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Preface

This volume is the result of academic and scientific collaboration between the mathematical communities from Spain and Brazil.

In December 2015 the Federal University of Ceará (Brazil) organized the first meeting between the Brazilian Mathematical Society (SBM), the Brazilian Society of Applied and Computational Mathematics (SBMAC), and the Real Sociedad Matemática Española (RSME). As a consequence of the success of that first edition, the respective national societies decided to host a second meeting in Spain. The University of Cádiz was chosen to organize the Second Meeting of the Brazilian and Spanish mathematical societies, which took place from December 11 to 14, 2018. Moreover, in this second edition, the Spanish Society of Applied Mathematics (SEMA) also participated in the organization of the event.

Of course, the main objective of this second meeting was to continue to strengthen the collaboration between researchers and institutions from Spain and Brazil. To this end, plenary conferences, special sessions, and posters were included on a wide range of topics in Pure and Applied Mathematics. All of them were submitted to the demands of a Scientific Committee of the highest level, comprising chosen representatives of the four participating societies.

The meeting took place in the "Constitution of 1812" building and in the Faculty of Philosophy and Letters of the University of Cádiz, both historical buildings in the ancient city of Cádiz. It is worth mentioning that Cádiz was chosen as the venue for this meeting as it is considered by all to be a historical city, with great links to the other side of the Atlantic and with a university that has innumerable links with institutions on the American continent.

The Bay of Cádiz, whose beauty could be appreciated by all participants from the congress venue, has been a place of reference for countless ships and cruises. From here, trade and maritime transport was practiced with the West and East Indies, the shores of the Mediterranean Sea, the great Atlantic ports of Europe, and the coasts of Africa. Cádiz has been a cosmopolitan city since the seventeenth century.

Building upon the success of this second meeting, we launch this volume in which a varied selection of works presented at the meeting have been included and in which quality is the predominant note. The lines of the selected works range from abstract algebra, such as Lie algebras, commutative semi-groups, and differential geometry, to more applied works in which mathematical modeling by means of boundary value problems governed by PDEs is the subject of study. All the works in the volume have been submitted to a blind peer review process. In addition, this collection offers a good summary of the recent activity of the different Spanish and Brazilian research groups interested in both the applications of mathematics and pure mathematics.

We would like to conclude this introduction by thanking all those who, in one way or another, participated in the organization of the congress and especially the local organizing committee along with all the volunteers. Not only did they make the congress possible but the publication of this volume is largely the result of their work. Finally, we would like to thank the authors themselves for submitting their valuable works.

Puerto Real, Spain — Juan Ignacio García García
Puerto Real, Spain — Francisco Ortegón Gallego
December 15, 2019

Contents

On the Control of the Navier-Stokes Equations and Related Systems

Enrique Fernández-Cara

Abstract This contribution relies on two main concepts: the *Navier-Stokes equations* and *control theory*. Our aim is to justify their relevance and, also, to describe some results that can be obtained by working simultaneously with both them. In particular, we will recall some recent results concerning related bi-objective optimal control and controllability problems.

1 Introduction

1.1 The Navier-Stokes Equations

The Navier-Stokes equations describe the behavior of an incompressible fluid under realistic conditions. Thus, they can be used to model a lot of atmospheric, oceanic, and climatological phenomena, the flow of a fluid around a body of any kind, flows in channels and associated jets, etc. See for instance [3, 67, 79] for some details.

In its simplest form, the equations are the following:

$$\begin{cases} \partial_t u_1 - \nu \left(\partial_1^2 u_1 + \partial_2^2 u_1 + \partial_3^2 u_1\right) + u_1 \partial_1 u_1 + u_2 \partial_2 u_1 + u_3 \partial_3 u_1 + \dfrac{1}{\rho} \partial_1 p = f_1 \\ \partial_t u_2 - \nu \left(\partial_1^2 u_2 + \partial_2^2 u_2 + \partial_3^2 u_2\right) + u_1 \partial_1 u_2 + u_2 \partial_2 u_2 + u_3 \partial_3 u_2 + \dfrac{1}{\rho} \partial_2 p = f_2 \\ \partial_t u_3 - \nu \left(\partial_1^2 u_3 + \partial_2^2 u_3 + \partial_3^2 u_3\right) + u_1 \partial_1 u_3 + u_2 \partial_2 u_3 + u_3 \partial_3 u_3 + \dfrac{1}{\rho} \partial_3 p = f_3 \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0. \end{cases}$$

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In a more compact way, the previous system, can be written as follows:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1}$$

The unknowns are $\mathbf{u} = (u_1, u_2, u_3)$ (the velocity field of the fluid) and p (the pressure); $\mathbf{u}$ and p are functions of $\mathbf{x} = (x_1, x_2, x_3)$ (the spatial variable) and t (the time) and (1) must be solved for $(\mathbf{x}, t) \in \Omega \times (0, T)$, where $\Omega \subset \mathbf{R}^3$ is a nonempty open set.

In (1), ν and ρ are positive constants, characteristic of the fluid. They are respectively called the *kinematic viscosity* and the *mass density.* On the other hand, $\mathbf{f} = (f_1, f_2, f_3)$ is given; in general, it is also a function of $\mathbf{x}$ and t and must be viewed as a density of external forces.

The first equality in (1) is a formulation of the conservation law of linear momentum (second Newton's law); it is usually known as the *motion equation.* The second identity indicates that the volume of a set of particles does not change with time; accordingly, it is called the *incompressibility condition;* see [12, 62, 66] for details.

The system (1) is thus composed of 4 scalar partial differential equations. The three first of them are nonlinear, due to the so called inertia term $(\mathbf{u}\cdot\nabla)\mathbf{u}$.

As usual, this differential system does not suffice by itself to identify the unknowns $\mathbf{u}$ and p. It has to be complemented with additional conditions of two kinds:

- *Initial conditions* at $t = 0$ and
- *Boundary conditions* on $\partial\Omega$ along $(0, T)$.

Generally (but not always) these requirements affect $\mathbf{u}$. Accordingly, the mathematical task is to find a couple $(\mathbf{u}, p)$ that satisfies (1) in $\Omega \times (0, T)$ such that the initial velocity field $\mathbf{u}|_{t=0}$ and the boundary data $\mathbf{u}|_{\partial\Omega\times(0,T)}$ are prescribed.

1.2 A Little Bit of History

The Navier-Sokes equations were deduced after the work of many people. Among them, let us mention the following:

Newton [59]
Mainly, he described and modelled internal forces among particles.

A fluid can be viewed as a conglomerate of particles that travel together. Particles interact; their motion is determined by internal efforts, among other things due to *friction* (also called viscosity). Newton claimed that friction forces are proportional to the spatial derivatives of the velocity. Actually, this is the reason of the occurrence of the so called viscous term $-\nu\Delta\mathbf{u}$.

Euler [22]

He was the first to use a partial differential system to describe the behavior of $\mathbf{u}$ and p. In particular, he introduced the pressure. Unfortunately, he arrived at an incomplete system, nowadays called the Euler equations, where he "forgot" the viscous term:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{2}$$

Nevertheless, there are many situations where (2) provides an acceptable description of the behavior of the fluid. This happens in particular when we consider the flow of a fluid far from solid walls.

Navier and Stokes [57, 72]

They finally deduced, independently of each other, the correct system. To this purpose, they incorporated Newton's term to the Euler system and they successfully interpreted each term separately, to give a complete explanation of all common realistic fluids: air, water, oil, etc.

Fluids governed by the Navier-Stokes equations are usually called *Newtonian.* It should be noted that not all fluids found in nature or resulting from industrial processes are Newtonian. Indeed, in many cases, the molecular structure of the fluid is too complex and the particles behave as elastic balls. Accordingly, they deform after interaction and then try to recover their original shape. This makes it necessary to account for additional (elastic or memory-like) terms in the equations. This is the case of magma, blood, shampoos and many other industrial fluids; see more details for example in [36, 65, 69].

Needless to say, the achievements of these scientists are not reduced to their work in fluid mechanics.

Newton contributed to many other areas of physics and mathematics. He can be considered the first applied mathematician of History. Thus, his interest to understand gravitational forces and planet motion led him to introduce fundamental mathematical tools (functions, derivatives, differential equations) that serve to us even today.[1]

Euler is also at the origin of a lot of mathematical concepts, results and conjectures. Among them, let us recall the well known *Riemann hypothesis,* that could actually be called the Euler hypothesis. It is the following:

Let us set

$$\zeta(x+iy) := \left(1 - \frac{2}{2^{x+iy}}\right)^{-1} \sum_{n\geq 1} \frac{(-1)^n}{n^{x+iy}} \quad \textit{for } x > 0.$$

Then we can have $\zeta(x+iy) = 0$ *only for* $x = 1/2$.

[1] Do not forget, however, his famous dispute with Leibniz, where each of them claimed to be the "inventor" of differential calculus.

This is one of the 1900 Hilbert's Problems and also one of the 2000 Clay Millenium Problems, see [13, 34]. It has led to more than 140 significative contributions in 2009–2018.

As first noticed by Euler, a positive answer would imply many interesting results for prime numbers. For example, if the previous conjecture holds, then one has:

$$\forall x > 2 \; \exists p \text{ (prime) with } x - \frac{4}{\pi} x^{1/2} \log x < p \leq x$$

(see [20]).

Until now, all computed zeros satisfy $x = 1/2$, as Table 1 shows.

Navier was a famous French engineer and physicist. Independently of his scientific career, he was the chief constructor of several bridges in Choisy, Asnières, Argenteuil and Paris. Also, he contributed to the development of *Elasticity*. In fact, he is nowadays considered one of the founders of the field *Structural Analysis*.

On the other hand, Stokes was an Irish mathematician and physicist specially characterized by precision and rigor. His numerous contributions to *Mathematical Analysis* are well known. It is also well known that he deduced many unpublished results that were later attributed to others. After being nominated for the Lucassian Chair, he decided to dedicate all his time to teach and assist lots of students and colleagues in their work.

After the formulation of the Navier-Stokes equations, it was soon understood that the computation of explicit solutions was essentially impossible, except in a very reduced number of cases. Consequently, in order to advance in the description of fluid flows, something had to be done. The related activities have produced a lot of benefits in many areas and can be grouped as follows:

1. Research in the theoretical analysis of linear and nonlinear partial differential equations and close fields (since the 30's). This favored in a very significant way works on distributions, Sobolev spaces, integral transforms, etc. The contributions began with Carl W. Oseen [61] and Jean Leray [49] and were followed by the work of Hopf, Jacques-Louis and Pierre-Louis Lions, Serrin, Ladyzhenskaya, Nirenberg and others; see [48] for a complete list.

Table 1 A summary of computed zeros of the ζ function

Year	Zeros	Found by
1859	3	Riemann
1936	1041	Titchmarsh and Comrie
1983	$3 \cdot 10^8 + 1$	Van de Lune and Riele
1988	$\sim 10^{12}$	Odlyzko and Schönhage (introducing fast Fourier transform methods)
2004	$\sim 10^{24}$	Gourdon and Demichel (using the Odlyzko-Schönhage algorithm)

Titchmarsh and Comrie were the last to compute zeros by hand

2. Research in the numerical analysis and solution of partial differential problems (since the 50's). The birth and progress of computers opened new possibilities in fluid mechanics. Probably, the first significative advances correspond to the contributions of John Von Neumann [81], considered by many people the father of *computational fluid dynamics.* They continued with Greenspan, Hughes, Chorin, Fortin, Glowinski, Rannacher and others; some information can be found in [30, 31, 58, 68, 76].

Unfortunately (or not), there are still many things to do in both directions.

In what concerns theoretical advances and, more precisely, on the existence and uniqueness of a solution to (1) complemented with initial and boundary conditions on $\mathbf{u}$, we are able to prove completely satisfactory results only in some particular situations. Specifically, to get this, we have either to simplify the geometrical context or to impose restrictions on the size of the data. In the early 30's, Leray proved the existence of what he called "turbulent" solutions (nowadays known as *weak solutions*), but he left open the important questions of uniqueness and regularity. Since then, a lot of effort has been paid to this and many partial results have been deduced. However, not much more is known at present!

On the other hand, Von Neumann, was convinced of the success of numerics in this setting, at the point of believing in 1946 that theoretical analysis and even physical experimentation would become obsolete in a few years. Unfortunately, he passed away too soon, before the arrival of parallel and vector computation. Without any doubt, he would have paid a major role in this direction.

Of course, we know today that Von Neumann's prediction was false. Indeed, it has been understood that the resolution of many problems of industrial origin requires a precision level that cannot be reached with our tools. Moreover, it is possible to observe experimentally many phenomena that we are not able to explain with the numerical results we have at hand. However, it is clear that the computation of approximate solutions is of fundamental help to understand and render useful the theoretical analysis (Figs. 1 and 2); see [30].

If we speak of the analysis of the Navier-Stokes equations, we must also refer to the work of Olga A. Ladyzhenskaya, one of the great names in the twentieth century.

She was an outstanding mathematician that contributed to several fields, with special dedication to the analysis of partial differential equations; see [45, 46]. Being the daughter of a descendent of the Russian noble class, she found severe difficulties to complete her career. At 1939 she applied to enter Leningrad University but was not admitted; it was only in 1943 when she was allowed to enter Moscow University in the difficult period of wartime and, although she had completed her Thesis in 1951, she still had to wait to defend it for 2 years. She could have easily been the first female Field medallist in 1958 (the awards finally went to Klaus Roth and René Thom).

In the context of the Navier-Stokes equations, among other achievements, she was the first to understand the relevance of the inequalities

$$\|v\|_{L^4} \leq C\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2} \quad \text{if } N = 2$$

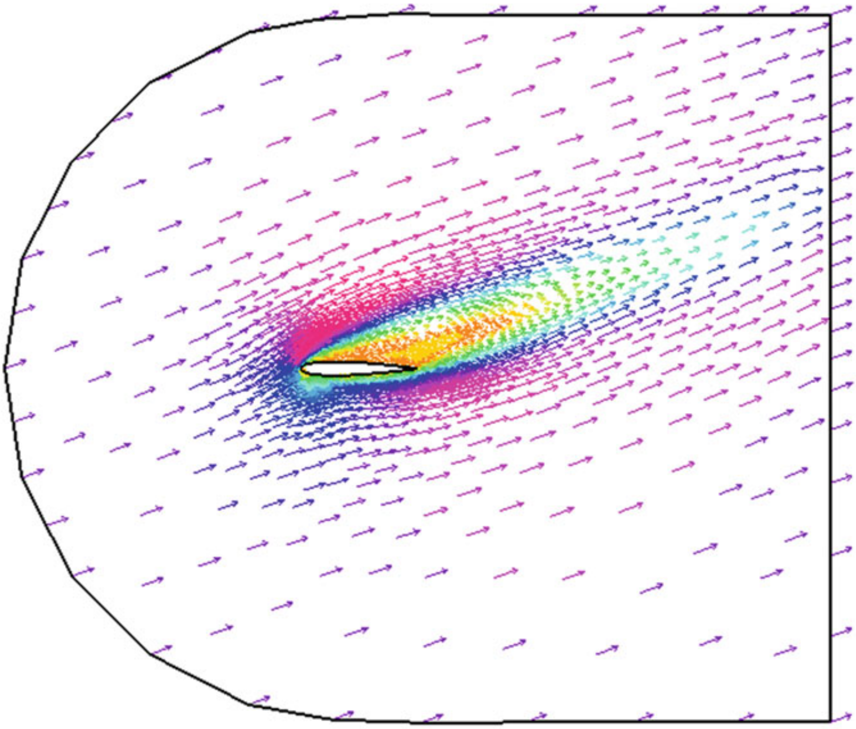

Fig. 1 Numerical computation of the flow of the air around an aerodynamic body: the velocity field. Results obtained with the FreeFem++ package (see [33])

and

$$\|v\|_{L^4} \leq C\|v\|_{L^2}^{1/4}\|\nabla v\|_{L^2}^{3/4} \quad \text{if } N = 3,$$

that must hold for any compactly supported smooth function; see [40, 43]. She was also the first to consider the notion of attractor for infinite dimensional dynamical systems, see [41, 44]. Also, she explored successfully alternative fluid models, different from the Navier-Stokes equations, with the aim to model and understand turbulence, see [42].

As indicated by Michael Struwe in [73], “there is only one explanation why in spite of such adversity Ladyzhenskaya was able to rise to the top of the Steklov Institute and become the uncontested head of the Leningrad School of partial differential equations, and this is her work.”

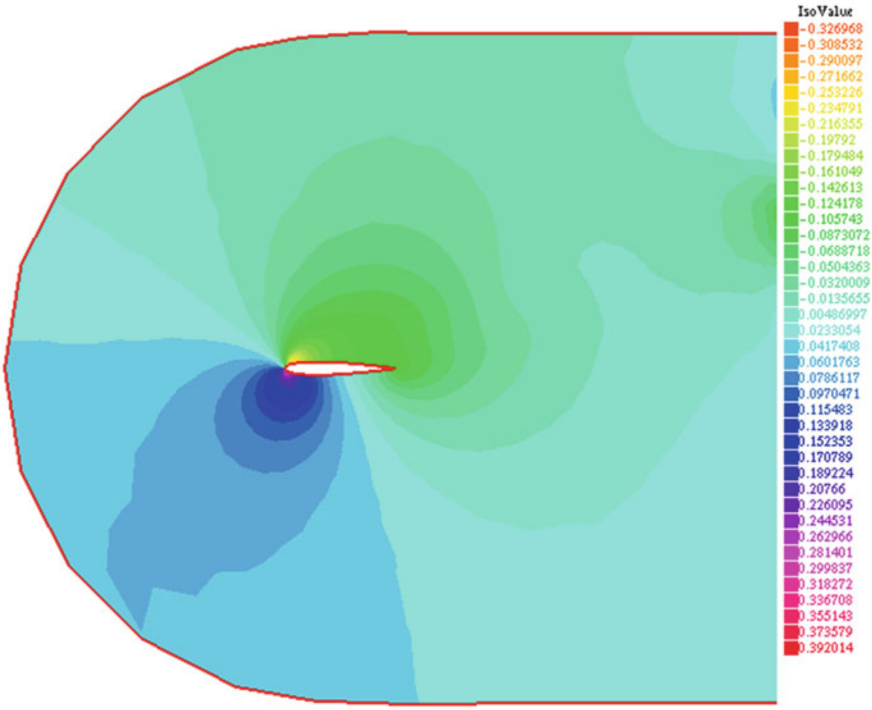

Fig. 2 Numerical computation of the flow of the air around an aerodynamic body: the pressure. Results obtained with the FreeFem++ package (see [33])

1.3 Fundamentals of Control Theory

The second main concept in this paper is *control theory.* This is a branch of mathematics with important connection and interaction with other sciences dedicated to govern the behavior of models of all kinds.

A typical problem in control theory can be described as follows. We consider a (linear or onlinear) system of the form

$$\begin{cases} \mathcal{A}(y) = B(f) \\ + \dots, \end{cases}$$

where f is a datum (the control), y is the solution (the state) and the dots may contain some additional information. We assume that f can be chosen in a (subset of a) space $\mathcal{F}$ and that, for each f, there exists at least one solution $y \in \mathcal{Y}$. Roughly speaking, the control problem consists of finding f such that an associated solution y *satisfies good properties.*

There are two usual ways to give a sense to the desired "good properties" of y. They correspond to the following classical approaches (see for instance [15, 51]):

- Optimal control: Find f such that (f, y) minimizes a cost $J = J(f, y)$.
- Controllability: Find f such that $R(y)$ takes a desired value, where $R : \mathcal{Y} \mapsto \mathcal{Z}$ is a prescribed mapping ($\mathcal{Z}$ is the space of observations).

Control problems are natural in the real world. For instance, the organisms of many living species are endowed with task regulating mechanisms to guarantee optimal regimes, keep the species alive and allow them to grow and reproduce.

The first *rigorous* mathematical formulation of a control problem was given by Maxwell in his pioneering paper [54], devoted to describe the behavior of "governed" steam engines. In that paper, the author considered what he called *centrifugal governors,* that had been already conceived for the control of windmills.[2]

1.4 Some Applications of Control Theory

There are many situations in the real world where the techniques of control theory can be applied and provide satisfactory results. Let us mention some of them.

Sound Therapies for Speech Problems In many cases, some children cannot speak correctly because they cannot hear or understand well. Accordingly, a speech therapy process based on auditory training must be designed to enable hearing and processing sound accurately, see [6, 38, 75, 78].

Usually, this is accomplished by filtering classical music and spoken stories through an appropriate electronic ear device.

In order to optimize understanding, we must choose this training program adequately and this task corresponds to a very interesting control problem.

Control of Diabetes Diabetes is excess of high blood sugar due to the absence or scarcity of insulin. Unfortunately, in many cases, diabetes leads to other critical diseases, like heart failure, obesity, etc. It is not curable, but can be controlled (in the usual sense) with regular exercise, diet plans, etc.

Thus, we find here an important application of control theory to health improvement. Some results can be found in [1, 19, 39, 55, 60]. There, the simulations provide results that show that a well programmed physical exercise, along with knowledge about how to modify daily insulin dosage (to prevent hypoglycemia) improves blood glucose control and enhance insulin sensitivity index.

Control Oriented to Cancer Therapies Since several decades, it has been possible to model tumor growth with (ordinary or partial) differential systems where

[2]The notion led him to the most important rule in control theory: the feedback principle. Roughly speaking, this principle is an inversion-like law asserting that the "good" or "best" control v must be computed from information furnished by the associated y.

some data (right hand sides, boundary data, etc.) can be viewed as therapy actuators; see for instance [2, 5, 7, 56, 70, 82] and the references therein. For example, this allows to describe radiotherapy strategies. These may be curative in many types of cancer if they are applied in a well chosen area of the body. Also, they can be used as part of an additional or auxiliary therapy, after surgery or in combination with chemotherapy.

Obviously, if radiation therapy is required, several relevant questions corresponding to specific control problems are in order: Where must it be applied? Which doses are appropriate? How long should the treatment last?

A more recent technique, still unexplored from the control theory viewpoint, is *nanotherapy.* Standard therapy drugs cannot always reach the cancer cells because of existing barriers. However, after encapsulation in microscale agents, they can pass, reach and act on the cells, leaving no time for reaction; see [10, 11].

Once more, questions about where, how many and how long nanoparticles must be placed in a therapy process can be answered or at least clarified after the formulation and solution of adequate control problems.

2 Controlling the Navier-Stokes System

There are many reasons to consider control problems for the Navier-Stokes equations (1). Among them, we can find motivations in physics, engineering, environmental sciences, biology, etc. In the following section we will review some of them. Then, we will recall two particular control problems for which recent achievements have been obtained.

2.1 Three Motivations

A first class of control problems concerns *optimum design.*

A classical (general) formulation is the following: find a domain $D \subset \mathbf{R}^3$ such that a solution to

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \mathbf{f} \;\text{ in } (\Omega\setminus D)\times(0,T), \\ \nabla\cdot\mathbf{u} = 0 \;\text{ in } (\Omega\setminus D)\times(0,T), \\ \dots \end{cases} \tag{3}$$

satisfies desired properties.

For instance, if D is viewed as an obstacle to the flow of the fluid, we can look for a domain that minimizes the drag, electromagnetic and/or turbulence effects, etc.; see for example [63, 64].

Another important family of control problems deals with fluid insulating/suction processes. They belong to the class of boundary optimal control or boundary controllability problems and their general formulation is as follows: find $\mathbf{h}$ in an appropriate space such that a solution to

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \mathbf{f} \ \text{ in } \Omega \times (0,T), \\ \nabla\cdot\mathbf{u} = 0 \ \text{ in } \Omega \times (0,T), \\ \mathbf{u} = \mathbf{h} \ \text{ on } \partial\Omega \times (0,T), \\ \dots \end{cases}$$

has good behavior. Again, we can search for boundary controls leading to minimal drag, "maximal" stability, etc.

An illustration of three-dimensional suction on a rectangular NACA 0012 wing is depicted in [83]. For example, the lift to drag ratio can be maximized with a suitable (optimal) suction jet length. The good choice leads to a "controlled" fluid near the wing surface.

The control of the Navier-Stokes and other similar equations is also related to other purposes. For instance, it appears in a natural way when we try to minimize contamination. An acceptable (simplified) controlled system for a contaminated fluid is the following:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \mathbf{B}(f,\psi), \\ \nabla\cdot\mathbf{u} = 0, \\ \psi_t - \kappa\Delta\psi + \mathbf{u}\cdot\nabla\psi = 0, \\ \dots \end{cases} \tag{4}$$

where $\psi = \psi(\mathbf{x},t)$ is a contamination density function, $\kappa > 0$ is the corresponding diffusion coefficient and $f = f(\mathbf{x},t)$ is a control function that allows to identify the cleaning strategy. In (4), the equations must be solved in a set of the usual form $\Omega \times (0,T)$ and the dots contain boundary and initial conditions for $\mathbf{u}$ and ψ. The particular structure of the term $\mathbf{B}(f,\psi)$ changes with the model; for more details, see [4, 8].

A natural related controllability problem is to find f in an appropriate space such that ψ is "small" at the final time T.

A very serious environmental problem of our times is plastic trash accumulation in the oceans. Five giant garbage patches have been identified as a consequence of the current action. The largest is the *Great Pacific Garbage Patch,* located between Hawaii and California, with a surface triplicating the surface of France; see [21, 37, 47, 80].

In 2013, a solution has been proposed: the Ocean Cleanup System 001 (OCS-001), by Boyan Slat (see [71]). The main tool is a system consisting of a 600 m-long floater designed to lie at the surface of the water and a tapered 3 m-deep skirt attached below. The structure intends to surround the Great Pacific plastic island and push trash to appropriate locations. It is assumed that, later, trash will be picked up from boats, transported and then adequately processed.

Clearly, several nontrivial control problems remain behind this proposal: Where do we have to begin operations? Which is the optimal shape for the OCS-001? Where should we fix the accumulation points? ...

Once again, the answers can be furnished by methods and techniques from control theory.

2.2 A Bi-objective Control Problem

This section is devoted to recall some recent results obtained in [23] for an optimal control problem where we try to improve simultaneously the values of two different objective (cost) functions.

For simplicity, we will simplify (1) and we will consider the stationary Navier-Stokes system:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p = \mathbf{f}\mathbb{1}_\omega \ \text{ in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{5}$$

Here, $\omega \subset \Omega$ is a nonempty open set (the control domain), $\mathbb{1}_\omega$ is the characteristic function of ω and $\mathbf{f}$ is the control. It is assumed to belong to the space of controls $\mathcal{F} := L^2(\omega)^3$.

Let us introduce the notation

$$V := \{\mathbf{v} \in H_0^1(\Omega)^3 : \mathbf{v} = 0 \ \text{ on } \ \partial\Omega\},$$

where $H_0^1(\Omega)$ denotes the usual Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives belong to $L^2(\Omega)$ and vanish on $\partial\Omega$. For each $\mathbf{f} \in \mathcal{F}$, there exists at least one state, i.e. one solution $(\mathbf{u}, p)$ to (5), with

$$\mathbf{u} \in V, \quad p \in L^2(\Omega).$$

Unfortunately, it is not known in general whether this solution is unique; see for instance [50, 77].

We will consider the cost functionals

$$J_i(\mathbf{f},\mathbf{u}) := \frac{1}{2}\int_{O_i} |\mathbf{u}-\mathbf{u}_{id}|^2\,d\mathbf{x} + \frac{\mu}{2}\int_{\omega} |\mathbf{f}|^2\,d\mathbf{x}, \quad i=1,2,$$

where the $O_i \subset \Omega$ are nonempty open sets, the $\mathbf{u}_{id} \in L^2(O_i)^3$ and $\mu > 0$.

It is in general impossible to look for a couple $(\mathbf{f},\mathbf{u})$ that minimizes simultaneously J_1 and J_2. Therefore, our goal will be to find an *equilibrium* for J_1 and J_2. More precisely, we will look for a *Pareto equilibrium,* that is, a control-state pair $(\mathbf{f},\mathbf{u}) \in \mathcal{F} \times V$ such that there is no other pair $(\mathbf{g},\mathbf{v})$ satisfying (5) and

$$J_i(\mathbf{g},\mathbf{v}) \le J_i(\mathbf{f},\mathbf{u}) \text{ for } i=1,2, \tag{6}$$

at least one inequality being strict.

Our goals are to prove the existence of Pareto equilibria, to get a characterization of them and, also, to set up convergent algorithms for their computation. The main difficulties found in this control problem are its bi-objective structure and the multiplicity of the control-to-state mapping. Accordingly, the standard tools of calculus of variations do not work here.

For each $\alpha \in [0,1]$, let us set $J_{(\alpha)} := \alpha J_1 + (1-\alpha) J_2$. The main results in [23] are the following.

Theorem 1 *For each $\alpha \in [0,1]$, there exists at least one solution to the following extremal problem:*

$$\begin{cases} \text{Minimize } J_{(\alpha)}(\mathbf{f},\mathbf{u}) \\ \text{Subject to } \mathbf{f} \in \mathcal{F},\ \mathbf{u} \in V,\ (\mathbf{u},p) \text{solves } (5). \end{cases}$$

Each minimizer $(\mathbf{f}_\alpha,\mathbf{u}_\alpha)$ is a Pareto equilibrium for J_1 and J_2.

The proof is not difficult, since the J_i are coercive and lower semicontinuous for the weak convergence in $\mathcal{F} \times L^2(\Omega)^3$ and the set

$$\{(\mathbf{f},\mathbf{u}) \in \mathcal{F} \times V : \exists p \in L^2(\Omega) \text{ such that } (\mathbf{u},p) \text{ solves } (5)\}$$

is nonempty and sequentially weakly closed in the same space.

Theorem 2 *Let $(\mathbf{f},\mathbf{u})$ be one of the Pareto equilibria for J_1 and J_2 furnished by Theorem 1. Then, $(\mathbf{f},\mathbf{u})$ is also a Pareto quasi-equilibrium, that is, there exists $\alpha \in [0,1]$ such that $(\mathbf{f},\mathbf{u})$ solves, together with some $p \in L^2(\Omega)$ and some*

$(\mathbf{w}, q) \in V \times L^2(\Omega)$*, the following coupled optimality system:*

$$\begin{cases} -\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\dfrac{1}{\rho}\nabla p=\mathbf{f}\mathbb{1}_{\omega} \ \text{ in } \Omega, \\ \nabla\cdot\mathbf{u}=0 \ \text{ in } \Omega, \\ \mathbf{u}=0 \ \text{ on } \partial\Omega, \\ -\nu\Delta\mathbf{w}-(\mathbf{u}\cdot\nabla)\mathbf{w}+(\nabla\mathbf{u})^t\mathbf{w}+\nabla q=\alpha(\mathbf{u}-\mathbf{u}_{1d})\mathbb{1}_{O_1}+(1-\alpha)(\mathbf{u}-\mathbf{u}_{2d})\mathbb{1}_{O_2} \ \text{ in } \Omega, \\ \nabla\cdot\mathbf{w}=0 \ \text{ in } \Omega, \\ \mathbf{w}=0 \ \text{ on } \partial\Omega, \\ \mathbf{f}=-\dfrac{1}{\mu}\mathbf{w}\mathbb{1}_{\omega}. \end{cases} \tag{7}$$

The proof is rather technical and relies on the Dubovitsky-Milyoutin formalism (see [29]). Roughly speaking, we use the fact that, at a minimizer of $J_{(\alpha)}$, the descent cones of J_1 and J_2 must be disjoint to the tangent space to (5); see [23] for the details; see also [9, 28, 74] for other related results.

For the computation of Pareto equilibria, we can use the following algorithm, based on Newton's method:

ALG We fix a decreasing factor $a \in (0, 1)$ and we do as follows.

(a) Choose $\mathbf{f}^0 \in \mathcal{F}$ and $\nu^0 \in \mathbf{R}^+$ and compute the solution $(\mathbf{u}^0, p^0)$ to

$$\begin{cases} -\nu^0\Delta\mathbf{u}^0+\nabla p^0=\mathbf{f}^0\mathbb{1}_{\omega}, \quad \mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{u}^0=0, \quad \mathbf{x}\in\Omega, \\ \mathbf{u}^0=0, \quad \mathbf{x}\in\partial\Omega, \end{cases} \tag{8}$$

and the solution $(\mathbf{w}^0, q^0)$ to

$$\begin{cases} -\nu^0\Delta\mathbf{w}^0+\nabla q^0=\alpha(\mathbf{u}^0-\mathbf{u}_{1d})\mathbb{1}_{O_1}+(1-\alpha)(\mathbf{u}^0-\mathbf{u}_{2d})\mathbb{1}_{O_2}, \quad \mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{w}^0=0, \quad \mathbf{x}\in\Omega, \\ \mathbf{w}^0=0, \quad \mathbf{x}\in\partial\Omega \end{cases} \tag{9}$$

and take

$$\mathbf{f}^0=-\frac{1}{\mu}\,\mathbf{w}^0\Big|_{\omega}.$$

(b) For given $n \geq 0$, ν^n and $\mathbf{f}^n \in \mathcal{F}$, $(\mathbf{u}^n, p^n)$ and $(\mathbf{w}^n, q^n)$, do the following:

(b.1) Take $\mathbf{f}^{n,0}=-\dfrac{1}{\mu}\mathbf{w}^n\,|_{\omega}$, $\mathbf{u}^{n,0}=\mathbf{u}^n$, $\mathbf{w}^{n,0}=\mathbf{w}^n$ and $\nu^{n+1}=\max(a\nu^n, \nu)$.

(b.2) Then, for given $k \geq 0$, $\mathbf{f}^{n,k}$, $\mathbf{u}^{n,k}$, $\mathbf{w}^{n,k}$, set

$$\mathbf{F}^{n,k} := -\nu^{n+1}\Delta\mathbf{u}^{n,k}+(\mathbf{u}^{n,k}\cdot\nabla)\mathbf{u}^{n,k}-\mathbf{f}^{n,k}\mathbb{1}_{\omega}$$

and

$$\mathbf{G}^{n,k} := -\nu^{n+1}\Delta\mathbf{w}^{n,k} - (\mathbf{u}^{n,k}\cdot\nabla)\mathbf{w}^{n,k} + (\nabla\mathbf{u}^{n,k})^t\mathbf{w}^{n,k} \\ - \alpha(\mathbf{u}^{n,k} - \mathbf{u}_{1d})\mathbb{1}_{O_1} - (1-\alpha)(\mathbf{u}^{n,k} - \mathbf{u}_{2d})\mathbb{1}_{O_2},$$

compute the solution $(\mathbf{v}^k, h^k, \mathbf{z}^k, \eta^k)$ to

$$\begin{cases} -\nu^{n+1}\Delta\mathbf{v}^k + (\mathbf{u}^{n,k}\cdot\nabla)\mathbf{v}^k + (\mathbf{v}^k\cdot\nabla)\mathbf{u}^{n,k} + \nabla h^k = \mathbf{F}^{n,k}, \quad \mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{v}^k = 0, \quad \mathbf{x}\in\Omega, \\ -\nu^{n+1}\Delta\mathbf{z}^k - (\mathbf{u}^{n,k}\cdot\nabla)\mathbf{z}^k - (\mathbf{v}^k\cdot\nabla)\mathbf{w}^{n,k} \\ \qquad + (\nabla\mathbf{u}^{n,k})^t\mathbf{z}^k + (\nabla\mathbf{v}^k)^t\mathbf{w}^{n,k} + \nabla\eta^k = \mathbf{G}^{n,k}, \quad \mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{z}^k = 0, \quad \mathbf{x}\in\Omega, \\ \mathbf{v}^k = 0, \ \ \mathbf{z}^k = 0, \quad \mathbf{x}\in\partial\Omega \end{cases} \tag{10}$$

and take:

$$\mathbf{u}^{n,k+1} = \mathbf{u}^{n,k} - \mathbf{v}^k, \quad \mathbf{w}^{n,k+1} = \mathbf{w}^{n,k} - \mathbf{z}^k. \tag{11}$$

Note that these iterates are conceived to compute a solution to the optimality system (7) that, maybe, is not a minimizer of $J_{(\alpha)}$. Thus, they are expected to furnish numerical approximations of Pareto quasi-equilibria.

Let us illustrate the behavior of this algorithm with a numerical experiment in two spatial dimensions. Specifically, we will try to compute a minimizer of the functional in (6) for $\alpha = 0.5$ and $\mu = 1$. The domain Ω is composed by two rectangles O_1 and O_2 and we assume that the controls act on a narrow band ω. In order to solve numerically the systems (8), (9) and (10), we have to fix a mesh and a finite element method. We have used to this purpose meshes of the kind indicated in Fig. 3 and a mixed finite element formulation with continuous piecewise $\mathbb{P}_1$-bubble and $\mathbb{P}_1$ functions respectively for the velocity field and the pressure; for details, see [27, 30].

The data $\mathbf{u}_{id}$ are the following: $\mathbf{u}_{1d} = \nabla\times\psi_{1d}$, where ψ_{1d} is the solution to the problem

$$\begin{cases} -\Delta\psi_{1d} = 1, \quad \mathbf{x}\in O_1, \\ \psi_{1d} = 0, \quad \mathbf{x}\in\partial O_1 \end{cases}$$

and $\mathbf{u}_{2d} \equiv 0$. That means that the "desired" configuration corresponds to a uniformly rotating flow in O_1 and a fluid at rest in O_2 (see Fig. 3).

For the external iterates in the previous Newton algorithm, the stopping test has been

$$\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^\infty} + \|p^{n+1} - p^n\|_{L^\infty} \le \varepsilon,$$

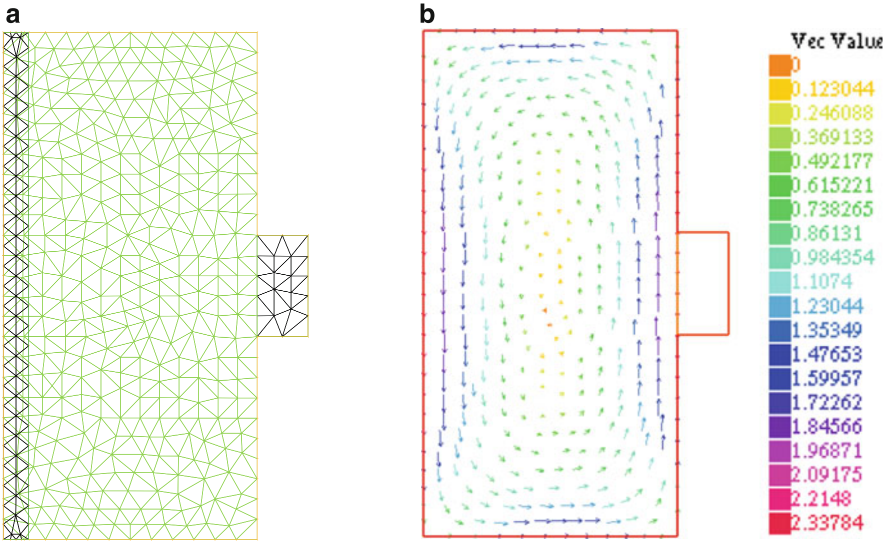

Fig. 3 (**a**) The domain and the "rough" mesh; Ω is composed of the band ω, the large rectangle $\mathcal{O}_1$ and the small rectangle $\mathcal{O}_2$; number of nodes: 1519; number of triangles: 2876. (**b**) The target velocity field $\mathbf{u}_{1d}$; $\mathbf{u}_{2d} = 0$

with $\varepsilon = 10^{-6}$. For the internal loops, the stopping test has been

$$\|\mathbf{u}^{n,k+1} - \mathbf{u}^{n,k}\|_{L^\infty} + \|\mathbf{w}^{n,k+1} - \mathbf{w}^{n,k}\|_{L^\infty} \leq \varepsilon.$$

The computations have been performed with the FreeFem++ package (see [33]). We have used three different meshes: a "rough" mesh with 1519 nodes, a "reasonable" mesh with 3449 nodes and, also, a "fine" mesh with 6003 nodes. Some results are depicted in Fig. 4.

2.3 A Null Controllability Problem

In the framework of the control of the time-dependent Navier-Stokes system, a very relevant question is whether one is able to drive any solution to rest working, i.e. acting, only at the points of a small set $\omega \times (0, T)$.

This is the so called *null controllability problem* for (1). A positive answer was conjectured by J.-L. Lions in [52, 53]. Since then, a lot of related contributions have appeared, but the problem still remains open; see [14, 16, 18, 24, 26, 32, 35] for the best results known at present. In particular, in [32] it is proved that, in the context of boundary control in a cube, by modifying slightly the right hand side of the equations, it is possible to find controls and associated states that vanish at the

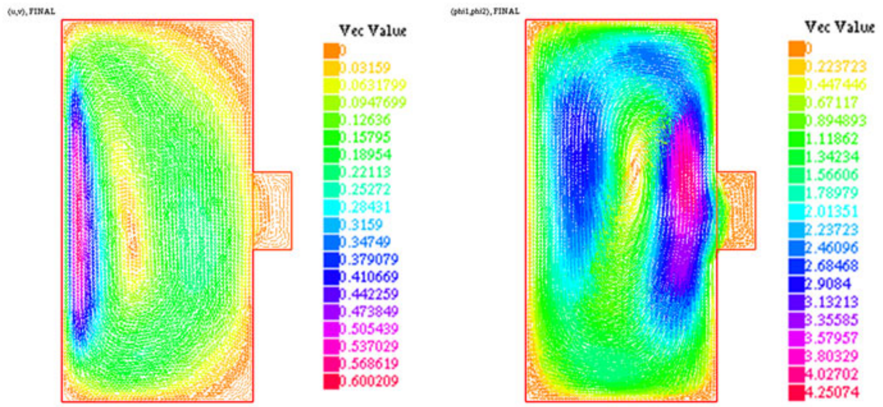

Fig. 4 Final velocity field $\mathbf{u}$ and adjoint $\mathbf{w}$ computed with a Newton-like method for $\alpha = 0.5$. Number of nodes: 6003. Number of triangles: 11684

final time $t = T$. In this section, we are going to recall a similar result for a system of the kind (4) that is established in [25].

More precisely, let us suppose that Ω is the unit cube in $\mathbb{R}^3$, Γ_0 is the face $\{\mathbf{x} \in \partial\Omega : x_1 = 0\}$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ and let us consider the so called Boussinesq system

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho}\nabla p - \theta\mathbf{k} = 0 \ \text{ in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} = 0 \ \text{ in } \Omega\times(0,T), \\ \theta_t - c\Delta\theta + \mathbf{u}\cdot\nabla\theta = 0 \ \text{ in } \Omega\times(0,T), \\ \mathbf{u}(\mathbf{x},t) = 0, \quad \theta(\mathbf{x},t) = 0 \ \text{ on } \Gamma_0\times(0,T), \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) \ \text{ in } \Omega, \end{cases} \tag{12}$$

where $\mathbf{k} \in \mathbb{R}^3$, $c > 0$ and the initial data $\mathbf{u}_0$ and θ_0 are prescribed.

It will be said that (12) is null-controllable if there exist boundary data for $\mathbf{u}$ and θ on $\Gamma_1 \times (0,T)$ such that an associated solution satisfies

$$\mathbf{u}(\mathbf{x},T) = 0 \ \text{ and } \ \theta(\mathbf{x},T) = 0 \ \text{ in } \ \Omega.$$

The following result holds:

Theorem 3 *Let $\mathbf{u}_0 \in V$ and $\theta_0 \in L^2(\Omega)$ be given. Then, there exist sequences $\{\mathbf{f}_n\}$ and $\{g_n\}$ with*

$$\mathbf{f}_n \to 0 \ \text{ in } \ L^r(0,T;H^{-1}(\Omega)^3) \ \text{ and } \ g_n \to 0 \ \text{ in } \ L^r(0,T;H^{-1}(\Omega)) \ \text{ as } n \to +\infty$$

for all $r \in [1, 4/3)$ such that, for each n, the coupled system (12) with right hand sides $\mathbf{f}_n$ and g_n respectively in the first and third equation is null-controllable.

The result says that, although maybe (12) is not null-controllable, we can always find "small" forces $\mathbf{f}_n$ and heat sources g_n such that, if they are imposed, the resulting system can be controlled exactly to zero.

The proof is inspired by the ideas in [32]. Essentially, what we have to do is to design the $(\mathbf{f}_n, g_n)$ and the boundary data in such a way that, at an intermediate time $t = T_0$, the velocity field possesses a specially simple structure. After this, in (T_0, T), the task is reduced to the proof of the boundary null controllability of a linear parabolic system, one-dimensional in space. In view of known results, this can be established. See [25] for more details; see also [17] for other similar results.

Remark 1 At present, no uniform estimate is known for the cost of controllability of the modified systems. In other words, it is unknown if the boundary controls furnished by Theorem 3 are uniformly bounded (independently of n) in a "good" space. If this were the case, we would probably be able to deduce the desired null controllability of (12). □

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Asymptotic Stability in Some Generic Classes of Three-Dimensional Discontinuous Dynamical Systems

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Abstract In this paper aspects of local asymptotic and Lyapunov stability of 3D piecewise smooth vector fields are studied.

1 Introduction

Asymptotically stability for smooth systems has classically been well developed. Asymptotic stability questions have probably motivated the introduction of many mathematical concepts (tools) in engineering, particularly in control engineering and it has had a stimulating impact on these fields, see [1–3, 7, 10, 12], for example. However, as far as the authors know, there are no specific or advanced techniques or tools within the piecewise smooth vector fields, PSVF for short.

This paper is part of a general program involving the study of PSVF in $\mathbb{R}^n$ of the form

$$\dot{u} = f(u) + \text{sign}(u_1)g(u),$$

where $u = (u_1, \ldots, u_n)$, and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. Note that we have two distinct differential systems, one in the half-space $u_1 > 0$ and the other in $u_1 < 0$.

Our main concern is to discuss the local Asymptotic Stability of PSVF on $\mathbb{R}^3$ around typical singularities. Our strategy is: 1-provide a codimension one

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classification of the systems around the origin and their respective normal forms and generic unfoldings, 2-exhibit conditions for the stability.

Let $\mathfrak{X}^r$ be the space of all germs of C^r vector fields on $(\mathbb{R}^3, 0)$ endowed with the C^r-topology, with $r \geq 1$. It seems natural to think that a qualitative study of stability of such systems, in general, needs to combine the dynamics of each smooth vector fields that compose the PSVF with the behavior of sliding modes.

Let Z be a PSVF. In our approach several ingredients and tools are used, such as elements of the contact between a vector field and the boundary of a manifold, sliding vector fields, first return map associated to Z and the interaction between these ingredients. As we are interested in the characterization of the asymptotic or Lyapunov stability at a typical singularity of Z, only the forward orbits of Z are considered.

The organization of the paper is as follows: In Sect. 2 some preliminaries, definitions and the notations are presented. In Sect. 3 the main results of the paper are discussed, in Sects. 4 and 5 we prove results on the asymptotic and Lyapunov stability at a typical singularity for a $3D$ family of PSVF.

2 Preliminaries

2.1 Filippov's Convention

Let be $M = \{(x, y, z); z = 0\}$ and we define a PSVF given by

$$Z(q) = \begin{cases} X(q), z \geq 0 \\ Y(q), z \leq 0, \end{cases}$$

where $X, Y \in \mathfrak{X}^r$. Call $\Omega^r = \mathfrak{X}^r \times \mathfrak{X}^r$ the space of all germs of vector fields Z at 0 endowed with the product topology and we denote $Z = (X, Y)$. Consider the map $h : (x, y, z) \mapsto z$. For each $X \in \mathfrak{X}^r$ we define a smooth function $Xh : \mathbb{R}^3 \to \mathbb{R}$ by $Xh = \langle X, \nabla h \rangle$ where $\langle ., . \rangle$ is the canonical inner product in $\mathbb{R}^3$. We denote $X^n h(p) = X(X^{n-1}h)(p), n \geq 2$. Let be $Z = (X, Y) \in \Omega^r$. In M, we distinguish the following open regions:

- **Sewing Region:** $M^c = \{p \in M; (Xh)(p)(Yh)(p) > 0\}$. We denote $M^c_+ = \{p \in M; (Xh)(p) > 0, (Yh)(p) > 0\}$ and $M^c_- = \{p \in M; (Xh)(p) < 0, (Yh)(p) < 0\}$.
- **Escaping Region:** $M^e = \{p \in M; (Xh)(p) > 0, (Yh)(p) < 0\}$.
- **Sliding Region:** $M^s = \{p \in M; (Xh)(p) < 0, (Yh)(p) > 0\}$.

Let $\mathcal{O} = M^c \cup M^e \cup M^s$. Observe that for any $p \in \mathcal{O}$ we get $X(p) \neq 0$ and $Y(p) \neq 0$ and if $p \in M^s$ then $\langle (Y - X)(p), \nabla h(p) \rangle > 0$.

The following definitions of orbit-solutions at points in the switching manifold $M = \{(x, y, z); z = 0\}$ are given by Filippov's convention (stated in [6]). The *sliding vector field* associated to Z is the vector field Z^M tangent to M^s and defined at $q \in M^s$ by

$$Z^M = \frac{1}{Y_3 - X_3}(X_1Y_3 - Y_1X_3, X_2Y_3 - Y_2X_3), \quad (1)$$

where $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$. It is clear that the vector field Z^M is orbitally equivalent, on M^s, to the *rescaling sliding vector field* given by

$$\widetilde{Z}^M = (X_1Y_3 - Y_1X_3, X_2Y_3 - Y_2X_3) \quad (2)$$

and therefore Z^M can be C^r extended beyond the boundary of M^s. Throughout the paper the forward orbit of Z through a point $p \in Cl(M^s)$ is given by the orbit of $\widetilde{Z}^M$ (and therefore is contained in M^s).

Definition 1 The point $p \in M^s$ is said a pseudo-equilibrium if $Z^M(p) = 0$.

As said above, Filippov's rule describes three basic forms of dynamics that would occur on the switching manifold: sewing, sliding and escaping. The trajectories of X and Y are denoted by ϕ_X and ϕ_Y, respectively.

2.2 Contact Between a Smooth Vector Field X and a Codimension One Submanifold M of $\mathbb{R}^3$

Details on the concepts and results treated in this subsection are in [9] and [13].

Definition 2 Let $X \in \mathfrak{X}^r$. We say that:

(a) 0 is M-regular point of X if $Xh(0) \neq 0$.
(b) 0 is a fold singularity of X if $Xh(0) = 0$ and $X^2h(0) \neq 0$.
(c) 0 is a cusp singularity of X if $Xh(0) = X^2h(0) = 0, X^3h(0) \neq 0$ and the vectors $dh(0)$, $dXh(0)$ and $dX^2h(0)$ are linearly independent.

Consider $S_X = \{p \in M; Xh(p) = 0\}$ the *tangential singularities* of X. In the following we characterize the generic behavior of smooth vector fields in $(\mathbb{R}^3, 0)$ relative to a codimension one submanifold M.

Definition 3 We call Σ_0 the set of elements $X \in \mathfrak{X}^r$ satisfying one of the following conditions:

(a) 0 is a M-regular point of X $(Xh(0) \neq 0)$.
(b) 0 is a fold singularity of X.
(c) 0 is a cusp singularity of X.

In the following we roughly define the classical notion of codimension of vector fields.

Definition 4 Consider $\Theta(Z)$ the set of all small perturbations of Z, defined on a compact set K such that $0 \in K$. We say that the codimension of Z at the singularity 0 is k if it appears exactly k distinct topological types of vector fields in $\Theta(Z)$.

In [9], it was proved that the subset Σ_0 is open, dense and it characterizes the structural stability in $\mathfrak{X}^r$. Following the Thom-Smale program the next step is to study the bifurcation set $\mathfrak{X}_1^r = \mathfrak{X}^r - \Sigma_0$. We will follow such strategy in $Z = (X, Y) \in \Omega^r$. For simplicity, when we say that Z is of codimension k means that the codimension of Z at the singularity 0 is k. The codimension of $Z = (X, Y)$ at 0 is, at least, the sum of codimensions of X and Y at 0. This is due to the fact that the dynamics of Z is composed by the dynamics of X, Y, the sliding vector fields Z^M, the first return map φ_Z and all the iteration of these ingredients.

As usual, the characterization of $\Sigma_1 \subset \mathfrak{X}_1^r$, the subset composed by PSVFs that are of codimension one, is based in certain issues involving unstable vector fields without rejecting a generic context. It means that certain conditions imposed on the definition of Σ_0 are violated but quasi-generic assumptions are considered.

Definition 5 Call Σ_1 the set $\Sigma_1(a) \cup \Sigma_1(b) \subset \mathfrak{X}^r$, where:

1. $\Sigma_1(a) \subset \mathfrak{X}^r$ is the set of smooth vector fields X such that 0 is a hyperbolic singular point, the eigenvectors are transverse to M at 0, the eigenvalues of $DX(0)$ are of algebraic multiplicity 1. Moreover the real parts of the non-conjugated eigenvalues are distinct, i.e., if $\lambda_i \neq \lambda_j$ and $\lambda_i \neq \overline{\lambda_j}$ then $Re(\lambda_i) \neq Re(\lambda_j)$, where λ_i, λ_j are eigenvalues of $DX(0)$ and $\overline{\lambda_j}$ denotes the conjugated of the number λ_j.
2. $\Sigma_1(b) \subset \mathfrak{X}^r$ is the set of smooth vector fields X such that $X(0) \neq 0$, $(Xh)(0) = 0 = (X^2h)(0)$ and one of the following conditions are valid:

   $(b.1)$ $(X^3h)(0) \neq 0$, $\dim\{dh(0), d(Xh)(0), d(X^2h)(0)\} = 2$ and 0 is a non degenerate critical point (Morse) of $Xh\mid_M$.

   $(b.2)$ $(X^3h)(0) = 0$, $(X^4h)(0) \neq 0$ and 0 is a regular point of $Xh\mid_M$.

Let $H(h)$ be the Hessian matrix of the function h.

Definition 6 If $X \in \Sigma_1(a) \subset \mathfrak{X}^r$ then we distinguish the cases:

$(a.1)$ **Node**: $X(0) = 0$, the eigenvalues of $DX(0)$, λ_j, $j = 1, 2, 3$ are real distinct, have the same sign and the corresponding eigenvectors are transversal to M at 0.

$(a.2)$ **Saddle**: $X(0) = 0$, the eigenvalues of $DX(0), \lambda_j, j = 1, 2, 3$ are real distinct, one of then has opposite sign in relation to the others and the corresponding eigenvectors are transversal to M at 0.

$(a.3)$ **Focus**: $X(0) = 0$, 0 is a hyperbolic critical point of X, the eigenvalues of $DX(0)$ are $\lambda_{12} = a \pm bi$, $\lambda_3 = c$ with a, b, c distinct of zero, $a \neq c$ and the corresponding eigenvector is transversal to M at 0.

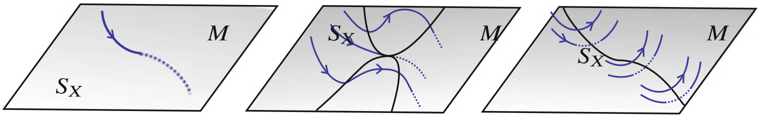

Fig. 1 The family $\Sigma_1(b)$ of 3D PSVF

If $X \in \Sigma_1(b) \subset \mathfrak{X}^r$ then we distinguish the cases:

$(b.1.1)$ **Lips** given in Definition 5 $(b.1)$ with $\det(H(Xh|_M(0))) > 0$, see Fig. 1.
$(b.1.2)$ **Beak to beak** given in Definition 5 $(b.1)$ with $det(H(Xh|_M(0))) < 0$.
$(b.1.3)$ **Dove's tail** given in Definition 5 $(b.2)$.

2.3 Classification of Singularities of Z ∈ Ω^r

Definition 7 We say that $0 \in \mathbb{R}^3$ is:

(a) a two-fold singularity of Z if it is a fold singularity of both vector fields X and Y.
(b) fold-cusp singularity of Z if it is a fold singularity of X and a cusp singularity of Y.

Definition 8 We say that the origin is an M-singularity or just a singularity of Z if:

- either it is a singular point of X or Y ($X(0) = 0$ or $Y(0) = 0$).
- or $X(0) \cdot Y(0) \neq 0$ and it is a tangential singularity of X or Y.

Lemma 1 *Suppose that $X(0) \neq 0$ and $Y(0) \neq 0$. If 0 is a tangential singularity of both, X and Y, then $\widetilde{Z}^M(0) = 0$. On the other hand, if the origin is a tangential singularity of X and a regular point of Y, or vice-versa, then $\widetilde{Z}^M(0) \neq 0$.*

Proof First, we assume that the origin is a tangential singularity of X and Y then $Xh(0) = X_3(0) = 0$ and $Yh(0) = Y_3(0) = 0$. Therefore, by (2) we get $\widetilde{Z}^M(0) = (X_1(0)Y_3(0) - Y_1(0)X_3(0), X_2(0)Y_3(0) - Y_2(0)X_3(0)) = (0, 0)$.

Suppose now that X is tangent and Y is transversal to M at $(0, 0, 0)$. So, $Xh(0) = X_3(0) = 0$ and $Yh(0) = Y_3(0) \neq 0$. From the expression of $\widetilde{Z}^M$, we get

$$\widetilde{Z}^M(0) = (X_1(0)Y_3(0), X_2(0)Y_3(0)).$$

In this way, we derive that the origin is a pseudo equilibrium of $\widetilde{Z}^M$ if and only if $X(0) = 0$, which contradicts the hypothesis. □

2.4 Properties of Some Families of 3D PSVF's

In the following relevant intrinsic properties of some classes of 3D PSVF's are discussed. Consider the subsets $\Omega_0 = \bigcup_{i=1}^{4} \Omega_0(i)$ and $\Omega_1 = \bigcup_{i=1}^{6} \Omega_1(i)$ where

- $Z = (X, Y) \in \Omega_0(1)$ if 0 is a regular point of both X and Y (**regular-regular**).
- $Z = (X, Y) \in \Omega_0(2)$ if 0 is a fold singularity of X and a regular point of Y (**fold-regular**).
- $Z = (X, Y) \in \Omega_0(3)$ if 0 is a cusp singularity of X and a regular point of Y (**cusp-regular**).
- $Z = (X, Y) \in \Omega_0(4)$ if 0 is a two-fold singularity of Z, S_X and S_Y are transverse at 0, the eigenspaces of $\widetilde{Z}^M$ are transverse to S_X and S_Y at 0 and 0 is a hyperbolic singular point of $\widetilde{Z}^M$ (**two-fold**).
- $Z = (X, Y) \in \Omega_1(1)$ if 0 is a lips singularity of X_0 and a regular point of Y_0, (see Definition 6) (**lips-regular**).
- $Z = (X, Y) \in \Omega_1(2)$ if 0 is a beak to beak singularity of X_0 and a regular point of Y_0 (**beak to beak-regular**).
- $Z = (X, Y) \in \Omega_1(3)$ if 0 is a dove's tail singularity of X_0 and a regular point of Y_0 (**dove's tail-regular**).
- $Z = (X, Y) \in \Omega_1(4)$ if 0 is a two-fold singularity and the contact between S_{X_0} and S_{Y_0} is quadratic at 0, $X_0^2 h(0) \neq Y_0^2 h(0)$. Moreover the eigenspaces of $D\widetilde{Z}_0^M(0)$ are transversal to S_{X_0} and S_{Y_0} at 0 (**1-degenerate two-fold**).
- $Z = (X, Y) \in \Omega_1(5)$ if 0 is a fold singularity, cusp singularity, resp., of X_0, Y_0, resp. and S_{X_0} is transversal to S_{Y_0} at 0 (**fold-cusp**).
- $Z = (X, Y) \in \Omega_1(6)$ if 0 is a two-fold singularity such that S_{X_0} is transversal to S_{Y_0} at 0, the origin is a saddle-node singularity of $\widetilde{Z}_0^M$, the eigenspaces of $D\widetilde{Z}_0^M(0)$ are transversal to S_{X_0} and S_{Y_0} at 0 and the center manifold intercepts M^s (**two-fold-saddle-node**).

A detailed analysis of such classes can be found in [9] and [11].

2.5 Subclasses of Ω_0 and Ω_1

As usual, we define the sign function on $\mathbb{R}$ as sign(0) = 0, sign$(x) = 1$ if $x > 0$, sign$(x) = -1$ if $x < 0$. We now make explicit some subclasses of codimension zero 3D PSVF, namely Ω_0. Our purpose is to characterize the asymptotic and Lyapunov stability at the origin for $Z \in \Omega_0$ that is C^0 M-equivalent to one of these subclasses. The approach to get these subclasses is given in [9] and [11]. In [5] are exhibited all the normal forms of structurally stable piecewise smooth vector fields (codimension zero). These normal forms are contained in the subsets $\widetilde{\Omega}_0(i) \subset \Omega_0$, $i = 1, \ldots, 4$ constituted by $Z = (X, Y) \in \Omega_0(i)$ such that:

- $\widetilde{\Omega}_0(1)$: $X(x, y, z) = (a, b, c)$ and $Y(x, y, z) = (d, e, f)$, where $a = X_1(0)$, $b = X_2(0)$, $d = Y_1(0)$, $e = Y_2(0)$, $Xh(0) = c \neq 0$ and $Yh(0) = f \neq 0$.

- $\widetilde{\Omega}_0(2)$: $X(x,y,z) = (a,0,x)$ and $Y(x,y,z) = (1,0,b)$ where $a = X^2h(0) \neq 0$ and $b = Yh(0) \neq 0$.
- $\widetilde{\Omega}_0(3)$: $X(x,y,z) = (a,0,b(y+x^2))$ and $Y(x,y,z) = (1,0,c)$ where $a = X_1(0)$, $\text{sign}(b) = \text{sign}(X^3h(0)) \neq 0$ and $c = Yh(0) \neq 0$.
- $\widetilde{\Omega}_0(4)$: $X(x,y,z) = (a,b,x)$ and $Y(x,y,z) = (c,d,y)$ where $a = X^2h(0) \neq 0$, $b = X(Yh)(0)$, $c = Y(Xh)(0)$ and $d = Y^2h(0) \neq 0$.

Notice that, as far as the authors know, there is not a complete classification of $3D$ codimension one singularities, namely Ω_1. In what follows, we exhibit a list of distinct topological types of codimension one singularities which are obtained via normal forms tools. However, it is not the main purpose of the present paper, to exhibit explicitly a complete list of normal forms of codimension one three dimensional PSVF. Consider the subsets $\widetilde{\Omega}_1(i) \subset \Omega_1(i)$, $i = 1,\dots,6$ constituted by the PSVF $Z_\lambda = (X_\lambda, Y_\lambda) \in \Omega_1(i)$ such that:

- $\widetilde{\Omega}_1(1)$: $X_\lambda(x,y,z) = (a,b,x^2+y^2+\lambda)$ and $Y_\lambda(x,y,z) = (1,0,c)$ where $a^2+b^2 = X_0^3h(0) \neq 0$ and $c = Y_0h(0) \neq 0$.
- $\widetilde{\Omega}_1(2)$: $X_\lambda(x,y,z) = (a,b,x^2-y^2+\lambda)$ and $Y_\lambda(x,y,z) = (1,0,c)$ where $a^2-b^2 = X_0^3h(0) \neq 0$ and $c = Y_0h(0) \neq 0$.
- $\widetilde{\Omega}_1(3)$: $X_\lambda(x,y,z) = (a,0,x^3+y+\lambda x)$ and $\widetilde{Y}_\lambda(x,y,z) = (1,0,b)$ where $\text{sign}(a) = \text{sign}(X_0^4h(0)) \neq 0$ and $b = Y_0h(0) \neq 0$.
- $\widetilde{\Omega}_1(4)$: $X_\lambda(x,y,z) = (a,b,\varepsilon_1(x-(y^2+\lambda)))$ and $Y_\lambda(x,y,z) = (c,d,\varepsilon_2(x+y^2))$ where $a = X_0^2h(0) \neq 0$, $b = X_0^2(0)$, $c = Y_0^2(0)$, $d = Y_0^2h(0) \neq 0$ and $\varepsilon_i = \pm 1$.
- $\widetilde{\Omega}_1(5)$: $X_\lambda(x,y,z) = (a,\lambda,b(y+x^2))$ and $Y_\lambda(x,y,z) = (c,d,x)$ where $a = X_{1,0}(0) \neq 0$, $\text{sign}(b) = \text{sign}(X_0^3h(0)) \neq 0$, $c = Y_0^2h(0) \neq 0$ and $d = Y_{2,0}(0)$.
- $\widetilde{\Omega}_1(6)$: $X_\lambda(x,y,z) = (a+a\lambda-a^2y, b+c\lambda-acy, x)$ and $Y_\lambda(x,y,z) = (c+b\lambda+b^2x-2aby, d+d\lambda+bdx-2bcy, y)$ where $a = X_0^2h(0)$, $b = X_0(Y_0h)(0)$, $c = Y_0(X_0h)(0)$, $d = Y_0^2h(0)$ are not zero and $ad = bc$.

3 Main Results

In this section we provide some results about the asymptotic/Lyapunov stability at the origin of subsets of Ω_0 and Ω_1 defined in Sect. 2.5.

Theorem 1 *Consider $Z = (X,Y) \in \Omega_0$. The following statements are true:*

(*a*) *$Z \in \widetilde{\Omega}_0(1)$ is asymptotically stable (resp. Lyapunov stable) at 0 if and only if $X(0)$ and $Y(0)$ are linearly dependent, $\Sigma = \Sigma^s$ and $\widetilde{Z}^\Sigma$ is asymptotically stable (resp. Lyapunov stable) at 0.*

(*b*) *If $Z \in \widetilde{\Omega}_0(2) \cup \widetilde{\Omega}_0(3) \cup \widetilde{\Omega}_0(4)$ then Z is not Lyapunov stable at 0.*

The proof of Theorem 1 is presented in Sect. 4, as consequence of Lemmas 2–4. Let us define:

$$\Omega_0^A(4) = \{Z \in \widetilde{\Omega}_0(4);\ X^2h(0) < 0,\ Y^2h(0) > 0,\ X(Yh)(0) - Y(Xh)(0) < 0,\\ X(Yh)(0)Y(Xh)(0) < X^2h(0)Y^2h(0)\}.$$

In the following we characterize the basin of attraction of 0 of $Z \in \Omega_0^A(4)$. The proof of this result is given in Theorem B and Proposition 2 in [8].

Proposition 1 *If $Z \in \Omega_0^A(4)$ then the basin of attraction of 0 is:*

$$\mathcal{B}_Z = \Sigma^s \cup \Sigma^e \cup [\Sigma^c \backslash R_0]$$

where $R_0 = [\{(x, y, 0) \in \Sigma;\ x > 0,\ y < 2X(Yh)(0)[X^2h(0)]^{-1}x\} \cap \Sigma^c]$.

Let $\Delta = (a-d)^2 + 4c(b-1)$. Consider the sets

$$\begin{aligned} \Omega_1^A(5) = \{Z_0 \in \widetilde{\Omega}_1(5);\ & X_0^1(0) < 0,\ X_0^3h(0) < 0,\ Y_0^2h(0) > 0,\\ & Y_0^2(0) < 0,\ X_0^1(0) + \frac{X_0^3h(0)Y_0^2(0)}{2(X_0^1(0))^2} > 0\},\\ \Omega_1^A(6) = \{Z_0 \in \widetilde{\Omega}_1(7);\ & X_0^2h(0) < 0,\ X_0(Y_0h)(0) < 0,\\ & X_0^2h(0)Y_0^2h(0) = X_0(Y_0h)(0)Y_0(X_0h)(0),\\ & Y_0^2h(0) > 0,\ X_0(Y_0h)(0) - Y_0(X_0h)(0) < 0\}. \end{aligned} \tag{3}$$

Next results deal with local asymptotic stability at the origin for some classes of codimension one PSVFs.

Theorem 2 *Let $Z_0 \in \Omega_1$.*

(*a*) *If $Z_0 \in \widetilde{\Omega}_1(1) \cup \widetilde{\Omega}_1(2) \cup \widetilde{\Omega}_1(3) \cup \widetilde{\Omega}_1(4) \cup \widetilde{\Omega}_1(6)$ then Z_0 is not Lyapunov stable at* 0.
(*b*) *If $Z_0 \in \Omega_1^A(5)$ then Z_0 is asymptotically stable at* 0*;*

The proof of Theorem 2 follows from Lemmas 5–9. As done in the case of $Z \in \Omega_0^A(4)$, the next proposition provides us the basin of attraction of the origin for the case where $Z \in \Omega_1^A(6)$.

Proposition 2 *If $Z \in \Omega_1^A(6) \subset \widetilde{\Omega}_1(6)$, then the basin of attraction of* 0 *is expressed as*

$$\begin{aligned} \mathcal{B}_{Z_0} = \Sigma^s & \bigcup \left\{(x, y, 0);\ x < [X_0^2h(0)][2X_0(Y_0h)(0)]^{-1}y,\ y > 0\right\}\\ & \bigcup \left\{(x, y, 0);\ x < [2X_0^2h(0)][X_0(Y_0h)(0)]^{-1}y,\ y < 0\right\}. \end{aligned} \tag{4}$$

The proof of Proposition 2 follows by Lemma 9.

4 Asymptotic Stability in $\Omega_0 \subset \Omega^r$

In this section, the asymptotic stability of $Z \in \widetilde{\Omega}_0(1)$ at the origin is characterized. If $Z \in \widetilde{\Omega}_0(2) \cup \widetilde{\Omega}_0(3)$ then we prove that Z is not Lyapunov stable at 0. The local dynamics of $Z \in \widetilde{\Omega}_0(4)$ is provided in [8].

Lemma 2

(a) *If $X(0)$ and $Y(0)$ are linearly independent then the origin is not an M-singularity of Z.*
(b) *Assume that $X(0)$ and $Y(0)$ are linearly dependent. Then*

(i) *If $\Sigma = \Sigma^c \cup \Sigma^e$ then Z is not Lyapunov stable at* 0.
(ii) *If $\Sigma = \Sigma^s$ and $\widetilde{Z}^\Sigma$ is asymptotically stable, or Lyapunov stable at* 0 *then Z is asymptotically stable, or Lyapunov stable, respectively, at* 0.

Proof The switching manifold coincides with the regions Σ^c, Σ^s or Σ^e, according to the signs of $Xh(0)$ and $Yh(0)$. In fact, if $Xh(0)Yh(0) > 0$ then $\Sigma = \Sigma^c$. If $Xh(0) < 0$ and $Yh(0) > 0$ then $\Sigma = \Sigma^s$ and if $Xh(0) > 0$ and $Yh(0) < 0$ then $\Sigma = \Sigma^e$.

We prove initially Item (a). Consider $U \subset \mathbb{R}^3$ an arbitrary neighborhood of the origin. If $X(0)$ and $Y(0)$ are linearly independent and observing that $Xh(0) \neq 0$ and $Yh(0) \neq 0$ then from Lemma 1 we get that 0 it is not a M-singularity of Z. Therefore, through the flow box construction we get that any trajectory of Z, with initial condition in U, leaves U.

Consider now Item (b). If $X(0)$ and $Y(0)$ are linearly dependent and $\Sigma = \Sigma^e \cup \Sigma^c$ then any trajectory of Z passing through $p \in U \cap \Sigma$ does not return to Σ (locally speaking). Finally, if $X(0)$ and $Y(0)$ are linearly dependent and $\Sigma = \Sigma^s$ then the origin is a pseudo equilibrium of $\widetilde{Z}^\Sigma$. Therefore, if 0 is an attractor for $\widetilde{Z}^\Sigma$ then 0 is an attractor for Z. □

Lemma 3 *If $Z \in \widetilde{\Omega}_0(2)$ then Z is not Lyapunov stable at* 0.

Proof The proof follows from Lemma 1 applying the Flow Box Theorem for $\widetilde{Z}^M$ and considering the contact between X and M at the origin. □

Lemma 4 *If $Z \in \widetilde{\Omega}_0(3)$ then Z is not Lyapunov stable at* 0.

Proof As the previous case, the proof of this lemma follows from Lemma 1 and considering the contact between X and M at the origin. □

The proofs of Theorem 1 and Proposition 1 for the case where $Z \in \widetilde{\Omega}_0(4)$ follow from Theorem B and Proposition 2 in [8]. Finally, the proof of Theorem 1 follows from Lemmas 2–4.

5 Asymptotic Stability in $\Omega_1 \subset \Omega^r$

In summary, the elements in $\widetilde{\Omega}_1(1)$, $\widetilde{\Omega}_1(2)$, $\widetilde{\Omega}_1(3)$ are trivially not Lyapunov stable at the origin. The analysis of the subclasses $\widetilde{\Omega}_1(4)$, $\widetilde{\Omega}_1(5)$ are provided in [4, 8], respectively. For the subclass $\widetilde{\Omega}_1(6)$ we are able to show the asymptotic stability or exhibit at least a basin of attraction at 0.

Lemma 5 *If* $Z_0 \in \widetilde{\Omega}_1(1)$ *then* Z_0 *is not Lyapunov stable at* 0.

Proof Observe that $X_0h(x, y, 0) = x^2+y^2 > 0$. Therefore, the switching manifold Σ coincides with Σ^c or Σ^e, according to the sign of $Y_0h(0)$. In this way, applying the flow box construction for the smooth vector field X_0 (in M_+) we get that the trajectories of X_0 leave any small neighborhood U of origin. Therefore, we conclude that Z_0 is not Lyapunov stable at the origin. □

Lemma 6 *If* $Z_0 \in \widetilde{\Omega}_1(2)$ *then* Z_0 *is not Lyapunov stable at* 0.

Proof We consider the case where $Y_0h(0) = c > 0$. So, by a time rescaling we can suppose that $c = 1$. The flow of Z_0 is given by: $\phi_{X_0}(t, (x, y, z)) = (x + at, y + bt, z + (x^2 - y^2)t + (ax - by)t^2 + 1/3(a^2 - b^2)t^3)$ and $\phi_{Y_0}(t, (x, y, z)) = (x + t, y, z + t)$. We get the regions in Σ: $\Sigma^s = \{(x, y, 0); x^2 - y^2 < 0\}$ and $\Sigma^c = \{(x, y, 0); x^2 - y^2 > 0\}$.

By (2) we obtain: $\widetilde{Z}_0^{\Sigma}(x, y) = (a - (x^2 - y^2), b)$ which is regular in a neighborhood of the origin. The dynamics of $\widetilde{Z}_0^{\Sigma}$ depends on a and b. In fact, we consider the cases: (i) $|a| \leq |b|$; (ii) $|a| > |b|$.

Given $p = (x, y, 0) \in \Sigma^c \subset \Sigma$ and if $a^2 - b^2 \leq 0$ (case (i)) then there exists a positive time of trajectory of X_0 passing through p to Σ: $t_1(p) = [-3ax + 3by - \sqrt{\Theta}][2(a^2 - b^2)]^{-1} > 0$ and $\phi_{\widetilde{X}_0}(t_1(p), p) \in \Sigma$, where $\Theta = -12(a^2 - b^2)(x^2 - y^2) + (3ax - 3by)^2$. Therefore, from Lemma 1, we get that $\widetilde{Z}_0^{\Sigma}$ is regular at origin and any trajectory does not intercept the set of Σ-singularities S_{X_0}. So, we conclude that all trajectories of $\widetilde{Z}_0^{\Sigma}$ leave any neighborhood of the origin.

If $a^2 - b^2 = X_0^2h(0) > 0$ (case (ii)) then there is no forward first return for the trajectories of X_0 passing through $p \in \Sigma$. So, applying the flow box construction for X_0 we get that $Z_0 \in \widetilde{\Omega}_1(2)$ is not Lyapunov stable at 0. □

Lemma 7 *If* $Z_0 \in \widetilde{\Omega}_1(3)$ *then* Z_0 *is not Lyapunov stable at* 0.

Proof The proof is analogous to the previous lemma and we omit. □

The proof of Theorem 2 for the case where $Z_0 \in \widetilde{\Omega}_1(4)$ follows from Theorem C and Proposition 2 in [8] and finally the proof of Theorem 2 for the case where $Z_0 \in \widetilde{\Omega}_1(5)$ follows from Theorem A in [4].

We remain that the subset $\Omega_1^A(6)$ is defined in (3).

Lemma 8 *If* $Z_0 \in \Omega_1^A(6) \subset \widetilde{\Omega}_1(6)$ *then* $\widetilde{Z}_0^{\Sigma}$ *is asymptotically stable at* 0.

Proof The linear part of the rescaling sliding vector field is given by $(ay - cx, by - dx)$. As the origin is a saddle-node for $\widetilde{Z}_0^{\Sigma}$, we must require that

$$ad = bc. \tag{5}$$

The regions on Σ are: $\Sigma^s = \{(x, y, 0); x < 0, y > 0\}$, $\Sigma^c = \{(x, y, 0); xy > 0\}$ and $\Sigma^e = \{(x, y, 0); x > 0, y < 0\}$. The non zero eigenvalue of $D\widetilde{Z}_0^{\Sigma}(0)$ is $\eta = b - c$. We suppose that $\eta < 0$. Therefore, by a time rescaling procedure we can suppose that $\eta = -1$, i.e.,

$$Y_0(X_0 h)(0) = X_0(Y_0 h)(0) + 1. \tag{6}$$

Observe that the Eqs. (5) and (6) are satisfied since the number $ad \in (-1/4, 0)$. In fact, if $b - c = -1$ and $ad = bc$ then $b = (-1 \pm \sqrt{1 + 4ad})/2$ which is a real positive number since $ad \in (-1/4, 0)$.

Let $v_1 = (1, c/a), v_2 = (1, b/a)$ be the tangent vectors to W^c, W^s where W^c, W^s are the central, stable, manifold of $\widetilde{Z}_0^{\Sigma}$, respectively. If $c/a < 0$ and $b/a > 0$ then W^c intercepts Σ^s and W^s intercepts Σ^c. In this way, we obtain the subset $\Omega_1^A(7)$ defined in (3). So, we get the asymptotic stability at the origin for $\widetilde{Z}_0^{\Sigma}$ in this case. □

We recall that the definition of the basin of attraction of $Z_0 \in \Omega_1^A(6)$, denoted by $\mathcal{B}_{Z_0} \subset \Sigma$, is given in (4).

Lemma 9 *If* $Z_0 \in \widetilde{\Omega}_1(6)$ *then* Z_0 *is not Lyapunov stable at* 0. *Moreover, if* $Z_0 \in \Omega_1^A(6)$ *then the basin of attraction* $\mathcal{B}_{Z_0} \subset \Sigma$ *is completely characterized.*

Proof Let $Z_0 \in \Omega_1^A(6)$ and $p = (x, y, 0) \in \Sigma^c$. We get the flows: $\phi_{\widetilde{X}_0}(t, p) = (x + at, y + bt, z + xt + a/2t^2) + O(t^2, t^2, t^3)$ and $\phi_{\widetilde{Y}_0}(t, p) = (x + ct, y + dt, z + yt + 1/2dt^2) + O(t^2, t^2, t^3)$. We denote $t_1(p) = -2x/a + O(||(x, y, z)||^2)$ and $t_2(p_1) = (4bx - 2ay)(ad)^{-1} + O(||(x, y, z)||^2)$. We have $\phi_{X_0}(t_1(p), p) = p_1 \in \Sigma$ and $\phi_{Y_0}(t_2(p_1), p_1) \in \Sigma$. If $a = X_0^2 h(0) < 0$ and $d = Y_0^2 h(0) > 0$ then t_1, t_2 are real positive numbers. The first return map is given by:

$$\varphi_{Z_0}(x, y) = \left(3x - \frac{2c}{d}y, \frac{2b}{a}x - y\right) + O(||(x, y)||^2). \tag{7}$$

We define $R_1 = \{(x, y, 0); x < \dfrac{a}{2b}y, y > 0\} \subset \Sigma^c$ and $R_2 = \{(x, y, 0); x < \dfrac{2a}{b}y, y < 0\} \subset \Sigma^c$. In Lemma 8 it is proved that $\widetilde{Z}_0^{\Sigma}$ is asymptotically stable at origin provided $Z_0 \in \Omega_1^A(6)$. Now we prove that if $p = (x, y, 0) \in R_1$ (resp. $p \in R_2$) then $\phi_{X_0}(t_1, p) \in \Sigma^s$ (resp. $\phi_{Y_0}(t_2(p), p) \in \Sigma^s$). Note that if $p \in R_1$ then $\phi_{X_0}(t_1, p) = (-x, y - \dfrac{2b}{a}x) + O(||(x, y)||^2) = (x_1, y_1) = p_1$, with $x_1 < 0$

and $y_1 > 0$. Therefore, $p_1 \in \Sigma^s$. Similarly, we get that $\phi_{Y_0}(t_2(p), p) \in \Sigma^s$, with $p \in R_2$. So the basin of attraction is given by $\mathcal{B}_{Z_0} = \Sigma^s \cup R_1 \cup R_2$.

To prove that $Z_0 \in \widetilde{\Omega}_1(6)$ is not Lyapunov stable at the origin, we analyse the behavior of the first return map. Considering the linear part, $L_{Z_0}(x, y)$, of expression (7), we get

$$L^n_{Z_0}(x, y) = ((2n + 1)x - 2an(b)^{-1}y,\quad 2bn(a)^{-1}x - (2n - 1)y) = (x_n, y_n). \tag{8}$$

Consider $F = \{(x, y, 0); x > \frac{a}{b}y, y > 0\} \subset \Sigma^c$ and $E = \{(x, y, 0); x > \frac{2a}{b}y, y > 0\} \subset F$. The iterates of L_{Z_0} satisfies the following properties:

(a) $L^n_{Z_0}(p) \in F$, for all $p \in F$;
(b) Consider the sequences of real numbers $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ given in (8) where $(x, y, 0) \in E$. These are monotone and increasing sequences.

We define the half-straight lines: $r_1 = \{(x, y, 0); x = \frac{a}{b}y, y > 0\}$ and $r_2 = \{(x, y, 0); x = \frac{2a}{b}y, y > 0\}$, that are the frontiers of the sets F, E, respectively. Note that the properties **(a)** and **(b)** show us that the iterates of L_{Z_0} leave any neighborhood of the origin. So, if $Z_0 \in \widetilde{\Omega}_1(6)$ then it is not Lyapunov stable at 0. Recall that $ad = bc$ and $a/c < 0$ (the center manifold $W^c = \{(x, y); y = cx/a + \mathcal{O}(x^2)\}$ intercepts the $\Sigma^s = \{(x, y, 0);\ x < 0, y > 0\}$). Therefore, $b = X_0(Y_0h)(0) < 0$ and $c = Y_0(X_0h)(0) > 0$.

Now we prove that L_{Z_0} satisfies the property **(a)**. We have $y_n = \frac{2nb}{a}x - (2n - 1)y = 2n\frac{b}{a}(x - \frac{a}{b}y) + y > 0$, because $(x, y, 0) \in F$ and $a = X_0^2h(0) < 0, b = X_0(Y_0h)(0) < 0$. Besides, $x_n - \frac{a}{b}y_n = (2n+1)x - 2n\frac{a}{b}y - \frac{a}{b}(2n\frac{b}{a}x - (2n-1)y) = x - \frac{a}{b}y > 0$, since $(x, y) \in F$. Therefore, the set F is invariant by L_{Z_0}.

To prove **(b)**, we get $x_{n+1} - x_n = [(2n + 1)x - 2n\frac{a}{b}y] - [2nx - (2n - 1)\frac{a}{b}y] = x - \frac{2a}{b}y > 0$ and $y_{n+1} - y_n = [\frac{2nb}{a}x - 2(n - 1)y] - [2(n - 1)\frac{b}{a}x - (2n - 3)y] = \frac{2b}{a}(x - \frac{a}{b}y) > 0$, because $\frac{b}{a} > 0$ and $(x, y) \in E \subset F$. □

Finally, the proof of Proposition 2 and Theorem 2 follows from Lemmas 5–9.

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Regularisation in Ejection-Collision Orbits of the RTBP

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Abstract Numerical explorations confirm well-known analytical results on the existence of ejection-collision orbits in the restricted three-body problem for very restrictive values of the Jacobi constant C. For different values of C some new types of ejection-collision orbits are found. The concept of n-ejection-collision orbit is introduced and numerical explorations are carried out which show a very rich dynamics when Hill regions contain both primaries. Complete details on the numerical methods and the bifurcations of the different families of orbits are given in the references.

1 Introduction

The object of this piece of work is to contribute with some results on ejection-collision (EC) orbits in the restricted three-body problem (RTBP). Orbits which eject from or collide with one of the primaries are of particular interest because they are relevant to astronomical problems such as the determination of regions of capture of irregular moons by giant planets (see [1]) or explaining the formation of Kuiper-belt binaries by means of physical collisions between the binary and intruders (see [2]). They are also relevant for some microscopic scale problems. The study of the hydrogen atom subject to a circularly polarised microwave field, where the collisions between the electron and the core play an important role to explain ionisation, is an instance of such an application (see [4] and [14]).

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Concerning analytical studies of EC orbits in the RTBP (planar, spatial, circular and elliptic cases), a perturbation approach is usually considered and the McGehee regularisation [13] is typically used. We emphasise that in all the papers published so far, the EC orbits analysed are what below we will call 1-EC orbits, i.e. orbits that eject from the primary, reach *one* maximum distance and come back to collision with the same primary. The references to be mentioned are: (i) in the planar RTBP, Llibre [10] (existence of at least two EC orbits for the mass parameter $\mu > 0$ small enough and the energy H small enough), Lacomba and Llibre [9] (by means of the existence of transversal EC orbits the authors prove that both the Hill problem and the RTBP have no C^1-extensible regular integrals), and Chenciner and Llibre [5] (existence of four EC orbits for any value of $\mu \in (0, 0.5]$ and H small enough). (ii) In the spatial RTBP, Llibre and Martinez Alfaro [11] (existence of EC orbits for small enough values of the mass parameter). (iii) In the planar elliptic RTBP, Llibre and Pinyol [12] and Pinyol [16] (existence of EC orbits for both the mass parameter and the eccentricity small enough). We remark that in all the mentioned references, only the case $n = 1$ is considered. We plan to prove the existence of n-EC orbits, for $n > 1$, i.e. EC orbits that reach n maxima in the distance before going back to collision. This is not done in the present note but will be published in a future paper.

Focussing on numerical results, there are some isolated computations published: we mention Henon's paper (see [7]) about the computation of EC orbits obtained along the continuation of some families of symmetric periodic—non-collision—orbits in the Copenhagen problem (that is $\mu = 0.5$) and also for Hill's problem (see [8]). Finally, the evolution of 16 particular collision periodic orbits obtained from the $\mu = 0.5$ case was numerically studied for various values of the mass ratio μ in [3].

The present piece of work has two main goals. First, it summarises the results put forward by the authors in [15]: (i) the existence of only four 1-EC orbits for any value of $\mu > 0$ and very small values of H (known analytical results) is confirmed and this result is numerically extended to less restrictive values of the energy. For higher values of the energy the Hill region contains the other primary and some new EC orbits appear. On the other hand, the concept of a family of n-EC orbits is introduced and some bifurcations along these families appear. The reader is referred to [15] for complete details on the numerical methods and the description of the bifurcations of the different families of orbits (not described here). We emphasise that in the mentioned paper only the McGehee regularisation was taken into account and therefore each EC orbit is regarded as a heteroclinic orbit, so the time from ejection to collision is infinity. However, in the present work, we will also consider the Levi-Civita regularisation and we will comment on the pros and cons when comparing both regularisations. This is precisely the second goal of this note.

2 Ejection-Collision in the RTBP

In this section, (i) we recall some properties of the RTBP, (ii) we introduce two regularisations, (iii) we analyse the collision manifold, and (iv) we present some results for n-EC orbits, $n \geq 1$.

2.1 The RTBP

We use the standard setting of the planar restricted three-body problem (RTBP): the primaries of masses $m_1 = 1 - \mu$ and $m_2 = \mu$ occupy, respectively, the positions $(-\mu, 0)$ and $(1-\mu, 0)$ on the x-axis of a rotating frame (the *synodical* frame). With these assumptions, the equations of motion for the particle in this rotating are given by

$$
\begin{aligned}
\ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x}(x, y) \\
\ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y}(x, y),
\end{aligned}
\tag{1}
$$

where $\dot{} = d/dt$ and

$$
\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2}} + \frac{1}{2}\mu(1-\mu).
\tag{2}
$$

It is well known that this system of ODE has the following properties (see [18] for details).

1. There exists a first integral, the Jacobi integral, given by

$$
C = 2\Omega(x, y) - \dot{x}^2 - \dot{y}^2.
\tag{3}
$$

2. The equations of motion are invariant under the symmetry

$$
(t, x, y, \dot{x}, \dot{y}) \longrightarrow (-t, x, -y, -\dot{x}, \dot{y}).
\tag{4}
$$

   which translates into the well-known symmetry of the orbits.
3. There exist five equilibrium points: the collinear ones, L_i, $i = 1, 2, 3$ on the x axis, and the triangular ones L_i, $i = 4, 5$ located at the vertices of an equilateral triangle with the primaries. We denote by x_{L_i} the abscissa of point L_i. We assume $\mu \leq 1/2$ and $x_{L_2} \geq 1 - \mu \geq x_{L_1} \geq -\mu \geq x_{L_3}$, so that L_1 is between the

primaries, L_2 is on the right of the small one and L_3 on the left of the large one. We denote by $C_{L_i}(\mu)$ the value of the Jacobi constant at L_i for a given μ.

4. The equations of motion can be written as a Hamiltonian system in the coordinates (x, y) and associated momenta (p_x, p_y). The Hamiltonian function is

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\mu(1-\mu),$$

with $r_1 = \sqrt{(x+\mu)^2 + y^2}$ and $r_2 = \sqrt{(x+\mu-1)^2 + y^2}$, and the relation between C and H is given by

$$H = -\frac{C}{2}. \tag{5}$$

We denote by $H_{L_i}(\mu)$, the associated value of the Hamiltonian at L_i for a given μ.

5. Given a value of the Jacobi constant C (or the Hamiltonian H), the motion is allowed to take place only in the Hill region defined by

$$\mathcal{R}(C) = \{(x, y) \in \mathbb{R}^2 \mid 2\Omega(x, y) \geq C\}.$$

In this paper we will restrict the values of C to the range $C \geq C_{L_2}(\mu)$ (equivalently $H \leq H_{L_2}(\mu)$, see in Fig. 1 the corresponding Hill regions). More precisely, we first study the existence of EC orbits with the big primary for $C \geq C_{L_1}(\mu)$ (see Fig. 1 bottom), where only the bounded region around the big primary is taken into account. Later on, we consider also $C \geq C_{L_2}(\mu)$, where the motion can take place in a bounded region containing both primaries, and therefore, there also exist orbits that eject from one primary and collide with the other one. Actually, the dynamics is very rich because of the Lyapunov periodic orbits around L_1 and their associated invariant manifolds. Specific values of H can be translated to values of C through the relation (5).

2.2 Two Regularisations

As the objective is to study the ejection-collision orbits of the big primary, we will deal with the singularity $r_1 = 0$ in the equations introducing two types of regularisations: following the work of McGehee (see [6, 13]) and Levi-Civita coordinates (see [18]).

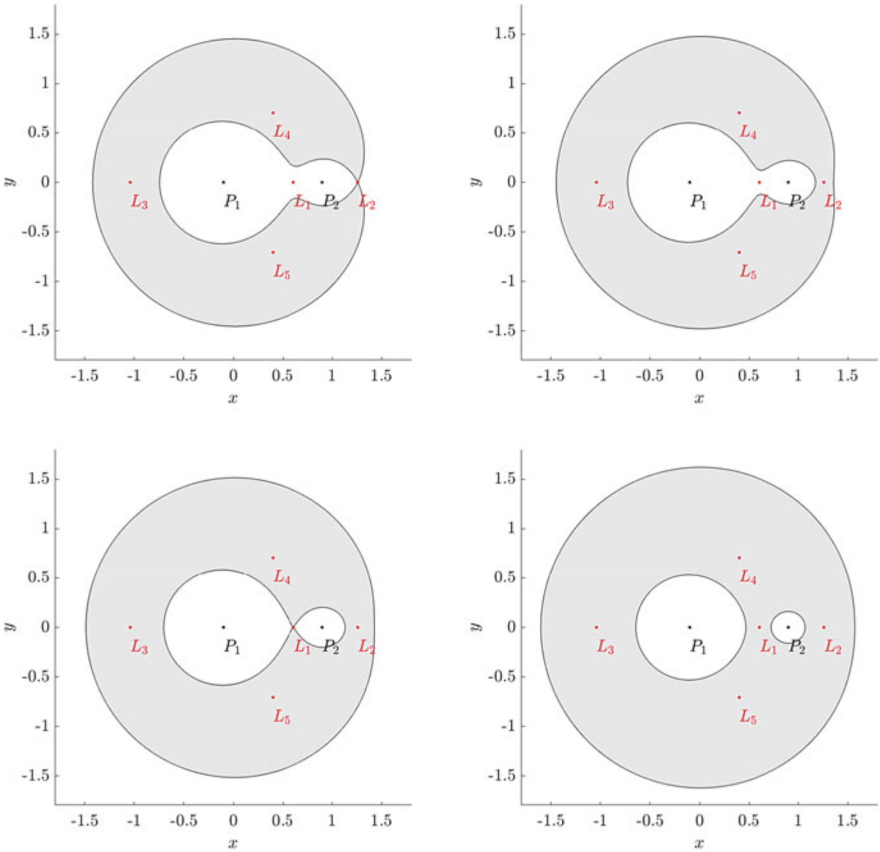

Fig. 1 Hill's region according to the Jacobi constant C. Top left: $C = C_{L_2}$; top right: $C_{L_2} < C < C_{L_1}$; bottom left: $C = C_{L_1}$; bottom right: $C > C_{L_1}$

2.2.1 McGehee Regularisation

We apply the translation $q_1 = x + \mu$, $q_2 = y$, to locate the primary of mass $1 - \mu$ at the origin of coordinates and that of mass μ at $(1, 0)$ and we introduce the canonical change of polar coordinates

$$q_1 = r\cos\theta \qquad p_1 = p_r\cos\theta - \frac{p_\theta}{r}\sin\theta$$

$$q_2 = r\sin\theta \qquad p_2 = p_r\sin\theta + \frac{p_\theta}{r}\cos\theta$$

which changes the Hamiltonian into

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - p_\theta - \frac{1-\mu}{r} + \mu\left(p_r \sin\theta + \frac{p_\theta}{r}\cos\theta\right) - \frac{\mu}{\sqrt{1+r^2-2r\cos\theta}} - \frac{1}{2}\mu(1-\mu) \tag{6}$$

Then we introduce the new variables

$$v = \dot{r}r^{1/2} \qquad u = r^{3/2}\dot{\theta} \tag{7}$$

and a change of time $dt/d\tau = r^{3/2}$, such that the system of ODE becomes

$$\begin{aligned} r' &= vr \\ \theta' &= u \\ v' &= \frac{1}{2}v^2 + u^2 + 2ur^{3/2} + r^3 - (1-\mu) \\ &\quad - \mu r^2 \cos\theta - \mu r^2 \frac{r - \cos\theta}{(1+r^2-2r\cos\theta)^{3/2}} \\ u' &= -\frac{1}{2}uv - 2vr^{3/2} + \mu r^2 \sin\theta\left(1 - \frac{1}{(1+r^2-2r\cos\theta)^{3/2}}\right), \end{aligned} \tag{8}$$

where $' = d/d\tau$.

2.2.2 Levi-Civita Regularisation

The Levi-Civita regularisation consists of the transformation (see Fig. 2) given by

$$\begin{cases} x = -\mu + u^2 - v^2 \\ y = 2uv \\ \dfrac{dt}{ds} = 4\left(u^2 + v^2\right). \end{cases} \tag{9}$$

Under this transformation the system (1) becomes (see details in [18]):

$$\begin{cases} u'' - 8(u^2+v^2)v' = \left(4U(u^2+v^2)\right)_u \\ v'' + 8(u^2+v^2)u' = \left(4U(u^2+v^2)\right)_v \end{cases} \tag{10}$$

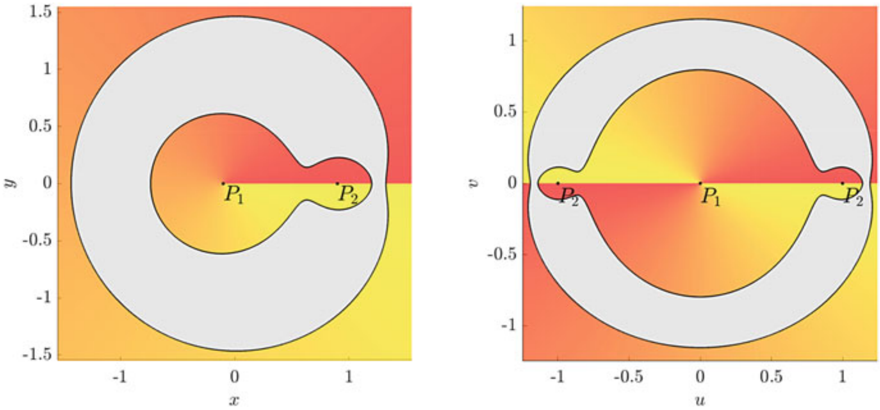

Fig. 2 Levi-Civita transformation for $\mu = 0.1$. Left. (x, y) variables. Right. Levi-Civita ones (u, v). In light grey the annular forbidden Hill's region for $C = 3.58$

where $' = d/ds$ and

$$U = \frac{1}{2}\left[(1-\mu)\left(u^2+v^2\right)^2 + \mu r_1^2\right] + \frac{1-\mu}{u^2+v^2} + \frac{\mu}{r_1} - \frac{C}{2}.$$

with $r_1 = \sqrt{(-1+u^2-v^2)^2 + 4u^2v^2}$.

2.3 *The Collision Manifold*

Two advantages in using McGehee regularisation are that the system of ODE in these variables is simpler, and that we have the so-called collision manifold that describes both the motion at the ejection/collision (by means of a blow-up) and it gives insight into the motion close to ejection/collision.

System (8) has an invariant manifold Λ defined by $r = 0$, called the collision manifold. This manifold Λ is a torus (see Fig. 3) given by

$$\Lambda = \{u^2 + v^2 = 2(1-\mu), \quad \theta \in [0, 2\pi]\} \tag{11}$$

and the dynamics on this torus is governed by the equations

$$\begin{aligned} \theta' &= u \\ v' &= \frac{1}{2}v^2 + u^2 - (1-\mu) \\ u' &= -\frac{1}{2}uv. \end{aligned} \tag{12}$$

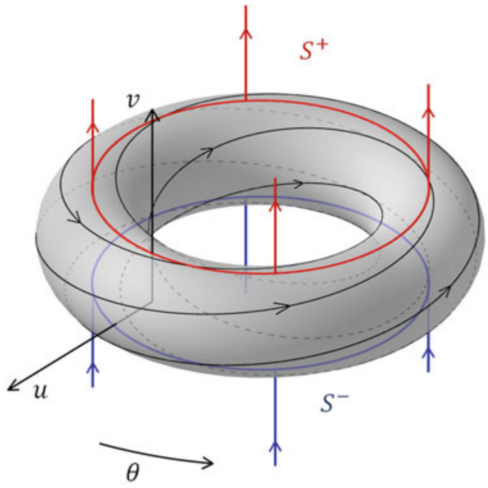

Fig. 3 The collision manifold

There exist two circles of equilibrium points on Λ defined by $S^+ = \{r = 0, \theta, v = v_0, u = 0, \theta \in [0, 2\pi]\}$ and $S^- = \{r = 0, \theta, v = -v_0, u = 0, \theta \in [0, 2\pi]\}$ with $v_0 = +\sqrt{2(1-\mu)}$.

For a value of the Jacobi constant fixed, each equilibrium point $P \in S^+$ has a 1-d unstable manifold $W^u(P)$ and a 1-d stable one $W^s(P)$. Similarly, each equilibrium point $Q \in S^-$, has a 1-d stable manifold $W^s(Q)$ and a 1-d unstable one $W^u(Q)$. In Fig. 3, $W^u(P)$ and $W^s(Q)$ are symbolically represented by arrows.

We distinguish 3 types of orbits: (i) ejection, (ii) collision and (iii) ejection-collision orbits.

(i) The set of ejection orbits—those which are ejected from collision with the big primary—is the set of orbits on the unstable manifold $W^u(P)$, for any $P = (0, \theta, v_0, 0) \in S^+$. So each ejection orbit may be regarded as an orbit such that $r > 0$ for all finite time τ and asymptotically tends to an equilibrium point $P \in S^+$ as $\tau \to -\infty$.

(ii) The set of collision orbits—those which arrive at collision with the big primary—is the set of orbits on the stable manifold $W^s(Q)$, for any $Q = (0, \theta, -v_0, 0) \in S^-$. So each collision orbit may be regarded as an orbit such that $r > 0$ for all finite time τ and asymptotically tends to an equilibrium point $Q \in S^-$ as $\tau \to +\infty$.

(iii) The set of ejection-collision orbits—those which eject from the big primary and then collide with it—is the set of orbits obtained from the intersection $W^u(S^+) \cap W^s(S^-)$. So they may be regarded as heteroclinic orbits between $P \in S^+$ and $Q \in S^-$.

Finally we define *n-ejection-collision orbits*, simply denoted by n-EC orbits, as those orbits that eject from the big primary, reach n times a relative maximum of the distance r, with $n - 1$ close approaches in between, before colliding with the big primary.

At this point it is worthwhile to compare the two regularisations contemplated in this note (McGehee and Levi-Civita) when applied to the study of ejection/collision orbits. When we consider the Levi-Civita regularisation, ejection/collision orbits are simply orbits that leave from/arrive at the origin, which is now a regular point, so it takes a finite range of time to describe an EC orbit and we do not have the collision manifold. By contrast, it takes an infinite time to describe an EC orbit in McGehee coordinates, since they are asymptotic (heteroclinic) connections. From this point of view, although the system of ODE in Levi-Civita variables is more intricated, the numerical computations are really faster. Moreover, the initial conditions of an ejection orbit are on invariant manifolds of equilibrium points when using McGehee variables, whereas in Levi-Civita variables, since the collision with the big primary takes place at $u = v = 0$ and the relation for the velocity components, $u'^2 + v'^2 = 8(1 - \mu)$, must be satisfied, we simply take the set of initial conditions

$$u = v = 0, \quad u' = \sqrt{8(1-\mu)}\cos\theta, \quad v' = \sqrt{8(1-\mu)}\sin\theta, \tag{13}$$

where, varying $\theta \in [0, 2\pi]$, we obtain all the possible initial conditions for an ejection orbit (see [18]).

A simple numerical method to detect EC orbits in McGehee variables can be implemented: we take a set of initial conditions on $W^u(S^+)$, integrate forward in time up to the $2n$-th crossing with the Poincaré section $v = 0$ and detect singularities in time due to the asymptotic behaviour which characterises EC orbits (see [15] for details).

When using Levi-Civita variables, we integrate the set of initial conditions (13) up to the n-th crossing with the Poincaré section $r = r_{max}$ and obtain a curve on this section. Then we proceed the same way backwards in time, obtaining another curve (this last task is not actually computed due to the symmetry (4). The intersection points between both curves belong to EC-orbits.

In summary, McGehee regularisation applied to the study of ejection/collision orbits has three main drawbacks. First, it requires integrating for a long time, since ejection/collision orbits are asymptotic orbits. Second, the initial conditions of ejection/collision orbits should be, ideally, on the unstable/stable manifolds of S^+/S^- respectively, but numerically we take initial conditions on the linear approximation of such manifolds. Therefore such initial conditions are not exactly *on* the invariant manifold itself. Finally a third inconvenience is due to the fact that successive passages through collision are numerically very badly conditioned because of the impossibility to reach an infinite time. On the other hand, with Levi-Civita regularisation only finite shorter ranges of time are required, the passage through collision is a regular point and the initial conditions are exact and there is no problem at all to consider integration spans with successive collisions. So, from the numerical point of view, Levi-Civita is really preferable.

2.4 *Results for* 1-*EC Orbits*

2.4.1 Existence of Four 1-EC Orbits

Although the analytical papers concerning the existence of 1-EC orbits typically consider very restrictive values of H and small values of μ, we have done extensive numerical explorations on a grid of values of μ in the interval $[0.01, 0.5]$ and $\theta_0 \in [0, 2\pi]$ for $H \le H_{L_1}(\mu)$. The simulations done confirm the existence of only four ejection-collision orbits. See in Fig. 4 examples of n-EC orbits for $n = 1, 2, 3$. In Fig. 5 left, we plot the four 4-EC orbits existing for particular values of H and $\mu = 0.5$.

Figure 5 shows the curves $W^u(S^+) \cap \Sigma_1$ and $W^s(S^-) \cap \Sigma_1$ (where Σ_1 denotes the first intersection with the Poincaré section $v = 0$) for $\mu = 0.5$ (for other values of μ

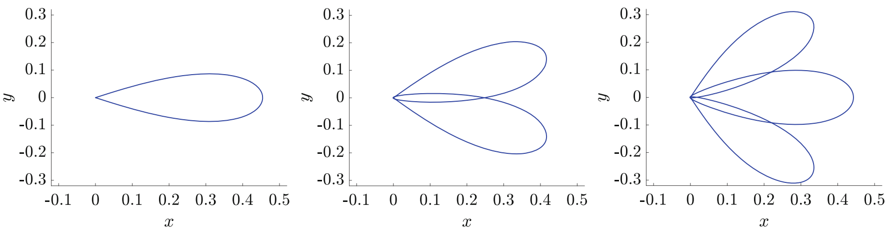

Fig. 4 Examples of n-ejection-collision orbits for $n = 1, 2, 3$ (*from left to right*). For $n = 2$ ($n = 3$), there are 1 (2) close passages to collision between ejection and collision

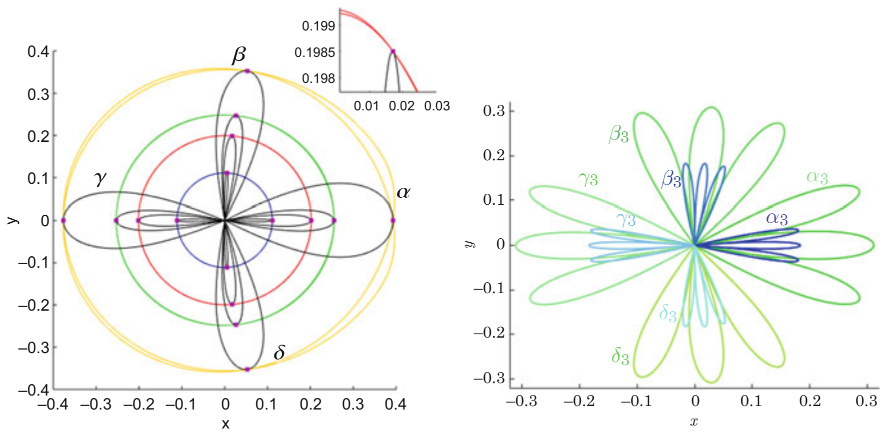

Fig. 5 *Left*. $\mu = 0.5$, $W^u(S^+) \cap \Sigma_1$ and $W^s(S^-) \cap \Sigma_1$ for values of H -5.25, -3.25, -2.75 and $H_{L_1}(\mu)$ (circles from small to large). In black the 1-EC orbits for such values of H. Also shown are the points of the EC orbits at Σ_1 and the almond-shaped projections of the EC orbits on the configuration plane (x, y). *Right*. The four n-ejection-collision orbits for $\mu = 0.1$ (*left*) and $n = 3$ for $H = -5.05$ (small ones) and $H = -3.05$ (large ones)

see [17]) and different values of H (Fig. 5 left). Also shown are the corresponding 1-EC orbits on the (x, y)-plane.

The existence of only four 1-EC orbits is no longer true for higher values of the energy H, since new ones show up. We refer the reader to the paper [15] for the details and the description of appearing bifurcations.

2.4.2 Results for *n*-EC Orbits

We have also done extensive numerical simulations on a grid of values for $\mu \in [0.01, 0.5]$ and energy levels $H < H_{L_1}(\mu)$ and we can conclude that for all n there exists a value $\hat{H}(\mu, n)$ such that for $H \leq \hat{H}(\mu, n)$ there are four n-ejection-collision orbits, which can be characterised in a way similar to the characterisation of the 1-ejection-collision orbits. For example, see the four 3-EC orbits for $\mu = 0.1$ and different values of H in Fig. 5 right.

Further details on the bifurcations of n-EC orbits and the numerical methods can be found in [15].

We remark that for high values of n, applying McGehee or Levi-Civita really makes a difference: using McGehee variables becomes a problem for large n. This effect is shown in Fig. 6: on the x axis we take the angle θ to characterise an ejection orbit, on the y axis the time it takes for such orbit to cross for the $2n$-th time the Poincaré section $v = 0$ (in McGehee variables). The small cusps on each curve correspond to very close approaches to collision, so that if the grid of θ values is refined the spike grows higher and tends to a vertical asymptote (infinite time to reach the collision). Each singularity (vertical asymptote) in time corresponds to an EC orbit and for the curve (θ_0, T_2) only four singularities appear, corresponding to the four 1-EC orbits. For the curve (θ_0, T_4) we observe 8 singularities: there are the previous existing 1-EC orbits and four new ones corresponding to the 2-EC orbits we are looking for. It is clear that for large n it is really difficult to detect the new EC orbits and to distinguish them from any previous ones with a smaller n.

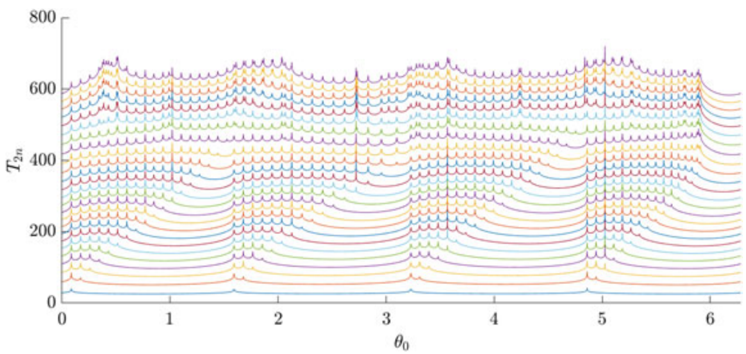

Fig. 6 (θ_0, T_{2n}), $n = 1, \ldots, 25$ for $\mu = 0.5$ and $H = -4$

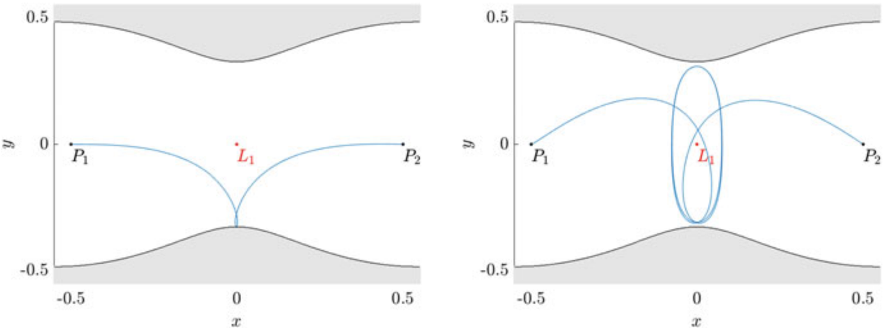

Fig. 7 $\mu = 0.5$, (x, y) coordinates. An orbit ejecting from the big primary and colliding with the small one. Left. A direct trajectory. Right. A trajectory describing a turn around the Lyapunov PO

We remark also the big intervals of time needed for the largest values of n. These drawbacks completely disappear when using Levi-Civita for the same simulation: one would see almost straight lines, due to the regular ODE, and the ranges of time are sensitively smaller.

Finally we remark that for higher values of H, say $H < H_{L_2}(\mu)$, the dynamics is richer due to two effects. The first one is that the bounded Hill region allows connections between both primaries. An example is shown in Fig. 7 left, where an ejection orbit from the big primary collides with the small one. A second effect is due to the Lyapunov periodic orbit PO around L_1 and its stable and unstable manifolds, which play a role when going from the neighborhood of one primary to the neighborhood of the other one. In Fig. 7 right we show an orbit which ejects from the big primary, describes a turn around the PO and collides with the small primary.

A detailed description of the variety of motions for higher values of H (where the particle can leave a neighborhood of the primaries and even reach infinity) will appear in a future paper.

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On Local Algebras of Maximal Algebras of Jordan Quotients

F. Montaner and I. Paniello

Abstract We study local algebras of maximal algebras of quotients of strongly prime Jordan algebras at nonzero elements of the Jordan algebra that become von Neumann regular in the maximal algebra of quotients. We prove that for such elements both constructions (local algebras and maximal algebras of quotients) commute. As a consequence we obtain that maximal algebras of quotients of strongly prime Jordan algebras with nonzero local PI-algebras are rationally primitive.

1 Introduction

Algebras of quotients of Jordan algebras in the general sense of Utumi's associative algebras of quotients were described by Montaner in [19], where a detailed description of how these algebras generalize the different notions of quotients that had formerly appeared in the literature can also be found.

Algebras of quotients of Jordan algebras as considered in [19] inherit regularity conditions [19, 22] as well as many other properties such as being PI or having nonzero PI-elements [20, 23]. A different question is how algebras of quotients interact with local algebras [15].

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This problem, how local algebras interact with algebras of quotients, has been studied by Gómez Lozano and Siles Molina in the associative setting [6]. There the authors considered the behaviour of different algebras of quotients of semiprime associative rings when taking local algebras at very particular elements. They proved that when the element at which the local algebra is considered becomes von Neumann regular in the algebra of quotients, taking the local algebra at such an element and considering different constructions of quotients commute. This result was proved not only for maximal rings of one-sided quotients ($Q^l_{max}(R)$ and $Q^r_{max}(R)$) of a semiprime ring R but also for the Martindale symmetric ring of quotients $Q_s(R)$ and the maximal symmetric ring of quotients $Q_\sigma(R)$ [3, 9].

It is in particular this last result, involving local algebras and maximal symmetric rings of quotients of semiprime associative algebras, the one which motivates this paper, as maximal symmetric rings of quotients of (semi)prime rings play a fundamental role in the description of maximal algebras of quotients of some special Jordan algebras. Indeed the maximal algebra of quotients of any nondegenerate Jordan algebra having essential hermitian part is nothing else that an ample subspace of symmetric elements of the maximal symmetric ring of quotients of any of its $*$-tight associative $*$-envelopes [19, 21].

After a section of preliminaries, Sects. 3 and 4 are devoted to show that a similar result to that obtained by Gómez Lozano and Siles Molina in [6] for semiprime associative rings can be achieved for Jordan algebras. We will first consider, in Sect. 3, the case when the element at which the local algebra is considered does not produce a Jordan PI-algebra but still becomes von Neumann regular in the maximal algebra of quotients, considering in Sect. 4 the case when this element does produce a local Jordan PI-algebra. Despite the more general results given in [6] for semiprime associative rings, here we will restrict ourselves to the case of strongly prime Jordan algebras.

Considering strongly prime Jordan algebras will be enough to obtain the last result of this paper, a Martindale-like theorem for Jordan algebras having nonzero local PI-algebras. More precisely in the last section we show that maximal algebras of quotients of strongly prime Jordan algebras with nonzero PI-elements are rationally primitive [16, 4.1].

2 Preliminaries

We will work with Jordan algebras over a unital commutative ring Φ which will be fixed throughout. We refer to [7, 8, 14] for notation, terminology and basic results.

Definition 1 A Jordan algebra J has products $U_x y$ and x^2, quadratic in x and linear in y, whose linearizations are $U_{x,z}y = V_{x,y}z = \{x, y, z\} = U_{x+z}y - U_x y - U_z y$ and $x \circ y = V_x y = (x + y)^2 - x^2 - y^2$.

2.1 Essential Ideals

A Φ-submodule K of a Jordan algebra J is an *inner ideal* if $U_k\widehat{J} \subseteq K$ for all $k \in K$, where $\widehat{J}$ denotes the free unital hull of J. An inner ideal $I \subseteq J$ is an *ideal* if $\{I, J, \widehat{J}\} + U_J I \subseteq I$. An (inner) ideal of J is *essential* if it has nonzero intersection with any nonzero (inner) ideal of J. Essential ideals of nondegenerate Jordan algebras are exactly those having zero annihilator [5, Lemma 1.13(i)]. The annihilator $ann_J(I)$ of an ideal I of a nondegenerate Jordan algebra J is the set $ann_J(I) = \{x \in J \mid U_x I = 0\} = \{x \in J \mid U_I x = 0\}$ (see [13, Proposition 1.7(i)] and [17, Lemma 1.3]). We denote by $I^{(1)}$ the *derived ideal* of an ideal I being $I^{(1)} = U_I I$. Similarly we can define $I^{(m)} = U_{I^{(m-1)}} I^{(m-1)}$ for all integers $m \geq 1$ (and $I^{(0)} = I$). For any ideal I of a nondegenerate Jordan algebra J it holds that $ann_J(I) = ann_J(I^{(m)})$ for all m [5, Lemma 1.13(ii)]. Therefore if I is an essential ideal so are all its derived ideals $I^{(m)}$, $m \geq 0$. Recall that nonzero ideals of strongly prime Jordan algebras are essential ideals.

2.2 Special Jordan Algebras

Any associative algebra R gives rise to a Jordan algebra $R^{(+)}$ just by defining $U_x y = xyx$ and $x^2 = xx$. A Jordan algebra is *special* if it is isomorphic to a subalgebra of an algebra of the form $R^{(+)}$ and it is called *i-special* if it satisfies all the identities satisfied by all special Jordan algebras.

Examples of special Jordan algebras are algebras of symmetric elements $H(R, *)$ of associative algebras with involution $(R, *)$ and their *ample subspaces* $H_0(R, *)$ i.e. subspaces of symmetric elements which contain the traces $x + x^*$ and norms xx^* of all elements $x \in R$ and such that $xH(R, *)x^* \subseteq H_0(R, *)$ for all $x \in R$.

An associative algebra with involution $(R, *)$ is an *associative $*$-envelope* of J if $J \subseteq H(R, *)$ and J generates R as an associative algebra and it is called *$*$-tight* if any nonzero $*$-ideal I of R hits J nontrivially i.e. $I \cap J \neq 0$.

An i-special Jordan algebra J is of *hermitian type* if $ann_J(\sum_{\mathcal{H}} \mathcal{H}(J)) = 0$, where $\sum_{\mathcal{H}}$ denotes the sum on the set of all hermitian ideals. We will also consider Jordan algebras hermitian in the stronger sense that there exists a hermitian ideal $\mathcal{H}(X)$ of the free special Jordan algebra $FSJ(X)$ such that $ann_J(\mathcal{H}(J)) = 0$, equivalently, that there exists a hermitian ideal $\mathcal{H}(X)$ whose values $\mathcal{H}(J)$ in J give an essential ideal of J. (See [19, Remark 4.9] and [21, Remark 4.8] for further details.)

Definition 2 ([19, 1.3]) Let K be an inner ideal of a nondegenerate Jordan algebra J. For any element $a \in J$ the set $(K : a) = \{x \in K \mid x \circ a,\ U_a x \in K\}$ is an inner ideal of J and so are the sets

$$(K : a_1 : a_2 : \ldots : a_n) = ((K : a_1 : a_2 : \ldots : a_{n-1}) : a_n)$$

for all $a_1, \ldots, a_n \in J$. An inner ideal K of J is said to be *dense* if $U_c(K : a_1 : a_2 : \ldots : a_n) \neq 0$ for any finite collection of elements $a_1, \ldots, a_n \in J$ and any $0 \neq c \in J$.

Definition 3 Let J be a subalgebra of a Jordan algebra $\widetilde{J}$ and let $\widetilde{a} \in \widetilde{J}$. An element $x \in J$ is a *J-denominator* of $\widetilde{a}$ if the following multiplications take $\widetilde{a}$ back into J:

$$\begin{array}{lll} \text{(Di) } U_x\widetilde{a} & \text{(Dii) } U_{\widetilde{a}}x & \text{(Diii) } U_{\widetilde{a}}U_x\widehat{J} \\ \text{(Diii') } U_xU_{\widetilde{a}}\widehat{J} & \text{(Div) } V_{x,\widetilde{a}}\widehat{J} & \text{(Div') } V_{\widetilde{a},x}\widehat{J}. \end{array}$$

We will denote the set of J-denominators of $\widetilde{a}$ by $\mathcal{D}_J(\widetilde{a})$. It was proved in [17, Lemma 4.2] that $\mathcal{D}_J(\widetilde{a})$ is an inner ideal of J.

Definition 4 Let J be a subalgebra of a Jordan algebra Q. We will say that Q is an *algebra of quotients of* J if the following conditions hold:

(i) $\mathcal{D}_J(q)$ is a dense inner ideal of J for all $q \in Q$.
(ii) $U_q\mathcal{D}_J(q) \neq 0$ for any nonzero $q \in Q$.

Nondegenerate Jordan algebras are its own algebras of quotients. Moreover for any Jordan algebra the existence of an algebra of quotients implies nondegenerancy.

Definition 5 An algebra of quotients Q of a Jordan algebra J is said to be *maximal* if for any other algebra of quotients Q' of J there exists a homomorphism $\alpha : Q' \to Q$ whose restriction to J is the identity map.

The existence (up to isomorphism) of maximal algebras of quotients of nondegenerate Jordan algebras was proved in [19, Theorem 5.8].

Theorem 1 ([19, Theorem 5.8]) *Any nondegenerate Jordan algebra J has a maximal algebra of quotients $Q_{max}(J)$.*

Remark 1 A similar construction of algebras of quotients, but based on essential inner ideals of denominators was given in [21] for strongly prime Jordan algebras satisfying the condition of being strongly nonsingular.

Remark 2 The description of maximal algebras of quotients of nondegenerate Jordan algebras is given in [19, Theorem 3.11] and [19, Theorem 4.10].

(i) The maximal algebra of quotients of a nondegenerate Jordan PI-algebra is (isomorphic to) its *almost classical algebra of quotients*

$$J_{\mathcal{E}(J)} = \varinjlim\{Hom_\Gamma(\mathfrak{a}, J) \mid \mathfrak{a} \in \mathcal{E}(J)\},$$

the direct limit of the directed system $\{Hom_\Gamma(\mathfrak{a}, J) \mid \mathfrak{a} \in \mathcal{E}(J)\}$ where $\mathcal{E}(J)$ denotes the set of essential ideals of the centroid $\Gamma = \Gamma(J)$ of J. We recall that if J is a strongly prime Jordan PI-algebra, then $Q_{max}(J) \cong \Gamma(J)J$ the central closure of J.

(ii) Special nondegenerate Jordan algebras which are hermitian in the strong sense that there exists a hermitian ideal whose values in the Jordan algebra J define an essential ideal (see Sect. 2.2) have maximal algebras of quotients which are ample subspaces of symmetric elements of the maximal algebra of symmetric quotients $Q_\sigma(R)$ of any $*$-tight associative $*$-envelope R of J [19, Proposition 4.8, Theorem 4.10]:

$$Q_{max}(J) = H_0(Q_\sigma(R), *) = \{q \in H(Q_\sigma(R), *) \mid \mathcal{D}_J(q) \text{ is dense in } J\}.$$

2.3 Algebras of Quotients and Local Algebras

Algebras of quotients inherit regularity conditions such as being nondegenerate or strongly prime [19, Lemma 2.4]. Moreover maximal algebras of quotients of nondegenerate Jordan algebras are unital [19, Remark 5.9], and if J is a nondegenerate unital Jordan algebra, then any algebra of quotients of J has the same unit as J [21, Lemma 3.2]. It is also known that any nondegenerate Jordan algebra of finite capacity is its own maximal algebra of quotients [21, Lemma 3.2].

Definition 6 Let J be a Jordan algebra and let a be an element of J. The *local algebra* of J at a is the quotient of the a-homotope algebra $J^{(a)}$ by the ideal $Ker_J(a)$ of $J^{(a)}$ of all the elements $x \in J$ such that $U_a x = U_a U_x a = 0$. If J is nondegenerate, the two previous conditions reduce to $U_a x = 0$.

Local algebras of associative algebras are defined similarly. Jordan and associative local algebras inherit regularity conditions as those of being nondegenerate or strongly prime from the original algebras [1, Theorem 4.1]. Local algebras also interact well with algebras of quotients.

Lemma 1 ([22, Lemma 2.2]) *Let Q be an algebra of quotients of a Jordan algebra J. For any element $a \in J$, Q_a is an algebra of quotients of J_a.*

Definition 7 An element a of a Jordan algebra J is said to be a *PI-element* if the local algebra of J at a is a PI-algebra.

The set of all PI-elements of a nondegenerate Jordan algebra J is an ideal denoted by $PI(J)$ [16, 3.1]. A Jordan algebra having no nonzero PI-elements is said to be a *PI-less* Jordan algebra. Nondegenerate PI-less Jordan algebras are special of hermitian type [19, Lemma 4.4].

2.4 The Socle of Nondegenerate Jordan Algebras

The socle $Soc(J)$ of a nondegenerate Jordan algebra J was characterized in [16, Lemma 0.7] as the set of elements $a \in J$ whose local algebra J_a has finite capacity.

Recall that any element $a \in Soc(J)$ is von Neumann regular [12] and that, as a result, the local algebra J_a is a unital Jordan algebra [4, Regular Example 1.7]

3 Algebras of Quotients of Local Algebras at Non-PI-elements

In this section we consider local algebras of strongly prime Jordan algebras at elements which are not PI-elements (that is their local algebras are not PI-algebras), but which become von Neumann regular in the maximal algebra of quotients. It is then proved that, in this case, taking the local algebra and considering the maximal algebra of quotients are commuting constructions. Some of the technical results contained in this section will be however provided in the more general setting of nondegenerate Jordan algebras for further application [23].

Strongly prime Jordan algebras having nonzero elements failing to be PI-elements are special Jordan algebras. Moreover their local algebras at such elements are special Jordan algebras, hermitian in the strong sense that there exist hermitian ideals whose values in the local algebras define nonzero, therefore essential, ideals.

For any element a of a special Jordan algebra J with a $*$-tight associative $*$-envelope R, the local algebra R_a of R at a remains to be an associative $*$-envelope of J_a, as clearly $J_a \subseteq H(R_a, *)$, where $(r + Ker_R(a))^* = r^* + Ker_R(a)$ for all $r \in R$, and J_a still generates R_a. However R_a does not necessarily retain the $*$-tightness over J_a. This drawback can be nonetheless avoided by restricting ourselves to suitable subalgebras of the Jordan algebra J containing the element a at which the local algebra is considered.

Lemma 2 ([23]) *Let J be a nondegenerate special Jordan algebra and let R be a $*$-tight associative envelope of J. If $\mathcal{H}(X)$ is a hermitian ideal of $FSJ(X)$ such that $ann_J(\mathcal{H}(J)) = 0$, then:*

(i) for each $a \in J$ the subalgebra S of J generated by the ideal $I = \mathcal{H}(J)^{(1)}$ and the element a is also nondegenerate of hermitian type. Moreover $S = H_0(A, *)$, where $A = subalg_R(S)$ is a $*$-tight associative envelope of S and a semiprime associative algebra.

(ii) $subalg_{A_a}(S_a)$ is a symmetric subring of A_a and therefore $subalg_{A_a}(S_a)$ is semiprime.

(iii) $subalg_{A_a}(S_a)$ is a $*$-tight associative $*$-envelope of S_a.

Proof Since $ann_J(I) = 0$ by [5, Lemma 1.13], a similar proof to that of the first part of [16, Theorem 6.5] works here to prove (i).

(ii) It follows from [1, Lemma 2.3] that S_a is an ample subspace of A_a, i.e., $S_a = H_0(A_a, *)$. Thus to prove that $subalg_{A_a}(S_a)$ is a symmetric subring (see [10, 11]) of A_a it suffices to check that $\overline{x}\ subalg_{A_a}(S_a)\ \overline{x}^* \subseteq subalg_{A_a}(S_a)$ for all $\overline{x} \in A_a$, where the bars denote the projection on the local algebra A_a of A at a. But this can be found in the proof of [25, Theorem 4.4]. Hence $subalg_{A_a}(S_a)$

is a symmetric subring of A_a and therefore it is semiprime by [11, Theorem 5, Theorem 8]. Finally (iii), that is, the $*$-tightness of $subalg_{A_a}(S_a)$ over S_a follows as in [2, Theorem 1.3] and [1, Lemma 2.3]. □

Lemma 3 *Let J be a nondegenerate special Jordan algebra with $ann_J(\mathcal{H}(J)) = 0$ for some hermitian ideal $\mathcal{H}(X)$ of $FSJ(X)$. Let S be the subalgebra of J generated by the ideal $\mathcal{H}(J)^{(1)}$ and a nonzero element a of J. Then $Q_{max}(S) = Q_{max}(J)$ and $Q_{max}(S_a) = Q_{max}(J_a)$.*

Proof We note that S contains the essential ideal $\mathcal{H}(J)^{(1)}$ of J since by [5, Lemma 1.13(ii)] we have $ann_J(\mathcal{H}(J)^{(1)}) = ann_J(\mathcal{H}(J)) = 0$, so that the essentiality of $\mathcal{H}(J)^{(1)}$ follows from [5, Lemma 1.13(i)]. Then $Q_{max}(S) = Q_{max}(J)$ as a result of [22, Lemma 3.3] and [19, Proposition 2.8]. Analogously, as S_a contains the essential ideal $\left(\mathcal{H}(J)^{(1)} + Ker_J(a)\right)/Ker_J(a)$ of J_a, the second assertion holds.

□

Remark 3 Let R be a semiprime ring with maximal symmetric ring of quotients $Q_\sigma(R)$ [9]. It is known that involutions of R straightforwardly extend to $Q_\sigma(R)$. Moreover for any $a \in H(R,*)$, the local algebra R_a inherits an involution, also denoted by $*$, which again extends to $Q_\sigma(R_a)$. The same applies to $Q_\sigma(R)_a$. If the element $a \in R$ at which the local algebra is considered, is not only symmetric (with respect to $*$) but also becomes von Neumann regular in $Q_\sigma(R)$, it was proved in [6, Theorem 4] the existence of an isomorphism between $Q_\sigma(R_a)$ and $Q_\sigma(R)_a$ and it is easy to check that this isomorphism preserves the involution $*$.

Lemma 4 *Let $(R,*)$ be a $*$-tight associative $*$-envelope of a nondegenerate special Jordan algebra J and let $a \in J$ be such that:*

(a) R_a is a $*$-tight associative envelope of J_a,
(b) $\mathcal{H}(J_a)$ is an essential ideal of J_a for some hermitian ideal $\mathcal{H}(X)$ of $FSJ(X)$.

Then, if a is von Neumann regular in $Q_{max}(J)$, for any dense inner ideal K of J_a, we have:

(i) $\widehat{R_a} \cdot K$ is a dense left ideal of R_a, where $\cdot$ represents the associative product in the (associative) local algebra R_a and $\widehat{R_a}$ the unital hull of R_a.
(ii) $RaDa \oplus lann_R(b)$ is a dense left ideal of R, where $b \in Q_{max}(J)$ is such that $U_a b = a$ and $U_b a = b$ and $D = \{d \in J \mid \overline{d} \in K\}$.

Proof (i) follows from [19, Theorem 4.6] and the proof of (ii) is similar to that of [6, Proposition 2] once one considers that any semiprime associative algebra is its own algebra of left quotients [28] and taking into account that $Q_{max}(J)$ is an ample subspace of the maximal algebra of symmetric quotients $Q_\sigma(R)$ of R [19, Proposition 4.8]. □

Proposition 1 *Let J be a strongly prime Jordan algebra. For any (nonzero) element $a \in J$ such that $a \notin PI(J)$ the following are equivalent:*

(i) a is von Neumann regular in $Q_{max}(J)$.
(ii) $Q_{max}(J_a) \cong Q_{max}(J)_a$ under an isomorphism extending the identity on J_a.

Proof Suppose first that (ii) holds. Since J_a is strongly prime and maximal algebras of quotients of strongly prime Jordan algebras are unital (see Sect. 2.3), to prove the regularity von Neumann of a in $Q_{max}(J)$ it suffices to show that the unit element $\overline{x}$ of $\overline{Q_{max}(J)}$ satisfies $U_a x = a$, where we denote with bars $-$ the projection of $Q_{max}(J)$ in $Q_{max}(J)_a = \overline{Q_{max}(J)}$. Otherwise (see [6, Theorem 1 $(ii) \Rightarrow (i)$]) $y = U_a x - a$ is a nonzero element of $Q_{max}(J)$ and then, since $Q_{max}(J)$ is strongly prime (see Sect. 2.3), there exists $q \in Q_{max}(J)$ such that $0 \neq U_y q \in U_y Q_{max}(J)$. But since $\overline{x}$ is the unit of $\overline{Q_{max}(J)}$, we have $U_y q = U_{U_a x - a} q = U_{U_a x} q + U_a q - \{U_a x, q, a\} = U_a U_x U_a q + U_a q - U_a \{x, a, q\} = U_a(U_x^{(a)} q + q - V_x^{(a)} q) = 0$ which contradicts the choice of q and gives (i).

Next let a be an element of J such that $a \notin PI(J)$ and assume that a becomes von Neumann regular in $Q_{max}(J)$, the maximal algebra of quotients of J.

Since $a \notin PI(J)$, neither J nor J_a are PI-algebras, thus both J and J_a are special Jordan algebras and there exist hermitian ideals whose values in these algebras produce nonzero (i.e. essential) ideals. Let R be a $*$-tight associative $*$-envelope of J. By Lemma 2 replacing J by the subalgebra generated by the element a and the derived ideal of an ideal of the form $\mathcal{H}(J)$ with $ann_J(\mathcal{H}(J)) = 0$ (see Lemma 3) we can also assume that R_a is a $*$-tight associative $*$-envelope of J_a. Then the maximal algebra of quotients $Q_{max}(J)$ of J is an ample subspace of $Q_\sigma(R)$ the maximal algebra of symmetric quotients of R (see [19, Proposition 4.8]).

On the other hand $Q_{max}(J)_a$ is an algebra of quotients of J_a (see Lemma 1), and thus, by the maximality of $Q_{max}(J_a)$, it holds that $Q_{max}(J)_a \subseteq Q_{max}(J_a)$, where

$$Q_{max}(J_a) = H_0(Q_\sigma(R_a), *) = \{\mathfrak{y} \in H(Q_\sigma(R_a), *) \mid \mathcal{D}_{J_a}(\mathfrak{y}) \text{ is dense in } J_a\}.$$

Take now an element $\mathfrak{p} \in Q_{max}(J_a)$. Being a von Neumann regular in $Q_{max}(J)$, so is in $Q_\sigma(R)$. Hence by [6, Theorem 4] $Q_\sigma(R_a) \cong Q_\sigma(R)_a$, and there exists an element $q \in Q_\sigma(R)$ such that $\overline{q} = \mathfrak{p}$ in $Q_\sigma(R)_a$, where the bar $-$ here denotes the projection of $Q_\sigma(R)$ in $Q_\sigma(R)_a$.

Finally to prove that $q \in Q_{max}(J)$ it suffices to combine together Lemma 4, [6, Theorem 4 $(i) \Rightarrow (ii)$] and [6, Proposition 2]. □

4 Algebras of Quotients of Local Algebras at PI-Elements

In this section we first consider maximal algebras of quotients of local algebras of strongly prime Jordan algebras at nonzero PI-elements, to finally address the first goal of this paper, that is to prove that when the elements at which the local algebras are considered become von Neumann regular in $Q_{max}(J)$, taking local algebras and maximal algebras of quotients are commuting constructions.

Definition 8 An element a of a Jordan algebra J is a *Lesieur-Croisot element* (or an LC-element, for short) if the local algebra J_a is a Lesieur-Croisot algebra (again an LC-algebra), i.e., if J_a is a classical order in a nondegenerate Jordan algebra of finite capacity [25, Definition 3.2].

The set $LC(J)$ of all LC-elements of a nondegenerate Jordan algebra J is an ideal of J [25, Theorem 5.13]. Moreover, if J is strongly prime and has nonzero PI-elements, then $LC(J) = PI(J)$ [25, Proposition 3.3].

Proposition 2 *Let J be a strongly prime Jordan algebra. For any nonzero element $a \in J$ such that $a \in PI(J)$ the following are equivalent:*

(i) a is von Neumann regular in $Q_{max}(J)$.
(ii) $Q_{max}(J_a) \cong Q_{max}(J)_a$ under an isomorphism extending the identity on J_a.

Proof The proof that (ii) implies (i) works as in the case when a is not a PI-element (see Proposition 1).

Assume now that (i) holds, i.e. that a is a nonzero PI-element becoming von Neumann regular in $Q_{max}(J)$. We first recall that for any $0 \neq a \in J$ it holds that $J_a \subseteq Q_{max}(J)_a \subseteq Q_{max}(J_a)$ by [22, Lemma 2.2] (see Lemma 1) and [19, Proposition 2.8]. Now since $0 \neq a \in PI(J)$, by [25, Proposition 3.3], the strong primeness of J implies that $a \in PI(J) = LC(J)$ and therefore that $a \in PI(J) = LC(J) = J \cap Soc(Q_{max}(J))$ as a result of [22, Theorem 3.5]. Thus $J_a \subseteq Q_{max}(J)_a$, which is an algebra of quotients of J_a by Lemma 1 and has finite capacity by [16, Lemma 0.7(b)]. Hence, by [21, Lemma 3.2(c)], $Q_{max}(J)_a$ is its own maximal algebra of quotients, so that $Q_{max}(J_a) \cong Q_{max}(J)_a$, again by [19, Proposition 2.8]. □

Theorem 2 *Let J be a strongly prime Jordan algebra. For any nonzero element $a \in J$ the following are equivalent:*

(i) a is von Neumann regular in $Q_{max}(J)$.
(ii) $Q_{max}(J_a) \cong Q_{max}(J)_a$ under an isomorphism extending the identity on J_a.

Proof If a is not a PI-element the above equivalence follows from Proposition 1. Otherwise a is a PI-element and then it follows from Proposition 2. □

5 Maximal Algebras of Quotients of Jordan Algebras with Nonzero PI-Elements

In this final section we take advantage of Theorem 2 above to prove that maximal algebras of quotients of strongly prime Jordan algebras having nonzero PI-elements are rationally primitive, that is, (see [16, 4.1]), for such a Jordan algebra J, $Q_{max}(J)$ is primitive and has nonzero PI-elements. This result can be understood as a version of Martindale Theorem for associative algebras satisfying generalized polynomial identities. A different version of Martindale Theorem involving the extended central closure of strongly prime Jordan algebras with nonzero PI-elements can be found in [17, Theorem 5.1].

Definition 9 ([16, 4.1]) A Jordan algebra J is *rationally primitive* if it is primitive and has a nonzero PI-element.

The notion of rationally primitive Jordan system was introduced in [16] as a jordanification of that of strong primitivity for associative algebras [26, p. 48] (also [27, p. 281]). For associative systems different from algebras, more precisely for associative pairs, the notion of strong primitivity has been recently considered in [24]. We refer the reader to [16, p. 317] for motivation about this slightly different terminology between the associative and Jordan notions.

Rationally primitive Jordan algebras were characterized in [16, Theorem 4.6] as strongly prime Jordan algebras with nonzero socle equal to the set of PI-elements or equivalently as strongly prime Jordan algebras having simple unital local PI-algebras.

Theorem 3 *Let J be a strongly prime Jordan algebra with nonzero PI-elements. Then the maximal algebra of quotients $Q_{max}(J)$ of J is rationally primitive.*

Proof We first note that it suffices to consider the case when $J = PI(J)$. Indeed, being $PI(J) \neq 0$ and J strongly prime, $PI(J)$ is an essential ideal of J and then, by [22, Lemma 3.3], we have $Q_{max}(J) = Q_{max}(PI(J))$. Moreover, clearly $PI(PI(J)) = PI(J)$.

Suppose that $J = PI(J)$. Then we have $J = PI(J) = LC(J) = J \cap Soc(Q_{max}(J))$ as a result of [25, Proposition 3.3] and [22, Theorem 3.5].

Take now an arbitrary nonzero element $x \in J = PI(J)$. Then J_x is a strongly prime Jordan PI-algebra, moreover J_x is an LC-algebra (see Definition 8). Therefore $Q_{max}(J_x) \cong \Gamma(J_x)J_x$, the central closure $\Gamma(J_x)J_x$ of J_x (see Remark 2), is a simple unital Jordan PI-algebra of finite capacity [18, Theorem 1.10] (see also [20]). Hence, since by Theorem 2 we have $Q_{max}(J)_x \cong Q_{max}(J_x)$, $Q_{max}(J)$ is rationally primitive as a result of the characterization of rational primitivity given in [16, Theorem 4.6]. □

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On the Numerical Behavior of a Chemotaxis Model with Linear Production Term

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Abstract We consider a chemorepulsion model with production, which is a nonlinear parabolic system for two variables: the cell density and the chemical concentration, defined in a bounded domain. Firstly, we comment some properties of the models considering different production terms of potential type. Afterwards, for the linear production case, we construct some fully discrete finite element schemes approaching this model, for which we analyze several properties, such that: mass-conservation, positivity of the variables, existence and uniqueness of solution, unconditional energy-stability, and uniform in time energy estimates.

1 The Model

The famous Keller-Segel model for chemotaxis has been widely studied in the literature, specially from a theoretical point of view. However, the numerical analysis is more scarce. Here, we want to review and summarize the main results obtained by the authors in [8], where the analysis of a model of Keller-Segel type with repulsive chemotaxis is made from a numerical point of view, presenting several schemes approaching the continuous model and preserving its main properties (although the positivity is obtained only in the limit). All technical details must be found in [8].

Chemotaxis is the biological process of the movement of living organisms in response to a chemical stimulus which can be given towards a higher (attractive)

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or lower (repulsive) concentration of a chemical substance. At the same time, the presence of living organisms can produce or consume chemical substance.

The classical Keller-Segel system (1970–1971, [9]) can be written as follows:

$$\begin{cases} \partial_t u - D_u \Delta u + \chi \nabla \cdot (u \nabla v) = 0 \ \text{ in } \Omega, \ t > 0, \\ \partial_t v - D_v \Delta v + v = u \ \text{ in } \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ v(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \geq 0, \ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0 \ \text{ in } \Omega, \end{cases} \tag{1}$$

where v is the chemical concentration and u denotes the cell density. The term $\chi \nabla \cdot (u \nabla v)$ models the transport of cells towards the higher concentrations of chemical signal if $\chi > 0$, and towards the lower concentrations of chemical signal if $\chi < 0$.

The classical Keller-Segel model is then an attractive system ($\chi > 0$). A deeper study on this system reveals that blow-up phenomena can appear. However, the repulsive model ($\chi < 0$) behaves in such a way that there exist global in time weak solutions (u, v) (see Cieslak et al. [3]) that tend asymptotically in time to a constant state. Moreover, for $2D$ domains, one has the existence and uniqueness of global in time regular solutions.

1.1 Chemorepulsion Production Systems

Assume now that we are in the repulsive framework considering that the equation of v has a production term with a potential structure (by simplicity, we take $\chi = -1$). In this case, the model could be written as:

$$\begin{cases} \partial_t u - D_u \Delta u - \nabla \cdot (u \nabla v) = 0 \ \text{ in } \Omega, \ t > 0, \\ \partial_t v - D_v \Delta v + v = f(u) \ \text{ in } \Omega, \ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \ t > 0, \\ v(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \geq 0, \ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0 \ \text{ in } \Omega, \end{cases} \tag{2}$$

where $f(u) \geq 0$ if $u \geq 0$. The analysis of the existence of solution will be different depending on the form of $f(u)$. We have considered the following cases:

- $f(u) = u$: linear term,
- $f(u) = u^2$: quadratic term,
- $f(u) = u^p$ ($p \in (1, 2)$): potential term.

Although the classical Keller-Segel model corresponds to $p = 1$, the fact of considering other kinds of production terms responds to an attempt of comparing the properties of these models in a variational framework. In this sense, it can be seen that the existence of a dissipative energy of these models falls in a hilbertian L^2-framework (for the quadratic case), a non-hilbertian L^p-framework (for the potential case) or a logarithmic case (linear case). In fact, although the models have different energies, the analytical results are rather similar, namely existence of global weak solutions in $3D$ domains and existence and uniqueness of global strong solutions in $2D$. Some theoretical results about the potential and quadratic cases can be found in [4, 6, 7]. These results are based on the properties described in the next Subsection.

1.2 Some Properties

Model (2) possesses some properties, and the main ones are summarized below:

- **Positivity:** $u \geq 0$ and $v \geq 0$, as biological variables.
- **Blow-up is not expected:** if cells move towards low chemical concentration, thus in the zone of low chemical concentration its quantity increases due to the production term. In such situation, it is not strange to expect that both variables tend to a stationary constant regime when time goes to $+\infty$. However, for $3D$ domains, there is not a rigorous proof of this fact.
- **Mass-conservation:** The problem is conservative in u, as we can check integrating $(2)_1$ in Ω,

$$\frac{d}{dt}\left(\int_\Omega u\right) = 0, \quad \text{i.e.} \quad \int_\Omega u(t) = \int_\Omega u_0, \quad \forall t > 0.$$

  Also, integrating $(2)_2$ in Ω, we deduce the following behavior of v,

$$\frac{d}{dt}\left(\int_\Omega v\right) = \int_\Omega f(u) - \int_\Omega v.$$

- **Energy inequality:** Writing the three cases in a common form, $f(u) = u^p$, for $p \in [1, 2]$, the problem satisfies an energy inequality:

$$\mathcal{E}(u(t_1), v(t_1)) - \mathcal{E}(u(t_0), v(t_0)) + \int_{t_0}^{t_1} D(u(s), v(s))\, ds \leq 0, \tag{3}$$

  for a.e. $t_0, t_1 \colon t_1 \geq t_0 \geq 0$, where $\mathcal{E}(\cdot,\cdot)$ and $D(\cdot,\cdot)$ denote the dissipative energy and the "physical" dissipation, respectively.

  In the linear case ($p = 1$), the obtention of (3) is made formally using $(\ln(u), -\Delta v)$ as test functions and taking into account that the chemotactic and production terms vanish. Concretely,

$$\mathcal{E}(u(t), v(t)) = \int_\Omega \left(F(u(t)) + \frac{1}{2}|\nabla v(t)|^2\right) dx \tag{4}$$

for $F(u) = u\,ln(u) - u$ a convex potential function, and

$$D(u(t), v(t)) = \int_\Omega F''(u(t))\,|\nabla u(t)|^2\,dx + \int_\Omega \left(|\nabla v(s)|^2 + |\Delta v(s)|^2\right).$$

In the quadratic case ($p = 2$), using $(2u, -\Delta v)$ as the test functions, (4) is obtained for $F(u) = u^2$. The potential case ($p \in (1,2)$) uses $\left(\frac{p}{p-1}\,u^{p-1}, -\Delta v\right)$ as test functions, and (4) is obtained for $F(u) = u^p/(p-1)$. In particular, in all cases the corresponding energy is time decreasing.

2 Numerical Results

We look for designing some numerical schemes with the aim of conserving at the discrete level the main theoretical properties of problem (2) for $p = 1$. A similar analysis is made in [4–6] for $p = 2$, and in [7] for $p \in (1,2)$.

2.1 The Numerical Difficulties

When numerical approach of the solution want to be made for a PDE system, sometimes Finite Element methods (FEM) are used. In this case, the solution is searched in a locally polynomial space, where the logarithm function is not included.

Therefore, the standard procedure of FEM approximation cannot be applied directly if we want to maintain good properties with respect to a discrete energy law. In particular, we want that discrete energy to also be decreasing in time. This property is called the energy-stability.

One possible solution could be to write another problem that coincides with the original one in some sense (for instance in the continuous framework) and makes possible that its FEM-approximation be energy-stable.

On the other hand, the mass conservation property is not difficult to maintain when FE are applied. However, the positivity for the discrete approximation of u is not evident. Therefore, some truncated and regularized functions will be used in order to treat this problem.

A first approximation of problem (2) for $f(u) = u$ is the problem (P_ε):

$$(P_\varepsilon)\ \begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon - \nabla\cdot(\lambda_\varepsilon(u_\varepsilon)\nabla v_\varepsilon) = 0 \ \text{ in } \Omega,\ t > 0, \\ \partial_t v_\varepsilon - \Delta v_\varepsilon + v_\varepsilon = u_\varepsilon \ \text{ in } \Omega,\ t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = \dfrac{\partial v_\varepsilon}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega,\ t > 0, \\ u_\varepsilon(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0,\ v_\varepsilon(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \geq 0 \ \text{ in } \Omega, \end{cases} \tag{5}$$

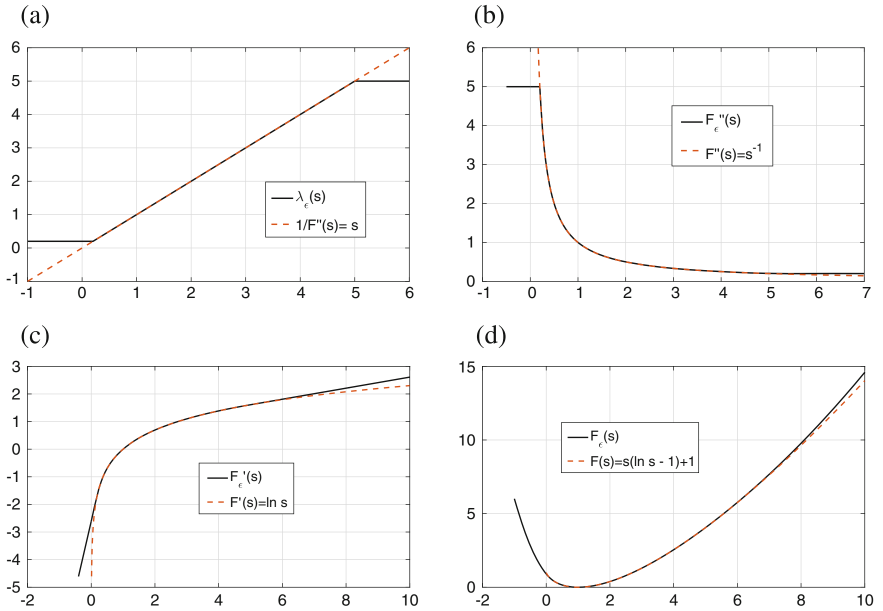


Fig. 1 Functions λ_ε and F_ε and its derivatives. (**a**) $\lambda_\varepsilon(s)$ vs $\frac{1}{F''(s)} := s$. (**b**) $F''_\varepsilon(s)$ vs $F''(s) := \frac{1}{s}$. (**c**) $F'_\varepsilon(s)$ vs $F'(s) := ln(s)$. (**d**) $F_\varepsilon(s)$ vs $F(s) := s(ln(s) - 1) + 1$

where $\lambda_\varepsilon(s)$ is a truncation of the function s (see Fig. 1a). In general, $\varepsilon \in (0, 1)$ is a parameter depending on k and h, $\varepsilon = \varepsilon(k, h)$, being k and h the time step and the mesh size of the discrete approximation, and such that $\varepsilon(k, h) \to 0$ when $(k, h) \to 0$.

Using the function $F_\varepsilon : \mathbb{R} \to [0, +\infty)$ (see Fig. 1d) given by

$$F_\varepsilon(s) \quad := \quad \begin{cases} \frac{s^2-\varepsilon^2}{2\varepsilon} + s(ln(\varepsilon) - 1) + 1 & \text{if } \ s \le \varepsilon, \\ s(ln(s) - 1) + 1 & \text{if } \ \varepsilon \le s \le \varepsilon^{-1}, \\ \frac{\varepsilon(s^2-\varepsilon^{-2})}{2} + s(ln(\varepsilon^{-1}) - 1) + 1 & \text{if } \ s \ge \varepsilon^{-1}, \end{cases}$$

(note that $F_\varepsilon(s)$ is a regularized/truncated function approaching the energy potential $F(s) = s\ln(s) - s + 1$, with $s > 0$), whose derivatives appear in Fig. 1b,c, and testing (P_ε) by $(F'_\varepsilon(u_\varepsilon), -\Delta v_\varepsilon)$, the following (approximate) energy law holds:

$$\frac{d}{dt}\int_\Omega \Big(F_\varepsilon(u_\varepsilon) + \frac{1}{2}|\nabla v_\varepsilon|^2\Big) d\boldsymbol{x} + \int_\Omega F''_\varepsilon(u_\varepsilon)\,|\nabla u_\varepsilon|^2\, d\boldsymbol{x} + \int_\Omega \Big(|\nabla v_\varepsilon|^2 + |\Delta v_\varepsilon|^2\Big) = 0.$$

2.2 Numerical Approach Using (P_ε)

Once the problem (P_ε) is written, a numerical approximation must be designed. First, we write a variational formulation for (5), and we approximate the space $H^1(\Omega)$ (where the solution (u, v) is being searched) by the FE-continuous spaces U_h and V_h (in principle, the space U_h and V_h must not be the same). Assuming a backward Euler discretization for the derivative in time, i.e. denoting $\delta_t u^n = (u^n - u^{n-1})/k$, we arrive to the scheme:

$$\begin{cases} (\delta_t u_\varepsilon^n, \bar{u}) + (\nabla u_\varepsilon^n, \nabla \bar{u}) = -(\lambda_\varepsilon(u_\varepsilon^n)\nabla v_\varepsilon^n, \nabla \bar{u}), \quad \forall \bar{u} \in U_h, \\ \\ (\delta_t v_\varepsilon^n, \bar{v}) + (\nabla v_\varepsilon^n, \nabla \bar{v}) + (v_\varepsilon^n, \bar{v}) - (u_\varepsilon^n, \bar{v}) = 0, \quad \forall \bar{v} \in V_h. \end{cases} \tag{6}$$

Hereafter, $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ scalar product.

Now, the spatial operator must be discretized. In the equation for v_ε^n, the fact of taking $-\Delta_h v_n \in V_h$ as test function gives similar estimates to the continuous case, where the linear operator $-\Delta_h : V_h \to V_h$ is defined as follows:

$$(-\Delta_h v^h, \bar{v}) = (\nabla v^h, \nabla \bar{v}), \quad \forall \bar{v} \in V_h.$$

However, in order to obtain a discrete energy inequality, the chemotactic part in the equation for u_ε^n must vanish with the production part in the equation for v_ε^n. Taking into account that $F_\varepsilon'(u_\varepsilon^n)$ has to be interpolated (in order to belong to the FE-space), we would need:

$$(\lambda_\varepsilon(u_\varepsilon^n)\,\nabla v_\varepsilon^n, \nabla \Pi^h(F_\varepsilon'(u_\varepsilon^n))) = (\nabla u_\varepsilon^n, \nabla v_\varepsilon^n), \tag{7}$$

being Π^h the Lagrange interpolator from $C^0(\overline{\Omega})$ onto U_h. However, it is not clear that equality (7) holds for λ_ε given in Fig. 1a. Thus, instead of the function λ_ε, we will use a matrix operator $\Lambda_\varepsilon : U_h \to L^\infty(\Omega)^{d\times d}$ (designed by Barret and Blowey, see [2]), which satisfies the following properties:

- $\Lambda_\varepsilon u^h$ is symmetric and positive definite for all $u^h \in U_h$ and it is an approximation of $\mathcal{I} u^h$ as $\varepsilon \to 0$, where $\mathcal{I}$ denotes the identity matrix.
- It holds, for all $u^h \in U_h$,

$$(\Lambda_\varepsilon u^h)\nabla \Pi^h(F_\varepsilon'(u^h)) = \nabla u^h \quad \text{in } \Omega.$$

In the construction of this matrix operator Λ_ε, the following hypotheses are required:

- **(H)** The triangulation is structured in the sense that all simplices of the domain discretization have a right angle.
- U_h is generated by $\mathbb{P}_1$-continuous FE. There is not constraints about the choice of V_h, that could be generated by $\mathbb{P}_k$-continuous FE ($k \geq 1$).

Using this operator Λ_ε, we can modify system (6), obtaining the following first order in time, nonlinear and coupled scheme (called scheme **UV** from now on):

$$\begin{cases} (\delta_t u_\varepsilon^n, \bar{u})^h + (\nabla u_\varepsilon^n, \nabla \bar{u}) = -(\Lambda_\varepsilon(u_\varepsilon^n)\nabla v_\varepsilon^n, \nabla \bar{u}), \quad \forall \bar{u} \in U_h, \\ \\ (\delta_t v_\varepsilon^n, \bar{v}) + (\nabla v_\varepsilon^n, \nabla \bar{v}) + (v_\varepsilon^n, \bar{v}) - (u_\varepsilon^n, \bar{v}) = 0, \quad \forall \bar{v} \in V_h, \end{cases} \tag{8}$$

where $(\cdot,\cdot)^h$ is the mass lumping discrete scalar product, which will serve to obtain unconditional energy-stability, which yields to energy estimates. Concretely, the following properties are deduced:

- **Well-posedness:** Unconditional existence of solution of (8), and conditional uniqueness.
- **Mass conservation**:

$$\int_\Omega u_\varepsilon^n = \int_\Omega u_\varepsilon^0, \quad \forall n \geq 1. \tag{9}$$

- **Unconditional energy-stability:** In fact, the following discrete energy law holds:

$$\delta_t \left((F_\varepsilon(u_\varepsilon^n), 1)^h + \frac{1}{2}\|\nabla v_\varepsilon^n\|_{L^2}^2 \right) + \varepsilon \|\nabla u_\varepsilon^n\|_{L^2}^2 + \|\Delta_h v_\varepsilon^n\|_{L^2}^2 + \|\nabla v_\varepsilon^n\|_{L^2}^2 \leq 0.$$

- Uniform in time **energy estimates.**
- **Approximated positivity of u_ε^n and v_ε^n**: the negative part of u_ε^n and v_ε^n tends to zero when $\varepsilon \to 0$:

$$\|((u_\varepsilon^n)_-, (v_\varepsilon^n)_-)\|_{L^2} \to 0 \quad \text{when } \varepsilon \to 0, \tag{10}$$

where, in general, we denote the negative part of w by $w_- := \min\{w, 0\}$.

2.3 Numerical Approach Using the Scheme US

A different approach is deduced based on the idea appearing in the work of Zhang et al. (see [11]) by using the equation satisfied by $\boldsymbol{\sigma} = \nabla v$, first solving the system satisfied by $(u, \boldsymbol{\sigma})$ and, given u, recovering v a posteriori from the equation of v. The resulting system must be rewritten in order to obtain that the numerical approach satisfies an energy inequality, and it is well-posed in a FE-framework. In this case, the problem (2) for $f(u) = u$ is approximated by the following problem $(Q)_\varepsilon$:

$$(Q_\varepsilon) \begin{cases} \partial_t u_\varepsilon - \nabla \cdot (\lambda_\varepsilon(u_\varepsilon)\nabla(F_\varepsilon'(u_\varepsilon))) - \nabla \cdot (\lambda_\varepsilon(u_\varepsilon)\boldsymbol{\sigma}_\varepsilon) = 0 \text{ in } \Omega, \ t > 0, \\ \partial_t \boldsymbol{\sigma}_\varepsilon + \text{rot}(\text{rot}\, \boldsymbol{\sigma}_\varepsilon) - \nabla(\nabla \cdot \boldsymbol{\sigma}_\varepsilon) + \boldsymbol{\sigma}_\varepsilon = \lambda_\varepsilon(u_\varepsilon)\nabla(F_\varepsilon'(u_\varepsilon)), \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \ t > 0, \\ \boldsymbol{\sigma}_\varepsilon \cdot \mathbf{n} = 0, \quad [\text{rot}\, \boldsymbol{\sigma}_\varepsilon \times \mathbf{n}]_{tang} = 0 \quad \text{on } \partial\Omega, \ t > 0, \\ u_\varepsilon(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0, \ \boldsymbol{\sigma}_\varepsilon(\boldsymbol{x}, 0) = \nabla v_0(\boldsymbol{x}), \quad \text{in } \Omega. \end{cases} \tag{11}$$

The energy law is now obtained testing (11) by $(F'_\varepsilon(u_\varepsilon), \boldsymbol{\sigma}_\varepsilon)$ (again chemotactic term cancels with the production term):

$$\frac{d}{dt}\int_\Omega \Big(F_\varepsilon(u_\varepsilon) + \frac{1}{2}|\boldsymbol{\sigma}_\varepsilon|^2\Big)d\boldsymbol{x} + \int_\Omega \lambda_\varepsilon(u_\varepsilon)|\nabla(F'_\varepsilon(u_\varepsilon))|^2 d\boldsymbol{x} + \|\boldsymbol{\sigma}_\varepsilon\|^2_{H^1} = 0.$$

Then, approximating the space $\mathbf{H}^1_\sigma(\Omega) := \{\boldsymbol{\sigma} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{\sigma}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$ (where the solution $\boldsymbol{\sigma}$ is sought) by the FE-continuous space $\boldsymbol{\Sigma}_h$, we propose the following fully discrete scheme associated to problem $(Q)_\varepsilon$ (called scheme **US** from now on):

- Given $(u_\varepsilon^{n-1}, \boldsymbol{\sigma}_\varepsilon^{n-1}) \in U_h \times \boldsymbol{\Sigma}_h$, compute $(u_\varepsilon^n, \boldsymbol{\sigma}_\varepsilon^n) \in U_h \times \boldsymbol{\Sigma}_h$ solving

$$\begin{cases} (\delta_t u_\varepsilon^n, \bar{u})^h + (\lambda_\varepsilon(u_\varepsilon^n)\nabla\Pi^h(F'_\varepsilon(u_\varepsilon^n)), \nabla\bar{u}) = -(\lambda_\varepsilon(u_\varepsilon^n)\boldsymbol{\sigma}_\varepsilon^n, \nabla\bar{u}), \ \forall \bar{u} \in U_h, \\ (\delta_t \boldsymbol{\sigma}_\varepsilon^n, \bar{\boldsymbol{\sigma}}) + (B_h\boldsymbol{\sigma}_\varepsilon^n, \bar{\boldsymbol{\sigma}}) = (\lambda_\varepsilon(u_\varepsilon^n)\nabla\Pi^h(F'_\varepsilon(u_\varepsilon^n)), \bar{\boldsymbol{\sigma}}), \ \forall \bar{\boldsymbol{\sigma}} \in \Sigma_h, \end{cases} \tag{12}$$

  with

$$(B_h\boldsymbol{\sigma}_\varepsilon^n, \bar{\boldsymbol{\sigma}}) = (\mathrm{rot}\,\boldsymbol{\sigma}_\varepsilon^n, \mathrm{rot}\,\bar{\boldsymbol{\sigma}}) + (\nabla\cdot\boldsymbol{\sigma}_\varepsilon^n, \nabla\cdot\bar{\boldsymbol{\sigma}}) + (\boldsymbol{\sigma}_\varepsilon^n, \bar{\boldsymbol{\sigma}}),$$

  and where the auxiliary variable $\boldsymbol{\sigma}_\varepsilon^n$ tries to approximate ∇v_ε^n.
- We recover $v_\varepsilon^n = v_\varepsilon^n(u_\varepsilon^n) \in V_h$ solving:

$$(\delta_t v_\varepsilon^n, \bar{v}) + (\nabla v_\varepsilon^n, \nabla\bar{v}) + (v_\varepsilon^n, \bar{v}) = (u_\varepsilon^n, \bar{v}), \ \forall \bar{v} \in V_h. \tag{13}$$

Again, we can prove the **well-posedness** of the scheme **US** (unconditional existence and conditional uniqueness), satisfying the **mass conservation** property (9), the **approximated positivity** of the solution $(u_\varepsilon^n, v_\varepsilon^n)$ (see (10)), and the following **discrete energy inequality**:

$$\delta_t\Big((F_\varepsilon(u_\varepsilon^n), 1)^h + \frac{1}{2}\|\boldsymbol{\sigma}_\varepsilon^n\|^2_{L^2}\Big) + \varepsilon\|\nabla\Pi^h(F'_\varepsilon(u_\varepsilon^n))\|^2_{L^2} + \|\boldsymbol{\sigma}_\varepsilon^n\|^2_{H^1} \le 0,$$

from which we deduce uniform in time **energy estimates**.

Note that, in this scheme, the hypothesis (**H**) assumed for Scheme **UV** is not required. Moreover, there is not constraints about the choice of discrete spaces U_h, V_h and $\boldsymbol{\Sigma}_h$.

2.4 Numerical Approach Using the Scheme UZSW

The last scheme that we are going to present, uses the energy quadratization technique (see for instance [1, 10]), that introduces more unknowns in the system in order to linearize the problem and rewrite the energy in a quadratic form, obtaining an energy-stable scheme with respect to a modified energy. Concretely, by introducing the auxiliary variables $z_\varepsilon = F'_\varepsilon(u_\varepsilon)$, $\boldsymbol{\sigma}_\varepsilon = \nabla v_\varepsilon$ and $w_\varepsilon = \sqrt{F_\varepsilon(u_\varepsilon) + A}$ with $A > 0$, we can rewrite the problem as:

$$\begin{cases} \partial_t u_\varepsilon - \nabla \cdot (\lambda_\varepsilon(u_\varepsilon)\nabla z_\varepsilon) - \nabla \cdot (u_\varepsilon \boldsymbol{\sigma}_\varepsilon) = 0 \text{ in } \Omega,\ t > 0, \\ \partial_t \boldsymbol{\sigma}_\varepsilon + \text{rot}(\text{rot}\,\boldsymbol{\sigma}_\varepsilon) - \nabla(\nabla \cdot \boldsymbol{\sigma}_\varepsilon) + \boldsymbol{\sigma}_\varepsilon = \lambda_\varepsilon(u_\varepsilon)\nabla z_\varepsilon \ \text{ in } \Omega,\ t > 0, \\ \partial_t w_\varepsilon = \dfrac{1}{2\sqrt{F_\varepsilon(u_\varepsilon) + A}} F'_\varepsilon(u_\varepsilon)\, \partial_t u_\varepsilon \ \text{ in } \Omega,\ t > 0, \\ z_\varepsilon = \dfrac{1}{\sqrt{F_\varepsilon(u_\varepsilon) + A}} F'_\varepsilon(u_\varepsilon)\, w_\varepsilon \ \text{ in } \Omega,\ t > 0, \\ \dfrac{\partial z_\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,\ t > 0, \\ \boldsymbol{\sigma}_\varepsilon \cdot \mathbf{n} = 0, \ \ [\text{rot}\,\boldsymbol{\sigma}_\varepsilon \times \mathbf{n}]_{tang} = 0 \quad \text{on } \partial\Omega,\ t > 0, \\ u_\varepsilon(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0,\ \boldsymbol{\sigma}_\varepsilon(\boldsymbol{x}, 0) = \nabla v_0(\boldsymbol{x}), \\ w_\varepsilon(\boldsymbol{x}, 0) = \sqrt{F_\varepsilon(u_0(\boldsymbol{x})) + A} \quad \text{in } \Omega, \end{cases} \tag{14}$$

The advantages and disadvantages that Scheme **UZSW** presents with respect to Schemes **UV** and **US** are summarized in Tables 1 and 2.

2.5 Numerical Simulations

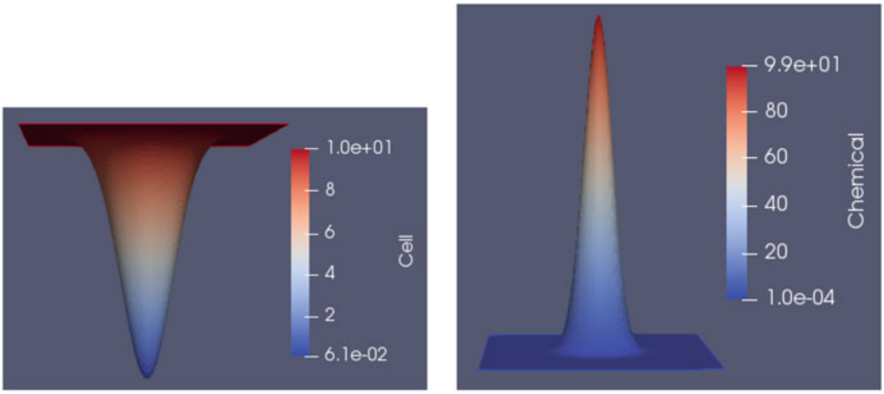
We make some numerical simulations in order to validate numerically the theoretical results obtained. Here, we compare the three schemes analyzed (schemes **UV**, **US**, and **UZSW**) with the classical Backward Euler scheme **BE** associated to (2) for $f(u) = u$. We chose the spaces for $(u, z, \boldsymbol{\sigma}, w, v)$ generated by $\mathbb{P}_1$-continuous FE, the domain $\Omega = [0, 2]^2$ using a structured mesh, and all the simulations were

Table 1 Overview for the numerical analysis of the schemes

Schemes	**UV**	**US**	**UZSW**
Linear or nonlinear	Nonlinear	Nonlinear	Linear
Well-posedness	Yes	Yes	Yes
Hypothesis (**H**) required?	Yes	No	No
Approximated positivity of u_ε^n and v_ε^n	Yes	Yes	No

Table 2 Overview for the numerical simulations of the schemes

Schemes	**UV**	**US**	**UZSW**
Energy stability w.r.t. $\mathcal{E}_e(u, v)$	Yes	Yes	No
Mass-conservation	Yes	Yes	Yes
Approximated positivity of u_ε^n and v_ε^n	Yes	Yes	No
CPU time	Less	Intermediate	Higher



(a) (b)

Fig. 2 Initial conditions. (**a**) Initial cell density u_0. (**b**) Initial chemical concentration v_0

carried out using **FreeFem++** software. The linear iterative method used to approach the nonlinear schemes was a Picard Method.

2.5.1 Approximated Positivity

In order to test the approximated positivity of the schemes, we choose $k = 10^{-5}$, $h = \frac{1}{40}$ and the very exigent initial conditions (see Fig. 2):

$$u_0 = -10xy(2-x)(2-y)exp(-10(y-1)^2 - 10(x-1)^2) + 10.0001$$

and

$$v_0 = 100xy(2-x)(2-y)exp(-30(y-1)^2 - 30(x-1)^2) + 0.0001.$$

We observe that for the schemes **UV** and **US**, the approximated positivity holds for $(u_\varepsilon^n, v_\varepsilon^n)$; while for the scheme **UZSW**, this behavior is not observed (see Fig. 3). For the scheme **BE** we have also observed negative values for the minimum of u_ε^n in some times $t_n > 0$, with values more negative than in the schemes **UV** and **US**.

2.5.2 Energy Stability

We compare the energy stability of the schemes with respect to the "exact energy" (which comes from the continuous problem):

$$\mathcal{E}_e(u, v) := \int_\Omega u_+ \left(\ln(u_+) - 1\right) d\boldsymbol{x} + \frac{1}{2}\|\nabla v\|_{L^2}^2$$

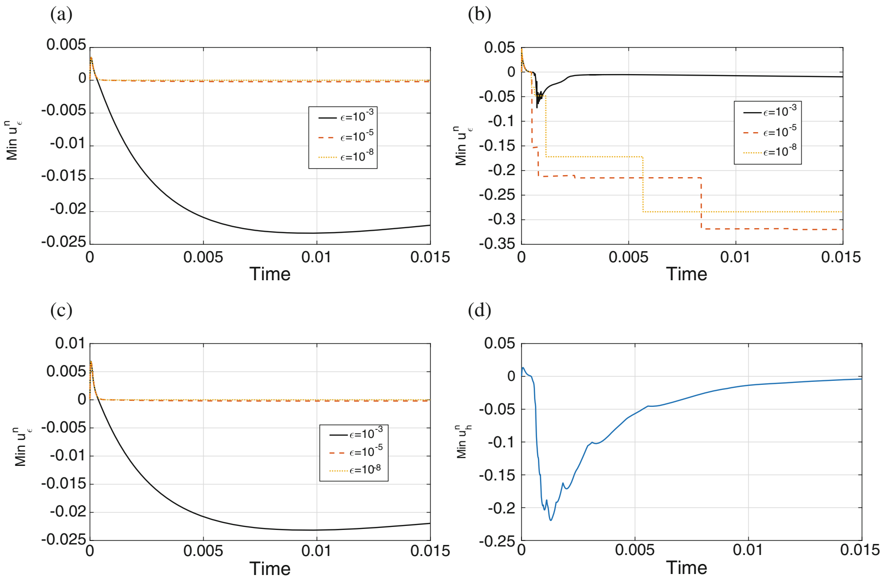


Fig. 3 Approximate positivity of u_ε^n. (**a**) Minimum values of u_ε^n computed using the scheme **UV**. (**b**) Minimum values of u_ε^n computed using the scheme **UZSW**. (**c**) Minimum values of u_ε^n computed using the scheme **US**. (**d**) Minimum values of u^n computed using the scheme **BE**

and the behaviour of the discrete residual of the energy law:

$$RE_e(u^n, v^n) := \delta_t \mathcal{E}_e(u^n, v^n) + 4\int_\Omega |\nabla\sqrt{u_+^n}|^2 d\boldsymbol{x} + \|\Delta_h v^n\|_{L^2}^2 + \|\nabla v^n\|_{L^2}^2 \le 0,$$

where $u_+^n := \max\{u^n, 0\}$.

We observe that the schemes **UV**, **US** and **BE** have decreasing in time energy $\mathcal{E}_e(u, v)$, while the scheme **UZSW** evidences increasing energies for some times $t_n > 0$. On the other hand, the schemes **UV** and **US** evidence $RE_e(u^n, v^n) \le 0$ for all $n \ge 0$, while the schemes **BE** and **UZSW** have $RE_e(u^n, v^n) \ge 0$ for some times t_n (see Fig. 4).

3 Conclusions

The three schemes (**UV**, **US** and **UZSW**) are **mass-conservative** and **unconditionally energy-stable**, but with respect to different discrete energies. The main theoretical and numerical results obtained for the schemes are sumarized in the Tables 1 and 2.

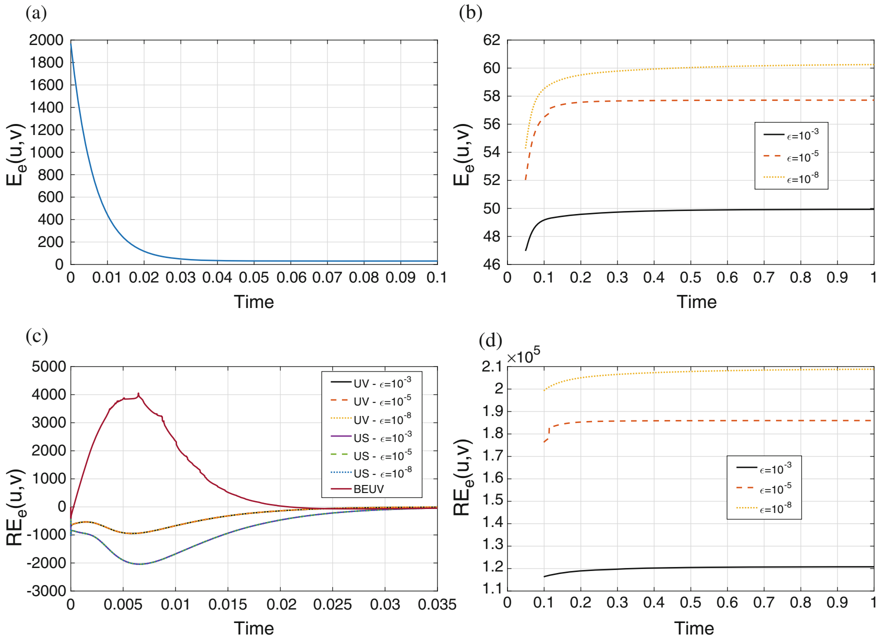


Fig. 4 Energy-stability of the schemes. (**a**) Energy $\mathcal{E}_e(u^n, v^n)$ of the scheme **BE**, **UV**, **US**. (**b**) Energy $\mathcal{E}_e(u^n_\varepsilon, v^n_\varepsilon)$ of the scheme **UZSW** for diff. values of ε. (**c**) $RE_e(u^n, v^n)$ of the schemes **BEUV**, **UV**, **US**. (**d**) $RE_e(u^n_\varepsilon, v^n_\varepsilon)$ of the scheme **UZSW** for diff. values of ε

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The Thin-Sandwich Problem in General Relativity

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Abstract This paper serves a review on the history and results on the reduced thin-sandwich equations of general relativity (GR). These equations arise within the initial value formulation of GR as a method of solving the Einstein constraint equations. The program of analysing the generality of this procedure was first started by J. A. Wheeler, which put forward clear physical motivations for this approach, and conjectured that this program could generically work. Through this paper, we will review the main contributions to this problem, going from the original conjecture to recent results. Finally, we will also highlight how problems related with the Einstein constraint equations, in particular with the thin-sandwich problem, can be used as tools when analysing interesting and seemingly unrelated problems in Lorentzian geometry. Namely, we will prove a theorem on the existence of isometric embeddings of low regularity for compact Lorentzian manifolds into Ricci-flat spaces.

1 Introduction

It is a well-known fact that the vacuum Einstein field equations of general relativity (GR) admit a well-posed initial value formulation, as long as the initial data satisfies a system of constraint equations [1]. The same thing can be said for different matter models of interest in physics [1, 2]. Furthermore, since the Einstein constraint equations (ECE) arise from the usual Gauss–Codazzi equations for hypersurfaces, it is a *necessary condition* that they must be satisfied in order for the space-time Einstein equations to have a well-posed Cauchy problem. It is a remarkable fact that the constraint equations are also a sufficient condition for this problem to be

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well-posed, at least in most cases of interest. There is no surprise then in the fact that these equations have drawn a great deal of attention.

During this paper, we will center on one problem related with the ECE, which coined the name of *thin-sandwich problem* (TSP) [3–5]. This problem comes about as a simplification of what came to be known as the *sandwich conjecture*. Both problems were initially posed by the famous physicist John A. Wheeler. In the following sections we will make explicit the nature of these problems, but it is important to remark that both these problems were motivated by physical situations [3, 6], and translated into problems in geometry and geometric analysis [4, 5, 7–9]. It is interesting to note that claims concerning the well-posedness of TSP can be found in the literature in very influential papers, such as [3, 6], in the early sixties. It was not until the end of such decade that a more rigorous investigation of the problem came about, and it was clearly shown that the TSP could not be well-posed in general [7], as might have been previously expected. The first existence results were due to Robert Bartnik and Gyula Fodor, which gave sufficient conditions for the TSP to well-posed in the case of 3-dimensional closed manifolds [4]. Their result was generalized by Domenico Giulinni to incorporate more realistic physical sources into the equations [8]. It is important to realize that, in spite of the fact that the main analytical tools used by Bartnik and Fodor in [4] do not rely on the dimensionality of the manifold, their proof took great advantage of the fact that they were working in low dimensions in order to manipulate different expressions. Furthermore, besides particular examples, the issue of whether this method of solving the constraint equations was general enough to be able to parametrize a sufficiently large subset of the space of solutions on *any* closed manifold, had not been addressed at all. The first result in these direction was [5], were the Bartnik-Fodor theorem is generalized for arbitrary dimensions, and it is shown that, on any closed manifold, the solutions of the TSP parametrize an open set in the space of solutions of the constraint equations [5]. Further analysis of the subset of the space of solutions of the ECE which can be parametrized via the thin-sandwich approach has been carried out in [10], where, for the first time, this problem is also posed on asymptotically euclidean (AE) manifolds.

In the following sections, this paper will be organized as follows. First, we will review the main definitions and results concerning the initial value formulation of GR. Then, we will introduce the TSP and the reduced thin-sandwich equations (RTSE) as a method of solving the constraint equations, which carries clear physical motivations. Then, we will analyse the well-posedness of the RTSE on closed manifolds together with some *genericity* issues related with these equations, and comment on how the problem translates to AE manifolds. Finally, we will show how results concerning the ECE can play a central role in analysing somewhat unrelated problems in Lorentzian geometry. In order to do this, we will show how our results on the TSP can be used as the main tool in proving existence of isometric embeddings of compact Lorentzian manifolds in Ricci-flat spaces [11]. This last problem has its own quite interesting motivations, both in physics and mathematics.

2 The Initial Value Formulation of GR

In order to introduce the Einstein equations within the context of their initial value formulation in the most direct possible way, let us consider the following definition.

Definition 1 An $(n+1)$-dimensional **globally hyperbolic space-time** is defined to be an $(n+1)$-dimensional Lorentzian manifold $(V \doteq M^n \times \mathbb{R}, \bar{g})$ satisfying the Einstein equations:

$$Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = T(\bar{g}, \bar{\psi}) \tag{1}$$

where, $Ric(\bar{g})$ and $R(\bar{g})$ represent the Ricci tensor and Ricci scalar of $\bar{g}$, respectively, and T represents some $(0, 2)$-tensor field, called the energy-momentum tensor, depending on $\bar{g}$ and (possibly) on a collection of tensor fields, collectively denoted by $\bar{\psi}$, representing other *physical* fields.

In order to formulate the Cauchy problem for GR, define an **initial data set** for the Einstein equations to be given by (M, g, K, ψ), where M is an n-dimensional smooth Riemannian manifold with metric g, K is a symmetric second rank tensor field, and ψ collectively denotes the necessary initial data for the non-geometric fields $\bar{\psi}$. Then, **The Cauchy problem for GR** consists in finding an isometric embedding of such an initial data set into a globally hyperbolic space-time $(V, \bar{g})$, $\iota : M \mapsto V$, such that K is the second fundamental form of M seen as a submanifold of V.

The Cauchy problem for GR is translated into an existence problem for a set of partial differential equations (PDEs) by trivially embedding $M \hookrightarrow M \times \mathbb{R}$, and writing the set of Eq. (1) as a hyperbolic system for the metric and the other fields involved. In this process, it is standard to consider local co-frames where we can write the metric $\bar{g}$ in a convenient way, such that we have a "space-time splitting". In order to do this, a vector field β, which is constructed so as to be tangent to each hypersurface $M \times \{t\}$, is used to define the following local coframe

$$\theta^i = dx^i + \beta^i dt \ , \quad i = 1, \cdots, n$$
$$\theta^0 = dt.$$

Then we can write the metric $\bar{g}$ in the following way

$$\bar{g} = -N^2\theta^0 \otimes \theta^0 + g_{ij}\theta^i \otimes \theta^j$$

where the function N is a positive function referred to as the **lapse** function, while the vector field β is called the **shift** vector. Using this frame, the extrinsic curvature to each submanifold M_t is given by:

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - (\nabla_i \beta_j + \nabla_j \beta_i)). \tag{2}$$

For most sources of interest, it is known that GR has a well posed Cauchy problem for initial data satisfying the following set of constraint equations on M (see [2, 12] for updated reviews on this topic):

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 2\epsilon \\ \mathrm{div}_g K - \nabla \mathrm{tr}_g K &= S \end{aligned} \tag{3}$$

where (ϵ, S) denote the induced energy and momentum densities on M, respectively, $R(g)$ represents the scalar curvature of g, $|\cdot|_g$ denotes the pointwise-tensor norm in the metric g and $\mathrm{div}_g K$ denotes the divergence of K in the metric g.

At this point, it should be noted that the above systems of constraints, as a system for the initial data (g, K) would seem to be highly underdetermined, since we have a scalar and a vector equation coupled on M (which could *naively* be thought of as $(n+1)$-equations), and we need to fix two $(0, 2)$-symmetric tensor field on M. In fact, this underdetermined property of the system can be expressed more rigorously, for instance within the *conformal method* of solving the constraint equations. Within this method, the proposal is to fix the conformal class of the metric g together with the mean curvature $\tau \doteq \mathrm{tr}_g K$, and solve the constraint system for the conformal factor and the traceless part of the extrinsic curvature. In this scenario, if the physical sources are known, the system (3) can be posed as an elliptic system for the conformal factor and a vector field, whose conformal lie derivative is used to construct the traceless part of K (see [2] for a detailed description of this procedure, together with further references). In particular, in the constant mean curvature scenario (CMC), on closed manifolds, there is a complete classification, due to James Isenberg, of the CMC conformal initial data which allow the constraint equations to be solved [13].

Taking into account the above paragraph, it becomes quite clear that trying to *choose* appropriately which part of the initial data (g, K) is to be set as *prescribed initial data* and which part is to be solved for, might be an issue of physical interest. In fact, this was the origin of Wheeler's sandwich and thin-sandwich conjectures. In the first one, the idea was to try to prescribe two Riemannian metrics on a 3-manifold M, and try to find a solution of the Einstein space-time equations which would result as the evolution of one of these metrics *towards* the other, filling the *sandwich* with the space-time in the middle. This idea comes about as an attempt of generalizing the usual variational problem in classical particle physics, where the ends of the trajectories are fixed, to the setting of GR. The TSP comes about as a simplification of the sandwich problem, where the idea is to try to prescribe the initial Riemannian metric g, together with its initial time-derivative $\dot{g}$ (see [3, 14]). Again, this correlates with well-understood problems in classical particle physics, where one is allowed to prescribe the initial position and velocities of the particles in the system. Notice from (2), that if $(g, \dot{g})$ are taken as prescribed, then the constraint equations should be posed for the lapse function and shift vector field on M. This was expected to be a determined system which could be solved for generic initial data, as can be read out from [14].

3 The Reduced Thin Sandwich Equations

As we have explained in the previous section, the RTSE appear when trying to solve the constraints in terms of the lapse function and shift-vector field. This is achieved by using the expression for K in terms of N and β. In fact, given a solution of the constraints, satisfying $2\epsilon - R(g) \neq 0$ over all M, the lapse function can be equated from the hamiltonian constraint to get:

$$N = \sqrt{\frac{(\mathrm{tr}_g \gamma)^2 - |\gamma|_g^2}{2\epsilon - R_g}}. \tag{4}$$

where the tensor γ has components

$$\gamma_{ij} = \frac{1}{2}\big(\dot{g}_{ij} - (\nabla_i \beta_j + \nabla_j \beta_i)\big). \tag{5}$$

Replacing (4) in the momentum constraint shows that the shift vector satisfies the following equation, **which is referred to as the RTSE**:

$$\mathrm{div_g}\left(\sqrt{\frac{2\epsilon - R_g}{(\mathrm{tr}_g \gamma)^2 - |\gamma|_g^2}}\,(\gamma - \mathrm{tr}_g \gamma\, g)\right) = S. \tag{6}$$

The main idea in what follows is that we can reverse the above process: If, for given data $\psi \doteq (g, \dot{g}, \epsilon, S)$, Eq. (6) is well posed for β (makes sense), and has a solution, then, taking (4) as a definition for the lapse, (N, β) will solve the constraint equations for the freely chosen initial data ψ.

In order to study the reduced Eq. (6), we will suppose that for some initial data set $\psi_0 \doteq (g_0, \dot{g}_0, \epsilon_0, S_0)$, we have a reference solution of the RTSE β_0, giving rise to a reference solution for the constraints. We will analyse under what conditions the RTSE admit a solution β for initial data $\psi \doteq (g, \dot{g}, \epsilon, S)$ sufficiently close to ψ_0. Afterwards, we will analyse whether it is possible to prescribe families of such reference solutions $(\psi_\alpha, \beta_\alpha)$ satisfying these conditions, where α is some real parameter. If we can, then we will produce an open subset in the space of solutions of the ECE which can be parametrized via the thin-sandwich formulation.

3.1 The Compact Case

Let M be a compact (without boundary) n-dimensional manifold and write this set of non-linear PDE for the shift vector in the following way. Let

$$H_s(T^p_q(M)), \quad s > \frac{n}{2}, \; s > 2,$$

denote the Sobolev space of (p, q)-tensor fields with s generalized derivatives in L^2. Denote

$$\mathcal{E}_1 \doteq H_{s+3}(T_2^0 M) \times H_{s+1}(T_2^0 M) \times H_{s+1}(M) \times H_s(T_1^0 M)$$

which is a Banach space with the norm $\|\cdot\|_{\mathcal{E}_1} : \mathcal{E}_1 \to \mathbb{R}$ given by

$$||(g, \dot{g}, \epsilon, S)||_{\mathcal{E}_1} = \|g\|_{H_{s+3}} + \|\dot{g}\|_{H_{s+1}} + \|\epsilon\|_{H_{s+1}} + \|S\|_{H_s}$$

and let

$$\mathcal{E}_2 \doteq H_{s+2}(T_0^1 M) \quad \text{and} \quad \mathcal{F} \doteq H_s(T_1^0 M).$$

Now suppose that for given data $\psi_0 \doteq (g_0, \dot{g}_0, \epsilon_0, S_0) \in \mathcal{E}_1$ we have a solution $\beta_0 \in \mathcal{E}_2$. Then, the continuity of all the maps involved guarantees that (6) is well-defined in a neighbourhood $\mathcal{U}$ of (ψ_0, β_0) in $\mathcal{E}_1 \times \mathcal{E}_2$. With this in mind, we define the map

$$\Phi : \mathcal{U} \subset \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{F}$$

given by

$$\Phi(\psi, \beta) \doteq \mathrm{div_g}\left(\sqrt{\frac{2\epsilon - R_g}{(\mathrm{tr}_g \gamma)^2 - |\gamma|_g^2}} \left(\gamma - \mathrm{tr}_g \gamma \, g\right)\right) - S \tag{7}$$

where we have denoted $\psi = (g, \dot{g}, \epsilon, S)$, and we are using β to denote the shift. Then the RTSE can be written as

$$\Phi(\psi, \beta) = 0. \tag{8}$$

Now our problem reduces to the following: we want to see if there are open sets $\mathcal{V} \subset \mathcal{E}_1$, $\mathcal{W} \subset \mathcal{E}_2$, with $\psi_0 \in \mathcal{V}$ and $\beta_0 \in \mathcal{W}$, and a unique map

$$g : \mathcal{V} \to \mathcal{W}$$

such that

$$\Phi(\psi, g(\psi)) = 0 \quad \text{for all} \quad \psi \in \mathcal{V}.$$

In this case, $\beta = g(\psi) \in \mathcal{W}$ would be the solution to our problem. In order to address this issue, we intend to use the Implicit Function Theorem. Hence, we need to show that

$$L \doteq D_2 \Phi_{(\psi_0, \beta_0)} : \mathcal{E}_2 \to \mathcal{F} \tag{9}$$

is an isomorphism. A few computations give:

$$D_2\Phi_{(\psi,\beta)}Y = \mathrm{div_g}\left(\frac{1}{N}\left(\mathrm{div_g}Y g - {}^S\nabla Y - \frac{1}{2\epsilon - R_g}\langle \pi, \nabla Y\rangle \pi\right)\right) \tag{10}$$

where π is the tensor

$$\pi \doteq \frac{1}{N}(\gamma - \mathrm{tr}_g\gamma\, g) = K - \mathrm{tr}_g K\ g \tag{11}$$

and

$${}^S\nabla Y_{ij} = \frac{1}{2}\big(\nabla_i Y_j + \nabla_j Y_i\big)$$

What we would like to do at this point, is to apply standard methods in elliptic theory to determine sufficient conditions which guarantee that the operator L is an isomorphism. Nevertheless, the first point that we have to deal with, is that L **is not generally elliptic**. Nonetheless, we have the following consequences [5]:

- If π is a definite operator all over M, then the linear operator L is elliptic.
- L is (formally) self-adjoint: $L^* = L$.
- Thus, assuming that π is definite and using standard elliptic theory, in order to show that L is an isomorphism, we just need to show that it is injective.

In the following proposition, we analyse the kernel of the linear operator L.

Proposition 1 *Consider a reference solution* (ψ, β) *for the TSP on a compact n-dimensional manifold M satisfying that:* (i) π *is a definite operator on M;* (ii) $2\epsilon - R_g > 0$ *on M;* (iii) *given a function* μ, *the equation*

$${}^S\nabla Y = \mu K \tag{12}$$

has only the solution $Y = 0$, $\mu = 0$. *Then L is injective.*

Proof Several computations show that, under hypotheses (i) and (ii), the kernell of L is characterized by the solutions Y of the equation (see [5]):

$${}^S\nabla Y = \frac{1}{|K|_g^2}\langle \nabla Y, K\rangle K \tag{13}$$

So if (13) has only the trivial solution $Y = 0$, then L is injective. □

Using the above results plus the implicit function theorem we prove the following:

Theorem 1 *Suppose* $(\psi_0, \beta_0) \in \mathcal{E}_1 \times \mathcal{E}_2$ *satisfies* $\Phi(\psi_0, \beta_0) = 0$. *Then, if* π *is a definite operator at each point of* M, $2\epsilon - R_g > 0$ *everywhere on* M, *and if for a given function* μ *on* M *the equation*

$$^S\nabla Y = \mu K$$

has only the solution $Y = 0$, $\mu = 0$, *then there are open neighbourhoods* $V \subset \mathcal{E}_1$ *and* $W \subset \mathcal{E}_2$ *of* ψ_0 *and* β_0 *respectively, and a unique mapping*

$$g : \mathcal{V} \to \mathcal{W}$$

such that $\Phi(\psi, g(\psi)) = 0$ *for all* $\psi \in \mathcal{V}$.

3.1.1 Existence of Reference Solutions

What we intend to show is that the constraint equations (3) on a compact manifold M always admit a solution (g, K), satisfying all the hypotheses of the theorem. We will proceed as follows:

- Look for a solution of the constraint equations of the form $(h, \alpha h)$.
- We will restrict ourselves to solutions of (3) with $S = 0$, i.e., with zero momentum density.
- In this set up, the momentum constraint is automatically satisfied: $\mathrm{div}_{\mathrm{h}} K - \nabla \mathrm{tr}_h K = 0$.
- What remains to be solved is the Hamiltonian constraint, which reads as

$$R_h = 2\epsilon - \alpha^2 n(n-1). \tag{14}$$

In order to guarantee the existence of solutions for (14), we will appeal to the following well-established theorem [15].

Theorem 2 (Kazdan–Warner) *Let* M *be a* C^∞ *compact manifold of dimension* $n \geq 3$. *If* $f \in C^\infty(M)$ *is negative somewhere, then there is a* C^∞ *Riemannian metric on* M *with* f *as its scalar curvature.*

Using this theorem and considering $\alpha^2 > \min \frac{2\epsilon}{n(n-1)}$, we see that (14) always admits a smooth solution. A solution constructed in this way satisfies two of the three conditions required by (1), that is, it satisfies

- $2\epsilon - R_h > 0$, which comes from (14).
- π is definite, since from $K = \alpha h$ we get that $\pi = \alpha(1-n)h$.

In this context, the last condition of Theorem 1 becomes the statement that h does not admit conformal Killing fields (CKF). We can guarantee that we can find a solution to (14) with this property by analysing with some care the proof of Kazdan–Warner theorem, and putting it together with some results concerning negative Ricci curvature, due to J. Lohkamp [16]. The idea in this procedure is to first produce a metric g' with $R_{g'} = -1$ which does not possess any CKF, and then follow the proof of the Kazdan–Warner theorem as it is laid out in [17] in order to find a conformal deformation g of g' which solves the prescribed scalar curvature equation (14). Then, the non-existence of CKF for g' implies the same result for g. Putting all this together, gives us the following theorem.

Theorem 3 *On any smooth compact n-dimensional manifold M, $n \geq 3$, there is an open subset in the space of solutions of the constraint equations* (3) *with $S = 0$, which can be parametrized via the thin-sandwich formulation.*

3.1.2 Solutions Around Symmetric Data

The aim of this section and the following one is to study the behaviour of the non linear operator Φ around solutions where its linearisation $D_2\Phi$ is not an isomorphism. We still consider reference solutions of the form treated above, that is, pairs (ψ_0, β_0) induced from a solution of the form $(g_0, K = \alpha g_0)$, with $S_0 = 0$. Under these conditions, the linearization $L \doteq D_2\Phi_{(\psi_0,\beta_0)} : \mathcal{E}_2 \to \mathcal{Z}$ is elliptic and formally self adjoint, and $\text{Ker}(L)$ is the space of conformal killing vector fields of g_0. In particular, we have

$$\begin{aligned} H_{s+2} &= \text{Ker}(L) \oplus \text{Ker}(L)^{\perp}, \\ H_s &= \text{Ker}(L) \oplus \text{Im}(L), \end{aligned}$$

and thus $L : \text{Ker}(L)^{\perp} \mapsto \text{Im}(L)$ is an isomorphism and both $\text{Ker}(L)^{\perp}$ and $\text{Im}(L)$ are closed subsets, and thus Banach. Thus, we can prove the following lemma, which has been extracted from [10].

Lemma 1 *Let M be an n-dimensional compact smooth manifold. Consider the following map*

$$\begin{aligned} &\widetilde{\Phi}_{g_0} : U \subset H_{s+1} \times H_{s+1} \times \text{Im}(L) \times \text{Ker}(L)^{\perp} \to H_s \\ &(\dot{g}, \epsilon, S, \beta) \mapsto \text{div}_{g_0}\left(\sqrt{\frac{2\epsilon_{\Lambda} - R_{g_0}}{(\text{tr}_{g_0}\gamma)^2 - |\gamma|^2_{g_0}}}(\gamma - \text{tr}_{g_0}\gamma\, g_0)\right) - S, \end{aligned} \tag{15}$$

where U is a neighbourhood of $(\tilde{\psi}_0 = (\dot{g}_0, \epsilon_0, 0), \beta_0)$ and $s > \frac{n}{2}$. If the space of conformal Killing vector fields of g_0 consists merely of Killing vector fields, then there are open subsets $U_1 \subset \widetilde{\mathcal{E}}_1 \doteq H_{s+1} \times H_{s+1} \times \text{Im}(L)$ and $U_2 \subset \text{Ker}(L)^{\perp}$, with $\tilde{\psi}_0 \in U_1$ and $\beta_0 \in U_2$ such that the equation

$$\widetilde{\Phi}_{g_0}(\tilde{\psi}, \beta) = 0 \tag{16}$$

has a unique solution $\beta = \beta(\tilde{\psi}) \in U_2$ for all $\tilde{\psi} \in U_1$.

Proof By definition, we have $\widetilde{\Phi} : U \subset \widetilde{\mathcal{E}}_1 \times \mathrm{Ker}L^{\perp} \to H_s$, $s > \frac{n}{2}$. Thus, if $\mathrm{Im}(\widetilde{\Phi}) \subset \mathrm{Im}(L)$, then, because of the above arguments, $D_2\widetilde{\Phi}_{(\psi_0,\beta_0)} : \mathrm{Ker}L^{\perp} \to \mathrm{Im}(L)$ is an isomorphism, and thus the implicit function theorem finishes the proof. Thus, we need to show that $\mathrm{Im}(\widetilde{\Phi}) \subset \mathrm{Im}(L)$. This is a rather straightforward computation, that can be found in [10]. □

The above lemma shows that, by restricting the functional spaces appropriately, we can still analyse the RTSE in a neighbourhood of our reference solution, even if it possesses continuous symmetries. All that is required is that every conformal Killing field must be in fact a Killing field. Some relevant examples of these types of symmetric initial data can be found in [10].

3.1.3 Neighbourhoods of Umbilical Reference Solutions with Conformal Killing Fields

The idea of this section is to consider the case where we have an umbilical reference solution for the vacuum constraint equations, which admits (genuine) conformal Killing fields. Since in this situation the implicit function argument fails, we would like to see whether we can find umbilical solutions of the ECE which are as close to the original one as we want, but without conformal Killing fields, so that we can apply the implicit function argument around them. Notice that, intuitively, this claim ought to hold, since metrics without continuous symmetries are *generic* [18, 19]. The tricky part is to show that, within this dense subset of metrics without symmetries, we can chose another dense subset which are solutions of the ECE. Furthermore, we will show that we can make such choice preserving the scalar and mean curvature of the initial data set.

Thus, we consider a smooth solution of the vacuum constraint equations of the form $(g_0, K = \frac{\tau}{n} g_0)$, where τ is constant which represents the mean curvature of the embedded hypersurface $M \hookrightarrow M \times \mathbb{R}$. We suppose that g_0 has non-trivial conformal Killing fields. Since we are considering vacuum, g_0 satisfies

$$R_{g_0} = -\frac{\tau^2}{n}(n-1).$$

Our aim is to find another solution of the form $(\bar{g}, \frac{\tau}{n}\bar{g})$, with $R_{\bar{g}} = R_{g_0}$ such that $\bar{g}$ is close to g_0 and $\mathrm{Ker}(\Delta_{\bar{g},\mathrm{conf}}) = \{0\}$, so that $(\bar{g}, \frac{\tau}{n}\bar{g})$ induces a reference solution $(\bar{\psi}_0, \bar{\beta}_0)$ where the implicit function argument can be applied. In order to do this, we will need some auxiliary results.

First of all, we need to make precise the statement concerning the *genericity* of metrics without conformal Killing field. We will make use of the following results. Consider the set of Riemannian metrics on a closed manifolds, of dimension greater or equal to three, and denote it by $\mathcal{M}$. It is possible to define a Fréchet topology on this set by considering the distance induced by the C^k (semi) norms

$\{p_k = ||\cdot||_{C^k}\}_{k=0}^{\infty}$, and denote such distance function by d_0. This distance is not complete, since we can have sequences which run out of $\mathcal{M}$ into the space of (non necessarily positive definite) symmetric $(0, 2)$-tensor fields. That is why it is necessary to supplement this distance function with another distance d_1 tailored so that sequences do not run out of $\mathcal{M}$. Then, the distance function $d = d_0 + d_1$ can be shown to be complete on $\mathcal{M}$. How to build up d_1 and to proove the completeness of d can be seen in [20]. Using these ideas, it has been shown in [19] that there is a residual subset in $(\mathcal{M}, d)$ without conformal Killing fields. The key observation to be made here is that the density of such subset in $\mathcal{M}$ implies that, given any C^k neighborhood U of an element $g_0 \in \mathcal{M}$ (with respect to the semi-norms p_k), we can always find a metric $g \in \mathcal{M}$ which is sufficiently close to g_0 in the distance d, so that $g \in U$ and g does not possess any conformal Killing fields. The proof of the following theorem relies on this density argument plus an implicit function argument. The detailed proof can be found in [10].

Theorem 4 *Given any umbilical smooth solution to the vacuum constraint equations (g_0, K_0) on a compact n-dimensional manifold M satisfying, with $n \geq 3$, there is another smooth solution $(\bar{g}, \bar{K})$, which is as close as we want to (g_0, K_0) in any C^k-topology and has the same mean and scalar curvatures as (g_0, K_0), for which the induced solution $(\bar{\psi}_0, \bar{\beta}_0)$ for the RTSE admits a neighborhood where the RTSE are well-posed.*

The above theorem, for instance, shows that any umbilical solution of the vacuum (without cosmological constant) constraint equations, either produces reference solutions of the RTSE such that these equations are well-posed in a neighborhood of this data, or there is a another solution close to it, such that the previous claim holds.

3.2 The Asymptotically Euclidean Case

In this section the idea is to review how some of the discussion of the RTSE on closed manifolds gets translated into the case of asymptotically euclidean manifolds, which are highly interesting objects in physics, since they are used to model initial data for isolated gravitational systems. These are complete non-compact manifolds which consist in a compact core and a finite number of ends which are diffeomorphic to the exterior of the unit ball in $\mathbb{R}^n$. Clearly, we can always introduce a complete Riemannian metric e which is isometric to the euclidean metric on each end.

Although many of the analytic aspects valid for closed manifolds are still valid in this scenario, we need to make some adjustments. For instance, we will need to work with *weighted Sobolev spaces*. We will follow the conventions adopted in [21], and thus denote by $H_{s,\delta}$ the weighted Sobolev space of sections of some vector bundle over M which has s-weak derivatives in some tempered L^2-spaces, which are specified by the parameter $\delta \in \mathbb{R}$. These are Banach spaces with the norm

$$||u||^2_{H_{s,\delta}} \doteq \sum_{m=0}^{s} \int_M |D^m u|_e^2 (1 + d_e^2)^{(m+\delta)} \mu_e \tag{17}$$

where d_e denotes the Riemannian distance function in the metric e with respect to a fixed origin $0 \in M$. These functional spaces share several of the properties of the usual Sobolev spaces, such as the embedding theorems and continuous multiplications properties [21].

In order to formulate the TSP in this context, we will again consider the linearisation of the RTSE around umbilical solutions of the constraint equations $(g, K = \alpha g)$. We will then set the momentum density to zero, *i.e.*, $S = 0$, and thus we have a prescribed scalar curvature equation for g, of the form

$$R_g + \alpha^2 n(n-1) = 2\epsilon \tag{18}$$

Since we will impose decaying conditions at infinity for both the allowed energy sources and the metric, we need to deal with the constant term in the left-hand side of (18). That is why we will introduce a *cosmological constant* term as part of the energy sources, which will fix the umbilicity constant α.[1] That is, we will consider that the energy sources are decomposed as $\epsilon = \Lambda + \epsilon'$, where Λ is a constant which represents the cosmological constant, and ϵ' represents the energy density induced by matter fields. Then, fix $\alpha^2 = \frac{2\Lambda}{n(n-1)}$, which means that (18) is equivalent to $R_g = \epsilon'$. In fact, for our reference solutions, we will consider vacuum. That is, $\epsilon' = 0$:

$$R_g = 0. \tag{19}$$

Now, let (M, e) be an n-dimensional manifold euclidean at infinity, $n \geq 3$, let (ψ_0, β_0) be a reference solution for the RTSE, where $\psi_0 = (g_0, \dot{g}_0, \Lambda, \epsilon_0 = 0, S_0 = 0)$. Also, assume that g_0 is $H_{s+3,\delta+1}$-asymptotically flat with $R_{g_0} = 0$, for some $s > \frac{n}{2}$ and $\delta > -\frac{n}{2}$. Finally, we fix $K_0 = \alpha g_0$, with $\alpha^2 = \frac{2\Lambda}{n(n-1)}$. As discussed above these choices would provide a solution of the constraint equations. Furthermore, from this solution, by choosing a lapse function $N_0 = 1$ and a shift vector $\beta_0 \in H_{s+2,\delta}$, we get

$$\dot{g}_0 = -2\alpha g_0 + L_{\beta_0} g_0. \tag{20}$$

Standard arguments show that $\dot{g}_0 + 2\alpha e \in H_{s+1,\delta+1}$.

[1] The introduction a positive cosmological constant can be very easily motivated from physical arguments.

Since we are restricted to a neighbourhood of the fixed reference solution, we only consider metrics g which are $H_{s+3,\delta+1}$-asymptotically flat and such that $||g - g_0||_{H_{s+3,\delta+1}}$ is small enough. In a similar way, we will consider $\dot{g}$ with the same asymptotic behaviour as $\dot{g}_0$, that is, $\dot{g} + 2\alpha e \in H_{s+1,\delta+1}$, such that $||\dot{g} - \dot{g}_0||_{H_{s+1,\delta+1}}$ is sufficiently small. In this way it will be useful for us to rewrite

$$\begin{aligned} \dot{g} &= -2\alpha e - 2\alpha(g - e) + L_\beta g, \\ &= -2\alpha e + \delta\dot{g}, \end{aligned} \tag{21}$$

and take $\delta\dot{g} \in H_{s+1,\delta+1}$ as part of the data that is actually freely specified. Since the same can be done with g, we will also take $\delta g \doteq g - e \in H_{s+3,\delta+1}$ as a freely given datum. In this way, it holds that γ is a continuous function of δg, $\delta\dot{g}$ and β, using the previously discussed functional spaces, with $\gamma + \alpha e \in H_{s+1,\delta+1}$. Furthermore, we get that $\text{tr}_g\gamma + n\alpha \in H_{s+1,\delta+1}$ and $|\gamma|_g^2 - n\alpha^2 \in H_{s+1,\delta+1}$. Having fixed this setting, one gets that N^{-1} is well defined in a neighbourhood of the reference solution, and furthermore, $N^{-1} - 1 \in H_{s+1,\delta+1}$ on it.

Now, by replacing the Sobolev spaces by weighted Sobolev spaces with appropriately chosen weights, it is possible to analyse the behaviour of the linearisation of the RTSE. In fact we find that (see [10] for the details):

$$D_2\Phi_{(\psi_0,\beta_0)} \cdot Y = -\frac{1}{2}\Delta_{g_0,\text{conf}}Y, \tag{22}$$

where

$$\Delta_{g_0,\text{conf}}Y \doteq \text{div}_{g_0}\left(L_Y g_0 - \frac{2}{n} g_0 \,\text{div}_{g_0} Y\right) \tag{23}$$

is the *conformal Killing Laplacian*. It has been shown, for instance in [22], that the following theorem holds:

Theorem 5 *Let (M, g_0) be a $H_{s,\rho}$-asymptotically euclidean manifold with $s > \frac{n}{2}$ and $\rho > -\frac{n}{2}$. Then, $\Delta_{g_0,\text{conf}}$ is an isomorphism from $H_{s,\delta}$ to $H_{s-2,\delta+2}$ for any $-\frac{n}{2} < \delta < \frac{n}{2} - 2$.*

Applying the above theorem to $D_2\Phi_{(\psi_0,\beta_0)}$, followed by the implicit function theorem, we prove

Theorem 6 *Given a solution (g_0, K_0) of the constraint equations fixed as above, there exists an $\mathcal{E}_1$-neighbourhood of the initial data $\psi_0 \in \mathcal{E}_1$, such that the reduced thin-sandwich equations are well-posed, i.e., there is a unique solution $\beta = \beta(\psi) \in H_{s+2,\delta}$, $s > \frac{n}{2}$ and $-\frac{n}{2} < \delta < \frac{n}{2} - 2$, for each ψ sufficiently close to ψ_0.*[2]

[2]In this theorem we have that $\mathcal{E}_1 \doteq H_{s+3,\delta+1} \times H_{s+2,\delta+1} \times H_{s+1,\delta+1} \times H_{s+1,\delta+3} \times H_{s,\delta+3}$, $\mathcal{E}_2 \doteq H_{s+2,\delta}$ and $\mathcal{Z} \doteq H_{s,\delta+2}$. The freely specified initial data are denoted by $\psi \doteq (\delta g, \delta\dot{g}, \epsilon', S) \in \mathcal{E}_1$; $\mathcal{E}_2$ is the space where we look for solutions β and $\mathcal{Z}$ is the range of the RTS operator Φ.

3.3 Existence of Reference Solutions

The issue of whether reference solutions of the type we have been proposing can be obtained on any AE manifold is not a simple one. Since we would need to produce solutions of (19), results concerning the Yamabe classification of AE [23] manifolds and obstructions to positive scalar curvature [24] can be used to produce obstructions to the solvavility of (19). A detailed discussion about this matter can be found in [10], where the final conclusion is the following one.

Theorem 7 *On any n-dimensional manifold euclidean at infinity, $n \geq 4$, which admits a $H_{s+3,\delta+1}$-Yamabe positive metric, with $s > \frac{n}{2}$ and $-\frac{n}{2} < \delta < \frac{n}{2} - 3$, there is an open subset in the space of solutions of the Einstein constraint equations where the thin sandwich problem is well-posed.*

It is interesting to point out that more explicit hypotheses that guarantee the resolvability of (19) can be found. For instance, if a manifold euclidean at infinity M admits a sufficiently regular AE-metric g whose scalar curvature R_g has *sufficiently small* negative part, then it admits a conformal deformation into zero scalar curvature [10]. This can be used to establish interesting examples. For instance, small compactly supported perturbations of Schwarchild's (initial data) metric can be conformally deformed into zero scalar curvature.

4 Ricci-Flat Embedding of Lorentzian Manifolds

The idea in this section is to relate the problem of embeddings of Lorentzian manifolds into Ricci-flat spaces with the constraint equations of GR. In particular, with the partial resolution of the TSP on closed manifolds offered above.

The main motivation for analysing this problem comes from extra-dimensional theories in physics. In some of them, the usual 4-dimensional space-time of GR is seen as a submanifold of a space with some specified geometric property (Ricci-flatness, Einstein space, etc). In this sense, determining whether generic solutions of the space-time Einstein equations can be isometrically embedded into these kind of structures becomes a natural problem. Even more ambitiously, we could think about whether general Lorentzian manifolds admit these embeddings. Thought in these terms, this problem has quite some resemblance with very important problems in Riemannian geometry, such as Whitney's embedding theorem or Nash's theorems on isometric embeddings of general Riemannian manifolds into some higher dimensional Euclidean space. In our case, the Riemann-flat condition is replaced by a weaker condition, such as Ricci-flatness, and we would aim to achieving embeddings of lower codimension that in the Riemann-flat case. A detailed review of both the physical motivation for this problem, together with the mathematical history related with similar problems can be found in [11].

Before moving to the statement of the main theorem, we should highlight that some local and analytic results concerning existence of such embeddings have been known for some time. These result are not ideal, even if we only think about the motivations coming from physics, since it is essential to admit non-analytic solutions of the Einstein equations in order to preserve causality. Furthermore, stability of the embedding with respect to the space-time metric is typically desired. Since the analytic results are proven via the Cauchy–Kovalevskaya theorem, there is no guarantee of stability. Also, higher dimensional models in physics demand global results. Thus, global and non-analytic results are well-motivated by some modern physical scenarios in theoretical physics.

Theorem 8 *Any n-dimensional compact Lorentzian manifold* (M, g)*, with* $n \geq 3$ *and* $g \in H_{s+3}$*,* $s > \frac{n}{2}$*, admits an isometric embedding in a Ricci-flat* $(2n+2)$*-dimensional semi-Riemannian manifold with index* $n+1$ *(that is, with n + 1 time-like dimensions).*

Sketch of the Proof[3] Compactness gives us that we can write the Lorentzian metric g as following

$$g = \lambda(\tilde{g} - g_0),$$

where $\bar{g}$ and g_0 are Riemannian metrics, and λ is a constant, which depends on g. Furthermore, we can guarantee that we can always pick $\tilde{g}$ sufficiently close to g_0. Also, from Theorem 3 and the discussion preceding it, we know that we can pick g_0 as part of a solution of the vacuum constraint equations for GR, and we can guarantee that for any g' sufficiently close to g_0 we can associate another solution of the vacuum constraint equations. Thus, we get that for appropriately chosen $\tilde{g}$ and λ, both g_0 and $\tilde{g}$ are part of a solution of the vacuum constraint equations on M. Hence, as a standard consequence of the evolution problem in GR, we get isometric embeddings of $(M, \tilde{g})$ and (M, g_0) in Lorentzian Ricci-flat spaces, say (V_1, h_1) and (V_2, h_2) respectively. We can then embed (M, g) into the product $V_1 \times V_2$ equipped with the metric

$$h = \lambda(\pi^* h_1 - \sigma^* h_2), \tag{24}$$

where π and σ denote the projections onto the first and second factors of $V_1 \times V_2$ respectively. This space is Ricci-flat, of index $n+1$, and the embedding is isometric.

Clearly, the above theorem demands only low regularity for the space-time metric.[4] Also, the above theorem is global. Furthermore, stability with respect of the space-time metric can be extracted as a corollary (see [11]).

[3] For the details of the proof see [11].

[4] Since $H_{s+3} \hookrightarrow C^3$ for $s > \frac{n}{2}$, we are demanding at least C^3-metrics. In particular, the results always holds for smooth metrics.

It is also worth to note that, despite the fact that compact space-times are not the most relevant object of study in GR, embeddings of arbitrary large closed strips in globally hyperbolic space-times, with compact space-slices, can also be extracted as a corollary. Finally, in spite of the codimension needed for the embedding being high, it is, as far as we are aware of, much lower that the best results available that can be applied for general compact Lorentzian manifolds: $(n^2 + 3n - 2)$-fewer dimensions are needed.[5]

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[5]Compare with [25].

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Parametric Solutions to a Static Fourth-Order Euler–Bernoulli Beam Equation in Terms of Lamé Functions

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Abstract The exact general solution to a static fourth-order Euler–Bernoulli beam equation has been obtained and it has been written in terms of a fundamental set of solutions to a Lamé equation. This permits to express the general solution in parametric form in terms of Weierstrass elliptic functions. Three-parameter families of solutions have been also reported by setting particular values to one of the arbitrary constants of integration in the general solution. One of these families is expressed in terms of the Weierstrass $\wp$-function and ζ-function whereas two of them are given in terms of either trigonometric or hyperbolic functions. Graphical representations of particular solutions are also shown for different values of the arbitrary constants of integration.

1 Introduction

One of the most important problems in both mechanical and civil engineering consists on studying the transverse motion of an elastic thin beam bearing a load. This problem was firstly addressed by Daniel Bernoulli and Leonard Euler [1] and it can be modelled [2] by the fourth-order partial differential equation

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)+\mu\frac{\partial^2 y}{\partial t^2}=f(y), \tag{1}$$

where $y(t, x)$ denotes the transverse displacement at time t and position x, E is the elastic modulus (also known as the Young's modulus), I is the second moment of area, μ represents the mass per unit length, and the function f, which is assumed to be smooth, describes the applied load.

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The determination of some exact solutions to Eq. (1) is of great importance due to its multiple applications. For this reason, different techniques have been applied in the recent literature with the aim of addressing the problem of the integrability of such equation. One of the most successful tools to find exact solutions is the Lie symmetry approach [3, 4]. The complete classification of the Lie symmetries of Eq. (1) was performed in [5] assuming that E, μ and I are constant, whereas in [6] the case of variable mass density was studied.

In the static case, if E and I are constant, Eq. (1) becomes (up to constant)

$$y_4 = f(y),$$

where $y_4 = \frac{d^4y}{dx^4}$. A remarkable particular case is

$$y_4 = \delta y^{-5/3}, \quad \delta = \pm 1, \tag{2}$$

whose Lie symmetry algebra is three-dimensional and isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since this Lie symmetry algebra is nonsolvable, the standard Lie reduction method cannot be applied to integrate equation (2) by quadrature. In fact, only partial results on the integrability of such equation have been obtained by using Lie and Noether symmetry methods [7–10].

Some recent theoretical results [11, 12], based on solvable structures [13], were applied by the authors in [14] to study Eq. (2). The general solution of such equation was obtained in parametric form and expressed in terms of a fundamental set of solutions to the following one-parameter family of linear second-order equations:

$$\phi''(r; K_0) - 2\wp(r; g_2, g_3)\phi(r; K_0) = 0, \tag{3}$$

where $\wp(r) = \wp(r; g_2, g_3)$ stands for the Weierstrass $\wp$-function with parameter values

$$g_2 = \frac{16^2}{3}\delta \quad \text{and} \quad g_3 = -16^2 K_0, \quad K_0 \in \mathbb{R}.$$

In this work it is addressed the analysis of Eq. (3) with the aim of obtaining exact solutions to Eq. (2). Equation (3) is a particular case of the Lamé equation and its solutions are called Lamé functions [15–17]. In Sect. 2 we prove, by using Lamé functions, that for $K_0 \neq 0$ the exact solution to Eq. (2) can be expressed in parametric form and in terms of the Weierstrass $\wp$-function, ζ-function and σ-function.

The case $K_0 = 0$ is studied in Sect. 3. For this case, two linearly independent solutions to Eq. (3) can be expressed in terms of the Weierstrass $\wp$ and ζ functions. This provides a three-parameter family of solutions in parametric form and expressed in terms of such elliptic functions.

The case in which the discriminant $g_2^3 - 27g_3^2$ of the Weierstrass $\wp$-function $\wp(r; g_2, g_3)$ equals zero corresponds with the values $K_0 = \pm\frac{16}{27}$. For these specific values, the Weierstrass $\wp$-function appearing in the Lamé equation (3) becomes either a trigonometric or a hyperbolic function [15, 16]. The corresponding Lamé functions are then elementary functions that provide a three-parameter family of solutions to Eq. (2) in terms of either trigonometric or hyperbolic functions [14]. These results are collected in Sect. 4 for the sake of completeness of the study of Eq. (2).

Finally, in Sect. 5 we include graphical representations of some particular solutions to Eq. (2) by choosing different values for the four independent constant of integration K_i, $i = 0, 1, 2, 3$.

2 Exact General Solution to Eq. (2)

Throughout this section

$$\wp(r) = \wp(r; g_2, g_3), \quad \zeta(r) = \zeta(r; g_2, g_3), \quad \sigma(r) = \sigma(r; g_2, g_3)$$

denote the Weierstrass $\wp$, ζ, and σ functions [15, 16, 18], respectively, with parameters values

$$g_2 = \frac{16^2}{3}\delta, \quad g_3 = -16^2 K_0, \quad K_0 \in \mathbb{R}. \tag{4}$$

In order to simplify the notation, from this point on the parameters g_2 and g_3 given in (4) will be omitted in the argument of the corresponding Weierstrass elliptic function.

In [14] it was proved that if $\{\phi_1, \phi_2\}$ is a fundamental set of solutions to the Lamé equation (3) such that the corresponding Wronskian becomes $W(\phi_1, \phi_2)(r) = 1$, then the general solution to the static Euler–Bernoulli beam equation (2) is given in parametric form by

$$\begin{aligned} x(r) &= K_3(K_1 - K_2)\frac{\phi_1(r; K_0) - K_2\phi_2(r; K_0)}{\phi_1(r; K_0) - K_1\phi_2(r; K_0)}, \\ y(r) &= \pm\left(\frac{K_3^{1/2}\,(K_1 - K_2)}{2\Big(\phi_1(r; K_0) - K_1\phi_2(r; K_0)\Big)}\right)^3, \end{aligned} \tag{5}$$

where $K_i \in \mathbb{R}$ for $i = 0, 1, 2, 3$, $K_3 > 0$ and $K_1 \neq K_2$. The case $K_1 = K_2$ leads to the two-parameter family of singular solutions [9, 10, 14]:

$$y(x) = \pm\big(ax^2 + bx + c\big)^{3/2},$$

where the constants a, b and c satisfy

$$3(b^2-4ac)^2-\frac{16}{3}\delta=0.$$

A fundamental set of solutions to the Lamé equation (3) for $K_0\neq 0$ can be found in [17, pag. 572]. These solutions can be expressed in terms of elliptic functions as follows:

$$\Phi_1(r)=\frac{\wp(r)}{\exp(r\zeta(\gamma))}\frac{\sigma(r)\sigma(\gamma)}{\sigma(r-\gamma)}\quad\text{and}\quad\Phi_2(r)=\exp(r\zeta(\gamma))\frac{\sigma(r-\gamma)}{\sigma(r)\sigma(\gamma)},\tag{6}$$

where γ is a value such that $\wp(\gamma)=0$. It can be checked that

$$W(\Phi_1,\Phi_2)(r)=\wp'(\gamma).$$

Therefore, a fundamental set of solutions $\{\phi_1,\phi_2\}$ to Eq. (3) for $K_0\neq 0$ such that $W(\phi_1,\phi_2)(r)=1$ is given by

$$\phi_1(r)=\frac{\wp(r)}{\exp(r\zeta(\gamma))}\frac{\sigma(r)\sigma(\gamma)}{\sigma(r-\gamma)}\quad\text{and}\quad\phi_2(r)=\frac{\exp(r\zeta(\gamma))}{\wp'(\gamma)}\frac{\sigma(r-\gamma)}{\sigma(r)\sigma(\gamma)}.\tag{7}$$

By using (5) and the functions ϕ_1 and ϕ_2, the general solution to Eq. (2) can be obtained in parametric form and in terms of Weierstrass elliptic functions. Such expression of the general solution is presented in the next theorem:

Theorem 1 *The four-parameter general solution to the Euler–Bernoulli beam equation*

$$y_4=\delta y^{-5/3},\quad \delta=\pm 1,$$

is given in parametric form through

$$x(r)=K_3(K_1-K_2)\frac{f_2(r)}{f_1(r)},$$

$$y(r)=\pm\left(\frac{K_3^{1/2}(K_1-K_2)\sigma(\gamma)\sigma(r)\sigma(r-\gamma)\wp'(\gamma)}{2f_1(r)}\right)^3,$$

where

$$f_i(r)=\sigma(\gamma)^2\sigma(r)^2\wp(r)\wp'(\gamma)exp(-r\zeta(\gamma))-K_i\,exp(r\zeta(\gamma))\sigma(r-\gamma)^2,\ \ i=1,2,$$

$K_1, K_2, K_3\in\mathbb{R}$*,* $K_1\neq K_2$*,* $K_3>0$*,* $\wp(r)=\wp(r;g_1,g_2)$*,* $\zeta(r)=\zeta(r;g_1,g_2)$*,* $\sigma(r)=\sigma(r;g_1,g_2)$ *stand for the Weierstrass* $\wp$*-function,* ζ*-function and* σ*-function, respectively, with parameters* $g_2=\frac{16^2}{3}\delta$ *and* $g_3=-16^2K_0$*,* $K_0\neq 0$*, and the value* γ *is such that* $\wp(\gamma)=0$*.*

3 A Three-Parametric Family of Solutions Corresponding to $K_0 = 0$

In this section we study separately the case $K_0 = 0$, which it is not included in the result presented in Theorem 1. For the value $K_0 = 0$, the Lamé equation (3) becomes

$$\phi''(r) - 2\wp(r; g_2, 0)\phi(r) = 0, \quad \text{where} \quad g_2 = \frac{16^2}{3}\delta. \tag{8}$$

It can be checked that a fundamental set of solutions $\{\phi_1, \phi_2\}$ to Eq. (8) such that $W(\phi_1, \phi_2)(r) = 1$ is given by

$$\phi_1(r) = \sqrt{\wp(r)} \quad \text{and} \quad \phi_2(r) = \sqrt{\wp(r)}H(r), \tag{9}$$

where

$$H'(r) = \frac{1}{\wp(r)}.$$

By using Formula 6.b in [18, p. 162] we have that

$$H(r) = \frac{6}{16^2\delta}\frac{\wp'(r)}{\wp(r)} + \frac{12}{16^2\delta}\zeta(r),$$

therefore, by (9), a fundamental set of solutions to Eq. (8) can be expressed as follows:

$$\phi_1(r) = \sqrt{\wp(r)} \quad \text{and} \quad \phi_2(r) = \sqrt{\wp(r)}\left(\frac{6}{16^2\delta}\frac{\wp'(r)}{\wp(r)} + \frac{12}{16^2\delta}\zeta(r)\right). \tag{10}$$

The fundamental set of solutions $\{\phi_1, \phi_2\}$ given in (10) leads, through (5), to the following three-parameter solution to Eq. (2):

$$x(r) = K_3(K_1 - K_2)\left(\frac{16^2\delta\wp(r) - 6K_2\left(\wp'(r) + 2\wp(r)\zeta(r)\right)}{16^2\delta\wp(r) - 6K_1\left(\wp'(r) + 2\wp(r)\zeta(r)\right)}\right),$$

$$y(r) = \pm\left(\frac{K_3^{1/2}(K_1 - K_2)16^2\delta\wp(r)}{2\sqrt{\wp(r)}\left(16^2\delta\wp(r) - 6K_1\left(\wp'(r) + 2\zeta(r)\wp(r)\right)\right)}\right)^3,$$

where $K_i \in \mathbb{R}$ for $i = 1, 2, 3$, $K_3 > 0$, $K_1 \neq K_2$.

4 Three-Parameter Families of Solutions Corresponding to $K_0 = \pm\frac{16}{27}$

The discriminant $g_2^3 - 27g_3^2$ of the Weierstrass $\wp$-function appearing in the Lamé equation (3) equals zero for the values $K_0 = \mp\frac{16}{27}$. In such cases, the Weierstrass $\wp$-function becomes either a trigonometric or a hyperbolic function and then a fundamental set of solutions to Eq. (3) can be expressed in terms of elementary functions [14]. With the aim of collecting all the results concerning the integrability of Eq. (2), we include in this section the explicit expression to such fundamental set of solutions:

- If $K_0 = -\frac{16}{27}$ then two linearly independent solutions to (3) verifying $W(\phi_1, \phi_2)(r) = 1$ become

$$\begin{aligned}\phi_1(r) &= \frac{\sqrt{3}}{8}\cot(\alpha_1(r))\sin(\alpha_2(r)) - \frac{\sqrt{2}}{8}\cos(\alpha_2(r)),\\ \phi_2(r) &= \csc(\alpha_1(r))\Big((3+\sqrt{6})\cos(\beta_1(r)) + (3-\sqrt{6})\cos(\beta_2(r))\Big),\end{aligned} \tag{11}$$

  where

$$\begin{aligned}\alpha_1(r) &= 2\sqrt{2}r, & \alpha_2(r) &= \frac{1}{\sqrt{6}}\big(4\sqrt{2}r - \pi\big),\\ \beta_1(r) &= \frac{1}{\sqrt{6}}\big(4(\sqrt{2}-\sqrt{3})r - \pi\big), & \beta_2(r) &= \frac{1}{\sqrt{6}}\big(4(\sqrt{2}+\sqrt{3})r - \pi\big).\end{aligned} \tag{12}$$

- If $K_0 = \frac{16}{27}$ then two linearly independent solutions to the corresponding Eq. (3) satisfying $W(\phi_1, \phi_2)(r) = 1$ are

$$\begin{aligned}\phi_1(r) &= \operatorname{csch}(\alpha_1(r))\Big((3+\sqrt{6})\cosh(\beta_1(r)) + (3-\sqrt{6})\cosh(\beta_2(r))\Big),\\ \phi_2(r) &= \frac{\sqrt{3}}{8}\coth(\alpha_1(r))\sinh(\alpha_2(r)) - \frac{\sqrt{2}}{8}\cosh(\alpha_2(r)),\end{aligned} \tag{13}$$

  where $\alpha_1(r)$, $\alpha_2(r)$, $\beta_1(r)$ and $\beta_2(r)$ are given in (12).

The functions ϕ_1 and ϕ_2 given in (11) (resp. (13)) provide through (5) a three-parameter family of solutions expressed in terms of trigonometric (resp. hyperbolic) functions. We omit here the expression of such solution due to its length.

5 Graphical Representations of Some Solutions

In this section we present some graphical representations of particular solutions to Eq. (2) by setting the integration constants to particular values. It can be observed how the qualitative behaviour of the solutions changes depending on the different choices of the integration constants and the domain of the parameter r (Figs. 1, 2, 3, 4, 5, 6, and 7).

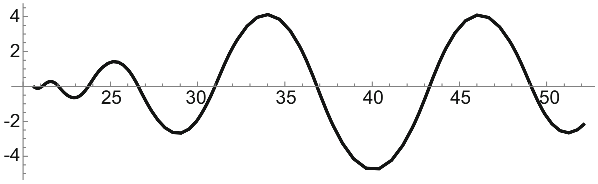


Fig. 1 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = 0.1$, $K_1 = -1$, $K_2 = -3$, $K_3 = 10$, $r \in]-100, 100[$

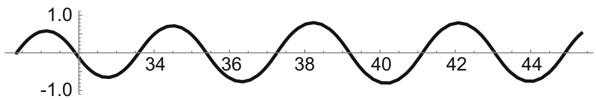


Fig. 2 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = 0.01$, $K_1 = -1$, $K_2 = -3$, $K_3 = 10$, $r \in]-100, 100[$

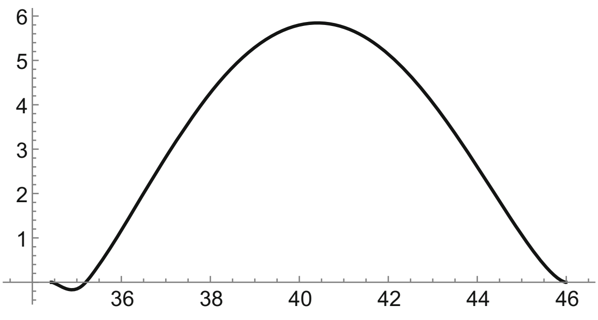


Fig. 3 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = \frac{16}{27}$, $K_1 = 1$, $K_2 = -3$, $K_3 = 10$, $r \in]-15, 15[$

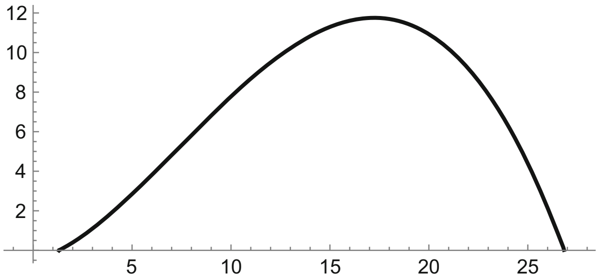


Fig. 4 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = 0$, $K_1 = 0.3$, $K_2 = -1$, $K_3 = 1$

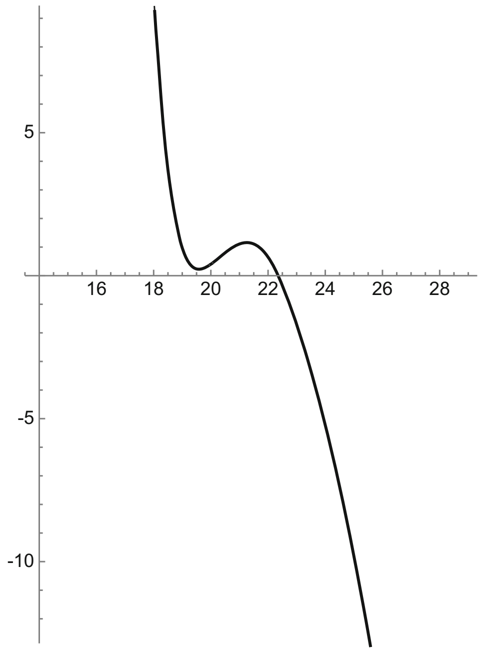


Fig. 5 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = 1$, $K_1 = -1$, $K_2 = -3$, $K_3 = 10$, $r \in]1, 4[$

Fig. 6 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = 1$, $K_1 = -1$, $K_2 = -3$, $K_3 = 10$, $r \in]1, 4[$

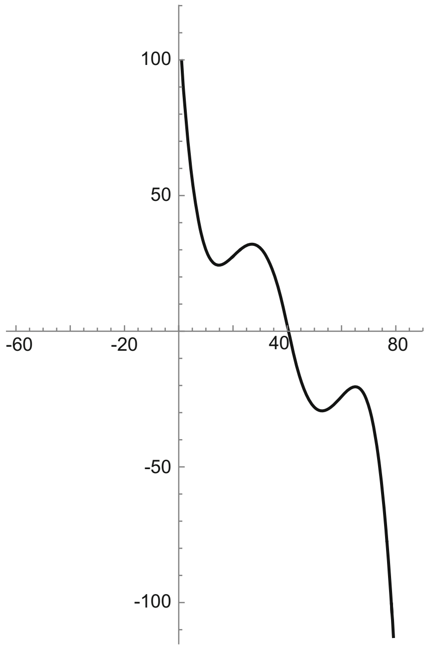
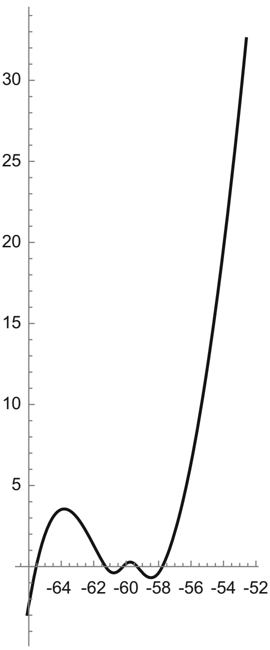


Fig. 7 Solution of Eq. (2) for the values $\delta = 1$, $K_0 = -\frac{16}{27}$, $K_1 = -1$, $K_2 = 3$, $K_3 = 10$, $r \in]0.89, 6.2[$



6 Concluding Remarks

The exact general solution to the static Euler–Bernoulli beam equation (2) has been obtained in terms of Weierstrass elliptic functions, as far as we know, for the first time in the literature. This has been achieved by considering an appropiate fundamental set of solutions to the Lamé equation (3). For the particular case $K_0 = 0$ a new three-parameter family of solutions has been also provided in terms of the Weierstrass $\wp$-function. The case in which the discriminant of the Weierstrass $\wp$ function appearing in the Lamé equation (3) equals zero has been also studied in this work. In this case two three-parameter family of solutions can be derived in parametric form and expressed in terms of either trigonometric or hyperbolic functions.

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On Large Orbits of Actions of Finite Soluble Groups: Applications

A. Ballester-Bolinches, R. Esteban-Romero, and H. Meng

Abstract The main aim of this survey paper is to present two orbit theorems and to show how to apply them to obtain a result that can be regarded as a significant step towards the solution of Gluck's conjecture on large character degrees of finite soluble groups. We also show how to apply them to solve questions about intersections of some conjugacy families of subgroups of finite soluble groups.

Keywords Finite groups · Soluble groups · Linear groups · Regular orbits · Formations · Prefrattini subgroups · System normalisers

Mathematics Subject Classification (2010) 20C15, 20D10, 20D20, 20D45

1 Introduction

The main aim of this paper is to present some results on regular orbits in finite and soluble groups and show how they can be used to solve or to progress in the solution of some open problems in finite group theory. Hence *all sets, groups, fields and modules considered here are finite*, and we assume this without further comment.

Recall that if a group G is acting on a non-empty set Ω, an element w of Ω is in a *regular* orbit if $C_G(w) = \{g \in G \mid wg = w\} = 1$, i.e., the orbit of w is as large as possible and it has size $|G|$. The study of regular orbits of linear groups actions,

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that is, regular orbits of actions of subgroups of $\mathrm{GL}(V)$ on a vector space V, plays an important role in many branches of group theory, particularly that of soluble groups. In fact, the solution of some well-known open problems such as the so-called $k(GV)$-problem ([17]) depends on the existence of such orbits. Consequently, the problem of the existence of regular orbits has attracted the attention of several authors and it is an active and interesting research area in Group Theory.

Our interest in this topic arose in trying to solve some questions about intersections of system normalisers and prefrattini subgroups of soluble groups raised by Kamornikov and Shemetkov and Vasil'ev in the *Kourovka Notebook* [12].

A general way to present this kind of problems is by means of k-conjugate systems.

Definition 1 A 3-tuple (G, X, Y) is said to be a *k-conjugate system* if G is a group, X, Y are subgroups of G with $Y = \mathrm{Core}_G(X)$, and there exist k elements $g_1, \ldots, g_k$ such that $Y = \bigcap_{i=1}^{k} X^{g_i}$.

There are some interesting examples of conjugate systems in the literature. Dolfi [4] proved that if π is a set of primes and G is a π-soluble group, then $(G, H, \mathrm{O}_\pi(G))$ is a 3-conjugate system, where H is a Hall π-subgroup and $\mathrm{O}_\pi(G)$ is the largest normal π-subgroup of G. This result extends earlier theorems of Passman [16] (case $|\pi| = 1$) and Zenkov [22] (case H nilpotent). On the other hand, as Mann pointed out in [11], the results of Passman imply that if H is a nilpotent injector of a soluble group G and $\mathrm{F}(G)$ is the Fitting subgroup of G, then $(G, H, \mathrm{F}(G))$ is a 3-conjugate system.

Due to the above results and the important role played by the system normalisers and prefrattini subgroups in the structural study of soluble groups, the following questions turn out to be natural and interesting:

Problem 1 ([12, Kamornikov, Problem 17.55]) Does there exist an absolute constant k such that the Frattini subgroup $\Phi(G)$ of a soluble group G is the intersection of k G-conjugates of any prefrattini subgroup H of G?

Problem 2 ([12, Shemetkov and Vasil'ev, Problem 17.39]) Is there a positive integer k such that the hypercentre of any finite soluble group coincides with the intersection of k system normalisers of that group? What is the least number with this property?

In the language of conjugate systems, these problems can be restated as follows.

Restatement of Problem 1 Does there exist an absolute constant k such that if G is a soluble group with Frattini subgroup $\Phi(G)$ and H is a prefrattini subgroup of G, then $(G, H, \Phi(G))$ is a k-conjugate system?

Restatement of Problem 2 Is there a positive integer k such that if G is a finite soluble group with hypercentre $\mathrm{Z}_\infty(G)$ and D is a system normaliser of G, then $(G, D, \mathrm{Z}_\infty(G))$ is a k-conjugate system? What is the least number with this property?

Kamornikov gave in [10] the solution to Problem 1. He also solved in [9] a similar question about the prefrattini subgroups associated to the formation of all nilpotent groups.

This kind of questions can be reduced in many cases to a problem about regular orbits in faithful actions: assume we want to prove that (G, X, Y) is a k-conjugate system. If we argue by induction on the order of G and the conjugate system has nice closure properties, we can often reduce the problem to $k > 1$, $Y = 1$, $G = NX$, and N is a faithful X-module. Assume that the natural action of X on $N \oplus \overset{(k-1)}{\cdots} \oplus N$ has a regular orbit. Then there exist $n_1, \ldots, n_{k-1} \in N$ such that $\mathrm{C}_X(n_1) \cap \cdots \cap \mathrm{C}_X(n_{k-1}) = 1$. Since $X \cap X^n = \mathrm{C}_X(n)$, for all $n \in N$, it follows that $X \cap X^{n_1} \cap \cdots \cap X^{n_{k-1}} = 1$.

Another interesting problem where the regular orbits play an important role is the following:

Let G be a group, and denote by

$$b(G) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\},$$

where $\mathrm{Irr}(G)$ is the set of all irreducible complex characters of G; $b(G)$ is the largest degree of an irreducible character of G.

Gluck [8] showed that if G is soluble, then $|G : \mathrm{F}(G)| \le b(G)^{13/2}$ and made the following conjecture.

Conjecture 1 (Gluck [8]) If G is a soluble group, then

$$|G : \mathrm{F}(G)| \le b(G)^2.$$

Note that the bound $|G : \mathrm{F}(G)| \le b(G)^2$ fails for many non-soluble groups (e.g. for G simple and non-abelian).

Gluck's strategy for producing irreducible characters of large degree consists in considering the action of $G/\mathrm{F}(G)$ on the faithful and completely reducible $G/\mathrm{F}(G)$-module V of all linear characters of the section $\mathrm{F}(G)/\Phi(G)$. It follows that large orbits of $G/\mathrm{F}(G)$ on V give large character degrees.

To prove Gluck's conjecture in this way, it is enough to prove that if V is a faithful completely reducible G-module, then there exists an orbit in V of length at least $\sqrt{|G|}$. We could get such an orbit by means of a regular orbit of G on $V \oplus V$: if $a, b \in V$ and $\mathrm{C}_G(a) \cap \mathrm{C}_G(b) = 1$, then

$$|\mathrm{C}_G(a)\mathrm{C}_G(b)| = |\mathrm{C}_G(a)||\mathrm{C}_G(b)| \le |G|$$

and so $|\mathrm{C}_G(a)| \le \sqrt{|G|}$ or $|\mathrm{C}_G(b)| \le \sqrt{|G|}$.

This is unfortunately not true in general as Gluck already noted in [8]: the group $G = S_3 \wr S_3$ acts faithfully and irreducibly on a vector space of dimension 6 on GF(2), the finite field of two elements but it does not have any regular orbit on $V \oplus V$.

It would be therefore interesting to find out sufficient conditions to guarantee the existence of a regular orbit on $V \oplus V$, for a faithful and completely reducible G-module V.

2 Regular Orbits

Espuelas (see [6, Theorem 3.1]) proved that if G is a group of odd order and V is a faithful and completely reducible G-module of odd characteristic, then G has a regular orbit on $V \oplus V$. Dolfi and Jabara [5, Theorem 2] extended Espuelas' result to the case where the Sylow 2-subgroups of the semidirect product $[V]G$ of V and the soluble group G are abelian, and Yang [20, Theorem 2.3] proved that the same is true if 3 does not divide the order of the soluble group G. A result of Wolf [19, Theorem A] shows that a similar result holds if G is supersoluble (see also [15, Theorem 3.1] for an improved result when G is nilpotent).

Dolfi [4, Theorem 1.4], reproving a result of Seress [18, Theorem 2.1], proved that any soluble group G has a regular orbit on $V \oplus V \oplus V$ and if either $(|V|, |G|) = 1$ or G is of odd order, then G has also a regular orbit on $V \oplus V$ [4, Theorems 1.1 and 1.5].

More recently, Yang [21] extends some of these results to the case when H is a subgroup of the soluble group G by proving that if V is a faithful completely reducible G-module (possibly of mixed characteristic) and if either H is nilpotent or 3 does not divide the order of H, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product $[V]H$ are abelian, then H has at least two regular orbits on $V \oplus V$.

We prove that all previous results on regular orbits are consequences of the following surprising theorems.

Theorem A ([13]) *Let G be a soluble group and let V be a faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a subgroup of G such that the semidirect product VH is S_4-free. Then H has at least two regular orbits on $V \oplus V$. Furthermore, if H is $\Gamma(2^3)$-free and* SL(2, 3)*-free, then H has at least three regular orbits on $V \oplus V$.*

Recall that if G and X are groups, then G is said to be X-*free* if X cannot be obtained as a quotient of a subgroup of G; $\Gamma(2^3)$ denotes the semilinear group of the Galois field of 2^3 elements.

Theorem B ([14]) *Let G be a soluble group and V be a faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a supersoluble subgroup of G. Then H has at least one regular orbit on $V \oplus V$.*

We now draw a series of conclusions from Theorem A.

Corollary 1 ([21]) *Let G be a soluble group acting completely reducibly and faithfully on an odd order module V. Suppose that H is a subgroup of G. If H*

is nilpotent or $3 \nmid |H|$*, then* H *has at least three regular orbits on* $V \oplus V$*. If the Sylow* 2*-subgroup of the semidirect product* VH *is abelian, then* H *has at least two regular orbits on* $V \oplus V$*.*

Corollary 2 (see [4, Theorem 1.1]) *Let* G *be a soluble group and* V *be a faithful completely reducible* G*-module. Suppose that* $(|G|, |V|) = 1$*. Then* G *has at least two regular orbits on* $V \oplus V$*.*

Theorems A and B have found an application to Gluck's conjecture about large character degrees. In fact, our next theorem not only extends almost all known results on Gluck's conjecture, but it could also be very useful to solve Gluck's conjecture in the future.

Theorem C ([13, 14]) *Let* G *be a soluble group satisfying one of the following conditions:*

1. G *is* S_4*-free;*
2. $G/\mathrm{F}(G)$ *is* S_4*-free and* $\mathrm{F}(G)$ *is of odd order;*
3. $G/\mathrm{F}(G)$ *is* S_3*-free;*
4. $G/\mathrm{F}(G)$ *is supersoluble.*

Then Gluck's conjecture is true for G*.*

3 Prefrattini Subgroups and Normalisers

This section has as its main theme the study of intersections of normalisers and prefrattini subgroups of finite soluble groups associated to saturated formations, and provides answers to the aforesaid questions of Kamornikov and Shemetkov and Vasil'ev.

Recall that a formation is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and such that every group G has an smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*residual* of G and denoted by $G^{\mathfrak{F}}$. A maximal subgroup M of a group G containing $G^{\mathfrak{F}}$ is called $\mathfrak{F}$-*normal* in G; otherwise, M is said to be $\mathfrak{F}$-*abnormal*.

We say that $\mathfrak{F}$ is *saturated* if it is closed under Frattini extensions. In such case, by a well-known theorem of Gaschütz-Lubeseder-Schmid [3, Theorem IV.4.6], there exists a collection of formations $\mathrm{F}(p) \subseteq \mathfrak{F}$, one for each prime p, such that $\mathfrak{F}$ coincides with the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in \mathrm{F}(p)$ for all primes p dividing $|H/K|$. In this case, we say that H/K is $\mathfrak{F}$-*central* in G and $\mathfrak{F}$ is *locally defined* by the $\mathrm{F}(p)$. We say that H/K is $\mathfrak{F}$-*eccentric* if it is not $\mathfrak{F}$-central.

Note that a chief factor H/K supplemented by a maximal subgroup M is $\mathfrak{F}$-central in G if and only if M is $\mathfrak{F}$-normal in G.

Every group G has a largest normal subgroup such that every chief factor of G below it is $\mathfrak{F}$-central in G. This subgroup is called the $\mathfrak{F}$-*hypercentre* of G and it is denoted by $\mathrm{Z}_{\mathfrak{F}}(G)$ (see [3, Section IV.6]).

Let Σ be a Hall system of the soluble group G (see [3, Chapter I, Section 1.4]). Let S^p be the p-complement of G contained in Σ, and denote by $\mathrm{W}^p(G)$ the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p ($\mathrm{W}^p(G) = G$ if the set of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p is empty). Then

$$\mathrm{W}(G, \Sigma, \mathfrak{F}) = \bigcap_{p \in \pi(G)} \mathrm{W}^p(G)$$

is called the $\mathfrak{F}$-*prefrattini subgroup* of G associated to Σ. The $\mathfrak{F}$-prefrattini subgroups of G form a characteristic class of G-conjugate subgroups (see [1, Section 4.3] for an exhaustive study of $\mathfrak{F}$-prefrattini subgroups).

According to [1, Proposition 4.3.17], the intersection $\mathrm{L}_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-abnormal maximal subgroups of a soluble group G is the core of every $\mathfrak{F}$-prefrattini subgroup of G and

$$\mathrm{L}_{\mathfrak{F}}(G)/\Phi(G) = \mathrm{Z}_{\mathfrak{F}}(G/\Phi(G))$$

for every group G. In fact, we have:

Theorem D ([2]) *Let $\mathfrak{F}$ be a saturated formation and let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

If $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{L}(G)$ is the intersection of all self-normalising maximal subgroups of G. It is a characteristic nilpotent subgroup of G that was introduced by Gaschütz in [7]. If $\mathfrak{F}$ is the trivial formation, then $\mathrm{L}_{\mathfrak{F}}(G) = \Phi(G)$, the Frattini subgroup of G. Hence:

Corollary 3 ([9]) *If G is soluble and H is an $\mathfrak{N}$-prefrattini subgroup of G, then $(G, H, \mathrm{L}(G))$ is a 3-conjugate system.*

Corollary 4 ([10]) *If G is soluble and H is a prefrattini subgroup of G, then $(G, H, \Phi(G))$ is a 3-conjugate system.*

To describe our next result, we shall give a review of the definition of the $\mathfrak{F}$-normalisers of a soluble group.

Let $\mathrm{F}(p)$ be a particular family of formations locally defining $\mathfrak{F}$ and such that $\mathrm{F}(p) \subseteq \mathfrak{F}$ for all primes p. Let

$$\pi = \{p \mid \mathrm{F}(p) \neq \emptyset\}.$$

For an arbitrary soluble group G and a Hall system Σ of G, choose for any prime p the p-complement $K^p = S^p \cap G^{\mathrm{F}(p)}$ of the $\mathrm{F}(p)$-residual $G^{\mathrm{F}(p)}$ of G, where S^p is the p-complement of G in Σ. Then

$$D_{\mathfrak{F}}(\Sigma) = G_\pi \cap \left(\bigcap_{p \in \pi} \mathrm{N}_G(K^p) \right),$$

where G_π is the Hall π-subgroup of G in Σ, is the $\mathfrak{F}$-*normaliser* of G associated to Σ. The $\mathfrak{F}$-normalisers of G are a characteristic class of G-conjugate subgroups. There were introduced by Carter and Hawkes and coincide with the classical system normalisers of Hall when $\mathfrak{F}$ is the formation of all nilpotent groups (see [3, Sections V.2 and V.3] for details).

According to [1, Proposition 4.2.6], if D is an $\mathfrak{F}$-normaliser of G, then $\mathrm{Core}_G(D) = \mathrm{Z}_{\mathfrak{F}}(G)$. We prove:

Theorem E ([2]) *Let $\mathfrak{F}$ be a saturated formation and let D be an $\mathfrak{F}$-normaliser of a soluble group G such that $\Phi(G) = 1$. Then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Recall that if $\mathfrak{F} = \mathfrak{N}$ is the formation of all nilpotent groups, then the $\mathfrak{N}$-normalisers of a soluble group G are exactly the system normalisers of G and $\mathrm{Z}_{\mathfrak{N}}(G) = \mathrm{Z}_\infty(G)$ is the hypercentre of G. Therefore the answer of Problem 2 for groups with trivial Frattini subgroup is contained in the following:

Corollary 5 *Let G be a soluble group with $\Phi(G) = 1$. If D is a system normaliser of G, then $(G, D, \mathrm{Z}_\infty(G))$ is a 3-conjugate system.*

Our next example shows that $(G, D, \mathrm{Z}_\infty(G))$ is not a 2-conjugate system in general.

Example 1 Let D be the dihedral group of order 8. Then D has an irreducible and faithful module V of dimension 2 over the field of 3-elements such that $\mathrm{C}_D(v) \neq 1$ for all $v \in V$. Let $G = [V]D$ be the corresponding semidirect product. Then D is a system normaliser of G and $\mathrm{Z}_\infty(G) = 1$. By [3, Lemma A.16.3], $D \cap D^v = \mathrm{C}_D(v) \neq 1$ for all $v \in V$. Hence $(G, D, \mathrm{Z}_\infty(G))$ is not a 2-conjugate system.

Our next theorem has the aforesaid result of Mann as starting point and analyses the intersections of injectors associated to Fitting classes of soluble groups. A class of groups $\mathfrak{F}$ is said to be a *Fitting class* if $\mathfrak{F}$ is a class closed under taking subnormal subgroups and such that every group G has a largest normal $\mathfrak{F}$-subgroup called $\mathfrak{F}$-*radical* and denoted by $G_{\mathfrak{F}}$. Every soluble group G has a conjugacy class of subgroups, called $\mathfrak{F}$-*injectors*, which are defined to be those subgroups I of G such that if S is a subnormal subgroup of G, then $I \cap S$ is $\mathfrak{F}$-maximal subgroup of S [3, Theorem IX.1.4]. Note that, in this case, $\mathrm{Core}_G(I) = G_{\mathfrak{F}}$. We prove:

Theorem F ([2]) *Let $\mathfrak{F}$ be a Fitting class and let I be an $\mathfrak{F}$-injector of a soluble group G. Then $(G, I, G_{\mathfrak{F}})$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, I, G_{\mathfrak{F}})$ is a 3-conjugate system.*

Theorem A is the key result used in the proofs of the theorems of this section: we need to get a regular orbit of an action of an $\mathfrak{F}$-prefrattini subgroup and an $\mathfrak{F}$-injector over a completely reducible module.

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Poisson Algebras and Graphs

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Abstract A non-commutative Poisson algebra is a Lie algebra endowed with a, not necessarily commutative, associative product in such a way that the Lie and associative products are compatible via the Leibniz identity. If we consider a non-commutative Poisson algebra $\mathcal{P}$ of arbitrary dimension, over an arbitrary base field $\mathbb{F}$, a basis $\mathcal{B} = \{v_i\}_{i \in I}$ of $\mathcal{P}$ is called multiplicative if for any $i, j \in I$ we have that $[v_i, v_j] \in \mathbb{F}v_r$ and that $v_i v_j \in \mathbb{F}v_s$ for some $r, s \in I$. We associate an adequate graph (V, E) to $\mathcal{P}$ relative to $\mathcal{B}$. By arguing on this graph we show that $\mathcal{P}$ decomposes as a direct sum of ideals, each one being associated to one connected component of (V, E). Also the minimality of $\mathcal{P}$ and the division property of $\mathcal{B}$ are characterized in terms of the weak symmetry of the graph.

Keywords Poisson algebra · Graph · Multiplicative basis · Non-commutative algebra · Decomposition theorem

MSC2010: 05C50, 05C25, 17B70, 16xx; 17B63

1 Introduction

An interesting problem in graph theory and in abstract algebra consists in characterizing the structure of an algebraic object by the properties satisfied for some graph associated to it (see for instance [1–4]). In this framework, the present paper is devoted to study non-commutative Poisson algebras admitting a multiplicative basis throughout an adequate associated graph.

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Definition 1 A *non-commutative Poisson algebra* $\mathcal{P}$ is a Lie algebra $(\mathcal{P}, [\cdot, \cdot])$ over an arbitrary base field $\mathbb{F}$, endowed with an associative product, denoted by juxtaposition, in such a way the following Leibniz identity

$$[xy, z] = [x, z]y + x[y, z]$$

holds for any $x, y, z \in \mathcal{P}$.

Definition 2 Let $\mathcal{P}$ be a non-commutative Poisson algebra. A *subalgebra* of $\mathcal{P}$ is a linear subspace closed by the Lie and the associative products. An *ideal* $\mathcal{I}$ of $\mathcal{P}$ is a subalgebra satisfying $[\mathcal{I}, \mathcal{P}] + \mathcal{I}\mathcal{P} + \mathcal{P}\mathcal{I} \subset \mathcal{I}$.

Definition 3 A direct sum of linear subspaces of a non-commutative Poisson algebra

$$\mathcal{P} = \bigoplus_{\alpha} \mathcal{P}_{\alpha},$$

is called *orthogonal* if $[\mathcal{P}_{\alpha}, \mathcal{P}_{\beta}] = \mathcal{P}_{\alpha}\mathcal{P}_{\beta} = \{0\}$ for any $\alpha \neq \beta$.

We also recall that an *automorphism* of a Poisson algebra $\mathcal{P}$ is a linear isomorphism $f : \mathcal{P} \to \mathcal{P}$ satisfying $[f(x), f(y)] = f([x, y])$ and $f(x)f(y) = f(xy)$ for any $x, y \in \mathcal{P}$.

Multiplicative bases were considered for algebras endowed with exactly one product in [5]. In this reference, a basis $\mathcal{B} = \{v_i\}_{i \in I}$ of an arbitrary algebra $\mathcal{A}$ is called multiplicative if for any $i, j \in I$ we have either $v_i v_j = 0$ or $0 \neq v_i v_j \in \mathbb{F}v_k$ for some (unique) $k \in I$.

From here, we can introduce in a natural way the next concept:

Definition 4 A basis $\mathcal{B} = \{v_i\}_{i \in I}$ of a non-commutative Poisson algebra $\mathcal{P}$ is said to be *multiplicative* if for any $i, j \in I$ we have that $[v_i, v_j] \in \mathbb{F}v_r$ and that $v_i v_j \in \mathbb{F}v_s$ for some $r, s \in I$.

Since it is usual in the literature to describe an algebra by exhibiting a multiplicative table among the elements of a fixed basis, we can find many classical examples of Poisson algebras admitting a multiplicative basis.

Example 1 Consider the complex linear space $\mathcal{V}$ with basis $\mathcal{B} = \{e_1, e_2, e_3\}$. Let us define the Lie algebras $\mathcal{P} = (\mathcal{V}, [\cdot, \cdot])$ and $\mathcal{P}' = (\mathcal{V}, [\cdot, \cdot])$ where the nonzero Lie products are given by $[e_1, e_2] = -[e_2, e_1] = e_3$ and $[e_1, e_2]' = -[e_2, e_1]' = e_2$ respectively.

By defining on $\mathcal{P}$ and $\mathcal{P}'$ the associative product as $e_1 e_1 = e_3$ and zero otherwise, we get that $\mathcal{P}$ and $\mathcal{P}'$ become Poisson algebras admitting both of them to $\mathcal{B}$ as a multiplicative basis.

The paper is organized as follows. In Sect. 2 we will introduce the (directed) graph associated to a non-commutative Poisson algebra $\mathcal{P}$ admitting a multiplicative basis $\mathcal{B}$. By utilizing this graph we prove that $\mathcal{P}$ decomposes as a direct sum

$$\mathcal{P} = \bigoplus_k \mathcal{I}_k$$

of ideals with a multiplicative basis contained in $\mathcal{B}$, each one being associated to one connected component of the associated graph.

In Sect. 3, we discuss the relation among the previous decompositions of $\mathcal{P}$ given by different choices of bases of $\mathcal{P}$.

Finally, in Sect. 4, we relate the weak symmetry of the associated graph with some properties of $\mathcal{P}$. Then, the minimality of $\mathcal{P}$ and the division property of $\mathcal{B}$ are characterized in terms of weak symmetry. It is also shown that the above decomposition is by means of the family of its minimal ideals if and only if $\mathcal{B}$ is of division, that is, if and only if the associated graph is weak symmetric.

2 Poisson Algebras Admitting a Multiplicative Basis, and Graphs

We begin this section by recalling that a (directed) graph is a pair (V, E) where V is a (non-empty) set of vertices and $E \subset V \times V$ a set of (directed) edges connecting the vertices.

Definition 5 Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a multiplicative basis of a non-commutative Poisson algebra $\mathcal{P}$. The graph $\Gamma(\mathcal{P}, \mathcal{B}) := (V, E)$ where $V = \mathcal{B}$ and

$$E = \Big\{(v_i, v_k) \in V \times V : \{v_i v_j, v_j v_i, [v_i, v_j]\} \cap \mathbb{F}^{\times} v_k \neq \emptyset \text{ for some } j \in I\Big\},$$

where $\mathbb{F}^{\times} := \mathbb{F} \setminus \{0\}$, is called the (directed) *graph associated* to $\mathcal{P}$ relative to $\mathcal{B}$.

Example 2 If we consider the Poisson algebras given in Example 1, we have that the associated graphs to $\mathcal{P}$ and $\mathcal{P}'$ relative to the multiplicative basis $\mathcal{B}$ are:

$\Gamma(\mathcal{P}, \mathcal{B})$ $\Gamma(\mathcal{P}', \mathcal{B})$

Given two vertices $v_i, v_j \in V$ an *undirected path* from v_i to v_j is an ordered family of vertices $\{v_{i_1}, \dots, v_{i_n}\} \subset V$ with $v_{i_1} = v_i$, $v_{i_n} = v_j$, and such that either $(v_{i_r}, v_{i_{r+1}}) \in E$ or $(v_{i_{r+1}}, v_{i_r}) \in E$ for $1 \le r \le n-1$.

We can introduce an equivalence relation in V defined by $v_i \sim v_j$ if and only if either $v_i = v_j$ or there exists an undirected path from v_i to v_j. Then it is said that v_i and v_j are *connected* and the equivalence class of v_i, (denoted by $[v_i] \in V/\sim$), corresponds to a connected component $C_{[v_i]}$ of the graph $\Gamma(\mathcal{P}, \mathcal{B})$. Then we have

$$\Gamma(\mathcal{P}, \mathcal{B}) = \dot{\bigcup_{[v_i] \in V/\sim}} C_{[v_i]}. \tag{1}$$

We also can associate to any $C_{[v_i]}$ the linear subspace

$$\mathcal{P}_{C_{[v_i]}} := \bigoplus_{v_j \in [v_i]} \mathbb{F} v_j, \tag{2}$$

and assert the next result:

Theorem 1 *Let $\mathcal{P}$ be a non-commutative Poisson algebra with a multiplicative basis $\mathcal{B} = \{v_i\}_{i \in I}$. Then $\mathcal{P}$ decomposes as the orthogonal direct sum*

$$\mathcal{P} = \bigoplus_{[v_i] \in V/\sim} \mathcal{P}_{C_{[v_i]}},$$

being any $\mathcal{P}_{C_{[v_i]}}$ an ideal of $\mathcal{P}$, admitting the set $[v_i] \subset \mathcal{B}$ as multiplicative basis.

Proof From Eqs. (1) and (2) we can assert that $\mathcal{P}$ is the direct sum of the family of linear subspaces $\mathcal{P}_{C_{[v_i]}}$ with $[v_i] \in V/\sim$, admitting each one the set $[v_i] \subset \mathcal{B}$ as multiplicative basis.

Let us suppose that there exist $v_j \in \mathcal{P}_{C_{[v_i]}}$ and $v_k, v_l \in \mathcal{B}$ such that

$$\{v_j v_k, v_k v_j, [v_j, v_k]\} \cap \mathbb{F}^\times v_l \neq \emptyset$$

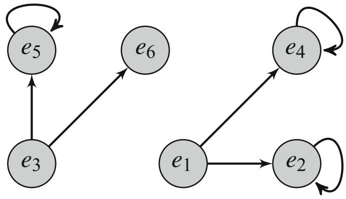
for some $j \in I$. Then (v_j, v_l) and (v_k, v_l) are edges of $C_{[v_i]}$, and then $v_k, v_l \in \mathcal{P}_{C_{[v_i]}}$. From here we conclude that the direct sum is orthogonal and that $\mathcal{P}_{C_{[v_i]}}$ is an ideal of $\mathcal{P}$. □

Observe that to identify the components of the decomposition given in Theorem 1 we only need to focus on the connected components of the associated graph.

Example 3 Consider the complex six-dimensional Poisson algebra $\mathcal{P}$ with basis

$$\mathcal{B} = \{e_1, e_2, e_3, e_4, e_5, e_6\},$$

nonzero Lie products $[e_1, e_2] = -[e_2, e_1] = e_2$ and $[e_3, e_5] = -[e_5, e_3] = e_5$; and nonzero associative products $e_1e_1 = e_4$, $e_1e_4 = e_4e_1 = e_4$, $e_4e_4 = e_4$ and $e_3e_3 = e_6$. Then, $\Gamma(\mathcal{P}, \mathcal{B})$ is:



Note that the corresponding decomposition of $\mathcal{P}$ given by Theorem 1 can be easily recovered from the above graph by writing

$$\mathcal{P} = \mathcal{P}_{C_{[e_1]}} \oplus \mathcal{P}_{C_{[e_3]}},$$

where $\mathcal{P}_{C_{[e_1]}} = \text{span}\{e_1, e_2, e_4\}$ and $\mathcal{P}_{C_{[e_3]}} = \text{span}\{e_3, e_5, e_6\}$.

Corollary 1 *If $\mathcal{P}$ is simple, then any two vertices of $\Gamma(\mathcal{P}, \mathcal{B})$ are connected.*

3 Relating the Graphs Given by Different Choices of Bases

In general, two different multiplicative bases of $\mathcal{P}$ have two different associated graphs, which give rise to two different decompositions of $\mathcal{P}$ (see Theorem 1).

Example 4 Let $\mathcal{P}$ be the complex six-dimensional Poisson algebra with basis

$$\mathcal{B} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

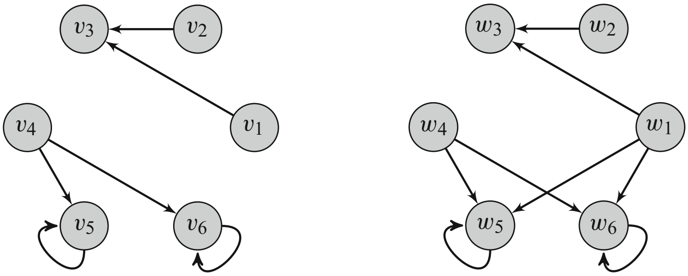
and nonzero products $[v_1, v_2] = -[v_2, v_1] = v_3$, $[v_4, v_5] = -[v_5, v_4] = v_5$, $[v_4, v_6] = -[v_6, v_4] = v_6$ and $v_1v_1 = v_3$.

Consider the basis

$$\mathcal{B}' = \{w_1, w_2, w_3, w_4, w_5, w_6\}$$

where $w_1 = v_1 + v_4$ and $w_i = v_i$ for $2 \leq i \leq 6$. Then we have that the nonzero products among these elements are $[w_1, w_2] = -[w_2, w_1] = w_3$, $[w_1, w_5] = -[w_5, w_1] = w_5$, $[w_1, w_6] = -[w_6, w_1] = w_6$, $[w_4, w_5] = -[w_5, w_4] = w_5$, $[w_4, w_6] = -[w_6, w_4] = w_6$ and $w_1w_1 = w_3$.

From here, we have that $\mathcal{B}$ and $\mathcal{B}'$ are multiplicative bases. The corresponding associated graphs are:



$\Gamma(\mathcal{P}, \mathcal{B})$ $\Gamma(\mathcal{P}, \mathcal{B}')$

Of course, these graphs give rise to different decompositions of $\mathcal{P}$. Namely

$$\mathcal{P} = \mathcal{P}_{C_{[v_1]}} \oplus \mathcal{P}_{C_{[v_4]}}$$

being $\mathcal{P}_{C_{[v_1]}} = \mathbb{F}v_1 \oplus \mathbb{F}v_2 \oplus \mathbb{F}v_3$ and $\mathcal{P}_{C_{[v_4]}} = \mathbb{F}v_4 \oplus \mathbb{F}v_5 \oplus \mathbb{F}v_6$ for the basis $\mathcal{B}$, and

$$\mathcal{P} = \mathcal{P}_{C_{[w_1]}}$$

for the basis $\mathcal{B}'$.

In this section, we give a condition under which the graphs associated to $\mathcal{P}$ and two different multiplicative bases $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic. As a consequence, we establish a condition under which two decompositions of $\mathcal{P}$, induced by two different multiplicative bases $\mathcal{B}$ and $\mathcal{B}'$, are equivalent.

We begin by recalling that two graphs (V, E) and (V', E') are *isomorphic* if there exists a bijection $f : V \to V'$ such that $(v_i, v_j) \in E$ if and only if $(f(v_i), f(v_j)) \in E'$.

Definition 6 Let $\mathcal{P}$ be a non-commutative Poisson algebra. Two bases $\mathcal{B} = \{v_i\}_{i \in I}$ and $\mathcal{B}' = \{v'_j\}_{j \in J}$ of $\mathcal{P}$ are said to be *equivalent* if there exists an automorphism $f : \mathcal{P} \to \mathcal{P}$ satisfying $f(\mathcal{B}) = \mathcal{B}'$.

Lemma 1 *Let $\mathcal{P}$ be a non-commutative Poisson algebra with multiplicative bases $\mathcal{B}$ and $\mathcal{B}'$. If $\mathcal{B}$ and $\mathcal{B}'$ are equivalent, then $\Gamma(\mathcal{P}, \mathcal{B})$ and $\Gamma(\mathcal{P}, \mathcal{B}')$ are isomorphic.*

Proof Let us suppose that $\mathcal{B} = \{v_i\}_{i \in I}$ and $\mathcal{B}' = \{w_j\}_{j \in J}$ are two equivalent multiplicative bases of $\mathcal{P}$. Then, there exists a linear isomorphism $f : \mathcal{P} \to \mathcal{P}$ satisfying

$$f(x)f(y) = f(xy) \text{ and } [f(x), f(y)] = f([x, y]) \tag{3}$$

for any $x, y \in \mathcal{P}$ and such that $f(\mathcal{B}) = \mathcal{B}'$.

Let us denote by (V, E) and (V', E') the set of vertices and edges of $\Gamma(\mathcal{P}, \mathcal{B})$ and $\Gamma(\mathcal{P}, \mathcal{B}')$ respectively. Taking into account that $V = \mathcal{B}$ and $V' = \mathcal{B}'$, and the fact that $f(\mathcal{B}) = \mathcal{B}'$, we have that the map f defines a bijection from V onto V'.

Finally, observe that for any $x, y \in V$, if $(x, y) \notin E$ then $xy = yx = [x, y] = 0$ and so $f(x)f(y) = f(y)f(x) = [f(x), f(y)] = 0$. Hence $(f(x), f(y)) \notin E'$ and we only need to show that $(f(x), f(y)) \in E'$ for every $(x, y) \in E$. By Definition 5, given $(x, y) \in E$, we have

$$\{xz, zx, [x, z]\} \cap \mathbb{F}^{\times} y \neq \emptyset$$

for some $z \in \mathcal{B}$. Then, by Eq. (3)

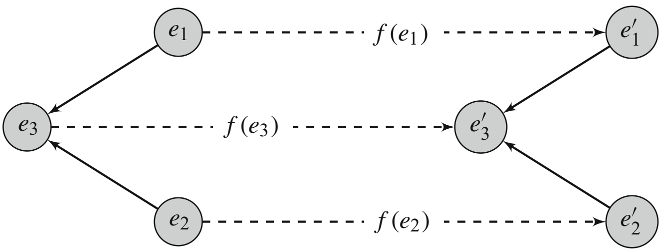
$$\{f(x)f(z), f(z)f(x), [f(x), f(z)]\} \cap \mathbb{F}^{\times} f(y) \neq \emptyset,$$

which means that $(f(x), f(y)) \in E'$. We can conclude that $\Gamma(\mathcal{P}, \mathcal{B})$ and $\Gamma(\mathcal{P}, \mathcal{B}')$ are isomorphic by means of f. □

Example 5 Let us consider the Poisson algebra $\mathcal{P}$ given in Example 1, with the basis $\mathcal{B} = \{e_1, e_2, e_3\}$ and nonzero products $[e_1, e_2] = -[e_2, e_1] = e_3$ and $e_1 e_1 = e_3$.

Let us also consider the basis $\mathcal{B}' = \{e'_1 = e_1 + e_2, e'_2 = e_2, e'_3 = e_3\}$ and the linear isomorphism $f : \mathcal{P} \to \mathcal{P}$ defined by $f(e_i) = e'_i$ for any $1 \leq i \leq 3$.

Taking into account that the nonzero products over the elements of the basis $\mathcal{B}'$ are $[e'_1, e'_2] = -[e'_2, e'_1] = e'_3$ and $e'_1 e'_1 = e'_3$, we have that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent. So the graphs $\Gamma(\mathcal{P}, \mathcal{B})$ and $\Gamma(\mathcal{P}, \mathcal{B}')$ are isomorphic (see the diagram below):



The following concept is taking borrowed from the theory of graded algebras (see for instance [6]).

Definition 7 Let $\mathcal{P}$ be a non-commutative Poisson algebra and let

$$\Upsilon := \mathcal{P} = \bigoplus_{i \in I} \mathcal{P}_i \text{ and } \Upsilon' := \mathcal{P} = \bigoplus_{j \in J} \mathcal{P}_j$$

be two decompositions of $\mathcal{P}$ as orthogonal direct sums of ideals. It is said that Υ and Υ' are *equivalent* if there exists an automorphism $f : \mathcal{P} \to \mathcal{P}$, and a bijection $\sigma : I \to J$ such that $f(\mathcal{P}_i) = \mathcal{P}_{\sigma(i)}$ for any $i \in I$.

Let us observe that if $f : V \to V'$ defines an isomorphism between two graphs (V, E) and (V', E'), then v is a vertex of $C_{[v]}$ if and only if $f(v)$ is a vertex of $C_{[f(v)]}$, for every $v \in V$. That means $f([v]) = [f(v)]$ for every $v \in V$. Hence, taking into account this observation and Lemma 1, we get the next result.

Proposition 1 *Let $\mathcal{P}$ be a non-commutative Poisson algebra with multiplicative bases $\mathcal{B} = \{v_i\}_{i \in I}$ and $\mathcal{B}' = \{v'_j\}_{j \in J}$. If $\mathcal{B}$ and $\mathcal{B}'$ are equivalent, then the decompositions*

$$\mathcal{P} = \bigoplus_{[v_i] \in V/\sim} \mathcal{P}_{C_{[v_i]}} \quad \text{and} \quad \mathcal{P} = \bigoplus_{[v'_j] \in V'/\sim} \mathcal{P}_{C_{[v'_j]}},$$

corresponding to the choices of $\mathcal{B}$ and $\mathcal{B}'$ in Theorem 1 respectively, are also equivalent.

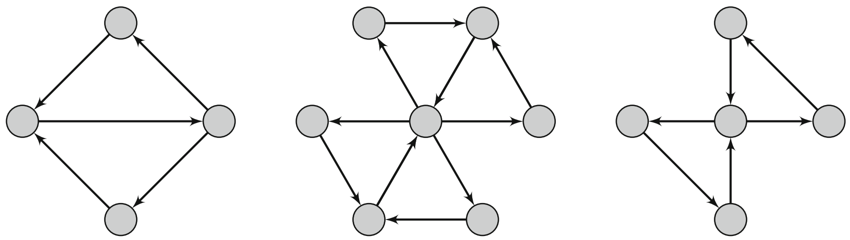
4 Minimality and Weak Symmetry

We begin this section by recalling that given a graph (V, E) and $v_i, v_j \in V$, an ordered family $\{v_{i_1}, \dots, v_{i_n}\} \subset V$ is called a *directed path* from v_i to v_j if $v_{i_1} = v_i$, $v_{i_n} = v_j$ and $(v_{i_r}, v_{i_{r+1}}) \in E$ for every $1 \leq r \leq n - 1$.

We also recall that a graph (V, E) is said to be *symmetric* if $(v_i, v_j) \in E$ for every $(v_j, v_i) \in E$. Then, a weaker concept can be introduced as follows:

Definition 8 We will say that a graph (V, E) is *weakly symmetric* if for every $(v_j, v_i) \in E$ there exists a directed path from v_i to v_j.

Example 6 The following graphs are examples of non-symmetric weakly symmetric graphs:



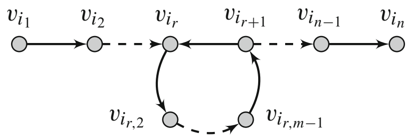
Taking into account that given a graph (V, E), two vertices $v_i, v_j \in V$ are said to be *strongly connected* if there exists a directed path from v_i to v_j and from v_j to v_i, we can assert the following result.

Lemma 2 *A graph (V, E) is weakly symmetric if and only if each pair of connected (different) vertices are strongly connected.*

Proof Suppose that (V, E) is weakly symmetric and consider $v_i, v_j \in V$, $v_i \neq v_j$ such that they are connected by an undirected path

$$\{v_{i_1}, \ldots, v_{i_n}\}. \tag{4}$$

In case that $(v_{i_r}, v_{i_{r+1}}) \notin E$ for some $1 \leq r \leq n-1$, then $(v_{i_{r+1}}, v_{i_r}) \in E$. By weak symmetry, there exists a directed path $\{v_{i_{r,1}}, \ldots, v_{i_{r,m}}\}$ from v_{i_r} to $v_{i_{r+1}}$ (see the diagram).



Hence, we can replace the undirected path (4) by the one

$$\{v_{i_1}, \ldots, v_{i_{r-1}}, v_{i_{r,1}}, \ldots, v_{i_{r,m}}, v_{i_{r+2}}, \ldots v_{i_n}\}$$

which solves this problem. From here, we always can find a directed path from v_i to v_j. Since the undirected path (4) is also from v_j to v_i, we have similarly that there exists a directed path from v_j to v_i and so v_i, v_j are strongly connected.

The converse is a direct consequence of the fact that any directed path from v_i to v_j is an undirected path, and so v_i and v_j are connected. □

We refer to the smallest ideal of $\mathcal{P}$ that contains $v \in \mathcal{V}$, denoted by $\mathcal{I}(v)$, as the *ideal of $\mathcal{P}$ generated by v*.

Lemma 3 *Let $\mathcal{B}$ be a multiplicative basis of $\mathcal{P}$ and $v_i, v_j \in \mathcal{B}$. We have that $v_j \in \mathcal{I}(v_i)$ if and only if there exists a directed path from v_i to v_j in $\Gamma(\mathcal{P}, \mathcal{B})$.*

Proof Suppose $v_j \in \mathcal{I}(v_i)$ for some $v_i \in \mathcal{B}$, taking into account that $\mathcal{B}$ is a multiplicative basis, we have that $v_j = \lambda f(\cdots(f(f(v_i, v_1), v_2), \cdots), v_n)$ for $v_1, \ldots, v_n \in \mathcal{B}$ and $0 \neq \lambda \in \mathbb{F}$, being any $f(v_r, v_s) \in \{v_r v_s, v_s v_r, [v_r, v_s]\}$. From here, by writing

$$w_k := \mathbb{F} f(\cdots(f(f(v_i, v_1), v_2), \cdots), v_k) \cap \mathcal{B}$$

for $k \in \{1, \ldots, n-1\}$ we get that $\{v_i, w_1, \ldots, w_{n-1}, v_j\}$ is a directed path from v_i to v_j.

Conversely, if $\{v_i, v_1, \ldots, v_{n-1}, v_j\}$ is a directed path from v_i to v_j then $0 \neq f(v_i, w_1) \in \mathbb{F}v_1$, $0 \neq f(v_1, w_2) \in \mathbb{F}v_2$, $\ldots$, $0 \neq f(v_{n-1}, w_n) \in \mathbb{F}v_j$ for some $w_1, \ldots, w_n \in \mathcal{B}$, where $f(\cdot, \cdot)$ is defined as above. Hence

$$v_j = \lambda f(\cdots(f(f(v_i, w_1), w_2), \cdots), w_n) \in \mathcal{I}(v_i),$$

for some $0 \neq \lambda \in \mathbb{F}$. □

Now, we will show that some notions of Poisson algebras admitting a multiplicative basis can be characterized by the weak division property of its associated graph.

Definition 9 Let $\mathcal{P}$ be a non-commutative Poisson algebra admitting a multiplicative basis $\mathcal{B} = \{v_i\}_{i \in I}$. It is said that $\mathcal{B}$ is of *division* if for any $v_i, v_j \in \mathcal{B}$ such that $0 \neq v_i v_j \in \mathbb{F}v_k$ or $0 \neq [v_i, v_j] \in \mathbb{F}v_k$ we have that $v_i, v_j \in \mathcal{I}(v_k)$, being $\mathcal{I}(v_k)$ the ideal of $\mathcal{P}$ generated by v_k.

Proposition 2 *Let $\mathcal{P}$ be a non-commutative Poisson algebra admitting a multiplicative basis $\mathcal{B} = \{v_i\}_{i \in I}$. Then $\mathcal{B}$ is of division if and only if $\Gamma(\mathcal{P}, \mathcal{B})$ is weakly symmetric.*

Proof Let us suppose that the multiplicative basis $\mathcal{B}$ is of division. Given an edge (v_i, v_j) of $\Gamma(\mathcal{P}, \mathcal{B})$ we have that

$$\lambda v_j \in \{v_i v_k, v_k v_i, [v_i, v_k]\}$$

for some $v_k \in \mathcal{B}$ and $0 \neq \lambda \in \mathbb{F}$. Hence, since $\mathcal{B}$ is of division, $v_i \in \mathcal{I}(v_j)$ and so (see Lemma 3) there exists a directed path from v_j to v_i. So $\Gamma(\mathcal{P}, \mathcal{B})$ is weakly symmetric.

Conversely, given $v_i, v_j, v_k \in \mathcal{B}$ such that $0 \neq v_i v_j \in \mathbb{F}v_k$ or $0 \neq [v_i, v_j] \in \mathbb{F}v_k$, we have that (v_i, v_k) or (v_j, v_k) are edges of $\Gamma(\mathcal{P}, \mathcal{B})$. By weak symmetry, there exist directed paths from v_k to v_i or from v_k to v_j. Then, by Lemma 3 we have that $v_i \in \mathcal{I}(v_k)$ or $v_j \in \mathcal{I}(v_k)$. So $\mathcal{B}$ is of division. □

Definition 10 A non-commutative Poisson algebra $\mathcal{P}$ admitting a multiplicative basis $\mathcal{B}$ is called *minimal* if its only nonzero ideal admitting a multiplicative basis contained in $\mathcal{B}$ is $\mathcal{P}$.

Proposition 3 *Let $\mathcal{P}$ be a non-commutative Poisson algebra admitting a multiplicative basis $\mathcal{B} = \{v_i\}_{i \in I}$. Then*

$$\mathcal{P} = \bigoplus_k \mathcal{I}_k$$

is the orthogonal direct sum of the family of its minimal ideals, each one admitting a multiplicative basis contained in $\mathcal{B}$, if and only if $\Gamma(\mathcal{P}, \mathcal{B})$ is weakly symmetric.

Proof Let us suppose that $\mathcal{P}$ decomposes as the orthogonal direct sum of the minimal ideals $\mathcal{I}_k$ with a multiplicative basis contained in $\mathcal{B}$. Let $(V, E) := \Gamma(\mathcal{P}, \mathcal{B})$ be the associated graph to $\mathcal{P}$ relative to $\mathcal{B}$ and take some $(v_i, v_j) \in E$. By orthogonality, $v_i, v_j \in \mathcal{I}_k$ for some k and, by the minimality of $\mathcal{I}_k$, we have that $v_i \in \mathcal{I}_k = \mathcal{I}(v_j)$. Taking now into account Lemma 3, there exists a directed path from v_j to v_i and, consequently, the graph $\Gamma(\mathcal{P}, \mathcal{B})$ is weakly symmetric.

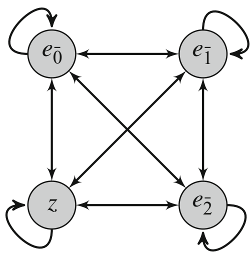
Conversely, from Theorem 1 we have that $\mathcal{P}$ is the orthogonal direct sum of the ideals $\mathcal{P}_{C_{[v_i]}}$. Let us consider a nonzero ideal $\mathcal{I}$ of $\mathcal{P}_{C_{[v_i]}}$ with a basis $\mathcal{B}_{\mathcal{I}}$ contained in $[v_i]$. For any $v_j \in \mathcal{B}_{\mathcal{I}}$ and $v_k \in [v_i]$ we have that v_j is connected to v_k and so, by Lemma 2, v_j and v_k are also strongly connected. So $v_k \in \mathcal{I}(v_j) \subset \mathcal{I}$ and then $\mathcal{I} = \mathcal{P}_{C_{[v_i]}}$. Then, any ideal $\mathcal{P}_{C_{[v_i]}}$ of the decomposition is minimal.

Finally, let us suppose that I is a nonzero minimal ideal of $\mathcal{P}$ with a multiplicative basis $\mathcal{B}_{\mathcal{I}} \subset \mathcal{B}$. Then, given $v \in \mathcal{B}_{\mathcal{I}}$ we have that $v \in \mathcal{I} \cap \mathcal{P}_{C_{[v]}} \neq \{0\}$. Then $\mathcal{I} \cap \mathcal{P}_{C_{[v]}}$ is a nonzero ideal of $\mathcal{P}$ with a multiplicative basis contained in $\mathcal{B}$. By minimality, $\mathcal{I} = \mathcal{P}_{C_{[v]}}$. So the ideal $\mathcal{I}$ appears in the decomposition given by Theorem 1. □

The next result is now immediate.

Corollary 2 *Let $\mathcal{P}$ be a non-commutative Poisson algebra admitting a multiplicative basis $\mathcal{B} = \{v_i\}_{i \in I}$. Then $\mathcal{P}$ is minimal if and only if the associated graph $\Gamma(\mathcal{P}, \mathcal{B})$ is weakly symmetric and has just one connected component.*

Example 7 Consider the complex four-dimensional Poisson algebra $\mathcal{P}$ with multiplicative basis $\mathcal{B} = \{z\} \cup \{e_{\bar{i}}\}_{i \in \mathbb{Z}_3}$, nonzero Lie products among the elements of $\mathcal{B}$ given by $[e_{\bar{0}}, e_{\bar{1}}] = -[e_{\bar{1}}, e_{\bar{0}}] = 2e_{\bar{2}}$, $[e_{\bar{0}}, e_{\bar{2}}] = -[e_{\bar{2}}, e_{\bar{0}}] = -2e_{\bar{1}}$ and $[e_{\bar{1}}, e_{\bar{2}}] = -[e_{\bar{2}}, e_{\bar{1}}] = 2e_0$; and nonzero associative products $zz = z$, $e_{\bar{i}}e_{\bar{i}} = -z$, $e_{\bar{i}}e_{\overline{i+1}} = -e_{\overline{i+1}}e_{\bar{i}} = e_{\overline{i+2}}$ and $ze_{\bar{i}} = e_{\bar{i}}z = e_{\bar{i}}$ for any $\bar{i} \in \mathbb{Z}_3$. The associated graph to $\mathcal{P}$ respectively to $\mathcal{B}$ is:



Then, Proposition 2 and Corollary 2 apply to get that $\mathcal{B}$ is of division and that $\mathcal{P}$ is minimal respectively.

Taking into account the Propositions 2 and 3, we have that the weak division property, allows us to state a context in which a second Wedderburn-type theorem (see for instance [7, pp. 137–139]) holds in the category of non-commutative Poisson algebras.

Theorem 2 *Let $\mathcal{P}$ be a non-commutative Poisson algebra admitting a multiplicative basis $\mathcal{B} = \{v_i\}_{i \in I}$. Then, the division property of $\mathcal{B}$ is a necessary and sufficient condition to state that*

$$\mathcal{P} = \bigoplus_k \mathcal{I}_k$$

is the orthogonal direct sum of the family of its minimal ideals, each one admitting a multiplicative basis contained in $\mathcal{B}$.

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Graphs with Weight of Fold Gauss Maps

C. Mendes de Jesus and E. Sanabria-Codesal

Abstract Our goal is to characterize the graphs associated to fold Gauss map of a closed orientable surface immersed in three-dimensional space. In this work we extends our previous result for graphs with weight.

1 Introduction

The singular set of a stable Gauss map of a orientable surface M generically immersed in Euclidean 3-space, called parabolic set, was described in [2]. It is formed by fold curves with isolated cusp points on them and each curve separates a hyperbolic region from an elliptic region of the surface. In order to codify the topology of the regular set, in [7] the authors introduce graphs with weight $\mathcal{G}_W(V, E)$, associated to stable Gauss maps, where E denote the number of edges, which corresponds to the path-components of the parabolic set of M, V is the number of vertices, that represent the different regions on the surface with non vanishing Gaussian curvature. Finally W is the weight of the graph W, defined as the sum of the genus of the regions corresponding to the vertices. They shown that *any weighted bipartite graph $\mathcal{G}_W(V, E)$ can be associated to a stable Gauss map from an appropriate closed orientable surface.* In the paper [6], the graphs are associated to stable Gauss map whose parabolic set has no cusp points, called fold

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Gauss maps, are characterized by *a graph with total weight equal to zero is a graph corresponding to a fold Gauss map of a closed orientable surface if and only if it is a 2-negative graph.*

Our aim here is to describe how these results can be extended in order to include graphs with total weight greater than zero.

2 Stable Gauss Maps and its Graphs

Let M, P be smooth orientable surfaces and $f, g : M \to P$ smooth maps. f and g are $\mathcal{A}$-*equivalent* if there are orientation-preserving diffeomorphisms l, k, such that $g \circ l = k \circ f$.

Given an immersion $f : M \to \mathbb{R}^3$ of a closed orientable surface M, let $\mathcal{N}_f : M \to S^2$ be its Gauss map. This map $\mathcal{N}_f$ is said to be *stable* if there exists a neighborhood $\mathcal{U}_f$ of f in the space $\mathcal{I}(M, \mathbb{R}^3)$ of immersions of M into $\mathbb{R}^3$, with Whitney C^∞-topology [4], such that for all $g \in \mathcal{U}_f$, the Gauss map $\mathcal{N}_g$ associated to g is $\mathcal{A}$-equivalent to $\mathcal{N}_f$. It can be seen that this condition is equivalent to stating that the family of height functions associated to f :

$$\begin{aligned} \lambda(f) : M \times S^2 &\longrightarrow \mathbb{R} \\ (x, v) &\longmapsto \langle f(x), v \rangle = f_v(x) \end{aligned}$$

is structurally stable [2, 10]. Then, two Gauss maps are $\mathcal{A}$-equivalent if and only if their corresponding families $\lambda(f)$ are $\mathcal{R}^+$-equivalent [1].

We remind that a point of M is a regular point of f if the map f is a local diffeomorphism around it and singular otherwise. We denote by Σf the *singular set* of f and its image $f(\Sigma f)$ is called the *branch set* of f.

For stable germs of Gauss maps, the regular points of $\mathcal{N}_f$ correspond to elliptic or hyperbolic points of M, i.e. points where the height function in the normal direction has a stable singularity of Morse type (A_1). The singular points of $\mathcal{N}_f$ are the parabolic points of M, i.e. points where the height function in the normal direction has a non stable singularity. In this case we may have *fold point* of $\mathcal{N}_f$, corresponding to an A_2 singularity of the height function in the normal direction or *cusp point* of $\mathcal{N}_f$, when the height function in the normal direction has a singularity of type A_3.

So the singular set $\Sigma\mathcal{N}_f$ for a stable Gauss maps $\mathcal{N}_f$ is the parabolic set of M associated to the immersion f. By Whitney's theorem (see [4]), this singular set consists of curves of fold points, possibly containing isolated cusp points. Then, the image of $\Sigma\mathcal{N}_f$, called the branch set of $\mathcal{N}_f$, consists in a collection of closed curves immersed in S^2 with possible isolated cusps and whose self-intersections (double points) correspond to parabolic points with parallel normals of the same orientation. This branch set is oriented as follows: as we transverse a branch curve following the orientation, nearby points on our left have two more inverse images than those on our right. The regular set, immersed into the surface S^2 by the map $\mathcal{N}_f$, consists of a finite number of regions.

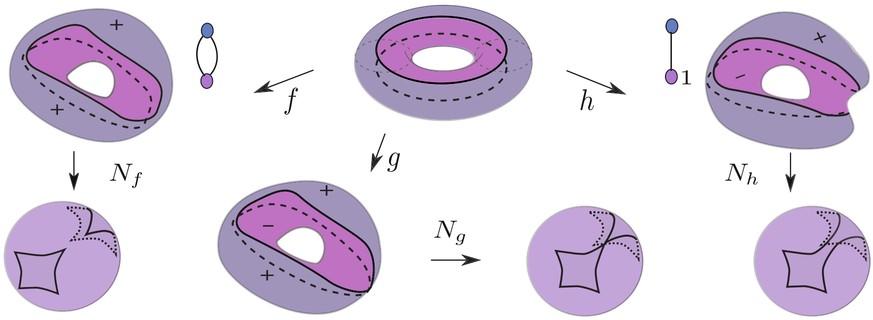

Fig. 1 Transition between two stable Gauss maps $\mathcal{N}_f$ and $\mathcal{N}_g$ from torus surface

Since every surface decomposes into a disjoint union of connected surfaces, we will assume that the surfaces with which we work are connected. We can encode the information of $(M, \Sigma N_f)$ over a *weighted graph* as follows [6, 7]:

- Each pathcomponent of $M \setminus \Sigma\mathcal{N}_f$ determines a *vertex* v and each curve of $\Sigma\mathcal{N}_f$ an *edge* e. A vertex v and an edge e are incident if and only if the curve represented by e lies in the boundary of the region represented by v.
- A vertex v receives a weight w if v represents a region with genus w.

Then, we denote this graph by $\mathcal{G}_W(V, E)$, where V, E are the number of vertices and edges, respectively, and W is the total weight.

We remind that a graph is *bipartite* if its vertices can be divided into two disjoint sets such that every edge connects vertices with opposite labels. Since M is orientable, each point of the parabolic set is in the frontier of a positive and a negative region, and consequently the corresponding graph is bipartite.

Figure 1 illustrates two stable Gauss map of the torus with their corresponding graphs: $\mathcal{G}_0(2, 2)$ is associated to $\mathcal{N}_f$ and the tree $\mathcal{G}_1(2, 1)$ to $\mathcal{N}_h$. The branch set of $\mathcal{N}_f$ has two curves with 4 cusp points each one with alternate signs, nevertheless the branch set of $\mathcal{N}_h$ has one curve with 6 cusps (see [2] and [9]).

Theorem 1 ([7]) *Any weighted bipartite graph $\mathcal{G}_W(V, E)$ can be associated to a stable Gauss map $\mathcal{N}_f$ of a closed orientable surface M whose genus is given by $g(M) = 1 - V + E + W$.*

3 Fold Gauss Maps and its Graph

In this section, we pursue to characterize bipartite graphs which can be associated to stable Gauss maps without cusp points of closed orientable surfaces, called *fold Gauss map*. If we denote by $V^{\pm}$, the total number of vertices corresponding to ellip-

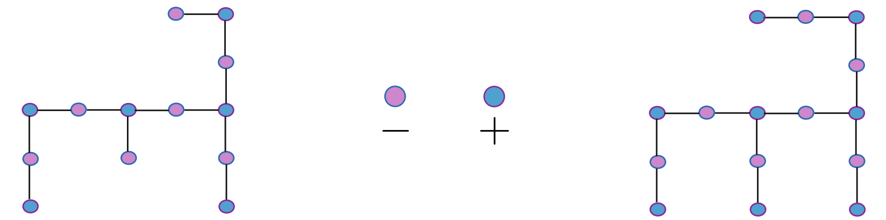

Fig. 2 2-Negative trees: (1) Not positive, (2) positive

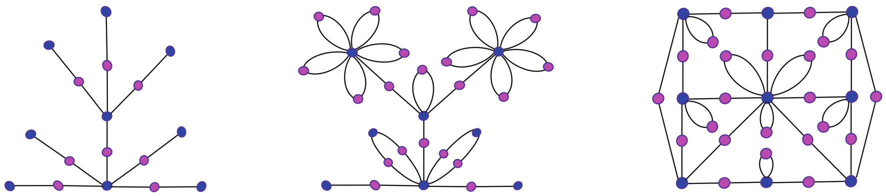

Fig. 3 Example of 2-negative graphs

tic and parabolic regions of M, respectively, and by $W^{\pm}$ their total corresponding genus, we know the following result:

Corollary 1 ([6]) *If $\mathcal{G}_W(V, E)$ is a graph of a fold Gauss map, then $E = 2\,(V^- - W^-)$.*

We remind that the degree of a vertex v in a graph is the number of edges incident to it. A vertex v is said to be *extremal* if v has degree one.

Definition 1 A bipartite graph, labeled positive and negative at its vertices, without extremal vertices or such that all extremal vertices are positive is called **positive graph** and is called 2**-negative** if all its negative vertices have degree two (see Fig. 2).

The properties of this special type of graphs were analyzed in [6].

Lemma 1 ([6]) *A bipartite graph $\mathcal{G}_W(V, E)$ is 2-negative if and only if it is positive and $E = 2V^-$ (see Fig. 3).*

We observe that only one of the above conditions (to be positive or $E = 2V^-$) does not guarantee a 2-negative graph.

Theorem 2 ([6]) *A connected bipartite graph $\mathcal{G}_W(V, E)$ is 2-negative if and only if it is positive and $V^+ - V^- = 1 - \beta_1(\mathcal{G})$, where $\beta_1(\mathcal{G})$ is the first Betti number of the graph.*

New properties of this type of graphs was obtained in [8]

Lemma 2 ([8]) *A graph $\mathcal{G}_W(V, E)$ corresponding to a fold Gauss map satisfies $W^- = 0$ and it is positive.*

Theorem 3 ([8]) *A graph $\mathcal{G}_W(V, E)$ corresponding to a fold Gauss map is* 2-*negative graph.*

Now, our goal is to determine sufficient conditions for graphs $\mathcal{G}_W(V, E)$ associated to fold Gauss map, generalizing the results obtained in [6] for zero weight graphs. To do this, we will start summarizing the necessary techniques.

4 Lips and Beaks Transition and Surgeries

A codimension one transition corresponds to a generic isotopy from a given stable map to another one, lying in a different pathcomponent of the set of the stable Gauss maps (see [3]).

Here we make a decomposition of the codimension one transition on the immersions, following the study of stable maps between surfaces and their invariants (see [5]), that alter the singular set of their Gauss maps and hence their graphs in convenient ways.

According to this study, the local codimension one phenomena are the following:

1. Morse transitions of the parabolic curve at a non-versal A_3. This corresponds to lips and beaks transitions in the Gauss map.
2. Birth/annihilation of a pair of cusps of the Gauss map on a smooth parabolic curve (at an A_4 point of the height function). This corresponds to a swallowtail type singularity in the Gauss map.
3. Cone sections at a $D_4^{\pm}$ point of the height function (flat umbilic). Correspondingly, we have the purse and the pyramid transitions for the Gauss map.

In this case, we are interested in lips and beaks transitions. The *lips transition*, that we denote by L, corresponds to a Morse transition of maximum or minimum type in the parabolic curve. It may be done in a region of positive (or negative) curvature, X of M giving rise to a new region with negative (positive) curvature Z. Their common boundary is a connected component of the parabolic set whose image through the Gauss map is a closed curve with two cusp points in S^2. The effect of this on the graph of $\mathcal{N}_f$ corresponds to adding a new edge attached to the positive (negative) vertex corresponding to the initial region, now renamed X_1 (see Fig. 4).

The *beaks transitions* correspond to a Morse transition of saddle type in the parabolic set. Such a transition occurs when we approach two arcs of the parabolic set until they join in a common point *beaks point* and break again giving rise to a new pair of arcs and as a result, a couple of cusp points are introduced in the branch set. This process, in the sense to increase the cusp points, can be separated into four different cases (see Fig. 4): B_v^+-transition increases by 1 the number of regular regions, i. e. adds a vertex and an edge on the graph of $\mathcal{N}_f$, B_v^--transition decreases by 1 the number of regular regions, therefore removes 1 vertex and 1 edge on the graph, B_w^+-transition increases by 1 the weight, maintains the number of

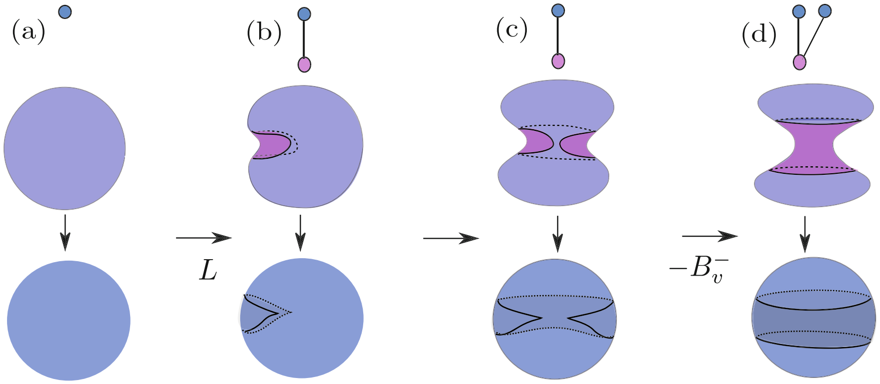

Fig. 4 Realization of the 2-negative graph $\mathcal{G}_0(3, 2)$ associated to the fold Gauss map of the sphere (**d**), which can be obtained from the fold Gauss map (**a**), by shown sequence of lip and beak transitions from (**a**) to (**d**)

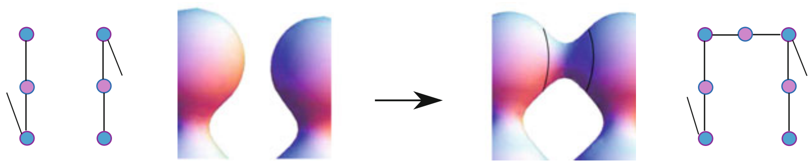

Fig. 5 Local examples of surgeries H_0^+ and H_1^+ of fold Gauss maps

regular regions (vertices) but decreases by 1 the number of edges and B_w^--transition decreases by 1 the weight, maintains the number of regular regions (vertices) but increases by 1 the number of edges.

In [6, 7] was introduced the definition of stable surgeries of stable Gauss maps, which contributes to characterize the graphs associated to them, as we will see below.

The *Surgery* on a closed surface M consists in joining two elliptic regions of M with an intermediary hyperbolic region. This process is done by removing two discs, one in each elliptic region and connecting a hyperbolic tube to their boundaries and it can be done, clearly, in a smooth way (see Fig. 5).

5 Graphs with Weight of Fold Gauss Maps

In order to analyze the necessary and sufficient conditions so that a graph with weight can be associated with a fold Gauss map of some closed orientable surface, we shall use lip and beaks transitions and surgeries in the immersion of the corresponding surface M associated to it. In [6], we proved the result for graph with total weight zero (Fig. 6):

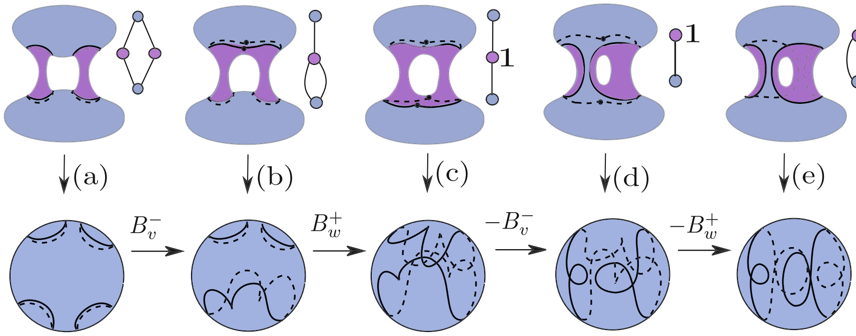

Fig. 6 Realization of the 2-negative graph $\mathcal{G}_0(2,2)$ associated to a fold Gauss map of the torus (**e**), which can be obtained from the fold Gauss map (**a**) by beak transitions from (**a**) to (**e**)

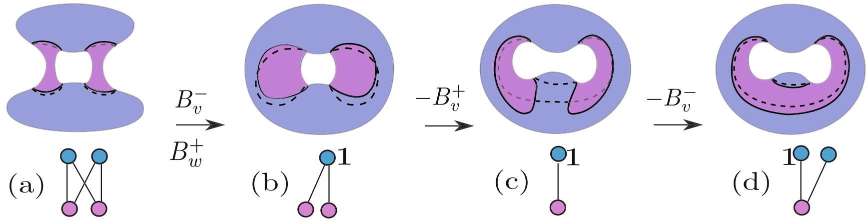

Fig. 7 Realization of the 2-negative graph $\mathcal{G}_1(3,2)$ associated to a fold Gauss map of the torus (**d**), which can be obtained from the fold Gauss map (**a**) by beak transitions from (**a**) to (**d**)

Theorem 4 ([6]) *$\mathcal{G}_0(V,E)$ is a graph corresponding to a fold Gauss map of a closed orientable surface if and only if $\mathcal{G}_0(V,E)$ is a 2-negative graph.*

In [8] we defined the simplest graph in this particular case and we found the corresponding fold Gauss maps associated to these basic graphs.

Definition 2 The graphs which vertices have degree at most 2 are called *basic graphs* (Fig. 7).

Lemma 3 ([8]) *All basic 2-negative trees can be associated to a fold Gauss map.*

Proposition 1 ([8]) *Any 2-negative tree $\mathcal{G}_{W^+}(V, V-1)$, with $V \geq 3$ can be associated to a fold Gauss map $\mathcal{N}_f : M \to S^2$, where the genus of M is given by W^+.*

Proposition 2 ([8]) *Any 2-negative graph with only one positive vertex, called positive daisy, can be associated to a fold Gauss maps $\mathcal{N}_f : M \to S^2$, where the genus of M is given by $V^- + W^+$.*

Proposition 3 ([8]) *Any 2-negative graph can be decomposed into basic 2-negative graphs (see Fig. 8).*

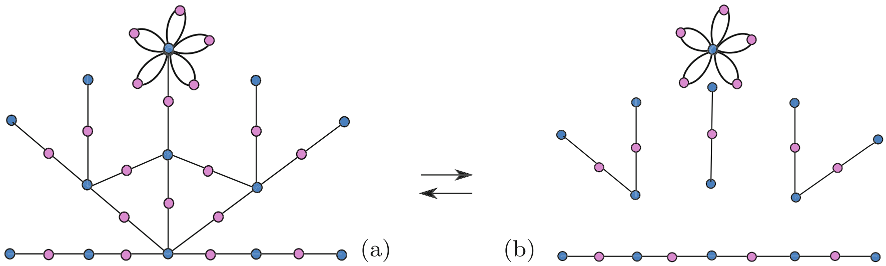

Fig. 8 **(a)** 2-Negative graph and **(b)** decomposition of **(a)** in four basic 2-negative trees and one daisy

By using the previous propositions, we obtain the general result:

Theorem 5 ([8]) *$\mathcal{G}_W(V, E)$ is a graph corresponding to a fold Gauss map of a closed orientable surface if and only if $\mathcal{G}_W(V, E)$ is 2-negative with $W^- = 0$.*

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A Note on Spacelike Hypersurfaces and Timelike Conformal Vectors

Giulio Colombo, José A. S. Pelegrín, and Marco Rigoli

Abstract Any compact spacelike hypersurface immersed in a doubly warped product spacetime $I_h \times_\rho \mathbb{P}$ with nondecreasing warping factor ρ must be a spacelike slice, provided that the mean curvature satisfies $H \geq \rho'/h\rho$ everywhere on the hypersurface. The conclusion also holds, under suitable assumptions on the immersion, when the hypersurface is complete and noncompact. A similar rigidity property is shown for compact hypersurfaces in spacetimes carrying a conformal, strictly expanding, timelike vector field.

1 Introduction

Spacelike hypersurfaces play a crucial role in the understanding of the geometry of a Lorentzian spacetime. Roughly speaking, they describe the physical space that can be measured in a given instant of time. For instance, they serve as initial data in the Cauchy problem for Einstein's field equations [18] and they play a privileged role in determining the causal properties of the spacetime. Indeed, a spacetime is globally hyperbolic if and only if it admits a Cauchy hypersurface [14]. Even more, any globally hyperbolic spacetime is diffeomorphic to $\mathbb{R} \times S$, being S a smooth spacelike Cauchy hypersurface [6].

In this article we will study the geometry of spacelike hypersurfaces in certain spacetimes that present a particular symmetry. In General Relativity, symmetry arises from the existence of a one-parameter group of transformations generated by a conformal vector field. This infinitesimal symmetry is usually assumed when

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searching for exact solutions of Einstein's field equations [13]. In fact, there is a vast literature concerning the study of spacelike hypersurfaces in spacetimes that admit different causal symmetries (see [8–10], for instance). We will focus on the study of spacelike hypersurfaces in doubly warped product spacetimes, where natural conformal vector field pertains to this structure.

Let $(\mathbb{P}, \sigma)$ be a (connected) Riemannian manifold of dimension m and $I \subseteq \mathbb{R}$ be an open interval. The product manifold $\overline{M} = I \times \mathbb{P}$ can be endowed with the Lorentzian metric $\overline{g}$ given at each point $(t, x) \in \overline{M}$ by

$$\overline{g}_{(t,x)} = -h(x)^2 \pi_I^*(\mathrm{d}t^2) + \rho(t)^2 \pi_{\mathbb{P}}^*(\sigma), \tag{1}$$

where $h \in C^\infty(\mathbb{P})$, $\rho \in C^\infty(I)$ are positive functions and $\pi_I : \overline{M} \to I$, $\pi_{\mathbb{P}} : \overline{M} \to \mathbb{P}$ are the projections onto the factors of $I \times \mathbb{P}$. We call $(\overline{M}, \overline{g})$ a doubly warped product spacetime, which will be denoted here by $\overline{M} = I_h \times_\rho \mathbb{P}$. The time orientation of $\overline{M}$ is the one given by the timelike vector field $\partial_t := \partial/\partial t$. The causal symmetry in these ambient spacetimes is given by the timelike vector field $\rho \partial_t$, which is conformal, as we will prove in Lemma 2 below. As a consequence, the family of spacelike slices $\Sigma_t = \{t\} \times \mathbb{P}$, $t \in I$, provides a foliation of $\overline{M}$ by totally umbilical hypersurfaces; in particular, the mean curvature of Σ_t at the point $(t, x) \in \overline{M}$ is given by $\mathcal{H}(t, x) = \rho'(t)/\rho(t)h(x)$.

Doubly warped product spacetimes were introduced in [5] as an extension of the singly warped product spacetimes defined in [16] and include several models such as standard static spacetimes and Generalized Robertson–Walker spacetimes (see [7] for the original definition of singly warped product spaces in the Riemannian setting). Causal properties of doubly warped product spacetimes were studied in [4], obtaining several conditions to guarantee their global hyperbolicity. Moreover, in [15] the authors studied the decomposition of an ambient spacetime as a doubly warped product.

Our article is organized as follows. Section 2 is devoted to proving some preliminary results for spacelike hypersurfaces in general Lorentzian manifolds as well as for the particular case of doubly warped product spacetimes. These results will enable us to prove our main theorems in Sect. 3. In particular, we prove in Theorem 2 that a compact spacelike hypersurface $\psi : M \to \overline{M}$ immersed in a doubly warped product spacetime $\overline{M} = I_h \times_\rho \mathbb{P}$ must be a spacelike slice if $H \geq \mathcal{H} \circ \psi \geq 0$ on M, with H the mean curvature of ψ in the direction of the normal unit vector with the same time-orientation as ∂_t. Theorem 3 partially extends this observation to the case of a general spacetime carrying a conformal timelike vector field. Theorems 5, 6 and 7 provide examples of cases where the conclusion of Theorem 2 still holds true for M complete and non-compact, under additional assumptions on ψ.

2 The Geometric Setting

Let $\psi : (M^m, g) \to (\overline{M}^{m+1}, \overline{g})$ be an isometric immersion between (connected) semi-Riemannian manifolds and denote with ∇ and $\overline{\nabla}$, respectively, the Levi-Civita connections for g and $\overline{g}$. For any given point $p \in M$, there exist sufficiently small neighbourhoods $U \subseteq M$ and $\overline{U} \subseteq \overline{M}$, respectively, of p and $\psi(p)$, such that $\psi(U) \subseteq \overline{U}$, $\psi|_U : U \to \overline{U}$ is an embedding and $\overline{U}$ supports a nowhere vanishing vector field $Z \in \mathfrak{X}(\overline{U})$ satisfying $\overline{g}(Z_{\psi(q)}, (\mathrm{d}\psi)_q V) = 0$ for every $q \in U$, $V \in T_q M$. We say that Z is orthogonal to ψ on U. Note that we are not requiring $\overline{g}(Z, Z)$ to be constant on $\overline{U}$. From now on, for the sake of simplicity, we shall omit writing pullbacks or pushforwards via ψ since the embedding $\psi|_U$ provides a natural identification between U and a regular submanifold of $\overline{U}$. Following Chapter 4 of [16], the second fundamental form $\mathbb{I}$ of ψ is defined by

$$\mathbb{I}(V, W) = \overline{\nabla}_V W - \nabla_V W \tag{2}$$

for every couple of vectors V, W tangent to U. $\mathbb{I}$ is a symmetric bilinear form taking values in the normal bundle $TU^{\perp} \subseteq T\overline{U}$ of vectors orthogonal to U and satisfies Weingarten's equation

$$\overline{g}(\mathbb{I}(V, W), Z) = \overline{g}(\overline{\nabla}_V W, Z) = -\overline{g}(\overline{\nabla}_V Z, W) \tag{3}$$

For $\overline{X} \in \mathfrak{X}(M)$, $V, W \in T_q\overline{M}$, $q \in \overline{M}$, we have the differential identity

$$(\mathcal{L}_{\overline{X}}\overline{g})(V, W) = \overline{g}(\overline{\nabla}_V \overline{X}, W) + \overline{g}(\overline{\nabla}_W \overline{X}, V) \tag{4}$$

for the Lie derivative $\mathcal{L}$ of the metric $\overline{g}$, so Eq. (3) can be rewritten as

$$\overline{g}(\mathbb{I}(V, W), Z) = -\frac{1}{2}(\mathcal{L}_Z\overline{g})(V, W). \tag{5}$$

The mean curvature vector $\mathbf{H}$ of ψ is the normalized trace of $\mathbb{I}$ with respect to the metric g, that is, for every $q \in U$ and for every g-orthonormal basis $\{e_i\}_{1\le i\le m}$ of T_qU,

$$\mathbf{H}_q = \frac{1}{m}\mathrm{trace}_g(\mathbb{I}_q) = \frac{1}{m}\sum_{i=1}^{m} g(e_i, e_i)\mathbb{I}(e_i, e_i) \tag{6}$$

For a generic vector field $\overline{X} \in \mathfrak{X}(\overline{M})$ we let $X \in \mathfrak{X}(M)$ be its tangential part along ψ, defined as follows: for every $q \in M$, X_q is the orthogonal projection of $\overline{X}_{\psi(q)}$ onto the tangent subspace $T_qM \subseteq T_{\psi(q)}\overline{M}$.

Lemma 1 *Let $\psi : (M^m, g) \to (\overline{M}^{m+1}, \overline{g})$ be an isometric immersion between semi-Riemannian manifolds, $\overline{X} \in \mathfrak{X}(\overline{M})$ a vector field and $X \in \mathfrak{X}(M)$ its tangential* *part along* ψ*. For any couple of tangent vectors* $V, W \in T_pM$*,* $p \in M$*,*

$$g(\nabla_V X, W) = \overline{g}(\overline{\nabla}_V \overline{X}, W) + \overline{g}(\overline{X}, \mathbb{I}(V, W)) \tag{7}$$

and for any choice of a local unit normal vector field N *on* M*,*

$$\operatorname{div}(X) = \overline{\operatorname{div}}(\overline{X}) + \overline{g}(m\mathbf{H}, \overline{X}) - \overline{g}(N, N)\overline{g}(\overline{\nabla}_N \overline{X}, N), \tag{8}$$

where div *and* $\overline{\operatorname{div}}$*, respectively, are the divergence operators induced by* ∇ *and* $\overline{\nabla}$*.*

Proof Let $U, \overline{U}, Z$ be as above. Then $X = \overline{X} - \frac{\overline{g}(\overline{X}, Z)}{\overline{g}(Z, Z)} Z$ on M, so

$$g(\nabla_V X, W) = \overline{g}(\overline{\nabla}_V X, W) = \overline{g}(\overline{\nabla}_V \overline{X}, W) - \overline{g}\left(\overline{\nabla}_V \left(\frac{\overline{g}(\overline{X}, Z)}{\overline{g}(Z, Z)} Z\right), W\right).$$

Since $\overline{g}(Z, W) = 0$ and $\mathbb{I}(V, W)$ is a multiple of Z,

$$-\overline{g}\left(\overline{\nabla}_V \left(\frac{\overline{g}(\overline{X}, Z)}{\overline{g}(Z, Z)} Z\right), W\right) = -\frac{\overline{g}(\overline{X}, Z)}{\overline{g}(Z, Z)} \overline{g}(\overline{\nabla}_V Z, W) = \frac{\overline{g}(\overline{X}, Z)\overline{g}(\mathbb{I}(V, W), Z)}{\overline{g}(Z, Z)}$$
$$= \overline{g}(\overline{X}, \mathbb{I}(V, W)).$$

Now let us consider a g-orthonormal basis $\{e_i\}_{1 \le i \le m}$ of T_pM. If $\overline{g}(Z, Z) = \pm 1$, that is, if $N = Z$ is a unit normal vector for ψ at p, then

$$\begin{aligned} \operatorname{div}(X) &= \sum_{i=1}^{m} g(e_i, e_i) g(\nabla_{e_i} X, e_i), \\ \overline{\operatorname{div}}(\overline{X}) &= \sum_{i=1}^{m} \overline{g}(e_i, e_i)\overline{g}(\overline{\nabla}_{e_i} \overline{X}, e_i) + \overline{g}(N, N)\overline{g}(\overline{\nabla}_N \overline{X}, N). \end{aligned} \tag{9}$$

Then (8) follows from (7), (9) and the definition (6) of $\mathbf{H}$. □

In this note we are interested in the case where $(\overline{M}, \overline{g})$ is a Lorentzian manifold, $\overline{X}$ is a timelike conformal vector field and $\psi : M \to \overline{M}$ is a spacelike hypersurface, that is, g is a Riemannian metric. Conformality of $\overline{X}$ means that there exists $\eta \in C^\infty(\overline{M})$ such that $\mathcal{L}_{\overline{X}}\overline{g} = 2\eta\overline{g}$ on $\overline{M}$, that is, $\overline{g}(\overline{\nabla}_V \overline{X}, W) = \eta\overline{g}(V, W)$ for every $V, W \in T_q\overline{M}$, $q \in \overline{M}$. Let us set the notation

$$\alpha = \sqrt{-\overline{g}(\overline{X}, \overline{X})}, \qquad \mathcal{H} = \frac{\eta}{\alpha}. \tag{10}$$

If $\hat{\psi} : \Sigma^m \to \overline{M}$ is a (necessarily spacelike) hypersurface such that $\overline{X}$ is orthogonal to $\hat{\psi}$ at some point $x \in \Sigma$, then by (5) the second fundamental form $\hat{\mathbb{I}}$ of $\hat{\psi}$ satisfies $\overline{g}(\hat{\mathbb{I}}(V, W), \overline{X}) = -\eta \hat{g}(V, W)$ for every $V, W \in T_x\Sigma$, with $\hat{g} = \hat{\psi}^*\overline{g}$. $\hat{N} = \alpha^{-1}\hat{X}$ is a unit normal vector for $\hat{\psi}$ at x and therefore $\hat{\mathbb{I}}(V, W) = \overline{g}(\hat{N}, \hat{N})\overline{g}(\hat{\mathbb{I}}(V, W), \hat{N})\hat{N} = \eta\alpha^{-1}\hat{g}(V, W)\hat{N}$. So, $\hat{\mathbb{I}} = \mathcal{H}\hat{g} \otimes \hat{N}$ at x, that is, $\hat{\psi}$ is umbilical at x with mean curvature vector

$$\hat{\mathbf{H}} = \mathcal{H}\hat{N} = \frac{\eta}{\alpha^2}\overline{X}. \tag{11}$$

The existence of a global timelike vector field $\overline{X}$ implies that $\psi : M \to \overline{M}$ is two-sided, that is, M admits a global (timelike) unit normal vector field N. In fact, on every open subset $U \subseteq M$ admitting a local unit normal vector field $N_U : U \to TU^{\perp}$ the function $\overline{g}(N_U, \overline{X})$ is always nonzero since N_U and $\overline{X}$ are both timelike and therefore cannot be orthogonal at any point. So, consider a family $\{N_a\}_{a\in I}$ of local unit normal vector fields defined on the elements of an open cover $\{U_a\}_{a\in I}$ of M and such that $\overline{g}(N_a, \overline{X}) < 0$ on U_a. The conditions $\overline{g}(N_a, N_a) = -1$ and $\overline{g}(N_a, \overline{X}) < 0$ uniquely determine N_a at every point of U_a, so $N_a = N_b$ on $U_a \cap U_b$ for every $a, b \in I$ and therefore we can glue together these vectors to obtain a global unit normal vector field $N : M \to TM^{\perp}$ satisfying $\overline{g}(N, \overline{X}) < 0$ on M.

In the following, we will always assume that N is chosen so that $\overline{g}(N, \overline{X}) < 0$, that is, with the same time-orientation of $\overline{X}$. In this case, we will also say that N is future-pointing. The mean curvature vector $\mathbf{H}$ then induces a mean curvature function $H = -\overline{g}(\mathbf{H}, N) \in C^{\infty}(M)$ for which $\mathbf{H} = HN$. By the wrong-way Cauchy-Schwarz inequality, $\overline{g}(N, \alpha^{-1}\overline{X}) \leq -1$, so we can introduce the hyperbolic angle function $\theta \in C^{\infty}(M)$ via its hyperbolic cosine

$$\cosh\theta = -\overline{g}(N, \alpha^{-1}\overline{X}) = -\frac{\overline{g}(N, \overline{X})}{\alpha}.$$

In this setting, recalling (4) we can express formulas (7) and (8) as

$$g(\nabla_V X, W) = \eta g(V, W) + g(AV, W)\alpha\cosh\theta, \tag{12}$$

with $A = -\overline{\nabla}_{(\,.\,)}N$ the shape operator of ψ induced by N, and

$$\mathrm{div}(X) = m\eta - mH\alpha\cosh\theta = m\alpha(\mathcal{H} - H\cosh\theta). \tag{13}$$

2.1 Doubly Warped Product Spacetimes

Examples of spacetimes admitting a timelike conformal vector field include doubly warped Lorentzian product spacetimes. As we have previously said, by a doubly warped product spacetime $\overline{M} = I_h \times_\rho \mathbb{P}$ we mean a product manifold $\overline{M} = I \times \mathbb{P}$, where $(\mathbb{P}, \sigma)$ is a connected m-dimensional Riemannian manifold and $I \subseteq \mathbb{R}$ is an open interval, endowed with the Lorentzian metric $\overline{g}$ given at $(t,x) \in \overline{M}$ by

$$\overline{g}_{(t,x)} = -h(x)^2 \pi_I^*(\mathrm{d}t^2) + \rho(t)^2 \pi_{\mathbb{P}}^*(\sigma) \tag{14}$$

with $h \in C^\infty(\mathbb{P})$, $\rho \in C^\infty(I)$ positive functions and $\pi_I : \overline{M} \to I$, $\pi_{\mathbb{P}} : \overline{M} \to \mathbb{P}$ the projections onto the factors of $I \times P$. In the following, with an abuse of notation we write ρ, ρ', h to denote the functions $\rho \circ \pi_I, \rho' \circ \pi_I, h \circ \pi_{\mathbb{P}} \in C^\infty(\overline{M})$. A time orientation for $\overline{M}$ is given by the timelike vector $\partial_t := \partial/\partial t$. Note that from these models we can reobtain a standard static spacetime by setting $\rho(t) \equiv 1$, as well as a Generalized Robertson–Walker spacetime when $h(x) \equiv 1$.

Lemma 2 *The timelike vector field* $\overline{X} = \rho\partial_t$ *is conformal on* $\overline{M} = I_h \times_\rho \mathbb{P}$ *and*

$$\mathcal{L}_{\overline{X}}\overline{g} = 2\rho'\overline{g}. \tag{15}$$

Proof Let $(t_0, x_0) \in \overline{M}$ be a given point and let $\{x^i\}$ be a coordinate system $(\mathbb{P}, \sigma)$ on a neighbourhood $U \subseteq \mathbb{P}$ of x_0. Then $\{t, x^i\}$ is a coordinate system for $\overline{M}$ defined on $I \times U \ni (t_0, x_0)$ and the $(m+1)$-ple $\{e_1, \ldots, e_{m+1}\} := \left\{\rho^{-1}\partial_1, \ldots, \rho^{-1}\partial_m, h^{-1}\partial_t\right\}$ is a local frame for $\overline{M}$ on $I \times U$. A direct computation shows that $[\overline{X}, e_\mu] = -\rho' e_\mu$ on $I \times U$ for $1 \le \mu \le m+1$. For every $1 \le \mu_1, \mu_2 \le m+1$, the product $\overline{g}(e_{\mu_1}, e_{\mu_2})$ is constant along the curve $I \times \{x_0\}$, so we have

$$\begin{aligned}(\mathcal{L}_{\overline{X}}\overline{g})(e_{\mu_1}, e_{\mu_2}) &= \overline{X}(\overline{g}(e_{\mu_1}, e_{\mu_2})) - \overline{g}(e_{\mu_1}, [\overline{X}, e_{\mu_2}]) - \overline{g}(e_{\mu_2}, [\overline{X}, e_{\mu_1}]) \\ &= 0 - \overline{g}(e_{\mu_1}, -\rho' e_{\mu_2}) - \overline{g}(e_{\mu_2}, -\rho' e_{\mu_1}) = 2\rho'\overline{g}(e_{\mu_1}, e_{\mu_2})\end{aligned}$$

at (t_0, x_0). Since $(\mathcal{L}_{\overline{X}}\overline{g})_{(t_0,x_0)}$ is a bilinear form on $T_{(t_0,x_0)}\overline{M}$, (15) follows. □

A doubly warped product $\overline{M} = I_h \times_\rho \mathbb{P}$ is foliated by the level sets $\Sigma_t = \{t\} \times \mathbb{P}$, $t \in I$, of the coordinate function $\pi_I : \overline{M} \to I$. They are always orthogonal to $\overline{X}$, so all of them are totally umbilical hypersurfaces by the previous discussion (see also Prop. 2.2 in [19]). In this setting, we have

$$\alpha = \sqrt{-\overline{g}(\overline{X}, \overline{X})} = h\rho, \qquad \mathcal{H} = \frac{\eta}{\alpha} = \frac{\rho'}{h\rho}.$$

Now, let $\psi : M \to I_h \times_\rho \mathbb{P}$ be a spacelike immersed hypersurface, that is, assume that $g = \psi^*\overline{g}$ is a Riemannian metric on M. Again, with a little abuse of notation, we denote with ρ, ρ', h the functions $\rho \circ \pi_I \circ \psi, \rho' \circ \pi_I \circ \psi, h \circ \pi_{\mathbb{P}} \circ \psi \in C^\infty(M)$. Since the coordinate function π_I has gradient $\overline{\nabla}\pi_I = -h^{-2}\partial_t = -\rho^{-1}h^{-2}\overline{X}$, if we introduce the height function $\tau = \pi_I \circ \psi$ then the tangential part X of $\overline{X}$ along ψ satisfies $-X = \rho h^2 \nabla\tau$. Then, Eq. (13) can be restated as

$$\operatorname{div}(\rho h^2 \nabla\tau) = mh\rho(H\cosh\theta - \mathcal{H}) \tag{16}$$

or, equivalently,

$$\Delta_{-\log(\rho h^2)}\tau = \frac{m}{h}(H\cosh\theta - \mathcal{H}) \tag{17}$$

where, for $f \in C^1(M)$, Δ_f is the symmetric diffusion operator

$$\Delta_f = e^f \operatorname{div}(e^{-f}\nabla \quad) = \Delta - g(\nabla f, \nabla \quad).$$

If we let $\mathcal{R}$ be an antiderivative of ρ on I, (16) is also equivalent to

$$\Delta_{-\log(h^2)}(\mathcal{R}\circ\tau) = \frac{m\rho}{h}(H\cosh\theta - \mathcal{H}) \tag{18}$$

3 Rigidity of H-Hypersurfaces with $H \geq \mathcal{H}$

The aim of this section is to prove some rigidity results for compact or complete spacelike hypersurfaces immersed in a spacetime $\overline{M}$ carrying a timelike conformal vector field $\overline{X}$. We shall refer again to the notation

$$\mathcal{L}_{\overline{X}}\overline{g} = 2\eta\overline{g}, \qquad \alpha = \sqrt{-\overline{g}(\overline{X}, \overline{X})}, \qquad \mathcal{H} = \frac{\eta}{\alpha} \tag{19}$$

introduced in the previous section. We first consider the case of a doubly warped product spacetime $\overline{M} = I_h \times_\rho \mathbb{P}$, where $\overline{X} = \rho\partial_t$ and

$$\mathcal{L}_{\overline{X}}\overline{g} = 2\rho'\overline{g}, \qquad \alpha = h\rho, \qquad \mathcal{H} = \frac{\rho'}{h\rho}. \tag{20}$$

Let us remark some facts about spacelike immersions into doubly warped product spacetimes. Given $\psi : M \longrightarrow \overline{M}$ a spacelike hypersurface, we can define its projection on $\mathbb{P}$ by $\varphi_{\mathbb{P}} := \pi_{\mathbb{P}} \circ \psi : M \longrightarrow \mathbb{P}$. Note that

$$\varphi_{\mathbb{P}}^*(\sigma) = \rho^{-2}\psi^*(\overline{g} + h^2 \mathrm{d}t^2) \geq \rho^{-2}\psi^*\overline{g} = \rho^{-2}g \tag{21}$$

with $g = \psi^*\overline{g}$ the metric induced by ψ on M. Since $\varphi_{\mathbb{P}}$ is a local diffeomorphism, by Lemma 3.3 in Chapter 7 of [12].

Lemma 3 *Let $\psi : M \longrightarrow \overline{M}$ be a complete spacelike hypersurface in a doubly warped product spacetime $\overline{M} = I_h \times_\rho \mathbb{P}$ and let $\varphi_{\mathbb{P}} := \pi_{\mathbb{P}} \circ \psi$ be its projection on $\mathbb{P}$. Then $\varphi_{\mathbb{P}}$ is a local diffeomorphism which is a covering map when ρ is bounded on M.*

The following is a direct consequence of Lemma 3.

Proposition 1 *Let $\overline{M} = I_h \times_\rho \mathbb{P}$ be a doubly warped product spacetime.*

1. *If $\overline{M}$ admits a compact spacelike hypersurface, then $\mathbb{P}$ is compact.*
2. *If the universal covering of $\mathbb{P}$ is compact, then any complete spacelike hypersurface immersed in $\overline{M}$ where ρ is bounded is compact.*

We now use Eq. (16) to deduce the following

Theorem 1 *Let $\overline{M} = I_h \times_\rho \mathbb{P}$ be a spatially closed doubly warped product spacetime, that is, assume that $\mathbb{P}$ is compact. Then, $\overline{M}$ admits a compact maximal hypersurface if and only if it admits a totally geodesic spacelike slice.*

Proof Any totally geodesic spacelike slice of a spatially closed doubly warped product is clearly a maximal compact hypersurface. Vice versa, let $\psi : M \to I_h \times_\rho \mathbb{P}$ be a compact hypersurface with mean curvature H. From (16) and (20) we get

$$\operatorname{div}\left(\rho h^2 \nabla \tau\right) = m\rho h H \cosh\theta - m\rho'. \tag{22}$$

Integrating (22) on a compact hypersurface M we infer the integral formula

$$\int_M \left(\rho h H \cosh\theta - \rho'\right) = \int_M \operatorname{div}(\rho h^2 \nabla \tau) = 0. \tag{23}$$

If M is maximal, then $H \equiv 0$ and (23) reduces to

$$\int_M \rho' = 0.$$

Thus, there must exist $t_0 \in I$ such that $\rho'(t_0) = 0$. Since the spacelike slices are totally umbilic and $\mathcal{H}(t_0, \cdot\,) \equiv 0$, $\Sigma_{t_0} = \{t_0\} \times \mathbb{P}$ is a totally geodesic spacelike slice. □

The above theorem extends to the present ambient spacetimes a result obtained by Choquet-Bruhat in [11] for Robertson–Walker spacetimes (see also Prop. 4.1 in [2]). As a second consequence of Eq. (16) we prove the next rigidity result, which generalizes Theorem 1 of [17] to doubly warped product spacetimes.

Theorem 2 *Let $\overline{M} = I_h \times_\rho \mathbb{P}$ be doubly warped spacetime satisfying $\rho' \geq 0$ on I. If $\psi : M \to \overline{M}$ is a connected compact spacelike hypersurface whose mean curvature in the direction of the future-pointing normal satisfies*

$$H \geq \mathcal{H} \circ \psi = \frac{\rho'}{h\rho} \circ \psi$$

then $\psi(M)$ is a spacelike slice.

Proof Let $\psi : M \to \overline{M}$ be a compact spacelike hypersurface satisfying $H \geq \mathcal{H}$. By Eq. (16), we have

$$\operatorname{div}(\rho h^2 \nabla \tau) = mh\rho(H \cosh\theta - \mathcal{H}) \geq m\rho\mathcal{H}(\cosh\theta - 1) \geq 0 \tag{24}$$

since $\mathcal{H} \geq 0$. Integrating (24) on the compact manifold M and applying the divergence theorem we get

$$0 = \int_M \operatorname{div}(\rho h^2 \nabla \tau)$$

so $\operatorname{div}(\rho h^2 \nabla \tau) = 0$ on M. Then τ is constant on M by the strong maximum principle. This is clearly equivalent to saying that $\psi(M)$ is contained in a slice $\Sigma_{t_0} = \{t_0\} \times \mathbb{P}$ for some $t_0 \in I$. Since M is compact, $\psi(M) \subseteq \Sigma_{t_0}$ is closed. The map $\psi : M \to \Sigma_{t_0}$ is an immersion between manifolds of equal dimension, so it is a local diffeomorphism and therefore an open map. Hence, the nonempty image $\psi(M)$ is both open and closed in the connected slice Σ_{t_0} and we conclude that $\psi(M) = \Sigma_{t_0}$. □

Clearly, the conclusion of Theorem 2 is still true if we replace the assumption $H \geq \mathcal{H} \circ \psi \geq 0$ with $H \leq \mathcal{H} \circ \psi \leq 0$. A similar property holds in the more general case of a spacetime admitting a timelike conformal vector field, provided that the stronger condition $H \geq \mathcal{H} \circ \psi > 0$ is satisfied.

Theorem 3 *Let $(\overline{M}^{m+1}, \overline{g})$ be a spacetime carrying a conformal timelike vector field $\overline{X} \in \mathfrak{X}(M)$ and suppose that $\mathcal{L}_{\overline{X}}\overline{g} = 2\eta\overline{g}$ with $\eta > 0$. Let $\psi : M^m \to \overline{M}$ be a compact immersed spacelike hypersurface whose mean curvature function in the direction of the future-pointing normal satisfies $H \geq \mathcal{H} \circ \psi$. Then $\overline{X}$ is orthogonal to ψ and $H = \mathcal{H} \circ \psi$ on M.*

Proof By Eq. (13), the tangential part X of $\overline{X}$ along ψ satisfies

$$\operatorname{div}(X) = -m\alpha(H \cosh\theta - \mathcal{H}) \leq -m\alpha\mathcal{H}(\cosh\theta - 1) \leq 0, \tag{25}$$

because $\mathcal{H} = \alpha^{-1}\eta > 0$. As in the proof of Theorem 2, an application of the divergence theorem yields $\operatorname{div}(X) \equiv 0$ on M. Since $\mathcal{H} > 0$, by (25) we conclude that $\cosh\theta \equiv 1$ on M. □

Remark 1 Note that in the hypotheses of Theorem 3, the orthogonal distribution of $\overline{X}$ is not assumed to be integrable, that is, the *a priori* existence of immersed hypersurfaces orthogonal to $\overline{X}$ is not assumed. Also note that the conclusion of the theorem is false, in general, if vanishing of η is allowed. For instance, the Lorentzian surface $(\overline{M}, \overline{g})$ obtained by endowing the cylinder $\overline{M} = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ with the Lorentzian metric $\overline{g} = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$ induced by the restriction to $\overline{M}$ of the Lorentz-Minkowski metric of $\mathbb{R}^3$ is foliated by compact spacelike geodesics $\psi_{t_0} : \mathbb{S}^1 \to \overline{M} : \theta \mapsto (\cos\theta, \sin\theta, t_0)$, $t_0 \in \mathbb{R}$. However, $\overline{M}$ carries a family of timelike vector fields

$$\overline{X}_a = \partial_t - ay\partial_x + ax\partial_y, \qquad -1 < a < 1$$

satisfying $\mathcal{L}_{\overline{X}_a}\overline{g} = 0$ for every $-1 < a < 1$, but, for every $t_0 \in \mathbb{R}$, ψ_{t_0} is orthogonal to $\overline{X}_a$ if and only if $a = 0$. In fact, when $a \neq 0$, the maximal codimension 1 spacelike submanifolds orthogonal to $\overline{X}_a$ are the noncompact geodesics $\gamma_{a,t_0} : \mathbb{R} \to \overline{M} : s \mapsto (\cos s, \sin s, as + t_0)$, $t_0 \in \mathbb{R}$.

Note that when $\overline{M}$ is a doubly warped product spacetime, the values of H on a compact spacelike hypersurface naturally relate with those of $\mathcal{H}$ as a consequence of the maximum principle. In fact, we have the following

Theorem 4 *Let $\psi : M \longrightarrow \overline{M}$ be a spacelike hypersurface in a doubly warped product spacetime $\overline{M} = I_h \times_\rho \mathbb{P}$. Suppose that there exist two points p_0 and p^0, respectively, where the height function τ attains local minimum and maximum values. Then,*

$$\mathcal{H}(\psi(p_0)) \leq H(p_0), \qquad \mathcal{H}(\psi(p^0)) \geq H(p^0). \tag{26}$$

Proof Since p_0 and p^0 are locally extremal for τ, at these points we have $\nabla\tau = 0$, so $\rho h^2 \Delta\tau = \operatorname{div}(\rho h^2 \nabla\tau)$ and $X = 0$, implying that $N = \alpha^{-1}\overline{X}$ and therefore $\cosh\theta = 1$. Since p_0 is a local minimum point for τ, we have $\Delta\tau(p_0) \geq 0$ and then

$$0 \leq \frac{\operatorname{div}(\rho h^2 \nabla\tau)}{mh\rho} = H - \mathcal{H} \qquad \text{at } p_0.$$

Similarly, we deduce $0 \geq H - \mathcal{H}$ at p^0. □

When H is constant on M and $\rho \equiv 1$, that is, when $\overline{M}$ is a standard static spacetime, the inequalities (26) are clearly satisfied if and only if $H \equiv 0$.

Corollary 1 *Let $\overline{M} = I_h \times \mathbb{P}$ be a standard static spacetime, $\psi : M \to \overline{M}$ a connected spacelike hypersurface with constant mean curvature. If the height function attains both locally minimal and maximal values on M, then $\psi(M)$ is contained in a totally geodesic spacelike slice.*

Proof Since H is constant on M and $\mathcal{H} \equiv 0$ on $\overline{M}$, by Theorem 4 it must be $H \equiv 0$ on M. So τ satisfies $\Delta_{-\log(h^2)}\tau = 0$ on M and attains a local maximum at some point of M. By the strong maximum principle together with connectedness of M and the unique continuation property for the equation $\Delta_{-\log(h^2)}u = 0$, τ is constant on M. So, $\psi(M)$ is contained in a spacelike slice. □

We conclude by giving different versions of Theorem 2 in the complete noncompact case, replacing compactness of M with different assumptions. The first one relies on the following result due to Alías, Caminha, do Nascimento, see Theorem 2.1 of [1].

Proposition 2 *Let (M, g) be a complete, noncompact Riemannian manifold, $V \in \mathfrak{X}(M)$. Assume that there exists $f \in C^\infty(M)$, $f \geq 0$, $f \not\equiv 0$ such that*

$$g(X, \nabla f) \geq 0 \quad \text{on } M \tag{27}$$

and

$$f(x) \to 0 \quad \text{as } x \to \infty \text{ in } M. \tag{28}$$

If $\operatorname{div}(X) \geq 0$ *then*

$$(i) \;\; \operatorname{div}(X) \equiv 0 \quad \text{on } M \setminus f^{-1}(0) \qquad \text{and} \qquad (ii) \;\; g(X, \nabla f) \equiv 0 \quad \text{on } M. \tag{29}$$

Remark 2 If div is replaced everywhere in Proposition 2 by the weighted divergence operator $\operatorname{div}_\varphi$ defined by $\operatorname{div}_\varphi(X) = \operatorname{div}(X) - g(\nabla\varphi, X) = e^\varphi \operatorname{div}(e^{-\varphi}X)$, with $\varphi \in C^\infty(M)$, then the resulting statement is also true. In fact, it suffices to apply the above Proposition to the vector field $e^{-\varphi}X$.

Theorem 5 *Let $\psi : M \to \overline{M} = I_h \times_\rho \mathbb{P}$ be a spacelike complete, noncompact, connected hypersurface in a doubly warped product spacetime satisfying $\rho' \geq 0$. Let $t_0 \in I$ and suppose that $\psi(M)$ is above the slice Σ_{t_0} and asymptotic to it at infinity. If $H \geq \mathcal{H} \circ \psi$ on M, then $\psi(M) = \Sigma_{t_0}$.*

Proof As in the proof of Theorem 2, the assumption $H \geq \mathcal{H}$ together with Eq. (16) yields

$$\operatorname{div}(\rho h^2 \nabla\tau) = mh\rho(H\cosh\theta - \mathcal{H}) \geq 0$$

for $\tau = \pi_I \circ \psi$ the height function of ψ. Since $\psi(M)$ is above the slice $\Sigma_{t_0} = \{t_0\} \times \mathbb{P}$ we have $\tau \geq t_0$ on M. Furthermore, $\tau(x) \to t_0$ as $x \to \infty$ in M. Thus the function $f = \tau - t_0$ satisfies $f \geq 0$ on M and, for $X = \rho h^2 \nabla\tau$,

$$g(X, \nabla f) = \rho h^2 |\nabla\tau|^2 \geq 0.$$

We reason by contradiction and we suppose that $\psi(M) \neq \Sigma_{t_0}$. Then for some $x \in M$ we have $\varphi(x) > 0$, so that $\varphi \not\equiv 0$ on M. We can thus apply Proposition 2 to deduce that

$$0 \equiv g(X, \nabla f) = \rho h^2 |\nabla\tau|^2 \quad \text{on } M.$$

Since $\rho h^2 > 0$ on M, it follows that $\varphi \equiv 0$ on M, contradiction. □

In the next two results, the role of compactness is played by the parabolicity of a certain operator.

Theorem 6 *Let $\overline{M} = I_h \times_\rho \mathbb{P}$ be a doubly warped product spacetime satisfying $\rho' \geq 0$ on I and let $\psi : M \to \overline{M}$ be a spacelike complete, noncompact hypersurface. Suppose that*

$$\int_1^{+\infty} \left(\int_{\partial B_r(o)} \rho h^2 \right)^{-1} \mathrm{d}r = +\infty \tag{30}$$

for some reference point $o \in M$. If $H \geq \mathcal{H} \circ \psi$ on M and the height function τ is bounded above, then $\psi(M)$ is a spacelike slice.

Proof Under the assumption $H \geq \mathcal{H} \geq 0$, Eq. (17) yields

$$\Delta_{-\log \rho h^2} \tau = \frac{m}{h}(H \cosh \theta - \mathcal{H}) \geq \frac{m\mathcal{H}}{h}(\cosh \theta - 1) \geq 0. \tag{31}$$

Next completeness of M and (30) imply that the operator $\Delta_{-\log \rho h^2}$ is parabolic on M, see Chapter 4 in [3]. Thus since τ is bounded above from (31) we deduce that τ is constant, and this implies that $\psi(M)$ is contained in a spacelike slice. By completeness of M, $\psi(M)$ must in fact be a slice. □

Observe that condition (30) involves both ρ and h. We can get rid of ρ by considering Eq. (18) instead of (17). If $\mathcal{R}$ is an antiderivative of ρ on I, then $\mathcal{R}' = \rho > 0$ and therefore

$$\sup_M \mathcal{R}(\tau) = \lim_{t \to \sup_M \tau} \mathcal{R}(t)$$

when τ is the height function of an immersion $\psi : M \to \overline{M}$. So, for instance, if $\sup_M \tau \in I$ then $\mathcal{R}(\tau)$ is bounded above. By applying the argument used in the proof of Theorem 6 to Eq. (18) we have the following Theorem. Note that condition (32) below is equivalent to (30) if ρ is bounded above and stays away from 0, but otherwise the two seem independent.

Theorem 7 *Let $\overline{M} = I_h \times_\rho \mathbb{P}$ be a doubly warped product spacetime satisfying $\rho' \geq 0$ on I. Let $\psi : M \to \overline{M}$ be a spacelike complete hypersurface such that*

$$\int_1^{+\infty} \left(\int_{\partial B_r(o)} h^2 \right)^{-1} \mathrm{d}r = +\infty \tag{32}$$

for some reference point $o \in M$. If $H \geq \mathcal{H} \circ \psi$ on M and $\sup_M \tau \in I$, then $\psi(M)$ is a spacelike slice.

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Naturally Graded Quasi-Filiform Associative Algebras

I. A. Karimjanov and M. Ladra

Abstract In this paper, we classify naturally graded complex quasi-filiform nilpotent associative algebras described using the characteristic sequence $C(\mathcal{A}) = (n-2, 1, 1)$ or $C(\mathcal{A}) = (n-2, 2)$.

1 Introduction

The classification of associative algebras is an old problem which has appeared periodically throughout the last years. Associative algebras form one class of classical algebras that have been comprehensively studied, and they were found to be related to other classical algebras like Lie and Jordan algebras.

Most classification problems of finite-dimensional associative algebras have been studied when different properties of the algebras hold, but the complete classification of associative algebras, in general, is still an open problem. The theory of finite-dimensional associative algebras is one of the ancient areas of modern algebra. It originates primarily from the work of Hamilton, who discovered the famous quaternions, and Cayley, who developed the theory of matrices. Later, the structural theory of finite-dimensional associative algebras has been treated by several mathematicians, notably B. Peirce, C. S. Peirce, Clifford, Weierstrass, Dedekind, Jordan, Frobenius. At the end of the nineteenth century, T. Molien and E. Cartan described semi-simple algebras over the fields of the complex and real numbers.

The purpose of this paper is to study naturally graded arbitrary dimensional associative algebras of index of nilpotency n and $n-1$. Similar results for Lie and Leibniz algebras were obtained in the works [1–6, 9].

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This paper is organized as follows. In Sect. 2, we provide some basic concepts needed for this study. The last section is devoted to classifying naturally graded complex quasi-filiform nilpotent associative algebras, which are described through the characteristic sequence $C(\mathcal{A}) = (n-2, 1, 1)$ or $C(\mathcal{A}) = (n-2, 2)$.

2 Preliminaries

For an algebra $\mathcal{A}$ of an arbitrary variety, we consider the series

$$\mathcal{A}^1 = \mathcal{A}, \qquad \mathcal{A}^{i+1} = \sum_{k=1}^{i} \mathcal{A}^k \mathcal{A}^{i+1-k}, \qquad i \geq 1.$$

We say that an algebra $\mathcal{A}$ is *nilpotent* if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest positive integer satisfying $\mathcal{A}^i = 0$ is called the *index of nilpotency* or *nilindex* of $\mathcal{A}$.

Definition 1 An n-dimensional algebra $\mathcal{A}$ is called null-filiform if $\dim \mathcal{A}^i = (n+1) - i,\ 1 \leq i \leq n+1$.

It is easy to see that an algebra has a maximum nilpotency index if and only if it is null-filiform. For a nilpotent algebra, the condition of null-filiformity is equivalent to the condition that the algebra is one-generated.

All null-filiform associative algebras were described in [8, Theorem 2.1].

Theorem 1 ([8]) *An arbitrary n-dimensional null-filiform associative algebra is isomorphic to the algebra:*

$$\mu_0^n: \quad e_i e_j = e_{i+j}, \quad 2 \leq i + j \leq n,$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra $\mathcal{A}$ and the omitted products vanish.

Definition 2 An n-dimensional algebra is called filiform if $\dim \mathcal{A}^i = n - i,\ 2 \leq i \leq n$.

Definition 3 An n-dimensional associative algebra $\mathcal{A}$ is called quasi-filiform algebra if $\mathcal{A}^{n-2} \neq 0$ and $\mathcal{A}^{n-1} = 0$.

Definition 4 Given a nilpotent associative algebra $\mathcal{A}$ with nilindex $k+1$, put $\mathcal{A}_i = \mathcal{A}^i / \mathcal{A}^{i+1},\ 1 \leq i \leq k-1$, and $\operatorname{gr} \mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_k$. Then $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ and we obtain the graded algebra $\operatorname{gr} \mathcal{A}$. If $\operatorname{gr} \mathcal{A}$ and $\mathcal{A}$ are isomorphic, denoted by $\operatorname{gr} \mathcal{A} \cong \mathcal{A}$, we say that the algebra $\mathcal{A}$ is naturally graded.

For any element x of $\mathcal{A}$, we define the left multiplication operator as

$$L_x : \mathcal{A} \to \mathcal{A}, \quad z \mapsto xz,\ z \in \mathcal{A}.$$

Let us take $x \in \mathcal{A}\backslash\mathcal{A}^2$ and for the nilpotent left multiplication operator L_x, define the decreasing sequence $C(x) = (n_1, n_2, \ldots, n_k)$ that consists of the dimensions of the Jordan blocks of the operator L_x. Endow the set of these sequences with the lexicographic order, i.e. $C(x) = (n_1, n_2, \ldots, n_k) \leq C(y) = (m_1, m_2, \ldots, m_s)$ means that there is an $i \in \mathbb{N}$ such that $n_j = m_j$ for all $j < i$ and $n_i < m_i$.

Definition 5 The sequence $C(\mathcal{A}) = \max_{x\in\mathcal{A}\backslash\mathcal{A}^2} C(x)$ is defined to be the characteristic sequence of the algebra $\mathcal{A}$.

Definition 6 The left annihilator of $\mathcal{A}$ is denoted by $\mathrm{lAnn}(\mathcal{A}) = \{x \in \mathcal{A} \mid xa = 0 \text{ for all } a \in \mathcal{A}\}$ and the right annihilator of $\mathcal{A}$ is denoted by $\mathrm{rAnn}(\mathcal{A}) = \{x \in \mathcal{A} \mid ax = 0 \text{ for all } a \in \mathcal{A}\}$. The annihilator of $\mathcal{A}$ is $\mathrm{Ann}(\mathcal{A}) = \mathrm{lAnn}(\mathcal{A}) \cap \mathrm{rAnn}(\mathcal{A})$.

Now, we define filiform algebras of degree p.

Definition 7 An n-dimensional associative algebra $\mathcal{A}$ is called filiform of degree p if $\dim \mathcal{A}^i = n - p + 1 - i,\ 1 \leq i \leq n - p + 1$.

The following theorems give the classification of the filiform associative algebras of degree p (see [7, Theorem 3.2 and Theorem 3.2]).

Theorem 2 ([7]) *Let $\mathcal{A}$ be a naturally graded filiform associative algebra of dimension n $(n > p + 2)$ of degree p over a field $\mathbb{F}$ characteristic zero. Then, $\mathcal{A}$ is isomorphic to $\mu_0^{n-p} \oplus \mathbb{F}^p$.*

Theorem 3 ([7]) *Every filiform associative algebra of dimension $n > p + 2$ of degree p over a field $\mathbb{F}$ of characteristic zero is isomorphic to the algebra*

$$\mu' : \begin{cases} e_i e_j = e_{i+j}, & 2 \leq i + j \leq n - p, \\ e_1 f_1 = \alpha e_{n-p}, & \alpha \in \{0, 1\}, \\ f_i f_j = \beta_{i,j} e_{n-p}, & 1 \leq i, j \leq p, \end{cases}$$

where $\beta_{i,j} \in \mathbb{F}$, and the other products vanish.

3 Naturally Graded Quasi-Filiform Associative Algebras

Let $\mathcal{A}$ be a naturally graded n-dimensional quasi-filiform associative algebra. Then, there are two possibilities for the characteristic sequence, either $C(\mathcal{A}) = (n - 2, 1, 1)$ or $C(\mathcal{A}) = (n - 2, 2)$.

3.1 Associative Algebra with Characteristic Sequence (n − 2, 2)

So we start considering the case $C(\mathcal{A}) = (n - 2, 2)$.

According to the definition of the characteristic sequence, $C(\mathcal{A}) = (n-2, 2)$, it follows the existence of a basis element $e_1 \in \mathcal{A} \setminus \mathcal{A}^2$ and a basis $\{e_1, e_2, \dots, e_n\}$ such that the left multiplication operator L_{e_1} has one of the following forms:

$$\begin{pmatrix} J_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}, \qquad \begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}.$$

Let us suppose that the operator L_{e_1} has the first form. Then we have the next multiplications

$$\begin{cases} e_1e_1 = e_2, \\ e_1e_2 = 0, \\ e_1e_i = e_{i+1}, \qquad 3 \le i \le n-1 \\ e_1e_n = 0. \end{cases}$$

From the chain of equalities

$$\begin{aligned} 0 &= (e_1e_2)e_3 = (e_1(e_1e_1))e_3 = ((e_1e_1)e_1)e_3 = (e_2e_1)e_3 = e_2(e_1e_3) \\ &= e_2e_4 = (e_1e_1)e_4 = e_1(e_1e_4) = e_1e_5 = e_6, \end{aligned}$$

we obtain a contradiction.

Thus, we can reduce the study to the following form of the matrix L_{e_1}:

$$\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}.$$

Theorem 4 *Let $\mathcal{A}$ be 5-dimensional complex naturally graded associative algebra with characteristic sequence $C(\mathcal{A}) = (3, 2)$. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\pi_1(\alpha): \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_5, \\ e_4e_1 = \alpha e_5, \ \alpha \in \mathbb{C} \end{cases} \quad \pi_2: \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_4e_1 = e_5, \\ e_4e_4 = e_5 \end{cases} \quad \pi_3: \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_5, \\ e_4e_4 = e_5 \end{cases}$$

$$\pi_4: \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_5, \\ e_4e_1 = e_2 - e_5, \\ e_5e_1 = e_3 \end{cases} \quad \pi_5: \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_5, \\ e_4e_1 = e_2 + e_5, \\ e_4e_2 = 2e_3, \\ e_4e_4 = 2e_5, \\ e_5e_1 = e_3 \end{cases} \quad \pi_6: \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_4e_1 = e_5, \\ e_4e_4 = e_2, \\ e_4e_5 = e_5e_4 = e_3 \end{cases}$$

$$\pi_7 : \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = -e_4e_1 = e_5, \\ e_4e_4 = e_2, \\ e_5e_4 = -e_4e_5 = e_3 \end{cases} \quad \pi_8(\alpha) : \begin{cases} e_1e_1 = e_2, & e_1e_2 = e_2e_1 = e_3, \\ e_1e_4 = e_5, & e_4e_1 = (1-\alpha)e_2 + \alpha e_5, \\ e_4e_2 = (1-\alpha^2)e_3, & e_4e_4 = -\alpha e_2 + (1+\alpha)e_5, \\ e_4e_5 = -\alpha^2 e_3, & e_5e_1 = (1-\alpha)e_3, \\ e_5e_4 = -\alpha e_3, & \alpha \in \mathbb{C}. \end{cases}$$

Proof We can assume that

$$\mathcal{A}_1 = \langle e_1, e_4\rangle, \quad \mathcal{A}_2 = \langle e_2, e_5\rangle, \quad \mathcal{A}_3 = \langle e_3\rangle.$$

By the associativity identities and from the left multiplication operator L_{e_1} we deduce

$$\begin{array}{lll} e_1e_1 = e_2, & e_4e_1 = \alpha_1e_2 + \alpha_2e_5, & e_4e_5 = \alpha_2\beta_1e_3, \\ e_1e_2 = e_2e_1 = e_3, & e_4e_2 = \alpha_1(\alpha_2+1)e_3, & e_5e_1 = \alpha_1e_3, \\ e_1e_4 = e_5, & e_4e_4 = \beta_1e_2 + \beta_2e_5, & e_5e_4 = \beta_1e_3, \end{array}$$

with the following restrictions

$$\alpha_1\beta_2 = \alpha_1^2(\alpha_2+1) + \beta_1(\alpha_2^2 - 1), \quad \beta_1(\alpha_1(\alpha_2+1) + \beta_2(\alpha_2-1)) = 0. \tag{1}$$

Let us suppose $\boxed{e_5 \in \mathrm{Ann}(\mathcal{A})}$. It follows $\alpha_1 = \beta_1 = 0$ and then we have

$$e_1e_1 = e_2, \quad e_1e_2 = e_2e_1 = e_3, \quad e_1e_4 = e_5, \quad e_4e_1 = \alpha_2e_5, \quad e_4e_4 = \beta_2e_5.$$

We make the following change of generator basis elements

$$e_1' = A_1e_1 + A_2e_4, \quad e_4' = B_1e_1 + B_2e_4, \quad A_1(A_1 + \beta_2A_2)(A_1B_2 - A_2B_1) \neq 0.$$

The equality $e_1'e_5' = 0$, implies $B_1 = 0$. Calculating new parameters we obtain:

$$\alpha_2' = \frac{\alpha_2A_1 + \beta_2A_2}{A_1 + \beta_2A_2}, \quad \beta_2' = \frac{\beta_2B_2}{A_1 + \beta_2A_2}.$$

If $\beta_2 = 0$, then $\beta_2' = 0$ and $\alpha_2' = \alpha_2$, and so we obtain $\pi_1(\alpha)$.

If $\beta_2 \neq 0$, then by choosing $B_2 = \frac{A_1+\beta_2A_2}{\beta_2}$ we deduce $\beta_2' = 1$. So we have the next invariant expression

$$\alpha_2' - 1 = \frac{(\alpha_2 - 1)A_1}{A_1 + \beta_2A_2}.$$

- If $\alpha_2 = 1$, then $\alpha_2' = 1$ and we have π_2.
- If $\alpha_2 \neq 1$, then by choosing $A_2 = -\frac{\alpha_2A_1}{\beta_2}$ we infer $\alpha_2' = 0$, and so we have π_3.

Let us suppose $\boxed{e_5 \notin \mathrm{Ann}(\mathcal{A})}$. It follows $(\alpha_1, \beta_1) \neq (0, 0)$. Then we consider the next cases.

Case 1 Let $\boxed{e_5 \in \mathrm{rAnn}(\mathcal{A})}$. Then $\alpha_2\beta_1 = 0$.

Case 1.1 If $\beta_1 = 0$, then $\alpha_1 \neq 0$ and $\beta_2 = \alpha_1(\alpha_2+1)$. Analogously as the previous case, we make the change of generator basis

$$e_1' = A_1e_1 + A_2e_4, \qquad e_4' = B_1e_1 + B_2e_4,$$

$$A_1(A_1 + \alpha_1 A_2)(A_1 + \alpha_1 A_2 + \alpha_1\alpha_2 A_2)(A_1B_2 - A_2B_1) \neq 0,$$

and we find the restriction $B_1 = 0$ on the coefficients.

Calculating new parameters we obtain

$$\alpha_1' = \frac{\alpha_1 B_2}{A_1 + \alpha_1 A_2}, \quad \alpha_2' = \frac{\alpha_2 A_1}{A_1 + \alpha_1 A_2 + \alpha_1\alpha_2 A_2}, \quad \alpha_2' + 1 = \frac{(\alpha_2 + 1)(A_1 + \alpha_1 A_2)}{A_1 + \alpha_1 A_2 + \alpha_1\alpha_2 A_2}.$$

By choosing $B_2 = \frac{A_1+\alpha_1 A_2}{\alpha_1}$ we obtain that $\alpha_1' = 1$.
If $\alpha_2 = 0$ then $\alpha_2' = 0$, and we have $\pi_8(0)$.
If $\alpha_2 \neq 0$ then we consider the next cases:

- If $\alpha_2 = -1$ then $\alpha_2' = -1$ and we obtain π_4.
- If $\alpha_2 \neq -1$ then by choosing $A_2 = \frac{(\alpha_2-1)A_1}{\alpha_1(\alpha_2+1)}$, we deduce $\alpha_2' = 1$ and we have π_5.

Case 1.2 If $\beta_1 \neq 0$ then $\alpha_2 = 0$. By restrictions (1) we obtain that $\beta_2 = \alpha_1$ and $\beta_1 = 0$, that is a contradiction with the condition.

Case 2 Let $\boxed{e_5 \notin \mathrm{rAnn}(\mathcal{A})}$. Then $\alpha_2\beta_1 \neq 0$. Taking the next change of basis

$$e_i' = e_i, \quad 1 \leq i \leq 3, \quad e_4' = \frac{1}{\sqrt{\beta_1}}e_4, \; e_5' = \frac{1}{\sqrt{\beta_1}}e_5,$$

we can assume that $\beta_1 = 1$.

Case 2.1 Let $\alpha_1 = 0$. Then from restriction (1) we have $\begin{cases} \alpha_2^2 - 1 = 0, \\ (\alpha_2 - 1)\beta_2 = 0. \end{cases}$

Case 2.1.1 Let $\alpha_2 = 1$. Then we have

$$e_1e_1 = e_2, \qquad e_1e_2 = e_2e_1 = e_3, \quad e_1e_4 = e_4e_1 = e_5,$$
$$e_4e_4 = e_2 + \beta_2 e_5, \quad e_4e_5 = e_5e_4 = e_3.$$

By making the change of generator basis elements, calculating restrictions on the coefficients and new parameters and applying similar arguments as the previous cases, we obtain $\pi_8(1)$ and π_6.

Case 2.1.2 If $\alpha_2 \neq 1$ then $\alpha_2 = -1$ and $\beta_2 = 0$. Hence we obtain π_7.

Note that the algebra π_6 is commutative and algebra π_7 is noncommutative. Thus, algebras π_6 and π_7 are non-isomorphic.

Case 2.2 Let $\alpha_1 \neq 0$. Then we consider the next cases.

Case 2.2.1 If $\alpha_2 = -1$ then $\beta_2 = 0$. We make the change of generator basis elements

$$e_1' = A_1e_1 + A_2e_4, \quad e_4' = B_1e_1 + B_2e_4, \quad A_1(A_1^2 + \alpha_1A_1A_2 + A_2^2)(A_1B_2 - A_2B_1) \neq 0.$$

Calculating restrictions on the coefficients and new parameters we obtain:

$$B_1 = 0, \quad B_2 = (A_1^2 + \alpha_1A_1A_2 + A_2^2)^{1/2}, \quad \alpha_1' = \frac{\alpha_1A_1 + 2A_2}{(A_1^2 + \alpha_1A_1A_2 + A_2^2)^{1/2}}.$$

It is easy to check that the next expression is an invariant expression

$$(\alpha_1')^2 - 4 = \frac{(\alpha_1^2 - 4)A_1^2}{A_1^2 + \alpha_1A_1A_2 + A_2^2}.$$

If $\alpha_1^2 = 4$ then $\alpha_1' = \pm 2$, and we have $\pi_8(-1)$.

If $\alpha_1^2 \neq 4$ then by choosing $A_2 = -\frac{1}{2}\alpha_1A_1$ we obtain $\alpha_1' = 0$, and it follows we have π_7.

Case 2.2.2 Let $\alpha_2 \neq -1$. Then from restriction (1) we get $\alpha_2 \neq 1$, $\beta_2 \neq 0$ and

$$\alpha_1 = \frac{(1 - \alpha_2)\beta_2}{1 + \alpha_2}, \quad (\alpha_2 + 1)^2 + \alpha_2\beta_2^2 = 0.$$

By making the following change of basis

$$e_i' = e_i, \quad 1 \leq i \leq 3, \quad e_4' = \frac{1 + \alpha_2}{\beta_2}e_4, \quad e_5' = \frac{1 + \alpha_2}{\beta_2}e_5,$$

we obtain $\pi_8(\alpha)$ where $\alpha \notin \{\pm 1, 0\}$.

By making the general change of basis, using the multiplication table on a new basis and calculating new parameters, we obtain that $\alpha' = \alpha$. It means that the obtained algebras are non-isomorphic for different values of α. □

Theorem 5 *Let $\mathcal{A}$ be n-dimensional $(n > 5)$ complex naturally graded associative algebra with the characteristic sequence $C(\mathcal{A}) = (n - 2, 2)$. Then it is isomorphic* *to one of the following pairwise non-isomorphic algebras:*

$$\mu_{2,2}^n(\alpha):\begin{cases} e_ie_j = e_{i+j}, \\ e_1e_{n-1} = e_n, \\ e_{n-1}e_1 = \alpha e_n, \end{cases} \qquad \mu_{2,3}^n:\begin{cases} e_ie_j = e_{i+j}, \\ e_1e_{n-1} = e_n, \\ e_{n-1}e_1 = e_n, \\ e_{n-1}e_{n-1} = e_n, \end{cases} \qquad \mu_{2,4}^n:\begin{cases} e_ie_j = e_{i+j}, \\ e_1e_{n-1} = e_n, \\ e_{n-1}e_{n-1} = e_n, \end{cases}$$

where $\alpha \in \mathbb{C}$ *and* $2 \le i+j \le n-2$.

Proof According to the condition of the theorem, we have the following multiplication:

$$\begin{cases} e_1e_i = e_{i+1}, & 1 \le i \le n-3, \\ e_1e_{n-2} = 0, \\ e_1e_{n-1} = e_n, \\ e_1e_n = 0. \end{cases}$$

It is easily seen that $\mathcal{A}_1 = \langle e_1, e_{n-1}\rangle, \mathcal{A}_2 = \langle e_2, e_n\rangle, \mathcal{A}_i = \langle e_i\rangle$ for $3 \le i \le n-2$. It follows $e_{n-2} \in \operatorname{Ann}(\mathcal{A})$.

Considering the next identities

$$e_1e_2 = e_1(e_1e_1) = (e_1e_1)e_1 = e_2e_1, \quad e_1e_3 = e_1(e_2e_1) = (e_1e_2)e_1 = e_3e_1,$$

we obtain that

$$e_1e_2 = e_2e_1 = e_3, \quad e_1e_3 = e_3e_1 = e_4.$$

Now, we will prove the following equalities by induction on i:

$$e_1e_i = e_ie_1 = e_{i+1}, \quad 1 \le i \le n-3. \tag{2}$$

Obviously, the equality holds for $i = 2, 3$. Let us assume that the equality (2) holds for $2 < i < n-3$, and we shall prove it for $i+1$:

$$e_1e_{i+1} = e_1(e_ie_1) = (e_1e_i)e_1 = e_{i+1}e_1,$$

so the induction proves the equalities (2) for any $i,\ 2 \le i \le n-3$.

From the following chain of equalities

$$\begin{aligned} e_ie_j &= e_i(e_1e_{j-1}) = (e_ie_1)e_{j-1} = e_{i+1}e_{j-1} = \cdots = e_{i+j-2}e_2 \\ &= e_{i+j-2}(e_1e_1) = (e_{i+j-2}e_1)e_1 = e_{i+j}, \end{aligned}$$

we derive that

$$e_i e_j = e_{i+j}, \quad 2 \le i + j \le n - 2.$$

Let us introduce the notations:

$$e_{n-1}e_1 = \gamma_1 e_2 + \alpha_1 e_n, \quad e_{n-1}e_{n-1} = \gamma_2 e_2 + \alpha_2 e_n, \quad e_{n-1}e_n = \beta_1 e_3,$$

$$e_n e_1 = \beta_2 e_3, \quad e_n e_{n-1} = \beta_3 e_3, \quad e_n e_n = \beta_4 e_4.$$

From the identities, we have

Identity		Constraint
$(e_1e_n)e_1 = e_1(e_ne_1)$	$\Longrightarrow$	$\beta_2 = 0,$
$(e_1e_n)e_{n-1} = e_1(e_ne_{n-1})$	$\Longrightarrow$	$\beta_3 = 0,$
$(e_{n-1}e_n)e_1 = e_{n-1}(e_ne_1)$	$\Longrightarrow$	$\beta_1 = 0,$
$(e_1e_{n-1})e_n = e_1(e_{n-1}e_n)$	$\Longrightarrow$	$\beta_4 = 0,$
$(e_1e_{n-1})e_1 = e_1(e_{n-1}e_1)$	$\Longrightarrow$	$\gamma_1 = 0,$
$(e_1e_{n-1})e_{n-1} = e_1(e_{n-1}e_{n-1})$	$\Longrightarrow$	$\gamma_2 = 0.$

Considering the identities

$$\begin{aligned} e_ie_n &= (e_{i-1}e_1)e_n = e_{i-1}(e_1e_n) = 0, \\ e_ne_i &= e_n(e_1e_{i-1}) = (e_ne_1)e_{i-1} = 0, && 2 \le i \le n-4, \\ e_je_{n-1} &= (e_{j-1}e_1)e_{n-1} = e_{j-1}(e_1e_{n-1}) = e_{j-1}e_n = 0, \\ e_{n-1}e_j &= e_{n-1}(e_1e_{j-1}) = (e_{n-1}e_1)e_{j-1} = \alpha_1e_ne_{j-1} = 0, && 2 \le j \le n-3, \end{aligned}$$

we deduce that

$$\begin{cases} e_ie_j = e_{i+j}, & 2 \le i+j \le n-2, \\ e_1e_{n-1} = e_n, \\ e_{n-1}e_1 = \alpha_1e_n, \\ e_{n-1}e_{n-1} = \alpha_2e_n. \end{cases}$$

We make the following general transformation of generator basis elements:

$$e_1' = \sum_{k=1}^{n} A_ke_k, \quad e_{n-1}' = \sum_{k=1}^{n} B_ke_k, \quad A_1(A_1+\alpha_2A_{n-1})(A_1B_{n-1}-A_{n-1}B_1) \neq 0,$$

while the other elements of the new basis are obtained as products of the above elements.

From the next table of multiplications in this new basis

$$e'_1 e'_n = 0, \quad e'_{n-1} e'_1 = \alpha'_1 e'_n, \quad e'_{n-1} e'_{n-1} = \alpha'_2 e'_n,$$

we have the following restrictions on the coefficients:

$$B_i = 0, \quad 1 \leq i \leq n-3,$$

and calculating new parameters we obtain:

$$\alpha'_1 = \frac{\alpha_1 A_1 + \alpha_2 A_{n-1}}{A_1 + \alpha_2 A_{n-1}}, \qquad \alpha'_2 = \frac{\alpha_2 B_{n-1}}{A_1 + \alpha_2 A_{n-1}}.$$

If $\alpha_2 = 0$, then $\alpha'_2 = 0$ and $\alpha'_1 = \alpha_1$, and so we have $\mu^n_{2,2}(\alpha)$.

If $\alpha_2 \neq 0$, then by choosing $B_{n-1} = \frac{A_1 + \alpha_2 A_{n-1}}{\alpha_2}$ we deduce $\alpha'_2 = 1$. It is easy to see that the next expression

$$\alpha'_1 - 1 = \frac{(\alpha_1 - 1) A_1}{A_1 + \alpha_2 A_{n-1}}$$

is an invariant expression.

- If $\alpha_1 = 1$, then $\alpha'_1 = 1$. Hence, we obtain $\mu^n_{2,3}$.
- If $\alpha_1 \neq 1$, then by choosing $A_{n-1} = -\frac{\alpha_1 A_1}{\alpha_2}$, we infer $\alpha'_1 = 0$ and we have $\mu^n_{2,4}$.

□

3.2 Associative Algebra with Characteristic Sequence (n − 2, 1.1)

Now we will consider the case of the characteristic sequence equal to $(n-2, 1, 1)$.

Definition 8 An associative algebra $\mathcal{A}$ is called 2-filiform if $C(\mathcal{A}) = (n-2, 1, 1)$.

By definition of the characteristic sequence, the operator L_{e_1} has in the Jordan form one block J_{n-2} of size $n-2$ and two blocks J_1 (where $J_1 = \{0\}$) of size one.

The possible forms for the operator L_{e_1} are the following:

$$\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_{n-2} & 0 \\ 0 & 0 & J_1 \end{pmatrix}, \qquad \begin{pmatrix} J_{n-2} & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{pmatrix}.$$

Let us suppose that the operator L_{e_1} has the first form. Then we have the next multiplications

$$\begin{cases} e_1e_1 = 0, \\ e_1e_i = e_{i+1}, \quad 2 \le i \le n-2, \\ e_1e_n = 0. \end{cases}$$

From the chain of equalities

$$e_4 = e_1e_3 = e_1(e_1e_2) = (e_1e_1)e_2 = 0$$

we obtain a contradiction.

Thus, we can reduce the study to the following form of the matrix L_{e_1}:

$$\begin{pmatrix} J_{n-2} & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{pmatrix}.$$

So, there exists a basis $\{e_1, e_2, \dots, e_{n-2}, f_1, f_2\}$ such that

$$\begin{aligned} & e_1e_i = e_{i+1}, \quad 1 \le i \le n-3, \\ & e_1f_1 = 0, \\ & e_1f_2 = 0. \end{aligned}$$

Applying arguments similar to Case 1 of Theorem 2 (see [7, Theorem 3.2]), we obtain the products:

$$\begin{aligned} & e_ie_j = e_{i+j}, \qquad 2 \le i+j \le n-2, \\ & e_kf_s = 0, \qquad 1 \le k \le n-2, \quad 1 \le s \le 2. \end{aligned}$$

By the previous multiplication table we have

$$e_1 \in \mathcal{A}_1, \quad e_2 \in \mathcal{A}_2, \ \dots, \ e_{n-2} \in \mathcal{A}_{n-2},$$

but we do not know about the places of the basis $\{f_1, f_2\}$.

Let denote by t_1, t_2 the places of the basis elements $\{f_1, f_2\}$ in the natural graduation corresponding, that is $f_i \in \mathcal{A}_{t_i}$, where $1 \le i \le 2$. Further will denote by $\mu(t_1, t_2)$ the law of the algebra with set $\{t_1, t_2\}$. We can suppose that $1 \le t_1 \le t_2$.

Proposition 1 *Let $\mathcal{A}$ be a naturally graded 2-filiform associative algebra. Then $t_i \le i$ for any $1 \le i \le 2$.*

Note that the case $\mu(1,1)$ is a filiform algebra of degree 2. Thus, it is sufficient to consider the case $\mu(1,2)$. Since the proofs of the next results are carried out by applying the arguments used above we shall omit them.

Proposition 2 *Let $\mathcal{A}$ be a five-dimensional complex naturally graded 2-filiform non-split associative algebra of type $\mu(1,2)$. Then $\mathcal{A}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\lambda_1 : \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_4e_1 = e_5 \end{cases} \qquad \lambda_2 : \begin{cases} e_1e_1 = e_2, \\ e_1e_2 = e_2e_1 = e_3, \\ e_4e_1 = e_5, \\ e_4e_2 = e_5e_1 = e_3 \end{cases}$$

Theorem 6 *Let $\mathcal{A}$ be an n-dimensional ($n > 5$) complex naturally graded 2-filiform non-split associative algebra of type $\mu(1,2)$. Then $\mathcal{A}$ is isomorphic to the following algebra:*

$$\mu_{2,1}^n : \begin{cases} e_ie_j = e_{i+j}, & 2 \le i+j \le n-2, \\ e_{n-1}e_1 = e_n. \end{cases}$$

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Spacelike Hypersurfaces in Conformally Stationary Spacetimes

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Abstract In this paper we obtain new characterizations of totally geodesic hypersurfaces in conformally stationary spacetimes, under the assumption that the future second fundamental form is positive semidefinite. In the compact case, our results will be a consequence of certain integral formulas, while in the complete noncompact case they will be an application of a new maximum principle at infinity for vector fields.

1 Introduction

Our purpose in this paper is to establish certain Bernstein type results for spacelike hypersurfaces in conformally stationary spacetimes. Recall that a spacetime $(\overline{M}, \langle\, ,\rangle)$ is said to be conformally stationary if it possesses a globally defined timelike conformal vector field $K \in \mathfrak{X}(\overline{M})$. In particular, when K is Killing then we say that $\overline{M}$ is a stationary spacetime. The reason for this terminology is due to the fact that $\overline{M}$, endowed with the conformal metric $\langle\, ,\rangle^* = (1/|K|)\langle\, ,\rangle$, is in fact a stationary spacetime, since the timelike field K is Killing for the conformal metric $\langle\, ,\rangle^*$. The class of conformally stationary spacetimes includes the family of generalized Robertson–Walker spacetimes, as defined in [2], and in this case the conformal field K is also closed, in the sense that its metrically equivalent 1-form is closed.

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The study of uniqueness results for spacelike hypersurfaces in conformally stationary spacetimes has been a topic of increasing interest in recent years. A basic question on this topic is the uniqueness of spacelike hypersurfaces with certain natural geometric properties, like the vanishing or constancy of the mean curvature. The first relevant result in this direction was the proof of the Calabi–Bernstein conjecture for maximal hypersurfaces in the Lorentz-Minkowski spacetime, given by Cheng and Yau in [12]. After this seminal result, many other authors have approached several Bernstein type results in other ambient spacetimes, looking for the characterization of totally geodesic hypersurfaces (and, more generally, totally umbilical hypersurfaces) among the class of spacelike hypersurfaces with vanishing or constant mean curvature. For instance, in [2–4] Alías, Romero and Sánchez studied the uniqueness of compact spacelike hypersurfaces with constant mean curvature in conformally stationary spacetimes. See also [5] and [1] for other uniqueness results in the same direction, for the case of compact spacelike hypersurfaces with constant higher order mean curvature, or [10], for the complete noncompact case in the Lorentz-Minkowski space. More recently, and as an application of a generalized version of the Omori-Yau maximum principle, these results were extended in [6] to the case of complete noncompact spacelike hypersurfaces in the family of generalized Robertson–Walker spacetimes (see also Chapter 9 in [7]).

In this paper we obtain new characterizations of totally geodesic hypersurfaces in conformally stationary spacetimes, under the assumption that the future second fundamental form is positive semidefinite. In the compact case, our results will be a consequence of certain integral formulas, while in the complete noncompact case they will be an application of a new maximum principle at infinity for vector fields, recently given by the authors in [8] (see Theorem 3 below).

2 Preliminaries

An $(n+1)$-dimensional $(n \geq 2)$ orientable manifold $\overline{M}^{n+1}$ endowed with a Lorentzian metric tensor $\langle \cdot, \cdot \rangle$ is said to be a conformally stationary spacetime provided it possesses a globally defined timelike conformal vector field $K \in \mathfrak{X}(\overline{M})$. In particular, if K is Killing then we say that $\overline{M}$ is a stationary spacetime. Since K is globally defined on $\overline{M}$, it determines a time-orientation on it; in this situation, we will always consider on $\overline{M}$ the time-orientation given by K.

Recall that K being conformal means that the Lie derivative of the Lorentzian metric $\langle \cdot, \cdot \rangle$ with respect to K satisfies $\mathcal{L}_K \langle \cdot, \cdot \rangle = 2\phi \langle \cdot, \cdot \rangle$ for a certain smooth function $\phi \in C^\infty(\overline{M})$. In other words,

$$\langle \overline{\nabla}_V K, W \rangle + \langle V, \overline{\nabla}_W K \rangle = 2\phi \langle V, W \rangle, \tag{1}$$

for all vector fields $V, W \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}$. The case of K being Killing corresponds to ϕ vanishing identically on $\overline{M}$. It follows from (1) that the function ϕ can be characterized as

$$\phi = \frac{1}{n+1}\mathrm{Div}\ K, \tag{2}$$

where Div stands for the divergence of $\overline{M}$.

Also in this setting, the conformal timelike vector field K is said to be *closed* if its metrically equivalent 1-form is closed. In other words, this means that

$$\overline{\nabla}_V K = \phi V, \tag{3}$$

for every vector field $V \in \mathfrak{X}(\overline{M})$.

From now on, whenever the conformal timelike vector field of $\overline{M}$ is closed, we will say that $\overline{M}$ is a conformally stationary-closed spacetime. Note that this includes the case of generalized Robertson–Walker Lorentz spacetimes. By a generalized Robertson–Walker Lorentz spacetime, and following [2], we mean a Lorentzian warped product $I \times_{\varrho} M^n$ with complete Riemannian fiber $(M^n, \langle\cdot,\cdot\rangle_M)$ and warping function $\varrho : I \to (0, +\infty)$, so that its Lorentzian metric is given by

$$\langle\cdot,\cdot\rangle = -dt^2 + \varrho(t)^2\langle\cdot,\cdot\rangle_M. \tag{4}$$

In a generalized Robertson–Walker spacetime, the vector field given by $K(t,x) = \varrho(t)\partial_t|_{(t,x)}$ is a closed conformal vector field with $\phi(t,x) = \varrho'(t)$.

A smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ of an n-dimensional connected manifold Σ is said to be a spacelike hypersurface if the metric induced on Σ via ψ is a Riemannian metric, which, as usual, we also denote $\langle\cdot,\cdot\rangle$. Thus, for a given spacelike hypersurface $\psi : \Sigma \to \overline{M}$, there exists a unique timelike unit normal vector field N, globally defined on Σ and having the same time-orientation as K, so that $\langle K, N\rangle \le -|K| = -\sqrt{-\langle K, K\rangle} < 0$ holds everywhere on Σ. We will refer to N as the *future pointing* Gauss map of Σ.

If ∇ denotes the Levi-Civita connection of Σ, then the Gauss and Weingarten formulae for the hypersurface in $\overline{M}$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N, \quad \text{and} \quad A(X) = -\overline{\nabla}_X N, \tag{5}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ defines the *future second fundamental form* of Σ.

3 The Case of Compact Spacelike Hypersurfaces

We begin this section by deriving some general formulae for spacelike hypersurfaces in a conformally stationary spacetime $\overline{M}$. In order to do that, let us consider $K^\top \in \mathfrak{X}(\Sigma)$, the vector field obtained on the hypersurface Σ by taking the tangential component of K, that is,

$$K^{\top} = K + \langle K, N\rangle N. \tag{6}$$

Our first idea here is to compute the divergence $\mathrm{div}(K^{\top})$,

$$\mathrm{div}(K^{\top}) = \sum_{i=1}^{n} \langle \nabla_{E_i} K^{\top}, E_i\rangle, \tag{7}$$

where $\{E_1, \dots, E_n\}$ is a local orthonormal frame on Σ. By taking covariant derivatives in (6) and using (5), we obtain from (1) that

$$\frac{1}{2}\left(\langle \nabla_X K^{\top}, Y\rangle + \langle X, \nabla_Y K^{\top}\rangle\right) = \phi\langle X, Y\rangle - \langle K, N\rangle\langle AX, Y\rangle, \tag{8}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Hence, $\langle \nabla_{E_i} K^{\top}, E_i\rangle = \phi - \langle K, N\rangle\langle AE_i, E_i\rangle$ and

$$\mathrm{div}(K^{\top}) = n\phi - \langle K, N\rangle \mathrm{tr}(A) = n\phi + nH\langle K, N\rangle, \tag{9}$$

where

$$H = -\frac{1}{n}\mathrm{trace}(A) \tag{10}$$

is the *future mean curvature* of Σ. The choice of the sign -1 in our definition of H is motivated by the fact that, in doing so, the mean curvature vector is given by $\overrightarrow{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\overrightarrow{H}(p)$ is future-pointing.

On the other hand, for every $X \in \mathfrak{X}(\Sigma)$ we have

$$X(\langle K, N\rangle) = \langle \overline{\nabla}_X K, N\rangle + \langle K, \overline{\nabla}_X N\rangle = -\langle X, (\overline{\nabla}_N K)^{\top}\rangle - \langle AK^{\top}, X\rangle, \tag{11}$$

where we have used (1) to deduce that $\langle \overline{\nabla}_X K, N\rangle + \langle X, \overline{\nabla}_N K\rangle = 2\phi\langle X, N\rangle = 0$. At this point, assume that the timelike conformal vector field K is closed, that is, that (3) holds. Then, $(\overline{\nabla}_N K)^{\top} = 0$ and (11) becomes

$$\nabla\langle K, N\rangle = -AK^{\top}. \tag{12}$$

Therefore, using (9) and (12) we can compute

$$\begin{aligned} \mathrm{div}(\langle K, N\rangle K^{\top}) &= \langle K, N\rangle \mathrm{div}(K^{\top}) + \langle \nabla\langle K, N\rangle, K^{\top}\rangle \\ &= n\phi\langle K, N\rangle + nH\langle K, N\rangle^2 - \langle AK^{\top}, K^{\top}\rangle. \end{aligned} \tag{13}$$

We are now ready to present our first main result. Hereafter, recall that a differentiable manifold is said to be *closed* if it is compact and without boundary.

Theorem 1 *Let $\overline{M}$ be a conformally stationary-closed spacetime, with closed timelike conformal field K, and let Σ be a closed spacelike hypersurface immersed into $\overline{M}$. If* $\mathrm{Div} K \geq 0$ *on Σ and the future second fundamental form of Σ is positive semidefinite, then Σ is totally geodesic in $\overline{M}$ and* $\mathrm{Div} K \equiv 0$ *on Σ.*

As a direct consequence of the theorem above, we obtain the following

Corollary 1 *Let $\overline{M}$ be a conformally stationary-closed spacetime endowed with a closed timelike conformal field K such that* $\mathrm{Div} K > 0$ *on $\overline{M}$. There exists no closed spacelike hypersurface immersed into $\overline{M}$ with positive semidefinite future second fundamental form.*

On the other hand, observe that when K is Killing, being closed is equivalent to being parallel. Therefore, Theorem 1 for the case of stationary-closed spacetimes reads as follows.

Corollary 2 *Let $\overline{M}$ be a stationary-closed spacetime endowed with a parallel timelike vector field K, and let Σ be a closed spacelike hypersurface immersed into $\overline{M}$. If the future second fundamental form of Σ is positive semidefinite, then Σ is totally geodesic.*

Proof of Theorem 1 Orient $\overline{M}$ and use the fact that K is timelike to endow Σ with the induced orientation. Letting $d\Sigma$ denote the corresponding volume form and integrating (13) on Σ, the divergence theorem gives

$$0 = \int_\Sigma [n\phi\langle K, N\rangle + nH\langle K, N\rangle^2 - \langle AK^\top, K^\top\rangle] d\Sigma.$$

Now, since $\mathrm{Div} K \geq 0$ on Σ, relation (2) gives $\phi \geq 0$ on Σ. Then, the positive semidefiniteness of A, together with $\langle K, N\rangle < 0$ on Σ, assure that the above integrand is nonpositive on Σ, so that it must vanish identically. In turn, this shows that $\phi \equiv 0$ and $H \equiv 0$ on Σ, whence $\mathrm{Div} K \equiv 0$ on Σ and $A \equiv 0$ on Σ. □

As a consequence of Theorem 1 in the particular case where $\overline{M} = -I \times_\varrho M^n$ is a generalized Robertson–Walker spacetime, we obtain the following characterization of totally geodesic slices. First of all, recall from [2, Proposition 3.2 (i)] that if a generalized Robertson–Walker spacetime admits a closed spacelike hypersurface, then it must be *spatially closed*, in the sense that the Riemannian fiber M^n must be closed. On the other hand, observe that, for a spatially closed generalized Robertson–Walker spacetime $-I \times_\varrho M^n$, the family of slices $\Sigma_t = \{t\} \times M^n$ constitutes a foliation of the ambient space by closed spacelike totally umbilical leaves with constant mean curvature $\mathcal{H}(t) = \varrho'(t)/\varrho(t)$.

Corollary 3 *Let $\overline{M} = -I \times_\varrho M^n$ be a spatially closed generalized Robertson–Walker spacetime such that its warping function ϱ satisfies the condition $\varrho'(t) \geq 0$, with equality only at isolated points. The only closed spacelike hypersurfaces* *immersed into* $-I \times_{\varrho} M^n$ *with positive semidefinite future second fundamental form are the totally geodesic slices of the form* $\Sigma_{t_0} = \{t_0\} \times M^n$, *with* $\varrho'(t_0) = 0$.

Proof The proof of Theorem 1 shows that $\phi \equiv 0$ on Σ. Hence $\varrho' \equiv 0$ on Σ, and our hypothesis on the zeroes of ϱ' guarantee that Σ is a totally geodesic slice of the stated form. □

We assume henceforth that K is a timelike conformal vector field, though not necessarily closed. In this case, using (9) and (11) we can compute

$$\begin{aligned}\operatorname{div}(\langle K, N\rangle K^{\top}) &= \langle K, N\rangle \operatorname{div}(K^{\top}) + \langle \nabla\langle K, N\rangle, K^{\top}\rangle \\ &= n\phi\langle K, N\rangle + nH\langle K, N\rangle^2 - \langle AK^{\top}, K^{\top}\rangle - \langle(\overline{\nabla}_N K)^{\top}, K^{\top}\rangle.\end{aligned} \tag{14}$$

Now, observe that

$$\langle(\overline{\nabla}_N K)^{\top}, K^{\top}\rangle = \langle\overline{\nabla}_N K, K\rangle + \langle K, N\rangle\langle\overline{\nabla}_N K, N\rangle = \frac{1}{2}N(\langle K, K\rangle) - \langle K, N\rangle\phi,$$

where we have used the fact that, by (1), $\langle\overline{\nabla}_N K, N\rangle = -\phi$. Inserting this into (14) we get

$$\operatorname{div}(\langle K, N\rangle K^{\top}) = (n+1)\phi\langle K, N\rangle + nH\langle K, N\rangle^2 - \langle AK^{\top}, K^{\top}\rangle - \frac{1}{2}N(\langle K, K\rangle). \tag{15}$$

This allows us to state our second main result.

Theorem 2 *Let* $\overline{M}$ *be a conformally stationary spacetime endowed with a timelike conformal vector field* K *such that* $\langle K, K\rangle$ *is constant, and let* Σ *be a closed spacelike hypersurface immersed into* $\overline{M}$. *If* $\operatorname{Div}K \geq 0$ *on* Σ *and the future second fundamental form of* Σ *is positive semidefinite, then* Σ *is totally geodesic in* $\overline{M}$ *and* $\operatorname{Div}K \equiv 0$ *on* Σ.

This result immediately yields the following

Corollary 4 *Let* $\overline{M}$ *be a conformally stationary spacetime endowed with a timelike conformal vector field* K *such that* $\langle K, K\rangle$ *is constant and* $\operatorname{Div}K > 0$ *on* $\overline{M}$. *There exists no closed spacelike hypersurface immersed into* $\overline{M}$ *with positive semidefinite future second fundamental form.*

The proof of Theorem 2 is the same as that of Theorem 1, after observing that $\langle K, K\rangle$ being constant implies, by (15),

$$\operatorname{div}(\langle K, N\rangle K^{\top}) = (n+1)\phi\langle K, N\rangle + nH\langle K, N\rangle^2 - \langle AK^{\top}, K^{\top}\rangle.$$

Finally, Theorem 2 for the case of stationary spacetimes reads as follows.

Corollary 5 *Let $\overline{M}$ be a stationary spacetime endowed with a unit timelike Killing vector field K, and let Σ be a closed spacelike hypersurface immersed into $\overline{M}$. If the future second fundamental form of Σ is positive semidefinite, then Σ is totally geodesic.*

Corollary 5 is nothing but the corresponding version of Theorem 3.2 in [8] for the case of closed spacelike hypersurfaces (see also Theorem 5 below). Observe that the hypothesis in Theorem 3.2 (and in Corollary 3.4) of [8] about the spacelike hypersurface being transversal to the unit timelike Killing field K is redundant, since one gets it for free from the fact that Σ is spacelike.

4 The Case of Complete Noncompact Spacelike Hypersurfaces

In this section, we extend the results of the previous one to complete, noncompact spacelike hypersurfaces. We first recall Proposition 2.1 of [11]. To this end, given an oriented, complete noncompact Riemannian manifold Σ, we let $\mathcal{L}^1(\Sigma)$ denote the space of Lebesgue integrable functions on Σ. If $X \in \mathfrak{X}(\Sigma)$ is such that $|X| \in \mathcal{L}^1(\Sigma)$ and $\operatorname{div}(X)$ does not change sign on Σ, then Proposition 2.1 of [11] shows that $\operatorname{div}(X) \equiv 0$ on Σ. The coming result is a useful variation of this.

Theorem 3 (Theorem 2.2 of [8]) *Let Σ be an oriented, complete noncompact Riemannian manifold, and let $X \in \mathfrak{X}(\Sigma)$ be a vector field on Σ. Assume that there exists a nonnegative, non identically vanishing function $f \in C^\infty(\Sigma)$ such that $\langle \nabla f, X\rangle \geq 0$ and $\lim_{r(x)\to+\infty} f(x) = 0$, where $r(x) = d(x, o)$ stands for the Riemannian distance function on Σ, from a fixed origin $o \in \Sigma$. If $\operatorname{div}(X) \geq 0$ on Σ, then:*

(a) $\langle \nabla f, X\rangle \equiv 0$ *on* Σ.
(b) $\operatorname{div}(X) \equiv 0$ *on* $\Sigma \setminus f^{-1}(0)$.
(c) $\operatorname{div}(X) \equiv 0$ *on* Σ *if* $f^{-1}(0)$ *has zero Lebesgue measure.*

As already hinted, we want to use the previous versions of maximum principle to establish results similar to those of the previous section for an oriented, complete noncompact spacelike immersed hypersurface Σ of a conformally-closed stationary spacetime $\overline{M}$. If K stands for the closed conformal timelike vector field of $\overline{M}$, and N for the unit normal vector field on Σ having the same time-orientation as K, we consider the hyperbolic angle-function $\theta : \Sigma \to [0, +\infty)$, given at each point of Σ by the equality

$$\langle K, N\rangle = -|K| \cosh\theta.$$

In particular, for $x \in \Sigma$ we have from the reverse Cauchy–Schwarz inequality that $\theta(x) = 0$ if and only if $K^\top(x) = 0$.

It is also worth recalling (cf. Proposition 1 of [14]) that, thanks to the closed conformal character of K, the distribution $K^{\perp}$ of vector fields $X \in \mathfrak{X}(\overline{M})$ such that $\langle K, X \rangle = 0$ is smooth and integrable. If $\mathcal{F}$ is a leaf of it, then ϕ and $|K|$ are constant on $\mathcal{F}$ and the second fundamental form of $\mathcal{F}$ in the direction of $K/|K|$ is $-(\phi/|K|)\mathrm{Id}$. In particular, $\mathcal{F}$ is totally umbilical in $\overline{M}$, with constant future mean curvature and equal to $\phi/|K|$. We are finally in position to state the following

Theorem 4 *Let $\overline{M}$ be a conformally stationary-closed spacetime with closed timelike conformal vector field K, and let Σ be a complete noncompact spacelike hypersurface immersed into $\overline{M}$. Assume that $|K|$ is bounded above on Σ and the hyperbolic angle θ between K and N satisfies* $\lim_{r(x)\to+\infty} \theta(x) = 0$, *where $r(x) = d(x, o)$ stands for the Riemannian distance function on Σ measured from a fixed origin $o \in \Sigma$. If* $\mathrm{Div} K \geq 0$ *on Σ and the future second fundamental form of Σ is positive semidefinite, then Σ is totally geodesic in $\overline{M}$ and* $\mathrm{Div} K \equiv 0$ *on Σ.*

As before, this immediately gives a corresponding nonexistence result.

Corollary 6 *Let $\overline{M}$ be a conformally stationary-closed spacetime with closed timelike conformal vector field K, such that $|K|$ is bounded and* $\mathrm{Div} K > 0$ *on $\overline{M}$. There exists no complete noncompact spacelike hypersurface Σ immersed into $\overline{M}$ and satisfying the following two conditions:*

(a) *Σ has positive semidefinite future second fundamental form.*
(b) *The hyperbolic angle θ between K and the future pointing unit normal vector field N on Σ satisfies* $\lim_{r(x)\to+\infty} \theta(x) = 0$, *where $r(x) = d(x, o)$ stands for the Riemannian distance function on Σ measured from a fixed origin $o \in \Sigma$.*

Proof of Theorem 4 Firstly, if $\theta \equiv 0$ on Σ, then $N \parallel K$ and Σ is contained in a leaf of $K^{\perp}$. Therefore, as observed above, $A = -(\phi/|K|)\mathrm{Id}$, and the fact that A is positive semidefinite gives $\phi \leq 0$ on Σ. On the other hand, (2) gives $\phi \geq 0$ on Σ, so that $\phi \equiv 0$ on Σ and Σ is totally geodesic.

We then assume that θ does not vanish identically on Σ, and shall apply the previous proposition to $X = K^{\top}$ and $f = -|K| - \langle K, N \rangle$. To this end, start by letting Σ have the orientation induced by that of $\overline{M}$ and K, and notice that $f = |K|\big(\cosh\theta - 1\big) \geq 0$ on Σ. Since $\lim_{r(x)\to+\infty} \theta(x) = 0$ and $|K|$ is bounded on Σ, we get $\lim_{r(x)\to+\infty} f(x) = 0$. We now compute

$$2|K|X(|K|) = X(|K|^2) = -X\langle K, K\rangle = -2\langle \overline{\nabla}_X K, K\rangle = -2\phi\langle X, K\rangle = -2\phi\langle K^{\top}, K^{\top}\rangle,$$

so that, with the aid of (11),

$$|K|\langle \nabla f, X\rangle = -|K|X(|K|) - |K|X\langle K, N\rangle = \phi\langle K^{\top}, K^{\top}\rangle + |K|\langle AK^{\top}, K^{\top}\rangle. \tag{16}$$

Our hypotheses guarantee that the right hand side of the above expression is nonnegative on Σ, so that $\langle \nabla f, X\rangle \geq 0$ on Σ. On the other hand, relation (2), the hypothesis on A and (9) give $\mathrm{div}(X) = n\phi + nH\langle K, N\rangle \geq 0$ on Σ.

Theorem 3 thus gives $\langle \nabla f, X\rangle \equiv 0$ on Σ and $\mathrm{div}(X) \equiv 0$ on $\Sigma \setminus f^{-1}(0)$. In turn, this assures that $\phi \equiv 0$ and $H \equiv 0$ on $\Sigma \setminus f^{-1}(0)$, and the positive semidefinite character of A shows that $A \equiv 0$ on $\Sigma \setminus f^{-1}(0)$. Now, notice that

$$f^{-1}(0) = \theta^{-1}(0) = \{x \in \Sigma;\, K(x) \perp T_x\Sigma\}.$$

Thus, if x lies in the interior of $f^{-1}(0)$, then $N \parallel K$ in a neighbourhood of x, whence, in such a neighbourhood, Σ is contained in the leaf of $K^{\perp}$ passing through x. This in particular gives, as in the first paragraph of the proof, $A \equiv 0$ and $\phi \equiv 0$ in that neighbourhood. Thus, A and ϕ vanish both in $\Sigma \setminus f^{-1}(0)$ and in the interior of $f^{-1}(0)$, so that they vanish on Σ. □

On the other hand, assume now that K is a timelike conformal vector field, not necessarily closed, with $|K|$ constant. In this case, the condition that $\lim_{r(x)\to+\infty} \theta(x) = 0$ is equivalent to the fact that N converges to K at infinity. Without loss of generality, we may assume that $\langle K, K\rangle = -1$. It then follows from (1) that K is necessarily Killing, since

$$0 = K(\langle K, K\rangle) = 2\phi\langle K, K\rangle = -2\phi$$

implies that $\phi \equiv 0$. This allows us to extend Theorem 3.2 of [8] to the case of conformally stationary spacetimes in the following way.

Theorem 5 *Let $\overline{M}$ be a conformally stationary spacetime endowed with a unit timelike conformal vector field K. Then K is necessarily Killing. Assume that Σ is a complete noncompact spacelike hypersurface immersed into $\overline{M}$ with positive semidefinite future second fundamental form and such that N converges to K at infinity, then Σ is totally geodesic in $\overline{M}$.*

For the sake of completeness and for the reader convenience, we include here a detailed proof of Theorem 5, which was not given in [8].

Proof As observed above, K is necessarily Killing. Setting $f = -1 - \langle K, N\rangle = -1 + \cosh\theta$ on Σ, we get $f \geq 0$. If f vanishes identically, then $N \equiv K$. In such a case, for every $x \in \Sigma$ and every $u, v \in T_x\Sigma$, Eq. (1) implies

$$\langle A_x u, v\rangle = -\langle \overline{\nabla}_u N, v\rangle = -\langle \overline{\nabla}_u K, v\rangle = \langle \overline{\nabla}_v K, u\rangle = -\langle A_x v, u\rangle. \tag{17}$$

Since A_x is symmetric, this gives $A_x = 0$ for every $x \in \Sigma$; that is, Σ is totally geodesic. Therefore, we may assume that f does not vanish identically on Σ; in other words $\Sigma \setminus f^{-1}(0) \neq \emptyset$. As observed above, the condition that N converges to K at infinity means that $\lim_{r(x)\to+\infty} f(x) = 0$.

Let $X = K^{\top}$. From (9) we have

$$\mathrm{div}X = nH\langle K, N\rangle \geq 0, \tag{18}$$

since $\phi \equiv 0$ and $H \le 0$ from the hypothesis on A. On the other hand, from (11) we also have

$$\begin{aligned}\langle \nabla f, X\rangle &= -X(\langle K, N\rangle) = \langle AX, X\rangle + \langle X, \overline{\nabla}_N K\rangle \\ &= \langle AX, X\rangle + \langle K, \overline{\nabla}_N K\rangle + \langle K, N\rangle\langle N, \overline{\nabla}_N K\rangle.\end{aligned}$$

Observe that

$$\langle K, \overline{\nabla}_N K\rangle = \frac{1}{2}N(\langle K, K\rangle)= 0,$$

since we are assuming $\langle K, K\rangle = -1$. Moreover, by (1) we have $\langle N, \overline{\nabla}_N K\rangle = -\phi = 0$, which yields

$$\langle \nabla f, X\rangle = \langle AX, X\rangle. \tag{19}$$

Therefore, we may apply Theorem 3 to conclude that $\langle \nabla f, X\rangle \equiv 0$ on Σ and $\mathrm{div} X \equiv 0$ on $\Sigma \setminus f^{-1}(0)$. By (18) and the second conclusion, we have that $H \equiv 0$ on $\Sigma \setminus f^{-1}(0)$, and the positive semidefinite character of A yields $A \equiv 0$ on $\Sigma \setminus f^{-1}(0)$. Now, recall that $x \in f^{-1}(0)$ if and only if $N(x) = K(x)$. If the interior of $f^{-1}(0)$ is empty, by continuity we have $A \equiv 0$ on Σ, and we are done. On the other hand, if the interior of $f^{-1}(0)$ is not empty, then $N \equiv K$ in the interior of $f^{-1}(0)$ and (17) gives $A_x = 0$ for every x in the interior of $f^{-1}(0)$. Thus, A vanishes both on $\Sigma \setminus f^{-1}(0)$ and on the interior of $f^{-1}(0)$, so that $A \equiv 0$ on Σ. This finishes the proof. □

5 The Case of Higher Order Mean Curvatures

In this last section we continue to investigate complete noncompact spacelike hypersurfaces Σ^n of a conformally stationary-closed spacetime $\overline{M}^{n+1}$, this time focusing on higher order mean curvatures.

As before, we orient and time orient $\overline{M}$, endow Σ with the induced orientation and let $N \in \mathfrak{X}(\Sigma)^\perp$ be the future pointing unit normal vector field with (future) second fundamental form A. For every $1 \le k \le n$, $k \in \mathbb{N}$, let $S_k : \Sigma \to \mathbb{R}$ denote the smooth function that associates to each $p \in \Sigma$ the $k-$th elementary symmetric function of the eigenvalues of $A_p : T_p\Sigma \to T_p\Sigma$. Letting $S_0 = 1$ and I stand for the identity operator, we set $P_0 = I$ (the identity operator) and, for $1 \le k \le n$,

$$P_k = (-1)^k S_k I + A P_{k-1}. \tag{20}$$

A trivial induction shows that

$$P_k = (-1)^k (S_k I - S_{k-1} A + S_{k-2} A^2 - \cdots + (-1)^k A^k), \tag{21}$$

so that Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_k is a polynomial in A for every k, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM, diagonalizing A at $p \in M$, also diagonalize all of the P_k at p.

With the aid of such a basis, it is a standard fact (cf. [5] or [9], for instance) that, for $0 \leq k < n$ and upon the convention that $S_{n+1} = 0$,

$$\mathrm{tr}(P_k) = (-1)^k(n-k)S_k, \quad \mathrm{tr}(AP_k) = (-1)^k(k+1)S_{k+1}$$

and

$$\mathrm{tr}(A^2P_k) = (-1)^k\big(S_1S_{k+1} - (k+2)S_{k+2}\big).$$

For $0 \leq k \leq n$, the $k-$th mean curvature H_k of Σ is defined by

$$\binom{n}{k}H_k = (-1)^kS_k.$$

According to (10), H_1 is simply the (future) mean curvature of Σ. Also, a simple computation gives

$$\mathrm{tr}(P_k) = c_kH_k, \qquad \mathrm{tr}(AP_k) = -c_kH_{k+1}, \tag{22}$$

with $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$, and

$$\mathrm{tr}(A^2P_k) = n\binom{n}{k+1}H_1H_{k+1} - n\binom{n-1}{k+1}H_{k+2}. \tag{23}$$

Also concerning higher order mean curvatures, item (c) of Proposition 2.3 of [10] shows that, if $H_k(p) = H_{k+1}(p) = 0$ for some $1 \leq k < n$ and $p \in \Sigma$, then $H_j(p) = 0$ for every $k \leq j \leq n$. In particular, the second fundamental form A has at most $k-1$ nonzero eigenvalues at p.

Following the computations in [5], we shall now compute $\mathrm{div}(P_kK^\top)$ in the more general setting of K being only conformal (i.e., not necessarily closed). To this end, we fix $p \in \Sigma$ and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field on Σ, geodesic at p and diagonalizing A at p. Then, we get at p

$$\begin{aligned}\mathrm{div}(P_kK^\top) &= \sum_{i=1}^n\langle\nabla_{E_i}P_kK^\top, E_i\rangle = \sum_{i=1}^n\langle(\nabla_{E_i}P_k)K^\top + P_k(\nabla_{E_i}K^\top), E_i\rangle \\ &= \langle\sum_{i=1}^n(\nabla_{E_i}P_k)E_i, K^\top\rangle + \sum_{i=1}^n\langle P_kE_i, \nabla_{E_i}K^\top\rangle\end{aligned}$$

If $\overline{M}$ is a Lorentzian space form, i.e., if it has constant sectional curvature, it follows from Lemma 3.1 of [5] that $\sum_{i=1}^n(\nabla_{E_i}P_k)E_i = 0$. Therefore, writing $K^\top = K +$

$\langle K, N\rangle N$, we obtain at p

$$\begin{aligned}\operatorname{div}(P_k K^\top) &= \sum_{i=1}^n \langle P_k E_i, \nabla_{E_i} K\rangle + \sum_{i=1}^n \langle P_k E_i, \nabla_{E_i}\big(\langle K, N\rangle N\big)\rangle \\ &= \frac{1}{2}\sum_{i=1}^n \big(\langle P_k E_i, \nabla_{E_i} K\rangle + \langle E_i, \nabla_{P_k E_i} K\rangle\big) + \langle K, N\rangle \sum_{i=1}^n \langle P_k E_i, \nabla_{E_i} N\rangle \\ &= \sum_{i=1}^n \phi\langle P_k E_i, E_i\rangle - \langle K, N\rangle \sum_{i=1}^n \langle P_k E_i, A E_i\rangle \\ &= \phi\,\operatorname{tr}(P_k) - \langle K, N\rangle \operatorname{tr}(A P_k).\end{aligned}$$

Thanks to (22), this gives

$$\operatorname{div}(P_k K^\top) = c_k\big(\phi\, H_k + \langle K, N\rangle H_{k+1}\big). \tag{24}$$

With the previous preliminaries at our disposal, we have the following extension of Theorem 4.

Theorem 6 *Let $\overline{M}^{n+1}$ be a Lorentzian space form with no compact, proper, totally geodesic submanifold of positive dimension, let K be a closed timelike conformal vector field on $\overline{M}$ and Σ be a connected, complete noncompact spacelike hypersurface immersed into $\overline{M}$, such that $|K|$ is bounded and* $\operatorname{Div} K \geq 0$ *on Σ. Let A stand for the future second fundamental form of Σ and assume that, for some $0 \leq k < n$, $k \in \mathbb{N}$, we have P_k positive definite and $A P_k$ positive semidefinite on Σ. If the hyperbolic angle θ between K and N satisfies* $\lim_{r(x)\to+\infty}\theta(x) = 0$, *where $r(x) = d(x, o)$ stands for the Riemannian distance function on Σ from a fixed origin $o \in \Sigma$, then:*

(a) $k = 0$ *and* $\operatorname{Div} K \equiv 0$ *on* Σ.
(b) Σ *is totally geodesic on* $\overline{M}$.

Proof As in the proof of Theorem 4, if $\theta \equiv 0$ on Σ, then $N \parallel K$ and Σ is contained in a leaf of $K^\perp$. Therefore, $A = -\frac{\phi}{|K|}\mathrm{Id}$, whence Σ is totally umbilical with $H_j = (\phi/|K|)^j$ for every $j \geq 0$. A simple induction using (20) shows that $P_k = \frac{c_k\phi^k}{n|K|^k} I$. In turn, this furnishes $A P_k = -\frac{c_k\phi^{k+1}}{n|K|^{k+1}} I$. Now, Eq. (2) gives $\phi \geq 0$ on Σ, and the fact that P_k is positive definite (hence $H_k > 0$) and $A P_k$ is positive semidefinite, assure that $k = 0$ and $\phi \equiv 0$ on Σ. Hence, Σ is totally geodesic.

Our objective now is to see that θ must vanish identically. We then assume that θ does not vanish identically on Σ, and shall once more apply Theorem 3, though this time to $X = P_k K^\top$ and $f = -|K| - \langle K, N\rangle$. First note that, as in the proof of Theorem 4, the facts that $f = |K|\big(\cosh\theta - 1\big) \geq 0$, $\lim_{r(x)\to+\infty}\theta(x) = 0$ and $|K|$ is bounded on Σ, give $\lim_{r(x)\to+\infty} f(x) = 0$. On the other hand, since $\phi \geq 0$ and $H_k > 0 \geq H_{k+1}$ (for $-c_k H_{k+1} = \operatorname{tr}(A P_k) \geq 0$), relation (24) yields $\operatorname{div}(X) \geq 0$.

Also, reworking the steps that led to (16) (with $X = P_k K^\top$ instead of $X = K^\top$), the hypotheses on P_k and AP_k give

$$|K|\langle \nabla f, X\rangle = \phi\langle P_k K^\top, K^\top\rangle + |K|\langle AP_k K^\top, K^\top\rangle \geq 0,$$

whence $\langle \nabla f, X\rangle \geq 0$.

Theorem 3 thus gives that $\operatorname{div}(X) = 0$ on $\Sigma \setminus f^{-1}(0)$. Therefore, it follows from (24) that $\phi \equiv 0$ and $H_{k+1} \equiv 0$ on $\Sigma \setminus f^{-1}(0)$; hence, $AP_k \equiv 0$ on $\Sigma \setminus f^{-1}(0)$. However, since $\theta \equiv 0$ on the interior of $f^{-1}(0)$, the same argument as in the first paragraph of this proof (applied to the interior of $f^{-1}(0)$, instead of Σ) gives $\phi \equiv 0$ and $A \equiv 0$ the interior of $f^{-1}(0)$. Hence, $\phi \equiv 0$ on Σ and $AP_k \equiv 0$ also on the interior of $f^{-1}(0)$, so that $AP_k \equiv 0$ on Σ.

If $k = n - 1$, then the relative nullity ν of Σ is identically 1. If $k \leq n - 2$, then $AP_k = 0$ implies $A^2 P_k = 0$. Thus, (23) gives $H_{k+2} = 0$ and, as we have previously observed, the equalities $H_{k+1} = H_{k+2} = 0$ furnish $H_j \equiv 0$ on Σ, for every $k < j \leq n$, whence $\nu \geq n - k$ on Σ. However, since $H_k > 0$ on Σ, we conclude that $\nu \equiv n - k$ on Σ. Hence, a theorem of Ferus (cf. [13], Chapter 5) assures that the distribution of relative nullity is smooth and integrable, and the leaves are complete and totally geodesic in Σ and in $\overline{M}$. By hypothesis, all of such leaves are noncompact. Since $\phi \equiv 0$ on Σ and the leaves are totally geodesic in $\overline{M}$, a simple computation shows that f is constant along each leaf. Hence, the only way by which we could have $\lim_{r(x)\to+\infty} f(x) = 0$ is if $f \equiv 0$. In turn, this implies $\theta \equiv 0$, which we are assuming not to be the case. □

Remark 1 Note that $P_n = 0$ is never positive definite. This shows why one has to rule out the case $k = n$, in the previous result.

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Geodesic Completeness and the Quasi-Einstein Equation for Locally Homogeneous Affine Surfaces

P. B. Gilkey and X. Valle-Regueiro

Abstract Let $\mathcal{M}$ be a Type $\mathcal{A}$ affine surface. We show that $\mathcal{M}$ is linearly strongly projectively flat. We use the quasi-Einstein equation together with the condition that $\mathcal{M}$ is strongly projectively flat to examine the geodesic completeness of $\mathcal{M}$.

Keywords Strongly projectively flat · Quasi-Einstein equation · Geodesic completeness · Locally homogeneous affine surface

Subject Classification 53C21, 35R01, 58J60, 58D27

1 Affine Geometry

A pair $\mathcal{M} = (M, \nabla)$ is said to be an *affine surface* if ∇ is a torsion free connection on the tangent bundle of a smooth surface M. A map from one affine surface to another is said to be an *affine map* if it intertwines the two connections. An affine surface is said to be *locally homogeneous* if given any two points of the surface, there is the germ of an affine diffeomorphism taking one point to the other. Let (x^1, x^2) be local coordinates on an affine surface. Adopt the *Einstein convention* and sum over repeated indices to expand $\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}{}^k \partial_{x^k}$ in terms of the *Christoffel symbols*; the condition that ∇ is torsion free is equivalent to the symmetry $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$. We have the following classification result due to Opozda [8].

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Theorem 1 *Let $\mathcal{M} = (M, \nabla)$ be a locally homogeneous affine surface. At least one of the following three possibilities holds for the local geometry:*

$\mathcal{A}$. *There exist local coordinates (x^1, x^2) so that $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$ is constant.*
$\mathcal{B}$. *There exist local coordinates (x^1, x^2) so that $\Gamma_{ij}{}^k = (x^1)^{-1} C_{ij}{}^k$ where $C_{ij}{}^k = C_{ji}{}^k$ is constant.*
$\mathcal{C}$. *∇ is the Levi-Civita connection of the round sphere.*

We say that $\mathcal{M}$ is a Type $\mathcal{A}$ model if $\mathcal{M} = (\mathbb{R}^2, \nabla)$ where ∇ is Type $\mathcal{A}$. This means that the Christoffel symbols $\Gamma_{ij}{}^k$ are constant. Let $\mathbb{R}^2$ be the group of translations acting on itself; a connection ∇ on $\mathbb{R}^2$ is Type $\mathcal{A}$ if ∇ is left-invariant, i.e. the translations are affine maps. Since ∇ is torsion free, $\Gamma_{12}{}^1 = \Gamma_{21}{}^1$ and $\Gamma_{12}{}^2 = \Gamma_{21}{}^2$. Thus there are 6 free parameters. We identify the set of *Type $\mathcal{A}$ models* with $\mathbb{R}^6$ by setting $\mathcal{M}(a, b, c, d, e, f) = (\mathbb{R}^2, \nabla)$ where the Christoffel symbols are given by

$$\Gamma_{11}{}^1 = a, \Gamma_{11}{}^2 = b, \Gamma_{12}{}^1 = \Gamma_{21}{}^1 = c, \Gamma_{12}{}^2 = \Gamma_{21}{}^2 = d, \Gamma_{22}{}^1 = e, \Gamma_{22}{}^2 = f\,.$$

The notion of a Type $\mathcal{B}$ or Type $\mathcal{C}$ model is defined similarly. The general linear group $\mathrm{Gl}(2, \mathbb{R})$ acts on the set of Type $\mathcal{A}$ models by change of variables; we say that two Type $\mathcal{A}$ models are *linearly equivalent* if they differ by a linear action. There are surfaces which are both Type $\mathcal{A}$ and Type $\mathcal{B}$ which are not flat. Any such geometry is, up to linear equivalence, one of the structures $\mathcal{M}_1^1$, $\mathcal{M}_2^1(c_1)$, $\mathcal{M}_3^1(c_1)$, or $\mathcal{M}_4^1(c)$ to be described presently in Definition 1; we refer to [2] for further details. The Type $\mathcal{C}$ geometry is neither Type $\mathcal{A}$ nor Type $\mathcal{B}$. The curvature operator R and the Ricci tensor ρ of an affine surface are given by

$$\begin{aligned} &R(\xi_1, \xi_2) = \nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}, \\ &\rho(\xi_1, \xi_2) = \mathrm{Tr}\{\xi_3 \to R(\xi_3, \xi_1)\xi_2\}\,. \end{aligned}$$

In general, the Ricci tensor of an affine surface need not be symmetric. However, in the Type $\mathcal{A}$ setting, the Ricci tensor is symmetric and is given by

$$\begin{aligned} &\rho_{11} = (\Gamma_{11}{}^1 - \Gamma_{12}{}^2)\Gamma_{12}{}^2 + \Gamma_{11}{}^2(\Gamma_{22}{}^2 - \Gamma_{12}{}^1), \\ &\rho_{12} = \rho_{21} = \Gamma_{12}{}^1\Gamma_{12}{}^2 - \Gamma_{11}{}^2\Gamma_{22}{}^1, \\ &\rho_{22} = -(\Gamma_{12}{}^1)^2 + \Gamma_{22}{}^2\Gamma_{12}{}^1 + (\Gamma_{11}{}^1 - \Gamma_{12}{}^2)\Gamma_{22}{}^1\,. \end{aligned} \tag{1}$$

We say that a curve σ in an affine surface is a *geodesic* if $\nabla_{\dot\sigma}\dot\sigma = 0$, i.e. $\ddot\sigma^i + \Gamma_{jk}{}^i\dot\sigma^j\dot\sigma^k = 0$ for all i. If ∇ is the Levi-Civita connection of a Riemannian metric, geodesics locally minimize length. There is no such interpretation in affine geometry. An affine surface is said to be *geodesically complete* if every maximal geodesic σ is defined for all $t \in \mathbb{R}$; otherwise the surface is said to be *geodesically incomplete*. We shall concentrate on the Type $\mathcal{A}$ geometries so that the geodesic equation is a pair of quadratic ODEs with constant coefficients. However, even with this restriction, it is still difficult to solve these equations directly. Instead, we shall first discuss the notion of strongly projectively flat geometries and show in Lemma 1 that any Type $\mathcal{A}$ geometry is strongly projectively flat. We shall then introduce the quasi-Einstein equation and present its basic properties in Theorem 2. This will enable us to give a classification of the Type $\mathcal{A}$ geometries in Theorem 3 which we will use to determine which Type $\mathcal{A}$ geometries are geodesically complete in Theorem 5; this gives a different treatment of a result originally established by D'Ascanio et al. [4] using different methods.

2 Strongly Projectively Flat Geometries

Two affine connections ∇ and $\tilde{\nabla}$ are said to be *projectively equivalent* if there exists a smooth 1-form ω so

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X \text{ for all } X, Y.$$

We remark that ∇ and $\tilde{\nabla}$ have the same unparametrized geodesics if and only if they are projectively equivalent (see Kobayashi and Nomizu [7]); reparametrization can, of course, affect geodesic completeness. If $\omega = dg$ for some smooth function g, then ∇ and $\tilde{\nabla}$ are said to be *strongly projectively equivalent*. If $\mathcal{M} = (M, \nabla)$, then we set ${}^g\mathcal{M} := (M, \tilde{\nabla})$ in this setting. If ∇ is strongly projectively equivalent to a flat connection, then $\mathcal{M}$ is said to be *strongly projectively flat*.

Lemma 1 *Let $\mathcal{M} = (\mathbb{R}^2, \nabla)$ be a Type $\mathcal{A}$ model. There exists a linear function $g(x^1, x^2) = a_1 x^1 + a_2 x^2$ which provides a strong projective equivalence from $\mathcal{M}$ to a flat Type $\mathcal{A}$ model.*

We remark that results of Eisenhart [5] showed that an affine surface is strongly projectively flat if and only if both ρ and $\nabla\rho$ are symmetric. Let $\mathcal{M}$ be a Type $\mathcal{A}$ model. Equation (1) shows that ρ is symmetric and one can make a similar direct computation to show $\nabla\rho$ is symmetric. However, this does not yield that the 1-form in question has constant coefficients so Lemma 1 does not follow from general theory.

Proof Let $\mathcal{M} = (\mathbb{R}^2, \nabla)$ be a Type $\mathcal{A}$ model. We work modulo linear equivalence. We use Eq. (1) to study the Ricci tensor ρ of $\mathcal{M}$. Let $g(x^1, x^2) = w_1 x^1 + w_2 x^2$ for $(w_1, w_2) \in \mathbb{R}^2$ and let ${}^g\mathcal{M}$ be the resulting strong projective deformation. We then have

$$\begin{aligned} &{}^g\Gamma_{11}{}^1 = \Gamma_{11}{}^1 + 2w_1,\ {}^g\Gamma_{11}{}^2 = \Gamma_{11}{}^2,\ {}^g\Gamma_{12}{}^1 = \Gamma_{12}{}^1 + w_2, \\ &{}^g\Gamma_{12}{}^2 = \Gamma_{12}{}^2 + w_1,\quad {}^g\Gamma_{22}{}^1 = \Gamma_{22}{}^1,\ {}^g\Gamma_{22}{}^2 = \Gamma_{22}{}^2 + 2w_2\,. \end{aligned}$$

Let ${}^g\rho$ be the Ricci tensor of ${}^g\mathcal{M}$. In dimension 2, the Ricci tensor carries the geometry; ${}^g\mathcal{M}$ is flat if and only if ${}^g\rho = 0$.

Case 1 Suppose ${\Gamma_{11}}^2 \neq 0$. Rescale x^2 to ensure ${\Gamma_{11}}^2 = 1$. We have

$$^g\rho_{11} = -{\Gamma_{12}}^1 - ({\Gamma_{12}}^2)^2 + {\Gamma_{22}}^2 + w_1^2 + {\Gamma_{11}}^1({\Gamma_{12}}^2 + w_1) + w_2\,.$$

We set $w_2 := {\Gamma_{12}}^1 - {\Gamma_{11}}^1{\Gamma_{12}}^2 + ({\Gamma_{12}}^2)^2 - {\Gamma_{22}}^2 - {\Gamma_{11}}^1 w_1 - w_1^2$ to ensure $^g\rho_{11} = 0$. Then $^g\rho_{12} = -w_1^3 + O(w_1^2)$ and $^g\rho_{22} = ({\Gamma_{11}}^1 - {\Gamma_{12}}^2 + w_1)^g\rho_{12}$. Since $^g\rho_{12}$ is cubic in w_1, we can find w_1 so $^g\rho_{12} = 0$. This forces $^g\rho_{22} = 0$.

Case 2 Suppose ${\Gamma_{11}}^2 = 0$. We set $w_1 = {\Gamma_{12}}^2 - {\Gamma_{11}}^1$ and $w_2 = -{\Gamma_{12}}^1$ to see $^g\rho(\mathcal{M}) = 0$. □

3 The Quasi-Einstein Equation

Let $\mathcal{H}f := (\partial_{x^i}\partial_{x^j}f - {\Gamma_{ij}}^k\partial_{x^k}f)\,dx^i \otimes dx^j$ be the *Hessian*. Let ρ_s be the symmetric Ricci tensor and let $\mathcal{Q} := \ker\{\mathcal{H} + \rho_s\}$. We refer to Brozos-Vázquez et al. [3] for a discussion of the context in which this operator arises and for applications to four-dimensional geometry arising from the modified Riemannian extension. We refer to [6] for the proof of the following result.

Theorem 2 *Let $\mathcal{M}$ be an affine surface.*

1. *If dg provides a strong projective equivalence between $\mathcal{M}$ and $^g\mathcal{M}$, then $\mathcal{Q}(^g\mathcal{M}) = e^g\mathcal{Q}(\mathcal{M})$.*
2. *We have that $\dim\{\mathcal{Q}(\mathcal{M})\} \leq 3$; equality holds if and only if $\mathcal{M}$ is strongly projectively flat.*
3. *If ∇ and $\tilde\nabla$ are two strongly projectively flat structures on a surface M, then $\nabla = \tilde\nabla$ if and only if $\mathcal{Q}(M,\nabla) = \mathcal{Q}(M,\tilde\nabla)$.*
4. *Suppose $^g\mathcal{M}$ is flat. Then $\mathcal{Q}(\mathcal{M}) = e^g\,\mathrm{Span}\{\mathbb{1},\phi^1,\phi^2\}$ and $\Phi := (\phi^1,\phi^2)$ provides local coordinates so that the unparameterized geodesics of $\mathcal{M}$ take the form $\Phi^{-1}(at + a_0, bt + b_0)$.*

Define distinguished Type $\mathcal{A}$ geometries and function spaces as follows. To simplify the notation, let $\mathcal{S}(f_1, f_2, f_3) := \mathrm{Span}_{\mathbb{R}}\{f_1, f_2, f_3\}$.

Definition 1 Let $c_1 \notin \{0,-1\}$ and $c_2 \neq 0$.

$$\begin{array}{ll}
\mathcal{M}_0^0 := \mathcal{M}(0,0,0,0,0,0), & \mathcal{Q}_0^0 = \mathcal{S}(\mathbb{1}, x^1, x^2),\\
\mathcal{M}_1^0 := \mathcal{M}(1,0,0,1,0,0), & \mathcal{Q}_1^0 = \mathcal{S}(\mathbb{1}, e^{x^1}, x^2e^{x^1}),\\
\mathcal{M}_2^0 := \mathcal{M}(-1,0,0,0,0,1), & \mathcal{Q}_2^0 = \mathcal{S}(\mathbb{1}, e^{x^2}, e^{-x^1}),\\
\mathcal{M}_3^0 := \mathcal{M}(0,0,0,0,0,1), & \mathcal{Q}_3^0 = \mathcal{S}(\mathbb{1}, x^1, e^{x^2}),\\
\mathcal{M}_4^0 := \mathcal{M}(0,0,0,0,1,0), & \mathcal{Q}_4^0 = \mathcal{S}(\mathbb{1}, x^2, (x^2)^2 + 2x^1),\\
\mathcal{M}_5^0 := \mathcal{M}(1,0,0,1,-1,0), & \mathcal{Q}_5^0 = \mathcal{S}(\mathbb{1}, e^{x^1}\cos(x^2), e^{x^1}\sin(x^2)),\\
\mathcal{M}_1^1 := \mathcal{M}(-1,0,1,0,0,2), & \mathcal{Q}_1^1 = e^{x^2}\mathcal{S}(\mathbb{1}, x^2, e^{-x^1}),\\
\mathcal{M}_2^1(c_1) := \mathcal{M}(-1,0,c_1,0,0,1+2c_1), & \mathcal{Q}_2^1(c_1) = e^{c_1x^2}\mathcal{S}(\mathbb{1}, e^{x^2}, e^{-x^1}),
\end{array}$$

$\mathcal{M}_3^1(c_1) := \mathcal{M}(0, 0, c_1, 0, 0, 1 + 2c_1), \qquad \mathcal{Q}_3^1(c_1) = e^{c_1 x^2}\mathcal{S}(\mathbb{1}, e^{x^2}, x^1),$
$\mathcal{M}_4^1(c) := \mathcal{M}(0, 0, 1, 0, c, 2), \qquad \mathcal{Q}_4^1(c) = e^{x^2}\mathcal{S}(\mathbb{1}, x^2, c(x^2)^2 + 2x^1),$
$\mathcal{M}_5^1(c) := \mathcal{M}(1, 0, 0, 0, 1 + c^2, 2c),\ \mathcal{Q}_5^1(c) = \mathcal{S}(e^{cx^2}\cos(x^2), e^{cx^2}\sin(x^2), e^{x^1}),$
$\mathcal{M}_1^2(a_1, a_2) := \mathcal{M}\left(\frac{a_1^2+a_2-1, a_1^2-a_1, a_1a_2, a_1a_2, a_2^2-a_2, a_1+a_2^2-1}{a_1+a_2-1}\right),$
$\mathcal{Q}_1^2(a_1, a_2) = \mathcal{S}(e^{x^1}, e^{x^2}, e^{a_1x^1+a_2x^2})$ for $a_1a_2 \neq 0$ and $a_1 + a_2 \neq 1$,
$\mathcal{M}_2^2(b_1, b_2) := \mathcal{M}\left(1 + b_1, 0, b_2, 1, \frac{1+b_2^2}{b_1-1}, 0\right)$, for $b_1 \neq 1$ and $(b_1, b_2) \neq (0, 0)$,
$\mathcal{Q}_2^2(b_1, b_2) = \mathcal{S}(e^{x^1}\cos(x^2), e^{x^1}\sin(x^2), e^{b_1x^1+b_2x^2}),$
$\mathcal{M}_3^2(c_2) := \mathcal{M}(2, 0, 0, 1, c_2, 1), \qquad \mathcal{Q}_3^2(c_2) = e^{x^1}\mathcal{S}(\mathbb{1}, x^1 - c_2x^2, e^{x^2}),$
$\mathcal{M}_4^2(\pm 1) := \mathcal{M}(2, 0, 0, 1, \pm 1, 0),\ \mathcal{Q}_4^2(\pm 1) = \mathcal{S}(e^{x^1}, x^2e^{x^1}, (2x^1 \pm (x^2)^2)e^{x^1}).$

Theorem 3 *If $\mathcal{M}$ is a Type $\mathcal{A}$ model, then $\mathcal{M}$ is linearly equivalent to one of the models $\mathcal{M}_i^\nu(\cdot)$ of Definition 1. We have that $\mathcal{Q}(\mathcal{M}_i^\nu(\cdot)) = \mathcal{Q}_i^\nu(\cdot)$ and that the Ricci tensor of $\mathcal{M}_i^\nu(\cdot)$ has rank ν.*

Proof Let $\mathcal{M}$ be a Type $\mathcal{A}$ model. By Lemma 1, $\mathcal{M}$ is strongly projectively flat. Thus by Theorem 2, $\mathcal{M}$ is determined by $\mathcal{Q}(\mathcal{M})$. Since the Christoffel symbols of $\mathcal{M}$ are constant, the translation group acts by affine diffeomorphisms. This implies that ∂_{x^1} and ∂_{x^2} are affine Killing vector fields. Consequently, $\mathcal{Q}(\mathcal{M})$ is a finite dimensional ∂_{x^1} and ∂_{x^2} module. Let $\mathcal{Q}_{\mathbb{C}}(\mathcal{M}) := \mathcal{Q}(\mathcal{M}) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. This 3-dimensional space of functions invariant under the action of $\{\partial_{x^1}, \partial_{x^2}\}$. By examining the generalized simultaneious eigenvalues of this action, we can conclude that $\mathcal{Q}_{\mathbb{C}}(\mathcal{M})$ is generated by functions of the form $e^{a_1x^1+a_2x^2}p(x^1, x^2)$ where p is polynomial and $(a_1, a_2) \in \mathbb{C}^2$. With a bit of additional work, one can classify the possible solution spaces $\mathcal{Q}$ up to linear equivalence and show they are linearly equivalent to $\mathcal{Q}_i^\nu(\cdot)$ for some value of the parameters; we refer to [6] for further details. By Theorem 2, $\dim\{\mathcal{Q}(\mathcal{M}_i^\nu(\cdot))\} \le 3$. A direct computation shows that $\mathcal{Q}_i^\nu(\cdot) \subset \mathcal{Q}(\mathcal{M}_i^\nu(\cdot))$ and thus equality holds for dimensional reasons. Finally, a direct computation determines $\rho(\mathcal{M}_i^\nu(\cdot))$ and shows that the Ricci tensor has rank ν. □

We have the following relations amongst the models $\mathcal{M}_i^\nu(\cdot)$.

Theorem 4 *The following are affine maps.*

1. $\Phi_1^0(x^1, x^2) := (e^{x^1}, x^2e^{x^1})$ *embeds* $\mathcal{M}_1^0$ *in* $\mathcal{M}_0^0$.
2. $\Phi_2^0(x^1, x^2) := (e^{x^2}, e^{-x^1})$ *embeds* $\mathcal{M}_2^0$ *in* $\mathcal{M}_0^0$.
3. $\Phi_3^0(x^1, x^2) := (x^1, e^{x^2})$ *embeds* $\mathcal{M}_3^0$ *in* $\mathcal{M}_0^0$.
4. $\Phi_4^0(x^1, x^2) := (x^2, (x^2)^2 + 2x^1)$ *defines* $\mathcal{M}_4^0 \approx \mathcal{M}_0^0$.
5. $\Phi_5^0(x^1, x^2) := (e^{x^1}\cos(x^2), e^{x^1}\sin(x^2))$ *immerses* $\mathcal{M}_5^0$ *in* $\mathcal{M}_0^0$.
6. $\Phi_1^1(x^1, x^2) := (e^{-x^1}, x^2)$ *embeds* $\mathcal{M}_1^1$ *in* $\mathcal{M}_4^1(0)$.
7. $\Phi_2^1(x^1, x^2) := (e^{-x^1}, x^2)$ *embeds* $\mathcal{M}_2^1(c_1)$ *in* $\mathcal{M}_3^1(c_1)$.

8. $\Phi_3^1(x^1,x^2) := (x^1e^{-x^2}, -x^2)$ *defines* $\mathcal{M}_3^1(c_1) \approx \mathcal{M}_3^1(-c_1-1)$.
9. $\Phi_4^1(c)(x^1,x^2) := (x^1+\frac{1}{2}c(x^2)^2, x^2)$ *defines* $\mathcal{M}_4^1(c) \approx \mathcal{M}_4^1(0)$.
10. $\Phi_5^1(x^1,x^2) := (x^1, -x^2)$ *is an isomorphism* $\mathcal{M}_5^1(c) \approx \mathcal{M}_5^1(-c)$.

Proof By Lemma 1, the Type $\mathcal{A}$ models $\mathcal{M}_i^\nu(\cdot)$ are strongly projectively flat. Thus, by Theorem 2, affine morphisms between them correspond to local diffeomorphisms which intertwine their corresponding spaces $\mathcal{Q}$. One verifies immediately that this condition is satisfied by the maps $\Phi_i^j(\cdot)$ of the Theorem and the desired result now holds. □

We can draw the following consequence.

Lemma 2 *Let $\mathcal{M}$ be a Type $\mathcal{A}$ flat geometry. Then $\mathcal{M}$ is geodesically complete if and only if $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_0^0$ or to $\mathcal{M}_4^0$.*

Proof By Theorem 3, $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_i^0$ for some i. $\mathcal{M}_1^0$, $\mathcal{M}_2^0$, and $\mathcal{M}_3^0$ have affine embeddings into $\mathcal{M}_0^0$ which are not surjective; they are therefore not geodesically complete. $\mathcal{M}_4^0$ is affine diffeomorphic to the flat affine plane $\mathcal{M}_0^0$ and thus is geodesically complete. $\mathcal{M}_5^0$ has an affine immersion into $\mathcal{M}_0^0$ which is not surjective; it is not geodesically complete. □

We use Theorem 3 to express $\mathcal{Q}_i^\nu(\cdot) = e^g \operatorname{Span}\{\mathbb{1}, \phi_1, \phi_2\}$ for g linear. Let $\Phi = (\phi_1, \phi_2)$. By Theorem 2, the unparameterized geodesics of $\mathcal{M}_i^\nu(\cdot)$ take the form $\Phi^{-1}(a_0 + a_1t, b_0 + b_1t)$. This reduces the problem of finding the geodesics of $\mathcal{M}_i^\nu(\cdot)$ to solving a single ODE defining the reparametrization. This fact informed our subsequent investigations; we did not simply proceed mechanically to solve the ODEs in question. We say a Type $\mathcal{A}$ model $\mathcal{M}$ *can be geodesically completed* if there is an affine embedding of $\mathcal{M}$ in a homogeneous geodesically complete surface; otherwise $\mathcal{M}$ is said to be *essentially geodesically incomplete*. The following is a useful criteria.

Lemma 3 *Let $\mathcal{M}$ be a Type $\mathcal{A}$ model. Assume there exists a geodesic $\sigma(t)$ for $t \in (t_-, t_+)$ so that* $\lim_{t\to\tau} |\rho(\dot\sigma(t), \partial_{x^i})| = \infty$ *where $\tau = t_+ < \infty$ or $\tau = t_- > -\infty$. Then $\mathcal{M}$ is essentially geodesically incomplete.*

Proof Suppose to the contrary that there exists an affine surface $\mathcal{M}_1$ which is locally modeled on $\mathcal{M}$. Copy a small piece of the given geodesic σ into $\mathcal{M}_1$ to define a geodesic σ_1 in $\mathcal{M}_1$. We may assume without loss of generality that $\mathcal{M}_1$ is simply connected and extend the vector field ∂_{x^i} to a globally defined affine Killing vector field X_i on $\mathcal{M}_1$. Results of [2] show that $\mathcal{M}_1$ is real analytic. Thus the function $f(t) := \rho_{\mathcal{M}}(\dot\sigma, \partial_{x^i})(t)$ defined for $t \in (t_-, t_+)$ extends to a real analytic function $f_1(t) := \rho_{\mathcal{M}_1}(\dot\sigma_1(t), X_i(t))$ for $t \in \mathbb{R}$. This is not possible since by assumption $f(t)$ blows up at a finite value. □

If the Ricci tensor of a Type $\mathcal{A}$ model $\mathcal{M}$ has rank 1, then $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_i^1(\cdot)$ for some value of the parameters. Thus it suffices to study these examples.

Lemma 4 *$\mathcal{M}_1^1$, $\mathcal{M}_2^1(c_1)$ for $c_1 \neq -\frac{1}{2}$, $\mathcal{M}_3^1(c_1)$ for $c_1 \neq -\frac{1}{2}$, $\mathcal{M}_4^1(c)$ for any c, and $\mathcal{M}_5^1(c)$ for $c \neq 0$ are essentially geodesically incomplete. $\mathcal{M}_3^1(-\frac{1}{2})$ is geodesically complete. $\mathcal{M}_2^1(-\frac{1}{2})$ and $\mathcal{M}_5^1(0)$ can be geodesically completed.*

Proof A direct computation shows

$$\begin{aligned} &\rho_{\mathcal{M}_1^1} = dx^2 \otimes dx^2, &&\rho_{\mathcal{M}_2^1(c_1)} = (c_1 + c_1^2)dx^2 \otimes dx^2,\\ &\rho_{\mathcal{M}_3^1(c_1)} = (c_1 + c_1^2)dx^2 \otimes dx^2, &&\rho_{\mathcal{M}_4^1(c)} = dx^2 \otimes dx^2,\\ &\rho_{\mathcal{M}_5^1(c)} = (1 + c^2)dx^2 \otimes dx^2. \end{aligned}$$

We apply the criteria of Lemma 3 with $\partial_{x^i} = \partial_{x^2}$ to study these geometries.

Case 1 Let $\mathcal{M} = \mathcal{M}_1^1$. A direct computation shows $\sigma(t) = (0, \frac{1}{2}\log(t))$ is a geodesic for $t \in (0, \infty)$. Since $\lim_{t\to 0} |\rho(\dot\sigma, \partial_{x^2})| = \infty$, $\mathcal{M}_1^1$ is essentially geodesically incomplete.

Case 2 Let $\mathcal{M} = \mathcal{M}_2^1(c_1)$ or $\mathcal{M} = \mathcal{M}_3^1(c_1)$ for $c_1 \neq -\frac{1}{2}$. A direct computation shows $\sigma(t) := (0, \frac{1}{1+2c}\log(t))$ is a geodesic for $t \in (0, \infty)$. Since $\lim_{t\to 0} |\rho(\dot\sigma, \partial_{x^2})| = \infty$, $\mathcal{M}$ is essentially geodesically incomplete.

Case 3 Let $\mathcal{M} = \mathcal{M}_3^1(-\frac{1}{2})$. Suppose $b \neq 0$. Let $\sigma_{a,b}(t) = (\frac{a}{b}(e^{bt} - 1), bt)$. Then σ is a geodesic with $\sigma(0) = (0, 0)$ and $\dot\sigma(0) = (a, b)$. If $b = 0$, let $\sigma_{a,b}(t) = (at, 0)$. Then σ is a geodesic with $\sigma(0) = (0, 0)$ and $\dot\sigma(0) = (a, 0)$. Thus every geodesic starting at (0, 0) extends for infinite time. Since $\mathcal{M}$ is homogeneous, $\mathcal{M}$ is geodesically complete.

Case 4 Let $\mathcal{M} = \mathcal{M}_2^1(-\frac{1}{2})$. A direct computation shows $\sigma(t) := (-\log(t), 0)$ is a geodesic for $t \in (0, \infty)$. This geodesic can not be continued to $t = 0$ and thus $\mathcal{M}_2^1(-\frac{1}{2})$ is geodesically incomplete. By Theorem 4, $\mathcal{M}_2^1(-\frac{1}{2})$ has an affine embedding in $\mathcal{M}_3^1(-\frac{1}{2})$. Thus by Case 3, $\mathcal{M}_2^1(-\frac{1}{2})$ can be geodesically completed.

Case 5 Let $\mathcal{M} = \mathcal{M}_4^1(c)$. Let $\sigma(t) := (-\frac{c}{8}\log(t)^2, \frac{1}{2}\log(t))$. A direct computation shows this is a geodesic for $t \in (0, \infty)$. Since $\lim_{t\to 0} |\rho(\dot\sigma, \partial_{x^2})| = \infty$, $\mathcal{M}$ is essentially geodesically incomplete.

Case 6 Let $\mathcal{M} = \mathcal{M}_5^1(c)$. Suppose that $c \neq 0$. A direct computation shows $\sigma(t) = (\log(\cos(\frac{\log(t)}{2c})) + \frac{\log(t)}{2}, \frac{\log(t)}{2c})$ is a geodesic for $t \in (0, \infty)$. Since $\lim_{t\to 0} |\rho(\dot\sigma, \partial_{x^2})| = \infty$, $\mathcal{M}$ is essentially geodesically incomplete.

Case 7 If $c = 0$, the curve $\sigma(t) = (\log(\cos(t)), t)$ is a geodesic for $\mathcal{M}_5^1(c)$ which does not extend to $\mathbb{R}$. Thus $\mathcal{M}_5^1(0)$ is geodesically incomplete. We complete the proof by showing $\mathcal{M}_5^1(0)$ can be geodesically completed. Let $\mathcal{N} = (\mathbb{R}^2, \nabla)$ be the affine surface where the only non-zero Christoffel symbol of ∇ is ${\Gamma_{22}}^1 = x^1$. We compute $\{\cos(x^2), \sin(x^2), x^1)\} \subset \mathcal{Q}(\mathcal{N})$ and thus by Theorem 2 and for dimensional reasons we have $\mathcal{Q}$ is spanned by these elements and $\mathcal{N}$ is strongly projectively flat. Let

$$\Psi_{a,b,c,d}(x^1, x^2) := (e^a x^1 + b\cos(x^2) + c\sin(x^2), x^2 + d)\,.$$

Then $\Psi^*_{a,b,c,d}\mathcal{Q}(\mathcal{N}) = \mathcal{Q}(\mathcal{N})$ so $\Psi_{a,b,c,d}$ is an affine diffeomorphism of $\mathcal{N}$. Since these diffeomorphisms act transitively on $\mathcal{N}$, $\mathcal{N}$ is homogeneous. If $b \neq 0$, let $\sigma_{a,b}(t) := (\frac{a}{b}\sin(bt), bt)$; this is a geodesic with $\sigma_{a,b}(0) = (0,0)$ and $\dot\sigma_{a,b}(0) = (a,b)$. If $b = 0$, let $\sigma_{a,0}(t) := (at, 0)$; this is a geodesic with $\sigma_{a,0}(0) = (0,0)$ and $\dot\sigma_{a,0}(0) = (a,0)$. Thus $\mathcal{N}$ is geodesically complete at $(0,0)$ and, since $\mathcal{N}$ is homogeneous, $\mathcal{N}$ is geodesically complete. The map $\Phi(x^1, x^2) = (e^{x^1}, x^2)$ embeds $\mathbb{R}^2$ in $\mathbb{R}^2$ and satisfies $\Phi^*\mathcal{Q}(\mathcal{N}) = \mathcal{Q}(\mathcal{M}_5^1(0))$. Thus Φ is an affine embedding of $\mathcal{M}_5^1(0)$ in $\mathcal{N}$ so $\mathcal{N}$ provides the desired geodesic completion of $\mathcal{M}_5^1(0)$. □

We begin our discussion of the geometries where the Ricci tensor has rank 2 with the following result.

Lemma 5 *$\mathcal{M}_1^2(a_1, a_2)$, $\mathcal{M}_2^2(b_1, b_2)$ for $b_1 \neq -1$, $\mathcal{M}_3^2(c_2)$, and $\mathcal{M}_4^2(\pm 1)$ are essentially geodesically incomplete.*

Proof A direct computation shows the Ricci tensor for the Type $\mathcal{A}$ models $\mathcal{M}_i^2(\cdot)$ has rank 2. Consequently the criteria of Lemma 3 for essential geodesic incompleteness is simply the existence of a geodesic so that $\lim_{t\to\tau}|\dot x(t)| = \infty$ or $\lim_{t\to\tau}|\dot y(t)| = \infty$ for some finite value τ.

Case 1 Let $\mathcal{M} = \mathcal{M}_1^2(a_1, a_2)$. Let

$$\sigma_1(t) := \log(t)\tfrac{(1-a_2, a_1)}{1+a_1-a_2} \text{ if } 1 + a_1 - a_2 \neq 0,$$
$$\sigma_2(t) := \log(t)\tfrac{(a_2, 1-a_1)}{1+a_2-a_1} \text{ if } 1 + a_2 - a_1 \neq 0.$$

Since $(1+a_2-a_1)+(1-a_2+a_1) = 2$, at least one of these curves is well defined. A direct computation shows such a curve is a geodesic and hence $\mathcal{M}$ is essentially geodesically incomplete.

Case 2 Let $\mathcal{M} = \mathcal{M}_2^2(b_1, b_2)$ for $b_1 \neq -1$. The curve $\sigma(t) = \frac{1}{1+b_1}(\log(t), 0)$ is a geodesic. Consequently, $\mathcal{M}$ is essentially geodesically incomplete.

Case 3 Let $\mathcal{M} = \mathcal{M}_3^2(c_2)$ or $\mathcal{M} = \mathcal{M}_4^2(\pm)$. The curve $\sigma(t) = \frac{1}{2}(\log(t), 0)$ is a geodesic; consequently, $\mathcal{M}$ is essentially geodesically incomplete. □

Before considering the geometry $\mathcal{M}_2^2(-1, b_2)$, we must establish a preliminary result.

Lemma 6 *Let P be a point of an affine manifold $\mathcal{M}$. Let $\sigma : [0, T) \to M$ be an affine geodesic. Suppose $\lim_{t\to T}\sigma(t) = P$ exists. Then there exists $\epsilon > 0$ so that σ can be extended to the parameter range $[0, T+\epsilon)$ as an affine geodesic.*

Proof Put a positive definite inner product $\langle\cdot,\cdot\rangle$ on T_PM to act as a reference metric. Let B_r be the ball of radius r about the origin in T_PM. Since the exponential map is a local diffeomorphism, we can use $\exp_P$ to identify B_ε with a neighborhood of P in M for some small ε. We use this identification to define a flat Riemannian metric near P on M so that $\exp_P$ is an isometry from B_ε to M. Let $d(\cdot,\cdot)$ be the associated distance function on M. Let $B_r(P) := \exp_P(B_r) = \{Q : d(P,Q) \leq r\}$ for $r \leq \varepsilon$. Choose linear coordinates on T_PM to put coordinates on $B_\varepsilon(P)$. This identifies T_QM with T_PM and extends $\langle\cdot,\cdot\rangle$ to T_QM for $Q \in B_\varepsilon(P)$. Compactness shows that there exists $0 < \tau < \frac{1}{2}\varepsilon$ so that if $Q \in B_{\frac{\varepsilon}{2}}(P)$ and if $\xi \in T_QM$ satisfies $\|\xi\| = 1$, then the geodesic $\sigma_{Q,\xi}(t) := \exp_Q(t\xi)$ exists for $t \in [0,\tau]$ and belongs to $B_\varepsilon(P)$. By continuity, we can choose $0 < \delta < \frac{1}{4}\tau$ so that if $Q \in B_\delta(P)$ and $\|\xi\| = 1$, then $d(\sigma_{Q,\xi}(\tau), \sigma_{P,\xi}(\tau)) < \frac{\tau}{2}$. Since $d(P, \sigma_{P,\xi}(\tau)) = \tau$, this implies $d(P, \sigma_{Q,\xi}(\tau)) \geq \frac{1}{2}\tau$. We conclude from these estimates that any non-trivial geodesic which begins in $B_\delta(P)$ continues to exist at least until it exits from $B_{\frac{1}{2}\tau}(P)$ and that it does in fact exit from $B_{\frac{1}{2}\tau}(P)$. We assumed $\lim_{t\to T}\sigma(t) = P$. Choose $T_0 < T$ so $\sigma(T_0, T) \subset B_\delta(P)$. Then σ continues to exist until σ exits from $B_{\frac{1}{2}\tau}(P)$. Furthermore, $\sigma(T) = P$ and σ extends to a geodesic defined on $(T_0, T+\epsilon)$ for some ϵ. □

We complete our discussion with the following result.

Lemma 7 *$\mathcal{M}_2^2(-1, b_2)$ is geodesically complete.*

Proof Let $\mathcal{M} = \mathcal{M}_2^2(-1, b_2)$. Suppose, to the contrary, that $\mathcal{M}$ is geodesically incomplete. Let σ be a geodesic in $\mathcal{M}$ which is defined on a parameter range (t_0, t_1) where $t_1 < \infty$ (resp. $-\infty < t_0$) which can not be extended to a parameter range $(t_0, t_1 + \varepsilon)$ (resp. $t_0 - \varepsilon$) for any $\varepsilon > 0$. By Lemma 6, this implies that $\lim_{t\downarrow t_0}\sigma(t)$ (resp. $\lim_{t\uparrow t_1}\sigma(t)$) does not exist. We argue for a contradiction. The non-zero Christoffel symbols of $\mathcal{M}$ are $\Gamma_{12}{}^1 = b_2$, $\Gamma_{12}{}^2 = 1$, and $\Gamma_{22}{}^1 = -\frac{1}{2}(1 + b_2^2)$. We work in the tangent bundle and introduce variables $u^1(t) := \dot{x}^1(t)$ and $u^2(t) := \dot{x}^2(t)$. This yields the geodesic equations

$$\dot{u}^1 + 2b_2u^1u^2 - \tfrac{1}{2}(1 + b_2^2)u^2u^2 = 0 \text{ and } \dot{u}^2 + 2u^1u^2 = 0\,. \qquad (2)$$

If $u^2(s) = 0$ for any $s \in (t_0, t_1)$, then $\dot{u}^1(s) = 0$ and $\dot{u}^2(s) = 0$. Consequently, $u^1(t) = u^1(s)$ and $u^2(t) = u^2(s)$ solves this ODE and (u^1, u^2) is constant on the interval (t_0, t_1). We may therefore assume u^2 does not change sign on the interval (t_0, t_1). We want initial conditions $u^1(0) = a$ and $u^2(0) = b$. Let τ be an unknown function with $\tau(0) = 1$. Set

$$u^1(t) := e^{-b_2\tau(t)}\left(\tfrac{1}{2}\left(-2ab_2 + bb_2^2 + b\right)\sin(\tau(t)) + a\cos(\tau(t))\right),$$
$$u^2(t) := e^{-b_2\tau(t)}((bb_2 - 2a)\sin(\tau(t)) + b\cos(\tau(t)))$$

We then have $u^1(0) = a$ and $u^2(0) = b$. Equation (2) then gives rise to a single ODE to be satisfied:

$$\dot{\tau}(t) = e^{-b_2\tau(t)}(-2a\sin(\tau(t)) + bb_2\sin(\tau(t)) + b\cos(\tau(t)))$$

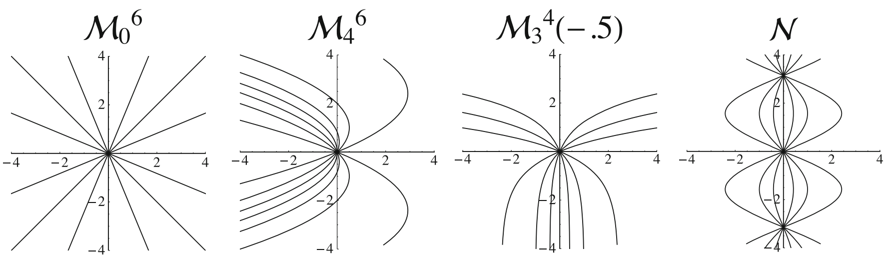
or equivalently $\dot{\tau}(t) = u^2(\tau(t))$. Since u^2 does not change sign, $\tau(t)$ is restricted to a parameter interval of length at most π. Thus u^1 and u^2 are bounded. If u^2 is positive (resp negative), then $\dot{\tau}(t)$ is positive (resp. negative) and bounded so $\tau(t)$ is monotonically increasing (resp. decreasing) and bounded on the interval (t_0, t_1). Thus $\lim_{t\downarrow t_0}\tau(t)$ and $\lim_{t\uparrow t_1}\tau(t)$ exist so $\lim_{t\downarrow t_0}\dot{\sigma}(t)$ and $\lim_{t\uparrow t_1}\dot{\sigma}(t)$ exist. We integrate to conclude $\lim_{t\downarrow t_0}\dot{\sigma}(t)$ and $\lim_{t\uparrow t_1}\dot{\sigma}(t)$ exist which provides the desired contradiction and completes the proof. We remark that work of Bromberg and Medina [1] can also be used to establish this result. □

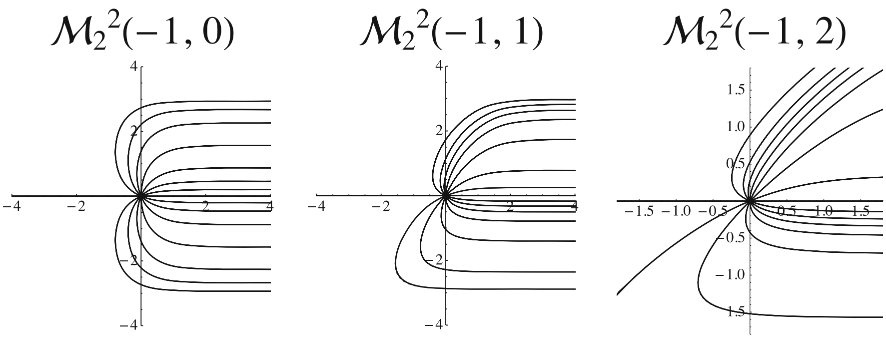
We summarize our results as follows; this result for non-flat connections was derived previously by D'Ascanio et al. [4] using an entirely different approach and the flat setting follows from Theorem 4.

Theorem 5 *Let $\mathcal{M}$ be a Type $\mathcal{A}$ affine surface.*

1. *Suppose $\mathcal{M}$ is flat. Then $\mathcal{M}$ is geodesically complete if and only if $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_0^0$ or to $\mathcal{M}_4^0$.*
2. *Suppose the Ricci tensor of $\mathcal{M}$ has rank 1. Then $\mathcal{M}$ is geodesically complete if and only if $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_3^1(-\frac{1}{2})$. If $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_5^1(0)$, then $\mathcal{M}$ is geodesically incomplete but has a geodesic completion $\mathcal{N}$. If $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_2^1(-\frac{1}{2})$, then $\mathcal{M}$ is geodesically incomplete but has the geodesic completion $\mathcal{M}_3^1(-\frac{1}{2})$. Otherwise $\mathcal{M}$ is essentially geodesically incomplete.*
3. *Suppose that the Ricci tensor has rank 2. If $\mathcal{M}$ is linearly equivalent to $\mathcal{M}_2^1(-1, b_2)$, then $\mathcal{M}$ is geodesically complete. Otherwise $\mathcal{M}$ is essentially geodesically incomplete.*

The geodesic structures of these models is pictured below





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