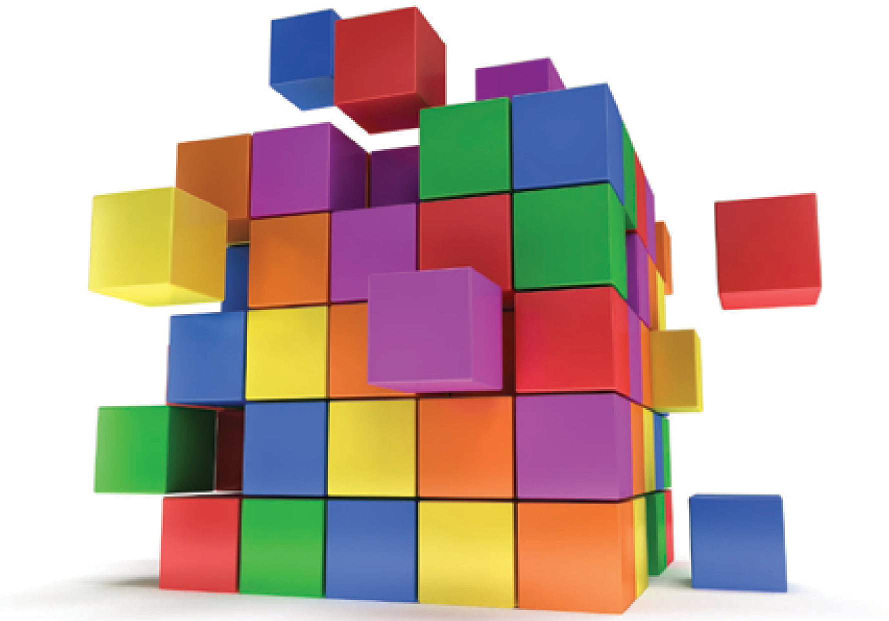
MATHEMATICAL BRAINTEASERS

with Surprising Solutions



Owen O'Shea

Foreword by Colm Mulcahy

MATHEMATICAL BRAINTEASERS with *SURPRISING* SOLUTIONS

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To Dermot and Kathleen,
Tony and Nora,
Noreen and Jerry,
Oliver and Ann,
Ann and Con,
Teresa and Denis,
Breda and Frank,
My late sister-in-law, Vera,
and to my twin brother, Michael, and his son, Kevin.

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FOREWORD

Owen O'Shea is a highly effective communicator of recreational mathematics, as this new tour de force for general readers makes clear. He has a terrific eye for numerical curiosities and an infectious love of puzzles and their solutions that comes across in all his writing. When he was barely a teenager, he discovered the following little gem, which he has never published:

There are $(1 + 2 + 3)$ primes less than $(4 + 5 + 6)$,
and the 7th prime equals $(8 + 9)$.

Owen likes counterintuitive results in probability theory. For example, he once told me that if one repeatedly tosses a fair coin, the expected waiting time for two consecutive heads is six tosses. However, the expected waiting time for a tail followed by a head is only four tosses. This result is counterintuitive because the probability that two consecutive heads will fall is the same probability as tossing a tail followed by a head. Owen likes to point out that this result can be used as a handy little "earner": ask a friend to repeatedly toss a coin. Bet that a tail followed by a head will turn up before two consecutive heads. You are likely to win the bet.

In the pages that follow, he shares his irrepressible enthusiasm with a new generation of readers.

Owen hails from Cobh (pronounced "cove") near the city of Cork in the south of Ireland, the port from which *Titanic* sailed on its ill-fated voyage in 1912 and the birthplace (a year later, in 1913) of famed boxer Jack (the "Gorgeous Gael") Doyle, whose 1978 funeral Owen proudly attended many years later as a young man.

Around 1990, Owen started corresponding with the great American writer Martin Gardner (1914–2010), who is remembered for his long stint writing the column "Mathematical Games" for *Scientific American*. Gardner was renowned for his gentle encouragement of inquiry in a playful spirit among amateurs and professionals alike and for his disdain for irrationality and pseudoscience. His wide circle included high-profile names such as Isaac Asimov, Paul Kurtz, Carl Sagan, James Randi, numerous magicians of note, and scores of mathematicians, including Sol Golomb, John H. Conway, Benoit Mandelbrot, and Roger Penrose.

Gardner and O'Shea shared a love of the puzzles of the great American puzzle master Sam Loyd (1841–1911)—indeed, Gardner had published two compendia of Loyd's puzzles in the 1960s—and Owen first wrote to Gardner to alert him of an article (*Reminiscences of Sam Loyd's Family*) that had not appeared in his extensive Loyd bibliography. The two soon became regular correspondents, discussed recreational mathematics, the philosophy of mathematics, philosophy itself, the alleged existence of God, religion, science, pseudoscience, literature, the paranormal, poetry, and myriad other subjects. Martin sometimes shared Owen's insights with his own readers, such as in *When You Were a Tadpole and I Was a Fish*, the last of his 100 or so books to appear in his lifetime.

Among the original discoveries that Martin was happy to receive from Owen over the years was the following numerical nugget, which is so typical of Owen's fertile mind:

$$1!^2 + 2!^2 + 3!^2 + 4!^2 + 1!^2 + 2!^2 + 3!^2 + 4!^2 = 1234.$$

(Recall 3! means $3 \times 2 \times 1 = 6$ etc.). Gardner had been collecting such novelties since the 1950s, often disseminating them via tall tales involving his alleged friend Dr. I. J. Matrix, numerologist extraordinaire, one of the relatively few fictional characters to have his own Wikipedia page.

Another item O'Shea sent to Gardner mines a vein totally in synch with the spirit of Dr. Matrix:

> "Cold" is the opposite meaning of "hot." Using the usual alphabet code where A = 1, B = 2, the letters of cold sum to 34, while the letters of hot sum to 43, which is the opposite of 34.

Following Gardner's death in 2010, there was a flurry of tribute puzzles in the *New York Times*, where it was correctly pointed out that "Owen [O'Shea] seems to be the heir apparent to Dr. I. J. Matrix in numerological acumen" (https://wordplay.blogs.nytimes.com/2010/06/21/numberplay-spooky-action-at-a-distance). Significantly, Prometheus published Gardner's *The Magic Numbers of Dr. Matrix* (1985).

Martin Gardner was in fact instrumental in getting the Mathematical Association of America interested in publishing Owen's first book, *The*

Magic Numbers of the Professor, coauthored with Underwood Dudley, for which he also wrote the preface. In January 2008, I had the pleasure of hand delivering a pristine copy of this volume to Martin at his retirement home in Oklahoma.

It was actually thanks to Martin that I first learned about Owen; he thought that two Irishmen who loved mathematics should get to know each other. Although a very private person with a low profile—despite his occasional articles in the Irish newspapers on assorted subjects over the years—Owen agreed to meet me in Cork in the summer of 2007. I can attest to his lively intellect, sparkling wit, and extraordinary ability to find patterns everywhere. Calculator ever at the ready, he can conjure up something interesting within minutes given almost any raw material. Here's one that he recalls once telling Martin over the phone. Consider the ten-digit array as it appears on an electronic calculator:

789
456
123
0

Note that $12 + 34 + 56 + 78 + 95 - 59 - 87 - 65 - 43 - 21 = 0$.

Like his mentor, whom Owen unfortunately never met, it's all done with a twinkle in the eye. Like Martin, Owen is under no illusions that there is any paranormal significance to the coincidences he repeatedly stumbles on. And yet, there are eerie examples.

When the terrible Madrid bombings happened on March 11, 2004, days before a national election in Spain, it wasn't long before Owen noticed that exactly 911 days had elapsed *between* the catastrophic World Trade Center attacks in New York City on September 11, 2001, and the Spanish atrocity. In the days immediately following the 2004 incident, nobody claimed responsibility for it, and the Spanish government was suggesting that the Basque group ETA was to blame. This turned out to be completely false, and the ruling party was promptly ousted from office that month.

Owen contacted the newsroom of an initially skeptical local newspaper to alert them of his observation and heard no more about it. About a week later, he learned that a satellite television broadcast on March 12 had mentioned that 911 days had passed between the 9/11 attack and 3/11 attacks. The story had spread fast in Ireland and farther afield, and soon thereafter the *Cork Examiner* belatedly credited him with his original insight.

Since early childhood, Owen has been fascinated with mathematical puzzles and number patterns. He is enchanted by the primes and how they are distributed along the number line. He considers himself a mathematical journalist, as did Martin Gardner. He is somewhat akin to the soccer reporter writing for the local newspaper: one who tries to convey the excitement of the match when writing his reports, giving various facts and figures to help the reader understand the game, but he may not be a soccer player himself. Owen does something similar when he writes about mathematics. He is not a mathematician per se but rather someone whom some others consider to be skilled in writing about the subject, particularly recreational mathematics.

Owen's writing has appeared in the *Cork Examiner*, the *Evening Echo* (Cork), *EdgeScience*, the *Journal of Recreational Mathematics*, and the *College Mathematics Journal*. He has written this book with a view to making mathematical puzzles attractive and interesting to the layperson. There is no difficult-to-understand mathematics in this collection. The aim of the book is to educate and amuse and to reveal to the layperson in a fun way the manner in which puzzlers solve mathematical brainteasers.

Owen thinks of the integers (the whole numbers) as his friends. They are faithful friends in the sense that their properties never change. Owen believes that the theorems of mathematics are timeless, eternal, unchanging, universal truths that we *discover* rather than *invent*.

Readers will discover a great deal of thought-provoking material in the pages that follow. Enjoy!

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INTRODUCTION

Recreational mathematics is an umbrella term used to describe the pursuit of mathematics for self-entertainment and educational purposes. Professional mathematicians and amateurs alike are counted among its many followers.

Recreational mathematics involves the investigation of numerous number patterns, magic tricks with numbers, paradoxes, fallacies, mathematical games, and, of course, mathematical puzzles. No specialized knowledge in advanced mathematics is required, and that is why so many amateurs get interested in the hobby. Many of these amateurs have been inspired to make further inquiries into mathematics generally, and some of them have made substantial and deep discoveries in this field that may not have been made were it not for their initial interest in the recreational side of mathematics.

One way of defining recreational mathematics is to say that it involves taking a playful approach to the whole concept of mathematics. Recreational mathematics is what it says it is: *recreational*. This book concentrates on mathematical puzzles—but not just ordinary mathematical puzzles. This book is written with a view to including *only* mathematical puzzles that have a *surprising* solution.

I have always believed that recreational mathematics should be used to gain a student's interest in mathematics. My opinion on how mathematics should be taught rests on a basic and core belief: the pupil must *like* mathematics to be good at it. If a young person likes, say, baseball, there is a good chance that he or she will be good at baseball. Similarly, if a young person can be coaxed to enjoy tackling interesting mathematical puzzles, there is a good chance that that young person or adult will become good at mathematics in general and certainly good at clear, logical thinking in particular.

Thus, one of the spin-offs of encouraging people to pursue puzzle solving as a hobby is that society as a whole benefits if a sizable proportion of its members can solve mathematical puzzles because that in turn means that a reasonable number of individuals in society can use logic and reasoning to confront and solve the daily problems of life that we all face. All right-minded persons know that we need to use logic and reasoning in the world now more than ever before to solve the many problems that face humanity.

Even if a child or student finds it difficult to solve a mathematical puzzle, he or she should be encouraged and guided step-by-step through the solution. The student may then encounter what Martin Gardner (1914–2010), the famous popularizer of mathematical recreations, described as an *aha moment*. When the step-by-step solution is shown to the student, he or she will see not only the solution but also, it is hoped, *why* there is that particular solution.

When one has hooked a student on the subject of mathematics, then of course that student should gradually be introduced to the various and more serious branches of the subject. But his or her initial interest in and perhaps love of puzzles will instill in him or her a sense of wonder that will last a lifetime.

There are those, of course, who say that mathematical puzzles are frivolous and that valuable time should not be wasted on giving these puzzles to teenagers or young adults.

I would strongly argue against this point of view.

Some of the greatest minds in history have pursued recreational mathematics as a hobby. In the twelfth century, the Italian mathematician Leonardo Fibonacci (ca. 1175–1250) offered a puzzle concerning how quickly rabbits reproduce.[1] The solution of the puzzle popularized the famous Fibonacci sequence. The great English scientist and mathematician Isaac Newton (1642–1727), who could not by any stretch of the imagination be described as a congenial character, formulated a few mathematical puzzles that can be described as recreational. The outstanding American theoretical physicist Richard Feynman (1918–1988) enjoyed solving mathematical puzzles, including the famous puzzle about the floating hat, which appears in this collection as puzzle 115.[2] The recreational mathematics popularizer Martin Gardner once wrote that the Danish engineer and inventor Piet Hein told him that whenever he visited Albert Einstein (1879–1955), Piet found a section of Einstein's bookshelves stocked with mathematical games and puzzles.[3]

It should not, however, be assumed that recreational mathematics is the sole preserve of the great minds among us. The average layperson who may have no previous experience solving mathematical puzzles can begin to take an interest in this hobby. As the old adage says, the start of a thousand-mile journey begins with the first step. The great thing about solving a mathematical problem is that like all other exercises, the more of it one does, the better one gets at it. Also, once one gets started, one will begin to learn to actually *enjoy* solving mathematical puzzles.

We all tackle puzzles of one sort or another on a daily basis: Should I bring my umbrella with me on my way to work when the morning is dark and cloudy? Should I take this particular job that pays such and such a salary or perhaps take the better-paid job that is sixty miles from my home, which will cost a lot in gasoline per week and make my workday much longer? Should I vote for this politician who promises this and that, or should I vote for the other politician who seems more grounded in reality?

This is the type of puzzle solving that we are all familiar with. To have a successful and happy life depends to some (perhaps a great) degree on our ability to solve puzzles.

Of course, the difference between these everyday puzzles and the mathematical problems found in puzzle books is that generally the mathematical puzzles found in the literature are a pleasure to work on. Most successful mathematical puzzlers admit that there is a wonderful sense of achievement (sometimes exhilaration!) in solving an interesting mathematical problem. It is this great thrill, this sense of achievement that the amateur feels, that is similar to the feelings of a professional mathematician when he or she conquers a more advanced problem that has long resisted attempts at yielding a solution. It is this sense of discovering something previously unknown to the amateur that is similar to the feeling the scientist experiences when he or she solves a puzzle posed by nature.

Mathematical puzzles can be divided into two groups: those that have a mundane solution and those that have an interesting solution. It is puzzles only from the second category that will be found in this book.

To see the difference between the two types of puzzle, consider the following problem, which has appeared in various forms online and in puzzle books over the years:

> Alec can build a wall by himself in 3 days. Brian can build the same size wall in 4 days, while Charlie takes 5 days to build a similar wall. All three men work at a constant rate. If the three men worked together, how long would it take to build the wall?[4]

I would solve this puzzle by first finding a number that is a multiple of the three numbers, 3, 4, and 5. The smallest such multiple is 60. I would then recognize that Alec can build 20 such walls in 60 days; Brian can build 15 walls in 60 days, and Charlie can build 12 walls in 60 days. Thus, all three of them working can build 20 + 15 + 12, or 47, walls in 60 days. Therefore, if all three men work together, they will build one wall in $^{60}/_{47}$, or 1 and $^{13}/_{47}$ days.

The answer to this problem is *not* surprising. No one can look at this little puzzle and truthfully say, "Gosh! What a wonderful puzzle! What a surprising answer! I was certain that the answer was such and such. I am really surprised that the answer is 1 and $^{13}/_{47}$ days."

This mathematical problem and others similar to it may be interesting to some. But these types of puzzles will not be found in this little book. The problems in this book will be more like the following ancient puzzle that is suitably dressed to fit in with the modern world:

> Two wealthy men, Smith and Jones, decide to invest some money in property. Smith invests $500,000, and Jones invests $300,000. They buy three beautiful, identical houses. They each take possession of one house and decide to sell the third house for $800,000. How should they fairly divide the money they received for the house?[5]

The solution of this puzzle is not at all obvious to the casual reader.

Many people will, it appears, say that the sum of money the two men received when they sold the house should be divided in a 5-to-3 ratio because that was the ratio of the sums of money initially invested by both men. Thus, Smith should receive $500,000 and Jones $300,000.

But, surprisingly, this division of the money is unfair and incorrect!

Here is the correct way to solve this puzzle. Both men bought three identical houses. These three houses cost a total of $800,000. Therefore, each house costs $800,000 divided by 3, or $266,666.67 each.

Smith contributed $500,000 to the investment project. Therefore, he has contributed $233,333.33 *more* than the cost of each house to the investment project. Jones has contributed $300,000 to the investment project. Therefore, Jones has contributed $33,333.33 *more* than the cost of each house to the project.

Each man received a house in return for his investment. When this is accounted for, the amount of money left in the two investments looks like this:

Smith	Jones
\$233,333.33	\$33,333.33

Thus, at this point, Smith has seven times more money invested in the project than Jones has.

Consequently, the investment return of \$800,000 that they received when they sold the house should be divided in a 7-to-1 ratio in favor of Smith. In other words, Smith should receive \$700,000 and Jones \$100,000.

Now, this puzzle does have a surprising and totally unexpected solution! I can almost hear the reader say that he or she was certain that the correct way to divide the money obtained when the third house was sold was in a 5-to-3 ratio in favor of Smith!

When the above solution is presented to many people, they still find it difficult to believe. Generally, though, those who are good at solving mathematical puzzles will obtain this solution themselves or will gratefully and pleasantly accept the surprising solution when it is shown to them.

This is the type of problem you will find in the pages of this book. Either the solution itself will be surprising or the method used in finding the solution will be surprising.

* * *

Two of the great inventors of mathematical puzzles were the American Sam Loyd (1841–1911)[6] and the Englishman Henry Ernest Dudeney (1857–1930).[7] Both men were mathematical puzzle geniuses. Many of the puzzles found in Dudeney's books are also found in Loyd's works and vice versa. It is almost impossible to say who borrowed the most from each other. In addition to mathematical puzzles, Loyd was able to produce mechanical puzzles that caught the world's imagination in a way that Dudeney's puzzles did not. However, many of Dudeney's mathematical puzzles are intellectually superior to those of Loyd. Because of that, many puzzle buffs believe that of the two, Dudeney was probably the better mathematician. There is hardly a mathematical puzzle book in print today in any part of the world that does not contain some of the work of either of these two outstanding pioneers in the art of puzzle making.

Sam Loyd had a son named Walter (1873–1934). He adopted his father's name and became known as Sam Loyd Jr. in adulthood. It had been originally thought that Loyd Jr.—although interested in mathematical puzzles—did not possess the same inventiveness as his famous father. But recent research appears to indicate that Sam Loyd Jr. was also a mathematical genius, equally as good at puzzle making as the elder Loyd. The younger Loyd is said to have created 10,000 puzzles in his lifetime.[8] When Sam Loyd Jr. died in 1934, he was laid to rest near his father in Maple Grove Cemetery in Kew Gardens in New York City.[9]

Both Sam Loyd Sr. and Henry Dudeney contributed their puzzles to newspapers and magazines to entertain the readers of those publications. Both began contributing puzzles from an early age and continued to do so throughout their lives. Their problems were of a remarkably high standard, and since their time, no one has come anywhere near equaling the quality and output of their puzzles.

Both Loyd and Dudeney and Loyd Jr. were always open to the idea that the solutions to the puzzles they offered could be improved on. One of Sam Loyd Sr.'s puzzles, *Bixley to Quixley* (puzzle 56 in this collection), was correctly solved by Loyd. However, decades later, a correspondent of the late Martin Gardner named Ronald C. Read of Kingston, Jamaica, offered a surprisingly simpler solution to Loyd's problem. Both Loyd's and Read's solutions are included in this collection.

There were, of course, other inventors of mathematical puzzles besides the men mentioned above. The inventors of many mathematical problems (it is usually quite difficult to state with certainty who first stated or invented a particular puzzle) produced these puzzles in bygone days to meet people's expectations to be challenged and entertained simultaneously. Many of these mathematical problems were passed down from one generation to the next in a similar way that songs were often passed on through the generations. The fact that these puzzles are still enjoyed today in the modern world is testimony to their quality and appeal.

Being able to solve mathematical puzzles depends to a great degree on the ability to think clearly and to reason correctly rather than having any particular knowledge of mathematics. There are of course puzzles, particularly the more difficult and advanced puzzles, that are best tackled by someone with an extensive knowledge of mathematics and mathematical techniques. But, by and large, mathematical puzzles can be tackled by the average layperson.

The puzzles in this book are suited to the intelligent and curious reader who wants to try his or her hand at some of the puzzles that have occupied and indeed captivated the minds of numerous people down through the years. The reader will not require a PhD to solve any of the puzzles in this book.

I have graded the puzzles in this collection into three parts: *Easy Puzzles, Easy Teasers*; *Challenging Teasers*; and *Slightly More Challenging Puzzles*. My selections, of course, are subjective, but I trust that most readers will concur with my choices. The *Easy Puzzles, Easy Teasers* are offered to lure the novice into the hobby of puzzle solving. The *Challenging Teasers* are presented for the serious puzzle solver. The *Slightly More Challenging Puzzles* are offered to those who like their puzzles a little more difficult than average.

Of course, while the primary aim of the puzzle solver is to obtain the solution to the puzzle at hand, it should be remembered that often finding the correct *solution method* is equally as—if not more important than—finding the answer. By finding the correct method of solution, the puzzler is arming him- or herself so as to be able to conquer a similar puzzle should one be encountered at a later stage.

* * *

I wish to take this opportunity to acknowledge here the outstanding contribution to recreational mathematics that the late Martin Gardner made in his long and illustrious career. He wrote a monthly column on the subject for *Scientific American* for twenty-five years beginning in January 1957. Martin's columns have been collected into fifteen volumes, which are still available today. They are essential reading for the serious recreational mathematician. Persi Diaconis, professor of mathematics at Stanford University, has said that Martin Gardner was responsible for turning dozens of innocent youngsters into math professors and thousands of math professors into innocent youngsters.[10] Martin also wrote many books on science and philosophy. All of his books, no matter what the subject, are worth reading.

It is my wish that you will enjoy solving some (if not all) the puzzles found between the covers of this book. But even if you do not succeed in solving all the problems, your attempts to crack the puzzles will perhaps bring you some insight into the world of mathematics. If you take the time to read through the solutions, you will almost certainly pick up some of the skills that puzzlers and mathematicians use in solving mathematical problems, and that will almost certainly make you a better puzzle solver. If you become a better puzzle solver, your life will almost certainly improve because when you think about it, life is all about solving puzzles.

EASY PUZZLES, EASY TEASERS

I have selected the puzzles in this section because the question asked in each problem is easily understood, even to a child of ten. There is very little mathematical ability needed to solve these puzzles. But clear thinking is required. The puzzles are a gentle warm-up to some of the harder puzzles that are found later in the book. However, a word of warning! The answer to each of the problems might appear obvious, but that does not mean that the obvious answer is the correct one.

1

THE ROPE LADDER

A ship with a rope ladder hanging over its side is at anchor. The rungs on the rope ladder are spaced exactly 1 foot apart. It is low tide, and the bottom rung of the ladder is exactly 1 foot above the level of the seawater. The tide is rising at the rate of 1 foot every hour. It will be high tide in 6 hours. How many of the rungs of the rope ladder will be submerged in the water at high tide?

SOLUTION

None! The ship and the attached rope ladder will rise with the tide. Therefore, none of the rungs of the rope ladder will be submerged in the water.

SOURCE

"Answer to the Friday Puzzle," http://www.richardwiseman.wordpress.com/2013/07/22/answer

2

THE BOTTLE AND THE CORK

A bottle and a cork cost \$1.10. The bottle costs \$1.00 more than the cork. How much did the bottle cost?

SOLUTION

Many people answer this by saying the bottle costs $1.00 and the cork 10 cents. But that means that the bottle costs 90 cents more than the cork. We were told that the bottle costs $1.00 more than the cork. So the solution must be that the bottle costs $1.05 and that the cork costs 5 cents.

SOURCE

"Puzzle Playground—A Bottle and a Cork," https://riddlesbrainteasers.com/bottle-cork/

3

THE NEWSPAPER ROUND

Young Johnny obtained a job delivering newspapers in a specific estate in his neighborhood.

Last week, Johnny's boss told him that his store was going to give out some free newspapers as a plan to increase sales. He told Johnny that on the following Friday, he was to deliver 1 newspaper—and 1 newspaper only—to every house that has the number 9 written on the hall door.

There are 100 numbered houses in the relevant estate. How many newspapers will Johnny deliver if he carries out his instructions?

SOLUTION

Surprisingly, Johnny will deliver 19 newspapers. When presented with this puzzle, most people will calculate as follows: Johnny will put a newspaper in houses numbered 9, 19, 29, 39, 49, 59, 69, 79, 89, and 99. That's 10 houses. Since Johnny delivers 1 newspaper to each house, he will deliver 10 newspapers. But he will also deliver 1 newspaper to each of the houses numbered 90, 91, 92, 93, 94, 95, 96, 97, and 98. That is an additional 9 newspapers. Therefore, Johnny will deliver a total of 19 newspapers.

SOURCE

James F. Fixx, *Games for the Super-Intelligent* (London: Frederick Muller Ltd, 1977), pp. 8, 70.

4

CIGARETTE BUTTS

A vagabond who likes smoking collects the discarded ends of cigarettes, which he calls *butts*.

He has a little machine that can make 1 whole cigarette from 6 butts. One morning within the space of 1 hour, he collected 36 butts.

How many cigarettes can he make from those 36 butts?

SOLUTION

Most people will say 6 cigarettes. But, surprisingly, the vagabond can make 7 cigarettes from the 36 butts. You see, he first makes 6 cigarettes from the 36 butts. He then smokes these cigarettes at his leisure. He will have 1 butt left over from each cigarette. He can then make another cigarette from these 6 butts. Thus, the total number of cigarettes he can make from the 36 butts is 7.

SOURCE

"Can You Solve the Cigarette Miser Puzzle?," http://www.curiosity.com

5

WHICH TANK WILL EMPTY FIRST?

There are 2 large identical tanks in an industrial estate. Underneath 1 tank, in the middle of the bottom surface, a circular hole 8 inches in diameter is cut. The hole is then plugged. Underneath the second tank, 2 circular holes, each 4 inches in diameter, are cut in the middle of the bottom surface. Each of these 2 holes is plugged as well.

The 2 tanks are filled. Each tank now contains an equal amount of water.

The 1 plug in the first tank and the 2 plugs in the second tank are all withdrawn simultaneously.

Which tank will empty first?

SOLUTION

Consider the tank with the 1 circular hole drilled into the bottom of it. Its diameter is 8 inches, so its radius is 4 inches. The area of a circle is $Pi \times r^2$, where *Pi* equals 3.14159 . . . and *r* is the radius. Thus, the radius of that hole is $Pi \times 4^2$, or 16*Pi*.

The radius of each of the 2 holes in the second tank is $Pi \times 2^2$, or 4*Pi*. Thus, the area of these 2 holes is 8*Pi*.

Therefore, the area of the 8-inch-diameter hole is twice the area of the combined 2 holes in the second tank. Consequently, the tank with the 8-inch-diameter hole will empty twice as fast as the second tank with the 2 holes drilled in it.

SOURCE

The author.

6

HOW LONG IS THE LINE AB?

Figure 1 shows a quadrant of a circle whose radius is 10 units. Inside the quadrant there is a rectangle drawn. The line AB is drawn inside the rectangle. How long is the line AB?

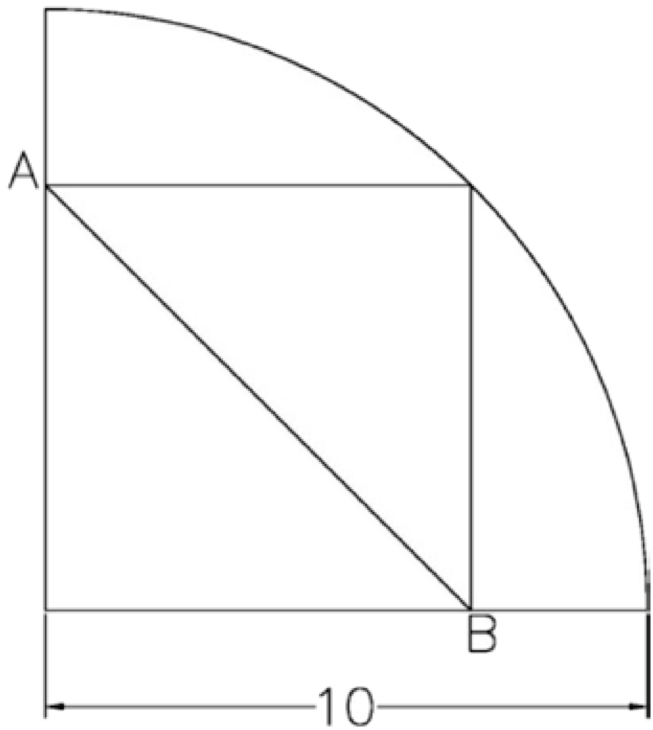

Figure 1

SOLUTION

Consider Figure 2. It can clearly be seen that AB and CD are diagonals of the rectangle inside the quadrant. It can also be seen that CD is the length of the radius of the circle, which is 10 units. Since AB is equal in length to CD, we conclude that AB is 10 units long.

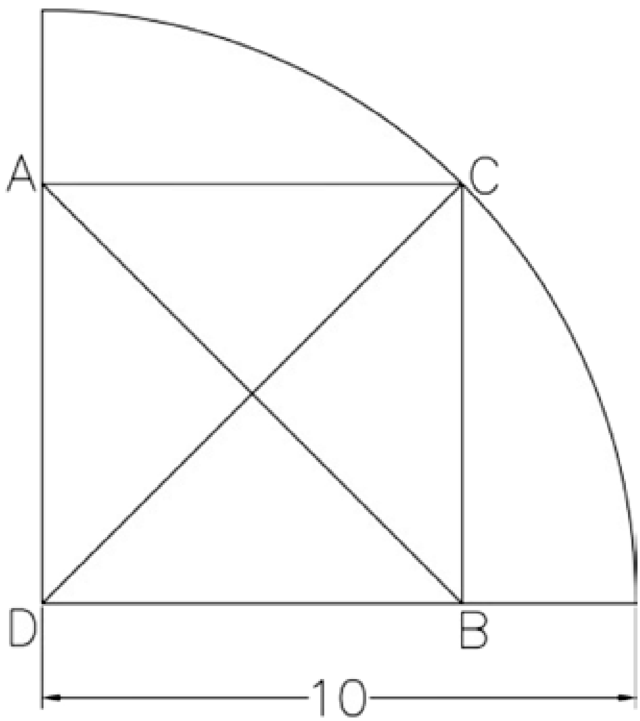

Figure 2

SOURCE

Martin Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions* (New York: Simon & Schuster, 1959), pp. 99, 100, 105.

7

BLANEY TO SLANEY

Smith goes on a car journey from Blaney to Slaney.

Exactly halfway there, he sees a friend, Jones, standing on the side of the road. He picks up Jones and drives to Slaney. Two hours later, Smith and Jones head off on the return trip to Blaney. Smith drops off Jones at precisely the place he had picked him up earlier in the day. Jones offers to pay his share of the costs of the trip. Smith tells Jones he will use $60 worth of gasoline for the entire trip.

How much should Jones pay for his share of the costs of the journey?

SOLUTION

Many people will answer this by saying that Jones should pay half of the costs; that is, he should pay $30 to Smith. But this answer is incorrect.

Others will argue that Jones traveled half of the total distance of the journey and thus should pay half of what Smith pays for the trip. Since Smith believes the journey required $60 worth of gasoline, Smith should pay $40, and Jones should pay $20 toward the cost.

But this second answer is also incorrect.

Here is the correct solution.

Jones has been in Smith's car for exactly half the journey. Therefore, Smith and Jones share equally half the trip, and therefore each should pay half of what half the trip costs.

Since Smith says that he believes he used $60 worth of gas on the entire journey, half that journey costs $30. Thus, Jones should pay Smith one-half of $30, or $15, for the gas used.

In other words, the cost of the gasoline used for the entire trip should be divided by 4.

Three-quarters of the cost of the gasoline should be borne by Smith and the other quarter by Jones.

This answer surprises many people.

SOURCE

Samuel Evans Clark, *Mental Nuts: Can You Crack 'Em?*, rev. ed. (New York: The Home Monthly, 1900), problem 95, pp. 26, 32.

8

THE CASE OF THE MISSING DOLLAR

Three men ordered dinner in a restaurant. The waiter told the 3 men that each dinner would cost $10. Each of the men paid $10. The waiter went to the cashier with $30.

The cashier told the waiter that there had been a mistake in pricing the dinners. The 3 dinners, he said, cost a total of only $25. The cashier took the $30 from the waiter. He then gave the waiter 5 $1 bills and instructed him to give the $5 back to the 3 men. However, the waiter was dishonest, so he kept $2 for himself. He then handed $1 back to each of the 3 men.

That meant that in reality, the 3 men had paid $27. The waiter had kept $2. That makes a total of $29.

But initially, the 3 men had paid $30. Where did the missing dollar go?

SOLUTION

The addition of the waiter's $2 to the $27 paid by the men is a meaningless operation.

Look at it this way. The men initially paid $30. The cashier gave the waiter $5, instructing him to return this $5 to the 3 customers. It is important to recognize that at this point, there is now only $25 in the cashier's till.

Now since each man had $1 returned to him, it meant in reality that each of the men paid $9. That is a total of $27. But there is only $25 in the till. Thus, the $27 that the 3 customers paid must consist of the $25 in the till *and* the $2 that the waiter has kept. That accounts for $27.

The $3 that was returned to the customers is added to the $27, making a total of $30.

When the question was originally put, one is told that each of the men had $1 returned to him, which meant that each customer paid, in reality, $9. So far, so good. But now adding the $2 that the waiter kept to the $27 the men paid is a meaningless calculation. The $27 the men paid *already consists* of the $2 the waiter had kept *and* the $25 that is in the till. It is an incorrect mathematical procedure to add the $2 the waiter stole to the $27 paid by the 3 customers.

SOURCE

Jerome Sydney Meyer, *The Big Fun Book* (New York: Grolier Society, 1948), pp. 55, 195.

9

HOW LONG IS THE SIDE OF THE INNER SQUARE?

The following is a little problem that can easily be solved given the right approach.

Figure 3 shows 2 squares of unequal size that are placed inside a circle. The radius of the circle is 7 units. How long is the side of the smaller square?

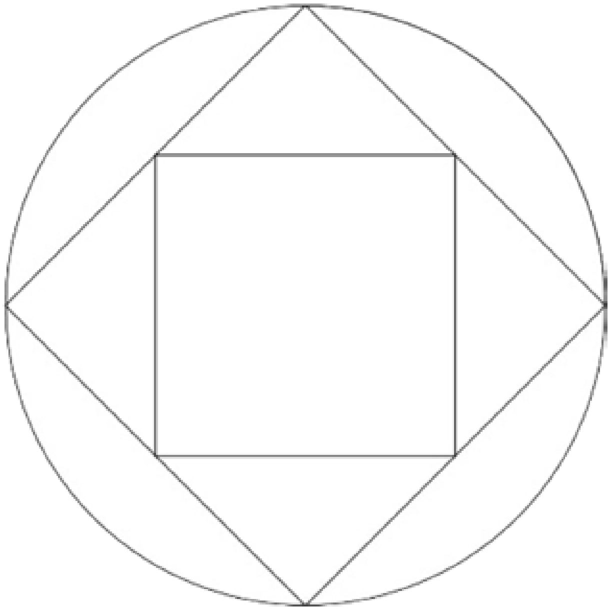

Figure 3

SOLUTION

The illustration in Figure 4 shows that the broken line from the center of the circle to the circumference is equal to the radius of the circle, which is 7 units. Since this line is equal in length to the side of the smaller square, we know that the length of the side of the smaller square is also 7 units. The solution is surprisingly obvious when one looks at Figure 4.

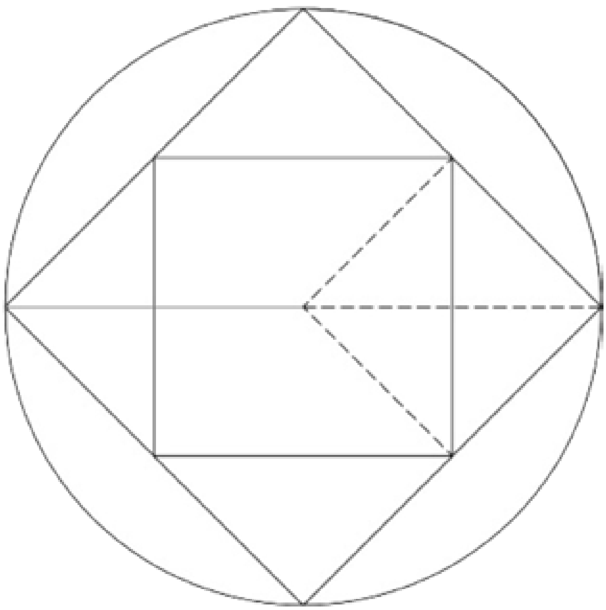

Figure 4

SOURCE

Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (New York: W. H. Freeman, 1989), pp. 302–03.

10

THE TRAIN LEAVING AND RETURNING TO THE STATION

A train leaves a station traveling at a constant speed of 20 miles per hour. The train reaches its destination. It then travels back to the original station traveling at a constant speed of 30 miles per hour.

What is the average speed of the train?

SOLUTION

Most people will answer 25 miles per hour when given this puzzle. But, surprisingly, the correct answer is that the train's average speed is 24 miles per hour!

We are not given any distance that the train travels in the puzzle. The distance the train travels is actually irrelevant.

Let's assume any distance. Assume the train travels a distance of 60 miles. Traveling at a constant speed of 20 miles per hour, the train will take 3 hours to get to its destination.

Traveling at a constant speed of 30 miles per hour, the return journey will take 2 hours.

Altogether, the train travels on the round trip a distance of 120 miles. It takes the train 5 hours to do this. Therefore, the average speed of the train is 24 miles per hour.

If you still doubt this, try out any distance yourself, and you will find that the train travels at an average speed of 24 miles per hour.

SOURCE

"Average Speed," http://www.braingle.com/brainteasers/960/average-speed.html

11

THE HEIGHT OF THE STACK OF PAPER

Suppose one has a giant square piece of paper that is one-hundredth of 1 inch thick. Suppose also that one cuts this large piece of paper into equal parts and then stacks one-half on top of the other half. Then one cuts these 2 pieces of paper down the middle, producing 4 equal-sized pieces. You then stack these 4 pieces one on top of the other and then cut these pieces down the middle, producing 8 equal pieces.

Suppose you continue this cutting, stacking, and cutting procedure for a total of 50 times. How high is the eventual stack after the 50th cut?

SOLUTION

Believe it or not, after the 50th cut, the stack will be more than 179 million miles high!

Most people refuse to believe this most surprising answer when they first encounter it. But it is true nevertheless.

When you make the first cut, it will produce 2 pieces of paper. When you make the second cut, it will produce 4 stacks of paper. When the third cut is made, it will result in 8 pieces of paper. There is a pattern here. After the first cut, there will 2^1, or 2, pieces of paper; the second cut will produce 2^2, or 4, pieces; the third cut will produce 2^3, or 8, pieces of paper. Thus, after the 50th cut, there will be 2^{50} pieces of paper.

If you have a scientific calculator nearby, punch in the value of 2^{50}. The result will be a 16-digit number. That is the number of pieces of paper in the stack after 50 cuts. Every 100 pieces of paper equals 1 inch in thickness. So divide 2^{50} by 100. The result is the height of the stack in inches, which is a 14-digit number. Divide that number by 12. The result is the height of the stack in feet. Divide that result by 3. The result is the height of the stack in yards. Divide that result by 1760 to find out how high the stack is in miles.

The answer is 177,698,848.93 miles! That is nearly twice the distance as the Earth is from the sun!

SOURCE

Owen O'Shea, *The Magic Numbers of the Professor* (Washington, DC: Mathematical Association of America, 2007), pp. 121–29.

12

THE CATERPILLAR

A caterpillar is climbing from the bottom of a 20-foot-deep well at the rate of 3 feet per day, but he falls back 2 feet each night. How long does it take the caterpillar to get out of the well?

SOLUTION

On being presented with this problem, most people will reason as follows: Since the caterpillar is making progress at the rate of 1 foot per day and since the caterpillar is at the bottom of a well that is 20 feet deep, it will take the caterpillar 20 days to get out of the well.

That answers seems plausible. But that answer is incorrect!

The caterpillar is indeed making progress at the rate of 1 foot per day. Therefore, at the end of 17 days, the caterpillar will have climbed 17 feet up the well. On the 18th day, he will climb another 3 feet and will be out of the 20-foot-deep well.

Consequently, the surprising answer is that it will take just 18 days for the caterpillar to climb out of the 20-foot-deep well.

SOURCE

"A Snail Is Climbing Out of a Well," http://www.interviewsansar.com/2016/05/24/a-snail-is-climbing

13

BROOKLYN VERSUS BRONX

Joe Kelly, who lives in New York, has 2 girlfriends. One lives in Brooklyn, and the other lives in the Bronx. He likes both of them equally well. Once every week, Kelly decides to visit one of them. He goes to his local underground station at a random time every Saturday. Kelly finds that trains going to Brooklyn leave the station every 10 minutes. Similarly, trains going to the Bronx leave every 10 minutes but obviously go in the opposite direction.

For some odd reason, Kelly finds that he is visiting his girlfriend in Brooklyn a lot more often than he visits his girlfriend in the Bronx. He takes note of this odd occurrence and finds that on average, 9 of 10 visits are to his girlfriend in Brooklyn compared to the 1 visit in 10 he makes to his girlfriend in the Bronx.

Can you think of a reason why this happens?

SOLUTION

The surprising answer is that it is all a matter of train schedules and the laws of probability. The trains for Brooklyn and the Bronx leave the station at intervals of 10 minutes. But the trains for Brooklyn leave on the hour and at 10-minute intervals after the hour. On the other hand, the trains for the Bronx leave at 1 minute past the hour and every 10 minutes after that.

Since Kelly arrives in the station every Saturday at a random time, it is much more likely that he will arrive during the 10-minute interval where the next train leaving the station is bound for Brooklyn. Only if he arrives in the 1-minute interval after the train for Brooklyn has left the station will the next train to leave be the train bound for the Bronx. The probability that Kelly arrives in this 1-minute interval and not in the other 10-minute interval is 1 chance in 10. Therefore, there are 9 chances in 10 that he will arrive in the 10-minute interval where the next train to leave the station is bound for Brooklyn. That explains why he ends up visiting his girlfriend in Brooklyn about 9 times more often than he visits his girlfriend in the Bronx.

SOURCE

"Brainteaser #15—Love Train," http://www.puzzlesandriddles.com/Brainteasers.html

14

THE COLOR OF THE BEAR

An old puzzle goes as follows: A man walks 1 mile due south, then he walks 1 mile due east, then he walks 1 mile due north. He is now back where he started. He shoots a bear.

What color is the bear?

The solution given in the puzzle books was that the man had to be at the North Pole. Every direction from the North Pole leads south; thus, at that spot, the conditions of the puzzle are met. Therefore, the bear had to be colored white.

Then around the middle of the twentieth century, it was discovered that the North Pole is not the only starting point that satisfies the given conditions in the problem. Can you figure out any other place on Earth where a person might start walking a mile south, a mile east, and a mile north and find himself back at the spot where he started his walk?

SOLUTION

The surprising answer is that there are an infinite number of places on Earth where a person could start a walk and go 1 mile due south, then 1 mile due east, and finally 1 mile due north and end up at the spot where he started. If a person started her walk at a distance of 1.16 miles from the South Pole, she could walk 1 mile due south, 1 mile due east (the walk due east would bring her one full circuit around the pole), and 1 mile due north, and she would end up at the starting point of her journey. Thus, the person's starting point could be any one of an infinite number of points on the circle around the South Pole with a radius of about 1.16 miles from the South Pole.

In fact, a person could start her walk closer to the South Pole so that the walk 1 mile due east would carry her not once but just twice around the Pole. Or she could start her walk closer still to the South Pole so that the walk due east would carry her three times around the pole, four times around the pole, and so on.

The interested reader may be curious as to why the explorer should begin their walk 1.16 miles from the South Pole. The starting point for the walk from the pole is 1.16 miles, which is equivalent to $1 + 1/(2\pi)$ miles from the South Pole. By choosing this starting point, it ensures that when the explorer has walked 1 mile due south, she will then be 0.159154943 miles from the South Pole. The walk around the pole will then be the circumference of a circle whose radius is 0.159154943 miles (or a diameter of 0.318309886 miles) around the South Pole. The walk due east (which brings the explorer around the pole in one complete circuit) is then equal to $\pi \times 0.318309886$ miles, which equals 1 mile. The explorer would then walk 1 mile due north, which would bring her back to her starting position.

SOURCE

Martin Gardner, *Mathematical Puzzles and Diversions* (London: Penguin Books, 1965), pp. 30, 34.

15

9 COINS

You have a double-pan weighing scale, where there is 1 pan on each side of the scale. The 2 pans are balanced against each other. The scales operate like a seesaw. You have been given 9 coins. Eight of the genuine coins weigh 5 grams. But 1 coin is counterfeit. It weighs 6 grams.

You are allowed 2 weighings on the scales. How do you identify the counterfeit coin?

SOLUTION

Surprising as it may seem, the counterfeit coin can be identified in just 2 weighings of the pans.

Here's how to do it. First, choose any 6 coins and place 3 in the pan on one side of the scale and place the other 3 coins in the other pan. One of 2 things will now happen: either the pans balance, or they don't. Let's assume for the moment that the pans balance. In that case, the 6 coins you weighed are all genuine. Discard these 6 genuine coins. Now select any 2 of the other 3 coins. Place 1 in each pan and weigh them. If one pan goes down, the coin in that pan is the counterfeit coin. If the pans balance, the coin that you did not weigh is the counterfeit coin.

Now go back to the first weighing, where you placed 3 coins in each pan. What happens if one pan goes down? In that case, that pan contains the counterfeit coin, along with 2 genuine coins. Discard all the other coins except these 3. Now select any 2 of these 3 coins and weigh 1 against the other. If one pan goes down, that pan contains the counterfeit coin. If the pans balance, then the coin that you did not weigh is the counterfeit coin.

SOURCE

"Answer to Riddle #65: 9 Coins, 1 Odd One, 2 Weighings," http://www.puzzles.nigelcoldwell.co.uk/sixtyfive.htm

16

2 TEENAGERS

Two teenagers, Mary and Peter, were having a chat. Mary said: “I was 14 years old the day before yesterday. But next year I will be 17.” If Mary spoke the truth, how was that possible?

SOLUTION

Suppose the chat was taking place on New Year's Day. Mary's birthday was on December 31 of the previous year (1 day previous) when she reached the age of 15. Thus, Mary was correct in saying that "she was 14 years old the day before yesterday." Thus, Mary was 15 years old on December 31 (1 day previous). She will reach the age of 16 on December 31 of this coming year. On December 31 of next year, Mary will reach the age of 17.

SOURCE

Erwin Brecher, *The Ultimate Book of Puzzles: Mathematical Diversions and Brainteasers* (New York: St. Martin's Griffin, 1996), problem 320, pp. 103, 249.

17

THE UNLUCKY STOREKEEPER

A stranger came into the local store one day last week. He asked Jones, the storekeeper, for a box of chocolates, which was priced at $5. The stranger tendered Jones a $50 bill. Jones told the man that he had no change for the $50 bill but that he would go next door to a pharmacy and obtain the necessary change there.

Jones went to the pharmacy and obtained the change from Smith, the owner of the pharmacy. Jones gave the chocolates and the change to the stranger, who left swiftly and was not seen in that village again.

Thirty minutes later, the pharmacist, Smith, came to Jones and told him that the $50 bill he had given him was a forgery.

"I'm sorry if you have been swindled," Smith said to Jones, "but I will have to ask you for a legal tender $50 bill." Jones had no option but to comply with Smith's request. He reached for his wallet and handed over a $50 bill to Smith.

The question is, how much did Jones lose over this unlucky transaction?

SOLUTION

Many people get the answer to this one wrong. The surprising answer is that Jones's loss in this transaction was a box of chocolates and $45.

Look at it this way.

The stranger tendered the forged $50 bill. Jones went next door to change the bill. He came back to his store with $50 in change. He gave the stranger the box of chocolates he had bought, plus $45 in change.

Thirty minutes later, the pharmacist, Smith, came to Jones and declared that the $50 bill Jones had given him was a counterfeit note. Jones accepted that and handed Smith $50. Thus, Jones repaid the $50 to Smith that he had obtained 30 minutes previously.

Thus, the situation now between Jones and Smith is that neither owes the other anything. Therefore, the loss incurred by Jones amounts to a box of chocolates and $45 that he gave the swindler in change.

SOURCE

Gilbert Wilkinson, *The Complete Home Entertainer* (London: Odhams Press, 1940), pp. 122, 123, 475.

18

BACTERIA IN A JAR

A jar in a science laboratory contains a number of bacteria. One morning at 11:00 a.m., a scientist in the laboratory notices that the bacteria in the jar are doubling every minute. The jar is completely full at noon. At what time was the jar half full of bacteria?

SOLUTION

We are told that the number of bacteria in a jar is doubling every minute and that the jar is completely full at noon. Therefore, the jar must have been half full of bacteria at 11:59 a.m. The answer surprises most people who have not come across this puzzle previously.

SOURCE

"Bacteria in a Jar," https://www.careercup.com/question?id=5216964236541952

19

3 CATS

If 3 cats can kill 3 rats in 3 minutes, how long will it take 100 cats to kill 100 rats?

SOLUTION

Three minutes!

Each cat takes 3 minutes to kill 1 rat.

Perhaps the following shows the results a little more clearly:

3 cats can kill 3 rats in 3 minutes.
1 cat can kill 1 rat in 3 minutes.
100 cats can kill 100 rats in 3 minutes.

SOURCE

"If 3 Cats Can Kill 3 Rats in 3 Minutes How Long Will It Take for 100 Cats to Kill 100 Rats?," https://riddles.tips/riddle-516

20

THE THIEF IN THE STORE

A thief went into a store and stole $100 from the cashier till and then left the store.

About 30 minutes later, the thief returned to the store and purchased an item that cost him $70. The thief paid for the item with the stolen cash and received $30 in change from the cashier.

The thief then left the store for a second and final time.

How much did the store lose?

SOLUTION

It is amazing how many people will answer this puzzle incorrectly. The surprising answer is that the store has lost *only* the $100 that was initially stolen.

Suppose the thief did not steal the $100 from the store. Suppose that having looked around the store for a few minutes, he then left the store and returned about 30 minutes later. He then purchased an item for $70. He tendered a genuine $100 bill. Thus, he received $30 in change. He then left the store.

If this were the case, the store would not have lost anything.

However, we are told that the man had initially entered the store and stole $100 from the cashier till. Consequently, the amount the store has lost is the $100 that was initially stolen.

The subsequent transaction by the thief only blurs the situation and confuses matters. For example, suppose another customer rather than the thief had purchased in the store the item for $70 and had received $30 in change and then left the store. This would have had no bearing on the loss to the store. The loss to the store would be the amount initially stolen, that is, $100.

SOURCE

"Thief in Grocery Store Riddle," http://www.puzzlefry.com/puzzles/thief-in-grocery-store-riddle

21

COUNTING BLONDE STUDENTS

There are 100 students in Richville High School. A number of the students in the school are blonde. Last Monday morning 99 percent of the students were present. However, only 98 percent of the students with blonde hair were present.

How many students in Richville High School have blonde hair?

SOLUTION

We are told that 99 of the 100 students were present, which means that 1 student was absent. We are also told that only 98 percent of the blonde students were present, so we know that the absent student was blonde (otherwise, 100 percent of blonde students would have been present).

We know from the question that 2 percent of blonde students were absent and that this 2 percent equals 1 student. If 2 percent equals 1 student, then 50 times 2 percent, or 100 percent, equals 50 students.

Thus, there were 50 blonde students at Richville High School. Last Monday morning, 49 of the blonde students, or 98 percent of them, were present. All 50 of the non-blonde students were present.

SOURCE

"Number Puzzle #08—How Many Blondes," http://puzzlesandriddles.com/NumberPuzzle08.html

22

THE COST OF SHOEING A HORSE

A farmer brings his horse to the blacksmith to be shod. The blacksmith tells the farmer that he will need to use 32 nails to shoe the horse. The farmer accepts the blacksmith's figure and asks how much that will cost.

The blacksmith answers by saying that his terms will be easy on the farmer and so proposes the following method of payment: the blacksmith will charge 1 cent for the first nail, 2 cents for the second nail, 4 cents for the third nail, 8 cents for the fourth nail, and so on, each nail costing twice the price of the previous nail.

The farmer thought about it for a moment and then agreed to the business proposal.

How much did it cost to shoe the horse?

SOLUTION

There is a very surprising answer to this problem.

The blacksmith will charge 1 cent for the first nail, 2 cents for the second nail, 4 cents for the third nail, 8 cents for the fourth nail, and so on, each nail costing twice the price of the previous nail.

Thus, the cost in cents of shoeing the horse forms the doubling series, beginning with 1. This sum of the doubling series, beginning with 2^0, equals

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + \ldots + 2^{29} + 2^{30} + 2^{31} = 2^{32} - 1.$$

We can see that the first nail will cost 2^0 cents. Any number raised to the power of 0 equals 1. So the cost of the first nail is 1 cent. Notice that 1 equals $2^1 - 1$.

We can see that the cost of the first 2 nails nail is $2^0 + 2^1$ cents. This equals 1 + 2 cents, or 3 cents. So the cost of the first 2 nails is 3 cents. Notice that 3 equals $2^2 - 1$.

We can see that the cost of the first three nails is $2^0 + 2^1 + 2^2$ cents. This equals 1 + 2 + 4 cents, or 7 cents. So the cost of the first 3 nails is 7 cents. Notice that 7 equals $2^3 - 1$.

You will notice the pattern here.

We can see that the cost of the first 32 nails is $2^0 + 2^1 + 2^2 + \ldots + 2^{31}$ cents. This equals $2^{32} - 1$, or 4,294,967,295 cents, or \$42,949,672.95. That means that the cost of shoeing the horse with 32 nails will cost more than \$42 million!

Readers who are unfamiliar with the rate of growth of the doubling series of numbers will find this answer not only surprising but also counterintuitive. If you care to check, you will see that the cost of the 10th nail *alone* is 2^9 cents, or \$5.12. The cost of the 20th nail *alone* is 2^{19}, or 524,288 cents, or \$5,242.88. The cost of the 30th nail *alone* is 2^{29} cents, or 536,870,912 cents, or \$5,368,709.12. The cost of the 32nd nail *alone* is 2^{31} cents, or 2,147,483,648 cents, or \$21,474,836.48. That means the 32nd nail *alone* costs more than \$21 million!

Of course, the price of each individual nail must be summed to get the total cost of shoeing the horse. As we have seen, that sum equals \$42,949,672.95!

This answer appears unbelievable to some folks, but it is true nevertheless!

SOURCE

"The Math Mom's Puzzles," http://www.themathmompuzzles.blogspot.com/2010/05/easy-money.html

23

THE 16-INCH RULER

Mr. Johnson was showing me a long ruler that he had bought. It was exactly 16 inches in length. The ruler had marks along its surface indicating various distances, such as inches, half inches, quarter inches, and so on.

Can the reader tell how many quarter-inch marks there are on the 16-inch ruler?

SOLUTION

Along every inch of the ruler, there will be 2 quarter-inch marks. (The half-inch marks are not quarter-inch marks.) Thus, there are 32 quarter-inch marks on the 16-inch ruler.

This answer comes as a surprise to most people when they hear it for the first time.

SOURCE

John Paul Adams, *We Dare You to Solve This!* (New York: Berkley, 1957), puzzle 161, p. 59.

24

PAY RAISE AND PAY CUT

An employee has been told by his employer that because of rescheduling and reorganization in the firm the employee's salary was going to be increased by 10 percent but that the following day he would receive a 10 percent pay cut. Thus, the employee was told that his wages would be back to the level they were before the raise and cut in pay and therefore that he had nothing to worry about. The employee was apparently happy with this arrangement.

Should the employee have been content with the company's proposal?

SOLUTION

Most people are surprised to learn that a 10 percent pay raise followed by a 10 percent pay cut will not restore the level of pay that was applicable before the pay cut.

Suppose the employee was earning \$800 per week. The 10 percent pay raise means his salary is now 110 percent of 800, or \$880 per week. The 10 percent pay cut that was implemented 1 day later means he will be earning 11 percent *less* than what he had been earning the previous day. In other words, he would be earning 99 percent of \$800, which is \$792 per week.

This means that the employee is earning \$8 less per week because of the pay raise and pay cut.

Generally, a 10 percent pay raise per week followed by a 10 percent pay cut leaves the employee 1 percent worse off per week.

Thus, the employee should feel aggrieved at the firm's proposal.

Incidentally, a 10 percent pay cut per week, followed by a 10 percent pay raise per week, will also leave the employee worse off by 1 percent per week.

SOURCE

Herbert McKay, *Party Night* (New York: Oxford University Press, 1940), puzzle 32, p. 183.

25

THE CLEANING LADY

Mrs. Jones has obtained part-time work cleaning offices in a skyscraper. She is told that her working hours are 3 days per week from 5:00 p.m. to 8:00 p.m. She is required to clean the floors numbered 10 to 13 and is paid $10 per hour to do it. Thus, she believes that she has 1 hour to clean each floor.

Why was Mrs. Jones dissatisfied with her job?

SOLUTION

Mrs. Jones presumed that she would have 1 hour to clean each floor. But this presumption is incorrect. She is expected to clean floors numbered 10 to 13 in 3 hours. But Mrs. Jones neglected to observe that the number of floors she had to clean was 4, not 3. Consequently, she had three-quarters of an hour to clean each floor, not 1 hour.

Presumably, Mrs. Jones was unhappy with this discovery.

SOURCE

"Learning How to Count (Avoiding the Fencepost Problem)," https://betterexplained.com/articles/learning-how-to-count-avoiding-the-fencepost-problem

CHALLENGING TEASERS

The puzzles in this section are a little more difficult than those previously given. However, the reader does not need to be a mathematics major to solve any of them! You can be assured that clear thinking will go a long way in helping to solve these problems. You will find some of the puzzles relatively easy, while others will prove a little more challenging. Each one has a surprising answer. Hopefully, you will have fun trying your hand at these conundrums.

26

THE LONELY MONK

One morning at 6:00 a.m., a monk ascends a mountain by way of a spiraling path that is at most 2 feet wide. He stays on this path at all times as he climbs the mountain. As he ascends, he stops every now and then for a rest. He reaches the mountain peak at 6:00 p.m.

He stays on the mountain top overnight.

The next morning at 6:00 a.m., the monk descends from the peak of the mountain. He stays at all times on the 2-foot-wide path as he descends from the peak. Every now and then, he stops for a rest. He reaches the bottom of the mountain at 6:00 p.m.

Is there a spot on the mountain that the monk occupied at precisely the same time of day on both days? If there is, prove it. If there isn't such a spot, prove that that is the case.

SOLUTION

We are told that at 6:00 a.m., the monk ascends the mountain by way of a narrow 2-foot-wide path that spirals around the mountain. He stops intermittently for a rest as he ascends. He reaches the top of the mountain at 6:00 p.m.

The next day at 6:00 a.m., the monk begins his descent of the mountain, staying on the narrow 2-foot-wide path as he does so. As he descends, he stops intermittently for a rest. He reaches the bottom of the mountain at 6:00 p.m.

Imagine now that as the monk descends the mountain, he meets a second climber who has taken the *exact* same ascending path as the monk did on the previous day ascending the mountain. No matter how complicated the path the 2 climbers takes, the 2 climbers will have to meet at some point on the mountain.

Therefore, there is a single spot on the mountain that the monk—both ascending and descending—must have occupied at precisely the same time on each of the 2 days.

SOURCE

Ivan L. Morris, *The Lonely Monk and Other Puzzles* (London: The Bodley Head, 1970).

27

THE TOWN COUNCIL CLOCK

The local town council has a clock that chimes every hour. At 1:00 a.m., it chimes once; at 2:00 a.m., it chimes twice; at 3:00 a.m., it chimes three times. And so on. At 6:00 a.m., the clock takes 6 seconds to chime 6 times. How long will it take the clock to chime 10 times at 10:00 a.m.?

SOLUTION

The surprising answer is that it takes 12 seconds for the clock to chime at 10:00 a.m.

At 6:00 a.m., the clock chimes 6 times, but there are only 5 intervals between the first chime and the sixth chime. Therefore, each interval equals $\frac{6}{5}$, or 1 and $\frac{1}{5}$ seconds. Consequently, the 6 chimes take a period of 5 times 1 and $\frac{1}{5}$ seconds, or 6 seconds. At 10:00 a.m., the clock chimes 10 times. Each interval between chimes equals $\frac{6}{5}$, or 1 and $\frac{1}{5}$ seconds. Thus, the total time for the 10 chimes is 1 and $\frac{1}{5}$ multiplied by 10. This equals 12 seconds. Thus, it takes 12 seconds for the clock to chime 10 times at 10:00 a.m.

SOURCE

Sam Loyd, *Sam Loyd's Cyclopedia of 5,000 Puzzles, Tricks, and Conundrums, with Answers* (New York: Morningside Press, 1914), pp. 318, 382.

28

THE BOOKS ON THE BOOKSHELF

Six different books are arranged on a bookshelf. The owner of the books, Atkins, decides to change the order of the books once every minute. He never repeats the order of the books. He finds that it takes him exactly 12 hours.

Atkins then arranges 9 different books on the shelf. Once again, he decides to order the books once every minute, without repeating the order. He finds that this takes him exactly 252 days. (For the sake of the puzzle, we assume Atkins can work 24 hours per day, 7 days per week.)

Atkins decides now to do the same thing again, but this time he has 15 books on the shelf.

How long does it take Atkins to order the books once every minute, without repeating the order of the books?

SOLUTION

The surprising answer is that it would take Atkins nearly 2.5 million years to make all the possible arrangements of the 15 books!

The number of ways 6 objects can be arranged is 6! The exclamation sign after the 6 is the factorial sign. It merely means that all the integers from 6 down to 1 are multiplied together. Thus, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$. This product equals 720. Thus, there are 720 ways 6 books can be arranged. Since each arrangement takes 1 minute, the 720 arrangements can be achieved in exactly 12 hours.

Similarly, 9 books can be arranged in 9!, or 362,880, ways. Making 1 arrangement of all 9 books every minute takes Atkins exactly 252 days.

Fifteen books can be arranged in 15! different arrangements. This equals 1,307,674,368,000 different ways. Making a different arrangement every minute takes the owner 908,107,200 days. Assuming a year consists of 365.25 days, this equals 2,486,262.012 years. In other words, it would take nearly 2½ million years to complete the task!

SOURCE

Isaac Asimov, *Book of Facts* (New York: Random House, 1997), p. 297.

29

THE BLACK AND WHITE MARBLES

Suppose you are presented with 3 sealed boxes. Three black and 3 white marbles are distributed in each of the 3 boxes so that 2 marbles are in each box. One of the 3 boxes is labeled B/B to indicate that there are 2 black marbles in that box. A second box is marked W/W to indicate that there are 2 white marbles in that box. The third box is labeled B/W to indicate that there is 1 black and 1 white marble in that box. You have been told that the 3 boxes are each labeled incorrectly. For example, the box marked B/B does not contain 2 black marbles. Similarly, the box marked W/W does not contain 2 white marbles. And so on. You are allowed to select 1 marble from 1 box, without seeing the color of the other marble in that box. Your job is to look at the color of the marble you selected and then determine the color of the marbles in each box.

SOLUTION

The surprising answer to this puzzle is that one need only take 1 marble from the box marked B/W to determine the color of marbles in all 3 boxes. Suppose the marble you choose from the B/W box is white. Then the color of the other marble in the box must be white also. If it were not, then the label on the box would be correct, which would contradict the information you were given.

Thus, you now have identified that the box marked B/W contains 2 white marbles. You now know that there are 4 other marbles, 3 of which are black, distributed in the other 2 boxes. Therefore, there must be 2 black marbles together in 1 box. These cannot be in the box marked B/B, or else the label on that box would be correct. Therefore, the 2 black marbles are in the box marked W/W. Consequently, the black and white marbles are in the box marked B/B.

Of course, similar reasoning would apply if the first marble you select from the box marked B/W is black.

SOURCE

"Puzzle about 3 Boxes with 2 Balls Inside (Black or White) with Mixed Labels on Them," http://www.math.stackexchange.com/questions/818597/puzzle

30

HOW FAR HAS THE FLY TRAVELED?

Two trains are traveling at a constant speed of 30 miles per hour toward each other. When the 2 trains are exactly 60 miles apart, a fly flies directly from the front of one train to the second train. On reaching the front of the second train, the fly immediately flies back toward the first train. The fly continues to do this, flying back and forth to the front of each train. The fly is traveling at a constant speed of 20 miles per hour. The 2 trains are approaching each other at a constant speed. Eventually, of course, the 2 trains collide, and the fly is killed.

How far has the fly traveled before it is killed?

SOLUTION

The straightforward solution to this little puzzle surprises most puzzlers. The 2 trains are 60 miles apart. They are traveling toward each other at a constant speed of 30 miles per hour. Thus, the 2 trains will collide in 1 hour. In that period of time, the fly has been traveling at a constant speed of 20 miles per hour.

Therefore, the fly travels a total of 20 miles.

No matter how complicated the fly's route may be, if he is traveling at a constant speed of 20 miles per hour, then in 1 hour he will travel 20 miles.

SOURCE

Martin Gardner, *Entertaining Mathematical Puzzles* (New York: Dover Publications, 1961), pp. 27–28.

31

5 LOAVES AND 3 GENTLEMEN

Two men, Alan and Brian, were out on a country walk. Alan had 3 small loaves of bread. Brian had 2 loaves. They sat down in a shady spot to eat. Just then, they were joined by a third man, named Charles. He had no bread.

The 3 men ate a similar amount. Charles left $5 to the other 2 men as payment. Alan took $3 since he had provided 3 loaves of bread to the gathering. Brian took the other $2 since he had provided 2 loaves.

Was this division of the $5 fair and correct?

SOLUTION

The surprising answer is that it was not fair. There were a total of 5 loaves shared equally by the 3 men. Therefore, each had eaten $\frac{5}{3}$ of a loaf. Alan had provided 3 whole loaves, but he has eaten only $\frac{5}{3}$ of a loaf. So he contributed another $\frac{4}{3}$ of a loaf to Charles. Brian had provided 2 whole loaves, but he had eaten only $\frac{5}{3}$ of a loaf. Therefore, Brian contributed $\frac{1}{3}$ of a loaf to Charles.

Thus, Alan contributes $\frac{4}{3}$ for every $\frac{1}{3}$ that Brian contributes. Therefore, their contributions are in the ratio of 4 to 1. Thus, of the $5 paid by Charles, Alan should receive $4, and Brian should receive $1.

SOURCE

Maurice Kraitchik, *Mathematical Recreations* (New York: Norton, 1942), problem 28, p. 28.

32

2 RED QUEENS AND A BLACK ACE

Suppose you are approached by a stranger who offers you the following bet. The stranger places 2 red queens and a black ace facedown on a table. You are told by the hustler that 2 of the cards are red queens and 1 of them is a black ace. The con artist also tells you that he knows where the black ace is located among the 3 cards on the table. You do not, of course, know how the 3 cards are distributed.

Let the position of the cards be designated A, B and C. Let us assume that the card at position B is the black ace. (Of course, you do not know this, but the hustler does.) The hustler now bets you $5 at even-money odds that you cannot point to the black ace. The probability you will correctly do this is ⅓. Suppose you point to position A as the card you believe is the black ace. The hustler now turns over 1 of the other 2 cards (in this case the card at position C) and reveals it to be a red queen. He removes that card from the table.

There are now only 2 cards on the table: A and B. One of the cards, A, is the card you chose. The hustler asks you whether you want to stick with your original choice or change your mind and choose the other facedown card, at position B, on the table.

What should you do? Should you stick with your original selection, or should you switch?

SOLUTION

It comes as a surprise to many people that if you decide to stick with your original choice, you will win this bet about 1 time in every 3 because the probability of finding the black ace is ⅓. The chances of you losing the bet are therefore ⅔. Thus, if you decide to stick with your initial choice, the odds favor the hustler winning the bet.

However if you decide to switch from your initial choice, A, to the other unturned card, the chances of you winning rises to ⅔, and the chance of you losing is ⅓.

Most people on being given this choice believe it does not matter whether one sticks with the initial choice or whether one changes one's mind and switches to the other unturned card.

However, it does matter!

Here's the thing. If you do not switch, your chance of picking the black ace is ⅓, so think of the other unturned card as the "winning card" with probability of ⅔. Therefore, if you *always* choose to switch, you will find that by doing so, ⅔ of the time you will switch to the unturned "winning card." Consequently, by switching, you double your chances from ⅓ to ⅔ of picking the black ace.

The true probability behind this proposition bet is so cleverly hidden that one of the greatest mathematicians who ever lived, Paul Erdős (1913–1996), was fooled by it.

SOURCE

Owen O'Shea, *The Call of the Primes* (Amherst, NY: Prometheus Books, 2016), pp. 77–86.

33

THE CURIOUS TRANSACTION THAT RESULTED IN A MISSING DOLLAR

Mr. Jones, accompanied with 2 of his pals, goes into a clothing store to purchase a shirt. He sees a shirt that he likes, which is priced at $97. However, Jones realizes that he has left his wallet at home and has no cash or credit card with him to purchase the shirt. He asks each of his 2 pals for a loan of $50. Each pal gives Jones a $50 bill. Jones purchases the shirt and receives $3 in change. He puts $1 in his pocket and gives $1 to each of his 2 pals, telling them that he will pay each of them the remaining $49 that he owes each of them when he gets home.

Driving home in his car, Jones did a quick mental calculation relating to the purchase of the shirt that day. He figured he owed each of his 2 pals $49. That equals $98. He had $1 relating to the transaction in his pocket. Adding that to the $98 made $99. But Jones knew he had borrowed $100. What Jones couldn't make out is what happened to the missing dollar.

Can you figure out what happened to the missing dollar?

SOLUTION

The surprising answer is that there is no missing dollar.

Jones borrowed 2 $50 bills, or $100. The shirt cost $97. When Jones purchased the shirt, he tendered the 2 $50 bills and received $3 in change. He put $1 in his own pocket and gave his 2 pals $1 each.

Thus, he now owed his 2 pals $49 each.

When Jones did his mental calculation relating to the transaction, he multiplied 49 by 2, obtaining 98. This told him that he owed $98 to his pals. This is correct. He then mistakenly added the $1 in his pocket to these $98 that he owes to obtain a figure of $99. However, this method of accounting is a bogus operation. It is meaningless to add the $1 in his pocket to the $98 that he owes.

Jones is correct in figuring that he owes $98 to his 2 pals, but this $98 that is owed already consists of the $1 in his pocket. So it is meaningless to add the $1 a second time in the calculation.

Recall that the shirt cost $97. The correct method of accounting would be to recognize that the shirt cost $97, plus there is $1 in Jones's pocket. This makes $98, plus the $2 that Jones gave to each of his pals, which makes $100.

Thus, there is no missing dollar.

SOURCE

https://www.quora.com/I-borrowed-50-from-mum-and-50-from-dad-to-buy-a-bag-costing-97-After-the-purchase-I-had-3-left-I-returned-1-to-dad-and-1-to-mum-and-reserved-1-for-myself-I-now-owe-49+-49-98-plus-the-1-I-reserved-for-myself-which-is-99-Where-is-the-missing-1

34

COUNTING THE NUMBER OF EARRINGS

Many years ago, a group of explorers in Africa encountered a native tribe. There were 800 women in the tribe. Some of them were wearing 2 earrings, some were wearing 1 earring, and others were wearing no earring. The explorers asked a member of their own group, a man named Jenkins, who was good at mathematics, to count the number of earrings the women in the tribe were wearing.

Jenkins, who was fond of mathematical puzzles, reported back but did not give the answer in a straightforward manner. Instead, he said that 3 percent of the women in the tribe were wearing 1 earring. Then he smiled and said that within the remainder of the group of women, half were wearing 2 earrings, and half were wearing none.

The question is, how many earrings in total were being worn by the 800 women?

SOLUTION

One way of tackling the problem is as follows. From the information given by Jenkins, we know that 3 percent of the 800 women, which equals 24 women, wore only 1 earring. That accounts for 24 earrings. Of the remaining 97 percent of the 800 women, one-half wore 2 earrings, and the other half wore none. Thus, one-half of 776, or 388, women wore 2 earrings. That accounts for 338 multiplied by 2, or 776 earrings. Thus, the total number of earrings worn by the 800 women is 776 + 24, or 800.

The surprisingly simpler solution to this problem is as follows. We are told that 3 percent of the women wore 1 earring. That accounts for 24 earrings. Since one-half of the remaining women wore 2 earrings and one-half wore no earring, that is the same as if each woman wore 1 earring. Therefore, all 100 percent of the 800 women were wearing 1 earring.

Therefore, the number of earrings worn by the 800 women is 800.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 4.9, pp. 87, 94.

35

THE BICYCLE RACE

Two brothers, Tom and Harry, decided to have a bicycle race. However, they had only 1 bicycle.

"I have been thinking about the fact that we have only 1 bike," Tom said to Harry. "However, I have worked out a plan. I hope you will be happy with it. The road between Rusty and Dusty has a very good surface all the way along and is completely straight and flat. Also, there are milestones along the road. I propose that I ride the bicycle from the first milestone to the fifth and that you ride the bike from the fifth milestone to the tenth. We will take our respective times with a stopwatch."

"That seems like a great idea," Harry said.

The race, however, was a failure. Can you spot the flaw in the young boys' plan?

SOLUTION

Many people will be surprised to learn that the race was a failure because the distance from the first milestone to the fifth is 4 miles, while the distance from the fifth milestone to the tenth is 5 miles.

SOURCE

Gilbert Wilkinson, *The Complete Home Entertainer* (London: Odhams Press, 1940), puzzle 4, pp. 119, 474.

36

THE WILL

A wealthy and very charitable man named John Brown died in our local small town recently. Mr. Brown stipulated in his will that every male resident of the town who was over 70 years of age was to each receive $1,000 from his estate. In addition, every female over 70 years was to receive $750. However, the sums of money were not to be paid automatically. Each eligible resident would have to complete special forms and prove their age to the satisfaction of the executor of Mr. Brown's will.

As things transpired, not every eligible male or female applied for this windfall. Two of every 3 eligible men made a claim, while 8 of every 9 eligible women completed the necessary forms.

In total, there were 30,000 residents in the town eligible for a legacy.

How much money was distributed to the senior citizens by Mr. Brown's will?

SOLUTION

We do not know how many eligible males or females applied for the windfall provided by Brown's will. Therefore, it appears that there is no way to calculate how much money was distributed by the will.

Surprisingly, the puzzle can be solved without this information.

This is how we can solve the problem.

We are told that 2 of every 3 eligible males applied for the windfall; therefore, the *average* amount paid to every eligible man in the town is (\$1,000/3) multiplied by 2, or \$666.67 (to the nearest penny).

We are also told that 8 of 9 eligible females applied for a payment. Therefore, the *average* amount paid to every eligible woman in the town is (\$750/9) multiplied by 8, or \$666.67.

Therefore, the *average* amount of money paid to the 30,000 eligible residents was \$666.67. If the average paid to all 30,000 residents was \$1,000, the total paid out would have been \$30,000. Since the average paid out to each of the eligible citizens was two-thirds of \$1,000, or \$666.67, the total sum of money that was paid out had to be \$20,000.

Hence, the numbers of each male or female who applied for the legacy is a matter of indifference. The puzzle can surprisingly be solved without knowing this information.

SOURCE

Gilbert Wilkinson, *The Complete Home Entertainer* (London: Odhams Press, 1940), chapter 8, pp. 124, 475, 476.

37

THE 2 CROSSING ROPES

Two ropes are tied to 2 posts as shown in Figure 5. The heights of the 2 posts are 84 feet and 60 feet. The crossing point of the 2 ropes is exactly 35 feet off the ground.

How far apart are the 2 posts?

SOLUTION

The surprising answer is that the posts can be any distance apart! Their distance apart does not affect the height of the point where the 2 ropes cross.

If the height of 1 post is a and the height of the second post is b, then the height of the crossing point is equal to $ab/a + b$.

Thus, in our example, to find the crossing point of the 2 ropes, we multiply the height of the 2 vertical posts: $84 \times 60 = 5{,}040$. Then we add the height of the 2 posts: $84 + 60 = 144$. Now we divide (84×60) by $(84 + 60)$. The result is 35.

Thus, the height of the crossing point is 35 feet.

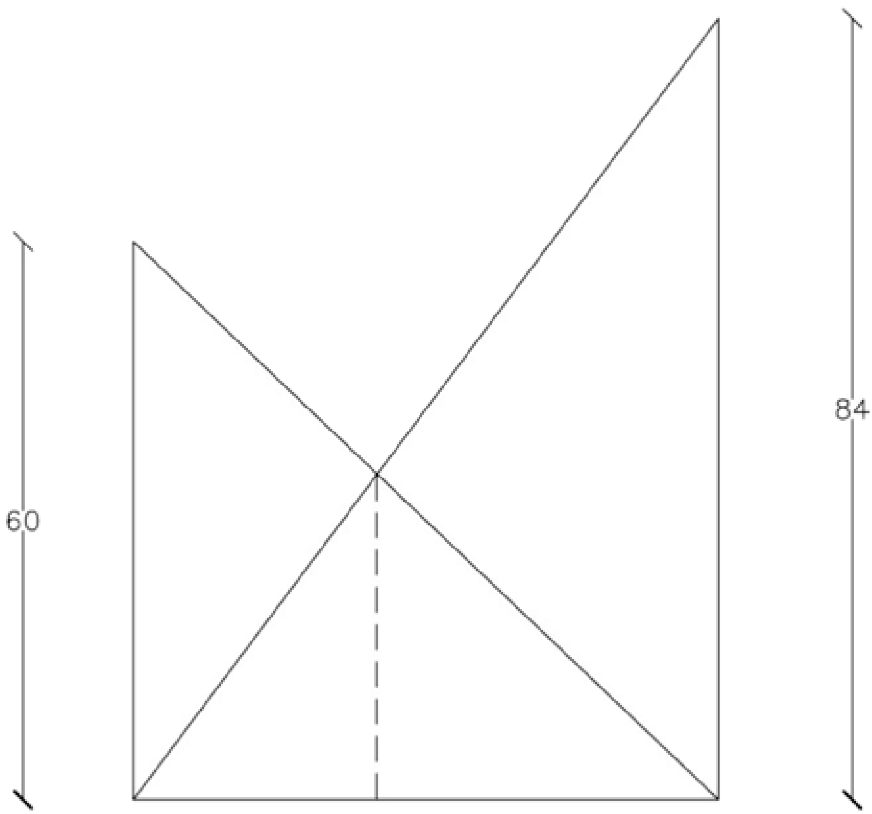

Figure 5

The curious reader may wish to know how the above rule is derived.

By similar triangles, AF/AB = b/d and BF/AB = b/e (see Figures 5 and 6).

The sum of AF/AB and BF/AB = 1.

Therefore, $1 = b/d + b/e$.

Dividing by b gives $1/b = 1/d + 1/e$.

Consequently, $de = be + bd$.

It is easy to rearrange this equation to obtain the following result: $de/(e + d) = b$.

This last algebra equation expresses the rule that we have been given.

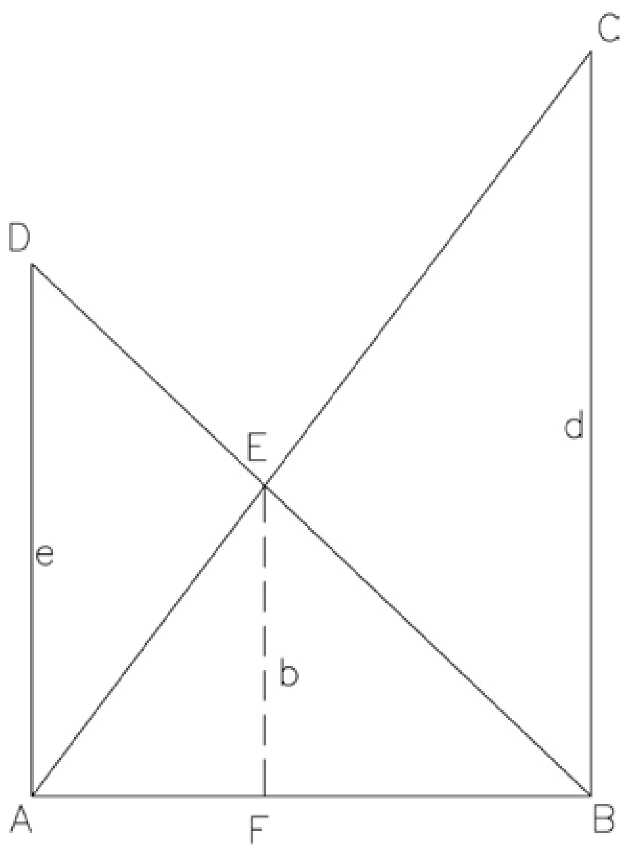

Figure 6

SOURCE

Henry Ernest Dudeney, *Amusements in Mathematics* (New York: Dover Publications, 1958), puzzle 186, pp. 50, 182, 183.

38

THE 2 BULLETS IN A REVOLVER

In a movie, a man is kidnapped by a terrorist. The terrorist produces a revolver containing 6 chambers and shows the kidnapped man that all 6 chambers are empty. The terrorist then loads 2 bullets into 2 consecutive chambers of the revolver and then spins the revolver. All of this is done in front of the kidnapped man.

The terrorist then holds the gun to the kidnapped man's head and fires the revolver. Fortunately for the kidnapped victim, no bullet is fired. The terrorist now offers the kidnapped victim 1 of 2 choices. The terrorist says that he can either (1) fire the revolver again without spinning the barrel or (2) spin the barrel of the revolver and then fire the revolver. The terrorist tells the kidnapped victim that if he should survive this second attempt on his life, he will be allowed to go free.

What choice should the kidnaped victim make in order to maximize his chances of survival?

SOLUTION

The kidnapped victim should choose (1).

Let's analyze the situation. The terrorist has loaded 2 bullets into 2 *consecutive* chambers of the revolver. Therefore, there are 4 consecutive chambers that are empty. Call these 4 empty chambers A, B, C, and D. Thus, the situation with the revolver is that there are 2 loaded chambers followed by 4 empty chambers: A, B, C, and D.

The terrorist fires the revolver, but no bullet exits the gun. Thus, 1 of the 4 empty chambers was fired. But which chamber was fired?

Consider first the situation where the barrel is not spun after the first shot. If it was chamber A that was fired first, then the next shot fired will not contain a bullet. Similarly, if the first shot fired was from chamber B or chamber C, then the next shot fired will not contain a bullet. However, if the first shot fired was from chamber D, then the next shot fired *will* contain a bullet. The probability that the first shot fired was from chamber D is 1 in 4, or 25 percent. Consequently, the probability that the next shot fired will contain a bullet is 1 in 4, or 25 percent.

Consider now the situation where the terrorist spins the barrel after the first blank shot is fired and then fires the revolver again. Having fired the first blank shot, there are 2 bullets in the revolver and 4 empty chambers. Thus, 2 of the 6 chambers are loaded. Therefore, the probability that a live round will be fired is 2 in 6, which simplifies to 1 in 3. This is equivalent to $33\frac{1}{3}$ percent.

We have therefore obtained 2 probabilities. If the revolver is fired a second time without spinning the barrel, the probability a live round will be fired is 1 in 4. If the revolver is fired a second time *after* spinning the barrel, the probability that a live round will be fired is 1 in 3. Therefore, the probability is significantly smaller that a live round will be fired *without* spinning the barrel.

Consequently, to maximize his chances of survival, the kidnapped victim in the movie should choose that the second shot is fired *without* spinning the chamber in the revolver.

SOURCE

"Loaded Revolver," http://www.braingle.com/brainteasers/20227/loaded-revolver.html

39

THE BOOKWORM

Three encyclopedia books written in English are sitting in an ordered fashion on a bookshelf in a library. Each volume is 4 inches thick.

A bookworm starts eating at the back of volume 1 and eats his way through to the front of volume 3.

How far has the bookworm traveled?

SOLUTION

The surprising answer is that the bookworm has traveled a distance of just 4 inches!

We are told that the books are sitting on the bookshelf in an orderly fashion. It is customary that the when the books are ordered in such a fashion, Volume 3 will be to the far left, Volume 2 in the middle, and Volume 1 on the right, as in the following way:

Volume 3; Volume 2; Volume 1

As the bookworm starts eating at the back of Volume 1 and eats his way through to the front of Volume 3, we see that he will only have to eat his way through Volume 2.

Thus, the bookworm will travel a total distance equal to the thickness of Volume 2, which is 4 inches.

SOURCE

Eugene P. Northrop, *Riddles in Mathematics* (London: Penguin Books, 1960), p. 11.

40

HOW MANY WAYS CAN THE 20 NUMBERS ON A DARTBOARD BE ARRANGED?

The game of darts is popular in both Ireland and the United Kingdom. The game is played by each of 2 players throwing little arrows at a dartboard. The integers from 1 to 20 are arranged on the dartboard in a circular and apparently haphazard formation, with higher numbers adjacent to lower numbers. Actually, though, the numbers appear to be arranged on the dartboard so that there is a relatively high number beside a low number and any 2 numerically close integers are separated as far as possible.

This information, though interesting in its own right, doesn't really concern our readers here.

The question that will interest our puzzlers is this: how many different arrangements of the 20 numbers on the dartboard are possible?

SOLUTION

The number of ways n objects in a circular formation can be arranged is $(n - 1)!$ That means that the 20 numbers on a dartboard can be arranged in $(20 - 1)!$, or 19!, ways. The number 19! equals the product of all of the integers from 19 down to 1. This number equals 121,645,100,408,832,000.

Many readers will be surprised to learn that if one were to change each arrangement of the 20 numbers once every minute, 24 hours per day, taking a year to consist of 365.25 days, and not duplicate any arrangement, it would take more than 231,282 million years to complete the task!

SOURCE

Owen O'Shea, *The Magic Numbers of the Professor* (Washington, DC: Mathematical Association of America, 2007), p. 121.

41

THE TOWER OF HANOI PUZZLE

Obtain a deck of cards and remove 9 cards, from the ace to the 9, of 1 suit. Let the ace equal 1, the deuce equal 2, the 3 equal 3, and so on.

Now place any 3 cards from the remaining 39 facedown on a table so that the 3 cards form the vertices of a triangle. These 3 cards are the "spots" that you will lay cards on faceup in a moment. Now arrange 9 of the cards, from the ace through to the 9, in order. Place the cards so that the largest in value is at the bottom faceup, the next largest in value is faceup second from the bottom, and so on. You will thus have the 9 on the bottom, the 8 will be second from the bottom, the 7 will be third from the bottom, and so on. The ace will, of, course, be on top. All these cards are placed faceup.

Now what you must do is to transfer the "tower" of 9 cards to either of the other 2 "spots" so that the order of the 9 cards at the finish is the same as it was at the start. Two rules must be followed. First, you must move only 1 card at a time. Second, you cannot place a card down on another card that has a smaller value than the card that is being placed on it.

To begin, take the first card (the ace) and place it on either 1 of the 2 spots. Now take the deuce. According to the rules, you cannot place this down on top of the ace, so the deuce must go down on the other spot. Now consider the 3. That cannot go down on the ace or the deuce, so that cannot be moved at this moment in time. That means you must take the ace and place it on the deuce and so on.

The question is, can you transfer the tower, and, if you can, what is the minimum number of moves required?

SOLUTION

This is the famous Tower of Hanoi puzzle played with cards. Transferring a tower consisting of 9 cards (or disks) requires a minimum of 511 moves. This surprises most people who have not come across the puzzle before.

The original puzzle, consisting of 8 disks, was invented by the French mathematician Édouard Lucas (1842–1891). It was first sold as a toy in 1883. The puzzle, which can still be bought online and in toy stores around the world, consists of 3 pegs. On 1 peg, 8 disks are placed so that the largest is on the bottom, the second largest is second from the bottom, the third largest is third from the bottom, and so on. Thus, the smallest disk is on top. The puzzle is to transfer the tower of 8 disks to either 1 of the other 2 vacant pegs in a minimum number of moves. Only 2 rules need to be obeyed: one can move only 1 disk at a time, and one cannot place a larger disk down on a smaller one.

It can be proved that the puzzle can always be solved no matter how many disks are in the tower. If n equals the number of disks, the minimum number of moves required to solve the puzzle is $2^n - 1$. Thus, if there are 8 disks, the minimum number of moves required to transfer the tower is $2^8 - 1$, which equals 255. Most people are surprised that this is the minimum number of moves required to solve an 8-disk Tower of Hanoi.

An interesting aspect of the puzzle when it is played with cards is that the even-numbered cards and odd-numbered cards always go in the opposite direction. For example, if the ace is initially moved in a clockwise direction from one spot to the next spot, then you will find that in order to solve the puzzle in the minimum number of moves, all odd-numbered cards will travel in a clockwise direction and all even-numbered cards in the opposite direction.

An interesting story was told by Édouard Lucas about the toy to help publicize the puzzle when it first came on the market. Lucas stated that the Tower of Hanoi puzzle was based on the mythical Tower of Brahma, which was located in a temple in the Indian city of Benares. That tower had 64 golden disks placed on 1 large needle, and 2 other vacant needles were situated nearby. The temple priests decided to transfer the 64-disk tower to either 1 of the 2 vacant needles, moving 1 golden disk at a time and never placing a disk down on top of a smaller one.

It was said that before the temple priests had accomplished their task, the temple would crumble into dust and the world would disappear in a clap of thunder. The disappearance of the world may be questioned, but the crumbling of the temple into dust certainly cannot. The minimum number of moves to transfer the 64-disk tower is $2^{64} - 1$. This equals 18,446,744,073,709,551,615. If the temple priests were working 24 hours per day, 365.25 days per year, making a correct move every second, it would take more than 584,542 million years to complete the task!

SOURCE

Martin Gardner, *Mathematical Puzzles and Diversions* (New York: Simon & Schuster, 1959), pp. 56–61.

42

THE KIDNAPPER

Suppose you are kidnapped and your kidnapper plays a little game with you. The kidnapper produces 2 boxes. Inside 1 box are 50 white balls, and inside the other box are 50 black balls. The kidnapper tells you that you can choose to distribute the 100 balls any way you like between the 2 boxes provided that the following 3 rules are obeyed:

(1) Every ball must be placed in 1 of the 2 boxes.
(2) No balls can be excluded from the game.
(3) Each box must contain at least 1 ball.

Having distributed the 100 balls, the kidnapper now randomly chooses 1 ball from 1 box. If the ball the kidnapper draws from one of the boxes is white, you will be unharmed and allowed to go free. If, however, the ball the kidnapper draws is a black ball, you will be killed.

Now it is clear that if you put 25 black and 25 white balls in each of the 2 boxes, your chances of survival are 50 percent. But can you distribute the 100 balls between the 2 boxes in such a way that you will dramatically increase your chances of surviving above 50 percent?

SOLUTION

Surprisingly, you can increase your chances of survival well above 50 percent. In fact, almost unbelievably, you can bring the probability of survival to just under 75 percent!

How do you do that? Call the 2 boxes Box A and Box B. Simply put 50 black balls and 49 white balls into Box A. Put the remaining single white ball into Box B. The kidnapper now has to choose a box. He has a 50 percent chance of choosing Box B. If he does so, you are 100 percent certain of going free.

If, however, he chooses Box A, you still have 49 chances in 99 of going free.

Therefore, your overall probability of going free is 50 percent + 50 × (49/99) percent. This equals 74.7474. . . percent.

SOURCE

https://www.braingle.com/brainteasers/15446/life-or-death-the-emperors-proposition.html

43

THE 5 LETTERS AND ENVELOPES

Suppose 5 secretaries type 5 letters and also type the 5 corresponding addresses on envelopes in which the letters are to be sent. Someone then enters the office and mixes up the 5 letters. Then a third person randomly places each of the 5 letters into each of the 5 envelopes.

What is the probability that exactly 4 of the letters will be placed in their corresponding envelopes?

SOLUTION

The probability is 0. It cannot happen! If 4 of the letters are placed in the correct envelopes, the fifth letter has to be placed in the correct envelope also!

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 2.2, pp. 39, 49.

44

THE RATE OF SPEED THAT MARY SWIMS

Mary is a reasonably good swimmer. She can swim from the shore to a buoy in 10 minutes when the tide is in her favor. Mary can swim exactly the same distance in still water in 20 minutes. How long does it take Mary to swim from the buoy to the shore when swimming against the tide?

SOLUTION

The surprising answer is that Mary can never reach the shore! From the question, we are told that Mary can swim to the buoy in still water in 20 minutes, but with the tide in her favor, it takes her just 10 minutes. Therefore, the current must be traveling at the same rate of speed that Mary can swim. That is why she can swim to the buoy in half the time it would take if there were no current.

Thus, when she is swimming against the tide, Mary can make no progress whatsoever.

SOURCE

"Swimmer," http://www.braingle.com/brainteasers/4630/*s*wimmer.html

45

THE FLOATING ICE CUBE

An ice cube is floating in a glass of water. When the ice melts, does the water level in the glass rise, fall, or stay the same?

SOLUTION

Many people are surprised to learn that the water level in the glass will stay the same when the ice melts.

Liquid water has a density of 1 gram per cubic centimeter. Suppose the ice cube has a mass of 10 grams. When the ice is floating, it displaces an amount of water equal to the ice cube's *mass*. It will therefore displace an amount of water equal to 10 grams. Now 10 grams of liquid water will take up 10 cubic centimeters of volume. Water expands by about 9 percent when it freezes. Thus, a 10-centimeter ice cube will have a *volume* close to 10.9 cubic centimeters. This is a larger volume than 10 cubic centimeters. It is impossible to compact 10.9 cubic centimeters of ice into a 10-cubic-centimeter space. As a consequence, part of the ice cube (about 9 percent) will be above the surface of the water as the ice floats. As the ice continues to melt, its water content (its mass) is conserved, but its density increases. When the ice is completely melted, its density will (naturally!) be the same as liquid water. At that stage, the 10-gram cubic centimeter will displace its *volume*, which is 10 cubic centimeters. Thus, the water level will remain the same as when the ice cube was floating in the glass of water.

SOURCE

"Melting Ice and Its Effect on Water Levels," http://www.smithplanet.com/stuff/iceandwater.htm

46

THE 10 STACKS OF COINS

Suppose you have a pointer weighing scale. You are given 10 stacks of coins. Nine of the 10 stacks contain genuine coins. However, 1 entire stack contains counterfeit coins. Each genuine coin weighs 10 grams. Each counterfeit coin weighs 11 grams. You can weigh as many coins as you wish on the pointer scale. How many weighings are necessary in order to identify the counterfeit stack?

SOLUTION

Surprisingly, only 1 weighing is necessary to identify the stack of false coins. To identify the stack consisting of the false coins, take 1 coin from stack number 1, 2 coins from stack number 2, 3 coins from stack number 3, and so on, right up to stack number 10, from which you take 10 coins.

You will now have 55 coins. Put all these coins into 1 stack. Weigh that stack. If all the 55 coins in this stack were genuine, they would weigh 550 grams. However, the pointer weighing scale will not give 550 grams as the weight of the stack. For example, the pointer scale may indicate that the stack of 55 coins weighs 551 grams. In that case, you know that the false coin must have come from stack number 1. Why? Because the 55 coins in the stack weighs 1 gram more than a stack containing only genuine coins. Therefore, stack number 1 must contain the false coins because you took 1 coin from that stack. If the stack of 55 coins weighs 552 grams, you know that there are 2 false coins in the stack, and you know that you took 2 coins from stack number 2. Therefore, stack number 2 contains the false coins and so on.

Another surprising and little-known feature of this puzzle is that the puzzle can be solved if there are 11 stacks of coins! You proceed as before to identify the stack of counterfeit coins, but as you do so, you ensure that you do not take any coins from the 11th stack. When you weigh the 55 coins that you have taken from the first 10 stacks, you will know by the method just explained how to identify the counterfeit stack. If there is no excess weight when you weigh the 55 coins, you know that the counterfeit stack is the 11th stack because you took *no* coins from that stack.

SOURCE

"Answer to Riddle #65: 9 Coins, 1 Odd One, 2 Weighings," http://www.puzzles.nigelcoldwell.co.uk/sixtyfive.htm

47

SIGNS ALONG THE HIGHWAY

Mr. Smith and his wife are on a driving trip. Mr. Smith comments on the annoying advertising signs for Puff cigars that were spaced evenly along the side of the highway. He wondered aloud how far apart the signs were. Mrs. Smith used her wristwatch to time the appearance of the signs as the car traveled down the highway at a constant speed.

When they were just halfway between 2 signs, she started counting the signs that they passed in exactly 1 minute and mentioned the result to her husband.

"What an extraordinary coincidence!" declared Mr. Smith. "If you multiply that number by 10, you get the exact speed that we are traveling in miles per hour."

How far apart along the highway are the equally spaced advertising signs?

SOLUTION

It is surprising to learn that we do not need to know the speed of the Smiths car to determine the distance between any 2 equally spaced signs along the side of the highway.

Let x equal the number of highway signs passed in 1 minute. The number of signs passed in 1 hour is therefore $60x$. From the question, we know that the speed of the car is equal to $10x$ miles per hour. Thus, the number of signs passed in 1 mile must equal $60x/10x = 6$.

Since the signs are equally spaced along the side of the highway, the signs must be $\frac{1}{6}$ of a mile, or 880 feet, apart.

Having obtained this information, we can then determine that the Smiths car is traveling at a constant speed of 60 miles per hour.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 4.19, pp. 90–99.

48

CHOOSE 2 GOOD EGGS FROM 6

A man has placed 6 eggs on a table. Two of the eggs are bad. All the other 4 eggs are good. You have to choose any 2 eggs. What is the probability that you will pick 2 good eggs?

SOLUTION

The answer to this puzzle surprises most people when they first hear it. The probability that you will pick 2 good eggs is $2/5$. Therefore, the probability that you will not choose 2 good eggs is $3/5$.

These probabilities are calculated as follows. Let the 2 bad eggs be labeled A and B. Let the 4 good eggs be numbered 1, 2, 3, and 4. There are 6 ways of choosing 2 objects from 4. Thus, there are 6 different ways that 2 numbered eggs can be chosen from the 4 numbered eggs.

Consider now all 6 eggs. There are 15 ways of choosing 2 eggs from the 6 eggs. Six of those ways contain 2 good eggs. Therefore, there must be 9 different ways of choosing 2 eggs that contain at least 1 bad egg. These 9 ways are the following:

A, 1; A, 2; A, 3; A, 4; B, 1; B, 2; B, 3; B, 4; A, B

Thus, the probability of choosing 2 good eggs is $6/15$, which reduces to $2/5$. Therefore, the probability that you will choose 2 eggs in which at least 1 of the 2 eggs is bad is $3/5$.

The workings of this puzzle are often used profitably—generally with playing cards—by con artists to swindle money from honest folk.

SOURCE

Owen O’Shea, *The Call of the Primes* (Amherst, NY: Prometheus Books, 2016), pp. 85–86.

49

THE PARTY

John and Mary decide to hold a party. John invites 2 guests, and Mary invites 3. Both John and Mary pay for their guests. The party costs a total of \$35. How much should John and Mary pay toward the cost of the party?

SOLUTION

On being asked this puzzle, most people state that John should pay two-fifths of the cost, or $14, and that Mary should pay three-fifths, or $21.

But this division of the costs is incorrect!

The correct division of the $35 is that John should pay three-sevenths, or $15, and that Mary should pay four-sevenths, or $20.

This answer surprises most people. But it is the correct answer.

Look at it this way. There were altogether 5 guests, plus the 2 hosts at the party. Therefore, there were 7 people at the party. Thus, the costs of the party should be divided by 7. The result is a cost of $5 for each partygoer. John should pay for his 2 guests plus himself. Therefore, he should pay $15. Mary should pay for her 3 guests plus herself. Therefore, she should pay $20.

SOURCE

Herbert McKay, *Party Night* (New York: Oxford University Press, 1940), puzzle 25, p. 182.

50

WHOM SHOULD JOHNNY PLAY FIRST?

Young Johnny was learning how to play chess and was making good progress. His mother and father also played chess. Johnny's father was a good chess player, but his mother was not. In order to encourage his son in learning chess, Johnny's father promised to double his weekly allowance for a month if he could win 2 consecutive games out of 3 chess games against his parents. Johnny's dad also gave Johnny the choice of which parent he wished to play first.

Which parent should Johnny play first in order to maximize his chances of winning 2 consecutive games out of 3?

SOLUTION

Johnny has to play 1 player once and 1 player twice. Intuition may tell you that it is to Johnny's advantage that he play the weaker player twice. Therefore, he should ensure that he plays the weaker opponent in the first and third games.

But, surprisingly, this is not the best strategy to adopt!

To win 2 consecutive games of the 3 chess matches, Johnny *must* win his second game. Thus, he should play his weaker opponent (mother) second. Consequently, Johnny should play his stronger opponent (father) first and third. In this strategy, he has 2 chances of winning against the stronger opponent. Therefore, with this game plan, Johnny maximizes his chances of winning 2 consecutive games; either game 1 and game 2 or game 2 and game 3.

Mathematically, we can tackle the puzzle as follows. Suppose the son plays the mother/father/mother pattern. Let us assume that the probability of beating the mother is 0.6 and that that of beating the father is 0.4.

The probability of the son winning all 3 games is 0.096. (This result is obtained by multiplying 0.4, 0.6, and 0.4 together.) The probability of the son winning the first 2 games is $0.6 \times 0.4 \times 1 - 0.6$, which equals 0.096.

The probability of the son winning the second 2 games is $1 - 0.6 \times 0.4 \times 0.6$, which equals 0.096.

These 3 probabilities are then added to find the overall probability of the son winning 2 consecutive games: $0.144 + 0.096 + 0.096$. The result is 0.336. This is the overall probability of the son winning 2 consecutive games playing the mother/father/mother pattern.

Consider now the second option open to the son. Suppose the son plays the father/mother/father pattern.

The probability of the son winning all 3 games is 0.096. The probability of the son winning the first 2 games is $0.4 \times 0.6 \times 0.6$, which equals 0.144. The probability of the son winning the last 2 games is $0.4 \times 0.6 \times 0.6$, which equals 0.144. Adding these 3 probabilities gives 0.384. This is the overall probability of the son winning 2 consecutive games playing the father/mother/father pattern.

Thus, we see that the son has a greater probability of winning 2 consecutive games out of 3 games if he chooses to play the weaker opponent once and the stronger opponent twice!

The result appears to be counterintuitive, but it is correct nevertheless.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), pp. 42, 43, 51, 52, 53.

51

THE ENGAGEMENT RING

Two lovers, Mary and John, are planning to get engaged. They live a long distance from each other in a country that has a strange postal system. Apparently, anything sent by mail will be stolen unless that item is enclosed in a box that is then padlocked. Both John and Mary have plenty of padlocks and the relevant keys for each padlock. However, Mary does not have any keys to open John's padlocks, and, similarly, John possesses no keys that will open Mary's padlocks.

John wants to securely send Mary an engagement ring by mail. How does he do it?

SOLUTION

It appears at first that there is no way that John can securely send the engagement ring to Mary. But, surprisingly, this is not the case.

Here is how he does it.

John places the engagement ring in a box and locks it securely by putting one of his padlocks on it, retaining the key for himself. He sends this box by mail to Mary. On receipt of the box, Mary secures one of her padlocks on the box, retains the key, and sends the box back to John. On receipt of the box, John removes his padlock and sends the box back to Mary.

Mary is now able to open the box with the relevant key.

(This puzzle was originated by Caroline Calderbank, who is a daughter of the Belgian physicist and mathematician Ingrid Daubechies and the British mathematician and electrical engineer Rob Calderbank.)

SOURCE

"Love in Kleptopia," http://www.testyourlogic.blogspot.com/2016/07/love-in-kleptopia.html

52

THE 2 WOMEN SELLING APPLES

Long ago, there were 2 women, Mrs. Smith and Mrs. Jones, selling apples from a cart on the side of the street. Mrs. Smith sold her apples at the rate of 2 for a penny. Mrs. Jones sold her apples at the rate of 3 for a penny.

One day, when both women had just 30 apples each left in their carts, each of them were called away. They asked a friend if he would sell all 60 apples for them.

If Mrs. Smith had sold her 30 apples at the rate of 2 for a penny, she would have earned 15 pence. If Mrs. Jones had sold her 30 apples at the rate of 3 for a penny, she would have earned 10 pence. So, by selling their apples separately, they would have had combined earnings of 25 pence.

However, when the 2 women had left, the friend decided to simplify the selling procedure. He decided to put all 60 apples together and sell them at the rate of 5 for 2 pence. He believed that this was the same as selling 3 apples for a penny and 2 apples for a penny. In this way, he earned 24 pence for selling the 60 apples.

At the end of the day when the 2 women returned, the friend explained what he had done and offered both women 12 pence as their share of the earnings for the selling of the 60 apples. In other words, the friend offered the women a total of 24 pence.

But if the friend had sold the apples at the 2 different rates as the 2 women had originally planned, the combined earnings would have been 25 pence.

Where did the missing penny go!

SOLUTION

There is a penny missing in the amount earned because the 2 methods of selling the 60 apples are different. This comes as a surprise to many people.

Because of the friend's method of selling the apples, Mrs. Jones ended up with an extra 2 pennies—she received 12 pence instead of the 10 pence she would have originally earned under her original method of selling. The opposite happened to Mrs. Smith. She received 3 pence less than she would have originally received under her method of selling. Thus, their combined earnings would be reduced by 1 penny.

If the amount of apples were in the ratio of 3 to 2, then either method of selling the apples would produce the same earnings. Thus, if Mrs. Jones had 36 apples and she sold them at the rate of 3 for a penny, she would have earned 12 pence. If Mrs. Smith had 24 apples and she sold them at the rate of 2 for a penny, she also would have earned 12 pence. Their combined earnings would have been 24 pence. If the friend had taken all 60 apples from the 2 women and sold them at the rate of 5 for 2 pence, he would also have made 24 pence.

However, when the 2 women have the same amount of apples left to sell, a discrepancy arises. Think of it this way. The friend is selling 30 apples belonging to Mrs. Smith and 30 apples belonging to Mrs. Jones at the rate of 5 for 2 pence. He will eventually reach the point where he has sold 10 batches of apples at 2 pence per batch (each batch consisting of 3 of Mrs. Jones's apples and 2 of Mrs. Smith's apples). This will yield earnings of 20 pence. At this point, all of Mrs. Jones apples are sold.

There remains to be sold the 10 apples belonging to Mrs. Smith. The friend will sell these 10 apples at the rate of 5 for 2 pence. Thus, he will get 4 pence for the 10 apples.

However, under Mrs. Smith's original selling plan (selling 2 apples for 1 penny), she would have earned 5 pence for selling these 10 apples. The selling of these 10 apples at 2 different selling rates creates the discrepancy and explains where the missing penny went.

This puzzle never fails to surprise people when they hear it for the first time.

SOURCE

Kobon Fujimura, *The Tokyo Puzzles* (London: Frederick Muller Ltd, 1979), problem 74, pp. 80, 164.

53

TOMMY TAKES A WALK

A few of the local men in our small town have their weekly poker game each Friday night. Two of the players, Smith and Jones, are known for their liking of recreational mathematics. Consequently, each of them sometimes poses puzzles for the other to solve.

Recently, at the end of the card session, Smith posed the following puzzle to Jones:

Suppose a man (call him Tommy) is standing with his girlfriend at a bus stop waiting for their bus to bring them into town. Tommy, who is fond of gambling, began thinking about the following problem. He knows that the spot where he is standing (call it point A) is exactly 100 yards to the right of the entrance to his favorite restaurant at point O (see Figure 7).

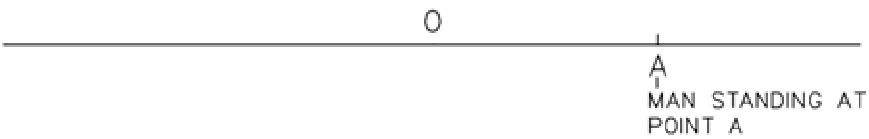

Figure 7

As Tommy stood waiting, he took out a deck of cards and removed 5 black cards and 5 red cards from the pack. He then shuffled these 10 cards and held them facedown. Tommy asked his girlfriend, Mary, to select any card from the 10. After the color of the card is noted, it is discarded. He tells his girlfriend that if the card is red, he will walk half the distance from the restaurant to where he is standing (50 yards) to the right of where he is standing so that he will be 150 yards from the restaurant at the end of his walk. If, however, the card is black, he will walk halfway between him and the restaurant (50 yards) toward the restaurant so that he will be just 50 yards from the restaurant at the end of the walk.

When that was done, Tommy repeated the procedure with the remaining 9 cards. Thus, Mary would again select a card. After its color is noted, it is discarded. If it is red, Tommy will walk half the distance between him and the restaurant but in a direction *away* from the restaurant toward his right. If the card is black, Tommy will walk halfway between him and the restaurant *toward* the restaurant.

Tommy continues with this procedure until the 10th card is noted. The question is this: will he then be to the right or to the left of the spot from which he started?

From what I have been told, Jones gave the correct answer to this puzzle. Can you do the same?

SOLUTION

The answer is very surprising. Tommy will certainly be to the left of the spot from which he started. However, regardless of the order in which the cards are drawn, at the end of the procedure, Tommy will always end up at precisely the same spot! That spot is 76.26953125 . . . yards to the left of the spot from which he started.

The reader is urged to take 5 red and 5 black cards and see this result for him- or herself.

The exact distance left of the point from which Tommy started is given by the following formula:

$$a - \left[\ a\left(\frac{3}{4}\right)^{n}\ \right]$$

In the formula, *a* equals the starting point of Tommy, and *n* equals the number of red (or black) cards in the packet. In the puzzle given here, *a* equals 100, and *n* equals 5.

The walk performed by Tommy can be translated into a betting game. Suppose a gambler starts with \$100. The cards from the packet are drawn and discarded as before. Each red card is a win, and each black card is a loss. Before each card is drawn, the gambler bets half of his capital. It is most surprising that at the end of any such game, regardless of the cards drawn, the gambler will have lost exactly \$76.26953125.

This amount increases as the value of *n* increases. Suppose Tommy started with \$100 capital as before but with a standard deck of 52 cards instead of the 10 cards previously. Before each card is drawn, Tommy bets exactly half of his capital. His loss now will be \$99.94355924 regardless of the order in which the cards are drawn!

In our example, Tommy chose to bet a fixed amount—half of his capital—before each card is drawn. However, he could have chosen to bet a smaller fixed amount at each stage. Tommy could have chosen to bet, say, one-fifth of his capital after each card is drawn. Or he could have chosen to bet one-quarter of his capital after each card is selected. The larger the fixed fraction is that he chooses to bet, the bigger his losses will be. For example, commencing with \$100, if he chooses to bet one-fifth of his capital after each card is drawn, he will lose exactly \$18.46273024. If he chooses to bet one-quarter of his capital at each stage, he will lose \$27.5803566. The formula tells us that if Tommy chooses to bet all of his capital after each card is drawn, he is certain to lose all of his capital.

In the general case, let *a* equal the amount of capital at the beginning of the game, let 1/*k* equal the fixed fraction, and let *n* equal the number of red (or black) cards. The amount lost by the gambler is found by the following formula:

$$a - \left[\ a\left(1 - \frac{1}{k^2}\right)^n\ \right]$$

These types of problems are intimately related to a branch of mathematics that deals with *random walks* and the *gambler's ruin estimate*. Random walks are used in describing various natural phenomena. For example, the random walk describes the statistical properties of genetic drift, they are used in computer science to estimate the size of the World Wide Web, and they are used by Twitter to make suggestions to its users as to whom they should follow (https://en.wikipedia.org/wiki/random walk).

SOURCE

Martin Gardner. *Mathematical Circus* (London: Penguin Books, 1979), chapter 6.

54

THE STORY OF LINDA

Puzzles involving probability theory are often extremely tricky. Consider the 2 following statements:

(1) Linda is a bank teller.
(2) Linda Smith is 31 years old, single, outspoken, and considered to be very intelligent. She attended college following high school, in which she majored in philosophy. As a college student, she was deeply concerned with issues of discrimination and social justice. She always stood up for the equality rights for citizens and also participated in antinuclear demonstrations.

Given this information, which of the following 2 statements is more likely to be true?

(1) Linda is a bank teller.
(2) Linda is a bank teller *and* is active in the feminist movement.

SOLUTION

On being presented with the 2 statements above, it appears that most people declare that statement 2 is more likely to be true than statement 1.

Surprisingly, this conclusion is a fallacy and hence is incorrect. The problem demonstrates what is known to statisticians as the *conjunction fallacy*.

The word *and* in the second sentence makes all the difference. Its inclusion means that the sentence is describing 2 separate, independent events. The probability of 2 independent events occurring is found by multiplying together the probabilities of each event occurring.

The 2 statements are what probability experts call *conjunctions*. The fact is that the probability of the conjunctions is never greater than that of its conjuncts. In simple English, this means that the probability of the two events simultaneously occurring as described in the second sentence that Linda gave is always less than or equal to the probability of either event occurring alone.

The reader can satisfy him- or herself of the truth of this by considering a couple of examples.

Suppose someone makes a statement stating that the probability that Linda is a bank teller is 0.9 percent. Suppose now that that person makes a second statement stating that the probability that Linda is active in the feminist movement is 0.05 percent. Thus, the probability that Linda is a bank teller *and* is active in the feminist movement equals 0.9 multiplied by 0.05. The result is 0.045 percent. This is less than 0.05 percent. Therefore, the probability that Linda is a bank teller *and* is active in the feminist movement is less than the probability that Linda is a bank teller.

Here's a second example. Suppose the probability that Linda is a bank teller is 0.95 percent. Suppose also that the probability that Linda is active in the feminist movement is 0.4 percent. Given these figures, the probability that Linda is a bank teller *and* is active in the feminist movement is 0.95 multiplied by 0.4. This equals 0.38 percent, which is less than the probability that Linda is active in the feminist movement.

The fallacy arises because in the puzzle, Linda's interests are representative of the way of life of a feminist as perceived in the public's mind and is unrepresentative of the typical bank teller.

The conjunction fallacy crops up on a daily basis in many areas of life. The fallacy is so prevalent in modern society that numerous people every day get taken by it.

SOURCE

Daniel Kahneman, *Thinking, Fast and Slow* (London: Penguin Books, 2011), pp. 419–32.

55

THE PROBLEM OF THE AVERAGE SPEED ON THE ROUND-TRIP

Mr. Smith decides to drive from city A to city B and then back home again. He hopes to attain an overall average speed of 70 miles per hour for the round-trip. However, when he reaches city B, he realizes that he has attained an average speed of just 35 miles per hour.

How fast must he go on the journey from city B to city A to attain an average speed of 70 miles per hour for the entire round-trip?

SOLUTION

It comes as a surprise to most people that it is impossible for Mr. Smith to attain an average speed of 70 miles per hour for the entire round-trip! The first half of the journey from city A to city B was completed by traveling at an average speed of 35 miles per hour. To bring his average speed for the entire trip up to 70 miles per hour, Mr. Smith would have to return to city A instantaneously!

To see this, let us assume that Mr. Smith's journey is 210 miles. (Any distance can be used.) On his outward journey from city A to city B, averaging 35 miles per hour, the trip would take Mr. Smith 6 hours. If Mr. Smith wants to attain an average speed of 70 miles per hour for the entire *round-trip*, he has to travel 420 miles in 6 hours. But he has already traveled 210 miles in 6 hours. So it is impossible for Mr. Smith to travel a total distance of 420 miles in 6 hours.

Even if Mr. Smith traveled the homeward trip from city B to city A at 210 miles per minute (12,600 miles per hour), the entire round-trip would have taken him 360 minutes plus 1 minute. Therefore, he would have traveled on the round-trip a total of 420 miles in 361 minutes, which equals an average of 1.163434903 . . . miles per minute. This equals an average of 69.80609418 . . . miles per hour.

Or, if Mr. Smith had traveled from city B to city A at 210 miles per second, the entire round-trip would have taken him 360 minutes and 1 second. His average speed for the entire round-trip would then have been 1.166612657 . . . miles per minute, or 69.99675941 miles per hour.

It can be seen that given the conditions of the problem, it is impossible for Mr. Smith to attain a speed of 70 miles per hour for the entire round-trip.

SOURCE

Martin Gardner, *Entertaining Mathematical Puzzles* (New York: Dover Publications, 1986), pp. 31–32.

56

BIXLEY TO QUIXLEY

In the *Cyclopedia*, Sam Loyd offered the following puzzle.

Sam told the reader that at one time in his life, he had occasion to be riding on a mule from Bixley to Quixley. The man guiding the mule was Don Pedro. The mule was going very slowly, and Sam asked Pedro if the mule had another gait. “He has,” says Pedro, “but it is much slower!”

In order to encourage Pedro to get the mule to move a little quicker, Sam mentioned that they would be passing through Pixley and that they could get liquid refreshments there. From that moment on, Pedro, it seems, could think of nothing but Pixley.

Having traveled for 40 minutes, Sam asked how far they had gone. Pedro replied, “Just half as far as it is to Pixley.”

After creeping along for 7 more miles, Sam asked how far it is to Quixley. Pedro replied, “Just half as far as it is to Pixley.”

They arrived in Quixley 1 hour later.

Sam then posed the following question to the reader: can you figure out how far it is from Bixley to Quixley?

SOLUTION

Sam Loyd solved the puzzle as follows:

They had traveled for 40 minutes when he asked Pedro how far they had gone. Pedro replied, “Just half as far as it is to Pixley.” Therefore, from the start of the journey to Pixley must have taken 120 minutes.

Then they traveled for another 7 miles when Loyd asked how far it is to Quixley. Pedro replied, “Just half as far as it is to Pixley.” Then they reached Quixley 1 hour later.

Therefore, from Pixley to Quixley must have taken a total of 180 minutes.

Thus, the total journey took 120 + 180 minutes, or 300 minutes.

From the time Loyd asked the first question to the time they reached Pixley must have taken 80 minutes. From the time taken to ride from Pixley to the point when Loyd asked his second question must have taken 120 minutes. Therefore, the time taken to travel from where the first question was asked to the point where the second question was asked was 200 minutes.

From the question, this was a distance of 7 miles. Since the whole journey took 300 minutes, the total distance from Bixley to Quixley was 10½ miles.

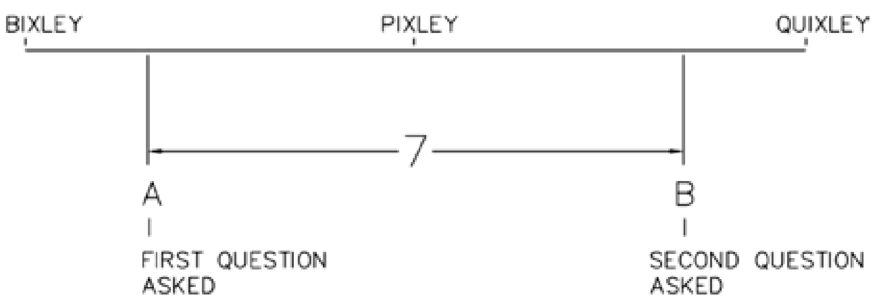

Figure 8

Many years after Loyd’s problem appeared in the *Cyclopedia*, one of Martin Gardner’s correspondents, Ronald C. Read, of Kingston, Jamaica, surprisingly pointed out that there is a much easier way to tackle the problem.

Consider Figure 8. It is clear that the distance from the point when Loyd asked his first question (A) to Pixley is two-thirds the distance from Pixley to Bixley. It is also clear that from Pixley to the point where Loyd asked his second question (B) is two-thirds of the way from Pixley to Quixley. Therefore, from A to B is two-thirds of the distance from Bixley to Quixley. We are told that this distance is 7 miles.

Therefore, the total distance from Bixley to Quixley is 10½ miles.

SOURCE

Sam Loyd, *Sam Loyd's Cyclopedia of 5,000 Puzzles, Tricks, and Conundrums, with Answers* (New York: Morningside Press, 1914), pp. 220, 368.

57

THE BOAT AND ITS CARGO

A man is in a rowing boat floating in a lake. There is a brick in the boat. For some reason, the man throws the brick overboard. Does the water level stay the same, rise, or fall in the lake as a result of the man hurtling the brick overboard?

SOLUTION

When the brick is in the boat, the amount of water displaced by the brick is equal to its *weight*. When the brick is thrown overboard, the amount of water displaced by the brick is equal to its *volume*. Therefore, to the surprise of many puzzlers, the water level will fall when the brick is thrown overboard.

Since the brick sinks in the water, we know that the volume of the brick is denser than that of water. Thus, the volume of water equivalent to the *mass* of the brick is greater than the *volume* of the brick. Therefore, when the brick is thrown overboard and into the water, less water is displaced than when the brick was in the boat.

To help clarify the matter, recall that 1 liter of water weighs 1 kilogram. Suppose the brick weighs 2 kilograms and is 1 liter in volume.

When the brick is lying inside the boat, it is forcing 2 kilograms or 2 liters of water to be displaced.

When the brick is thrown overboard and is in the lake, the brick is displacing its own volume, or 1 liter of water. From this, it is clear that more water is being displaced when the brick is in the boat. Therefore, when the brick is thrown overboard, less water is being displaced, and consequently the water level will fall.

SOURCE

"Answer to Puzzle #21: The Brick, the Boat, and the Lake," http://www.puzzles.nigelcoldwell.co.uk/twentyone.htm

58

THE MUTILATED CHECKERBOARD

Suppose you have an 8-by-8 checkerboard consisting of 64 alternating red and white squares and a set of dominoes that measure 2 by 1. Each domino exactly covers 2 adjacent squares on the checkerboard. Thus, you will find it easy to cover all 64 squares with 32 dominoes.

Suppose now that 2 opposite diagonal corner squares on the checkerboard are cut off (see Figure 9). Is it possible to cover the remaining 62 squares of the chessboard with 31 dominoes?

SOLUTION

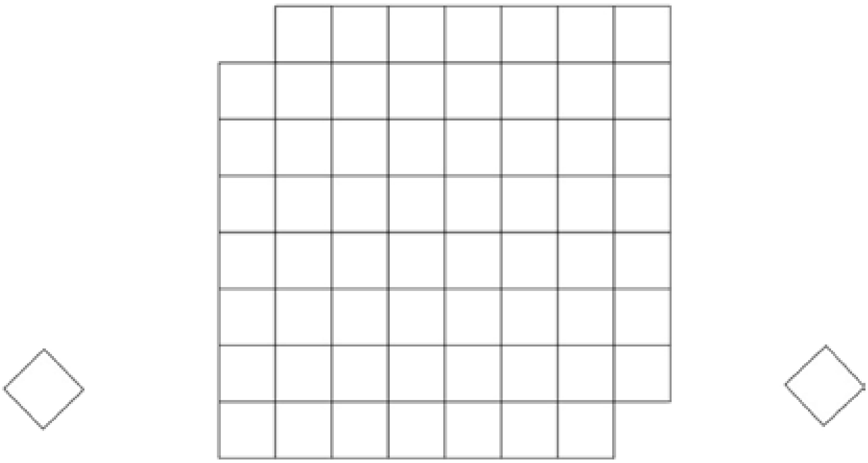

Figure 9

The surprising answer is that it is not possible to cover the mutilated checkerboard with 31 dominoes.

To see why, consider how the domino is placed on the checkerboard. Each time a domino is placed on the checkerboard, it covers a white square and an adjacent red square.

We are told that 2 opposite diagonal corner squares have been cut off from the checkerboard. These 2 corners will have identical colors. Let's say that both are white. The checkerboard therefore now consists of 32 red squares and 30 white squares. Since each domino covers a white square and a red square, it means that we can place at most 30 dominoes on the board to cover the white squares. Therefore, there will be 2 red squares that we cannot cover with the dominoes.

Similar reasoning, of course, applies if the 2 opposite diagonal squares that are cut off are red.

SOURCE

Max Black, *Critical Thinking: An Introduction to Logic and Scientific Method* (New York: Prentice Hall, 1946).

59

THE ROSE GARDEN

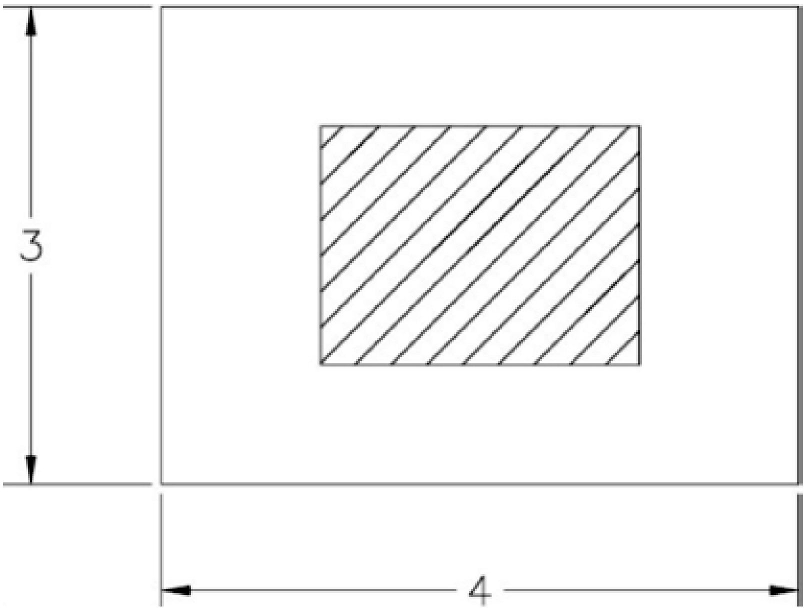

Figure 10

Figure 10 shows a drawing of John Smith's back garden. Its dimensions are 4 yards long and 3 yards wide. Smith wishes to place a rectangular path inside the back garden so that it will be equal in area to the inner rectangular rose garden he wants to develop. Having done some calculations, Smith has worked out that the width of the path should be 0.5 yards. This will ensure that the area of the path is $2 \times 4 \times 0.5 + 2 \times 2 \times 0.5$. This equals 6 square yards. The dimensions of the inner rectangle created as a result of the inner path are 3 yards by 2 yards. Its area is also 6 square yards.

Had John Smith being interested in recreational mathematics, it is quite possible that he would have been able to solve this little problem about areas rather easily. He would probably have known that the American puzzle genius Sam Loyd gave a simple rule that will help find the width of the path no matter what the dimensions of the rectangular garden are so that the area of the path will equal the area of the inner rectangle.

Can you find that surprisingly simple rule? It is a handy rule to know that applies to all rectangles that puzzlists should know.

SOLUTION

The surprisingly simple rule that Sam Loyd gave to find the width of the path is to subtract the diagonal of the garden from the sum of the 2 sides of the garden and divide that result by 4. That answer is the width of the path.

Thus, in our problem, we use the Pythagorean theorem to find that the diagonal of the garden is the square root of $(3^2 + 4^2)$, which equals 5. The sum of the two sides is 3 + 4, which is 7. Subtract 5 from 7, obtaining 2, and divide 2 by 4. The answer is 0.5 yards. Thus, the width of the path inside the garden should be 0.5 yards. Such a path inside the 4-by-3 rectangle will create a smaller rectangle, measuring 3 by 2, which is half the area of the larger rectangle. Thus, the area of the path and the area of the inner rectangle will be equal.

The width of the path can be determined by algebra as follows: let x equal the width of the path. The area of the inner triangle is then equal to $(4 - 2x) \times (3 - 2x)$. This equals $12 - 14x + 4x^2$. We know that this equals 6. Therefore, we can write $4x^2 - 14x + 6 = 0$.

This is a quadratic equation, which can be solved by one of the usual methods. There are 2 solutions: $x = 3$ or $x = 0.5$. The only solution that fits the problem's conditions is that $x = 0.5$.

Thus, the width of the path is 0.5 yards.

SOURCE

Sam Loyd, *More Mathematical Puzzles of Sam Loyd*, selected and edited by Martin Gardner (New York: Dover Publications, 1960), puzzle 90, pp. 63, 149.

60

THE DECK OF CARDS PROBLEM

Playing cards have been around since the 12th century. People all over the world enjoy playing various games with the 52 *pasteboards*, as magicians like to call a deck of cards.

It is safe to assume that decks of playing cards must have been shuffled an enormous number of times through the past 8 centuries or so.

Given that fact, is it likely that when a deck of 52 cards is shuffled, that particular order in which the cards are arranged has happened before in history?

SOLUTION

Many people will be surprised to learn that it is virtually certain that when a deck of cards is thoroughly shuffled and dealt, that particular order of the cards has never occurred before in history!

The number of ways 52 playing cards can be ordered is 52 factorial. This is usually written as 52! It simply means $52 \times 51 \times 50 \times 49 \times \ldots 4 \times 3 \times 2 \times 1$. This number approximately equals 10^{67}. To give an idea of how large a number 10^{67} is, consider this: the number of seconds that have elapsed in the past 4½ billion years is approximately 10^{17}.

Let's assume that Earth was formed 4½ billion years ago. Let's also assume that from the moment Earth was formed to the present day, 8 billion people (8×10^9 people, the approximate population of the world in the near future) were constantly shuffling cards at the rate of 1 shuffle per second. To calculate the total number of shuffles in the past 4½ billion years, we multiply 10^{17} by 8×10^9. We obtain approximately 10^{27}.

The total number of ways of ordering a deck of 52 cards is approximately 10^{67}. Thus, the probability that the cards are arranged as in a previous way after a shuffle is approximately $10^{27}/10^{67}$. The answer is approximately 10^{-40}. That equals 0.000 . . . , where there are 39 zeros after the decimal point! That result is extremely close to 0.

In other words, it is virtually certain that when a deck of cards is thoroughly shuffled and dealt, that particular order of the cards has never before occurred in history.

SOURCE

Owen O'Shea, *The Magic Numbers of the Professor* (Washington, DC: Mathematical Association of America, 2007), pp. 153–64.

61

THE WEIGHT OF THE POTATOES

Suppose you buy 100 pounds of potatoes and you are told that 99 percent of the potatoes consist of water.

You bring the potatoes home and leave them outside to dehydrate until the amount of water in the potatoes is 98 percent. What is the weight of the potatoes now?

SOLUTION

The surprising answer is that the potatoes now weigh only 50 pounds.

Many people find this solution incredible.

But look at it this way. When you bought the 100 pounds of potatoes, 1 percent of them, or 1 pound in weight, is nonwater. Thus, 1 percent of the potatoes, which are solids, weighs 1 pound. When the potatoes dehydrated so that 98 percent of the potatoes were water, the other 2 percent in the potatoes must consist of the nonwater component. This 2 percent nonwater component weighs 1 pound. If 2 percent of the nonwater solids weighs 1 pound, then 100 percent must weigh 50 pounds.

Thus, the potatoes weigh 50 pounds.

SOURCE

"The Potato Paradox," http://www.puzzlefry.com/puzzles/the-potato-paradox

62

CALCULATING THE WIDTH OF A RIVER

An explorer out in the bush in Australia comes to a short river crossing. Directly across from him, he sees a tree trunk on the opposite edge of the river. He wishes to calculate the width of the river. He has no tools in his possession. How does he measure the width of the river?

SOLUTION

On being offered this puzzle, many people are surprised that the explorer can determine the width of the river without actually crossing it.

The explorer places a small stick in the ground where he is standing and designates that spot A (see Figure 11). He calls the spot where the tree trunk is B. He then walks southward parallel to the bank of the river a distance of 20 yards and puts a small stick in the ground there. He designates that spot C. He then walks another 16 yards southward and places a small stick in the ground at that spot. He calls that spot E. He now walks 12 yards westward and places a small stick at that spot, which he calls D. He now lines up D with C and B.

He now knows that he has formed 2 similar triangles—a southern one and a northern one—on the ground. The Pythagorean theorem tells him that the distance from D to C is the square root of the sum of $12^2 + 16^2$. This equals 20 yards. He now knows that the ratio of the length of the longest leg of the southern triangle to its hypotenuse is $\frac{16}{20}$, which reduces to $\frac{4}{5}$. Because the 2 triangles are similar, the ratio between the longest leg of the northern triangle and its hypotenuse must also equal $\frac{4}{5}$. Since the longer leg of the northern triangle is 20, its length must equal $\frac{4}{5}$ of the length of the hypotenuse. Therefore, the hypotenuse must be 25 yards in length.

The explorer now knows that the distance from C to B is 25 yards. The Pythagorean theorem tells him that the width of the river equals the square root of $(25^2 - 20^2)$. This equals 15.

Therefore, the width of the river is 15 yards.

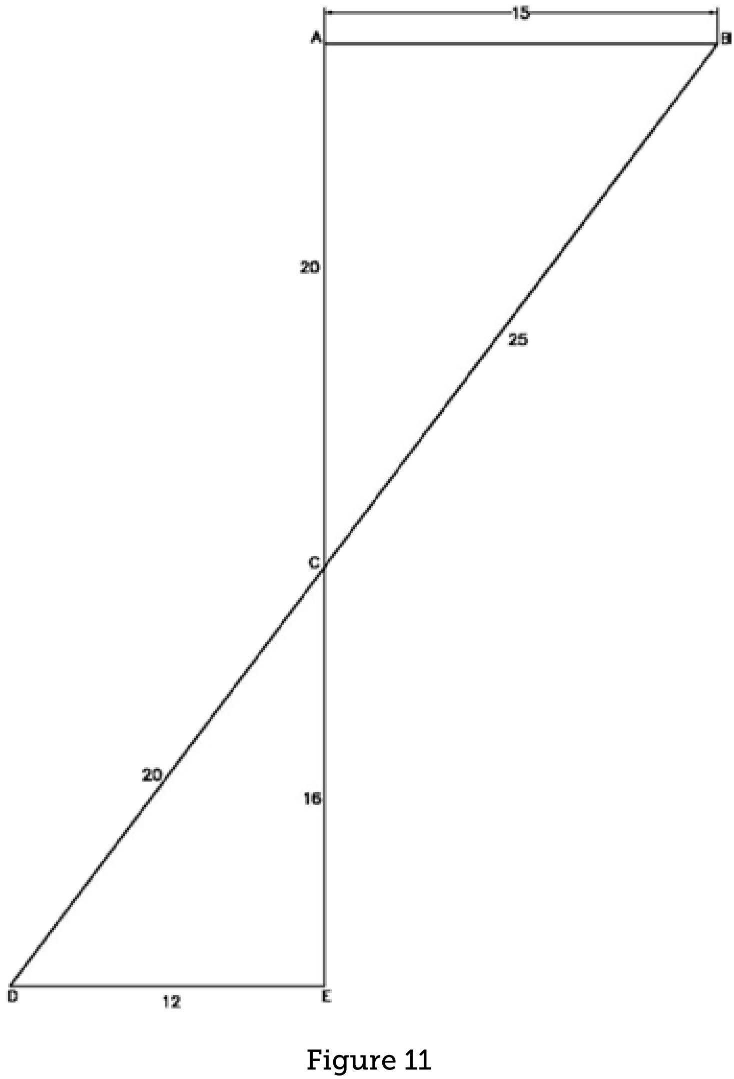

Figure 11

SOURCE

Henry Ernest Dudeney, *More Puzzles and Curious Problems*, edited by Martin Gardner (London: Fontana Books, 1970), pp. 57, 146.

63

THE GREEN AND BLUE CABS

In a particular city, there are 2 cab companies: the Green Cab Company, which operates green-colored cabs, and the Blue Cab Company, which operates blue-colored cabs. The Blue Cab Company consists of 85 percent of the total cabs in the city, while the Green Cab Company consists of 15 percent of the city's cabs.

One night last month, one of the city's cabs was involved in a hit-and-run accident. The only witness to the accident identified the cab as being green. The court tested this witness under the approximate weather conditions that were prevailing on the night of the accident and found that the witness is accurate in distinguishing the colors of the cabs in those conditions in the city 80 percent of the time.

What is the probability that the cab involved in the accident was in fact colored green?

SOLUTION

Suppose that there were 100 hit-and-run accidents in the city and that each one of these accidents was witnessed by the same observer as above. On average, 85 of these accidents would involve a blue cab, and 15 percent would involve a green cab.

Of the 85 accidents involving a blue cab, the observer would inadvertently state that 17 of these accidents involved a green cab. Of the 15 accidents involving a green cab, the observer would state that 12 of these accidents involved a green cab. Thus, the total number of reports stating that a green cab was involved in a hit-and-run accident would be 17 + 12, or 29. But of these 29 reports, only 12 green cabs were *actually* involved in a hit-and-run accident.

Therefore, the probability that a green cab was involved in the hit-and-run accident is $\frac{12}{29}$, or 41.37 percent.

Thus, it is more likely (58.63 percent) that the cab that was involved in the hit-and-run accident was colored blue.

A case like the one outlined above could easily end up in court. The correct but counterintuitive answer illustrates that a strong argument exists that judges should be trained in probability theory.

SOURCE

Daniel Kahneman, Paul Slovic, and Amos Tversky, eds., *Judgment under Uncertainty: Heuristics and Biases* (Cambridge: Cambridge University Press, 1982).

64

THE 3 DICE

Three players, Mary, Ann, and Jane, are playing with some unusual dice. Each rolls 6 dice, and the person getting the highest number wins.

The dice are numbered as follows:

Mary	Ann	Jane
2, 2, 4, 4, 9, 9	1, 1, 6, 6, 8, 8	3, 3, 5, 5, 7, 7

One can see that Mary is likely to roll higher than Ann and that Ann is likely to roll higher than Jane.

Therefore, it seems logical that Mary will roll higher than Jane? Is this true?

SOLUTION

The answer is surprising. It is *not* more likely that Mary will roll higher than Jane. This counterintuitive result astonishes most people when they are presented with the solution.

It is obvious from the numbers on the dice that Mary beats Ann two-thirds of the time. Ann beats Jane two-thirds of the time. But here is the surprise. Jane beats Mary two-thirds of the time!

The dice used here are what are called *nontransitive dice*.

SOURCE

"Nontransitive Dice," en.wikipedia.org/wiki/nontransitive dice

65

THE 3 VAGABONDS

Three vagabonds, Alec, Brian, and Charles, who were aged 50, 30, and 20, were walking along a country road when they stumbled on 3 bags. Each of the bags contained a number of fancy but inexpensive writing pens. One bag contained 50 pens, the second contained 30 pens, and the third contained 10 pens.

Each of the 2 older vagabonds decided to keep the bag that contained the number of pens that corresponded with each of their ages. All 3 vagabonds decided to take the pens to the nearest village and sell them there.

The youngest vagabond objected to this, saying that he would come out the worse off since he had only 10 pens.

But the oldest vagabond, who despite his years on the road had maintained an interest in mathematics over the years, said, “I have devised a little plan. We will all sell the pens at the same rate. That way, we will all end up having the same income from selling the pens.” He then proceeded to tell his pals of his plan.

The plan worked perfectly, and all 3 vagabonds made exactly $10 each from the various transactions.

Can you figure out the system in which the vagabonds sold the pens?

SOLUTION

It seems most surprising that all 3 men should end up with $10 each for selling their pens since all 3 had a different number of pens to begin with.

However, here is what the 3 men agreed on. Each of the 3 would sell 49, 28, and 7 pens, respectively, at the rate of 7 for $1. Then each would sell 1, 2, and 3 pens, respectively, at $3 per pen. Surprising as it may seem, each vagabond made a total of $10 on his transaction.

Here is how they fared:

Alec sold 49 of his 50 pens for $7. He then sold the leftover pen for $3. Total income: $10.

Brian sold 28 of his 30 pens for $4. He then sold the two leftover pens for $3 each. Total income: $10.

Charles sold 7 of his 10 pens for $1. He then sold 3 pens for $3 each. Total income: $10.

SOURCE

Samuel Evans Clark, *Mental Nuts: Can You Crack 'Em?*, rev. ed. (New York: The Home Monthly, 1900), puzzle 4, p. 4.

66

THE TENNIS TOURNAMENT

Mr. Smith and his wife and their 10-year-old son, John, were sitting down to breakfast one Saturday morning. Mr. Smith is secretary of the local tennis club, and he is telling his wife about the National Tennis Tournament, which is scheduled to be played at his local tennis club during the forthcoming summer months.

"It promises to be a very exciting tournament," Mr. Smith said. "Some very good tennis players will be participating. Of course, the tournament is a single-elimination competition. In other words, if a player is unlucky enough to lose, she is immediately eliminated from the tournament. The winner goes through to the next round. This will continue until the final 2 players are left in the competition. They will then play in the final match to determine the outright winner of the competition."

"The tournament will be fantastic," Mrs. Smith said. "I am confident that it will attract large crowds. Incidentally, how many players will be participating in the competition?"

"There will be exactly 256," said Mr. Smith. "I have received all the necessary paperwork from the various clubs. I must calculate later how many games in total that will need to be played in the tournament. I need that information to let the club's insurance company know. They insist on knowing exactly how many games are played in any competition we hold at the club grounds."

Mr. Smith's 10-year-old son, John, was listening to all of this. Suddenly he piped up: "There is no need to do any calculating, Dad," he said. "The total number of matches in the tennis tournament is ———."

Mr. and Mrs. Smith were flabbergasted. "How," they asked, "could you possibly know that?"

That is the question for the reader. How did John Smith know how many matches there would be in the tennis tournament?

SOLUTION

Young John Smith heard his dad say that the tennis tournament is a single-elimination competition. Thus, every player who loses is immediately eliminated, and the winner of that particular match continues to the next round. John Smith heard his dad say that there are 256 players scheduled to play in the tournament. Of these 256 players, 255 will be eliminated, leaving 1 outright winner. That means that there will have to be 255 matches.

The young lad's answer is easily verified. The 128 matches in the first round will eliminate 128 players. The remaining 128 players will play 64 matches in the second round, which will eliminate 64 players. These 64 players will play 32 matches in the third round. The 32 players remaining will play 16 matches, eliminating 16 players. The remaining 16 players will play 8 matches. The winning 8 players will play 4 matches. Four winning players will play 2 matches, leaving only 2 players in the tournament. These 2 players will play in the final match. Thus, the total number of matches is 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1, which equals 255.

SOURCE

"Tournament Scheduling," http://www.nrich.maths.org/1443

67

2 FOOTBALLS

Mr. Clancy sells sports goods, including footballs. He sells the footballs at a price according to their volume. The larger the volume, the more expensive is the football.

Last week, I went into his store to purchase a football. There were only 2 sizes left. One ball had a radius of 3 inches, while the second ball had a radius of 6 inches.

The price of the smaller football was $10. What is the price of the larger football?

SOLUTION

Surprisingly, the price of the second ball is $80.

We are told that Mr. Clancy prices the footballs according to their volume. The volume of the first ball is

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 3^3 = 36\pi$$

The volume of the second ball is

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 6^3 = 288\pi$$

Thus, the volume of the second ball is 8 times that of the first ball. Therefore, the price will be 8 times that of the first ball.

SOURCE

The author.

68

THE BIRTHDAY PARADOX

Suppose you and 29 other people are at a party. As the evening proceeds, one of the other guests says he is prepared to make the following wager. He asks all 30 people to write down on a slip of paper their dates of birth (month and day only) and to then fold each slip and place it in a box. He then bets $20 at even-money odds that 2 of those birthdays will match.

On hearing this bet, you and the others may be inclined to reason as follows. There are 30 of us here at this gathering. We have all written down our birthdays (day and month only), and this individual bets that 2 of these birthdays will match. But there are 365 different dates in a normal year. The odds must be low that 2 birthdays match. Therefore, the bet favors me, and I will accept the wager. It will be a handy way of picking up $20.

Whom does the bet favor, and what precisely is that probability?

SOLUTION

This is one of the classic proposition bets. It has a truly counterintuitive solution.

With 30 people in the room, the probability is more than 70 percent that the con artist will win the bet! Many people find this result so surprising that they refuse to believe it.

Here's how this probability is calculated.

The probability that a second person does *not* have the same birthday as the first is 364/365.

The probability that a third person does *not* have the same birthday as the previous 2 persons is 363/365. The probability that a fourth person does *not* have the same birthday as any of the previous 3 is 362/365. And so on.

Thus, the probability that any of the 30 people at the gathering does *not* have a matching birthday is

$$\frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{337}{365} \times \frac{336}{365} = 0.2936+$$

In percentage terms, this equals 29.36+ percent. Thus, there is a 29.36 percent probability that there will be *no* birthday match among 30 randomly selected persons.

Of course, this means the probability that there *will* be 2 matching birthdays among 30 randomly chosen persons chosen is 70.63+ percent.

The truly surprising thing about this bet is that with as little as 23 people chosen at random, the odds are just about even that the birthdays of 2 people will match. With 23 people, the probability is 0.50729 that 2 birthdays will match, and with 22 people, the probability is 0.47569 that 2 birthdays will match. Thus, 23 people is the critical point for obtaining an even chance that at least 2 birthdays will match.

This is a very counterintuitive result. Many people find it difficult to believe even when the mathematics involved in the solution is explained to them.

Of course, the probability that there will be a match between 2 birthdays increases as the number of people at the party increases.

With 35 people present, the probability of matching birthdays is 81.44 percent; with 40 people, it is 89.12 percent; and with 50 people, it is over 97 percent probable that there will be a match!

With 100 people present, there is only 1 chance in 3,254,690 that there will be *no* match. Thus, there are 3,254,689 chances to 1 that there *will* be a match. In other words, there is only 1 chance in over 3,000,000 that there will *not* be a matching birthday with 100 people picked at random! If someone proposes the bet with 100 randomly selected people, it is almost certain that there will be a matching birthday!

SOURCE

Owen O'Shea, *The Call of the Primes* (Amherst, NY: Prometheus Books, 2016), pp. 27–76.

69

THE SECOND CHILD PUZZLE

In this puzzle, we assume that the likelihood of the gender of any child born to a couple is exactly 50 percent and that that probability is independent of previous births of children.

You live in a small village that is located many miles from the nearest town. You know that all the couples in the village have families consisting of 2 children and that in each family, at least 1 of these 2 children is a boy.

Suppose you are out for a walk in the village on the first Sunday in May. You meet a couple with whom you have never spoken before and get engaged in conversation with them. You ask them how many children they have. They reply that they have 2 children, of which at least 1 is a boy.

The question now arises: what is the probability that both children are boys?

The solution to this lies in recognizing that the only 4 possibilities that exist in relation to any couple with 2 children where the eldest child is listed first are the following:

(1) Boy, Boy
(2) Boy, Girl
(3) Girl, Boy
(4) Girl, Girl

Since you know that all the families in the village consist of at least 1 boy, the fourth possibility above does not apply. In the other 3 cases, in which we assume that the gender of all children born occurs independently and with equal probability, a family with 2 boys occurs only once in 3 possible cases. Where the firstborn child is a boy, the second child is a boy in only 1 case. Therefore, the answer to the question of what the probability is that the family consists of 2 boys is 1 in 3, or $\frac{1}{3}$.

Suppose you are out walking in the village the following Sunday. You meet a couple on your travels with whom you have never spoken before. You ask them how many children they have. They reply that they have 2 children and say that the older child is a boy.

What is the probability that both children are boys?

SOLUTION

It is hard to believe at first, but the probability that the second child is a boy has now changed from the previous answer of ⅓ to ½.

Many people find the fact that this solution is different to the first incredible. Why, they ask, should specifying one child as the oldest change the probability that the second child is a boy?

But, astonishingly, it does!

By specifying one child as the *oldest*, the *tallest*, the *heaviest*, the most *curious*, or the most *athletic* reduces the *sample space* in the puzzle and consequently changes the probability that the second child is a boy.

Let's look at the puzzle a little more closely.

You ask the couple if they have any children. They reply that they have and add that the older child is a boy. You are now asking yourself, what is the probability that the second child is a boy also?

In any family of 2 children, the following 4 possibilities arise, where the first listed is the older child:

(1) Boy, Boy
(2) Boy, Girl
(3) Girl, Boy
(4) Girl, Girl

You know that every couple in the village has 2 children and that at least 1 child in every family is a boy. Therefore, you know that the fourth possibility above does not apply

Therefore, there are now only 3 cases to consider, where we write the specified *elder* child first:

(1) Elder child (Boy), Boy
(2) Elder child (Boy), Girl
(3) Elder child (Girl), Boy

Here we see that the third possibility also does not apply because the couple stated that their *elder* child is a boy.

Therefore, there are only 2 possibilities to consider:

(1) Elder child (Boy), Boy
(2) Elder child (Boy), Girl

We see that the second child is just as likely to be a boy as a girl. Therefore, the probability that the second child is a boy is 1 in 2, or ½.

SOURCE

Martin Gardner, *More Mathematical Puzzles and Diversions* (London: Penguin Books, 1966), p. 175.

70

THE 4 BRIDGE PLAYERS

Four players sit down to play a game of bridge. Each player is dealt 13 cards. Which event is more likely: that 2 partners are dealt all the clubs or that 2 partners are dealt none of the clubs?

SOLUTION

The 2 probabilities are equal! This surprising fact usually astonishes those who try their hand at this puzzle.

Surprisingly, we can solve the puzzle without resorting to long and tedious calculations involving the theory of probability. For 2 players to be dealt *all* the clubs, their opposing partners must be dealt *none* of the clubs. If 1 event happens, that is, if 2 partners are dealt all the clubs, the second event must simultaneously happen; that is, the opposing two partners must be dealt *none* of the clubs.

Therefore, the probability that 2 partners will be dealt all the clubs exactly equals the probability that 2 partners will be dealt none of the clubs.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 2.1, pp. 39, 49.

71

THE 2 END DIGITS OF 20 NUMBERS

What is the probability that 2 of 20 randomly selected 2-digit numbers from 00 to 99 will match?

SOLUTION

This is a similar type of puzzle as the matching birthday problem (see problem 68) and is solved in a similar way.

There are 100 2-digit numbers from 00 to 99.

The probability that the second number does *not* match the first number on the list is $^{99}\!/_{100}$. The probability that the third number does not match the first or second number is $^{98}\!/_{100}$. The probability that the fourth number does not match the previous 3 numbers on the list is $^{97}\!/_{100}$. And so the calculations proceed, each time calculating the probability that each successive number does not match any of the preceding numbers. The calculations look like this:

$$\frac{99}{100} \times \frac{98}{100} \times \frac{97}{100} \times \ldots \times \frac{82}{100} \times \frac{81}{100} = 0.13039\ldots$$

By the time the 20th number is reached, the probability that all 20 numbers end with 2 *different* digits equals 0.1303+.

Therefore, the probability that at least 2 numbers end with 2 *similar* digits is 1 − 0.1303+. This equals 0.8696+.

In percentage terms, this equals 86.96+ percent.

Therefore, the probability that at least 2 of the 20 numbers *will* end with 2 *similar* digits is nearly 7 times more likely than that any of the 2 numbers will *not* end with *similar* digits.

This astonishing fact never fails to surprise people.

It is a very counterintuitive result!

SOURCE

Oswald Jacoby, *How to Figure the Odds* (Garden City, NY: Doubleday, 1947), pp. 103–17.

72

LEWIS CARROLL'S PROBLEM IN PROBABILITY

A stranger has a bag containing either a single white marble or a single black marble. He then places a black marble into the bag. There are now only 2 marbles in the bag. He then shakes the marbles in the bag around to mix them up. You then draw a marble from the bag. It is colored black.

What is the probability that the second marble in the bag is black?

SOLUTION

On being given this puzzle, most people, it seems, answer that the probability is even that the color of the marble in the bag is black. They reason that the marble is just as likely to be colored black as it is to be colored white.

Surprisingly, this answer is incorrect.

We know that initially, the marble in the bag is either white or black. Let us assume for a moment that this marble is black, and we designate it black 1. The stranger then places a black marble, call it black 2, into the bag. He then shakes the marbles around in the bag to mix them up.

When you draw 1 marble from the bag and it is black, 1 of 3 possible states exist:

You have drawn the black 1 marble. The marble in the bag is black 2.
You have drawn the black 2 marble. The marble in the bag is black 1.
You have drawn the black 2 marble. The marble in the bag is white.

In only 1 of these 3 possible states is the marble in the bag white. In 2 of the 3 possible states, the marble in the bag is black.

Therefore, the probability that the marble in the bag is black is $\frac{2}{3}$.

This is quite a surprising result. It is believed that Charles Dodgson (1832–1898), who was better known to the world as Lewis Carroll, was the first to pose this puzzle.

SOURCE

Martin Gardner, *Mathematical Circus* (London: Penguin Books, 1981), pp. 188–90.

73

AGAINST THE WIND

You are told that a cyclist on a 1-mile journey rode his bicycle at the rate of 1 mile in 3 minutes with the wind to his back, but on the return trip with the wind blowing against him, he managed to cycle at the rate of only 1 mile in 4 minutes. If he always applied the same force to his pedals, how long would it take him to ride 1 mile if there were no wind?

SOLUTION

On being asked this puzzle, most people, it appears, answer it as follows. The cyclist went on a journey of 1 mile. He rode the outward part of the journey at 3 miles per minute and rode the return part of the trip in 4 minutes. Therefore, if there were no wind, his speed must have been one-half of 3 plus 4, or 3½ miles per minute.

This answer appears plausible, but it is incorrect!

The surprising answer is that if there were no wind, the cyclist would ride 1 mile in 3 and $\frac{3}{7}$ minutes.

The incorrect answer is arrived at by assuming that the effects of the wind on the cyclist are the same for the outward and homeward trips. But this is not the case. On the outward journey, the wind has helped the cyclist for only 3 minutes. On the homeward trip, the wind has hindered the cyclist for 4 minutes.

One method of obtaining the correct solution is as follows.

On the outward journey, with the wind blowing behind him, the cyclist travels 1 mile in 3 minutes. On the homeward trip, with the wind blowing against him, the cyclist travels 1 mile in 4 minutes, which is equivalent to three-quarters of a mile in 3 minutes. It is this 3-minute time period in both directions that we are interested in. Since the wind is blowing in the cyclist's favor for 3 minutes on the outward trip and the wind is blowing against him for 3 minute on the return trip, the effects of the wind on the cyclist cancel.

We know that in this 6-minute period, with the effects of the wind cancelled, the cyclist travels 1¾ miles. This is equivalent to the cyclist—in 24 minutes—traveling 7 miles, which is equivalent to cycling 1 mile in $3\frac{3}{7}$ minutes.

Thus, with no wind assisting him or hindering him, the cyclist will cycle 1 mile in $3\frac{3}{7}$ minutes.

SOURCE

Martin Gardner, *More Mathematical Puzzles of Sam Loyd* (New York: Dover Publications, 1960), puzzle 49, pp. 34, 137.

74

THE 2 EXPRESS TRAINS

An express train starts out on a Monday morning at 8 a.m. on a journey from Rosetown to Petaltown. We assume that the train travels at a constant speed of 60 miles per hour. At precisely the same time, an express train starts out on a trip from Petaltown to Rosetown. We assume that that train travels at a constant speed of 40 miles per hour.

How far apart are the 2 trains 1 hour before they meet?

SOLUTION

We are not told how far apart the 2 towns of Rosetown and Petaltown are. But the surprising thing about this little puzzle is that we do not need to know this information.

Wherever the 2 trains meet, just imagine each of the 2 trains going back in the direction they came from for 1 hour. One train will go back 60 miles, while the other train will go back 40 miles. They will then be 100 miles apart. That is precisely where they were 1 hour before they met.

This is the answer to our little puzzle. The 2 trains will have to be 100 miles apart 1 hour before they meet.

SOURCE

Henry Ernest Dudeney, *Puzzles and Curious Problems* (London: Thomas Nelson and Sons Ltd, 1932), puzzle 80, pp. 30, 137.

75

THE MAN PACKING A SUITCASE

A man, working alone, can pack a suitcase in 20 minutes. With his wife working alongside him, it takes 1 hour to pack 1 suitcase.

If the man's wife works on her own, how long will it take her to pack a suitcase?

SOLUTION

The surprising answer is that the man's wife, working alone, will never pack a suitcase no matter how long she works on it.

From the question, it can be seen that the man, working alone, can pack a suitcase in 20 minutes. Therefore, working alone, he can pack 3 suitcases in 60 minutes. However, we are told that the man and his wife working together can pack only 1 suitcase in 60 minutes.

This allows us to write the following equations:

Man working alone	3 suitcases = 1 hour
Man and his wife working together	1 suitcase = 3 – 2 suitcases = 1 hour

Consequently, his wife must have packed 2 *negative* suitcases in 1 hour, which equals 1 *negative* suitcase in 30 minutes.

We can therefore write

Wife working alone: −1 suitcase = 30 minutes
Multiply across the equation by negative 1 to obtain
1 suitcase = −30 minutes

Therefore, the man's wife working alone can pack 1 positive suitcase in −30 minutes. But this period of time, −30 minutes, is 30 minutes *less* than 0.

Therefore, the man's wife can never pack 1 positive suitcase no matter how much time is allocated to her!

SOURCE

This puzzle was passed along to me by Dr. David Singmaster, retired professor of mathematics at South Bank University, London. It is used here with his kind permission.

SLIGHTLY MORE CHALLENGING TEASERS

This last section of math teasers contains a few problems that are more difficult than those already encountered in the book. The last two are especially more difficult, but both of those, similar to all the other puzzles, have solutions that are most surprising. Try your hand at them, and hopefully you will enjoy tackling the problems. Have fun!

76

THE LINKS IN THE GOLD CHAIN

An old puzzle tells of a traveler in a strange town who has a valuable gold watch chain consisting of 7 links. The traveler has no funds but is expecting a large check to arrive in 7 days. To pay for a room, he arranges with the landlady to give her as collateral 1 link per day for each of 7 days.

The traveler naturally wants to minimize any damage done to the gold watch chain by cutting. What is the minimum number of cuts he must make so that he can meet his financial obligations to the landlady?

SOLUTION

The surprising answer is that he needs to make only 1 cut to the chain. He cuts the chain at the third link. He will now have 2 segments consisting of 3 links and 4 links. However, he will be able to remove the link he has cut from the 3-link segment. Thus, he now has 3 segments of chain: 1 link, 2 links, and 4 links.

Instead of giving 1 separate link of the chain every day to the landlady, he gives her 1 link on the first day, then on the second day, he will take that first link back and give the landlady a segment of chain consisting of 2 links. On the third day, he can give her the first link, so the landlady now has payment for 3 days. On the fourth day, he can take back the 3 links and give her a chain of 4 links. On the fifth day, he gives the landlady 1 more link. On the sixth day, he takes back the 1-link segment and gives her the 2-link segment. On the seventh and final day, he gives her 1 link. In this way, the landlady is in possession each day of a number of links of the gold watch chain that corresponds to the number of days the traveler has stayed in the room.

This puzzle has cropped up in various forms over the years. Sometimes the gold chain will consist of 23 links. Once again, the traveler wants to use the chain to pay 1 link per day for each of the 23 days he stays in the room and wishes to minimize the damage done to the chain. Surprisingly, this can be achieved by cutting the chain just twice: at the fourth link and at the 11th link. Even more surprising is that a 69-link chain requires just three cuts: at links 5, 14, and 31.

The formula to find the maximum number of links in the chain so that the above conditions can be adhered to, where n equals the number of cuts, is

$$\left[\ (n+1)2^{n+1}\ \right] - 1$$

Thus, when n equals 1, the maximum number of links in the chain is 7; when n equals 2, the maximum number of links is 23; when n equals 3, the maximum number of links is 63; and so on. This series of numbers runs as follows: 1, 7, 23, 69, 159, 383, 895, 2047, 4607 . . . .

SOURCE

Martin Gardner, *Martin Gardner's Sixth Book of Mathematical Diversions from Scientific American* (Chicago: University of Chicago Press, 1971), puzzle 4, pp. 50, 51, 57, 58.

77

THE ROTATING COIN

Suppose you have 2 identical coins. Call them coin A and coin B. You place coin A on a flat surface. You place coin B so that it is touching the rim of coin A. You then rotate coin B around the circumference of coin A, making sure that there is contact at all times between the 2 coins and that there is no slipping, until coin B returns to its starting point.

The question is, how many times does coin B rotate?

SOLUTION

The obvious answer to this little puzzle is to state that coin B rotates once as it goes around the circumference of coin A.

But this answer is incorrect!

The very surprising answer is that coin B rotates twice as it goes around the circumference of coin A.

I urge the reader to try this little experiment herself. When she has concluded the experiment, she will still probably find it difficult to understand why the second coin rotates twice as it revolves around the first coin.

Let the radius of each of the 2 coins equal r. The circumference of each of the coins will then equal $2\pi r$. Suppose you draw a line on a sheet of paper that is equal in length to $2\pi r$ units. Now rotate coin B along this line. You will find that coin B rotates just once along the line that measures $2\pi r$ units.

But why does coin B rotate twice as it goes around coin A?

When you rotate coin B around the rim of coin A, there are 2 different motions involving coin B. Coin B rotates around its own center, but as it rolls around the curved rim of coin A, it simultaneously rotates around the center of coin A also.

Therefore, surprising as it may seem, coin B will rotate *twice* as it rotates around coin A.

SOURCE

"Puzzle: A Coin Rolls without Slipping around Another Coin," http://www.math.stackexchange.com/questions/149234/puzzle-a-coin-rolls-without-slipping-around-another-coin

78

DIVIDING THE SPOILS

Two women, Smith and Jones, have bought a very successful company between them. Smith owns more of the company than Jones. A new investor, Robinson, invests $50,000 in the company, which results in all 3 investors having an equal share of the business. Smith and Jones are wondering how to correctly divide Robinson's investment between them.

James, the 12-year-old son of Smith, is a fairly good mathematician for his age, and he tells his mother that she should take $40,000 of Robinson's investment and that Jones should take the remaining $10,000 of Robinson's money. But other members of the 2 families disagree. Assuming that the young lad's advice is the correct division of the new investment, how much of the company did Smith own compared to Jones?

SOLUTION

Many people will answer this by saying that if young James's calculation is correct, where he advises that the new investment is divided in a 4-to-1 ratio in favor of Smith, it is obvious that Smith owned 4 times as much of the company as Jones did.

This answer appears very plausible, but it is incorrect!

The surprising answer is that if the young lad's calculation is correct, Smith owned 1½ times as much of the company as Jones did.

Here is how this solution is reached.

We are informed in the question that when Robinson invested $50,000 in the firm, it resulted in each of the 3 investors, Smith, Jones, and Robinson, having an equal share of the company. In that case, the firm must be valued at 3 times $50,000, or $150,000.

We are also told that Smith initially owned more of the company than Jones. But how much more? Since the new investment is to be divided in a 4-to-1 ratio, the amount in *excess* of $50,000 each that Smith and Jones invested must be in a 4-to-1 ratio, in favor of Smith.

Thus, Smith invested $50,000 + $40,000, or $90,000, and Jones invested $50,000 + $10,000, or $60,000. In this way, Smith owned 1½ times as much of the firm as Jones did.

We also see that Smith invested $40,000 *more* than her share of what the company is now worth and that Jones invested $10,000 *more* than her share of the company.

Consequently, the new investment of $50,000 should be divided in 4-to-1 ratio so that Smith gets $40,000 and Jones gets $10,000.

Thus, Smith's young lad, James, was correct in his calculations.

SOURCE

Martin Gardner, *More Mathematical Puzzles of Sam Loyd* (New York: Dover Publications, 1960), puzzle 151, pp. 107, 170, 171.

79

COUNTING THE VOTES

There were 5,219 votes cast in an election for 4 candidates. The victor exceeded his opponents by 22, 30, and 73 votes. This information was given to young Tommy, who was fond of solving mathematical puzzles. Given this information, Tommy asked the candidates how many votes were cast for each of them, but none of them was able to calculate the exact number. But Tommy was able to calculate it.

On the basis of the information given to Tommy, could you have figured how many votes were cast for each candidate?

SOLUTION

The surprising method to solve this puzzle is to add 22, 30, and 73 together, obtaining 125; add this result to 5,219, obtaining 5,344, and then divide that result by 4. The answer obtained is 1,336, which tells us that that is the number of votes that the most successful candidate obtained. All one need do to obtain the number of votes that each of the other candidates received is to successively subtract 22, 30, and 73 from 1,336.

Thus, each of the candidates received the following votes: 1,336, 1,314, 1,306, and 1,263.

The procedure used to solve this puzzle yields easily to analysis.

Let x equal the number of votes the most successful candidate obtained. Thus, each candidate must have received x, $x - 73$, $x - 30$, and $x - 22$. The sum of these votes equals $4x - 125$. This must equal 5,219, which is the total amount of votes cast. Thus, we can write $4x - 125 = 5{,}219$. Therefore, $4x = 5{,}219 + 125 = 5{,}344$. Thus, $x = 1{,}336$.

Thus, the most successful candidate received 1,336 votes. The number of votes the other candidates received is then easily obtained as explained above.

SOURCE

Martin Gardner, *Mathematical Puzzles of Sam Loyd* (New York: Dover Publications, 1959), puzzle 90, pp. 87, 152.

80

THE FENCE-POST PROBLEM

Mr. Atkins lives at the end of a lane that is exactly 100 feet long. He wishes to erect a fence all the way down the lane, on both sides of the lane, with posts every 10 feet.

How many posts does he need to complete the job?

SOLUTION

Many people will answer this by saying 20 posts. They reason that 10 posts will be required on each side of the lane to measure off a distance of 100 feet.

But that answer is incorrect!

Eleven posts are required on each side of the lane, where each post is 10 feet apart, to measure off a distance of 100 feet.

Therefore, a total of 22 posts are required to complete the job.

SOURCE

John Paul Adams, *We Dare You to Solve This!* (New York: Berkley, 1955), puzzle 134, p. 57.

81

MR. BLUEBERRY'S WILL

Joe Blueberry was unfortunately killed in a road accident. Two weeks previously, Joe and his wife, who had just been informed by her doctor that she was expecting a baby, had made a will. In the will, Joe said that in the event of his death, should his wife give birth to a son, the little boy should get two-thirds of his estate, and his wife, Mary, should get one-third. If, however, Mary gave birth to a daughter, the little girl should get one-quarter of the estate, and Mary should get three-quarters.

As things turned out, Mary gave birth to twins: 1 boy and 1 girl. How should Joe Blueberry's estate be divided between his wife and 2 children?

SOLUTION

It is clear that Joe Blueberry wanted his son to get twice as much as his wife from the estate. He also wanted his wife to get 3 times as much as his daughter from his estate.

Let the value of the whole estate equal S. Let Mary's share equal x, let the son's share equal y, and let the daughter's share equal z:

We know that $x + y + z = S$.
According to Joe's wishes, $x = y/2$ and $x = 3z$.
Thus, $x + 2x + x/3 = S$.
Therefore, $3x + 6x + x = 3S$ or $10x = 3S$ or $x = 3/10S$.

Therefore, Joe Blueberry's wife, Mary, should receive $\frac{3}{10}$ of the estate. Joe Blueberry's son should receive twice that amount, or $\frac{6}{10}$ of the estate. Joe Blueberry's daughter should receive $\frac{1}{10}$ of the estate.

In this way, Mr. Blueberry's wishes are carried out.

SOURCE

Erwin Brecher, *The Ultimate Book of Puzzles, Mathematical Diversions, and Brainteasers: A Definitive Collection of the Best Puzzles Ever Devised* (New York: St. Martin's Griffin, 1996), puzzle 251, pp. 51, 186, 187.

82

THE JEWELER AND THE CHAIN

A young lady brought 4 pieces of gold chain, each piece consisting of 3 links, to the jeweler's to see if the chain could be mended.

The jeweler looked at the 4 pieces.

"I can repair the chain for you," he said. "Let me see now. To put the chain back together, I will have to break and rejoin four links. Now I charge \$2 for breaking a link and \$2 to mend each link again. As regards the cost, I'm afraid I will have to charge you \$16.

The young lady looked disappointed. "I'm afraid," she said, "I have not got \$16. I only have \$12. I will have to take it elsewhere and see if I can have it repaired more cheaply."

The jeweler took another hard look at the 4 pieces of chain. Eventually, he said, "Today, young lady, must be your lucky day! I have a little plan. I have thought of another way in which I can mend your chain. I will be able to repair the chain for you for just \$12."

The young lady was both surprised and delighted.

What was the plan that the jeweler had?

SOLUTION

The jeweler cut all 3 links of 1 of the 3 pieces and then used the broken links to join the other segments. Thus, the total cost of the repair work was $12.

SOURCE

Erwin Brecher, *The Ultimate Book of Puzzles, Mathematical Diversions, and Brainteasers: A Definitive Collection of the Best Puzzles Ever Devised* (New York: St. Martin's Griffin, 1996), puzzle 142, pp. 51, 186, 187.

83

THE PRICE OF A BOOK

The price of a book is $10 plus half its selling price. How much did the book sell for?

SOLUTION

This little puzzle never fails to perplex people. Some answer the question by saying that the book sold for \$10; others say that the selling price was \$15. The correct solution, which surprises many people, is that the book sold for \$20. Thus, the book sold for \$10 plus half its selling price.

SOURCE

Boris A. Kordemsky, *The Moscow Puzzles*, edited by Martin Gardner (New York: Dover Publications, 1992), problem 37, pp. 15, 195.

84

VISITING A FRIEND

You wish to visit a friend who lives in the sixth floor of a 6-story building. Each story is the same height and has a stair the same length going from story to story. How many times as long is the ascent from the first floor to the sixth floor as is the ascent from the first floor to the third floor?

SOLUTION

Most people tackle this problem by reasoning as follows. You will have to climb the stairs 3 times to reach the third floor. But you will have to climb the stairs 6 times to reach the sixth floor. Therefore, the ascent to the sixth floor is $\frac{6}{3}$, or twice the length as the ascent to the third floor.

It all appears very plausible, but this solution is incorrect!

The surprising answer is that the ascent to the sixth floor is $2\frac{1}{2}$ times as long as the ascent to the third floor!

Here is how to work this out. To go from the first floor to the third involves two ascents: you must ascend from the first floor to the second floor and then from the second floor ascend to the third floor.

To go from the first floor to the sixth floor involves five ascents: (1) you climb from the first floor to the second, (2) you ascend from the second to the third, (3) you climb from the third to the fourth, (4) you ascend from the fourth to the fifth, and (5) you climb from the fifth to the sixth.

Therefore, your ascent to the sixth floor involved 5 ascents, while your ascent to the third floor involved only 2 ascents. Therefore, the climb to the sixth floor is $\frac{5}{2}$, or $2\frac{1}{2}$, times as long as the ascent to the third floor.

SOURCE

Boris A. Kordemsky, *The Moscow Puzzles*, edited by Martin Gardner (New York: Dover Publications, 1992), problem 67, pp. 25, 202.

85

THE TYPING TEENAGER

Mrs. Taylor, a busy mother of 5 children, is secretary of the local Parents Association. She is preparing her annual report. She has asked her teenage son, Tom, to type up the lengthy report, stating that she needs 20 pages typed per day for a specific period.

Tom agrees to do the task. But like most teenage boys, he has his mind on other things. He decides to type up 10 pages of the report every day until he has half the report completed. Tom then intends to type the other half of the report at the rate of 30 pages per day. In that way, he reasoned, he would achieve an average of 20 pages of the report typed per day.

Was Tom's reasoning correct?

SOLUTION

Tom's reasoning seems to be plausible, but, alas, it is not correct.

Suppose there are 100 pages in the report. Tom wants to average 20 pages per day. At the rate of typing 10 pages per day, it would take 5 days to complete typing the first half of the report. To average 20 pages per day, Tom would have to complete the entire task in 5 days. But he has already used up 5 days typing the first half of the report. For Tom to average 20 pages per day, he would now have to type the second half of the report in 0 time! This is impossible.

Therefore, Tom's reasoning was not correct.

Many people are surprised at this finding.

SOURCE

Boris A. Kordemsky, *The Moscow Puzzles*, edited by Martin Gardner (New York: Dover Publications, 1992), problem 222, pp. 98, 259.

86

MR. SIMPSON'S INFLUENCE

Suppose the national executive of a labor union is seeking to fill the positions of various regional secretaries and administrative assistants within the organization.

Twenty men apply for the post of regional secretary, and 4 men are successful in obtaining the positions. That is a success rate of 20 percent. Thirty-two women apply for the post of regional secretary, and 8 women are successful in getting the position. That is a success rate of 25 percent. It appears that women are more favored than men for the positions in the labor union.

Now consider the situation in relation to the position of administrative assistant.

Thirty-two men apply for the position of administrative assistant, and 24 men are successful in getting the position. That is a success rate of 75 percent. Twenty women apply for the position of administrative assistant, and 16 women are successful in getting the position. That is a success rate of 80 percent. Once again, it appears that women have been favored over men in obtaining positions in the labor union.

However, further analysis of the figures reveals that 52 candidates applied for the 2 vacancies and that 28 men and 24 women were hired. Now it seems that the labor union has favored men over women!

Consider the following table, which shows the success rate of men and women:

Position	Men	Women	
Regional secretary	4/20	8/32	Women are treated more favorably than men.
	↓	↓	
	20%	25%	
Administrative assistant	24/32	16/20	Women are treated more favorably than men.
	↓	↓	
	75%	80%	
Total numbers employed	28	24	Men are treated more favorably than women!

It is clear that women are treated more favorably than men when one considers the *percentage rate* of successful candidates for each position. But when one considers the *number* of successful candidates for each position, it is clear that the figures indicate that the men have been treated more favorably than women.

What is wrong here?

SOLUTION

This is an example of what is known as *Simpson's paradox*. The paradox is named for the English statistician and civil servant Edward Simpson (1922–2019), who described the effect in an article in the *Journal of the Royal Statistical Society* in 1951. The effect demonstrates that numbers can be used to apparently support one conclusion, but when another variable is entered into the data, a different conclusion is obtained. The paradox is taught as part of most mathematical statistics courses in an attempt to ensure that the results of statisticians interpreting data are not skewed by the paradox.

SOURCE

"Simpson's Paradox," http://www.squishlikegrape.com/content/simpsons-paradox

87

THE POTENTIAL EMPLOYEE

Mr. Snoggs, the human resources manager of a large multinational company, has an interest in recreational mathematics. In recent weeks, he has been interviewing potential employees for the position of accountant with the firm. Snoggs had narrowed the number of potential employees down to 2 female applicants. He interviewed each applicant separately a second time and addressed each of them as follows:

"You are an intelligent lady. You know that if your monthly income increases by 30 percent and the prices of goods remain stationary, your purchasing power will also increase by 30 percent. Isn't that correct?"

"It certainly is, sir."

"Good," Snoggs said. "I was hoping that you would say that." Snoggs was enjoying himself. "Now tell me this," he asked in a rather casual manner. "Suppose your monthly income stays the same and the prices of goods in the economy decreased by 30 percent. By what percentage does your purchasing power increase?"

I have been told that the first applicant answered this question incorrectly. But the second applicant gave the correct answer. She consequently got the job.

If the question were put to you, could you have given the correct answer?

SOLUTION

Surprisingly, if the prices of goods decrease by 30 percent, your purchasing power increases by nearly 43 percent. More precisely, your purchasing power will increase by 10/7, or 1.42857 . . . percent.

An example will make it clear. Suppose you wish to purchase a coat in a store that costs $100. You decide to wait until payday to purchase the item. But when payday comes around, the prices of goods have decreased by 30 percent. The coat now costs $70. Therefore, your purchasing power has increased by 100/70, or 1.42857 times that of what it was.

Thus, your purchasing power has increased by 42.875 percent.

SOURCE

Boris A. Kordemsky, *The Moscow Puzzles* (New York: Dover Publications, 1992), problem 234 (B), pp. 101, 260.

88

THE PRICE OF BOOKS

The boss of a reputable firm, Mr. Muggins, was having a Christmas drink with a couple of friends named Lisa and Marge. Muggins was known to have an interest in recreational mathematics. Before long, the subject of Christmas shopping arose in the conversation.

"I always do my shopping weeks ahead of Christmas" Marge said. "I usually purchase a book for my husband. He is an avid reader."

"Good for him!" Lisa said. "A present of a book is a great gift to receive. I always buy books for my kids."

"Glad to hear that," Muggins said. "Speaking of books, a curious little problem arose the other day in a bookstore not too far from here. I went there to see an old friend who works in the store. I asked her how the books went last Christmas and New Year. She told me that they had a 10 percent price reduction on every book sold in last January's sales, but they were still able to make an 8 percent profit on each book sold."

"What was the profit margin before the sale?" Lisa asked.

"I was hoping," Muggins said, "that one of you would be able to tell me."

Can you figure out what the profit margin was before the 10 percent reduction took effect?

SOLUTION

On being given this puzzle, many people will reason as follows: each book is reduced in price by 10 percent. Each book is then sold at an 8 percent profit to the bookstore. If the bookstore did not reduce the selling price of the book by 10 percent, the overall profit the bookstore would have made would have been 10 + 8 or 18 percent. Thus, the overall profit before the 10 percent price reduction was implemented was 18 percent.

It comes as a surprise to many people that that analysis—hence the solution—is not correct.

The overall profit margin before the 10 percent price reduction was implemented was 20 percent.

Here is the correct way to solve this puzzle.

Let the original percentage profit of all the books—before the price reduction was implemented—equal x.

When the price reduction has been made the percentage profit of all the books can be expressed as $90x/100$. From the question, we see that this price still represents a profit of 8 percent to the bookstore.

This allows us to write the following equation:

$$\frac{90x}{100} = \frac{108}{100}$$

Therefore

$$9{,}000x = 10{,}800$$

This equals

$$x = 1.2$$

Since x equals the percentage profit, we see that the percentage profit was 1.2 times the cost of the book.

Let's see how this works in practice.

For every batch of \$100 worth of books the bookstore originally purchased, it had intended to sell each of those batches for \$120. This would represent a profit margin of 20 percent. However, in a sale, the bookstore lowered the price of their books by 10 percent. Thus, the batches of books selling at \$120 were lowered by \$12 in price to \$108. This selling price still represented a profit margin of 8 percent for the bookstore.

Thus, the overall profit margin for the bookstore *before* the price reduction took effect was 20 percent.

SOURCE

Boris A. Kordemsky, *The Moscow Puzzles* (New York: Dover Publications, 1992), problem 234 (C), pp. 101, 260.

89

THE LADY IN THE JEWELRY STORE

A woman went into a jewelry store and bought a watch for $100. She paid the correct amount for the watch. Just as she was about to leave, she turned back and said, "I think I prefer that watch over there that is priced at $200. Can I exchange this watch for it?"

"Certainly madam," the store assistant said as he handed the watch to the customer. "That will be another $100 please."

The customer took hold of the $200 watch, gave back the $100 watch to the store assistant, and said, "You must be mistaken. Just 5 minutes ago I gave you $100. I have now just given you a watch that is worth another $100. That makes $200. Therefore, I owe you nothing."

With that, she stormed out of the shop with the $200 watch in her handbag.

Was this transaction fair?

SOLUTION

No, it most certainly was not fair. The jewelry store assistant was cheated out of $100.

Look at it this way.

The situation when the customer walked in to the store was as follows:

Customer	Jewelry Store
$100	Watch

When the watch is purchased, the situation is as follows:

Customer	Jewelry Store
$0 + watch	$100 – watch

When the customer takes hold of the $200 watch and hands back the $100 watch, the situation is as follows:

Customer	Jewelry Store
$0 + $200 watch	$100 + $100 watch – $200 watch

At this point the jeweler has the rightful possession of the $100 watch. He also has possession of $100 that the customer gave him for the $100 watch. But he has given the customer a $200 watch. The store requires another $100 from the customer for the purchase of the more expensive watch.

Thus, when the customer walks out of the store, the store has been cheated of $100.

SOURCE

Rob Eastaway, *How Long Is a Piece of String? More Hidden Mathematics of Everyday Life* (Stevenage: Portico Publishing, 2003), chapter 2.

90

PROVING DR. KRIEGER'S EQUATION CANNOT HOLD

Mr. White was reading a newspaper in his sitting room with his wife and 14-year-old son, Ben, who was good at mathematics.

"Listen to this," Mr. White said. "It states here that on March 7, 1938, *Time* magazine ran a story about a certain Samuel Isaac Krieger. Apparently, sometime in 1937, Dr. Krieger was taking a mineral bath in Buffalo, New York, when he suddenly realized that he had found a number greater than 2, call it n, such that the following equation holds: $1{,}324^n + 731^n = 1{,}961^n$. That is a counterexample of Fermat's Last Theorem."

"You mean," said Mrs. White, "in 1937 someone believed they had found an example that disproved Fermat's Last Theorem?

"It seems that way," said Mr. White. "There's more. On making the discovery, Krieger jumped out of the bath and ran naked into the next room. He began to scribble figures. Krieger said he was certain that his equation was correct, but he refused to reveal what the value of n was. Krieger's claim first came to light when the *New York Times* ran a story about it on February 22, 1938. Thirteen days later, *Time* magazine followed up with its own story on the *New York Times* piece on Krieger. *Time* magazine said that a newspaper reporter at the *New York Times* easily proved that Krieger was mistaken."

"That is an interesting story," Mrs. White said. "I wonder how the reporter proved that Dr. Krieger had made a mistake in his calculations."

Young Ben was listening attentively to all of this.

"That's obvious, Mom," he said. "That equation of Dr. Krieger's could not possibly hold because . . ."

"That's very clever of you, Ben," said Mrs. White, on hearing her son's explanation.

"Yes, it certainly is," said Mr. White, agreeing with his wife.

The question for our puzzlers is this: can you show why the equation that Krieger came up with could not possibly be true?

SOLUTION

The equation Krieger came up with is

$$1{,}324^n + 731^n = 1{,}961^n$$

Consider the last digit of the number 1,324. It is 4. If any number ending in 4 is raised to any power, the last digit of the answer must be 4 or 6. Consider now the last digit of the numbers 731 and 1,961. If any number ending in 1 is raised to any power, the end digit of the result will be 1. Thus, the only possible end digit of the 2 numbers on the left of the equation is 4 or 6 and 1. Thus, when these 2 powers are added together, the result must be a number that ends in either 5 or 7. But the number on the right ends in 1. Consequently, the equation cannot be correct.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 4.8, pp. 86, 94.

91

THE PROBLEM INVOLVING PERCENTAGES

If 35 percent of 70 is 24.5, what is 70 percent of 35?

SOLUTION

The answer is 24.5.

The following identity holds: x percent of y = y percent of x.

Let x percent of $y = (x/100) \times y$. Let y percent of $x = (y/100) \times x$.

We wish to prove that

$(x/100) \times y = (y/100) \times x$
If $(x/100) \times y = (y/100) \times x$ (Equation 1)
then we can rewrite equation 1 as
$(xy/100) = (xy/100)$ (Equation 2)

Equation 2 is just another way of writing equation 1. Equation 2 is clearly true. Therefore, equation 1 is true also.

Knowing this identity can prove useful. For example, suppose it is necessary for you to calculate 139 percent of 40, and there is no electronic calculator nearby. Simply calculate 40 percent of 139. If you are good at figures you will be able to do this mentally (and quickly) by dividing 139 by 10, obtaining 13.9, and multiplying that result by 4, obtaining 55.6 as the answer.

Incidentally, here is a curiosity concerning percentages which I have not seen published elsewhere:

0.12 equals 3 percent of 4 and .56 equals 7 percent of 8.

Note how the digits from 0 to 8 are in order in the above sentence!

This result is derived from the following identity, where once again the digits are in order, which was contributed by Everett W. Comstock to *Recreational Mathematics Magazine* (April 1962, page 33).

$$12 = 3 \times 4 \text{ and } 56 = 7 \times 8$$

SOURCE

"What Is the Reason for x% of y Being the Same As y% of x?," https://www.quora.com/What-is-the-reason-for-x-of-y-being-the-same-as-y-of-x

92

THE HIV TEST

A heterosexual patient with no known risk behavior goes to his doctor because he is worried in case he has been infected with HIV.

The doctor tells him that he should undergo a test to see if he is infected with the virus.

The doctor assures the patient that only 0.0001 percent of the population who has no risk behavior is infected with HIV. The patient is also told that if a patient has the virus, there is a 99.99 percent chance that the test result will be positive. If a patient is not infected, there is a 99.99 percent chance that the test result will be negative.

What is the chance that a patient who tests positive actually has the virus?

SOLUTION

On being presented with this problem, most people would believe that the probability that the patient has HIV is somewhere near 99 percent.

Surprisingly, this answer is incorrect. The correct answer is that the probability that the patient has HIV is 50 percent!

How is this answer arrived at?

One way of getting the correct result is to imagine that 10,000 patients go for the test. One patient will have HIV. Therefore, his test will prove positive.

We are told that if a patient does not have the virus, there is a 99.99 percent chance that the test will prove negative. That means that if there are 10,000 patients undergoing the test, one of them, who does not have the virus, will prove positive.

Therefore, of the 10,000 patients who undergo the test, 2 patients will have results that prove positive. Of course, only 1 of those 2 patients will have the virus.

Therefore, if a patient's test result is positive, the probability that that patient has the virus is 50 percent.

SOURCE

Gerd Gigerenzer, *Calculated Risks: How to Know When Numbers Deceive You* (New York: Simon & Schuster, 2002), pp. 124–25.

93

THE MAMMOGRAM TEST

A group of physicians are asked what are the chances of a female patient truly having breast cancer given that the mammogram test has a 90 percent chance of accurately spotting the cancer if a patient has it. The physicians are also told that the mammogram test has a 93 percent accuracy of a correct reading if the patient does not have breast cancer.

Let's assume that 0.008 percent of the female population actually have breast cancer.

Suppose a lady goes for a mammogram test. The results prove positive.

Should the lady in question be seriously worried?

SOLUTION

The surprising answer is no; the lady should not be unduly worried.

Let's look at the situation this way. In a group of 1,000 women, 8 will have breast cancer. Of these 8 women, 90 percent of them, or 7, will have a positive result. Unfortunately, the test will state that 1 of the 8 women does not have cancer, when in fact she does. This woman is given what is known in the medical profession as a *false negative*.

Consider the 992 women among the 1,000 women who do not have breast cancer. The test will incorrectly state that 7 percent of them have cancer. In other words, 69.44, or 70, women will be told that they have breast cancer when in fact they don't. Putting it another way, these 70 women have been given a false positive.

Therefore, the test will state that a total of 7 plus 70, or 77, women have breast cancer. However, only 7 of these women *actually* have breast cancer.

Thus, if a lady undergoes a mammogram in these circumstances and the test states that she has breast cancer, the probability that she actually has breast cancer is $\frac{7}{70}$, which equals 10 percent.

In other words, there is only a 10 percent probability that the lady in question actually has breast cancer.

This means the chance the lady has breast cancer if the test is positive is really only 1 chance in 10.

SOURCE

Gerd Gigerenzer, *Calculated Risks: How to Know When Numbers Deceive You* (New York: Simon & Schuster, 2002), pp. 5–6.

94

BOYS AND GIRLS

A sultan in a far-off land was trying to increase the number of women available for harems in his country. He passed a law forbidding every mother to have another child after she gave birth to her first son; as long as her children were girls, she would be allowed to continue childbearing.

"As a consequence of this new law," the sultan told his people, "you will see women having families such as 8 girls and 1 boy, 5 girls and 1 boy, 8 girls and 1 boy, perhaps having a solitary boy. This will have the effect that the ratio of girls to boys in our fair land will increase."

Is the sultan's mathematical reasoning correct?

SOLUTION

Surprisingly, the sultan's reasoning is not correct.

On being given this puzzle, many people will argue that there will be a lot more girls than boys in the population in that country. But to the surprise of many, that conclusion is incorrect!

Look at it this way. The probability that the first child, let us call it *B*, born to a couple will be a boy is 50 percent, or $\frac{1}{2}$. The probability that of 2 children born to a couple the second will be a boy, *GB*, is $\frac{1}{2}$ multiplied by $\frac{1}{2}$, or 25 percent, or $\frac{1}{4}$. The probability that of 3 children born to a couple, *GGB*, the third child will be a boy is $12\frac{1}{2}$ percent, or $\frac{1}{8}$. Therefore, the probability that the total number of boys born is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. This infinite series has a sum equal to 1. Of course, the same principle applies to girls. The probability that the total number of girls born is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$, which also sums to 1. Therefore, the ratio of boys to girls is 1 to 1. The proportion is therefore equal.

There is a simpler way to approach the problem. One does not need to sum an infinite series to solve this little teaser. Here's the easy solution. Imagine you are the census taker in the relevant country. Let us assume that every birth is recorded and that the details are sent to your office. For each birth, there is a 50 percent chance that the child will be a boy and a 50 percent chance that the child will be a girl. No matter what social convention exists in the country, the ratio for boys to girls must be 50 to 50, or 1 to 1.

Suppose, for example, that there are 256 couples in the country. For the first birth, 128 couples will have a boy, and they will not have any more children; 128 couples will have a girl. At this point, there are 128 boys and 128 girls. Consider now the 128 couples who have had a girl. For the second birth, each couple who had a girl now produces a second child; 64 of these children will be boys, and 64 will be girls. At this point, the total number of boys is 192, and the total number of girls is 192. For the third round of births, 64 couples will produce a child. Half of these, or 32, will be boys, and 32 will be girls. At this point, the total number of boys is 224, and the total number of girls is 224. For the fourth round of births, 32 couples will have a child, half of which will be boys and half of which will be girls. The total number of boys now is 240, and the total number of girls is also 240. And so on. At all stages, the proportion of boys to girls is 1 to 1. Therefore, the proportion of boys to girls is equal.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 2.14, pp. 44, 45, 54.

95

THE DISTANCE TO THE HORIZON

Young Johnny was standing on the seashore. He had been told by his dad that if a man who is 6 feet tall looked out to sea on a clear day, the horizon would appear to be 3 miles away. Johnny then reasoned that if he doubled the height of the observer to 12 feet, the horizon should appear to be 6 miles away.

Was Johnny correct in his reasoning?

SOLUTION

Surprisingly, Johnny was not correct.

In order to see twice as far out to sea, Johnny would have to increase his height above sea level to 4 times the previous height. A very good formula to calculate the approximate distance to the horizon is

$$\sqrt{\frac{3\times h}{2}} = d$$

In the formula, h represents the height in feet of the observer above sea level, and d represents the distance of the horizon in miles.

Thus, if an observer on the seashore is 6 feet above sea level, the formula is

$$\sqrt{\frac{3\times 6}{2}} = 3$$

The formula tells us that in that situation, the horizon is approximately 3 miles away.

If the height of the observer is 12 feet above sea level, the formula is

$$\sqrt{\frac{3\times 12}{2}} = 4.2426+$$

Thus, the horizon will appear to be 4.2426+ miles away. The distance 4.2426 equals 3 multiplied by $\sqrt{2}$.

If the observer is 24 feet above sea level, the distance in miles to the horizon will equal

$$\sqrt{\frac{3\times 24}{2}} = 6$$

Thus, the horizon will appear to be 6 miles away.

Therefore, one has to increase the height by a factor of 4 to double the distance to the horizon.

One can also check that one has to increase the height by a factor of 9 to triple the distance to the horizon. In general, one has to increase the height by a factor of n^2 to see n times as far out to sea.

If one goes online to Wolfram Alpha (https://www.wolframalpha.com), one can check these figures. If the height of the observer above sea level is 6 feet, the distance to the horizon is given as 3.001 miles. If the height of the observer above sea level is 24 feet, the distance to the horizon is given as 5.999 miles.

The formula given here is an excellent one. Suppose we wish to use the formula to find the distance of the horizon looking out the front window of a jet airliner that is, say, 6 miles above sea level. Six miles above sea level equals 5,280 multiplied by 6, or 31,680 feet. The formula then is

$$\sqrt{\frac{3\times 31680}{2}} = 217.990$$

The formula tells us that the distance to the horizon if one is 31,680 feet above sea level is 217.990 miles. If one uses Wolfram Alpha to calculate this distance, the answer given is 218 miles as the *direct horizon distance*. The Wolfram Alpha computational engine also tells us that if one is 31,680 feet above sea level, the horizon distance along the *curved surface of Earth* is 217.8 miles.

SOURCE

Lancelot Hogben, *Mathematics for the Million* (New York: Norton, 1993), pp. 157, 158.

96

TRAVERSING 50 MILES OF TRACK

A train travels 500 miles along a straight track without stopping. The train is traveling at various speeds during its journey. However, it completed its trip of 500 miles in exactly 10 hours; therefore, its average speed per hour for the entire journey is 50 miles.

It seems reasonable to assume that the train did not complete any 50-mile section of the track in exactly 1 hour.

Is this assumption correct?

SOLUTION

Surprisingly, this assumption is not correct.

To see why, divide the 500 miles of track into 10 50-mile segments. If the train actually traversed any one of these segments in 1 hour, then the assumption is false. If it did not traverse any segment in 1 hour, then we can conclude that the train took less than 1 hour or more than 1 hour to traverse each segment.

Consequently, there will be at least 1 pair of adjacent segments, one (call it segment A) traversed in less than 1 hour and the other (call it segment B) traversed in more than 1 hour.

In your mind, imagine a giant measuring rod 50 miles in length being placed over segment A. Imagine the rod sliding slowly along segment A toward segment B. As the imaginary rod slides toward segment B, the time taken by the train to travel the distance of the rod (50 miles) varies continuously from less than 1 hour (for segment A) to more than 1 hour (for segment B). Therefore, there must be some 50-mile length of track where the time taken by the train to travel 50 miles along it must be exactly 1 hour.

Consequently, the assumption is false.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 2.14, pp. 44, 45, 54.

97

2 STEAMBOATS

A steamboat leaving pier 1 takes 20 hours to go against the current upriver to pier 2. It can return downriver with the current from pier 2 to pier 1 in 15 hours. If there were no current, how long would it take the steamboat to travel between the 2 piers?

SOLUTION

On being asked this question, many people will add 20 and 15 together and then divide the result by 2, obtaining 17.5. They will then say that it would take the steamboat 17.5 hours to complete the trip if there were no tide.

It comes as a surprise to many to learn that this result is incorrect.

The surprising answer is that if there were no tide, the steamboat would complete the trip between piers in 17.142857 hours.

On the upriver trip, the steamboat would travel three-quarters of the journey in 15 hours. On the downriver trip, the steamboat traveled the entire journey in 15 hours. Since the steamboat is traveling with the tide and against the tide for the same period of time, the effect of the tide cancels. Therefore, the steamboat would cover $1\frac{3}{4}$ trips in 30 hours. It would therefore do one entire trip in $30/1\frac{3}{4}$, or 17.142857, hours if there were no tide.

This puzzle is interesting because, in addition to obtaining the required solution, we can also learn the distance between the 2 piers and the speed of the flow of the river.

We know from the question that the steamboat took a total of 35 hours to do the upriver trip and the downriver journey.

Let x equal the distance from pier to pier. This allows us to write the following equation:

$$\frac{x}{20} + \frac{x}{15} = 35$$

This may be rewritten as

$$15x + 20x = 10500$$

$$x = 300$$

Therefore, the distance between both piers is 300 miles. The total journey upriver and downriver is 600 miles, and this is accomplished in 35 hours. Thus, the average speed is 600 divided by 35, or 17.142857 miles per hour. Therefore, if there were no current, the steamboat would be traveling at

17.142857 miles per hour, which confirms the answer that we have already arrived at. It would therefore complete the 600-mile trip in 600/17.142857, or 35, hours.

What is the speed of the river? From the question, we know that the steamboat took 20 hours to go 300 miles upriver and 15 hours to go 300 miles downriver. The steamboat traveling at 17.142857 miles per hour would do the upriver trip in 17.5 hours if there were no tide. But it takes the steamboat 20 hours, or an extra 2.5 hours, to do this journey with the tide flowing against it. Thus, the tide is flowing at the rate of 2.5 miles per hour. On the downriver trip, the steamboat traveling at the rate 17.142857 miles per hour in no tide would do the journey in 17.5 hours, but with the tide in its favor, it does the trip in 2.5 hours less, or 15 hours.

SOURCE

Henry Ernest Dudeney, *More Puzzles and Curious Problems*, edited by Martin Gardner (London: Fontana Books, 1970), puzzle 42, pp. 20, 118.

98

THE BRIDGE PLAYERS

Four players sit down to play a game of bridge. Each player is dealt 13 cards. What is the probability that each player is dealt a complete suit of cards?

SOLUTION

The answer to this question is most surprising! But before we reveal the answer, let's first do a little calculating.

The number of hands available to the first player is 52*C*13. This is a mathematical way of describing the number of ways 13 objects (in this case, cards) can be selected from 52. This equals 635,013,559,600.

The number of hands the second player can be dealt is 39*C*13. This equals 8,122,425,444. The number of hands the third player can be dealt is 26*C*13. This equals 10,400,600. Finally, the fourth player has no choice but to accept the 13 cards that are left in the deck. So the number of hands that can be dealt to the fourth player is 1.

Therefore, the number of possible bridge deals is the product of these 4 numbers: 635,013,559,600 × 8,122,425,444 × 10,400,600 × 1. This product equals 53,644,737,765,488,792,839,237,440,000. This is the number of bridge deals that are possible!

We now calculate the number of ways all four players can each be dealt a complete suit. There are 4!, or 24, ways each of the players can be dealt a complete suit. (The number 4! is just the mathematician's way of writing 4 multiplied by 3 multiplied by 2 multiplied by 1. It equals 24.)

Thus, we divide 53,644,737,765,488,792,839,237,440,000 by 24 to obtain the number of ways each of the 4 players can be dealt a complete suit.

This result is 2,235,197,406,895,366,368,301,560,000. Thus, the odds of all four players being dealt a perfect hand in bridge is 2,235,197,406,895, 366,368,301,559,999 to 1.

To get an idea of how large a number this is, consider the following. Assume the world's population is 7 billion people. Assume that all of these people are divided into bridge foursomes and that each of these 7 billion people played bridge 12 hours per day, 365 days of the year, and had a hand of 13 cards dealt once every 10 minutes in those 12-hour periods. One would expect a deal consisting of a complete suit for all 4 players to happen only once in about 12 trillion years!

Every now and then, one reads in the newspapers that at some bridge game or other, a complete suit for all 4 players was dealt (https://aperiodical.com/2011/12/four-perfect-hands-an-event-never-seen-before-right). I would suggest that given how unlikely this is, one should take such stories with a large grain of salt.

I will mention in passing that there are 635,013,559,600 different ways 13 cards can be dealt from a deck of 52 cards. If one dealt a different 13-card hand from a shuffled 52-card deck once every 10 minutes, 12 hours per day, 365 days per year, it would take more than 24 million years to deal all possible hands.

To be more precise, it would take 24,163,377 years to deal all the possible hands!

SOURCE

Hugh Kelsey and Michael Glauert, *Bridge Odds for Practical Players* (London: Victor Gollancz in association with Peter Crawley, 1980), pp. 10–14.

99

THE MOST LIKELY POSITION OF THE FIRST BLACK ACE IN A SHUFFLED DECK

Suppose a deck of 52 playing cards is thoroughly shuffled. You are asked to guess in advance the position of the first black ace. This procedure is repeated many times; each time, the deck is thoroughly shuffled, and each time, you guess the position of the first black ace.

What position in the deck should you guess so that if the procedure is repeated many times, you would maximize your chances of guessing correctly the position of the black ace?

SOLUTION

The very surprising answer is that you should guess that the top card is the first black ace in the deck!

To see that this is the case, consider 2 black aces and, say, a red queen. There are 3 ways these 3 cards can be ordered: *AAQ*, *AQA*, and *QAA*. In 2 of the 3 orderings, the first top card is an ace. So the probability that the top card is the first black ace is $\frac{2}{3}$. The probability that the second top card is the first black ace is seen to be $\frac{1}{3}$.

The same reasoning can be applied to a larger deck. Suppose you have 8 cards, 2 of which are the black aces. The probability that the top card is the first black ace is $\frac{1}{4}$, or 25 percent.

The probability that the second card from the top is the first black ace depends on the first card from the top *not* being the first black ace. The probability of this occurring is $\frac{6}{8}$.

Given that this is the case, the probability that the second card from the top is the first black ace is $\frac{6}{8} \times \frac{2}{7}$ since there are 2 aces among the remaining 7 cards. This simplifies to $\frac{3}{14}$, which equals 21.4 percent. Similarly, the probability that the third card from the top is the first black ace is $\frac{6}{8} \times \frac{5}{7} \times \frac{2}{6}$, which is approximately 17.8 percent.

In general, in a deck consisting of n cards, the probability that the top card is the first of two black aces is $n - 1$ over the sum of the integers from 1 to $n - 1$. Thus, in a deck consisting of 3 cards, the probability that the top card is the first black ace is $\frac{2}{3}$. In a deck of 4 cards, the probability is $\frac{3}{6}$, or $\frac{1}{2}$; with 8 cards, the probability is $\frac{7}{28}$, or $\frac{1}{4}$, with 12, it is $\frac{1}{6}$; and so on. With 52 cards, it is $\frac{51}{1{,}326}$.

Of course, by symmetry, it can be argued that the most likely position for the second black ace is the bottom card of the deck.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 2.17, pp. 47, 56, 57.

100

GOLD AND SILVER COINS

Jack Jones was showing his son, Paul, 3 identical-looking chests that were in the family's possession for generations. Let's call them chest A, chest B, and chest C. Each chest contains 2 drawers. Now Jack told Paul that each of the 2 drawers of chest A contains a gold coin and that in Chest B, 1 drawer contains a gold coin and 1 drawer contains a silver coin. He also said that in chest C, each drawer contains a silver coin.

"Now my man," said Jack. "I want you to randomly choose any 1 of the 3 chests. When you have done that, I want you to randomly choose any 1 of its 2 drawers. Open that drawer and look at the coin inside the drawer."

"Okay Dad," Paul said. The young lad looked at the 3 chests. Eventually, he chose one of them. He then opened 1 of its 2 drawers. The coin inside the drawer was gold.

"Good man, Paul," his father said. "You have found a gold coin. Now here is a little puzzle for you. What is the probability that the other drawer in the same chest also contains a gold coin?"

Paul was a smart young lad. He thought about the problem for a while and then gave the correct answer to this little puzzle.

The question is, can you give the correct answer also?

SOLUTION

The correct solution to this little puzzle surprises most people.

Apparently, most people will tackle this problem by reasoning along the following lines. Since we have discovered a gold coin in one of the drawers in the chest that we chose, we can eliminate the possibility that we chose chest C. Therefore, we have chosen chest A or chest B. The probability of choosing chest A is ½, and that of choosing Chest B is ½. If we chose chest A, the other coin is definitely gold. However, if we chose chest B, the other coin is definitely silver. The 2 probabilities are equal. Thus, the probability that the coin in the other drawer is gold is 50 percent.

This reasoning seems very plausible. But it is entirely wrong!

Recall that there are 3 gold coins and 3 silver coins distributed among the 3 chests. Let's label each of these coins as follows: chest A contains gold coin_1, chest A also contains gold coin_2, chest B contains gold coin_3, chest B contains silver coin_1, chest C contains silver coin_2, and chest C contains silver coin_3.

Paul has selected a chest and opened one of its drawers. He has found 1 gold coin in that drawer. That gold coin can be any 1 of 3 coins: it can be gold coin_1, gold coin_2, or gold coin_3. In 2 of these 3 cases, the coin in the other drawer in the chest is gold also.

Therefore, the probability that the coin in the other drawer is gold is ⅔.

SOURCE

Warren Weaver, *Lady Luck: The Theory of Probability* (New York: Dover Publications, 1982), p. 123.

101

THE PARADOX OF THE SECOND ACE

You do not have to understand the rules of bridge to attempt this pretty puzzle.

Suppose you are playing bridge. You ask one of the players if she holds at least 1 ace. She answers yes. The probability that she holds a second ace can now be calculated. It is precisely 5,359/14,498, which is slightly less than 37 percent.

Later in the game, you ask the same player if she holds the ace of spades. She answers yes. Given that she holds the ace of spades, the probability that she holds a second ace can now be calculated. It is exactly 11,686/20,825, which is slightly more than 56 percent.

Why should naming the particular ace change the probability?

SOLUTION

The computations of the exact probabilities that are given above are long and tedious. However to see why this surprising result occurs, let's consider a much simpler situation.

Consider a game of poker where each of 2 players is dealt 2 cards from a deck consisting of the following 4 cards: the ace of hearts; the ace of spades, the 2 of diamonds, and the king of clubs.

There are 6 ways to choose 2 of these 4 cards:

ace of hearts, ace of spades	ace of spades, 2 of diamonds
ace of hearts, 2 of diamonds	ace of spades, king of clubs
ace of hearts, king of clubs,	2 of diamonds, king of clubs,

Suppose now that 2 hands of 2 cards are dealt. You ask 1 player if she has an ace. She answers yes. There are 5 hands containing an ace that she may have, but only 1 of those hands contains a second ace. Thus, if a player announces that she has an ace, the probability that she has a second ace is 1 chance in 5.

However, suppose you ask a player if she has an ace of spades. The player answers yes. There are 3 hands containing the ace of spades that she may have, but only 1 of them contains a second ace. Therefore, if a player states that she has the ace of spades, the probability that she has a second ace is 1 in 3. The odds have now increased from 1 in 5 to 1 in 3 that she has a second ace. (A mathematician would explain this phenomenon by stating that the sample space in the problem is different when a player is asked if she has an ace or if she is asked if she has a *specific* ace.)

Something similar happens in a game of bridge when using a full deck of 52 cards. The potential number of hands that contains the ace of spades is *less* than the potential number of hands that contain *any* ace. The potential hands that contain *any* ace have a specific percentage of them that contains more than 1 ace. But of the potential number of hands that contains the ace of spades, a larger percentage of those hands contains a second ace. This is what gives rise to the paradox.

Thus, if a player at bridge is asked if she has the ace of spades and answers yes, the probability that she has a second ace is greater than if she had responded positively to the question of whether her hand contained *any* ace.

The solution is so counterintuitive that on encountering the paradox for the first time, many people refuse to believe it.

SOURCE

Owen O'Shea, *The Magic Numbers of the Professor* (Washington, DC: Mathematical Association of America, 2007), pp. 21, 22.

102

THE PAY RAISE

A company was in the process of hiring an employee to fill a very important position. The firm's human resources manager wanted to ensure that the prospective employee had a good head for figures and also was able to rationalize any given situation reasonably quickly.

He had interviewed each of the candidates and had narrowed his choices down to 3 possible persons. He then interviewed each of these 3 candidates separately a second time and in that second interview put the following terms to each candidate.

Suppose the company can offer you one of the following terms: your salary begins at $100,000 per year. The company will then give you a choice: you can have a yearly raise of $5,000 per year, paid yearly. Or you can choose to be paid $50,000 semiannually and have an annual raise of $2,500 every year, paid semiannually. In other words, that increase would be equivalent to a semiannual raise of $1,250. I want you to tell me: within the next 5 minutes, what offer you would take if given these 2 offers?

The first two applicants, Adam and Brian, informed the human resources manager that they would take the first offer; that is, they would take the offer of a yearly salary of $100,000 and a yearly increase of $5,000.

At her interview, the third applicant, Ann, thought about the proposition for 5 minutes. She then informed the human resources manager that if given the choice, she would take the second offer; that is, she would choose to take a semiannual salary of $50,000 and would take an annual raise of $2,500, paid at the rate of $1,250 semiannually.

The third applicant got the job.

Why?

SOLUTION

Surprisingly, the second offer is the more rewarding. Therefore, the human resources manager assumed that the third applicant was the best at figures and also the best at solving problems in a logical fashion. The third applicant therefore got the job.

At the interview, the third applicant had taken her notebook from her pocket and within a minute had written down details of the 2 offers of a pay raise as follows:

\$5,000 raise yearly At the end of each year I will have earned	\$1,250 raise semiannually At the end of each year I will have earned
Year 1 = \$100,000	\$50,000 + \$51,250 = \$101,250
Year 2 = \$105,000	\$52,500 + \$53,750 = \$106,250
Year 3 = \$110,000	\$55,000 + \$56,250 = \$111,250
Year 4 = \$115,000	\$57,500 + \$58,750 = \$116,250
Year 5 = \$120,000	\$60,000 + \$61,250 = \$121,250

And so on.

It can be seen that the second offer, giving a raise of \$2,500 every year, paid at the rate of \$1,250 at semiannual intervals and, beginning after 6 months of commencing employment, is more profitable for the employee. That second proposal works out better than the first offer by \$1,250 every year, beginning at year 1.

The result is difficult to believe at first, even when one scrutinizes the figures. Perhaps the best way to see why the second proposal is the better offer is to consider the following table, which shows how Ann's salary increases during the first 5 years:

Ann's salary	First 6 months	Second 6 months	Total
Year 1	50,000	50,000 + \$1,250	100,000 + \$1,250
Year 2	50,000 + \$2,500	50,000 + \$3,750	100,000 + \$6,250
Year 3	50,000 + \$5,000	50,000 + \$6,250	100,000 + \$11,250
Year 4	50,000 + \$7,500	50,000 + \$8,750	100,000 + \$16,250
Year 5	50,000 + \$10,000	50,000 + \$11,250	100,000 + \$21,250

Surprisingly, we see that Ann's salary is increasing by \$5,000 each year!

We can see what happens generally under the 2 offers by using a little algebra. If we let Ann's 6-month salary of \$50,000 equal x and the 6-month salary increase of \$1,250 equal y, the following table shows Ann's earnings for the first 5 years:

Ann's salary	First 6 months	Second 6 months	Total per year
Year 1	$\$x$	$\$x + \$y$	$\$2x + \$y$
Year 2	$\$x + \$2y$	$\$x + \$3y$	$\$2x + \$5y$
Year 3	$\$x + \$4y$	$\$x + \$5y$	$\$2x + \$9y$
Year 4	$\$x + \$6y$	$\$x + \$7y$	$\$2x + \$13y$
Year 5	$\$x + \$8y$	$\$x + \$9y$	$\$2x + \$17y$

It is clear from the table that Ann's earnings are increasing by $\$4y$ each successive year.

Compare Ann's salary with the earnings of either Adam or Brian for the first 5 years, where $2x$ equals the yearly salary of \$100,000 and $4y$ equals the \$5,000 annual increase in Adam or Brian's salary:

Adam's or Brian's salary	Total per year
Year 1	$\$2x$
Year 2	$\$2x + \$4y$
Year 3	$\$2x + \$8y$
Year 4	$\$2x + \$12y$
Year 5	$\$2x + \$16y$

Thus, Adam and Brian's salary will increase by $\$4y$ each year.

However, in year 1 under her plan, Ann received $\$y$ more than either of the 2 men. Therefore, Ann is working from a higher base figure.

Overall, we see that Ann is surprisingly better off each year by $\$y$ under the plan she chose.

SOURCE

Gilbert Wilkinson, *The Complete Home Entertainer* (London: Odhams Press, 1940), puzzle 10, pp. 123, 475.

103

THE DAILY COMMUTER

Every evening, a commuter arrives at his suburban railroad station at exactly 5:00 p.m. His wife also arrives at the railroad station exactly at 5:00 p.m. and drives her husband home. One day, he takes an earlier train and arrives at the station at precisely 4:00 p.m. The weather is pleasant, so instead of telephoning home, he decides to walk along the route always taken by his wife. They meet somewhere along the way. He gets into the car, and they drive home. They arrive home 10 minutes earlier than usual. Assuming that the wife always drives at a constant speed and that on this occasion she left just in time to meet the arrival of the train at 5:00 p.m., can you figure out how long the man was walking before he was picked up?

SOLUTION

This problem appears to be very difficult to solve. We are not given the man's speed of walking or the wife's speed of driving, and we are not given the distance between the man's home and the railroad station. But these variables are not required to solve the puzzle! Indeed, if you tried to solve the problem using various figures for these variables, you probably found yourself in an algebraic mess reasonably quickly.

The insight into solving this puzzle is to realize that if the wife and husband arrived home 10 minutes earlier than usual, she must have cut 10 minutes from her usual travel time to and from the railroad station, or 5 minutes from her travel time to the station. Consequently, she must have met her husband 5 minutes before his usual pickup time of 5:00 p.m. Thus, she met her husband at 4:55 p.m. He started walking at exactly 4:00 p.m.; therefore, he walked for exactly 55 minutes.

The simple reasoning used in reaching the solution to this puzzle surprises many people.

SOURCE

Martin Gardner, *Mathematical Puzzles and Diversions* (London: Penguin Books, 1965), puzzle 8, pp. 33, 39, 40.

104

HOW FAR IS IT FROM FLAG D TO FLAG A?

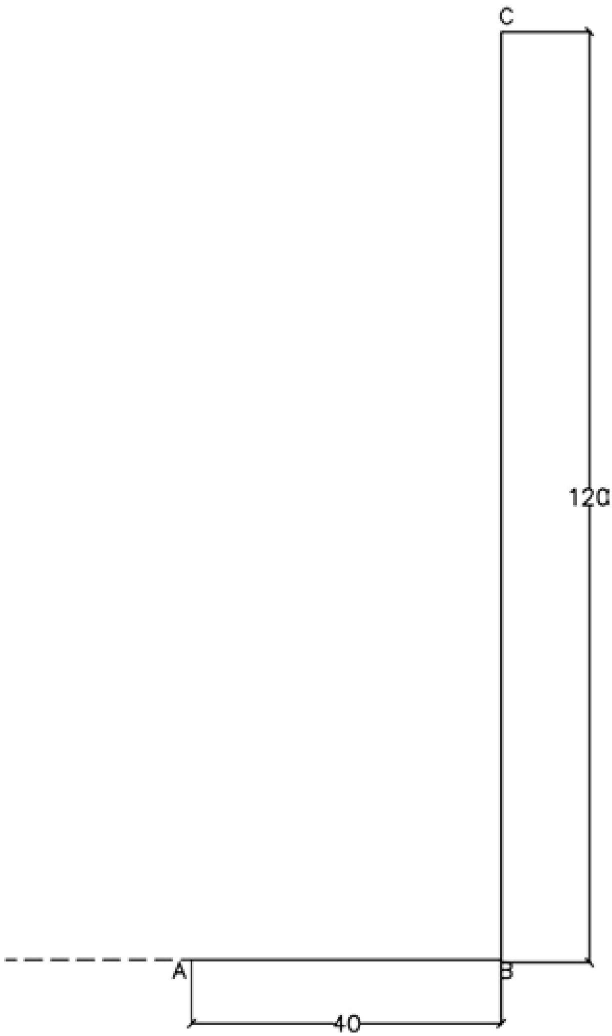

Figure 12

Figure 12 shows the location of 3 flags in one of the fields on a neighbor's farm. The angle ABC is a right angle. Flag A is 40 yards from flag B. Flag B is 120 yards from flag C. Thus, if one was to walk from A to B and then on to C, one would walk a total of 160 yards.

Now there is a point, marked by flag D, to the left of flag A, that is not shown in Figure 12. Curiously, if one were to walk from flag A to flag D and then diagonally across to flag C, one would also walk a total distance of 160 yards.

The question for our puzzlers is this: how far is it from flag D to flag A?

SOLUTION

One can use algebra to solve this problem. However, there is a surprisingly simple rule that can be used to find the distance from flag D to flag A.

This rule is as follows.

Divide the distance BC by AB. Add 2 to the result. Then divide that sum into BC again. The answer will be the distance from flag D to flag A. In our example, we divide 120 by 40. The result is 3. Now add 2 to 3 to obtain 5. Now divide 120 by 5. The result is 24. Therefore, the distance from flag D to flag A is 24 yards. Thus, the total length of the base of the triangle, from flag D to flag B, is 64 yards.

We can check if this result is correct by using the Pythagorean theorem. That theorem tells us that the length of the diagonal equals the square root of $64^2 + 120^2$. This equals 136.

Thus, the 3 sides of the triangle are 64, 120, and 136.

We see that if one were to walk from flag A to flag D and then diagonally across to flag C, one would walk 24 + 136, or 160, yards. This is the same result as if one were to walk from flag A to flag B and then on to flag C (see Figure 13).

Thus, our answer is correct.

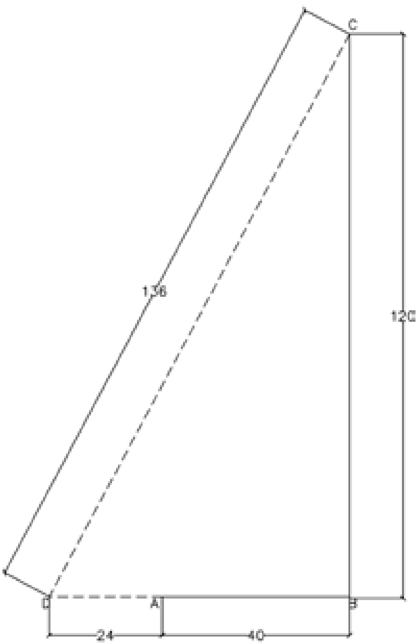

Figure 13

The simple rule to solve this problem applies to all right-angle triangles.

To solve the problem by algebra, let the distance from flag A to flag B equal a and let the distance from flag B to flag C equal c. Let the hypotenuse, from flag C to flag D, equal d and let the distance from flag D to flag A equal b.

The Pythagorean theorem tells us that $(a + b)^2 + c^2 = d^2$ (Equation 1).

This equals $a^2 + 2ab + b^2 + c^2 = d^2$ (Equation 2).

From the question, we know that $b + d = a + c$ or $d = a + \mathrm{c} - b$ (Equation 3).

Therefore, $d^2 = a^2 + 2ac - 2ab + c^2 - 2cb + b^2 = d^2$ (Equation 4).

The left-hand sides of equations 2 and 4 are equal because both equal d^2. Therefore, we can write

$$a^2 + 2ac - 2ab + \mathrm{c}^2 - 2cb + b^2 = a^2 + 2ab + b^2 + c^2$$

All the squared terms cancel, leaving

$$2ac - 2ab - 2cb = 2ab$$

or

$$ac - ab - cb = ab$$

Thus,

$$\frac{ac}{b} = 2a + c$$

or

$$c = b\left(2 + \frac{c}{a}\right)$$

This may be rewritten as

$$\frac{c}{2+\frac{c}{a}} = b$$

This is the equation that produces the handy rule. The rule can be found in the works of both Sam Loyd and Henry E. Dudeney.

SOURCE

Sam Loyd, *Sam Loyd's Cyclopedia of 5,000 Puzzles, Tricks, and Conundrums, with Answers* (New York: Morningside Press, 1914), pp. 138, 357.

105

THE 2 TENNIS PLAYERS

Two expert tennis players, Atkins and Burns, both equally good, play a series of 5 games. The first to win 3 games wins the prize of $300. At the end of 3 games, Atkins was leading 2 games to 1.

Both men were then called away on urgent business.

How should the prize of $300 be divided between the 2 men?

SOLUTION

Many people answer this puzzle by saying that Atkins should get ⅔ of the prize ($200) and Burns ⅓ ($100). Their reasoning for this division of the spoils is that Atkins was leading Burns 2 to 1 at the time both players were called away, and therefore the cash prize should be divided in a 2-to-1 ratio.

This answer seems very plausible. But, surprisingly, this solution is incorrect!

Let's look at how the games could have proceeded if they had not been interrupted:

(1) Atkins is leading 2 to 1; possibility of Atkins winning 3 to 1.
(2) Atkins is leading 2 to 1; possibility of Burns evening the match, making it 2 to 2, then Atkins winning 3 to 2.
(3) Atkins is leading 2 to 1; the possibility of Burns evening the match, making it 2 to 2, then Burns winning 2 to 3.

The possibility of (1) occurring is ½ since Atkins has 1 chance in 2 of winning.

Now consider (2).The possibility of Burns evening the match is ½. The possibility of Atkins then winning *after* Burns has evened the match is also ½. Thus, the possibility of (2) occurring is ½ multiplied by ½, which equals ¼.

Now consider (3). The possibility of Burns evening the match is ½. The possibility that Burns will then go on to win the match *after* drawing even is also ½. Thus, the possibility of (3) occurring is ½ multiplied by ½, which equals ¼.

Therefore, the possibility of (1) occurring is ½. The possibility of (2) occurring is ¼. Thus, the possibility of (3) occurring is ¼.

Thus, there are 3 different possible ways the match could have continued to its conclusion.

According to (1), Atkins has ½ chance of winning the match and then, according to (2), ¼ chance of winning the match.

Therefore, Atkins has ¾ chance of winning the match.

Consequently, Burns has ¼ chance of winning the match.

Therefore, the prize money of $300 should be divided in the ratio of 3 to 1 as follows:

Atkins: $225
Burns: $75

Many people are very surprised at this counterintuitive answer.

SOURCE

Maurice Kraitchik, *Mathematical Recreations* (New York: Norton, 1942), pp. 117–18.

106

THE COURIER PROBLEM

A column of troops on horseback 50 miles long is advancing forward at a constant speed. A courier is sent on horseback from the rear of the column to the front to deliver a message.

On reaching the front, the courier delivers the message and then immediately turns around and rides toward the rear of the column. He reaches the rear of the column just as the column has advanced 50 miles.

Assume that the courier rode at a constant speed at all times. How far did the courier travel?

SOLUTION

The courier has traveled 50 miles, plus 50 times the $\sqrt{2}$ miles, or 120.710678 . . . miles.

The surprising thing about this puzzle is the appearance of the $\sqrt{2}$ in the solution. The $\sqrt{2}$ is a famous constant in mathematics. It equals 1.4142135. . . . The decimal expansion of the $\sqrt{2}$ shows no apparent pattern and goes on forever. Yet we know that this strange number, whose decimals continue to infinity, when multiplied by itself, exactly equals 2.

When Sam Loyd and Henry E. Dudeney offered the courier puzzle to their readers, both men gave the correct solution to the problem but did not explain how to find that solution!

There are at least 2 methods known to me that can be used to solve this problem. One method uses the fact that the ratios of the speeds of both the column and the dispatch rider on the forward and return trips are equal. An alternative method of solution (which hitherto had apparently been unknown) was supplied to Martin Gardner in the 1960s by one of his readers named Robert F. Jackson. At the time, Jackson was employed in the Computing Center at the University of Delaware.

The following is Jackson's very clever method of solving this puzzle:

Let the length of the column equal 1. Let the length of time it takes the column to march its own length also equal 1. Therefore, the speed of the column will be also 1. Let x equal the total distance the courier has traveled. Let x also equal the courier's speed.

On the forward trip, the speed of the courier, relative to the moving column, is $x - 1$. On the return trip, his speed relative to the advancing column is $x + 1$. The two trips are completed in a time period equal to 1. This allows us to write the following equation:

$$\frac{1}{x-1}+\frac{1}{x+1}=1$$

This can be expressed as the following quadratic equation:

$$x^2-2x-1=0$$

The 2 solutions to this quadratic are $x = 1 + \sqrt{2}$ and $x = -0.4142135. . . .$

The only value that fits the problem's conditions is that $x = 1 + \sqrt{2}$. Multiply this by 50 to get the total distance traveled by the courier, which equals 120.710678. . . miles.

The following table explains clearly the situation.

On the forward and return trips, the distances traveled by the courier and the column of troops is as follows:

Trip	Courier	Column of Troops
Forward trip	85.3553396 . . . miles	35.3553396 . . . miles
Return trip	35.3553396 . . . miles	14. 64466094 . . . miles
Total	120.7106781 . . . miles	50.00000054 . . . miles

Thus, the courier travels a total distance equal to the length of the column plus that same length times $\sqrt{2}$.

SOURCE

Martin Gardner, *Martin Gardner's New Mathematical Diversions* (Chicago: University of Chicago Press, 1966), problem 4, pp. 36, 37, 44.

107

THE LAKE PROBLEM

The following is a puzzle that appears in the works of both Henry E. Dudeney and Sam Loyd. It is a beautiful problem, with a very surprising method of solution. Here is the puzzle.

Figure 14 shows the map of 3 plots of ground that recently went up for sale in our town. The large square piece of ground has an area of 370 acres. The 2 smaller square parcels of land have areas of 110 and 74 acres, respectively. The small triangular piece resting between these 3 squares is a lake. The lake was not for sale. For some curious reason, I wondered what was the exact area of that lake. I did a little calculation on a scrap of paper. I realized that the dimensions (in yards) of the 3 sides of the lake were $\sqrt{370}$, which equals 19.2353840. . . ; $\sqrt{110}$, which equals 10.4880884. . . ; and $\sqrt{74}$, which equals 8.6023252. . . .

Now I am aware that there is a well-known formula in mathematics called *Heron's formula*, which can be used to obtain the *exact* area of the triangular piece. But I realized that perhaps some of our puzzlers might not be familiar with Heron's formula. This got me thinking. I was curious if there was another way to tackle the problem. Surprisingly, there is! Believe it or not, the exact area of the triangular lake can be determined from the information given in Figure 14 without using Heron's formula.

Can you figure out the exact area of the lake?

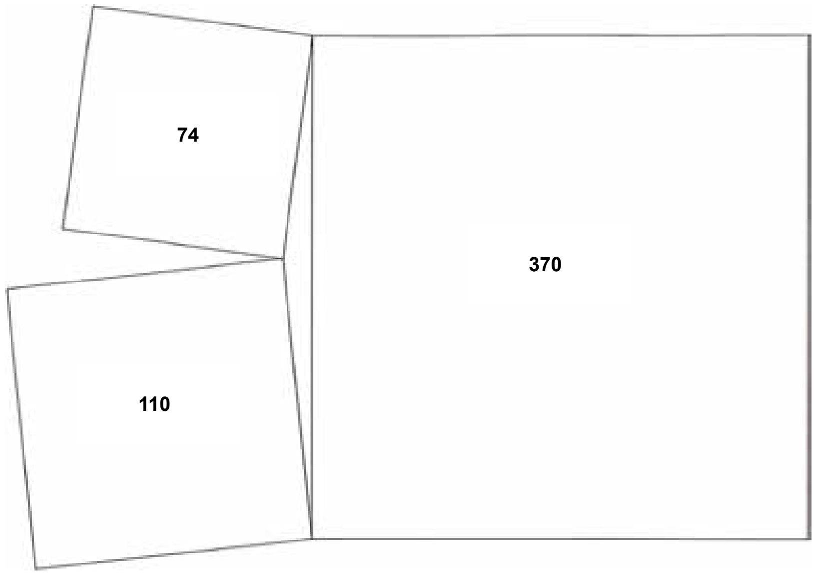

Figure 14

SOLUTION

The method of solution given by Dudeney and Loyd to the problem of the triangular lake is ingenious and very surprising. This is how the 2 great puzzle makers went about solving this famous puzzle.

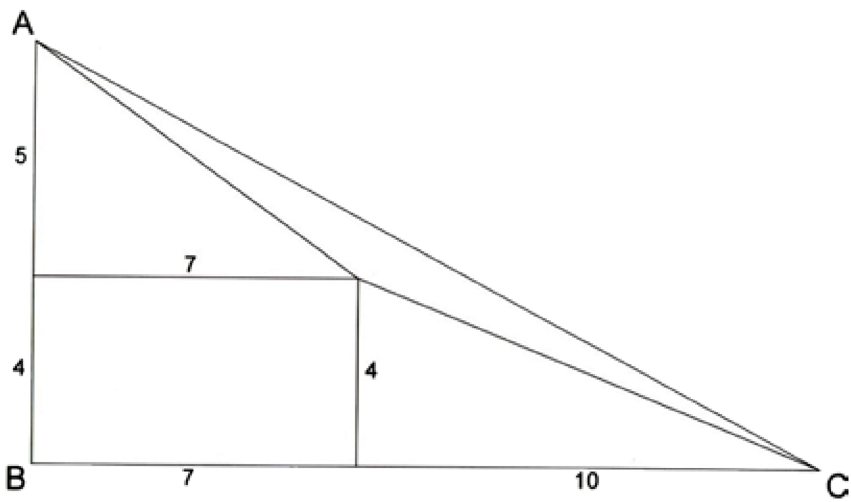

Figure 15

Consider Figure 15. It shows a right-angle triangle, ABC, whose two sides measure 9 and 17 units. From the Pythagorean theorem, we know that the square of the hypotenuse is $9^2 + 17^2$, or 370, square units in area. The area of the large triangle, ABC, is $(9 \times 17)/2$, which equals 76.5, square units.

Consider the small triangle within ABC, whose two sides measure 5 and 7 units in length. The square of the hypotenuse of this triangle is $5^2 + 7^2$, or 74, square units. The area of this triangle is $(5 \times 7)/2$, or 16.5, square units.

Consider the other small right-angle triangle within ABC. This triangle measures 4 by 10 units in length. The square of the hypotenuse of this triangle is $4^2 + 10^2$, or 116, square units.

The area of this triangle is $(4 \times 10)/2$, or 20, square units.

The area of the little rectangle within ABC is 4×7, or 28, square units.

As one can see, the constructions of the 2 small triangles and the little rectangle within the triangle ABC accurately reproduce the little triangular lake that was found in Figure 14.

From Figure 15, we see that the areas of the 2 small triangles, plus the little rectangle, within ABC are 17.5, 20, and 28 square units. This is a total of 65.5 square units.

The area of triangle ABC is $(9 \times 17)/2$, or 76.5, square units. If we subtract from this the area of the 2 small triangles and the area of the rectangle, we will be left with the *exact* area of the triangular piece representing the triangular lake.

$76.5 - 65.5 = 11$ square units.

Therefore, the little triangular piece contains exactly 11 square units.

Thus, the triangular lake contains exactly 11 acres.

SOURCE

Henry E. Dudeney, *Amusements in Mathematics* (New York: Dover Publications, 1958), problem 189, pp. 51, 183, 184.

108

THE SPOT IN A RECTANGLE

The following puzzle illustrates a beautiful mathematical relationship involving a rectangle of any size and a random point within that rectangle that most people, including mathematicians, are unaware of.

Figure 16 shows a rectangular room. There is a matchbox located 6 feet from one corner of the room and 27 feet from the opposite corner. The matchbox is also located 21 feet from a third corner.

How far is the matchbox from the fourth corner?

SOLUTION

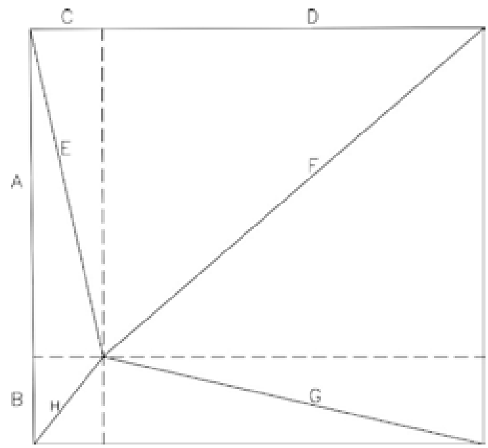

Figure 16

From the illustration, we see that the Pythagorean theorem gives the following equations:

Because $E^2 = A^2 + C^2$ and $G^2 = B^2 + D^2$, we know that $E^2 + G^2 = A^2 + C^2 + B^2 + D^2$

Since $F^2 = A^2 + D^2$ and $H^2 = B^2 + C^2$, we know that $F^2 + H^2 = A^2 + D^2 + B^2 + C^2$. We see that the right sides of both equations are identical. Therefore, $E^2 + G^2 = F^2 + H^2$.

Thus, very surprisingly, we find that the sum of the squares of the distances from the matchbox to 2 opposite corners of the rectangle is equal to the sum of the squares of its distances from the other 2 corners.

From the question, we know that the matchbox is 6 feet from one corner and 27 feet from the opposite corner. The matchbox is also 21 feet from a third corner. Let the distance from the matchbox to the fourth corner equal x. Then we know from our analysis of the problem that

$6^2 + 27^2 = 21^2 + x^2$
or
$6^2 + 27^2 - 21^2 = x^2$
$324 = x^2$
or
$x = 18$

Therefore, the matchbox is located 18 feet from the fourth corner of the room.

SOURCE

Martin Gardner, *The Colossal Book of Short Puzzles and Problems*, edited by Dana Richards (New York: Norton, 2006), problem 6.16, pp. 147, 159, 160.

109

HOW SHOULD THE INVESTMENT BE DIVIDED?

Atkins and Brown have invested in a business. Atkins invested exactly twice as much as Brown. Two years after setting up the business, Clancy bought in to the company and obtained an equal share as the other 2 investors. He paid $60,000 for an equal share of the firm. How should this $60,000 be divided between Atkins and Brown?

SOLUTION

Many people attempt to solve this problem by saying that since Atkins has twice the amount of money invested in the business as Brown, the investment of $60,000 by Clancy should be divided by Atkins and Brown in a 2-to-1 ratio. Thus, Atkins should receive $40,000 and Brown $20,000.

This approach appears very plausible, but it is entirely incorrect!

We know that the business is valued at $180,000 because Clancy paid $60,000 for an equal share of the company, which amounted to one-third of the value of the firm. Since Atkins's and Brown's investments are in a ratio of 2 to 1, it is clear that Atkins must have invested $120,000 in the business and that Brown invested half of that, or $60,000.

Therefore, Brown's investment exactly equals one-third of the value of the business.

However, Atkins's investment equals one-third of the value of the firm *plus* $60,000.

Consequently, the whole of the investment by Clancy, which is $60,000, should be paid to Atkins.

That then leaves a situation where each of the 3 men has $60,000 invested in the business, each holding a one-third share of the company.

Many people are surprised by this answer.

SOURCE

The author.

110

SQUARE FIELDS AND FENCE HURDLES

The following problem appears in the works of Henry Ernest Dudeney and, in a slightly different form, in the works of Sam Loyd. The puzzle is interesting because of a curious shortcut that Dudeney gives and also because of the very counterintuitive solution.

Here is the puzzle.

Suppose one intends to enclose a square field with rails. Each rail is 8.25 feet in length, and one wishes to have the fence 7 rails high. The field is of such size that if one were to proceed with one's plan, there would be as many rails in the fence as there are acres in the field. What is the area of the field?

Loyd's version of the problem results in a surprising answer also. He presented the puzzle as follows. A square field is to be enclosed with a rail fence. Each rail is 12 feet long, and the fence is required to be 3 rails high. When the task is completed, it is found that the number of rails enclosing the square field equals the number of acres in the field.

How large is the field?

SOLUTION

Dudeney's version of the problem is solved as follows.

Let the length of the field in miles equal x. Since there are 640 acres in a square mile, the area of the field in acres is $640x^2$. Since there are 5,280 feet in 1 mile, the length of the side of the field in feet is $5{,}280x$.

We are told that each rail is 8.25 feet in length and that each portion of the fence (known as a hurdle) is 7 rails high. The number of rails on each side of the square field is therefore $(5{,}280x \times 7)/8.25$. Therefore the *total* number of rails around the square field will be 4 times this, or $(5{,}280x \times 7 \times 4)/ 8.25$.

We are also informed that the number of acres enclosed in the field equals the total number of rails around the field. This allows us to write the following equation:

$$640x^2 = (5{,}280x \times 4 \times 7)/8.25$$

This equals

$$5{,}280x^2 = 5280x \times 28.$$

Dividing by $5{,}280x$, we obtain

$$x = 1 \times 28$$

Thus, $x = 28$. In other words, the length of the square field is 28 miles! A square field of this length contains 501,760 acres. The number of rails—each 8.25 feet in length and 7 rails high in each hurdle—enclosing the field would be 501,760.

The shortcut in solving this puzzle—where each rail is 8.25 feet in length—is as follows. To find the length of the field in miles, multiply the number of rails high in each hurdle by 4.

The answer is the length of the square field in miles.

For example, in the version of the puzzle given here, each hurdle is 7 rails high. Multiply 7 by 4, obtaining 28. Therefore, the square field will have to be 28 miles in length to meet the conditions of the problem. If the fence were 6 rails high, multiply 6 by 4, obtaining 24. Such a field, surrounded by a fence 6 rails high and each length of fence being 8.25 feet, would have to be 24 miles in length to meet the conditions of the problem. And so on.

A proof of this general result is as follows.

Let x equal the length of the field in miles. Let the number of fences in each hurdle equal n.

From the conditions of the problem, we can write

$$640x^2 = (5{,}280x \times 4 \times n)/8.25$$
$$5{,}280x^2 = 5{,}280x \times 4n$$
$$5{,}280x = 5{,}280 \times 4n$$

Therefore,

$$x = 4n.$$

This tells us that the length of the field in miles equals 4 times the number of rails high in each hurdle, where the length of each rail is 8.25 feet.

Sam Loyd's version of the problem can be approached in a similar way. It turns out that in Loyd's version, the length of the square field is 8.25 miles. This length will accommodate 3,630 rails, each 12 feet in length. Since the fence is 3 rails high, there will be 3 × 3,360, or 10,890, rails on each side of the square field. Thus, there are 4 × 10,890, or 43,560, rails enclosing the field.

Since the square field measures 8.25 miles in length, the area of the field is 8.25^2, or 68.0625, square miles. There are 640 acres in a square mile. Therefore, the field contains 640 × 68.0625, or 43,560, acres.

Thus, the number of rails surrounding the field equals the area of the field in acres.

Incidentally, Loyd tantalizingly pointed out in his proposition of the puzzle in the *Cyclopedia* that there are 43,560 square feet in 1 acre!

SOURCE

Henry Ernest Dudeney, *Amusements in Mathematics* (New York: Dover Publications, 1958), problem 117, pp. 21, 162.

111

THE ROPE AROUND THE EARTH

Assume that the Earth is a perfect sphere and that the Earth's circumference at the equator is exactly 25,000 miles. Suppose a rope is placed taut around the Earth at the equator so that the rope is just touching the Earth's surface at all points. Suppose the length of the rope is increased by 2π feet and lifted uniformly off the surface of the earth. (The number π equals approximately 3.1415.) This increased length of rope is once again replaced all around the earth at the equator.

How much above the surface of the Earth all around the equator will the rope now be?

SOLUTION

The surprising answer is that the rope will be 1 foot above the surface of the Earth at all points around the equator! Many people are astounded when they hear this. But the result is correct.

Here is how the puzzle is solved.

Let the radius of the Earth equal R. Thus, the radius of the rope before it is lengthened is also R.

The circumference of the Earth is $2\pi r$.

Let the extra radius of the rope (after it has been lengthened) equal r.

When the length of the rope has been increased by 2π feet, the new circumference is $2\pi R + 2\pi$.

This allows us to write

$$2\pi R + 2\pi = \text{New Circumference}$$

or

$$2\pi\,(R + 1) = \text{New Circumference}$$

Divide by 2π to obtain the new radius, r. This gives

$$R + 1 = r$$

We see from this that the new radius is 1 foot more than the previous radius! Therefore, the new diameter will be 2 feet greater than the previous diameter!

In other words, the rope will be 1 foot above the surface of the Earth at all points! This will be the case no matter what the size of the sphere is! Every 2π of extra feet in the circumference will add 1 foot to the length of the radius. Whether the sphere is the size of an orange or whether the sphere is the size of the sun, every 2π of extra feet in the circumference will add 1 foot to the length of the radius.

It is a totally unexpected and counterintuitive solution.

SOURCE

Owen O'Shea, *The Call of the Primes* (Amherst, NY: Prometheus Books, 2016), pp. 222–24.

112

THE ROPE IN THE SOCCER PLAYING FIELD

Suppose you enter a soccer playing field that is 120 yards in length. You tie a rope at the bottom of one of the goalposts. With the other end of the rope in your hand, you then walk to the corresponding goalpost at the other end of the field. You pull the rope taut and tie it to the corresponding goalpost so that the rope is in a straight line across the length of the field.

You then add 2 feet of rope to the length of rope. You find you are now able to lift the rope vertically at the center spot of the soccer field. How high can you lift the rope at the center spot?

SOLUTION

The surprising answer is that you will be able to lift the rope to a vertical height of precisely 19 feet! Most people are astonished with this result and find it difficult to believe.

This problem is simply a problem involving the Pythagorean theorem.

The initial length of rope is 120 yards, or 360 feet. When 2 feet of slack is added to this length, it becomes 362 feet. Half of this length is 181 feet. Thus, the *length of rope* from the goalpost to the center spot is 181 feet.

When you lift the rope vertically at the center spot, the portion of rope measuring 181 feet becomes the hypotenuse of a right-angle triangle.

The *distance* from the goalpost to the center spot is 180 feet. Consider this as 1 leg of the same right angle triangle. The height that you can vertically lift the rope at the center spot is the third length of the same right triangle. This length in feet equals the square root of $(181^2 - 180^2)$. This equals exactly 19 feet.

Thus, you will be able to lift the rope to a height of 19 feet!

This counterintuitive answer never fails to surprise people.

SOURCE

“Football Field,” http://www.math.hmc.edu/funfacts/ffiles/10010.2.shtml

113

WHICH SCENARIO SHOULD YOU CHOOSE?

Suppose you are a participant on a TV game show. The host of the show presents the following opportunity to you to win $10,000 in cash. He presents 2 different scenarios to you. You are to choose the 1 scenario that you believe will maximize your chances of winning the prize money.

The first scenario is as follows. The host places 4 identical envelopes on a table in front of you. The host tells you that inside each envelope is a slip of paper. The word "cash" or "lose" is written on each slip of paper. The host tells you that 3 of the slips contain the word "cash." Just 1 of the slips of paper contains the word "lose." Of course, you do not know which slips of paper are in any of the envelopes. The host tells you that your task is to choose 2 slips of paper, 1 each from 2 different envelopes, so that each slip contains the word "cash." If you do that, you win $10,000. But if one of the slips of paper you choose is the slip marked "lose," you win nothing.

The second scenario is almost identical to the first. In this scenario, the host presents you with 21 identical envelopes, each envelope containing a slip of paper. Fifteen slips of paper with the word "cash" written on them are in 15 envelopes (1 in each envelope). Also, 6 slips of paper, each with the word "lose" written on them, are in 6 envelopes (1 in each envelope). You do not know which envelopes contain the slips marked "cash" or "lose."

Once again, the host tells you that your job is to choose 2 slips of paper, 1 each from 2 different envelopes, and each slip containing the word "cash." If you do that, you win $10,000. But if one of the slips of paper you choose is the slip marked "lose," you win nothing.

The host now gives you the opportunity to pick either the first or the second scenario. You obviously want to maximize your chances of winning the cash.

Which scenario should you choose?

SOLUTION

Many people, on being given this puzzle, will choose the first scenario. Their thinking is that in that scenario, there are only 4 envelopes containing 4 slips of paper. Three of the slips are favorable, and all one need do is to choose 2 favorable slips.

But consider the second scenario. In this scenario, there are 21 envelopes, of which 15 are favorable. It appears that many people somehow believe that it is more difficult to choose 2 favorable slips of paper in the second scenario than it is in the first scenario.

The surprising truth is that the probability of success is identical in both scenarios!

In the first scenario, the probability that you will pick 2 favorable slips is

$$\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

In the second scenario, the probability that you will pick 2 favorable slips is

$$\frac{15}{21} \times \frac{14}{20} = \frac{1}{2}$$

In either scenario, the probability of success is ½, or 50 percent.

Thus, it does not matter which scenario you choose. Your chance of success in either situation is just 1 in 2.

This usually comes as a surprise to most people who encounter the puzzle for the first time.

SOURCE

The author.

114

WHAT IS THE PROBABILITY THAT YOU WILL GET YOUR ASSIGNED SEAT?

Let's say you are taking an airplane trip. You are told that there are 100 seats on the plane and that there are 100 passengers traveling.

The first person getting on board the plane loses his ticket, so he takes a seat at random. From that moment on, every passenger getting on board the airplane sits in the seat that has been assigned to him—if it is free; if it's not free, he selects a seat at random.

You are the last person to get on board the plane. What is the probability that the seat assigned to you is free?

SOLUTION

Most people believe that because of the possibility of various movements of the passengers in the airplane, the probability that you (the last person) will get the seat that has been assigned to you is tiny. However, believe it or not, the chance that you will get the seat that is assigned to you is 50 percent!

This answer surprises most people when they hear it for the first time.

When you board the airplane, the only possibilities are that your seat is vacant or the seat assigned to the first passenger is vacant! No other seat can be vacant.

To see this, suppose that as you get on board the airplane (recall that you are the last passenger to board), you notice that seat 23 is free. This cannot happen, because when the 23rd passenger boarded the airplane, that seat would have been free then, and he would have sat in seat 23 because that was *his* assigned seat. Nor could it be that, say, seat 45 is free when you board the airplane. Why? Because when the passenger assigned to seat 45 boarded the airplane, that seat would have been vacant, and he would have taken that seat. The same reasoning applies to all the other seats. Therefore, there is only 1 of 2 seats available when you board the airplane: the seat assigned to the first passenger or the seat assigned to you. There are no other possibilities.

Thus, the probability that you will get the seat assigned to you is 1 out of 2, or 50 percent.

SOURCE

"Taking Seats on a Plane," https://math.stackexchange.com/questions/5595/taking-seats-on-a-plane

115

THE FLOATING HAT

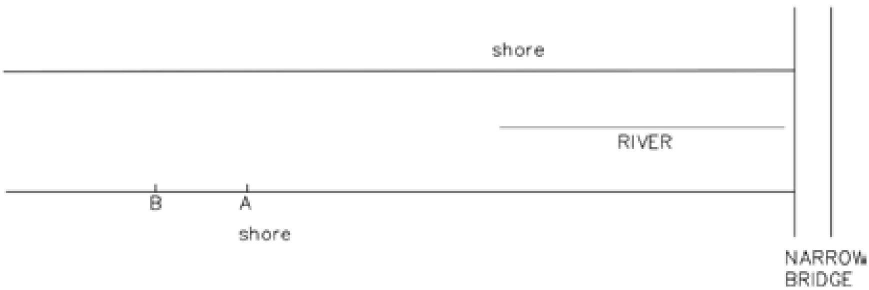

Figure 17

A man was sitting in his paddleboat in a river and was situated directly under a bridge. The man started paddling the boat upstream. When he is 1 mile upstream from the bridge, his hat (unnoticed to the man) fell into the water (see Figure 17). Ten minutes later, the man realized his hat was in the water and flowing downstream toward the bridge. The man instantly turned his boat around and paddled downstream in pursuit of his hat. The man caught up with his hat at the same spot under the bridge where he had commenced his trip.

How fast is the river flowing downstream?

SOLUTION

If one tries to solve this problem by algebra, one may easily find oneself getting tangled in an algebraic mess. Consequently, you will probably find it astonishing when you are informed that the problem can be solved without algebra! The only thing that is required to find a solution to this brainteaser is clear logical thinking!

Here is that surprising method of solution.

Let's look at what is happening in the river from the perspective of the floating hat. The speed of the river has the same effect on the man's boat and hat. Therefore, we can imagine that the river is not flowing at all. Instead, imagine that the shoreline on each side of the river is moving at the speed of the river and in the opposite direction to the direction that the man is rowing in.

From the perspective of the hat—which is 1 mile from the bridge—the man paddled away from the hat for 10 minutes and then paddled toward the hat for 10 minutes. From the hat's perspective, the water is still, so the man would have rowed the same distance away from the hat for 10 minutes and then back to the hat for 10 minutes. Thus, the man has rowed for 20 minutes *with respect to the water*. From the question, we know that in that 20-minute period, the shoreline would have traveled upstream by 1 mile so that the hat was directly under the bridge when the man retrieved it.

Thus, the shoreline is traveling at 1 mile in 20 minutes, which is equivalent to 3 miles per hour.

Of course, we know that the shoreline is not moving at all! We just imagined that and used it as a device so as to easily solve the puzzle. It is the river that is flowing.

Therefore, the river is flowing at a rate of 3 miles per hour.

A picture now emerges concerning the events in the river. The man rowed upstream from the bridge at 4 miles per hour. The current downstream is flowing at a rate of 3 miles per hour. Thus, the man is rowing upstream at the rate of 1 mile per hour in relation to the shoreline. The man's hat blows off when he is 1 mile from the bridge. He continues rowing upstream for 10 minutes, while his hat is flowing downstream at the rate of 3 miles per hour. At the end of this 10-minute period, the hat has traveled $\frac{3}{6}$, or $\frac{1}{2}$, of a mile. The hat is now $\frac{3}{6}$, or $\frac{1}{2}$, of a mile from the bridge, while the man is $\frac{7}{6}$ of a mile from the bridge.

When the man realizes that his hat has blown off, he is $\frac{7}{6}$ of a mile from the bridge, while the floating hat is $\frac{1}{2}$ of a mile from the bridge traveling at 3 miles per hour. The hat will reach the bridge in 10 minutes. On noticing that his hat is gone, the man immediately turns his boat around and rows downstream. His speed downstream (with the tide in his favor) is 4 + 3, or 7, miles per hour. Thus, he reaches the bridge in $(\frac{7}{6})/7$ hours, which equals 10 minutes.

Thus, the man and the hat will reach the bridge together.

SOURCE

Martin Gardner, *Entertaining Mathematical Puzzles* (New York: Dover Publications, 1986), pp. 29–30.

116

THE 2 RIVERBOATS

Two riverboats start to cross a river at right angles to each of the 2 banks of the river. One boat is traveling faster than the other. Each boat travels across the river at a constant speed. Both riverboats meet for the first time at a spot that is 720 yards from the near shore.

On reaching each shore, both boats are moored for 10 minutes to allow passengers to disembark and to take on board new passengers. Then each boat travels back across the river.

Both boats again travel at a constant speed, but one is traveling faster than the other. They meet for the second time at a point that is 400 yards from the far shore (see Figure 18).

The question for our puzzlers is this: how wide is the river?

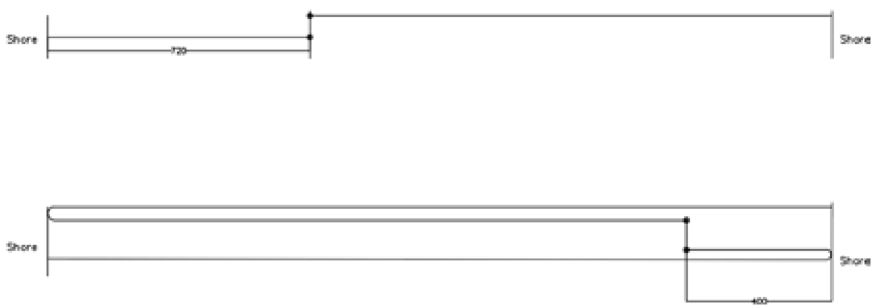

Figure 18

SOLUTION

There is a surprisingly simple method to solve this puzzle without algebra.

Here it is.

We are told that the 2 boats are traveling at a constant speed and at right angles to the shore. When they meet for the first time, the combined distances that the 2 boats have traveled is equal to 1 river length.

When both boats reach shore for the first time, the combined distances that both boats have traveled is equal to 2 river lengths.

When both boats meet for the second time, the combined distances that the 2 boats have traveled is equal to 3 river lengths. Therefore, at that stage, each boat must have travelled 3 times as far as when they had first met. The second boat that had traveled 720 yards when it met the first boat has therefore traveled a total of 3 times 720, or 2,160, yards. We can see from the diagram that this boat has traveled at that stage a distance equal to 1 river width, plus 400 yards. Thus, 2,160 yards must be 400 yards more than the river width.

Therefore, the width of the river is 1,760 yards, or exactly 1 mile.

To solve the problem by algebra, let x equal the river width in yards. We know that both boats are traveling at a constant speed but that one is traveling faster than the other. We know that the ratio of the speed of the boats on the first trip across the river is

$$\frac{x-720}{720}$$

The ratio of the speed of boats on the second trip across the river is

$$\frac{2x-400}{x+400}$$

Since the boats are traveling at a constant speed, these ratios are equal. Thus, we can write

$$\frac{x-720}{720}=\frac{2x-400}{x+400}$$

This equals

$$x^2-320x-28800=1440x-288{,}000$$

This simplifies to

$$x=1{,}760$$

Thus, the width of the river is 1,760 yards, or exactly 1 mile.

SOURCE

Sam Loyd, *Sam Loyd's Cyclopedia of 5,000 Puzzles, Tricks, and Conundrums, with Answers* (New York: Morningside Press, 1914), pp. 80, 349.

117

WHAT IS THE LARGEST NUMBER THAT CAN BE EXPRESSED USING 3 DIGITS?

Mr. Smith likes brainteasers. Thus, he was pleased when he recently read the following problem in a magazine: what is the largest number that can be expressed using only 3 digits? No mathematical symbols are allowed to be used.

Mr. Smith thought about the problem for some considerable time, but the solution eluded him. That night, however, as he lay in bed, the solution came to him.

Can you solve the problem that Mr. Smith solved?

SOLUTION

The largest number that can be expressed using 3 digits is

$$9^{9^9}$$

To find the value of this number, we calculate it from the top down. Thus, we first calculate 9^9. This equals 387,420,489. Then we calculate the number that equals 9 raised to the power of 387,420,489. (This number equals 9 multiplied by itself 387,420,488 times.)

How do we find the number of digits in this rather large number? We first obtain the logarithm (base 10) of 9. The logarithm of 9 is 0.9542425094 . . . . We multiply this by 387,420,488 to obtain 369,693,098.6.

Now the number whose logarithm (base 10) is 369,693,098 is $10^{369,693,098}$, and the number whose logarithm (base 10) is 369,693,099 is $10^{369,693,099}$. The number whose logarithm (base 10) is 369,693,098.6 is somewhere between the 2 numbers; therefore, the number whose logarithm is 369,693,098.6 contains 369,693,099 digits. Such a number equals $10^{369,693,099}$.

This tells us that

$$9^{9^9} = 10^{369,693,099}$$

In other words, the number $10^{369,693,098.6}$ contains 369,693,099 digits.

It will surely come as a surprise to those who have not come across this problem before that there are more than 369 million digits in the number

$$9^{9^9}$$

SOURCE

Karl J. Smith, *Nature of Mathematics* (Pacific Grove, California: Brooks Cole, 2011), pp. 487–88.

118

LOTTERY ODDS

Mr. Jones liked to play the number lottery every week. To win the lottery jackpot, one had to successfully pick 6 numbers from the numbers 1 to 49 inclusive.

Mr. Jones had read over the year that there are precisely 13,983,816 ways of choosing 6 numbers from 49. He thus knows that the odds of success are low, but he continues to play once every week in the hope that he will literally hit the jackpot.

Recently, Jones read in his local newspaper that the organization that runs the lottery is considering changing the rules to win the jackpot. The lottery chiefs were considering implementing the rule that instead of choosing 6 numbers from 49 to win the jackpot, a player now would have to choose 43 numbers from 49 to claim the jackpot.

Of course, this was causing consternation amongst many lottery players.

Should Jones be concerned about the new rule that the lottery organization is considering introducing?

SOLUTION

No, Jones should not be concerned. The introduction of the new rule does not change the probability of winning the jackpot. In other words,

$$49C6 = 13{,}983{,}816 = 49C43$$

When a player chooses 6 numbers from 49, he is, ipso facto, selecting 43 numbers that he does *not* want to play in the lottery. Therefore, the probabilities are equal.

In general, $nCr = nCn - r$.

SOURCE

The author.

119

WATER AND BRANDY

Suppose you have 2 containers. One container, A, contains water. The second container, B, contains brandy.

You now obtain an empty glass. You fill the glass with water taken from container A. You then pour the contents of this glass into container B. You mix the contents of container B. You then fill the glass a second time with liquid taken from container B and pour it into container A.

The question now arises, is there more brandy in container A than there is water in container B, or is there more water in container A than there is brandy in container B?

SOLUTION

The solution to this puzzle is so counterintuitive that many people fail to believe it when presented with it.

The key point to remember is that at the end of the process, the 2 containers have the same amount of liquid that they had at the beginning of the process. The simple fact, then, is that at the end of the filling and pouring operations, the amount of brandy in container A takes the place of the amount of water that was originally in container A but has now been poured into container B. Therefore, the amount of brandy in container A must exactly equal the amount of water in container B.

Here is another way of looking at what is happening. You first fill the glass with brandy from container B and pour the brandy into container A. You then mix the contents of container A. You now fill the glass with the mixture of brandy and water taken from container A and pour it into container B. Thus, when you fill the glass this second time, it is partially filled with water. The amount of water in the glass *has* to be exactly equal to the amount of brandy left behind in container A. Consequently, the amount of brandy in container A has to be equal to the amount of water in container B.

The truth of all this can easily be demonstrated with playing cards. Get, say, 20 black cards to represent water and, say, 15 red cards to represent brandy and lay them facedown in 2 vertical columns so that the 20 black cards (column A) are in one column and the 15 red cards (column B) are in the second column. Now take as many red cards as you like (let x represent that number of cards) from column B and transfer them to column A. Now thoroughly shuffle *all* of the cards in column A. Then take x number of cards from column A (representing the mixture of water and brandy) and transfer them to column B. You will then find that there are as many red cards in column A as there are black cards in column B.

SOURCE

David Wells, *The Penguin Book of Curious and Interesting Puzzles* (London: Penguin Books, 1992), puzzle 249, pp. 68, 251.

120

THE HOLE THROUGH A SPHERE

A 6-inch-long cylindrical hole is drilled straight through the center of a sphere. What is the volume of material left in the sphere?

SOLUTION

It appears from the question that one is not given sufficient information to solve this problem. We are not given the diameter of the hole or the diameter of the sphere. This is what makes this beautiful problem so attractive.

The reader who has not encountered this puzzle previously will find it difficult to accept the solution because it is counterintuitive! Nevertheless, the result is true.

The volume of material remaining in the sphere after the 6-inch hole is drilled straight through its center is 36π cubic inches. Incredible as it may seem, this is *always* the case provided that the diameter of the sphere is 6 inches or more! (If the diameter of the sphere is less than 6 inches, then obviously a 6-inch hole cannot be drilled through its center.)

L. A. Graham showed how this problem can be solved using calculus. A method of solution not involving calculus is obtained as follows. Let R equal the radius of the sphere. The radius of the 6-inch-diameter cylinder is the square root of $R^2 - 3^2$. The altitude of the spherical caps at each end of the cylindrical hole is $R - 3$. The volume of the cylinder is $6\pi(R^2 - 9)$.

The volume of a sphere, where R equals its radius, is $4/3\pi R^3$, and the volume of the cylindrical hole is $6\pi(R^2 - 9)$. If the radius of the sphere is R and the height of the spherical cap is h, the volume of the spherical cap is $\pi h^2(3R - h)/3$.

The amount of material left in the sphere after the hole is drilled equals the volume of the sphere minus the volume of the cylinder minus the volume of the 2 cylindrical caps at each end of the cylindrical hole.

This equals $4/3\pi R^3 - 6\pi(R^2 - 9) - 2\pi h^2(3R - h)/3$.

We can substitute, in the formula, $R - 3$ for h, the height of the spherical cap. Our formula then becomes

$$\pi(4R^3/3 - 6R^2 + 54 - 2\,(R - 3)^2\,(3R - (R - 3)/3)$$

Expanding this, we obtain

$$\pi(4R^3/3 - 6R^2 + 54 - 2(2R^3 - 9R^2 + 27)/3)$$

This equals

$$\pi(4R^3/3 - 6R^2 + 54 - 4R^3/3 + 6R^2 - 18)$$

All the terms in the parentheses cancel, except for + 54 − 18, which equals 36.

Thus, we are left with

$$\pi(36)$$

Thus, the volume of material remaining in the sphere after the 6-inch hole is drilled straight through its center is 36π cubic inches. This is the case regardless of the size of the sphere or the diameter of the hole!

The puzzle is sometimes referred to in the literature and online as the *napkin ring problem* because the residue left in the sphere after the hole is drilled resembles a napkin ring. The counterintuitive fact is that the volume of the napkin ring does not depend on the volume of the sphere, only on the height of the cylindrical hole drilled through the sphere.

Consider, for example, a sphere the size of the Earth. If a 6-inch-long cylindrical hole is drilled straight through the center of the Earth, the hole would have to be nearly 8,000 miles in diameter! The drilled hole would have to have this enormous diameter in order to make the length of the drilled hole 6 inches. The drilling of such a large hole would remove the material in the caps at the top and bottom of the sphere plus the material that was formerly in the hole. What remains of the sphere resembles a napkin ring. That volume of material left in the sphere would equal 36π cubic inches.

Figure 19 shows two spheres, one much larger than the other. Both spheres have a 6-inch hole drilled straight through the center. The illustration shows that no matter how large the sphere is, it is still possible to drill a 6-inch hole through its center. (In other words, the center of the sphere and the center of the hole coincide.) Of course, the diameter of the hole is very nearly the diameter of the sphere. As the sphere increases, the diameter of the hole necessarily increases as well in order to maintain the length of the hole at 6 inches. The residue in each sphere (often referred to as the band) after the 6-inch hole is drilled is constant and is *not* dependent on the size

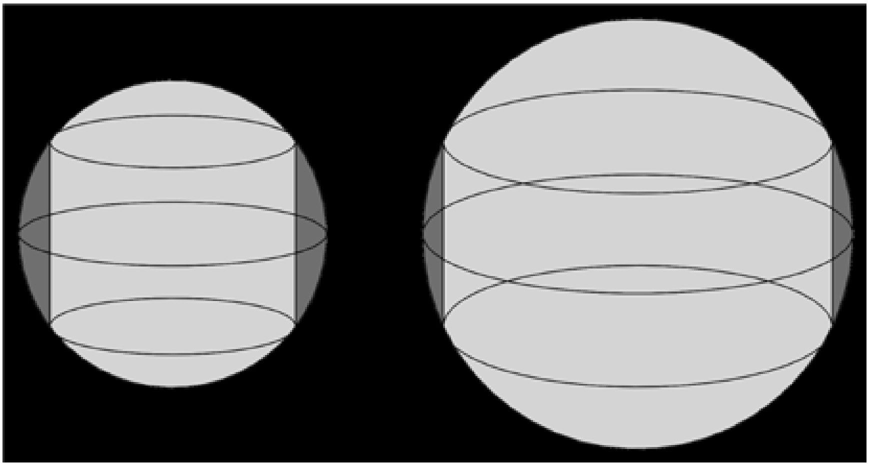

Figure 19

of the sphere. The residue is in fact 36π cubic inches, which works out as 113.0973 cubic inches. In other words, the residue is equal to the volume of a 6-inch sphere.

It is truly amazing and counterintuitive that this constant holds, whether the sphere is the size of an orange or the size of a planet!

According to the late Martin Gardner, a second method of solution to this problem was proposed by John J. Campbell Jr., who was the editor of *Astounding Science Fiction*. Campbell solved the problem by reasoning as follows. Since we are not given the diameter of the sphere or the diameter of the hole in it, we can assume that these dimensions are irrelevant to obtaining a solution. This means that if the problem has a solution, it must be a unique solution that holds regardless of the size of the sphere or the size of the diameter of the hole. We know that the sphere must be at least 6 inches in diameter because a cylindrical hole 6 inches long has been drilled straight through the sphere's center. A sphere that has a diameter of 6 inches has a volume of 36π cubic inches. If a theoretical hole 6 inches in length is drilled straight through the center of such a sphere, the volume remaining must be a constant, even if the diameter of the hole is zero! Of course, if the diameter of the hole is zero, then the residue of the sphere must equal the volume of the sphere, which is 36π cubic inches. Hence, the residue of the 6-inch-diameter (or bigger) sphere, when a 6-inch cylindrical hole is drilled straight through its center, will always equal 36π cubic inches!

SOURCE

L. A. Graham, *Ingenious Mathematical Problems and Methods* (New York: Dover Publications, 1959), problem 34, pp. 23, 145–47.

121

THE ANT ON A RUBBER ROPE

For the final puzzle in this collection, I offer a puzzle that has an answer that is extremely surprising.

Imagine a rubber rope that is 1 kilometer long. The rope can be stretched indefinitely. A worm is at the end of the rope. The worm begins to crawl at a constant speed of 1 centimeter per second toward the other end of the rope. At the end of each second, the rope is stretched in length by another kilometer. Thus, at the end of the first second, the ant has crawled 1 centimeter, but then the rope instantly stretches in length by 1 kilometer so that now the rope is 2 kilometers in length. At the end of the second second, the ant has crawled another centimeter. Then the rope instantly stretches by another kilometer so that now the rope is 3 kilometers in length. And so on. It should be noted that when each expansion occurs, the expansion of the rope is over its entire length.

The stretching of the rope results in an additional kilometer being added to the length of the rope at the end of each subsequent second. The stretching happens only at the end of each second. Units of length and time remain constant. For the purpose of the puzzle, we assume that the ant will live forever.

The question is, does the ant ever reach the end of the rope?

SOLUTION

One of the key aspects of this puzzle is that the rubber rope is being stretched uniformly. Therefore, the rubber rope in front of the ant and behind the ant is being stretched at the same rate. Thus, at the beginning of the ant's journey, the ant has to traverse 100 percent of the length of the rope. At the end of the first second, the ant has to traverse 99,999/1,000,000 of the rope. So even at the end of the first second, he has less of the rope to traverse, proportionally speaking, than he had had at the beginning of his journey! The same principle applies at the end of each second: the proportion of the rope to be traversed has decreased. Thus, we can be assured that the ant will—eventually—get to the end of the rope.

Let's look at the problem a little more closely.

Because the ant is traveling at a constant speed of 1 centimeter per second along the rope and because the rope is stretched by an additional kilometer, it seems at first sight that the ant will never reach the end of the rope. But, surprisingly, this conclusion is incorrect. The ant *will* reach the end of the rope in a finite time, although the duration of that time period is enormous.

The ant crawls at a speed of 1 centimeter per second. Since there are 100,000 centimeters in a kilometer, at the end of the first second, the ant has traveled 1/100,000th the length of the rope. Then the rope is instantly stretched by an additional kilometer to 2 kilometers in length. The ant now crawls along another centimeter of the rope in 1 second. At the end of that second second, the rope is instantly stretched to 3 kilometers in length. At that instant in time, the ant has crawled a total of 1/100,000th plus 1/200,000th the length of the rope. The ant now crawls another centimeter along the rope in 1 second. At the end of that third second, the rope is instantly stretched by an additional kilometer. At that instant, the ant has crawled a total of 1/100,000th plus 1/200,000th plus 1/300,000th the length of the rope. Thus, the ant is making progress as it continues crawling 1 centimeter each second, and at the end of each second, the rope is stretched by an additional 1 kilometer.

The ant's progress each second can be expressed as

$$\frac{1}{100000}+\frac{1}{200000}+\frac{1}{300000}+\ldots$$

This equals

$$\frac{1}{100000}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\right)$$

The fractions inside the parentheses are part of a famous infinite series in mathematics known as the *harmonic series*. (Beyond the first term in the harmonic series, the partial sum of the harmonic series is never an integer.) The harmonic series diverges. This means that the sum of the harmonic series grows larger and larger as more and more terms of the series are summed.

However, the harmonic series grows extremely slowly. The sum of the first 100,000 terms, 1 + ½ + ⅓ + . . . , is just above 12. The sum of the first 1,000,000 terms is 14.39, and the sum of the first 10,000,000 terms is 16.69. The sum of the harmonic series, where each term is divided by 100,000, grows ever more slowly still. Once the partial sum of the series 1/100,000 (1 + ½ + ⅓ + . . .) exceeds 100,000, the above expression will exceed 1. This tells us that the ant has then reached the end of the rope.

Let the number of terms in this partial harmonic series equal *n* so that the sum of the terms from the first term to *n* tells us that the ant has reached the end of the rope in *n* seconds.

Since the ant crawls 1 centimeter per second and since the rope is stretched by an additional kilometer per second, the final length of the rope in kilometers also equals *n*.

The duration of time in seconds for the ant to reach the end of the rope is

$$\frac{1}{100000}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$$

An excellent approximation of this sum is

$$\ln n+\gamma \approx \frac{1}{100000}\ 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

(Here, γ equals the *Euler–Mascheroni constant*, which to 5 decimal places equals 0.57721.)

This expression can be rearranged as

$$\ln n \approx \frac{1}{100000} \quad 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \quad -\gamma$$

The above sum approximately equals $e^{100{,}000-\gamma} \pm 1$, where e equals the transcendental number 2.71828. . . .

According to the online Wolfram-Alpha computational facility, this result is approximately equal to $1.57 \times 10^{43{,}429}$. This number of seconds is the length of time it will take the ant to reach the end of the rope. This time period is vastly longer than the age of the universe! (If we take the age of the universe as 13.8 billion years, we find that the number of seconds in that period is approximately 4.35×10^{17}.)

How long will the stretched rope eventually be by the time the ant reaches the end of it?

Since the ant will traverse the rope in $2.80 \times 10^{43{,}429}$ seconds and since the rope is extending by 1 kilometer per second, the rope will be $2.80 \times 10^{43{,}429}$ kilometers in length.

It is believed that the observable universe is about 93 billion light-years in diameter. This distance approximately equals 9.176908×10^{23} kilometers. Thus, we can see that the length of the rope is much, much longer than the length of the observable universe.

This is most certainly a very counterintuitive result!

SOURCE

Martin Gardner, *Time Travel and Other Mathematical Bewilderments* (New York: W. H. Freeman and Company), pp. 111, 118, 119.

SHORT BIOGRAPHIES

We now come to the section of the book where I give three relatively short biographies of the three giants of recreational mathematics: Sam Loyd, Henry Ernest Dudeney, and Martin Gardner.

SAMUEL LOYD

Samuel Loyd, better known as Sam Loyd, was born on Saturday, January 30, 1841, in Philadelphia. Sam was the youngest of eight children, born, he said, to "wealthy but honest parents." Sam's father, Isaac Loyd, was a real estate operator. Loyd's mother, Elizabeth Singer Sergeant, was a first cousin of the famous American portrait painter, John Singer Sargent (1856–1925).[1]

Shortly after his marriage, Sam's grandfather, Thomas Loyd, dropped the second "L" in the name "Lloyd" because of a neighbor—also named Thomas Loyd—who had earned an unpleasant reputation.[2]

When Sam was three years old, his father moved the family to New York. The Loyds spent their summers in Bordentown, on the Delaware River, in New Jersey. The young Sam had a natural tendency to like puzzles and tricks. He became skilled at ventriloquism and mimicry. He also enjoyed magic tricks. Sam and his brother learned the game of chess from the deckhands on the steamboats on the Delaware River while traveling with their father on his business excursions from their home in Bordentown to Philadelphia.

Sam became fascinated with the game and soon began to compose chess puzzles. By the time he was fourteen, he had his first chess problem published. Two years later, he became the problem editor of *Chess Monthly*. As a result, he became very well known to chess players throughout America. He also wrote chess columns for *Scientific American Supplement*.

In 1870, Loyd married Adeline J. Coombs (1854–1939) of Utica, New York. They had one son, Walter, and three daughters.[3] Walter later changed his name to Sam Loyd Jr. when his father died. Loyd Jr. was also a puzzle creator.

Loyd had studied to be an engineer and held a license for steam and mechanical engineering.[4] He ran a variety of business enterprises. Loyd also owned a printing press in Elizabeth, New Jersey, from which he published many of his advertising puzzles.

If Sam Loyd had decided to go to college to study mathematics, he probably would have become a famous mathematician and perhaps would have had achieved major discoveries in that discipline. But if he did so, the world of recreational mathematics would have lost one of its great and most famous pioneers.

In 1858, when he was only seventeen, Loyd brought out one of his most famous puzzles, titled the *Trick Donkeys*. The famous circus owner P. T. Barnum bought the naming rights to the puzzle from Loyd to advertise his circus around the United States.[5] It was said that he paid the young Sam Loyd $10,000 for the puzzle in the space of a few weeks. The $10,000 that Loyd earned in 1858 would be equivalent in value in 2019 to more than $300,000.[6]

In his youth, Loyd gained considerable success as a chess player. He was said to have been one of the best players in the United States and was ranked the fifteenth best chess player in the world in 1868, when he was twenty-seven years old.[7] Sam Loyd was inducted into the U.S. Chess Hall of Fame in 1987.[8]

It was, however, as a composer of chess puzzles rather than as a chess player that Loyd truly excelled. In 1878, he published *Chess Strategy*. The book contained Loyd's chess problems. (This work was republished in 2015. The original edition is a collector's item.) Today, chess masters and chess problem composers around the world recognize Loyd's chess puzzles as being the work of a genius.[9]

When Loyd turned thirty, his interest in composing chess problems began to wane, and he turned his attention to creating mathematical and mechanical puzzles. He also excelled at these arts. He was able to create cardboard puzzles, such as *The Horse of a Different Color*, which caught the world's imagination. He created his famous *Get Off the Earth Paradox* in 1896 and brought out a number of variations of this amazing puzzle in the following years.

Sam Loyd was undoubtedly a puzzle genius. However, he was also a self-promoter and was not above taking credit for puzzles he did not invent. He falsely claimed to his dying day that he invented the famous *Fifteen Puzzle*. However, he had nothing to do with the invention of the puzzle. The *Fifteen Puzzle* was originally created by Noyes Palmer Chapman (1829–1892), a local postmaster in Canastota, New York.[9] The *Fifteen Puzzle* became a craze in the United States in January 1880.[10]

To be fair to Loyd, he did popularize his own particular version of the *Fifteen Puzzle*, which he first proposed on January 4, 1896, in an article in *The Illustrated American*.[11] The original *Fifteen Puzzle* consisted of fifteen numbered blocks that were randomly placed in a suitable box that accommodated them, with one empty space available so that one could slide a block into that space. The object of the puzzle was to slide all fifteen numbered blocks into numerical order, with the empty space in the bottom right-hand corner. The curious thing about the puzzle was that if someone attempted to solve it, they found that sometimes it was an easy matter to slide the blocks into numerical order, but at other times it seemed to be impossible to do so.

Loyd mathematically analyzed the puzzle. The number of permutations of the sliding blocks in the *Fifteen Puzzle* is more than 20 billion. From any given starting position, exactly half of those approximately 20 billion permutations can be reached by sliding the blocks around inside the little box. The other half of the 20 billion permutations or so cannot be reached because they are of a different *parity*. Loyd knew that one can change the parity of the puzzle, from an unsolvable state to a solvable state (or vice versa), by lifting any two blocks in the puzzle and exchanging them and then placing them back into the puzzle.

Loyd's version of the puzzle amounted to this: he started with all fifteen numbered blocks in the correct numerical order, with the empty space in the bottom right-hand corner. Thus, the puzzle was in a solvable state. Loyd then simply lifted blocks 14 and 15, exchanged their places, and put them back in the box and kept the empty space in the bottom right-hand corner. This changed the parity of the puzzle; the puzzle had now been changed to an unsolvable state. Thus, the starting position of Loyd's version of the puzzles had all fifteen blocks in numerical order, except the last two, 14 and 15, which had changed places.

To solve Loyd's version of the *Fifteen Puzzle*, one had to slide all fifteen blocks into numerical order, finishing with the empty space in the bottom right-hand corner, where it had been at the beginning. Loyd offered a $1,000 reward in the 1896 article in *The Illustrated American* to anyone who could solve his version of the *Fifteen Puzzle*.[12] Thousands claimed to have solved the puzzle, but none could remember the sequence of moves necessary to achieve the solution and receive Loyd's $1,000 reward. Because his version of the puzzle could not be solved, Loyd knew his reward was safe.

Loyd was able to create extremely interesting arithmetic puzzles based on simple algebraic equations. (A classic example is puzzle 79 in this collection.) The arithmetic puzzles' algebraic foundations were very well disguised by coating them with interesting story lines that attracted readers in great numbers. Also, the puzzles published by Loyd were usually accompanied by charming illustrations that caught the fancy of the reader. At first glance, the puzzles might appear to be not too difficult, but one usually needed sharp reasoning abilities and considerable mathematical acumen to obtain the correct solutions to his problems. He also offered the odd mathematical puzzle that could be solved only by calculus. But generally, his puzzles were such that the average layperson found them extremely interesting and entertaining.

In 1914, three years after his death, his son, who was now calling himself Sam Loyd, published a huge collection of his father's problems that he called *Sam Loyd's Cyclopedia of 5,000 Puzzles, Tricks, and Conundrums, with Answers*. The collection was misnamed, for it did not contain 5,000 puzzles. It was also hastily assembled and contained many typographical errors and even incorrect solutions to some of the problems. Nevertheless, the *Cyclopedia*, as it subsequently became known to mathematical puzzle buffs around the world, is a wonderful collection of puzzles. The marvelous drawings accompanying many of the mathematical puzzles add to the book's charm.

The *Cyclopedia* was out of print for decades and in time became a prized collector's item. The book came back into print in 2007.[13]

In the late 1950s, Dover Publications in New York commissioned Martin Gardner to select the best mathematical puzzles from the *Cyclopedia*. Gardner's selections were published in two volumes, which are still in print today.[14]

For years, it was believed that the *Cyclopedia* contained most (if not all) of Sam Loyd's mathematical puzzles. But in recent decades, researchers have discovered long-lost puzzles that they have confirmed are the work of Sam Loyd.[15] These puzzles appeared in newspapers and magazines during Loyd's lifetime but for one reason or other did not find their way into the *Cyclopedia*. It is believed that there are many gems among the long-lost Loyd problems.

Similar to all of us, Sam Loyd had human failings and shortcomings. But he was also a fantastic creator of amazing chess, mechanical, and mathematical puzzles. He deserves to be remembered and honored for the outstanding puzzles he created. He was indisputably one of the greatest puzzle makers—if not the greatest—who ever lived.

Sam Loyd died at his home in Brooklyn, New York, at age seventy on Monday, April 10, 1911.

HENRY ERNEST DUDENEY

Henry Ernest Dudeney was born in the village of Mayfield, East Sussex, England, on Friday, April 24, 1857. Henry was one of six children born to Gilbert and Lucy Ann Dudeney (née Rich). His father was a schoolmaster. Henry's grandfather, John Dudeney, had begun life as a shepherd but later taught himself mathematics and astronomy. He too eventually became a school master in Lewes. Similar to Sam Loyd, Henry Dudeney learned to play chess at an early age. He became fascinated with the game and began to construct chess puzzles from the age of nine that were published in a local newspaper.[1]

Henry Dudeney had a very basic education. He never attended college, and he worked all of his life, beginning at age thirteen as a clerk in the English Civil Service. His hobbies included billiards, bowling, and croquet. He was also a skilled pianist and organist.

Dudeney married Alice Louisa Whiffin in 1884. Alice was a novelist and wrote under the name "Mrs. Henry Dudeney." She was more famous in her day than Henry. The income she earned from writing made the Dudeney's financially well-off. Henry and Alice had two children: Phyllis Mary, who died at age four months, and Margery Janet, who married John Christopher Fulleylove and moved to Canada and later to the United States. Margery Dudeney died in 1977 at age eighty-seven. Henry's wife, Alice, died on November 21, 1945, at age eighty-one and is buried beside her husband in Lewes Cemetery.[2]

Dudeney left school at age thirteen and thus had a very limited formal education. Surprisingly, he became very interested in mathematics and in constructing mathematical puzzles in particular. In this art, he has never been equaled, except perhaps by Sam Loyd. Dudeney began to submit puzzles to newspapers. He began a puzzle column, "Perplexities," which was published in a magazine titled *The Strand*. The column ran for twenty years.[3]

Henry Ernest Dudeney's first book, *The Canterbury Puzzles*, was published in 1907. The book is one of the most famous in recreational mathematics. The first half of the book purports to tell the puzzle tales of the pilgrims from Geoffrey Chaucer's famous book *The Canterbury Tales*.

A few examples show how good Dudeney was at constructing and solving puzzles in number theory. He found the following unexpected result: $(17/21)^3 + (37/21)^3 = 6$. In puzzle 61 in *The Canterbury Puzzles*, Dudeney posed the difficult problem of finding two fractions whose cubes sum to exactly 17. The answer he provided was 104940/40831 and 11663/40831. If you have a scientific calculator nearby, you can punch the cubes of these two fractions into it and find that their sum is indeed exactly 17 in a matter of seconds. Such fractions are not easily discovered. They can be found using elliptic curves, which involve mathematics not normally accessible or comprehensible to a youth who had left school at age thirteen.

In puzzle 20 in *The Canterbury Puzzles*, Dudeney offers the more difficult problem of finding two cubes (besides 1^3 and 2^3) that sum to 9. He gave the answer that he discovered: the cubes of (415280564497/348671682660) and (676702467503/348671682660) added together exactly equal 9. It was an extraordinary achievement to obtain this result. The fact that Dudeney was able to find these fractions illustrates just how good a mathematician he was.

Dudeney was not confined to inventing problems involving number theory. He was a prolific creator of puzzles, and his range was wide: he produced puzzles in arithmetic, algebra, geometry, combinatorics, logic, and cryptarithmetic and also produced many chess problems.

In *The Haberdasher's Puzzle*, Dudeney asked to find the minimum number of pieces in which an equilateral triangle can be dissected and the pieces reassembled to form a square. The puzzle was published in *The Weekly Dispatch* in 1902. It was later published as puzzle number 26 in *The Canterbury Puzzles* in 1907. Many readers wrote in to *The Weekly Dispatch* and said that they could find a five-part dissection. Only one reader found the required four-part dissection. Dudeney showed a mahogany version of the puzzle—where the four dissected pieces are hinged together—to the Royal Society in May 1905.[4] With the four pieces hinged together at specific points, one can form either an equilateral triangle or a square. The hinged version of the puzzle is sold today in the United Kingdom by Grand Illusions. Many recreational mathematicians consider this geometric dissection to be Dudeney's greatest mathematical discovery.

Dudeney invented what he termed verbal arithmetic, which is now referred to as cryptarithms. In these puzzles, letters are used in addition sums as substitutes for digits. (The same letters are used to represent the same digits [and vice versa], and a number cannot begin with zero.) An elegant cryptarithm is one defined as one that has a unique solution and one in which the words formed by the letters constitute a logical phrase. Dudeney's most famous cryptarithm is the following, which appeared in the July 1924 issue of *The Strand Magazine*:[5]

$$\begin{array}{c} \text{SEND} \\ \underline{\text{MORE}} \\ \text{MONEY} \end{array}$$

The puzzle is solved by making a number of logical deductions concerning the possible values of the various letters. For example, it will be found that the "M" in the word "MONEY" can only be 1. The remaining letters have the following values: Y = 2, E = 5, N = 6, D = 7, R = 8, S = 9, and O = 0. The unique solution of this particular puzzle is as follows:

$$\begin{array}{c} 9567 \\ \underline{1085} \\ 10652 \end{array}$$

Dudeney published his famous *Amusements in Mathematics* in 1917; it is a splendid collection of mathematical puzzles across an enormous range. There are geometry puzzles, money puzzles, combination puzzles, and chess board problems, to name but a few. The book is still in print today. In 1926, Dudeney published *Modern Puzzles*. All of Dudeney's books contain a wealth of material and are eagerly sought to this day by enthusiasts of recreational mathematics.

When Dudeney died in 1930, his wife Alice edited two collections of his problems. *Puzzles and Curious Problems* appeared in 1931 and *A Puzzle Mine* in 1940. Two decades later, Dudeney's daughter, Mrs. Margery Fulleylove, acquired the rights to Dudeney's books, and in 1967, Scribner's published *536 Puzzles and Curious Problems*. The book was edited by Martin Gardner. In England in 1970, Fontana Books published *Puzzles and Curious Problems* and *More Puzzles and Curious Problems*. (Both books between them contain the contents of the Scribner's book published in

1967.) *The World's Best Word Puzzles*, also by Dudeney, was published in London by the Daily News in 1925. Dudeney continued creating puzzles up to shortly before his death.

Today, the self-taught mathematician Henry Ernest Dudeney is recognized as one of the greatest puzzle creators who ever lived. He stands alongside Sam Loyd at the very top of the "roll of honor" of mathematical puzzle geniuses.

Henry Ernest Dudeney died at his home in Sussex at age seventy-three on Thursday, April 24, 1930.

MARTIN GARDNER

Martin Gardner was born in Tulsa, Oklahoma, on Wednesday, October 21, 1914. His father was a geologist and had started his own small oil company. Martin's mother was a primary school teacher. When she married and had children, she decided to stay at home and look after the young ones. Martin was the oldest of three children. He had a younger brother, Jim, and a sister, Judith.[1]

The family was financially well-off. They had their own tennis court by the time Martin was old enough to play. Martin related in later life that his mother used to read *The Wizard of Oz* to him before he had even started to go to school. He would look at the words on the page as his mother read them, and it was in this way that Martin learned to read before entering primary school.

Martin's interest in mathematical puzzles was sparked when as a young boy, his father gave him a present: Sam Loyd's *Cyclopedia of Puzzles*. Martin was good at mathematics and physics at school. He was also interested in conjuring and had his first magic trick published in a magic magazine when he was still a teenager. Martin also learned to play chess, and he retained a love of the game all his life.[2]

Martin wanted to study physics at the California Institute of Technology. But to get into that institute, he was required to have two years of study at a university. So Martin went to the University of Chicago with the intention of staying there for two years and then applying to the California Institute of Technology to study physics. However, at Chicago, he became fascinated with philosophy and decided to stay there to major in the subject.

He graduated from Chicago with a BA in philosophy in 1936. It was the midst of the Great Depression, and after graduating, Martin worked in a variety of odd jobs. He soon took up employment as a reporter with the *Tulsa Tribune*. He also worked as a public relations officer with the University of Chicago. When World War II broke out, he enlisted in the U.S. Navy. He served on board USS *Pope*, which was a naval destroyer that escorted convoys across the Atlantic Ocean.[3]

When the war ended, Martin returned to Chicago. He could have taken up his old job as public relations officer at the University of Chicago, but he didn't. He recommenced his writing career and sold his first short story to *Esquire* magazine. He also wrote for the children's magazine *Humpty Dumpty*.[4]

At that time, Martin also attended graduate courses in philosophy at the University of Chicago. He was particularly impressed by one course given by the German-born philosopher Rudolph Carnap (1891–1970).[5] In 1974 and again in 1995, Martin edited Carnap's book *An Introduction to the Philosophy of Science*.

In 1947, Martin moved to New York City, where he continued to make his living by writing articles that he sold to various magazines. He also got to know many influential magicians there. One of those was a man named Bill Simon (1927–1988). He introduced Martin to Charlotte Greenwald. Martin married Charlotte in 1952. They had two children, Jim and Tom.[6]

In December 1956, Martin had an article on hexaflexagons published in *Scientific American*. (Hexaflexagons are paper toys that can be manipulated to show different faces.) The magazine's editor, Gerard Piel (1915–2004), liked the article so much that he called Martin into his office and asked him if he thought he could write a monthly column on recreational mathematics for the magazine. Martin told the editor that he could.[7]

Martin later in life related how at that time, he knew of a number of famous books on recreational mathematics but that he did not own one of them! On getting the job offer from *Scientific American*, he immediately toured the secondhand bookstores in New York City and purchased as many books on the subject as possible.

His first column in *Scientific American* appeared in January 1957. His monthly columns were titled "Mathematical Games." Coincidentally, the title bears the same initials, MG, as Martin's full name. The column appeared every month for twenty-five years. Many subscribers to *Scientific American* reportedly said that Martin's column was their favorite piece in the magazine each month.[8]

Martin's contract with *Scientific American* allowed him to republish collections of his columns from that publication in the form of recreational mathematics books that were published over the years. He also wrote books on many other subjects, including mathematical puzzles, physics, magic, poetry, literature, philosophy, skepticism, and religion. He had an intense interest in religion all his life. Martin had been brought up by his parents as a devout Methodist but had lost his Christian faith as a young adult as a result of his extensive reading.[9]

In the late 1950s, Dover Publications commissioned Martin to write two books filled with Sam Loyd puzzles, taken from the long-out-of-print *Cyclopedia of Puzzles*. The books were very well received by the general public and are still in print today.

Martin had a natural love for fun and games. His column in April 1975 was a famous April Fool's Day joke. (The column is reprinted as chapter 10 in Gardner's *Time Travel and Other Mathematical Bewilderments*, published by W. H. Freeman and Company, New York, in 1988.) Gardner's column was titled "Six Sensational Discoveries That Somehow or Another Have Escaped Public Attention." It famously declared that Albert Einstein's (1879–1955) theory of relativity was flawed and that the valve flush toilet was invented by Leonardo da Vinci (1452–1519).

That April column also contained a map stating that it required five colors to color, if adjacent countries were to receive a different color. (If this last item were true, it would have been a counterexample of the then long-standing four-color-map conjecture. That conjecture was proved in 1976.) Martin received more than a thousand letters from readers pointing out that they could color the map displayed in his column using four colors. They obviously had not realized that the entire column that month was a hoax!

Although Martin had a lifelong interest in conjuring, he never performed as a professional magician. But he was so well versed in conjuring that in 1999, *MAGIC* magazine nominated him as one of the "100 Most Influential Magicians of the Twentieth Century."[10]

Martin's column was enhanced by the brilliant and regular feedback of his many loyal readers. Some of these were the world's top mathematicians. One was John Horton Conway (1937–). Martin wrote a column in October 1970 about *The Game of Life*, which Conway had invented. Martin's article about Conway's game brought it to the attention of millions. Later in life, John Horton Conway said that Martin Gardner was the most learned man he had ever met.[11]

Martin also corresponded with the English mathematician and physicist Sir Roger Penrose (1931–). He is famous for many discoveries, including the Penrose stairway, which is a drawing of a stairway that one could traverse forever without getting higher or lower. Penrose also invented Penrose tiling, which is a form of nonperiodic tiling. Martin wrote about Penrose tiling in his January 1977 column. The article is reprinted as chapter 1 in his *From Penrose Tiling to Trapdoor Spiders* (published by W. H. Freeman, New York, in 1989).

In his column of April 1961, Martin wrote a positive review of *Introduction to Geometry*, which was written by the British-born Canadian mathematician Harold Scott MacDonald Coxeter (1907–2003). Coxeter is regarded as one of the greatest geometers of the twentieth century. Martin's positive review helped to popularize his book.

Gardner was recognized as an expert on Charles Lutwidge Dodgson (1832–1898), who is better known to the world as Lewis Carroll, author of the classic *Alice's Adventures in Wonderland* and its sequel *Through the Looking Glass*. Martin's *The Annotated Alice* was published in 1960, and *More Annotated Alice* followed in 1990. In 1998, *The Annotated Alice: The Definitive Edition* was published by W. W. Norton and Company. Through the years, more than a million copies of the various Alice books have been sold.[12]

His interest in the philosophy of science led Gardner to write his first book, *Fads and Fallacies in the Name of Science*, in 1952 to debunk pseudoscience. Martin was a skeptic of the paranormal all his life and helped to expose con men and con women who preyed on the public with their scientific fraud and quackery. He publicly criticized those who propounded such silly notions as faith healing, the mystical powers of the Great Pyramid, a flat Earth, alien abductions, and UFO sightings. His strong interest in debunking pseudoscience led Gardner, along with science authors Isaac Asimov (1920–1992) and Carl Sagan (1934–1996), to establish the Committee for the Scientific Investigation of Claims of the Paranormal. Martin regularly wrote for its magazine, the *Skeptical Inquirer*, which he helped found.

Martin believed that it is the duty of scientists and science writers to expose the errors of bad science, especially in the field of medicine, in which false beliefs often lead to needless suffering and death. Martin always expressed alarm at the number of crank books that promote crazy ideas (such as the practice of speaking to plants or sharpening razor blades by putting them inside little pyramids) and that constantly outsell good books on rational science. The noted biologist Stephen Jay Gould described Martin as "the single brightest beacon defending rationality and good science against the mysticism and anti-intellectualism that surround us."[13]

Martin lived at 10 Euclid Avenue in Hastings-on-Hudson, New York, for several years, an address that many thought apt for someone who loved mathematics. In 1981, he ceased writing the monthly column for *Scientific American* so that he would have more time to write books, and he and Charlotte moved to Hendersonville, North Carolina.

In 1993, one of Martin's fans, Thomas Rodgers (1945–2012), organized an event in Atlanta, Georgia, in Martin's honor. It was called the Gathering for Gardner (G4G). Since then, the event has been held every two years in Atlanta. Many leading figures in mathematics and magic, including Persi Diaconis, Ron Graham, and John Horton Conway, attend the biennial event.

In 2000, Charlotte died. Two years later, Martin moved to Norman, Oklahoma, to be near his son, Jim, who was professor of education at the University of Oklahoma. Martin continued writing. At the end of his long life, he had written or edited more than one hundred books as well as numerous pamphlets and articles.

Martin Gardner did not consider himself a mathematician. He once said that he did not consider himself good at solving puzzles![14] Instead, he looked on himself as a mathematical journalist who reported and explained some of the findings of professional mathematicians. But most mathematicians would probably disagree with Martin's modest description of himself.

Martin had strong views about what mathematics actually is. Many professional mathematicians believe that the structure we call mathematics is something that has been invented by human beings and that it has no reality outside human minds. But Martin disagreed with that viewpoint. He was a mathematical Platonist; he believed that mathematics is *discovered*, not invented. In defending his view on mathematics, he once wrote,

> We know matter is made of molecules and that molecules are made of atoms, and atoms are made of electrons, protons, and neutrons. Protons and neutrons are made of quarks, and quarks and electrons *may* be made of vibrating loops of superstrings. And what are the strings made of? They seem to be made of pure mathematics. A friend once said that the universe appears to be made out of nothing, yet somehow it manages to exist. It is even possible that matter has an infinity of levels.[15]

The book titled *The Mathematical Gardner* was published in 1981 by Prindle, Weber & Schmidt/Wadsworth International of Belmont, California, as a tribute to Martin. The editor of that book, David A. Klarner (1940–1999), wrote in the preface, “Very few people in the general population who are capable of understanding and enjoying the beauty of mathematics actually get the opportunity to do so. Martin Gardner’s writing has contributed enormously to improve the accessibility of the subject. The authors of this book together with a larger group of fans praise and thank him.”[16]

Martin Gardner was a philosopher who fearlessly spoke out in defense of his beliefs. On the subject of human knowledge, he believed that what human beings know is much less than what they do not know and that that will *always* be the case. One of his favorite short stories illustrates this viewpoint succinctly. The story is an Indian legend that comes from a 300-page book-length poem, first published in 1936 by Carl Sandburg (1878–1967) titled *The People, Yes*. The poem contains references to American culture, phrases, and stories. The short story in the book that Martin liked goes as follows:

> The white man drew a small circle in the sand and told the red man, “This is what the Indian knows.” The white man then drew a bigger circle around the smaller one and said, “This is what the white man knows.” The Indian took the stick and swept an immense ring around both circles and said, “This is where the white man and the red man know nothing.”[17]

Martin was influenced by many great philosophers and writers: Plato (428/427–348/347 BCE), Immanuel Kant (1724–1804), G. K. Chesterton (1874–1936), and H. G. Wells (1866–1946), to name but a few.[18] He had a wonderful sense of wonder about the universe and why it exists and could never take the existence of the universe for granted. He believed that there are truths as far beyond our grasp as calculus is beyond the grasp of a cat. He once said that “he was impressed, perhaps overwhelmed is more accurate, by the vastness of the universe and the even greater vastness of the darkness that extends beyond the farthest frontiers of scientific knowledge.”[19] He believed that all the evils of the world are a small price to pay for the privilege of existing.[20]

Many youngsters who had read Martin’s monthly column in *Scientific American* over the years went on to become professional mathematicians and scientists. Many of them later in life said that they had chosen their particular careers as a result of reading Martin’s columns.

Many are surprised when they learn that the only course in mathematics that Martin Gardner ever did was in high school. It is astonishing that such an individual, through his columns and books, has brought the beauty and joy of recreational mathematics to millions of readers all over the world.

Martin Gardner, who had given so much to so many, died peacefully in Norman, Oklahoma, at age ninety-five on Saturday, May 22, 2010. His body was cremated. The location of his ashes is unknown to the general public.

NOTES

INTRODUCTION

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