 1 = 0$

56. $at^2 + 24t + 16 = 0$ **57.** $9x^2 - 30x + c = 0$ **58.** $4m^2 + 12m + c = 0$

59. One solution of $4x^2 + bx - 3 = 0$ is $-\frac{5}{2}$. Find b and the other solution.

60. One solution of $3x^2 - 7x + c = 0$ is $\frac{1}{3}$. Find c and the other solution.

9.3 Equations Quadratic in Form

OBJECTIVES

1. Solve an equation with fractions by writing it in quadratic form.
2. Use quadratic equations to solve applied problems.
3. Solve an equation with radicals by writing it in quadratic form.
4. Solve an equation that is quadratic in form by substitution.

We have introduced four methods for solving quadratic equations written in standard form $ax^2 + bx + c = 0$. The following table lists some advantages and disadvantages of each method.

AUDIO

Methods for Solving Quadratic Equations

Method	*Advantages*	*Disadvantages*
Factoring	This is usually the fastest method.	Not all polynomials are factorable; some factorable polynomials are hard to factor.
Square root property	This is the simplest method for solving equations of the form $(ax + b)^2 = c$.	Few equations are given in this form.
Completing the square	This method can always be used, although most people prefer the quadratic formula.	It requires more steps than other methods.
Quadratic formula	This method can always be used.	It is more difficult than factoring because of the square root, although calculators can simplify its use.

OBJECTIVE 1 Solve an equation with fractions by writing it in quadratic form. A variety of nonquadratic equations can be written in the form of a quadratic equation and solved by using one of the methods in the table. As you solve the equations in this section, try to decide which method is best for each equation.

VIDEO YOU TRY IT

EXAMPLE 1 Solving an Equation with Fractions that Leads to a Quadratic Equation

Solve $\dfrac{1}{x} + \dfrac{1}{x-1} = \dfrac{7}{12}$.

Clear fractions by multiplying each term by the least common denominator, $12x(x - 1)$. (Note that the domain must be restricted to $x \neq 0$ and $x \neq 1$.)

$$12x(x-1)\frac{1}{x} + 12x(x-1)\frac{1}{x-1} = 12x(x-1)\frac{7}{12}$$

$$\begin{aligned} 12(x-1) + 12x &= 7x(x-1) \\ 12x - 12 + 12x &= 7x^2 - 7x && \text{Distributive property} \\ 24x - 12 &= 7x^2 - 7x && \text{Combine terms.} \end{aligned}$$

Recall that a quadratic equation must be in standard form before it can be solved by factoring or the quadratic formula. Combine and rearrange terms so that one side is 0. Then factor to solve the resulting equation.

$$7x^2 - 31x + 12 = 0 \qquad \text{Standard form}$$
$$(7x - 3)(x - 4) = 0 \qquad \text{Factor.}$$

Using the zero-factor property gives the solutions $\frac{3}{7}$ and 4. Check by substituting these solutions in the original equation. The solution set is $\left\{\frac{3}{7}, 4\right\}$.

Now Try Exercise 19.

OBJECTIVE 2 Use quadratic equations to solve applied problems. Earlier we solved distance-rate-time (or motion) problems that led to linear equations or rational equations. Now we solve motion problems that lead to quadratic equations. We continue to use the six-step problem-solving method from Chapter 2.

VIDEO

EXAMPLE 2 Solving a Motion Problem

A riverboat for tourists averages 12 mph in still water. It takes the boat 1 hr 4 min to go 6 mi upstream and return. Find the speed of the current. See Figure 1.

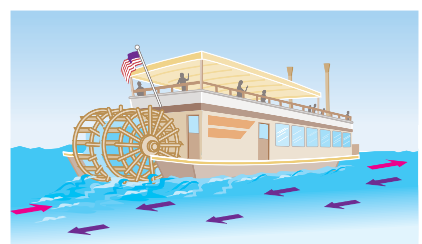


FIGURE 1

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.** Let x = the speed of the current. The current slows down the boat when it is going upstream, so the rate (or speed) of the boat going upstream is its speed in still water less the speed of the current, or $12 - x$. Similarly, the current speeds up the boat as it travels downstream, so its speed downstream is $12 + x$. Thus,

$$12 - x = \text{the rate upstream;}$$
$$12 + x = \text{the rate downstream.}$$

Use the distance formula, $d = rt$, solved for time t.

$$t = \frac{d}{r}$$

This information can be used to complete a table.

	d	r	t
Upstream	6	$12 - x$	$\frac{6}{12 - x}$
Downstream	6	$12 + x$	$\frac{6}{12 + x}$

Times in hours

Step 3 **Write an equation.** The total time of 1 hr 4 min can be written as

$$1 + \frac{4}{60} = 1 + \frac{1}{15} = \frac{16}{15} \text{ hr.}$$

Because the time upstream plus the time downstream equals $\frac{16}{15}$ hr,

$$\underset{\downarrow}{\text{Time upstream}} + \underset{\downarrow}{\text{Time downstream}} = \underset{\downarrow}{\text{Total time}}$$

$$\frac{6}{12 - x} + \frac{6}{12 + x} = \frac{16}{15}.$$

Step 4 **Solve** the equation. Multiply each side by $15(12 - x)(12 + x)$, the LCD, and solve the resulting quadratic equation.

$$15(12 + x)6 + 15(12 - x)6 = 16(12 - x)(12 + x)$$

$$90(12 + x) + 90(12 - x) = 16(144 - x^2)$$

$$1080 + 90x + 1080 - 90x = 2304 - 16x^2 \qquad \text{Distributive property}$$

$$2160 = 2304 - 16x^2 \qquad \text{Combine terms.}$$

$$16x^2 = 144$$

$$x^2 = 9 \qquad \text{Divide by 16.}$$

$$x = 3 \quad \text{or} \quad x = -3 \qquad \text{Square root property}$$

Step 5 **State the answer.** The speed of the current cannot be -3, so the answer is 3 mph.

Step 6 **Check** that this value satisfies the original problem.

Now Try Exercise 31.

CAUTION As shown in Example 2, when a quadratic equation is used to solve an applied problem, sometimes only *one* answer satisfies the application. *Always* check each answer in the words of the original problem.

In Chapter 7 we solved problems about work rates. Recall that a person's work rate is $\frac{1}{t}$ part of the job per hour, where t is the time in hours required to do the complete job. Thus, the part of the job the person will do in x hr is $\frac{1}{t}x$.

EXAMPLE 3 Solving a Work Problem

In takes two carpet layers 4 hr to carpet a room. If each worked alone, one of them could do the job in 1 hr less time than the other. How long would it take each carpet layer to complete the job alone?

Photo not available

Step 1 **Read** the problem again. There will be two answers.

Step 2 **Assign a variable.** Let x represent the number of hours for the slower carpet layer to complete the job alone. Then the faster carpet layer could do the entire job in $(x - 1)$ hr. The slower person's rate is $\frac{1}{x}$, and the faster person's rate is $\frac{1}{x - 1}$. Together, they do the job in 4 hr. Complete a table as shown.

	Rate	Time Working Together	Fractional Part of the Job Done
Slower Worker	$\frac{1}{x}$	4	$\frac{1}{x}(4)$
Faster Worker	$\frac{1}{x-1}$	4	$\frac{1}{x-1}(4)$

Sum is 1 whole job.

Step 3 **Write an equation.** The sum of the fractional parts done by the workers should equal 1 (the whole job).

$$\underbrace{\frac{4}{x}}_{\text{Part done by slower worker}} + \underbrace{\frac{4}{x-1}}_{\text{Part done by faster worker}} = \underbrace{1}_{\text{1 whole job}}$$

Step 4 **Solve** the equation. Multiply each side by the LCD, $x(x-1)$.

$$4(x-1) + 4x = x(x-1)$$

$$4x - 4 + 4x = x^2 - x \qquad \text{Distributive property}$$

$$x^2 - 9x + 4 = 0 \qquad \text{Standard form}$$

This equation cannot be solved by factoring, so use the quadratic formula.

$$x = \frac{9 \pm \sqrt{81 - 16}}{2} = \frac{9 \pm \sqrt{65}}{2} \qquad a = 1,\ b = -9,\ c = 4$$

To the nearest tenth,

$$x = \frac{9 + \sqrt{65}}{2} \approx 8.5 \quad \text{or} \quad x = \frac{9 - \sqrt{65}}{2} \approx .5. \qquad \text{Use a calculator.}$$

Step 5 **State the answer.** Only the solution 8.5 makes sense in the original problem. (Why?) Thus, the slower worker could do the job in about 8.5 hr and the faster in about $8.5 - 1 = 7.5$ hr.

Step 6 **Check** that these results satisfy the original problem.

Now Try Exercise 37.

OBJECTIVE 3 Solve an equation with radicals by writing it in quadratic form.

YOU TRY IT

EXAMPLE 4 Solving Radical Equations That Lead to Quadratic Equations

Solve each equation.

(a) $k = \sqrt{6k - 8}$

This equation is not quadratic. However, squaring both sides of the equation gives a quadratic equation that can be solved by factoring.

$$k^2 = 6k - 8 \qquad \text{Square both sides.}$$

$$k^2 - 6k + 8 = 0 \qquad \text{Standard form}$$

$$(k - 4)(k - 2) = 0 \qquad \text{Factor.}$$

$$k - 4 = 0 \quad \text{or} \quad k - 2 = 0 \qquad \text{Zero-factor property}$$

$$k = 4 \quad \text{or} \quad k = 2 \qquad \text{Potential solutions}$$

Recall from our work with radical equations in Section 8.6 that squaring both sides of an equation can introduce extraneous solutions that do not satisfy the original equation. Therefore, *all potential solutions must be checked in the original (not the squared) equation.*

Check: If $k = 4$, then

$$\begin{aligned} k &= \sqrt{6k - 8} \\ 4 &= \sqrt{6(4) - 8} \quad ? \\ 4 &= \sqrt{16} \quad ? \\ 4 &= 4. \quad \text{True} \end{aligned}$$

If $k = 2$, then

$$\begin{aligned} k &= \sqrt{6k - 8} \\ 2 &= \sqrt{6(2) - 8} \quad ? \\ 2 &= \sqrt{4} \quad ? \\ 2 &= 2. \quad \text{True} \end{aligned}$$

Both solutions check, so the solution set is $\{2, 4\}$.

(b) $x + \sqrt{x} = 6$

$$\begin{aligned} \sqrt{x} &= 6 - x && \text{Isolate the radical on one side.} \\ x &= 36 - 12x + x^2 && \text{Square both sides.} \\ 0 &= x^2 - 13x + 36 && \text{Standard form} \\ 0 &= (x - 4)(x - 9) && \text{Factor.} \\ x - 4 = 0 \quad &\text{or} \quad x - 9 = 0 && \text{Zero-factor property} \\ x = 4 \quad &\text{or} \quad x = 9 && \text{Potential solutions} \end{aligned}$$

Check both potential solutions in the *original* equation.

If $x = 4$, then

$$\begin{aligned} x + \sqrt{x} &= 6 \\ 4 + \sqrt{4} &= 6 \quad ? \\ 6 &= 6. \quad \text{True} \end{aligned}$$

If $x = 9$, then

$$\begin{aligned} x + \sqrt{x} &= 6 \\ 9 + \sqrt{9} &= 6 \quad ? \\ 12 &= 6. \quad \text{False} \end{aligned}$$

Only the solution 4 checks, so the solution set is $\{4\}$.

Now Try Exercises 41 and 47.

OBJECTIVE 4 Solve an equation that is quadratic in form by substitution. A nonquadratic equation that can be written in the form $au^2 + bu + c = 0$, for $a \neq 0$ and an algebraic expression u, is called **quadratic in form.**

EXAMPLE 5 Solving Equations That Are Quadratic in Form

Solve each equation.

(a) $x^4 - 13x^2 + 36 = 0$

Because $x^4 = (x^2)^2$, we can write this equation in quadratic form with $u = x^2$ and $u^2 = x^4$. (Any letter except x could be used instead of u.)

$$\begin{aligned} x^4 - 13x^2 + 36 &= 0 \\ (x^2)^2 - 13x^2 + 36 &= 0 && x^4 = (x^2)^2 \\ u^2 - 13u + 36 &= 0 && \text{Let } u = x^2. \\ (u - 4)(u - 9) &= 0 && \text{Factor.} \end{aligned}$$

$$u - 4 = 0 \quad \text{or} \quad u - 9 = 0 \qquad \text{Zero-factor property}$$
$$u = 4 \quad \text{or} \quad u = 9 \qquad \text{Solve.}$$

To find x, we substitute x^2 for u.

$$x^2 = 4 \quad \text{or} \quad x^2 = 9$$
$$x = \pm 2 \quad \text{or} \quad x = \pm 3 \qquad \text{Square root property}$$

The equation $x^4 - 13x^2 + 36 = 0$, a fourth-degree equation, has four solutions.* The solution set is $\{-3, -2, 2, 3\}$, which can be verified by substituting into the original equation for x.

(b) $4x^6 + 1 = 5x^3$

This equation is quadratic in form with $u = x^3$ and $u^2 = x^6$.

$$4x^6 + 1 = 5x^3$$
$$4(x^3)^2 + 1 = 5x^3$$
$$4u^2 + 1 = 5u \qquad \text{Let } u = x^3.$$
$$4u^2 - 5u + 1 = 0 \qquad \text{Standard form}$$
$$(4u - 1)(u - 1) = 0 \qquad \text{Factor.}$$
$$4u - 1 = 0 \quad \text{or} \quad u - 1 = 0 \qquad \text{Zero-factor property}$$
$$u = \frac{1}{4} \quad \text{or} \quad u = 1 \qquad \text{Solve.}$$
$$x^3 = \frac{1}{4} \quad \text{or} \quad x^3 = 1 \qquad u = x^3$$

From these equations,

$$x = \sqrt[3]{\frac{1}{4}} = \frac{\sqrt[3]{1}}{\sqrt[3]{4}} = \frac{1}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{\sqrt[3]{2}}{2} \qquad \text{or} \qquad x = \sqrt[3]{1} = 1.$$

There are other complex solutions for this equation, but finding them involves trigonometry. The real number solution set of $4x^6 + 1 = 5x^3$ is $\left\{\frac{\sqrt[3]{2}}{2}, 1\right\}$.

(c) $x^4 = 6x^2 - 3$

First write the equation as

$$x^4 - 6x^2 + 3 = 0 \qquad \text{or} \qquad (x^2)^2 - 6(x^2) + 3 = 0,$$

which is quadratic in form with $u = x^2$. Substitute u for x^2 and u^2 for x^4 to get

$$u^2 - 6u + 3 = 0.$$

*In general, an equation in which an nth-degree polynomial equals 0 has n complex solutions, although some of them may be repeated.

Since this equation cannot be solved by factoring, use the quadratic formula.

$$\begin{aligned}
u &= \frac{6 \pm \sqrt{36 - 12}}{2} && a = 1,\ b = -6,\ c = 3\\
u &= \frac{6 \pm \sqrt{24}}{2} &&\\
u &= \frac{6 \pm 2\sqrt{6}}{2} && \sqrt{24} = \sqrt{4}\cdot\sqrt{6} = 2\sqrt{6}\\
u &= \frac{2\left(3 \pm \sqrt{6}\right)}{2} && \text{Factor.}\\
u &= 3 \pm \sqrt{6} && \text{Lowest terms}
\end{aligned}$$

Since $u = x^2$, find x by using the square root property.

$$\begin{aligned}
x^2 &= 3 + \sqrt{6} \quad &\text{or}\quad x^2 &= 3 - \sqrt{6}\\
x &= \pm\sqrt{3 + \sqrt{6}} \quad &\text{or}\quad x &= \pm\sqrt{3 - \sqrt{6}}
\end{aligned}$$

The solution set contains four numbers:

$$\left\{\sqrt{3 + \sqrt{6}}, -\sqrt{3 + \sqrt{6}}, \sqrt{3 - \sqrt{6}}, -\sqrt{3 - \sqrt{6}}\right\}.$$

Now Try Exercises 55, 79, and 83.

NOTE Some students prefer to solve equations like those in Examples 5(a) and (b) by factoring directly. For example,

$$\begin{aligned}
x^4 - 13x^2 + 36 &= 0 && \text{Example 5(a) equation}\\
(x^2 - 9)(x^2 - 4) &= 0 && \text{Factor.}\\
(x + 3)(x - 3)(x + 2)(x - 2) &= 0. && \text{Factor again.}
\end{aligned}$$

Using the zero-factor property gives the same solutions obtained in Example 5(a). Equations that cannot be solved by factoring, like that in Example 5(c), must be solved using the method of substitution and the quadratic formula.

YOU TRY IT

EXAMPLE 6 Solving Equations That Are Quadratic in Form

Solve each equation.

(a) $2(4m - 3)^2 + 7(4m - 3) + 5 = 0$

Because of the repeated quantity $4m - 3$, this equation is quadratic in form with $u = 4m - 3$.

$$\begin{aligned}
2(4m - 3)^2 + 7(4m - 3) + 5 &= 0 &&\\
2u^2 + 7u + 5 &= 0 && \text{Let } 4m - 3 = u.\\
(2u + 5)(u + 1) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
2u + 5 &= 0 \quad &\text{or}\quad u + 1 &= 0 && \text{Zero-factor property}\\
u &= -\frac{5}{2} \quad &\text{or}\quad u &= -1 &&\\
4m - 3 &= -\frac{5}{2} \quad &\text{or}\quad 4m - 3 &= -1 && \text{Substitute } 4m - 3 \text{ for } u.
\end{aligned}$$

$$4m = \frac{1}{2} \quad \text{or} \quad 4m = 2 \qquad \text{Solve for } m.$$

$$m = \frac{1}{8} \quad \text{or} \quad m = \frac{1}{2}$$

Check that the solution set of the original equation is $\left\{\frac{1}{8}, \frac{1}{2}\right\}$.

(b) $2a^{2/3} - 11a^{1/3} + 12 = 0$

Let $a^{1/3} = u$; then $a^{2/3} = (a^{1/3})^2 = u^2$. Substitute into the given equation.

$$2u^2 - 11u + 12 = 0 \qquad \text{Let } a^{1/3} = u;\ a^{2/3} = u^2.$$

$$(2u - 3)(u - 4) = 0 \qquad \text{Factor.}$$

$$2u - 3 = 0 \quad \text{or} \quad u - 4 = 0 \qquad \text{Zero-factor property}$$

$$u = \frac{3}{2} \quad \text{or} \quad u = 4$$

$$a^{1/3} = \frac{3}{2} \quad \text{or} \quad a^{1/3} = 4 \qquad u = a^{1/3}$$

$$(a^{1/3})^3 = \left(\frac{3}{2}\right)^3 \quad \text{or} \quad (a^{1/3})^3 = 4^3 \qquad \text{Cube each side.}$$

$$a = \frac{27}{8} \quad \text{or} \quad a = 64$$

Check that the solution set is $\left\{\frac{27}{8}, 64\right\}$.

Now Try Exercises 59 and 65.

CAUTION A common error when solving problems like those in Examples 5 and 6 is to stop too soon. Once you have solved for u, remember to substitute and solve for the values of the *original* variable.

9.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 14/Videotape 14

Refer to the box at the beginning of this section. Decide whether factoring, the square root property, *or* the quadratic formula *is most appropriate for solving each quadratic equation. Do not actually solve the equations.*

1. $(2x + 3)^2 = 4$

2. $4x^2 - 3x = 1$

3. $z^2 + 5z - 8 = 0$

4. $2k^2 + 3k = 1$

5. $3m^2 = 2 - 5m$

6. $p^2 = 5$

Write a sentence describing the first step you would take to solve each equation. Do not actually solve.

7. $\dfrac{14}{x} = x - 5$

8. $\sqrt{1 + x} + x = 5$

9. $(r^2 + r)^2 - 8(r^2 + r) + 12 = 0$

10. $3t = \sqrt{16 - 10t}$

11. What is wrong with the following "solution"?

$$
\begin{aligned}
x &= \sqrt{3x+4} \\
x^2 &= 3x+4 \quad \text{Square both sides.} \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x - 4 = 0 \quad &\text{or} \quad x + 1 = 0 \\
x = 4 \quad &\text{or} \quad x = -1
\end{aligned}
$$

Solution set: $\{4, -1\}$

12. What is wrong with the following "solution"?

$$
\begin{aligned}
2(m-1)^2 - 3(m-1) + 1 &= 0 \\
2u^2 - 3u + 1 &= 0 \quad \text{Let } u = m - 1. \\
(2u-1)(u-1) &= 0 \\
2u - 1 = 0 \quad &\text{or} \quad u - 1 = 0 \\
u = \frac{1}{2} \quad &\text{or} \quad u = 1
\end{aligned}
$$

Solution set: $\left\{\frac{1}{2}, 1\right\}$

Solve each equation. Check your solutions. See Example 1.

13. $\dfrac{14}{x} = x - 5$

14. $\dfrac{-12}{x} = x + 8$

15. $1 - \dfrac{3}{x} - \dfrac{28}{x^2} = 0$

16. $4 - \dfrac{7}{r} - \dfrac{2}{r^2} = 0$

17. $3 - \dfrac{1}{t} = \dfrac{2}{t^2}$

18. $1 + \dfrac{2}{k} = \dfrac{3}{k^2}$

19. $\dfrac{1}{x} + \dfrac{2}{x+2} = \dfrac{17}{35}$

20. $\dfrac{2}{m} + \dfrac{3}{m+9} = \dfrac{11}{4}$

21. $\dfrac{2}{x+1} + \dfrac{3}{x+2} = \dfrac{7}{2}$

22. $\dfrac{4}{3-p} + \dfrac{2}{5-p} = \dfrac{26}{15}$

23. $\dfrac{3}{2x} - \dfrac{1}{2(x+2)} = 1$

24. $\dfrac{4}{3x} - \dfrac{1}{2(x+1)} = 1$

25. $3 = \dfrac{1}{t+2} + \dfrac{2}{(t+2)^2}$

26. $1 + \dfrac{2}{3z+2} = \dfrac{15}{(3z+2)^2}$

27. $\dfrac{6}{p} = 2 + \dfrac{p}{p+1}$

28. $\dfrac{k}{2-k} + \dfrac{2}{k} = 5$

Use the concepts of this section to answer each question.

29. A boat goes 20 mph in still water, and the rate of the current is t mph.

(a) What is the rate of the boat when it travels upstream?
(b) What is the rate of the boat when it travels downstream?

30. If it takes m hr to grade a set of papers, what is the grader's rate (in job per hour)?

Solve each problem. See Examples 2 and 3.

31. On a windy day Yoshiaki found that he could go 16 mi downstream and then 4 mi back upstream at top speed in a total of 48 min. What was the top speed of Yoshiaki's boat if the current was 15 mph?

	d	*r*	*t*
Upstream	4	$x - 15$	
Downstream	16		

32. Lekesha flew her plane for 6 hr at a constant speed. She traveled 810 mi with the wind, then turned around and traveled 720 mi against the wind. The wind speed was a constant 15 mph. Find the speed of the plane.

	d	*r*	*t*
With Wind	810		
Against Wind	720		

33. In Canada, Medicine Hat and Cranbrook are 300 km apart. Harry rides his Honda 20 km per hr faster than Yoshi rides his Yamaha. Find Harry's average speed if he travels from Cranbrook to Medicine Hat in $1\frac{1}{4}$ hr less time than Yoshi. (*Source: State Farm Road Atlas.*)

34. In California, the distance from Jackson to Lodi is about 40 mi, as is the distance from Lodi to Manteca. Rico drove from Jackson to Lodi during the rush hour, stopped in Lodi for a root beer, and then drove on to Manteca at 10 mph faster. Driving time for the entire trip was 88 min. Find his speed from Jackson to Lodi. (*Source: State Farm Road Atlas.*)

35. Working together, two people can cut a large lawn in 2 hr. One person can do the job alone in 1 hr less time than the other. How long (to the nearest tenth) would it take the faster person to do the job? (*Hint: x* is the time of the faster person.)

	Rate	*Time Working Together*	*Fractional Part of the Job Done*
Faster Worker	$\frac{1}{x}$	2	
Slower Worker		2	

36. A janitorial service provides two people to clean an office building. Working together, the two can clean the building in 5 hr. One person is new to the job and would take 2 hr longer than the other person to clean the building alone. How long (to the nearest tenth) would it take the new worker to clean the building alone?

	Rate	*Time Working Together*	*Fractional Part of the Job Done*
Faster Worker			
Slower Worker			

37. Rusty and Nancy Brauner are planting flats of spring flowers. Working alone, Rusty would take 2 hr longer than Nancy to plant the flowers. Working together, they do the job in 12 hr. How long would it have taken each person working alone?

38. Jay Beckenstein can work through a stack of invoices in 1 hr less time than Colleen Manley Jones can. Working together they take $1\frac{1}{2}$ hr. How long would it take each person working alone?

39. A washing machine can be filled in 6 min if both the hot and cold water taps are fully opened. Filling the washer with hot water alone takes 9 min longer than filling it with cold water alone. How long does it take to fill the washer with cold water?

40. Two pipes together can fill a large tank in 2 hr. One of the pipes, used alone, takes 3 hr longer than the other to fill the tank. How long would each pipe take to fill the tank alone?

Solve each equation. Check your solutions. See Example 4.

EXERCISES

41. $x = \sqrt{7x - 10}$ **42.** $z = \sqrt{5z - 4}$ **43.** $2x = \sqrt{11x + 3}$ **44.** $4x = \sqrt{6x + 1}$

45. $3x = \sqrt{16 - 10x}$ **46.** $4t = \sqrt{8t + 3}$ **47.** $p - 2\sqrt{p} = 8$ **48.** $k + \sqrt{k} = 12$

49. $m = \sqrt{\dfrac{6 - 13m}{5}}$ **50.** $r = \sqrt{\dfrac{20 - 19r}{6}}$

Solve each equation. Check your solutions. See Examples 5 and 6.

EXERCISES

51. $t^4 - 18t^2 + 81 = 0$ **52.** $x^4 - 8x^2 + 16 = 0$ **53.** $4k^4 - 13k^2 + 9 = 0$

54. $9x^4 - 25x^2 + 16 = 0$ **55.** $x^4 + 48 = 16x^2$ **56.** $z^4 = 17z^2 - 72$

57. $(x + 3)^2 + 5(x + 3) + 6 = 0$ **58.** $(k - 4)^2 + (k - 4) - 20 = 0$

59. $3(m + 4)^2 - 8 = 2(m + 4)$ **60.** $(t + 5)^2 + 6 = 7(t + 5)$

61. $2 + \dfrac{5}{3k - 1} = \dfrac{-2}{(3k - 1)^2}$ **62.** $3 - \dfrac{7}{2p + 2} = \dfrac{6}{(2p + 2)^2}$

63. $2 - 6(m - 1)^{-2} = (m - 1)^{-1}$ **64.** $3 - 2(x - 1)^{-1} = (x - 1)^{-2}$

65. $x^{2/3} + x^{1/3} - 2 = 0$ **66.** $x^{2/3} - 2x^{1/3} - 3 = 0$

67. $r^{2/3} + r^{1/3} - 12 = 0$ **68.** $3x^{2/3} - x^{1/3} - 24 = 0$

69. $4k^{4/3} - 13k^{2/3} + 9 = 0$ **70.** $9m^{2/5} = 16 - 10m^{1/5}$

71. $2\left(1 + \sqrt{r}\right)^2 = 13\left(1 + \sqrt{r}\right) - 6$ **72.** $(k^2 + k)^2 + 12 = 8(k^2 + k)$

73. $2x^4 + x^2 - 3 = 0$ **74.** $4k^4 + 5k^2 + 1 = 0$

The equations in Exercises 75–84 are not grouped by type. Decide which method of solution applies, and then solve each equation. Give only real solutions. See Examples 1 and 4–6.

75. $12x^4 - 11x^2 + 2 = 0$ **76.** $\left(x - \dfrac{1}{2}\right)^2 + 5\left(x - \dfrac{1}{2}\right) - 4 = 0$

77. $\sqrt{2x + 3} = 2 + \sqrt{x - 2}$ **78.** $\sqrt{m + 1} = -1 + \sqrt{2m}$

79. $2m^6 + 11m^3 + 5 = 0$ **80.** $8x^6 + 513x^3 + 64 = 0$

81. $6 = 7(2w - 3)^{-1} + 3(2w - 3)^{-2}$ **82.** $m^6 - 10m^3 = -9$

83. $2x^4 - 9x^2 = -2$ **84.** $8x^4 + 1 = 11x^2$

RELATING CONCEPTS (EXERCISES 85–90)

For Individual or Group Work

Consider the following equation, and ***work Exercises 85–90 in order.***

$$\frac{x^2}{(x - 3)^2} + \frac{3x}{x - 3} - 4 = 0.$$

85. Why must 3 be excluded from the domain of this equation?

86. Multiply both sides of the equation by the LCD, $(x - 3)^2$, and solve. There is only one solution—what is it?

87. Write the equation so that it is quadratic in form using the rational expression $\frac{x}{x - 3}$.

88. Explain why the expression $\frac{x}{x - 3}$ cannot equal 1.

89. Solve the equation from Exercise 87 by making the substitution $t = \frac{x}{x - 3}$. You should get two values for t. Why is one of them impossible for this equation?

90. Solve the equation $x^2(x - 3)^{-2} + 3x(x - 3)^{-1} - 4 = 0$ by letting $s = (x - 3)^{-1}$. You should get two values for s. Why is this impossible for this equation?

9.4 Formulas and Further Applications

OBJECTIVES

1. Solve formulas for variables involving squares and square roots.
2. Solve applied problems using the Pythagorean formula.
3. Solve applied problems using area formulas.
4. Solve applied problems using quadratic functions as models.

OBJECTIVE 1 Solve formulas for variables involving squares and square roots. The methods presented earlier can be used to solve such formulas.

EXAMPLE 1 Solving for Variables Involving Squares or Square Roots

YOU TRY IT

Solve each formula for the given variable.

(a) $w = \dfrac{kFr}{v^2}$ for v

$$w = \frac{kFr}{v^2} \qquad \text{Get } v \text{ alone on one side.}$$

$$v^2 w = kFr \qquad \text{Multiply by } v^2.$$

$$v^2 = \frac{kFr}{w} \qquad \text{Divide by } w.$$

$$v = \pm\sqrt{\frac{kFr}{w}} \qquad \text{Square root property}$$

$$v = \frac{\pm\sqrt{kFr}}{\sqrt{w}} \cdot \frac{\sqrt{w}}{\sqrt{w}} = \frac{\pm\sqrt{kFrw}}{w} \qquad \text{Rationalize the denominator.}$$

(b) $d = \sqrt{\dfrac{4A}{\pi}}$ for A

$$d = \sqrt{\frac{4A}{\pi}} \qquad \text{Get } A \text{ alone on one side.}$$

$$d^2 = \frac{4A}{\pi} \qquad \text{Square both sides.}$$

$$\pi d^2 = 4A \qquad \text{Multiply by } \pi.$$

$$\frac{\pi d^2}{4} = A \qquad \text{Divide by 4.}$$

Now Try Exercises 9 and 19.

NOTE In many formulas like $v = \dfrac{\pm\sqrt{kFrw}}{w}$ in Example 1(a), we choose the positive value. In our work here, we will include both positive and negative values.

VIDEO

EXAMPLE 2 Solving for a Variable that Appears in First- and Second-Degree Terms

Solve $s = 2t^2 + kt$ for t.

Since the given equation has terms with t^2 and t, write it in standard form $ax^2 + bx + c = 0$, with t as the variable instead of x.

$$2t^2 + kt - s = 0$$

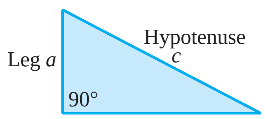
Now use the quadratic formula with $a = 2$, $b = k$, and $c = -s$.

$$t = \frac{-k \pm \sqrt{k^2 - 4(2)(-s)}}{2(2)} \qquad \text{Solve for } t.$$

$$t = \frac{-k \pm \sqrt{k^2 + 8s}}{4}$$

The solutions are $t = \dfrac{-k + \sqrt{k^2 + 8s}}{4}$ and $t = \dfrac{-k - \sqrt{k^2 + 8s}}{4}$.

Now Try Exercise 15.



$c^2 = a^2 + b^2$
Pythagorean Formula

OBJECTIVE 2 Solve applied problems using the Pythagorean formula. The Pythagorean formula $a^2 + b^2 = c^2$, illustrated by the figure in the margin, was introduced in Chapter 8 and is used to solve applications involving right triangles. Such problems often require solving quadratic equations.

VIDEO

EXAMPLE 3 Using the Pythagorean Formula

Two cars left an intersection at the same time, one heading due north, the other due west. Some time later, they were exactly 100 mi apart. The car headed north had gone 20 mi farther than the car headed west. How far had each car traveled?

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.** Let x be the distance traveled by the car headed west. Then $(x + 20)$ is the distance traveled by the car headed north. See Figure 2. The cars are 100 mi apart, so the hypotenuse of the right triangle equals 100.

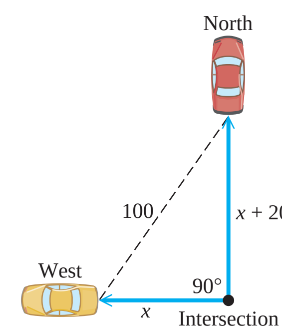


FIGURE 2

Step 3 **Write an equation.** Use the Pythagorean formula.

$$c^2 = a^2 + b^2$$
$$100^2 = x^2 + (x + 20)^2$$

Step 4 **Solve.**

$$10{,}000 = x^2 + x^2 + 40x + 400 \qquad \text{Square the binomial.}$$
$$0 = 2x^2 + 40x - 9600 \qquad \text{Standard form}$$
$$0 = x^2 + 20x - 4800 \qquad \text{Divide by 2.}$$
$$0 = (x + 80)(x - 60) \qquad \text{Factor.}$$
$$x + 80 = 0 \quad \text{or} \quad x - 60 = 0 \qquad \text{Zero-factor property}$$
$$x = -80 \quad \text{or} \quad x = 60$$

Step 5 **State the answer.** Since distance cannot be negative, discard the negative solution. The required distances are 60 mi and $60 + 20 = 80$ mi.

Step 6 **Check.** Since $60^2 + 80^2 = 100^2$, the answer is correct.

Now Try Exercise 31.

OBJECTIVE 3 Solve applied problems using area formulas.

EXAMPLE 4 Solving an Area Problem

A rectangular reflecting pool in a park is 20 ft wide and 30 ft long. The park gardener wants to plant a strip of grass of uniform width around the edge of the pool. She has enough seed to cover 336 ft^2. How wide will the strip be?

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.** The pool is shown in Figure 3. If x represents the unknown width of the grass strip, the width of the large rectangle is given by $20 + 2x$ (the width of the pool plus two grass strips), and the length is given by $30 + 2x$.

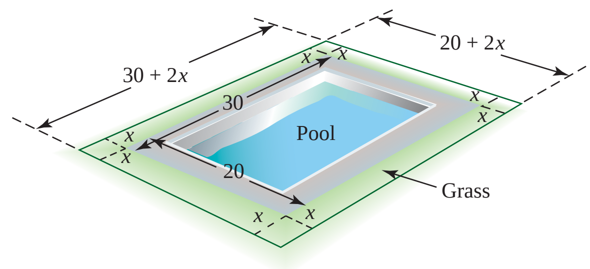


FIGURE 3

Step 3 **Write an equation.** The area of the large rectangle is given by the product of its length and width, $(30 + 2x)(20 + 2x)$. The area of the pool is $30 \cdot 20 = 600$ ft^2. The area of the large rectangle minus the area of the pool should equal the area of the grass strip. Since the area of the grass strip is to be 336 ft^2, the equation is

$$\overset{\substack{\text{Area}\\\text{of}\\\text{rectangle}\\\downarrow}}{(30 + 2x)(20 + 2x)} \overset{-}{-} \overset{\substack{\text{Area}\\\text{of}\\\text{pool}\\\downarrow}}{600} \overset{=}{=} \overset{\substack{\text{Area}\\\text{of}\\\text{grass}\\\downarrow}}{336}.$$

Step 4 **Solve.**

$$\begin{aligned} 600 + 100x + 4x^2 - 600 &= 336 && \text{Multiply.}\\ 4x^2 + 100x - 336 &= 0 && \text{Standard form}\\ x^2 + 25x - 84 &= 0 && \text{Divide by 4.}\\ (x + 28)(x - 3) &= 0 && \text{Factor.}\\ x = -28 \quad \text{or} \quad x &= 3 && \text{Zero-factor property} \end{aligned}$$

Step 5 **State the answer.** The width cannot be -28 ft, so the grass strip should be 3 ft wide.

Step 6 **Check.** If $x = 3$, then the area of the large rectangle (which includes the grass strip) is

$$(30 + 2 \cdot 3)(20 + 2 \cdot 3) = 36 \cdot 26 = 936 \text{ ft}^2. \qquad \text{Area of pool and strip}$$

The area of the pool is $30 \cdot 20 = 600$ ft^2. So, the area of the grass strip is $936 - 600 = 336$ ft^2, which is the area the gardener had enough seed to cover. The answer is correct.

Now Try Exercise 37.

OBJECTIVE 4 Solve applied problems using quadratic functions as models. Some applied problems can be modeled by *quadratic functions,* which can be written in the form

$$f(x) = ax^2 + bx + c,$$

for real numbers a, b, and c, $a \neq 0$.

EXAMPLE 5 Solving an Applied Problem Using a Quadratic Function

VIDEO

YOU TRY IT

If an object is propelled upward from the top of a 144-ft building at 112 ft per sec, its position (in feet above the ground) is given by

$$s(t) = -16t^2 + 112t + 144,$$

where t is time in seconds after it was propelled. When does it hit the ground?

When the object hits the ground, its distance above the ground is 0. We must find the value of t that makes $s(t) = 0$.

$$0 = -16t^2 + 112t + 144 \qquad \text{Let } s(t) = 0.$$
$$0 = t^2 - 7t - 9 \qquad \text{Divide by } -16.$$
$$t = \frac{7 \pm \sqrt{49 + 36}}{2} \qquad \text{Quadratic formula}$$
$$t = \frac{7 \pm \sqrt{85}}{2} \approx \frac{7 \pm 9.2}{2} \qquad \text{Use a calculator.}$$

The solutions are $t \approx 8.1$ or $t \approx -1.1$. Time cannot be negative, so we discard the negative solution. The object hits the ground about 8.1 sec after it is propelled.

Now Try Exercise 43.

EXAMPLE 6 Using a Quadratic Function to Model Company Bankruptcy Filings

Photo not available

The number of companies filing for bankruptcy was high in the early 1990s due to an economic recession. The number then declined during the middle 1990s, and in recent years has increased again. The quadratic function defined by

$$f(x) = 3.37x^2 - 28.6x + 133$$

approximates the number of company bankruptcy filings during the years 1990–2001, where x is the number of years since 1990. (*Source:* www.BankruptcyData.com)

(a) Use the model to approximate the number of company bankruptcy filings in 1995.

For 1995, $x = 5$, so find $f(5)$.

$$f(5) = 3.37(5)^2 - 28.6(5) + 133 \qquad \text{Let } x = 5.$$
$$= 74.25$$

There were about 74 company bankruptcy filings in 1995.

(b) In what year did company bankruptcy filings reach 150?

Find the value of x that makes $f(x) = 150$.

$$f(x) = 3.37x^2 - 28.6x + 133$$
$$150 = 3.37x^2 - 28.6x + 133 \qquad \text{Let } f(x) = 150.$$
$$0 = 3.37x^2 - 28.6x - 17 \qquad \text{Standard form}$$

Now use $a = 3.37$, $b = -28.6$, and $c = -17$ in the quadratic formula.

$$x = \frac{28.6 \pm \sqrt{(-28.6)^2 - 4(3.37)(-17)}}{2(3.37)}$$

$$x \approx 9.0 \quad \text{or} \quad x \approx -.56 \qquad \text{Use a calculator.}$$

The positive solution is $x \approx 9$, so company bankruptcy filings reached 150 in the year $1990 + 9 = 1999$. (Reject the negative solution since the model is not valid for negative values of x.) Note that company bankruptcy filings doubled from about 74 in 1995 to 150 in 1999.

Now Try Exercises 55 and 57.

9.4 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 15/Videotape 15

Answer each question in Exercises 1–4.

1. What is the first step in solving a formula that has the specified variable in the denominator?

2. What is the first step in solving a formula like $gw^2 = 2r$ for w?

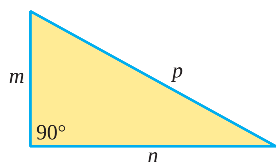
3. What is the first step in solving a formula like $gw^2 = kw + 24$ for w?

4. Why is it particularly important to check all proposed solutions to an applied problem against the information in the original problem?

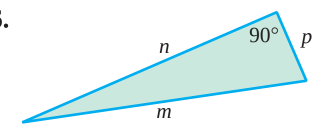
In Exercises 5 and 6, solve for m in terms of the other variables ($m > 0$).

EXERCISES

5.



6.



Solve each equation for the indicated variable. (Leave ± in your answers.) See Examples 1 and 2.

EXERCISES

7. $d = kt^2$ for t

8. $s = kwd^2$ for d

9. $I = \dfrac{ks}{d^2}$ for d

10. $R = \dfrac{k}{d^2}$ for d

11. $F = \dfrac{kA}{v^2}$ for v

12. $L = \dfrac{kd^4}{h^2}$ for h

13. $V = \dfrac{1}{3}\pi r^2 h$ for r

14. $V = \pi(r^2 + R^2)h$ for r

15. $At^2 + Bt = -C$ for t

16. $S = 2\pi rh + \pi r^2$ for r

17. $D = \sqrt{kh}$ for h

18. $F = \dfrac{k}{\sqrt{d}}$ for d

19. $p = \sqrt{\dfrac{k\ell}{g}}$ for ℓ

20. $p = \sqrt{\dfrac{k\ell}{g}}$ for g

21. If g is a positive number in the formula of Exercise 19, explain why k and ℓ must have the same sign in order for p to be a real number.

22. Refer to Example 2 of this section. Suppose that k and s both represent positive numbers.

(a) Which one of the two solutions given is positive?
(b) Which one is negative? **(c)** How can you tell?

Solve each equation for the indicated variable.

23. $p = \dfrac{E^2R}{(r + R)^2}$ for R $\quad (E > 0)$

24. $S(6S - t) = t^2$ for S

25. $10p^2c^2 + 7pcr = 12r^2$ for r

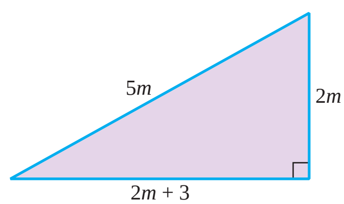
26. $S = vt + \dfrac{1}{2}gt^2$ for t

27. $LI^2 + RI + \dfrac{1}{c} = 0$ for I

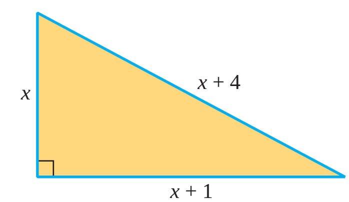
28. $P = EI - RI^2$ for I

Solve each problem. When appropriate, round answers to the nearest tenth. See Example 3.

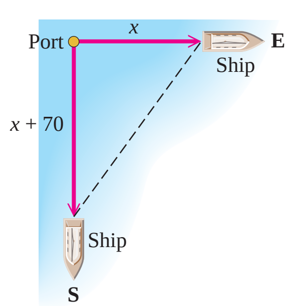
29. Find the lengths of the sides of the triangle.



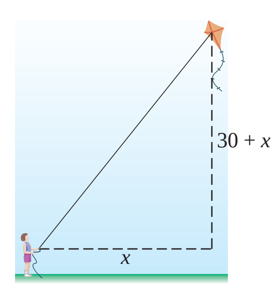
30. Find the lengths of the sides of the triangle.



31. Two ships leave port at the same time, one heading due south and the other heading due east. Several hours later, they are 170 mi apart. If the ship traveling south traveled 70 mi farther than the other ship, how many miles did they each travel?



32. Allyson Pellissier is flying a kite that is 30 ft farther above her hand than its horizontal distance from her. The string from her hand to the kite is 150 ft long. How high is the kite?

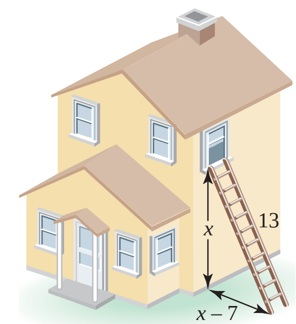


33. A toy manufacturer needs a piece of plastic in the shape of a right triangle with the longer leg 2 cm more than twice as long as the shorter leg, and the hypotenuse 1 cm more than the longer leg. How long should the three sides of the triangular piece be?

34. Michael Fuentes, a developer, owns a piece of land enclosed on three sides by streets, giving it the shape of a right triangle. The hypotenuse is 8 m longer than the longer leg, and the shorter leg is 9 m shorter than the hypotenuse. Find the lengths of the three sides of the property.

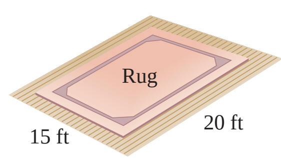
35. Two pieces of a large wooden puzzle fit together to form a rectangle with length 1 cm less than twice the width. The diagonal, where the two pieces meet, is 2.5 cm in length. Find the length and width of the rectangle.

36. A 13-ft ladder is leaning against a house. The distance from the bottom of the ladder to the house is 7 ft less than the distance from the top of the ladder to the ground. How far is the bottom of the ladder from the house?

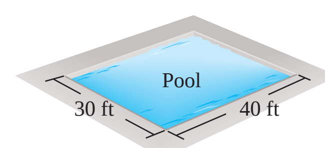


Solve each problem. See Example 4.

37. A couple wants to buy a rug for a room that is 20 ft long and 15 ft wide. They want to leave an even strip of flooring uncovered around the edges of the room. How wide a strip will they have if they buy a rug with an area of 234 ft^2?



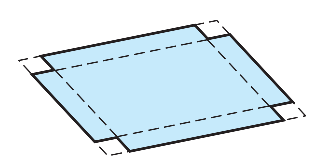
38. A club swimming pool is 30 ft wide and 40 ft long. The club members want an exposed aggregate border in a strip of uniform width around the pool. They have enough material for 296 ft^2. How wide can the strip be?



39. Arif's backyard is 20 m by 30 m. He wants to put a flower garden in the middle of the backyard, leaving a strip of grass of uniform width around the flower garden. Arif must have 184 m^2 of grass. Under these conditions, what will the length and width of the garden be?

40. A rectangle has a length 2 m less than twice its width. When 5 m are added to the width, the resulting figure is a square with an area of 144 m^2. Find the dimensions of the original rectangle.

41. A rectangular piece of sheet metal has a length that is 4 in. less than twice the width. A square piece 2 in. on a side is cut from each corner. The sides are then turned up to form an uncovered box of volume 256 $in.^3$. Find the length and width of the original piece of metal.



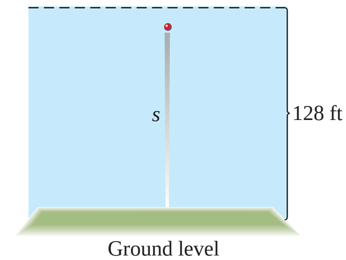
42. Another rectangular piece of sheet metal is 2 in. longer than it is wide. A square piece 3 in. on a side is cut from each corner. The sides are then turned up to form an uncovered box of volume 765 $in.^3$. Find the dimensions of the original piece of metal.

Solve each problem. When appropriate, round answers to the nearest tenth. See Example 5.

43. An object is projected directly upward from the ground. After t sec its distance in feet above the ground is

$$s(t) = 144t - 16t^2.$$

After how many seconds will the object be 128 ft above the ground? (*Hint:* Look for a common factor before solving the equation.)

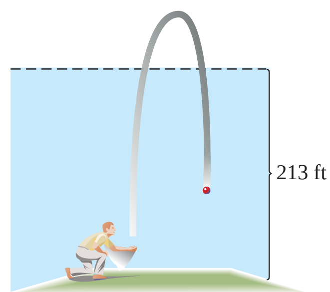


44. When does the object in Exercise 43 strike the ground?

45. A ball is projected upward from the ground. Its distance in feet from the ground in t sec is given by

$$s(t) = -16t^2 + 128t.$$

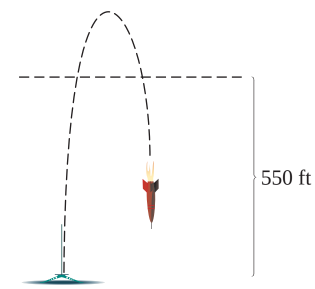
At what times will the ball be 213 ft from the ground?



46. A toy rocket is launched from ground level. Its distance in feet from the ground in t sec is given by

$$s(t) = -16t^2 + 208t.$$

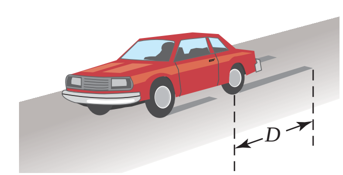
At what times will the rocket be 550 ft from the ground?



47. The function defined by

$$D(t) = 13t^2 - 100t$$

gives the distance in feet a car going approximately 68 mph will skid in t sec. Find the time it would take for the car to skid 180 ft.



48. The function given in Exercise 47 becomes

$$D(t) = 13t^2 - 73t$$

for a car going 50 mph. Find the time for this car to skid 218 ft.

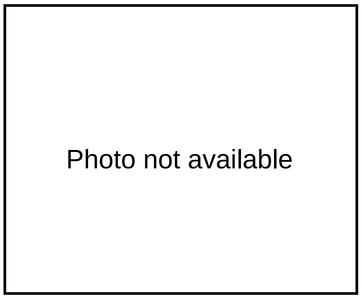
A rock is projected upward from ground level, and its distance in feet from the ground in t sec is given by $s(t) = -16t^2 + 160t$. Use algebra and a short explanation to answer Exercises 49 and 50.

49. After how many seconds does it reach a height of 400 ft? How would you describe in words its position at this height?

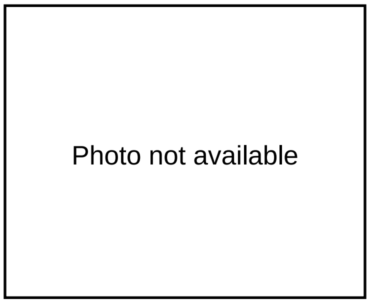
50. After how many seconds does it reach a height of 425 ft? How would you interpret the mathematical result here?

Solve each problem using a quadratic equation.

51. A certain bakery has found that the daily demand for bran muffins is $\frac{3200}{p}$, where p is the price of a muffin in cents. The daily supply is $3p - 200$. Find the price at which supply and demand are equal.



52. In one area the demand for compact discs is $\frac{700}{P}$ per day, where P is the price in dollars per disc. The supply is $5P - 1$ per day. At what price does supply equal demand?



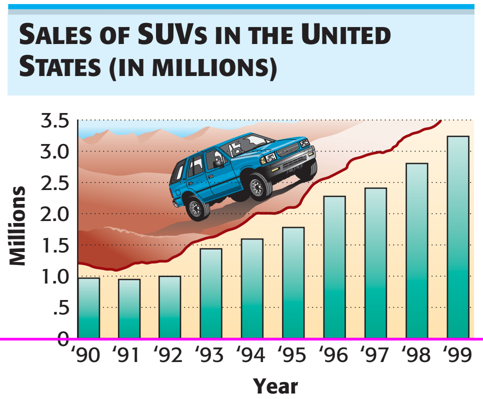
53. The formula $A = P(1 + r)^2$ gives the amount A in dollars that P dollars will grow to in 2 yr at interest rate r (where r is given as a decimal), using compound interest. What interest rate will cause \$2000 to grow to \$2142.25 in 2 yr?

54. If a square piece of cardboard has 3-in. squares cut from its corners and then has the flaps folded up to form an open-top box, the volume of the box is given by the formula $V = 3(x - 6)^2$, where x is the length of each side of the original piece of cardboard in inches. What original length would yield a box with volume 432 in.^{3}?

Sales of SUVs (sport utility vehicles) in the United States (in millions) for the years 1990 through 1999 are shown in the bar graph and can be modeled by the quadratic function defined by

$$f(x) = .016x^2 + .124x + .787.$$

Here, $x = 0$ represents 1990, $x = 1$ represents 1991, and so on. Use the graph and the model to work Exercises 55–58. See Example 6.



Source: CNW Marketing Research of Bandon, OR, based on automakers' reported sales.

55. (a) Use the graph to estimate sales in 1997 to the nearest tenth.

(b) Use the model to approximate sales in 1997 to the nearest tenth. How does this result compare to your estimate from part (a)?

56. (a) Use the model to estimate sales in 2000 to the nearest tenth.

(b) Sales through October 2000 were about 2.9 million. Based on this, is the sales estimate for 2000 from part (a) reasonable? Explain.

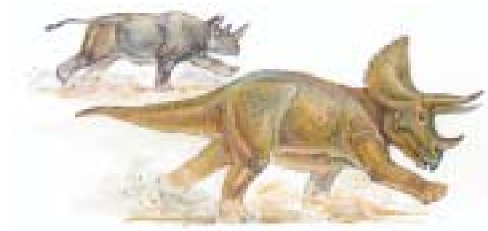
57. Based on the model, in what year did sales reach 2 million? (Round down to the nearest year.) How does this result compare to the sales shown in the graph?

58. Based on the model, in what year did sales reach 3 million? (Round down to the nearest year.) How does this result compare to the sales shown in the graph?

William Froude was a 19th century naval architect who used the expression

$$\frac{v^2}{g\ell}$$

in shipbuilding. This expression, known as the Froude number, was also used by R. McNeill Alexander in his research on dinosaurs. (Source: *"How Dinosaurs Ran,"* Scientific American, *April 1991.) In Exercises 59 and 60, find the value of v (in meters per second), given* $g = 9.8$ m per sec^2.

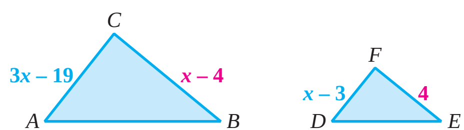


59. Rhinoceros: $\ell = 1.2$;
Froude number $= 2.57$

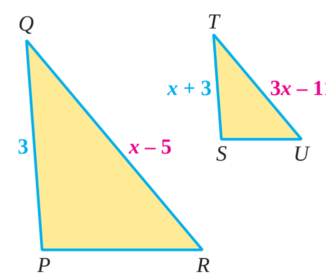
60. Triceratops: $\ell = 2.8$;
Froude number $= .16$

Recall that corresponding sides of similar triangles are proportional. Use this fact to find the lengths of the indicated sides of each pair of similar triangles. Check all possible solutions in both triangles. Sides of a triangle cannot be negative (and are not drawn to scale here).

61. Side AC



62. Side RQ



9.5 Graphs of Quadratic Functions

OBJECTIVES

1. Graph a quadratic function.
2. Graph parabolas with horizontal and vertical shifts.
3. Predict the shape and direction of a parabola from the coefficient of x^2.
4. Find a quadratic function to model data.

OBJECTIVE 1 Graph a quadratic function. Polynomial functions were defined in Chapter 5, where we graphed a few simple second-degree polynomial functions by point-plotting. In Figure 4, we repeat a table of ordered pairs for the simplest quadratic function, defined by $y = x^2$, and the resulting graph.

x	y
-2	4
-1	1
0	0
1	1
2	4

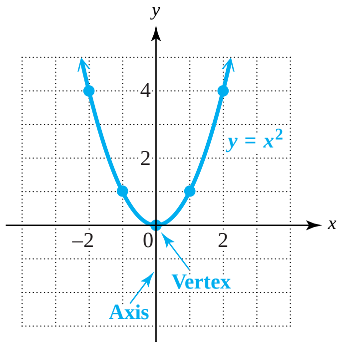


FIGURE 4

As mentioned in Chapter 5, this graph is called a **parabola.** The point (0, 0), the lowest point on the curve, is the **vertex** of this parabola. The vertical line through the vertex is the **axis** of the parabola, here $x = 0$. A parabola is **symmetric about its axis;** that is, if the graph were folded along the axis, the two portions of the curve would coincide. As Figure 4 suggests, x can be any real number, so the domain of the function defined by $y = x^2$ is $(-\infty, \infty)$. Since y is always nonnegative, the range is $[0, \infty)$.

In Section 9.4, we solved applications modeled by quadratic functions. In this section and the next, we consider graphs of more general quadratic functions as defined here.

Quadratic Function

A function that can be written in the form

$$f(x) = ax^2 + bx + c$$

for real numbers a, b, and c, with $a \neq 0$, is a **quadratic function.**

The graph of any quadratic function is a parabola with a vertical axis.

NOTE We use the variable y and function notation $f(x)$ interchangeably when discussing parabolas. Although we use the letter f most often to name quadratic functions, other letters can be used. We use the capital letter F to distinguish between different parabolas graphed on the same coordinate axes.

Photo not available

Parabolas, which are a type of *conic section* (Chapter 11), have many applications. The large dishes seen on the sidelines of televised football games, which are used by television crews to pick up the shouted signals of players on the field, have cross sections that are parabolas. Cross sections of satellite dishes and automobile headlights also form parabolas. The cables that are used to support suspension bridges are shaped like parabolas.

OBJECTIVE 2 Graph parabolas with horizontal and vertical shifts. Parabolas need not have their vertices at the origin, as does the graph of $f(x) = x^2$. For example, to graph a parabola of the form $F(x) = x^2 + k$, start by selecting sample values of x like those that were used to graph $f(x) = x^2$. The corresponding values of $F(x)$ in $F(x) = x^2 + k$ differ by k from those of $f(x) = x^2$. For this reason, the graph of $F(x) = x^2 + k$ is *shifted*, or *translated*, k units vertically compared with that of $f(x) = x^2$.

EXAMPLE 1 Graphing a Parabola with a Vertical Shift

Graph $F(x) = x^2 - 2$.

This graph has the same shape as that of $f(x) = x^2$, but since k here is -2, the graph is shifted 2 units down, with vertex $(0, -2)$. Every function value is 2 less than the corresponding function value of $f(x) = x^2$. Plotting points on both sides of the vertex gives the graph in Figure 5 on the next page. Notice that since the parabola is symmetric about its axis $x = 0$, the plotted points are "mirror images" of each other. Since x can be any real number, the domain is still $(-\infty, \infty)$; the value of y (or $F(x)$) is always greater than or equal to -2, so the range is $[-2, \infty)$. The graph of $f(x) = x^2$ is shown in Figure 5 for comparison.

x	$f(x) = x^2$	$F(x) = x^2 - 2$
-2	4	2
-1	1	-1
0	0	-2
1	1	-1
2	4	2

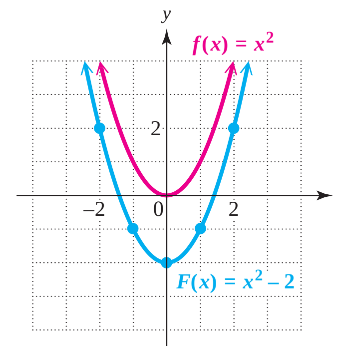


FIGURE 5

Now Try Exercise 23.

Vertical Shift

The graph of $\boldsymbol{F(x) = x^2 + k}$ is a parabola with the same shape as the graph of $f(x) = x^2$. The parabola is shifted k units up if $k > 0$, and $|k|$ units down if $k < 0$. The vertex is $(0, k)$.

The graph of $F(x) = (x - h)^2$ is also a parabola with the same shape as that of $f(x) = x^2$. Because $(x - h)^2 \geq 0$ for all x, the vertex of $F(x) = (x - h)^2$ is the lowest point on the parabola. The lowest point occurs here when $F(x)$ is 0. To get $F(x)$ equal to 0, let $x = h$ so the vertex of $F(x) = (x - h)^2$ is $(h, 0)$. Based on this, the graph of $F(x) = (x - h)^2$ is shifted h units horizontally compared with that of $f(x) = x^2$.

EXAMPLE 2 Graphing a Parabola with a Horizontal Shift

Graph $F(x) = (x - 2)^2$.

If $x = 2$, then $F(x) = 0$, giving the vertex $(2, 0)$. The graph of $F(x) = (x - 2)^2$ has the same shape as that of $f(x) = x^2$ but is shifted 2 units to the right. Plotting several points on one side of the vertex and using symmetry about the axis $x = 2$ to find corresponding points on the other side of the vertex gives the graph in Figure 6. Again, the domain is $(-\infty, \infty)$; the range is $[0, \infty)$.

x	$F(x) = (x - 2)^2$
0	4
1	1
2	0
3	1
4	4

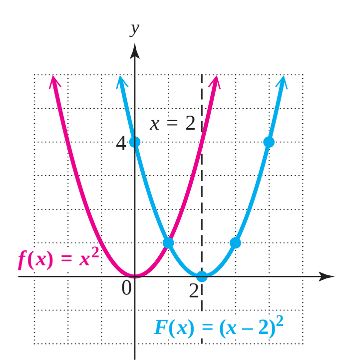


FIGURE 6

Now Try Exercise 27.

Horizontal Shift

The graph of $F(x) = (x - h)^2$ is a parabola with the same shape as the graph of $f(x) = x^2$. The parabola is shifted h units horizontally: h units to the right if $h > 0$, and $|h|$ units to the left if $h < 0$. The vertex is $(h, 0)$.

CAUTION Errors frequently occur when horizontal shifts are involved. To determine the direction and magnitude of a horizontal shift, find the value that would cause the expression $x - h$ to equal 0. For example, the graph of $F(x) = (x - 5)^2$ would be shifted 5 units to the *right*, because $+5$ would cause $x - 5$ to equal 0. On the other hand, the graph of $F(x) = (x + 5)^2$ would be shifted 5 units to the *left*, because -5 would cause $x + 5$ to equal 0.

A parabola can have both horizontal and vertical shifts.

EXAMPLE 3 Graphing a Parabola with Horizontal and Vertical Shifts

Graph $F(x) = (x + 3)^2 - 2$.

This graph has the same shape as that of $f(x) = x^2$, but is shifted 3 units to the left (since $x + 3 = 0$ if $x = -3$) and 2 units down (because of the -2). As shown in Figure 7, the vertex is $(-3, -2)$, with axis $x = -3$. This function has domain $(-\infty, \infty)$ and range $[-2, \infty)$.

x	$F(x)$
-5	2
-4	-1
-3	-2
-2	-1
-1	2

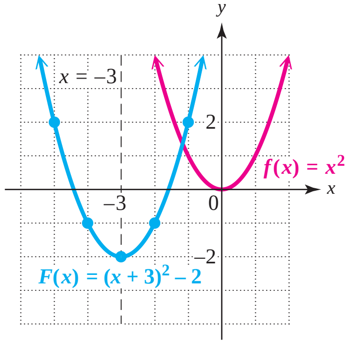


FIGURE 7

Now Try Exercise 29.

The characteristics of the graph of a parabola of the form $F(x) = (x - h)^2 + k$ are summarized as follows.

Vertex and Axis of a Parabola

The graph of $F(x) = (x - h)^2 + k$ is a parabola with the same shape as the graph of $f(x) = x^2$, but with vertex (h, k). The axis is the vertical line $x = h$.

OBJECTIVE 3 Predict the shape and direction of a parabola from the coefficient of x^2.

Not all parabolas open up, and not all parabolas have the same shape as the graph of $f(x) = x^2$.

EXAMPLE 4 Graphing a Parabola That Opens Down

Graph $f(x) = -\dfrac{1}{2}x^2$.

This parabola is shown in Figure 8. The coefficient $-\frac{1}{2}$ affects the shape of the graph; the $\frac{1}{2}$ makes the parabola wider $\left(\text{since the values of } \frac{1}{2}x^2 \text{ increase more slowly than those of } x^2\right)$, and the negative sign makes the parabola open down. The graph is not shifted in any direction; the vertex is still $(0, 0)$ and the axis is $x = 0$. Unlike the parabolas graphed in Examples 1–3, the vertex here has the *largest* function value of any point on the graph. The domain is $(-\infty, \infty)$; the range is $(-\infty, 0]$.

x	$f(x)$
-2	-2
-1	$-\frac{1}{2}$
0	0
1	$-\frac{1}{2}$
2	-2

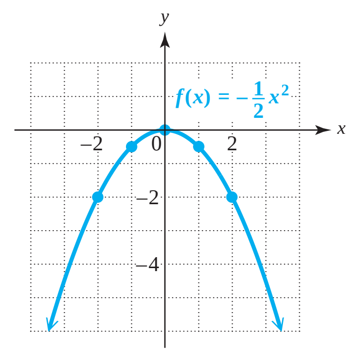


FIGURE 8

Now Try Exercise 21.

Some general principles concerning the graph of $F(x) = a(x - h)^2 + k$ are summarized as follows.

General Principles

1. The graph of the quadratic function defined by
$$F(x) = a(x - h)^2 + k, \quad a \neq 0$$
is a parabola with vertex (h, k) and the vertical line $x = h$ as axis.
2. The graph opens up if a is positive and down if a is negative.
3. The graph is wider than that of $f(x) = x^2$ if $0 < |a| < 1$. The graph is narrower than that of $f(x) = x^2$ if $|a| > 1$.

VIDEO EXPLORE

EXAMPLE 5 Using the General Principles to Graph a Parabola

Graph $F(x) = -2(x + 3)^2 + 4$.

The parabola opens down (because $a < 0$) and is narrower than the graph of $f(x) = x^2$, since $|-2| = 2 > 1$, causing values of $F(x)$ to decrease more quickly than those of $f(x) = -x^2$. This parabola has vertex $(-3, 4)$, as shown in Figure 9. To complete the graph, we plotted the ordered pairs $(-4, 2)$ and, by symmetry, $(-2, 2)$. Symmetry can be used to find additional ordered pairs that satisfy the equation, if desired.

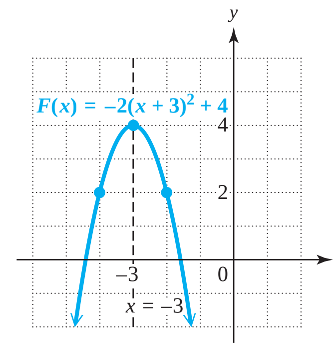


FIGURE 9

Now Try Exercise 33.

OBJECTIVE 4 Find a quadratic function to model data.

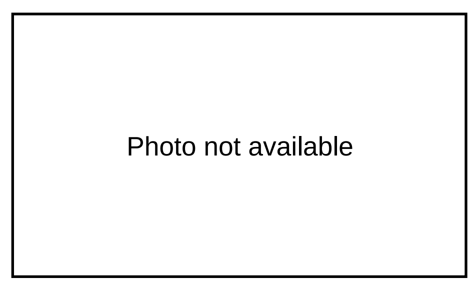
VIDEO

EXAMPLE 6 Finding a Quadratic Function to Model the Rise in Multiple Births

The number of higher-order multiple births in the United States is rising. Let x represent the number of years since 1970 and y represent the rate of higher-order multiples born per 100,000 births since 1971. The data are shown in the following table.

Year	x	y
1971	1	29.1
1976	6	35.0
1981	11	40.0
1986	16	47.0
1991	21	100.0
1996	26	152.6

Source: National Center for Health Statistics.



U.S. HIGHER-ORDER MULTIPLE BIRTHS

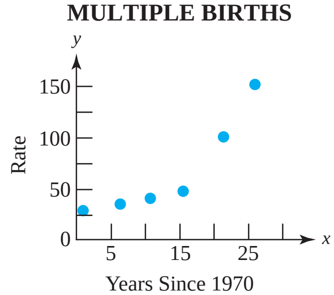


FIGURE 10

Find a quadratic function that models the data.

A scatter diagram of the ordered pairs (x, y) is shown in Figure 10. Notice that the graphed points do not follow a linear pattern, so a linear function would not model the data very well. Instead, the general shape suggested by the scatter diagram indicates that a parabola should approximate these points, as shown by the dashed curve in Figure 11. The equation for such a parabola would have a positive coefficient for x^2 since the graph opens up. To find a quadratic function of the form

$$y = ax^2 + bx + c$$

that models, or *fits*, these data, we choose three representative ordered pairs and use them to write a system of three equations. Using (1, 29.1), (11, 40), and (21, 100), we substitute the x- and y-values from the ordered pairs into the quadratic form $y = ax^2 + bx + c$ to get the following three equations.

$$a(1)^2 + b(1) + c = 29.1 \quad \text{or} \quad a + b + c = 29.1 \quad (1)$$
$$a(11)^2 + b(11) + c = 40 \quad \text{or} \quad 121a + 11b + c = 40 \quad (2)$$
$$a(21)^2 + b(21) + c = 100 \quad \text{or} \quad 441a + 21b + c = 100 \quad (3)$$

U.S. HIGHER-ORDER MULTIPLE BIRTHS

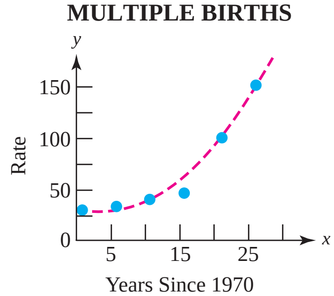


FIGURE 11

We can find the values of a, b, and c by solving this system of three equations in three variables using the methods of Section 4.2. Multiplying equation (1) by -1 and adding the result to equation (2) gives

$$120a + 10b = 10.9. \qquad (4)$$

Multiplying equation (2) by -1 and adding the result to equation (3) gives

$$320a + 10b = 60. \qquad (5)$$

We eliminate b from this system of two equations in two variables by multiplying equation (4) by -1 and adding the result to equation (5) to obtain

$$200a = 49.1$$
$$a = .2455. \qquad \text{Use a calculator.}$$

We substitute .2455 for a in equation (4) or (5) to find that $b = -1.856$. Substituting the values of a and b into equation (1) gives $c = 30.7105$. Using these values of a, b, and c, our model is defined by

$$y = .2455x^2 - 1.856x + 30.7105.$$

Now Try Exercise 49.

NOTE In Example 6, if we had chosen three different ordered pairs of data, a slightly different model would result. The *quadratic regression* feature on a graphing calculator can also be used to generate the quadratic model that best fits given data. See your owner's manual for details.

9.5 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 15/Videotape 15

1. Match each quadratic function with its graph from choices A–D.

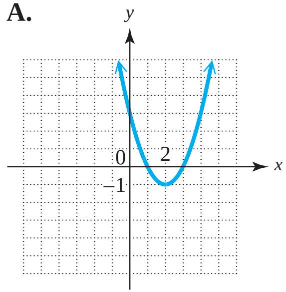
(a) $f(x) = (x + 2)^2 - 1$

(b) $f(x) = (x + 2)^2 + 1$

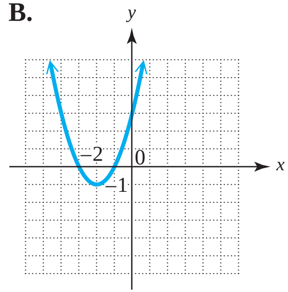
(c) $f(x) = (x - 2)^2 - 1$

(d) $f(x) = (x - 2)^2 + 1$

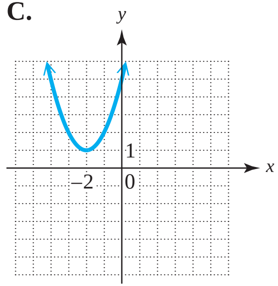
A.



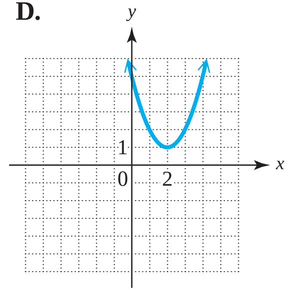
B.



C.



D.



2. Match each quadratic function with its graph from choices A–D.

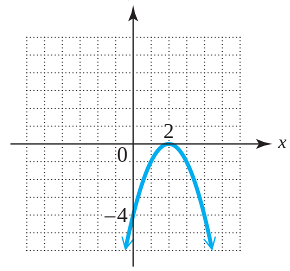
(a) $f(x) = -x^2 + 2$

(b) $f(x) = -x^2 - 2$

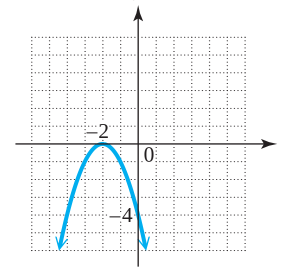
(c) $f(x) = -(x + 2)^2$

(d) $f(x) = -(x - 2)^2$

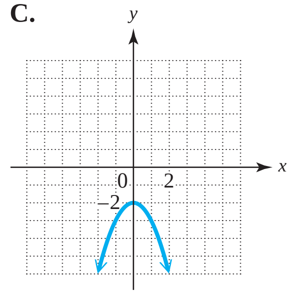
A.



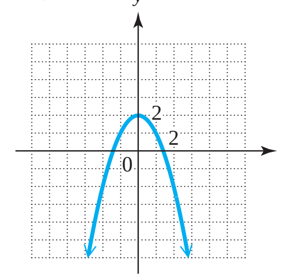
B.



C.



D.



Identify the vertex of each parabola. See Examples 1–4.

3. $f(x) = -3x^2$ **4.** $f(x) = \frac{1}{2}x^2$ **5.** $f(x) = x^2 + 4$

6. $f(x) = x^2 - 4$ **7.** $f(x) = (x - 1)^2$ **8.** $f(x) = (x + 3)^2$

9. $f(x) = (x + 3)^2 - 4$ **10.** $f(x) = (x - 5)^2 - 8$

11. Describe how each of the parabolas in Exercises 9 and 10 is shifted compared to the graph of $f(x) = x^2$.

12. What does the value of a in $F(x) = a(x - h)^2 + k$ tell you about the graph of the function compared to the graph of $f(x) = x^2$?

For each quadratic function, tell whether the graph opens up or down and whether the graph is wider, narrower, or the same shape as the graph of $f(x) = x^2$. See Examples 4 and 5.

13. $f(x) = -\frac{2}{5}x^2$ **14.** $f(x) = -2x^2$

15. $f(x) = 3x^2 + 1$ **16.** $f(x) = \frac{2}{3}x^2 - 4$

17. For $f(x) = a(x - h)^2 + k$, in what quadrant is the vertex if

(a) $h > 0, k > 0$; **(b)** $h > 0, k < 0$; **(c)** $h < 0, k > 0$; **(d)** $h < 0, k < 0$?

18. (a) What is the value of h if the graph of $f(x) = a(x - h)^2 + k$ has vertex on the y-axis?

(b) What is the value of k if the graph of $f(x) = a(x - h)^2 + k$ has vertex on the x-axis?

19. Match each quadratic function with the description of the parabola that is its graph.

(a) $f(x) = (x - 4)^2 - 2$	**A.** Vertex $(2, -4)$, opens down
(b) $f(x) = (x - 2)^2 - 4$	**B.** Vertex $(2, -4)$, opens up
(c) $f(x) = -(x - 4)^2 - 2$	**C.** Vertex $(4, -2)$, opens down
(d) $f(x) = -(x - 2)^2 - 4$	**D.** Vertex $(4, -2)$, opens up

20. Explain in your own words the meaning of each term.

(a) Vertex of a parabola **(b)** Axis of a parabola

Graph each parabola. Plot at least two points in addition to the vertex. Give the vertex, axis, domain, and range in Exercises 27–36. See Examples 1–5.

EXERCISES

21. $f(x) = -2x^2$

22. $f(x) = \frac{1}{3}x^2$

23. $f(x) = x^2 - 1$

24. $f(x) = x^2 + 3$

25. $f(x) = -x^2 + 2$

26. $f(x) = 2x^2 - 2$

27. $f(x) = (x - 4)^2$

28. $f(x) = -2(x + 1)^2$

29. $f(x) = (x + 2)^2 - 1$

30. $f(x) = (x - 1)^2 + 2$

31. $f(x) = 2(x - 2)^2 - 4$

32. $f(x) = -3(x - 2)^2 + 1$

33. $f(x) = -\frac{1}{2}(x + 1)^2 + 2$

34. $f(x) = -\frac{2}{3}(x + 2)^2 + 1$

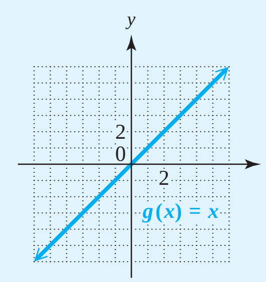
35. $f(x) = 2(x - 2)^2 - 3$

36. $f(x) = \frac{4}{3}(x - 3)^2 - 2$

RELATING CONCEPTS (EXERCISES 37–42)

For Individual or Group Work

The procedures that allow the graph of $y = x^2$ to be shifted vertically and horizontally apply to other types of functions. In Section 3.5 we introduced linear functions of the form $g(x) = ax + b$. Consider the graph of the simplest linear function defined by $g(x) = x$, shown here. ***Work Exercises 37–42 in order.***



37. Based on the concepts of this section, how does the graph of $F(x) = x^2 + 6$ compare to the graph of $f(x) = x^2$ if a *vertical* shift is considered?

38. Graph the linear function defined by $G(x) = x + 6$.

39. Based on the concepts of Chapter 3, how does the graph of $G(x) = x + 6$ compare to the graph of $g(x) = x$ if a vertical shift is considered? (*Hint:* Look at the y-intercept.)

40. Based on the concepts of this section, how does the graph of $F(x) = (x - 6)^2$ compare to the graph of $f(x) = x^2$ if a *horizontal* shift is considered?

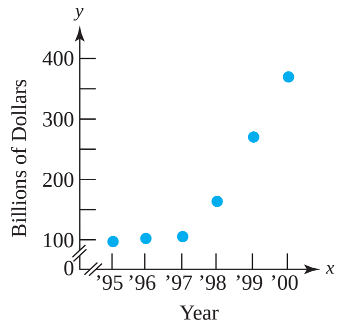
41. Graph the linear function $G(x) = x - 6$.

42. Based on the concepts of Chapter 3, how does the graph of $G(x) = x - 6$ compare to the graph of $g(x) = x$ if a horizontal shift is considered? (*Hint:* Look at the x-intercept.)

In Exercises 43–48, tell whether a linear or quadratic function would be a more appropriate model for each set of graphed data. If linear, tell whether the slope should be positive or negative. If quadratic, tell whether the coefficient a of x^2 should be positive or negative. See Example 6.

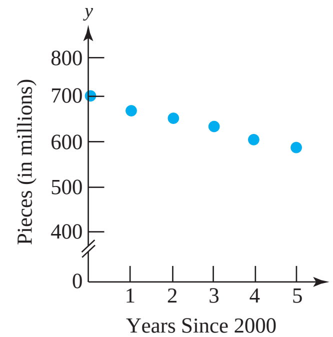
43.

U.S. TRADE DEFICIT



Source: U.S. Department of Commerce.

44. **AVERAGE DAILY VOLUME OF FIRST-CLASS MAIL***

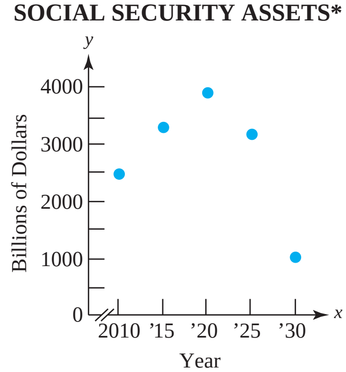


*Projected

Source: General Accounting Office.

45.

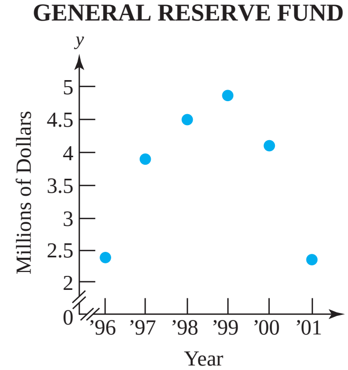
SOCIAL SECURITY ASSETS*



*Projected

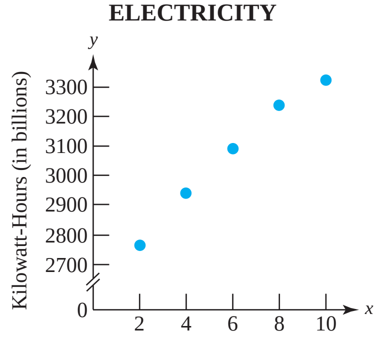
Source: Social Security Administration.

46. **CEDAR RAPIDS SCHOOLS—GENERAL RESERVE FUND**



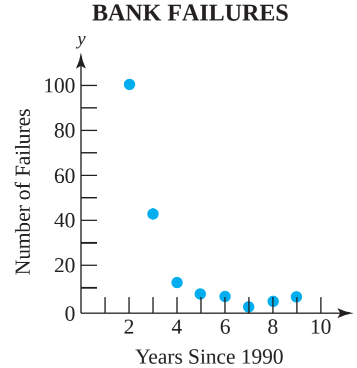
Source: Cedar Rapids School District.

47. **CONSUMER DEMAND FOR ELECTRICITY**



Source: U.S. Department of Energy.

48. **U.S. COMMERCIAL BANK FAILURES**



Source: www.ABA.com

Solve each problem. See Example 6.

EXERCISES

49. The number of publicly traded companies filing for bankruptcy for selected years between 1990 and 2000 are shown in the table. In the year column, 0 represents 1990, 2 represents 1992, and so on.

Year	*Number of Bankruptcies*
0	115
2	91
4	70
6	84
8	120
10	176

Source: www.BankruptcyData.com

Photo not available

(a) Use the ordered pairs (year, number of bankruptcies) to make a scatter diagram of the data.

(b) Use the scatter diagram to decide whether a linear or quadratic function would better model the data. If quadratic, should the coefficient a of x^2 be positive or negative?

(c) Use the ordered pairs (0, 115), (4, 70), and (8, 120) to find a quadratic function that models the data. Round the values of a, b, and c in your model to three decimal places, as necessary.

(d) Use your model from part (c) to approximate the number of company bankruptcy filings in 2002. Round your answer to the nearest whole number.

(e) The number of company bankruptcy filings through August 16, 2002 was 129. Based on this, is your estimate from part (d) reasonable? Explain.

50. In a study, the number of new AIDS patients who survived the first year for the years from 1991 through 1997 are shown in the table. In the year column, 1 represents 1991, 2 represents 1992, and so on.

Year	*Number of Patients*
1	55
2	130
3	155
4	160
5	155
6	150
7	115

Source: HIV Health Services Planning Council.

(a) Use the ordered pairs (year, number of patients) to make a scatter diagram of the data.

(b) Would a linear or quadratic function better model the data?

(c) Should the coefficient a of x^2 in a quadratic model be positive or negative?

(d) Use the ordered pairs (2, 130), (3, 155), and (7, 115) to find a quadratic function that models the data.

(e) Use your model from part (d) to approximate the number of AIDS patients who survived the first year in 1994 and 1996. How well does the model approximate the actual data from the table?

51. In Example 6, we determined that the quadratic function defined by

$$y = .2455x^2 - 1.856x + 30.7105$$

modeled the rate of higher-order multiple births, where x represents the number of years since 1970.

(a) Use this model to approximate the rate of higher-order births in 1999 to the nearest tenth.

(b) The actual rate of higher-order births in 1999 was 184.9. (*Source:* National Center for Health Statistics.) How does the approximation using the model compare to the actual rate for 1999?

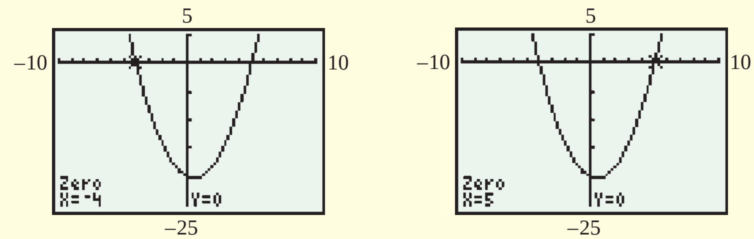
TECHNOLOGY INSIGHTS (EXERCISES 52–56)

Recall from Chapter 3 that the x-value of the x-intercept of the graph of the line $y = mx + b$ is the solution of the linear equation $mx + b = 0$. In the same way, the x-values of the x-intercepts of the graph of the parabola $y = ax^2 + bx + c$ are the real solutions of the quadratic equation $ax^2 + bx + c = 0$.

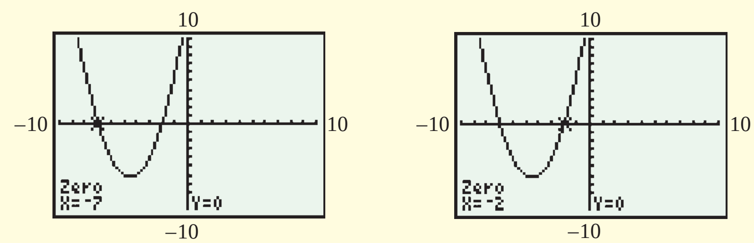
In Exercises 52–55, the calculator graphs show the x-values of the x-intercepts of the graph of the polynomial in the equation. Use the graphs to solve each equation.

EXERCISES

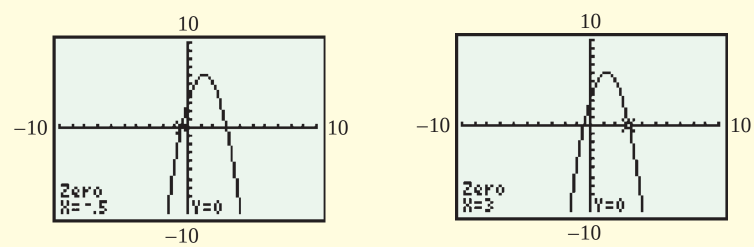
52. $x^2 - x - 20 = 0$



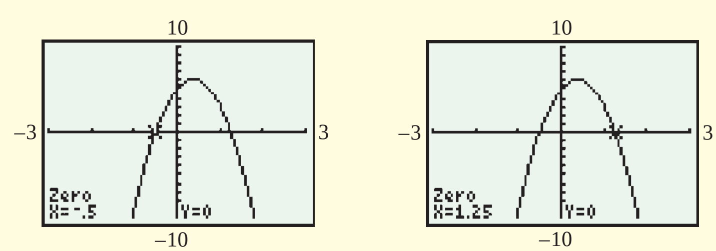
53. $x^2 + 9x + 14 = 0$



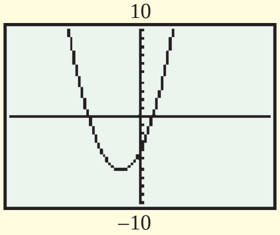
54. $-2x^2 + 5x + 3 = 0$



55. $-8x^2 + 6x + 5 = 0$



56. The graph of a quadratic function defined by $y = f(x)$ is shown in the standard viewing window, without x-axis tick marks. Which one of the following choices would be the only possible solution set for the equation $f(x) = 0$?



A. $\{-4, 1\}$ **B.** $\{1, 4\}$ **C.** $\{-1, -4\}$ **D.** $\{4, -1\}$

Explain your answer.

9.6 More About Parabolas; Applications

OBJECTIVES

1. Find the vertex of a vertical parabola.
2. Graph a quadratic function.
3. Use the discriminant to find the number of *x*-intercepts of a vertical parabola.
4. Use quadratic functions to solve problems involving maximum or minimum value.
5. Graph horizontal parabolas.

OBJECTIVE 1 Find the vertex of a vertical parabola. When the equation of a parabola is given in the form $f(x) = ax^2 + bx + c$, we need to locate the vertex to sketch an accurate graph. There are two ways to do this: complete the square as shown in Examples 1 and 2, or use a formula derived by completing the square.

EXAMPLE 1 Completing the Square to Find the Vertex

Find the vertex of the graph of $f(x) = x^2 - 4x + 5$.

To find the vertex, we need to express $x^2 - 4x + 5$ in the form $(x - h)^2 + k$. We do this by completing the square on $x^2 - 4x$, as in Section 9.1. The process is slightly different here because we want to keep $f(x)$ alone on one side of the equation. Instead of adding the appropriate number to each side, we *add and subtract* it on the right. This is equivalent to adding 0.

YOU TRY IT

$$\begin{aligned}
f(x) &= x^2 - 4x + 5 \\
&= (x^2 - 4x \qquad) + 5 && \text{Group the variable terms.} \\
& \left[\tfrac{1}{2}(-4)\right]^2 = (-2)^2 = 4 \\
&= (x^2 - 4x + 4 - 4) + 5 && \text{Add and subtract 4.} \\
&= (x^2 - 4x + 4) - 4 + 5 && \text{Bring } -4 \text{ outside the parentheses.} \\
f(x) &= (x - 2)^2 + 1 && \text{Factor; combine terms.}
\end{aligned}$$

The vertex of this parabola is (2, 1).

Now Try Exercise 5.

VIDEO

EXAMPLE 2 Completing the Square to Find the Vertex When $a \neq 1$

Find the vertex of the graph of $f(x) = -3x^2 + 6x - 1$.

We must complete the square on $-3x^2 + 6x$. Because the x^2-term has a coefficient other than 1, we factor that coefficient out of the first two terms and then proceed as in Example 1.

$$\begin{aligned}
f(x) &= -3x^2 + 6x - 1 \\
&= -3(x^2 - 2x) - 1 && \text{Factor out } -3. \\
& \left[\tfrac{1}{2}(-2)\right]^2 = (-1)^2 = 1 \\
&= -3(x^2 - 2x + 1 - 1) - 1 && \text{Add and subtract 1 within the parentheses.}
\end{aligned}$$

Bring -1 outside the parentheses; be sure to multiply it by -3.

$$= -3(x^2 - 2x + 1) + (-3)(-1) - 1 \qquad \text{Distributive property}$$
$$= -3(x^2 - 2x + 1) + 3 - 1$$
$$f(x) = -3(x - 1)^2 + 2 \qquad \text{Factor; combine terms.}$$

The vertex is (1, 2).

Now Try Exercise 7.

To derive a formula for the vertex of the graph of the quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$, complete the square.

$$f(x) = ax^2 + bx + c \qquad \text{Standard form}$$
$$= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \text{Factor } a \text{ from the first two terms.}$$
$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \qquad \text{Add and subtract } \tfrac{b^2}{4a^2}.$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c \qquad \text{Distributive property}$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \qquad \text{Factor; combine terms.}$$
$$f(x) = a\left[x - \underbrace{\left(\frac{-b}{2a}\right)}_{h}\right]^2 + \underbrace{\frac{4ac - b^2}{4a}}_{k} \qquad f(x) = (x - h)^2 + k$$

Thus, the vertex (h, k) can be expressed in terms of a, b, and c. However, it is not necessary to remember this expression for k, since it can be found by replacing x with $\frac{-b}{2a}$. Using function notation, if $y = f(x)$, then the y-value of the vertex is $f\left(\frac{-b}{2a}\right)$.

Vertex Formula

The graph of the quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ has vertex

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right),$$

and the axis of the parabola is the line

$$x = \frac{-b}{2a}.$$

EXAMPLE 3 Using the Formula to Find the Vertex

Use the vertex formula to find the vertex of the graph of

$$f(x) = x^2 - x - 6.$$

For this function, $a = 1$, $b = -1$, and $c = -6$. The x-coordinate of the vertex of the parabola is given by

$$\frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2}.$$

The y-coordinate is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right)$.

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 6 = \frac{1}{4} - \frac{1}{2} - 6 = -\frac{25}{4}$$

The vertex is $\left(\frac{1}{2}, -\frac{25}{4}\right)$.

Now Try Exercise 9.

OBJECTIVE 2 Graph a quadratic function. We give a general approach for graphing any quadratic function here.

Graphing a Quadratic Function f

Step 1 **Determine whether the graph opens up or down.** If $a > 0$, the parabola opens up; if $a < 0$, it opens down.

Step 2 **Find the vertex.** Use either the vertex formula or completing the square.

Step 3 **Find any intercepts.** To find the x-intercepts (if any), solve $f(x) = 0$. To find the y-intercept, evaluate $f(0)$.

Step 4 **Complete the graph.** Plot the points found so far. Find and plot additional points as needed, using symmetry about the axis.

EXAMPLE 4 Using the Steps to Graph a Quadratic Function

Graph the quadratic function defined by

$$f(x) = x^2 - x - 6.$$

Step 1 From the equation, $a = 1$, so the graph of the function opens up.

Step 2 The vertex, $\left(\frac{1}{2}, -\frac{25}{4}\right)$, was found in Example 3 using the vertex formula.

Step 3 Find any intercepts. Since the vertex, $\left(\frac{1}{2}, -\frac{25}{4}\right)$, is in quadrant IV and the graph opens up, there will be two x-intercepts. To find them, let $f(x) = 0$ and solve.

$$\begin{aligned}
f(x) &= x^2 - x - 6 & \\
0 &= x^2 - x - 6 & \text{Let } f(x) = 0. \\
0 &= (x - 3)(x + 2) & \text{Factor.} \\
x - 3 = 0 \quad &\text{or} \quad x + 2 = 0 & \text{Zero-factor property} \\
x = 3 \quad &\text{or} \quad x = -2 &
\end{aligned}$$

The x-intercepts are $(3, 0)$ and $(-2, 0)$.

To find the y-intercept, evaluate $f(0)$.

$$f(x) = x^2 - x - 6$$
$$f(0) = 0^2 - 0 - 6 \qquad \text{Let } x = 0.$$
$$f(0) = -6$$

The y-intercept is $(0, -6)$.

Step 4 Plot the points found so far and additional points as needed using symmetry about the axis, $x = \frac{1}{2}$. The graph is shown in Figure 12. The domain is $(-\infty, \infty)$, and the range is $\left[-\frac{25}{4}, \infty\right)$.

x	y
-2	0
-1	-4
0	-6
$\frac{1}{2}$	$-\frac{25}{4}$
2	-4
3	0

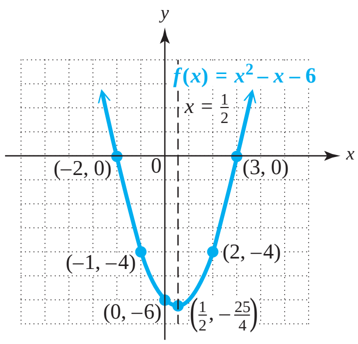


FIGURE 12

Now Try Exercise 17.

OBJECTIVE 3 Use the discriminant to find the number of x-intercepts of a vertical parabola. The graph of a quadratic function may have two x-intercepts, one x-intercept, or no x-intercepts, as shown in Figure 13. Recall from Section 9.2 that $b^2 - 4ac$ is called the *discriminant* of the quadratic equation $ax^2 + bx + c = 0$ and that we can use it to determine the number of real solutions of a quadratic equation.

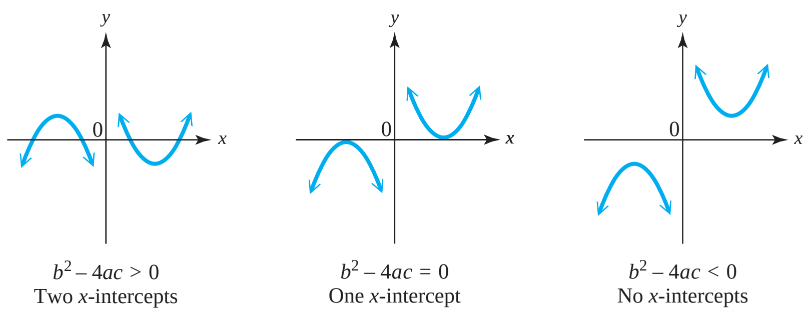


FIGURE 13

In a similar way, we can use the discriminant of a quadratic *function* to determine the number of x-intercepts of its graph. If the discriminant is positive, the parabola will have two x-intercepts. If the discriminant is 0, there will be only one x-intercept, and it will be the vertex of the parabola. If the discriminant is negative, the graph will have no x-intercepts.

EXAMPLE 5 Using the Discriminant to Determine the Number of *x*-Intercepts

Find the discriminant and use it to determine the number of x-intercepts of the graph of each quadratic function.

(a) $f(x) = 2x^2 + 3x - 5$

The discriminant is $b^2 - 4ac$. Here $a = 2$, $b = 3$, and $c = -5$, so

$$b^2 - 4ac = 9 - 4(2)(-5) = 49.$$

Since the discriminant is positive, the parabola has two x-intercepts.

(b) $f(x) = -3x^2 - 1$

Here, $a = -3$, $b = 0$, and $c = -1$. The discriminant is

$$b^2 - 4ac = 0 - 4(-3)(-1) = -12.$$

The discriminant is negative, so the graph has no x-intercepts.

(c) $f(x) = 9x^2 + 6x + 1$

Here, $a = 9$, $b = 6$, and $c = 1$. The discriminant is

$$b^2 - 4ac = 36 - 4(9)(1) = 0.$$

The parabola has only one x-intercept (its vertex) because the value of the discriminant is 0.

Now Try Exercises 11 and 13.

OBJECTIVE 4 Use quadratic functions to solve problems involving maximum or minimum value. The vertex of the graph of a quadratic function is either the highest or the lowest point on the parabola. The y-value of the vertex gives the maximum or minimum value of y, while the x-value tells where that maximum or minimum occurs.

PROBLEM SOLVING

In many applied problems we must find the largest or smallest value of some quantity. When we can express that quantity in terms of a quadratic function, the value of k in the vertex (h, k) gives that optimum value.

VIDEO

EXAMPLE 6 Finding the Maximum Area of a Rectangular Region

A farmer has 120 ft of fencing. He wants to put a fence around a rectangular field next to a building. Find the maximum area he can enclose.

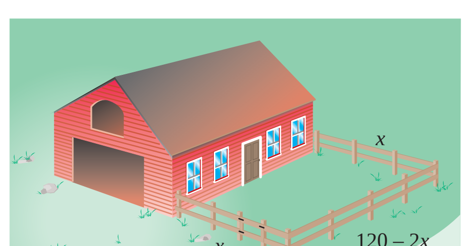


FIGURE 14

Figure 14 on the preceding page shows the field. Let x represent the width of the field. Since he has 120 ft of fencing,

$$
\begin{aligned}
x + x + \text{length} &= 120 && \text{Sum of the sides is 120 ft.}\\
2x + \text{length} &= 120 && \text{Combine terms.}\\
\text{length} &= 120 - 2x. && \text{Subtract } 2x.
\end{aligned}
$$

The area is given by the product of the width and length, so

$$
\begin{aligned}
A(x) &= x(120 - 2x)\\
&= 120x - 2x^2.
\end{aligned}
$$

To determine the maximum area, find the vertex of the parabola given by $A(x) = 120x - 2x^2$ using the vertex formula. Writing the equation in standard form as $A(x) = -2x^2 + 120x$ gives $a = -2$, $b = 120$, and $c = 0$, so

$$h = \frac{-b}{2a} = \frac{-120}{2(-2)} = \frac{-120}{-4} = 30;$$

$$A(30) = -2(30)^2 + 120(30) = -2(900) + 3600 = 1800.$$

The graph is a parabola that opens down, and its vertex is $(30, 1800)$. Thus, the maximum area will be 1800 ft^2. This area will occur if x, the width of the field, is 30 ft.

Now Try Exercise 35.

CAUTION Be careful when interpreting the meanings of the coordinates of the vertex. The first coordinate, x, gives the value for which the *function value* is a maximum or a minimum. Be sure to read the problem carefully to determine whether you are asked to find the value of the independent variable, the function value, or both.

EXAMPLE 7 Finding the Maximum Height Attained by a Projectile

If air resistance is neglected, a projectile on Earth shot straight upward with an initial velocity of 40 m per sec will be at a height s in meters given by

$$s(t) = -4.9t^2 + 40t,$$

where t is the number of seconds elapsed after projection. After how many seconds will it reach its maximum height, and what is this maximum height?

For this function, $a = -4.9$, $b = 40$, and $c = 0$. Use the vertex formula.

$$h = \frac{-b}{2a} = \frac{-40}{2(-4.9)} \approx 4.1 \qquad \text{Use a calculator.}$$

This indicates that the maximum height is attained at 4.1 sec. To find this maximum height, calculate $s(4.1)$.

$$
\begin{aligned}
s(4.1) &= -4.9(4.1)^2 + 40(4.1)\\
&\approx 81.6 && \text{Use a calculator.}
\end{aligned}
$$

The projectile will attain a maximum height of approximately 81.6 m.

Now Try Exercise 37.

OBJECTIVE 5 Graph horizontal parabolas. If x and y are interchanged in the equation $y = ax^2 + bx + c$, the equation becomes $x = ay^2 + by + c$. Because of the interchange of the roles of x and y, these parabolas are horizontal (with horizontal lines as axes), compared with the vertical ones graphed previously.

AUDIO

Graph of a Horizontal Parabola

The graph of

$$x = ay^2 + by + c \qquad \text{or} \qquad x = a(y - k)^2 + h$$

is a parabola with vertex (h, k) and the horizontal line $y = k$ as axis. The graph opens to the right if $a > 0$ and to the left if $a < 0$.

EXPLORE

EXAMPLE 8 Graphing a Horizontal Parabola

Graph $x = (y - 2)^2 - 3$.

This graph has its vertex at $(-3, 2)$, since the roles of x and y are reversed. It opens to the right because $a = 1 > 0$, and has the same shape as $y = x^2$. Plotting a few additional points gives the graph shown in Figure 15. Note that the graph is symmetric about its axis, $y = 2$. The domain is $[-3, \infty)$, and the range is $(-\infty, \infty)$.

x	y
-3	2
-2	3
-2	1
1	4
1	0

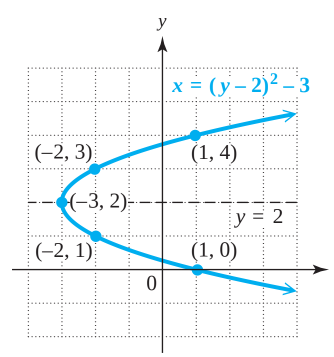


FIGURE 15

Now Try Exercise 21.

When a quadratic equation is given in the form $x = ay^2 + by + c$, completing the square on y allows us to find the vertex.

EXAMPLE 9 Completing the Square to Graph a Horizontal Parabola

Graph $x = -2y^2 + 4y - 3$. Give the domain and range of the relation.

$$x = -2y^2 + 4y - 3$$

$$= -2(y^2 - 2y) - 3 \qquad \text{Factor out } -2.$$

$$= -2(y^2 - 2y + 1 - 1) - 3 \qquad \text{Complete the square within the parentheses; add and subtract 1.}$$

$$= -2(y^2 - 2y + 1) + (-2)(-1) - 3 \qquad \text{Distributive property}$$

$$x = -2(y - 1)^2 - 1 \qquad \text{Factor; simplify.}$$

Because of the negative coefficient (-2) in $x = -2(y-1)^2 - 1$, the graph opens to the left (the negative x-direction) and is narrower than the graph of $y = x^2$ because $|-2| > 1$. As shown in Figure 16, the vertex is $(-1, 1)$ and the axis is $y = 1$. The domain is $(-\infty, -1]$, and the range is $(-\infty, \infty)$.

x	y
-3	2
-3	0
-1	1

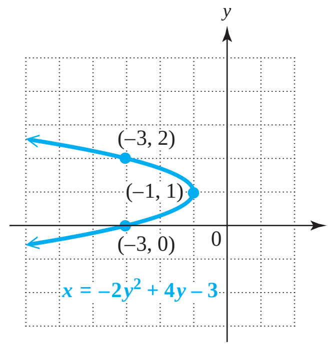


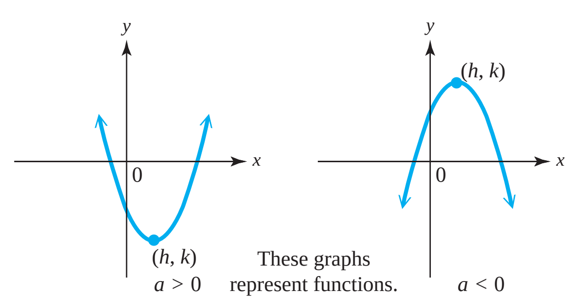
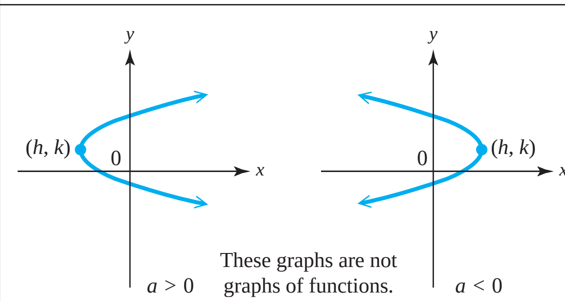
FIGURE 16

Now Try Exercise 25.

CAUTION Only quadratic equations solved for y (whose graphs are vertical parabolas) are examples of functions. *The horizontal parabolas in Examples 8 and 9 are not graphs of functions*, because they do not satisfy the vertical line test. Furthermore, the vertex formula given earlier does not apply to parabolas with horizontal axes.

In summary, the graphs of parabolas studied in this section and the previous one fall into the following categories.

Graphs of Parabolas

Equation	*Graph*
$y = ax^2 + bx + c$ $y = a(x-h)^2 + k$	
$x = ay^2 + by + c$ $x = a(y-k)^2 + h$	

9.6 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 15/Videotape 15

1. How can you determine just by looking at the equation of a parabola whether it has a vertical or a horizontal axis?

2. Why can't the graph of a quadratic function be a parabola with a horizontal axis?

3. How can you determine the number of x-intercepts of the graph of a quadratic function without graphing the function?

4. If the vertex of the graph of a quadratic function is $(1, -3)$, and the graph opens down, how many x-intercepts does the graph have?

Find the vertex of each parabola. See Examples 1–3.

5. $f(x) = x^2 + 8x + 10$ **6.** $f(x) = x^2 + 10x + 23$ **7.** $f(x) = -2x^2 + 4x - 5$

8. $f(x) = -3x^2 + 12x - 8$ **9.** $f(x) = -\frac{1}{2}x^2 + 2x - 3$ **10.** $f(x) = 4x^2 - x + 5$

Find the vertex of each parabola. For each equation, decide whether the graph opens up, down, to the left, or to the right, and whether it is wider, narrower, or the same shape as the graph of $y = x^2$. If it is a vertical parabola, find the discriminant and use it to determine the number of x-intercepts. See Examples 1–3, 5, 8, and 9.

11. $f(x) = 2x^2 + 4x + 5$ **12.** $f(x) = 3x^2 - 6x + 4$ **13.** $f(x) = -x^2 + 5x + 3$

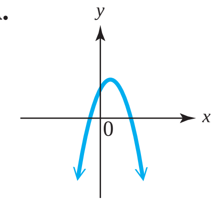
14. $x = -y^2 + 7y + 2$ **15.** $x = \frac{1}{3}y^2 + 6y + 24$ **16.** $x = \frac{1}{2}y^2 + 10y - 5$

Use the concepts of this section to match each equation in Exercises 17–22 with its graph in A–F.

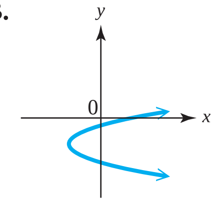
17. $y = 2x^2 + 4x - 3$ **18.** $y = -x^2 + 3x + 5$ **19.** $y = -\frac{1}{2}x^2 - x + 1$

20. $x = y^2 + 6y + 3$ **21.** $x = -y^2 - 2y + 4$ **22.** $x = 3y^2 + 6y + 5$

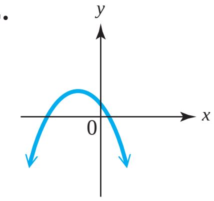
A.



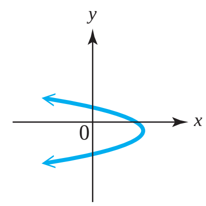
B.



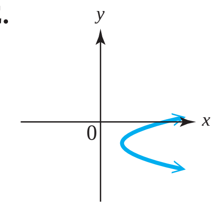
C.



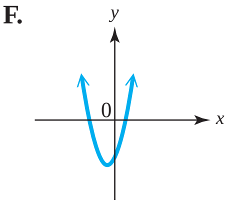
D.



E.



F.



Graph each parabola. (Use the results of Exercises 5–8 to help graph the parabolas in Exercises 23–26.) Give the vertex, axis, domain, and range. See Examples 4, 8, and 9.

23. $f(x) = x^2 + 8x + 10$ **24.** $f(x) = x^2 + 10x + 23$ **25.** $f(x) = -2x^2 + 4x - 5$

26. $f(x) = -3x^2 + 12x - 8$ **27.** $x = (y + 2)^2 + 1$ **28.** $x = (y + 3)^2 - 2$

29. $x = -\frac{1}{5}y^2 + 2y - 4$ **30.** $x = -\frac{1}{2}y^2 - 4y - 6$ **31.** $x = 3y^2 + 12y + 5$

32. $x = 4y^2 + 16y + 11$

Solve each problem. See Examples 6 and 7.

33. Find the pair of numbers whose sum is 60 and whose product is a maximum. (*Hint:* Let x and $60 - x$ represent the two numbers.)

34. Find the pair of numbers whose sum is 40 and whose product is a maximum.

35. Morgan's Department Store wants to construct a rectangular parking lot on land bordered on one side by a highway. It has 280 ft of fencing that is to be used to fence off the other three sides. What should be the dimensions of the lot if the enclosed area is to be a maximum? What is the maximum area?

36. Keisha Hughes has 100 m of fencing material to enclose a rectangular exercise run for her dog. What width will give the enclosure the maximum area?

37. If an object on Earth is propelled upward with an initial velocity of 32 ft per sec, then its height after t sec is given by

$$h(t) = 32t - 16t^2.$$

Find the maximum height attained by the object and the number of seconds it takes to hit the ground.

38. A projectile on Earth is fired straight upward so that its distance (in feet) above the ground t sec after firing is given by

$$s(t) = -16t^2 + 400t.$$

Find the maximum height it reaches and the number of seconds it takes to reach that height.

39. After experimentation, two Pacific Institute physics students find that when a bottle of California wine is shaken several times, held upright, and uncorked, its cork travels according to the function defined by

$$s(t) = -16t^2 + 64t + 3,$$

where s is its height in feet above the ground t sec after being released. After how many seconds will it reach its maximum height? What is the maximum height?

40. Professor Levy has found that the number of students attending his intermediate algebra class is approximated by

$$S(x) = -x^2 + 20x + 80,$$

where x is the number of hours that the Campus Center is open daily. Find the number of hours that the center should be open so that the number of students attending class is a maximum. What is this maximum number of students?

Photo not available

41. The annual percent increase in the amount pharmacies paid wholesalers for drugs in the years 1990 through 1999 can be modeled by the quadratic function defined by

$$f(x) = .228x^2 - 2.57x + 8.97,$$

where $x = 0$ represents 1990, $x = 1$ represents 1991, and so on. (*Source: IMS Health,* Retail and Provider Perspective.)

(a) Since the coefficient of x^2 in the model is positive, the graph of this quadratic function is a parabola that opens up. Will the y-value of the vertex of this graph be a maximum or minimum?

(b) In what year was the minimum percent increase? (Round down to the nearest year.) Use the actual x-value of the vertex, to the nearest tenth, to find this increase.

42. The U.S. domestic oyster catch (in millions) for the years 1990 through 1998 can be approximated by the quadratic function defined by

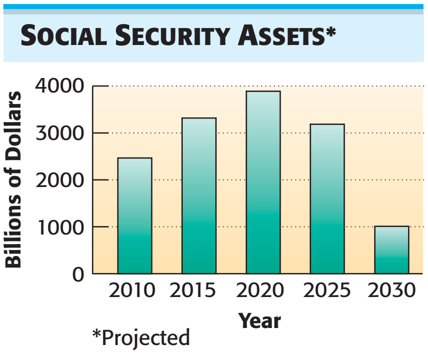
$$f(x) = -.566x^2 + 5.08x + 29.2,$$

where $x = 0$ represents 1990, $x = 1$ represents 1991, and so on. (*Source:* National Marine Fisheries Service.)

(a) Since the coefficient of x^2 in the model is negative, the graph of this quadratic function is a parabola that opens down. Will the y-value of the vertex of this graph be a maximum or minimum?

(b) In what year was the maximum domestic oyster catch? (Round down to the nearest year.) Use the actual x-value of the vertex, to the nearest tenth, to find this catch.

43. The graph shows how Social Security assets are expected to change as the number of retirees receiving benefits increases.



Source: Social Security Administration.

The graph suggests that a quadratic function would be a good fit to the data. The data are approximated by the function defined by

$$f(x) = -20.57x^2 + 758.9x - 3140.$$

In the model, $x = 10$ represents 2010, $x = 15$ represents 2015, and so on, and $f(x)$ is in billions of dollars.

(a) Explain why the coefficient of x^2 in the model is negative, based on the graph.

(b) Algebraically determine the vertex of the graph, with coordinates to four significant digits.

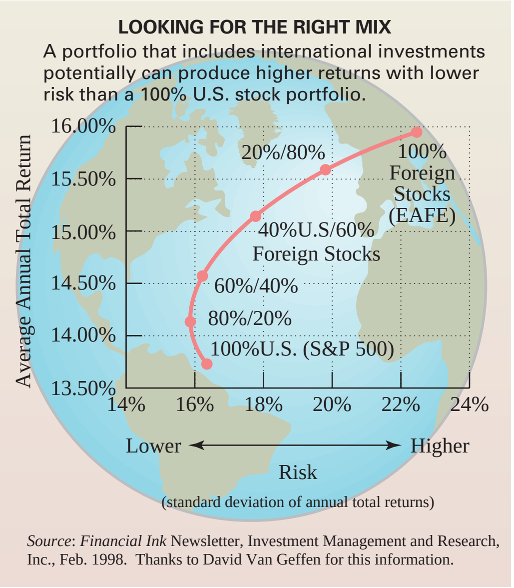
(c) Interpret the answer to part (b) as it applies to the application.

44. The graph shows the performance of investment portfolios with different mixtures of U.S. and foreign investments for the period January 1, 1971, to December 31, 1996.

(a) Is this the graph of a function? Explain.

(b) What investment mixture shown on the graph appears to represent the vertex? What relative amount of risk does this point represent? What return on investment does it provide?

(c) Which point on the graph represents the riskiest investment mixture? What return on investment does it provide?



Source: *Financial Ink* Newsletter, Investment Management and Research, Inc., Feb. 1998. Thanks to David Van Geffen for this information.

45. A charter flight charges a fare of \$200 per person, plus \$4 per person for each unsold seat on the plane. If the plane holds 100 passengers and if x represents the number of unsold seats, find the following.

(a) A function defined by $R(x)$ that describes the total revenue received for the flight (*Hint:* Multiply the number of people flying, $100 - x$, by the price per ticket, $200 + 4x$.)

(b) The graph of the function from part (a)

(c) The number of unsold seats that will produce the maximum revenue

(d) The maximum revenue

46. For a trip to a resort, a charter bus company charges a fare of \$48 per person, plus \$2 per person for each unsold seat on the bus. If the bus has 42 seats and x represents the number of unsold seats, find the following.

(a) A function defined by $R(x)$ that describes the total revenue from the trip (*Hint:* Multiply the total number riding, $42 - x$, by the price per ticket, $48 + 2x$)

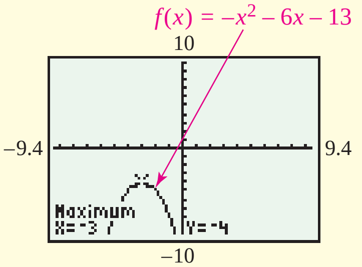
(b) The graph of the function from part (a)

(c) The number of unsold seats that produces the maximum revenue

(d) The maximum revenue

TECHNOLOGY INSIGHTS (EXERCISES 47–50)

Graphing calculators are capable of determining the coordinates of "peaks" and "valleys" of graphs. In the case of quadratic functions, these peaks and valleys are the vertices and are called maximum and minimum points. For example, the vertex of the graph of $f(x) = -x^2 - 6x - 13$ *is* $(-3, -4)$, *as indicated in the display at the bottom of the screen. In this case, the vertex is a maximum point.*



In Exercises 47–50, match the function with its calculator graph in A–D by determining the vertex and using the display at the bottom of the screen.

EXERCISES

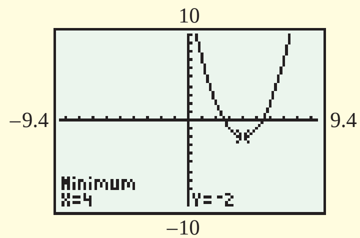
47. $f(x) = x^2 - 8x + 18$

48. $f(x) = x^2 + 8x + 18$

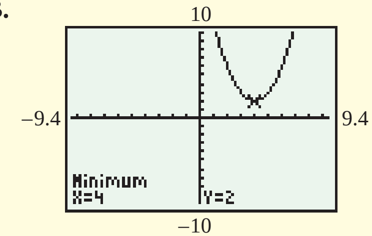
49. $f(x) = x^2 - 8x + 14$

50. $f(x) = x^2 + 8x + 14$

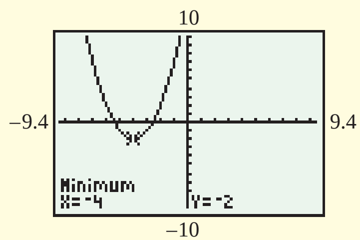
A.



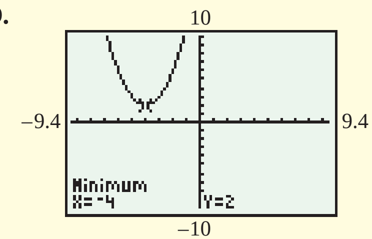
B.



C.

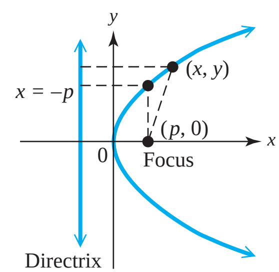


D.



In the following exercise, the distance formula is used to develop the equation of a parabola.

51. A parabola can be defined as the set of all points in a plane equally distant from a given point and a given line not containing the point. (The point is called the *focus* and the line is called the *directrix*.) See the figure.



(a) Suppose (x, y) is to be on the parabola. Suppose the directrix has equation $x = -p$. Find the distance between (x, y) and the directrix. (The distance from a point to a line is the length of the perpendicular from the point to the line.)

(b) If $x = -p$ is the equation of the directrix, why should the focus have coordinates $(p, 0)$? (*Hint:* See the figure.)

(c) Find an expression for the distance from (x, y) to $(p, 0)$.

(d) Find an equation for the parabola in the figure. (*Hint:* Use the results of parts (a) and (c) and the fact that (x, y) is equally distant from the focus and the directrix.)

52. Use the equation derived in Exercise 51 to find an equation for a parabola with focus $(3, 0)$ and directrix with equation $x = -3$.

9.7 Quadratic and Rational Inequalities

OBJECTIVES

1 Solve quadratic inequalities. (continued)

We discussed methods of solving linear inequalities in Chapter 3 and methods of solving quadratic equations in this chapter. Now we combine these ideas to solve *quadratic inequalities.*

OBJECTIVES (continued)

2. Solve polynomial inequalities of degree 3 or more.
3. Solve rational inequalities.

Quadratic Inequality

A **quadratic inequality** can be written in the form

$$ax^2 + bx + c < 0 \quad \text{or} \quad ax^2 + bx + c > 0,$$

where a, b, and c are real numbers, with $a \neq 0$.

AUDIO

As before, the symbols $<$ and $>$ may be replaced with $\leq$ and $\geq$.

OBJECTIVE 1 Solve quadratic inequalities. One method for solving a quadratic inequality is by graphing the related quadratic function.

EXAMPLE 1 Solving Quadratic Inequalities by Graphing

Solve each inequality.

(a) $x^2 - x - 12 > 0$

To solve the inequality, we graph the related quadratic function defined by $f(x) = x^2 - x - 12$. We are particularly interested in the x-intercepts, which are found as in Section 9.6 by letting $f(x) = 0$ and solving the quadratic equation

$$x^2 - x - 12 = 0.$$

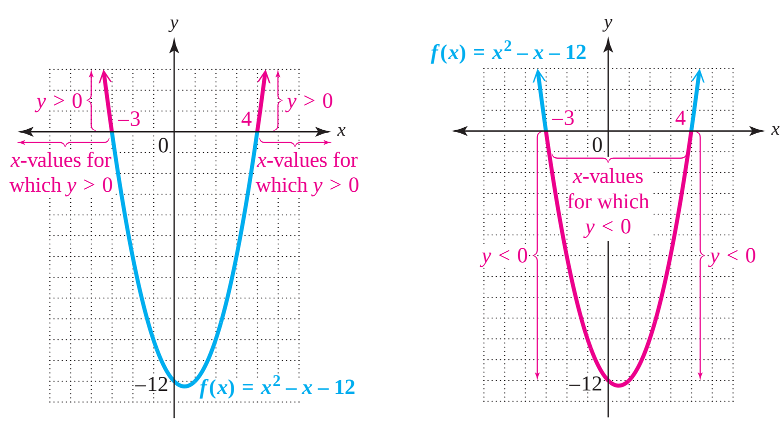
$$(x - 4)(x + 3) = 0 \qquad \text{Factor.}$$

$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Zero-factor property}$$

$$x = 4 \quad \text{or} \quad x = -3$$

Thus, the x-intercepts are (4, 0), and $(-3, 0)$. The graph, which opens up since the coefficient of x^2 is positive, is shown in Figure 17(a). Notice from this graph that x-values less than -3 or greater than 4 result in y-values *greater than* 0. Therefore, the solution set of $x^2 - x - 12 > 0$, written in interval notation, is

$$(-\infty, -3) \cup (4, \infty).$$



The graph is *above* the x-axis for $(-\infty, -3) \cup (4, \infty)$.

(a)

The graph is *below* the x-axis for $(-3, 4)$.

(b)

FIGURE 17

(b) $x^2 - x - 12 < 0$

Here we want values of y that are *less than* 0. Referring to Figure 17(b), we notice from the graph that x-values between -3 and 4 result in y-values less than 0. Therefore, the solution set of the inequality $x^2 - x - 12 < 0$, written in interval notation, is $(-3, 4)$.

Now Try Exercise 1.

NOTE If the inequalities in Example 1 had used $\geq$ and $\leq$, the solution sets would have included the x-values of the intercepts and been written in interval notation as $(-\infty, -3] \cup [4, \infty)$ for Example 1(a) and $[-3, 4]$ for Example 1(b).

In Example 1, we used graphing to divide the x-axis into intervals. Then using the graphs in Figure 17, we determined which x-values resulted in y-values that were either greater than or less than 0. Another method for solving a quadratic inequality uses these basic ideas without actually graphing the related quadratic function.

EXAMPLE 2 Solving a Quadratic Inequality Using Test Numbers

Solve $x^2 - x - 12 > 0$.

Solve the quadratic equation $x^2 - x - 12 = 0$ by factoring, as in Example 1(a).

$$(x - 4)(x + 3) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

The numbers 4 and -3 divide a number line into the three intervals shown in Figure 18. Be careful to put the smaller number on the left. (Notice the similarity between Figure 18 and the x-axis with intercepts $(-3, 0)$ and $(4, 0)$ in Figure 17(a).)

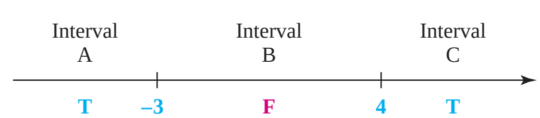


FIGURE 18

The numbers 4 and -3 are the only numbers that make the expression $x^2 - x - 12$ equal to 0. All other numbers make the expression either positive or negative. The sign of the expression can change from positive to negative or from negative to positive only at a number that makes it 0. Therefore, if one number in an interval satisfies the inequality, then all the numbers in that interval will satisfy the inequality.

To see if the numbers in Interval A satisfy the inequality, choose any number from Interval A in Figure 18 (that is, any number less than -3). Substitute this test number for x in the original inequality $x^2 - x - 12 > 0$. If the result is *true*, then all numbers in Interval A satisfy the inequality.

Try -5 from Interval A. Substitute -5 for x.

$$x^2 - x - 12 > 0 \qquad \text{Original inequality}$$
$$(-5)^2 - (-5) - 12 > 0 \quad ?$$
$$25 + 5 - 12 > 0 \quad ?$$
$$18 > 0 \qquad \text{True}$$

Because -5 from Interval A satisfies the inequality, all numbers from Interval A are solutions.

Now try 0 from Interval B. If $x = 0$, then

$$0^2 - 0 - 12 > 0 \quad ?$$
$$-12 > 0. \quad \text{False}$$

The numbers in Interval B are *not* solutions. Verify that the test number 5 satisfies the inequality, so the numbers in Interval C are also solutions.

Based on these results (shown by the colored letters in Figure 18), the solution set includes the numbers in Intervals A and C, as shown on the graph in Figure 19. The solution set is written in interval notation as

$$(-\infty, -3) \cup (4, \infty).$$

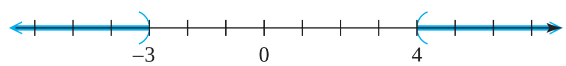


FIGURE 19

This agrees with the solution set we found by graphing the related quadratic function in Example 1(a).

Now Try Exercise 11.

In summary, follow these steps to solve a quadratic inequality.

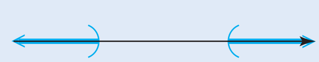
AUDIO

Solving a Quadratic Inequality

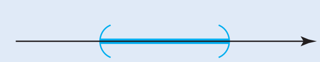
Step 1 **Write the inequality as an equation and solve it.**

Step 2 **Use the solutions from Step 1 to determine intervals.** Graph the numbers found in Step 1 on a number line. These numbers divide the number line into intervals.

Step 3 **Find the intervals that satisfy the inequality.** Substitute a test number from each interval into the original inequality to determine the intervals that satisfy the inequality. All numbers in those intervals are in the solution set. A graph of the solution set will usually look like one of these. (Square brackets might be used instead of parentheses.)



or



Step 4 **Consider the endpoints separately.** The numbers from Step 1 are included in the solution set if the inequality symbol is $\leq$ or $\geq$; they are not included if it is $<$ or $>$.

Special cases of quadratic inequalities may occur, as in the next example.

EXAMPLE 3 Solving Special Cases

Solve $(2t - 3)^2 > -1$. Then solve $(2t - 3)^2 < -1$.

Because $(2t - 3)^2$ is never negative, it is always greater than -1. Thus, the solution for $(2t - 3)^2 > -1$ is the set of all real numbers, $(-\infty, \infty)$. In the same way, there is no solution for $(2t - 3)^2 < -1$ and the solution set is $\emptyset$.

Now Try Exercises 25 and 27.

OBJECTIVE 2 Solve polynomial inequalities of degree 3 or more. Higher-degree polynomial inequalities that can be factored are solved in the same way as quadratic inequalities.

VIDEO

YOU TRY IT

EXAMPLE 4 Solving a Third-Degree Polynomial Inequality

Solve $(x - 1)(x + 2)(x - 4) \leq 0$.

This is a *cubic* (third-degree) inequality rather than a quadratic inequality, but it can be solved using the method shown in the box by extending the zero-factor property to more than two factors. Begin by setting the factored polynomial *equal* to 0 and solving the equation. (Step 1)

$$(x - 1)(x + 2)(x - 4) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = 1 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 4$$

Locate the numbers -2, 1, and 4 on a number line, as in Figure 20, to determine the Intervals A, B, C, and D. (Step 2)

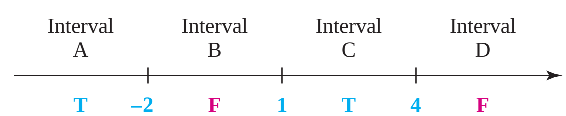


FIGURE 20

Substitute a test number from each interval in the *original* inequality to determine which intervals satisfy the inequality. (Step 3) Use a table to organize this information.

Interval	*Test Number*	*Test of Inequality*	*True or False?*
A	-3	$-28 \leq 0$	T
B	0	$8 \leq 0$	F
C	2	$-8 \leq 0$	T
D	5	$28 \leq 0$	F

Verify the information given in the table and graphed in Figure 21 on the next page. The numbers in Intervals A and C are in the solution set, which is written in interval notation as

$$(-\infty, -2] \cup [1, 4].$$

The three endpoints are included since the inequality symbol is $\leq$. (Step 4)

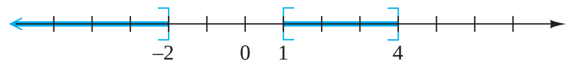


FIGURE 21

Now Try Exercise 29.

OBJECTIVE 3 Solve rational inequalities. Inequalities that involve rational expressions, called **rational inequalities,** are solved similarly using the following steps.

Solving a Rational Inequality

Step 1 **Write the inequality** so that 0 is on one side and there is a single fraction on the other side.

Step 2 **Determine the numbers that make the numerator or denominator equal to 0.**

Step 3 **Divide a number line into intervals.** Use the numbers from Step 2.

Step 4 **Find the intervals that satisfy the inequality.** Test a number from each interval by substituting it into the *original* inequality.

Step 5 **Consider the endpoints separately.** Exclude any values that make the denominator 0.

CAUTION As indicated in Step 5, any number that makes the denominator 0 *must* be excluded from the solution set.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{-1}{p-3} > 1$.

Write the inequality so that 0 is on one side. (Step 1)

$$\frac{-1}{p-3} - 1 > 0 \qquad \text{Subtract 1.}$$

$$\frac{-1}{p-3} - \frac{p-3}{p-3} > 0 \qquad \text{Use } p-3 \text{ as the common denominator.}$$

$$\frac{-1-p+3}{p-3} > 0 \qquad \text{Write the left side as a single fraction; be careful with signs in the numerator.}$$

$$\frac{-p+2}{p-3} > 0 \qquad \text{Combine terms.}$$

The sign of the rational expression $\frac{-p+2}{p-3}$ will change from positive to negative or negative to positive only at those numbers that make the numerator or denominator 0. The number 2 makes the numerator 0, and 3 makes the denominator 0. (Step 2) These two numbers, 2 and 3, divide a number line into three intervals. See Figure 22. (Step 3)

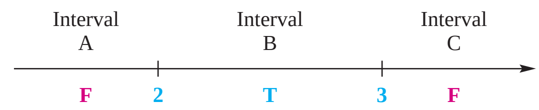


FIGURE 22

Testing a number from each interval in the *original* inequality, $\frac{-1}{p-3} > 1$, gives the results shown in the table. (Step 4)

Interval	*Test Number*	*Test of Inequality*	*True or False?*
A	0	$\frac{1}{3} > 1$	F
B	2.5	$2 > 1$	T
C	4	$-1 > 1$	F

The solution set is the interval (2, 3). This interval does not include 3 since it would make the denominator of the original equality 0; 2 is not included either since the inequality symbol is $>$. (Step 5) A graph of the solution set is given in Figure 23.

FIGURE 23

Now Try Exercise 37.

YOU TRY IT

EXAMPLE 6 Solving a Rational Inequality

Solve $\frac{m-2}{m+2} \le 2$.

Write the inequality so that 0 is on one side. (Step 1)

$$\frac{m-2}{m+2} - 2 \le 0 \qquad \text{Subtract 2.}$$

$$\frac{m-2}{m+2} - \frac{2(m+2)}{m+2} \le 0 \qquad \text{Use } m+2 \text{ as the common denominator.}$$

$$\frac{m-2-2m-4}{m+2} \le 0 \qquad \text{Write as a single fraction.}$$

$$\frac{-m-6}{m+2} \le 0 \qquad \text{Combine terms.}$$

The number -6 makes the numerator 0, and -2 makes the denominator 0. (Step 2) These two numbers determine three intervals. (Step 3) Test one number from each interval (Step 4) to see that the solution set is

$$(-\infty, -6] \cup (-2, \infty).$$

The number -6 satisfies the original inequality, but -2 cannot be used as a solution since it makes the denominator 0. (Step 5) A graph of the solution set is shown in Figure 24.

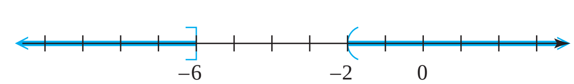


FIGURE 24

Now Try Exercise 41.

9.7 EXERCISES

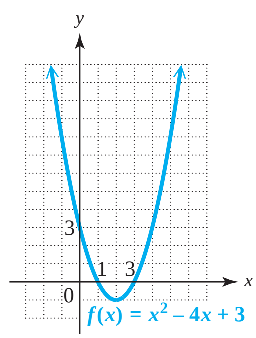
For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 15/Videotape 15

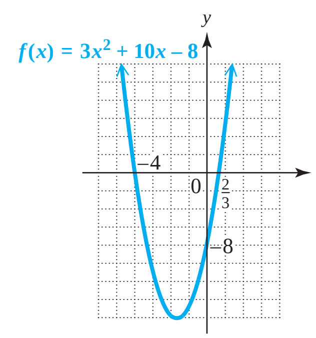
In Example 1, we determined the solution sets of the quadratic inequalities $x^2 - x - 12 > 0$ and $x^2 - x - 12 < 0$ by graphing $f(x) = x^2 - x - 12$. The x-intercepts of this graph indicated the solutions of the equation $x^2 - x - 12 = 0$. The x-values of the points on the graph that were above the x-axis formed the solution set of $x^2 - x - 12 > 0$, and the x-values of the points on the graph that were below the x-axis formed the solution set of $x^2 - x - 12 < 0$.

In Exercises 1–4, the graph of a quadratic function f is given. Use the graph to find the solution set of each equation or inequality. See Example 1.

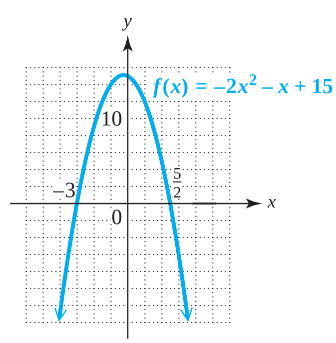
1. **(a)** $x^2 - 4x + 3 = 0$
(b) $x^2 - 4x + 3 > 0$
(c) $x^2 - 4x + 3 < 0$



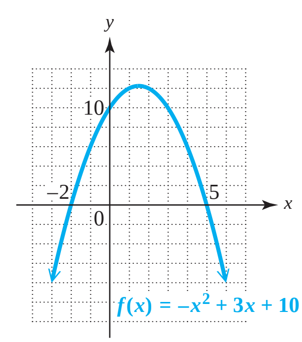
2. **(a)** $3x^2 + 10x - 8 = 0$
(b) $3x^2 + 10x - 8 \geq 0$
(c) $3x^2 + 10x - 8 < 0$



3. **(a)** $-2x^2 - x + 15 = 0$
(b) $-2x^2 - x + 15 \geq 0$
(c) $-2x^2 - x + 15 \leq 0$



4. **(a)** $-x^2 + 3x + 10 = 0$
(b) $-x^2 + 3x + 10 \geq 0$
(c) $-x^2 + 3x + 10 \leq 0$



5. Explain how to determine whether to include or exclude endpoints when solving a quadratic or higher-degree inequality.

6. The solution set of the inequality $x^2 + x - 12 < 0$ is the interval $(-4, 3)$. Without actually performing any work, give the solution set of the inequality $x^2 + x - 12 \geq 0$.

Solve each inequality and graph the solution set. See Example 2. (Hint: In Exercises 23 and 24, use the quadratic formula.)

7. $(x + 1)(x - 5) > 0$ **8.** $(m + 6)(m - 2) > 0$ **9.** $(r + 4)(r - 6) < 0$

10. $(x + 4)(x - 8) < 0$ **11.** $x^2 - 4x + 3 \geq 0$ **12.** $m^2 - 3m - 10 \geq 0$

13. $10t^2 + 9t \geq 9$
14. $3r^2 + 10r \geq 8$
15. $4x^2 - 9 \leq 0$
16. $9x^2 - 25 \leq 0$
17. $6x^2 + x \geq 1$
18. $4p^2 + 7p \geq -3$
19. $z^2 - 4z \geq 0$
20. $x^2 + 2x < 0$
21. $3k^2 - 5k \leq 0$
22. $2z^2 + 3z > 0$
23. $x^2 - 6x + 6 \geq 0$
24. $3k^2 - 6k + 2 \leq 0$

Solve each inequality. See Example 3.

EXERCISES

25. $(4 - 3x)^2 \geq -2$
26. $(6z + 7)^2 \geq -1$
27. $(3x + 5)^2 \leq -4$
28. $(8t + 5)^2 \leq -5$

Solve each inequality and graph the solution set. See Example 4.

29. $(p - 1)(p - 2)(p - 4) < 0$
30. $(2r + 1)(3r - 2)(4r + 7) < 0$
31. $(x - 4)(2x + 3)(3x - 1) \geq 0$
32. $(z + 2)(4z - 3)(2z + 7) \geq 0$

Solve each inequality and graph the solution set. See Examples 5 and 6.

EXERCISES

33. $\dfrac{x - 1}{x - 4} > 0$
34. $\dfrac{x + 1}{x - 5} > 0$
35. $\dfrac{2n + 3}{n - 5} \leq 0$
36. $\dfrac{3t + 7}{t - 3} \leq 0$
37. $\dfrac{8}{x - 2} \geq 2$
38. $\dfrac{20}{x - 1} \geq 1$
39. $\dfrac{3}{2t - 1} < 2$
40. $\dfrac{6}{m - 1} < 1$
41. $\dfrac{g - 3}{g + 2} \geq 2$
42. $\dfrac{m + 4}{m + 5} \geq 2$
43. $\dfrac{x - 8}{x - 4} < 3$
44. $\dfrac{2t - 3}{t + 1} > 4$
45. $\dfrac{4k}{2k - 1} < k$
46. $\dfrac{r}{r + 2} < 2r$
47. $\dfrac{2x - 3}{x^2 + 1} \geq 0$
48. $\dfrac{9x - 8}{4x^2 + 25} < 0$
49. $\dfrac{(3x - 5)^2}{x + 2} > 0$
50. $\dfrac{(5x - 3)^2}{2x + 1} \leq 0$

RELATING CONCEPTS (EXERCISES 51–54)

For Individual or Group Work

A rock is projected vertically upward from the ground. Its distance s in feet above the ground after t sec is given by the quadratic function defined by

$$s(t) = -16t^2 + 256t.$$

Work Exercises 51–54 in order, *to see how quadratic equations and inequalities are related.*

51. At what times will the rock be 624 ft above the ground? (*Hint:* Let $s(t) = 624$ and solve the quadratic *equation*.)

52. At what times will the rock be more than 624 ft above the ground? (*Hint:* Set $s(t) > 624$ and solve the quadratic *inequality*.)

53. At what times will the rock be at ground level? (*Hint:* Let $s(t) = 0$ and solve the quadratic *equation*.)

54. At what times will the rock be less than 624 ft above the ground? (*Hint:* Set $s(t) < 624$, solve the quadratic *inequality*, and observe the solutions in Exercises 52 and 53 to determine the smallest and largest possible values of t.)

10.1 Inverse Functions

OBJECTIVES

1. Decide whether a function is one-to-one and, if it is, find its inverse.
2. Use the horizontal line test to determine whether a function is one-to-one.
3. Find the equation of the inverse of a function.
4. Graph f^{-1} from the graph of f.
5. Use a graphing calculator to graph inverse functions.

In this chapter we will study two important types of functions, *exponential* and *logarithmic*. These functions are related in a special way: They are *inverses* of one another. We begin by discussing inverse functions in general.

OBJECTIVE 1 Decide whether a function is one-to-one and, if it is, find its inverse. Suppose we define the function

$$G = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}.$$

We can form another set of ordered pairs from G by interchanging the x- and y-values of each pair in G. We can call this set F, so

$$F = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}.$$

To show that these two sets are related, F is called the *inverse* of G. For a function f to have an inverse, f must be a *one-to-one function.*

One-to-One Function

In a **one-to-one function,** each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.

The function shown in Figure 1(a) is not one-to-one because the y-value 7 corresponds to *two* x-values, 2 and 3. That is, the ordered pairs (2, 7) and (3, 7) both belong to the function. The function in Figure 1(b) is one-to-one.

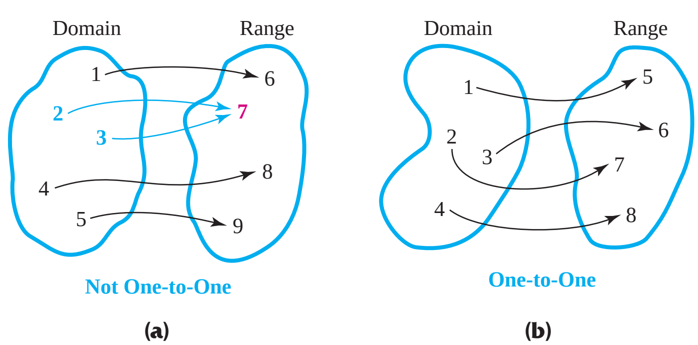


FIGURE 1

The *inverse* of any one-to-one function f is found by interchanging the components of the ordered pairs of f. The inverse of f is written f^{-1}. Read f^{-1} as "the inverse of f" or "f-inverse."

CAUTION The symbol $f^{-1}(x)$ does not represent $\dfrac{1}{f(x)}$.

The definition of the inverse of a function follows.

Inverse of a Function

The **inverse** of a one-to-one function f, written f^{-1}, is the set of all ordered pairs of the form (y, x), where (x, y) belongs to f. Since the inverse is formed by interchanging x and y, the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}.

For inverses f and f^{-1}, it follows that

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

EXAMPLE 1 Finding the Inverses of One-to-One Functions

Find the inverse of each one-to-one function.

(a) $F = \{(-2, 1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$

Each x-value in F corresponds to just one y-value. However, the y-value 2 corresponds to two x-values, 1 and 2. Also, the y-value 1 corresponds to both -2 and 0. Because some y-values correspond to more than one x-value, F is not one-to-one and does not have an inverse.

(b) $G = \{(3, 1), (0, 2), (2, 3), (4, 0)\}$

Every x-value in G corresponds to only one y-value, and every y-value corresponds to only one x-value, so G is a one-to-one function. The inverse function is found by interchanging the x- and y-values in each ordered pair.

$$G^{-1} = \{(1, 3), (2, 0), (3, 2), (0, 4)\}$$

Notice how the domain and range of G become the range and domain, respectively, of G^{-1}.

(c) The U.S. Environmental Protection Agency has developed an indicator of air quality called the Pollutant Standard Index (PSI). If the PSI exceeds 100 on a particular day, then that day is classified as unhealthy. The table shows the number of unhealthy days in Chicago for the years 1991 through 1997.

Year	*Number of Unhealthy Days*
1991	21
1992	4
1993	3
1994	8
1995	21
1996	6
1997	9

Source: U.S. Environmental Protection Agency.

Photo not available

Let f be the function defined in the table, with the years forming the domain and the numbers of unhealthy days forming the range. Then f is not one-to-one, because in two different years (1991 and 1995), the number of unhealthy days was the same, 21.

Now Try Exercises 1, 9, and 11.

OBJECTIVE 2 Use the horizontal line test to determine whether a function is one-to-one. It may be difficult to decide whether a function is one-to-one just by looking at the equation that defines the function. However, by graphing the function and observing the graph, we can use the *horizontal line test* to tell whether the function is one-to-one.

Horizontal Line Test

If any horizontal line intersects the graph of a function in no more than one point, then the function is one-to-one.

The horizontal line test follows from the definition of a one-to-one function. Any two points that lie on the same horizontal line have the same y-coordinate. No two ordered pairs that belong to a one-to-one function may have the same y-coordinate, and therefore no horizontal line will intersect the graph of a one-to-one function more than once.

EXAMPLE 2 Using the Horizontal Line Test

Use the horizontal line test to determine whether the graphs in Figures 2 and 3 are graphs of one-to-one functions.

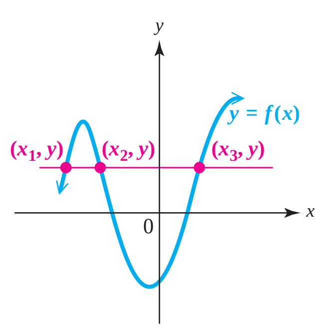


FIGURE 2

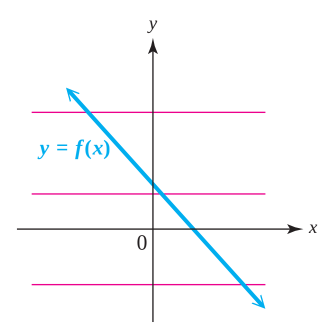


FIGURE 3

Because the horizontal line shown in Figure 2 intersects the graph in more than one point (actually three points), the function is not one-to-one.

Every horizontal line will intersect the graph in Figure 3 in exactly one point. This function is one-to-one.

Now Try Exercise 7.

OBJECTIVE 3 Find the equation of the inverse of a function. By definition, the inverse of a function is found by interchanging the x- and y-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows.

Step 1 **Interchange** x and y.

Step 2 **Solve** for y.

Step 3 **Replace** y with $f^{-1}(x)$.

EXAMPLE 3 Finding Equations of Inverses

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(a) $f(x) = 2x + 5$

The graph of $y = 2x + 5$ is a nonvertical line, so by the horizontal line test, f is a one-to-one function. To find the inverse, let $y = f(x)$ so that

$$\begin{aligned} y &= 2x + 5 \\ x &= 2y + 5 && \text{Interchange } x \text{ and } y. \text{ (Step 1)} \\ 2y &= x - 5 && \text{Solve for } y. \text{ (Step 2)} \\ y &= \frac{x-5}{2} \\ f^{-1}(x) &= \frac{x-5}{2}. && \text{(Step 3)} \end{aligned}$$

Thus, f^{-1} is a linear function. In the function defined by $y = 2x + 5$, the value of y is found by starting with a value of x, multiplying by 2, and adding 5. The equation for the inverse has us *subtract* 5, and then *divide* by 2. This shows how an inverse is used to "undo" what a function does to the variable x.

(b) $y = x^2 + 2$

This equation has a vertical parabola as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the x^2-term, there are many pairs of x-values that correspond to the same y-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

If this is not noticed, then following the steps for finding the equation of an inverse leads to

$$\begin{aligned} y &= x^2 + 2 \\ x &= y^2 + 2 && \text{Interchange } x \text{ and } y. \\ x - 2 &= y^2 && \text{Solve for } y. \\ \pm\sqrt{x-2} &= y. && \text{Square root property} \end{aligned}$$

The last step shows that there are two y-values for each choice of $x > 2$, so the given function is not one-to-one and cannot have an inverse.

(c) $f(x) = (x - 2)^3$

Refer to Section 5.3 to see that the graphs of cubing functions are one-to-one.

$$\begin{aligned}
y &= (x - 2)^3 && \text{Replace } f(x) \text{ with } y.\\
x &= (y - 2)^3 && \text{Interchange } x \text{ and } y.\\
\sqrt[3]{x} &= \sqrt[3]{(y - 2)^3} && \text{Take the cube root on each side.}\\
\sqrt[3]{x} &= y - 2 &&\\
\sqrt[3]{x} + 2 &= y && \text{Solve for } y.\\
f^{-1}(x) &= \sqrt[3]{x} + 2 && \text{Replace } y \text{ with } f^{-1}(x).
\end{aligned}$$

Now Try Exercises 13, 17, and 19.

OBJECTIVE 4 Graph f^{-1} from the graph of f. One way to graph the inverse of a function f whose equation is known is to find some ordered pairs that belong to f, interchange x and y to get ordered pairs that belong to f^{-1}, plot those points, and sketch the graph of f^{-1} through the points. A simpler way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.

For example, suppose the point (a, b) shown in Figure 4 belongs to a one-to-one function f. Then the point (b, a) belongs to f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$. For this reason we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line $y = x$.

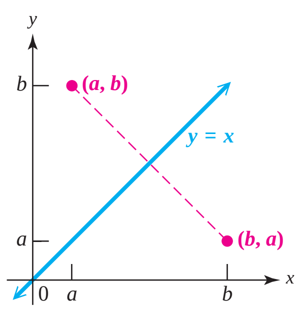


FIGURE 4

EXAMPLE 4 Graphing the Inverse

Graph the inverses of the functions f (shown in blue) in Figure 5.

In Figure 5 the graphs of two functions f are shown in blue. Their inverses are shown in red. In each case, the graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$.

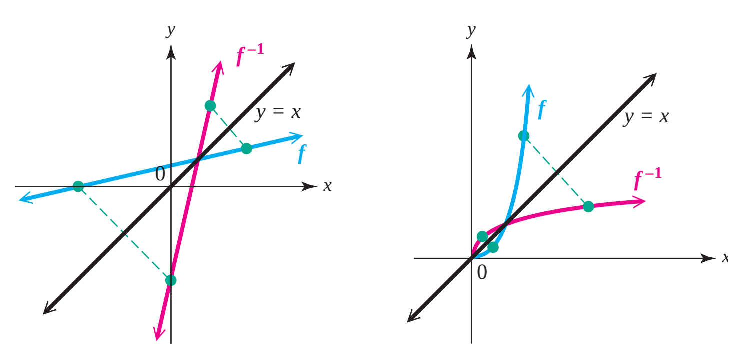


FIGURE 5

Now Try Exercises 25 and 29.

OBJECTIVE 5 Use a graphing calculator to graph inverse functions. We have described how inverses of one-to-one functions may be determined algebraically. We also explained how the graph of a one-to-one function f compares to the graph of its inverse f^{-1}: It is a reflection of the graph of f^{-1} across the line $y = x$. In Example 3 we showed that the inverse of the one-to-one function defined by $f(x) = 2x + 5$ is given by $f^{-1}(x) = \frac{x-5}{2}$. If we use a square viewing window of a graphing calculator and graph $y_1 = f(x) = 2x + 5$, $y_2 = f^{-1}(x) = \frac{x-5}{2}$, and $y_3 = x$, we can see how this reflection appears on the screen. See Figure 6.

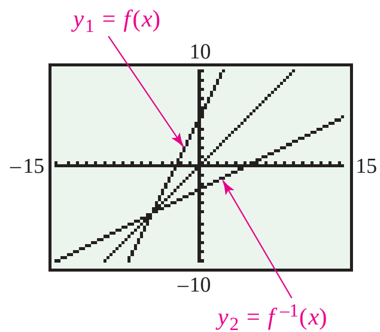


FIGURE 6

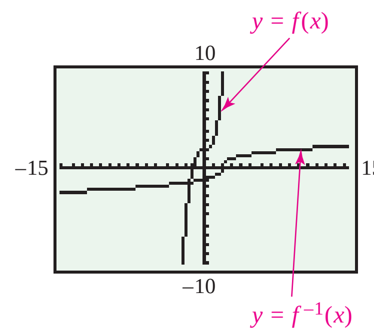


FIGURE 7

Some graphing calculators have the capability to "draw" the inverse of a function. Figure 7 shows the graphs of $f(x) = x^3 + 2$ and its inverse in a square viewing window.

10.1 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 16/Videotape 16

In Exercises 1–4, write a few sentences of explanation. See Example 1.

1. The table shows the number of uncontrolled hazardous waste sites that require further investigation to determine whether remedies are needed under the Superfund program. The seven states listed are ranked in the top ten in the United States.

If this correspondence is considered to be a function that pairs each state with its number of uncontrolled waste sites, is it one-to-one? If not, explain why. (See Example 1(c).)

State	*Number of Sites*
New Jersey	108
Pennsylvania	101
California	94
New York	79
Florida	53
Illinois	40
Wisconsin	40

Source: U.S. Environmental Protection Agency.

Photo not available

2. The table shows emissions of a major air pollutant, carbon monoxide, in the United States for the years 1992 through 1998.

If this correspondence is considered to be a function that pairs each year with its emissions amount, is it one-to-one? If not, explain why.

Year	*Amount of Emissions (in thousands of tons)*
1992	97,630
1993	98,160
1994	102,643
1995	93,353
1996	95,479
1997	94,410
1998	89,454

Source: U.S. Environmental Protection Agency.

3. The road mileage between Denver, Colorado, and several selected U.S. cities is shown in the table. If we consider this as a function that pairs each city with a distance, is it a one-to-one function? How could we change the answer to this question by adding 1 mile to one of the distances shown?

City	*Distance to Denver (in miles)*
Atlanta	1398
Dallas	781
Indianapolis	1058
Kansas City, MO	600
Los Angeles	1059
San Francisco	1235

4. Suppose you consider the set of ordered pairs (x, y) such that x represents a person in your mathematics class and y represents that person's mother. Explain how this function might not be a one-to-one function.

In Exercises 5–8, choose the correct response from the given list.

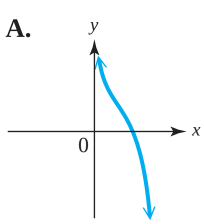
5. If a function is made up of ordered pairs in such a way that the same y-value appears in a correspondence with two different x-values, then

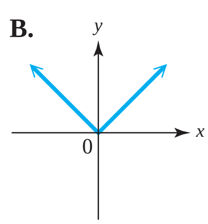
A. the function is one-to-one
B. the function is not one-to-one
C. its graph does not pass the vertical line test
D. it has an inverse function associated with it.

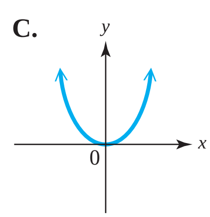
6. Which equation defines a one-to-one function? Explain why the others are not, using specific examples.

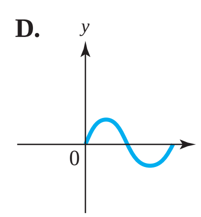
A. $f(x) = x$ **B.** $f(x) = x^2$ **C.** $f(x) = |x|$ **D.** $f(x) = -x^2 + 2x - 1$

7. Only one of the graphs illustrates a one-to-one function. Which one is it? (See Example 2.)

A.


B.


C.


D.


8. If a function f is one-to-one and the point (p, q) lies on the graph of f, then which point *must* lie on the graph of f^{-1}?

A. $(-p, q)$ **B.** $(-q, -p)$ **C.** $(p, -q)$ **D.** (q, p)

If the function is one-to-one, find its inverse. See Examples 1–3.

9. $\{(3, 6), (2, 10), (5, 12)\}$

10. $\left\{(-1, 3), (0, 5), (5, 0), \left(7, -\frac{1}{2}\right)\right\}$

11. $\{(-1, 3), (2, 7), (4, 3), (5, 8)\}$

12. $\{(-8, 6), (-4, 3), (0, 6), (5, 10)\}$

13. $f(x) = 2x + 4$

14. $f(x) = 3x + 1$

15. $g(x) = \sqrt{x - 3}, \quad x \geq 3$

16. $g(x) = \sqrt{x + 2}, \quad x \geq -2$

17. $f(x) = 3x^2 + 2$

18. $f(x) = -4x^2 - 1$

19. $f(x) = x^3 - 4$

20. $f(x) = x^3 - 3$

Let $f(x) = 2^x$. We will see in the next section that this function is one-to-one. Find each value, always working part (a) before part (b).

21. (a) $f(3)$ **(b)** $f^{-1}(8)$

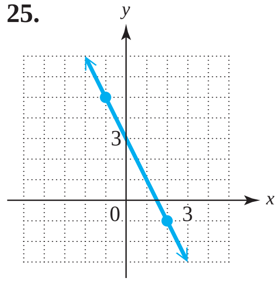
22. (a) $f(4)$ **(b)** $f^{-1}(16)$

23. (a) $f(0)$ **(b)** $f^{-1}(1)$

24. (a) $f(-2)$ **(b)** $f^{-1}\left(\frac{1}{4}\right)$

*The graphs of some functions are given in Exercises 25–30. **(a)** Use the horizontal line test to determine whether the function is one-to-one. **(b)** If the function is one-to-one, then graph the inverse of the function. (Remember that if f is one-to-one and (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}.) See Example 4.*

25.



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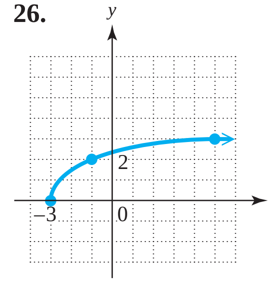

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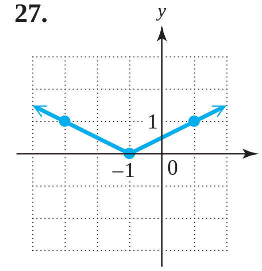

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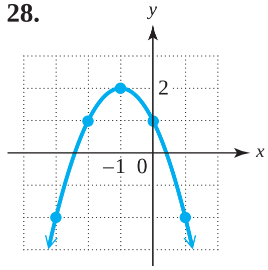

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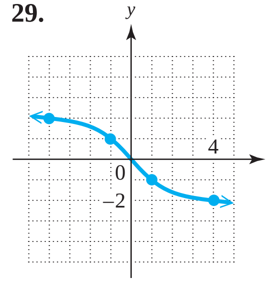
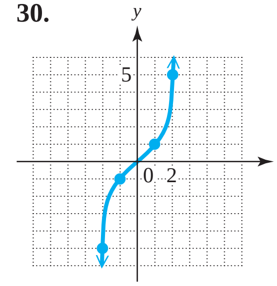


30.



Each function defined in Exercises 31–38 is a one-to-one function. Graph the function as a solid line (or curve) and then graph its inverse on the same set of axes as a dashed line (or curve). In Exercises 35–38 you are given a table to complete so that graphing the function will be a bit easier. See Example 4.

31. $f(x) = 2x - 1$

32. $f(x) = 2x + 3$

33. $g(x) = -4x$

34. $g(x) = -2x$

35. $f(x) = \sqrt{x}, \quad x \geq 0$

x	$f(x)$
0	
1	
4	

36. $f(x) = -\sqrt{x}, \quad x \geq 0$

x	$f(x)$
0	
1	
4	

37. $f(x) = x^3 - 2$

x	$f(x)$
-1	
0	
1	
2	

38. $f(x) = x^3 + 3$

x	$f(x)$
-2	
-1	
0	
1	

RELATING CONCEPTS (EXERCISES 39–42)

For Individual or Group Work

Inverse functions are used by government agencies and other businesses to send and receive coded information. The functions they use are usually very complicated. A simple example might use the function defined by $f(x) = 2x + 5$. *(Note that it is one-to-one.) Suppose that each letter of the alphabet is assigned a numerical value according to its position, as follows:*

A	1	G	7	L	12	Q	17	V	22
B	2	H	8	M	13	R	18	W	23
C	3	I	9	N	14	S	19	X	24
D	4	J	10	O	15	T	20	Y	25
E	5	K	11	P	16	U	21	Z	26
F	6								

This is an Enigma machine, used by the Germans in World War II to send coded messages.

Using the function, the word ALGEBRA *would be encoded as*

$$7 \quad 29 \quad 19 \quad 15 \quad 9 \quad 41 \quad 7,$$

because

$$f(\text{A}) = f(1) = 2(1) + 5 = 7, \quad f(\text{L}) = f(12) = 2(12) + 5 = 29,$$

and so on. The message would then be decoded by using the inverse of f, defined by $f^{-1}(x) = \frac{x-5}{2}$. *For example,*

$$f^{-1}(7) = \frac{7-5}{2} = 1 = \text{A}, \quad f^{-1}(29) = \frac{29-5}{2} = 12 = \text{L},$$

and so on. ***Work Exercises 39–42 in order.***

39. Suppose that you are an agent for a detective agency and you know that today's function for your code is defined by $f(x) = 4x - 5$. Find the rule for f^{-1} algebraically.

40. You receive the following coded message today. (Read across from left to right.)

47 95 23 67 −1 59 27 31 51 23 7 −1 43 7 79 43 −1 75 55 67
31 71 75 27 15 23 67 15 −1 75 15 71 75 75 27 31 51
23 71 31 51 7 15 71 43 31 7 15 11 3 67 15 −1 11

Use the letter/number assignment described earlier to decode the message.

41. Why is a one-to-one function essential in this encoding/decoding process?

42. Use $f(x) = x^3 + 4$ to encode your name, using the letter/number assignment described earlier.

10.2 Exponential Functions

OBJECTIVES

1. Define an exponential function.
2. Graph an exponential function.
3. Solve exponential equations of the form $a^x = a^k$ for x.
4. Use exponential functions in applications involving growth or decay.

OBJECTIVE 1 Define an exponential function. In Section 8.2 we showed how to evaluate 2^x for rational values of x. For example,

$$2^3 = 8, \qquad 2^{-1} = \frac{1}{2}, \qquad 2^{1/2} = \sqrt{2}, \qquad \text{and} \qquad 2^{3/4} = \sqrt[4]{2^3} = \sqrt[4]{8}.$$

In more advanced courses it is shown that 2^x exists for all real number values of x, both rational and irrational. (Later in this chapter, we will see how to approximate the value of 2^x for irrational x.) The following definition of an exponential function assumes that a^x exists for all real numbers x.

Exponential Function

For $a > 0$, $a \neq 1$, and all real numbers x,

$$f(x) = a^x$$

defines the **exponential function with base *a*.**

NOTE The two restrictions on a in the definition of an exponential function are important. The restriction that a must be positive is necessary so that the function can be defined for all real numbers x. For example, letting a be negative ($a = -2$, for instance) and letting $x = \frac{1}{2}$ would give the expression $(-2)^{1/2}$, which is not real. The other restriction, $a \neq 1$, is necessary because 1 raised to any power is equal to 1, and the function would then be the linear function defined by $f(x) = 1$.

OBJECTIVE 2 Graph an exponential function. We graph an exponential function by finding several ordered pairs that belong to the function, plotting these points, and connecting them with a smooth curve.

CAUTION Be sure to plot enough points to see how rapidly the graph rises.

EXAMPLE 1 Graphing an Exponential Function with $a > 1$

Graph $f(x) = 2^x$.

Choose some values of x, and find the corresponding values of $f(x)$.

x	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Plotting these points and drawing a smooth curve through them gives the blue graph shown in Figure 8. This graph is typical of the graphs of exponential functions of the form $F(x) = a^x$, where $a > 1$. The larger the value of a, the faster the graph rises. To see this, compare the red graph of $F(x) = 5^x$ with the graph of $f(x) = 2^x$ in Figure 8.

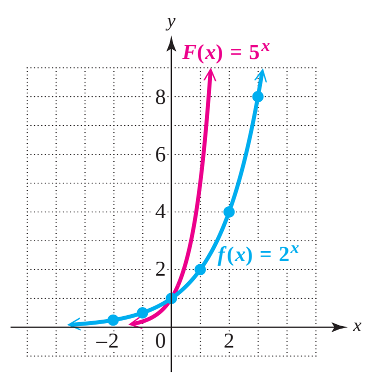


FIGURE 8

By the vertical line test, the graphs in Figure 8 represent functions. As these graphs suggest, the domain of an exponential function includes all real numbers. Because y is always positive, the range is $(0, \infty)$. Figure 8 also shows an important characteristic of exponential functions where $a > 1$: As x gets larger, y increases at a faster and faster rate.

Now Try Exercise 5.

YOU TRY IT

EXAMPLE 2 Graphing an Exponential Function with $a < 1$

Graph $g(x) = \left(\frac{1}{2}\right)^x$.

Again, find some points on the graph.

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The graph, shown in Figure 9, is very similar to that of $f(x) = 2^x$ (Figure 8) with the same domain and range, except that here as x gets larger, y *decreases*. This graph is typical of the graph of a function of the form $F(x) = a^x$, where $0 < a < 1$.

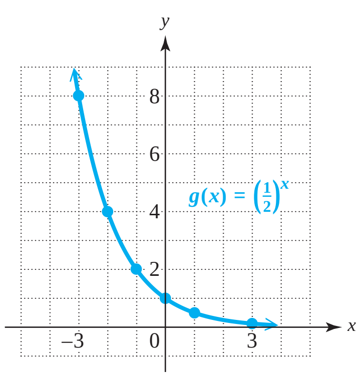


FIGURE 9

Now Try Exercise 7.

Based on Examples 1 and 2, we make the following generalizations about the graphs of exponential functions of the form $F(x) = a^x$.

AUDIO

Graph of $F(x) = a^x$

1. The graph contains the point $(0, 1)$.
2. When $a > 1$, the graph will *rise* from left to right. When $0 < a < 1$, the graph will *fall* from left to right. In both cases, the graph goes from the second quadrant to the first.
3. The graph will approach the x-axis, but never touch it. (Recall from Chapter 7 that such a line is called an *asymptote*.)
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

EXAMPLE 3 Graphing a More Complicated Exponential Function

Graph $f(x) = 3^{2x-4}$.

Find some ordered pairs.

$$\text{If } x = 0, \text{ then } y = 3^{2(0)-4} = 3^{-4} = \frac{1}{81}.$$

$$\text{If } x = 2, \text{ then } y = 3^{2(2)-4} = 3^{0} = 1.$$

These ordered pairs, $\left(0, \frac{1}{81}\right)$ and $(2, 1)$, along with the other ordered pairs shown in the table, lead to the graph in Figure 10. The graph is similar to the graph of $f(x) = 3^x$ except that it is shifted to the right and rises more rapidly.

x	y
0	$\frac{1}{81}$
1	$\frac{1}{9}$
2	1
3	9

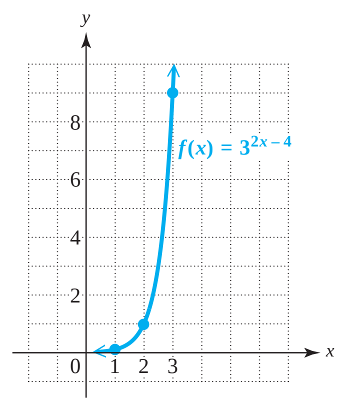


FIGURE 10

Now Try Exercise 11.

OBJECTIVE 3 Solve exponential equations of the form $a^x = a^k$ for x. Until this chapter, we have solved only equations that had the variable as a base, like $x^2 = 8$; all exponents have been constants. An **exponential equation** is an equation that has a variable in an exponent, such as

$$9^x = 27.$$

By the horizontal line test, the exponential function defined by $F(x) = a^x$ is a one-to-one function, so we can use the following property to solve many exponential equations.

AUDIO

Property for Solving an Exponential Equation

For $a > 0$ and $a \neq 1$, if $\boldsymbol{a^x = a^y}$ then $\boldsymbol{x = y}$.

This property would not necessarily be true if $a = 1$.

To solve an exponential equation using this property, follow these steps.

Solving an Exponential Equation

Step 1 **Each side must have the same base.** If the two sides of the equation do not have the same base, express each as a power of the same base if possible.

Step 2 **Simplify exponents** if necessary, using the rules of exponents.

Step 3 **Set exponents equal** using the property given in this section.

Step 4 **Solve** the equation obtained in Step 3.

NOTE These steps cannot be applied to an exponential equation like

$$3^x = 12$$

because Step 1 cannot easily be done. A method for solving such equations is given in Section 10.6.

EXAMPLE 4 Solving an Exponential Equation

Solve the equation $9^x = 27$.

We can use the property given at the bottom of the previous page if both sides are written with the same base.

$$\begin{aligned}
9^x &= 27 \\
(3^2)^x &= 3^3 && \text{Write with the same base; } 9 = 3^2 \text{ and } 27 = 3^3. \text{ (Step 1)} \\
3^{2x} &= 3^3 && \text{Power rule for exponents (Step 2)} \\
2x &= 3 && \text{If } a^x = a^y, \text{ then } x = y. \text{ (Step 3)} \\
x &= \frac{3}{2} && \text{(Step 4)}
\end{aligned}$$

Check that the solution set is $\left\{\frac{3}{2}\right\}$ by substituting $\frac{3}{2}$ for x in the original equation.

Now Try Exercise 17.

EXAMPLE 5 Solving Exponential Equations

Solve each equation.

(a) $4^{3x-1} = 16^{x+2}$

$$\begin{aligned}
(2^2)^{3x-1} &= (2^4)^{x+2} && \text{Write with the same base; } 4 = 2^2 \text{ and } 16 = 2^4. \\
2^{6x-2} &= 2^{4x+8} && \text{Power rule for exponents} \\
6x - 2 &= 4x + 8 && \text{Set exponents equal.} \\
2x &= 10 && \text{Subtract } 4x\text{; add 2.} \\
x &= 5 && \text{Divide by 2.}
\end{aligned}$$

Verify that the solution set is $\{5\}$.

(b) $6^x = \dfrac{1}{216}$

$6^x = \dfrac{1}{6^3}$ $\quad$ $216 = 6^3$

$6^x = 6^{-3}$ $\quad$ Write with the same base; $\dfrac{1}{6^3} = 6^{-3}$.

$x = -3$ $\quad$ Set exponents equal.

Verify that the solution set is $\{-3\}$.

(c) $\left(\dfrac{2}{3}\right)^x = \dfrac{9}{4}$

$\left(\dfrac{2}{3}\right)^x = \left(\dfrac{4}{9}\right)^{-1}$ $\quad$ $\dfrac{9}{4} = \left(\dfrac{4}{9}\right)^{-1}$

$\left(\dfrac{2}{3}\right)^x = \left[\left(\dfrac{2}{3}\right)^2\right]^{-1}$ $\quad$ Write with the same base.

$\left(\dfrac{2}{3}\right)^x = \left(\dfrac{2}{3}\right)^{-2}$ $\quad$ Power rule for exponents

$x = -2$ $\quad$ Set exponents equal.

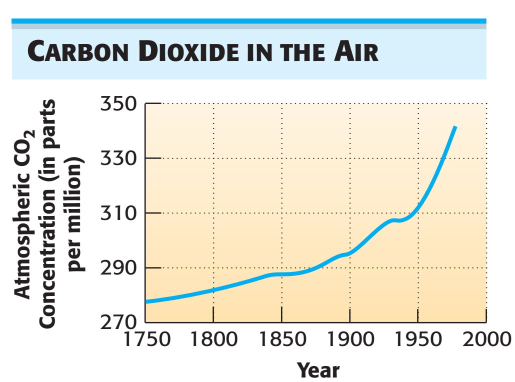
Check that the solution set is $\{-2\}$.

Now Try Exercises 19, 21, and 25.

OBJECTIVE 4 Use exponential functions in applications involving growth or decay.

EXAMPLE 6 Solving an Application Involving Exponential Growth

One result of the rapidly increasing world population is an increase of carbon dioxide in the air, which scientists believe may be contributing to global warming. Both population and amounts of carbon dioxide in the air are increasing exponentially. This means that the growth rate is continually increasing. The graph in Figure 11 shows the concentration of carbon dioxide (in parts per million) in the air.



Source: *Sacramento Bee*, Monday, September 13, 1993.

FIGURE 11

The data are approximated by the function defined by

$$f(x) = 278(1.00084)^x,$$

where x is the number of years since 1750. Use this function and a calculator to approximate the concentration of carbon dioxide in parts per million for each year.

(a) 1900
Because x represents the number of years since 1750, in this case

$$x = 1900 - 1750 = 150.$$

Thus, evaluate $f(150)$.

$$\begin{aligned} f(150) &= 278(1.00084)^{150} && \text{Let } x = 150. \\ &\approx 315 \text{ parts per million} && \text{Use a calculator.} \end{aligned}$$

(b) 1950
Use $x = 1950 - 1750 = 200$.

$$\begin{aligned} f(200) &= 278(1.00084)^{200} \\ &\approx 329 \text{ parts per million} \end{aligned}$$

Now Try Exercise 39.

EXAMPLE 7 Applying an Exponential Decay Function

The atmospheric pressure (in millibars) at a given altitude x, in meters, can be approximated by the function defined by

$$f(x) = 1038(1.000134)^{-x},$$

for values of x between 0 and 10,000. Because the base is greater than 1 and the coefficient of x in the exponent is negative, the function values decrease as x increases. This means that as the altitude increases, the atmospheric pressure decreases. (*Source:* Miller, A. and J. Thompson, *Elements of Meteorology*, Fourth Edition, Charles E. Merrill Publishing Company, 1993.)

(a) According to this function, what is the pressure at ground level?
At ground level, $x = 0$, so

$$f(0) = 1038(1.000134)^{-0} = 1038(1) = 1038.$$

The pressure is 1038 millibars.

(b) What is the pressure at 5000 m?
Use a calculator to find $f(5000)$.

$$\begin{aligned} f(5000) &= 1038(1.000134)^{-5000} \\ &\approx 531 \end{aligned}$$

The pressure is approximately 531 millibars.

Now Try Exercise 41.

10.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 16/Videotape 16

Choose the correct response in Exercises 1–4.

1. Which point lies on the graph of $f(x) = 2^x$?

A. $(1, 0)$ **B.** $(2, 1)$ **C.** $(0, 1)$ **D.** $\left(\sqrt{2}, \frac{1}{2}\right)$

2. Which statement is true?

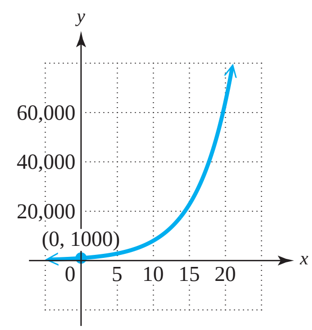
A. The y-intercept of the graph of $f(x) = 10^x$ is $(0, 10)$.
B. For any $a > 1$, the graph of $f(x) = a^x$ falls from left to right.
C. The point $\left(\frac{1}{2}, \sqrt{5}\right)$ lies on the graph of $f(x) = 5^x$.
D. The graph of $y = 4^x$ rises at a faster rate than the graph of $y = 10^x$.

3. The asymptote of the graph of $F(x) = a^x$

A. is the x-axis **B.** is the y-axis
C. has equation $x = 1$ **D.** has equation $y = 1$.

4. Which equation is graphed here?

A. $y = 1000\left(\frac{1}{2}\right)^{.3x}$ **B.** $y = 1000\left(\frac{1}{2}\right)^{x}$
C. $y = 1000(2)^{.3x}$ **D.** $y = 1000^x$



Graph each exponential function. See Examples 1–3.

5. $f(x) = 3^x$ **6.** $f(x) = 5^x$ **7.** $g(x) = \left(\frac{1}{3}\right)^x$ **8.** $g(x) = \left(\frac{1}{5}\right)^x$

9. $y = 4^{-x}$ **10.** $y = 6^{-x}$ **11.** $y = 2^{2x-2}$ **12.** $y = 2^{2x+1}$

13. (a) For an exponential function defined by $f(x) = a^x$, if $a > 1$, the graph __________ (rises/falls) from left to right. If $0 < a < 1$, the graph __________ (rises/falls) from left to right.

(b) Based on your answers in part (a), make a conjecture (an educated guess) concerning whether an exponential function defined by $f(x) = a^x$ is one-to-one. Then decide whether it has an inverse based on the concepts of Section 10.1.

14. In your own words, describe the characteristics of the graph of an exponential function. Use the exponential function defined by $f(x) = 3^x$ (Exercise 5) and the words *asymptote, domain,* and *range* in your explanation.

Solve each equation. See Examples 4 and 5.

15. $6^x = 36$ **16.** $8^x = 64$ **17.** $100^x = 1000$ **18.** $8^x = 4$

19. $16^{2x+1} = 64^{x+3}$ **20.** $9^{2x-8} = 27^{x-4}$ **21.** $5^x = \frac{1}{125}$ **22.** $3^x = \frac{1}{81}$

23. $5^x = .2$ **24.** $10^x = .1$ **25.** $\left(\frac{3}{2}\right)^x = \frac{8}{27}$ **26.** $\left(\frac{4}{3}\right)^x = \frac{27}{64}$

Use the exponential key of a calculator to find an approximation to the nearest thousandth.

27. $12^{2.6}$ **28.** $13^{1.8}$ **29.** $.5^{3.921}$ **30.** $.6^{4.917}$ **31.** $2.718^{2.5}$ **32.** $2.718^{-3.1}$

33. Try to evaluate $(-2)^4$ on a scientific calculator. You may get an error message, since the exponential function key on many calculators does not allow negative bases. Discuss the concept introduced in this section that is closely related to this "peculiarity" of many scientific calculators.

34. Explain why the exponential equation $4^x = 6$ cannot be solved using the method explained in this section. Change 6 to another number that *will* allow the method of this section to be used, and then solve the equation.

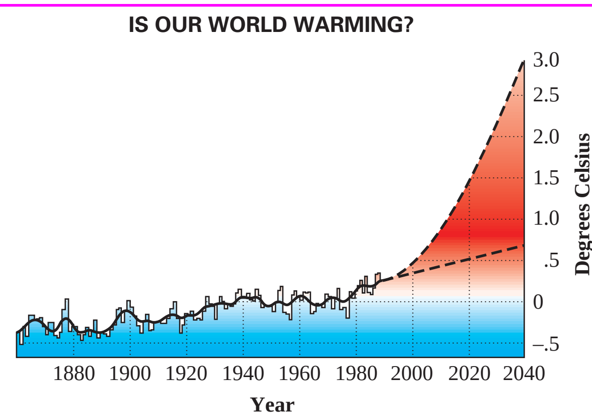
The graph shown here accompanied the article "Is Our World Warming?" which appeared in the October 1990 issue of National Geographic. *It shows projected temperature increases using two graphs: one an exponential-type curve, and the other linear. From the graph, approximate the increase* **(a)** *for the exponential curve and* **(b)** *for the linear graph for each year.*

35. 2000

36. 2010

37. 2020

38. 2040



Graph, "Zero Equals Average Global Temperature for the Period 1950–1979." Dale D. Glasgow, © National Geographic Society. Reprinted by permission.

Solve each problem. See Examples 6 and 7.

39. Based on figures from 1970 through 1998, the worldwide carbon monoxide emissions in thousands of tons are approximated by the exponential function defined by

$$f(x) = 132{,}359(1.0124)^{-x},$$

where $x = 0$ corresponds to 1970, $x = 5$ corresponds to 1975, and so on. (*Source:* U.S. Environmental Protection Agency.)

(a) Use this model to approximate the emissions in 1970.

(b) Use this model to approximate the emissions in 1995.

(c) In 1998, the actual amount of emissions was 89,454 million tons. How does this compare to the number that the model provides?

40. Based on figures from 1980 through 1999, the municipal solid waste generated in millions of tons can be approximated by the exponential function defined by

$$f(x) = 157.28(1.0204)^x,$$

where $x = 0$ corresponds to 1980, $x = 5$ corresponds to 1985, and so on. (*Source:* U.S. Environmental Protection Agency.)

(a) Use the model to approximate the number of tons of this waste in 1980.
(b) Use the model to approximate the number of tons of this waste in 1995.
(c) In 1999, the actual number of millions of tons of this waste was 229.9. How does this compare to the number that the model provides?

41. A small business estimates that the value $V(t)$ of a copy machine is decreasing according to the function defined by

$$V(t) = 5000(2)^{-.15t},$$

where t is the number of years that have elapsed since the machine was purchased, and $V(t)$ is in dollars.

(a) What was the original value of the machine?
(b) What is the value of the machine 5 yr after purchase? Give your answer to the nearest dollar.
(c) What is the value of the machine 10 yr after purchase? Give your answer to the nearest dollar.
(d) Graph the function.

42. The amount of radioactive material in an ore sample is given by the function defined by

$$A(t) = 100(3.2)^{-.5t},$$

where $A(t)$ is the amount present, in grams, of the sample t months after the initial measurement.

(a) How much was present at the initial measurement? (*Hint:* $t = 0$.)
(b) How much was present 2 months later?
(c) How much was present 10 months later?
(d) Graph the function.

43. Refer to the function in Exercise 41. When will the value of the machine be \$2500? (*Hint:* Let $V(t) = 2500$, divide both sides by 5000, and use the method of Example 4.)

44. Refer to the function in Exercise 41. When will the value of the machine be \$1250?

RELATING CONCEPTS (EXERCISES 45–50)

For Individual or Group Work

In these exercises we examine several methods of simplifying the expression $16^{3/4}$. ***Work Exercises 45–50 in order.***

45. Write $16^{3/4}$ as a radical expression with the exponent outside the radical. Then simplify the expression.

46. Write $16^{3/4}$ as a radical expression with the exponent under the radical. Then simplify the expression.

47. Use a calculator to find the square root of 16^3. Now find the square root of that result.

48. Explain why the result in Exercise 47 is equal to $16^{3/4}$.

49. Predict the result a calculator will give when 16 is raised to the .75 power. Then check your answer by actually performing the operation on your calculator.

50. Write $\sqrt[100]{16^{75}}$ as an exponential expression. Then write the exponent in lowest terms, rewrite as a radical, and evaluate this radical expression.

10.3 Logarithmic Functions

OBJECTIVES

1. Define a logarithm.
2. Convert between exponential and logarithmic forms.
3. Solve logarithmic equations of the form $\log_a b = k$ for a, b, or k.
4. Define and graph logarithmic functions.
5. Use logarithmic functions in applications involving growth or decay.

The graph of $y = 2^x$ is the curve shown in blue in Figure 12. Because $y = 2^x$ defines a one-to-one function, it has an inverse. Interchanging x and y gives $x = 2^y$, the inverse of $y = 2^x$. As we saw in Section 10.1, the graph of the inverse is found by reflecting the graph of $y = 2^x$ about the line $y = x$. The graph of $x = 2^y$ is shown as a red curve in Figure 12.

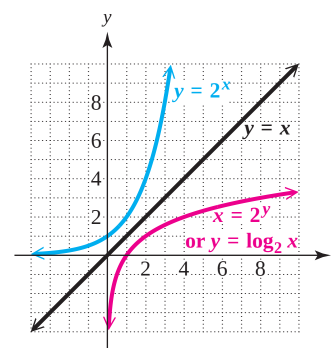


FIGURE 12

OBJECTIVE 1 Define a logarithm. We cannot solve the equation $x = 2^y$ for the dependent variable y with the methods presented up to now. The following definition is used to solve $x = 2^y$ for y.

Logarithm

For all positive numbers a, $a \neq 1$, and all positive numbers x,

$$\mathbf{y = \log_a x} \quad \text{means the same as} \quad \mathbf{x = a^y}.$$

This key statement should be memorized. The abbreviation **log** is used for the word **logarithm.** Read $\log_a x$ as "the logarithm of x to the base a" or "the base a logarithm of x." To remember the location of the base and the exponent in each form, refer to the following diagrams.

$$\text{Logarithmic form: } \overset{\text{Exponent}}{y} = \log_{\underset{\text{Base}}{a}} x \qquad \text{Exponential form: } x = \underset{\text{Base}}{a}^{\overset{\text{Exponent}}{y}}$$

In working with logarithmic form and exponential form, remember the following.

Meaning of $\log_a x$

A logarithm is an exponent; $\log_a x$ is the exponent to which the base a must be raised to obtain x.

OBJECTIVE 2 Convert between exponential and logarithmic forms. We can use the definition of logarithm to write exponential statements in logarithmic form and logarithmic statements in exponential form. The following table shows several pairs of equivalent statements.

Exponential Form	*Logarithmic Form*
$3^2 = 9$	$\log_3 9 = 2$
$\left(\frac{1}{5}\right)^{-2} = 25$	$\log_{1/5} 25 = -2$
$10^5 = 100{,}000$	$\log_{10} 100{,}000 = 5$
$4^{-3} = \frac{1}{64}$	$\log_4 \frac{1}{64} = -3$

OBJECTIVE 3 Solve logarithmic equations of the form $\log_a b = k$ for a, b, or k. A **logarithmic equation** is an equation with a logarithm in at least one term. We solve logarithmic equations of the form $\log_a b = k$ for any of the three variables by first writing the equation in exponential form.

EXAMPLE 1 Solving Logarithmic Equations

Solve each equation.

(a) $\log_4 x = -2$

By the definition of logarithm, $\log_4 x = -2$ is equivalent to $x = 4^{-2}$. Solve this exponential equation.

$$x = 4^{-2} = \frac{1}{16}$$

The solution set is $\left\{\frac{1}{16}\right\}$.

(b) $\log_{1/2}(3x + 1) = 2$

$$3x + 1 = \left(\frac{1}{2}\right)^2 \quad \text{Write in exponential form.}$$

$$3x + 1 = \frac{1}{4}$$

$$12x + 4 = 1 \quad \text{Multiply by 4.}$$

$$12x = -3 \quad \text{Subtract 4.}$$

$$x = -\frac{1}{4} \quad \text{Divide by 12.}$$

The solution set is $\left\{-\frac{1}{4}\right\}$.

(c) $\log_x 3 = 2$

$$x^2 = 3 \quad \text{Write in exponential form.}$$

$$x = \pm\sqrt{3} \quad \text{Take square roots.}$$

Notice that only the principal square root satisfies the equation since the base must be a positive number. The solution set is $\left\{\sqrt{3}\right\}$.

(d) $\log_{49} \sqrt[3]{7} = x$

$$49^x = \sqrt[3]{7} \quad \text{Write in exponential form.}$$
$$(7^2)^x = 7^{1/3} \quad \text{Write with the same base.}$$
$$7^{2x} = 7^{1/3} \quad \text{Power rule for exponents}$$
$$2x = \frac{1}{3} \quad \text{Set exponents equal.}$$
$$x = \frac{1}{6} \quad \text{Divide by 2.}$$

The solution set is $\left\{\frac{1}{6}\right\}$.

Now Try Exercises 21, 25, 37, and 39.

For any real number b, we know that $b^1 = b$ and for $b \neq 0$, $b^0 = 1$. Writing these two statements in logarithmic form gives the following two properties of logarithms.

Properties of Logarithms

For any positive real number b, $b \neq 1$,

$$\log_b b = 1 \quad \text{and} \quad \log_b 1 = 0.$$

EXAMPLE 2 Using Properties of Logarithms

Use the preceding two properties of logarithms to evaluate each logarithm.

(a) $\log_7 7 = 1$

(b) $\log_{\sqrt{2}} \sqrt{2} = 1$

(c) $\log_9 1 = 0$

(d) $\log_{.2} 1 = 0$

Now Try Exercise 1.

OBJECTIVE 4 Define and graph logarithmic functions. Now we define the logarithmic function with base a.

Logarithmic Function

If a and x are positive numbers, with $a \neq 1$, then

$$G(x) = \log_a x$$

defines the **logarithmic function with base a.**

The graph of $x = 2^y$ in Figure 12, which is equivalent to $y = g(x) = \log_2 x$, is typical of graphs of logarithmic functions with base $a > 1$. To graph a logarithmic function, it is helpful to write it in exponential form first. Then plot selected ordered pairs to determine the graph.

EXAMPLE 3 Graphing a Logarithmic Function

Graph $f(x) = \log_{1/2} x$.

By writing $y = f(x) = \log_{1/2} x$ in exponential form as $x = \left(\frac{1}{2}\right)^y$, we can identify ordered pairs that satisfy the equation. Here it is easier to choose values for y and find the corresponding values of x. See the table of ordered pairs.

x	y
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2

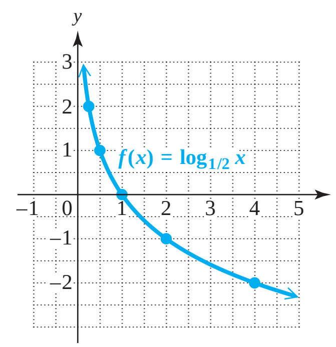


FIGURE 13

Plotting these points (be careful to get the x- and y-values in the right order) and connecting them with a smooth curve gives the graph in Figure 13. This graph is typical of logarithmic functions with $0 < a < 1$.

Now Try Exercise 43.

Based on the graphs of the functions defined by $y = \log_2 x$ in Figure 12 and $y = \log_{1/2} x$ in Figure 13, we make the following generalizations about the graphs of logarithmic functions of the form $G(x) = \log_a x$.

Graph of $G(x) = \log_a x$

1. The graph contains the point $(1, 0)$.
2. When $a > 1$, the graph will *rise* from left to right, from the fourth quadrant to the first. When $0 < a < 1$, the graph will *fall* from left to right, from the first quadrant to the fourth.
3. The graph will approach the y-axis, but never touch it. (The y-axis is an asymptote.)
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

Compare these generalizations to the similar ones for exponential functions found in Section 10.2.

OBJECTIVE 5 Use logarithmic functions in applications involving growth or decay. Logarithmic functions, like exponential functions, can be applied to growth or decay of real-world phenomena.

EXPLORE

YOU TRY IT

VIDEO

Photo not available

EXAMPLE 4 Solving an Application of a Logarithmic Function

The function defined by

$$f(x) = 27 + 1.105 \log_{10}(x + 1)$$

approximates the barometric pressure in inches of mercury at a distance of x mi from the eye of a typical hurricane. (*Source:* Miller, A. and R. Anthes, *Meteorology*, Fifth Edition, Charles E. Merrill Publishing Company, 1985.)

(a) Approximate the pressure 9 mi from the eye of the hurricane.

Let $x = 9$, and find $f(9)$.

$$\begin{aligned} f(9) &= 27 + 1.105 \log_{10}(9 + 1) && \text{Let } x = 9. \\ &= 27 + 1.105 \log_{10} 10 && \text{Add inside parentheses.} \\ &= 27 + 1.105(1) && \log_{10} 10 = 1 \\ &= 28.105 \end{aligned}$$

The pressure 9 mi from the eye of the hurricane is 28.105 in.

(b) Approximate the pressure 99 mi from the eye of the hurricane.

$$\begin{aligned} f(99) &= 27 + 1.105 \log_{10}(99 + 1) && \text{Let } x = 99. \\ &= 27 + 1.105 \log_{10} 100 && \text{Add inside parentheses.} \\ &= 27 + 1.105(2) && \log_{10} 100 = 2 \\ &= 29.21 \end{aligned}$$

The pressure 99 mi from the eye of the hurricane is 29.21 in.

Now Try Exercise 55.

Photo not available

CONNECTIONS

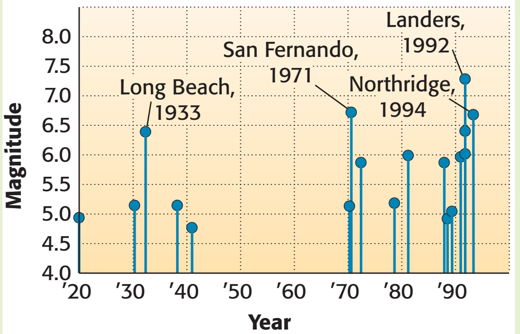
In the United States, the intensity of an earthquake is rated using the *Richter scale.* The Richter scale rating of an earthquake of intensity x is given by

$$R = \log_{10} \frac{x}{x_0},$$

where x_0 is the intensity of an earthquake of a certain (small) size. The graph here shows Richter scale ratings for major Southern California earthquakes since 1920. As the graph indicates, earthquakes "come in bunches," and the 1990s were an especially busy time.

MAJOR SOUTHERN CALIFORNIA EARTHQUAKES

Earthquakes with magnitudes greater than 4.8



Source: Caltech; U.S. Geological Survey.

(continued)

For Discussion or Writing

Writing the given logarithmic equation in exponential form, we obtain

$$10^R = \frac{x}{x_0} \qquad \text{or} \qquad x = 10^R x_0.$$

1. The 1994 Northridge earthquake had a Richter scale rating of 6.7; the Landers earthquake had a rating of 7.3. How much more powerful was the Landers earthquake than the Northridge earthquake?
2. Compare the smallest rated earthquake in the figure (at 4.8) with the Landers quake. How much more powerful was the Landers quake?

10.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 16/Videotape 16

1. By definition, $\log_a x$ is the exponent to which the base a must be raised in order to obtain x. Use this definition to match the logarithm in Column I with its value in Column II. (*Example:* $\log_3 9$ is equal to 2 because 2 is the exponent to which 3 must be raised in order to obtain 9.)

I	II
(a) $\log_4 16$	**A.** -2
(b) $\log_3 81$	**B.** -1
(c) $\log_3\left(\frac{1}{3}\right)$	**C.** 2
(d) $\log_{10} .01$	**D.** 0
(e) $\log_5\sqrt{5}$	**E.** $\frac{1}{2}$
(f) $\log_{13} 1$	**F.** 4

2. Match the logarithmic equation in Column I with the corresponding exponential equation from Column II.

I	II
(a) $\log_{1/3} 3 = -1$	**A.** $8^{1/3} = \sqrt[3]{8}$
(b) $\log_5 1 = 0$	**B.** $\left(\frac{1}{3}\right)^{-1} = 3$
(c) $\log_2\sqrt{2} = \frac{1}{2}$	**C.** $4^1 = 4$
(d) $\log_{10} 1000 = 3$	**D.** $2^{1/2} = \sqrt{2}$
(e) $\log_8\sqrt[3]{8} = \frac{1}{3}$	**E.** $5^0 = 1$
(f) $\log_4 4 = 1$	**F.** $10^3 = 1000$

Write in logarithmic form. See the table in Objective 2.

EXERCISES

3. $4^5 = 1024$ **4.** $3^6 = 729$ **5.** $\left(\frac{1}{2}\right)^{-3} = 8$ **6.** $\left(\frac{1}{6}\right)^{-3} = 216$

7. $10^{-3} = .001$ **8.** $36^{1/2} = 6$ **9.** $\sqrt[4]{625} = 5$ **10.** $\sqrt[3]{343} = 7$

Write in exponential form. See the table in Objective 2.

EXERCISES

11. $\log_4 64 = 3$ **12.** $\log_2 512 = 9$ **13.** $\log_{10}\frac{1}{10{,}000} = -4$ **14.** $\log_{100} 100 = 1$

15. $\log_6 1 = 0$ **16.** $\log_\pi 1 = 0$ **17.** $\log_9 3 = \frac{1}{2}$ **18.** $\log_{64} 2 = \frac{1}{6}$

19. When a student asked his teacher to explain how to evaluate $\log_9 3$ without showing any work, his teacher told him, "Think radically." Explain what the teacher meant by this hint.

20. A student told her teacher, "I know that $\log_2 1$ is the exponent to which 2 must be raised in order to obtain 1, but I can't think of any such number." How would you explain to the student that the value of $\log_2 1$ is 0?

Solve each equation for x. See Examples 1 and 2.

21. $x = \log_{27} 3$ **22.** $x = \log_{125} 5$ **23.** $\log_x 9 = \frac{1}{2}$ **24.** $\log_x 5 = \frac{1}{2}$

25. $\log_x 125 = -3$ **26.** $\log_x 64 = -6$ **27.** $\log_{12} x = 0$ **28.** $\log_4 x = 0$

29. $\log_x x = 1$ **30.** $\log_x 1 = 0$ **31.** $\log_x \frac{1}{25} = -2$ **32.** $\log_x \frac{1}{10} = -1$

33. $\log_8 32 = x$ **34.** $\log_{81} 27 = x$ **35.** $\log_\pi \pi^4 = x$ **36.** $\log_{\sqrt{2}} \sqrt{2^9} = x$

37. $\log_6 \sqrt{216} = x$ **38.** $\log_4 \sqrt{64} = x$

39. $\log_4(2x + 4) = 3$ **40.** $\log_3(2x + 7) = 4$

If the point (p, q) is on the graph of $f(x) = a^x$ (for $a > 0$ and $a \neq 1$), then the point (q, p) is on the graph of $f^{-1}(x) = \log_a x$. Use this fact, and refer to the graphs required in Exercises 5–8 in Section 10.2 to graph each logarithmic function. See Example 3.

41. $y = \log_3 x$ **42.** $y = \log_5 x$ **43.** $y = \log_{1/3} x$ **44.** $y = \log_{1/5} x$

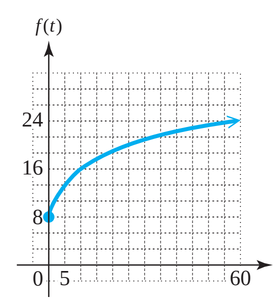
45. Explain why 1 is not allowed as a base for a logarithmic function.

46. Compare the summary of facts about the graph of $F(x) = a^x$ in Section 10.2 with the similar summary of facts about the graph of $G(x) = \log_a x$ in this section. Make a list of the facts that reinforce the concept that F and G are inverse functions.

47. The domain of $F(x) = a^x$ is $(-\infty, \infty)$, while the range is $(0, \infty)$. Therefore, since $G(x) = \log_a x$ defines the inverse of F, the domain of G is __________, while the range of G is __________.

48. The graphs of both $F(x) = 3^x$ and $G(x) = \log_3 x$ rise from left to right. Which one rises at a faster rate?

Use the graph at the right to predict the value of $f(t)$ for the given value of t.



49. $t = 0$

50. $t = 10$

51. $t = 60$

52. Show that the points determined in Exercises 49–51 lie on the graph of $f(t) = 8 \log_5(2t + 5)$.

Solve each application of a logarithmic function. See Example 4.

53. According to selected figures from 1981 through 1995, the number of Superfund hazardous waste sites in the United States can be approximated by the function defined by

$$f(x) = 11.34 + 317.01 \log_2 x,$$

where $x = 1$ corresponds to 1981, $x = 2$ to 1982, and so on. (*Source:* U.S. Environmental Protection Agency.) Use the function to approximate the number of sites in each year.

(a) 1984 **(b)** 1988 **(c)** 1996

54. According to selected figures from 1980 through 1993, the number of trillion cubic feet of dry natural gas consumed worldwide can be approximated by the function defined by

$$f(x) = 51.47 + 6.044 \log_2 x,$$

where $x = 1$ corresponds to 1980, $x = 2$ to 1981, and so on. (*Source:* Energy Information Administration.) Use the function to approximate consumption in each year.

(a) 1980 **(b)** 1987 **(c)** 1995

55. Sales (in thousands of units) of a new product are approximated by the function defined by

$$S(t) = 100 + 30 \log_3(2t + 1),$$

where t is the number of years after the product is introduced.

(a) What were the sales after 1 yr?
(b) What were the sales after 13 yr?
(c) Graph $y = S(t)$.

56. A study showed that the number of mice in an old abandoned house was approximated by the function defined by

$$M(t) = 6 \log_4(2t + 4),$$

where t is measured in months and $t = 0$ corresponds to January 1998. Find the number of mice in the house in

(a) January 1998 **(b)** July 1998 **(c)** July 2000.
(d) Graph the function.

57. A supply of hybrid striped bass were introduced into a lake in January 1990. Biologists researching the bass population found that the number of bass in the lake was approximated by the function defined by

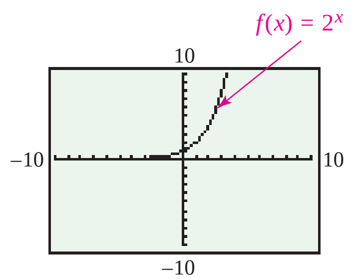
$$B(t) = 500 \log_3(2t + 3),$$

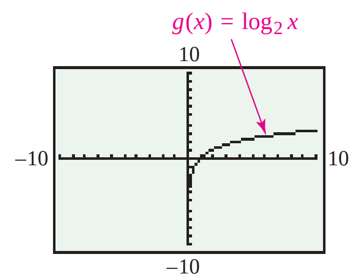
where $t = 0$ corresponds to January 1990, $t = 1$ to January 1991, $t = 2$ to January 1992, and so on. Use this function to find the bass population in

(a) January 1990 **(b)** January 1993 **(c)** January 2002.
(d) Graph the function for $0 \le t \le 12$.

58. Use the exponential key of your calculator to find approximations for the expression $\left(1 + \frac{1}{x}\right)^x$, using x values of 1, 10, 100, 1000, and 10,000. Explain what seems to be happening as x gets larger and larger.

As mentioned in Section 10.1, some graphing calculators have the capability of drawing the inverse of a function. For example, the two screens that follow show the graphs of $f(x) = 2^x$ and $g(x) = \log_2 x$. The graph of g was obtained by drawing the graph of f^{-1}, since $g(x) = f^{-1}(x)$. (Compare to Figure 12 in this section.)





Use a graphing calculator with the capability of drawing the inverse of a function to draw the graph of each logarithmic function. Use the standard viewing window.

59. $g(x) = \log_3 x$ (Compare to Exercise 41.)

60. $g(x) = \log_5 x$ (Compare to Exercise 42.)

61. $g(x) = \log_{1/3} x$ (Compare to Exercise 43.)

62. $g(x) = \log_{1/5} x$ (Compare to Exercise 44.)

10.4 Properties of Logarithms

OBJECTIVES

1. Use the product rule for logarithms.
2. Use the quotient rule for logarithms.
3. Use the power rule for logarithms.
4. Use properties to write alternative forms of logarithmic expressions.

Logarithms have been used as an aid to numerical calculation for several hundred years. Today the widespread use of calculators has made the use of logarithms for calculation obsolete. However, logarithms are still very important in applications and in further work in mathematics.

OBJECTIVE 1 Use the product rule for logarithms. One way in which logarithms simplify problems is by changing a problem of multiplication into one of addition. We know that $\log_2 4 = 2$, $\log_2 8 = 3$, and $\log_2 32 = 5$. Since $2 + 3 = 5$,

$$\log_2 32 = \log_2 4 + \log_2 8$$
$$\log_2 (4 \cdot 8) = \log_2 4 + \log_2 8.$$

This is true in general.

Product Rule for Logarithms

If x, y, and b are positive real numbers, where $b \neq 1$, then

$$\mathbf{\log_b xy = \log_b x + \log_b y.}$$

That is, the logarithm of a product is the sum of the logarithms of the factors.

NOTE The word statement of the product rule can be restated by replacing "logarithm" with "exponent." The rule then becomes the familiar rule for multiplying exponential expressions: The *exponent* of a product is the sum of the *exponents* of the factors.

To prove this rule, let $m = \log_b x$ and $n = \log_b y$, and recall that

$$\log_b x = m \quad \text{means} \quad b^m = x.$$
$$\log_b y = n \quad \text{means} \quad b^n = y.$$

Now consider the product xy.

$$xy = b^m \cdot b^n \qquad \text{Substitute.}$$
$$xy = b^{m+n} \qquad \text{Product rule for exponents}$$
$$\log_b xy = m + n \qquad \text{Convert to logarithmic form.}$$
$$\log_b xy = \log_b x + \log_b y \qquad \text{Substitute.}$$

The last statement is the result we wished to prove.

EXAMPLE 1 Using the Product Rule

Use the product rule to rewrite each expression. Assume $x > 0$.

(a) $\log_5(6 \cdot 9)$

By the product rule,

$$\log_5(6 \cdot 9) = \log_5 6 + \log_5 9.$$

(b) $\log_7 8 + \log_7 12 = \log_7(8 \cdot 12) = \log_7 96$

(c) $\log_3(3x) = \log_3 3 + \log_3 x$

$= 1 + \log_3 x$ $\quad$ $\log_3 3 = 1$

(d) $\log_4 x^3 = \log_4(x \cdot x \cdot x)$ $\quad$ $x^3 = x \cdot x \cdot x$

$= \log_4 x + \log_4 x + \log_4 x$ $\quad$ Product rule

$= 3 \log_4 x$

Now Try Exercises 7 and 21.

OBJECTIVE 2 Use the quotient rule for logarithms. The rule for division is similar to the rule for multiplication.

Quotient Rule for Logarithms

If x, y, and b are positive real numbers, where $b \neq 1$, then

$$\log_b \frac{x}{y} = \log_b x - \log_b y.$$

That is, the logarithm of a quotient is the difference between the logarithm of the numerator and the logarithm of the denominator.

The proof of this rule is very similar to the proof of the product rule.

EXAMPLE 2 Using the Quotient Rule

Use the quotient rule to rewrite each logarithm.

(a) $\log_4 \frac{7}{9} = \log_4 7 - \log_4 9$

(b) $\log_5 6 - \log_5 x = \log_5 \frac{6}{x}, \quad x > 0$

(c) $\log_3 \frac{27}{5} = \log_3 27 - \log_3 5$

$= 3 - \log_3 5$ $\quad$ $\log_3 27 = 3$

Now Try Exercises 9 and 23.

CAUTION Remember that there is no property of logarithms to rewrite the logarithm of a *sum* or *difference*. For example, we *cannot* write $\log_b(x + y)$ in terms of $\log_b x$ and $\log_b y$. Also,

$$\log_b \frac{x}{y} \neq \frac{\log_b x}{\log_b y}.$$

OBJECTIVE 3 Use the power rule for logarithms. An exponential expression such as 2^3 means $2 \cdot 2 \cdot 2$; the base is used as a factor 3 times. Thus, it seems reasonable that the product rule can be extended to rewrite the logarithm of a power as the product of the exponent and the logarithm of the base. For example, by the product rule for logarithms,

$$\begin{aligned}\log_5 2^3 &= \log_5(2 \cdot 2 \cdot 2)\\ &= \log_5 2 + \log_5 2 + \log_5 2\\ &= 3 \log_5 2.\end{aligned}$$

Also,

$$\begin{aligned}\log_2 7^4 &= \log_2(7 \cdot 7 \cdot 7 \cdot 7)\\ &= \log_2 7 + \log_2 7 + \log_2 7 + \log_2 7\\ &= 4 \log_2 7.\end{aligned}$$

Furthermore, we saw in Example 1(d) that $\log_4 x^3 = 3 \log_4 x$. These examples suggest the following rule.

AUDIO

Power Rule for Logarithms

If x and b are positive real numbers, where $b \neq 1$, and if r is any real number, then

$$\boldsymbol{\log_b x^r = r \log_b x.}$$

That is, the logarithm of a number to a power equals the exponent times the logarithm of the number.

As further examples of this result,

$$\log_b m^5 = 5 \log_b m \qquad \text{and} \qquad \log_3 5^4 = 4 \log_3 5.$$

To prove the power rule, let

$\log_b x = m.$	
$b^m = x$	Convert to exponential form.
$(b^m)^r = x^r$	Raise to the power r.
$b^{mr} = x^r$	Power rule for exponents
$\log_b x^r = mr$	Convert to logarithmic form.
$\log_b x^r = rm$	
$\log_b x^r = r \log_b x$	$m = \log_b x$

This is the statement to be proved.

As a special case of the power rule, let $r = \frac{1}{p}$, so

$$\log_b \sqrt[p]{x} = \log_b x^{1/p} = \frac{1}{p} \log_b x.$$

For example, using this result, with $x > 0$,

$$\log_b \sqrt[5]{x} = \log_b x^{1/5} = \frac{1}{5} \log_b x \quad \text{and} \quad \log_b \sqrt[3]{x^4} = \log_b x^{4/3} = \frac{4}{3} \log_b x.$$

Another special case is

$$\log_b \frac{1}{x} = \log_b x^{-1} = -\log_b x.$$

NOTE For a review of rational exponents, refer to Section 8.2.

EXAMPLE 3 Using the Power Rule

Use the power rule to rewrite each logarithm. Assume $b > 0$, $x > 0$, and $b \neq 1$.

(a) $\log_5 4^2 = 2 \log_5 4$

(b) $\log_b x^5 = 5 \log_b x$

(c) $\log_b \sqrt{7}$

Begin by rewriting the radical expression with a rational exponent.

$$\begin{aligned} \log_b \sqrt{7} &= \log_b 7^{1/2} && \sqrt{x} = x^{1/2} \\ &= \frac{1}{2} \log_b 7 && \text{Power rule} \end{aligned}$$

(d) $$\begin{aligned} \log_2 \sqrt[5]{x^2} &= \log_2 x^{2/5} && \sqrt[5]{x^2} = x^{2/5} \\ &= \frac{2}{5} \log_2 x && \text{Power rule} \end{aligned}$$

Now Try Exercise 11.

Two special properties involving both exponential and logarithmic expressions come directly from the fact that logarithmic and exponential functions are inverses of each other.

Special Properties

If $b > 0$ and $b \neq 1$, then

$$\boldsymbol{b^{\log_b x} = x, \quad x > 0 \quad \text{and} \quad \log_b b^x = x.}$$

To prove the first statement, let

$$\begin{aligned} y &= \log_b x. \\ b^y &= x && \text{Convert to exponential form.} \\ b^{\log_b x} &= x && \text{Replace } y \text{ with } \log_b x. \end{aligned}$$

The proof of the second statement is similar.

EXAMPLE 4 Using the Special Properties

Find the value of each logarithmic expression.

(a) $\log_5 5^4 = 4$, since $\log_b b^x = x$. **(b)** $\log_3 9 = \log_3 3^2 = 2$

(c) $4^{\log_4 10} = 10$

Now Try Exercises 3 and 5.

Here is a summary of the properties of logarithms.

Properties of Logarithms

If x, y, and b are positive real numbers, where $b \neq 1$, and r is any real number, then

Product Rule	$\log_b xy = \log_b x + \log_b y$
Quotient Rule	$\log_b \dfrac{x}{y} = \log_b x - \log_b y$
Power Rule	$\log_b x^r = r \log_b x$
Special Properties	$b^{\log_b x} = x$ and $\log_b b^x = x$.

OBJECTIVE 4 Use properties to write alternative forms of logarithmic expressions. Applying the properties of logarithms is important for solving equations with logarithms and in calculus.

EXAMPLE 5 Writing Logarithms in Alternative Forms

Use the properties of logarithms to rewrite each expression if possible. Assume all variables represent positive real numbers.

(a)
$$\begin{aligned} \log_4 4x^3 &= \log_4 4 + \log_4 x^3 && \text{Product rule} \\ &= 1 + 3\log_4 x && \log_4 4 = 1;\ \text{power rule} \end{aligned}$$

(b)
$$\begin{aligned} \log_7 \sqrt{\frac{m}{n}} &= \log_7\left(\frac{m}{n}\right)^{1/2} \\ &= \frac{1}{2}\log_7 \frac{m}{n} && \text{Power rule} \\ &= \frac{1}{2}(\log_7 m - \log_7 n) && \text{Quotient rule} \end{aligned}$$

(c)
$$\begin{aligned} \log_5 \frac{a^2}{bc} &= \log_5 a^2 - \log_5 bc && \text{Quotient rule} \\ &= 2\log_5 a - \log_5 bc && \text{Power rule} \\ &= 2\log_5 a - (\log_5 b + \log_5 c) && \text{Product rule} \\ &= 2\log_5 a - \log_5 b - \log_5 c \end{aligned}$$

Notice the careful use of parentheses in the third step. Since we are subtracting the logarithm of a product and rewriting it as a sum of two terms, we must place parentheses around the sum.

(d) $4 \log_b m - \log_b n = \log_b m^4 - \log_b n$ Power rule

$= \log_b \dfrac{m^4}{n}$ Quotient rule

(e) $\log_b(x + 1) + \log_b(2x - 1) - \dfrac{2}{3} \log_b x$

$= \log_b(x + 1) + \log_b(2x - 1) - \log_b x^{2/3}$ Power rule

$= \log_b \dfrac{(x + 1)(2x - 1)}{x^{2/3}}$ Product and quotient rules

$= \log_b \dfrac{2x^2 + x - 1}{x^{2/3}}$

(f) $\log_8(2p + 3r)$ cannot be rewritten using the properties of logarithms.

Now Try Exercises 13, 15, 27, and 31.

In the next example, we use numerical values for $\log_2 5$ and $\log_2 3$. While we use the equals sign to give these values, they are actually just approximations since most logarithms of this type are irrational numbers. We use $=$ with the understanding that the values are correct to four decimal places.

EXAMPLE 6 Using the Properties of Logarithms with Numerical Values

Given that $\log_2 5 = 2.3219$ and $\log_2 3 = 1.5850$, evaluate the following.

(a) $\log_2 15 = \log_2(3 \cdot 5)$

$= \log_2 3 + \log_2 5$ Product rule

$= 1.5850 + 2.3219$

$= 3.9069$

(b) $\log_2 .6 = \log_2 \dfrac{3}{5}$ $\quad .6 = \frac{6}{10} = \frac{3}{5}$

$= \log_2 3 - \log_2 5$ Quotient rule

$= 1.5850 - 2.3219$

$= -.7369$

(c) $\log_2 27 = \log_2 3^3$

$= 3 \log_2 3$ Power rule

$= 3(1.5850)$

$= 4.7550$

Now Try Exercises 33, 35, and 43.

YOU TRY IT

EXAMPLE 7 Deciding Whether Statements about Logarithms Are True

Decide whether each statement is *true* or *false*.

(a) $\log_2 8 - \log_2 4 = \log_2 4$

Evaluate both sides.

Left side: $\log_2 8 - \log_2 4 = \log_2 2^3 - \log_2 2^2 = 3 - 2 = 1$

Right side: $\log_2 4 = \log_2 2^2 = 2$

The statement is false because $1 \neq 2$.

(b) $\log_3(\log_2 8) = \dfrac{\log_7 49}{\log_8 64}$

Evaluate both sides.

Left side: $\log_3(\log_2 8) = \log_3 3 = 1$

Right side: $\dfrac{\log_7 49}{\log_8 64} = \dfrac{\log_7 7^2}{\log_8 8^2} = \dfrac{2}{2} = 1$

The statement is true because $1 = 1$.

Now Try Exercises 45 and 51.

Napier's Rods

CONNECTIONS

Long before the days of calculators and computers, the search for making calculations easier was an ongoing process. Machines built by Charles Babbage and Blaise Pascal, a system of "rods" used by John Napier, and slide rules were the forerunners of today's electronic marvels. The invention of logarithms by John Napier in the sixteenth century was a great breakthrough in the search for easier methods of calculation.

Since logarithms are exponents, their properties allowed users of tables of common logarithms to multiply by adding, divide by subtracting, raise to powers by multiplying, and take roots by dividing. Although logarithms are no longer used for computations, they play an important part in higher mathematics.

For Discussion or Writing

1. To multiply 458.3 by 294.6 using logarithms, we add $\log_{10} 458.3$ and $\log_{10} 294.6$, then find 10 to the sum. Perform this multiplication using the $\boxed{\log x}$ key* and the $\boxed{10^x}$ key on your calculator. Check your answer by multiplying directly with your calculator.
2. Try division, raising to a power, and taking a root by this method.

*In this text, the notation log x is used to mean $\log_{10} x$. This is also the meaning of the log key on calculators.

10.4 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 17/Videotape 17

Use the indicated rule of logarithms to complete each equation in Exercises 1–5.

1. $\log_{10}(3 \cdot 4) =$ ______________ (product rule)

2. $\log_{10}\dfrac{3}{4} =$ ______________ (quotient rule)

3. $3^{\log_3 4} =$ ______________ (special property)

4. $\log_{10} 3^4 =$ ______________ (power rule)

5. $\log_3 3^4 =$ ______________ (special property)

6. Evaluate $\log_2(8 + 8)$. Then evaluate $\log_2 8 + \log_2 8$. Are the results the same? How could you change the operation in the first expression to make the two expressions equal?

Use the properties of logarithms to express each logarithm as a sum or difference of logarithms, or as a single number if possible. Assume all variables represent positive real numbers. See Examples 1–5.

7. $\log_7(4 \cdot 5)$

8. $\log_8(9 \cdot 11)$

9. $\log_5 \dfrac{8}{3}$

10. $\log_3 \dfrac{7}{5}$

11. $\log_4 6^2$

12. $\log_5 7^4$

13. $\log_3 \dfrac{\sqrt[3]{4}}{x^2y}$

14. $\log_7 \dfrac{\sqrt[3]{13}}{pq^2}$

15. $\log_3 \sqrt{\dfrac{xy}{5}}$

16. $\log_6 \sqrt{\dfrac{pq}{7}}$

17. $\log_2 \dfrac{\sqrt[3]{x} \cdot \sqrt[5]{y}}{r^2}$

18. $\log_4 \dfrac{\sqrt[4]{z} \cdot \sqrt[5]{w}}{s^2}$

19. A student erroneously wrote $\log_a(x + y) = \log_a x + \log_a y$. When his teacher explained that this was indeed wrong, the student claimed that he had used the distributive property. Write a few sentences explaining why the distributive property does not apply in this case.

20. Write a few sentences explaining how the rules for multiplying and dividing powers of the same base are similar to the rules for finding logarithms of products and quotients.

Use the properties of logarithms to write each expression as a single logarithm. Assume all variables are defined in such a way that the variable expressions are positive, and bases are positive numbers not equal to 1. See Examples 1–5.

21. $\log_b x + \log_b y$

22. $\log_b 2 + \log_b z$

23. $\log_a m - \log_a n$

24. $\log_b x - \log_b y$

25. $(\log_a r - \log_a s) + 3 \log_a t$

26. $(\log_a p - \log_a q) + 2 \log_a r$

27. $3 \log_a 5 - 4 \log_a 3$

28. $3 \log_a 5 + \dfrac{1}{2} \log_a 9$

29. $\log_{10}(x + 3) + \log_{10}(x - 3)$

30. $\log_{10}(y + 4) + \log_{10}(y - 4)$

31. $3 \log_p x + \dfrac{1}{2} \log_p y - \dfrac{3}{2} \log_p z - 3 \log_p a$

32. $\dfrac{1}{3} \log_b x + \dfrac{2}{3} \log_b y - \dfrac{3}{4} \log_b s - \dfrac{2}{3} \log_b t$

To four decimal places, the values of $\log_{10} 2$ *and* $\log_{10} 9$ *are*

$$\log_{10} 2 = .3010 \qquad \log_{10} 9 = .9542.$$

Evaluate each logarithm by applying the appropriate rule or rules from this section. ***DO NOT USE A CALCULATOR.*** *See Example 6.*

33. $\log_{10} 18$ **34.** $\log_{10} \frac{9}{2}$ **35.** $\log_{10} \frac{2}{9}$ **36.** $\log_{10} 4$

37. $\log_{10} 36$ **38.** $\log_{10} 162$ **39.** $\log_{10} 3$ **40.** $\log_{10} \sqrt[5]{2}$

41. $\log_2 10$ **42.** $\log_9 10$ **43.** $\log_{10} 9^5$ **44.** $\log_{10} 2^{19}$

Decide whether each statement is true *or* false. *See Example 7.*

45. $\log_2(8 + 32) = \log_2 8 + \log_2 32$

46. $\log_2(64 - 16) = \log_2 64 - \log_2 16$

47. $\log_3 7 + \log_3 7^{-1} = 0$

48. $\log_9 14 - \log_{14} 9 = 0$

49. $\log_6 60 - \log_6 10 = 1$

50. $\log_3 8 + \log_3 \frac{1}{8} = 0$

51. $\dfrac{\log_{10} 7}{\log_{10} 14} = \dfrac{1}{2}$

52. $\dfrac{\log_{10} 10}{\log_{10} 100} = \dfrac{1}{10}$

53. Refer to the Note following the word statement of the product rule for logarithms in this section. Now, state the quotient rule in words, replacing "logarithm" with "exponent."

54. Explain why the statement for the power rule for logarithms requires that x be a positive real number.

55. Refer to Example 7(a). Change the left side of the equation using the quotient rule so that the statement becomes true, and simplify.

56. What is wrong with the following "proof" that $\log_2 16$ does not exist? Explain.

$$\begin{aligned}\log_2 16 &= \log_2(-4)(-4)\\ &= \log_2(-4) + \log_2(-4)\end{aligned}$$

Since the logarithm of a negative number is not defined, the final step cannot be evaluated, and so $\log_2 16$ does not exist.

RELATING CONCEPTS (EXERCISES 57–62)

For Individual or Group Work

Work Exercises 57–62 in order.

57. Evaluate $\log_3 81$.

58. Write the *meaning* of the expression $\log_3 81$.

59. Evaluate $3^{\log_3 81}$.

60. Write the *meaning* of the expression $\log_2 19$.

61. Evaluate $2^{\log_2 19}$.

62. Keeping in mind that a logarithm is an exponent and using the results from Exercises 57–61, what is the simplest form of the expression $k^{\log_k m}$?

10.5 Common and Natural Logarithms

OBJECTIVES

1. Evaluate common logarithms using a calculator.
2. Use common logarithms in applications.
3. Evaluate natural logarithms using a calculator.
4. Use natural logarithms in applications.
5. Use the change-of-base rule.

As mentioned earlier, logarithms are important in many applications of mathematics to everyday problems, particularly in biology, engineering, economics, and social science. In this section we find numerical approximations for logarithms. Traditionally, base 10 logarithms were used most often because our number system is base 10. Logarithms to base 10 are called **common logarithms,** and $\log_{10} x$ is abbreviated as simply $\log x$, where the base is understood to be 10.

OBJECTIVE 1 Evaluate common logarithms using a calculator. We use calculators to evaluate common logarithms. In the next example we give the results of evaluating some common logarithms using a calculator with a LOG key. (This may be a second function key on some calculators.) For simple scientific calculators, just enter the number, then press the LOG key. For graphing calculators, these steps are reversed. We give all logarithms to four decimal places.

EXAMPLE 1 Evaluating Common Logarithms

Evaluate each logarithm using a calculator.

(a) $\log 327.1 \approx 2.5147$ **(b)** $\log 437{,}000 \approx 5.6405$

(c) $\log .0615 \approx -1.2111$

Now Try Exercises 7, 9, and 11.

Figure 14 shows how a graphing calculator displays the common logarithms in Example 1. The calculator is set to give four decimal places.

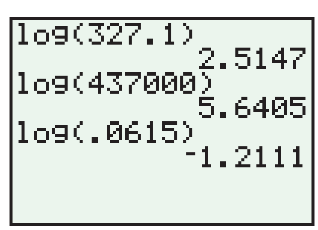


FIGURE 14

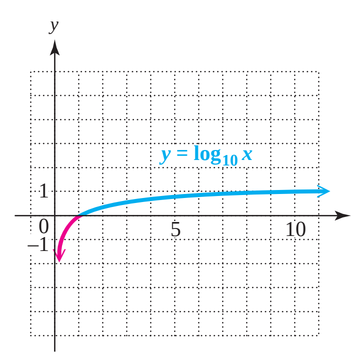


FIGURE 15

Notice that $\log .0615 \approx -1.2111$, a negative result. The common logarithm of a number between 0 and 1 is always negative because the logarithm is the exponent on 10 that produces the number. For example,

$$10^{-1.2111} \approx .0615.$$

If the exponent (the logarithm) were positive, the result would be greater than 1 because $10^0 = 1$. See Figure 15.

OBJECTIVE 2 Use common logarithms in applications. In chemistry, pH is a measure of the acidity or alkalinity of a solution; pure water, for example, has pH 7. In general, acids have pH numbers less than 7, and alkaline solutions have pH values greater than 7. The **pH** of a solution is defined as

$$\text{pH} = -\log[H_3O^+],$$

where $[H_3O^+]$ is the hydronium ion concentration in moles per liter. It is customary to round pH values to the nearest tenth.

EXAMPLE 2 Using pH in an Application

Photo not available

Wetlands are classified as *bogs*, *fens*, *marshes*, and *swamps*. These classifications are based on pH values. A pH value between 6.0 and 7.5, such as that of Summerby Swamp in Michigan's Hiawatha National Forest, indicates that the wetland is a "rich fen." When the pH is between 4.0 and 6.0, the wetland is a "poor fen," and if the pH falls to 3.0 or less, it is a "bog." (*Source:* Mohlenbrock, R., "Summerby Swamp, Michigan," *Natural History*, March 1994.) Suppose that the hydronium ion concentration of a sample of water from a wetland is 6.3×10^{-3}. How would this wetland be classified?

Use the definition of pH.

$$\begin{aligned} \text{pH} &= -\log(6.3 \times 10^{-3}) && \\ &= -(\log 6.3 + \log 10^{-3}) && \text{Product rule} \\ &= -[.7993 - 3(1)] && \text{Use a calculator to find log 6.3.} \\ &= -.7993 + 3 && \\ &\approx 2.2 && \end{aligned}$$

Since the pH is less than 3.0, the wetland is a bog.

Now Try Exercise 29.

EXAMPLE 3 Finding Hydronium Ion Concentration

Find the hydronium ion concentration of drinking water with pH 6.5.

$$\begin{aligned} \text{pH} &= -\log[H_3O^+] && \\ 6.5 &= -\log[H_3O^+] && \text{Let pH} = 6.5. \\ \log[H_3O^+] &= -6.5 && \text{Multiply by } -1. \end{aligned}$$

Solve for $[H_3O^+]$ by writing the equation in exponential form, remembering that the base is 10.

$$\begin{aligned} [H_3O^+] &= 10^{-6.5} && \\ [H_3O^+] &\approx 3.2 \times 10^{-7} && \text{Use a calculator.} \end{aligned}$$

Now Try Exercise 35.

Photo not available

OBJECTIVE 3 Evaluate natural logarithms using a calculator. The most important logarithms used in applications are **natural logarithms,** which have as base the number e. The number e is a fundamental number in our universe. For this reason e, like π, is called a *universal constant*. The letter e is used to honor Leonhard Euler, who published extensive results on the number in 1748. Since it is an irrational number, its decimal expansion never terminates and never repeats. The first few digits of the decimal value of e are 2.7182818285. A calculator key e^x or the two keys INV and ln x are used to approximate powers of e. For example, a calculator gives

$$e^2 \approx 7.389056099, \quad e^3 \approx 20.08553692, \quad \text{and} \quad e^{.6} \approx 1.8221188.$$

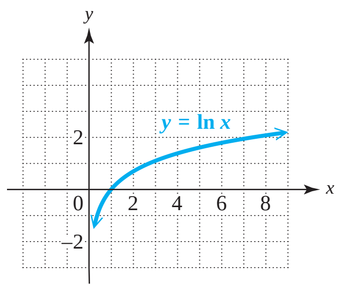


FIGURE 16

Logarithms to base e are called natural logarithms because they occur in biology and the social sciences in natural situations that involve growth or decay. The base e logarithm of x is written ln x (read "el en x"). A graph of $y = \ln x$, the equation that defines the natural logarithmic function, is given in Figure 16.

A calculator key labeled ln x is used to evaluate natural logarithms. If your calculator has an e^x key, but not a key labeled ln x, find natural logarithms by entering the number, pressing the INV key, and then pressing the e^x key. This works because $y = e^x$ defines the inverse function of $y = \ln x$ (or $y = \log_e x$).

EXAMPLE 4 Finding Natural Logarithms

Evaluate each logarithm with a calculator.

(a) $\ln .5841 \approx -.5377$

As with common logarithms, a number between 0 and 1 has a negative natural logarithm.

(b) $\ln 192.7 \approx 5.2611$ **(c)** $\ln 10.84 \approx 2.3832$

Figure 17 shows how a graphing calculator displays these natural logarithms to four decimal places.

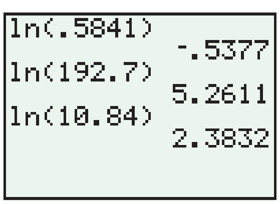


FIGURE 17

Now Try Exercises 15, 17, and 19.

OBJECTIVE 4 Use natural logarithms in applications. Some applications involve functions that use natural logarithms, as seen in the next example.

EXAMPLE 5 Applying a Natural Logarithmic Function

The altitude in meters that corresponds to an atmospheric pressure of x millibars is given by the logarithmic function defined by

$$f(x) = 51{,}600 - 7457 \ln x.$$

(*Source:* Miller, A. and J. Thompson, *Elements of Meteorology*, Fourth Edition, Charles E. Merrill Publishing Company, 1993.) Use this function to find the altitude when atmospheric pressure is 400 millibars.

Let $x = 400$ and substitute in the expression for $f(x)$.

$$f(400) = 51{,}600 - 7457 \ln 400$$
$$\approx 6900$$

Atmospheric pressure is 400 millibars at approximately 6900 m.

Now Try Exercise 39.

NOTE In Example 5, the final answer was obtained using a calculator *without* rounding the intermediate values. In general, it is best to wait until the final step to round the answer; otherwise, a buildup of round-off error may cause the final answer to have an incorrect final decimal place digit.

OBJECTIVE 5 Use the change-of-base rule. We have used a calculator to approximate the values of common logarithms (base 10) and natural logarithms (base e). However, some applications involve logarithms to other bases. For example, for the years 1980–1996, the percentage of women who had a baby in the last year and returned to work is given by

$$f(x) = 38.83 + 4.208 \log_2 x,$$

for year x. (*Source:* U.S. Bureau of the Census.) To use this function, we need to find a base 2 logarithm. The following rule is used to convert logarithms from one base to another.

Change-of-Base Rule

If $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, and $x > 0$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

NOTE Any positive number other than 1 can be used for base b in the change-of-base rule, but usually the only practical bases are e and 10 because calculators give logarithms only for these two bases.

To derive the change-of-base rule, let $\log_a x = m$.

$$\log_a x = m$$

$$a^m = x \qquad \text{Change to exponential form.}$$

Since logarithmic functions are one-to-one, if all variables are positive and if $x = y$, then $\log_b x = \log_b y$.

$$\log_b(a^m) = \log_b x$$

$$m \log_b a = \log_b x \qquad \text{Power rule}$$

$$(\log_a x)(\log_b a) = \log_b x \qquad \text{Substitute for } m.$$

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{Divide by } \log_b a.$$

The last step gives the change-of-base rule.

EXAMPLE 6 Using the Change-of-Base Rule

Find $\log_5 12$.

Use common logarithms and the change-of-base rule.

$$\log_5 12 = \frac{\log 12}{\log 5}$$

$$\approx 1.5440 \qquad \text{Use a calculator.}$$

Verify that the same value is found when using natural logarithms.

Now Try Exercise 47.

EXAMPLE 7 Using the Change-of-Base Rule in an Application

Use natural logarithms in the change-of-base rule and the function

$$f(x) = 38.83 + 4.208 \log_2 x$$

(given earlier) to find the percent of women who returned to work after having a baby in 1995. In the equation, $x = 0$ represents 1980.

Substitute $1995 - 1980 = 15$ for x in the equation.

$$f(15) = 38.83 + 4.208 \log_2 15$$

$$= 38.83 + 4.208\left(\frac{\ln 15}{\ln 2}\right) \qquad \text{Change-of-base rule}$$

$$\approx 55.3\% \qquad \text{Use a calculator.}$$

This is very close to the actual value of 55%.

Now Try Exercise 59.

CONNECTIONS

As previously mentioned, the number $e \approx 2.718281828$ is a fundamental number in our universe. If there are intelligent beings elsewhere, they too will have to use e to do higher mathematics.

The properties of e are used extensively in calculus and in higher mathematics. In Section 10.6 we see how it applies to growth and decay in the physical world.

For Discussion or Writing

The value of e can be expressed as

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots.$$

Approximate e using two terms of this expression, then three terms, four terms, five terms, and six terms. How close is the approximation to the value of e given above with six terms? Does this infinite sum approach the value of e very quickly?

10.5 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 17/Videotape 17

Choose the correct response in Exercises 1–4.

1. What is the base in the expression $\log x$?

A. e **B.** 1 **C.** 10 **D.** x

2. What is the base in the expression $\ln x$?

A. e **B.** 1 **C.** 10 **D.** x

3. Since $10^0 = 1$ and $10^1 = 10$, between what two consecutive integers is the value of $\log 5.6$?

A. 5 and 6 **B.** 10 and 11 **C.** 0 and 1 **D.** -1 and 0

4. Since $e^1 \approx 2.718$ and $e^2 \approx 7.389$, between what two consecutive integers is the value of $\ln 5.6$?

A. 5 and 6 **B.** 2 and 3 **C.** 1 and 2 **D.** 0 and 1

5. Without using a calculator, give the value of $\log 10^{19.2}$.

6. Without using a calculator, give the value of $\ln e^{\sqrt{2}}$.

You will need a calculator for the remaining exercises in this set.

Find each logarithm. Give an approximation to four decimal places. See Examples 1 and 4.

EXERCISES

7. $\log 43$	**8.** $\log 98$	**9.** $\log 328.4$
10. $\log 457.2$	**11.** $\log .0326$	**12.** $\log .1741$
13. $\log(4.76 \times 10^9)$	**14.** $\log(2.13 \times 10^4)$	**15.** $\ln 7.84$
16. $\ln 8.32$	**17.** $\ln .0556$	**18.** $\ln .0217$
19. $\ln 388.1$	**20.** $\ln 942.6$	**21.** $\ln(8.59 \times e^2)$
22. $\ln(7.46 \times e^3)$	**23.** $\ln 10$	**24.** $\log e$

25. Use your calculator to find approximations of the following logarithms:

(a) $\log 356.8$ **(b)** $\log 35.68$ **(c)** $\log 3.568$.

(d) Observe your answers and make a conjecture concerning the decimal values of the common logarithms of numbers greater than 1 that have the same digits.

26. Let k represent the number of letters in your last name.

(a) Use your calculator to find $\log k$.

(b) Raise 10 to the power indicated by the number you found in part (a). What is your result?

(c) Use the concepts of Section 10.1 to explain why you obtained the answer you found in part (b). Would it matter what number you used for k to observe the same result?

27. Try to find $\log(-1)$ using a calculator. (If you have a graphing calculator, it should be in real number mode.) What happens? Explain.

Refer to Example 2. In Exercises 28–30, suppose that water from a wetland area is sampled and found to have the given hydronium ion concentration. Determine whether the wetland is a rich fen, *a* poor fen, *or a* bog.

28. 2.5×10^{-5} **29.** 2.5×10^{-2} **30.** 2.5×10^{-7}

Find the pH *of the substance with the given hydronium ion concentration. See Example 2.*

31. Ammonia, 2.5×10^{-12}
32. Sodium bicarbonate, 4.0×10^{-9}
33. Grapes, 5.0×10^{-5}
34. Tuna, 1.3×10^{-6}

Use the formula for pH to find the hydronium ion concentration of the substance with the given pH. *See Example 3.*

35. Human blood plasma, 7.4
36. Human gastric contents, 2.0
37. Spinach, 5.4
38. Bananas, 4.6

Solve each problem. See Example 5.

39. The number of years, $N(r)$, since two independently evolving languages split off from a common ancestral language is approximated by

$$N(r) = -5000 \ln r,$$

where r is the percent of words (in decimal form) from the ancestral language common to both languages now. Find the number of years since the split for each percent of common words.

(a) 85% (or .85) **(b)** 35% (or .35) **(c)** 10% (or .10)

40. The time t in years for an amount increasing at a rate of r (in decimal form) to double is given by

$$t(r) = \frac{\ln 2}{\ln(1 + r)}.$$

This is called *doubling time.* Find the doubling time to the nearest tenth for an investment at each interest rate.

(a) 2% (or .02) **(b)** 5% (or .05) **(c)** 8% (or .08)

41. The loudness of sounds is measured in a unit called a *decibel*, abbreviated dB. A very faint sound, called the *threshold sound*, is assigned an intensity I_0. If a particular sound has intensity I, then the decibel level of this louder sound is

$$D = 10 \log\left(\frac{I}{I_0}\right).$$

Photo not available

Find the average decibel level for each popular movie with the given intensity I. For comparison, a motorcycle or power saw has a decibel level of about 95 dB, and the sound of a jackhammer or helicopter is about 105 dB. (*Source: World Almanac and Book of Facts*, 2001; www.lhh.org/noise/)

(a) *Armageddon;* $5.012 \times 10^{10} I_0$
(b) *Godzilla;* $10^{10} I_0$
(c) *Saving Private Ryan;* $6{,}310{,}000{,}000\ I_0$

42. The concentration of a drug injected into the bloodstream decreases with time. The intervals of time T when the drug should be administered are given by

$$T = \frac{1}{k} \ln \frac{C_2}{C_1},$$

where k is a constant determined by the drug in use, C_2 is the concentration at which the drug is harmful, and C_1 is the concentration below which the drug is ineffective. (*Source:* Horelick, Brindell and Sinan Koont, "Applications of Calculus to Medicine: Prescribing Safe and Effective Dosage," *UMAP Module 202*, 1977.) Thus, if $T = 4$, the drug should be administered every 4 hr. For a certain drug, $k = \frac{1}{3}$, $C_2 = 5$, and $C_1 = 2$. How often should the drug be administered? (*Hint:* Round down.)

43. The growth of outpatient surgeries as a percent of total surgeries at hospitals is approximated by

$$f(x) = -1317 + 304 \ln x,$$

where x represents the number of years since 1900. (*Source:* American Hospital Association.)

(a) What does this function predict for the percent of outpatient surgeries in 1998?
(b) When did outpatient surgeries reach 50%? (*Hint:* Substitute for y, then write the equation in exponential form to solve it.)

44. In the central Sierra Nevada of California, the percent of moisture p that falls as snow rather than rain is approximated reasonably well by

$$f(x) = 86.3 \ln x - 680,$$

where x is the altitude in feet.

(a) What percent of the moisture at 5000 ft falls as snow?
(b) What percent at 7500 ft falls as snow?

Photo not available

45. The *cost-benefit equation*

$$T = -.642 - 189 \ln(1 - p)$$

describes the approximate tax T, in dollars per ton, that would result in a p% (in decimal form) reduction in carbon dioxide emissions.

(a) What tax will reduce emissions 25%?
(b) Explain why the equation is not valid for $p = 0$ or $p = 1$.

46. The age in years of a female blue whale is approximated by

$$t = -2.57 \ln\left(\frac{87 - L}{63}\right),$$

where L is its length in feet.

(a) How old is a female blue whale that measures 80 ft?
(b) The equation that defines t has domain $24 < L < 87$. Explain why.

Use the change-of-base rule (with either common or natural logarithms) to find each logarithm to four decimal places. See Example 6.

47. $\log_3 12$	**48.** $\log_4 18$	**49.** $\log_5 3$
50. $\log_7 4$	**51.** $\log_3 \sqrt{2}$	**52.** $\log_6 \sqrt[3]{5}$
53. $\log_\pi e$	**54.** $\log_\pi 10$	**55.** $\log_e 12$

56. Explain why the answer to Exercise 55 is the same one that you get when you use a calculator to approximate ln 12.

57. Let m be the number of letters in your first name, and let n be the number of letters in your last name.

(a) In your own words, explain what $\log_m n$ means.
(b) Use your calculator to find $\log_m n$.
(c) Raise m to the power indicated by the number you found in part (b). What is your result?

58. The equation $5^x = 7$ cannot be solved using the methods described in Section 10.2. However, in solving this equation, we must find the exponent to which 5 must be raised in order to obtain 7: this is $\log_5 7$.

(a) Use the change-of-base rule and your calculator to find $\log_5 7$.
(b) Raise 5 to the number you found in part (a). What is your result?
(c) Using as many decimal places as your calculator gives, write the solution set of $5^x = 7$. (Equations of this type will be studied in more detail in Section 10.6.)

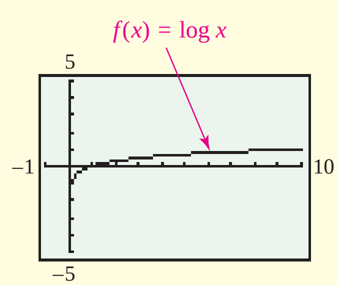
Solve each application of a logarithmic function. See Example 7.

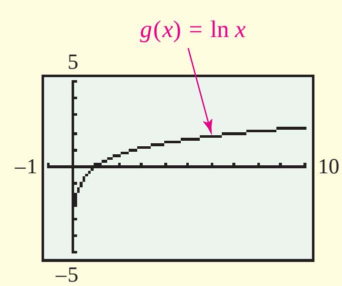
59. Refer to Exercise 53 in Section 10.3. Determine the number of waste sites in 1998.

60. Refer to Exercise 54 in Section 10.3. Determine the approximate consumption in 1998.

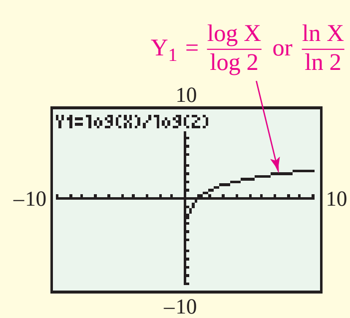
TECHNOLOGY INSIGHTS (EXERCISES 61–64)

Because graphing calculators are equipped with log x *and* ln x *keys, it is possible to graph the functions defined by* $f(x) = \log x$ *and* $g(x) = \ln x$ *directly, as shown in the figures that follow.*





To graph functions defined by logarithms to bases other than 10 or e, however, we must use the change-of-base rule. For example, to graph $y = \log_2 x$, *we may enter* Y_1 *as* $\dfrac{\log X}{\log 2}$ *or* $\dfrac{\ln X}{\ln 2}$. *This is shown in the figure at the right. (Compare it to the figure in Exercises 59–62 of Section 10.3, where it was drawn using the fact that* $y = \log_2 x$ *is the inverse of* $y = 2^x$.)



Use the change-of-base rule to graph each logarithmic function with a graphing calculator. Use a viewing window with Xmin $= -1$, Xmax $= 10$, Ymin $= -5$, *and* Ymax $= 5$.

61. $g(x) = \log_3 x$ **62.** $g(x) = \log_5 x$ **63.** $g(x) = \log_{1/3} x$ **64.** $g(x) = \log_{1/5} x$

10.6 Exponential and Logarithmic Equations; Further Applications

OBJECTIVES

1. Solve equations involving variables in the exponents.
2. Solve equations involving logarithms.
3. Solve applications of compound interest.
4. Solve applications involving exponential growth and decay.
5. Use a graphing calculator to solve exponential and logarithmic equations.

As mentioned earlier, exponential and logarithmic functions are important in many applications of mathematics. Using these functions in applications requires solving exponential and logarithmic equations. Some simple equations were solved in Sections 10.2 and 10.3. More general methods for solving these equations depend on the following properties.

Properties for Solving Exponential and Logarithmic Equations

For all real numbers $b > 0$, $b \neq 1$, and any real numbers x and y:

1. If $x = y$, then $b^x = b^y$.

2. If $b^x = b^y$, then $x = y$.

3. If $x = y$, and $x > 0$, $y > 0$, then $\log_b x = \log_b y$.

4. If $x > 0$, $y > 0$, and $\log_b x = \log_b y$, then $x = y$.

We used Property 2 to solve exponential equations in Section 10.2.

OBJECTIVE 1 Solve equations involving variables in the exponents. The first two examples illustrate the method for solving exponential equations using Property 3.

EXAMPLE 1 Solving an Exponential Equation

Solve $3^x = 12$.

$$\begin{aligned} 3^x &= 12 \\ \log 3^x &= \log 12 && \text{Property 3} \\ x \log 3 &= \log 12 && \text{Power rule} \\ x &= \frac{\log 12}{\log 3} && \text{Divide by log 3.} \end{aligned}$$

This quotient is the exact solution. To get a decimal approximation for the solution, use a calculator.

$$x \approx 2.262$$

The solution set is $\{2.262\}$. Check that $3^{2.262} \approx 12$.

Now Try Exercise 5.

CAUTION Be careful: $\frac{\log 12}{\log 3}$ is *not* equal to log 4 because $\log 4 \approx .6021$, but $\frac{\log 12}{\log 3} \approx 2.262$.

When an exponential equation has e as the base, it is easiest to use base e logarithms.

EXAMPLE 2 Solving an Exponential Equation with Base e

Solve $e^{.003x} = 40$.

Take base e logarithms on both sides.

$$\begin{aligned} \ln e^{.003x} &= \ln 40 & & \\ .003x \ln e &= \ln 40 & & \text{Power rule} \\ .003x &= \ln 40 & & \ln e = \ln e^1 = 1 \\ x &= \frac{\ln 40}{.003} & & \text{Divide by .003.} \\ x &\approx 1230 & & \text{Use a calculator.} \end{aligned}$$

The solution set is $\{1230\}$. Check that $e^{.003(1230)} \approx 40$.

Now Try Exercise 15.

General Method for Solving an Exponential Equation

Take logarithms to the same base on both sides and then use the power rule of logarithms or the special property $\log_b b^x = x$. (See Examples 1 and 2.)

As a special case, if both sides can be written as exponentials with the same base, do so, and set the exponents equal. (See Section 10.2.)

OBJECTIVE 2 Solve equations involving logarithms. The properties of logarithms from Section 10.4 are useful here, as is using the definition of a logarithm to change the equation to exponential form.

EXAMPLE 3 Solving a Logarithmic Equation

Solve $\log_2(x + 5)^3 = 4$. Give the exact solution.

$$\begin{aligned} (x+5)^3 &= 2^4 & & \text{Convert to exponential form.} \\ (x+5)^3 &= 16 & & \\ x + 5 &= \sqrt[3]{16} & & \text{Take the cube root on each side.} \\ x &= -5 + \sqrt[3]{16} & & \text{Subtract 5.} \\ x &= -5 + 2\sqrt[3]{2} & & \text{Simplify the radical.} \end{aligned}$$

Verify that the solution satisfies the equation, so the solution set is $\{-5 + 2\sqrt[3]{2}\}$.

Now Try Exercise 29.

CAUTION Recall that the domain of $y = \log_b x$ is $(0, \infty)$. For this reason, it is always necessary to check that the solution of an equation with logarithms yields only logarithms of positive numbers in the original equation.

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\log_2(x + 1) - \log_2 x = \log_2 7$.

$$\begin{aligned}
\log_2(x + 1) - \log_2 x &= \log_2 7 \\
\log_2 \frac{x + 1}{x} &= \log_2 7 && \text{Quotient rule} \\
\frac{x + 1}{x} &= 7 && \text{Property 4} \\
x + 1 &= 7x && \text{Multiply by } x. \\
1 &= 6x \\
\frac{1}{6} &= x
\end{aligned}$$

Check this solution by substituting in the original equation. Here, both $x + 1$ and x must be positive. If $x = \frac{1}{6}$, this condition is satisfied, so the solution set is $\left\{\frac{1}{6}\right\}$.

Now Try Exercise 35.

EXAMPLE 5 Solving a Logarithmic Equation

Solve $\log x + \log(x - 21) = 2$.

Write the left side as a single logarithm, write in exponential form, and solve the equation.

$$\begin{aligned}
\log x + \log(x - 21) &= 2 \\
\log x(x - 21) &= 2 && \text{Product rule} \\
x(x - 21) &= 10^2 && \log x = \log_{10} x\text{; write in exponential form.} \\
x^2 - 21x &= 100 \\
x^2 - 21x - 100 &= 0 && \text{Standard form} \\
(x - 25)(x + 4) &= 0 && \text{Factor.} \\
x - 25 = 0 \quad \text{or} \quad x + 4 &= 0 && \text{Zero-factor property} \\
x = 25 \quad \text{or} \quad x &= -4
\end{aligned}$$

The value -4 must be rejected as a solution since it leads to the logarithm of a negative number in the original equation:

$$\log(-4) + \log(-4 - 21) = 2. \qquad \text{The left side is undefined.}$$

The only solution, therefore, is 25, and the solution set is $\{25\}$.

Now Try Exercise 39.

CAUTION Do not reject a potential solution just because it is nonpositive. Reject any value that *leads to* the logarithm of a nonpositive number.

In summary, we use the following steps to solve a logarithmic equation.

Solving a Logarithmic Equation

Step 1 **Transform the equation so that a single logarithm appears on one side.** Use the product rule or quotient rule of logarithms to do this.

Step 2 **(a) Use Property 4.** If $\log_b x = \log_b y$, then $x = y$. (See Example 4.)

(b) Write the equation in exponential form. If $\log_b x = k$, then $x = b^k$. (See Examples 3 and 5.)

OBJECTIVE 3 Solve applications of compound interest. So far in this book, problems involving applications of interest have been limited to simple interest using the formula $I = prt$. In most cases, interest paid or charged is *compound interest* (interest paid on both principal and interest). The formula for compound interest is an important application of exponential functions.

Compound Interest Formula (for a Finite Number of Periods)

If a principal of P dollars is deposited at an annual rate of interest r compounded (paid) n times per year, the account will contain

$$\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{nt}}$$

dollars after t years. (In this formula, r is expressed as a decimal.)

EXAMPLE 6 Solving a Compound Interest Problem for *A*

How much money will there be in an account at the end of 5 yr if \$1000 is deposited at 6% compounded quarterly? (Assume no withdrawals are made.)

Because interest is compounded quarterly, $n = 4$. The other values given in the problem are $P = 1000$, $r = .06$ (because $6\% = .06$), and $t = 5$. Substitute into the compound interest formula to find the value of A.

$$A = 1000\left(1 + \frac{.06}{4}\right)^{4 \cdot 5}$$

$$A = 1000(1.015)^{20}$$

Now use the y^x key on a calculator and round the answer to the nearest cent.

$$A = 1346.86$$

The account will contain \$1346.86. (The actual amount of interest earned is \$1346.86 − \$1000 = \$346.86. Why?)

Photo not available

Now Try Exercise 45(a).

Interest can be compounded annually, semiannually, quarterly, daily, and so on. The number of compounding periods can get larger and larger. If the value of n is allowed to approach infinity, we have an example of *continuous compounding.* However, the compound interest formula above cannot be used for continuous compounding since there is no finite value for n. The formula for continuous compounding is an example of exponential growth involving the number e.

Continuous Compound Interest Formula

If a principal of P dollars is deposited at an annual rate of interest r compounded continuously for t years, the final amount on deposit is

$$A = Pe^{rt}.$$

EXAMPLE 7 Solving a Continuous Compound Interest Problem

In Example 6 we found that \$1000 invested for 5 yr at 6% interest compounded quarterly would grow to \$1346.86.

(a) How much would this same investment grow to if interest were compounded continuously?

Use the formula for continuous compounding with $P = 1000$, $r = .06$, and $t = 5$.

$$\begin{aligned} A &= Pe^{rt} && \text{Formula} \\ &= 1000e^{.06(5)} && \text{Substitute.} \\ &= 1000e^{.30} \\ &= 1349.86 && \text{Use a calculator and round to the nearest cent.} \end{aligned}$$

Continuous compounding would cause the investment to grow to \$1349.86. Notice that this is \$3.00 more than the amount in Example 6, when interest was compounded quarterly.

(b) How long would it take for the initial investment to double its original amount? (This is called the *doubling time.*)

We must find the value of t that will cause A to be $2(\$1000) = \$2000$.

$$\begin{aligned} A &= Pe^{rt} \\ 2000 &= 1000e^{.06t} && \text{Let } A = 2P = 2000. \\ 2 &= e^{.06t} && \text{Divide by 1000.} \\ \ln 2 &= .06t && \text{Take natural logarithms; } \ln e^k = k. \\ t &= \frac{\ln 2}{.06} && \text{Divide by .06.} \\ t &\approx 11.55 && \text{Use a calculator.} \end{aligned}$$

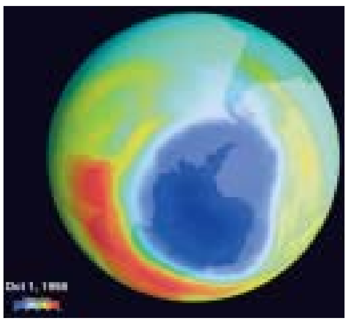
It would take about 11.55 yr for the original investment to double.

Now Try Exercise 47.

OBJECTIVE 4 Solve applications involving exponential growth and decay. One of the most common applications of exponential functions depends on the fact that in many situations involving growth or decay of a population, the amount or number of some quantity present at time t can be closely approximated by

$$y = y_0 e^{kt},$$

where y_0 is the amount or number present at time $t = 0$, k is a constant, and e is the base of natural logarithms.



EXAMPLE 8 Applying an Exponential Function

The *greenhouse effect* refers to the phenomenon whereby emissions of gases such as carbon dioxide, methane, and chlorofluorocarbons (CFCs) have the potential to alter the climate of the earth and destroy the ozone layer. Concentrations of CFC-12, used in refrigeration technology, in parts per billion (ppb) can be modeled by the exponential function defined by

$$f(x) = .48e^{.04x},$$

where $x = 0$ represents 1990. Use this function to approximate the concentration in 1998.

Since $x = 0$ represents 1990, $x = 8$ represents 1998. Evaluate $f(8)$ using a calculator.

$$f(8) = .48e^{.04(8)} = .48e^{.32} \approx .66$$

In 1998, the concentration of CFC-12 was about .66 ppb.

Now Try Exercise 53.

You have probably heard of the carbon 14 dating process used to determine the age of fossils. The method used involves a base e exponential decay function.

EXAMPLE 9 Solving an Exponential Decay Problem

Carbon 14 is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the radioactive carbon 14 disintegrates according to the function defined by

$$y = y_0e^{-.000121t},$$

where t is time in years, y is the amount of the sample at time t, and y_0 is the initial amount present at $t = 0$.

(a) If an initial sample contains $y_0 = 10$ g of carbon 14, how many grams will be present after 3000 yr?

Let $y_0 = 10$ and $t = 3000$ in the formula, and use a calculator.

$$y = 10e^{-.000121(3000)} \approx 6.96 \text{ g}$$

(b) How long would it take for the initial sample to decay to half of its original amount? (This is called the *half-life*.)

Let $y = \frac{1}{2}(10) = 5$, and solve for t.

$$5 = 10e^{-.000121t} \qquad \text{Substitute.}$$

$$\frac{1}{2} = e^{-.000121t} \qquad \text{Divide by 10.}$$

$$\ln\frac{1}{2} = -.000121t \qquad \text{Take natural logarithms; } \ln e^k = k.$$

$$t = \frac{\ln\frac{1}{2}}{-.000121} \qquad \text{Divide by } -.000121.$$

$$t \approx 5728 \qquad \text{Use a calculator.}$$

The half-life is just over 5700 yr.

Now Try Exercise 59.

OBJECTIVE 5 Use a graphing calculator to solve exponential and logarithmic equations. Earlier we saw that the x-intercepts of the graph of a function f correspond to the real solutions of the equation $f(x) = 0$. This idea was applied to linear and quadratic equations and can be extended to exponential and logarithmic equations as well. In Example 1, we solved the equation $3^x = 12$ algebraically using rules for logarithms and found the solution set to be $\{2.262\}$. This can be supported graphically by showing that the x-intercept of the graph of the function defined by $y = 3^x - 12$ corresponds to this solution. See Figure 18.

In Example 5, we solved $\log x + \log(x - 21) = 2$ and found the solution set to be $\{25\}$. (We rejected the apparent solution -4 since it led to the logarithm of a negative number.) Figure 19 shows that the x-intercept of the graph of the function defined by $y = \log x + \log(x - 21) - 2$ supports this result.

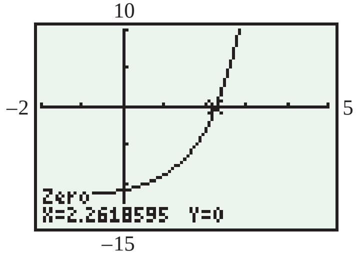


FIGURE 18

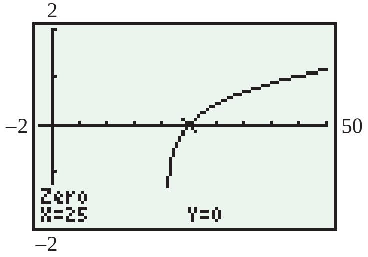


FIGURE 19

10.6 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 17/Videotape 17

RELATING CONCEPTS (EXERCISES 1–4)

For Individual or Group Work

In Section 10.2 we solved an equation such as $5^x = 125$ *by writing each side as a power of the same base, setting exponents equal, and then solving the resulting equation as follows.*

$5^x = 125$	Original equation
$5^x = 5^3$	$125 = 5^3$
$x = 3$	Set exponents equal.

Solution set: $\{3\}$

The method described in this section can also be used to solve this equation. ***Work Exercises 1–4 in order,*** *to see how this is done.*

1. Take common logarithms on both sides, and write this equation.

2. Apply the power rule for logarithms on the left.

3. Write the equation so that x is alone on the left.

4. Use a calculator to find the decimal form of the solution. What is the solution set?

Many of the problems in the remaining exercises require a scientific calculator.

Solve each equation. Give solutions to three decimal places. See Example 1.

EXERCISES

5. $7^x = 5$ **6.** $4^x = 3$ **7.** $9^{-x+2} = 13$

8. $6^{-t+1} = 22$
9. $3^{2x} = 14$
10. $5^{.3x} = 11$
11. $2^{x+3} = 5^x$
12. $6^{m+3} = 4^m$
13. $2^{x+3} = 3^{x-4}$

Solve each equation. Use natural logarithms. When appropriate, give solutions to three decimal places. See Example 2.

14. $e^{.006x} = 30$
15. $e^{.012x} = 23$
16. $e^{-.103x} = 7$
17. $e^{-.205x} = 9$
18. $\ln e^x = 4$
19. $\ln e^{3x} = 9$
20. $\ln e^{.04x} = \sqrt{3}$
21. $\ln e^{.45x} = \sqrt{7}$
22. $\ln e^{2x} = \pi$

23. Try solving one of the equations in Exercises 14–17 using common logarithms rather than natural logarithms. (You should get the same solution.) Explain why using natural logarithms is a better choice.

24. If you were asked to solve $10^{.0025x} = 75$, would natural or common logarithms be a better choice? Explain.

Solve each equation. Give the exact solution. See Example 3.

25. $\log_3(6x + 5) = 2$
26. $\log_5(12x - 8) = 3$
27. $\log_2(2x - 1) = 5$
28. $\log_6(4x + 2) = 2$
29. $\log_7(x + 1)^3 = 2$
30. $\log_4(x - 3)^3 = 4$

31. Suppose that in solving a logarithmic equation having the term $\log(x - 3)$ you obtain an apparent solution of 2. All algebraic work is correct. Explain why you must reject 2 as a solution of the equation.

32. Suppose that in solving a logarithmic equation having the term $\log(3 - x)$ you obtain an apparent solution of -4. All algebraic work is correct. Should you reject -4 as a solution of the equation? Explain why or why not.

Solve each equation. Give exact solutions. See Examples 4 and 5.

33. $\log(6x + 1) = \log 3$
34. $\log(7 - x) = \log 12$
35. $\log_5(3t + 2) - \log_5 t = \log_5 4$
36. $\log_2(x + 5) - \log_2(x - 1) = \log_2 3$
37. $\log 4x - \log(x - 3) = \log 2$
38. $\log(-x) + \log 3 = \log(2x - 15)$
39. $\log_2 x + \log_2(x - 7) = 3$
40. $\log(2x - 1) + \log 10x = \log 10$
41. $\log 5x - \log(2x - 1) = \log 4$
42. $\log_3 x + \log_3(2x + 5) = 1$
43. $\log_2 x + \log_2(x - 6) = 4$
44. $\log_2 x + \log_2(x + 4) = 5$

Solve each problem. See Examples 6 and 7.

45. **(a)** How much money will there be in an account at the end of 6 yr if \$2000 is deposited at 4% compounded quarterly? (Assume no withdrawals are made.)
(b) To one decimal place, how long will it take for the account to grow to \$3000?

46. **(a)** How much money will there be in an account at the end of 7 yr if \$3000 is deposited at 3.5% compounded quarterly? (Assume no withdrawals are made.)
(b) To one decimal place, when will the account grow to \$5000?

47. **(a)** What will be the amount A in an account with initial principal \$4000 if interest is compounded continuously at an annual rate of 3.5% for 6 yr?
(b) How long will it take for the initial amount to double?

48. Refer to Exercise 46. Does the money grow to a larger value under those conditions, or when invested for 7 yr at 3% compounded continuously?

49. Find the amount of money in an account after 12 yr if \$5000 is deposited at 7% annual interest compounded as follows.

(a) Annually **(b)** Semiannually **(c)** Quarterly
(d) Daily (Use $n = 365$.) **(e)** Continuously

50. How much money will be in an account at the end of 8 yr if \$4500 is deposited at 6% annual interest compounded as follows?

(a) Annually **(b)** Semiannually **(c)** Quarterly
(d) Daily (Use $n = 365$.) **(e)** Continuously

51. How much money must be deposited today to amount to \$1850 in 40 yr at 6.5% compounded continuously?

52. How much money must be deposited today to amount to \$1000 in 10 yr at 5% compounded continuously?

Solve each problem. See Examples 8 and 9.

53. The total expenditures in millions of current dollars for pollution abatement and control during the period from 1985 through 1993 can be approximated by the function defined by

$$P(x) = 70{,}967e^{.0526x},$$

where $x = 0$ corresponds to 1985, $x = 1$ to 1986, and so on. Approximate the expenditures for each year. (*Source:* U.S. Bureau of Economic Analysis, *Survey of Current Business, May 1995*.)

(a) 1987 **(b)** 1990 **(c)** 1993
(d) What were the approximate expenditures for 1985?

54. The emission of the greenhouse gas nitrous oxide increased yearly during the first half of the 1990s. Based on figures during the period from 1990 through 1994, the emissions in thousands of metric tons can be modeled by the function defined by

$$N(x) = 446.5e^{.0118x},$$

where $x = 0$ corresponds to 1990, $x = 1$ to 1991, and so on. Approximate the emissions for each year. (*Source:* U.S. Energy Information Administration, *Emission of Greenhouse Gases in the United States, annual*.)

(a) 1991 **(b)** 1992 **(c)** 1994
(d) What were the approximate emissions in 1990?

55. Based on selected figures obtained during the 1980s and 1990s, consumer expenditures on all types of books in the United States can be modeled by the function defined by

$$B(x) = 8768e^{.072x},$$

where $x = 0$ represents 1980, $x = 1$ represents 1981, and so on, and $B(x)$ is in millions of dollars. Approximate consumer expenditures for 1998. (*Source:* Book Industry Study Group.)

56. Based on selected figures obtained during the 1970s, 1980s, and 1990s, the total number of bachelor's degrees earned in the United States can be modeled by the function defined by

$$D(x) = 815{,}427e^{.0137x},$$

where $x = 1$ corresponds to 1971, $x = 10$ corresponds to 1980, and so on. Approximate the number of bachelor's degrees earned in 1994. (*Source:* U.S. National Center for Education Statistics.)

57. Suppose that the amount, in grams, of plutonium 241 present in a given sample is determined by the function defined by

$$A(t) = 2.00e^{-.053t},$$

where t is measured in years. Find the amount present in the sample after the given number of years.

(a) 4 **(b)** 10 **(c)** 20

(d) What was the initial amount present?

58. Suppose that the amount, in grams, of radium 226 present in a given sample is determined by the function defined by

$$A(t) = 3.25e^{-.00043t},$$

where t is measured in years. Find the amount present in the sample after the given number of years.

(a) 20 **(b)** 100 **(c)** 500

(d) What was the initial amount present?

59. A sample of 400 g of lead 210 decays to polonium 210 according to the function defined by

$$A(t) = 400e^{-.032t},$$

where t is time in years.

(a) How much lead will be left in the sample after 25 yr?

(b) How long will it take the initial sample to decay to half of its original amount?

60. The concentration of a drug in a person's system decreases according to the function defined by

$$C(t) = 2e^{-.125t},$$

where $C(t)$ is in appropriate units, and t is in hours.

(a) How much of the drug will be in the system after 1 hr?

(b) Find the time that it will take for the concentration to be half of its original amount.

61. Refer to Exercise 53. Assuming that the function continued to apply past 1993, in what year could we have expected total expenditures to have been 133,500 million dollars? (*Source:* U.S. Bureau of Economic Analysis, *Survey of Current Business, May 1995.)*

62. Refer to Exercise 54. Assuming that the function continued to apply past 1994, in what year could we have expected nitrous oxide emissions to have been 485 thousand metric tons? (*Source:* U.S. Energy Information Administration, *Emission of Greenhouse Gases in the United States, annual.*)

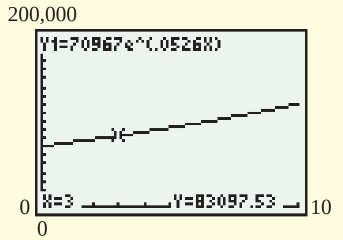
63. The number of ants in an anthill grows according to the function defined by

$$f(t) = 300e^{.4t},$$

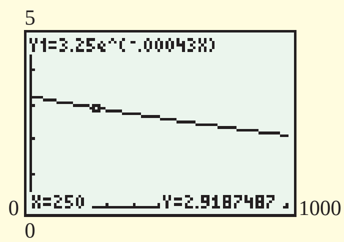
where t is time measured in days. Find the time it will take for the number of ants to double.

TECHNOLOGY INSIGHTS (EXERCISES 64–67)

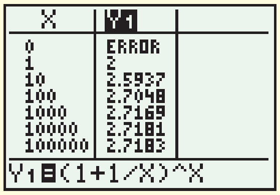
64. The function defined by $P(x) = 70{,}967e^{.0526x}$, described in Exercise 53, is graphed in the screen at the right. Interpret the meanings of X and Y in the display at the bottom of the screen in the context of Exercise 53.



65. The function defined by $A(x) = 3.25e^{-.00043x}$, with $x = t$, described in Exercise 58, is graphed in the following figure. Interpret the meanings of X and Y in the display at the bottom of the screen in the context of Exercise 58.



66. The screen shows a table of selected values for the function defined by $Y_1 = \left(1 + \frac{1}{X}\right)^X$.



(a) Why is there an error message for X = 0?

(b) What number does the function value seem to approach as X takes on larger and larger values?

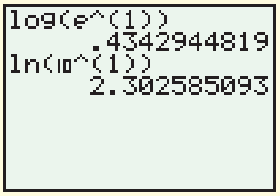
(c) Use a calculator to evaluate this function for X = 1,000,000. What value do you get? Now evaluate $e = e^1$. How close are these two values?

(d) Make a conjecture: As the values of x approach infinity, the value of $\left(1 + \frac{1}{x}\right)^x$ approaches _____.

67. Here is another property of logarithms: For $b > 0$, $x > 0$, $b \neq 1$, $x \neq 1$,

$$\log_b x = \frac{1}{\log_x b}.$$

Now observe the following calculator screen.



(a) Without using a calculator, give a decimal representation for $\frac{1}{.4342944819}$. Then support your answer using the reciprocal key of your calculator.

(b) Without using a calculator, give a decimal representation for $\frac{1}{2.302585093}$. Then support your answer using the reciprocal key of your calculator.

11.1 Additional Graphs of Functions; Composition

OBJECTIVES

1. Recognize the graphs of the elementary functions defined by $|x|$, $\frac{1}{x}$, and $\sqrt{x}$, and graph their translations.
2. Recognize and graph step functions.
3. Find the composition of functions.

In earlier chapters we introduced the function defined by $f(x) = x^2$, sometimes called the **squaring function.** This is one of the most important elementary functions in algebra.

OBJECTIVE 1 Recognize the graphs of the elementary functions defined by $|x|$, $\frac{1}{x}$, and $\sqrt{x}$, and graph their translations. Another one of the elementary functions, defined by $f(x) = |x|$, is called the **absolute value function.** Its graph, along with a table of selected ordered pairs, is shown in Figure 1. Its domain is $(-\infty, \infty)$, and its range is $[0, \infty)$.

x	y
0	0
± 1	1
± 2	2
± 3	3

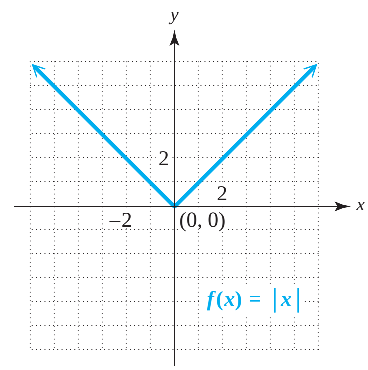


FIGURE 1

The **reciprocal function,** defined by $f(x) = \frac{1}{x}$, is a rational function. Rational functions were introduced in Chapter 7. The graph of this function is shown in Figure 2, along with a table of selected ordered pairs. Notice that x can never equal 0 for this function, and as a result, as x gets closer and closer to 0, $\frac{1}{x}$ approaches either ∞ or $-\infty$. Also, $\frac{1}{x}$ can never equal 0, and as x approaches ∞ or $-\infty$, $\frac{1}{x}$ approaches 0. The axes are called **asymptotes** for the function. (Asymptotes are studied in more detail in college algebra courses.) For the reciprocal function, the domain and the range are both $(-\infty, 0) \cup (0, \infty)$.

x	y
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$

x	y
$-\frac{1}{3}$	-3
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-3	$-\frac{1}{3}$

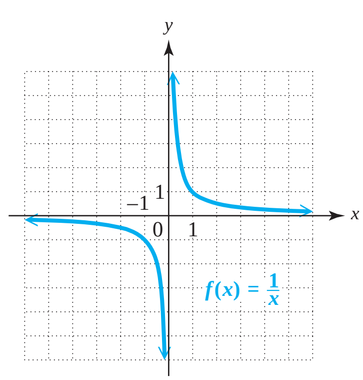


FIGURE 2

x	y
0	0
1	1
4	2

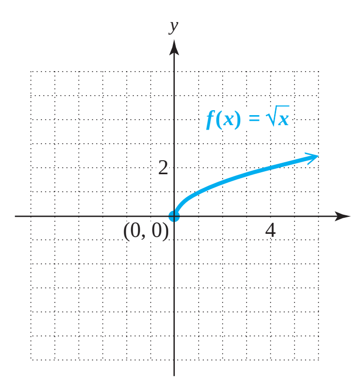


FIGURE 3

The **square root function,** defined by $f(x) = \sqrt{x}$, was introduced in Chapter 8. Its graph is shown in Figure 3. Notice that since we restrict function values to be real numbers, x cannot take on negative values. Thus, the domain of the square root function is $[0, \infty)$. Because the principal square root is always nonnegative, the range is also $[0, \infty)$. A table of values is shown along with the graph.

Just as the graph of $f(x) = x^2$ can be shifted, or translated, as we saw in Section 9.5, so can the graphs of these other elementary functions.

EXAMPLE 1 Applying a Horizontal Shift

Graph $f(x) = |x - 2|$.

The graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ two units to the right. In a similar manner, the graph of $f(x) = |x - 2|$ is found by shifting the graph of $y = |x|$ two units to the right, as shown in Figure 4. The table of ordered pairs accompanying the graph supports this, as you can see by comparing it to the table with Figure 1. The domain of this function is $(-\infty, \infty)$, and its range is $[0, \infty)$.

x	y
0	2
1	1
2	0
3	1
4	2

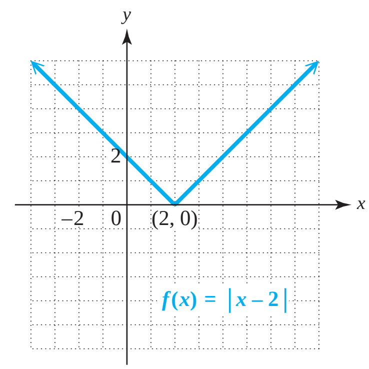


FIGURE 4

Now Try Exercise 9.

EXAMPLE 2 Applying a Vertical Shift

Graph $f(x) = \dfrac{1}{x} + 3$.

The graph of this function is found by shifting the graph of $y = \frac{1}{x}$ three units up. See Figure 5. The domain is $(-\infty, 0) \cup (0, \infty)$, and the range is $(-\infty, 3) \cup (3, \infty)$.

x	y
$\frac{1}{3}$	6
$\frac{1}{2}$	5
1	4
2	3.5

x	y
$-\frac{1}{3}$	0
$-\frac{1}{2}$	1
-1	2
-2	2.5

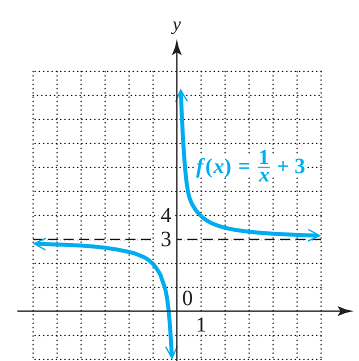


FIGURE 5

Now Try Exercise 11.

YOU TRY IT

EXAMPLE 3 Applying Both Horizontal and Vertical Shifts

Graph $f(x) = \sqrt{x+1} - 4$.

The graph of $y = (x+1)^2 - 4$ is obtained by shifting the graph of $y = x^2$ one unit to the left and four units down. Following this pattern here, we shift the graph of $y = \sqrt{x}$ one unit to the left and four units down to get the graph of $f(x) = \sqrt{x+1} - 4$. See Figure 6. The domain is $[-1, \infty)$, and the range is $[-4, \infty)$.

x	y
-1	-4
0	-3
3	-2

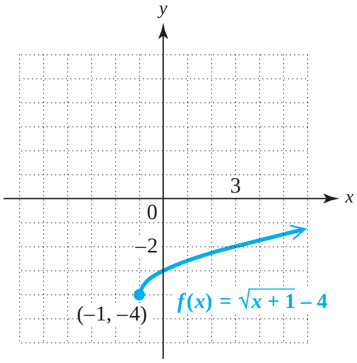


FIGURE 6

Now Try Exercise 17.

OBJECTIVE 2 Recognize and graph step functions. The **greatest integer function,** usually written $f(x) = [\![x]\!]$, is defined by saying that $[\![x]\!]$ denotes the largest integer that is less than or equal to x. For example, $[\![8]\!] = 8$, $[\![7.45]\!] = 7$, $[\![\pi]\!] = 3$, $[\![-1]\!] = -1$, $[\![-2.6]\!] = -3$, and so on.

EXAMPLE 4 Graphing the Greatest Integer Function

Graph $f(x) = [\![x]\!]$.

For $[\![x]\!]$, if $-1 \le x < 0$, then $[\![x]\!] = -1$. If $0 \le x < 1$, then $[\![x]\!] = 0$. If $1 \le x < 2$, then $[\![x]\!] = 1$, and so on. Thus, the graph, as shown in Figure 7, consists of a series of horizontal line segments. In each one, the left endpoint is included and the right endpoint is excluded. These segments continue infinitely following this pattern to the left and right. Since x can take any real number value, the domain is $(-\infty, \infty)$. The range is the set of integers $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. The shape of the graph is the reason that this function is called a **step function.**

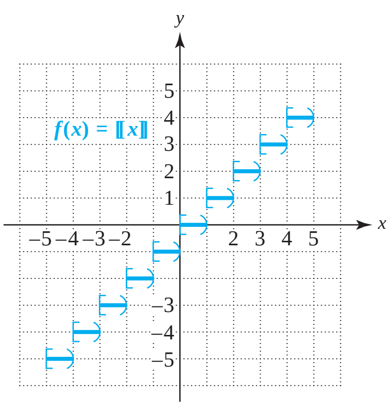


FIGURE 7

The graph of a step function also may be shifted. For example, the graph of $h(x) = [\![x-2]\!]$ is the same as the graph of $f(x) = [\![x]\!]$ shifted two units to the right. Similarly, the graph of $g(x) = [\![x]\!] + 2$ is the graph of $f(x)$ shifted two units up.

Now Try Exercise 19.

EXAMPLE 5 Applying a Greatest Integer Function

An overnight delivery service charges \$25 for a package weighing up to 2 lb. For each additional pound or fraction of a pound there is an additional charge of \$3. Let $D(x)$ represent the cost to send a package weighing x lb. Graph $D(x)$ for x in the interval $(0, 6]$.

For x in the interval $(0, 2]$, $y = 25$. For x in $(2, 3]$, $y = 25 + 3 = 28$. For x in $(3, 4]$, $y = 28 + 3 = 31$, and so on. The graph, which is that of a step function, is shown in Figure 8.

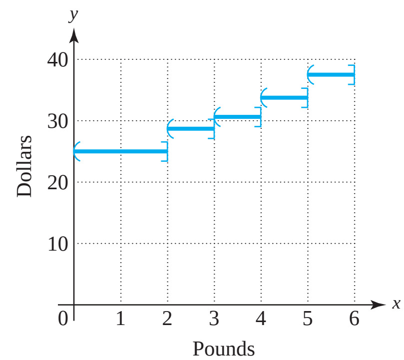


FIGURE 8

Now Try Exercise 21.

OBJECTIVE 3 Find the composition of functions. The diagram in Figure 9 shows a function f that assigns to each element x of set X some element y of set Y. Suppose that a function g takes each element of set Y and assigns a value z of set Z. Using both f and g, then, an element x in X is assigned to an element z in Z. The result of this process is a new function h, which takes an element x in X and assigns it an element z in Z.

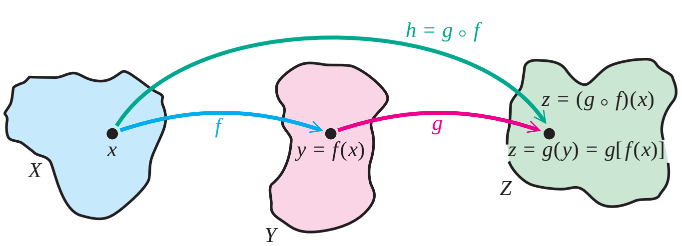


FIGURE 9

This function h is called the *composition* of functions g and f, written $g \circ f$, and is defined as follows.

Composition of Functions

If f and g are functions, then the **composite function,** or **composition,** of g and f is defined by

$$(g \circ f)(x) = g[f(x)]$$

for all x in the domain of f such that $f(x)$ is in the domain of g.

Read $g \circ f$ as "g of f."

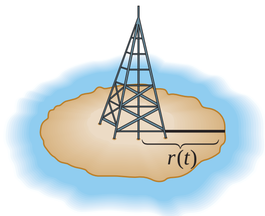


FIGURE 10

As a real-life example of how composite functions occur, suppose an oil well off the California coast is leaking, with the leak spreading oil in a circular layer over the surface. See Figure 10. At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick is given by $r(t) = 5t$ ft. Since $A(r) = \pi r^2$ gives the area of a circle of radius r, the area can be expressed as a function of time by substituting $5t$ for r in $A(r) = \pi r^2$ to get

$$A(r) = \pi r^2$$
$$A[r(t)] = \pi(5t)^2 = 25\pi t^2.$$

The function $A[r(t)]$ is a composite function of the functions A and r.

EXAMPLE 6 Finding a Composite Function

Let $f(x) = x^2$ and $g(x) = x + 3$. Find $(f \circ g)(4)$.

$$
\begin{aligned}
(f \circ g)(4) &= f[g(4)] && \text{Definition} \\
&= f(4 + 3) && \text{Use the rule for } g(x);\ g(4) = 4 + 3. \\
&= f(7) && \text{Add.} \\
&= 7^2 && \text{Use the rule for } f(x);\ f(7) = 7^2. \\
&= 49
\end{aligned}
$$

Now Try Exercise 23.

Notice in Example 6 that if we interchange the order of the functions, the composition of g and f is defined by $g[f(x)]$. Once again, letting $x = 4$, we have

$$
\begin{aligned}
(g \circ f)(4) &= g[f(4)] && \text{Definition} \\
&= g(4^2) && \text{Use the rule for } f(x);\ f(4) = 4^2. \\
&= g(16) && \text{Square 4.} \\
&= 16 + 3 && \text{Use the rule for } g(x);\ g(16) = 16 + 3. \\
&= 19.
\end{aligned}
$$

Here we see that $(f \circ g)(4) \neq (g \circ f)(4)$ because $49 \neq 19$. In general,

$$(f \circ g)(x) \neq (g \circ f)(x).$$

VIDEO YOU TRY IT

EXAMPLE 7 Finding Composite Functions

Let $f(x) = 4x - 1$ and $g(x) = x^2 + 5$. Find the following.

(a) $(f \circ g)(2)$

$$
\begin{aligned}
(f \circ g)(2) &= f[g(2)] \\
&= f(2^2 + 5) \\
&= f(9) \\
&= 4(9) - 1 \\
&= 35
\end{aligned}
$$

(b) $(f \circ g)(x)$

Here, use $g(x)$ as the input for the function f.

$$\begin{aligned} (f \circ g)(x) &= f[g(x)] \\ &= 4(g(x)) - 1 && \text{Use the rule for } f(x);\ f(x) = 4x - 1. \\ &= 4(x^2 + 5) - 1 && g(x) = x^2 + 5 \\ &= 4x^2 + 20 - 1 && \text{Distributive property} \\ &= 4x^2 + 19 && \text{Combine terms.} \end{aligned}$$

(c) Find $(f \circ g)(2)$ again, this time using the rule obtained in part (b).

$$\begin{aligned} (f \circ g)(x) &= 4x^2 + 19 && \text{From part (b)} \\ (f \circ g)(2) &= 4(2)^2 + 19 \\ &= 4(4) + 19 \\ &= 16 + 19 \\ &= 35 \end{aligned}$$

The result, 35, is the same as the result in part (a).

Now Try Exercises 25 and 29.

11.1 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 18/Videotape 18

Fill in each blank with the correct response.

1. For the reciprocal function defined by $f(x) = \frac{1}{x}$, ________ is the only real number not in the domain.

2. The range of the square root function, given by $f(x) = \sqrt{x}$, is ________.

3. The lowest point on the graph of $f(x) = |x|$ has coordinates (________, ________).

4. The range of $f(x) = [\![x]\!]$, the greatest integer function, is ________.

Without actually plotting points, match each function defined by the absolute value expression with its graph. See Example 1.

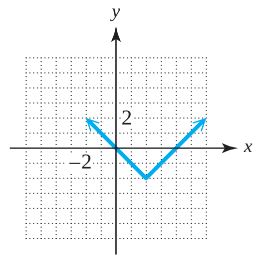
5. $f(x) = |x - 2| + 2$

6. $f(x) = |x + 2| + 2$

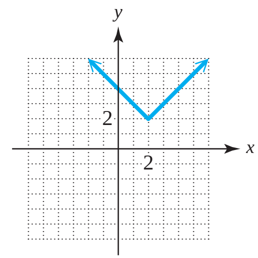
7. $f(x) = |x - 2| - 2$

8. $f(x) = |x + 2| - 2$

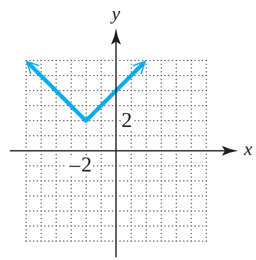
A.



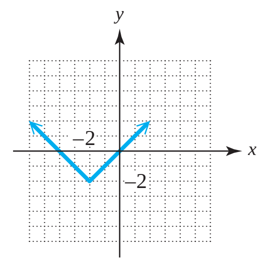
B.



C.



D.



Graph each function. Give the domain and range. See Examples 1–3.

9. $f(x) = |x + 1|$ **10.** $f(x) = |x - 1|$ **11.** $f(x) = \dfrac{1}{x} + 1$

12. $f(x) = \dfrac{1}{x} - 1$ **13.** $f(x) = \sqrt{x - 2}$ **14.** $f(x) = \sqrt{x + 5}$

15. $f(x) = \dfrac{1}{x - 2}$ **16.** $f(x) = \dfrac{1}{x + 2}$ **17.** $f(x) = \sqrt{x + 3} - 3$

18. Explain how the graph of $f(x) = \frac{1}{x-3} + 2$ is obtained from the graph of $g(x) = \frac{1}{x}$.

Graph each step function. See Examples 4 and 5.

19. $f(x) = [\![x - 3]\!]$ **20.** $g(x) = [\![x + 2]\!]$

21. Assume that postage rates are 37¢ for the first ounce, plus 23¢ for each additional ounce, and that each letter carries one 37¢ stamp and as many 23¢ stamps as necessary. Graph the function defined by $p(x) =$ the number of stamps on a letter weighing x oz. Use the interval $(0, 5]$.

Photo not available

22. The cost of parking a car at an airport hourly parking lot is \$3 for the first half-hour and \$2 for each additional half-hour or fraction thereof. Graph the function defined by $f(x) =$ the cost of parking a car for x hr. Use the interval $(0, 2]$.

Let $f(x) = x^2 + 4$, $g(x) = 2x + 3$, *and* $h(x) = x + 5$. *Find each value or expression. See Examples 6 and 7.*

23. $(h \circ g)(4)$ **24.** $(f \circ g)(4)$ **25.** $(g \circ f)(6)$ **26.** $(h \circ f)(6)$

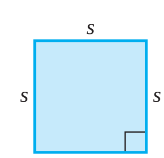
27. $(f \circ h)(-2)$ **28.** $(h \circ g)(-2)$ **29.** $(f \circ g)(x)$ **30.** $(g \circ h)(x)$

31. $(f \circ h)(x)$ **32.** $(g \circ f)(x)$ **33.** $(h \circ g)(x)$ **34.** $(h \circ f)(x)$

Solve each problem.

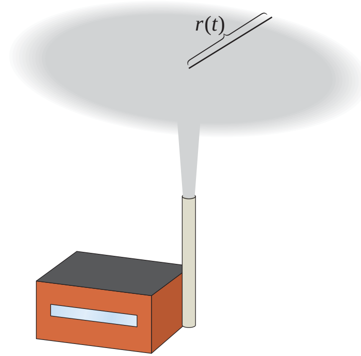
35. The function defined by $f(x) = 12x$ computes the number of inches in x ft and the function defined by $g(x) = 5280x$ computes the number of feet in x mi. What is $(f \circ g)(x)$ and what does it compute?

36. The perimeter x of a square with sides of length s is given by the formula $x = 4s$.



(a) Solve for s in terms of x.
(b) If y represents the area of this square, write y as a function of the perimeter x.
(c) Use the composite function of part (b) to find the area of a square with perimeter 6.

37. When a thermal inversion layer is over a city (as happens often in Los Angeles), pollutants cannot rise vertically but are trapped below the layer and must disperse horizontally. Assume that a factory smokestack begins emitting a pollutant at 8 A.M. Assume that the pollutant disperses horizontally over a circular area. Suppose that t represents the time, in hours, since the factory began emitting pollutants ($t = 0$ represents 8 A.M.), and assume that the radius of the circle of pollution is $r(t) = 2t$ mi. Let $A(r) = \pi r^2$ represent the area of a circle of radius r. Find and interpret $(A \circ r)(t)$.



38. An oil well off the Gulf Coast is leaking, with the leak spreading oil over the surface as a circle. At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = 4t$ ft. Let $A(r) = \pi r^2$ represent the area of a circle of radius r. Find and interpret $(A \circ r)(t)$.

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11.2 The Circle and the Ellipse

OBJECTIVES

1. Find the equation of a circle given the center and radius.
2. Determine the center and radius of a circle given its equation.
3. Recognize the equation of an ellipse.
4. Graph ellipses.
5. Graph circles and ellipses using a graphing calculator.

When an infinite cone is intersected by a plane, the resulting figure is called a **conic section.** The parabola is one example of a conic section; circles, ellipses, and hyperbolas may also result. See Figure 11.

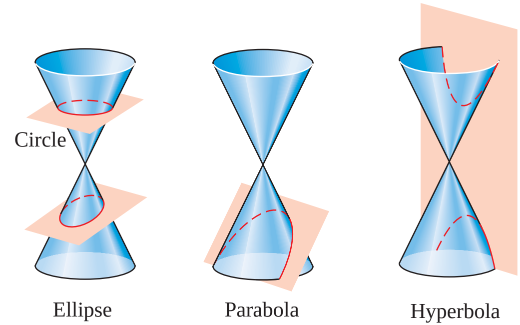


FIGURE 11

OBJECTIVE 1 Find the equation of a circle given the center and radius. A **circle** is the set of all points in a plane that lie a fixed distance from a fixed point. The fixed point is called the **center,** and the fixed distance is called the **radius.** We use the distance formula to find an equation of a circle.

EXAMPLE 1 Finding the Equation of a Circle and Graphing It

Find an equation of the circle with radius 3 and center at $(0, 0)$, and graph it.

If the point (x, y) is on the circle, then the distance from (x, y) to the center $(0, 0)$ is 3. By the distance formula,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$$
$$\sqrt{(x - 0)^2 + (y - 0)^2} = 3$$
$$x^2 + y^2 = 9. \qquad \text{Square both sides.}$$

An equation of this circle is $x^2 + y^2 = 9$. The graph is shown in Figure 12.

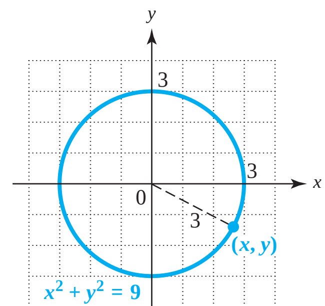


FIGURE 12

Now Try Exercise 1.

A circle may not be centered at the origin, as seen in the next example.

EXAMPLE 2 Finding an Equation of a Circle and Graphing It

Find an equation of the circle with center at $(4, -3)$ and radius 5, and graph it.

Use the distance formula again.

$$\sqrt{(x - 4)^2 + [y - (-3)]^2} = 5$$
$$(x - 4)^2 + (y + 3)^2 = 25 \qquad \text{Square both sides.}$$

To graph the circle, plot the center $(4, -3)$, then move 5 units right, left, up, and down from the center. Draw a smooth curve through these four points, sketching one quarter of the circle at a time. The graph of this circle is shown in Figure 13.

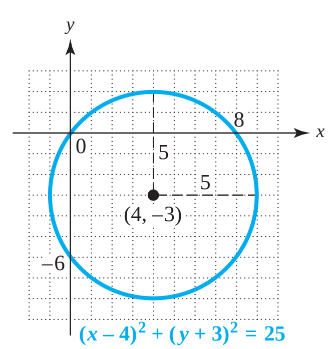


FIGURE 13

Now Try Exercises 7 and 23.

Examples 1 and 2 suggest the form of an equation of a circle with radius r and center at (h, k). If (x, y) is a point on the circle, then the distance from the center (h, k) to the point (x, y) is r. By the distance formula,

$$\sqrt{(x-h)^2+(y-k)^2}=r.$$

Squaring both sides gives the **center-radius form** of the equation of a circle.

Equation of a Circle (Center-Radius Form)

$$(x-h)^2+(y-k)^2=r^2$$

is an equation of the circle with radius r and center at (h, k).

EXAMPLE 3 Using the Center-Radius Form of the Equation of a Circle

Find an equation of the circle with center at $(-1, 2)$ and radius 4.

Use the center-radius form, with $h=-1$, $k=2$, and $r=4$.

$$(x-h)^2+(y-k)^2=r^2$$
$$[x-(-1)]^2+(y-2)^2=4^2$$
$$(x+1)^2+(y-2)^2=16$$

Now Try Exercise 9.

OBJECTIVE 2 Determine the center and radius of a circle given its equation. In the equation found in Example 2, multiplying out $(x-4)^2$ and $(y+3)^2$ gives

$$(x-4)^2+(y+3)^2=25$$
$$x^2-8x+16+y^2+6y+9=25$$
$$x^2+y^2-8x+6y=0.$$

This general form suggests that an equation with both x^2- and y^2-terms with equal coefficients may represent a circle. The next example shows how to tell, by completing the square. This procedure was introduced in Chapter 9.

EXAMPLE 4 Completing the Square to Find the Center and Radius

Find the center and radius of the circle $x^2+y^2+2x+6y-15=0$, and graph it.

Since the equation has x^2- and y^2-terms with equal coefficients, its graph might be that of a circle. To find the center and radius, complete the squares on x and y.

$x^2+y^2+2x+6y=15$	Get the constant on the right.
$(x^2+2x\qquad)+(y^2+6y\qquad)=15$	Rewrite in anticipation of completing the square.
$\left[\frac{1}{2}(2)\right]^2=1 \qquad \left[\frac{1}{2}(6)\right]^2=9$	Square half the coefficient of each middle term.
$(x^2+2x+1)+(y^2+6y+9)=15+1+9$	Complete the squares on both x and y.
$(x+1)^2+(y+3)^2=25$	Factor on the left; add on the right.
$[x-(-1)]^2+[y-(-3)]^2=5^2$	Center-radius form

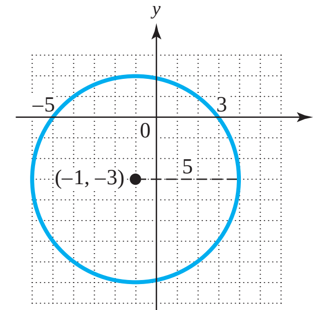
The final equation

$$[x - (-1)]^2 + [y - (-3)]^2 = 5^2$$

or

$$(x + 1)^2 + (y + 3)^2 = 5^2$$

shows that the graph is a circle with center at $(-1, -3)$ and radius 5. The graph is shown in Figure 14.



$x^2 + y^2 + 2x + 6y - 15 = 0$

FIGURE 14

Now Try Exercise 11.

NOTE If the procedure of Example 4 leads to an equation of the form $(x - h)^2 + (y - k)^2 = 0$, then the graph is the single point (h, k). If the constant on the right side is negative, then the equation has no graph.

OBJECTIVE 3 Recognize the equation of an ellipse. An **ellipse** is the set of all points in a plane the *sum* of whose distances from two fixed points is constant. These fixed points are called **foci** (singular: *focus*). Figure 15 shows an ellipse whose foci are $(c, 0)$ and $(-c, 0)$, with x-intercepts $(a, 0)$ and $(-a, 0)$ and y-intercepts $(0, b)$ and $(0, -b)$. It is shown in more advanced courses that $c^2 = a^2 - b^2$ for an ellipse of this type. The origin is the **center** of the ellipse.

An ellipse has the following equation.

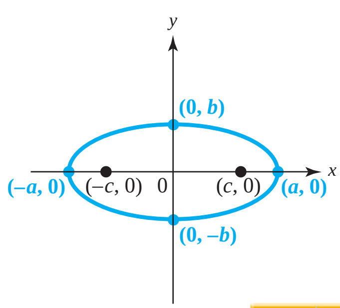


FIGURE 15

AUDIO

Equation of an Ellipse

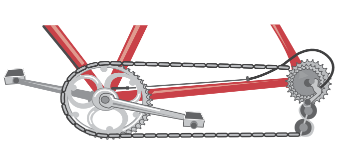
The ellipse whose x-intercepts are $(a, 0)$ and $(-a, 0)$ and whose y-intercepts are $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

NOTE A circle is a special case of an ellipse, where $a^2 = b^2$.

The paths of Earth and other planets around the sun are approximately ellipses; the sun is at one focus and a point in space is at the other. The orbits of communication satellites and other space vehicles are elliptical. Elliptical bicycle gears are designed to respond to the legs' natural strengths and weaknesses. At the top and bottom of the powerstroke, where the legs have the least leverage, the gear offers little resistance, but as the gear rotates, the resistance increases. This allows the legs to apply more power where it is most naturally available. See Figure 16.

FIGURE 16



OBJECTIVE 4 Graph ellipses. To graph an ellipse centered at the origin, we plot the four intercepts and then sketch the ellipse through those points.

EXAMPLE 5 Graphing Ellipses

Graph each ellipse.

(a) $\dfrac{x^2}{49} + \dfrac{y^2}{36} = 1$

Here, $a^2 = 49$, so $a = 7$, and the x-intercepts for this ellipse are $(7, 0)$ and $(-7, 0)$. Similarly, $b^2 = 36$, so $b = 6$, and the y-intercepts are $(0, 6)$ and $(0, -6)$. Plotting the intercepts and sketching the ellipse through them gives the graph in Figure 17.

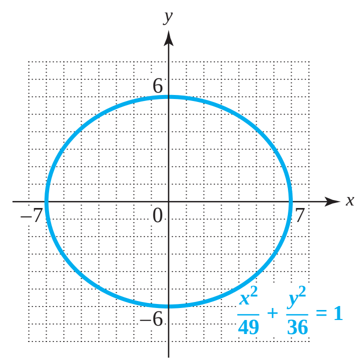


FIGURE 17

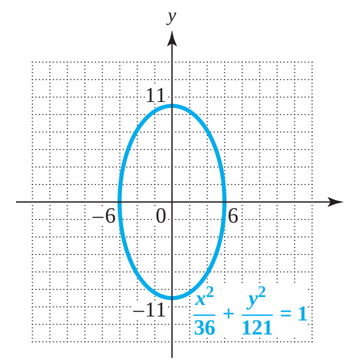


FIGURE 18

(b) $\dfrac{x^2}{36} + \dfrac{y^2}{121} = 1$

The x-intercepts for this ellipse are $(6, 0)$ and $(-6, 0)$, and the y-intercepts are $(0, 11)$ and $(0, -11)$. Join these with the smooth curve of an ellipse. The graph has been sketched in Figure 18.

Now Try Exercises 27 and 31.

As with the graphs of functions and circles, the graph of an ellipse may be shifted horizontally and vertically, as in the next example.

EXAMPLE 6 Graphing an Ellipse Shifted Horizontally and Vertically

Graph $\dfrac{(x-2)^2}{25} + \dfrac{(y+3)^2}{49} = 1.$

Just as $(x - 2)^2$ and $(y + 3)^2$ would indicate that the center of a circle would be $(2, -3)$, so it is with this ellipse. Figure 19 shows that the graph goes through the four points $(2, 4)$, $(7, -3)$, $(2, -10)$, and $(-3, -3)$. The x-values of these points are found by adding $\pm a = \pm 5$ to 2, and the y-values come from adding $\pm b = \pm 7$ to -3.

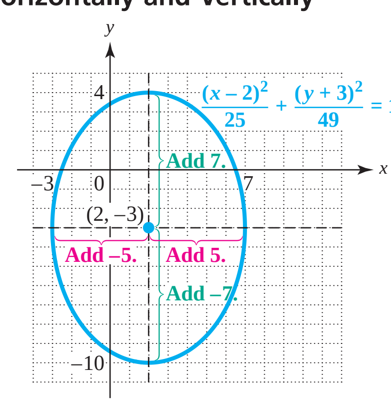


FIGURE 19

Now Try Exercise 35.

OBJECTIVE 5 Graph circles and ellipses using a graphing calculator. The only conic section whose graph is a function is the vertical parabola with equation $f(x) = ax^2 + bx + c$. Therefore, a graphing calculator in function mode cannot directly graph a circle or an ellipse. We must first solve the equation for y, getting two functions y_1 and y_2. The union of these two graphs is the graph of the entire figure. For example, to graph $(x + 3)^2 + (y + 2)^2 = 25$, begin by solving for y.

$$(x + 3)^2 + (y + 2)^2 = 25$$

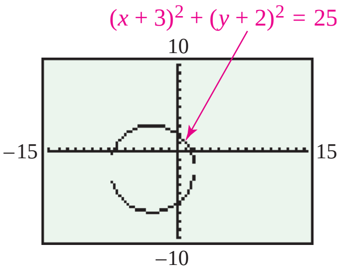
$$(y + 2)^2 = 25 - (x + 3)^2 \qquad \text{Subtract } (x + 3)^2.$$

$$y + 2 = \pm\sqrt{25 - (x + 3)^2} \qquad \text{Take square roots.}$$

$$y = -2 \pm \sqrt{25 - (x + 3)^2} \qquad \text{Subtract 2.}$$

The two functions to be graphed are

$$y_1 = -2 + \sqrt{25 - (x + 3)^2} \qquad \text{and} \qquad y_2 = -2 - \sqrt{25 - (x + 3)^2}.$$



Square Viewing Window

FIGURE 20

To get an undistorted screen, a *square viewing window* must be used. (Refer to your instruction manual for details.) See Figure 20. The two semicircles seem to be disconnected. This is because the graphs are nearly vertical at those points, and the calculator cannot show a true picture of the behavior there.

11.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 18/Videotape 18

1. See Example 1. Consider the circle whose equation is $x^2 + y^2 = 25$.

(a) What are the coordinates of its center? **(b)** What is its radius?
(c) Sketch its graph.

2. Explain why a set of points defined by a circle does not satisfy the definition of a function.

Match each equation with the correct graph. See Examples 1–3.

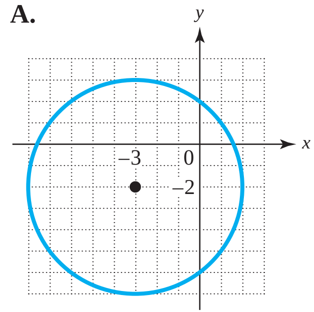
3. $(x - 3)^2 + (y - 2)^2 = 25$

4. $(x - 3)^2 + (y + 2)^2 = 25$

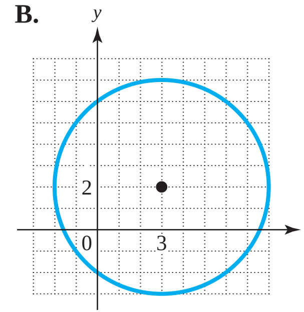
5. $(x + 3)^2 + (y - 2)^2 = 25$

6. $(x + 3)^2 + (y + 2)^2 = 25$

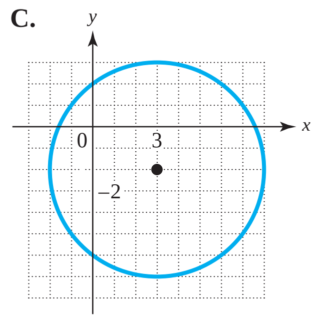
A.



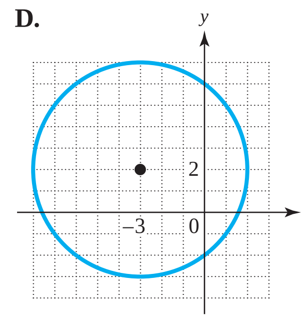
B.



C.



D.



Find the equation of a circle satisfying the given conditions. See Examples 2 and 3.

7. Center: $(-4, 3)$; radius: 2

8. Center: $(5, -2)$; radius: 4

9. Center: $(-8, -5)$; radius: $\sqrt{5}$

10. Center: $(-12, 13)$; radius: $\sqrt{7}$

Find the center and radius of each circle. (Hint: In Exercises 15 and 16, divide each side by a common factor.) See Example 4.

11. $x^2 + y^2 + 4x + 6y + 9 = 0$

12. $x^2 + y^2 - 8x - 12y + 3 = 0$

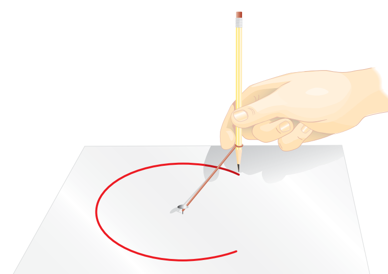
13. $x^2 + y^2 + 10x - 14y - 7 = 0$

14. $x^2 + y^2 - 2x + 4y - 4 = 0$

15. $3x^2 + 3y^2 - 12x - 24y + 12 = 0$

16. $2x^2 + 2y^2 + 20x + 16y + 10 = 0$

17. A circle can be drawn on a piece of posterboard by fastening one end of a string with a thumbtack, pulling the string taut with a pencil, and tracing a curve, as shown in the figure. Explain why this method works.



18. This figure shows how the crawfish race is held at the Crawfish Festival in Breaux Bridge, Louisiana. Explain why a circular "racetrack" is appropriate for such a race.

Graph each circle. Identify the center if it is not at the origin. See Examples 1, 2, and 4.

19. $x^2 + y^2 = 9$

20. $x^2 + y^2 = 4$

21. $2y^2 = 10 - 2x^2$

22. $3x^2 = 48 - 3y^2$

23. $(x + 3)^2 + (y - 2)^2 = 9$

24. $(x - 1)^2 + (y + 3)^2 = 16$

25. $x^2 + y^2 - 4x - 6y + 9 = 0$

26. $x^2 + y^2 + 8x + 2y - 8 = 0$

Graph each ellipse. See Examples 5 and 6.

27. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

28. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

29. $\dfrac{x^2}{36} = 1 - \dfrac{y^2}{16}$

30. $\dfrac{x^2}{9} = 1 - \dfrac{y^2}{4}$

31. $\dfrac{y^2}{25} = 1 - \dfrac{x^2}{49}$

32. $\dfrac{y^2}{9} = 1 - \dfrac{x^2}{16}$

33. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

34. $\dfrac{x^2}{49} + \dfrac{y^2}{81} = 1$

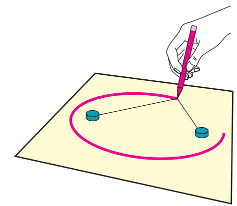
35. $\dfrac{(x + 1)^2}{64} + \dfrac{(y - 2)^2}{49} = 1$

36. $\dfrac{(x - 4)^2}{9} + \dfrac{(y + 2)^2}{4} = 1$

37. $\dfrac{(x - 2)^2}{16} + \dfrac{(y - 1)^2}{9} = 1$

38. $\dfrac{(x + 3)^2}{25} + \dfrac{(y + 2)^2}{36} = 1$

39. It is possible to sketch an ellipse on a piece of posterboard by fastening two ends of a length of string, pulling the string taut with a pencil, and tracing a curve, as shown in the figure. Explain why this method works.



40. Discuss the similarities and differences between the equations of a circle and an ellipse.

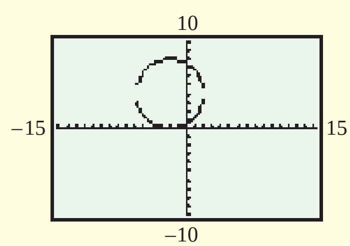
41. Explain why a set of ordered pairs whose graph forms an ellipse does not satisfy the definition of a function.

42. **(a)** How many points are there on the graph of $(x - 4)^2 + (y - 1)^2 = 0$? Explain.
(b) How many points are there on the graph of $(x - 4)^2 + (y - 1)^2 = -1$? Explain.

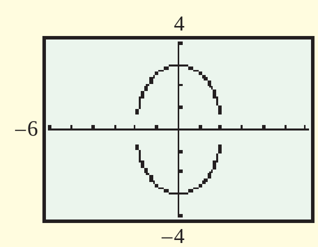
TECHNOLOGY INSIGHTS (EXERCISES 43 AND 44)

EXERCISES

43. The circle shown in the calculator graph was created using function mode, with a square viewing window. It is the graph of $(x + 2)^2 + (y - 4)^2 = 16$. What are the two functions y_1 and y_2 that were used to obtain this graph?



44. The ellipse shown in the calculator graph was graphed using function mode, with a square viewing window. It is the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$. What are the two functions y_1 and y_2 that were used to obtain this graph?



Use a graphing calculator in function mode to graph each circle or ellipse. Use a square viewing window. See Objective 5.

EXERCISES

45. $x^2 + y^2 = 36$

46. $(x - 2)^2 + y^2 = 49$

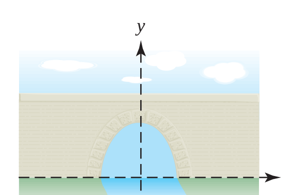
47. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

48. $\frac{(x - 3)^2}{25} + \frac{y^2}{9} = 1$

Solve each problem.

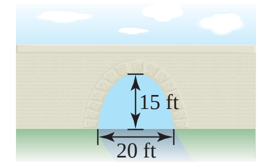
49. An arch has the shape of half an ellipse. The equation of the ellipse is $100x^2 + 324y^2 = 32{,}400$, where x and y are in meters.

(a) How high is the center of the arch?
(b) How wide is the arch across the bottom?



NOT TO SCALE

50. A one-way street passes under an overpass, which is in the form of the top half of an ellipse, as shown in the figure. Suppose that a truck 12 ft wide passes directly under the overpass. What is the maximum possible height of this truck?

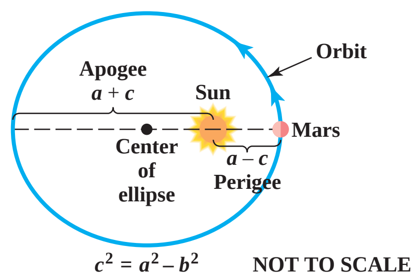


In Exercises 51 and 52, see Figure 15 and use the fact that $c^2 = a^2 - b^2$ *where* $a^2 > b^2$.

51. The orbit of Mars is an ellipse with the sun at one focus. For x and y in millions of miles, the equation of the orbit is

$$\frac{x^2}{141.7^2} + \frac{y^2}{141.1^2} = 1.$$

(*Source:* Kaler, James B., *Astronomy!*, Addison-Wesley, 1997.)



(a) Find the greatest distance (the *apogee*) from Mars to the sun.

(b) Find the smallest distance (the *perigee*) from Mars to the sun.

52. The orbit of Venus around the sun (one of the foci) is an ellipse with equation

$$\frac{x^2}{5013} + \frac{y^2}{4970} = 1,$$

where x and y are measured in millions of miles. (*Source:* Kaler, James B., *Astronomy!*, Addison-Wesley, 1997.)

(a) Find the greatest distance between Venus and the sun.

(b) Find the smallest distance between Venus and the sun.

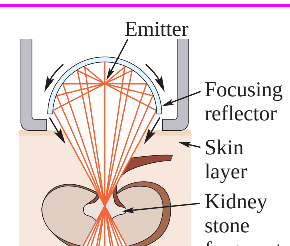
A lithotripter *is a machine used to crush kidney stones using shock waves. The patient is placed in an elliptical tub with the kidney stone at one focus of the ellipse. A beam is projected from the other focus to the tub, so that it reflects to hit the kidney stone.*

53. Suppose a lithotripter is based on the ellipse with equation

$$\frac{x^2}{36} + \frac{y^2}{9} = 1.$$

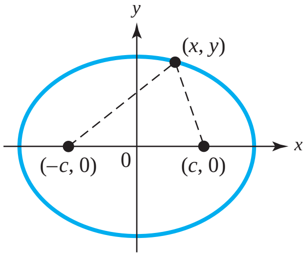
How far from the center of the ellipse must the kidney stone and the source of the beam be placed?

54. Rework Exercise 53 if the equation of the ellipse is $9x^2 + 4y^2 = 36$. (*Hint:* Write the equation in fraction form by dividing each term by 36, and use $c^2 = b^2 - a^2$, since $b > a$ here.)



The top of an ellipse is illustrated in this depiction of how a lithotripter crushes kidney stones.

Source: Adapted drawing of an ellipse in illustration of a lithotripter. The American Medical Association, *Encyclopedia of Medicine*, 1989.



55. (a) Suppose that $(c, 0)$ and $(-c, 0)$ are the foci of an ellipse and that the sum of the distances from any point (x, y) on the ellipse to the two foci is $2a$. See the figure. Show that the equation of the resulting ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

(b) Show that in the equation in part (a), the x-intercepts are $(a, 0)$ and $(-a, 0)$.

(c) Let $b^2 = a^2 - c^2$, and show that $(0, b)$ and $(0, -b)$ are the y-intercepts in the equation in part (a).

56. Use the result of Exercise 55(a) to find an equation of an ellipse with foci $(3, 0)$ and $(-3, 0)$, where the sum of the distances from any point of the ellipse to the two foci is 10.

11.3 The Hyperbola and Functions Defined by Radicals

OBJECTIVES

1. Recognize the equation of a hyperbola.
2. Graph hyperbolas by using asymptotes.
3. Identify conic sections by their equations.
4. Graph certain square root functions.

OBJECTIVE 1 Recognize the equation of a hyperbola. A **hyperbola** is the set of all points in a plane such that the absolute value of the *difference* of the distances from two fixed points (called *foci*) is constant. Figure 21 shows a hyperbola; using the distance formula and the definition above, we can show that this hyperbola has equation

$$\frac{x^2}{16} - \frac{y^2}{12} = 1.$$

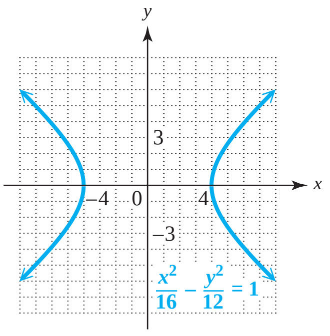


FIGURE 21

To graph hyperbolas centered at the origin, we need to find their intercepts. For the hyperbola in Figure 21, we proceed as follows.

x-Intercepts

Let $y = 0$.

$$\frac{x^2}{16} - \frac{0^2}{12} = 1 \qquad \text{Let } y = 0.$$

$$\frac{x^2}{16} = 1$$

$$x^2 = 16 \qquad \text{Multiply by 16.}$$

$$x = \pm 4$$

The x-intercepts are (4, 0) and (−4, 0).

y-Intercepts

Let $x = 0$.

$$\frac{0^2}{16} - \frac{y^2}{12} = 1 \qquad \text{Let } x = 0.$$

$$-\frac{y^2}{12} = 1$$

$$y^2 = -12 \qquad \text{Multiply by } -12.$$

Because there are no *real* solutions to $y^2 = -12$, the graph has no y-intercepts.

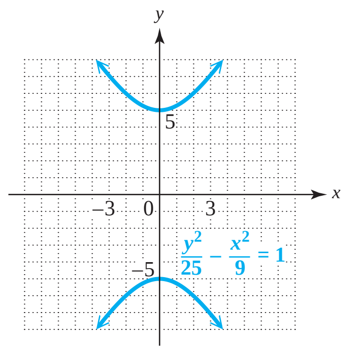


FIGURE 22

The graph of $\frac{x^2}{16} - \frac{y^2}{12} = 1$ has no y-intercepts. On the other hand, the hyperbola in Figure 22 has no x-intercepts. Its equation is

$$\frac{y^2}{25} - \frac{x^2}{9} = 1,$$

with y-intercepts (0, 5) and (0, −5).

AUDIO

Equations of Hyperbolas

A hyperbola with x-intercepts $(a, 0)$ and $(-a, 0)$ has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and a hyperbola with y-intercepts $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

OBJECTIVE 2 Graph hyperbolas by using asymptotes. The two branches of the graph of a hyperbola approach a pair of intersecting straight lines, which are its asymptotes. See Figure 23 on the next page. The asymptotes are useful for sketching the graph of the hyperbola.

AUDIO

Asymptotes of Hyperbolas

The extended diagonals of the rectangle with vertices (corners) at the points (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$ are the **asymptotes** of the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This rectangle is called the **fundamental rectangle.** Using the methods of Chapter 3, we could show that the equations of these asymptotes are

$$y = \frac{b}{a}x \qquad \text{and} \qquad y = -\frac{b}{a}x.$$

To graph hyperbolas, follow these steps.

AUDIO

Graphing a Hyperbola

Step 1 **Find the intercepts.** Locate the intercepts at $(a, 0)$ and $(-a, 0)$ if the x^2-term has a positive coefficient, or at $(0, b)$ and $(0, -b)$ if the y^2- term has a positive coefficient.

Step 2 **Find the fundamental rectangle.** Locate the vertices of the fundamental rectangle at (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$.

Step 3 **Sketch the asymptotes.** The extended diagonals of the rectangle are the asymptotes of the hyperbola, and they have equations $y = \pm\frac{b}{a}x$.

Step 4 **Draw the graph.** Sketch each branch of the hyperbola through an intercept and approaching (but not touching) the asymptotes.

VIDEO EXPLORE

EXAMPLE 1 Graphing a Horizontal Hyperbola

Graph $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1.$

Step 1 Here $a = 4$ and $b = 5$. The x-intercepts are $(4, 0)$ and $(-4, 0)$.

Step 2 The four points $(4, 5)$, $(-4, 5)$, $(-4, -5)$, and $(4, -5)$ are the vertices of the fundamental rectangle, as shown in Figure 23.

Steps 3 and 4 The equations of the asymptotes are $y = \pm\frac{5}{4}x$, and the hyperbola approaches these lines as x and y get larger and larger in absolute value.

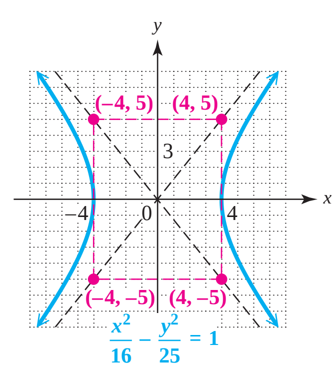


FIGURE 23

Now Try Exercise 7.

CAUTION When sketching the graph of a hyperbola, be sure that the branches do not touch the asymptotes.

EXPLORE

YOU TRY IT

EXAMPLE 2 Graphing a Vertical Hyperbola

Graph $\dfrac{y^2}{49} - \dfrac{x^2}{16} = 1.$

This hyperbola has y-intercepts $(0, 7)$ and $(0, -7)$. The asymptotes are the extended diagonals of the rectangle with vertices at $(4, 7)$, $(-4, 7)$, $(-4, -7)$, and $(4, -7)$. Their equations are $y = \pm\frac{7}{4}x$. See Figure 24.

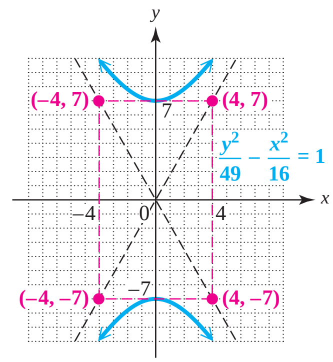
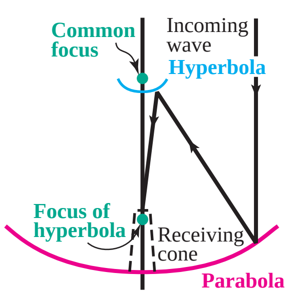


FIGURE 24

Now Try Exercise 9.

Hyperbolas are graphed with a graphing calculator in much the same way as circles and ellipses, by first writing the equations of two root functions whose union is equivalent to the equation of the hyperbola. A square window gives a truer shape for hyperbolas, too.



CONNECTIONS

A hyperbola and a parabola are used together in one kind of microwave antenna system. The cross sections of the system consist of a parabola and a hyperbola with the focus of the parabola coinciding with one focus of the hyperbola. See the figure.

The incoming microwaves that are parallel to the axis of the parabola are reflected from the parabola up toward the hyperbola and back to the other focus of the hyperbola, where the cone of the antenna is located to capture the signal.

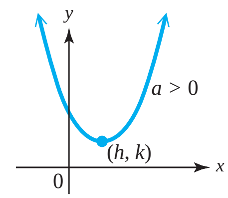
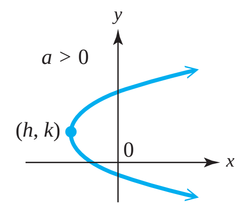
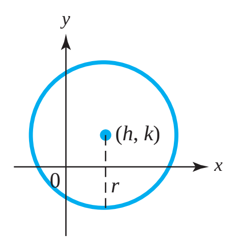
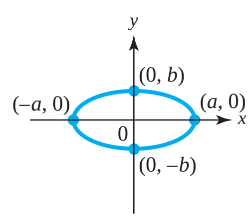
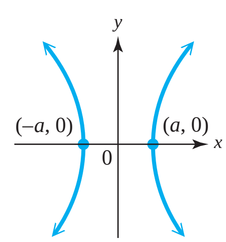
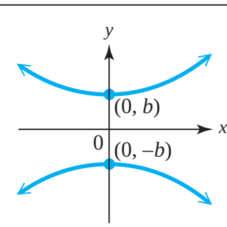
For Discussion or Writing

The property of the parabola and the hyperbola that is used here is a “reflection property” of the foci. Explain why this name is appropriate.

OBJECTIVE 3 Identify conic sections by their equations. Rewriting a second-degree equation in one of the forms given for ellipses, hyperbolas, circles, or parabolas makes it possible to determine when the graph is one of these.

AUDIO

Summary of Conic Sections

Equation	*Graph*	*Description*	*Identification*
$\boldsymbol{y = ax^2 + bx + c}$ **or** $\boldsymbol{y = a(x - h)^2 + k}$	 Parabola	It opens up if $a > 0$, down if $a < 0$. The vertex is (h, k).	It has an x^2-term. y is not squared.
$\boldsymbol{x = ay^2 + by + c}$ **or** $\boldsymbol{x = a(y - k)^2 + h}$	 Parabola	It opens to the right if $a > 0$, to the left if $a < 0$. The vertex is (h, k).	It has a y^2-term. x is not squared.
$\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$	 Circle	The center is (h, k), and the radius is r.	x^2- and y^2-terms have the same positive coefficient.
$\boldsymbol{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$	 Ellipse	The x-intercepts are $(a, 0)$ and $(-a, 0)$. The y-intercepts are $(0, b)$ and $(0, -b)$.	x^2- and y^2-terms have different positive coefficients.
$\boldsymbol{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$	 Hyperbola	The x-intercepts are $(a, 0)$ and $(-a, 0)$. The asymptotes are found from (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$.	x^2 has a positive coefficient. y^2 has a negative coefficient.
$\boldsymbol{\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1}$	 Hyperbola	The y-intercepts are $(0, b)$ and $(0, -b)$. The asymptotes are found from (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$.	y^2 has a positive coefficient. x^2 has a negative coefficient.

EXAMPLE 3 Identifying the Graphs of Equations

Identify the graph of each equation.

(a) $9x^2 = 108 + 12y^2$

Both variables are squared, so the graph is either an ellipse or a hyperbola. (This situation also occurs for a circle, which is a special case of an ellipse.) To see whether the graph is an ellipse or a hyperbola, rewrite the equation so that the x^2- and y^2-terms are on one side of the equation and 1 is on the other.

$$9x^2 - 12y^2 = 108 \qquad \text{Subtract } 12y^2.$$

$$\frac{x^2}{12} - \frac{y^2}{9} = 1 \qquad \text{Divide by 108.}$$

Because of the minus sign, the graph of this equation is a hyperbola.

(b) $x^2 = y - 3$

Only one of the two variables, x, is squared, so this is the vertical parabola $y = x^2 + 3$.

(c) $x^2 = 9 - y^2$

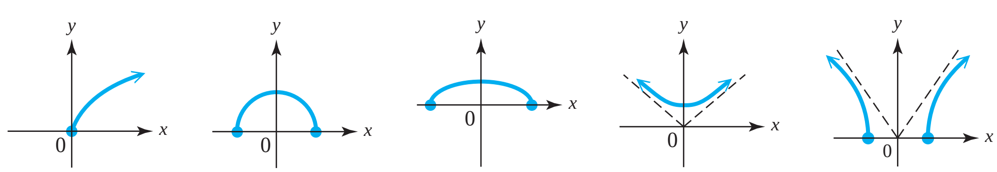
Get the variable terms on the same side of the equation.

$$x^2 + y^2 = 9 \qquad \text{Add } y^2.$$

The graph of this equation is a circle with center at the origin and radius 3.

Now Try Exercises 17 and 21.

OBJECTIVE 4 Graph certain square root functions. Recall that no vertical line will intersect the graph of a function in more than one point. Thus, horizontal parabolas, all circles and ellipses, and most hyperbolas discussed in this chapter are examples of graphs that do not satisfy the conditions of a function. However, by considering only a part of the graph of each of these we have the graph of a function, as seen in Figure 25.



(a) (b) (c) (d) (e)

FIGURE 25

In parts (a), (b), (c), and (d) of Figure 25, the top portion of a conic section is shown (parabola, circle, ellipse, and hyperbola, respectively). In part (e), the top two portions of a hyperbola are shown. In each case, the graph is that of a function since the graph satisfies the conditions of the vertical line test.

In Sections 8.1 and 11.1 we observed the square root function defined by $f(x) = \sqrt{x}$. To find equations for the types of graphs shown in Figure 25, we extend its definition.

Square Root Function

A function of the form

$$f(x) = \sqrt{u}$$

for an algebraic expression u, with $u \geq 0$, is called a **square root function.**

EXAMPLE 4 Graphing a Semicircle

Graph $f(x) = \sqrt{25 - x^2}$. Give the domain and range.

Replace $f(x)$ with y and square both sides to get the equation

$$y^2 = 25 - x^2 \quad \text{or} \quad x^2 + y^2 = 25.$$

This is the graph of a circle with center at $(0, 0)$ and radius 5. Since $f(x)$, or y, represents a principal square root in the original equation, $f(x)$ must be nonnegative. This restricts the graph to the upper half of the circle, as shown in Figure 26. Use the graph and the vertical line test to verify that it is indeed a function. The domain is $[-5, 5]$, and the range is $[0, 5]$.

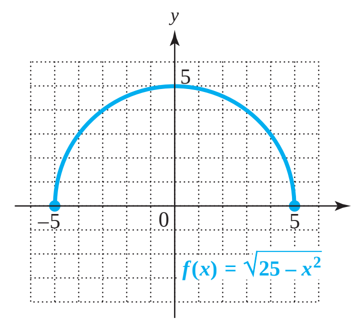


FIGURE 26

Now Try Exercise 25.

EXAMPLE 5 Graphing a Portion of an Ellipse

Graph $\frac{y}{6} = -\sqrt{1 - \frac{x^2}{16}}$. Give the domain and range.

Square both sides to get an equation whose form is known.

$$\frac{y^2}{36} = 1 - \frac{x^2}{16}$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1 \qquad \text{Add } \tfrac{x^2}{16}.$$

This is the equation of an ellipse with x-intercepts $(4, 0)$ and $(-4, 0)$ and y-intercepts $(0, 6)$ and $(0, -6)$. Since $\frac{y}{6}$ equals a negative square root in the original equation, y must be nonpositive, restricting the graph to the lower half of the ellipse, as shown in Figure 27. Verify that this is the graph of a function, using the vertical line test. The domain is $[-4, 4]$, and the range is $[-6, 0]$.

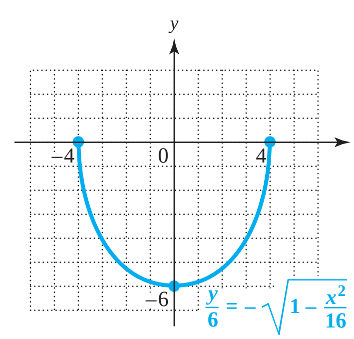


FIGURE 27

Now Try Exercise 27.

Root functions, since they are functions, can be entered and graphed directly with a graphing calculator.

11.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 18/Videotape 18

Based on the discussions of ellipses in the previous section and of hyperbolas in this section, match each equation with its graph.

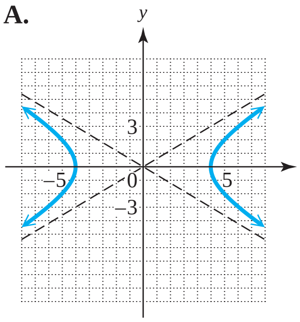
1. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

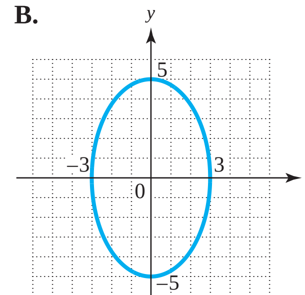
3. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

4. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

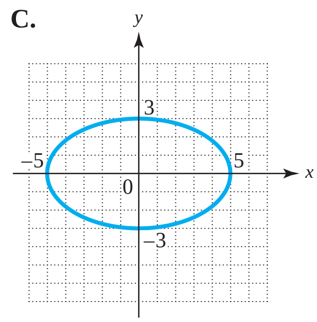
A.



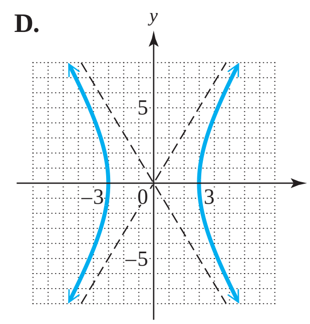
B.



C.



D.



5. Write an explanation of how you can tell from the equation whether the branches of a hyperbola open up and down or left and right.

6. Describe how the fundamental rectangle is used to sketch a hyperbola.

Graph each hyperbola. See Examples 1 and 2.

7. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

8. $\dfrac{y^2}{4} - \dfrac{x^2}{25} = 1$

9. $\dfrac{y^2}{9} - \dfrac{x^2}{9} = 1$

10. $\dfrac{x^2}{49} - \dfrac{y^2}{16} = 1$

11. $\dfrac{x^2}{25} - \dfrac{y^2}{36} = 1$

12. $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

13. $\dfrac{y^2}{16} - \dfrac{x^2}{16} = 1$

14. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

Identify the graph of each equation as a parabola, circle, ellipse, *or* hyperbola, *and then sketch. See Example 3.*

15. $x^2 - y^2 = 16$

16. $x^2 + y^2 = 16$

17. $4x^2 + y^2 = 16$

18. $x^2 - 2y = 0$

19. $y^2 = 36 - x^2$

20. $9x^2 + 25y^2 = 225$

21. $9x^2 = 144 + 16y^2$

22. $x^2 + 9y^2 = 9$

23. $y^2 = 4 + x^2$

24. State in your own words the major difference between the definitions of *ellipse* and *hyperbola.*

Graph each function defined by a radical expression. Give the domain and range. See Examples 4 and 5.

25. $f(x) = \sqrt{16 - x^2}$

26. $f(x) = \sqrt{9 - x^2}$

27. $f(x) = -\sqrt{36 - x^2}$

28. $f(x) = -\sqrt{25 - x^2}$

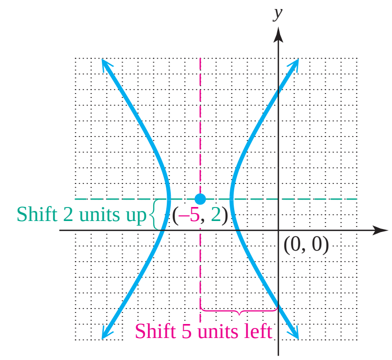
29. $\frac{y}{3} = \sqrt{1 + \frac{x^2}{9}}$

30. $y = \sqrt{\frac{x + 4}{2}}$

In Section 11.2, Example 6, we saw that the center of an ellipse may be shifted away from the origin. The same process applies to hyperbolas. For example, the hyperbola

$$\frac{(x+5)^2}{4} - \frac{(y-2)^2}{9} = 1,$$

shown at the right, has the same graph as $\frac{x^2}{4} - \frac{y^2}{9} = 1$, *but it is centered at* $(-5, 2)$. *Graph each hyperbola with center shifted away from the origin.*



31. $\frac{(x-2)^2}{4} - \frac{(y+1)^2}{9} = 1$

32. $\frac{(x+3)^2}{16} - \frac{(y-2)^2}{25} = 1$

33. $\frac{y^2}{36} - \frac{(x-2)^2}{49} = 1$

34. $\frac{(y-5)^2}{9} - \frac{x^2}{25} = 1$

Solve each problem.

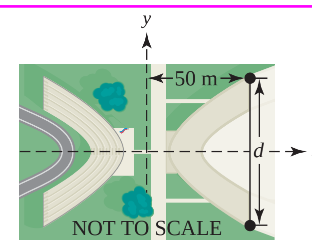
35. Two buildings in a sports complex are shaped and positioned like a portion of the branches of the hyperbola with equation

$$400x^2 - 625y^2 = 250{,}000,$$

where x and y are in meters.

(a) How far apart are the buildings at their closest point?

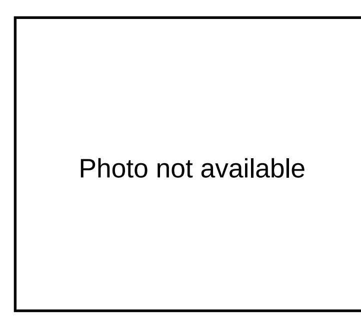
(b) Find the distance d in the figure.



36. In rugby, after a *try* (similar to a touchdown in American football) the scoring team attempts a kick for extra points. The ball must be kicked from directly behind the point where the try was scored. The kicker can choose the distance but cannot move the ball sideways. It can be shown that the kicker's best choice is on the hyperbola with equation

$$\frac{x^2}{g^2} - \frac{y^2}{g^2} = 1,$$

where $2g$ is the distance between the goal posts. Since the hyperbola approaches its asymptotes, it is easier for the kicker to estimate points on the asymptotes instead of on the hyperbola. What are the asymptotes of this hyperbola? Why is it relatively easy to estimate them? (*Source:* Isaksen, Daniel C., "How to Kick a Field Goal," *The College Mathematics Journal*, September 1996.)



37. When a satellite is launched into orbit, the shape of its trajectory is determined by its velocity. The trajectory will be hyperbolic if the velocity V, in meters per second, satisfies the inequality

$$V > \frac{2.82 \times 10^7}{\sqrt{D}},$$

where D is the distance, in meters, from the center of Earth. For what values of V will the trajectory be hyperbolic if $D = 4.25 \times 10^7$ m? (*Source:* Kaler, James B., *Astronomy!*, Addison-Wesley, 1997.)

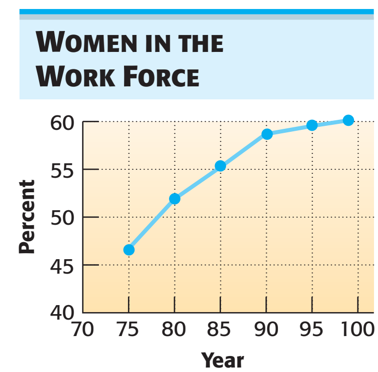
38. The percent of women in the work force has increased steadily for many years. The line graph shows the change for the period from 1975 through 1999, where $x = 75$ represents 1975, $x = 80$ represents 1980, and so on. The graph resembles the upper branch of a horizontal hyperbola. Using statistical methods, we found the corresponding square root equation

$$y = .607\sqrt{383.9 + x^2},$$

which closely approximates the line graph.

(a) According to the graph, what percent of women were in the work force in 1985?

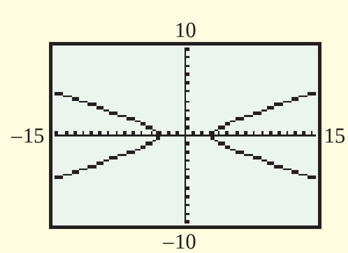
(b) According to the equation, what percent of women worked in 1985? (Round to the nearest percent.)



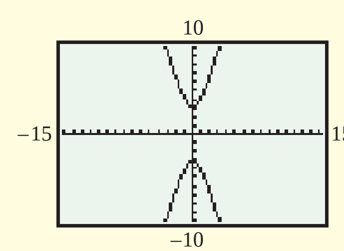
Source: U.S. Bureau of Labor Statistics.

TECHNOLOGY INSIGHTS (EXERCISES 39 AND 40)

39. The hyperbola shown in the figure was graphed in function mode, with a square viewing window. It is the graph of $\frac{x^2}{9} - y^2 = 1$. What are the two functions y_1 and y_2 that were used to obtain this graph?



40. Repeat Exercise 39 for the graph of $\frac{y^2}{9} - x^2 = 1$, shown in the figure.



Use a graphing calculator in function mode to graph each hyperbola. Use a square viewing window.

EXERCISES

41. $\dfrac{x^2}{25} - \dfrac{y^2}{49} = 1$

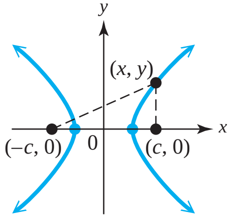
42. $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

43. $\dfrac{y^2}{9} - x^2 = 1$

44. $\dfrac{y^2}{36} - \dfrac{x^2}{4} = 1$

45. Suppose that a hyperbola has center at the origin, foci at $(-c, 0)$ and $(c, 0)$, and the absolute value of the difference between the distances from any point (x, y) of the hyperbola to the two foci is $2a$. See the figure. Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



46. Use the result of Exercise 45 to find an equation of a hyperbola with center at the origin, foci at $(-2, 0)$ and $(2, 0)$, and the absolute value of the difference between the distances from any point of the hyperbola to the two foci equal to 2.

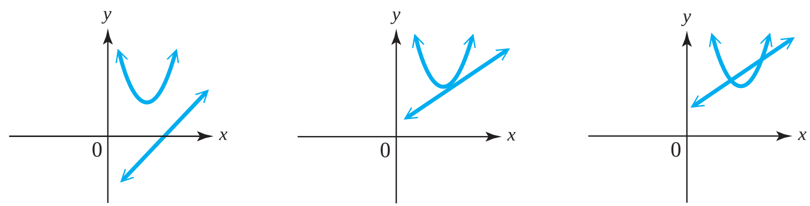
11.4 Nonlinear Systems of Equations

OBJECTIVES

1. Solve a nonlinear system by substitution.
2. Use the elimination method to solve a system with two second-degree equations.
3. Solve a system that requires a combination of methods.
4. Use a graphing calculator to solve a nonlinear system.

An equation in which some terms have more than one variable or a variable of degree 2 or greater is called a **nonlinear equation.** A **nonlinear system of equations** includes at least one nonlinear equation.

When solving a nonlinear system, it helps to visualize the types of graphs of the equations of the system to determine the possible number of points of intersection. For example, if a system includes two equations where the graph of one is a parabola and the graph of the other is a line, then there may be zero, one, or two points of intersection, as illustrated in Figure 28.



No points of intersection　　One point of intersection　　Two points of intersection

FIGURE 28

OBJECTIVE 1 Solve a nonlinear system by substitution. We solve nonlinear systems by the elimination method, the substitution method, or a combination of the two. The substitution method is usually best when one of the equations is linear.

VIDEO

EXAMPLE 1 Solving a Nonlinear System by Substitution

Solve the system

$$x^2 + y^2 = 9 \qquad (1)$$
$$2x - y = 3. \qquad (2)$$

The graph of (1) is a circle and the graph of (2) is a line. Visualizing the possible ways the graphs could intersect indicates that there may be zero, one, or two points of intersection. It is best to solve the linear equation first for one of the two variables; then substitute the resulting expression into the nonlinear equation to obtain an equation in one variable.

$$2x - y = 3 \qquad (2)$$
$$y = 2x - 3 \qquad (3)$$

Substitute $2x - 3$ for y in equation (1).

$$x^2 + (2x - 3)^2 = 9$$
$$x^2 + 4x^2 - 12x + 9 = 9$$
$$5x^2 - 12x = 0$$
$$x(5x - 12) = 0 \qquad \text{GCF is } x.$$
$$x = 0 \quad \text{or} \quad x = \frac{12}{5} \qquad \text{Zero-factor property}$$

Let $x = 0$ in equation (3) to get $y = -3$. If $x = \frac{12}{5}$, then $y = \frac{9}{5}$. The solution set of the system is $\left\{(0, -3), \left(\frac{12}{5}, \frac{9}{5}\right)\right\}$. The graph in Figure 29 confirms the two points of intersection.

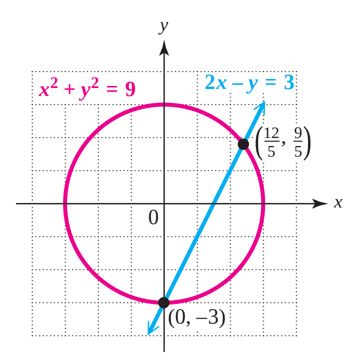


FIGURE 29

Now Try Exercise 19.

YOU TRY IT

EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve the system

$$6x - y = 5 \qquad (1)$$
$$xy = 4. \qquad (2)$$

The graph of (1) is a line. We have not specifically mentioned equations like (2); however, it can be shown by plotting points that its graph is a hyperbola. Visualizing a line and a hyperbola indicates that there may be zero, one, or two points of intersection. Since neither equation has a squared term, we can solve either equation for one of the variables and then substitute the result into the other equation. Solving $xy = 4$ for x gives $x = \frac{4}{y}$. We substitute $\frac{4}{y}$ for x in equation (1).

$$
\begin{aligned}
6\left(\frac{4}{y}\right) - y &= 5 && \text{Let } x = \tfrac{4}{y} \text{ in Equation (1).}\\
\frac{24}{y} - y &= 5 \\
24 - y^2 &= 5y && \text{Multiply by } y,\ y \neq 0.\\
0 &= y^2 + 5y - 24 && \text{Standard form}\\
0 &= (y-3)(y+8) && \text{Factor.}\\
y = 3 \quad \text{or} \quad y &= -8 && \text{Zero-factor property}
\end{aligned}
$$

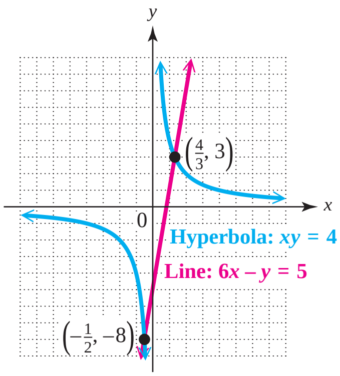


FIGURE 30

We substitute these results into $x = \frac{4}{y}$ to obtain the corresponding values of x.

$$\text{If } y = 3, \text{ then } x = \frac{4}{3}. \qquad \text{If } y = -8, \text{ then } x = -\frac{1}{2}.$$

The solution set of the system is $\left\{\left(\frac{4}{3}, 3\right), \left(-\frac{1}{2}, -8\right)\right\}$. The graph in Figure 30 shows that there are two points of intersection.

Now Try Exercise 21.

OBJECTIVE 2 Use the elimination method to solve a system with two second-degree equations. The elimination method is often used when both equations are second degree.

VIDEO

EXAMPLE 3 Solving a Nonlinear System by Elimination

Solve the system

$$
\begin{aligned}
x^2 + y^2 &= 9 && (1)\\
2x^2 - y^2 &= -6. && (2)
\end{aligned}
$$

The graph of (1) is a circle, while the graph of (2) is a hyperbola. By analyzing the possibilities we conclude that there may be zero, one, two, three, or four points of intersection. Adding the two equations will eliminate y, leaving an equation that can be solved for x.

$$
\begin{aligned}
x^2 + y^2 &= 9\\
\underline{2x^2 - y^2} &= \underline{-6}\\
3x^2 \quad &= 3\\
x^2 &= 1\\
x = 1 \quad \text{or} \quad x &= -1
\end{aligned}
$$

Each value of x gives corresponding values for y when substituted into one of the original equations. Using equation (1) gives the following.

If $x = 1$, then

$$
\begin{aligned}
1^2 + y^2 &= 9\\
y^2 &= 8\\
y = \sqrt{8} \quad \text{or} \quad y &= -\sqrt{8}\\
y = 2\sqrt{2} \quad \text{or} \quad y &= -2\sqrt{2}.
\end{aligned}
$$

If $x = -1$, then

$$
\begin{aligned}
(-1)^2 + y^2 &= 9\\
y^2 &= 8\\
y = 2\sqrt{2} \quad \text{or} \quad y &= -2\sqrt{2}.
\end{aligned}
$$

The solution set is $\{(1, 2\sqrt{2}), (1, -2\sqrt{2}), (-1, 2\sqrt{2}), (-1, -2\sqrt{2})\}$. Figure 31 shows the four points of intersection.

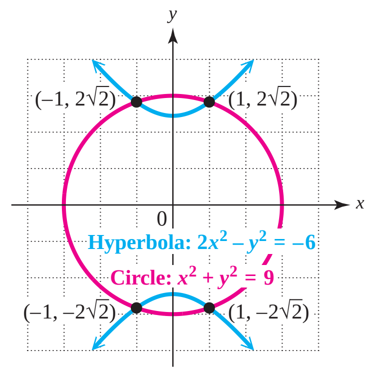


FIGURE 31

Now Try Exercise 35.

OBJECTIVE 3 Solve a system that requires a combination of methods. Solving a system of second-degree equations may require a combination of methods.

EXAMPLE 4 Solving a Nonlinear System by a Combination of Methods

EXPLORE YOU TRY IT VIDEO

Solve the system

$$x^2 + 2xy - y^2 = 7 \qquad (1)$$
$$x^2 - y^2 = 3. \qquad (2)$$

While we have not graphed equations like (1), its graph is a hyperbola. The graph of (2) is also a hyperbola. Two hyperbolas may have zero, one, two, three, or four points of intersection. We use the elimination method here in combination with the substitution method. We begin by eliminating the squared terms by multiplying each side of equation (2) by -1 and then adding the result to equation (1).

$$\begin{aligned} x^2 + 2xy - y^2 &= 7 \\ -x^2 \qquad\quad + y^2 &= -3 \\ \hline 2xy \qquad\quad &= 4 \end{aligned}$$

Next, we solve $2xy = 4$ for y. (Either variable would do.)

$$2xy = 4$$
$$y = \frac{2}{x} \qquad (3)$$

Now, we substitute $y = \frac{2}{x}$ into one of the original equations. It is easier to do this with equation (2).

$$x^2 - y^2 = 3 \qquad (2)$$
$$x^2 - \left(\frac{2}{x}\right)^2 = 3$$
$$x^2 - \frac{4}{x^2} = 3$$

$$x^4 - 4 = 3x^2 \qquad \text{Multiply by } x^2,\ x \neq 0.$$
$$x^4 - 3x^2 - 4 = 0 \qquad \text{Subtract } 3x^2.$$
$$(x^2 - 4)(x^2 + 1) = 0 \qquad \text{Factor.}$$
$$x^2 - 4 = 0 \quad \text{or} \quad x^2 + 1 = 0$$
$$x^2 = 4 \quad \text{or} \quad x^2 = -1$$
$$x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = i \quad \text{or} \quad x = -i$$

Substituting these four values of x into equation (3) gives the corresponding values for y.

If $x = 2$, then $y = 1$. If $x = i$, then $y = -2i$.
If $x = -2$, then $y = -1$. If $x = -i$, then $y = 2i$.

Note that if we substitute the x-values we found into equation (1) or (2) instead of into equation (3), we get extraneous solutions. It is always wise to check all solutions in both of the given equations. There are four ordered pairs in the solution set, two with real values and two with imaginary values. The solution set is

$$\{(2, 1), (-2, -1), (i, -2i), (-i, 2i)\}.$$

The graph of the system, shown in Figure 32, shows only the two real intersection points because the graph is in the real number plane. The two ordered pairs with imaginary components are solutions of the system, but do not appear on the graph.

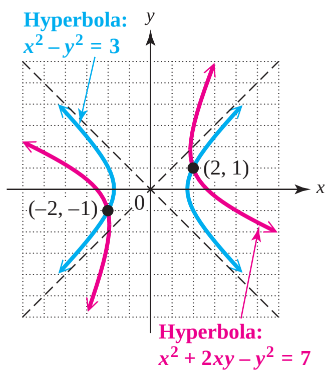


FIGURE 32

Now Try Exercise 39.

NOTE In the examples of this section, we analyzed the possible number of points of intersection of the graphs in each system. However, in Examples 2 and 4, we worked with equations whose graphs had not been studied. Keep in mind that it is not absolutely essential to visualize the number of points of intersection in order to solve the system. Furthermore, as in Example 4, there are sometimes imaginary solutions to nonlinear systems that do not appear as points of intersection in the real plane. Visualizing the geometry of the graphs is only an aid to solving these systems.

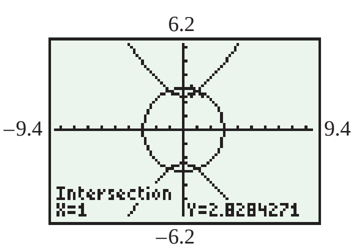


FIGURE 33

OBJECTIVE 4 Use a graphing calculator to solve a nonlinear system. If the equations in a nonlinear system can be solved for y, then we can graph the equations of the system with a graphing calculator and use the capabilities of the calculator to identify all intersection points. For instance, the two equations in Example 3 would require graphing the four separate functions

$$Y_1 = \sqrt{9 - X^2}, \quad Y_2 = -\sqrt{9 - X^2}, \quad Y_3 = \sqrt{2X^2 + 6}, \quad \text{and} \quad Y_4 = -\sqrt{2X^2 + 6}.$$

Figure 33 indicates the coordinates of one of the points of intersection.

11.4 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 18/Videotape 19

1. Write an explanation of the steps you would use to solve the system

$$x^2 + y^2 = 25$$
$$y = x - 1$$

by the substitution method. Why would the elimination method not be appropriate for this system?

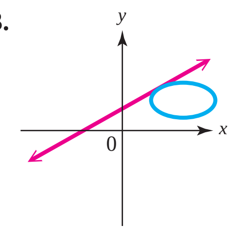
2. Write an explanation of the steps you would use to solve the system

$$x^2 + y^2 = 12$$
$$x^2 - y^2 = 13$$

by the elimination method.

Each sketch represents the graphs of a pair of equations in a system. How many points are in each solution set?

3.



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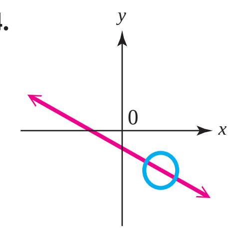

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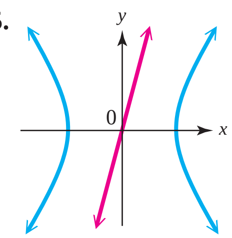

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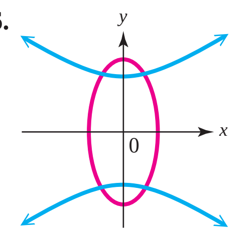


Suppose that a nonlinear system is composed of equations whose graphs are those described, and the number of points of intersection of the two graphs is as given. Make a sketch satisfying these conditions. (There may be more than one way to do this.)

7. A line and a circle; no points

8. A line and a circle; one point

9. A line and a hyperbola; one point

10. A line and an ellipse; no points

11. A circle and an ellipse; four points

12. A parabola and an ellipse; one point

13. A parabola and an ellipse; four points

14. A parabola and a hyperbola; two points

Solve each system by the substitution method. See Examples 1 and 2.

EXERCISES

15. $y = 4x^2 - x$
$y = x$

16. $y = x^2 + 6x$
$3y = 12x$

17. $y = x^2 + 6x + 9$
$x + y = 3$

18. $y = x^2 + 8x + 16$
$x - y = -4$

19. $x^2 + y^2 = 2$
$2x + y = 1$

20. $2x^2 + 4y^2 = 4$
$x = 4y$

21. $xy = 4$
$3x + 2y = -10$

22. $xy = -5$
$2x + y = 3$

23. $\begin{aligned} xy &= -3 \\ x + y &= -2 \end{aligned}$

24. $\begin{aligned} xy &= 12 \\ x + y &= 8 \end{aligned}$

25. $\begin{aligned} y &= 3x^2 + 6x \\ y &= x^2 - x - 6 \end{aligned}$

26. $\begin{aligned} y &= 2x^2 + 1 \\ y &= 5x^2 + 2x - 7 \end{aligned}$

27. $\begin{aligned} 2x^2 - y^2 &= 6 \\ y &= x^2 - 3 \end{aligned}$

28. $\begin{aligned} x^2 + y^2 &= 4 \\ y &= x^2 - 2 \end{aligned}$

29. $\begin{aligned} x^2 - xy + y^2 &= 0 \\ x - 2y &= 1 \end{aligned}$

30. $\begin{aligned} x^2 - 3x + y^2 &= 4 \\ 2x - y &= 3 \end{aligned}$

Solve each system by the elimination method or a combination of the elimination and substitution methods. See Examples 3 and 4.

31. $\begin{aligned} 3x^2 + 2y^2 &= 12 \\ x^2 + 2y^2 &= 4 \end{aligned}$

32. $\begin{aligned} 2x^2 + y^2 &= 28 \\ 4x^2 - 5y^2 &= 28 \end{aligned}$

33. $\begin{aligned} 2x^2 + 3y^2 &= 6 \\ x^2 + 3y^2 &= 3 \end{aligned}$

34. $\begin{aligned} 6x^2 + y^2 &= 9 \\ 3x^2 + 4y^2 &= 36 \end{aligned}$

35. $\begin{aligned} 5x^2 - 2y^2 &= -13 \\ 3x^2 + 4y^2 &= 39 \end{aligned}$

36. $\begin{aligned} x^2 + 6y^2 &= 9 \\ 4x^2 + 3y^2 &= 36 \end{aligned}$

37. $\begin{aligned} 2x^2 &= 8 - 2y^2 \\ 3x^2 &= 24 - 4y^2 \end{aligned}$

38. $\begin{aligned} 5x^2 &= 20 - 5y^2 \\ 2y^2 &= 2 - x^2 \end{aligned}$

39. $\begin{aligned} x^2 + xy + y^2 &= 15 \\ x^2 + y^2 &= 10 \end{aligned}$

40. $\begin{aligned} 2x^2 + 3xy + 2y^2 &= 21 \\ x^2 + y^2 &= 6 \end{aligned}$

41. $\begin{aligned} 3x^2 + 2xy - 3y^2 &= 5 \\ -x^2 - 3xy + y^2 &= 3 \end{aligned}$

42. $\begin{aligned} -2x^2 + 7xy - 3y^2 &= 4 \\ 2x^2 - 3xy + 3y^2 &= 4 \end{aligned}$

Use a graphing calculator to solve each system. Then confirm your answer algebraically.

43. $\begin{aligned} xy &= -6 \\ x + y &= -1 \end{aligned}$

44. $\begin{aligned} y &= 2x^2 + 4x \\ y &= -x^2 - 1 \end{aligned}$

Solve each problem by using a nonlinear system.

45. The area of a rectangular rug is 84 ft^2 and its perimeter is 38 ft. Find the length and width of the rug.

Photo not available

46. Find the length and width of a rectangular room whose perimeter is 50 m and whose area is 100 m^2.

47. A company has found that the price p (in dollars) of its scientific calculator is related to the supply x (in thousands) by the equation

$$px = 16.$$

The price is related to the demand x (in thousands) for the calculator by the equation

$$p = 10x + 12.$$

The *equilibrium price* is the value of p where demand equals supply. Find the equilibrium price and the supply/demand at that price by solving a system of equations. (*Hint:* Demand, price, and supply must all be positive.)

48. The calculator company in Exercise 47 has also determined that the cost y to make x (thousand) calculators is

$$y = 4x^2 + 36x + 20,$$

while the revenue y from the sale of x (thousand) calculators is

$$36x^2 - 3y = 0.$$

Find the *break-even point*, where cost equals revenue, by solving a system of equations.

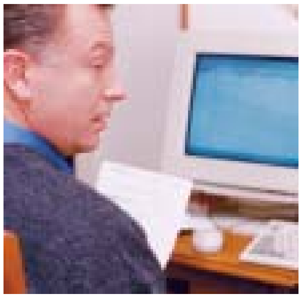
49. In the 1970s, the number of bachelor's degrees earned by men began to decrease. It stayed fairly constant in the 1980s, and then in the 1990s slowly began to increase again. Meanwhile, the number of bachelor's degrees earned by women continued to rise steadily throughout this period. Functions that model the situation are defined by the following equations, where y is the number of degrees (in thousands) granted in year x, with $x = 0$ corresponding to 1970.

Photo not available

$$\text{Men:} \quad y = .138x^2 + .064x + 451$$
$$\text{Women:} \quad y = 12.1x + 334$$

Solve this system of equations to find the year when the same number of bachelor's degrees were awarded to men and women. How many bachelor's degrees were awarded to each sex in that year? Give the answer to the nearest ten thousand. (*Source:* U.S. National Center for Education Statistics, *Digest of Education Statistics*, annual.)

50. Andy Grove, chairman of chip maker Intel Corp., once noted that decreasing prices for computers and stable prices for Internet access implied that the trend lines for these costs either have crossed or soon will. He predicted that the time is not far away when computers, like cell phones, may be given away to sell on-line time. To see this, assume a price of \$1000 for a computer, and let x represent the number of months it will be used. (*Source:* Corcoran, Elizabeth, "Can Free Computers Be That Far Away?", *Washington Post*, from *Sacramento Bee*, February 3, 1999.)



(a) Write an equation for the monthly cost y of the computer over this period.

(b) The average monthly on-line cost is about \$20. Assume this will remain constant and write an equation to express this cost.

(c) Solve the system of equations from parts (a) and (b). Interpret your answer in relation to the situation.

11.5 Second-Degree Inequalities and Systems of Inequalities

OBJECTIVES

1. Graph second-degree inequalities.
2. Graph the solution set of a system of inequalities.

OBJECTIVE 1 Graph second-degree inequalities. The linear inequality $3x + 2y \leq 5$ is graphed by first graphing the boundary line $3x + 2y = 5$. *Second-degree inequalities* are graphed in the same way. A **second-degree inequality** is an inequality with at least one variable of degree 2 and no variable with degree greater than 2. An example is $x^2 + y^2 \leq 36$. The boundary of the inequality $x^2 + y^2 \leq 36$ is the graph of the equation $x^2 + y^2 = 36$, a circle with radius 6 and center at the origin, as shown in Figure 34.

The inequality $x^2 + y^2 \leq 36$ will include either the points outside the circle or the points inside the circle, as well as the boundary. We decide which region to shade by substituting any test point not on the circle, such as (0, 0), into the original inequality. Since $0^2 + 0^2 \leq 36$ is a true statement, the original inequality includes the points inside the circle, the shaded region in Figure 34, and the boundary.

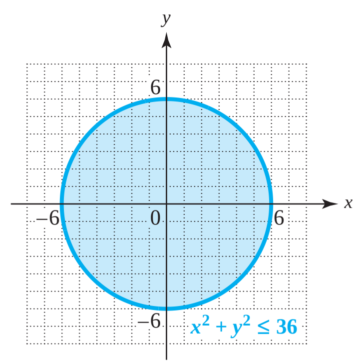


FIGURE 34

EXAMPLE 1 Graphing a Second-Degree Inequality

Graph $y < -2(x - 4)^2 - 3$.

The boundary, $y = -2(x - 4)^2 - 3$, is a parabola that opens down with vertex at $(4, -3)$. Using (0, 0) as a test point gives

$$0 \overset{?}{<} -2(0 - 4)^2 - 3$$
$$0 \overset{?}{<} -32 - 3$$
$$0 < -35. \quad \text{False}$$

Because the final inequality is a false statement, the points in the region containing (0, 0) do not satisfy the inequality. Figure 35 shows the final graph; the parabola is drawn as a dashed curve since the points of the parabola itself do not satisfy the inequality, and the region inside (or below) the parabola is shaded.

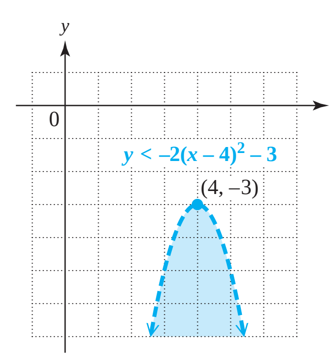


FIGURE 35

Now Try Exercise 11.

NOTE Since the substitution is easy, the origin is the test point of choice unless the graph actually passes through (0, 0).

EXAMPLE 2 Graphing a Second-Degree Inequality

Graph $16y^2 \le 144 + 9x^2$.

First rewrite the inequality as follows.

$$16y^2 - 9x^2 \le 144 \qquad \text{Subtract } 9x^2.$$

$$\frac{y^2}{9} - \frac{x^2}{16} \le 1 \qquad \text{Divide by 144.}$$

This form shows that the boundary is the hyperbola given by

$$\frac{y^2}{9} - \frac{x^2}{16} = 1.$$

Since the graph is a vertical hyperbola, the desired region will be either the region between the branches or the regions above the top branch and below the bottom branch. Choose (0, 0) as a test point. Substituting into the original inequality leads to $0 \le 144$, a true statement, so the region between the branches containing (0, 0) is shaded, as shown in Figure 36.

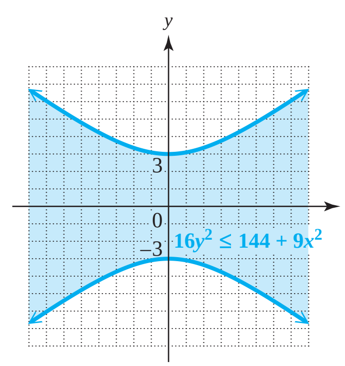


FIGURE 36

Now Try Exercise 17.

OBJECTIVE 2 Graph the solution set of a system of inequalities. If two or more inequalities are considered at the same time, we have a **system of inequalities.** To find the solution set of the system, we find the intersection of the graphs (solution sets) of the inequalities in the system.

EXAMPLE 3 Graphing a System of Two Inequalities

Graph the solution set of the system

$$2x + 3y > 6$$
$$x^2 + y^2 < 16.$$

Begin by graphing the solution set of $2x + 3y > 6$. The boundary line is the graph of $2x + 3y = 6$ and is a dashed line because of the symbol $>$. The test point (0, 0) leads to a false statement in the inequality $2x + 3y > 6$, so shade the region above the line, as shown in Figure 37. The graph of $x^2 + y^2 < 16$ is the interior of a dashed circle centered at the origin with radius 4. This is shown in Figure 38.

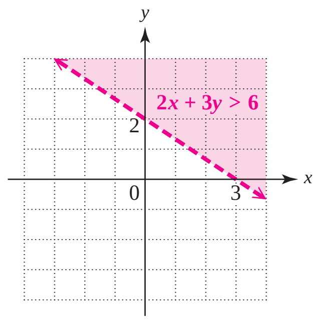


FIGURE 37

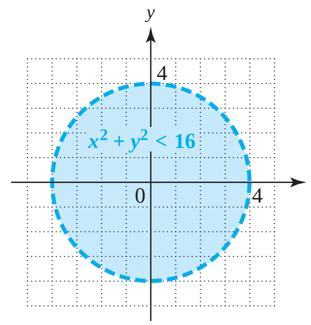


FIGURE 38

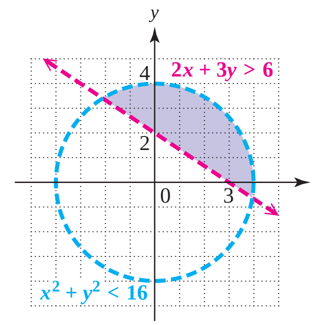


FIGURE 39

Finally, to get the graph of the solution set of the system, determine the intersection of the graphs of the two inequalities. The overlapping region in Figure 39 is the solution set.

Now Try Exercise 29.

EXAMPLE 4 Graphing a Linear System with Three Inequalities

VIDEO

Graph the solution set of the system

$$\begin{aligned} x + y &< 1 \\ y &\le 2x + 3 \\ y &\ge -2. \end{aligned}$$

Graph each inequality separately, on the same axes. The graph of $x + y < 1$ consists of all points below the dashed line $x + y = 1$. The graph of $y \le 2x + 3$ is the region that lies below the solid line $y = 2x + 3$. Finally, the graph of $y \ge -2$ is the region above the solid horizontal line $y = -2$.

The graph of the system, the intersection of these three graphs, is the triangular region enclosed by the three boundary lines in Figure 40, including two of its boundaries.

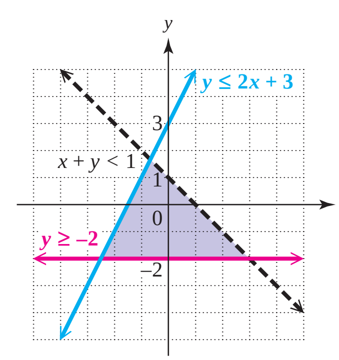


FIGURE 40

Now Try Exercise 31.

EXAMPLE 5 Graphing a System with Three Inequalities

VIDEO

Graph the solution set of the system

$$\begin{aligned} y &\ge x^2 - 2x + 1 \\ 2x^2 + y^2 &> 4 \\ y &< 4. \end{aligned}$$

The graph of $y = x^2 - 2x + 1$ is a parabola with vertex at (1, 0). Those points above (or in the interior of) the parabola satisfy the condition $y > x^2 - 2x + 1$.

Thus, points on the parabola or in the interior are the solution set of $y \geq x^2 - 2x + 1$. The graph of the equation $2x^2 + y^2 = 4$ is an ellipse. We draw it as a dashed curve. To satisfy the inequality $2x^2 + y^2 > 4$, a point must lie outside the ellipse. The graph of $y < 4$ includes all points below the dashed line $y = 4$. Finally, the graph of the system is the shaded region in Figure 41, which lies outside the ellipse, inside or on the boundary of the parabola, and below the line $y = 4$.

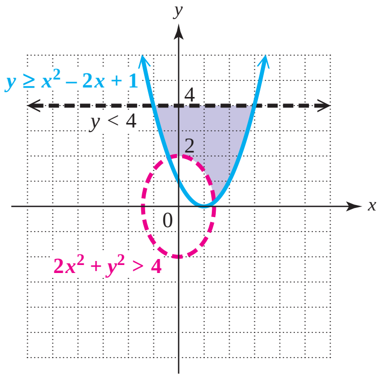


FIGURE 41

Now Try Exercise 33.

11.5 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 19/Videotape 19

1. Which one of the following is a description of the graph of the solution set of the following system?

$$x^2 + y^2 < 25$$
$$y > -2$$

A. All points outside the circle $x^2 + y^2 = 25$ and above the line $y = -2$
B. All points outside the circle $x^2 + y^2 = 25$ and below the line $y = -2$
C. All points inside the circle $x^2 + y^2 = 25$ and above the line $y = -2$
D. All points inside the circle $x^2 + y^2 = 25$ and below the line $y = -2$

2. Fill in each blank with the appropriate response. The graph of the system

$$y > x^2 + 1$$
$$\frac{x^2}{9} + \frac{y^2}{4} > 1$$
$$y < 5$$

consists of all points $\underset{\text{(above/below)}}{\underline{\hspace{3cm}}}$ the parabola $y = x^2 + 1$, $\underset{\text{(inside/outside)}}{\underline{\hspace{3cm}}}$ the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and $\underset{\text{(above/below)}}{\underline{\hspace{3cm}}}$ the line $y = 5$.

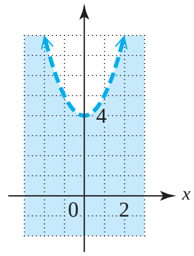
3. Explain how to graph the solution set of a nonlinear inequality.

4. Explain how to graph the solution set of a system of nonlinear inequalities.

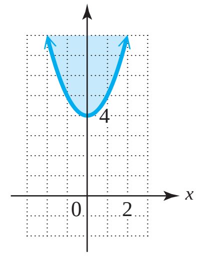
Match each nonlinear inequality with its graph.

5. $y \geq x^2 + 4$ **6.** $y \leq x^2 + 4$ **7.** $y < x^2 + 4$ **8.** $y > x^2 + 4$

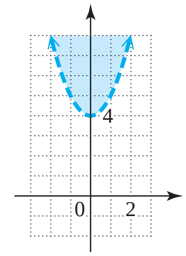
A.



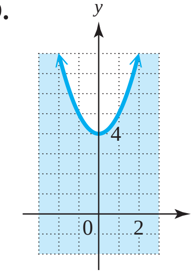
B.



C.



D.



Graph each nonlinear inequality. See Examples 1 and 2.

EXERCISES

9. $y^2 > 4 + x^2$
10. $y^2 \leq 4 - 2x^2$
11. $y + 2 \geq x^2$
12. $x^2 \leq 16 - y^2$
13. $2y^2 \geq 8 - x^2$
14. $x^2 \leq 16 + 4y^2$
15. $y \leq x^2 + 4x + 2$
16. $9x^2 < 16y^2 - 144$
17. $9x^2 > 16y^2 + 144$
18. $4y^2 \leq 36 - 9x^2$
19. $x^2 - 4 \geq -4y^2$
20. $x \geq y^2 - 8y + 14$
21. $x \leq -y^2 + 6y - 7$
22. $y^2 - 16x^2 \leq 16$

Graph each system of inequalities. See Examples 3–5.

EXERCISES

23. $2x + 5y < 10$
$x - 2y < 4$

24. $3x - y > -6$
$4x + 3y > 12$

25. $5x - 3y \leq 15$
$4x + y \geq 4$

26. $4x - 3y \leq 0$
$x + y \leq 5$

27. $x \leq 5$
$y \leq 4$

28. $x \geq -2$
$y \leq 4$

29. $y > x^2 - 4$
$y < -x^2 + 3$

30. $x^2 - y^2 \geq 9$
$\dfrac{x^2}{16} + \dfrac{y^2}{9} \leq 1$

31. $x^2 + y^2 \geq 4$
$x + y \leq 5$
$x \geq 0$
$y \geq 0$

32. $y^2 - x^2 \geq 4$
$-5 \leq y \leq 5$

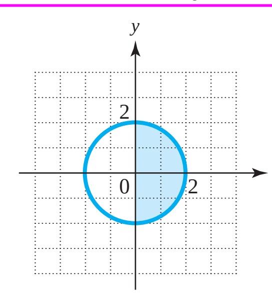
33. $y \leq -x^2$
$y \geq x - 3$
$y \leq -1$
$x < 1$

34. $y < x^2$
$y > -2$
$x + y < 3$
$3x - 2y > -6$

For each nonlinear inequality in Exercises 35–42, a restriction is placed on one or both variables. For example, the graph of

$$x^2 + y^2 \leq 4, \quad x \geq 0$$

would be as shown in the figure. Only the right half of the interior of the circle and its boundary is shaded, because of the restriction that x must be nonnegative. Graph each nonlinear inequality with the given restrictions.



EXERCISES

35. $x^2 + y^2 > 36, \quad x \geq 0$
36. $4x^2 + 25y^2 < 100, \quad y < 0$
37. $x < y^2 - 3, \quad x < 0$
38. $x^2 - y^2 < 4, \quad x < 0$
39. $4x^2 - y^2 > 16, \quad x < 0$
40. $x^2 + y^2 > 4, \quad y < 0$
41. $x^2 + 4y^2 \geq 1, \quad x \geq 0, y \geq 0$
42. $2x^2 - 32y^2 \leq 8, \quad x \leq 0, y \geq 0$

Use the shading feature of a graphing calculator to graph each system.

EXERCISES

43. $y \geq x - 3$
$y \leq -x + 4$

44. $y \geq -x^2 + 5$
$y \leq x^2 - 3$

45. $y < x^2 + 4x + 4$
$y > -3$

46. $y > (x - 4)^2 - 3$
$y < 5$

12.1 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 19/Videotape 20

Write out the first five terms of each sequence. See Example 1.

1. $a_n = n + 1$ **2.** $a_n = n - 4$ **3.** $a_n = \dfrac{n+3}{n}$ **4.** $a_n = \dfrac{n+2}{n+1}$

5. $a_n = 3^n$ **6.** $a_n = 1^{n-1}$ **7.** $a_n = \dfrac{1}{n^2}$ **8.** $a_n = \dfrac{n^2}{n+1}$

9. $a_n = (-1)^n$ **10.** $a_n = (-1)^{2n-1}$

Find the indicated term for each sequence. See Example 1.

11. $a_n = -9n + 2$; a_8 **12.** $a_n = 3n - 7$; a_{12}

13. $a_n = \dfrac{3n+7}{2n-5}$; a_{14} **14.** $a_n = \dfrac{5n-9}{3n+8}$; a_{16}

15. $a_n = (n+1)(2n+3)$; a_8 **16.** $a_n = (5n-2)(3n+1)$; a_{10}

Find a general term, a_n, for the given terms of each sequence. See Example 2.

17. $4, 8, 12, 16, \ldots$ **18.** $-10, -20, -30, -40, \ldots$

19. $\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \dfrac{1}{81}, \ldots$ **20.** $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$

Solve each applied problem by writing the first few terms of a sequence. See Example 3.

21. Anne borrows \$1000 and agrees to pay \$100 plus interest of 1% on the unpaid balance each month. Find the payments for the first 6 months and the remaining debt at the end of this period.

22. Larissa Perez is offered a new modeling job with a salary of $20{,}000 + 2500n$ dollars per year at the end of the nth year. Write a sequence showing her salary at the end of each of the first 5 yr. If she continues in this way, what will her salary be at the end of the tenth year?

23. Suppose that an automobile loses $\frac{1}{5}$ of its value each year; that is, at the end of any given year, the value is $\frac{4}{5}$ of the value at the beginning of that year. If a car costs \$20,000 new, what is its value at the end of 5 yr?

24. A certain car loses $\frac{1}{2}$ of its value each year. If this car cost \$40,000 new, what is its value at the end of 6 yr?

Write out each series and evaluate it. See Example 4.

25. $\sum_{i=1}^{5} (i + 3)$ **26.** $\sum_{i=1}^{6} (i + 9)$ **27.** $\sum_{i=1}^{3} (i^2 + 2)$ **28.** $\sum_{i=1}^{4} i(i + 3)$

29. $\sum_{i=1}^{6} (-1)^i$ **30.** $\sum_{i=1}^{5} (-1)^i \cdot i$ **31.** $\sum_{i=3}^{7} (i - 3)(i + 2)$ **32.** $\sum_{i=2}^{6} \dfrac{i^2 + 1}{2}$

Write out the terms of each series.

33. $\sum_{i=1}^{5} 2x \cdot i$ **34.** $\sum_{i=1}^{6} x^i$ **35.** $\sum_{i=1}^{5} i \cdot x^i$ **36.** $\sum_{i=2}^{6} \dfrac{x+i}{x-i}$

Write each series using summation notation. See Example 5.

37. $3 + 4 + 5 + 6 + 7$

38. $1 + 4 + 9 + 16$

39. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

40. $-1 + 2 - 3 + 4 - 5 + 6$

41. $1 + 4 + 9 + 16 + 25$

42. $1 + 16 + 81 + 256$

43. Suppose that f is a function with domain all real numbers, where $f(x) = 2x + 4$. Suppose that an infinite sequence is defined by $a_n = 2n + 4$. Discuss the similarities and differences between the function and the sequence. Give examples using each.

44. What is wrong with the following?
For the sequence defined by $a_n = 2n + 4$, find $a_{1/2}$.

45. Explain the basic difference between a sequence and a series.

46. Evaluate $\sum_{i=1}^{3} 5i$ and $5 \sum_{i=1}^{3} i$. Notice that the sums are the same. Explain how the distributive property plays a role in assuring us that the two sums are equal.

Find the arithmetic mean for each collection of numbers. See Example 6.

47. 8, 11, 14, 9, 3, 6, 8

48. 10, 12, 8, 19, 23

49. 5, 9, 8, 2, 4, 7, 3, 2

50. 2, 1, 4, 8, 3, 7

Solve each problem. See Example 6.

51. The number of mutual funds available to investors for each year during the period 1996 through 2000 is given in the table.

Year	*Number of Funds Available*
1996	6254
1997	6684
1998	7314
1999	7791
2000	8171

Source: Investment Company Institute.

Photo not available

To the nearest whole number, what was the average number of funds available per year during this period?

52. The total assets of mutual funds, in billions of dollars, for each year during the period 1992 through 1996 are shown in the table. To the nearest tenth (in billions of dollars), what were the average assets per year during this period?

Year	*Assets (in billions of dollars)*
1992	1646.3
1993	2075.4
1994	2161.5
1995	2820.4
1996	3539.2

Source: Investment Company Institute.

RELATING CONCEPTS (EXERCISES 53–60)

For Individual or Group Work

The following properties of series provide useful shortcuts for evaluating series.

If $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are two sequences, and c is a constant, then for every positive integer n,

(a) $\sum_{i=1}^{n} c = nc$

(b) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

(c) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(d) $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$.

Work Exercises 53–60 in order, *to see how these shortcuts can make work easier.*

53. Use property (c) to write $\sum_{i=1}^{6} (i^2 + 3i + 5)$ as the sum of three summations.

54. Use property (b) to rewrite the second summation from Exercise 53.

55. Use property (a) to rewrite the third summation from Exercise 53.

56. Rewrite $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$ using summation notation.

57. Rewrite $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ using summation notation.

58. Use the summations you wrote in Exercises 56 and 57 and the given properties to evaluate the three summations from Exercises 53–55. This gives the value of $\sum_{i=1}^{6} (i^2 + 3i + 5)$ without writing out all six terms.

59. Use the given properties and summations to evaluate $\sum_{i=1}^{12} (i^2 - i)$.

60. Use the given properties and summations to evaluate $\sum_{i=1}^{20} (2 + i - i^2)$.

12.2 Arithmetic Sequences

OBJECTIVES

1. Find the common difference of an arithmetic sequence.
2. Find the general term of an arithmetic sequence.
3. Use an arithmetic sequence in an application.
4. Find any specified term or the number of terms of an arithmetic sequence. (continued)

OBJECTIVE 1 Find the common difference of an arithmetic sequence. In this section we introduce a special type of sequence that has many applications.

Arithmetic Sequence

A sequence in which each term after the first differs from the preceding term by a constant amount is called an **arithmetic sequence** or **arithmetic progression.**

For example, the sequence

$$6, 11, 16, 21, 26, \ldots$$

is an arithmetic sequence, since the difference between any two adjacent terms is always 5. The number 5 is called the **common difference** of the arithmetic sequence.

OBJECTIVES (continued)

5 Find the sum of a specified number of terms of an arithmetic sequence.

The common difference, d, is found by subtracting any pair of terms a_n and a_{n+1}. That is,

$$d = a_{n+1} - a_n.$$

EXAMPLE 1 Finding the Common Difference

Find d for the arithmetic sequence.

$$-11, -4, 3, 10, 17, 24, \ldots$$

Since the sequence is arithmetic, d is the difference between any two adjacent terms. Choosing the terms 10 and 17 gives

$$\begin{aligned} d &= 17 - 10 \\ &= 7. \end{aligned}$$

The terms -11 and -4 would give $d = -4 - (-11) = 7$, the same result.

Now Try Exercise 7.

EXAMPLE 2 Writing the Terms of a Sequence from the First Term and Common Difference

Write the first five terms of the arithmetic sequence with first term 3 and common difference -2.

The second term is found by adding -2 to the first term 3, getting 1. For the next term, add -2 to 1, and so on. The first five terms are

$$3, 1, -1, -3, -5.$$

Now Try Exercise 9.

OBJECTIVE 2 Find the general term of an arithmetic sequence. Generalizing from Example 2, if we know the first term a_1 and the common difference d of an arithmetic sequence, then the sequence is completely defined as

$$a_1, \quad a_2 = a_1 + d, \quad a_3 = a_1 + 2d, \quad a_4 = a_1 + 3d, \ldots.$$

Writing the terms of the sequence in this way suggests the following rule.

General Term of an Arithmetic Sequence

The general term of an arithmetic sequence with first term a_1 and common difference d is

$$\boldsymbol{a_n = a_1 + (n - 1)d.}$$

Since $a_n = a_1 + (n - 1)d = dn + (a_1 - d)$ is a linear function in n, any linear expression of the form $kn + c$, where k and c are real numbers, defines an arithmetic sequence.

EXAMPLE 3 Finding the General Term of an Arithmetic Sequence

Find the general term for the arithmetic sequence.

$$-9, -6, -3, 0, 3, 6, \ldots$$

Then use the general term to find a_{20}.

Here the first term is $a_1 = -9$. To find d, subtract any two adjacent terms. For example,

$$d = -3 - (-6) = 3.$$

Now find a_n.

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Formula for } a_n \\ &= -9 + (n-1)(3) && \text{Let } a_1 = -9,\ d = 3. \\ &= -9 + 3n - 3 && \text{Distributive property} \\ a_n &= 3n - 12 && \text{Combine terms.} \end{aligned}$$

Thus, the general term is $a_n = 3n - 12$. To find a_{20}, let $n = 20$.

$$a_{20} = 3(20) - 12 = 60 - 12 = 48$$

Now Try Exercise 13.

OBJECTIVE 3 Use an arithmetic sequence in an application.

EXAMPLE 4 Applying an Arithmetic Sequence

Howie Sorkin's uncle decides to start a fund for Howie's education. He makes an initial contribution of \$3000 and each month deposits an additional \$500. Thus, after one month there will be \$3000 + \$500 = \$3500. How much will there be after 24 months? (Disregard any interest.)

The contributions can be described using an arithmetic sequence. After n months, the fund will contain

$$a_n = 3000 + 500n \text{ dollars.}$$

To find the amount in the fund after 24 months, find a_{24}.

$$\begin{aligned} a_{24} &= 3000 + 500(24) && \text{Let } n = 24. \\ &= 3000 + 12{,}000 && \text{Multiply.} \\ &= 15{,}000 && \text{Add.} \end{aligned}$$

The account will contain \$15,000 (disregarding interest) after 24 months.

Now Try Exercise 47.

OBJECTIVE 4 Find any specified term or the number of terms of an arithmetic sequence.

The formula for the general term has four variables: a_n, a_1, n, and d. If we know any three of these, the formula can be used to find the value of the fourth variable. The next example shows how to find a particular term.

VIDEO

EXAMPLE 5 Finding Specified Terms

Find the indicated term for each arithmetic sequence.

(a) $a_1 = -6,\ d = 12;\ a_{15}$

We use the formula $a_n = a_1 + (n - 1)d$. Since we want $a_n = a_{15}$, $n = 15$.

$$\begin{aligned} a_{15} &= a_1 + (15 - 1)d && \text{Let } n = 15. \\ &= -6 + 14(12) && \text{Let } a_1 = -6,\ d = 12. \\ &= 162 \end{aligned}$$

(b) $a_5 = 2$ and $a_{11} = -10;\ a_{17}$

Any term can be found if a_1 and d are known. Use the formula for a_n with the two given terms.

$$\begin{aligned} a_5 &= a_1 + (5 - 1)d \\ a_5 &= a_1 + 4d \\ 2 &= a_1 + 4d && a_5 = 2 \end{aligned}$$

$$\begin{aligned} a_{11} &= a_1 + (11 - 1)d \\ a_{11} &= a_1 + 10d \\ -10 &= a_1 + 10d && a_{11} = -10 \end{aligned}$$

This gives a system of two equations with two variables, a_1 and d. Find d by adding -1 times one equation to the other to eliminate a_1.

$$\begin{aligned} -10 &= a_1 + 10d \\ -2 &= -a_1 - 4d && \text{Multiply } 2 = a_1 + 4d \text{ by } -1. \\ \hline -12 &= 6d && \text{Add.} \\ -2 &= d && \text{Divide by 6.} \end{aligned}$$

Now find a_1 by substituting -2 for d into either equation.

$$\begin{aligned} -10 &= a_1 + 10(-2) && \text{Let } d = -2. \\ -10 &= a_1 - 20 \\ 10 &= a_1 \end{aligned}$$

Use the formula for a_n to find a_{17}.

$$\begin{aligned} a_{17} &= a_1 + (17 - 1)d && \text{Let } n = 17. \\ &= a_1 + 16d \\ &= 10 + 16(-2) && \text{Let } a_1 = 10,\ d = -2. \\ &= -22 \end{aligned}$$

Now Try Exercises 19 and 23.

Sometimes we need to find out how many terms are in a sequence, as shown in the following example.

EXAMPLE 6 Finding the Number of Terms in a Sequence

Find the number of terms in the arithmetic sequence.

$$-8, -2, 4, 10, \ldots, 52$$

Let n represent the number of terms in the sequence. Since $a_n = 52$, $a_1 = -8$, and $d = -2 - (-8) = 6$, use the formula $a_n = a_1 + (n - 1)d$ to find n. Substituting the known values into the formula gives

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 52 &= -8 + (n-1)6 && \text{Let } a_n = 52,\ a_1 = -8,\ d = 6. \\ 52 &= -8 + 6n - 6 && \text{Distributive property} \\ 66 &= 6n && \text{Combine terms.} \\ n &= 11. && \text{Divide by 6.} \end{aligned}$$

The sequence has 11 terms.

Now Try Exercise 25.

OBJECTIVE 5 Find the sum of a specified number of terms of an arithmetic sequence.

To find a formula for the sum, S_n, of the first n terms of an arithmetic sequence, we can write out the terms as

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d].$$

This same sum can be written in reverse as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n-1)d].$$

Now add the corresponding terms of these two expressions for S_n to get

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n).$$

The right-hand side of this expression contains n terms, each equal to $a_1 + a_n$, so

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n).$$

EXAMPLE 7 Finding the Sum of the First *n* Terms

Find the sum of the first five terms of the arithmetic sequence in which $a_n = 2n - 5$.

We can use the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to find the sum of the first five terms. Here $n = 5$, $a_1 = 2(1) - 5 = -3$, and $a_5 = 2(5) - 5 = 5$. From the formula,

$$S_5 = \frac{5}{2}(-3 + 5) = \frac{5}{2}(2) = 5.$$

Now Try Exercise 39.

It is sometimes useful to express the sum of an arithmetic sequence, S_n, in terms of a_1 and d, the quantities that define the sequence. We can do this as follows. Since

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{and} \qquad a_n = a_1 + (n-1)d,$$

by substituting the expression for a_n into the expression for S_n we obtain

$$S_n = \frac{n}{2}(a_1 + [a_1 + (n-1)d])$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d].$$

The summary on the next page gives both of the alternative forms that may be used to find the sum of the first n terms of an arithmetic sequence.

Sum of the First *n* Terms of an Arithmetic Sequence

The sum of the first n terms of the arithmetic sequence with first term a_1, nth term a_n, and common difference d is

$$\mathbf{S_n = \frac{n}{2}(a_1 + a_n)} \quad \text{or} \quad \mathbf{S_n = \frac{n}{2}[2a_1 + (n - 1)d].}$$

EXAMPLE 8 Finding the Sum of the First *n* Terms

Find the sum of the first eight terms of the arithmetic sequence having first term 3 and common difference -2.

Since the known values, $a_1 = 3$, $d = -2$, and $n = 8$, appear in the second formula for S_n, we use it.

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

$$S_8 = \frac{8}{2}[2(3) + (8 - 1)(-2)] \qquad \text{Let } a_1 = 3,\ d = -2,\ n = 8.$$

$$= 4[6 - 14]$$

$$= -32$$

Now Try Exercise 35.

As mentioned earlier, linear expressions of the form $kn + c$, where k and c are real numbers, define an arithmetic sequence. For example, the sequences defined by $a_n = 2n + 5$ and $a_n = n - 3$ are arithmetic sequences. For this reason,

$$\sum_{i=1}^{n} (ki + c)$$

represents the sum of the first n terms of an arithmetic sequence having first term $a_1 = k(1) + c = k + c$ and general term $a_n = k(n) + c = kn + c$. We can find this sum with the first formula for S_n, as shown in the next example.

EXAMPLE 9 Using S_n to Evaluate a Summation

Find $\sum_{i=1}^{12} (2i - 1)$.

This is the sum of the first 12 terms of the arithmetic sequence having $a_n = 2n - 1$. This sum, S_{12}, is found with the first formula for S_n,

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Here $n = 12$, $a_1 = 2(1) - 1 = 1$, and $a_{12} = 2(12) - 1 = 23$. Substitute these values into the formula to find

$$S_{12} = \frac{12}{2}(1 + 23) = 6(24) = 144.$$

Now Try Exercise 41.

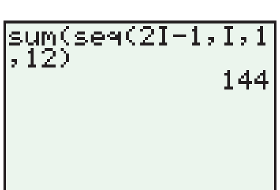


FIGURE 3

Figure 3 shows how a graphing calculator supports the result of Example 9.

12.2 EXERCISES

For Extra Help

- Student's Solutions Manual
- MyMathLab
- InterAct Math Tutorial Software
- AW Math Tutor Center
- MathXL
- Digital Video Tutor CD 19/Videotape 20

1. Using several examples, explain the meaning of *arithmetic sequence.*

2. Can any two terms of an arithmetic sequence be used to find the common difference? Explain.

If the given sequence is arithmetic, find the common difference, d. If the sequence is not arithmetic, say so. See Example 1.

3. 1, 2, 3, 4, 5, . . .
4. 2, 5, 8, 11, . . .
5. 2, -4, 6, -8, 10, -12, . . .
6. -6, -10, -14, -18, . . .
7. -10, -5, 0, 5, 10, . . .
8. 1, 2, 4, 7, 11, 16, . . .

Write the first five terms of each arithmetic sequence. See Example 2.

9. $a_1 = 5, d = 4$
10. $a_1 = 6, d = 7$
11. $a_1 = -2, d = -4$
12. $a_1 = -3, d = -5$

Use the formula for a_n to find the general term for each arithmetic sequence. See Example 3.

13. $a_1 = 2, d = 5$
14. $a_1 = 5, d = -3$
15. $3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4}, \ldots$
16. 4, 14, 24, . . .
17. -3, 0, 3, . . .
18. -10, -5, 0, 5, 10, . . .

Find the indicated term for each arithmetic sequence. See Examples 2 and 5.

19. $a_1 = 4, d = 3; a_{25}$
20. $a_1 = 1, d = -\frac{1}{2}; a_{12}$
21. $2, 4, 6, \ldots; a_{24}$
22. $1, 5, 9, \ldots; a_{50}$
23. $a_{12} = -45, a_{10} = -37; a_1$
24. $a_{10} = -2, a_{15} = -8; a_3$

Find the number of terms in each arithmetic sequence. See Example 6.

25. 3, 5, 7, . . . , 33
26. $2, \frac{3}{2}, 1, \frac{1}{2}, \ldots, -5$
27. $\frac{3}{4}, 3, \frac{21}{4}, \ldots, 12$
28. 4, 1, -2, . . . , -32

29. In the formula for S_n, what does n represent?

30. Explain when you would use each of the two formulas for S_n.

RELATING CONCEPTS (EXERCISES 31–34)

For Individual or Group Work

Exercises 31–34 show how to find the sum $1 + 2 + 3 + \cdots + 99 + 100$ *in an ingenious way.* ***Work them in order.***

31. Consider the following:

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + 99 + 100 \\ \underline{S} &\underline{= 100 + 99 + 98 + \cdots + 2 + 1.} \end{aligned}$$

Add the left sides of this equation. The result is ______. Add the columns on the right side. The sum ______ appears ______ times, so by multiplication, the sum of the right sides of the equations is ______. (continued)

32. Form an equation by setting the sum of the left sides equal to the sum of the right sides.

33. Solve the equation from Exercise 32 to find that the desired sum, S, is ______.

34. Find the sum $S = 1 + 2 + 3 + \cdots + 199 + 200$ using the procedure described in Exercises 31–33.

Find S_6 for each arithmetic sequence. See Examples 7 and 8.

35. $a_1 = 6, d = 3$

36. $a_1 = 5, d = 4$

37. $a_1 = 7, d = -3$

38. $a_1 = -5, d = -4$

39. $a_n = 4 + 3n$

40. $a_n = 9 + 5n$

Use a formula for S_n to evaluate each series. See Example 9.

41. $\sum_{i=1}^{10} (8i - 5)$

42. $\sum_{i=1}^{17} (i - 1)$

43. $\sum_{i=1}^{20} (2i - 5)$

44. $\sum_{i=1}^{10} \left(\frac{1}{2}i - 1\right)$

45. $\sum_{i=1}^{250} i$

46. $\sum_{i=1}^{2000} i$

Solve each applied problem. (Hint: *Determine whether you need to find a specific term of a sequence or the sum of the terms of a sequence immediately after reading the problem.) See Example 4.*

47. Nancy Bondy's aunt has promised to deposit $1 in her account on the first day of her birthday month, $2 on the second day, $3 on the third day, and so on for 30 days. How much will this amount to over the entire month?

48. Repeat Exercise 47, but assume that the deposits are $2, $4, $6, and so on, and that the month is February of a leap year.

49. Suppose that Randy Morgan is offered a job at $1600 per month with a guaranteed increase of $50 every 6 months for 5 yr. What will his salary be at the end of this period of time?

50. Repeat Exercise 49, but assume that the starting salary is $2000 per month, and the guaranteed increase is $100 every 4 months for 3 yr.

51. A seating section in a theater-in-the-round has 20 seats in the first row, 22 in the second row, 24 in the third row, and so on for 25 rows. How many seats are there in the last row? How many seats are there in the section?

52. José Valdevielso has started on a fitness program. He plans to jog 10 min per day for the first week, and then add 10 min per day each week until he is jogging an hour each day. In which week will this occur? What is the total number of minutes he will run during the first 4 weeks?

53. A child builds with blocks, placing 35 blocks in the first row, 31 in the second row, 27 in the third row, and so on. Continuing this pattern, can she end with a row containing exactly 1 block? If not, how many blocks will the last row contain? How many rows can she build this way?

54. A stack of firewood has 28 pieces on the bottom, 24 on top of those, then 20, and so on. If there are 108 pieces of wood, how many rows are there? (*Hint:* $n \leq 7$.)

12.3 Geometric Sequences

OBJECTIVES

1. Find the common ratio of a geometric sequence.
2. Find the general term of a geometric sequence.
3. Find any specified term of a geometric sequence.
4. Find the sum of a specified number of terms of a geometric sequence.
5. Apply the formula for the future value of an ordinary annuity.
6. Find the sum of an infinite number of terms of certain geometric sequences.

In an arithmetic sequence, each term after the first is found by *adding* a fixed number to the previous term. A *geometric sequence* is defined as follows.

Geometric Sequence

A **geometric sequence** or **geometric progression** is a sequence in which each term after the first is a constant multiple of the preceding term.

OBJECTIVE 1 Find the common ratio of a geometric sequence. We find the constant multiplier, called the **common ratio,** by dividing any term after the first by the preceding term. That is, the common ratio is

$$r = \frac{a_{n+1}}{a_n}.$$

For example,

$$2, 6, 18, 54, 162, \ldots$$

is a geometric sequence in which the first term, a_1, is 2 and the common ratio is

$$r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3.$$

EXAMPLE 1 Finding the Common Ratio

Find r for the geometric sequence.

$$15, \frac{15}{2}, \frac{15}{4}, \frac{15}{8}, \ldots$$

To find r, choose any two successive terms and divide the second one by the first. Choosing the second and third terms of the sequence gives

$$r = \frac{a_3}{a_2} = \frac{15}{4} \div \frac{15}{2} = \frac{1}{2}.$$

Any other two successive terms could have been used to find r. Additional terms of the sequence can be found by multiplying each successive term by $\frac{1}{2}$.

Now Try Exercise 3.

OBJECTIVE 2 Find the general term of a geometric sequence. The general term a_n of a geometric sequence $a_1, a_2, a_3, \ldots$ is expressed in terms of a_1 and r by writing the first few terms as

$$a_1, \quad a_2 = a_1 r, \quad a_3 = a_1 r^2, \quad a_4 = a_1 r^3, \ldots,$$

which suggests the rule on the next page.

General Term of a Geometric Sequence

The general term of the geometric sequence with first term a_1 and common ratio r is

$$a_n = a_1 r^{n-1}.$$

CAUTION Be careful to use the correct order of operations when finding $a_1 r^{n-1}$. The value of r^{n-1} must be found first. Then multiply the result by a_1.

EXAMPLE 2 Finding the General Term

Find the general term of the sequence in Example 1.

The first term is $a_1 = 15$ and the common ratio is $r = \frac{1}{2}$. Substituting into the formula for the general term gives

$$a_n = a_1 r^{n-1} = 15\left(\frac{1}{2}\right)^{n-1},$$

the required general term. Notice that it is not possible to simplify further, because the exponent must be applied before the multiplication can be done.

Now Try Exercise 11.

OBJECTIVE 3 Find any specified term of a geometric sequence. We can use the formula for the general term to find any particular term.

EXAMPLE 3 Finding Specified Terms

Find the indicated term for each geometric sequence.

(a) $a_1 = 4$, $r = -3$; a_6

Let $n = 6$. From the general term $a_n = a_1 r^{n-1}$,

$$\begin{aligned} a_6 &= a_1 \cdot r^{6-1} && \text{Let } n = 6. \\ &= 4 \cdot (-3)^5 && \text{Let } a_1 = 4, r = -3. \\ &= -972. && \text{Evaluate } (-3)^5 \text{ first.} \end{aligned}$$

(b) $\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots; a_7$

Here, $r = \frac{1}{2}$, $a_1 = \frac{3}{4}$, and $n = 7$.

$$a_7 = \frac{3}{4} \cdot \left(\frac{1}{2}\right)^6 = \frac{3}{4} \cdot \frac{1}{64} = \frac{3}{256}$$

Now Try Exercises 17 and 19.

EXAMPLE 4 Writing the Terms of a Sequence

Write the first five terms of the geometric sequence whose first term is 5 and whose common ratio is $\frac{1}{2}$.

Using the formula $a_n = a_1 r^{n-1}$,

$$a_1 = 5, \quad a_2 = 5\left(\frac{1}{2}\right) = \frac{5}{2}, \quad a_3 = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4},$$

$$a_4 = 5\left(\frac{1}{2}\right)^3 = \frac{5}{8}, \quad a_5 = 5\left(\frac{1}{2}\right)^4 = \frac{5}{16}.$$

Now Try Exercise 23.

OBJECTIVE 4 Find the sum of a specified number of terms of a geometric sequence. It is convenient to have a formula for the sum of the first n terms of a geometric sequence, S_n. We can develop a formula by first writing out S_n.

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}$$

Next, we multiply both sides by r.

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^n$$

We subtract the first result from the second.

$$\begin{aligned} rS_n - S_n = {} & (a_1 r - a_1) + (a_1 r^2 - a_1 r) + (a_1 r^3 - a_1 r^2) \\ & + (a_1 r^4 - a_1 r^3) + \cdots + (a_1 r^n - a_1 r^{n-1}) \end{aligned}$$

Using the commutative, associative, and distributive properties, we obtain

$$\begin{aligned} rS_n - S_n &= (a_1 r - a_1 r) + (a_1 r^2 - a_1 r^2) \\ &\quad + (a_1 r^3 - a_1 r^3) + \cdots + (a_1 r^n - a_1) \\ S_n(r - 1) &= a_1 r^n - a_1. \end{aligned}$$

If $r \neq 1$, then

$$S_n = \frac{a_1 r^n - a_1}{r - 1} = \frac{a_1(r^n - 1)}{r - 1}. \qquad \text{Divide by } r - 1.$$

A summary of this discussion follows.

Sum of the First n Terms of a Geometric Sequence

The sum of the first n terms of the geometric sequence with first term a_1 and common ratio r is

$$\mathbf{S_n = \frac{a_1(r^n - 1)}{r - 1} \quad (r \neq 1).}$$

If $r = 1$, then $S_n = a_1 + a_1 + a_1 + \cdots + a_1 = na_1$.

Multiplying the formula for S_n by $\frac{-1}{-1}$ gives an alternative form that is sometimes preferable.

$$S_n = \frac{a_1(r^n - 1)}{r - 1} \cdot \frac{-1}{-1} = \frac{a_1(1 - r^n)}{1 - r}$$

EXAMPLE 5 Finding the Sum of the First *n* Terms

Find the sum of the first six terms of the geometric sequence with first term -2 and common ratio 3.

Substitute $n = 6$, $a_1 = -2$, and $r = 3$ into the formula for S_n.

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

$$S_6 = \frac{-2(3^6 - 1)}{3 - 1} \qquad \text{Let } n = 6,\ a_1 = -2,\ r = 3.$$

$$= \frac{-2(729 - 1)}{2} \qquad \text{Evaluate } 3^6.$$

$$= -728$$

Now Try Exercise 27.

A series of the form

$$\sum_{i=1}^{n} a \cdot b^i$$

represents the sum of the first n terms of a geometric sequence having first term $a_1 = a \cdot b^1 = ab$ and common ratio b. The next example illustrates this form.

EXAMPLE 6 Using the Formula for S_n to Find a Summation

Find $\sum_{i=1}^{4} 3 \cdot 2^i$.

Since the series is in the form

$$\sum_{i=1}^{n} a \cdot b^i,$$

it represents the sum of the first n terms of the geometric sequence with $a_1 = a \cdot b^1$ and $r = b$. The sum is found by using the formula

$$S_n = \frac{a_1(r^n - 1)}{r - 1}.$$

Here $n = 4$. Also, $a_1 = 6$ and $r = 2$. Now substitute into the formula for S_n.

$$S_4 = \frac{6(2^4 - 1)}{2 - 1} \qquad \text{Let } n = 4,\ a_1 = 6,\ r = 2.$$

$$= \frac{6(16 - 1)}{1} \qquad \text{Evaluate } 2^4.$$

$$= 90$$

Now Try Exercise 31.

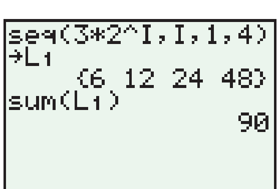


FIGURE 4

Figure 4 shows how a graphing calculator can store the terms in a list, and then find the sum of these terms. This supports the result of Example 6.

OBJECTIVE 5 Apply the formula for the future value of an ordinary annuity. A sequence of equal payments made at equal periods of time is called an **annuity.** If the payments are made at the end of the time period, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity.** The time between payments is the **payment period,** and the time from the beginning of the first payment period to the end of the last period is called the **term of the annuity.** The **future value of the annuity,** the final sum on deposit, is defined as the sum of the compound amounts of all the payments, compounded to the end of the term.

For example, suppose \$1500 is deposited at the end of the year for the next 6 yr in an account paying 8% per yr compounded annually. To find the future value of this annuity, look separately at each of the \$1500 payments. The first of these payments will produce a compound amount of

$$1500(1 + .08)^5 = 1500(1.08)^5.$$

Use 5 as the exponent instead of 6 since the money is deposited at the *end* of the first year and earns interest for only 5 yr. The second payment of \$1500 will produce a compound amount of $1500(1.08)^4$. Continuing in this way and finding the sum of all the terms gives

$$1500(1.08)^5 + 1500(1.08)^4 + 1500(1.08)^3 + 1500(1.08)^2 + 1500(1.08)^1 + 1500.$$

(The last payment earns no interest at all.) Reading in reverse order, we see that this expression is the sum of the first six terms of a geometric sequence with $a_1 = 1500$, $r = 1.08$, and $n = 6$. Therefore, the sum is

$$\frac{a_1(r^n - 1)}{r - 1} = \frac{1500[(1.08)^6 - 1]}{1.08 - 1} = 11{,}003.89$$

or \$11,003.89.

We state the following formula without proof.

Future Value of an Ordinary Annuity

$$S = R\left[\frac{(1 + i)^n - 1}{i}\right]$$

where

S is future value,

R is the payment at the end of each period,

i is the interest rate per period, and

n is the number of periods.

EXAMPLE 7 Applying the Formula for the Future Value of an Annuity

(a) Rocky Rhodes is an athlete who feels that his playing career will last 7 yr. To prepare for his future, he deposits \$22,000 at the end of each year for 7 yr in an account paying 6% compounded annually. How much will he have on deposit after 7 yr?

His payments form an ordinary annuity with $R = 22{,}000$, $n = 7$, and $i = .06$. The future value of this annuity (using the formula) is

$$S = 22{,}000\left[\frac{(1.06)^7 - 1}{.06}\right] = 184{,}664.43, \qquad \text{Use a calculator.}$$

or \$184,664.43.

(b) Experts say that the baby boom generation (born between 1946 and 1960) cannot count on a company pension or Social Security to provide a comfortable retirement, as their parents did. It is recommended that they start to save early and regularly. Judy Zahrndt, a baby boomer, has decided to deposit \$200 at the end of each month in an account that pays interest of 7.2% compounded monthly for retirement in 20 yr. How much will be in the account at that time?

Because the interest is compounded monthly, $i = \frac{.072}{12}$. Also, $R = 200$ and $n = 12(20)$. The future value is

$$S = 200\left[\frac{\left(1 + \frac{.072}{12}\right)^{12(20)} - 1}{\frac{.072}{12}}\right] = 106{,}752.47,$$

or \$106,752.47.

Now Try Exercise 35.

OBJECTIVE 6 Find the sum of an infinite number of terms of certain geometric sequences. Consider an infinite geometric sequence such as

$$\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \ldots.$$

Can the sum of the terms of such a sequence be found somehow? The sum of the first two terms is

$$S_2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} = .5.$$

In a similar manner,

$$S_3 = S_2 + \frac{1}{12} = \frac{1}{2} + \frac{1}{12} = \frac{7}{12} \approx .583, \quad S_4 = S_3 + \frac{1}{24} = \frac{7}{12} + \frac{1}{24} = \frac{15}{24} = .625,$$

$$S_5 = \frac{31}{48} \approx .64583, \quad S_6 = \frac{21}{32} = .65625, \quad S_7 = \frac{127}{192} \approx .6614583.$$

Each term of the geometric sequence is smaller than the preceding one, so each additional term is contributing less and less to the sum. In decimal form (to the nearest thousandth) the first seven terms and the tenth term are given in the table.

Term	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_{10}
Value	.333	.167	.083	.042	.021	.010	.005	.001

As the table suggests, the value of a term gets closer and closer to 0 as the number of the term increases. To express this idea, we say that as n increases without bound (written $n \to \infty$), the limit of the term a_n is 0, written

$$\lim_{n \to \infty} a_n = 0.$$

A number that can be defined as the sum of an infinite number of terms of a geometric sequence can be found by starting with the expression for the sum of a finite number of terms:

$$S_n = \frac{a_1(r^n - 1)}{r - 1}.$$

If $|r| < 1$, then as n increases without bound the value of r^n gets closer and closer to 0. For example, in the infinite sequence just discussed, $r = \frac{1}{2} = .5$. The following table shows how $r^n = (.5)^n$, given to the nearest thousandth, gets smaller as n increases.

n	1	2	3	4	5	6	7	10
r^n	.5	.25	.125	.063	.031	.016	.008	.001

As r^n approaches 0, $r^n - 1$ approaches $0 - 1 = -1$, and S_n approaches the quotient $\frac{-a_1}{r - 1}$. Thus,

$$\lim_{r^n \to 0} S_n = \lim_{r^n \to 0} \frac{a_1(r^n - 1)}{r - 1} = \frac{a_1(0 - 1)}{r - 1} = \frac{-a_1}{r - 1} = \frac{a_1}{1 - r}.$$

This limit is defined to be the sum of the infinite geometric sequence:

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots = \frac{a_1}{1 - r}, \quad \text{if } |r| < 1.$$

What happens if $|r| > 1$? For example, suppose the sequence is

$$6, 12, 24, \ldots, 3(2)^n, \ldots.$$

In this kind of sequence, as n increases, the value of r^n also increases and so does the sum S_n. Since each new term adds a larger and larger amount to the sum, there is no limit to the value of S_n, and the sum S_n does not exist. A similar situation exists if $r = 1$.

In summary, the sum of the terms of an infinite geometric sequence is defined as follows.

AUDIO

Sum of the Terms of an Infinite Geometric Sequence

The sum S of the terms of an infinite geometric sequence with first term a_1 and common ratio r, where $|r| < 1$, is

$$\boldsymbol{S = \frac{a_1}{1 - r}}.$$

If $|r| \geq 1$, then the sum does not exist.

EXAMPLE 8 Finding the Sum of the Terms of an Infinite Geometric Sequence

Find the sum of the terms of the infinite geometric sequence with $a_1 = 3$ and $r = -\frac{1}{3}$.

From the preceding rule, the sum is

$$S = \frac{a_1}{1 - r} = \frac{3}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{\frac{4}{3}} = \frac{9}{4}.$$

Now Try Exercise 39.

Using summation notation, the sum of an infinite geometric sequence is written as

$$\sum_{i=1}^{\infty} a_i.$$

For instance, the sum in Example 8 would be written

$$\sum_{i=1}^{\infty} 3\left(-\frac{1}{3}\right)^{i-1}.$$

EXAMPLE 9 Finding the Sum of the Terms of an Infinite Geometric Series

Find $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i}$.

This is the infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. Since $|r| < 1$, we find the sum as follows.

$$S = \frac{a_1}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Now Try Exercise 43.

12.3 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 19/Videotape 20

1. Using several examples, explain the meaning of *geometric sequence.*

2. Explain why the sequence 5, 5, 5, 5, . . . can be considered either arithmetic or geometric.

If the given sequence is geometric, find the common ratio, r. If the sequence is not geometric, say so. See Example 1.

EXERCISES

3. 4, 8, 16, 32, . . .

4. 5, 15, 45, 135, . . .

5. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$

6. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

7. $1, -3, 9, -27, 81, \ldots$

8. $1, -3, 7, -11, \ldots$

9. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

10. $\frac{2}{3}, \frac{2}{15}, \frac{2}{75}, \frac{2}{375}, \ldots$

Find a general term for each geometric sequence. See Example 2.

11. 5, 10, . . .

12. $-2, -6, \ldots$

13. $\frac{1}{9}, \frac{1}{3}, \ldots$

14. $-3, \frac{3}{2}, \ldots$

15. $10, -2, \ldots$

16. $-4, 8, \ldots$

Find the indicated term for each geometric sequence. See Example 3.

17. $a_1 = 2, r = 5; a_{10}$

18. $a_1 = -1, r = 3; a_{15}$

19. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots; a_{12}$

20. $\frac{2}{3}, -\frac{1}{3}, \frac{1}{6}, \ldots; a_{18}$

21. $a_3 = \frac{1}{2}, a_7 = \frac{1}{32}; a_{25}$

22. $a_5 = 48, a_8 = -384; a_{10}$

Write the first five terms of each geometric sequence. See Example 4.

23. $a_1 = 2, r = 3$

24. $a_1 = 4, r = 2$

25. $a_1 = 5, r = -\frac{1}{5}$

26. $a_1 = 6, r = -\frac{1}{3}$

Use the formula for S_n to find the sum for each geometric sequence. See Examples 5 and 6. In Exercises 29–34, give the answer to the nearest thousandth.

27. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$

28. $\frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{64}{3}, \frac{128}{3}$

29. $-\frac{4}{3}, -\frac{4}{9}, -\frac{4}{27}, -\frac{4}{81}, -\frac{4}{243}, -\frac{4}{729}$

30. $\frac{5}{16}, -\frac{5}{32}, \frac{5}{64}, -\frac{5}{128}, \frac{5}{256}$

31. $\sum_{i=1}^{7} 4\left(\frac{2}{5}\right)^i$

32. $\sum_{i=1}^{8} 5\left(\frac{2}{3}\right)^i$

33. $\sum_{i=1}^{10} (-2)\left(\frac{3}{5}\right)^i$

34. $\sum_{i=1}^{6} (-2)\left(-\frac{1}{2}\right)^i$

Solve each problem involving an ordinary annuity. See Example 7.

35. A father opened a savings account for his daughter on the day she was born, depositing \$1000. Each year on her birthday he deposits another \$1000, making the last deposit on her twenty-first birthday. If the account pays 9.5% interest compounded annually, how much is in the account at the end of the day on the daughter's twenty-first birthday?

Photo not available

36. A 45-year-old man puts \$1000 in a retirement account at the end of each quarter $\left(\frac{1}{4}\text{ of a year}\right)$ until he reaches age 60. If the account pays 11% annual interest compounded quarterly, how much will be in the account at that time?

37. At the end of each quarter a 50-year-old woman puts \$1200 in a retirement account that pays 7% interest compounded quarterly. When she reaches age 60, she withdraws the entire amount and places it in a mutual fund that pays 9% interest compounded monthly. From then on she deposits \$300 in the mutual fund at the end of each month. How much is in the account when she reaches age 65?

38. John Bray deposits \$10,000 at the beginning of each year for 12 yr in an account paying 5% compounded annually. He then puts the total amount on deposit in another account paying 6% compounded semiannually for another 9 yr. Find the final amount on deposit after the entire 21-yr period.

Find the sum, if it exists, of the terms of each infinite geometric sequence. See Examples 8 and 9.

39. $a_1 = 6,\ r = \frac{1}{3}$

40. $a_1 = 10,\ r = \frac{1}{5}$

41. $a_1 = 1000,\ r = -\frac{1}{10}$

42. $a_1 = 8500,\ r = \frac{3}{5}$

43. $\sum_{i=1}^{\infty} \frac{9}{8}\left(-\frac{2}{3}\right)^i$

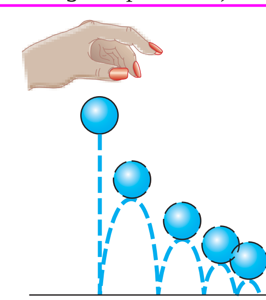
44. $\sum_{i=1}^{\infty} \frac{3}{5}\left(\frac{5}{6}\right)^i$

45. $\sum_{i=1}^{\infty} \frac{12}{5}\left(\frac{5}{4}\right)^i$

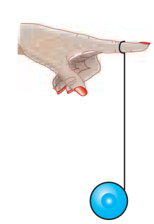
46. $\sum_{i=1}^{\infty} \left(-\frac{16}{3}\right)\left(-\frac{9}{8}\right)^i$

Solve each application. (Hint: *Determine whether you need to find a specific term of a sequence or the sum of the terms of a sequence immediately after reading the problem.)*

47. A certain ball when dropped from a height rebounds $\frac{3}{5}$ of the original height. How high will the ball rebound after the fourth bounce if it was dropped from a height of 10 ft?



48. A fully wound yo-yo has a string 40 in. long. It is allowed to drop and on its first rebound, it returns to a height 15 in. lower than its original height. Assuming this “rebound ratio” remains constant until the yo-yo comes to rest, how far does it travel on its third trip up the string?



49. A particular substance decays in such a way that it loses half its weight each day. In how many days will 256 g of the substance be reduced to 32 g? How much of the substance is left after 10 days?

50. A tracer dye is injected into a system with an input and an excretion. After one hour $\frac{2}{3}$ of the dye is left. At the end of the second hour $\frac{2}{3}$ of the remaining dye is left, and so on. If one unit of the dye is injected, how much is left after 6 hr?

51. In a certain community the consumption of electricity has increased about 6% per yr.

(a) If a community uses 1.1 billion units of electricity now, how much will it use 5 yr from now?

(b) Find the number of years it will take for the consumption to double.

52. Suppose the community in Exercise 51 reduces its increase in consumption to 2% per yr.

(a) How much will it use 5 yr from now?

(b) Find the number of years it will take for the consumption to double.

53. A machine depreciates by $\frac{1}{4}$ of its value each year. If it cost \$50,000 new, what is its value after 8 yr?

54. Refer to Exercise 48. Theoretically, how far does the yo-yo travel before coming to rest?

RELATING CONCEPTS (EXERCISES 55–60)

For Individual or Group Work

In Chapter 1 we learned that any repeating decimal is a rational number; that is, it can be expressed as a quotient of integers. Thus, the repeating decimal

$$.99999\ldots,$$

an endless string of 9s, *must be a rational number.*

Work Exercises 55–60 in order, *to discover the surprising simplest form of this rational number.*

55. Use long division or your previous experience to write a repeating decimal representation for $\frac{1}{3}$.

56. Use long division or your previous experience to write a repeating decimal representation for $\frac{2}{3}$.

57. Because $\frac{1}{3} + \frac{2}{3} = 1$, the sum of the decimal representations in Exercises 55 and 56 must also equal 1. Line up the decimals in the usual vertical method for addition, and obtain the repeating decimal result. The value of this decimal is exactly 1.

58. The repeating decimal .99999 . . . can be written as the sum of the terms of a geometric sequence with $a_1 = .9$ and $r = .1$:

$$.99999\ldots = .9 + .9(.1) + .9(.1)^2 + .9(.1)^3 + .9(.1)^4 + .9(.1)^5 + \cdots.$$

Since $|.1| < 1$, this sum can be found using the formula $S = \dfrac{a_1}{1 - r}$. Use this formula to support the result you found another way in Exercises 55–57.

59. Which one of the following is true, based on your results in Exercises 57 and 58?

A. $.99999\ldots < 1$ **B.** $.99999\ldots = 1$ **C.** $.99999\ldots \approx 1$

60. Show that $.49999\ldots = \frac{1}{2}$.

12.4 The Binomial Theorem

OBJECTIVES

1. Expand a binomial raised to a power.
2. Find any specified term of the expansion of a binomial.

OBJECTIVE 1 Expand a binomial raised to a power. Writing out the binomial expression $(x + y)^n$ for nonnegative integer values of n gives a family of expressions that is important in the study of mathematics and its applications. For example,

$$
\begin{aligned}
(x+y)^0 &= 1,\\
(x+y)^1 &= x + y,\\
(x+y)^2 &= x^2 + 2xy + y^2,\\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3,\\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,\\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.
\end{aligned}
$$

Inspection shows that these expansions follow a pattern. By identifying the pattern, we can write a general expression for $(x + y)^n$.

First, if n is a positive integer, each expansion after $(x + y)^0$ begins with x raised to the same power to which the binomial is raised. That is, the expansion of $(x + y)^1$ has a first term of x^1, the expansion of $(x + y)^2$ has a first term of x^2, the expansion of $(x + y)^3$ has a first term of x^3, and so on. Also, the last term in each expansion is y to this same power, so the expansion of $(x + y)^n$ should begin with the term x^n and end with the term y^n.

The exponents on x decrease by 1 in each term after the first, while the exponents on y, beginning with y in the second term, increase by 1 in each succeeding term. Thus, the *variables* in the expansion of $(x + y)^n$ have the following pattern.

$$x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad x^{n-3}y^3, \ldots, xy^{n-1}, \quad y^n$$

Photo not available

This pattern suggests that the sum of the exponents on x and y in each term is n. For example, in the third term above, the variable part is $x^{n-2}y^2$ and the sum of the exponents, $n - 2$ and 2, is n.

Now examine the pattern for the *coefficients* in the terms of the preceding expansions. Writing the coefficients alone in a triangular pattern gives **Pascal's triangle,** named in honor of the 17th century mathematician Blaise Pascal, one of the first to use it extensively.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\qquad \text{and so on}
$$

Arranging the coefficients in this way shows that each number in the triangle is the sum of the two numbers just above it (one to the right and one to the left). For example, in the fifth row from the top, 1 is the sum of 1 (the only number above it), 4 is the sum of 1 and 3, 6 is the sum of 3 and 3, and so on.

To obtain the coefficients for $(x + y)^6$, we attach the seventh row to the table by adding pairs of numbers from the sixth row.

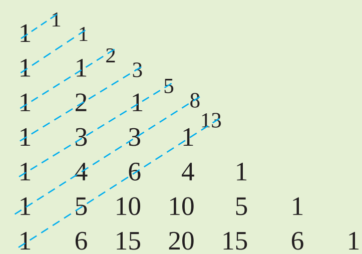
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

We then use these coefficients to expand $(x + y)^6$ as

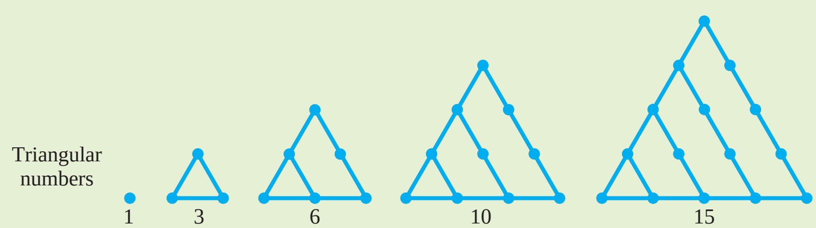
$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

CONNECTIONS

Over the years, many interesting patterns have been discovered in Pascal's triangle. In the following figure, the triangular array is written in a different form. The indicated sums along the diagonals shown are the terms of the *Fibonacci sequence,* mentioned in the chapter introduction. The presence of this sequence in the triangle apparently was not recognized by Pascal.



Triangular numbers are found by counting the number of points in triangular arrangements of points. The first few triangular numbers are shown in the figure below.

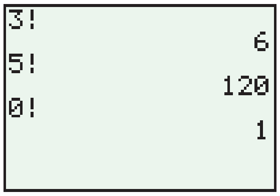


The number of points in these figures form the sequence 1, 3, 6, 10, . . . , a sequence that is found in Pascal's triangle, as shown in the next figure.

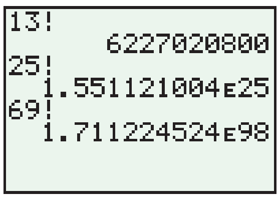
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

For Discussion or Writing

1. Predict the next two numbers in the sequence of sums of the diagonals of Pascal's triangle.
2. Predict the next five numbers in the list of triangular numbers.
3. Describe other sequences that can be found in Pascal's triangle.



(a)



A graphing calculator with a 10-digit display will give the exact value of $n!$ for $n \le 13$ and approximate values of $n!$ for $14 \le n \le 69$.

(b)

FIGURE 5

Although it is possible to use Pascal's triangle to find the coefficients of $(x + y)^n$ for any positive integer value of n, it is impractical for large values of n. A more efficient way to determine these coefficients uses a notational shorthand with the symbol $\boldsymbol{n!}$ (read "n factorial") defined as follows.

n Factorial ($n!$)

For any positive integer n,

$$n(n - 1)(n - 2)(n - 3) \cdots (2)(1) = n!.$$

For example,

$$3! = 3 \cdot 2 \cdot 1 = 6 \qquad \text{and} \qquad 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

From the definition of n factorial, $n[(n - 1)!] = n!$. If $n = 1$, then $1(0!) = 1! = 1$. Because of this, $0!$ is defined as

$$0! = 1.$$

Now Try Exercise 1.

Scientific and graphing calculators can compute factorials. The three example factorial expressions above are shown in Figure 5(a). Figure 5(b) shows some larger factorials.

EXAMPLE 1 Evaluating Expressions with $n!$

Find the value of each expression.

(a) $\dfrac{5!}{4!1!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(1)} = 5$

(b) $\dfrac{5!}{3!2!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$

(c) $\dfrac{6!}{3!3!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

(d) $\dfrac{4!}{4!0!} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(1)} = 1$

Now Try Exercises 3 and 7.

Now look again at the coefficients of the expansion

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

The coefficient of the second term is 5 and the exponents on the variables in that term are 4 and 1. From Example 1(a), $\frac{5!}{4!1!} = 5$. The coefficient of the third term is 10, and the exponents are 3 and 2. From Example 1(b), $\frac{5!}{3!2!} = 10$. Similar results hold true for the remaining terms. The first term can be written as $1x^5y^0$, and the last term can be written as $1x^0y^5$. Then the coefficient of the first term should be $\frac{5!}{5!0!} = 1$, and the coefficient of the last term would be $\frac{5!}{0!5!} = 1$. Generalizing, the coefficient for a term of $(x + y)^n$ in which the variable part is $x^r y^{n-r}$ will be

$$\frac{n!}{r!(n-r)!}.$$

NOTE The denominator factorials in the coefficient of a term are the same as the exponents on the variables in that term.

FIGURE 6

The expression $\frac{n!}{r!(n-r)!}$ is often represented by the symbol ${}_nC_r$. This comes from the fact that if we choose *combinations* of n things taken r at a time, the result is given by that expression. A graphing calculator can evaluate this expression for particular values of n and r. Figure 6 shows how a calculator evaluates ${}_5C_4$, ${}_5C_3$, and ${}_6C_3$. Compare these results to parts (a), (b), and (c) of Example 1.

Now Try Exercise 5.

Summarizing this work gives the **binomial theorem,** or the **general binomial expansion.**

Binomial Theorem

For any positive integer n,

$$(x+y)^n = x^n + \frac{n!}{(n-1)!1!}x^{n-1}y + \frac{n!}{(n-2)!2!}x^{n-2}y^2 + \frac{n!}{(n-3)!3!}x^{n-3}y^3 + \cdots + \frac{n!}{1!(n-1)!}xy^{n-1} + y^n.$$

The binomial theorem can be written in summation notation as

$$(x+y)^n = \sum_{i=0}^{n} \frac{n!}{(n-i)!i!}x^{n-i}y^i.$$

NOTE The letter i is used here instead of r because we are using summation notation. It is not the imaginary number i.

EXAMPLE 2 Using the Binomial Theorem

Expand $(2m + 3)^4$.

$$(2m+3)^4 = (2m)^4 + \frac{4!}{3!1!}(2m)^3(3) + \frac{4!}{2!2!}(2m)^2(3)^2 + \frac{4!}{1!3!}(2m)(3)^3 + 3^4$$

$$= 16m^4 + 4(8m^3)(3) + 6(4m^2)(9) + 4(2m)(27) + 81$$

$$= 16m^4 + 96m^3 + 216m^2 + 216m + 81$$

Now Try Exercise 17.

EXAMPLE 3 Using the Binomial Theorem

Expand $\left(a - \frac{b}{2}\right)^5$.

$$\left(a - \frac{b}{2}\right)^5 = a^5 + \frac{5!}{4!1!}a^4\left(-\frac{b}{2}\right) + \frac{5!}{3!2!}a^3\left(-\frac{b}{2}\right)^2 + \frac{5!}{2!3!}a^2\left(-\frac{b}{2}\right)^3$$

$$+ \frac{5!}{1!4!}a\left(-\frac{b}{2}\right)^4 + \left(-\frac{b}{2}\right)^5$$

$$= a^5 + 5a^4\left(-\frac{b}{2}\right) + 10a^3\left(\frac{b^2}{4}\right) + 10a^2\left(-\frac{b^3}{8}\right)$$

$$+ 5a\left(\frac{b^4}{16}\right) + \left(-\frac{b^5}{32}\right)$$

$$= a^5 - \frac{5}{2}a^4b + \frac{5}{2}a^3b^2 - \frac{5}{4}a^2b^3 + \frac{5}{16}ab^4 - \frac{1}{32}b^5$$

Now Try Exercise 19.

CAUTION When the binomial is the *difference* of two terms as in Example 3, the signs of the terms in the expansion will alternate. Those terms with odd exponents on the second variable expression $\left(-\frac{b}{2}\text{ in Example 3}\right)$ will be negative, while those with even exponents on the second variable expression will be positive.

OBJECTIVE 2 Find any specified term of the expansion of a binomial. Any single term of a binomial expansion can be determined without writing out the whole expansion. For example, if $n \geq 10$, then the tenth term of $(x + y)^n$ has y raised to the ninth power (since y has the power of 1 in the second term, the power of 2 in the third term, and so on). Since the exponents on x and y in any term must have a sum of n, the exponent on x in the tenth term is $n - 9$. These quantities, 9 and $n - 9$, determine the factorials in the denominator of the coefficient. Thus,

$$\frac{n!}{(n-9)!9!}x^{n-9}y^9$$

is the tenth term of $(x + y)^n$. A generalization of this idea follows.

***r*th Term of the Binomial Expansion**

If $n \geq r - 1$, then the rth term of the expansion of $(x + y)^n$ is

$$\frac{n!}{[n-(r-1)]!(r-1)!}x^{n-(r-1)}y^{r-1}.$$

In this general expression, remember to start with the exponent on y, which is 1 less than the term number r. Then subtract that exponent from n to get the exponent on x: $n - (r - 1)$. The two exponents are then used as the factorials in the denominator of the coefficient.

EXAMPLE 4 Finding a Single Term of a Binomial Expansion

Find the fourth term of $(a + 2b)^{10}$.

In the fourth term, $2b$ has an exponent of $4 - 1 = 3$ and a has an exponent of $10 - 3 = 7$. The fourth term is

$$\begin{aligned}\frac{10!}{7!3!}(a^7)(2b)^3 &= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}(a^7)(8b^3) \\ &= 120a^7(8b^3) \\ &= 960a^7b^3.\end{aligned}$$

Now Try Exercise 29.

12.4 EXERCISES

For Extra Help

Student's Solutions Manual

MyMathLab

InterAct Math Tutorial Software

AW Math Tutor Center

MathXL

Digital Video Tutor CD 19/Videotape 20

Evaluate each expression. See Example 1.

1. $6!$ **2.** $4!$ **3.** $\frac{6!}{4!2!}$ **4.** $\frac{7!}{3!4!}$

5. ${}_6C_2$ **6.** ${}_7C_4$ **7.** $\frac{4!}{0!4!}$ **8.** $\frac{5!}{5!0!}$

9. $4! \cdot 5$ **10.** $6! \cdot 7$ **11.** ${}_{13}C_{11}$ **12.** ${}_{13}C_2$

Use the binomial theorem to expand each expression. See Examples 2 and 3.

13. $(m + n)^4$ **14.** $(x + r)^5$ **15.** $(a - b)^5$ **16.** $(p - q)^4$

17. $(2x + 3)^3$ **18.** $\left(\frac{x}{3} + 2y\right)^5$ **19.** $\left(\frac{x}{2} - y\right)^4$ **20.** $(x^2 + 1)^4$

21. $(mx - n^2)^3$ **22.** $(2p^2 - q^2)^3$

Write the first four terms of each binomial expansion.

23. $(r + 2s)^{12}$ **24.** $(m - n)^{20}$ **25.** $(3x - y)^{14}$

26. $(2p + 3q)^{11}$ **27.** $(t^2 + u^2)^{10}$ **28.** $(x^2 - y^2)^{15}$

Find the indicated term of each binomial expansion. See Example 4.

29. $(2m + n)^{10}$; fourth term **30.** $(a - 3b)^{12}$; fifth term

31. $\left(x + \frac{y}{2}\right)^8$; seventh term **32.** $(3p - 2q)^{15}$; eighth term

33. $(k - 1)^9$; third term **34.** $(-4 - s)^{11}$; fourth term

35. The middle term of $(x^2 - 2y)^6$ **36.** The middle term of $(m^3 + 3)^8$

37. The term with x^9y^4 in $(3x^3 - 4y^2)^5$ **38.** The term with x^{10} in $\left(x^3 - \frac{2}{x}\right)^6$