JACARANDA

MATHS QUEST 10+10A

VICTORIAN CURRICULUM | REVISED EDITION

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MATHS QUEST 10+10A

VICTORIAN CURRICULUM | REVISED EDITION

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CONTENTS

Topic 1 Indices 1

Topic 2 Algebra and equations 31

Topic 3 Coordinate geometry 66

Topic 4 Simultaneous linear equations and inequalities 112

Topic 9 Non-linear relationships 329

Topic 10 Deductive geometry 394

Project: Getting the budget in order 437

Topic 11 Probability 439

Topic 12 Univariate data 491

Topic 13 Bivariate data 549

Topic 14 Statistics in the media 591

Project: Climate change 639

Topic 15 Financial mathematics 640

Topic 16 Real numbers 675

Topic 17 Polynomials 728

Topic 18 Functions and relations 759

Topic 19 Circle geometry 807

Topic 20 Trigonometry II 842

Topic 21 Programming

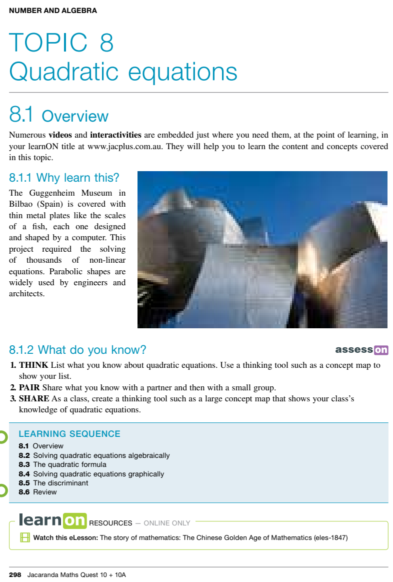
ABOUT THIS RESOURCE

Jacaranda Maths Quest 10+10A Victorian Curriculum Third Edition has been completely revised to help teachers and students navigate the Victorian Curriculum Mathematics syllabus. The suite of resources in the Maths Quest series is designed to enrich the learning experience and improve learning outcomes for all students.

Maths Quest is designed to cater for students of all abilities: no student is left behind and none is held back. Maths Quest is written with the specific purpose of helping students deeply understand mathematical concepts. The content is organised around a number of features, in both print and online through Jacaranda's *learnON* platform, to allow for seamless sequencing through material to scaffold every student's learning.

Topic introductions put the topic into a real-world context.

The learning sequence at a glance



NUMBER AND ALGEBRA

TOPIC 8
Quadratic equations

8.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

8.1.1 Why learn this?

The Guggenheim Museum in Bilbao (Spain) is covered with thin metal plates like the scales of a fish, each one designed and shaped by a computer. This project required the solving of thousands of non-linear equations. Parabolic shapes are widely used by engineers and architects.

8.1.2 What do you know?

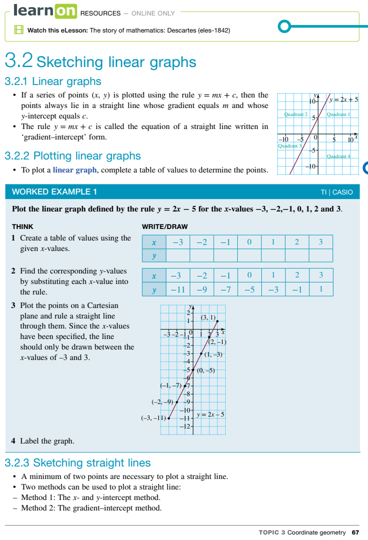
assess on

1. **THINK** List what you know about quadratic equations. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of quadratic equations.

LEARNING SEQUENCE
8.1 Overview
8.2 Solving quadratic equations algebraically
8.3 The quadratic formula
8.4 Solving quadratic equations graphically
8.5 The discriminant
8.6 Review

learn on RESOURCES — ONLINE ONLY
Watch this eLesson: The story of mathematics: The Chinese Golden Age of Mathematics (eles-1847)

298 Jacaranda Maths Quest 10 + 10A



learn on RESOURCES — ONLINE ONLY
Watch this eLesson: The story of mathematics: Descartes (eles-1842)

3.2 Sketching linear graphs

3.2.1 Linear graphs

- If a series of points (x, y) is plotted using the rule $y = mx + c$, then the points always lie in a straight line whose gradient equals m and whose y-intercept equals c.
- The rule $y = mx + c$ is called the equation of a straight line written in 'gradient–intercept' form.

3.2.2 Plotting linear graphs

- To plot a **linear graph**, complete a table of values to determine the points.

WORKED EXAMPLE 1 TI | CASIO

Plot the linear graph defined by the rule $y = 2x - 5$ for the x-values $-3, -2, -1, 0, 1, 2$ and 3.

THINK / WRITE/DRAW

1 Create a table of values using the given x-values.

x	−3	−2	−1	0	1	2	3
y							

2 Find the corresponding y-values by substituting each x-value into the rule.

x	−3	−2	−1	0	1	2	3
y	−11	−9	−7	−5	−3	−1	1

3 Plot the points on a Cartesian plane and rule a straight line through them. Since the x-values have been specified, the line should only be drawn between the x-values of –3 and 3.

4 Label the graph.

3.2.3 Sketching straight lines

- A minimum of two points are necessary to plot a straight line.
- Two methods can be used to plot a straight line:
– Method 1: The x- and y-intercept method.
– Method 2: The gradient–intercept method.

TOPIC 3 Coordinate geometry **67**

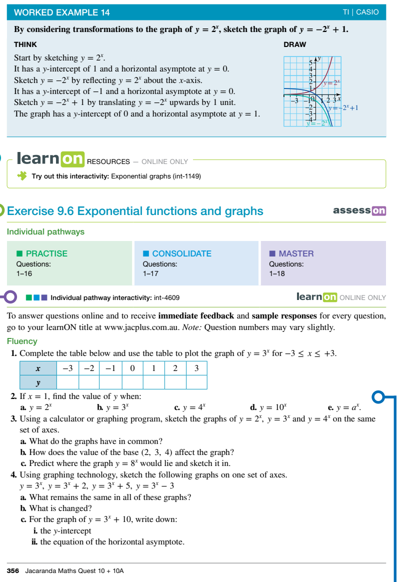
Visit your *learnON* title to watch videos which tell the story of mathematics.

An extensive glossary of mathematical terms in print, and as a hover-over feature in your *learnON* title

Fully worked examples throughout the text demonstrate key concepts.

Your FREE online *learnON* resources contain hundreds of videos, interactivities and traditional WorkSHEETs and SkillSHEETs to support and enhance learning.

Carefully graded questions cater for all abilities. Question types are classified according to strands of the Victorian Curriculum.



WORKED EXAMPLE 14 TI | CASIO

By considering transformations to the graph of $y = 2^x$, sketch the graph of $y = -2^x + 1$.

THINK

Start by sketching $y = 2^x$.
It has a y-intercept of 1 and a horizontal asymptote at $y = 0$.
Sketch $y = -2^x$ by reflecting $y = 2^x$ about the x-axis.
It has a y-intercept of -1 and a horizontal asymptote at $y = 0$.
Sketch $y = -2^x + 1$ by translating $y = -2^x$ upwards by 1 unit.
The graph has a y-intercept of 0 and a horizontal asymptote at $y = 1$.

DRAW

learn on RESOURCES — ONLINE ONLY
Try out this interactivity: Exponential graphs (int-1149)

Exercise 9.6 Exponential functions and graphs

assess on

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–16	Questions: 1–17	Questions: 1–18

■■■ **Individual pathway interactivity:** int-4609 learn on ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. Complete the table below and use the table to plot the graph of $y = 3^x$ for $-3 \le x \le +3$.

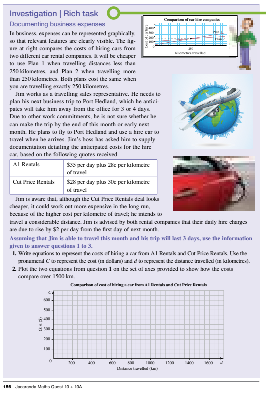
x	−3	−2	−1	0	1	2	3
y							

2. If $x = 1$, find the value of y when:
   a. $y = 2^x$ **b.** $y = 3^x$ **c.** $y = 4^x$ **d.** $y = 10^x$ **e.** $y = a^x$.
3. Using a calculator or graphing program, sketch the graphs of $y = 2^x$, $y = 3^x$ and $y = 4^x$ on the same set of axes.
   a. What do the graphs have in common?
   b. How does the value of the base (2, 3, 4) affect the graph?
   c. Predict where the graph $y = 8^x$ would lie and sketch it in.
4. Using graphing technology, sketch the following graphs on one set of axes.
   $y = 3^x$, $y = 3^x + 2$, $y = 3^x + 5$, $y = 3^x - 3$
   a. What remains the same in all of these graphs?
   b. What is changed?
   c. For the graph of $y = 3^x + 10$, write down:
      i. the y-intercept
      ii. the equation of the horizontal asymptote.

356 Jacaranda Maths Quest 10 + 10A

Individual pathway interactivities in each sub-topic ensure consolidation of learning for every skill level.

Fully worked solutions to every question are provided online, and answers are provided at the end of each print topic.



Investigation | Rich task

Documenting business expenses

In business, expenses can be represented graphically, so that relevant features are clearly visible. The figure at right compares the costs of hiring cars from two different car rental companies. It will be cheaper to use Plan 1 when travelling distances less than 250 kilometres, and Plan 2 when travelling more than 250 kilometres. Both plans cost the same when you are travelling exactly 250 kilometres.

Jim works as a travelling sales representative. He needs to plan his next business trip to Port Hedland, which he anticipates will take him away from the office for 3 or 4 days. Due to other work commitments, he is not sure whether he can make the trip by the end of this month or early next month. He plans to fly to Port Hedland and use a hire car to travel when he arrives. Jim's boss has asked him to supply documentation detailing the anticipated costs for the hire car, based on the following quotes received.

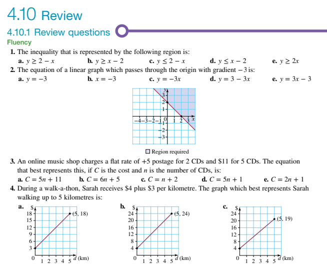
A1 Rentals	\$35 per day plus 28c per kilometre of travel
Cut Price Rentals	\$28 per day plus 30c per kilometre of travel

Jim is aware that, although the Cut Price Rentals deal looks cheaper, it could work out more expensive in the long run, because of the higher cost per kilometre of travel; he intends to travel a considerable distance. Jim is advised by both rental companies that their daily hire charges are due to rise by \$2 per day from the first day of next month.

Assuming that Jim is able to travel this month and his trip will last 3 days, use the information given to answer questions 1 to 3.

1. Write equations to represent the costs of hiring a car from A1 Rentals and Cut Price Rentals. Use the pronumeral C to represent the cost (in dollars) and d to represent the distance travelled (in kilometres).
2. Plot the two equations from question **1** on the set of axes provided to show how the costs compare over 1500 km.

156 Jacaranda Maths Quest 10 + 10A



4.10 Review

4.10.1 Review questions

Fluency

1. The inequality that is represented by the following region is:
   a. $y \ge 2 - x$ **b.** $y \ge x - 2$ **c.** $y \le 2 - x$ **d.** $y \le x - 2$ **e.** $y \ge 2x$
2. The equation of a linear graph which passes through the origin with gradient -3 is:
   a. $y = -3$ **b.** $x = -3$ **c.** $y = -3x$ **d.** $y = 3 - 3x$ **e.** $y = 3x - 3$
3. An online music shop charges a flat rate of +5 postage for 2 CDs and \$11 for 5 CDs. The equation that best represents this, if C is the cost and n is the number of CDs, is:
   a. $C = 5n + 11$ **b.** $C = 6n + 5$ **c.** $C = n + 2$ **d.** $C = 5n + 1$ **e.** $C = 2n + 1$
4. During a walk-a-thon, Sarah receives \$4 plus \$3 per kilometre. The graph which best represents Sarah walking up to 5 kilometres is:

Engaging Investigations at the end of each topic to deepen conceptual understanding

Each topic concludes with comprehensive Review questions, in both print and online.

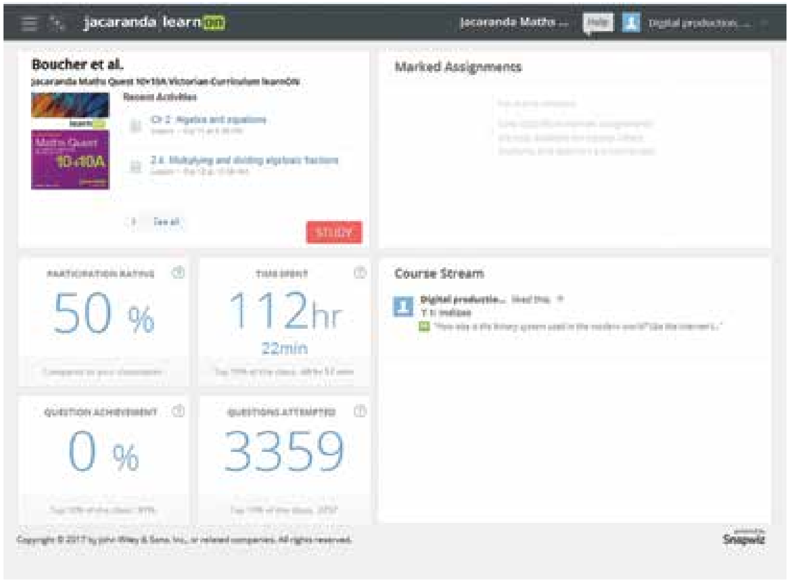
learnON

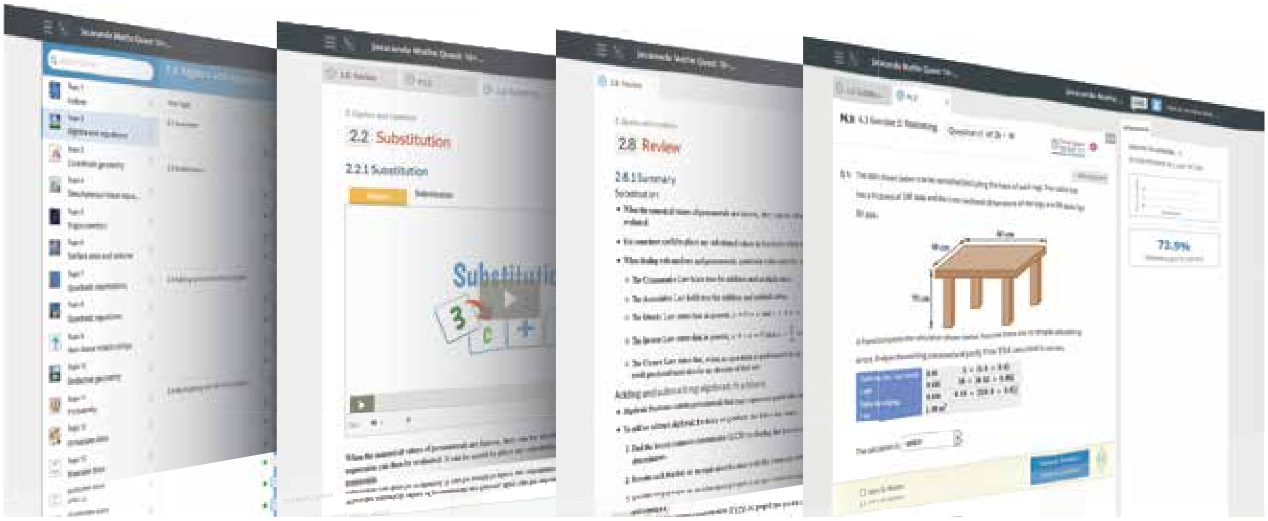
LearnON is Jacaranda's immersive and flexible digital learning platform that transforms trusted Jacaranda content to make learning more visible, personalised and social. Hundreds of engaging videos and interactivities are embedded just where you need them — at the point of learning. At Jacaranda, our 'learning made visible' framework ensures immediate feedback for students and teachers, with customisation and collaboration to drive engagement with learning.

Maths Quest contains a free activation code for *learnON* (please see instructions on the inside front cover), so students and teachers can take advantage of the benefits of both print and digital, and see how *learnON* enhances their digital learning and teaching journey.

learnON includes:

- Students and teachers connected in a class group
- Hundreds of videos and interactivities to bring concepts to life
- Fully worked solutions to every question
- Immediate feedback for students
- Immediate insight into student progress and performance for teachers
- Dashboards to track progress
- Collaboration in real time through class discussions
- Comprehensive summaries for each topic
- Code puzzles and dynamic interactivities to help students engage with and work through challenging concepts
- Formative and summative assessments
- And much more …





CAS SUPPORT IN MATHS QUEST

A CAS appendix has been added to this text showing a selection of Worked Examples from the text produced with CAS instructions and screenshots. A more comprehensive guide to using CAS is included in the About this course/prelim section of your online course.

TOPIC 2

WORKED EXAMPLE 13

Simplify $5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$.

TI | THINK

In a new problem, on a Calculator page, complete the entry line as:
$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
Then press ENTER enter.

WRITE

$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
$= 9\sqrt{3} + 2\sqrt{2}$

CASIO | THINK

Ensure the calculator is set to Standard mode.
On the Main screen, complete the entry line as:
$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
Then press EXE.

WRITE

$5\sqrt{75}-6\sqrt{12}+\sqrt{8}-4\sqrt{3}$
$9\cdot\sqrt{3}+2\cdot\sqrt{2}$

$5\sqrt{75} - 6\sqrt{12} + \sqrt{8} - 4\sqrt{3}$
$= 9\sqrt{3} + 2\sqrt{2}$

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TOPIC 1
Indices

1.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

1.1.1 Why learn this?

Don't you wish that your money could grow as quickly as a culture of bacteria? Perhaps it can — both financial investments and a culture of bacteria can grow exponentially, that is, according to the laws of indices. Indices are useful when a number is continually multiplied by itself, becoming very large, or perhaps very small.

1.1.2 What do you know?

assess**on**

1. **THINK** List what you know about indices. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of indices.

LEARNING SEQUENCE

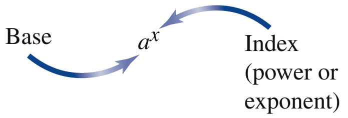
learn**on** RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Leibniz (eles-1840)

1.2 Review of index laws

1.2.1 Review of index laws

- When a number or pronumeral is repeatedly multiplied by itself, it can be written in a shorter form called index form.
- A number written in index form has two parts, the **base** and the **index**, and is written as:



- Another name for an index is an exponent or a power.
- Performing operations on numbers or pronumerals written in index form requires the application of the index laws.

First Index Law: When terms with the same base are multiplied, the indices are added.

$$a^m \times a^n = a^{m+n}$$

Second Index Law: When terms with the same base are divided, the indices are subtracted.

$$a^m \div a^n = a^{m-n}$$

WORKED EXAMPLE 1 TI | CASIO

Simplify each of the following.

a $m^4n^3p \times m^2n^5p^3$ **b** $2a^2b^3 \times 3ab^4$ **c** $\dfrac{2x^5y^4}{10x^2y^3}$

THINK	WRITE
a 1 Write the expression.	**a** $m^4n^3p \times m^2n^5p^3$
2 Multiply the terms with the same base by adding the indices. *Note:* $p = p^1$.	$= m^{4+2}n^{3+5}p^{1+3}$ $= m^6n^8p^4$
b 1 Write the expression.	**b** $2a^2b^3 \times 3ab^4$
2 Simplify by multiplying the coefficients, then multiply the terms with the same base by adding the indices.	$= 2 \times 3 \times a^{2+1} \times b^{3+4}$ $= 6a^3b^7$
c 1 Write the expression.	**c** $\dfrac{2x^5y^4}{10x^2y^3}$
2 Simplify by dividing both of the coefficients by the same factor, then divide terms with the same base by subtracting the indices.	$= \dfrac{1x^{5-2}y^{4-3}}{5}$ $= \dfrac{x^3y}{5}$

Third Index Law: Any term (excluding 0) with an index of 0 is equal to 1.

$$a^0 = 1,\ a \neq 0$$

WORKED EXAMPLE 2

Simplify each of the following.

a $(2b^3)^0$ **b** $-4(a^2b^5)^0$

THINK	WRITE
a **1** Write the expression.	**a** $(2b^3)^0$
2 Apply the Third Index Law, which states that any term (excluding 0) with an index of 0 is equal to 1.	$= 1$
b **1** Write the expression.	**b** $-4(a^2b^5)^0$
2 The entire term inside the brackets has an index of 0, so the bracket is equal to 1.	$= -4 \times 1$
3 Simplify.	$= -4$

Fourth Index Law: When a power (a^m) is raised to a power, the indices are multiplied.

$$(a^m)^n = a^{mn}$$

Fifth Index Law: When the base is a product, raise every part of the product to the index outside the brackets.

$$(ab)^m = a^m b^m$$

Sixth Index Law: When the base is a fraction, multiply the indices of both the numerator and denominator by the index outside the brackets.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

WORKED EXAMPLE 3

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Simplify each of the following.

a $(2n^4)^3$ **b** $(3a^2b^7)^3$ **c** $\left(\frac{2x^3}{y^4}\right)^4$ **d** $(-4)^3$

THINK	WRITE
a **1** Write the term.	**a** $(2n^4)^3$
2 Apply the Fourth Index Law and simplify.	$= 2^{1 \times 3} \times n^{4 \times 3}$ $= 2^3n^{12}$ $= 8n^{12}$
b **1** Write the expression.	**b** $(3a^2b^7)^3$
2 Apply the Fifth Index Law and simplify.	$= 3^{1 \times 3} \times a^{2 \times 3} \times b^{7 \times 3}$ $= 3^3a^6b^{21}$ $= 27a^6b^{21}$
c **1** Write the expression.	**c** $\left(\frac{2x^3}{y^4}\right)^4$
2 Apply the Sixth Index Law and simplify.	$= \frac{2^{1 \times 4} \times x^{3 \times 4}}{y^{4 \times 4}}$ $= \frac{16x^{12}}{y^{16}}$

d	**1** Write the expression.	**d** $(-4)^3$
	2 Write in expanded form.	$= -4 \times -4 \times -4$
	3 Simplify, taking careful note of the negative sign.	$= -64$

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Exercise 1.2 Review of index laws

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Questions: 1a–f, 2a–f, 3a–f, 4a–f, 6, 7a–f, 9, 10	Questions: 1d–i, 2d–i, 3a–f, 4e–l, 6, 7a–f, 9–11, 15	Questions: 1d–l, 2d–l, 3, 4d–o, 5, 6, 7d–i, 8–16

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Fluency

1. **WE1a, b** Simplify each of the following.

a. $a^3 \times a^4$
b. $a^2 \times a^3 \times a$
c. $b \times b^5 \times b^2$
d. $ab^2 \times a^3b^5$
e. $m^2n^6 \times m^3n^7$
f. $a^2b^5c \times a^3b^2c^2$
g. $mnp \times m^5n^3p^4$
h. $2a \times 3ab$
i. $4a^2b^3 \times 5a^2b \times \frac{1}{2}b^5$
j. $3m^3 \times 2mn^2 \times 6m^4n^5$
k. $4x^2 \times \frac{1}{2}xy^3 \times 6x^3y^3$
l. $2x^3y^2 \times 4x \times \frac{1}{2}x^4y^4$

2. **WE1c** Simplify each of the following.

a. $a^4 \div a^3$
b. $a^7 \div a^2$
c. $b^6 \div b^3$
d. $\dfrac{4a^7}{3a^3}$
e. $\dfrac{21b^6}{7b^2}$
f. $\dfrac{48m^8}{12m^3}$
g. $\dfrac{m^7n^3}{m^4n^2}$
h. $\dfrac{2x^4y^3}{4x^4y}$
i. $6x^7y \div 8x^4$
j. $7ab^5c^4 \div ab^2c^4$
k. $\dfrac{20m^5n^3p^4}{16m^3n^3p^2}$
l. $\dfrac{14x^3y^4z^2}{28x^2y^2z^2}$

3. **WE2** Simplify each of the following.

a. a^0
b. $(2b)^0$
c. $(3m^2)^0$
d. $3x^0$
e. $4b^0$
f. $-3 \times (2n)^0$
g. $4a^0 - \left(\dfrac{a}{4}\right)^0$
h. $5y^0 - 12$
i. $5x^0 - (5xy^2)^0$

4. **WE3** Simplify each of the following.

a. $(a^2)^3$
b. $(2a^5)^4$
c. $\left(\frac{m^2}{3}\right)^4$
d. $\left(\frac{2n^4}{3}\right)^2$
e. $(a^2b)^3$
f. $(3a^3b^2)^2$
g. $(2m^3n^5)^4$
h. $\left(\frac{3m^2n}{4}\right)^3$
i. $\left(\frac{a^2}{b^3}\right)^2$
j. $\left(\frac{5m^3}{n^2}\right)^4$
k. $\left(\frac{7x}{2y^5}\right)^3$
l. $\left(\frac{3a}{5b^3}\right)^4$
m. $(-3)^5$
n. $(-7)^2$
o. $(-2)^5$

5. **MC** **a.** $2m^{10}n^5$ is the simplified form of:

A. $m^5n^3 \times 2m^4n^2$
B. $\frac{6m^{10}n^4}{3n}$
C. $(2m^5n^2)^2$
D. $2n(m^5)^2 \times n^4$
E. $\left(\frac{2m^5}{n^3}\right)^2$

b. The value of $4 - (5a)^0$ is:

A. -1
B. 9
C. 1
D. 3
E. 5

6. **MC** **a.** $4a^3b \times b^4 \times 5a^2b^3$ simplifies to:

A. $9a^5b^8$
B. $20a^5b^7$
C. $20a^5b^8$
D. $9a^5b^7$
E. $21a^5b^8$

b. $\frac{15x^9 \times 3x^6}{9x^{10} \times x^4}$ simplifies to:

A. $5x^9$
B. $9x$
C. $5x^{29}$
D. $9x^9$
E. $5x$

c. $\frac{3p^7 \times 8q^9}{12p^3 \times 4q^5}$ simplifies to:

A. $2q^4$
B. $\frac{p^4q^4}{2}$
C. $\frac{q^4}{2}$
D. $\frac{p^4q^4}{24}$
E. $\frac{q^4}{24}$

d. $\frac{7a^5b^3}{5a^6b^2} \div \frac{7b^3a^2}{5b^5a^4}$ simplifies to:

A. $\frac{49a^3b}{25}$
B. $\frac{25a^3b}{49}$
C. a^3b
D. ab^3
E. $\frac{25ab^3}{49}$

Understanding

7. Evaluate each of the following.

a. $2^3 \times 2^2 \times 2$
b. $2 \times 3^2 \times 2^2$
c. $(5^2)^2$
d. $\frac{3^5 \times 4^6}{3^4 \times 4^4}$
e. $(2^3 \times 5)^2$
f. $\left(\frac{3}{5}\right)^3$
g. $\frac{4^4 \times 5^6}{4^3 \times 5^5}$
h. $(3^3 \times 2^4)^0$
i. $4(5^2 \times 3^5)^0$

8. Simplify each of the following.

a. $(x^y)^{3z}$
b. $a^b \times (p^q)^0$
c. $m^a \times n^b \times (mn)^0$
d. $\left(\frac{a^2}{b^3}\right)^x$
e. $\frac{n^3m^2}{n^pm^q}$
f. $(a^{m+n})^p$

Reasoning

9. Explain why $a^3 \times a^2 = a^5$ and not a^6.

10. Is $2x$ ever the same as x^2? Explain your reasoning using examples.

11. Explain the difference between $3x^0$ and $(3x)^0$.

12. a. In the following table, enter the values of $3a^2$ and $5a$ when $a = 0, 1, 2$ and 3.

a	0	1	2	3
$3a^2$				
$5a$				
$3a^2 + 5a$				
$3a^2 \times 5a$				

b. Enter the values of $3a^2 + 5a$ and $3a^2 \times 5a$ in the table.

c. What do you think will happen as a becomes very large?

13. Find algebraically the exact value of x if $4^{x+4} = 2^{x^2}$. Justify your answer.

14. Binary numbers (base 2 numbers) are used in computer operations. As the name implies, binary uses only two types of numbers, 0 and 1, to express all numbers. A binary number such as 101 (read one, zero, one) means $(1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = 4 + 0 + 1 = 5$ (in base 10, the base we are most familiar with).

The number 1010 (read one, zero, one, zero) means $(1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0) = 8 + 0 + 2 + 0 = 10$.

If we read the binary number from right to left, the index of 2 increases by one each time, beginning with a power of zero. Using this information, write out the numbers 1 to 10 in binary (base 2) form.

Problem solving

15. Solve for x:

a. $\dfrac{7^x \times 7^{1+2x}}{(7^x)^2} = 16\,807$

b. $2^{2x} - 5(2^x) = -4$

16. For the following:

a. calculate the correct answer

b. identify the error in the solution.

$$\begin{aligned}\left(\frac{a^2b^3c}{a^2b^2}\right)^3 \times \left(\frac{a^3b^2c^2}{a^2b^3}\right)^2 &= \left(\frac{b^3c}{b^2}\right)^3 \times \left(\frac{ab^2c^2}{b^3}\right)^2 \\ &= \left(\frac{bc}{1}\right)^3 \times \left(\frac{ac^2}{b}\right)^2 \\ &= \left(\frac{abc^3}{b}\right)^6 \\ &= \left(\frac{ac^3}{1}\right)^6 \\ &= a^6c^{18}\end{aligned}$$

Reflection

Why are these laws called index laws?

CHALLENGE 1.1

It was estimated that there were 4×10^{10} locusts in the largest swarm ever seen. If each locust can consume 2 grams of grain in a day, how long would it take the swarm to consume 1 tonne of grain?

1.3 Negative indices

1.3.1 Negative indices

- Consider the expression $\frac{a^3}{a^5}$. This expression can be simplified in two different ways.

  1. Written in expanded form: $\frac{a^3}{a^5} = \frac{a \times a \times a}{a \times a \times a \times a \times a}$

     $= \frac{1}{a \times a}$

     $= \frac{1}{a^2}$

  2. Using the Second Index Law: $\frac{a^3}{a^5} = a^{3-5}$

     $= a^{-2}$

     So, $a^{-2} = \frac{1}{a^2}$.

- In general, $\frac{1}{a^n} = \frac{a^0}{a^n}$ $(1 = a^0)$

  $= a^{0-n}$ (using the Second Index Law)

  $= a^{-n}$

  Seventh Index Law: $a^{-n} = \frac{1}{a^n}$

- The convention is that an expression should be written using positive indices and with pronumerals given in alphabetical order.

WORKED EXAMPLE 4

Express each of the following with positive indices.

a x^{-3} **b** $2m^{-4}n^2$ **c** $\frac{4}{a^{-3}}$

THINK	WRITE
a 1 Write the expression.	**a** x^{-3}
2 Apply the Seventh Index Law.	$= \frac{1}{x^3}$
b 1 Write the expression.	**b** $2m^{-4}n^2$
2 Apply the Seventh Index Law to write the expression with positive indices.	$= \frac{2n^2}{m^4}$
c 1 Write the expression and rewrite the fraction, using a division sign.	**c** $\frac{4}{a^{-3}} = 4 \div a^{-3}$
2 Apply the Seventh Index Law to write the expression with positive indices.	$= 4 \div \frac{1}{a^3}$
3 To divide the fraction, change fraction division into multiplication.	$= 4 \times \frac{a^3}{1}$ $= 4a^3$

- Part **c** from Worked example 4 demonstrates the **converse** of the Seventh Index Law $\frac{1}{a^{-n}} = a^n$.

WORKED EXAMPLE 5

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Simplify each of the following, expressing the answers with positive indices.

a $a^2b^{-3} \times a^{-5}b$ **b** $\dfrac{2x^4y^2}{3xy^5}$ **c** $\left(\dfrac{2m^3}{n^{-2}}\right)^{-2}$

THINK	WRITE
a **1** Write the expression.	**a** $a^2b^{-3} \times a^{-5}b$
2 Apply the First Index Law. Multiply terms with the same base by adding the indices.	$= a^{2+-5}b^{-3+1}$ $= a^{-3}b^{-2}$
3 Express the answer with positive indices.	$= \dfrac{1}{a^3b^2}$
b **1** Write the expression.	**b** $\dfrac{2x^4y^2}{3xy^5}$
2 Apply the Second Index Law. Divide terms with the same base by subtracting the indices.	$= \dfrac{2x^{4-1}y^{2-5}}{3}$ $= \dfrac{2x^3y^{-3}}{3}$
3 Express the answer with positive indices.	$= \dfrac{2x^3}{3y^3}$
c **1** Write the expression.	**c** $\left(\dfrac{2m^3}{n^{-2}}\right)^{-2}$
2 Apply the Sixth Index Law. Multiply the indices of both the numerator and denominator by the index outside the brackets.	$= \dfrac{2^{-2}m^{-6}}{n^4}$
3 Express all terms with positive indices.	$= \dfrac{1}{2^2m^6n^4}$
4 Simplify.	$= \dfrac{1}{4m^6n^4}$

- Numbers in index form can be easily evaluated if they are expressed with positive indices first. Consider the following example.

WORKED EXAMPLE 6

Evaluate 6×3^{-3} without using a calculator.

THINK	WRITE
1 Write the multiplication.	6×3^{-3}
2 Apply the Seventh Index Law to write 3^{-3} with a positive index.	$= 6 \times \dfrac{1}{3^3}$
3 Multiply the numerator of the fraction by the whole number.	$= \dfrac{6}{3^3}$

4 Evaluate the denominator.	$= \frac{6}{27}$
5 Cancel by dividing both the numerator and denominator by the highest common factor (3).	$= \frac{2}{9}$

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Exercise 1.3 Negative indices

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Fluency

1. **WE4** Express each of the following with positive indices.

a. x^{-5} **b.** y^{-4} **c.** $2a^{-9}$

d. $\frac{4}{5}a^{-3}$ **e.** $3x^2y^{-3}$ **f.** $2^{-2}m^{-3}n^{-4}$

g. $6a^3b^{-1}c^{-5}$ **h.** $\frac{1}{a^{-6}}$ **i.** $\frac{2}{3a^{-4}}$

j. $\frac{6a}{3b^{-2}}$ **k.** $\frac{7a^{-4}}{2b^{-3}}$ **l.** $\frac{2m^3n^{-5}}{3a^{-2}b^4}$

2. **WE5** Simplify each of the following, expressing the answers with positive indices.

a. $a^3b^{-2} \times a^{-5}b^{-1}$ **b.** $2x^{-2}y \times 3x^{-4}y^{-2}$ **c.** $3m^2n^{-5} \times m^{-2}n^{-3}$

d. $4a^3b^2 \div a^5b^7$ **e.** $2xy^6 \div 3x^2y^5$ **f.** $5x^{-2}y^3 \div 6xy^2$

g. $\frac{6m^4n}{2n^3m^6}$ **h.** $\frac{4x^2y^9}{x^7y^{-3}}$ **i.** $\frac{2m^2n^{-4}}{6m^5n^{-1}}$

j. $(2a^3m^4)^{-5}$ **k.** $4(p^7q^{-4})^{-2}$ **l.** $3(a^{-2}b^{-3})^4$

m. $\left(\frac{2p^2}{3q^3}\right)^{-3}$ **n.** $\left(\frac{a^{-4}}{2b^{-3}}\right)^2$ **o.** $\left(\frac{6a^2}{3b^{-2}}\right)^{-3}$

3. **WE6** Evaluate each of the following without using a calculator.

a. 2^{-3} **b.** 6^{-2} **c.** 3^{-4}

d. $3^{-2} \times 2^3$ **e.** $4^{-3} \times 2^2$ **f.** 5×6^{-2}

g. $\dfrac{6}{2^{-3}}$ **h.** $\dfrac{4 \times 3^{-3}}{2^{-3}}$ **i.** $\frac{1}{3} \times 5^{-2} \times 3^{4}$

j. $\dfrac{16^{0} \times 2^{4}}{8^{2} \times 2^{-4}}$ **k.** $\dfrac{5^{3} \times 25^{0}}{25^{2} \times 5^{-4}}$ **l.** $\dfrac{3^{4} \times 4^{2}}{12^{3} \times 15^{0}}$

4. Write each of these numbers as a power of 2.

a. 8 **b.** $\frac{1}{8}$ **c.** 32 **d.** $\frac{1}{64}$

5. Complete each statement by writing the correct index.

a. $125 = 5^{\cdots}$ **b.** $\frac{1}{16} = 4^{\cdots}$ **c.** $\frac{1}{7} = 7^{\cdots}$ **d.** $216 = 6^{\cdots}$

e. $0.01 = 10^{\cdots}$ **f.** $1 = 8^{\cdots}$ **g.** $64 = 4^{\cdots}$ **h.** $\frac{1}{64} = 4^{\cdots}$

i. $\frac{1}{64} = 2^{\cdots}$ **j.** $\frac{1}{64} = 8^{\cdots}$

6. Evaluate the following expressions.

a. $\left(\frac{2}{3}\right)^{-1}$ **b.** $\left(\frac{5}{4}\right)^{-1}$ **c.** $\left(3\frac{1}{2}\right)^{-1}$ **d.** $\left(\frac{1}{5}\right)^{-1}$

7. Write the following expressions with positive indices.

a. $\left(\dfrac{a}{b}\right)^{-1}$ **b.** $\left(\dfrac{a^{2}}{b^{3}}\right)^{-1}$ **c.** $\left(\dfrac{a^{-2}}{b^{-3}}\right)^{-1}$ **d.** $\left(\dfrac{m^{3}}{n^{-2}}\right)^{-1}$

8. Evaluate each of the following, using a calculator.

a. 3^{-6} **b.** 12^{-4} **c.** 7^{-5}

d. $\left(\frac{1}{2}\right)^{-8}$ **e.** $\left(\frac{3}{4}\right)^{-7}$ **f.** $(0.04)^{-5}$

9. **MC** **a.** x^{-5} is the same as:

A. $-x^{5}$ **B.** $-5x$ **C.** $5x$ **D.** $\dfrac{1}{x^{5}}$ **E.** $\dfrac{1}{x^{-5}}$

b. $\dfrac{1}{a^{-4}}$ is the same as:

A. $4a$ **B.** $-4a$ **C.** a^{4} **D.** $\dfrac{1}{a^{4}}$ **E.** $-a^{4}$

c. $\frac{1}{8}$ is the same as:

A. 2^{3} **B.** 2^{-3} **C.** 3^{2} **D.** 3^{-2} **E.** $\dfrac{1}{2^{-3}}$

10. **MC** **a.** Which of the following, when simplified, gives $\dfrac{3m^{4}}{4n^{2}}$?

A. $\dfrac{3m^{-4}n^{-2}}{4}$ **B.** $3 \times 2^{-2} \times m^{4} \times n^{-2}$ **C.** $\dfrac{3n^{-2}}{2^{-2}m^{-4}}$

D. $\dfrac{2^{2}n^{-2}}{3^{-1}m^{-4}}$ **E.** $3m^{4} \times 2^{2}n^{-2}$

b. When simplified, $3a^{-2}b^{-7} \div \frac{3}{4}a^{-4}b^{6}$ is equal to:

A. $\dfrac{4}{a^{6}b^{13}}$ **B.** $\dfrac{9b}{4a^{6}}$ **C.** $\dfrac{9a^{2}}{4b}$ **D.** $\dfrac{4a^{2}}{b^{13}}$ **E.** $\dfrac{4a^{2}}{b}$

c. When $(2x^{6}y^{-4})^{-3}$ is simplified, it is equal to:

A. $\dfrac{2x^{18}}{y^{12}}$ **B.** $\dfrac{x^{18}}{8y^{12}}$ **C.** $\dfrac{y^{12}}{8x^{18}}$ **D.** $\dfrac{8y^{12}}{x^{18}}$ **E.** $\dfrac{x^{18}}{6y^{12}}$

d. If $\left(\dfrac{2a^x}{b^y}\right)^3$ is equal to $\dfrac{8b^9}{a^6}$, then x and y (in that order) are:

A. -3 and -6 **B.** -6 and -3 **C.** -3 and 2
D. -3 and -2 **E.** -2 and -3

Understanding

11. Simplify, expressing your answer with positive indices.

a. $\dfrac{m^{-3}n^{-2}}{m^{-5}n^6}$ **b.** $\dfrac{(m^3n^{-2})^{-7}}{(m^{-5}n^3)^4}$ **c.** $\dfrac{5(a^3b^{-3})^2}{(ab^{-4})^{-1}} \div \dfrac{(5a^{-2}b)^{-1}}{(a^{-4}b)^3}$

12. Simplify, expanding any expressions in brackets.

a. $(r^3 + s^3)(r^3 - s^3)$ **b.** $(m^5 + n^5)^2$

c. $\dfrac{(x^{a+1})^b \times x^{a+b}}{x^{a(b+1)} \times x^{2b}}$ **d.** $\left(\dfrac{p^{x+1}}{p^{x-1}}\right)^{-4} \times \dfrac{p^{8(x+1)}}{(p^{2x})^4} \times \dfrac{p^2}{(p^{12x})^0}$

13. Write $\left(\dfrac{2^r \times 8^r}{2^{2r} \times 16}\right)$ in the form 2^{ar+b}.

14. Write $2^{-m} \times 3^{-m} \times 6^{2m} \times 3^{2m} \times 2^{2m}$ as a power of 6.

15. Solve for x if $4^x - 4^{x-1} = 48$.

Reasoning

16. Explain why each of these statements is false. Illustrate each answer by substituting a value for the pronumeral.

a. $5x^0 = 1$ **b.** $9x^5 \div 3x^5 = 3x$ **c.** $a^5 \div a^7 = a^2$ **d.** $2c^{-4} = \dfrac{1}{2c^4}$

Problem solving

17. Solve for x and y if $5^{x-y} = 625$ and $3^{2x} \times 3^y = 243$.

Hence, evaluate $\dfrac{35^x}{7^{-2y} \times 5^{-3y}}$.

18. Solve for n. Verify your answers.

a. $(2^n)^n \times (2^n)^3 \times 4 = 1$ **b.** $\dfrac{(3^n)^n \times (3^n)^{-3}}{81} = 1$

Reflection

Are there any index laws from Section 1.2 that do not apply to negative indices?

1.4 Fractional indices

1.4.1 Fractional indices

- Terms with fractional indices can be written as surds, using the following laws:

$$1.\ a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$\begin{aligned} 2.\ a^{\frac{m}{n}} &= \sqrt[n]{a^m} \\ &= (\sqrt[n]{a})^m \end{aligned}$$

- To understand how these laws are formed, consider the following numerical examples.

We know $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1$

and that $\begin{aligned} \sqrt{4} \times \sqrt{4} &= \sqrt{16} \\ &= 4 \end{aligned}$

It follows, then, that $4^{\frac{1}{2}} = \sqrt{4}$.

Similarly, we know that $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^1$

and that $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = \sqrt[3]{512}$

$$= 8$$

It follows, then, that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This observation can be generalised to $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Now consider:
$$\begin{aligned} a^{\frac{m}{n}} &= a^{m \times \frac{1}{n}} & \quad\text{or}\quad a^{\frac{m}{n}} &= a^{\frac{1}{n} \times m} \\ &= (a^m)^{\frac{1}{n}} & &= \left(a^{\frac{1}{n}}\right)^m \\ &= \sqrt[n]{a^m} & &= (\sqrt[n]{a})^m \end{aligned}$$

Eighth Index Law: $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

WORKED EXAMPLE 7

Evaluate each of the following without using a calculator.

a $9^{\frac{1}{2}}$

b $16^{\frac{3}{2}}$

THINK	WRITE
a 1 Rewrite the number using the Eighth Index Law.	**a** $9^{\frac{1}{2}} = \sqrt{9}$
2 Evaluate.	$= 3$
b 1 Rewrite the number using $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.	**b** $16^{\frac{3}{2}} = (\sqrt{16})^3$ $= 4^3$
2 Simplify and evaluate the result.	$= 64$

WORKED EXAMPLE 8

TI | CASIO

Simplify each of the following.

a $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$

b $(a^2b^3)^{\frac{1}{6}}$

c $\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

THINK	WRITE
a 1 Write the expression.	**a** $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$
2 Apply the First Index Law to multiply terms with the same base by adding the indices.	$= m^{\frac{3}{5}}$
b 1 Write the expression.	**b** $(a^2b^3)^{\frac{1}{6}}$
2 Use the Fourth Index Law to multiply each index inside the brackets by the index outside the brackets.	$= a^{\frac{2}{6}}b^{\frac{3}{6}}$
3 Simplify.	$= a^{\frac{1}{3}}b^{\frac{1}{2}}$

c **1** Write the expression.

c $\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

2 Use the Sixth Index Law to multiply the index in both the numerator and denominator by the index outside the brackets.

$=\dfrac{x^{\frac{1}{3}}}{y^{\frac{3}{8}}}$

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Exercise 1.4 Fractional indices

assess on

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–5, 6a, d, g, 7a, d, 8a, d, g, 9a, d, 10a, d, g, 11a, d, g, 12, 13, 14a, d, g, 15, 16	Questions: 1–5, 6a, b, e, h, i, 7a, b, c, f, 8a, b, d, e, g, h, 9a, b, d, e, 10b, e, h, 11b, e, h, 12, 13, 14b, e, h, 15, 16, 17	Questions: 1–5, 6c, f, i, 7c, f, 8c, f, i, 9b, c, e, f, 10c, f, i, 11c, f, i, 12–19

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Fluency

1. **WE7** Evaluate each of the following without using a calculator.

a. $16^{\frac{1}{2}}$ **b.** $25^{\frac{1}{2}}$ **c.** $81^{\frac{1}{2}}$ **d.** $8^{\frac{1}{3}}$ **e.** $64^{\frac{1}{3}}$ **f.** $81^{\frac{1}{4}}$

2. Write the following in surd form.

a. $15^{\frac{1}{2}}$ **b.** $m^{\frac{1}{4}}$ **c.** $7^{\frac{2}{5}}$ **d.** $7^{\frac{5}{2}}$

e. $w^{\frac{3}{8}}$ **f.** $w^{1.25}$ **g.** $5^{3\frac{1}{3}}$ **h.** $a^{0.3}$

3. Write the following in index form.

a. $\sqrt{t}$ **b.** $\sqrt[4]{5^7}$ **c.** $\sqrt[6]{6^{11}}$ **d.** $\sqrt[7]{x^6}$

e. $\sqrt[6]{x^7}$ **f.** $\sqrt[5]{w^{10}}$ **g.** $\sqrt[10]{w^5}$ **h.** $\sqrt{11^n}$

4. Without using a calculator, find the exact value of each of the following.

a. $8^{\frac{2}{3}}$ **b.** $8^{\frac{4}{3}}$ **c.** $32^{\frac{3}{5}}$ **d.** $32^{\frac{4}{5}}$

e. $25^{\frac{3}{2}}$ **f.** $27^{\frac{2}{3}}$ **g.** $27^{\frac{-2}{3}}$ **h.** $81^{\frac{3}{4}}$

i. $10^{\frac{6}{2}}$ **j.** $36^{1\frac{1}{2}}$ **k.** $7^{\frac{1}{2}}$ **l.** $12^{\frac{1}{3}}$

5. Using a calculator, evaluate each of the following. Give the answer correct to 2 decimal places.

a. $3^{\frac{1}{3}}$
b. $5^{\frac{1}{2}}$
c. $7^{\frac{1}{5}}$
d. $8^{\frac{1}{9}}$
e. $12^{\frac{3}{8}}$
f. $(0.6)^{\frac{4}{5}}$
g. $\left(\frac{2}{3}\right)^{\frac{3}{2}}$
h. $\left(\frac{3}{4}\right)^{\frac{3}{4}}$
i. $\left(\frac{4}{5}\right)^{\frac{2}{3}}$

6. **WE8a** Simplify each of the following.

a. $4^{\frac{3}{5}} \times 4^{\frac{1}{5}}$
b. $2^{\frac{1}{8}} \times 2^{\frac{3}{8}}$
c. $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$
d. $x^{\frac{3}{4}} \times x^{\frac{2}{5}}$
e. $5m^{\frac{1}{3}} \times 2m^{\frac{1}{5}}$
f. $\frac{1}{2}b^{\frac{3}{7}} \times 4b^{\frac{2}{7}}$
g. $-4y^2 \times y^{\frac{2}{9}}$
h. $\frac{2}{5}a^{\frac{3}{8}} \times 0.05a^{\frac{3}{4}}$
i. $5x^3 \times x^{\frac{1}{2}}$

7. Simplify each of the following.

a. $a^{\frac{2}{3}}b^{\frac{3}{4}} \times a^{\frac{1}{3}}b^{\frac{3}{4}}$
b. $x^{\frac{3}{5}}y^{\frac{2}{9}} \times x^{\frac{1}{5}}y^{\frac{1}{3}}$
c. $2ab^{\frac{1}{3}} \times 3a^{\frac{3}{5}}b^{\frac{4}{5}}$
d. $6m^{\frac{3}{7}} \times \frac{1}{3}m^{\frac{1}{4}}n^{\frac{2}{5}}$
e. $x^3y^{\frac{1}{2}}z^{\frac{1}{3}} \times x^{\frac{1}{6}}y^{\frac{1}{3}}z^{\frac{1}{2}}$
f. $2a^{\frac{2}{5}}b^{\frac{3}{8}}c^{\frac{1}{4}} \times 4b^{\frac{3}{4}}c^{\frac{3}{4}}$

8. Simplify each of the following.

a. $3^{\frac{1}{2}} \div 3^{\frac{1}{3}}$
b. $5^{\frac{2}{3}} \div 5^{\frac{1}{4}}$
c. $12^2 \div 12^{\frac{3}{2}}$
d. $a^{\frac{6}{7}} \div a^{\frac{3}{7}}$
e. $x^{\frac{3}{2}} \div x^{\frac{1}{4}}$
f. $\dfrac{m^{\frac{4}{5}}}{m^{\frac{5}{9}}}$
g. $\dfrac{2x^{\frac{3}{4}}}{4x^{\frac{3}{5}}}$
h. $\dfrac{7n^2}{21n^{\frac{4}{3}}}$
i. $\dfrac{25b^{\frac{3}{5}}}{20b^{\frac{1}{4}}}$

9. Simplify each of the following.

a. $x^3y^2 \div x^{\frac{4}{3}}y^{\frac{3}{5}}$
b. $a^{\frac{5}{9}}b^{\frac{2}{3}} \div a^{\frac{2}{5}}b^{\frac{2}{5}}$
c. $m^{\frac{3}{8}}n^{\frac{4}{7}} \div 3n^{\frac{3}{8}}$
d. $10x^{\frac{4}{5}}y \div 5x^{\frac{2}{3}}y^{\frac{1}{4}}$
e. $\dfrac{5a^{\frac{3}{4}}b^{\frac{3}{5}}}{20a^{\frac{1}{5}}b^{\frac{1}{4}}}$
f. $\dfrac{p^{\frac{7}{8}}q^{\frac{1}{4}}}{7p^{\frac{2}{3}}q^{\frac{1}{6}}}$

10. Simplify each of the following.

a. $\left(2^{\frac{3}{4}}\right)^{\frac{3}{5}}$
b. $\left(5^{\frac{2}{3}}\right)^{\frac{1}{4}}$
c. $\left(7^{\frac{1}{5}}\right)^6$
d. $(a^3)^{\frac{1}{10}}$
e. $\left(m^{\frac{4}{9}}\right)^{\frac{3}{8}}$
f. $\left(2b^{\frac{1}{2}}\right)^{\frac{1}{3}}$
g. $4\left(p^{\frac{3}{7}}\right)^{\frac{14}{15}}$
h. $\left(x^{\frac{m}{n}}\right)^{\frac{n}{p}}$
i. $\left(3m^{\frac{a}{b}}\right)^{\frac{b}{c}}$

11. **WE8b, c** Simplify each of the following.

a. $\left(a^{\frac{1}{2}}b^{\frac{1}{3}}\right)^{\frac{1}{2}}$
b. $(a^4b)^{\frac{3}{4}}$
c. $\left(x^{\frac{3}{5}}y^{\frac{7}{8}}\right)^2$
d. $\left(3a^{\frac{1}{3}}b^{\frac{3}{5}}c^{\frac{3}{4}}\right)^{\frac{1}{3}}$
e. $5\left(x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{2}{5}}\right)^{\frac{1}{2}}$
f. $\left(\dfrac{a^{\frac{3}{4}}}{b}\right)^{\frac{2}{3}}$
g. $\left(\dfrac{m^{\frac{4}{5}}}{n^{\frac{7}{8}}}\right)^2$
h. $\left(\dfrac{b^{\frac{3}{5}}}{c^{\frac{4}{9}}}\right)^{\frac{2}{3}}$
i. $\left(\dfrac{4x^7}{2y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

12. **MC** **a.** $y^{\frac{2}{5}}$ is equal to:

A. $(y^{\frac{1}{2}})^5$ **B.** $y \times \frac{2}{5}$ **C.** $(y^5)^{\frac{1}{2}}$ **D.** $2\sqrt[5]{y}$ **E.** $(y^{\frac{1}{5}})^2$

b. $k^{\frac{2}{3}}$ is not equal to:

A. $(k^{\frac{1}{3}})^2$ **B.** $\sqrt[3]{k^2}$ **C.** $(k^{\frac{1}{2}})^3$ **D.** $(\sqrt[3]{k})^2$ **E.** $(k^2)^{\frac{1}{3}}$

c. $\dfrac{1}{\sqrt[5]{g^2}}$ is equal to:

A. $g^{\frac{2}{5}}$ **B.** $g^{-\frac{2}{5}}$ **C.** $g^{\frac{5}{2}}$ **D.** $g^{-\frac{5}{2}}$ **E.** $2g^{\frac{1}{5}}$

13. **MC** **a.** If $(a^{\frac{3}{4}})^{\frac{m}{n}}$ is equal to $a^{\frac{1}{4}}$, then m and n could not be:

A. 1 and 3 **B.** 2 and 6 **C.** 3 and 8

D. 4 and 9 **E.** both **C** and **D**

b. When simplified, $\left(\dfrac{a^{\frac{m}{n}}}{b^{\frac{n}{p}}}\right)^{\frac{p}{m}}$ is equal to:

A. $\dfrac{a^{\frac{m}{p}}}{b^{\frac{n}{m}}}$ **B.** $\dfrac{a^{\frac{p}{n}}}{b^{\frac{n}{m}}}$ **C.** $\dfrac{a^{\frac{mp}{n}}}{b^{\frac{n}{m}}}$ **D.** $\dfrac{a^p}{b^m}$ **E.** $\dfrac{a^{\frac{m^2}{np}}}{b^{\frac{nm}{p^2}}}$

14. Simplify each of the following.

a. $\sqrt{a^8}$ **b.** $\sqrt[3]{b^9}$ **c.** $\sqrt[4]{m^{16}}$

d. $\sqrt{16x^4}$ **e.** $\sqrt[3]{8y^9}$ **f.** $\sqrt[4]{16x^8y^{12}}$

g. $\sqrt[3]{27m^9n^{15}}$ **h.** $\sqrt[5]{32p^5q^{10}}$ **i.** $\sqrt[3]{216a^6b^{18}}$

Understanding

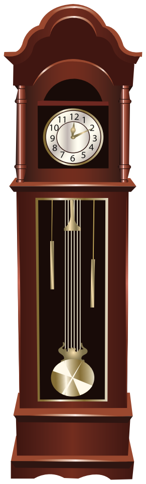
15. The relationship between the length of a pendulum (L) in a grandfather clock and the time it takes to complete one swing (T) in seconds is given by the following rule. Note that g is the acceleration due to gravity and will be taken as 9.8.

$$T = 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}}$$

a. Calculate the time it takes a 1 m long pendulum to complete one swing.

b. Calculate the time it takes the pendulum to complete 10 swings.

c. How many swings will be completed after 10 seconds?



Reasoning

16. Using the index laws, show that $\sqrt[5]{32a^5b^{10}} = 2ab^2$.

17. To rationalise a fraction means to remove all non-rational numbers from the denominator of the fraction. Rationalise $\dfrac{a^2}{3+\sqrt{b^3}}$ by multiplying the numerator and denominator by $3-\sqrt{b^3}$, and then evaluate if $b = a^2$ and $a = 2$. Show all of your working.

Problem solving

18. Simplify $\dfrac{m^{\frac{2}{5}} - 2m^{\frac{1}{5}}n^{\frac{1}{5}} + n^{\frac{2}{5}} - p^{\frac{2}{5}}}{m^{\frac{1}{5}} - n^{\frac{1}{5}} - p^{\frac{1}{5}}}$.

19. A scientist has discovered a piece of paper with a complex formula written on it. She thinks that someone has tried to disguise a simpler formula. The formula is:

$$\frac{\sqrt[4]{a^{13}}a^2\sqrt{b^3}}{\sqrt{a^1b}} \times b^3 \times \left(\frac{\sqrt{a^3}b}{ab^2}\right)^2 \times \left(\frac{b^2}{a^2\sqrt{b}}\right)^3$$

a. Simplify the formula using index laws so that it can be worked with.
b. From your simplified formula, can a take a negative value? Explain.
c. What is the smallest value for a for which the expression will give a rational answer? Consider only integers.

Reflection

Why is it easier to perform operations with fractional indices than with expressions using surds?

1.5 Combining index laws

1.5.1 Combining index laws

- When several steps are needed to simplify an expression, expand brackets first.
- When fractions are involved, it is usually easier to carry out all multiplications first, leaving one division as the final process.
- Final answers are conventionally written using positive indices.

WORKED EXAMPLE 9

Simplify each of the following.

a $\frac{(2a)^4b^4}{6a^3b^2}$ **b** $\frac{3^{n-2} \times 9^{n+1}}{81^{n-1}}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{(2a)^4b^4}{6a^3b^2}$
2 Apply the Fourth Index Law to remove the bracket.	$= \frac{16a^4b^4}{6a^3b^2}$
3 Apply the Second Index Law for each number and pronumeral to simplify.	$= \frac{8a^{4-3}b^{4-2}}{3}$
4 Write the answer.	$= \frac{8ab^2}{3}$
b 1 Write the expression.	**b** $\frac{3^{n-2} \times 9^{n+1}}{81^{n-1}}$
2 Rewrite each term in the expression so that it has a base of 3.	$= \frac{3^{n-2} \times (3^2)^{n+1}}{(3^4)^{n-1}}$
3 Apply the Fourth Index Law to expand the brackets.	$= \frac{3^{n-2} \times 3^{2n+2}}{3^{4n-4}}$
4 Apply the First and Second Index Laws to simplify and write your answer.	$= \frac{3^{3n}}{3^{4n-4}}$ $= \frac{1}{3^{n-4}}$

WORKED EXAMPLE 10

Simplify each of the following.

a $(2a^3b)^4 \times 4a^2b^3$ **b** $\dfrac{7xy^3}{(3x^3y^2)^2}$ **c** $\dfrac{2m^5n \times 3m^7n^4}{7m^3n^3 \times mn^2}$

THINK	WRITE
a **1** Write the expression.	**a** $(2a^3b)^4 \times 4a^2b^3$
2 Apply the Fourth Index Law. Multiply each index inside the brackets by the index outside the brackets.	$= 2^4a^{12}b^4 \times 4a^2b^3$
3 Evaluate the number.	$= 16a^{12}b^4 \times 4a^2b^3$
4 Multiply coefficients and multiply pronumerals. Apply the First Index Law to multiply terms with the same base by adding the indices.	$= 16 \times 4 \times a^{12+2}b^{4+3}$ $= 64a^{14}b^7$
b **1** Write the expression.	**b** $\dfrac{7xy^3}{(3x^3y^2)^2}$
2 Apply the Fourth Index Law in the denominator. Multiply each index inside the brackets by the index outside the brackets.	$= \dfrac{7xy^3}{9x^6y^4}$
3 Apply the Second Index Law. Divide terms with the same base by subtracting the indices.	$= \dfrac{7x^{-5}y^{-1}}{9}$
4 Use $a^{-m} = \dfrac{1}{a^m}$ to express the answer with positive indices.	$= \dfrac{7}{9x^5y}$
c **1** Write the expression.	**c** $\dfrac{2m^5n \times 3m^7n^4}{7m^3n^3 \times mn^2}$
2 Simplify each numerator and denominator by multiplying coefficients and then terms with the same base.	$= \dfrac{6m^{12}n^5}{7m^4n^5}$
3 Apply the Second Index Law. Divide terms with the same base by subtracting the indices.	$= \dfrac{6m^8n^0}{7}$
4 Simplify the numerator using $a^0 = 1$.	$= \dfrac{6m^8 \times 1}{7}$ $= \dfrac{6m^8}{7}$

WORKED EXAMPLE 11

TI | CASIO

Simplify each of the following.

a $\dfrac{(5a^2b^3)^2}{a^{10}} \times \dfrac{a^2b^5}{(a^3b)^7}$ **b** $\dfrac{8m^3n^{-4}}{(6mn^2)^3} \div \dfrac{4m^{-2}n^{-4}}{6m^{-5}n}$

THINK	WRITE
a **1** Write the expression.	**a** $\dfrac{(5a^2b^3)^2}{a^{10}} \times \dfrac{a^2b^5}{(a^3b)^7}$

	2 Remove the brackets in the numerator of the first fraction and in the denominator of the second fraction.	$= \dfrac{25a^4b^6}{a^{10}} \times \dfrac{a^2b^5}{a^{21}b^7}$
	3 Multiply the numerators and then multiply the denominators of the fractions. (Simplify across.)	$= \dfrac{25a^6b^{11}}{a^{31}b^7}$
	4 Divide terms with the same base by subtracting the indices. (Simplify down.)	$= 25a^{-25}b^4$
	5 Express the answer with positive indices.	$= \dfrac{25b^4}{a^{25}}$
b	**1** Write the expression.	**b** $\dfrac{8m^3n^{-4}}{(6mn^2)^3} \div \dfrac{4m^{-2}n^{-4}}{6m^{-5}n}$
	2 Remove the brackets.	$= \dfrac{8m^3n^{-4}}{216m^3n^6} \div \dfrac{4m^{-2}n^{-4}}{6m^{-5}n}$
	3 Change the division to multiplication.	$= \dfrac{8m^3n^{-4}}{216m^3n^6} \times \dfrac{6m^{-5}n}{4m^{-2}n^{-4}}$
	4 Multiply the numerators and then multiply the denominators. (Simplify across.)	$= \dfrac{48m^{-2}n^{-3}}{864mn^2}$
	5 Cancel common factors and divide pronumerals with the same base. (Simplify down.)	$= \dfrac{m^{-3}n^{-5}}{18}$
	6 Simplify and express the answer with positive indices.	$= \dfrac{1}{18m^3n^5}$

Note that the whole numbers in part **b** of Worked example 11 could be cancelled in step 3.

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Exercise 1.5 Combining index laws

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Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2a–d, 3a–d, 4a–d, 5a–d, 6, 7, 9, 10, 11a, d, 12	**Questions:** 1c–h, 2c–f, 3c–g, 4b–f, 5c–f, 6–10, 11b–e, 12	**Questions:** 1f–j, 2e–i, 3f–i, 4d–f, 5e–h, 6–10, 11c–f, 12–15

🟩🟦🟪 **Individual pathway interactivity:** int-4565 learnON ONLINE ONLY

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Fluency

1. **WE10a** Simplify each of the following.

a. $(3a^2b^2)^3 \times 2a^4b^3$ **b.** $(4ab^5)^2 \times 3a^3b^6$ **c.** $2m^3n^{-5} \times (m^2n^{-3})^{-6}$

d. $(2pq^3)^2 \times (5p^2q^4)^3$ **e.** $(2a^7b^2)^2 \times (3a^3b^3)^2$ **f.** $5(b^2c^{-2})^3 \times 3(bc^5)^{-4}$

g. $6x^{\frac{1}{2}}y^{\frac{1}{3}} \times (4x^{\frac{3}{4}}y^{\frac{4}{5}})^{\frac{1}{2}}$ **h.** $(16m^3n^4)^{\frac{3}{4}} \times (m^{\frac{1}{2}}n^{\frac{1}{4}})^3$ **i.** $2(p^{\frac{2}{3}}q^{\frac{1}{3}})^{-\frac{3}{4}} \times 3(p^{\frac{1}{4}}q^{-\frac{3}{4}})^{-\frac{1}{3}}$

j. $(8p^{\frac{1}{5}}q^{\frac{2}{3}})^{-\frac{1}{3}} \times (64p^{\frac{1}{3}}q^{\frac{3}{4}})^{\frac{2}{3}}$

2. **WE10b** Simplify each of the following.

a. $\dfrac{5a^2b^3}{(2a^3b)^3}$ **b.** $\dfrac{4x^5y^6}{(2xy^3)^4}$ **c.** $\dfrac{(3m^2n^3)^3}{(2m^5n^5)^7}$

d. $\left(\dfrac{4x^3y^{10}}{2x^7y^4}\right)^6$ **e.** $\dfrac{3a^3b^{-5}}{(2a^7b^4)^{-3}}$ **f.** $\left(\dfrac{3g^2h^5}{2g^4h}\right)^3$

g. $\dfrac{(5p^6q^{\frac{1}{3}})^2}{25(p^{\frac{1}{2}}q^{\frac{1}{4}})^{\frac{2}{3}}}$ **h.** $\left(\dfrac{3b^2c^3}{5b^{-3}c^{-4}}\right)^{-4}$ **i.** $\dfrac{(x^{\frac{1}{2}}y^{\frac{1}{4}}z^{\frac{1}{2}})^2}{(x^{\frac{2}{3}}y^{-\frac{1}{4}}z^{\frac{1}{3}})^{-\frac{3}{2}}}$

3. **WE10c** Simplify each of the following.

a. $\dfrac{2a^2b \times 3a^3b^4}{4a^3b^5}$ **b.** $\dfrac{4m^6n^3 \times 12mn^5}{6m^7n^6}$ **c.** $\dfrac{10m^6n^5 \times 2m^2n^3}{12m^4n \times 5m^2n^3}$

d. $\dfrac{6x^3y^2 \times 4x^6y}{9xy^5 \times 2x^3y^6}$ **e.** $\dfrac{(6x^3y^2)^4}{9x^5y^2 \times 4xy^7}$ **f.** $\dfrac{5x^2y^3 \times 2xy^5}{10x^3y^4 \times x^4y^2}$

g. $\dfrac{a^3b^2 \times 2(ab^5)^3}{6(a^2b^3)^3 \times a^4b}$ **h.** $\dfrac{(p^6q^2)^{-3} \times 3pq}{2p^{-4}q^{-2} \times (5pq^4)^{-2}}$ **i.** $\dfrac{6x^{\frac{3}{2}}y^{\frac{1}{2}} \times x^{\frac{4}{5}}y^{\frac{3}{5}}}{2(x^2y)^{\frac{1}{5}} \times 3x^{\frac{1}{2}}y^{\frac{1}{5}}}$

4. **WE11a** Simplify each of the following.

a. $\dfrac{a^3b^2}{5a^4b^7} \times \dfrac{2a^6b}{a^9b^3}$ **b.** $\dfrac{(2a^6)^2}{10a^7b^3} \times \dfrac{4ab^6}{6a^3}$ **c.** $\dfrac{(m^4n^3)^2}{(m^6n)^4} \times \dfrac{(m^3n^3)^3}{(2mn)^2}$

d. $\left(\dfrac{2m^3n^2}{3mn^5}\right)^3 \times \dfrac{6m^2n^4}{4m^3n^{10}}$ **e.** $\left(\dfrac{2xy^2}{3x^3y^5}\right)^4 \times \left(\dfrac{x^3y^9}{2y^{10}}\right)^2$ **f.** $\dfrac{4x^{-5}y^{-3}}{(x^2y^2)^{-2}} \times \dfrac{3x^5y^6}{2^{-2}x^{-7}y}$

g. $\dfrac{5p^6q^{-5}}{3q^{-4}} \times \left(\dfrac{5p^6q^4}{3p^5}\right)^{-2}$ **h.** $\dfrac{2a^{\frac{1}{2}}b^{\frac{1}{3}}}{6a^{\frac{1}{3}}b^{\frac{1}{2}}} \times \dfrac{(4a^{\frac{1}{4}}b)^{\frac{1}{2}}}{b^{\frac{1}{4}}a}$ **i.** $\dfrac{3x^{\frac{2}{3}}y^{\frac{1}{5}}}{9x^{\frac{1}{3}}y^{\frac{1}{4}}} \times \dfrac{4x^{\frac{1}{2}}}{x^{\frac{3}{4}}y}$

5. **WE11b** Simplify each of the following.

a. $\dfrac{5a^2b^3}{6a^7b^5} \div \dfrac{a^9b^4}{3ab^6}$ **b.** $\dfrac{7a^2b^4}{3a^6b^7} \div \left(\dfrac{3ab}{2a^6b^4}\right)^3$ **c.** $\left(\dfrac{4a^9}{b^6}\right)^3 \div \left(\dfrac{3a^7}{2b^5}\right)^4$

d. $\dfrac{5x^2y^6}{(2x^4y^5)^2} \div \dfrac{(4x^6y)^3}{10xy^3}$ **e.** $\left(\dfrac{x^5y^{-3}}{2xy^5}\right)^{-4} \div \dfrac{4x^6y^{-10}}{(3x^{-2}y^2)^{-3}}$ **f.** $\dfrac{3m^3n^4}{2m^{-6}n^{-5}} \div \left(\dfrac{2m^4n^6}{m^{-1}n}\right)^{-2}$

g. $4m^{\frac{1}{2}}n^{\frac{3}{4}} \div \dfrac{6m^{\frac{1}{3}}n^{\frac{1}{4}}}{8m^{\frac{3}{4}}n^{\frac{1}{2}}}$ **h.** $\left(\dfrac{4b^3c^{\frac{1}{3}}}{6c^{\frac{1}{5}}b}\right)^{\frac{1}{2}} \div (2b^3c^{-\frac{1}{5}})^{-\frac{3}{2}}$

Understanding

6. Evaluate each of the following.

a. $(5^2 \times 2)^0 \times (5^{-3} \times 2^0)^5 \div (5^6 \times 2^{-1})^{-3}$

b. $(2^3 \times 3^3)^{-2} \div \dfrac{(2^6 \times 3^9)^0}{2^6 \times (3^{-2})^{-3}}$

7. Evaluate the following for $x = 8$. (*Hint:* Simplify first.)

$$(2x)^{-3} \times \left(\frac{x}{2}\right)^2 \div \frac{2x}{(2^3)^4}$$

8. a. Simplify the following fraction. $\dfrac{a^{2y} \times 9b^y \times (5ab)^y}{(a^y)^3 \times 5(3b^y)^2}$

b. Find the value of y if the fraction is equal to 125.

9. **MC** Which of the following is not the same as $(4xy)^{\frac{3}{2}}$?

A. $8x^{\frac{3}{2}}y^{\frac{3}{2}}$ **B.** $(\sqrt{4xy})^3$ **C.** $\sqrt{64x^3y^3}$

D. $\dfrac{(2x^3y^3)^{\frac{1}{2}}}{(\sqrt{32})^{-1}}$ **E.** $4xy^{\frac{1}{2}} \times (2xy^2)^{\frac{1}{2}}$

10. **MC** The expression $\dfrac{x^2y}{(2xy^2)^3} \div \dfrac{xy}{16x^0}$ is equal to:

A. $\dfrac{2}{x^2y^6}$ **B.** $\dfrac{2x^2}{b^6}$ **C.** $2x^2y^6$ **D.** $\dfrac{2}{xy^6}$ **E.** $\dfrac{1}{128xy^5}$

11. Simplify the following.

a. $\sqrt[3]{m^2n} \div \sqrt{mn^3}$ **b.** $(g^{-2}h)^3 \times \left(\dfrac{1}{n^{-3}}\right)^{\frac{1}{2}}$ **c.** $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$

d. $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$ **e.** $\left(\dfrac{a^3b^{-2}}{3^{-3}b^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}a^{-2}b}{a^4b^{-2}}\right)^2$ **f.** $(\sqrt[5]{d^2})^{\frac{3}{2}} \times (\sqrt[3]{d^5})^{\frac{1}{5}}$

Reasoning

12. In a controlled breeding program at the Melbourne Zoo, the population (P) of koalas at t years is modelled by $P = P_0 \times 10^{kt}$. Given $P_0 = 20$ and $k = 0.3$:

a. calculate the number of koalas after 2 years

b. determine when the population will be equal to 1000.

13. The decay of uranium is modelled by $D = D_0 \times 2^{-kt}$. If it takes 6 years for the mass of uranium to halve, find the percentage remaining after:

a. 2 years

b. 5 years

c. 10 years.

Give your answers to the nearest whole number.

Problem solving

14. Simplify $\dfrac{7^{2x+1} - 7^{2x-1} - 48}{36 \times 7^{2x} - 252}$.

15. Simplify $\dfrac{z^4 + z^{-4} - 3}{z^2 + z^{-2} - 5^{\frac{1}{2}}}$.

Reflection

Do index laws need to be performed in a certain order?

CHALLENGE 1.2

Find an expression for x in terms of y, given that $(\sqrt{a^y})^x = a^x \times a^y$.

1.6 Review

1.6.1 Review questions

Fluency

1. $3d^{10}e^4$ is the simplified form of:

a. $d^6e^2 \times 3d^4e^3$
b. $\dfrac{6d^{10}e^5}{2e^2}$
c. $(3d^5e^2)^2$
d. $3e\,(d^5)^2 \times e^3$
e. $3\left(\dfrac{d^5}{e^2}\right)^2$

2. $8m^3n \times n^4 \times 2m^2n^3$ simplifies to:

a. $10m^5n^8$
b. $16m^5n^7$
c. $16m^5n^8$
d. $10m^5n^7$
e. $17m^5n^8$

3. $8x^3 \div 4x^{-3}$ is equal to:

a. 2
b. $2x^0$
c. $2x^6$
d. $2x^{-1}$
e. $\dfrac{2}{x^9}$

4. $\dfrac{12x^8 \times 2x^7}{6x^9 \times x^5}$ simplifies to:

a. $4x^5$
b. $8x$
c. $4x$
d. $8x^5$
e. $4x^{29}$

5. The expression $\dfrac{(a^2b^3)^5}{(2a^2b)^2}$ is equal to:

a. $\dfrac{a^6b^{13}}{4}$
b. $2a^6b^{13}$
c. $\dfrac{a^3b^6}{2}$
d. $\dfrac{a^6b^{13}}{2}$
e. $\dfrac{a^3b^6}{4}$

6. $\dfrac{(p^2q)^4}{(2p^5q^2)^3} \div \dfrac{(p^5q^2)^2}{2pq^5}$ can be simplified to:

a. $\dfrac{1}{4p^{16}q}$
b. $\dfrac{2^2}{p^{16}q}$
c. $\dfrac{1}{4p^8}$
d. $\dfrac{1}{2p^{16}q}$
e. $2^2p^{16}q$

7. $16^{-\frac{3}{4}} \div 9^{\frac{3}{2}}$ can be simplified to:

a. 2
b. $\dfrac{1}{216}$
c. $\dfrac{8}{27}$
d. $3\dfrac{3}{8}$
e. $\dfrac{1}{2}$

8. $\dfrac{\left(2l^{\frac{2}{9}}m^{-1}\right)^{-3}}{8\left(\frac{1}{16}lm^{-2}\right)^2}$ can be simplified to:

a. $\dfrac{8m^7}{l^{\frac{11}{3}}}$
b. $\dfrac{2m^7}{l^{\frac{7}{3}}}$
c. $\dfrac{4m^7}{l^{\frac{8}{3}}}$
d. $\dfrac{16m^7}{l^{\frac{5}{3}}}$
e. $\dfrac{m^7}{2l^{\frac{7}{3}}}$

9. $\sqrt[5]{32i^{\frac{10}{7}}j^{\frac{5}{11}}k^2}$ can be simplified to:

a. $\dfrac{32i^{\frac{2}{7}}j^{\frac{1}{11}}k^{\frac{2}{5}}}{5}$
b. $2i^{\frac{2}{7}}j^{\frac{1}{11}}k^{\frac{2}{5}}$
c. $\dfrac{32i^{\frac{10}{7}}j^{\frac{5}{11}}k^2}{5}$
d. $2i^{\frac{50}{7}}j^{\frac{25}{11}}k^{10}$
e. $\dfrac{2i^{\frac{2}{7}}j^{\frac{1}{11}}k^{\frac{2}{5}}}{5}$

10. Simplify each of the following.

a. $5x^3 \times 3x^5y^4 \times \frac{3}{5}x^2y^6$ **b.** $\frac{26a^4b^6c^5}{12a^3b^3c^3}$ **c.** $\left(\frac{20m^5n^2}{6}\right)^3$ **d.** $\left(\frac{14p^7}{21q^3}\right)^4$

11. Evaluate each of the following.

a. $5a^0 - \left(\frac{2a}{3}\right)^0 + 12$ **b.** $-(3b)^0 - \frac{(4b)^0}{2}$

12. Simplify each of the following and express your answer with positive indices.

a. $2a^{-5}b^2 \times 4a^{-6}b^{-4}$ **b.** $4x^{-5}y^{-3} \div 20x^{12}y^{-5}$ **c.** $(2m^{-3}n^2)^{-4}$

13. Evaluate each of the following without using a calculator.

a. $\left(\frac{1}{2}\right)^{-3}$ **b.** $2 \times (3)^{-3} \times \left(\frac{9}{2}\right)^2$ **c.** $4^{-3} \times \frac{5}{8^{-2}} - 5$

14. Simplify each of the following.

a. $2a^{\frac{4}{5}}b^{\frac{1}{2}} \times 3a^{\frac{1}{2}}b^{\frac{3}{4}} \times 5a^{\frac{3}{4}}b^{\frac{2}{5}}$ **b.** $\frac{4^3x^{\frac{3}{4}}y^{\frac{1}{9}}}{16xy}$ **c.** $\left(\frac{4a^{\frac{1}{3}}}{b^3}\right)^{\frac{1}{2}}$

15. Evaluate each of the following without using a calculator. Show all working.

a. $\frac{16^{\frac{3}{4}} \times 81^{\frac{1}{4}}}{6 \times 16^{\frac{1}{2}}}$ **b.** $\left(125^{\frac{2}{3}} - 27^{\frac{2}{3}}\right)^{\frac{1}{2}}$

16. Simplify:

a. $\sqrt[3]{a^9} + \sqrt[4]{16a^8b^2} - 3(\sqrt[5]{a})^{15}$ **b.** $\sqrt[5]{32x^5y^{10}} + \sqrt[3]{64x^3y^6}$

17. Simplify each of the following.

a. $\frac{(5a^{-2}b)^{-3} \times 4a^6b^{-2}}{2a^2b^3 \times 5^{-2}a^{-3}b^{-6}}$ **b.** $\frac{2x^4y^{-5}}{3y^6x^{-2}} \times \left(\frac{4xy^{-2}}{3x^{-6}y^3}\right)^{-3}$ **c.** $\left(\frac{2m^3n^4}{5m^{\frac{1}{2}}n}\right)^{\frac{1}{3}} \div \left(\frac{4m^{\frac{1}{3}}n^{-2}}{5^{-\frac{2}{3}}}\right)^{-\frac{1}{2}}$

18. Simplify each of the following and then evaluate.

a. $(3 \times 5^6)^{\frac{1}{2}} \times 3^{\frac{3}{2}} \times 5^{-2} + (3^6 \times 5^{-\frac{1}{2}})^0$ **b.** $(6 \times 3^{-2})^{-1} \div \frac{(3^{\frac{1}{2}} \times 6^{\frac{1}{3}})^6}{-6^2 \times (3^{-3})^0}$

Problem solving

19. If $m = 2$, determine the value of:

$$\frac{6a^{3m} \times 2b^{2m} \times (3ab)^{-m}}{(4b)^m \times (9a^{4m})^{\frac{1}{2}}}$$

20. Answer the following and explain your reasoning.

a. What is the tens digit of 3^{3^3}?

b. What is the ones digit of 6^{309}?

c. What is the ones digit of 8^{1007}?

21. For the work shown below:

a. calculate the correct answer

b. identify where the student has made mistakes.

$$\begin{aligned}\left(\frac{3a^3b^5c^3}{5a^2b}\right)^2 \div \left(\frac{2ab}{c}\right) &= \frac{3a^6b^{10}c^6}{10a^4b^2} \div \frac{2ab}{c} \\ &= \frac{3a^6b^{10}c^6}{10a^4b^2} \times \frac{c}{2ab} \\ &= \frac{3a^6b^{10}c^7}{20a^5b^3} \\ &= \frac{3ab^7c^7}{20}\end{aligned}$$

22. A friend is trying to calculate the volume of water in a reservoir amid fears there may be a severe water shortage. She comes up with the following expression:

$W=\dfrac{r^4u^2}{r^{\frac{3}{2}}d^2\sqrt{u}}\times\dfrac{ru\times d^2}{dr^3u^4}$, where r is the amount of rain, d is how dry the area is, u is the usage of water by the townsfolk, and W is the volume of water in kL.

a. Help her simplify the expression by simplifying each pronumeral one at a time.
b. Does the final expression contain any potential surds?
c. Express the fraction with a rational denominator.
d. List the requirements for the possible values of r, u and d to give a rational answer.
e. Calculate the volume of water in the reservoir when $r = 4$, $d = 60$ and $u = 9$. Write your answer in:
  i. kL **ii.** L **iii.** mL.
f. Does a high value for d mean the area is dry? Explain using working.

23. The speed of a toy plane can be modelled by the equation $S=\dfrac{p^2}{2+\sqrt{w^3}}$, where

w = wind resistance and
p = battery power (from 0 (empty) to 10 (full)).

a. Rationalise the denominator of the expression.
b. Using your knowledge of perfect squares, estimate the speed of a toy plane with its battery half full and a wind resistance of 2. Check your answer with a calculator.
c. How does the speed of the toy plane change with increasing wind resistance? Explain providing supportive calculations.

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Try out this interactivity: Word search: Topic 1 (int-2826)

Try out this interactivity: Crossword: Topic 1 (int-2827)

Try out this interactivity: Sudoku: Topic 1 (int-3588)

Complete this digital doc: Concept map: Topic 1 (doc-13802)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

base
constant
denominator
evaluate
exponent
expression
index
index law
negative
numerator
positive
power indices
pronumeral
simplify
substitute
surd

Investigation | Rich Task

Digital world: 'A bit of this and a byte of that'

'The digital world of today is run by ones and zeros.' What does this mean?

Data is represented on a modern digital computer using a base two (binary) system, that is, using the two digits 1 and 0, thought of as 'on' and 'off'. The smallest unit of data that is transferred on a computer is a **bit** (an abbreviation of binary digit). Computer and storage mechanisms need to hold much larger values than a bit. Units such as bytes, kilobytes (KB), megabytes (MB), gigabytes (GB), and terabytes (TB) are based on the conversion of 8 bits to 1 byte. Your text messages, graphics, music and photos are files stored in sequences of bytes, each byte being 8 bits (8b = 1B).

You may have heard the terms 'meg' and 'gig'. In computer terminology, these refer to gigabytes and megabytes. In the digital world, the prefixes kilo-, mega- and giga- express powers of two, where kilo- means 2^{10}, mega- means $(2^{10})^2$ and so on. Thus the number of bytes in a computer's memory builds in powers of 2, for example 1 kilobyte = 1024 bytes (2^{10} bytes). (This differs from the decimal system, in which the prefixes kilo-, mega- and giga- express powers of ten, with kilo- meaning 10^3, mega- meaning $(10^3)^2$ and so on.)

A byte (8 bits) is used to represent a single character. For example the letter 'A' is represented in binary as 01000001. A book of a thousand pages in print can be stored in millions of bits, but more commonly it would be described as being stored in megabytes with one byte per character.

1. Complete the table below to show the difference in value between the binary and decimal systems.

Unit	Symbol	Power of 2 and value in bytes	Power of 10 and value in bytes
Byte	B	$2^0 = 1$	$10^0 = 1$
Kilobyte	KB	$2^{10} = 1024$	$10^3 = 1000$
Megabyte		$2^{20} =$	
Gigabyte			
Terabyte			

2. The two numbering systems have led to some confusion, with some manufacturers of digital products thinking of a kilobyte as 1000 bytes rather than 1024 bytes. Similar confusion arises with megabytes, gigabytes, terabytes and so on. This means you might not be getting exactly the amount of storage that you think.

If you bought a device quoted as having 16 GB memory, what would be the difference in memory storage if the device had been manufactured using the decimal value of GB as opposed to the binary system?

Many devices allow you to check the availability of storage. On one such device, the iPhone, available storage is found by going to 'General' under the heading 'Settings'.

3. How much storage is left in MB on the following iPhone?
4. If each photo uses 3.2 MB of memory, how many photos can be added?

Storage

3.9 GB Available 9.5 GB Used

Photos & Camera	1.6 GB
Radio	1.6 GB
Maps	1.2 GB
My Movie	461 MB

Have you ever wondered about the capacity of our brain to store information and the speed at which information is transmitted inside it?

5. Discuss how the storage and speed of our brains compares to our current ability to send and store information in the digital world. The capacity of the human brain is 10–100 terabytes. On average 20 million billion bits of information are transmitted within the brain per second.
6. Investigate which country has the fastest internet speed and compare this to Australia.

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Complete this digital doc: Code puzzle: What historical event took place in France in 1783? (doc-15917)

Answers

TOPIC 1 Indices

Exercise 1.2 Review of index laws

1. a. a^7 **b.** a^6 **c.** b^8 **d.** a^4b^7 **e.** m^5n^{13} **f.** $a^5b^7c^3$
g. $m^6n^4p^5$ **h.** $6a^2b$ **i.** $10a^4b^9$ **j.** $36m^8n^7$ **k.** $12x^6y^6$ **l.** $4x^8y^6$

2. a. a **b.** a^5 **c.** b^3 **d.** $\frac{4}{3}a^4$ **e.** $3b^4$ **f.** $4m^5$
g. m^3n **h.** $\frac{1}{2}y^2$ **i.** $\frac{3}{4}x^3y$ **j.** $7b^3$ **k.** $\frac{5}{4}m^2p^2$ **l.** $\frac{1}{2}xy^2$

3. a. 1 **b.** 1 **c.** 1 **d.** 3 **e.** 4 **f.** -3
g. 3 **h.** -7 **i.** 4

4. a. a^6 **b.** $16a^{20}$ **c.** $\frac{1}{81}m^8$ **d.** $\frac{4}{9}n^8$ **e.** a^6b^3 **f.** $9a^6b^4$
g. $16m^{12}n^{20}$ **h.** $\frac{27}{64}m^6n^3$ **i.** $\frac{a^4}{b^6}$ **j.** $\frac{625m^{12}}{n^8}$ **k.** $\frac{343x^3}{8y^{15}}$ **l.** $\frac{81a^4}{625b^{12}}$
m. -243 **n.** 49 **o.** -32

5. a. D **b.** D

6. a. C **b.** E **c.** B **d.** D

7. a. 64 **b.** 72 **c.** 625 **d.** 48 **e.** 1600 **f.** $\frac{27}{125}$
g. 20 **h.** 1 **i.** 4

8. a. x^{3yz} **b.** a^b **c.** m^an^b **d.** $\frac{a^{2x}}{b^{3x}}$ **e.** $n^{3-p}m^{2-q}$ **f.** a^{mp+np}

9.
$$\begin{aligned} a^3 &= a \times a \times a \\ a^2 &= a \times a \\ a^3 \times a^2 &= a \times a \times a \times a \times a \\ &= a^5,\ \text{not } a^6 \end{aligned}$$

Explanations will vary.

10. They are equal when $x = 2$. Explanations will vary.

11. $3x^0 = 3$ and $(3x)^0 = 1$. Explanations will vary.

12. a, b

a	0	1	2	3
$3a^2$	0	3	12	27
$5a$	0	5	10	15
$3a^2 + 5a$	0	8	22	42
$3a^2 \times 5a$	0	15	120	405

c $3a^2 \times 5a$ will become much larger than $3a^2 + 5a$.

13. $x = -2$ or 4

14. $1 \equiv 1$ $2 \equiv 10$ $3 \equiv 11$ $4 \equiv 100$ $5 \equiv 101$ $6 \equiv 110$
$7 \equiv 111$ $8 \equiv 1000$ $9 \equiv 1001$ $10 \equiv 1010$

15. a. $x = 4$ **b.** $x = 0, 2$

16. a. a^2bc^7

b. The student made a mistake when multiplying the two brackets in line 3. Individual brackets should be expanded first.

Challenge 1.1

1.08 seconds

Exercise 1.3 Negative indices

1. a. $\frac{1}{x^5}$ **b.** $\frac{1}{y^4}$ **c.** $\frac{2}{a^9}$ **d.** $\frac{4}{5a^3}$ **e.** $\frac{3x^2}{y^3}$ **f.** $\frac{1}{4m^3n^4}$
g. $\frac{6a^3}{bc^5}$ **h.** a^6 **i.** $\frac{2a^4}{3}$ **j.** $2ab^2$ **k.** $\frac{7b^3}{2a^4}$ **l.** $\frac{2m^3a^2}{3b^4n^5}$

2. a. $\frac{1}{a^2b^3}$ **b.** $\frac{6}{x^6y}$ **c.** $\frac{3}{n^8}$ **d.** $\frac{4}{a^2b^5}$ **e.** $\frac{2y}{3x}$ **f.** $\frac{5y}{6x^3}$

g. $\frac{3}{m^2n^2}$ **h.** $\frac{4y^{12}}{x^5}$ **i.** $\frac{1}{3m^3n^3}$ **j.** $\frac{1}{32a^{15}m^{20}}$ **k.** $\frac{4q^8}{p^{14}}$ **l.** $\frac{3}{a^8b^{12}}$

m. $\frac{27q^9}{8p^6}$ **n.** $\frac{b^6}{4a^8}$ **o.** $\frac{1}{8a^6b^6}$

3. a. $\frac{1}{8}$ **b.** $\frac{1}{36}$ **c.** $\frac{1}{81}$ **d.** $\frac{8}{9}$ **e.** $\frac{1}{16}$ **f.** $\frac{5}{36}$

g. 48 **h.** $\frac{32}{27}$ **i.** $\frac{27}{25} = 1\frac{2}{25}$ **j.** 4 **k.** 125 **l.** $\frac{3}{4}$

4. a. 2^3 **b.** 2^{-3} **c.** 2^5 **d.** 2^{-6}

5. a. 3 **b.** −2 **c.** −1 **d.** 3 **e.** −2 **f.** 0

g. 3 **h.** −3 **i.** −6 **j.** −2

6. a. $\frac{3}{2}$ **b.** $\frac{4}{5}$ **c.** $\frac{2}{7}$ **d.** 5

7. a. $\frac{b}{a}$ **b.** $\frac{b^3}{a^2}$ **c.** $\frac{a^2}{b^3}$ **d.** $\frac{1}{m^3n^2}$

8. a. $\frac{1}{729}$ **b.** $\frac{1}{20\,736}$ **c.** 0.000 059499 or $\frac{1}{16\,807}$

d. 256 **e.** $\frac{16\,384}{2187}$ **f.** 9765625

9. a. D **b.** C **c.** B

10. a. B **b.** D **c.** C **d.** E

11. a. $\frac{m^2}{n^8}$ **b.** $\frac{n^2}{m}$ **c.** $\frac{25}{a^7b^6}$

12. a. $r^6 - s^6$ **b.** $m^{10} + 2m^5n^5 + n^{10}$ **c.** 1 **d.** p^2

13. 2^{2r-4}

14. 6^{3m}

15. $x = 3$

16. Answers will vary; check with your teacher.

17. $x = 3,\ y = -1$; 7

18. a. $n = -1, -2$ **b.** $n = -1, 4$

Exercise 1.4 Fractional indices

1. a. 4 **b.** 5 **c.** 9 **d.** 2 **e.** 4 **f.** 3

2. a. $\sqrt{15}$ **b.** $\sqrt[4]{m}$ **c.** $\sqrt[5]{7^2}$ **d.** $\sqrt{7^5}$ **e.** $\sqrt[8]{w^3}$ **f.** $\sqrt[4]{w^5}$

g. $\sqrt[3]{5^{10}}$ **h.** $\sqrt[10]{a^3}$

3. a. $t^{\frac{1}{2}}$ **b.** $5^{\frac{7}{4}}$ **c.** $6^{\frac{11}{6}}$ **d.** $x^{\frac{6}{7}}$ **e.** $x^{\frac{7}{6}}$ **f.** w^2

g. $w^{\frac{1}{2}}$ **h.** $11^{\frac{n}{x}}$

4. a. 4 **b.** 16 **c.** 8 **d.** 16 **e.** 125 **f.** 9

g. $\frac{1}{9}$ **h.** 27 **i.** 1000 **j.** 216 **k.** $\sqrt{7}$ **l.** $\sqrt[3]{12}$

5. a. 1.44 **b.** 2.24 **c.** 1.48 **d.** 1.26 **e.** 2.54 **f.** 0.66

g. 0.54 **h.** 0.81 **i.** 0.86

6. a. $4^{\frac{4}{5}}$ **b.** $2^{\frac{1}{2}}$ **c.** $a^{\frac{5}{6}}$ **d.** $x^{\frac{23}{20}}$ **e.** $10m^{\frac{8}{15}}$ **f.** $2b^{\frac{5}{7}}$

g. $-4y^{\frac{20}{9}}$ **h.** $0.02a^{\frac{9}{8}}$ **i.** $5x^{\frac{7}{2}}$

7. a. $ab^{\frac{3}{2}}$ **b.** $x^{\frac{4}{5}}y^{\frac{5}{9}}$ **c.** $6a^{\frac{8}{5}}b^{\frac{17}{15}}$ **d.** $2m^{\frac{19}{28}}n^{\frac{2}{5}}$ **e.** $x^{\frac{19}{6}}y^{\frac{5}{6}}z^{\frac{5}{6}}$ **f.** $8a^{\frac{2}{5}}b^{\frac{9}{8}}c$

8. a. $3^{\frac{1}{6}}$ **b.** $5^{\frac{5}{12}}$ **c.** $12^{\frac{1}{2}}$ **d.** $a^{\frac{3}{7}}$ **e.** $x^{\frac{5}{4}}$ **f.** $m^{\frac{11}{45}}$

g. $\frac{1}{2}x^{\frac{3}{20}}$ **h.** $\frac{1}{3}n^{\frac{2}{3}}$ **i.** $\frac{5}{4}b^{\frac{7}{20}}$

9. a. $x^{\frac{5}{3}}y^{\frac{7}{5}}$ **b.** $a^{\frac{7}{45}}b^{\frac{4}{15}}$ **c.** $\frac{1}{3}m^{\frac{3}{8}}n^{\frac{11}{56}}$ **d.** $2x^{\frac{2}{15}}y^{\frac{3}{4}}$ **e.** $\frac{1}{4}a^{\frac{11}{20}}b^{\frac{7}{20}}$ **f.** $\frac{1}{7}p^{\frac{5}{24}}q^{\frac{1}{12}}$

10. a. $2^{\frac{9}{20}}$ **b.** $5^{\frac{1}{6}}$ **c.** $7^{\frac{6}{5}}$ **d.** $a^{\frac{3}{10}}$ **e.** $m^{\frac{1}{6}}$ **f.** $2^{\frac{1}{3}}b^{\frac{1}{6}}$
g. $4p^{\frac{2}{5}}$ **h.** $x^{\frac{m}{p}}$ **i.** $3^{\frac{b}{c}}m^{\frac{a}{c}}$

11. a. $a^{\frac{1}{4}}b^{\frac{1}{6}}$ **b.** $a^3b^{\frac{3}{4}}$ **c.** $x^{\frac{6}{5}}y^{\frac{7}{4}}$ **d.** $3^{\frac{1}{3}}a^{\frac{1}{9}}b^{\frac{1}{5}}c^{\frac{1}{4}}$ **e.** $5x^{\frac{1}{4}}y^{\frac{1}{3}}z^{\frac{1}{5}}$ **f.** $\dfrac{a^{\frac{1}{2}}}{b^{\frac{2}{3}}}$
g. $\dfrac{m^{\frac{8}{5}}}{n^{\frac{7}{4}}}$ **h.** $\dfrac{b^{\frac{2}{5}}}{c^{\frac{8}{27}}}$ **i.** $\dfrac{2^{\frac{1}{2}}x^{\frac{7}{2}}}{y^{\frac{3}{8}}}$

12. a. E **b.** C **c.** B

13. a. E **b.** B

14. a. a^4 **b.** b^3 **c.** m^4 **d.** $4x^2$ **e.** $2y^3$ **f.** $2x^2y^3$
g. $3m^3n^5$ **h.** $2pq^2$ **i.** $6a^2b^6$

15. a. 2.007 s **b.** 20.07 s **c.** 4.98 swings

16. $(2^5a^5b^{10})^{\frac{1}{5}} = 2ab^2$

17. $\dfrac{a^2(3 - \sqrt{b^3})}{9 - b^3}; \dfrac{4}{11}$

18. $m^{\frac{1}{5}} - n^{\frac{1}{5}} + p^{\frac{1}{5}}$

19. a. $a^{-\frac{1}{4}} \times b^{\frac{13}{2}}$
b. No, because you can't take the fourth root of a negative number.
c. $a = 1$

Exercise 1.5 Combining index laws

1. a. $54a^{10}b^9$ **b.** $48a^5b^{16}$ **c.** $\dfrac{2n^{13}}{m^9}$ **d.** $500p^8q^{18}$ **e.** $36a^{20}b^{10}$ **f.** $\dfrac{15b^2}{c^{26}}$
g. $12x^{\frac{7}{8}}y^{\frac{11}{15}}$ **h.** $8m^{\frac{15}{4}}n^{\frac{15}{4}}$ **i.** $\dfrac{6}{p^{\frac{7}{12}}}$ **j.** $8p^{\frac{7}{45}}q^{\frac{5}{18}}$

2. a. $\dfrac{5}{8a^7}$ **b.** $\dfrac{x}{4y^6}$ **c.** $\dfrac{27}{128m^{29}n^{26}}$ **d.** $\dfrac{64y^{36}}{x^{24}}$ **e.** $24a^{24}b^7$ **f.** $\dfrac{27h^{12}}{8g^6}$
g. $p^{\frac{35}{3}}q^{\frac{1}{2}}$ **h.** $\dfrac{625}{81b^{20}c^{28}}$ **i.** $x^{\frac{5}{3}}y^{\frac{1}{8}}z^{\frac{3}{2}}$

3. a. $\dfrac{3a^2}{2}$ **b.** $8n^2$ **c.** $\dfrac{m^2n^4}{3}$ **d.** $\dfrac{4x^5}{3y^8}$ **e.** $\dfrac{36x^6}{y}$ **f.** $\dfrac{y^2}{x^4}$
g. $\dfrac{b^7}{3a^4}$ **h.** $\dfrac{75q^5}{2p^{11}}$ **i.** $x^{\frac{17}{10}}y^{\frac{7}{10}}$

4. a. $\dfrac{2}{5a^4b^7}$ **b.** $\dfrac{4a^3b^3}{15}$ **c.** $\dfrac{n^9}{4m^9}$ **d.** $\dfrac{4m^5}{9n^{15}}$ **e.** $\dfrac{4}{81x^2y^{14}}$ **f.** $48x^{11}y^6$
g. $\dfrac{3p^4}{5q^9}$ **h.** $\dfrac{2b^{\frac{1}{12}}}{3a^{\frac{17}{24}}}$ **i.** $\dfrac{4x^{\frac{1}{12}}}{3y^{\frac{21}{20}}}$

5. a. $\dfrac{5}{2a^{13}}$ **b.** $\dfrac{56a^{11}b^6}{81}$ **c.** $\dfrac{1024b^2}{81a}$ **d.** $\dfrac{25}{128x^{23}y^4}$ **e.** $\dfrac{4y^{36}}{27x^{16}}$ **f.** $6m^{19}n^{19}$
g. $\dfrac{16m^{\frac{11}{12}}n}{3}$ **h.** $\dfrac{4b^{\frac{11}{2}}}{3^{\frac{1}{2}}c^{\frac{7}{30}}}$

6. a. $\frac{125}{8}$ **b.** 1

7. 1

8. a. 5^{y-1} **b.** $y = 4$

9. E

10. A

11. a. $m^{\frac{1}{6}}n^{-\frac{7}{6}}$ or $\sqrt[6]{\frac{m}{n^7}}$
b. $g^{-6}h^3n^{\frac{3}{2}}$ **c.** $3^{-\frac{7}{3}} \times 5^{-\frac{7}{6}}$ **d.** 2^{-2} or $\frac{1}{4}$ **e.** a^6b^{-8} or $\frac{a^6}{b^8}$
f. $d^{\frac{14}{15}}$ or $\sqrt[15]{d^{14}}$

12. a. 80 koalas **b.** During the 6th year

13. a. 79% **b.** 56% **c.** 31%

14. $\frac{4}{21}$

15. $z^2 + z^{-2} + \sqrt{5}$

Challenge 1.2

$$x = \frac{2y}{y - z}$$

1.6 Review

1. D
2. C
3. C
4. C
5. A
6. A
7. B
8. C
9. B

10. a. $9x^{10}y^{10}$ **b.** $\frac{13ab^3c^2}{6}$ **c.** $\frac{1000m^{15}n^6}{27}$ **d.** $\frac{16p^{28}}{81q^{12}}$

11. a. 16 **b.** $-\frac{3}{2}$

12. a. $\frac{8}{a^{11}b^2}$ **b.** $\frac{y^2}{5x^{17}}$ **c.** $\frac{m^{12}}{16n^8}$

13. a. 8 **b.** $\frac{3}{2}$ **c.** 0

14. a. $30a^{\frac{41}{20}}b^{\frac{33}{20}}$ **b.** $\frac{4}{x^{\frac{1}{20}}y^{\frac{2}{9}}}$ **c.** $\frac{2a^{\frac{1}{6}}}{b^{\frac{3}{2}}}$

15. a. 1 **b.** 4

16. a. $-2a^3 + 2a^2b^{\frac{1}{2}}$ **b.** $6xy^2$

17. a. $\frac{2a^{13}}{5b^2}$ **b.** $\frac{9y^4}{32x^{15}}$ **c.** $2^{\frac{4}{3}}m$

18. a. 46 **b.** $-\frac{1}{18}$

19. $\frac{1}{36}$

20. a. 8 **b.** 6 **c.** 2

21. a. $\frac{9ab^7c^7}{50}$
b. The student has made two mistakes when squaring the left-hand bracket in line 1 : $3^2 = 9$, $5^2 = 25$.

22. a. $\dfrac{\sqrt{r}}{d\sqrt{u^3}}$ **b.** Yes, $\sqrt{r}$, $\sqrt{u^3}$ **c.** $\dfrac{\sqrt{ru^3}}{du^3}$

d. r should be a perfect square, u should be a perfect cube and d should be a rational number.

i. 0.0012346 kL **ii.** 1.2346 L **iii.** 1234.6 mL

e. A high value for d causes the expression to be smaller, as d only appears on the denominator of the fraction. This means that when d is high there is less water in the reservoir and the area is dry.

23. a. $\dfrac{p^2(2-\sqrt{w^3})}{4-w^3}$

b. Answers will vary; approximately 5.

c. speed decreases as wind resistance increases.

Investigation — Rich task

1.

Unit	Symbol	Power of 2 and value in bytes	Power of 10 and value in bytes
Byte	B	$2^0 = 1$	$10^0 = 1$
Kilobyte	KB	$2^{10} = 1024$	$10^3 = 1000$
Megabyte	MB	$2^{20} = 1\,048\,576$	$10^6 = 1\,000\,000$
Gigabyte	GB	$2^{30} = 1\,073\,741\,824$	$10^9 = 1\,000\,000\,000$
Terabyte	TB	$2^{40} = 1\,099\,511\,627\,776$	$10^{12} = 1\,000\,000\,000\,000$

2. Approximately 1.1 GB

3. 3993.6 MB

4. 1248 photos

5. Discuss with your teacher.

6. Discuss with your teacher. The discussion will depend on the latest information from the internet.

TOPIC 2
Algebra and equations

2.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

2.1.1 Why learn this?

Do you speak mathematics? Algebra is the language of mathematics; it holds the key to understanding the rules, formulas and relationships that summarise much of our understanding of the universe. Every student of mathematics needs a mathematical tool chest, a set of algebraic skills to manipulate and process mathematical information.

2.1.2 What do you know?

assesson

1. **THINK** List what you know about linear equations. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of linear equations.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Al-Khwarizmi (eles-1841)

2.2 Substitution

2.2.1 Substitution

When the numerical values of pronumerals are known, they can be substituted into an algebraic expression and the expression can then be evaluated. It can be useful to place any substituted values in brackets when evaluating an expression.

WORKED EXAMPLE 1

If $a = 4$, $b = 2$ and $c = -7$, evaluate the following expressions.

a $a - b$ **b** $a^3 + 9b - c$

THINK	WRITE
a **1** Write the expression.	**a** $a - b$
2 Substitute $a = 4$ and $b = 2$ into the expression.	$= 4 - 2$
3 Simplify.	$= 2$
b **1** Write the expression.	**b** $a^3 + 9b - c$
2 Substitute $a = 4$, $b = 2$ and $c = -7$ into the expression.	$= (4)^3 + 9(2) - (-7)$
3 Simplify.	$= 64 + 18 + 7$ $= 89$

WORKED EXAMPLE 2

TI | CASIO

If $c = \sqrt{a^2 + b^2}$, calculate c if $a = 12$ and $b = -5$.

THINK	WRITE
1 Write the expression.	$c = \sqrt{a^2 + b^2}$
2 Substitute $a = 12$ and $b = -5$ into the expression.	$= \sqrt{(12)^2 + (-5)^2}$
3 Simplify.	$= \sqrt{144 + 25}$ $= \sqrt{169}$ $= 13$

2.2.2 Number laws

- Recall from previous studies that when dealing with numbers and pronumerals, particular rules must be obeyed. Before progressing further, let us briefly review the Commutative, Associative, Identity and Inverse Laws.
- Consider any three pronumerals x, y and z, where x, y and z are elements of the set of Real numbers.

2.2.3 Commutative Law

1. $x + y = y + x$ (example: $3 + 2 = 5$ and $2 + 3 = 5$)
2. $x - y \neq y - x$ (example: $3 - 2 = 1$ but $2 - 3 = -1$)
3. $x \times y = y \times x$ (example: $3 \times 2 = 6$ and $2 \times 3 = 6$)
4. $x \div y \neq y \div x$ (example: $3 \div 2 = \frac{3}{2}$, but $2 \div 3 = \frac{2}{3}$)

Therefore, the **Commutative Law** holds true for addition and multiplication, since the order in which two numbers or pronumerals are added or multiplied does not affect the result. However, the Commutative Law does not hold true for subtraction or division.

2.2.4 Associative Law

1. $x + (y + z) = (x + y) + z$ [example: $2 + (3 + 4) = 2 + 7 = 9$ and $(2 + 3) + 4 = 5 + 4 = 9$]
2. $x - (y - z) \neq (x - y) - z$ [example: $2 - (3 - 4) = 2 - -1 = 3$ and $(2 - 3) - 4 = -1 - 4 = -5$]
3. $x \times (y \times z) = (x \times y) \times z$ [example: $2 \times (3 \times 4) = 2 \times 12 = 24$ and $(2 \times 3) \times 4 = 6 \times 4 = 24$]
4. $x \div (y \div z) \neq (x \div y) \div z$ [example: $2 \div (3 \div 4) = 2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{8}{3}$ but $(2 \div 3) \div 4 = \frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$]

The **Associative Law** holds true for addition and multiplication since grouping two or more numbers or pronumerals and calculating them in a different order does not affect the result. However, the Associative Law does not hold true for subtraction or division.

2.2.5 Identity Law

The **Identity Law** states that in general:

$$x + 0 = 0 + x = x$$
$$x \times 1 = 1 \times x = x$$

In both of the examples above, x has not been changed (that is, it has kept its identity) when zero is added to it or it is multiplied by 1.

2.2.6 Inverse Law

The **Inverse Law** states that in general:

$$x + -x = -x + x = 0$$
$$x \times \frac{1}{x} = \frac{1}{x} \times x = 1$$

That is, when the additive inverse of a number or pronumeral is added to itself, it equals 0. When the multiplicative inverse of a number or pronumeral is multiplied by itself, it equals 1.

2.2.7 Closure Law

A law that you may not yet have encountered is the Closure Law. The **Closure Law** states that, when an operation is performed on an element (or elements) of a set, the result produced must also be an element of that set. For example, addition is closed on natural numbers (that is, positive integers: 1, 2, 3, …) since adding a pair of natural numbers produces a natural number. Subtraction is not closed on natural numbers. For example, 5 and 7 are natural numbers and the result of adding them is 12, a natural number. However, the result of subtracting 7 from 5 is −2, which is not a natural number.

WORKED EXAMPLE 3

Find the value of the following expressions, given the integer values $x = 4$ and $y = -12$. Comment on whether the Closure Law for integers holds for each of the expressions when these values are substituted.

a $x + y$ **b** $x - y$ **c** $x \times y$ **d** $x \div y$

THINK	WRITE
a 1 Substitute each pronumeral into the expression.	**a** $x + y = 4 + -12$
2 Evaluate and write the answer.	$= -8$
3 Determine whether the Closure Law holds; that is, is the result an integer?	The Closure Law holds for these substituted values.

b Repeat steps **1**–**3** of part **a**.

b $x - y = 4 - -12$
$= 16$

The Closure Law holds for these substituted values.

c Repeat steps **1**–**3** of part **a**.

c $x \times y = 4 \times -12$
$= -48$

The Closure Law holds for these substituted values.

d Repeat steps **1**–**3** of part **a**.

d
$$\begin{aligned} x \div y &= 4 \div -12 \\ &= \frac{4}{-12} \\ &= -\frac{1}{3} \end{aligned}$$

The Closure Law does not hold for these substituted values since the answer obtained is a fraction, not an integer.

- It is important to note that, although a particular set of numbers may be closed under a given operation, for example multiplication, another set of numbers may not be closed under that same operation. For example, in part **c** of Worked example 3, integers were closed under multiplication. However, in some cases, the set of *irrational numbers* is not closed under multiplication, since $\sqrt{3} \times \sqrt{3} = \sqrt{9} = 3$. In this example, two irrational numbers produced a rational number under multiplication.

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Complete this digital doc: SkillSHEET: Collecting like terms (doc-5184)
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Complete this digital doc: SkillSHEET: Order of operations (doc-5189)

Exercise 2.2 Substitution

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–f, 2a–f, 3c–d, 4a–c, 5a–c, 6a–e, 7, 8, 10, 14	Questions: 1c–i, 2a–f, 3c–d, 4, 5a–c, 6d–j, 9, 10, 14	Questions: 1e–l, 2c–i, 3, 4, 5, 6d–j, 9–15

🟩🟦🟪 **Individual pathway interactivity:** int-4566

learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** If $a = 2$, $b = 3$ and $c = 5$, evaluate the following expressions.
   - **a.** $a + b$
   - **b.** $c - b$
   - **c.** $c - a - b$
   - **d.** $c - (a - b)$
   - **e.** $7a + 8b - 11c$
   - **f.** $\frac{a}{2} + \frac{b}{3} + \frac{c}{5}$
   - **g.** abc
   - **h.** $ab(c - b)$
   - **i.** $a^2 + b^2 - c^2$
   - **j.** $c^2 + a$
   - **k.** $-a \times b \times -c$
   - **l.** $2.3a - 3.2b$

2. If $d = -6$ and $k = -5$, evaluate the following.
   - **a.** $d + k$
   - **b.** $d - k$
   - **c.** $k - d$
   - **d.** kd
   - **e.** $-d(k + 1)$
   - **f.** d^2
   - **g.** k^3
   - **h.** $\frac{k - 1}{d}$
   - **i.** $3k - 5d$

3. If $x = \frac{1}{3}$ and $y = \frac{1}{4}$, evaluate the following.
   - **a.** $x + y$
   - **b.** $y - x$
   - **c.** xy
   - **d.** $\frac{x}{y}$
   - **e.** x^2y^3
   - **f.** $\frac{9x}{y^2}$

4. If $x = 3$, find the value of the following.
   - **a.** x^2
   - **b.** $-x^2$
   - **c.** $(-x)^2$
   - **d.** $2x^2$
   - **e.** $-2x^2$
   - **f.** $(-2x)^2$

5. If $x = -3$, find the value of the following.
   - **a.** x^2
   - **b.** $-x^2$
   - **c.** $(-x)^2$
   - **d.** $2x^2$
   - **e.** $-2x^2$
   - **f.** $(-2x)^2$

6. **WE2** Calculate the unknown variable in the following real-life mathematical formulas.
   - **a.** If $c = \sqrt{a^2 + b^2}$, calculate c if $a = 8$ and $b = 15$.
   - **b.** If $A = \frac{1}{2}bh$, determine the value of A if $b = 12$ and $h = 5$.
   - **c.** The perimeter, P, of a rectangle is given by $P = 2L + 2W$. Calculate the perimeter, P, of a rectangle, given $L = 1.6$ and $W = 2.4$.
   - **d.** If $T = \frac{C}{L}$, determine the value of T if $C = 20.4$ and $L = 5.1$.
   - **e.** If $K = \frac{n + 1}{n - 1}$, determine the value of K if $n = 5$.
   - **f.** Given $F = \frac{9C}{5} + 32$, calculate F if $C = 20$.
   - **g.** If $v = u + at$, evaluate v if $u = 16$, $a = 5$, $t = 6$.
   - **h.** The area, A, of a circle is given by the formula $A = \pi r^2$. Calculate the area of a circle, correct to 1 decimal place, if $r = 6$.
   - **i.** If $E = \frac{1}{2}mv^2$, calculate m if $E = 40$, $v = 4$.
   - **j.** Given $r = \sqrt{\frac{A}{\pi}}$, evaluate A to 1 decimal place if $r = 14.1$.

7. **MC** **a.** If $p = -5$ and $q = 4$, then pq is equal to:
   - **A.** 20
   - **B.** 1
   - **C.** -1
   - **D.** -20
   - **E.** $-\frac{5}{4}$

   b. If $c^2 = a^2 + b^2$, and $a = 6$ and $b = 8$, then c is equal to:
   - **A.** 28
   - **B.** 100
   - **C.** 10
   - **D.** 14
   - **E.** 44

   c. Given $h = 6$ and $k = 7$, then kh^2 is equal to:
   - **A.** 294
   - **B.** 252
   - **C.** 1764
   - **D.** 5776
   - **E.** 85

Understanding

8. Knowing the length of two sides of a right-angled triangle, the third side can be calculated using Pythagoras' theorem. If the two shorter sides have lengths of 1.5 cm and 3.6 cm, calculate the length of the hypotenuse.

9. The volume of a sphere can be calculated using the formula $\frac{4}{3}\pi r^3$. What is the volume of a sphere with a radius of 2.5 cm? Give your answer correct to 2 decimal places.
10. A rectangular park is 200 m by 300 m. If Blake runs along the diagonal of the park, how far will he run? Give your answer to the nearest metre.

Reasoning

11. **WE3** Determine the value of the following expressions, given the integer values $x = 1$, $y = -2$ and $z = -1$. Comment on whether the Closure Law for integers holds true for each of the expressions when these values are substituted.
   - **a.** $x + y$
   - **b.** $y - z$
   - **c.** $y \times z$
   - **d.** $x \div z$
   - **e.** $z - x$
   - **f.** $x \div y$
12. Find the value of the following expressions, given the natural number values $x = 8$, $y = 2$ and $z = 6$. Comment on whether the Closure Law for natural numbers holds true for each of the expressions.
   - **a.** $x + y$
   - **b.** $y - z$
   - **c.** $y \times z$
   - **d.** $x \div z$
   - **e.** $z - x$
   - **f.** $x \div y$
13. For each of the following, complete the relationship to illustrate the stated law. Justify your reasoning.
   - **a.** $(a + 2b) + 4c =$ ____________ Associative Law
   - **b.** $(x \times 3y) \times 5c =$ ____________ Associative Law
   - **c.** $2p \div q \neq$ ________________ Commutative Law
   - **d.** $5d + q =$ ________________ Commutative Law
   - **e.** $3z + 0 =$ ________________ Identity Law
   - **f.** $2x \times$ __________ $=$ __________ Inverse Law
   - **g.** $(4x \div 3y) \div 5z \neq$ ____________ Associative Law
   - **h.** $3d - 4y \neq$ _______________ Commutative Law

Problem solving

14. $s = ut + \frac{1}{2}at^2$
   where t is the time in seconds, s is the displacement in metres, u is the initial velocity and a is the acceleration due to gravity.
   - **a.** Calculate s when $u = 16.5$ m/s, $t = 2.5$ seconds and $a = 9.8$ m/s^2.
   - **b.** A body has an initial velocity of 14.7 m/s and after t seconds has a displacement of 137.2 metres. Find the value of t if $a = 9.8$ m/s^2.

15. Find the value of m if $n = p\sqrt{1 + \frac{1}{m}}$, when $n = 6$ and $p = 4$.

Reflection

Why is understanding of the Commutative Law useful?

CHALLENGE 2.1

The lowest common multiple of four terms is $24a^3bc^2d$. Three of the terms are $12a^2bc$, $8ab$ and $4a^2cd$. The fourth term contains only two pronumerals, and its coefficient is an odd prime number. What is the fourth term?

2.3 Adding and subtracting algebraic fractions

2.3.1 Algebraic fractions

- In an algebraic fraction, the denominator, the numerator or both are algebraic expressions. For example, $\frac{x}{2}$, $\frac{3x+1}{2x-5}$ and $\frac{1}{x^2+5}$ are all **algebraic fractions**.
- As with all fractions, algebraic fractions must have a common denominator if they are to be added or subtracted, so an important step is to find the lowest common denominator (LCD).

WORKED EXAMPLE 4 TI | CASIO

Simplify the following expressions.

a $\frac{2x}{3} - \frac{x}{2}$

b $\frac{x+1}{6} + \frac{x+4}{4}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{2x}{3} - \frac{x}{2}$
2 Rewrite each fraction as an equivalent fraction using the LCD of 3 and 2, which is 6.	$= \frac{2x}{3} \times \frac{2}{2} - \frac{x}{2} \times \frac{3}{3}$ $= \frac{4x}{6} - \frac{3x}{6}$
3 Express as a single fraction.	$= \frac{4x - 3x}{6}$
4 Simplify the numerator.	$= \frac{x}{6}$
b 1 Write the expression.	**b** $\frac{x+1}{6} + \frac{x+4}{4}$
2 Rewrite each fraction as an equivalent fraction using the LCD of 6 and 4, which is 12.	$= \frac{x+1}{6} \times \frac{2}{2} + \frac{x+4}{4} \times \frac{3}{3}$ $= \frac{2(x+1)}{12} + \frac{3(x+4)}{12}$
3 Express as a single fraction.	$= \frac{2(x+1) + 3(x+4)}{12}$
4 Simplify the numerator by expanding brackets and collecting like terms.	$= \frac{2x + 2 + 3x + 12}{12}$ $= \frac{5x + 14}{12}$

2.3.2 Pronumerals in the denominator

- If pronumerals appear in the denominator, the process involved in adding and subtracting the fractions is to find a lowest common denominator as usual.
- When there is an algebraic expression in the denominator of each fraction, we can obtain a common denominator by writing the product of the denominators. For example, if $x + 3$ and $2x - 5$ are in the denominator of each fraction, then a common denominator of the two fractions will be $(x + 3)(2x - 5)$.

WORKED EXAMPLE 5

Simplify $\frac{2}{3x} - \frac{1}{4x}$.

THINK	WRITE
1 Write the expression.	$\frac{2}{3x} - \frac{1}{4x}$
2 Rewrite each fraction as an equivalent fraction using the LCD of $3x$ and $4x$, which is $12x$. *Note:* $12x^2$ is not the lowest LCD.	$= \frac{2}{3x} \times \frac{4}{4} - \frac{1}{4x} \times \frac{3}{3}$ $= \frac{8}{12x} - \frac{3}{12x}$
3 Express as a single fraction.	$= \frac{8 - 3}{12x}$
4 Simplify the numerator.	$= \frac{5}{12x}$

WORKED EXAMPLE 6

Simplify $\frac{x + 1}{x + 3} + \frac{2x - 1}{x + 2}$ by writing it first as a single fraction.

THINK	WRITE
1 Write the expression.	$\frac{x + 1}{x + 3} + \frac{2x - 1}{x + 2}$
2 Rewrite each fraction as an equivalent fraction using the LCD of $x + 3$ and $x + 2$, which is the product $(x + 3)(x + 2)$.	$= \frac{(x + 1)}{(x + 3)} \times \frac{(x + 2)}{(x + 2)} + \frac{(2x - 1)}{(x + 2)} \times \frac{(x + 3)}{(x + 3)}$ $= \frac{(x + 1)(x + 2)}{(x + 3)(x + 2)} + \frac{(2x - 1)(x + 3)}{(x + 3)(x + 2)}$
3 Express as a single fraction.	$= \frac{(x + 1)(x + 2) + (2x - 1)(x + 3)}{(x + 3)(x + 2)}$
4 Simplify the numerator by expanding brackets and collecting like terms. *Note:* The denominator is generally kept in factorised form. That is, it is not expanded.	$= \frac{(x^2 + 2x + x + 2) + (2x^2 + 6x - x - 3)}{(x + 3)(x + 2)}$ $= \frac{(x^2 + 3x + 2 + 2x^2 + 5x - 3)}{(x + 3)(x + 2)}$ $= \frac{3x^2 + 8x - 1}{(x + 3)(x + 2)}$

WORKED EXAMPLE 7

TI | CASIO

Simplify $\frac{x+2}{x-3}+\frac{x-1}{(x-3)^2}$ by writing it first as a single fraction.

THINK	WRITE
1 Write the expression.	$\frac{x+2}{x-3}+\frac{x-1}{(x-3)^2}$
2 Rewrite each fraction as an equivalent fraction using the LCD of $x-3$ and $(x-3)^2$, which is $(x-3)^2$.	$=\frac{x+2}{x-3}\times\frac{x-3}{x-3}+\frac{x-1}{(x-3)^2}$ $=\frac{(x+2)(x-3)}{(x-3)^2}+\frac{x-1}{(x-3)^2}$ $=\frac{x^2-x-6}{(x-3)^2}+\frac{x-1}{(x-3)^2}$
3 Express as a single fraction.	$=\frac{x^2-x-6+x-1}{(x-3)^2}$
4 Simplify the numerator.	$=\frac{x^2-7}{(x-3)^2}$

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Complete this digital doc: SkillSHEET: Addition and subtraction of fractions (doc-5186)

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Exercise 2.3 Adding and subtracting algebraic fractions

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Individual pathways

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Questions: 1a–f, 2a–f, 3a–f, 4–6	Questions: 1d–i, 2a–f, 3a–i, 4–7, 9	Questions: 1g–i, 2e–j, 3d–i, 4–10

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Fluency

1. Simplify each of the following.

a. $\frac{4}{7}+\frac{2}{3}$
b. $\frac{1}{8}+\frac{5}{9}$
c. $\frac{3}{5}+\frac{6}{15}$
d. $\frac{4}{9}-\frac{3}{11}$
e. $\frac{3}{7}-\frac{2}{5}$
f. $\frac{1}{5}-\frac{x}{6}$
g. $\frac{5x}{9}-\frac{4}{27}$
h. $\frac{3}{8}-\frac{2x}{5}$
i. $\frac{5}{x}-\frac{2}{3}$

2. WE4 Simplify the following expressions.

a. $\frac{2y}{3}-\frac{y}{4}$
b. $\frac{y}{8}-\frac{y}{5}$
c. $\frac{4x}{3}-\frac{x}{4}$
d. $\frac{8x}{9}+\frac{2x}{3}$
e. $\frac{2w}{14}-\frac{w}{28}$
f. $\frac{y}{20}-\frac{y}{4}$
g. $\frac{12y}{5}+\frac{y}{7}$
h. $\frac{10x}{5}+\frac{2x}{15}$
i. $\frac{x+1}{5}+\frac{x+3}{2}$
j. $\frac{x+2}{4}+\frac{x+6}{3}$
k. $\frac{2x-1}{5}-\frac{2x+1}{6}$
l. $\frac{3x+1}{2}+\frac{5x+2}{3}$

3. WE5 Simplify the following.

a. $\frac{2}{4x}+\frac{1}{8x}$
b. $\frac{3}{4x}-\frac{1}{3x}$
c. $\frac{5}{3x}+\frac{1}{7x}$
d. $\frac{12}{5x}+\frac{4}{15x}$
e. $\frac{1}{6x}+\frac{1}{8x}$
f. $\frac{9}{4x}-\frac{9}{5x}$
g. $\frac{2}{100x}+\frac{7}{20x}$
h. $\frac{1}{10x}+\frac{5}{x}$
i. $\frac{4}{3x}-\frac{3}{2x}$

4. WE6, 7 Simplify the following by writing as single fractions.

a. $\frac{2}{x+4}+\frac{3x}{x-2}$
b. $\frac{2x}{x+5}+\frac{5}{x-1}$
c. $\frac{5}{2x+1}+\frac{x}{x-2}$
d. $\frac{2x}{x+1}-\frac{3}{2x-7}$
e. $\frac{4x}{x+7}+\frac{3x}{x-5}$
f. $\frac{x+2}{x+1}+\frac{x-1}{x+4}$
g. $\frac{x+8}{x+1}-\frac{2x+1}{x+2}$
h. $\frac{x+5}{x+3}-\frac{x-1}{x-2}$
i. $\frac{x+1}{x+2}-\frac{2x-5}{3x-1}$
j. $\frac{2}{x-1}-\frac{3}{1-x}$
k. $\frac{4}{(x+1)^2}+\frac{3}{x+1}$
l. $\frac{3}{x-1}-\frac{1}{(x-1)^2}$

Understanding

5. A classmate attempted to complete an algebraic fraction subtraction problem.

$$\frac{x}{x-1}-\frac{3}{x-2}=\frac{x}{x-1}\times\frac{(x-2)}{(x-2)}-\frac{3}{x-2}\times\frac{(x-1)}{(x-1)}$$
$$=\frac{x(x-2)-3(x-1)}{(x-1)(x+2)}$$
$$=\frac{x^2-2x-3x-1}{(x-1)(x+2)}$$
$$=\frac{x^2-5x-1}{(x-1)(x+2)}$$

a. What mistake did she make?
b. What is the correct answer?

Reasoning

Adding and subtracting algebraic fractions can become more complicated if you add a third fraction into the expression.

6. Simplify the following.

a. $\frac{1}{x+2}+\frac{2}{x+1}+\frac{1}{x+3}$
b. $\frac{1}{x-1}+\frac{4}{x+2}+\frac{2}{x-4}$
c. $\frac{3}{x+1}+\frac{2}{x+3}-\frac{1}{x+2}$
d. $\frac{2}{x-4}-\frac{3}{x-1}+\frac{5}{x+3}$

7. Why is the process that involves finding the lowest common denominator important in question **6**?

8. The reverse process of adding or subtracting algebraic fractions is quite complex. Use trial and error, or technology, to determine the value of a if $\frac{7x-4}{(x-8)(x+5)} = \frac{a}{x-8} + \frac{3}{x+5}$.

Problem solving

9. Simplify $\frac{3}{x^2+7x+12} - \frac{1}{x^2+x-6} + \frac{2}{x^2+2x-8}$.

10. Simplify $\frac{x^2+3x-18}{x^2-x-42} - \frac{x^2-3x+2}{x^2-5x+4}$.

Reflection

Why can't we just add the numerators and the denominators of fractions; for example, $\boldsymbol{\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}}$?

2.4 Multiplying and dividing algebraic fractions

2.4.1 Simplifying algebraic fractions

- Algebraic fractions can be simplified using the index laws and by cancelling factors common to the numerator and denominator.
- A fraction can only be simplified if:
  - – there is a common factor in the numerator and the denominator
  - – the numerator and denominator are both written in factorised form, that is, as the *product* of two or more factors.

$$\frac{3ab}{12a} = \frac{{}^{1}\cancel{3} \times {}^{1}\cancel{a} \times b}{{}^{4}\cancel{12} \times {}^{1}\cancel{a}} \begin{array}{l} \leftarrow \text{product of factors} \\ \leftarrow \text{product of factors} \end{array}$$

$$= \frac{b}{4}$$

$$\frac{3a+b}{12a} = \frac{3 \times a + b}{12 \times a} \begin{array}{l} \leftarrow \text{not a product of factors} \\ \leftarrow \text{product of factors} \end{array}$$

Cannot be simplified

2.4.2 Multiplying algebraic fractions

- Multiplication of algebraic fractions follows the same rules as multiplication of numerical fractions: multiply the numerators, then multiply the denominators.

WORKED EXAMPLE 8

Simplify each of the following.

a $\frac{5y}{3x} \times \frac{6z}{7y}$

b $\frac{2x}{(x+1)(2x-3)} \times \frac{x+1}{x}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{5y}{3x} \times \frac{6z}{7y}$
2 Cancel common factors in the numerator and denominator. The y can be cancelled in the denominator and the numerator. Also, the 3 in the denominator can divide into the 6 in the numerator.	$= \frac{5\cancel{y}^{1}}{{}_{1}\cancel{3}x} \times \frac{\cancel{6}^{2}z}{7\cancel{y}^{1}}$ $= \frac{5}{x} \times \frac{2z}{7}$

3 Multiply the numerators, then multiply the denominators.	$= \frac{10z}{7x}$
b 1 Write the expression.	**b** $\frac{2x}{(x+1)(2x-3)} \times \frac{x+1}{x}$
2 Cancel common factors in the numerator and the denominator. $(x + 1)$ and the x are both common in the numerator and the denominator and can therefore be cancelled.	$= \frac{2\cancel{x}^1}{{}^1\cancel{(x+1)}(2x-3)} \times \frac{\cancel{x+1}^1}{\cancel{x}^1}$ $= \frac{2}{2x-3} \times \frac{1}{1}$
3 Multiply the numerators, then multiply the denominators.	$= \frac{2}{2x-3}$

2.4.3 Dividing algebraic fractions

- When dividing algebraic fractions, follow the same rules as for division of numerical fractions: write the division as a multiplication and invert the second fraction. This process is sometimes known as multiplying by the **reciprocal**.

WORKED EXAMPLE 9

TI | CASIO

Simplify the following expressions.

a $\frac{3xy}{2} \div \frac{4x}{9y}$ **b** $\frac{4}{(x+1)(3x-5)} \div \frac{x-7}{x+1}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{3xy}{2} \div \frac{4x}{9y}$
2 Change the division sign to a multiplication sign and write the second fraction as its reciprocal.	$= \frac{3xy}{2} \times \frac{9y}{4x}$
3 Cancel common factors in the numerator and denominator and cancel. The pronumeral x is common to both the numerator and denominator and can therefore be cancelled.	$= \frac{3y}{2} \times \frac{9y}{4}$
4 Multiply the numerators, then multiply the denominators.	$= \frac{27y^2}{8}$
b 1 Write the expression.	**b** $\frac{4}{(x+1)(3x-5)} \div \frac{x-7}{x+1}$
2 Change the division sign to a multiplication sign and write the second fraction as its reciprocal.	$= \frac{4}{(x+1)(3x-5)} \times \frac{x+1}{x-7}$
3 Cancel common factors in the numerator and denominator and cancel. $(x + 1)$ is common to both the numerator and denominator and can therefore be cancelled.	$= \frac{4}{3x-5} \times \frac{1}{x-7}$
4 Multiply the numerators, then multiply the denominators.	$= \frac{4}{(3x-5)(x-7)}$

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Exercise 2.4 Multiplying and dividing algebraic fractions

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Fluency

1. **WE8a** Simplify each of the following.

a. $\frac{x}{5}\times\frac{20}{y}$

b. $\frac{x}{4}\times\frac{12}{y}$

c. $\frac{y}{4}\times\frac{16}{x}$

d. $\frac{x}{2}\times\frac{9}{2y}$

e. $\frac{x}{10}\times\frac{-25}{2y}$

f. $\frac{3w}{-14}\times\frac{-7}{x}$

g. $\frac{3y}{4x}\times\frac{8z}{7y}$

h. $\frac{-y}{3x}\times\frac{6z}{-7y}$

i. $\frac{x}{3z}\times\frac{-9z}{2y}$

j. $\frac{5y}{3x}\times\frac{x}{8y}$

k. $\frac{-20y}{7x}\times\frac{-21z}{5y}$

l. $\frac{y}{-3w}\times\frac{x}{2y}$

2. **WE8b** Simplify the following expressions.

a. $\frac{2x}{(x-1)(3x-2)}\times\frac{x-1}{x}$

b. $\frac{5x}{(x-3)(4x+7)}\times\frac{4x+7}{x}$

c. $\frac{9x}{(5x+1)(x-6)}\times\frac{5x+1}{2x}$

d. $\frac{(x+4)}{(x+1)(x+3)}\times\frac{x+1}{x+4}$

e. $\frac{2x}{x+1}\times\frac{x-1}{(x+1)(x-1)}$

f. $\frac{2}{x(2x-3)}\times\frac{x(x+1)}{4}$

g. $\frac{2x}{4(a+3)}\times\frac{3a}{15x}$

h. $\frac{15c}{12(d-3)}\times\frac{21d}{6c}$

i. $\frac{6x^2}{20(x-2)^2}\times\frac{15(x-2)}{16x^4}$

j. $\frac{7x^2(x-3)}{5x(x+1)}\times\frac{3(x-3)(x+1)}{14(x-3)^2(x-1)}$

3. **WE9a** Simplify the following expressions.

a. $\frac{3}{x}\div\frac{5}{x}$

b. $\frac{2}{x}\div\frac{9}{x}$

c. $\frac{4}{x}\div\frac{12}{x}$

d. $\frac{20}{y}\div\frac{20}{3y}$

e. $\frac{1}{5w}\div\frac{5}{w}$

f. $\frac{7}{2x}\div\frac{3}{5x}$

g. $\dfrac{3xy}{7} \div \dfrac{3x}{4y}$ **h.** $\dfrac{2xy}{5} \div \dfrac{5x}{y}$ **i.** $\dfrac{6y}{9} \div \dfrac{3x}{4xy}$

j. $\dfrac{8wx}{5} \div \dfrac{3w}{4y}$ **k.** $\dfrac{2xy}{5} \div \dfrac{3xy}{5}$ **l.** $\dfrac{10xy}{7} \div \dfrac{20x}{14y}$

4. **WE9b** Simplify the following expressions.

a. $\dfrac{9}{(x-1)(3x-7)} \div \dfrac{x+3}{x-1}$ **b.** $\dfrac{1}{(x+2)(2x-5)} \div \dfrac{x-9}{2x-5}$

c. $\dfrac{12(x-3)^2}{(x+5)(x-9)} \div \dfrac{4(x-3)}{7(x-9)}$ **d.** $\dfrac{13}{6(x-4)^2(x-1)} \div \dfrac{3(x+1)}{2(x-4)(x-1)}$

Reasoning

5. Is $\dfrac{3}{x+2}$ the same as $\dfrac{1}{x+2} + \dfrac{1}{x+2} + \dfrac{1}{x+2}$? Explain your reasoning.

6. a. Simplify $\dfrac{(x-4)(x+3)}{4x-x^2} \times \dfrac{x^2-x}{(x+3)(x-1)}$.

b. Find and describe the error in the following reasoning.

$$\frac{(x-4)(x+3)}{4x-x^2} \times \frac{x^2-x}{(x+3)(x-1)}$$
$$= \frac{(x-4)(x+3)}{x(4-x)} \times \frac{x(x-1)}{(x+3)(x-1)}$$
$$= 1$$

Problem solving

7. Simplify $\dfrac{x^2-2x-3}{x^4-1} \times \dfrac{x^2+4x-5}{x^2-5x+6} \div \dfrac{x^2+7x+10}{x^4-3x^2-4}$.

8. Simplify $\dfrac{x+1}{x-\dfrac{x}{\dfrac{x}{a}}}$ where $a = \dfrac{x-1}{x+1}$.

Reflection

How are multiplying and dividing algebraic fractions different to adding and subtracting them?

CHALLENGE 2.2

Simplify the expression: $\left(1 + \dfrac{\frac{1}{x}}{1+\frac{1}{x}}\right) \times \left(1 + \dfrac{\frac{1}{x}}{1-\frac{1}{x}}\right)$.

2.5 Solving simple equations

2.5.1 Equations

- **Equations** show the equivalence of two expressions.
- Equations can be solved using inverse operations.
- When solving equations, the last operation performed on the pronumeral when building the equation is the first operation undone by applying inverse operations to both sides of the equation. For example, the equation $2x + 3 = 5$ is built from x by multiplying x by 2 and then adding 3 to give the result of 5. To solve the equation, undo the adding 3 by subtracting 3, then undo the multiplying by 2 by dividing by 2.

+ and − are inverse operations
× and ÷ are inverse operations
^{2} and √ are inverse operations

2.5.2 One-step equations

- Equations that require one step to solve are called one-step equations.

WORKED EXAMPLE 10

Solve the following equations.

a $a + 27 = 71$ **b** $\frac{d}{16} = 3\frac{1}{4}$ **c** $\sqrt{e} = 0.87$ **d** $f^2 = \frac{4}{25}$

THINK			WRITE
a	**1**	Write the equation.	**a** $a + 27 = 71$
	2	27 has been added to a resulting in 71. The addition of 27 has to be reversed by subtracting 27 from both sides of the equation to obtain the solution.	$a + 27 - 27 = 71 - 27$ $a = 44$
b	**1**	Write the equation.	**b** $\frac{d}{16} = 3\frac{1}{4}$
	2	Express $3\frac{1}{4}$ as an improper fraction.	$\frac{d}{16} = \frac{13}{4}$
	3	The pronumeral d has been divided by 16 resulting in $\frac{13}{4}$. Therefore the division has to be reversed by multiplying both sides of the equation by 16 to obtain d.	$\frac{d}{16} \times 16 = \frac{13}{4} \times 16$ $d = 52$
c	**1**	Write the equation.	**c** $\sqrt{e} = 0.87$
	2	The square root of e has been taken to result in 0.87. Therefore, the square root has to be reversed by squaring both sides of the equation to obtain e.	$\left(\sqrt{e}\right)^2 = 0.87^2$ $e = 0.7569$
d	**1**	Write the equation.	**d** $f^2 = \frac{4}{25}$
	2	The pronumeral f has been squared, resulting in $\frac{4}{25}$. Therefore the squaring has to be reversed by taking the square root of both sides of the equation to obtain f. Note that there are two possible solutions, one positive and one negative, since two negative numbers can also be multiplied together to produce a positive result.	$f = \pm\sqrt{\frac{4}{25}}$ $f = \pm\frac{2}{5}$

2.5.3 Two-step equations

- Two-step equations involve the inverse of two operations in their solutions.

WORKED EXAMPLE 11

TI | CASIO

Solve the following equations.

a $\mathbf{5y - 6 = 79}$ **b** $\mathbf{\frac{4x}{9} = 5}$

THINK	WRITE
a **1** Write the equation.	**a** $5y - 6 = 79$
2 Step 1: Add 6 to both sides of the equation.	$5y - 6 + 6 = 79 + 6$ $5y = 85$
3 Step 2: Divide both sides of the equation by 5 to obtain *y*.	$\frac{5y}{5} = \frac{85}{5}$ $y = 17$
b **1** Write the equation.	**b** $\frac{4x}{9} = 5$
2 Step 1: Multiply both sides of the equation by 9.	$\frac{4x}{9} \times 9 = 5 \times 9$ $4x = 45$
3 Step 2: Divide both sides of the equation by 4 to obtain *x*.	$\frac{4x}{4} = \frac{45}{4}$ $x = \frac{45}{4}$
4 Express the improper fraction as a mixed number.	$x = 11\frac{1}{4}$

2.5.4 Equations where the pronumeral appears on both sides

- In solving equations where the pronumeral appears on both sides, subtract the smaller pronumeral term so that it is eliminated from both sides of the equation.

WORKED EXAMPLE 12

Solve the following equations.

a $\mathbf{5h + 13 = 2h - 2}$ **b** $\mathbf{14 - 4d = 27 - d}$ **c** $\mathbf{2(x - 3) = 5(2x + 4)}$

THINK	WRITE
a **1** Write the equation.	**a** $5h + 13 = 2h - 2$
2 Eliminate the pronumeral from the right-hand side by subtracting 2*h* from both sides of the equation.	$3h + 13 = -2$
3 Subtract 13 from both sides of the equation.	$3h = -15$
4 Divide both sides of the equation by 3 and write your answer.	$h = -5$
b **1** Write the equation.	**b** $14 - 4d = 27 - d$
2 Add 4*d* to both sides of the equation.	$14 = 27 + 3d$
3 Subtract 27 from both sides of the equation.	$-13 = 3d$
4 Divide both sides of the equation by 3.	$-\frac{13}{3} = d$

5 Express the improper fraction as a mixed number.	$-4\frac{1}{3} = d$
6 Write your answer so that *d* is on the left-hand side.	$d = -4\frac{1}{3}$
c **1** Write the equation.	**c** $2(x - 3) = 5(2x + 4)$
2 Expand the brackets on both sides of the equation.	$2x - 6 = 10x + 20$
3 Subtract $2x$ from both sides of the equation.	$2x - 2x - 6 = 10x - 2x + 20$
4 Subtract 20 from both sides of the equation.	$-6 - 20 = 8x + 20 - 20$ $-26 = 8x$
5 Divide both sides of the equation by 8.	$-\frac{26}{8} = x$
6 Simplify and write your answer with the pronumeral on the left-hand side.	$x = -\frac{13}{4}$

Exercise 2.5 Solving simple equations

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Individual pathways

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Questions: 1a–f, 2a–e, 3a–f, 4a–b, 5a–b, 6a–b, 7a–f, 8a–d, 9a–b, 10a–b, 11a–c, 12a–b, 13a–c, 15a–d, 16a–d, 17a–d, 19–21, 25	Questions: 1d–i, 2d–i, 3a–f, 4, 5a–b, 6a–b, 7d–i, 8c–f, 9c–g, 10a–d, 11c–f, 12, 13d–i, 15c–f, 16c–f, 17c–f, 19–21, 23, 25, 26	Questions: 3–6, 7d–i, 8c–f, 9e–i, 10d–f, 11d–f, 12, 14, 15g–i, 16g–i, 17g–i, 18–26

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Fluency

1. **WE10a** Solve the following equations.

a. $a + 61 = 85$ **b.** $k - 75 = 46$ **c.** $g + 9.3 = 12.2$
d. $r - 2.3 = 0.7$ **e.** $h + 0.84 = 1.1$ **f.** $i + 5 = 3$
g. $t - 12 = -7$ **h.** $q + \frac{1}{3} = \frac{1}{2}$ **i.** $x - 2 = -2$

2. **WE10b** Solve the following equations.

a. $\frac{f}{4} = 3$ **b.** $\frac{i}{10} = -6$ **c.** $6z = -42$
d. $9v = 63$ **e.** $6w = -32$ **f.** $\frac{k}{12} = \frac{5}{6}$
g. $4a = 1.7$ **h.** $\frac{m}{19} = \frac{7}{8}$ **i.** $\frac{y}{4} = 5\frac{3}{8}$

3. **WE10c, d** Solve the following equations.

a. $\sqrt{t} = 10$ **b.** $y^2 = 289$ **c.** $\sqrt{q} = 2.5$
d. $f^2 = 1.44$ **e.** $\sqrt{h} = \frac{4}{7}$ **f.** $p^2 = \frac{9}{64}$
g. $\sqrt{g} = \frac{15}{22}$ **h.** $j^2 = \frac{196}{961}$ **i.** $a^2 = 2\frac{7}{9}$

4. Solve the following equations.

a. $\sqrt{t} - 3 = 2$ **b.** $5x^2 = 180$ **c.** $3\sqrt{m} = 12$
d. $-2t^2 = -18$ **e.** $t^2 + 11 = 111$ **f.** $\sqrt{-5} = 0$

5. Solve the following equations.

a. $\sqrt[3]{x} = 2$ **b.** $x^3 = -27$ **c.** $\sqrt[3]{m} = \frac{1}{2}$

d. $x^3 = \frac{27}{64}$ **e.** $\sqrt[3]{m} = 0.2$ **f.** $w^3 = 15\frac{5}{8}$

6. Solve the following equations.

a. $x^3 + 1 = 0$ **b.** $3x^3 = -24$ **c.** $\sqrt[3]{m} + 5 = 6$

d. $-2 \times \sqrt[3]{w} = 16$ **e.** $\sqrt[3]{t} - 13 = -8$ **f.** $2x^3 - 14 = 2$

7. **WE11a** Solve the following.

a. $5a + 6 = 26$ **b.** $6b + 8 = 44$ **c.** $8i - 9 = 15$

d. $7f - 18 = 45$ **e.** $8q + 17 = 26$ **f.** $10r - 21 = 33$

g. $6s + 46 = 75$ **h.** $5t - 28 = 21$ **i.** $8a + 88 = 28$

8. Solve the following.

a. $\frac{f}{4} + 6 = 16$ **b.** $\frac{g}{6} + 4 = 9$ **c.** $\frac{r}{10} + 6 = 5$

d. $\frac{m}{9} - 12 = -10$ **e.** $\frac{n}{8} + 5 = 8.5$ **f.** $\frac{p}{12} - 1.8 = 3.4$

9. Solve the following.

a. $6(x + 8) = 56$ **b.** $7(y - 4) = 35$

c. $5(m - 3) = 7$ **d.** $3(2k + 5) = 24$

e. $5(3n - 1) = 80$ **f.** $6(2c + 7) = 58$

g. $2(x - 5) + 3(x - 7) = 19$ **h.** $3(x + 5) - 5(x - 1) = 12$

i. $3(2x - 7) - (x + 3) = -60$

10. **WE11b** Solve the following.

a. $\frac{3k}{5} = 15$ **b.** $\frac{9m}{8} = 18$ **c.** $\frac{7p}{10} = -8$

d. $\frac{8u}{11} = -3$ **e.** $\frac{11x}{4} = 2$ **f.** $\frac{4v}{15} = 0.8$

11. Solve the following.

a. $\frac{x - 5}{3} = 7$ **b.** $\frac{2m + 1}{3} = -3$ **c.** $\frac{3w - 1}{4} = 6$

d. $\frac{t - 5}{2} = 0$ **e.** $\frac{6 - x}{3} = -1$ **f.** $\frac{3n - 5}{4} = -6$

12. **MC** **a.** The solution to the equation $\frac{p}{5} + 2 = 7$ is:

A. $p = 5$ **B.** $p = 25$ **C.** $p = 45$ **D.** $p = 10$ **E.** $p = 1$

b. If $5h + 8 = 53$, then h is equal to:

A. $\frac{1}{5}$ **B.** 12.2 **C.** 225 **D.** 10 **E.** 9

c. The exact solution to the equation $14x = 75$ is:

A. $x = 5.357\ 142\ 857$ **B.** $x = 5.357$ (to 3 decimal places)

C. $x = 5\frac{5}{14}$ **D.** $x = 5.4$

E. $x = 5.5$

13. Solve the following equations.

a. $-x = 5$ **b.** $2 - d = 3$ **c.** $5 - p = -2$

d. $-7 - x = 4$ **e.** $-5h = 10$ **f.** $-6t = -30$

g. $-\frac{v}{5} = 4$ **h.** $-\frac{r}{12} = \frac{1}{4}$ **i.** $-4g = 3.2$

14. Solve the following equations.

a. $6 - 2x = 8$ **b.** $10 - 3v = 7$ **c.** $9 - 6l = -3$

d. $-3 - 2g = 1$ **e.** $-5 - 4t = -17$ **f.** $-\frac{3e}{5} = 14$

g. $-\frac{8j}{3} = 9$ **h.** $-\frac{k}{4} - 3 = 6$ **i.** $-\frac{4f}{7} + 1 = 8$

15. **WE12a** Solve the following equations.

a. $6x + 5 = 5x + 7$ **b.** $7b + 9 = 6b + 14$ **c.** $11w + 17 = 6w + 27$

d. $8f - 2 = 7f + 5$ **e.** $10t - 11 = 5t + 4$ **f.** $12r - 16 = 3r + 5$

g. $12g - 19 = 3g - 31$ **h.** $7h + 5 = 2h - 6$ **i.** $5a - 2 = 3a - 2$

16. **WE12b** Solve the following equations.

a. $5 - 2x = 6 - x$ **b.** $10 - 3c = 8 - 2c$ **c.** $3r + 13 = 9r - 3$

d. $k - 5 = 2k - 6$ **e.** $5y + 8 = 13y + 17$ **f.** $17 - 3g = 3 - g$

g. $14 - 5w = w + 8$ **h.** $4m + 7 = 8 - m$ **i.** $14 - 5p = 9 - 2p$

17. **WE12c** Solve the following equations.

a. $3(x + 5) = 2x$ **b.** $8(y + 3) = 3y$ **c.** $6(t - 5) = 4(t + 3)$

d. $10(u + 1) = 3(u - 3)$ **e.** $12(f - 10) = 4(f - 5)$ **f.** $2(4r + 3) = 3(2r + 7)$

g. $5(2d + 9) = 3(3d + 13)$ **h.** $5(h - 3) = 3(2h - 1)$ **i.** $2(4x + 1) = 5(3 - x)$

18. **MC** **a.** The solution to $8 - 4k = -2$ is:

A. $k = 2\frac{1}{2}$ **B.** $k = -2\frac{1}{2}$ **C.** $k = 1\frac{1}{2}$ **D.** $k = -1\frac{1}{2}$ **E.** $k = \frac{2}{5}$

b. The solution to $-\frac{6n}{5} + 3 = -7$ is:

A. $n = 3\frac{1}{3}$ **B.** $n = -3\frac{1}{3}$ **C.** $n = \frac{1}{3}$ **D.** $n = 8\frac{1}{3}$ **E.** $n = -8\frac{1}{3}$

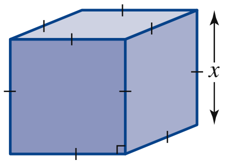
c. The solution to $p - 6 = 8 - 4p$ is:

A. $p = \frac{2}{5}$ **B.** $p = 2\frac{4}{5}$ **C.** $p = 4\frac{2}{3}$ **D.** $p = \frac{2}{3}$ **E.** $p = \frac{4}{5}$

Understanding

19. If the side length of a cube is x cm, then its volume V is given by $V = x^3$. What is the side length (correct to the nearest cm) of a cube that has a volume of:

a. 216 cm^3 **b.** 2 m^3?



20. The surface area of a cube with side length x cm is given by $A = 6x^2$. Find the side length (correct to the nearest cm) of a cube that has a surface area of:

a. 37.5 cm^2 **b.** 1 m^2.

21. A pebble is dropped down a well. In time t seconds it falls a distance of d metres, given by $d = 5t^2$.

a. How far does the pebble fall in 1 s?

b. How many seconds will it take the pebble to fall 40 m? (Answer correct to 1 decimal place.)

22. The surface area of a sphere is given by the formula $A = 4\pi r^2$, where r is the radius of the sphere.

a. Find the surface area of a sphere that has a radius of 5 cm.

b. What is the radius of a sphere that has a surface area equal to 500 cm^2? (Answer correct to the nearest mm.)

Reasoning

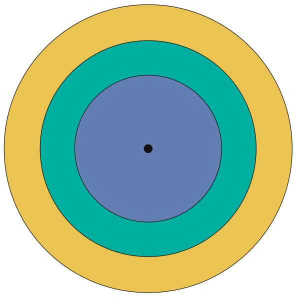
23. Find the radius of a circle of area 10 cm^2.

24. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. If the sphere can hold 1 litre of water, what is its radius correct to the nearest mm?

Problem solving

25. The width of a room is three-fifths of its length. When the width is increased by 2 metres and the length is decreased by 2 metres, the resultant shape is a square. Find the dimensions of the room.

26. A target board for a dart game has been designed as three concentric circles where each coloured region is the same area. If the radius of the purple circle is r cm and the radius of the outer circle is 10 cm, find the value of r.



Reflection

Describe in one sentence what it means to solve linear equations.

2.6 Solving multi-step equations

2.6.1 Equations with multiple brackets

Many equations need to be simplified by expanding brackets and collecting like terms before they are solved. Doing this reduces the equation to one of the basic types covered in the previous exercise.

WORKED EXAMPLE 13

TI | CASIO

Solve each of the following linear equations.

a $6(x + 1) - 4(x - 2) = 0$ **b** $7(5 - x) = 3(x + 1) - 10$

THINK	WRITE
a **1** Write the equation.	**a** $6(x + 1) - 4(x - 2) = 0$
2 Expand all the brackets. (Be careful with the −4.)	$6x + 6 - 4x + 8 = 0$
3 Collect like terms.	$2x + 14 = 0$
4 Subtract 14 from both sides of the equation.	$2x = -14$
5 Divide both sides of the equation by 2 to find the value of x.	$x = -7$
b **1** Write the equation.	**b** $7(5 - x) = 3(x + 1) - 10$
2 Expand all the brackets.	$35 - 7x = 3x + 3 - 10$
3 Collect like terms.	$35 - 7x = 3x - 7$
4 Create a single pronumeral term by adding $7x$ to both sides of the equation.	$35 = 10x - 7$
5 Add 7 to both sides of the equation.	$42 = 10x$
6 Divide both sides of the equation by 10 to solve for x and simplify.	$\frac{42}{10} = x$ $\frac{21}{5} = x$
7 Express the improper fraction as a mixed number fraction.	$4\frac{1}{5} = x$
8 Rewrite the equation so that x is on the left-hand side.	$x = 4\frac{1}{5}$

2.6.2 Equations involving algebraic fractions

- To solve an equation containing algebraic fractions, multiply both sides of the equation by the lowest common multiple (LCM) of the denominators. This gives an equivalent form of the equation without fractions.

WORKED EXAMPLE 14

Solve the equation $\frac{x-5}{3}=\frac{x+7}{4}$ and verify the solution.

THINK	WRITE
1 Write the equation.	$\frac{x-5}{3}=\frac{x+7}{4}$
2 The LCM is 3 × 4 = 12. Multiply both sides of the equation by 12.	$\frac{{}^{4}\cancel{12}(x-5)}{{}^{1}\cancel{3}}=\frac{{}^{3}\cancel{12}(x+7)}{{}^{1}\cancel{4}}$
3 Simplify the fractions.	$4(x-5)=3(x+7)$
4 Expand the brackets.	$4x-20=3x+21$
5 Subtract $3x$ from both sides of the equation.	$x-20=21$
6 Add 20 to both sides of the equation and write the answer.	$x=41$
7 To verify, check that the answer $x=41$ is true for both the left-hand side (LHS) and the right-hand side (RHS) of the equation by substitution.	
Substitute $x=41$ into the LHS.	$\begin{aligned}\text{LHS}&=\frac{41-5}{3}\\&=\frac{36}{3}\\&=12\end{aligned}$
Substitute $x=41$ into the RHS.	$\begin{aligned}\text{RHS}&=\frac{41+7}{4}\\&=\frac{48}{4}\\&=12\end{aligned}$
8 Write your answer.	Because the LHS = RHS, the solution $x=41$ is correct.

WORKED EXAMPLE 15

TI | CASIO

Solve each of the following equations.

a $\frac{5(x+3)}{6}=4+\frac{3(x-1)}{5}$

b $\frac{4}{3(x-1)}=\frac{1}{x+1}$

THINK	WRITE
a 1 Write the equation.	**a** $\frac{5(x+3)}{6}=4+\frac{3(x-1)}{5}$
2 The lowest common denominator of 5 and 6 is 30. Write each term as an equivalent fraction with a denominator of 30.	$\frac{25(x+3)}{30}=\frac{120}{30}+\frac{18(x-1)}{30}$
3 Multiply each term by 30. This effectively removes the denominator.	$25(x+3)=120+18(x-1)$
4 Expand the brackets and collect like terms.	$25x+75=120+18x-18$ $25x+75=102+18x$
5 Subtract $18x$ from both sides of the equation.	$7x+75=102$

	6 Subtract 75 from both sides of the equation.	$7x = 27$
	7 Divide both sides of the equation by 7 to solve for *x*.	$x = \frac{27}{7}$
	8 Express the improper fraction as a mixed number.	$x = 3\frac{6}{7}$
b	**1** Write the equation.	**b** $\frac{4}{3(x-1)} = \frac{1}{x+1}$
	2 The lowest common denominator of 3, $x + 1$ and $x - 1$ is $3(x - 1)(x + 1)$. Write each term as an equivalent fraction with a common denominator of $3(x - 1)(x + 1)$.	$\frac{4(x+1)}{3(x-1)(x+1)} = \frac{3(x-1)}{3(x-1)(x+1)}$
	3 Multiply each term by the common denominator.	$4(x + 1) = 3(x - 1)$
	4 Expand the brackets.	$4x + 4 = 3x - 3$
	5 Subtract $3x$ from both sides of the equation.	$x + 4 = -3$
	6 Subtract 4 from both sides of the equation to solve for *x*.	$x + 4 - 4 = -3 - 4$ $= -7$

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Exercise 2.6 Solving multi-step equations

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Individual pathways

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Questions: 1, 2a–c, 3a–e, 4a–d, 5, 6, 9	Questions: 1, 2a–f, 3b–i, 4c–g, 6, 7, 9, 11, 12	Questions: 1, 2c–h, 3g–l, 4g–l, 7–13

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Fluency

1. **WE13** Solve each of the following linear equations.

a. $6(4x - 3) + 7(x + 1) = 9$ **b.** $9(3 - 2x) + 2(5x + 1) = 0$

c. $8(5 - 3x) - 4(2 + 3x) = 3$ **d.** $9(1 + x) - 8(x + 2) = 2x$

e. $6(4 + 3x) = 7(x - 1) + 1$ **f.** $10(4x + 2) = 3(8 - x) + 6$

2. **WE14** Solve each of the following equations and verify the solutions.

a. $\frac{x+1}{2} = \frac{x+3}{3}$ **b.** $\frac{x-7}{5} = \frac{x-8}{4}$ **c.** $\frac{x-6}{4} = \frac{x-2}{2}$

d. $\frac{8x+3}{5} = 2x$ **e.** $\frac{2x-1}{5} = \frac{x-3}{4}$ **f.** $\frac{4x+1}{3} = \frac{x+2}{4}$

g. $\frac{6-x}{3} = \frac{2x-1}{5}$ **h.** $\frac{8-x}{9} = \frac{2x+1}{3}$

3. Solve each of the following linear equations.

a. $\frac{x}{3} + \frac{4x}{5} = \frac{1}{3}$ **b.** $\frac{x}{4} - \frac{x}{5} = \frac{3}{4}$ **c.** $\frac{x}{4} - \frac{4x}{7} = 2$

d. $\frac{-3x}{5}+\frac{x}{8}=\frac{1}{4}$ **e.** $\frac{2x}{3}-\frac{x}{6}=-\frac{3}{4}$ **f.** $\frac{5x}{8}-8=\frac{2x}{3}$

g. $\frac{2}{7}-\frac{x}{8}=\frac{3x}{8}$ **h.** $\frac{4}{x}-\frac{1}{6}=\frac{2}{x}$ **i.** $\frac{15}{x}-4=\frac{2}{x}$

j. $\frac{1}{3}+\frac{4}{x}=\frac{5}{x}$ **k.** $\frac{2x-4}{5}+6=\frac{x}{2}$ **l.** $\frac{4x-1}{2}-\frac{2x+5}{3}=0$

4. **WE15** Solve each of the following linear equations.

a. $\frac{3(x+1)}{2}+\frac{5(x+1)}{3}=4$ **b.** $\frac{2(x+1)}{7}+\frac{3(2x-5)}{8}=0$ **c.** $\frac{2(4x+3)}{5}-\frac{6(x-2)}{2}=\frac{1}{2}$

d. $\frac{8(x+3)}{5}=\frac{3(x+2)}{4}$ **e.** $\frac{5(7-x)}{2}=\frac{2(2x-1)}{7}+1$ **f.** $\frac{2(6-x)}{3}=\frac{9(x+5)}{6}+\frac{1}{3}$

g. $\frac{-5(x-2)}{3}-\frac{6(2x-1)}{5}=\frac{1}{3}$ **h.** $\frac{9(2x-1)}{7}=\frac{4(x-5)}{3}$ **i.** $\frac{1}{x-1}+\frac{3}{x+1}=\frac{8}{x+1}$

j. $\frac{3}{x+1}+\frac{5}{x-4}=\frac{5}{x+1}$ **k.** $\frac{1}{x-1}-\frac{3}{x}=\frac{-1}{x-1}$ **l.** $\frac{4}{2x-1}-\frac{5}{x}=\frac{-1}{x}$

Understanding

5. Last week Maya broke into her money box. She spent one-quarter of the money on a birthday present for her brother and one-third of the money on an evening out with her friends, leaving her with $75. How much money was in her money box?

6. At work Keith spends one-fifth of his time in planning and buying merchandise. He spends seven-twelfths of his time in customer service and one-twentieth of his time training the staff. This leaves him ten hours to deal with the accounts. How many hours does he work each week?

7. Last week's school fete was a great success, raising a good deal of money. Three-eighths of the profit came from sales of food and drink, and the market stalls recorded one-fifth of the total. A third of the profit came from the major raffle, and the jumping castle raised $1100. How much money was raised at the fete?

8. Lucy had half as much money as Mel, but since Grandma gave them each $20 she now has three-fifths as much. How much money does Lucy have?

Reasoning

9. a. Which numbers smaller than 100 have exactly 3 factors (including 1 and the number itself)?

b. Which two numbers smaller than 100 have exactly 5 factors?

c. Which number smaller than 100 has exactly 7 factors?

10. To raise money for a charity, a Year 10 class has decided to organise a school lunch. Tickets will cost $6 each. The students have negotiated a special deal for delivery of drinks and pizzas, and they have budgeted $200 for drinks and $250 for pizzas. If they raise $1000 or more, they qualify for a special award.

a. Write an equation to represent the minimum number of tickets required to be sold to qualify for the award.

b. Solve the equation to find the number of tickets they must sell to qualify for the award. Explain your answer.

11. If $\frac{x+7}{(x+2)(x+3)}\equiv\frac{a}{x+2}-\frac{4}{x+3}$, explain why a must be equal to 5.

(*Note:* '≡' means identically equal to.)

Problem solving

12. Solve for x:

$$\tfrac{2}{9}(x-1)-\tfrac{5}{8}(x-2)=\tfrac{2}{5}(x-4)-\tfrac{7}{12}.$$

13. If $\dfrac{2(4x+3)}{(x-3)(x+7)} \equiv \dfrac{a}{x-3}+\dfrac{b}{x+7}$, find the values of a and b.

Reflection

Do the rules for the order of operations apply to algebraic fractions? Explain.

2.7 Literal equations

2.7.1 Literal equations

- **Literal equations** are equations that include several pronumerals or variables. Solving literal equations involves changing the subject of the equation to a particular pronumeral.
- A formula is a literal equation that records an interesting or important real-life relationship.

WORKED EXAMPLE 16

Solve the following literal equations for x.

a $ax^2 + bd = c$ **b** $ax = cx + b$

THINK	WRITE
a 1 Write the equation.	**a** $ax^2 + bd = c$
2 Subtract bd from both sides of the equation.	$ax^2 = c - bd$
3 Divide both sides by a.	$x^2 = \dfrac{c-bd}{a}$
4 To solve for x, take the square root of both sides. This gives both a positive and negative result for x.	$x = \pm\sqrt{\dfrac{c-bd}{a}}$
b 1 Write the equation.	**b** $ax = cx + b$
2 Subtract cx from both sides.	$ax - cx = b$
3 Factorise by taking x as a common factor.	$x(a-c) = b$
4 To solve for x, divide both sides by $a - c$.	$x = \dfrac{b}{a-c}$

WORKED EXAMPLE 17

Make b the subject of the formula $D = \sqrt{b^2 - 4ac}$.

THINK	WRITE
1 Write the formula.	$D = \sqrt{b^2 - 4ac}$
2 Square both sides.	$D^2 = b^2 - 4ac$
3 Add $4ac$ to both sides of the equation.	$D^2 + 4ac = b^2$
4 Take the square root of both sides.	$\pm\sqrt{D^2 + 4ac} = b$
5 Make b the subject of the formula by solving for b.	$b = \pm\sqrt{D^2 + 4ac}$

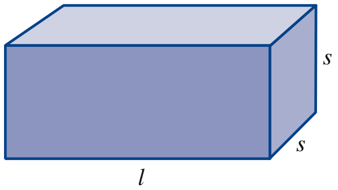
2.7.2 Restrictions on variables

- Some variables may have implicit restrictions on the values that they may be assigned in an equation or formula. For example:
  - if $V = \frac{d}{t}$, then t cannot equal zero, otherwise the value of V would be undefined.
  - if $d = \sqrt{x - 9}$, then:
    - the value of d will be restricted to positive values or 0
    - the value of $x - 9$ must be greater than or equal to zero because the square root of a negative number cannot be found.

      $x - 9 \geq 0$

      $x \geq 9$ (Hence x must be greater than or equal to 9.)
- Other restrictions may arise once a formula is rearranged. For example, if we look at the formula $V = ls^2$, there are no restrictions on the values that the variables l and s can be assigned. (However, the sign of V must always be the same as the sign of l because s^2 is always positive.) If the formula is transposed to make s the subject, then:

$$
\begin{aligned}
V &= ls^2 \\
\frac{V}{l} &= s^2 \\
\text{or } s &= \pm\sqrt{\frac{V}{l}}
\end{aligned}
$$



This shows the restrictions that $l \neq 0$ and $\frac{V}{l} \geq 0$.

If the formula $V = ls^2$ represents the volume of the rectangular prism shown, additional restrictions become evident: the variables l and s represent a length and must be positive numbers. Hence, when we make s the subject we get $s = \sqrt{\frac{V}{l}}$.

Exercise 2.7 Literal equations

assesson

Individual pathways

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Questions: 1a–f, 2a–d, 3a–d, 4a–d, 5, 7	Questions: 1a–g, 2a–f, 3a–f, 4a–f, 5, 7, 8	Questions: 1d–i, 2c–j, 3c–h, 4–8

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Fluency

1. **WE16** Solve the following literal equations for x.

   a. $\frac{ax}{bc} = d$ **b.** $\frac{x}{a} - bc = d$ **c.** $\sqrt{x} + n = m$ **d.** $acx^2 = w$

   e. $\frac{a}{x} = \frac{b}{y}$ **f.** $\frac{x + m}{n} = w$ **g.** $ab(x + b) = c$ **h.** $\frac{a}{x} = \frac{b}{c} + m$

   i. $mx = ay - bx$ **j.** $\frac{x}{m} + a = \frac{c}{d}$

2. **WE17** Rearrange each of the following literal equations to make the variable in brackets the subject.

   a. $V = lbh$ $[l]$ **b.** $P = 2l + 2b$ $[b]$ **c.** $A = \frac{1}{2}bh$ $[h]$

   d. $c = \sqrt{a^2 + b^2}$ $[a]$ **e.** $F = \frac{9C}{5} + 32$ $[C]$ **f.** $A = \pi r^2$ $[r]$

g. $v = u + at$ $[a]$ **h.** $I = \dfrac{PRN}{100}$ $[N]$ **i.** $E = \frac{1}{2}mv^2$ $[m]$

j. $E = \frac{1}{2}mv^2$ $[v]$ **k.** $v^2 = u^2 + 2as$ $[a]$ **l.** $v^2 = u^2 + 2as$ $[u]$

m. $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$ $[a]$ **n.** $x = \dfrac{nx_1 + mx_2}{m + n}$ $[x_1]$

3. Complete the following.

a. If $c = \sqrt{a^2 + b^2}$, calculate a if $c = 13$ and $b = 5$.

b. If $A = \frac{1}{2}bh$, find the value of h if $A = 56$ and $b = 16$.

c. If $F = \dfrac{9C}{5} + 32$, find the value of C if $F = 86$.

d. If $v = u + at$, find the value of a if $v = 83.6$, $u = 15$ and $t = 7$.

e. If $V = ls^2$, find the value of s if $V = 2028$ and $l = 12$.

f. If $v^2 = u^2 + 2as$, find the value of u if $v = 16$, $a = 10$ and $s = 6.75$.

g. If $A = \frac{1}{2}h(a + b)$, find the value of a if $A = 360$, $b = 15$ and $h = 18$.

h. If $x = \dfrac{nx_1 + mx_2}{m + n}$, find the value of x_2 if $x = 10$, $m = 2$, $n = 1$ and $x_1 = 4$.

Understanding

4. For the following equations:

i. list any restrictions on the variables in the equation.
ii. rearrange the equation to make the variable in brackets the subject.
iii. list any new restrictions on the variables in the equation formed in part **ii**.

a. $y = x^2 + 4$ $[x]$ **b.** $y = \dfrac{2}{x - 3}$ $[x]$

c. $v = u + at$ $[t]$ **d.** $c = \sqrt{a^2 + b^2}$ $[b]$

e. $s = \dfrac{a}{1 - r}$ $[r]$ **f.** $m = \dfrac{pb + qa}{p + q}$ $[b]$

g. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ [c] **h.** $m = \dfrac{pb + qa}{p + q}$ $[p]$

Reasoning

5. The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height of the cylinder.

a. State any restrictions on the values of the variables in this formula.
b. Make r the subject of the formula.
c. List any new restrictions on the variables in the formula.

6. T is the period of a pendulum whose length is l and g is the acceleration due to gravity. The formula relating these variables is $T = 2\pi\sqrt{\dfrac{l}{g}}$.

a. What restrictions are applied to the variables T and l?
b. Make l the subject of the equation.
c. Do the restrictions stated in part **a** still apply?
d. Find the length of a pendulum that has a period of 3 seconds, given that $g = 9.8$ m/s^2. Give your answer correct to 1 decimal place.

Problem solving

7. $F = 32 + \frac{9}{5}C$ is the formula relating degrees Celsius (C) to degrees Fahrenheit (F).

a. Transform the equation to make C the subject.

b. Find the temperature when degrees Celsius is equal to degrees Fahrenheit.

8. Jing Jing and Pieter live on the same main road but Jing Jing lives a kilometres to the east of Pieter. Both Jing Jing and Pieter set off on their bicycles at exactly the same time and both ride in a westerly direction. Jing Jing rides at j kilometres per hour and Pieter rides at p kilometres per hour. It is known that $j > p$. Find an equation in terms of a, j and p for the distance Jing Jing has ridden in order to catch up with Pieter.

Reflection

Why is it important to consider restrictions on variables when solving literal equations?

2.8 Review

2.8.1 Review questions

Fluency

1. Given $E = \frac{1}{2}mv^2$ where $m = 0.2$ and $v = 0.5$, the value of E is:

a. 0.000 625 **b.** 0.1 **c.** 0.005 **d.** 0.025 **e.** 0.0025

2. The expression $-6d + 3r - 4d - r$ simplifies to:

a. $2d + 2r$ **b.** $-10d + 2r$ **c.** $-10d - 4r$ **d.** $2d + 4r$ **e.** $-8dr$

3. The expression $5(2f + 3) + 6(4f - 7)$ simplifies to:

a. $34f + 2$ **b.** $34f - 4$ **c.** $34f - 27$ **d.** $34f + 14$ **e.** $116f - 14$

4. The expression $7(b - 1) - (8 - b)$ simplifies to:

a. $8b - 9$ **b.** $8b - 15$ **c.** $6b - 9$ **d.** $6b - 15$ **e.** $8b + 1$

5. If $14p - 23 = 6p - 7$ then p equals:

a. -3 **b.** -1 **c.** 1 **d.** 2 **e.** 4

6. Simplify the following by collecting like terms.

a. $3c - 5 + 4c - 8$ **b.** $-3k + 12m - 4k - 9m$

c. $-d + 3c - 8c - 4d$ **d.** $6y^2 + 2y + y^2 - 7y$

7. If $A = \frac{1}{2}bh$, determine the value of A if $b = 10$ and $h = 7$

8. For each of the following, complete the relationship to illustrate the stated law.

a. $(a + 3b) + 6c =$ __________ Associative Law

b. $12a - 3b \neq$ ______________ Commutative Law

c. $7p \times$ _____ $=$ ____________ Inverse Law

d. $(x \times 5y) \times 7z =$ ___________ Associative Law

e. $12p + 0 =$ ______________ Identity Law

f. $(3p \div 5q) \div 7r =$ __________ Associative Law

g. $9d + 11e =$ ______________ Commutative Law

h. $4a \div b \neq$ ________________ Commutative Law

9. Find the value of the following expressions given the natural number values $x = 12$, $y = 8$ and $z = 4$. Comment on whether the Closure Law holds for each of the expressions when the values are substituted.

a. $x \times y$ **b.** $z \div x$ **c.** $y - x$

10. Simplify the following.

a. $\frac{5y}{3} - \frac{y}{2}$ **b.** $\frac{x + 4}{5} + \frac{x + 2}{2}$ **c.** $\frac{5}{3x} - \frac{1}{5x}$ **d.** $\frac{x - 1}{x + 3} + \frac{2x - 5}{x + 2}$

11. Simplify the following.

a. $\frac{y}{4} \times \frac{32}{x}$ **b.** $\frac{20y}{7x} \times \frac{35z}{16y}$ **c.** $\frac{x+6}{(x+1)(x+3)} \times \frac{5(x+1)}{x+6}$

d. $\frac{25}{x} \div \frac{30}{x}$ **e.** $\frac{xy}{5} \div \frac{10x}{y}$ **f.** $\frac{2x}{(x+8)(x-1)} \div \frac{9x+1}{x+8}$

12. Solve the following equations.

a. $p - 20 = 68$ **b.** $s - 0.56 = 2.45$ **c.** $3b = 48$

d. $\frac{r}{7} = -5$ **e.** $\sqrt{x} = 12$ **f.** $2(x + 5) = -3$

g. $\frac{y}{4} - 3 = 12$ **h.** $a^2 = 36$ **i.** $5 - k = -7$

13. Solve the following.

a. $42 - 7b = 14$ **b.** $12t - 11 = 4t + 5$ **c.** $2(4p - 3) = 2(3p - 5)$

14. Solve each of the following linear equations.

a. $5(x - 2) + 3(x + 2) = 0$ **b.** $7(5 - 2x) - 3(1 - 3x) = 1$

c. $5(x + 1) - 6(2x - 1) = 7(x + 2)$ **d.** $8(3x - 2) + (4x - 5) = 7x$

e. $7(2x - 5) - 4(x + 20) = x - 5$ **f.** $3(x + 1) + 6(x + 5) = 3x + 40$

15. Solve each of the following equations.

a. $\frac{x}{2} + \frac{x}{5} = \frac{3}{5}$ **b.** $\frac{x}{3} - \frac{x}{5} = 3$

c. $-\frac{1}{21} = \frac{x}{7} - \frac{x}{6}$ **d.** $\frac{3}{x} + \frac{2}{5} = \frac{5}{x}$

e. $\frac{2x-3}{2} - \frac{3}{5} = \frac{x+3}{5}$ **f.** $\frac{2(x+2)}{3} = \frac{3}{7} + \frac{5(x+1)}{3}$

16. a. Make x the subject of $bx + cx = \frac{d}{2}$. **b.** Make r the subject of $V = \frac{4}{3}\pi r^3$.

Problem solving

17. A production is in town and many parents are taking their children. An adult ticket costs $15 and a child's ticket costs $8. Every child must be accompanied by an adult and each adult can have no more than 4 children with them. It costs the company $12 per adult and $3 per child to run the production. There is a seating limit of 300 people and all tickets are sold.

a. Determine how much profit the company makes on each adult ticket and on each child's ticket.

b. To maximise profit, the company should sell as many children's tickets as possible. Of the 300 available seats, determine how many should be allocated to children if there is a maximum of 4 children per adult.

c. Using your answer to part **b**, determine how many adults would make up the remaining seats.

d. Construct an equation to represent the profit that the company can make depending on the number of children and adults attending the production.

e. Substitute your values to calculate the maximum profit the company can make.

18. You are investigating prices for having business cards printed for your new games store. A local printing company charges a flat rate of $250 for the materials used and $40 per hour for labour.

a. If h is the number of hours of labour required to print the cards, construct an equation for the cost of the cards, C.

b. You have budgeted $1000 for the printing job. How many hours of labour can you afford? Give your answer to the nearest minute.

c. The printer estimates that it can print 1000 cards per hour of labour. How many cards will be printed with your current budget?

d. An alternative to printing is photocopying. The company charges 15 cents per side for the first 10 000 cards and then 10 cents per side for the remaining cards. Which is the cheaper option for 18 750 single-sided cards and by how much?

19. A scientist tried to use a mathematical formula to predict people's moods based on the number of hours of sleep they had the previous night. One formula that he used was what he called the 'grumpy formula', $g = 0.16(h - 8)^2$, which was valid on a 'grumpy scale' from 0 to 10 (least grumpy to most grumpy).

a. Calculate the number of hours needed to not be grumpy.

b. Evaluate the grumpy factor for somebody who has had:

- **i.** 4 hours of sleep
- **ii.** 6 hours of sleep
- **iii.** 10 hours of sleep.

c. Calculate the number of hours of sleep required to be most grumpy.

20. Another scientist already had his own grumpy formula and claims that the scientist from question **19** stole his idea and has just simplified it. The second scientist's grumpy formula was

$$g = \frac{0.16(h-8)}{8-h} \times \frac{2(8-h)}{3(h-8)} \div \frac{2h}{3(h-8)^2}.$$

a. Write the second scientist's formula in simplified form.

b. Are the second scientist's claims justified? Explain.

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Try out this interactivity: Word search: Topic 2 (int-2829)

Try out this interactivity: Crossword: Topic 2 (int-2830)

Try out this interactivity: Sudoku: Topic 2 (int-3589)

Complete this digital doc: Concept map: Topic 2 (doc-13706)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

algebra
denominator
equation
evaluate
expression
factor
formula
literal
multiple
numerator
pronumeral
restriction
simplify
substitution
undefined
variable

Investigation | Rich Task

Checking for data entry errors

When entering numbers into an electronic device, or even writing numbers down, errors frequently occur. A common type of error is a transposition error, which occurs when two digits are written in the reverse order. Take the number 2869, for example. With this type of error, it could be written as 8269, 2689 or 2896. A common rule for checking these errors is as follows.

If the difference between the correct number and the recorded number is a multiple of 9, a transposition error has occurred.

We can use algebraic expressions to check this rule. Let the digit in the thousands position be represented by a, the digit in the hundreds position by b, the digit in the tens position by c and the digit in the ones position by d. So the real number can be represented as $1000a + 100b + 10c + d$.

1. If the digits in the ones position and the tens position were written in the reverse order, the number would be $1000a + 100b + 10d + c$. The difference between the correct number and the incorrect one would then be: $1000a + 100b + 10c + d - (1000a + 100b + 10d + c)$.
   a. Simplify this expression.
   b. Is the expression a multiple of 9? Explain.
2. If a transposition error had occurred in the tens and hundreds position, the incorrect number would be $1000a + 100c + 10b + d$. Perform the procedure shown in question **1** to determine whether the difference between the correct number and the incorrect one is a multiple of 9.
3. Consider, lastly, a transposition error in the thousands and hundreds positions. Is the difference between the two numbers a multiple of 9?
4. Comment on the checking rule for transposition errors.

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Complete this digital doc: Code puzzle: In which country was the first practical ice-making machine and refrigerator produced in 1856? (doc-15919)

Answers

Topic 2 Algebra and equations

Exercise 2.2 Substitution

1. a. 5 **b.** 2 **c.** 0 **d.** 6 **e.** −17 **f.** 3
g. 30 **h.** 12 **i.** −12 **j.** 27 **k.** 30 **l.** −5

2. a. −11 **b.** −1 **c.** 1 **d.** 30 **e.** −24
f. 36 **g.** −125 **h.** 1 **i.** 15

3. a. $\frac{7}{12}$ **b.** $-\frac{1}{12}$ **c.** $\frac{1}{12}$ **d.** $1\frac{1}{3}$ **e.** $\frac{1}{576}$ **f.** 48

4. a. 9 **b.** −9 **c.** 9 **d.** 18 **e.** −18 **f.** 36

5. a. 9 **b.** −9 **c.** 9 **d.** 18 **e.** −18 **f.** 36

6. a. 17 **b.** 30 **c.** 8 **d.** 4 **e.** 1.5 **f.** 68
g. 46 **h.** 113.1 **i.** 5 **j.** 624.6

7. a. D **b.** C **c.** B

8. 3.9 cm

9. 65.45 cm^3

10. 361 m

11. a. −1 — in this case, addition is closed on integers.
b. −1 — in this case, subtraction is closed on integers.
c. 2 — in this case, multiplication is closed on integers.
d. −1 — in this case, division is closed on integers.
e. −2 — in this case, subtraction is closed on integers.
f. $-\frac{1}{2}$ — in this case, division is not closed on integers.

12. a. 10 — in this case, addition is closed on natural numbers.
b. −4 — in this case, subtraction is not closed on natural numbers.
c. 12 — in this case, multiplication is closed on natural numbers.
d. $\frac{4}{3}$ — in this case, division is not closed on natural numbers.
e. −2 — in this case, subtraction is not closed on natural numbers.
f. 4 — in this case, division is closed on natural numbers.

13. a. $(a + 2b) + 4c = a + (2b + 4c)$ **b.** $(x \times 3y) \times 5c = x \times (3y \times 5c)$
c. $2p \div q \neq q \div 2p$ **d.** $5d + q = q + 5d$
e. $3z + 0 = 0 + 3z = 3z$ **f.** $2x \times \frac{1}{2x} = \frac{1}{2x} \times 2x = 1$
g. $(4x \div 3y) \div 5z \neq 4x \div (3y \div 5z)$ **h.** $3d - 4y \neq 4y - 3d$

14. a. $s = 71.875$ metres **b.** $t = 4$ seconds

15. $m = \frac{4}{5}$

Challenge 2.1

$3a^3c^2$

Exercise 2.3 Adding and subtracting algebraic fractions

1. a. $\frac{26}{21}$ or $1\frac{5}{21}$ **b.** $\frac{49}{72}$ **c.** 1
d. $\frac{17}{99}$ **e.** $\frac{1}{35}$ **f.** $\frac{6-5x}{30}$
f. $\frac{15x-4}{27}$ **g.** $\frac{15-16x}{40}$ **h.** $\frac{15-2x}{3x}$

2. a. $\frac{5y}{12}$ **b.** $-\frac{3y}{40}$ **c.** $\frac{13x}{12}$ **d.** $\frac{14x}{9}$ **e.** $\frac{3w}{28}$ **f.** $-\frac{y}{5}$
g. $\frac{89y}{35}$ **h.** $\frac{32x}{15}$ **i.** $\frac{7x+17}{10}$ **j.** $\frac{7x+30}{12}$ **k.** $\frac{2x-11}{30}$ **l.** $\frac{19x+7}{6}$

3. a. $\frac{5}{8x}$ **b.** $\frac{5}{12x}$ **c.** $\frac{38}{21x}$ **d.** $\frac{8}{3x}$ **e.** $\frac{7}{24x}$ **f.** $\frac{9}{20x}$

g. $\frac{37}{100x}$ **h.** $\frac{51}{10x}$ **i.** $-\frac{1}{6x}$

4. a. $\frac{3x^2 + 14x - 4}{(x+4)(x-2)}$ **b.** $\frac{2x^2 + 3x + 25}{(x+5)(x-1)}$ **c.** $\frac{2x^2 + 6x - 10}{(2x+1)(x-2)}$ **d.** $\frac{4x^2 - 17x - 3}{(x+1)(2x-7)}$

e. $\frac{7x^2 + x}{(x+7)(x-5)}$ **f.** $\frac{2x^2 + 6x + 7}{(x+1)(x+4)}$ **g.** $\frac{-x^2 + 7x + 15}{(x+1)(x+2)}$ **h.** $\frac{x-7}{(x+3)(x-2)}$

i. $\frac{x^2 + 3x + 9}{(x+2)(3x-1)}$ **j.** $\frac{5 - 5x}{(x-1)(1-x)} = \frac{5}{x-1}$ **k.** $\frac{3x+7}{(x+1)^2}$ **l.** $\frac{3x-4}{(x-1)^2}$

5. a. The student transcribed the denominator incorrectly and wrote $(x + 2)$ instead of $(x - 2)$ in line 2.
Also, the student forgot that multiplying a negative number by a negative number gives a positive number.
Line 3 should have $+3$ in the numerator, not -1. They didn't multiply.

b. $\frac{x^2 - 5x + 3}{(x-1)(x-2)}$

6. a. $\frac{4x^2 + 17x + 17}{(x+2)(x+1)(x+3)}$ **b.** $\frac{7x^2 - 20x + 4}{(x-1)(x+2)(x-4)}$

c. $\frac{4x^2 + 17x + 19}{(x+1)(x+3)(x+2)}$ **d.** $\frac{2(2x^2 - 9x + 25)}{(x-4)(x-1)(x+3)}$

7. The lowest common denominator may not always be the product of the denominators. Each fraction must be multiplied by the correct multiple.

8. $a = 4$ **9.** $\frac{4(x-1)}{(x+3)(x+4)(x-2)}$ **10.** $\frac{2(x-1)}{(x-7)(x-4)}$

Exercise 2.4 Multiplying and dividing algebraic fractions

1. a. $\frac{4x}{y}$ **b.** $\frac{3x}{y}$ **c.** $\frac{4y}{x}$ **d.** $\frac{9x}{4y}$ **e.** $\frac{-5x}{4y}$ **f.** $\frac{3w}{2x}$

g. $\frac{6z}{7x}$ **h.** $\frac{2z}{7x}$ **i.** $\frac{-3x}{2y}$ **j.** $\frac{5}{24}$ **k.** $\frac{12z}{x}$ **l.** $\frac{-x}{6w}$

2. a. $\frac{2}{3x-2}$ **b.** $\frac{5}{x-3}$ **c.** $\frac{9}{2(x-6)}$ **d.** $\frac{1}{x+3}$ **e.** $\frac{2x}{(x+1)^2}$ **f.** $\frac{x+1}{2(2x-3)}$

g. $\frac{a}{10(a+3)}$ **h.** $\frac{35d}{8(d-3)}$ **i.** $\frac{9}{32x^2(x-2)}$ **j.** $\frac{3x}{10(x-1)}$

3. a. $\frac{3}{5}$ **b.** $\frac{2}{9}$ **c.** $\frac{1}{3}$ **d.** 3 **e.** $\frac{1}{25}$ **f.** $\frac{35}{6}$ or $5\frac{5}{6}$

g. $\frac{4y^2}{7}$ **h.** $\frac{2y^2}{25}$ **i.** $\frac{8y^2}{9}$ **j.** $\frac{32xy}{15}$ **k.** $\frac{2}{3}$ **l.** y^2

4. a. $\frac{9}{(3x-7)(x+3)}$ **b.** $\frac{1}{(x+2)(x-9)}$ **c.** $\frac{21(x-3)}{x+5}$ **d.** $\frac{13}{9(x-4)(x+1)}$

5. Yes, because all of the fractions have the same denominator and therefore can be added together.

6. a. -1

b. $4 - x$ considered to be the same as $x - 4$

7. 1

8. $\frac{(x+1)^2}{x^2+1}$

Challenge 2.2

$\frac{x(x+2)}{(x-1)(x+1)}$ or $\frac{x(x+2)}{x^2-1}$

Exercise 2.5 Solving simple equations

1. a. $a = 24$ **b.** $k = 121$ **c.** $g = 2.9$ **d.** $r = 3$ **e.** $h = 0.26$ **f.** $i = -2$

g. $t = 5$ **h.** $q = \frac{1}{6}$ **i.** $x = 0$

2. a. $f = 12$ **b.** $i = -60$ **c.** $z = -7$ **d.** $v = 7$ **e.** $w = -5\frac{1}{3}$ **f.** $k = 10$
g. $a = 0.425$ **h.** $m = 16\frac{5}{8}$ **i.** $y = 21\frac{1}{2}$

3. a. $t = 100$ **b.** $y = \pm 17$ **c.** $q = 6.25$ **d.** $f = \pm 1.2$ **e.** $h = \frac{16}{49}$ **f.** $p = \pm\frac{3}{8}$
g. $g = \frac{225}{484}$ **h.** $j = \pm\frac{14}{31}$ **i.** $a = \pm 1\frac{2}{3}$

4. a. $t = 25$ **b.** $x = \pm 6$ **c.** $m = 16$ **d.** $t = \pm 3$ **e.** $t = \pm 10$ **f.** $m = 25$

5. a. $x = 8$ **b.** $x = -3$ **c.** $m = \frac{1}{8}$ **d.** $x = \frac{3}{4}$ **e.** $m = 0.008$ **f.** $w = 2\frac{1}{2}$

6. a. $x = -1$ **b.** $x = -2$ **c.** $m = 1$ **d.** $w = -512$ **e.** $t = 125$ **f.** $x = 2$

7. a. $a = 4$ **b.** $b = 6$ **c.** $i = 3$ **d.** $f = 9$ **e.** $q = 1\frac{1}{8}$ **f.** $r = 5\frac{2}{5}$
g. $s = 4\frac{5}{6}$ **h.** $t = 9\frac{4}{5}$ **i.** $a = -7\frac{1}{2}$

8. a. $f = 40$ **b.** $g = 30$ **c.** $r = -10$ **d.** $m = 18$ **e.** $n = 28$ **f.** $p = 62.4$

9. a. $x = 1\frac{1}{3}$ **b.** $y = 9$ **c.** $m = 4\frac{2}{5}$ **d.** $k = 1\frac{1}{2}$ **e.** $n = 5\frac{2}{3}$ **f.** $c = 1\frac{1}{3}$
g. $x = 10$ **h.** $x = 4$ **i.** $x = -7\frac{1}{5}$

10. a. $k = 25$ **b.** $m = 16$ **c.** $p = -11\frac{3}{7}$ **d.** $u = -4\frac{1}{8}$ **e.** $x = \frac{8}{11}$ **f.** $v = 3$

11. a. $x = 26$ **b.** $m = -5$ **c.** $w = \frac{25}{3}$ **d.** $t = 5$ **e.** $x = 9$ **f.** $n = -\frac{19}{3}$

12. a. B **b.** E **c.** C

13. a. $x = -5$ **b.** $d = -1$ **c.** $p = 7$ **d.** $x = -11$ **e.** $h = -2$ **f.** $t = 5$
g. $v = -20$ **h.** $r = -3$ **i.** $g = -0.8$

14. a. $x = -1$ **b.** $v = 1$ **c.** $l = 2$ **d.** $g = -2$ **e.** $t = 3$ **f.** $e = -23\frac{1}{3}$
g. $j = -3\frac{3}{8}$ **h.** $k = -36$ **i.** $f = -12\frac{1}{4}$

15. a. $x = 2$ **b.** $b = 5$ **c.** $w = 2$ **d.** $f = 7$ **e.** $t = 3$ **f.** $r = 2\frac{1}{3}$
g. $g = -1\frac{1}{3}$ **h.** $h = -2\frac{1}{5}$ **i.** $a = 0$

16. a. $x = -1$ **b.** $c = 2$ **c.** $r = 2\frac{2}{3}$ **d.** $k = 1$ **e.** $y = -1\frac{1}{8}$ **f.** $g = 7$
g. $w = 1$ **h.** $m = \frac{1}{5}$ **i.** $p = 1\frac{2}{3}$

17. a. $x = -15$ **b.** $y = -4\frac{4}{5}$ **c.** $t = 21$ **d.** $u = -2\frac{5}{7}$ **e.** $f = 12\frac{1}{2}$ **f.** $r = 7\frac{1}{2}$
g. $d = -6$ **h.** $h = -12$ **i.** $x = 1$

18. a. A **b.** D **c.** B

19. a. 6 cm **b.** 1.26 m

20. a. 2.5 cm **b.** 41 cm

21. a. 5 m **b.** 2.8 s

22. a. 314 cm^2 **b.** 6.3 cm

23. 1.8 cm

24. 6.2 cm

25. Dimensions are 10 m by 6 m.

26. $\frac{10\sqrt{3}}{3}$ cm

Exercise 2.6 Solving multi-step equations

1. a. $x = \frac{20}{31}$ **b.** $x = 3\frac{5}{8}$ **c.** $x = \frac{29}{36}$ **d.** $x = -7$ **e.** $x = -2\frac{8}{11}$ **f.** $x = \frac{10}{43}$

2. a. $x = 3$ **b.** $x = 12$ **c.** $x = -2$ **d.** $x = \frac{3}{2}$ **e.** $x = -\frac{11}{3}$ or $x = -3\frac{2}{3}$
f. $x = \frac{2}{13}$ **g.** $x = 3$ **h.** $x = \frac{5}{7}$

3. a. $x = \frac{5}{17}$ **b.** $x = 15$ **c.** $x = -6\frac{2}{9}$ **d.** $x = -\frac{10}{19}$ **e.** $x = -1\frac{1}{2}$ **f.** $x = -192$
g. $x = \frac{4}{7}$ **h.** $x = 12$ **i.** $x = 3\frac{1}{4}$ **j.** $x = 3$ **k.** $x = 52$ **l.** $x = 1\frac{5}{8}$

4. a. $x = \frac{5}{19}$ **b.** $x = 1\frac{31}{58}$ **c.** $x = 4\frac{11}{14}$ **d.** $x = -3\frac{15}{17}$ **e.** $x = 5\frac{20}{43}$ **f.** $x = -1\frac{10}{13}$

g. $x = 1\frac{2}{61}$ **h.** $x = -4\frac{9}{26}$ **i.** $x = 1.5$ **j.** $x = -4\frac{1}{3}$ **k.** $x = 3$ **l.** $x = 1$

5. \$180

6. 60 hours

7. \$12 000

8. \$60

9. a. 4, 9, 25, 49 **b.** 16, 81 **b.** 64

10. a. $6x - 450 = 1000$

b. $241\frac{1}{3}$ tickets. This means they need to sell 242 tickets to qualify, as the number of tickets must be a whole number.

11. Teacher to check

12. 4

13. $a = 3$, $b = 5$

Exercise 2.7 Literal equations

1. a. $x = \frac{bcd}{a}$ **b.** $x = a(d + bc)$ **c.** $x = (m - n)^2$ **d.** $x = \pm\frac{w}{ac}$ **e.** $x = \frac{ay}{b}$ **f.** $x = nw - m$

g. $x = \frac{c}{ab} - b$ **h.** $x = \frac{ac}{b + mc}$ **i.** $x = \frac{ay}{m + b}$ **j.** $x = \frac{mc - amd}{d}$

2. a. $l = \frac{V}{bh}$ **b.** $b = \frac{P - 2l}{2}$ **c.** $h = \frac{2A}{b}$ **d.** $a = \pm\sqrt{c^2 - b^2}$ **e.** $C = \frac{5}{9}(F - 32)$ **f.** $r = \pm\sqrt{\frac{A}{\pi}}$

g. $a = \frac{v - u}{t}$ **h.** $N = \frac{100I}{PR}$ **i.** $m = \frac{2E}{v^2}$ **j.** $v = \pm\sqrt{\frac{2E}{m}}$ **k.** $a = \frac{v^2 - u^2}{2s}$

l. $u = \pm\sqrt{v^2 - 2as}$ **m.** $a = \frac{xb}{b - x}$ **n.** $x_1 = \frac{x(m + n) - mx_2}{n}$

3. a. $a = \pm 12$ **b.** $h = 7$ **c.** $C = 30$ **d.** $a = 9.8$ **e.** $s = \pm 13$ **f.** $u = \pm 11$

g. $a = 25$ **h.** $x_2 = 13$

4. a. i. No restrictions on x **ii.** $x = \pm\sqrt{y - 4}$ **iii.** $y \geq 4$

b. i. $x \neq 3$ **ii.** $x = \frac{2}{y} + 3$ **iii.** $y \neq 0$

c. i. No restrictions **ii.** $t = \frac{v - u}{a}$ **iii.** $a \neq 0$

d. i. $c \geq 0$ **ii.** $b = \pm\sqrt{c^2 - a^2}$ **iii.** $|c| \geq |a|$

e. i. $r \neq 1$ **ii.** $r = \frac{s - a}{s}$ **iii.** $s \neq 0$

f. i. $p \neq -q$ **ii.** $b = \frac{m(p + q) - qa}{p}$ **iii.** $p \neq 0$

g. i. $a \neq 0, b^2 \geq 4ac$ **ii.** $c = \frac{b^2 - (2ax + b)^2}{4a}$ or $c = -ax^2 - bx$ **iii.** No new restrictions

h. i. $p \neq -q$ **ii.** $p = \frac{q(a - m)}{m - b}$ **iii.** $m \neq b$

5. a. No restrictions, all values must be positive for a cylinder to exist.

b. $r = \sqrt{\frac{V}{\pi h}}$

c. $h \neq 0$, no new restrictions

6. a. T and l must be greater than zero.

b. $l = \frac{T^2 g}{4\pi^2}$ **c.** The restrictions still hold. **d.** 2.2 m

7. a. $C = \frac{5}{9}(F - 32)$ **b.** −40°

8. Distance Jing Jing has ridden is $\frac{ja}{j - p}$ kilometres.

2.8 Review

1. D

2. B

3. C

4. B

5. D

6. a. $7c - 13$ **b.** $-7k + 3m$ **c.** $-5c - 5d$ **d.** $7y^2 - 5y$

7. 35

8. a. $(a + 3b) + 6c = a + (3b + 6c)$ **b.** $12a - 3b \neq 3b - 12a$ **c.** $7p \times \frac{1}{7p} = \frac{1}{7p} \times 7p = 1$

d. $(x \times 5y) \times 7z = x \times (5y \times 7z)$ **e.** $12p + 0 = 0 + 12p = 12p$ **f.** $(3p \div 5q) \div 7r \neq 3p \div (5q \div 7r)$

g. $9d + 11e = 11e + 9d$ **h.** $4a \div b \neq b \div 4a$

9. a. 96 — in this case, multiplication is closed on natural numbers.

b. $\frac{1}{3}$ — in this case, division is not closed on natural numbers.

c. -4 — in this case, subtraction is not closed on natural numbers.

10. a. $\frac{7y}{6}$ **b.** $\frac{7x + 18}{10}$ **c.** $\frac{22}{15x}$ **d.** $\frac{3x^2 + 2x - 17}{(x + 3)(x + 2)}$

11. a. $\frac{8y}{x}$ **b.** $\frac{25z}{4x}$ **c.** $\frac{5}{x + 3}$ **d.** $\frac{5}{6}$ **e.** $\frac{y^2}{50}$ **f.** $\frac{2x}{(x - 1)(9x + 1)}$

12. a. $p = 88$ **b.** $s = 3.01$ **c.** $b = 16$ **d.** $r = -35$ **e.** $x = 144$ **f.** $x = -\frac{13}{2}$

g. $y = 60$ **h.** $a = \pm 6$ **i.** $k = 12$

13. a. $b = 4$ **b.** $t = 2$ **c.** $p = -2$

14. a. $x = \frac{1}{2}$ **b.** $x = 6\frac{1}{5}$ **c.** $x = -\frac{3}{14}$ **d.** $x = 1$ **e.** $x = 12\frac{2}{9}$ **f.** $x = 1\frac{1}{6}$

15. a. $x = \frac{6}{7}$ **b.** $x = 22\frac{1}{2}$ **c.** $x = 2$ **d.** $x = 5$ **e.** $x = 3\frac{3}{8}$ **f.** $x = -\frac{16}{21}$

16. a. $x = \frac{d}{2(b + c)}$ **b.** $r = \sqrt{\frac{3V}{4\pi}}$

17. a. \$3 per adult ticket; \$5 per child's ticket

b. 240

c. 60

d. $P = 3a + 5c$, where a = number of adults and c = number of children

e. \$1380

18. a. $C = 250 + 40h$

b. 18 hours 45 minutes

c. 18750

d. Printing is the cheaper option by \$1375.

19. a. 8 hours

b. i. 2.56 **ii.** 0.64 **iii.** 0.64

c. 0.094 hours or 15.9 hours

20. a. $g = \frac{0.16(h - 8)^2}{h}$ **b.** No, the formula is not the same.

Investigation — Rich task

1. a. $9(c - d)$

b. Yes, this is a multiple of 9 as the number that multiples the brackets is 9.

2. $90(b - c)$; 90 is a multiple of 9 so the difference between the correct and incorrect one is a multiple of 9.

3. $900(a - b)$; again 900 is a multiple of 9.

4. If two adjacent digits are transposed, the difference between the correct number and the transposed number is a multiple of 9.

TOPIC 3
Coordinate geometry

3.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learn ON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

3.1.1 Why learn this?

What did you weigh as a baby, and how tall were you? Did you grow at a steady (linear) rate, or were there periods in your life when you grew rapidly? What is the relationship between your height and your weight? We constantly seek to find relationships between variables, and coordinate geometry provides a picture, a visual clue as to what the relationships may be.

3.1.2 What do you know?

assesson

1. **THINK** List what you know about linear graphs and their equations. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a large concept map that shows your class's knowledge of linear graphs and their equations.

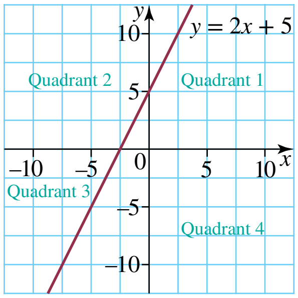
LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Descartes (eles-1842)

3.2 Sketching linear graphs

3.2.1 Linear graphs



- If a series of points (x, y) is plotted using the rule $y = mx + c$, then the points always lie in a straight line whose gradient equals m and whose y-intercept equals c.
- The rule $y = mx + c$ is called the equation of a straight line written in 'gradient–intercept' form.

3.2.2 Plotting linear graphs

- To plot a **linear graph**, complete a table of values to determine the points.

WORKED EXAMPLE 1

TI | CASIO

Plot the linear graph defined by the rule $y = 2x - 5$ for the x-values $-3, -2, -1, 0, 1, 2$ and 3.

THINK | **WRITE/DRAW**

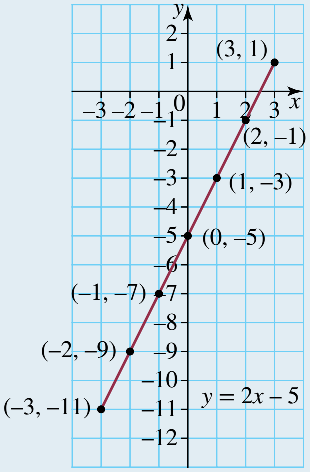
1 Create a table of values using the given x-values.

x	-3	-2	-1	0	1	2	3
y							

2 Find the corresponding y-values by substituting each x-value into the rule.

x	-3	-2	-1	0	1	2	3
y	-11	-9	-7	-5	-3	-1	1

3 Plot the points on a Cartesian plane and rule a straight line through them. Since the x-values have been specified, the line should only be drawn between the x-values of –3 and 3.



4 Label the graph.

3.2.3 Sketching straight lines

- A minimum of two points are necessary to plot a straight line.
- Two methods can be used to plot a straight line:
  - Method 1: The x- and y-intercept method.
  - Method 2: The gradient–intercept method.

3.2.4 Method 1: Sketching a straight line using the *x*- and *y*-intercepts

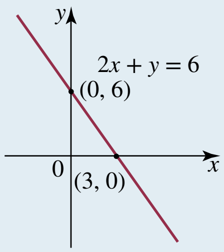
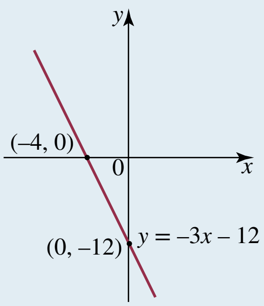
- As the name implies, this method involves plotting the x- and y-intercepts, then joining them to sketch the straight line.
- The line cuts the y-axis where $x = 0$ and the x-axis where $y = 0$.

WORKED EXAMPLE 2

Sketch graphs of the following linear equations by finding the x- and y-intercepts.

a $2x + y = 6$

b $y = -3x - 12$

THINK	WRITE/DRAW
a **1** Write the equation.	**a** $2x + y = 6$
2 Find the x-intercept by substituting $y = 0$.	x-intercept: when $y = 0$, $2x + 0 = 6$ $2x = 6$ $x = 3$ x-intercept is (3, 0).
3 Find the y-intercept by substituting $x = 0$.	y-intercept: when $x = 0$, $2(0) + y = 6$ $y = 6$ y-intercept is (0, 6).
4 Plot both points and rule the line.	
5 Label the graph.	
b **1** Write the equation.	**b** $y = -3x - 12$
2 Find the x-intercept by substituting $y = 0$ **i.** Add 12 to both sides of the equation. **ii.** Divide both sides of the equation by -3.	x-intercept: when $y = 0$, $-3x - 12 = 0$ $-3x = 12$ $x = -4$ x-intercept is (−4, 0).
3 Find the y-intercept. The equation is in the form $y = mx + c$, so compare this with our equation to find the y-intercept, c.	$c = -12$ y-intercept is (0, −12).
4 Plot both points and rule the line.	
5 Label the graph.	

3.2.5 Method 2: Sketching a straight line using the gradient–intercept method

- This method is often used if the equation is in the form $y = mx + c$, where m represents the gradient (slope) of the straight line, and c represents the y-intercept.
- The steps below outline how to use the gradient–intercept method to sketch a linear graph.

Step 1: Plot a point at the y-intercept.

Step 2: Write the gradient in the form $m = \frac{\text{rise}}{\text{run}}$. (To write a whole number as a fraction, place it over a denominator of 1.)

Step 3: Starting from the y-intercept, move up the number of units suggested by the rise (move down if the gradient is negative).

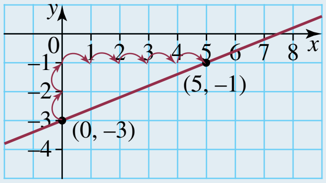
Step 4: Move to the right the number of units suggested by the run and plot the second point.

Step 5: Rule a straight line through the two points.

WORKED EXAMPLE 3

TI | CASIO

Sketch the graph of $y = \frac{2}{5}x - 3$ using the gradient–intercept method.

THINK	WRITE/DRAW
1 Write the equation of the line.	$y = \frac{2}{5}x - 3$
2 Identify the value of c (that is, the y-intercept) and plot this point.	$c = -3$, so y-intercept: $(0, -3)$.
3 Write the gradient, m, as a fraction.	$m = \frac{2}{5}$
4 $m = \frac{\text{rise}}{\text{run}}$, note the rise and run.	So, rise = 2; run = 5.
5 Starting from the y-intercept at $(0, -3)$, move 2 units up and 5 units to the right to find the second point $(5, -1)$. We have still not found the x-intercept.	

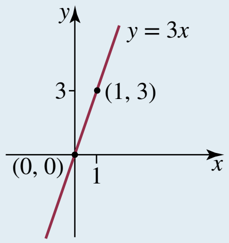
3.2.6 Sketching linear graphs of the form $y = mx$

- Graphs given by $y = mx$ pass through the origin $(0, 0)$, since $c = 0$.
- A second point may be determined using the rule $y = mx$ by substituting a value for x to find y.

WORKED EXAMPLE 4

Sketch the graph of $y = 3x$.

THINK	WRITE/DRAW
1 Write the equation.	$y = 3x$
2 Find the x- and y-intercepts. *Note:* By recognising the form of this linear equation, $y = mx$ you can simply state that the graph passes through the origin, $(0, 0)$.	x-intercept: when $y = 0$, $0 = 3x$ $x = 0$ y-intercept: $(0, 0)$ Both the x- and y-intercepts are at $(0, 0)$.

3 Find another point to plot by finding the y-value when $x = 1$.	When $x = 1$, $y = 3 \times 1$ $= 3$ Another point on the line is $(1, 3)$.
4 Plot the two points $(0, 0)$ and $(1, 3)$ and rule a straight line through them.	
5 Label the graph.	

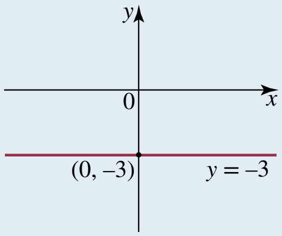
3.2.7 Sketching linear graphs of the form $y = c$ and $x = a$

- The line $y = c$ is parallel to the x-axis, having a gradient of zero and a y-intercept of c.
- The line $x = a$ is parallel to the y-axis and has an undefined (infinite) gradient.

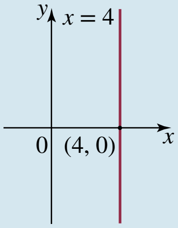
WORKED EXAMPLE 5

Sketch graphs of the following linear equations.

a $y = -3$ **b** $x = 4$

THINK	**WRITE/DRAW**
a 1 Write the equation.	**a** $y = -3$
2 The y-intercept is -3. As x does not appear in the equation, the line is parallel to the x-axis, such that all points on the line have a y-coordinate equal to -3. That is, this line is the set of points $(x, -3)$ where x is an element of the set of real numbers.	y-intercept $= -3$, $(0, -3)$
3 Sketch a horizontal line through $(0, -3)$.	
4 Label the graph.	
b 1 Write the equation.	**b** $x = 4$
2 The x-intercept is 4. As y does not appear in the equation, the line is parallel to the y-axis, such that all points on the line have an x-coordinate equal to 4. That is, this line is the set of points $(4, y)$ where y is an element of the set of real numbers.	x-intercept $= 4$, $(4, 0)$

3 Sketch a vertical line through (4, 0).



4 Label the graph.

3.2.8 Using linear graphs to model real-life contexts

- If a real-life situation involves a constant increase or decrease at regular intervals, then it can be modelled by a linear equation. Examples include water being poured from a tap into a container at a constant rate, or money being deposited into a savings account at regular intervals.
- To model a linear situation, we first need to determine which of the two given variables is the **independent variable** and which is the **dependent variable**.
- The independent variable does not depend on the value of the other variable, whereas the dependent variable takes its value depending on the value of the other variable. When plotting a graph of a linear model, the independent variable will be on the x-axis (horizontal) and the dependent variable will be on the y-axis (vertical).
- The following table contains a list of situations, with the independent and dependent variable being identified in each instance.

Situation	Independent variable	Dependent variable
Money being deposited into a savings account at regular intervals	Time	Money in account
The age of a person in years and their height in cm	Age in years	Height in cm
The temperature at a snow resort and the depth of the snow	Temperature	Depth of snow
The length of Pinocchio's nose and the amount of lies he told	Amount of lies Pinocchio told	Length of Pinocchio's nose
The number of workers building a house and the time taken to complete the project	Number of workers	Time

- Note that if time is one of the variables, it will usually be the independent variable. The final example above is a rare case of time being the dependent variable. Also, some of the above cases can't be modelled by linear graphs, as the increases or decreases aren't necessarily happening at constant rates.

WORKED EXAMPLE 6

Water is leaking from a bucket at a constant rate. After 1 minute there is 45 litres in the bucket; after 3 minutes there is 35 litres in the bucket; after 5 minutes there is 25 litres in the bucket; and after 7 minutes there is 15 litres in the bucket.

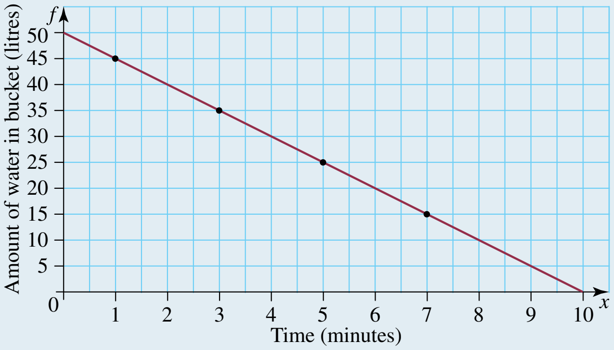
a **Define two variables to represent the given information.**

b **Determine which variable is the independent variable and which is the dependent variable.**

c **Represent the given information in a table of values.**

d **Plot a graph to represent how the amount of water in the bucket is changing.**

e **Use your graph to determine how much water was in the bucket at the start and how long it will take for the bucket to be empty.**

<table>
<tr><th>THINK</th><th>WRITE/DRAW</th></tr>
<tr><td>a Determine which two values change in the relationship given.</td><td>a The two variables are ‘time’ and ‘amount of water in bucket’.</td></tr>
<tr><td>b The dependent variable takes its value depending on the value of the independent variable.
In this situation the amount of water depends on the amount of time elapsed, not the other way round.</td><td>b Independent variable = time
Dependent variable = amount of water in bucket</td></tr>
<tr><td>c The independent variable should appear in the top row of the table of values, with the dependent variable appearing in the second row.</td><td>c
<table>
<tr><td>Time (minutes)</td><td>1</td><td>3</td><td>5</td><td>7</td></tr>
<tr><td>Amount of water in bucket (litres)</td><td>45</td><td>35</td><td>25</td><td>15</td></tr>
</table></td></tr>
<tr><td>d The values in the top row of the table represent the values on the horizontal axis, and the values in the bottom row of the table represent the values on the vertical axis. As the value for time can’t be negative and there can’t be a negative amount of water in the bucket, only the first quadrant needs to be drawn for the graph. Plot the 4 points and rule a straight line through them. Extend the graph to meet the vertical and horizontal axes.</td><td>d

</td></tr>
<tr><td>e The amount of water in the bucket at the start is the value at which the line meets the vertical axis, and the time taken for the bucket to be empty is the value at which the line meets the horizontal axis.</td><td>e There was 50 litres of water in the bucket at the start, and it will take 10 minutes for the bucket to be empty.</td></tr>
</table>

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Describing the gradient of a line (doc-5197)

Complete this digital doc: SkillSHEET: Plotting a line using a table of values (doc-5198)

Complete this digital doc: SkillSHEET: Stating the *y*-intercept from a graph (doc-5199)

Complete this digital doc: SkillSHEET: Solving linear equations that arise when finding *x*- and *y*-intercepts (doc-5200)

Complete this digital doc: SkillSHEET: Using Pythagoras’ theorem (doc-5201)

Complete this digital doc: SkillSHEET: Substitution into a linear rule (doc-5202)

Complete this digital doc: SkillSHEET: Transposing linear equations to standard form (doc-5203)

Exercise 3.2 Sketching linear graphs

assesson

Individual pathways

🟩 **PRACTISE**
Questions:
1, 2, 3a–h, 4a–e, 5a–d, 6a–f, 7a–d, 8a–d, 9, 10, 12

🟦 **CONSOLIDATE**
Questions:
1, 2, 3f–m, 4a–e, 5a–d, 6a–f, 7c–f, 8a–f, 9–12

🟪 **MASTER**
Questions:
1, 2, 3h–o, 4d–i, 5c–f, 6e–i, 7d–h, 8c–h, 9–13

🟩🟦🟪 **Individual pathway interactivity:** int-4572

learnon ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** Generate a table of values and then plot the linear graphs defined by the following rules for the given range of x-values.

Rule	**x-values**
a. $y = 10x + 25$	$-5, -4, -3, -2, -1, 0, 1$
b. $y = 5x - 12$	$-1, 0, 1, 2, 3, 4$
c. $y = -0.5x + 10$	$-6, -4, -2, 0, 2, 4$
d. $y = 100x - 240$	$0, 1, 2, 3, 4, 5$
e. $y = -5x + 3$	$-3, -2, -1, 0, 1, 2$
f. $y = 7 - 4x$	$-3, -2, -1, 0, 1, 2$

2. Plot the linear graphs defined by the following rules for the given range of x-values.

Rule **x-values**

a. $y = -3x + 2$

x	-6	-4	-2	0	2	4	6
y							

b. $y = -x + 3$

x	-3	-2	-1	0	1	2	3
y							

c. $y = -2x + 3$

x	-6	-4	-2	0	2	4	6
y							

3. **WE2** Sketch graphs of the following linear equations by finding the x- and y-intercepts.

a. $5x - 3y = 10$
b. $5x + 3y = 10$
c. $-5x + 3y = 10$
d. $-5x - 3y = 10$
e. $2x - 8y = 20$
f. $4x + 4y = 40$
g. $-x + 6y = 120$
h. $-2x + 8y = -20$
i. $10x + 30y = -150$
j. $5x + 30y = -150$
k. $-9x + 4y = 36$
l. $6x - 4y = -24$
m. $y = 2x - 10$
n. $y = -5x + 20$
o. $y = -\frac{1}{2}x - 4$

4. **WE3** Sketch graphs of the following linear equations using the gradient–intercept method.

a. $y = 4x + 1$
b. $y = 3x - 7$
c. $y = -2x + 3$
d. $y = -5x - 4$
e. $y = \frac{1}{2}x - 2$
f. $y = -\frac{2}{7}x + 3$
g. $y = 0.6x + 0.5$
h. $y = 8x$
i. $y = x - 7$

5. **WE4** Sketch the graphs of the following linear equations.

a. $y = 2x$
b. $y = 5x$
c. $y = -3x$
d. $y = \frac{1}{2}x$
e. $y = \frac{2}{3}x$
f. $y = -\frac{5}{2}x$

6. **WE5** Sketch the graphs of the following linear equations.

a. $y = 10$ **b.** $y = -10$ **c.** $x = 10$
d. $x = -10$ **e.** $y = 100$ **f.** $y = 0$
g. $x = 0$ **h.** $x = -100$ **i.** $y = -12$

7. Transpose each of the equations to gradient–intercept form (that is, $y = mx + c$). State the x- and y-intercept for each.

a. $5(y + 2) = 4(x + 3)$ **b.** $5(y - 2) = 4(x - 3)$ **c.** $2(y + 3) = 3(x + 2)$
d. $10(y - 20) = 40(x - 2)$ **e.** $4(y + 2) = -4(x + 2)$ **f.** $2(y - 2) = -(x + 5)$
g. $-5(y + 1) = 4(x - 4)$ **h.** $5(y + 2.5) = 2(x - 3.5)$ **i.** $2.5(y - 2) = -6.5(x - 1)$

Understanding

8. Find the x- and y-intercepts of the following lines.

a. $-y = 8 - 4x$ **b.** $6x - y + 3 = 0$ **c.** $2y - 10x = 50$

9. Explain why the gradient of a horizontal line is equal to zero and the gradient of a vertical line is undefined.

Reasoning

10. Determine whether $\frac{x}{3} - \frac{y}{2} = \frac{7}{6}$ is the equation of a straight line by rearranging into an appropriate form and hence sketch the graph, showing all relevant features.

11. **WE6** Your friend loves to download music. She earns \$50 and spends some of it buying music online at \$1.75 per song. She saves the remainder. Her saving is given by the function $f(x) = 50 - 1.75x$.

a. Determine which variable is the independent variable and which is the dependent variable.
b. Sketch the function.
c. How many songs can she buy and still save \$25?

Problem solving

12. A straight line has a general equation defined by $y = mx + c$. This line intersects the lines defined by the rules $y = 7$ and $x = 3$. The lines $y = mx + c$ and $y = 7$ have the same y-intercept while $y = mx + c$ and $x = 3$ have the same x-intercept.

a. On the one set of axes, sketch all three graphs.
b. Determine the y-axis intercept for $y = mx + c$.
c. Determine the gradient for $y = mx + c$.
d. **MC** The equation of the line defined by $y = mx + c$ is:

A. $x + y = 3$ **B.** $7x + 3y = 21$ **C.** $3x + 7y = 21$
D. $x + y = 7$ **E.** $7x + 3y = 7$

13. Water is flowing from a tank at a constant rate. The equation relating the volume of water in the tank, V litres, to the time the water has been flowing from the tank, t minutes, is given by $V = 80 - 4t$, $t \geq 0$.

a. Determine which variable is the independent variable and which is the dependent variable.
b. How much water is in the tank initially?
c. Why is it important that $t \geq 0$?
d. At what rate is the water flowing from the tank?
e. How long does it take for the tank to empty?
f. Sketch the graph of V versus t.

Reflection

What types of straight lines have an x-and y-intercept of the same value?

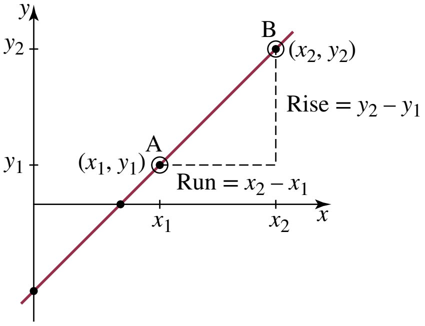
3.3 Determining linear equations

3.3.1 Finding a linear equation given two points

- The gradient of a straight line can be calculated from the coordinates of two points (x_1, y_1) and (x_2, y_2) that lie on the line.

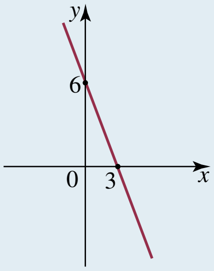
$$\text{Gradient} = m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

- The equation of the straight line can then be found in the form $y = mx + c$, where c is the y-intercept.



WORKED EXAMPLE 7

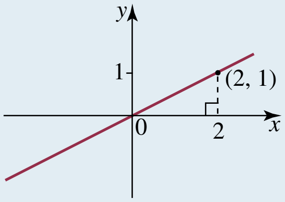
Find the equation of the straight line shown in the graph.



THINK	WRITE
1 There are two points given on the straight line: the x-intercept (3, 0) and the y-intercept (0, 6).	(3, 0), (0, 6)
2 Find the gradient of the line by applying the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$, where $(x_1, y_1) = (3, 0)$ and $(x_2, y_2) = (0, 6)$.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{6 - 0}{0 - 3}$ $= \frac{6}{-3}$ $= -2$ The gradient $m = -2$.
3 The graph has a y-intercept of 6, so $c = 6$. Substitute $m = -2$, and $c = 6$ into $y = mx + c$ to find the equation.	$y = mx + c$ $y = -2x + 6$

WORKED EXAMPLE 8

Find the equation of the straight line shown in the graph.



THINK	WRITE
1 There are two points given on the straight line: the x- and y-intercept $(0, 0)$ and another point $(2, 1)$.	$(0, 0), (2, 1)$
2 Find the gradient of the line by applying the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$, where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2, 1)$.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{1 - 0}{2 - 0}$ $= \frac{1}{2}$ The gradient $m = \frac{1}{2}$.
3 The y-intercept is 0, so $c = 0$. Substitute $m = \frac{1}{2}$ and $c = 0$ into $y = mx + c$ to determine the equation.	$y = mx + c$ $y = \frac{1}{2}x + 0$ $y = \frac{1}{2}x$

WORKED EXAMPLE 9

TI | CASIO

Find the equation of the straight line passing through $(-2, 5)$ and $(1, -1)$.

THINK	WRITE
1 Write the general equation of a straight line.	$y = mx + c$
2 Write the formula for calculating the gradient of a line between two points.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
3 Let (x_1, y_1) and (x_2, y_2) be the two points $(-2, 5)$ and $(1, -1)$ respectively. Substitute the values of the pronumerals into the formula to calculate the gradient.	$m = \frac{-1 - 5}{1 - -2}$ $= \frac{-6}{3}$ $= -2$
4 Substitute the value of the gradient into the general rule.	$y = -2x + c$
5 Select either of the two points, say $(1, -1)$, and substitute its coordinates into $y = -2x + c$.	Point $(1, -1)$: $-1 = -2 \times 1 + c$
6 Solve for c; that is, add 2 to both sides of the equation.	$-1 = -2 + c$ $1 = c$
7 State the equation by substituting the value of c into $y = -2x + c$.	The equation of the line is $y = -2x + 1$.

3.3.2 Finding the equation of a straight line using the gradient and one point

- If the gradient of a line is known, only one point is needed to determine the equation of the line.

WORKED EXAMPLE 10

Find the equation of the straight line with gradient of 2 and y-intercept of -5.

THINK	WRITE
1 Write the known information. The other point is the y-intercept, which makes the calculation of c straightforward.	Gradient = 2, y-intercept = -5
2 State the values of m and c.	$m = 2, c = -5$
3 Substitute these values into $y = mx + c$ to find the equation.	$y = mx + c$ $y = 2x - 5$

WORKED EXAMPLE 11

TI | CASIO

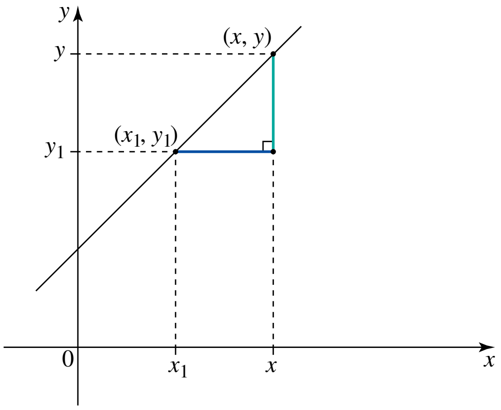
Find the equation of the straight line passing through the point $(5, -1)$ with a gradient of 3.

THINK	WRITE
1 Write the known information.	Gradient = 3, point $(5, -1)$.
2 State the values of m, x and y.	$m = 3, (x, y) = (5, -1)$
3 Substitute the values $m = 3$, $x = 5$ and $y = -1$ into $y = mx + c$ and solve to find c.	$y = mx + c$ $-1 = 3(5) + c$ $-1 = 15 + c$ $-16 = c$
4 Substitute $m = 3$ and $c = -16$ into $y = mx + c$ to determine the equation.	The equation of the line is $y = 3x - 16$.

3.3.3 A simple formula

- The diagram shows a line of gradient m passing through the point (x_1, y_1).
- If (x, y) is any other point on the line, then:

$$
\begin{aligned}
m &= \frac{\text{rise}}{\text{run}} \\
m &= \frac{y - y_1}{x - x_1} \\
m(x - x_1) &= y - y_1 \\
y - y_1 &= m(x - x_1)
\end{aligned}
$$



- The formula $y - y_1 = m(x - x_1)$ can be used to write down the equation of a line, given the gradient and the coordinates of one point.

WORKED EXAMPLE 12

Find the equation of the line with a gradient of -2 which passes through the point $(3, -4)$. Write the equation in general form, that is in the form $ax + by + c = 0$.

THINK	WRITE
1 Use the formula $y - y_1 = m(x - x_1)$. Write the values of x_1, y_1, and m.	$m = -2, x_1 = 3, y_1 = -4$ $y - y_1 = m(x - x_1)$
2 Substitute for x_1, y_1, and m into the equation.	$y - (-4) = -2(x - 3)$ $y + 4 = -2x + 6$
3 Transpose the equation into the form $ax + by + c = 0$.	$y + 4 + 2x - 6 = 0$ $2x + y - 2 = 0$

WORKED EXAMPLE 13

A printer prints pages at a constant rate. It can print 165 pages in 3 minutes and 275 pages in 5 minutes.

a Determine which variable is the independent variable (x) and which is the dependent variable (y).

b Determine the gradient of the equation and explain what this means in the context of the question.

c Write an equation, in algebraic form, linking the independent and dependent variables.

d Rewrite your equation in words.

e Using the equation, determine how many pages can be printed in 11 minutes.

THINK	WRITE/DRAW
a The dependent variable takes its value depending on the value of the independent variable. In this situation the number of pages depends on the time elapsed, not the other way round.	**a** Independent variable = time Dependent variable = number of pages
b 1 Determine the two points given by the information in the question.	**b** $(x_1, y_1) = (3, 165)$ $(x_2, y_2) = (5, 275)$
2 Substitute the values of these two points into the formula to calculate the gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{275 - 165}{5 - 3}$ $= \frac{110}{2}$ $= 55$
3 The gradient states how much the dependent variable increases for each increase of 1 unit in the independent variable.	In the context of the question, this means that each minute 55 pages are printed.
c The graph travels through the origin, as the time elapsed for the printer to print 0 pages is 0 seconds. Therefore, the equation will be in the form $y = mx$. Substitute in the value of m.	**c** $y = mx$ $y = 55x$

d Replace x and y in the equation with the independent and dependent variables.	**d** Number of pages = 55 × time
e **1** Substitute $x = 11$ into the equation.	**e** $y = 55x$ $= 55 \times 11$ $= 605$
2 Write the answer in words.	The printer can print 605 pages in 11 minutes.

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- **Complete this digital doc:** SkillSHEET: Measuring the rise and the run (doc-5196)
- **Complete this digital doc:** SkillSHEET: Finding the gradient given two points (doc-5204)
- **Complete this digital doc:** WorkSHEET: Gradient (doc-13849)

Exercise 3.3 Determining linear equations

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2, 3, 4, 5a–d, 7	Questions: 1a–f, 2, 3, 4, 5c–g, 7, 9	Questions: 1d–h, 2, 3, 4, 5e–j, 6–10

🟩🟦🟪 **Individual pathway interactivity:** int-4573

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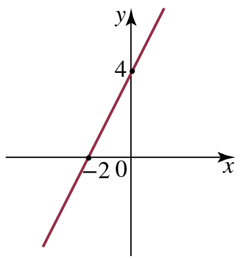
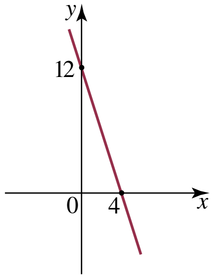
Fluency

1. **WE7** Determine the equation for each of the straight lines shown.

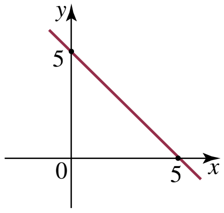
a.



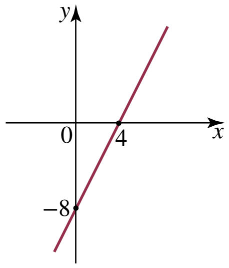
b.



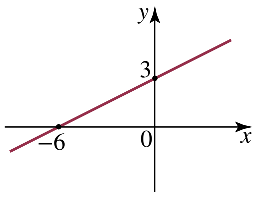
c.



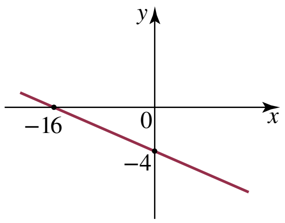
d.



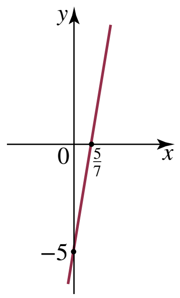
e.



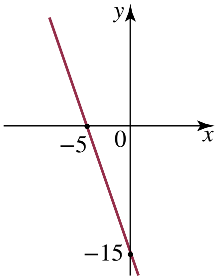
f.



g.

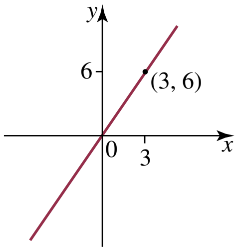


h.

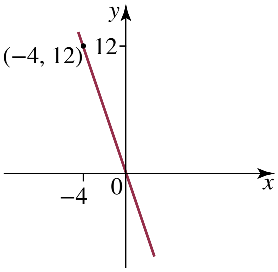


2. **WE8** Determine the equation of each of the straight lines shown.

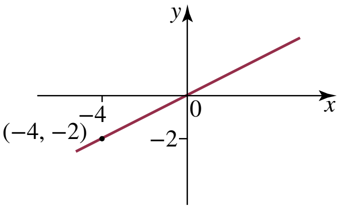
a.



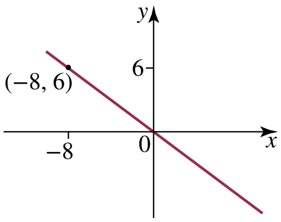
b.



c.



d.



3. **WE9** Find the equation of the straight line that passes through each pair of points.

a. (1, 4) and (3, 6) **b.** (0, −1) and (3, 5) **c.** (−1, 4) and (3, 2)
d. (3, 2) and (−1, 0) **e.** (−4, 6) and (2, −6) **f.** (−3, −5) and (−1, −7)

4. **WE10** Find the linear equation given the information in each case below.

a. Gradient = 3, y-intercept = 3
b. Gradient = −3, y-intercept = 4
c. Gradient = −4, y-intercept = 2
d. Gradient = 4, y-intercept = 2
e. Gradient = −1, y-intercept = −4
f. Gradient = 0.5, y-intercept = −4
g. Gradient = 5, y-intercept = 2.5
h. Gradient = −6, y-intercept = 3
i. Gradient = −2.5, y-intercept = 1.5
j. Gradient = 3.5, y-intercept = 6.5

5. **WE11, 12** For each of the following, find the equation of the straight line with the given gradient and passing through the given point.

a. Gradient = 5, point = (5, 6)
b. Gradient = −5, point = (5, 6)
c. Gradient = −4, point = (−2, 7)
d. Gradient = 4, point = (8, −2)
e. Gradient = 3, point = (10, −5)
f. Gradient = −3, point = (3, −3)
g. Gradient = −2, point = (20, −10)
h. Gradient = 2, point = (2, −0.5)
i. Gradient = 0.5, point = (6, −16)
j. Gradient = −0.5, point = (5, 3)

Understanding

6. **WE13** **a.** Determine which variable (time or cost) is the independent variable and which is the dependent variable.

b. If t represents the time in hours and C represents cost ($), construct a table of values for 0–3 hours for the cost of playing ten-pin bowling at the new alley.

Save $$$ with Supa-Bowl!!!
NEW Ten-Pin Bowling Alley
Shoe rental just $2 (fixed fee)
Rent a lane for ONLY $6/hour!

c. Use your table of values to plot a graph of time versus cost. (*Hint:* Ensure your time axis (horizontal axis) extends to 6 hours and your cost axis (vertical axis) extends to \$40.)

d. i. What is the y-intercept?

  ii. What does the y-intercept represent in terms of the cost?

e. Calculate the gradient and explain what this means in the context of the question.

f. Write a linear equation to describe the relationship between cost and time.

g. Use your linear equation from part **f** to calculate the cost of a 5-hour tournament.

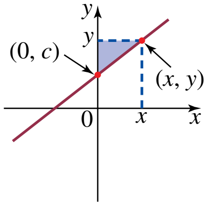
h. Use your graph to check your answer to part **g**.

Reasoning

7. When using the gradient to draw a line, does it matter if you rise before you run or run before you rise? Explain your answer.

8. a. Using the graph at right, write a general formula for the gradient m in terms of x, y and c.

b. Transpose your formula to make y the subject. What do you notice?

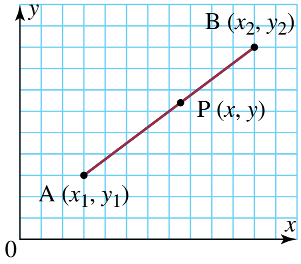


Problem solving

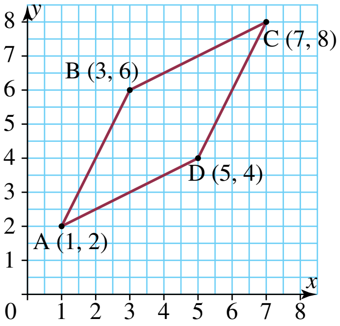
9. The points A (x_1, y_1), B (x_2, y_2) and P (x, y) are co-linear. P is a general point that lies anywhere on the line.

Show that an equation relating these three points is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$



10. Show that the quadrilateral ABCD is a parallelogram.

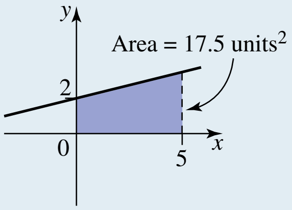


Reflection

What problems might you encounter when calculating the equation of a line whose graph is actually parallel to one of the axes?

CHALLENGE 3.1

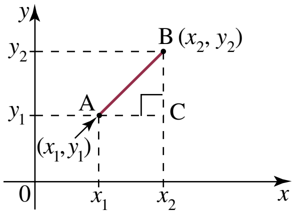
The graph of the straight line crosses the y-axis at (0, 2). The shaded section represents an area of 17.5 square units. Use this information to determine the equation of the line.



3.4 The distance between two points

3.4.1 The distance between two points

- The distance between two points can be calculated using Pythagoras' theorem.
- Consider two points A (x_1, y_1) and B (x_2, y_2) on the Cartesian plane as shown below.



- If point C is placed as shown, ABC is a right-angled triangle and AB is the hypotenuse.

$$AC = x_2 - x_1$$
$$BC = y_2 - y_1$$

By Pythagoras' theorem:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

Hence $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

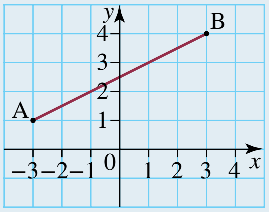
The distance between two points A (x_1, y_1) and B (x_2, y_2) is:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- This distance formula can be used to calculate the distance between any two points on the Cartesian plane.
- The distance formula has many geometric applications.

WORKED EXAMPLE 14

Find the distance between the points A and B in the figure below.
Answer correct to two decimal places.



THINK	WRITE
1 From the graph, locate points A and B.	A (−3, 1) and B (3, 4)
2 Let A have coordinates (x_1, y_1).	Let $(x_1, y_1) = (-3, 1)$
3 Let B have coordinates (x_2, y_2).	Let $(x_2, y_2) = (3, 4)$
4 Find the length AB by applying the formula for calculating the distance between two points.	$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[3 - (-3)]^2 + (4 - 1)^2} \\ &= \sqrt{(6)^2 + (3)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \\ &= 6.71 \text{ (correct to 2 decimal places)} \end{aligned}$

Note: If the coordinates were named in the reverse order, the formula would still give the same answer. Check this for yourself using $(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (-3, 1)$.

WORKED EXAMPLE 15

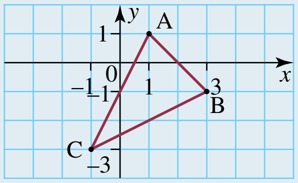
TI | CASIO

Find the distance between the points P (−1, 5) and Q (3, −2).

THINK	**WRITE**
1 Let P have coordinates (x_1, y_1).	Let $(x_1, y_1) = (-1, 5)$
2 Let Q have coordinates (x_2, y_2).	Let $(x_2, y_2) = (3, -2)$
3 Find the length PQ by applying the formula for the distance between two points.	$\text{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{[3 - (-1)]^2 + (-2 - 5)^2}$ $= \sqrt{(4)^2 + (-7)^2}$ $= \sqrt{16 + 49}$ $= \sqrt{65}$ $= 8.06$ (correct to 2 decimal places)

WORKED EXAMPLE 16

Prove that the points A (1, 1), B (3, −1) and (−1, −3) are the vertices of an isosceles triangle.

THINK	**WRITE/DRAW**
1 Plot the points and draw the triangle. *Note*: For triangle ABC to be isosceles, two sides must have the same magnitude.	
2 AC and BC seem to be equal. Find the length AC. A (1, 1) = (x_2, y_2) C (−1, −3) = (x_1, y_1)	$\text{AC} = \sqrt{[1 - (-1)]^2 + [1 - (-3)]^2}$ $= \sqrt{(2)^2 + (4)^2}$ $= \sqrt{20}$ $= 2\sqrt{5}$
3 Find the length BC. B (3, −1) = (x_2, y_2) C (−1, −3) = (x_1, y_1)	$\text{BC} = \sqrt{[3 - (-1)]^2 + [-1 - (-3)]^2}$ $= \sqrt{(4)^2 + (2)^2}$ $= \sqrt{20}$ $= 2\sqrt{5}$
4 Find the length AB. A (1, 1) = (x_1, y_1) B (3, −1) = (x_2, y_2)	$\text{AB} = \sqrt{[3 - (1)]^2 + [-1 - (1)]^2}$ $= \sqrt{(2)^2 + (-2)^2}$ $= \sqrt{4 + 4}$ $= 2\sqrt{2}$
5 State your proof.	Since AC = BC ≠ AB, triangle ABC is an isosceles triangle.

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Complete this digital doc: Spreadsheet: Distance between two points (doc-5206)

Exercise 3.4 The distance between two points

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 2a–d, 5, 8, 9	Questions: 1, 2c–f, 5, 7, 9, 11	Questions: 1, 2e–h, 3–7, 9–12

🟩🟦🟪 **Individual pathway interactivity:** int-4574

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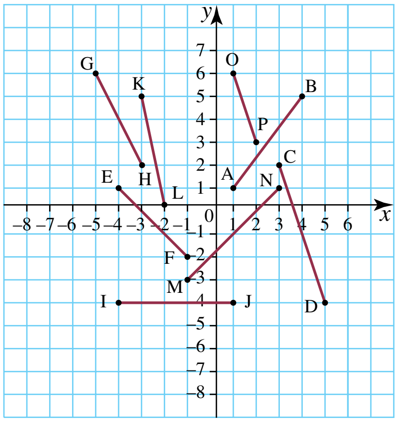
Fluency



1. **WE14** Find the distance between each pair of points shown at right.

2. **WE15** Find the distance between the following pairs of points.

a. $(2, 5), (6, 8)$
b. $(-1, 2), (4, 14)$
c. $(-1, 3), (-7, -5)$
d. $(5, -1), (10, 4)$
e. $(4, -5), (1, 1)$
f. $(-3, 1), (5, 13)$
g. $(5, 0), (-8, 0)$
h. $(1, 7), (1, -6)$
i. $(a, b), (2a, -b)$
j. $(-a, 2b), (2a, -b)$

Understanding

3. **MC** If the distance between the points $(3, b)$ and $(-5, 2)$ is 10 units, then the value of b is:

A. -8
B. -4
C. 4
D. 0
E. 2

4. **MC** A rhombus has vertices A $(1, 6)$, B $(6, 6)$, C $(-2, 2)$ and D (x, y). The coordinates of D are:

A. $(2, -3)$
B. $(2, 3)$
C. $(-2, 3)$
D. $(3, 2)$
E. $(3, -2)$

5. The vertices of a quadrilateral are A $(1, 4)$, B $(-1, 8)$, C $(1, 9)$ and D $(3, 5)$.

a. Find the lengths of the sides.
b. Find the lengths of the diagonals.
c. What type of quadrilateral is it?

Reasoning

6. **WE16** Prove that the points A $(0, -3)$, B $(-2, -1)$ and C $(4, 3)$ are the vertices of an isosceles triangle.

7. The points P $(2, -1)$, Q $(-4, -1)$ and R $(-1, 3\sqrt{3} - 1)$ are joined to form a triangle. Prove that triangle PQR is equilateral.

8. Prove that the triangle with vertices D $(5, 6)$, E $(9, 3)$ and F $(5, 3)$ is a right-angled triangle.

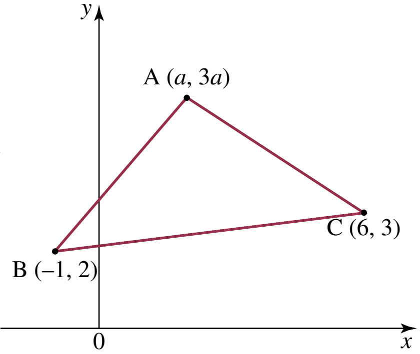
9. A rectangle has vertices A $(1, 5)$, B $(10.6, z)$, C $(7.6, -6.2)$ and D $(-2, 1)$. Find:

a. the length of CD
b. the length of AD
c. the length of the diagonal AC
d. the value of z.

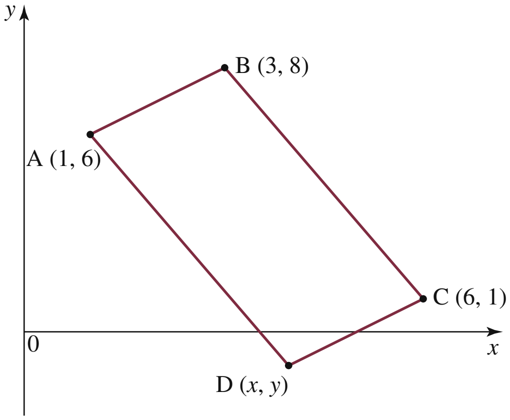
10. Show that the triangle ABC with coordinates A (a, a), B $(m, -a)$ and C $(-a, m)$ is isosceles.

Problem solving

11. Triangle ABC is an isosceles triangle where AB = AC, B is the point $(-1, 2)$, C is the point $(6, 3)$ and A is the point $(a, 3a)$ Find the value of the integer constant a.



12. ABCD is a parallelogram.
  a. Find the gradients of AB and BC.
  b. Find the coordinates of the point D (x, y).
  c. Show that the diagonals AC and BD bisect each other.



Reflection

How could you use the distance formula to show that a series of points lay on the circumference of a circle with centre *C*?

3.5 The midpoint of a line segment

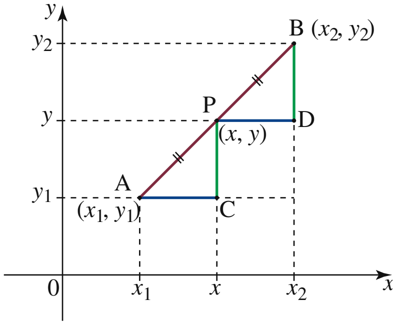
3.5.1 Midpoint of a line segment

- The **midpoint** of a **line segment** is the halfway point.
- The x- and y-coordinates of the midpoint are halfway between those of the coordinates of the end points.
- The following diagram shows the line interval AB joining points A (x_1, y_1) and B (x_2, y_2).

The midpoint of AB is P, so AP = PB.

Points C (x, y_1) and D (x_2, y) are added to the diagram and are used to make the two right-angled triangles ΔABC and ΔPBD.

The two triangles are congruent:



$$\begin{aligned} AP &= PB && \text{(given)} \\ \angle APC &= \angle PBD && \text{(corresponding angles)} \\ \angle CAP &= \angle DPB && \text{(corresponding angles)} \\ \text{So } \Delta APC &= \Delta PBD && \text{(ASA)} \end{aligned}$$

This means that AC = PD;

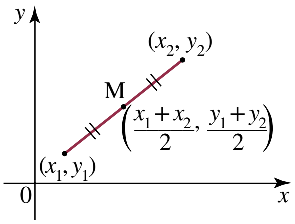
$$\begin{aligned} \text{i.e. } x - x_1 &= x_2 - x \qquad \text{(solve for } x\text{)} \\ \text{i.e. } 2x &= x_1 + x_2 \\ x &= \frac{x_1 + x_2}{2} \end{aligned}$$

In other words x is simply the average of x_1 and x_2.

Similarly, $y = \dfrac{y_1 + y_2}{2}$.

In general, the coordinates of the midpoint of a line segment joining the points (x_1, y_1) and (x_2, y_2) can be found by averaging the x- and y-coordinates of the end points, respectively.

The coordinates of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) are: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



WORKED EXAMPLE 17

TI | CASIO

Find the coordinates of the midpoint of the line segment joining (−2, 5) and (7, 1).

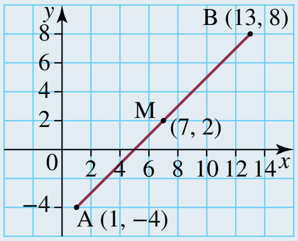
THINK	WRITE
1 Label the given points (x_1, y_1) and (x_2, y_2).	Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (7, 1)$
2 Find the x-coordinate of the midpoint.	$x = \frac{x_1 + x_2}{2}$ $= \frac{-2+7}{2}$ $= \frac{5}{2}$ $= 2^{\frac{1}{2}}$
3 Find the y-coordinate of the midpoint.	$y = \frac{y_1 + y_2}{2}$ $= \frac{5+1}{2}$ $= \frac{6}{2}$ $= 3$
4 Give the coordinates of the midpoint.	The midpoint is $(2\frac{1}{2}, 3)$.

WORKED EXAMPLE 18

The coordinates of the midpoint, M, of the line segment AB are (7, 2). If the coordinates of A are (1, −4), find the coordinates of B.

THINK	WRITE/DRAW
1 Let the start of the line segment be (x_1, y_1) and the midpoint be (x, y).	Let $(x_1, y_1) = (1, -4)$ and $(x, y) = (7, 2)$
2 The average of the x-coordinates is 7. Find the x-coordinate of the end point.	$x = \frac{x_1 + x_2}{2}$ $7 = \frac{1 + x_2}{2}$ $14 = 1 + x_2$ $x_2 = 13$
3 The average of the y-coordinates is 2. Find the y-coordinate of the end point.	$y = \frac{y_1 + y_2}{2}$ $2 = \frac{-4 + y_2}{2}$ $4 = -4 + y_2$ $y_2 = 8$
4 Give the coordinates of the end point.	The coordinates of the point B are (13, 8).

5 Check that the coordinates are feasible by drawing a diagram.



learnON RESOURCES — ONLINE ONLY

Complete this digital doc: Spreadsheet: Midpoint of a segment (doc-5207)

Complete this digital doc: WorkSHEET: Midpoint of a line segment (doc-13850)

Exercise 3.5 The midpoint of a line segment

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2, 3a, 4, 9, 11	Questions: 1a–d, 2–6, 9, 11	Questions: 1a–f, 2–12

🟩🟦🟪 **Individual pathway interactivity:** int-4575 **learnON** ONLINE ONLY

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Fluency

1. **WE17** Use the formula method to find the coordinates of the midpoint of the line segment joining the following pairs of points.

a. $(-5, 1), (-1, -8)$ **b.** $(4, 2), (11, -2)$ **c.** $(0, 4), (-2, -2)$
d. $(3, 4), (-3, -1)$ **e.** $(a, 2b), (3a, -b)$ **f.** $(a + 3b, b), (a - b, a - b)$

2. **WE18** The coordinates of the midpoint, M, of the line segment AB are $(2, -3)$. If the coordinates of A are $(7, 4)$, find the coordinates of B.

Understanding

3. A square has vertices A $(0, 0)$, B $(2, 4)$, C $(6, 2)$ and D $(4, -2)$. Find:

a. the coordinates of the centre **b.** the length of a side
c. the length of a diagonal.

4. **MC** The midpoint of the line segment joining the points $(-2, 1)$ and $(8, -3)$ is:

A. $(6, -2)$ **B.** $(5, 2)$ **C.** $(6, 2)$ **D.** $(3, -1)$ **E.** $(5, -2)$

5. **MC** If the midpoint of AB is $(-1, 5)$ and the coordinates of B are $(3, 8)$, then A has coordinates:

A. $(1, 6.5)$ **B.** $(2, 13)$ **C.** $(-5, 2)$ **D.** $(4, 3)$ **E.** $(7, 11)$

6. a. The vertices of a triangle are A $(2, 5)$, B $(1, -3)$ and C $(-4, 3)$. Find:
  i. the coordinates of P, the midpoint of AC
  ii. the coordinates of Q, the midpoint of AB
  iii. the length of PQ.
b. Show that BC = 2 PQ.

7. a. A quadrilateral has vertices A $(6, 2)$, B $(4, -3)$, C $(-4, -3)$ and D $(-2, 2)$. Find:
  i. the midpoint of the diagonal AC
  ii. the midpoint of the diagonal BD.
b. What can you infer about the quadrilateral?

8. **a.** The points A $(-5, 3.5)$, B $(1, 0.5)$ and C $(-6, -6)$ are the vertices of a triangle. Find:
   - **i.** the midpoint, P, of AB
   - **ii.** the length of PC
   - **iii.** the length of AC
   - **iv.** the length of BC.

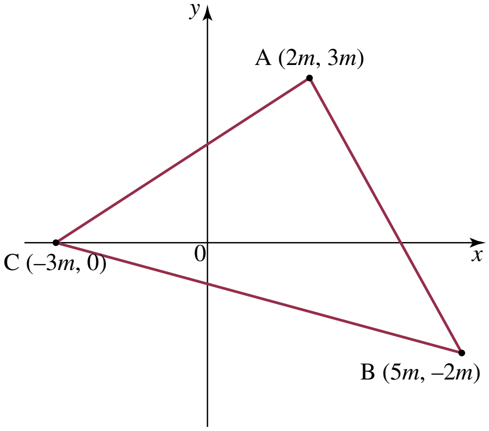
   b. Describe the triangle. What does PC represent?

Reasoning

9. Find the equation of the straight line that passes through the midpoint of A $(-2, 5)$ and B $(-2, 3)$, and has a gradient of -3.
10. Find the equation of the straight line that passes through the midpoint of A $(-1, -3)$ and B $(3, -5)$, and has a gradient of $\frac{2}{3}$.

Problem solving

11. The points A $(2m, 3m)$, B $(5m, -2m)$ and C $(-3m, 0)$ are the vertices of a triangle. Show that this is a right-angled triangle.



12. Write down the coordinates of the midpoint of the line joining the points $(3k - 1, 4 - 5k)$ and $(5k - 1, 3 - 5k)$. Show that this point lies on the line with equation $5x + 4y = 9$.

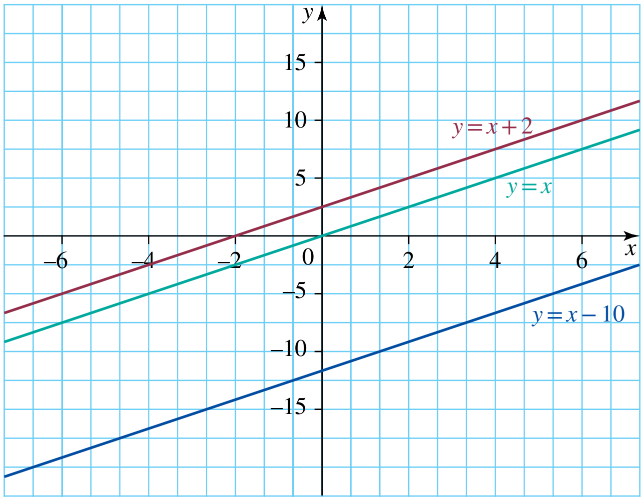
Reflection

If the midpoint of a line segment is the origin, what are the possible values of the x- and y-coordinates of the end points?

3.6 Parallel and perpendicular lines

3.6.1 Parallel lines

- Lines that have the same gradient are **parallel** lines. The three lines on the graph at right all have a gradient of 1 and are parallel to each other.



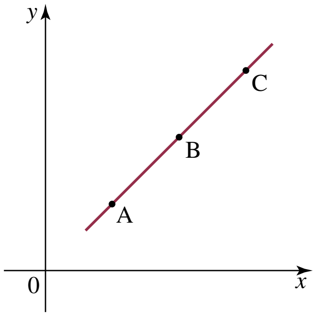
WORKED EXAMPLE 19

Show that AB is parallel to CD given that A has coordinates (−1, −5), B has coordinates (5, 7), C has coordinates (−3, 1) and D has coordinates (4, 15).

THINK	WRITE
1 Find the gradient of AB by applying the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.	Let A $(-1, -5) = (x_1, y_1)$ and B $(5, 7) = (x_2, y_2)$ Since $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m_{AB} = \frac{7 - (-5)}{5 - (-1)}$ $= \frac{12}{6}$ $= 2$
2 Find the gradient of CD.	Let C $(-3, 1) = (x_1, y_1)$ and D $(4, 15) = (x_2, y_2)$ $m_{CD} = \frac{15 - 1}{4 - (-3)}$ $= \frac{14}{7}$ $= 2$
3 Draw a conclusion. (*Note*: ‖ means ‘is parallel to’.)	Since $m_{AB} = m_{CD} = 2$, then AB ‖ CD.

3.6.2 Collinear points

- **Collinear points** are points that all lie on the same straight line.
- If A, B and C are collinear, then $m_{AB} = m_{BC}$.



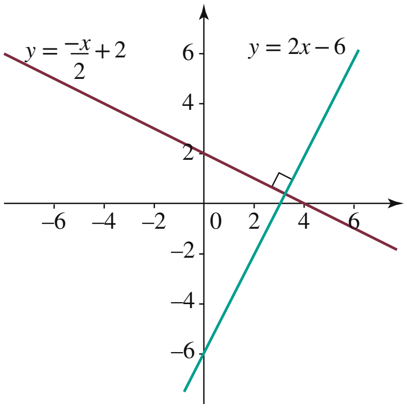
WORKED EXAMPLE 20

Show that the points A (2, 0), B (4, 1) and C (10, 4) are collinear.

THINK	WRITE
1 Find the gradient of AB.	Let A $(2, 0) = (x_1, y_1)$ and B $(4, 1) = (x_2, y_2)$ Since $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m_{AB} = \frac{1 - 0}{4 - 2}$ $= \frac{1}{2}$

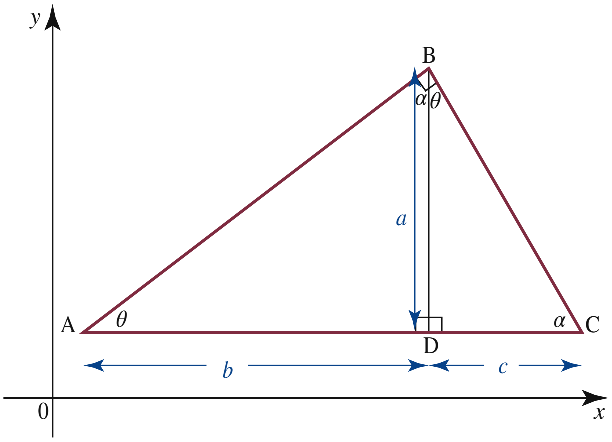
2 Find the gradient of BC.	Let B $(4, 1) = (x_1, y_1)$ and C $(10, 4) = (x_2, y_2)$ $m_{BC} = \dfrac{4-1}{10-4}$ $= \frac{3}{6}$ $= \frac{1}{2}$
3 Show that A, B and C are collinear.	Since $m_{AB} = m_{BC} = \frac{1}{2}$ and B is common to both line segments, A, B and C are collinear.

3.6.3 Perpendicular lines



- There is a special relationship between the gradients of two **perpendicular** lines.
  The graph at right shows two perpendicular lines. What do you notice about their gradients?
- Consider the diagram shown below, in which the line segment AB is perpendicular to the line segment BC, AC is parallel to the x-axis, and BD is the perpendicular height of the resulting triangle ABC.

$$\begin{aligned}
\text{In } \Delta\text{ABD, let } m_{AB} &= m_1\\
&= \frac{a}{b}\\
&= \tan(\theta)\\
\text{In } \Delta\text{BCD, let } m_{BC} &= m_2\\
&= -\frac{a}{c}\\
&= -\tan(\alpha)\\
\text{In } \Delta\text{ABC}, \tan(\alpha) &= \frac{b}{a}\\
\text{So} \qquad m_2 &= -\frac{b}{a}\\
&= \frac{-1}{m_1}\\
\text{Hence} \qquad m_2 &= \frac{-1}{m_1}\\
\text{or} \qquad m_1 m_2 &= -1
\end{aligned}$$



- Hence, if two lines are perpendicular to each other, then the product of their gradients is −1. Two lines are perpendicular if and only if:

$$m_1 m_2 = -1$$

- If two lines are perpendicular, then their gradients are $\frac{a}{b}$ and $-\frac{b}{a}$ respectively.

WORKED EXAMPLE 21

Show that the lines $y = -5x + 2$ and $5y - x + 15 = 0$ are perpendicular to one another.

THINK	**WRITE**
1 Find the gradient of the first line.	$y = -5x + 2$ Hence $m_1 = -5$

2 Find the gradient of the second line.	$5y - x + 15 = 0$ Rewrite in the form $y = mx + c$: $5y = x - 15$ $y = \frac{x}{5} - 3$ Hence $m_2 = \frac{1}{5}$ $m_1 m_2 = -5 \times \frac{1}{5}$ $= -1$
3 Test for perpendicularity. (The two lines are perpendicular if the product of their gradients is -1.)	Hence, the two lines are perpendicular.

3.6.4 Determining the equation of a line parallel or perpendicular to another line

- The gradient properties of parallel and perpendicular lines can be used to solve many problems.

WORKED EXAMPLE 22

Find the equation of the line that passes through the point (3, −1) and is parallel to the straight line with equation $y = 2x + 1$.

THINK	WRITE
1 Write the general equation.	$y = mx + c$
2 Find the gradient of the given line. The two lines are parallel, so they have the same gradient.	$y = 2x + 1$ has a gradient of 2 Hence $m = 2$
3 Substitute for m in the general equation.	so $y = 2x + c$
4 Substitute the given point to find c.	$(x, y) = (3, -1)$ $\therefore -1 = 2(3) + c$ $-1 = 6 + c$ $c = -7$ $y = 2x - 7$
5 Substitute for c in the general equation.	$y = 2x - 7$ or $2x - y - 7 = 0$

WORKED EXAMPLE 23

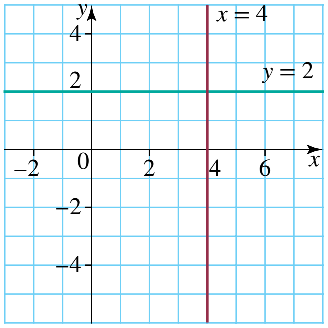
Find the equation of the line that passes through the point (0, 3) and is perpendicular to a straight line with a gradient of 5.

THINK	WRITE
1 For perpendicular lines, $m_1 \times m_2 = -1$. Find the gradient of the perpendicular line.	Given $m_1 = 5$ $m_2 = -\frac{1}{5}$

THINK	WRITE
2 Use the equation $y - y_1 = m(x - x_1)$ where $m = -\frac{1}{5}$ and $(x_1, y_1) = (0, 3)$.	$$\begin{aligned} \text{Since } y - y_1 &= m(x - x_1) \\ \text{and } (x_1, y_1) &= (0, 3) \\ \text{then } y - 3 &= -\tfrac{1}{5}(x - 0) \\ y - 3 &= -\frac{x}{5} \\ 5(y - 3) &= -x \\ 5y - 15 &= -x \\ x + 5y - 15 &= 0 \end{aligned}$$

3.6.5 Horizontal and vertical lines

- Horizontal lines are parallel to the x-axis, have a gradient of zero, are expressed in the form $y = c$ and have no x-intercept.
- Vertical lines are parallel to the y-axis, have an undefined (infinite) gradient, are expressed in the form $x = a$ and have no y-intercept.



WORKED EXAMPLE 24

Find the equation of:
a the vertical line that passes through the point (2, –3)
b the horizontal line that passes through the point (–2, 6).

THINK	WRITE
a The equation of a vertical line is $x = a$. The x-coordinate of the given point is 2.	**a** $x = 2$
b The equation of a horizontal line is $y = c$. The y-coordinate of the given point is 6.	**b** $y = 6$

WORKED EXAMPLE 25

Find the equation of the perpendicular bisector of the line joining the points (0, –4) and (6, 5). (A bisector is a line that crosses another line at right angles and cuts it into two equal lengths.)

THINK	WRITE
1 Find the gradient of the line joining the given points by applying the formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$.	Let $(0, -4) = (x_1, y_1)$. Let $(6, 5) = (x_2, y_2)$. $$\begin{aligned} m_1 &= \frac{y_2 - y_1}{x_2 - x_1} \\ m_1 &= \frac{5 - (-4)}{6 - 0} \\ &= \frac{9}{6} \\ &= \frac{3}{2} \end{aligned}$$
2 Find the gradient of the perpendicular line. $m_1 \times m_2 = -1$	$m_1 = \frac{3}{2}$ $m_2 = -\frac{2}{3}$

3 Find the midpoint of the line joining the given points. $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ where $(x_1, y_1) = (0, -4)$ and $(x_2, y_2) = (6, 5)$.	$x = \frac{x_1 + x_2}{2} = \frac{0 + 6}{2} = 3$ $y = \frac{y_1 + y_2}{2} = \frac{-4 + 5}{2} = \frac{1}{2}$ Hence $(3, \frac{1}{2})$ are the coordinates of the midpoint.
4 Find the equations of the line with gradient $-\frac{2}{3}$ that passes through $(3, \frac{1}{2})$.	Since $y - y_1 = m(x - x_1)$, then $y - \frac{1}{2} = -\frac{2}{3}(x - 3)$
5 Simplify by removing the fractions. Multiply both sides by 3. Multiply both sides by 2.	$3(y - \frac{1}{2}) = -2(x - 3)$ $3y - \frac{3}{2} = -2x + 6$ $6y - 3 = -4x + 12$ $4x + 6y - 15 = 0$

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Try out this interactivity: Parallel and perpendicular lines (int-2779)

Complete this digital doc: Spreadsheet: Perpendicular checker (doc-5209)

Complete this digital doc: Spreadsheet: Equation of a straight line (doc-5210)

Exercise 3.6 Parallel and perpendicular lines

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2, 5, 6a–c, 7, 8, 9a–c, 12, 13, 16a–b, 18, 20a, 21, 23, 26a, 27	Questions: 1a–d, 2–5, 6c–d, 7, 8, 9a–c, 12, 13, 15, 16a–b, 17a, 18, 20a, 21–23, 26–28, 30, 32	Questions: 1c–f, 2, 3, 4, 5, 6e–f, 7–19, 20b, 21, 22, 24–31, 33–37

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Fluency

1. **WE19** Find whether AB is parallel to CD given the following sets of points.
   - **a.** A (4, 13), B (2, 9), C (0, −10), D (15, 0)
   - **b.** A (2, 4), B (8, 1), C (−6, −2), D (2, −6)
   - **c.** A (−3, −10), B (1, 2), C (1, 10), D (8, 16)
   - **d.** A (1, −1), B (4, 11), C (2, 10), D (−1, −5)
   - **e.** A (1, 0), B (2, 5), C (3, 15), D (7, 35)
   - **f.** A (1, −6), B (−5, 0), C (0, 0), D (5, −4)

2. Which pairs of the following straight lines are parallel?

a. $2x + y + 1 = 0$	**b.** $y = 3x - 1$
c. $2y - x = 3$	**d.** $y = 4x + 3$
e. $y = \frac{x}{2} - 1$	**f.** $6x - 2y = 0$
g. $3y = x + 4$	**h.** $2y = 5 - x$

3. **WE20** Show that the points A $(0, -2)$, B $(5, 1)$ and C $(-5, -5)$ are collinear.

4. Show that the line that passes through the points $(-4, 9)$ and $(0, 3)$ also passes through the point $(6, -6)$.

5. **WE21** Show that the lines $y = 6x - 3$ and $x + 6y - 6 = 0$ are perpendicular to one another.

6. Determine whether AB is perpendicular to CD, given the following sets of points.

a. A $(1, 6)$, B $(3, 8)$, C $(4, -6)$, D $(-3, 1)$

b. A $(2, 12)$, B $(-1, -9)$, C $(0, 2)$, D $(7, 1)$

c. A $(1, 3)$, B $(4, 18)$, C $(-5, 4)$, D $(5, 0)$

d. A $(1, -5)$, B $(0, 0)$, C $(5, 11)$, D $(-10, 8)$

e. A $(-4, 9)$, B $(2, -6)$, C $(-5, 8)$, D $(10, 14)$

f. A $(4, 4)$, B $(-8, 5)$, C $(-6, 2)$, D $(3, 11)$

7. **WE22** Find the equation of the line that passes through the point $(4, -1)$ and is parallel to the line with equation $y = 2x - 5$.

8. **WE23** Find the equation of the line that passes through the point $(-2, 7)$ and is perpendicular to a line with a gradient of $\frac{2}{3}$.

9. Find the equations of the following lines.

a. Gradient 3 and passing through the point $(1, 5)$

b. Gradient -4 and passing through the point $(2, 1)$

c. Passing through the points $(2, -1)$ and $(4, 2)$

d. Passing through the points $(1, -3)$ and $(6, -5)$

e. Passing through the point $(5, -2)$ and parallel to $x + 5y + 15 = 0$

f. Passing through the point $(1, 6)$ and parallel to $x - 3y - 2 = 0$

g. Passing through the point $(-1, -5)$ and perpendicular to $3x + y + 2 = 0$

10. Find the equation of the line that passes through the point $(-2, 1)$ and is:

a. parallel to the line with equation $2x - y - 3 = 0$

b. perpendicular to the line with equation $2x - y - 3 = 0$.

11. Find the equation of the line that contains the point $(1, 1)$ and is:

a. parallel to the line with equation $3x - 5y = 0$

b. perpendicular to the line with equation $3x - 5y = 0$.

12. **WE24** Find the equation of:

a. the vertical line that passes through the point $(1, -8)$

b. the horizontal line that passes through the point $(-5, -7)$.

13. **MC** **a.** The vertical line passing through the point $(3, -4)$ is given by:

A. $y = -4$
B. $x = 3$
C. $y = 3x - 4$
D. $y = -4x + 3$
E. $x = -4$

b. Which of the following points does the horizontal line given by the equation $y = -5$ pass through?

A. $(-5, 4)$
B. $(4, 5)$
C. $(3, -5)$
D. $(5, -4)$
E. $(5, 5)$

c. Which of the following statements is true?

A. Vertical lines have a gradient of zero.
B. The y-coordinates of all points on a vertical line are the same.
C. Horizontal lines have an undefined gradient.
D. The x-coordinates of all points on a vertical line are the same.
E. A horizontal line has the general equation $x = a$.

d. Which of the following statements is false?

A. Horizontal lines have a gradient of zero.
B. The line joining the points (1, −1) and (−7, −1) is vertical.
C. Vertical lines have an undefined gradient.
D. The line joining the points (1, 1) and (−7, 1) is horizontal.
E. A horizontal line has the general equation $y = c$.

14. The triangle ABC has vertices A (9, −2), B (3, 6), and C (1, 4).

a. Find the midpoint, M, of BC.
b. Find the gradient of BC.
c. Show that AM is the perpendicular bisector of BC.
d. Describe triangle ABC.

15. **WE25** Find the equation of the perpendicular bisector of the line joining the points (1, 2) and (−5, −4).

16. Find the equation of the perpendicular bisector of the line joining the points (−2, 9) and (4, 0).

17. ABCD is a parallelogram. The coordinates of A, B and C are (4, 1), (1, −2) and (−2, 1) respectively. Find:

a. the equation of AD
b. the equation of DC
c. the coordinates of D.

Understanding

18. In each of the following, show that ABCD is a parallelogram.

a. A (2, 0), B (4, −3), C (2, −4), D (0, −1)
b. A (2, 2), B (0, −2), C (−2, −3), D (0, 1)
c. A (2.5, 3.5), B (10, −4), C (2.5, −2.5), D (−5, 5)

19. In each of the following, show that ABCD is a trapezium.

a. A (0, 6), B (2, 2), C (0, −4), D (−5, −9)
b. A (26, 32), B (18, 16), C (1, −1), D (−3, 3)
c. A (2, 7), B (1, −1), C (−0.6, −2.6), D (−2, 3)

20. **MC** The line that passes through the points (0, −6) and (7, 8) also passes through:

A. (4, 3)
B. (5, 4)
C. (−2, 10)
D. (1, −8)
E. (1, 4)

21. **MC** The point (−1, 5) lies on a line parallel to $4x + y + 5 = 0$. Another point on the same line as (−1, 5) is:

A. (2, 9)
B. (4, 2)
C. (4, 0)
D. (−2, 3)
E. (3, −11)

22. Find the equation of the straight line given the following conditions.

a. Passes through the point (−1, 3) and parallel to $y = -2x + 5$
b. Passes through the point (4, −3) and parallel to $3y + 2x = -3$

23. Determine which pairs of the following lines are perpendicular.

a. $x + 3y - 5 = 0$
b. $y = 4x - 7$
c. $y = x$
d. $2y = x + 1$
e. $y = 3x + 2$
f. $x + 4y - 9 = 0$
g. $2x + y = 6$
h. $x + y = 0$

24. Find the equation of the straight line that cuts the x-axis at 3 and is perpendicular to the line with equation $3y - 6x = 12$.

25. Calculate the value of m for which lines with the following pairs of equations are perpendicular to each other.

a. $2y - 5x = 7$ and $4y + 12 = mx$
b. $5x - 6y = -27$ and $15 + mx = -3y$

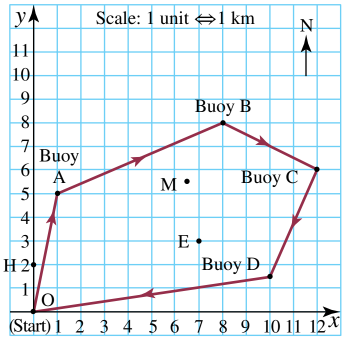
26. **MC** The gradient of the line perpendicular to the line with equation $3x - 6y = 2$ is:

A. 3
B. −6
C. 2
D. $\frac{1}{2}$
E. −2

27. **MC** Triangle ABC has a right angle at B. The vertices are A (−2, 9), B (2, 8) and C (1, z). The value of z is:

A. $8\frac{1}{4}$
B. 4
C. 12
D. $7\frac{3}{4}$
E. −4

Reasoning



28. The map shows the proposed course for a yacht race. Buoys have been positioned at A $(1, 5)$, B $(8, 8)$, C $(12, 6)$, and D $(10, w)$.

- **a.** How far is it from the start, O, to buoy A?
- **b.** The race marshall boat, M, is situated halfway between buoys A and C. What are the coordinates of the boat's position?
- **c.** Stage 4 of the race (from C to D) is perpendicular to stage 3 (from B to C). What is the gradient of CD?
- **d.** Find the linear equation that describes stage 4.
- **e.** Hence determine the exact position of buoy D.
- **f.** An emergency boat is to be placed at point E, $(7, 3)$. How far is the emergency boat from the hospital, located at H, 2 km north of the start?

29. Show that the following sets of points form the vertices of a right-angled triangle.

- **a.** A $(1, -4)$, B $(2, -3)$, C $(4, -7)$
- **b.** A $(3, 13)$, B $(1, 3)$, C $(-4, 4)$
- **c.** A $(0, 5)$, B $(9, 12)$, C $(3, 14)$

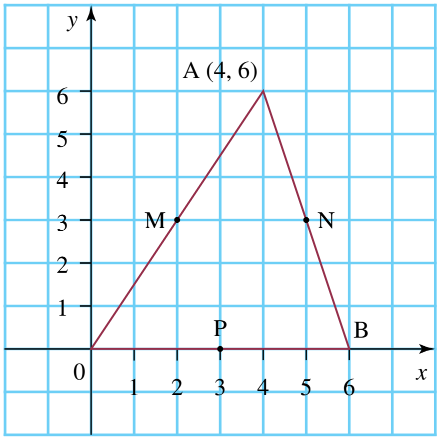
30. Prove that the quadrilateral ABCD is a rectangle when A is $(2, 5)$, B $(6, 1)$, C $(3, -2)$ and D $(-1, 2)$.

31. Prove that the quadrilateral ABCD is a rhombus, given A $(2, 3)$, B $(3, 5)$, C $(5, 6)$ and D $(4, 4)$.
Hint: A rhombus is a parallelogram with diagonals that intersect at right angles.

32.
- **a.** A square has vertices at $(0, 0)$ and $(2, 0)$. Where are the other 2 vertices? (There are 3 sets of answers.)
- **b.** An equilateral triangle has vertices at $(0, 0)$ and $(2, 0)$. Where is the other vertex? (There are 2 answers.)
- **c.** A parallelogram has vertices at $(0, 0)$ and $(2, 0)$. and $(1, 1)$. Where is the other vertex? (There are 3 sets of answers.)

33. A is the point $(0, 0)$ and B is the point $(0, 2)$.

- **a.** Find the perpendicular bisector of AB.
- **b.** Show that any point on this line is equidistant from A and B.



Questions 34 and 35 relate to the diagram.

M is the midpoint of OA.
N is the midpoint of AB.
P is the midpoint of OB.

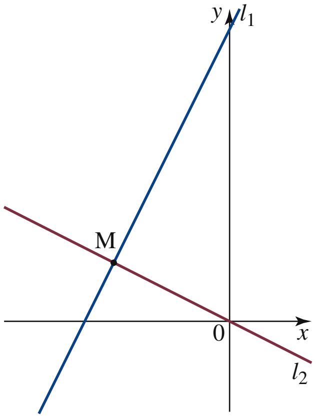
34. A simple investigation:

- **a.** Show that MN is parallel to OB.
- **b.** Is PN parallel to OA?
- **c.** Is PM parallel to AB?

35. A difficult investigation:

- **a.** Find the perpendicular bisectors of OA and OB.
- **b.** Find the point W where the two bisectors intersect.
- **c.** Show that the perpendicular bisector of AB also passes through W.
- **d.** Explain why W is equidistant from O, A and B.
- **e.** W is called the circumcentre of triangle OAB. Using W as the centre, draw a circle through O, A and B.

Problem solving



36. The lines l_1 and l_2 are at right angles to each other. The line l_1 has the equation $px + py + r = 0$. Show that the distance from M to the origin is given by $\dfrac{r}{\sqrt{p^2 + p^2}}$.

37. Line A is parallel to the line with equation $2x - y = 7$ and passes through the point $(2, 3)$. Line B is perpendicular to the line with equation $4x - 3y + 3 = 0$ and also passes through the point $(2, 3)$. Line C intersects with line A where it cuts the y-axis and intersects with line B where it cuts the x-axis.

a. Determine the equations for all three lines. Give answers in the form $ax + by + c = 0$.

b. Sketch all three lines on the one set of axes.

c. Determine whether the triangle formed by the three lines is scalene, isosceles or equilateral.

Reflection

How could you use coordinate geometry to design a logo for an organisation?

CHALLENGE 3.2

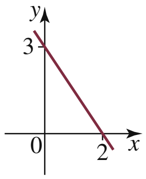
The first six numbers of a particular number pattern are 1, 2, 3, 6, 11 and 20. Given that this pattern continues, what will be the next four numbers? Describe the pattern.

3.7 Review

3.7.1 Review questions

Fluency

1. **MC** The equation of the following line is:



A. $3x + 2y = 6$ **B.** $3x - 2y = 6$ **C.** $2x + 3y = 6$
D. $2x - 3y = 6$ **E.** $2x - 3y = -6$

2. **MC** The equation of a linear graph with gradient -3 and x-intercept of 4 is:

A. $y = -3x - 12$ **B.** $y = -3x + 4$ **C.** $y = -3x - 4$
D. $y = -3x + 12$ **E.** $y = 4x - 3$

3. **MC** The equation of a linear graph which passes through $(2, -7)$ and $(-2, -2)$ is:

A. $4x - 5y + 18 = 0$ **B.** $5x + 4y + 18 = 0$ **C.** $5x + 4y - 18 = 0$
D. $5x - 4y - 18 = 0$ **E.** $4x + 5y + 18 = 0$

4. **MC** The distance between the points $(1, 5)$ and $(6, -7)$ is:

A. $\sqrt{53}$ **B.** $\sqrt{29}$ **C.** 13 **D.** $\sqrt{193}$ **E.** 12

5. **MC** The midpoint of the line segment joining the points $(-4, 3)$ and $(2, 7)$ is:

A. $(-1, 5)$ **B.** $(-2, 10)$ **C.** $(-6, 4)$ **D.** $(-2, 4)$ **E.** $(-1, 2)$

6. **MC** If the midpoint of the line segment joining the points A$(3, 7)$ and B(x, y) has coordinates $(6, 2)$, then the coordinates of B are:

A. $(15, 3)$ **B.** $(0, -6)$ **C.** $(9, -3)$ **D.** $(4.5, 4.5)$ **E.** $(-9, 3)$

7. **MC** If the points $(-6, -11)$, $(2, 1)$ and $(x, 4)$ are collinear, then the value of x is:

A. 4 **B.** 3.2 **C.** $\frac{1}{4}$ **D.** $\frac{5}{16}$ **E.** 3

8. **MC** The gradient of the line perpendicular to $3x - 4y + 7 = 0$ is:

A. $\frac{3}{4}$ **B.** $\frac{4}{3}$ **C.** $-\frac{4}{3}$ **D.** 3 **E.** -4

9. **MC** The equation of the line perpendicular to $2x + y - 1 = 0$ and passing through the point $(1, 4)$ is:

A. $2x + y - 6 = 0$ **B.** $2x + y - 2 = 0$ **C.** $x - 2y + 7 = 0$
D. $x + 2y + 9 = 0$ **E.** $x - 2y = 0$

10. Produce a table of values, and sketch the graph of the equation $y = -5x + 15$ for values of x between -10 and $+10$.

11. Sketch the graph of the following linear equations, labelling the x- and y-intercepts.

a. $y = 3x - 2$ **b.** $y = -5x + 15$ **c.** $y = -\frac{2}{3}x + 1$ **d.** $y = \frac{7}{5}x - 3$

12. Find the x- and y-intercepts of the following straight lines.

a. $y = -7x + 6$ **b.** $y = \frac{3}{8}x - 5$ **c.** $y = \frac{4}{7}x - \frac{3}{4}$ **d.** $y = 0.5x + 2.8$

13. Sketch graphs of the following linear equations by finding the x- and y-intercepts.

a. $2x - 3y = 6$ **b.** $3x + y = 0$ **c.** $5x + y = -3$ **d.** $x + y + 3 = 0$

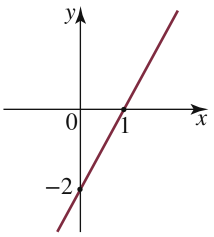
14. Sketch the graph of each of the following.

a. $y = \frac{1}{2}x$ **b.** $y = -4x$ **c.** $x = -2$ **d.** $y = 7$

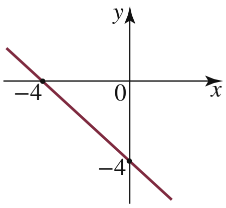
15. Sketch the graph of the equation $3(y - 5) = 6(x + 1)$.

16. Find the equations of the straight lines in the following graphs.

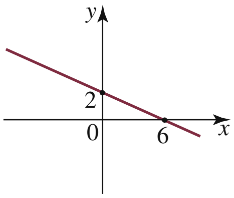
a.



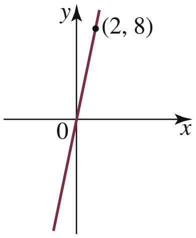
b.



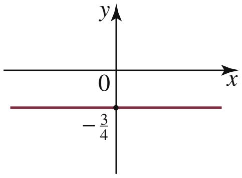
c.



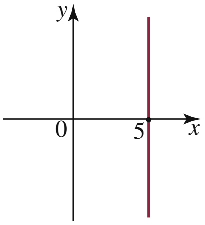
d.



e.



f.



17. Find the linear equation given the information in each case below.

a. gradient $= 3$, y-intercept $= -4$ **b.** gradient $= -2$, y-intercept $= -5$
c. gradient $= \frac{1}{2}$, y-intercept $= 5$ **d.** gradient $= 0$, y-intercept $= 6$

18. For each of the following, find the equation of the straight line with the given gradient and passing through the given point.

a. gradient $= 7$, point $(2, 1)$ **b.** gradient $= -3$, point $(1, 1)$
c. gradient $= \frac{1}{2}$, point $(-2, 5)$ **d.** gradient $= \frac{3}{5}$, point $(1, -3)$

19. Find the distance between the points $(1, 3)$ and $(7, -2)$ in exact form.

20. Prove that triangle ABC is isosceles given A $(3, 1)$, B $(-3, 7)$ and C $(-1, 3)$.

21. Show that the points A $(1, 1)$, B $(2, 3)$ and C $(8, 0)$ are the vertices of a right-angled triangle.

22. The midpoint of the line segment AB is $(6, -4)$. If B has coordinates $(12, 10)$, find the coordinates of A.

23. Show that the points A $(3, 1)$, B $(5, 2)$ and C $(11, 5)$ are collinear.

24. Show that the lines $y = 2x - 4$ and $x + 2y - 10 = 0$ are perpendicular to one another.

25. Find the equation of the straight line passing through the point $(6, -2)$ and parallel to the line $x + 2y - 1 = 0$.

26. Find the equation of the line perpendicular to $3x - 2y + 6 = 0$ and having the same y-intercept.

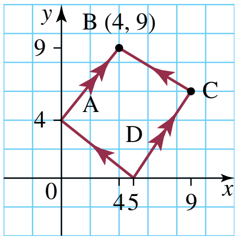
27. Find the equation of the perpendicular bisector of the line joining the points $(-2, 7)$ and $(4, 11)$.

28. Find the equation of the straight line joining the point $(-2, 5)$ and the point of intersection of the straight lines with equations $y = 3x - 1$ and $y = 2x + 5$.

29. Using the information given in the diagram:

a. find:

i. the gradient of AD
ii. the gradient of AB
iii. the equation of BC
iv. the equation of DC
v. the coordinates of C.



b. describe quadrilateral ABCD.

30. In triangle ABC, A is $(1, 5)$, B is $(-2, -3)$ and C is $(8, -2)$.

a. Find:

i. the gradient of BC
ii. the midpoint, P, of AB
iii. the midpoint, Q, of AC.

b. Hence show that:

i. PQ is parallel to BC
ii. PQ is half the length of BC.

Problem solving

31. John has a part-time job working as a gardener and is paid \$13.50 per hour.

a. Complete the following table of values relating the amount of money received to the number of hours worked.

Number of hours	0	2	4	6	8	10
Pay (\$)						

b. Find a linear equation relating the amount of money received to the number of hours worked.

c. Sketch the linear equation on a Cartesian plane over a suitable domain.

d. Using algebra, calculate the pay that John will receive if he works for $6\frac{3}{4}$ hours.

32. A fun park charges a \$12.50 entry fee and an additional \$2.50 per ride.

a. Complete the following table of values relating the total cost to the number of rides.

Number of rides	0	2	4	6	8	10
Cost (\$)						

b. Find a linear equation relating total cost to the number of rides.

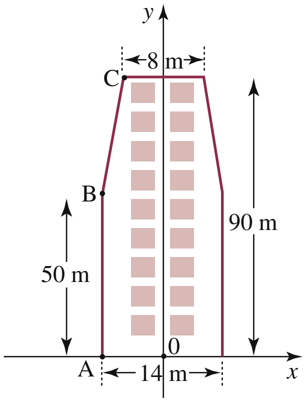
c. Sketch the linear equation on a Cartesian plane over a suitable domain.

d. Using algebra, calculate the cost for 7 rides.

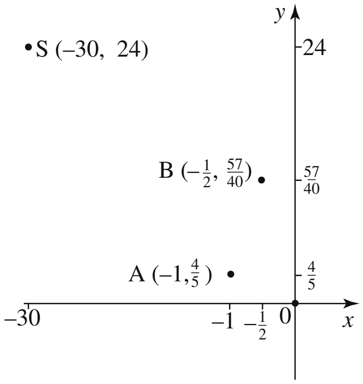
33. The cost of hiring a boat is \$160 plus \$22.50 per hour.
  a. Sketch a graph showing the total cost for between 0 and 12 hours.
  b. State the equation relating cost to time rented.
  c. Predict the cost of hiring a boat for 12 hours and 15 minutes.

34. ABCD is a quadrilateral with vertices A $(4, 9)$, B $(7, 4)$, C $(1, 2)$ and D $(a, 10)$. Given that the diagonals are perpendicular to each other, find:
  a. the equation of the diagonal AC
  b. the equation of the diagonal BD
  c. the value of a.

35. An architect decides to design a building with a 14-metre-square base such that the external walls are initially vertical to a height of 50 metres, but taper so that their separation is 8 metres at its peak height of 90 metres. A profile of the building is shown with the point $(0, 0)$ marked as a reference at the centre of the base.
  a. Write the equation of the vertical line connecting A and B.
  b. Write the coordinates of B and C.
  c. Find the length of the tapered section of wall from B to C.



36. In a game of lawn bowls, the object is to bowl a biased ball so that it gets as close as possible to a smaller white ball called a jack. During a game, a player will sometimes bowl a ball quite quickly so that it travels in a straight line in order to displace an opponent's 'guard balls'. In a particular game, player X has 2 guard balls close to the jack. The coordinates of the jack are $(0, 0)$ and the coordinates of the guard balls are A $\left(-1, \frac{4}{5}\right)$ and B $\left(-\frac{1}{2}, \frac{57}{40}\right)$. Player Y bowls a ball so that it travels in a straight line toward the jack. The ball is bowled from the position S, with the coordinates $(-30, 24)$.



(Not to scale)

  a. Will player Y displace one of the guard balls? If so, which one?
  b. Due to bias, the displaced guard ball is knocked so that it begins to travel in a straight line (at right angles to the path found in part **a**). Find the equation of the line of the guard ball.

c. Show that guard ball A is initially heading directly toward guard ball B.

d. Given its initial velocity, guard ball A can travel in a straight line for 1 metre before its bias affects it path. Calculate and explain whether guard ball A will collide with guard ball B.

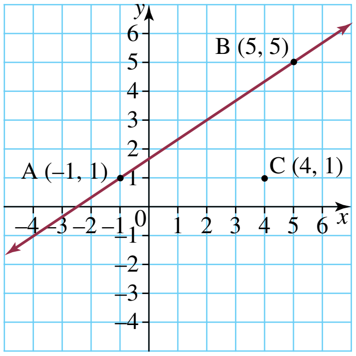
37. The graph shows the line p passing through the points A (–1, 1) and B (5, 5). Given that C is the point (4, 1), find:

a. the gradient of p

b. the equation of p

c. the area of ΔABC

d. the length BC, giving your answer correct to 2 decimal places.



38. The temperature of the air (T °C) is related to the height above sea level (h metres) by the formula $T = 18 - 0.005h$.

a. What is the temperature at the heights of:

  i. 600 m

  ii. 1000 m

  iii. 3000 m?

b. Draw a graph using the results from part a.

c. Use the graph to find the temperature at 1200 m and 2500 m.

d. Predict the height at which the temperature is 9 °C.

39. An old theory on the number of hours of sleep (h) that a child of c years of age should have each night is $h = 8 + \frac{18 - c}{2}$.

a. How many hours should a 10-year-old have?

b. How old is a child that requires 10 hours sleep?

c. For every year, how much less sleep does a child require?

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Try out this interactivity: Word search: Topic 3 (int-2832)

Try out this interactivity: Crossword: Topic 3 (int-2833)

Try out this interactivity: Sudoku: Topic 3 (int-3590)

Complete this digital doc: Concept map: Topic 3 (doc-13714)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

axes
bisect
Cartesian plane
collinear
coordinates
dependent variable
diagonal
general form
gradient
gradient–intercept form
horizontal
independent variable
linear graph
midpoint
origin
parallel
parallelogram
perpendicular
quadrilateral
rhombus
rise
run
segment
substitute
trapezium
vertical
vertices
***x*-intercept**
***y*-intercept**

Investigation | Rich Task

What common computer symbol is this?

On computer hardware, and on many different software applications, a broad range of symbols is used. These symbols help us to identify where things need to be plugged into, what buttons we need to push, or what option needs to be selected. The main focus of this task involves constructing a common symbol found on the computer. The instructions are given below. Grid lines have been provided on the opposite page for you to construct the symbol.

The construction part of this task requires you to graph nine lines to reveal a common computer symbol. Draw the scale of your graph to accommodate x- and y-values in the following ranges: $-10 \leq x \leq 16$ and $-10 \leq y \leq 16$. Centre the axes on the grid lines provided.

- Line 1 has an equation $y = x - 1$. Graph this line in the range $-7 \leq x \leq -2$.
- Line 2 is perpendicular to line 1 and has a y-intercept of -5. Determine the equation of this line, and then draw the line in the range $-5 \leq x \leq -1$.
- Line 3 is parallel to line 1, with a y-intercept of 3. Determine the equation of the line, and then graph the line in the range $-9 \leq x \leq -4$.

- Line 4 is parallel to line 1, with a y-intercept of -3. Determine the equation of the line, and then graph the line in the range $-1 \leq x \leq 2$.
- Line 5 has the same length as line 4 and is parallel to it. The point $(-2, 3)$ is the starting point of the line, which decreases in both x- and y-values from there.
- Line 6 commences at the same starting point as line 5, and then runs at right angles to line 5. It has an x-intercept of 1 and is the same length as line 2.
- Line 7 commences at the same starting point as both lines 5 and 6. Its equation is $y = 6x + 15$. The point $(-1, 9)$ lies at the midpoint.
- Line 8 has the equation $y = -x + 15$. Its midpoint is the point $(7, 8)$ and its extremities are the points where the line meets line 7 and line 9.
- Line 9 has the equation $6y - x + 8 = 0$. It runs from the intersection of lines 4 and 6 until it meets line 8.

1. What common computer symbol have you drawn?
2. The top section of your figure is a familiar geometric shape. Use the coordinates on your graph, together with the distance formula to determine the necessary lengths to calculate the area of this figure.
3. Using any symbol of interest to you, draw your symbol on grid lines and provide instructions for your design. Ensure that your design involves aspects of coordinate geometry that have been used throughout this task.

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Complete this digital doc: Code puzzle: Who won the inaugural 875 km Sydney to Melbourne marathon in 1983? (doc-15921)

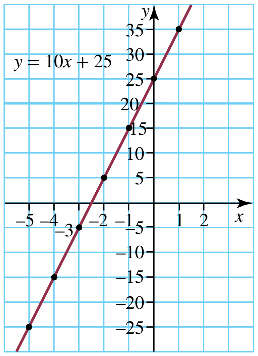
Answers

Topic 3 Coordinate geometry

Exercise 3.2 Sketching linear graphs

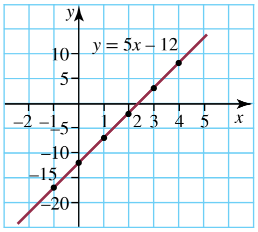
1 a.

x	y
−5	−25
−4	−15
−3	−5
−2	5
−1	15
0	25
1	35



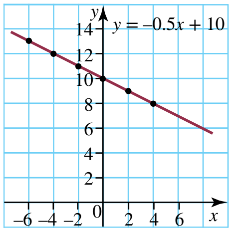
b.

x	y
−1	−17
0	−12
1	−7
2	−2
3	3
4	8



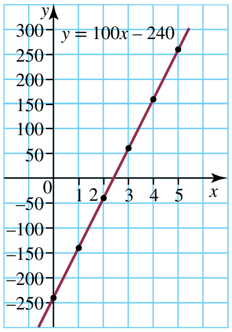
c.

x	y
−6	13
−4	12
−2	11
0	10
2	9
4	8



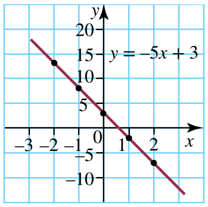
d.

x	y
0	−240
1	−140
2	−40
3	60
4	160
5	260



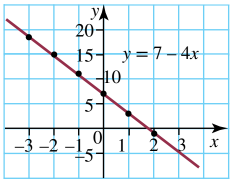
e.

x	y
−3	18
−2	13
−1	8
0	3
1	−2
2	−7



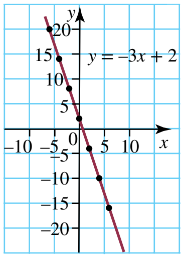
f.

x	y
−3	19
−2	15
−1	11
0	7
1	3
2	−1



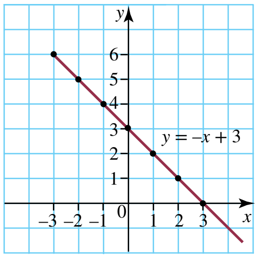
2 a.

x	y
−6	20
−4	14
−2	8
0	2
2	−4
4	−10
6	−16



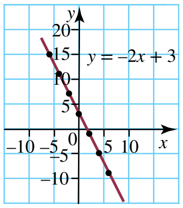
b.

x	y
−3	6
−2	5
−1	4
0	3
1	2
2	1
3	0

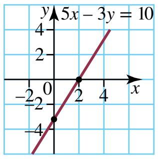


c.

x	y
−6	15
−4	11
−2	7
0	3
2	−1
4	−5
6	−9



3 a



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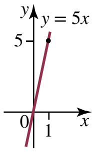

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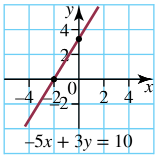

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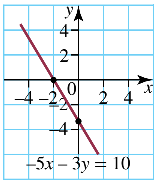

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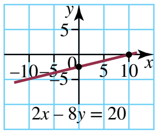

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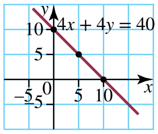

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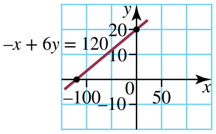

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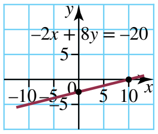

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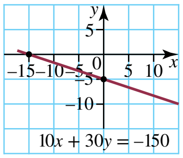

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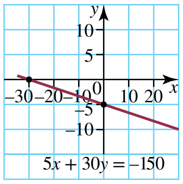

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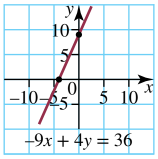

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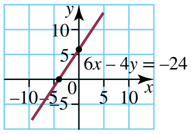

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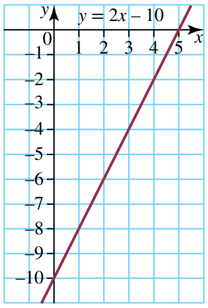

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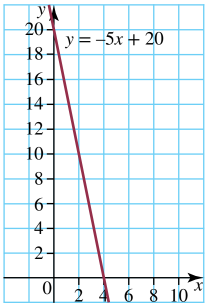
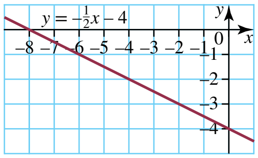
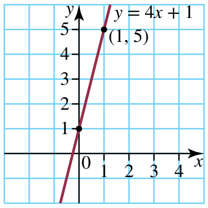


o



4 a



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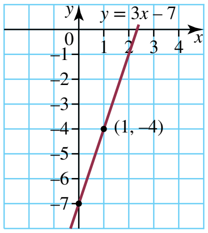

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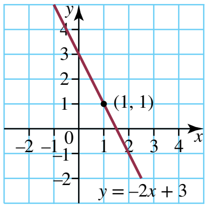

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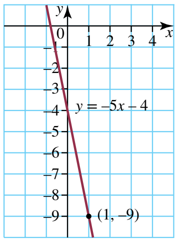

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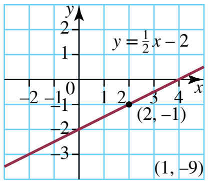

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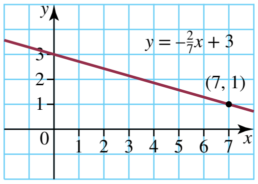

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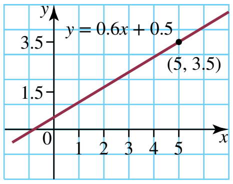

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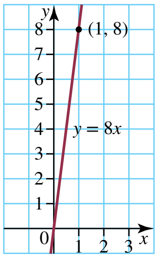

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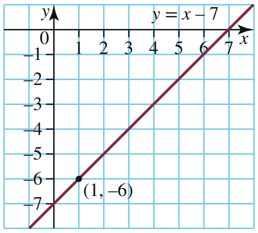

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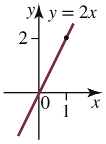

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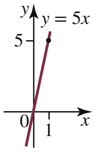

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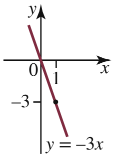

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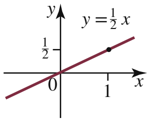

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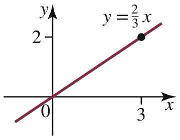

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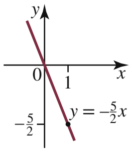
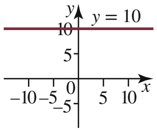


6 a



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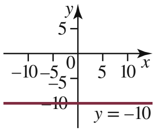

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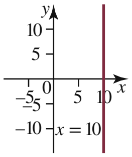

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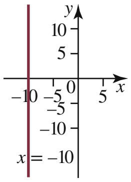

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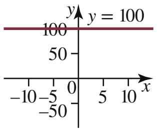

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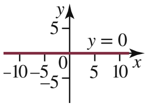

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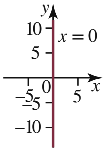

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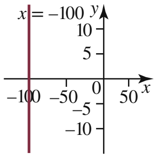
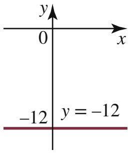


i



7 a x-intercept: −0.5; y-intercept: 0.4

b x-intercept: 0.5; y-intercept: −0.4

c x-intercept: 0; y-intercept: 0

d x-intercept: −3; y-intercept: 12

e x-intercept: −4; y-intercept: −4

f x-intercept: −1; y-intercept: −0.5

g x-intercept: 2.75; y-intercept: 2.2

h x-intercept: 9.75; y-intercept: −3.9

i x-intercept: $\frac{23}{13} \approx 1.77$; y-intercept: 4.6

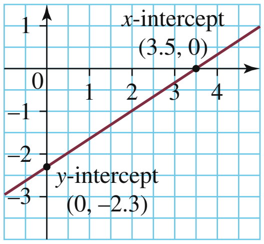
8 a (2, 0), (0, −8)

b $(-\frac{1}{2}, 0)$, (0, 3)

c (−5, 0), (0, 25)

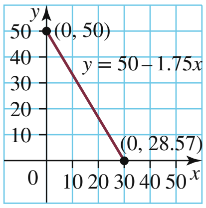
9 Answers will vary.

10 $y = \frac{2}{3}x - \frac{7}{3}$



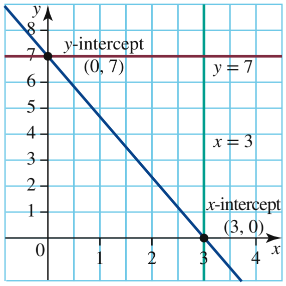
11 a Independent variable
= number of songs bought, dependent variable
= amount of money saved

b



c 14 songs

12 a



b 7

c $-\frac{7}{3}$

d B

13 a Independent variable = time, dependent variable = amount of water in the tank

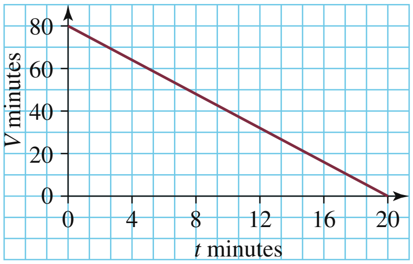
b Initially there are 80 litres of water.

c Time cannot be negative.

d 4 litres per minute

e 20 minutes

f



Exercise 3.3 Determining linear equations

1. a. $y = 2x + 4$ **b.** $y = -3x + 12$ **c.** $y = -x + 5$ **d.** $y = 2x - 8$ **e.** $y = \frac{1}{2}x + 3$

f. $y = -\frac{1}{4}x - 4$ **g.** $y = 7x - 5$ **h.** $y = -3x - 15$

2. a. $y = 2x$ **b.** $y = -3x$ **c.** $y = \frac{1}{2}x$ **d.** $y = -\frac{3}{4}x$

3. a. $y = x + 3$ **b.** $y = 2x - 1$ **c.** $y = -\frac{1}{2}x + \frac{7}{2}$ **d.** $y = \frac{1}{2}x + \frac{1}{2}$

e. $y = -2x - 2$ **f.** $y = -x - 8$

4. a. $y = 3x + 3$ **b.** $y = -3x + 4$ **c.** $y = -4x + 2$ **d.** $y = 4x + 2$ **e.** $y = -x - 4$

f. $y = 0.5x - 4$ **g.** $y = 5x + 2.5$ **h.** $y = -6x + 3$ **i.** $y = -2.5x + 1.5$ **j.** $y = 3.5x + 6.5$

5. a. $y = 5x - 19$ **b.** $y = -5x + 31$ **c.** $y = -4x - 1$ **d.** $y = 4x - 34$ **e.** $y = 3x - 35$

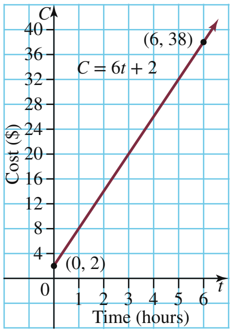
f. $y = -3x + 6$ **g.** $y = -2x + 30$ **h.** $y = 2x - 4.5$ **i.** $y = 0.5x - 19$ **j.** $y = -0.5x + 5.5$

6. a. Independent variable = time (in hours), dependent variable = cost (in \$)

b.

t	0	1	2	3
C	2	8	14	20

c.



d. i. (0, 2)

ii. The y-intercept represents the initial cost of bowling at the alley, which is the shoe rental.

e. $m = 6$, which represents the cost to hire a lane for an additional hour.

f. $C = 6t + 2$

g. \$32

h. Answers will vary.

7. It does not matter if you rise before you run or run before you rise, as long as you take into account whether the rise or run is negative.

8. a. $m = \frac{y - c}{x}$ **b.** $y = mx + c$ **9.** Teacher to check

10. $m_{AB} = m_{CD} = 2$ and $m_{BC} = m_{AD} = \frac{1}{2}$. As opposite sides have the same gradients, this quadrilateral is a parallelogram.

Challenge 3.1

$y = \frac{3}{5}x + 2$

Exercise 3.4 The distance between two points

1. AB = 5, CD = $2\sqrt{10}$ or 6.32, EF = $3\sqrt{2}$ or 4.24, GH = $2\sqrt{5}$ or 4.47, IJ = 5, KL = $\sqrt{26}$ or 5.10, MN = $4\sqrt{2}$ or 5.66, OP = $\sqrt{10}$ or 3.16

2. a. 5 **b.** 13 **c.** 10 **d.** 7.07 **e.** 6.71

f. 14.42 **g.** 13 **h.** 13 **i.** $\sqrt{a^2 + 4b^2}$ **j.** $3\sqrt{a^2 + b^2}$

3. B **4.** D

5. a. AB = 4.47, BC = 2.24, CD = 4.47, DA = 2.24 **b.** AC = 5, BD = 5 **c.** Rectangle

6, 7 and **8** Answers will vary.

9. a. 12 **b.** 5 **c.** 13 **d.** −2.2

10. Answers will vary.

11. $a = 2$

12. a. $m_{AB} = 1$ and $m_{BC} = -\frac{7}{3}$ **b.** D (4, −1) **c.** Teacher to check

Exercise 3.5 The midpoint of a line segment

1. a. $(-3, -3\frac{1}{2})$ **b.** $(7\frac{1}{2}, 0)$ **c.** $(-1, 1)$ **d.** $(0, 1\frac{1}{2})$ **e.** $(2a, \frac{1}{2}b)$ **f.** $(a + b, \frac{1}{2}a)$

2. $(-3, -10)$

3. a. $(3, 1)$ **b.** 4.47 **c.** 6.32

4. D

5. C

6. a. i. $(-1, 4)$ **ii.** $(1\frac{1}{2}, 1)$ **iii.** 3.91 **b.** BC = 7.8 = 2 PQ

7. a. i. $(1, -0.5)$ **ii.** $(1, -0.5)$ **b.** The diagonals bisect each other, so it is a parallelogram.

8. a. i. $(-2, 2)$ **ii.** 8.94 **iii.** 9.55 **iv.** 9.55

b. Isosceles. PC is the perpendicular height of the triangle.

9. $y = -3x - 2$ **10.** $3y - 2x + 14 = 0$ **11.** Teacher to check **12.** $(4k - 1, 3.5 - 5k)$

Exercise 3.6 Parallel and perpendicular lines

1. a. No **b.** Yes **c.** No **d.** No **e.** Yes **f.** No

2. b, f**;** c, e **3.** Answers will vary. **4.** Answers will vary. **5.** Answers will vary.

6. a. Yes **b.** Yes **c.** No **d.** Yes **e.** Yes **f.** No

7. $y = 2x - 9$

8. $3x + 2y - 8 = 0$

9. a. $y = 3x + 2$ **b.** $y = -4x + 9$ **c.** $3x - 2y - 8 = 0$ **d.** $5y + 2x + 13 = 0$ **e.** $x + 5y + 5 = 0$

f. $x - 3y + 17 = 0$ **g.** $x - 3y - 14 = 0$

10. a. $2x - y + 5 = 0$ **b.** $x + 2y = 0$

11. a. $3x - 5y + 2 = 0$ **b.** $5x + 3y - 8 = 0$

12. a. $x = 1$ **b.** $y = -7$

13. a. B **b.** C **c.** D **d.** B

14. a. $(2, 5)$ **b.** 1 **c.** Answers will vary. **d.** Isosceles triangle

15. $y = -x - 3$

16. $4x - 6y + 23 = 0$

17. a. $y = -x + 5$ **b.** $y = x + 3$ **c.** $(1, 4)$

18. Answers will vary. **19.** Answers will vary. **20.** B **21.** E

22. a. $y = -2x + 1$ **b.** $3y + 2x + 1 = 0$

23. a, e; b, f; c, h; d, g

24. $y = -\frac{1}{2}x + \frac{3}{2}$

25. a. $m = -\frac{8}{5}$ **b.** $m = \frac{18}{5}$

26. E

27. B

28. a. 5.10 km **b.** $(6.5, 5.5)$ **c.** 2 **d.** $y = 2x - 18$ **e.** $(10, 2)$ **f.** 7.07 km

29, 30, 31 Answers will vary.

32. a. $(2, 0), (2, 2)$ or $(-2, 0), (-2, 2)$ or $(1, 1), (1, -1)$

b. $(1, \sqrt{3})$ or $(1, -\sqrt{3})$ **c.** $(3, 1), (-1, 1)$ or $(1, -1)$

33. a. $x = 1$ **b.** Answers will vary.

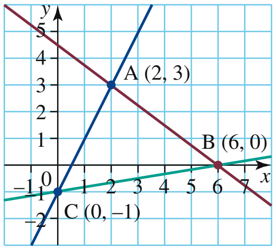
34. a. Answers will vary. **b.** Yes **c.** Yes

35. a. OA: $2x + 3y - 13 = 0$; OB: $x = 3$ **b.** $(3, \frac{7}{3})$ **c, d.** Answers will vary.

36. Teacher to check

37. a. Line A: $2x - y - 1 = 0$, Line B: $3x + 4y - 18 = 0$, Line C: $x - 6y - 6 = 0$

b.



c. Scalene

Challenge 3.2

37, 68, 125, 230. To find the next number, add the three preceding numbers.

3.7 Review

1. A

2. D

3. B

4. C

5. A

6. C

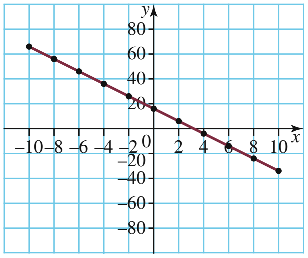
7. A

8. C

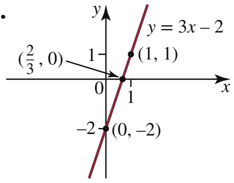
9. C

10.

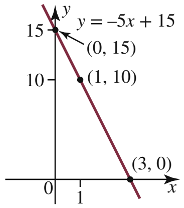
x	−10	−8	−6	−4	−2	0	2	4	6	8	10
y	65	55	45	35	25	15	5	−5	−15	−25	−35



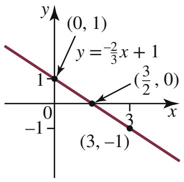
11. a.



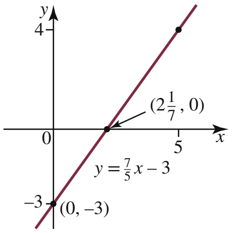
b.



c.



d.



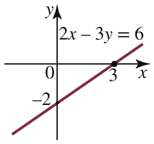
12. a. x-intercept $= \frac{6}{7}$, y-intercept $b = 6$

b. x-intercept $= \frac{40}{3}$, $\left(= 13\frac{1}{3}\right)$, y-intercept $b = -5$

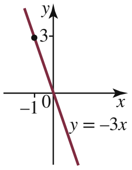
c. x-intercept $= \frac{21}{16}$, $\left(= 1\frac{5}{16}\right)$, y-intercept $b = -\frac{3}{4}$

d. x-intercept $= -5.6$, y-intercept $b = 2.8$

13. a.



b.



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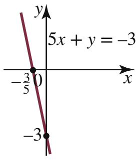
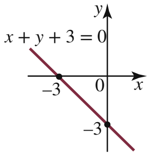
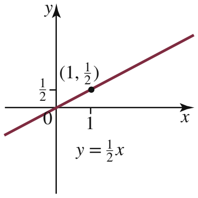


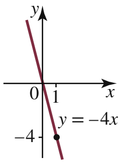
d.



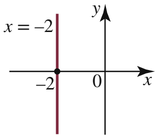
14. a.



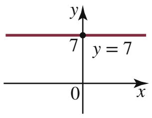
b.



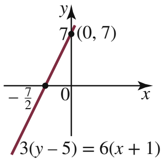
c.



d.



15.



16. a. $y = 2x - 2$ **b.** $y = -x - 4$ **c.** $y = -\frac{1}{3}x + 2$ **d.** $y = 4x$ **e.** $y = -\frac{3}{4}$ **f.** $x = 5$

17. a. $y = 3x - 4$ **b.** $y = -2x - 5$ **c.** $y = \frac{1}{2}x + 5$ **d.** $y = 6$

18. a. $y = 7x - 13$ **b.** $y = -3x + 4$ **c.** $y = \frac{1}{2}x + 6$ **d.** $y = \frac{3}{5}x - \frac{18}{15}$

19. $\sqrt{61}$ **20.** Answers will vary. **21.** Answers will vary.

22. $(0, -18)$ **23.** Answers will vary. **24.** Answers will vary.

25. $x + 2y - 2 = 0$ **26.** $2x + 3y - 9 = 0$ **27.** $3x + 2y - 21 = 0$ **28.** $3x - 2y + 16 = 0$

29. a. i. $-\frac{4}{5}$ **ii.** $\frac{5}{4}$ **iii.** $4x + 5y - 61 = 0$ **iv.** $5x - 4y - 25 = 0$ **v.** $(9, 5)$

b. Square

30. a. i. $\frac{1}{10}$ **ii.** $(-\frac{1}{2}, 1)$ **iii.** $(4\frac{1}{2}, 1\frac{1}{2})$

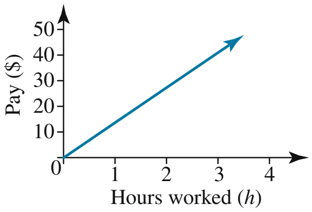
b. Answers will vary.

31. a.

Number of hours	0	2	4	6	8	10
Pay($)	0	27	54	81	108	135

b. Pay = $13.50 × (number of hours worked)

c.



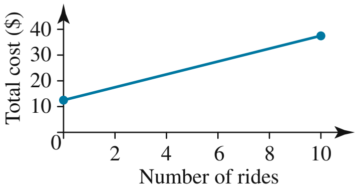
d. $91.13

32. a.

Number of rides	0	2	4	6	8	10
Cost($)	12.50	17.50	22.50	27.50	32.50	37.50

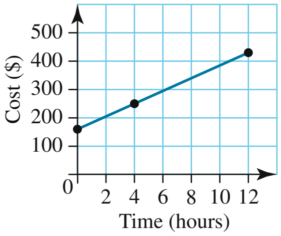
b. Cost = $2.50 × number of rides + $12.50

c.



d. $30

33. a.



b. $C = 22.50h + 160$ **c.** Approximately $436

34. a. $7x - 3y - 1 = 0$ **b.** $3x + 7y - 49 = 0$ **c.** –7

35. a. $x = -7$ **b.** B (–7, 50), C (–4, 90) **c.** 40.11 metres

36. a. Since the gradient of SA equals the gradient of SO = –0.8, the points S, A and O are collinear. Player Y will displace guard ball A.

b. $y = \frac{5}{4}x + \frac{41}{20}$ or $25x - 20y + 41 = 0$

c. Since the gradient of the path AB is $\frac{5}{4}$, which is the same as the gradient of the known path of travel from the common point A, the direction of travel is toward B.

d. $d_{AB} = 0.625$. Yes, guard ball A will collide with guard ball B as it will not be deviated from its linear path under 1 metre of travel.

37. a. Gradient $= m = \frac{5 - 1}{5 - -1} = \frac{4}{6} = \frac{2}{3}$

b. $y = mx + b$, $y = \frac{2}{3}x + b$

If $x = -1$ and $y = 1$, substitute in the question:

$1 = \frac{2}{3}(-1) + b$

$b = 1\frac{2}{3}$

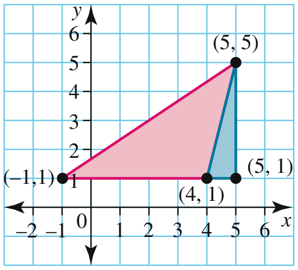
$y = \frac{2}{3}x + 1\frac{2}{3}$

c. Plot the point (5,1).

Area of large $\Delta = \frac{1}{2} \times 6 \times 4 = 12$

Area of small $\Delta = \frac{1}{2} \times 1 \times 4 = 2$

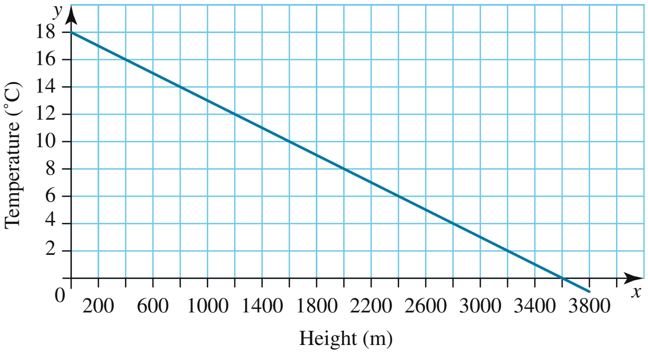
Area of $\Delta ABC = 12 - 2 = 10$ units^2



d. $BC^2 = 4^2 + 1^2$
$BC^2 = 16 + 1$
$BC^2 = 17$
$BC = \sqrt{17} \approx 4.12$ units

38. a. **i.** $T = 18 - 0.005(600) = 15$
ii. $T = 18 - 0.005(1000) = 13$
iii. $T = 18 - 0.005(3000) = 3$

b.



c. 1200 m = 12 °C, 2500 m = 5.5 °C

d. 1800 m

39. a. 12 hours

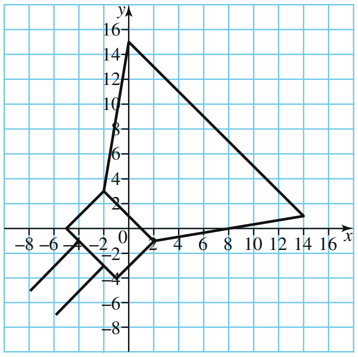
b. 14 years old

c. $h = 8 + \dfrac{18 - c}{2}$

$2h = 16 + 18 - c$
$2h = -c + 34$
$h = -\frac{1}{2}c + 17$

For every year, the child requires half an hour less sleep.

Investigation — Rich task



1. The symbol is the one used to represent a speaker.
2. The shape is a trapezium.

$$\begin{aligned} \text{Area} &= \frac{1}{2}(\text{length line 6} + \text{length line 8}) \times \\ &\quad \text{perpendicular distance between these lines.} \\ &= \frac{1}{2}(4\sqrt{2} + 14\sqrt{2}) \times 7\sqrt{2} \\ &= 126 \text{ units}^2 \end{aligned}$$

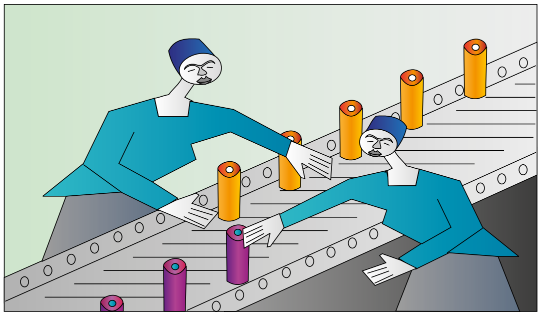
3. Teacher to check

TOPIC 4
Simultaneous linear equations and inequalities

4.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

4.1.1 Why learn this?



Picture this — you own a factory that produces two different products, and you are planning to buy some new machines. The big machines are more expensive than the small ones, take up more floor space and need more staff to operate, but they can produce more. Which machines should you buy?

Solving simultaneous equations will help you determine feasible solutions to questions like this.

4.1.2 What do you know?

assesson

1. **THINK** List what you know about linear equations and linear inequations. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of linear equations and linear inequations.

LEARNING SEQUENCE

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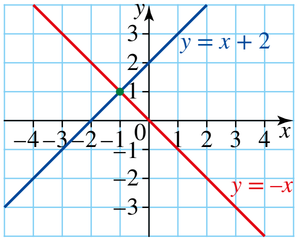
Watch this eLesson: The story of mathematics: Khayyam (eles-1843)

4.2 Graphical solution of simultaneous linear equations

4.2.1 Simultaneous linear equations

- **Simultaneous** means occurring at the same time.
- When a point belongs to more than one line, the coordinates of the point satisfy all equations. The equations of the lines are called **simultaneous equations**. An example is shown below.
- A **system of equations** is a set of two or more equations with the same variables.
- To solve simultaneous equations is to calculate the values of the variables that satisfy all equations in the system.
- Any two linear graphs will meet at a point, unless they are parallel.
- At this point, the two equations simultaneously share the same x- and y-coordinates, which are referred to as the solution.
- Simultaneous equations can be solved graphically or algebraically.



4.2.2 Graphical solution

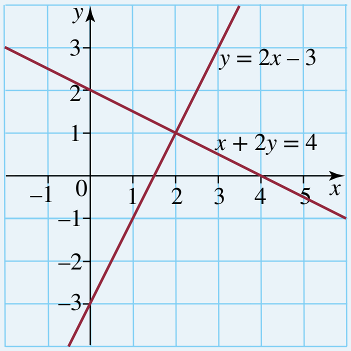
- The solution to a pair of simultaneous equations can be found by graphing the two equations and identifying the coordinates of the point of intersection.
- An accurate solution depends on drawing an accurate graph.
- Graph paper or graphing software can be used.

WORKED EXAMPLE 1

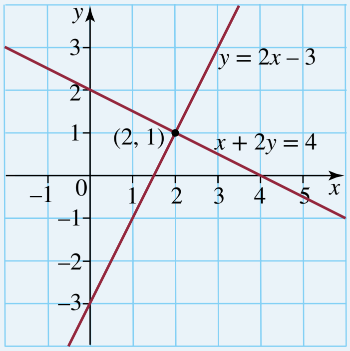
TI | CASIO

Use the graphs of the given simultaneous equations to determine the point of intersection and, hence, the solution of the simultaneous equations.

$$\begin{aligned} x &= 2y = 4 \\ y &= 2x - 3 \end{aligned}$$



THINK	WRITE/DRAW
1 Write the equations and number them.	$x + 2y = 4$ [1] $y = 2x - 3$ [2]
2 Locate the point of intersection of the two lines. This gives the solution.	Point of intersection (2, 1) Solution: $x = 2$ and $y = 1$



3 Check the solution by substituting $x = 2$ and $y = 1$ into the given equations. Comment on the results obtained.	Check equation [1]: $\text{LHS} = x + 2y \qquad \text{RHS} = 4$ $= 2 + 2(1)$ $= 4$ LHS = RHS Check equation [2]: $\text{LHS} = y \qquad \text{RHS} = 2x - 3$ $= 1 \qquad\qquad = 2(2) - 3$ $= 4 - 3$ $= 1$ LHS = RHS In both cases LHS = RHS, therefore the solution set (2, 1) is correct.

WORKED EXAMPLE 2

TI | CASIO

Check whether the given pair of coordinates, (5, −2), is the solution to the following pair of simultaneous equations.

$$\begin{aligned} 3x - 2y &= 19 \\ 4y + x &= -3 \end{aligned}$$

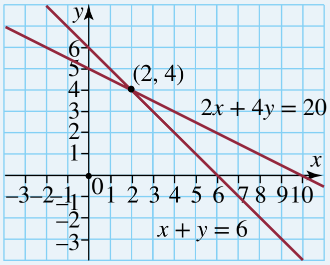
THINK	**WRITE**
1 Write the equations and number them.	$3x - 2y = 19 \qquad [1]$ $4y + x = -3 \qquad [2]$
2 Substitute $x = 5$ and $y = -2$ into equation [1].	Check equation [1]: $\text{LHS} = 3x - 2y \qquad \text{RHS} = 19$ $= 3(5) - 2(-2)$ $= 15 + 4$ $= 19$ LHS = RHS
3 Substitute $x = 5$ and $y = -2$ into equation [2].	Check equation [2]: $\text{LHS} = 4y + x \qquad \text{RHS} = -3$ $= 4(-2) + 5$ $= -8 + 5$ $= -3$ LHS = RHS Therefore, the solution set (5, −2) is a solution to both equations.

WORKED EXAMPLE 3

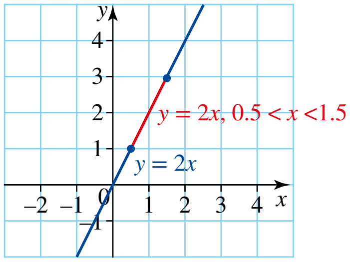
Solve the following pair of simultaneous equations using a graphical method.

$$\begin{aligned} x + y &= 6 \\ 2x + 4y &= 20 \end{aligned}$$

THINK	**WRITE/DRAW**
1 Write the equations, one under the other and number them.	$x + y = 6 \qquad [1]$ $2x + 4y = 20 \qquad [2]$

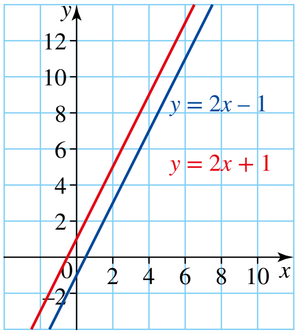
2	Calculate the x- and y-intercepts for equation [1]. For the x-intercept, substitute $y = 0$ into equation	Equation [1] x-intercept: when $y = 0$, $x + 0 = 6$ $x = 6$
	For the y-intercept, substitute $x = 0$ into equation [1].	The x-intercept is at $(6, 0)$. y-intercept: when $x = 0$, $0 + y = 6$ $y = 6$ The y-intercept is at $(0, 6)$.
3	Calculate the x- and y-intercepts for equation [2]. For the x-intercept, substitute $y = 0$ into equation [2]. Divide both sides by 2.	Equation [2] x-intercept: when $y = 0$, $2x + 0 = 20$ $2x = 20$ $x = 10$ The x-intercept is at $(10, 0)$.
	For the y-intercept, substitute $x = 0$ into equation [2].	y-intercept: when $x = 0$, $0 + 4y = 20$ $4y = 20$ $y = 5$
	Divide both sides by 4.	The y-intercept is at $(0, 5)$.
4	Use graph paper to rule up a set of axes and label the x-axis from 0 to 10 and the y-axis from 0 to 6.	
5	Plot the x- and y-intercepts for each equation.	
6	Produce a graph of each equation by ruling a straight line through its intercepts.	
7	Label each graph.	
8	Locate the point of intersection of the lines.	The point of intersection is $(2, 4)$.
9	Check the solution by substituting $x = 2$ and $y = 4$ into each equation.	Check [1]: $\text{LHS} = x + y$ $\quad$ $\text{RHS} = 6$ $= 2 + 4$ $= 6$ $\text{LHS} = \text{RHS}$ Check [2]: $\text{LHS} = 2x + 4y$ $\quad$ $\text{RHS} = 20$ $= 2(2) + 4(4)$ $= 4 + 16$ $= 20$ $\text{LHS} = \text{RHS}$
10	State the solution.	In both cases, LHS = RHS. Therefore, the solution set $(2, 4)$ is correct. The solution is $x = 2, y = 4$.

4.2.3 Equations with multiple solutions



- Two lines are **coincident** if they lie one on top of the other. For example, the line in blue and line segment in red at right are coincident.
- There are an infinite number of solutions to coincident equations. Every point where the lines coincide satisfies both equations and hence is a solution to the simultaneous equations.
- Coincident equations have the same equation, although the equations may have been transposed so they look different. For example, $y = 2x + 3$ and $2y - 4x = 6$ are coincident equations.

4.2.4 Equations with no solutions

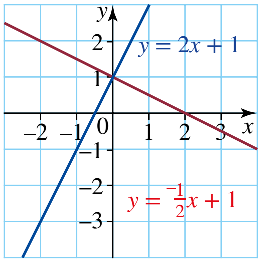


- If two lines do not intersect, there is no simultaneous solution to the equations. For example, the lines at right do not intersect, so there is no point that belongs to both lines.
- Parallel lines have the same gradient but a different y-intercept.
- For straight lines, the only situation in which the lines do not cross is if the lines are parallel *and* not coincident.
- Writing both equations in the form $y = mx + c$ confirms that the lines are parallel since the gradients are equal.

$$\begin{aligned} 2x - y &= 1 \qquad [1] \\ -y &= 1 - 2x \\ -y &= -2x + 1 \\ y &= 2x - 1 \\ \text{Gradient } m &= 2 \end{aligned}$$

$$\begin{aligned} 4x - 2y &= -2 \qquad [2] \\ -2y &= -2 - 4x \\ -2y &= -4x - 2 \\ y &= 2x + 1 \\ \text{Gradient } m &= 2 \end{aligned}$$

4.2.5 Perpendicular lines



- Perpendicular lines meet at right angles (90°).
- Perpendicular lines have negative reciprocal gradients: $m_1 = \dfrac{-1}{m_2}$ or $m_1 m_2 = -1$, where m_1 is the gradient of the first line and m_2 is the gradient of the second line. For example, for the two lines at right, $m_1 = 2$ and $m_2 = \frac{-1}{2}$.

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Exercise 4.2 Graphical solution of simultaneous linear equations

assess**on**

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 2a–d, 3a–d, 4a–d, 7	Questions: 1, 2c–g, 3a–d,, 4a–f, 5, 7, 9	Questions: 1, 2e–j, 3–10

🟩🟦🟪 **Individual pathway interactivity:** int-4577

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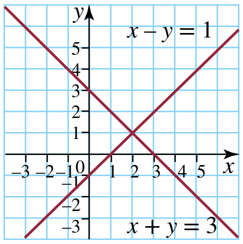
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

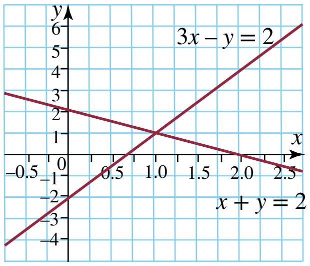
Fluency

1. **WE1** Use the graphs to find the solution of the simultaneous equations.

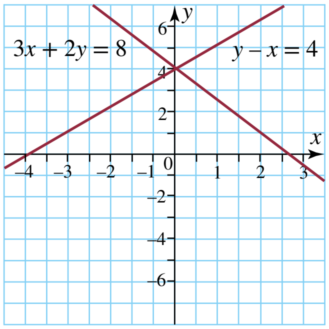
a. $x + y = 3$
$x - y = 1$



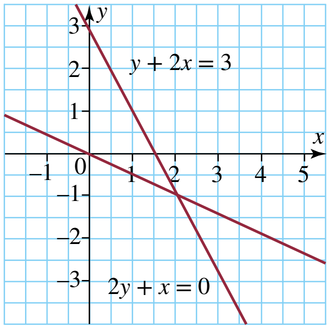
b. $x + y = 2$
$3x - y = 2$



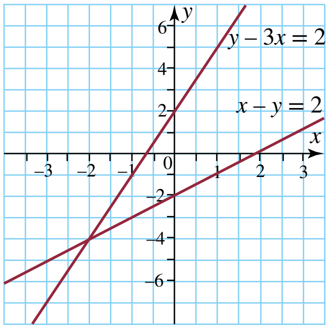
c. $y - x = 4$
$3x + 2y = 8$



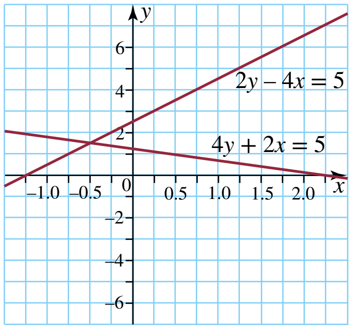
d. $y + 2x = 3$
$2y + x = 0$



e. $y - 3x = 2$
$x - y = 2$



f. $2y - 4x = 5$
$4y + 2x = 5$



2. **WE2** For the following simultaneous equations, use substitution to check if the given pair of coordinates is a solution.

a. (7, 5) $3x + 2y = 31$
$2x + 3y = 28$

b. (3, 7) $y - x = 4$
$2y + x = 17$

c. (9, 1) $x + 3y = 12$
$5x - 2y = 43$

d. (2, 5) $x - y = 7$
$2x + 3y = 18$

e. (4, −3) $y = 3x - 15$
$4x + 7y = -5$

f. (6, −2) $x - 2y = 2$
$3x + y = 16$

g. (4, −2) $2x + y = 6$
$x - 3y = 8$

h. (5, 1) $y - 5x = -24$
$3y + 4x = 23$

i. (−2, −5) $3x - 2y = -4$
$2x - 3y = 11$

j. (−3, −1) $y - x = 2$
$2y - 3x = 7$

3. **WE3** Solve each of the following pairs of simultaneous equations using a graphical method.

a. $x + y = 5$
$2x + y = 8$

b. $x + 2y = 10$
$3x + y = 15$

c. $2x + 3y = 6$
$2x - y = -10$

d. $x - 3y = -8$
$2x + y = -2$

e. $6x + 5y = 12$
$5x + 3y = 10$

f. $y + 2x = 6$
$2y + 3x = 9$

g. $y = 3x + 10$
$y = 2x + 8$

h. $y = 8$
$3x + y = 17$

i. $4x - 2y = -5$
$x + 3y = 4$

j. $3x + y = 11$
$4x - y = 3$

k. $3x + 4y = 27$
$x + 2y = 11$

l. $3y + 3x = 8$
$3y + 2x = 6$

Understanding

4. Using technology, determine which of the following pairs of simultaneous equations have no solutions. Confirm by finding the gradient of each line.

a. $y = 2x - 4$
$3y - 6x = 10$

b. $5x - 3y = 13$
$4x - 2y = 10$

c. $x + 2y = 8$
$5x + 10y = 45$

d. $y = 4x + 5$
$2y - 10x = 8$

e. $3y + 2x = 9$
$6x + 4y = 22$

f. $y = 5 - 3x$
$3y = -9x + 18$

g. $4y + 3x = 7$
$12y + 9x = 22$

h. $2y - x = 0$
$14y - 6x = 2$

5. Two straight lines intersect at the point $(3, -4)$. One of the lines has a y-intercept of 8. The second line is a mirror image of the first in the line $x = 3$. Determine the equation of the second line. (*Hint:* Draw a graph of both lines.)

Reasoning

6. At a well-known beach resort it is possible to hire a jet-ski by the hour in two different locations. On the northern beach the cost is \$20 plus \$12 per hour, while on the southern beach the cost is \$8 plus \$18 per hour. The jet-skis can be rented for up to 5 hours.

a. Write the rules relating cost to the length of rental.

b. On the same set of axes sketch a graph of cost (y-axis) against length of rental (x-axis) for 0–5 hours.

c. For what rental times, if any, is the northern beach rental cheaper than the southern beach rental? Use your graph to justify your answer.

d. For what length of rental time are the two rental schemes identical? Use the graph and your rules to justify your answer.

7. For each of the pairs of simultaneous equations below, determine whether they are the same line, parallel lines, perpendicular lines or intersecting lines. Show your working.

a. $2x - y = -9$
$-4x - 18 = -2y$

b. $x - y = 7$
$x + y = 7$

c. $x + 6 = y$
$2x + y = 6$

d. $x + y = -2$
$x + y = 7$

8. Which of the following problems has one solution, an infinite number of solutions or no solution? Explain your answers.

a. $x - y = 1$
$2x - 3y = 2$

b. $2x - y = 5$
$4x - 2y = -6$

c. $x - 2y = -8$
$4x - 8y = -16$

Problem solving

9. Line A is parallel to the line with equation $y - 3x - 3 = 0$ and passes through the point $(1, 9)$. Line B is perpendicular to the line with equation $2y - x + 6 = 0$ and passes through the point $(2, -3)$.

a. Find the equation of line A.

b. Find the equation of line B.

c. Sketch both lines on the one set of axes to find where they intersect.

10. Solve the system of three simultaneous equations graphically.

$3x - y = 2 \qquad y + 3x = 4 \qquad 2y - x = 1$

Reflection

What do you think is the major error made when solving simultaneous equations graphically?

4.3 Solving simultaneous linear equations using substitution

4.3.1 Solving simultaneous linear equations

- There are two algebraic methods that are commonly used to solve simultaneous equations.
- They are the **substitution method** and the **elimination method**.

4.3.2 Substitution method

- The substitution method is particularly useful when one (or both) of the equations is in a form where one of the two variables is the subject.
- This variable is then substituted into the other equation, producing a third equation with only one variable.
- This third equation can then be used to determine the value of the variable.

WORKED EXAMPLE 4

TI | CASIO

Solve the simultaneous equations $y = 2x - 1$ and $3x + 4y = 29$ using the substitution method.

THINK	WRITE
1 Write the equations, one under the other and number them.	$y = 2x - 1$ [1] $3x + 4y = 29$ [2]
2 y and $2x - 1$ are equal so substitute expression $(2x - 1)$ for y into equation [2].	Substituting $(2x - 1)$ into [2]: $3x + 4(2x - 1) = 29$
3 Solve for x.	
i Expand the brackets on the LHS of the equation.	$3x + 8x - 4 = 29$
ii Collect like terms.	
iii Add 4 to both sides of the equation.	$11x - 4 = 29$
iv Divide both sides by 11.	$11x = 33$ $x = 3$
4 Substitute $x = 3$ into either of the equations, say [1], to find the value of y.	Substituting $x = 3$ into [1]: $y = 2(3) - 1$ $= 6 - 1$ $= 5$
5 Write your answer.	Solution: $x = 3, y = 5$ or $(3, 5)$
6 Check the solution by substituting $(3, 5)$ into equation [2].	Check: Substitute $(3, 5)$ into $3x + 4y = 29$. LHS $= 3(3) + 4(5)$ RHS $= 29$ $= 9 + 20$ $= 29$ As LHS = RHS, the solution is correct.

WORKED EXAMPLE 5

Solve the pair of simultaneous equations $y = 5x - 8$ and $y = -3x + 16$ using the substitution method.

THINK	WRITE
1 Write the equations, one under the other and number them.	$y = 5x - 8$ [1] $y = -3x + 16$ [2]
2 Both equations are written with y as the subject, so equate them.	$5x - 8 = -3x + 16$

▶

3 Solve for x. **i** Add $3x$ to both sides of the equation. **ii** Add 8 to both sides of the equation. **iii** Divide both sides of the equation by 8.	 $8x - 8 = 16$ $8x = 24$ $x = 3$
4 Substitute the value of x into either of the original equations, say [1], and solve for y.	Substituting $x = 3$ into [1]: $y = 5(3) - 8$ $= 15 - 8$ $= 7$
5 Write your answer.	Solution: $x = 3, y = 7$ or $(3, 7)$
6 Check the answer by substituting the point of intersection into equation [2].	Check: Substitute into $y = -3x + 16$. LHS $= y$ $= 7$ RHS $= -3x + 16$ $= -3(3) + 16$ $= -9 + 16$ $= 7$ As LHS = RHS, the solution is correct.

Exercise 4.3 Solving simultaneous linear equations using substitution

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2a–d, 4, 6, 8	Questions: 1a–d, 2c–f, 5–9	Questions: 1–11

🟩🟦🟪 **Individual pathway interactivity:** int-4578

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE4** Solve the following simultaneous equations using the substitution method. Check your solutions using technology.

a. $x = -10 + 4y$
$3x + 5y = 21$

b. $3x + 4y = 2$
$x = 7 + 5y$

c. $3x + y = 7$
$x = -3 - 3y$

d. $3x + 2y = 33$
$y = 41 - 5x$

e. $y = 3x - 3$
$-5x + 3y = 3$

f. $4x + y = 9$
$y = 11 - 5x$

g. $x = -5 - 2y$
$5y + x = -11$

h. $x = -4 - 3y$
$-3x - 4y = 12$

i. $x = 7 + 4y$
$2x + y = -4$

j. $x = 14 + 4y$
$-2x + 3y = -18$

k. $3x + 2y = 12$
$x = 9 - 4y$

l. $y = 2x + 1$
$-5x - 4y = 35$

2. **WE5** Solve the following pairs of simultaneous equations using the substitution method. Check your solutions using technology.

a. $y = 2x - 11$ and $y = 4x + 1$
b. $y = 3x + 8$ and $y = 7x - 12$
c. $y = 2x - 10$ and $y = -3x$
d. $y = x - 9$ and $y = -5x$
e. $y = -4x - 3$ and $y = x - 8$
f. $y = -2x - 5$ and $y = 10x + 1$
g. $y = -x - 2$ and $y = x + 1$
h. $y = 6x + 2$ and $y = -4x$

i. $y = 0.5x$ and $y = 0.8x + 0.9$

j. $y = 0.3x$ and $y = 0.2x + 0.1$

k. $y = -x$ and $y = -\frac{2}{7}x + \frac{4}{7}$

l. $y = -x$ and $y = -\frac{3}{4}x - \frac{1}{4}$

Understanding

3. A small farm has sheep and chickens. There are twice as many chicken as sheep, and there are 104 legs between the sheep and the chickens. How many chickens are there?

4. Use substitution to solve each of the following pairs of simultaneous equations.

a. $5x + 2y = 17$, $y = \dfrac{3x - 7}{2}$

b. $2x + 7y = 17$, $x = \dfrac{1 - 3y}{4}$

c. $2x + 3y = 13$, $y = \dfrac{4x - 15}{5}$

d. $-2x - 3y = -14$, $x = \dfrac{2 + 5y}{3}$

e. $3x + 2y = 6$, $y = 3 - \dfrac{5x}{3}$

f. $-3x - 2y = -12$, $y = \dfrac{5x - 20}{3}$

5. Use substitution to solve each of the following pairs of simultaneous equations for x and y in terms of m and n.

a. $mx + y = n$, $y = mx$

b. $x + ny = m$, $y = nx$

c. $mx - y = n$, $y = nx$

d. $mx - ny = n$, $y = x$

e. $mx - ny = -m$, $x = y - n$

f. $mx + y = m$, $x = \dfrac{y + m}{n}$

6. Determine the values of a and b so that the pair of equations $ax + by = 17$ and $2ax - by = -11$ has a unique solution of $(-2, 3)$.

7. The earliest record of magic squares is from China in about 2200 BC. In magic squares the sums of the numbers of each row, column and diagonal are all equal to a magic number. Let z be the magic number. By creating a set of equations, solve to find the magic number and the missing values in the magic square.

m	11	7
9		
n	5	10

Reasoning

8. a. For the pair of simultaneous equations:

$$8x - 7y = 9$$
$$x + 2y = 4,$$

which of the equations is the logical choice to make x the subject of the equation?

b. Use the substitution method to solve the system of equations. Show all your working.

9. A particular chemistry book costs \$6 less than a particular physics book, while two such chemistry books and three such physics books cost a total of \$123. Construct two simultaneous equations and solve them using the substitution method. Show your working.

Problem solving

10. Use the substitution method to solve the following.

$$2x + y - 9 = 0$$
$$4x + 5y + 3 = 0$$

11. Use the substitution method to solve the following.

$$\frac{y-x}{2}-\frac{x+y}{3}=\frac{1}{6}$$

$$\frac{x}{5}+\frac{y}{2}=\frac{1}{2}$$

Reflection

When would you choose the substitution method in solving simultaneous equations?

4.4 Solving simultaneous linear equations using elimination

4.4.1 The elimination method

- The elimination method is an algebraic method to solve simultaneous equations without graphing.
- If two balanced equations contain the same variables, the equations can be added or subtracted to eliminate one of the variables. For example, the equations $2x + y = 5$ and $x + y = 3$ are shown at right on balance scales.

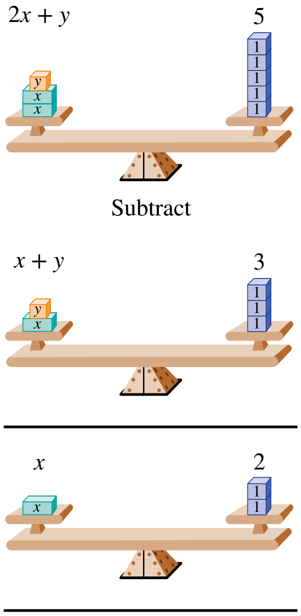
  If the left-hand side of the second equation is subtracted from the left-hand side of the first equation, and the right-hand side of the second equation is subtracted from the right-hand side of the first equation, the variable y is eliminated, leaving $x = 2$.

  Another way to represent this situation is:

$$\begin{array}{r} 2x + y \ = 5 \\ -\ (x + y = 3) \\ \hline x \qquad = 2 \\ \hline \end{array}$$

  In this example, the variable is eliminated by subtraction to reveal the value of x. The value of y can then be calculated by substituting $x = 2$ into either equation.

$$2(2) + y = 5 \ \Rightarrow \ y = 1$$



WORKED EXAMPLE 6

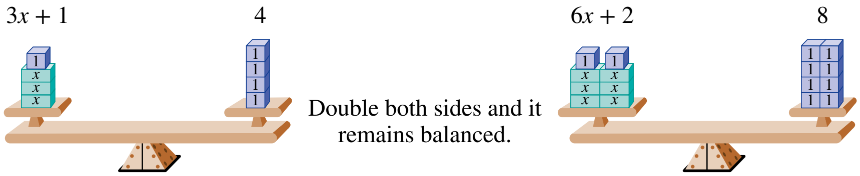
Solve the following pair of simultaneous equations using the elimination method.

$$-2x - 3y = -9 \qquad 2x + y = 7$$

THINK	WRITE
1 Write the equations, one under the other and number them.	$-2x - 3y = -9 \quad [1]$ $2x + y = 7 \quad [2]$
2 Look for an addition or subtraction that will eliminate either x or y. *Note:* Adding equations [1] and [2] in order will eliminate x.	[1] + [2]: $-2x - 3y + (2x + y) = -9 + 7$ $-2x - 3y + 2x + y = -2$ $-2y = -2$
3 Solve for y by dividing both sides of the equation by -2.	$y = 1$
4 Substitute the value of y into equation [2]. *Note:* $y = 1$ may be substituted into either equation.	Substituting $y = 1$ into [2]: $2x + 1 = 7$
5 Solve for x.	
i Subtract 1 from both sides of the equation.	$2x = 6$
ii Divide both sides of the equation by 2.	$x = 3$

6 Write the solution.	Solution: $x = 3$, $y = 1$ or $(3, 1)$
7 Check the solution by substituting $(3, 1)$ into equation [1] since equation [2] was used to find the value of x.	Check: Substitute into $-2x - 3y = -9$. $\text{LHS} = -2(3) - 3(1)$ $= -6 - 3$ $= -9$ $\text{RHS} = -9$ LHS = RHS, so the solution is correct.

- If a variable is not eliminated when the equations are simply added or subtracted, it may be necessary to multiply one or both equations by some number or numbers so that when the equations are added, one of the variables is then eliminated.
- If two equal quantities are multiplied by the same number, the results remain equal.



WORKED EXAMPLE 7

Solve the following pair of simultaneous equations using the elimination method.

$$x - 5y = -17 \qquad 2x + 3y = 5$$

THINK	WRITE
1 Write the equations, one under the other and number them.	$x - 5y = -17$ [1] $2x + 3y = 5$ [2]
2 Look for a single multiplication that will create the same coefficient of either x or y. Multiply equation [1] by 2 and call the new equation [3].	[1] × 2: $2x - 10y = -34$ [3]
3 Subtract equation [2] from [3] in order to eliminate x.	[3] − [2]: $2x - 10y - (2x + 3y) = -34 - 5$ $2x - 10y - 2x - 3y = -39$ $-13y = -39$
4 Solve for y by dividing both sides of the equation by -13.	$y = 3$
5 Substitute the value of y into equation [2].	Substituting $y = 3$ into [2]: $2x + 3(3) = 5$ $2x + 9 = 5$
6 Solve for x.	
i Subtract 9 from both sides of the equation.	$2x = -4$
ii Divide both sides of the equation by 2.	$x = -2$
7 Write the solution.	Solution: $x = -2$, $y = 3$ or $(-2, 3)$
8 Check the solution by substituting into equation [1].	Check: Substitute into $x - 5y = -17$. $\text{LHS} = (-2) - 5(3)$ $= -2 - 15$ $= -17$ $\text{RHS} = -17$ LHS = RHS, so the solution is correct.

Note: In this example, equation [1] could have been multiplied by -2 (instead of by 2), then the two equations added (instead of subtracted) to eliminate x.

WORKED EXAMPLE 8

Solve the following pair of simultaneous equations using the elimination method.

$$6x + 5y = 3 \qquad 5x + 4y = 2$$

THINK	WRITE
1 Write the equations, one under the other and number them.	$6x + 5y = 3$ [1] $5x + 4y = 2$ [2]
2 Decide which variable to eliminate, say y. Multiply equation [1] by 4 and call the new equation [3]. Multiply equation [2] by 5 and call the new equation [4].	Eliminate y. [1] × 4: $24x + 20y = 12$ [3] [2] × 5: $25x + 20y = 10$ [4]
3 Subtract equation [3] from [4] in order to eliminate y.	[4] − [3]: $25x + 20y - (24x + 20y) = 10 - 12$ $25x + 20y - 24x - 20y = -2$ $x = -2$
4 Substitute the value of x into equation [1].	Substituting $x = -2$ into [1]: $6(-2) + 5y = 3$ $-12 + 5y = 3$
5 Solve for y. **i** Add 12 to both sides of the equation. **ii** Divide both sides of the equation by 5.	$5y = 15$ $y = 3$
6 Write your answer.	Solution $x = -2$, $y = 3$ or $(-2, 3)$
7 Check the answer by substituting the solution into equation [2].	Check: Substitute into $5x + 4y = 2$. $\text{LHS} = 5(-2) + 4(3)$ $= -10 + 12$ $= 2$ $\text{RHS} = 2$ LHS = RHS, so the solution is correct.

Note: Equation [1] could have been multiplied by -4 (instead of by 4), then the two equations added (instead of subtracted) to eliminate y.

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Exercise 4.4 Solving simultaneous linear equations using elimination

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 3a–c, 4a–c, 5a–c, 6, 7	Questions: 1, 2, 3a–d, 4a–e, 5a–d, 7, 8, 9	Questions: 1, 2, 3d–f, 43e–i, 5c–f, 6–10

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE6** Solve the following pairs of simultaneous equations by adding equations to eliminate either x or y.

a. $x + 2y = 5$
$-x + 4y = 1$

b. $5x + 4y = 2$
$5x - 4y = -22$

c. $-2x + y = 10$
$2x + 3y = 14$

2. Solve the following pairs of equations by subtracting equations to eliminate either x or y.

a. $3x + 2y = 13$
$5x + 2y = 23$

b. $2x - 5y = -11$
$2x + y = 7$

c. $-3x - y = 8$
$-3x + 4y = 13$

3. Solve each of the following equations using the elimination method.

a. $6x - 5y = -43$
$6x - y = -23$

b. $x - 4y = 27$
$3x - 4y = 17$

c. $-4x + y = -10$
$4x - 3y = 14$

d. $-5x + 3y = 3$
$-5x + y = -4$

e. $5x - 5y = 1$
$2x - 5y = -5$

f. $4x - 3y - 1 = 0$
$4x + 7y - 11 = 0$

4. **WE7** Solve the following pairs of simultaneous equations.

a. $6x + y = 9$
$-3x + 2y = 3$

b. $x + 3y = 14$
$3x + y = 10$

c. $5x + y = 27$
$4x + 3y = 26$

d. $-6x + 5y = -14$
$-2x + y = -6$

e. $2x + 5y = 14$
$3x + y = -5$

f. $-3x + 2y = 6$
$x + 4y = -9$

g. $3x - 5y = 7$
$x + y = -11$

h. $2x + 3y = 9$
$4x + y = -7$

i. $-x + 5y = 7$
$5x + 5y = 19$

5. **WE8** Solve the following pairs of simultaneous equations.

a. $-4x + 5y = -9$
$2x + 3y = 21$

b. $2x + 5y = -6$
$3x + 2y = 2$

c. $2x - 2y = -4$
$5x + 4y = 17$

d. $2x - 3y = 6$
$4x - 5y = 9$

e. $\frac{x}{2} + \frac{y}{3} = 2$
$\frac{x}{4} + \frac{y}{3} = 4$

f. $\frac{x}{3} + \frac{y}{2} = \frac{3}{2}$
$\frac{x}{2} + \frac{y}{5} = -\frac{1}{2}$

Understanding

6. Solve the following simultaneous equations using an appropriate method. Check your answer using technology.

a. $7x + 3y = 16$
$y = 4x - 1$

b. $2x + y = 8$
$4x + 3y = 16$

c. $-3x + 2y = 19$
$4x + 5y = 13$

d. $-3x + 7y = 9$
$4x - 3y = 7$

e. $-4x + 5y = -7$
$x = 23 - 3y$

f. $y = -x$
$y = -\frac{2}{5}x - \frac{1}{5}$

Reasoning

7. Ann, Beth and Celine wanted to weigh themselves, but the scales they had were broken and would only give readings over 100 kg. They decided to weigh themselves in pairs and calculate their weights from the results.

- Ann and Beth weighed 119 kg
- Beth and Celine weighed 112 kg
- Celine and Ann weighed 115 kg

How much did each of the girls weigh? Show your working.

8. a. For the general case $ax + by = e$ [1]
$cx + dy = f$ [2]
y can be found by eliminating x.

i. Multiply equation [1] by c to create equation 3.
ii. Multiply equation [2] by a to create equation 4.
iii. Use the elimination method to find a general solution for y.

b. Use a similar process to that outlined above to find a general solution for x.

c. Use the general solution for x and y to solve each of the following.

i. $2x + 5y = 7$
$7x + 2y = 24$

ii. $3x - 5y = 4$
$x + 3y = 5$

Choose another method to check that your solutions are correct in each part.

d. For y to exist, it is necessary to state that $bc - ad \neq 0$. Why?

e. Is there a necessary condition for x to exist? Explain.

Problem solving

9. Use the method of elimination to solve

$$\frac{x-4}{3} + y = -2$$
$$\frac{2y-1}{7} + x = 6.$$

10. Use an appropriate method to solve

$$2x + 3y + 3z = -1$$
$$3x - 2y + z = 0$$
$$z + 2y = 0.$$

Reflection

How does eliminating one variable help to solve simultaneous equations?

CHALLENGE 4.1

If $x + y = 17$, $y + z = 15$ and $x + z = 14$, what is the value of z?

4.5 Applications of simultaneous linear equations

4.5.1 Applications of simultaneous linear equations

- There are many practical applications of simultaneous equations, some examples of which are shown below.
- When solving practical problems, the following steps can be useful.
  - Define the unknown quantities using appropriate pronumerals.
  - Use the information given in the problem to form two equations in terms of these pronumerals.
  - Solve these equations using an appropriate method.
  - Write the solution in words.
  - Check the solution.

WORKED EXAMPLE 9

Ashley received better results for his Mathematics test than for his English test. If the sum of the two marks is 164 and the difference is 22, calculate the mark he received for each subject.

THINK	WRITE
1 Define the two variables.	Let x = the Mathematics mark. Let y = the English mark.
2 Formulate two equations from the information given and number them. The sum of the two marks is $x + y$. The difference of the two marks is $x - y$.	$x + y = 164$ [1] $x - y = 22$ [2]
3 Use the elimination method by adding equations [1] and [2] to eliminate y.	[1] + [2]: $2x = 186$
4 Solve for x by dividing both sides of the equation by 2.	$x = 93$
5 Substitute the value of x into equation [1].	Substituting $x = 93$ into [1]: $x + y = 164$ $93 + y = 164$
6 Solve for y by subtracting 93 from both sides of the equation.	$y = 71$
7 Write the solution.	Solution: Mathematics mark $(x) = 93$ English mark $(y) = 71$
8 Check the solution by substituting $x = 93$ and $y = 71$ into equation [1].	Check: Substitute into $x + y = 164$. $\text{LHS} = 93 + 71 = 164$ $\quad \text{RHS} = 164$ As LHS = RHS, the solution is correct.

WORKED EXAMPLE 10

To finish a project, Genevieve buys a total of 25 nuts and bolts from a hardware store. If each nut costs 12 cents, each bolt costs 25 cents and the total purchase price is $4.30, how many nuts and how many bolts does Genevieve buy?

THINK	WRITE
1 Define the two variables.	Let x = the number of nuts. Let y = the number of bolts.
2 Formulate two equations from the information given and number them. *Note:* The total number of nuts and bolts is 25. Each nut cost 12 cents, each bolt cost 25 cents and the total cost is 430 cents ($4.30).	$x + y = 25$ [1] $12x + 25y = 430$ [2]
3 Solve simultaneously using the substitution method, since equation [1] is easy to rearrange.	
4 Rearrange equation [1] to make x the subject by subtracting y from both sides of equation [1].	Rearrange equation [1]: $x + y = 25$ $x = 25 - y$
5 Substitute the expression $(25 - y)$ for x into equation [2].	Substituting $(25 - y)$ into [2]: $12(25 - y) + 25y = 430$
6 Solve for y.	$300 - 12y + 25y = 430$ $300 + 13y = 430$ $13y + 300 = 430$ $13y = 130$ $y = 10$
7 Substitute the value of y into the rearranged equation $x = 25 - y$ from step **4**.	Substituting $y = 10$ into $x = 25 - y$: $x = 25 - 10$ $x = 15$
8 Write the solution.	Solution: The number of nuts (x) = 15. The number of bolts (y) = 10.
9 Check the solution by substituting $x = 15$ and $y = 10$ into equation [1].	Check: Substitute into $x + y = 25$. LHS = 15 + 10 RHS = 25 = 25 As LHS = RHS, the solution is correct.

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Exercise 4.5 Applications of simultaneous linear equations

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–3, 6, 8, 12, 14, 16, 19	Questions: 1, 2, 4, 7, 9, 11, 14, 16, 17, 19	Questions: 1, 2, 5, 7, 9, 10, 13, 15–20

🟩🟦🟪 **Individual pathway interactivity**: int-4580 learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE9** Rick received better results for his Maths test than for his English test. If the sum of his two marks is 163 and the difference is 31, find the mark for each subject.
2. **WE10** Rachael buys 30 nuts and bolts to finish a project. If each nut costs 10 cents, each bolt costs 20 cents and the total purchase price is \$4.20, how many nuts and how many bolts does she buy?

Understanding

3. Find two numbers whose difference is 5 and whose sum is 11.
4. The difference between two numbers is 2. If three times the larger number minus twice the smaller number is 13, find the two numbers.
5. One number is 9 less than three times a second number. If the first number plus twice the second number is 16, find the two numbers.
6. A rectangular house has a perimeter of 40 metres and the length is 4 metres more than the width. What are the dimensions of the house?
7. Mike has 5 lemons and 3 oranges in his shopping basket. The cost of the fruit is \$3.50. Voula, with 2 lemons and 4 oranges, pays \$2.10 for her fruit. How much does each type of fruit cost?

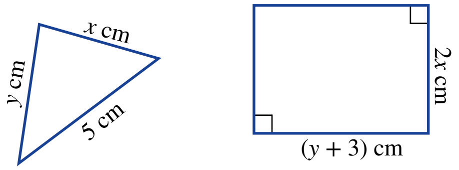
8. A surveyor measuring the dimensions of a block of land finds that the length of the block is three times the width. If the perimeter is 160 metres, what are the dimensions of the block?
9. Julie has \$3.10 in change in her pocket. If she has only 50 cent and 20 cent pieces and the total number of coins is 11, how many coins of each type does she have?
10. Mr Yang's son has a total of twenty-one \$1 and \$2 coins in his moneybox. When he counts his money, he finds that its total value is \$30. How many coins of each type does he have?
11. If three Magnums and two Paddlepops cost \$8.70 and the difference in price between a Magnum and a Paddlepop is 90 cents, how much does each type of ice-cream cost?
12. If one Redskin and 4 Golden Roughs cost \$1.65, whereas 2 Redskins and 3 Golden Roughs cost \$1.55, how much does each type of sweet cost?
13. A catering firm charges a fixed cost for overheads and a price per person. It is known that a party for 20 people costs \$557, whereas a party for 35 people costs \$909.50. What is the fixed cost and the cost per person charged by the company?

14. The difference between Sally's PE mark and Science mark is 12, and the sum of the marks is 154. If the PE mark is the higher mark, what did Sally get for each subject?

15. Mozza's Cheese Supplies sells six Mozzarella cheeses and eight Swiss cheeses to Munga's deli for \$83.60, and four Mozzarella cheeses and four Swiss cheeses to Mina's deli for \$48. How much does each type of cheese cost?

Reasoning

16. If the perimeter of the triangle in the diagram is 12 cm and the length of the rectangle is 1 cm more than the width, find the value of x and y. Show your working.



17. Mr and Mrs Waugh want to use a caterer for a birthday party for their twin sons. The manager says the cost for a family of four would be \$160. However, the sons want to invite 8 friends, making 12 people in all. The cost for this would be \$360. If the total cost in each case is made up of the same cost per person and the same fixed cost, find the cost per person and the fixed cost. Show your working.

18. Joel needs to buy some blank DVDs and zip disks to back up a large amount of data that has been generated by an accounting firm. He buys 6 DVDs and 3 zip disks for \$96. He later realises these are not sufficient and so buys another 5 DVDs and 4 zip disks for \$116. How much did each DVD and each zip disk cost? (Assume the same rate per item was charged for each visit.) Show your working.

Problem solving

19. At the football hot chips are twice as popular as meat pies and three times as popular as hot dogs. Over the period of half an hour during half time, a fast-food outlet serves 121 people who each bought one item. How many serves of each of the foods were sold during this half-hour period?

20. Three jet-skis in a 300 kilometre handicap race leave at two hour intervals. Jet-ski 1 leaves first and has an average speed of 25 kilometres per hour for the entire race. Jet-ski 2 leaves two hours later and has an average speed of 30 kilometres per hour for the entire race. Jet-ski 3 leaves last, two hours after jet-ski 2 and has an average speed of 40 kilometres per hour for the entire race.

  a. Sketch a graph to show each jet-ski's journey on the one set of axes.
  b. Determine who wins the race.
  c. Check your findings algebraically and describe what happened to each jet-ski during the course of the race.

Reflection

How do you decide which method to use when solving problems using simultaneous linear equations?

CHALLENGE 4.2

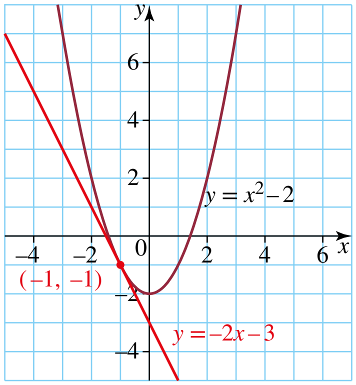
1 At a fun park, the cost of a rollercoaster ride and a Ferris wheel ride is \$10. The cost of a Gravitron ride and a Ferris wheel ride is \$12. The cost of a rollercoaster ride and a Gravitron ride is \$14. What is the cost of each ride?

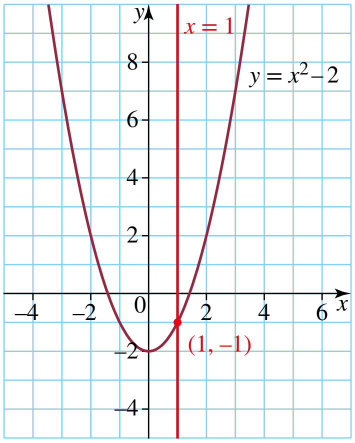
2 A number has five digits. The digit 6 is three places to the right of the digit 9. The digit 4 is somewhere to the left of digit 6 and to the right of digit 2. The digit 8 is three places to the left of the digit 4. What is the number?

4.6 Solving simultaneous linear and non-linear equations

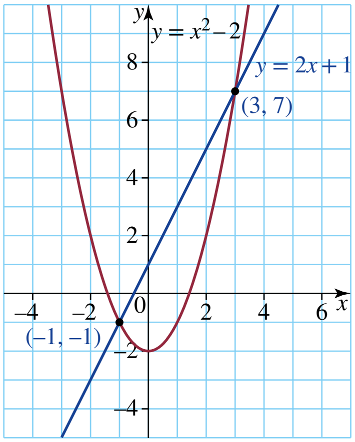
4.6.1 Solving simultaneous linear and quadratic equations

- The graph of a quadratic function is called a parabola.
- A parabola and a straight line may:
  - intersect at only one point, as shown in red

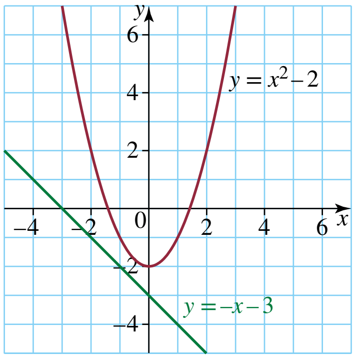




  - intersect at two points, as shown in blue



  - not intersect at all, as shown in green.



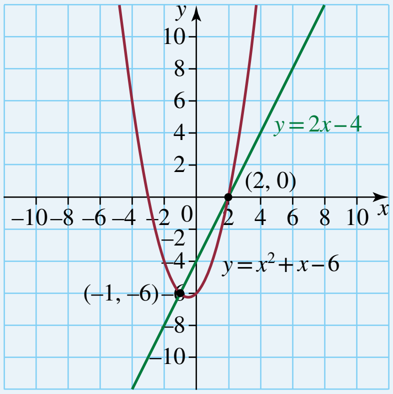
WORKED EXAMPLE 11

Find the points of intersection of $y = x^2 + x - 6$ and $y = 2x - 4$:

a algebraically **b graphically.**

THINK	WRITE/DRAW
a 1 Number the equations. Equate [1] and [2].	**a** $y = x^2 + x - 6$ [1] $y = 2x - 4$ [2] $x^2 + x - 6 = 2x - 4$
2 Collect all the terms on one side and simplify.	$x^2 + x - 6 - 2x + 4 = 2x - 4 - 2x + 4$ $x^2 + x - 6 - 2x + 4 = 0$ $x^2 - x - 2 = 0$
3 Factorise and solve the quadratic equation, using the null factor law.	$(x - 2)(x + 1) = 0$ $x - 2 = 0$ or $x + 1 = 0$ $x = 2 \qquad x = -1$
4 Identify the y-coordinate for each point of intersection by substituting each x-value into one of the equations.	When $x = 2$, $y = 2(2) - 4$ $= 4 - 4$ $= 0$ Intersection point (2, 0) When $x = -1$ $y = 2(-1) - 4$ $= -2 - 4$ $= -6$
5 Write the solution.	Intersection point$(-1, -6)$
b 1 To sketch the graph of $y = x^2 + x - 6$, find the x- and y-intercepts and the turning point.	**b** x-intercepts: $y = 0$ $0 = x^2 + x - 6$ $0 = (x + 3)(x - 2)$ $x = -3, x = 2$ The x-intercepts are $(-3, 0)$ and $(2, 0)$. y-intercept: $x = 0$ $y = -6$ The y-intercept is $(0, -6)$. x-value of TP: $\dfrac{-3 + 2}{2} = -0.5$ $y = (-0.5)^2 + (-0.5) - 6$ $y = -6.25$ The TP is $(-0.5, -6.25)$.
2 To sketch the graph of $y = 2x - 4$, find the x- and y-intercepts.	x-intercept: $y = 0$ $0 = 2x - 4$ $x = 2$ The x-intercept is (2, 0) y-intercept: $x = 0$ $y = -4$ The y-intercept is $(0, -4)$

3 On the same set of axes, sketch the graphs of $y = x^2 + x - 6$ and $y = 2x - 4$, labelling both.

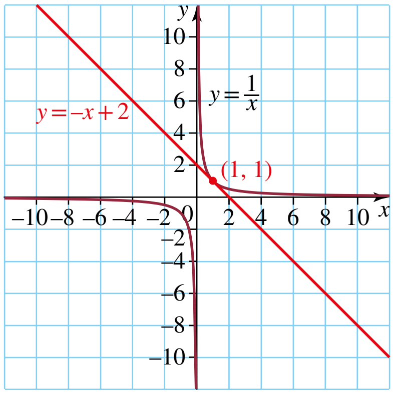


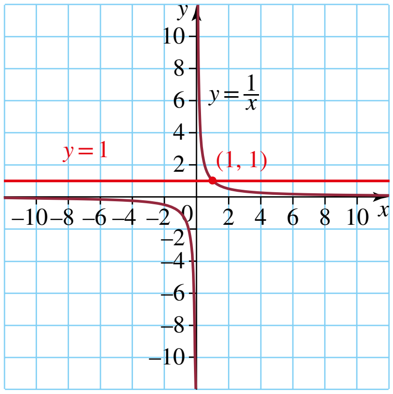
4 On the graph, locate the points of intersection and write the solutions.

The points of intersection are $(2, 0)$ and $(-1, -6)$.

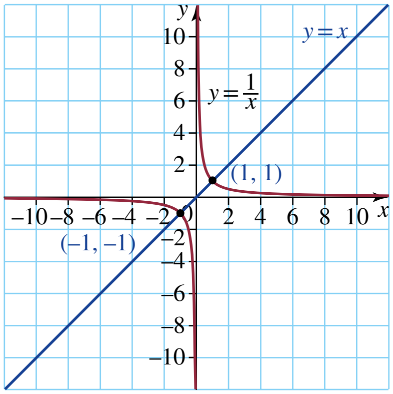
4.6.2 Solving simultaneous linear and hyperbolic equations

- A hyperbola and a straight line may:
  - intersect at only one point, as shown in red. In the first case, the line is a tangent to the curve.

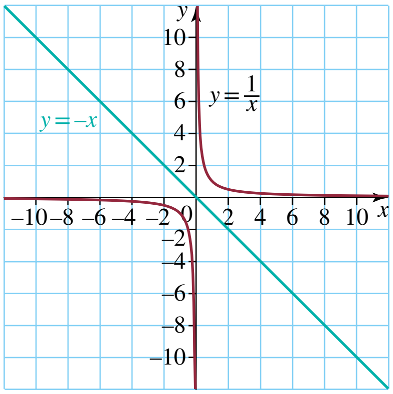




  - intersect at two points, as shown in blue



– not intersect at all, as shown in green.



WORKED EXAMPLE 12 TI | CASIO

Find the point(s) of intersection between $y = x + 5$ and $y = \frac{6}{x}$:

a algebraically **b graphically.**

THINK

WRITE/DRAW

a 1 Number the equations. Equate [1] and [2].

a $y = x + 5$ [1]

$y = \frac{6}{x}$ [2]

$$x + 5 = \frac{6}{x}$$
$$x(x+5) = 6$$

2 Collect all terms on one side, factorise and simplify.

$$x^2 + 5x - 6 = 0$$
$$(x+6)(x-1) = 0$$
$$x = -6,\ x = 1$$

3 To find the y-coordinates of the points of intersection, substitute the values of x into [1].

$x = -6$
$y = -6 + 5$
$y = -1$

$x = 1$
$y = 1 + 5$
$y = 6$

4 Write the solutions.

The points of intersection are $(-6, -1)$ and $(1, 6)$.

b 1 To sketch the graph of $y = \frac{6}{x}$, draw a table of values.

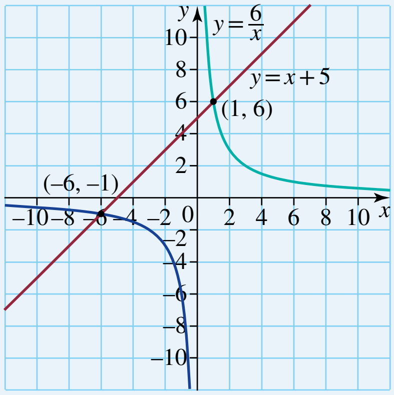
b

x	-6	-5	-4	-3	-2	-1	0	1	2
y	-1	$-1\frac{1}{5}$	$-1\frac{1}{2}$	-2	-3	-6	Undef.	6	3

2 To sketch the graph of $y = x + 5$, find the x- and y-intercepts.

x-intercept: $y = 0$
$0 = x + 5$
$x = -5$
The x-intercept is $(-5, 0)$.
y-intercept: $x = 0$
$y = 5$
The y-intercept is $(0, 5)$.

3 On the same set of axes, sketch the graphs of $y = x + 5$ and $y = \frac{6}{x}$, labelling both.

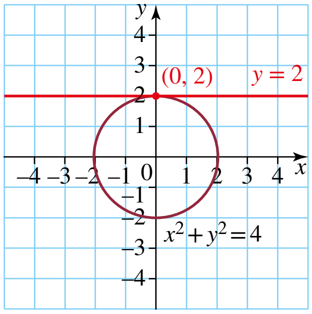


4 On the graph, locate the points of intersection and write the solutions.

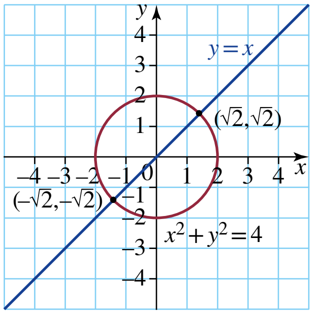
The points of intersection are $(1, 6)$ and $(-6, -1)$.

4.6.3 Solving simultaneous linear equations and circles

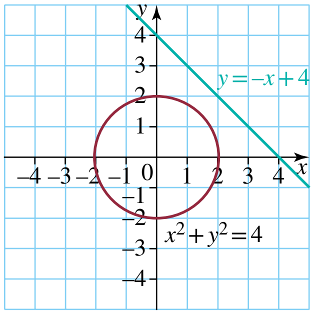
- A circle and a straight line may:
  - intersect at only one point, as shown in red. In this case, the line is a tangent to the curve.



  - intersect at two points, as shown in blue



  - not intersect at all, as shown in green.



Exercise 4.6 Solving simultaneous linear and non-linear equations

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 2a–c, 4, 5, 7a–b, 8a–b, 9, 12	Questions: 1, 2, 3c–e, 4–6, 7b–d, 8b–d, 9, 12	Questions: 1, 2, 3d–f, 4–6, 7c–d, 8c–d, 9–12

🟩🟦🟪 **Individual pathway interactivity**: int-4581

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Fluency

1. Describe how a parabola and straight line may intersect. Use diagrams to illustrate your explanation.

2. **WE11** Find the points of intersection of the following:

- **i.** using algebra
- **ii.** algebraically using a calculator
- **iii.** graphically using a calculator.

a. $y = x^2 + 5x + 4$ and $y = -x - 1$

b. $y = -x^2 + 2x + 3$ and $y = -2x + 7$

c. $y = -x^2 + 2x + 3$ and $y = -6$

3. Find the points of intersection of the following.

a. $y = -x^2 + 2x + 3$ and $y = 3x - 8$

b. $y = -(x - 1)^2 + 2$ and $y = x - 1$

c. $y = x^2 + 3x - 7$ and $y = 4x + 2$

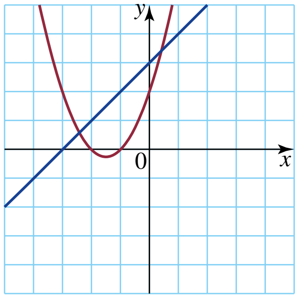
d. $y = 6 - x^2$ and $y = 4$

e. $y = 4 + x - x^2$ and $y = \dfrac{3 - x}{2}$

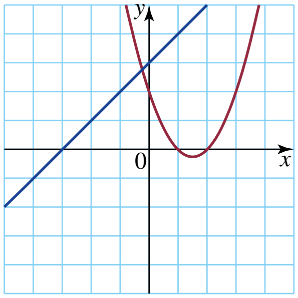
f. $x = 3$ and $y = 2x^2 + 7x - 2$

4. **MC** Which of the following graphs shows the parabola $y = x^2 + 3x + 2$, $x \in R$, and the straight line $y = x + 3$?

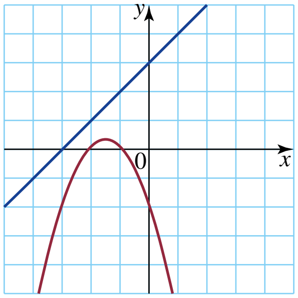
A.



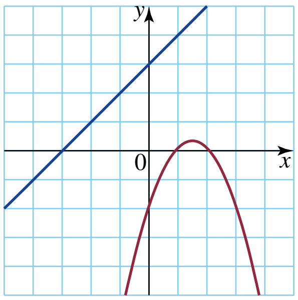
B.



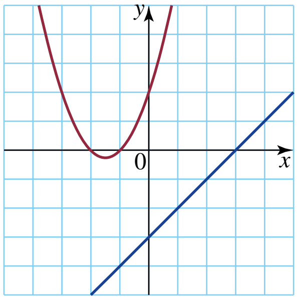
C.



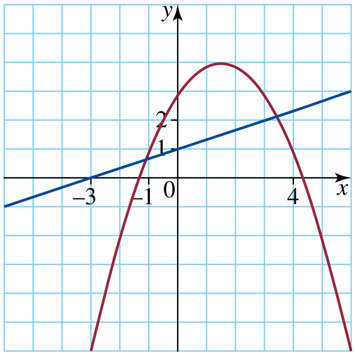
D.



E.



5. **MC** The diagram below could show which of the following?



A. $y = 0.5(x + 1.5)^2 + 4$ and $y = -\frac{1}{3}x + 1$

B. $y = -0.5(x + 1.5)^2 - 4$ and $y = -\frac{1}{3}x + 1$

C. $y = -0.5(x - 1.5)^2 + 4$ and $y = \frac{1}{3}x + 1$

D. $y = 0.5(x - 1.5)^2 + 4$ and $y = -\frac{1}{3}x + 1$

E. $y = 0.5(x - 1.5)^2 + 4$ and $y = -\frac{1}{3}x + 1$

6. Determine whether the following graphs intersect.

a. $y = -x^2 + 3x + 4$ and $y = x - 4$
b. $y = -x^2 + 3x + 4$ and $y = 2x + 5$
c. $y = -(x + 1)^2 + 3$ and $y = -4x - 1$
d. $y = (x - 1)^2 + 5$ and $y = -4x - 1$

7. **WE12** Find the point(s) of intersection between the following.

a. $y = x$
$y = \frac{1}{x}$

b. $y = x - 2$
$y = \frac{1}{x}$

c. $y = 3x$
$y = \frac{5}{x}$

d. $y = \frac{6}{x}$
$y = \frac{x}{2} + 2$

Understanding

8. Find the point(s) of intersection between the following.

a. $y = 3x$
$x^2 + y^2 = 10$

b. $x^2 + y^2 = 25$
$3x + 4y = 0$

c. $x^2 + y^2 = 50$
$y = 5 - 2x$

d. $x^2 + y^2 = 9$
$y = 2 - x$

Reasoning

9. Show that there is at least one point of intersection between the parabola $y = -2(x + 1)^2 - 5$, where $y = f(x)$, and the straight line $y = mx - 7$, where $y = f(x)$.

Problem solving

10. **a.** Find the point(s) of intersection between the circle $x^2 + y^2 = 50$ and the linear equation $y = 2x - 5$.
b. Confirm your solution to part **a** by plotting the equation of the circle and the linear equation on the same graph.

11. The sum of two positive numbers is 21. Twice the square of the larger number minus three times the square of the smaller number is 45. Find the value of the two numbers.

Reflection

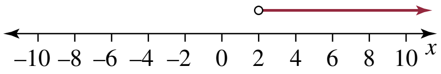
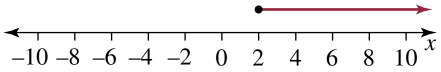
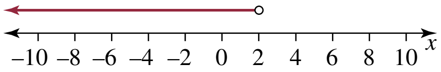
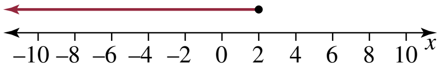
WHOLE CLASS

What does it mean if a straight line touches a curve only once?

4.7 Solving linear inequalities

4.7.1 Inequalities between two expressions

- An equation is a statement of *equality* such as $x = 2$; an **inequation** is a statement of **inequality** between two expressions such as $x < 2$ (x is less than 2).
- The solution to a linear equation is a single point on a number line, but the solution to an inequation is a portion of the number line. That is, there are an infinite number of solutions to an inequality.
- The following table shows examples of four types of simple inequalities and their corresponding representation on a number line.
- Note that an open circle placed over the 2 indicates that 2 is not included; that is, 2 does not satisfy the inequality. A closed or solid circle indicates that 2 is included; that is, it does satisfy the inequality.

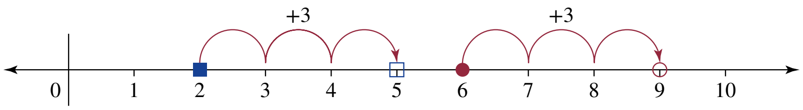
Mathematical statement	English statement	Number line diagram
$x > 2$	x is greater than 2	
$x \geq 2$	x is greater than or equal to 2	
$x < 2$	x is less than 2	
$x \leq 2$	x is less than or equal to 2	

4.7.2 Solving inequalities

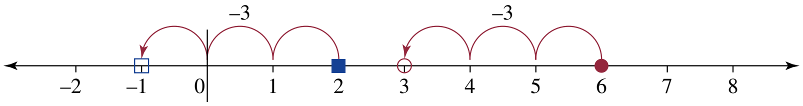
- The following things may be done to both sides of an inequality without affecting its truth.
  – A number can be added or subtracted from both sides of the inequality.

Adding or subtracting a number:

e.g. $6 > 2$ Add 3 to both sides:
$9 > 5$ (True)



e.g. $6 \geq -2$ Subtract 3 from both sides:
$3 \geq -1$ (True)

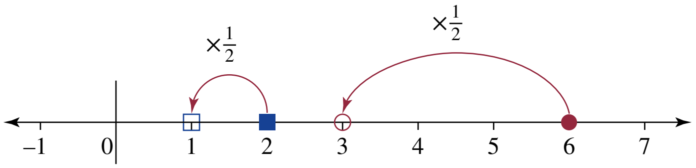


Adding or subtracting moves both numbers the same distance along the number line.

– A number can be multiplied or divided by a positive number.

Multiplying or dividing by a positive number:

e.g. $6 > 2$ Multiply both sides by $\frac{1}{2}$:
$3 > 1$ (True)

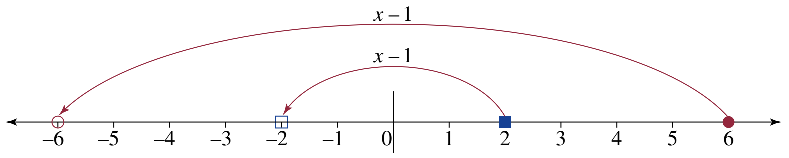


The distance between the numbers has changed, but their relative position has not.

- Care must be taken when multiplying or dividing by a negative number.

Multiplying or dividing by a negative number:

e.g. $6 > 2$ Multiply both sides by -1:
$-6 < -2$ (False)



Multiplying or dividing by a negative number reflects numbers about $x = 0$. Their relative positions are reversed.

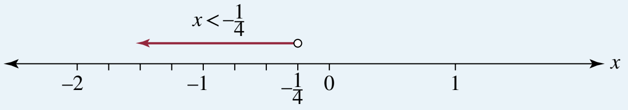
- **When solving inequalities, if both sides are multiplied or divided by a negative number, then the inequality sign must be reversed.**
  For example, $6 > 2$ implies that $-6 < -2$.

WORKED EXAMPLE 13

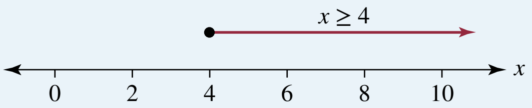
Solve each of the following linear inequalities and show the solution on a number line.

a $\mathbf{4x - 1 < -2}$ **b** $\mathbf{6x - 7 \geq 3x + 5}$

THINK	WRITE/DRAW
a 1 Write the inequality.	**a** $4x - 1 < -2$
2 Add 1 to both sides of the inequality.	$4x - 1 + 1 < -2 + 1$ $4x < -1$
3 Obtain x by dividing both sides of the inequality by 4.	$\frac{4x}{4} < -\frac{1}{4}$ $x < -\frac{1}{4}$



THINK	WRITE/DRAW
b 1 Write the inequality.	**b** $6x - 7 \geq 3x + 5$
2 Subtract $3x$ from both sides of the inequality.	$6x - 7 - 3x \geq 3x + 5 - 3x$ $3x - 7 \geq 5$
3 Add 7 to both sides of the inequality.	$3x - 7 + 7 \geq 5 + 7$ $3x \geq 12$
4 Obtain x by dividing both sides of the inequality by 3.	$\frac{3x}{3} \geq \frac{12}{3}$ $x \geq 4$



WORKED EXAMPLE 14

TI | CASIO

Solve each of the following linear inequalities.

a $\mathbf{-3m + 5 < -7}$ **b** $\mathbf{5(x - 2) \geq 7(x + 3)}$

THINK	WRITE
a 1 Write the inequality.	**a** $-3m + 5 < -7$

2 Subtract 5 from both sides of the inequality. (No change to the inequality sign.)

$$-3m + 5 - 5 < -7 - 5$$
$$-3m < -12$$

3 Obtain m by dividing both sides of the inequation by -3. Reverse the inequality sign, since you are dividing by a negative number.

$$\frac{-3m}{-3} > \frac{-12}{-3}$$
$$m > 4$$

b **1** Write the inequality.

b $$5(x - 2) \geq 7(x + 3)$$

2 Expand both brackets.

$$5x - 10 \geq 7x + 21$$

3 Subtract $7x$ from both sides of the inequality.

$$5x - 10 - 7x \geq 7x + 21 - 7x$$
$$-2x - 10 \geq 21$$

4 Add 10 to both sides of the inequation.

$$-2x - 10 + 10 \geq 21 + 10$$
$$-2x \geq 31$$

5 Divide both sides of the inequality by -2. Since we need to divide by a negative number, reverse the direction of the inequality sign.

$$\frac{-2x}{-2} \leq \frac{31}{-2}$$
$$x \leq \frac{-31}{2}$$
$$x \leq -15\tfrac{1}{2}$$

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Checking whether a given point makes the inequation a true statement (doc-5218)

Complete this digital doc: SkillSHEET: Writing equations from worded statements (doc-5219)

Exercise 4.7 Solving linear inequalities

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–f, 2, 3a–f, 4a–f, 5a–c, 6a–g, 7, 8a–c, 9a–f, 10, 13	Questions: 1e–l, 2, 3d–i, 4d–i, 5a–c, 6a–l, 7, 8a–c, 9a–i, 10, 12, 13	Questions: 1j–l, 2g–l, 3g–l, 4d–l, 5d–f, 6f–o, 8d–f, 9–14

🟩🟦🟪 **Individual pathway interactivity:** int-4582

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE13a** Solve each of the following inequalities.

a. $x + 1 > 3$ **b.** $a + 2 > 1$ **c.** $y - 3 \geq 4$ **d.** $m - 1 \geq 3$ **e.** $p + 4 < 5$
f. $x + 2 < 9$ **g.** $m - 5 \leq 4$ **h.** $a - 2 \leq 5$ **i.** $x - 4 > -1$ **j.** $5 + m \geq 7$
k. $6 + q \geq 2$ **l.** $5 + a > -3$

2. Solve each of the following inequalities. Check your solutions by substitution.

a. $3m > 9$ **b.** $5p \leq 10$ **c.** $2a < 8$ **d.** $4x \geq 20$ **e.** $5p > -25$
f. $3x \leq -21$ **g.** $2m \geq -1$ **h.** $4b > -2$ **i.** $\frac{m}{3} > 6$ **j.** $\frac{x}{2} < 4$
k. $\frac{a}{7} \leq -2$ **l.** $\frac{m}{5} \geq 5$

3. **WE13a** Solve each of the following inequalities.

a. $2m + 3 < 12$ **b.** $3x + 4 \geq 13$ **c.** $5p - 9 > 11$ **d.** $4n - 1 \leq 7$ **e.** $2b - 6 < 4$

f. $8y - 2 > 14$ **g.** $10m + 4 \leq -6$ **h.** $2a + 5 \geq -5$ **i.** $3b + 2 < -11$ **j.** $6c + 7 \leq 1$

k. $4p - 2 > -10$ **l.** $3a - 7 \geq -28$

4. **WE13b** Solve each of the following inequalities.

a. $2m + 1 > m + 4$ **b.** $2a - 3 \geq a - 1$ **c.** $5a - 3 < a - 7$

d. $3a + 4 \leq a - 2$ **e.** $5x - 2 > 40 - 2x$ **f.** $7x - 5 \leq 11 - x$

g. $7b + 5 < 2b + 25$ **h.** $2(a + 4) > a + 13$ **i.** $3(m - 1) < m + 1$

j. $5(2m - 3) \leq 3m + 6$ **k.** $3(5b + 2) \leq -10 + 4b$ **l.** $5(3m + 1) \geq 2(m + 9)$

5. Solve each of the following inequalities.

a. $\frac{x + 1}{2} \leq 4$ **b.** $\frac{x - 2}{5} \geq -4$ **c.** $\frac{x + 7}{3} < -1$

d. $\frac{2x + 3}{4} > 6$ **e.** $\frac{3x - 1}{7} \geq 2$ **f.** $\frac{5x + 9}{6} < 0$

6. **WE14** Solve each of the following inequalities.

a. $-2m > 4$ **b.** $-5p \leq 15$ **c.** $-2a \geq -10$ **d.** $-p - 3 \leq 2$

e. $10 - y \geq 13$ **f.** $14 - x < 7$ **g.** $1 - 6p > 1$ **h.** $2 - 10a \leq 0$

i. $2(3 - x) < 12$ **j.** $-4(a + 9) \geq 8$ **k.** $-15 \leq -3(2 + b)$ **l.** $2x - 3 > 5x + 6$

m. $k + 5 < 2k - 3$ **n.** $3(x - 4) < 5(x + 5)$ **o.** $7(a + 4) \geq 4(2a - 3)$

7. **MC** When solving the inequality $-2x > -7$ we need to:

A. change the sign to $\geq$ **B.** change the sign to $<$ **C.** change the sign to $=$

D. change the sign to $\leq$ **E.** keep the sign unchanged

8. Solve each of the following inequalities.

a. $\frac{2 - x}{3} > 1$ **b.** $\frac{5 - m}{4} \geq 2$ **c.** $\frac{-3 - x}{5} < -4$

d. $\frac{3 - 8a}{2} < -1$ **e.** $\frac{4 - 3m}{2} \leq 0$ **f.** $\frac{-2m + 6}{10} \leq 3$

9. Solve each of the following inequalities.

a. $3k > 6$ **b.** $-a - 7 < -2$ **c.** $5 - 3m \geq 0$ **d.** $x + 4 > 9$

e. $10 - y \leq 3$ **f.** $5 + 3d < -1$ **g.** $\frac{7p}{3} \geq -2$ **h.** $\frac{1 - x}{3} \leq 2$

i. $\frac{-4 - 2m}{5} > 0$ **j.** $5a - 2 < 4a + 7$ **k.** $6p + 2 \leq 7p - 1$ **l.** $2(3x + 1) > 2x - 16$

Understanding

10. Write linear inequalities for the following statements, using x to represent the unknown. (Do not attempt to solve the equations.)

a. The product of 5 and a certain number is greater than 10.

b. When three is subtracted from a certain number the result is less than or equal to 5.

c. The sum of seven and three times a certain number is less than 42.

Reasoning

11. Given the positive numbers a, b, c and d and the variable x, there is the following relationship: $-c < ax + b < -d$.

a. Find the possible range of values of x if $a = 2$, $b = 3$, $c = 10$ and $d = 1$.

b. Rewrite the original relationship in terms of x only (x by itself between the $<$ signs), using a, b, c and d.

12. Two speed boats are racing along a section of Lake Quikalong. The speed limit along this section of the lake is 50 km/h. Ross is travelling 6 km/h faster than Steven and the sum of the speeds at which they are travelling is greater than 100 km/h.

a. Write an inequation and solve it to describe all possible speeds that Steven could be travelling at.

b. At Steven's lowest possible speed, is he over the speed limit?

c. The water police issue a warning to Ross for exceeding the speed limit on the lake. Show that the police were justified in issuing a warning to Ross.

Problem solving

13. Mick the painter has fixed costs (e.g. insurance, equipment, etc) of \$3400 per year. His running cost to travel to jobs is based on \$0.75 per kilometre. Last year Mick had costs that were less than \$16 000.

a. Write an inequality to show this information and solve it to find how many kilometres Mick travelled for the year.

b. Explain the information you have found.

14. I have \$40 000 to invest. Part of this I intend to invest in a stable 5% simple interest account. The remainder will be invested in my friend's I.T. business. She has said that she will pay me 7.5% interest on any money I give to her. I am saving for a European trip so want the best return for my money. What is the least amount of money I should invest with my friend so that I receive at least \$2500 interest per year from my investments?

Reflection

What is similar and different when solving linear inequations to linear equations?

4.8 Inequalities on the Cartesian plane

4.8.1 Inequalities on the Cartesian plane

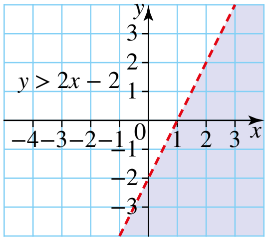
- A solution to a linear inequality is any ordered pair that makes the inequality true.
- When graphing a linear inequality on a Cartesian plane, the solution is a region on one side of the line called a **half plane**.

Inequality symbols	
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

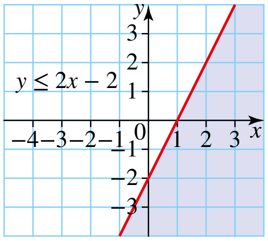
- There are an infinite number of solutions to a linear inequality. For example, some of the solutions to the inequality $x + y < 16$ are (5, 1), (8, 7) and (0, −2), whereas the point (9, 10) is not a solution.
- To indicate whether the points on a line satisfy the inequality, a specific type of **boundary line** is used.

Points on the line	Symbol	Type of boundary line used
Do not satisfy the inequality	$<$ or $>$	Dashed ---------------------
Satisfy the inequality	$\leq$ or $\geq$	Solid ________________

- The **required region** is the region that contains the points that satisfy the inequality.
- Shading or no shading is used to indicate which side of the line is the required region, and a key is shown to indicate the region.

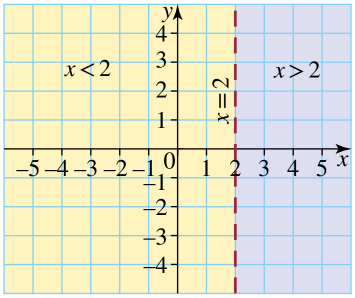


The required region is ▨.



The required region is ▨.

- Consider the line $x = 2$. It divides the **Cartesian plane** into two distinct regions or half-planes.



- The region on the left (shaded yellow) contains all the points whose x-coordinate is less than 2, for example (1, 3), so this region is given the name $x < 2$.
- The region on the right (shaded purple) contains all the points whose x-coordinate is greater than 2, for example (3, −2), so this region is given the name $x > 2$.
- There are three distinct parts to the graph:
  – the boundary line, where $x = 2$
  – the yellow region, where $x < 2$
  – the purple region, where $x > 2$.

WORKED EXAMPLE 15

TI | CASIO

Sketch a graph of each of the following regions.

a $x \geq -1$ **b** $y < 3$

THINK

a 1 $x \geq -1$ includes the line $x = -1$ and the region $x > -1$.

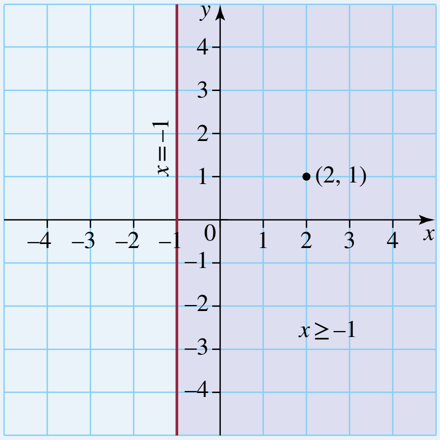
2 On a neat Cartesian plane sketch the line $x = -1$. Because the line is required, it will be drawn as a continuous (unbroken) line.

3 Find a point where $x > -1$, say $(2, 1)$.

4 Shade the region that includes this point. Label the graph $x \geq -1$.

DRAW

a



b 1 The line $y = 3$ is not included.

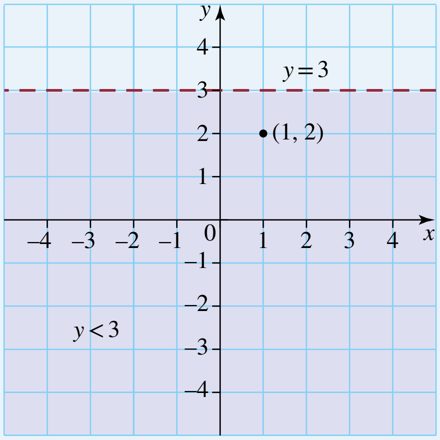
2 Sketch the line $y = 3$. Because the line is not included, show it as a dashed (broken) line.

3 Find a point where $y < 3$, say $(1, 2)$.

4 Shade the region where $y < 3$.

5 Label the graph.

b



- Note that the boundary line is drawn as a continuous line if it is included in the inequality (e.g. $x \leq 5$) but as a broken line if it is not included (e.g. $x < 5$).
- For more complex regions, first sketch the boundary line, then decide which half-plane satisfies the inequality.
- Choose a point and decide whether it satisfies the rule or not. The origin $(0, 0)$ is often an easy point to use.

WORKED EXAMPLE 16

Determine whether the points $(0, 0)$ and $(3, 4)$ satisfy either of the following inequalities.

a $x - 2y < 3$ **b** $y > 2x - 3$

THINK	WRITE
a 1 Substitute $(0, 0)$ for x and y.	**a** $x - 2y < 3$ Substitute $(0, 0)$:
2 Since the statement is true, $(0, 0)$ satisfies the inequality.	$0 - 0 < 3$ $0 < 3$ True

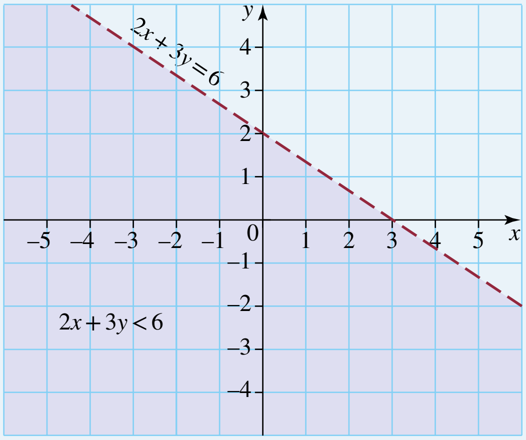
3 Substitute $(3, 4)$ for x and y.		$x - 2y < 3$ Substitute $(3, 4)$: $3 - 2(4) < 3$ $3 - 8 < 3$ $-5 < 3$ True
4 Since the statement is true, $(3, 4)$ satisfies the inequality.		The points $(0, 0)$ and $(3, 4)$ both satisfy the inequality.
b **1** Substitute $(0, 0)$ for x and y.	**b**	$y > 2x - 3$ Substitute $(0, 0)$: $0 > 0 - 3$ $0 > -3$ True
2 Since the statement is true, $(0, 0)$ satisfies the inequality.		
3 Substitute $(3, 4)$ for x and y.		$y > 2x - 3$ Substitute $(3, 4)$: $4 > 2(3) - 3$ $4 > 6 - 3$ $4 > 3$ True
4 Since the statement is true, $(3, 4)$ satisfies the inequality.		The points $(0, 0)$ and $(3, 4)$ both satisfy the inequality.

WORKED EXAMPLE 17

TI | CASIO

Sketch a graph of the region $2x + 3y < 6$.

THINK	WRITE/DRAW
1 Locate the boundary line $2x + 3y = 6$ by finding the x- and y-intercepts.	$x = 0$: $0 + 3y = 6$ $y = 2$ $y = 0$: $2x + 0 = 6$ $x = 3$
2 The line is not required, so rule a broken line.	
3 Test with the point $(0, 0)$. Does $(0, 0)$ satisfy $2x + 3y = 6$?	Test $(0, 0)$: $2(0) + 3(0) = 0$ As $0 < 6$, $(0, 0)$ is in the required region.
4 Shade the region that includes $(0, 0)$.	
5 Label the graph.	



Exercise 4.8 Inequalities on the Cartesian plane

assesson

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–8	Questions: 1–8, 10	Questions: 1–11

■■■ **Individual pathway interactivity:** int-4583

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Fluency

1. **WE15** Sketch a graph of each of the following regions.

a. $x < 1$ **b.** $y \geq -2$ **c.** $x \geq 0$ **d.** $y < 0$

2. **WE16** Determine which of the points A $(0, 0)$, B $(1, -2)$ and C $(4, 3)$ satisfy each of the following inequalities.

a. $x + y > 6$ **b.** $x - 3y < 2$ **c.** $y > 2x - 5$ **d.** $y < x + 3$

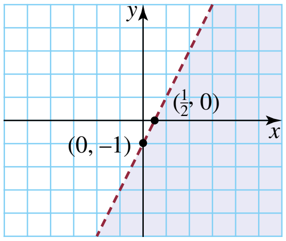
3. **WE17** Sketch the graphs for the regions given by each of the following inequations.

a. $y \geq x + 1$ **b.** $y < x - 6$ **c.** $y > -x - 2$ **d.** $y < 3 - x$
e. $y > x - 2$ **f.** $y < 4$ **g.** $2x - y < 6$ **h.** $y \leq x - 7$
i. $x - y > 3$ **j.** $y < x + 7$ **k.** $x + 2y \leq 5$ **l.** $y \leq 3x$

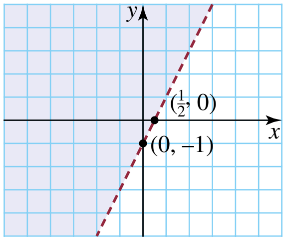
4. Verify your solutions to question **3** using technology.

5. **MC** **a.** The shaded region satisfying the inequality $y > 2x - 1$ is:

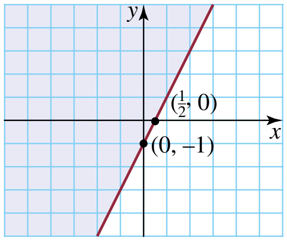
A.



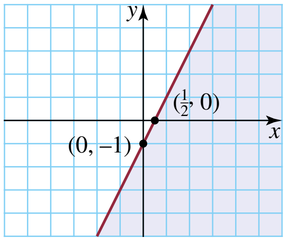
B.



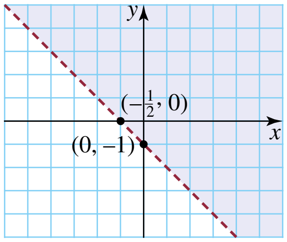
C.



D.

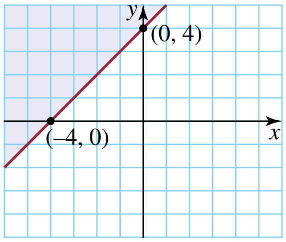


E.

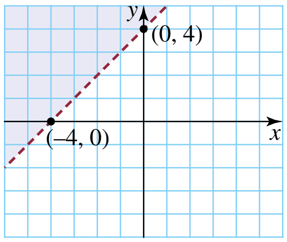


b. The shaded region satisfying the inequality $y \leq x + 4$ is:

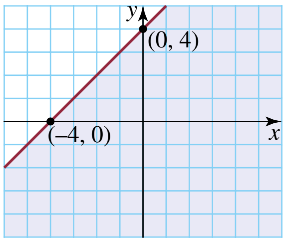
A.



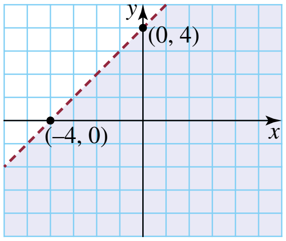
B.



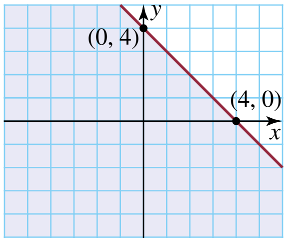
C.



D.

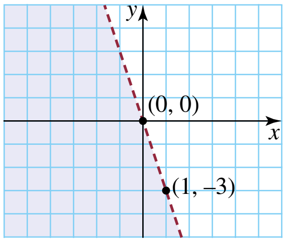


E.

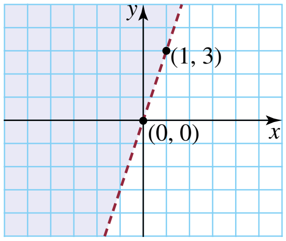


c. The region satisfying the inequality $y < -3x$ is:

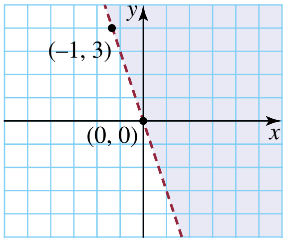
A.



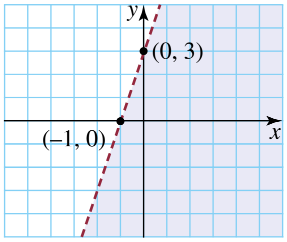
B.



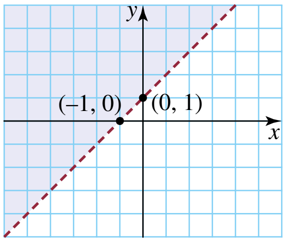
C.



D.



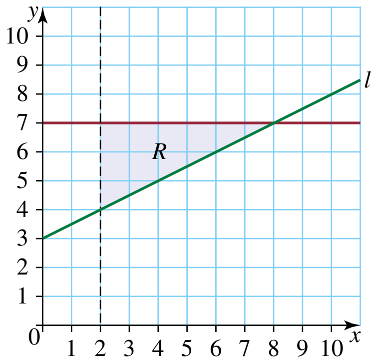
E.



Understanding

6. a. Find the equation of the line l shown in the diagram at right.

b. Write down three inequalities that define the region R.



7. Happy Yaps Dog Kennels charges \$35 per day for large dogs (dogs over 20 kg) and \$20 per day for small dogs (less than 20 kg). On any day, Happy Yaps Kennels can only accommodate a maximum of 30 dogs.

a. If l represents the number of large dogs and s represents the number of small dogs, write an inequality in terms of l and s that represents the total number of dogs at Happy Yaps.

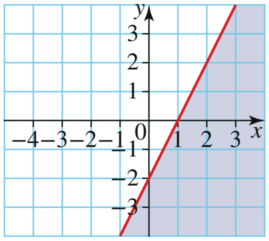
b. Another inequality can be written as $s \geq 12$. In the context of this problem, write down what this inequality represents.

c. The inequality $l \leq 15$ represents the number of large dogs that Happy Yaps can accommodate on any day. Draw a graph that represents this situation.

d. Explore the maximum number of small and large dogs Happy Yaps Kennels can accommodate to receive the maximum amount in fees.

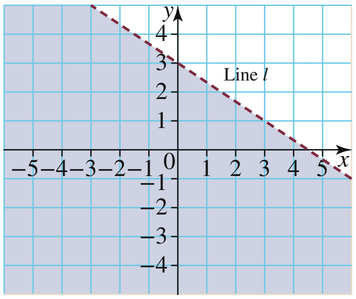
Reasoning

8. **a.** Given the following graph, state the inequality it represents.
   b. Choose a point from each half plane and show how this point confirms your answer to part **a**.



The required region is ▢.

9. **a.** Find the equation of the line, *l*.
   b. Write an inequation to represent the unshaded region.
   c. Write an inequation to represent the shaded region.



Problem solving

10. **a.** Sketch the graph of $\frac{x+1}{2} - \frac{x+1}{3} = 2 - y$.
    b. Shade the region that represents $\frac{x+1}{2} - \frac{x+1}{3} \le 2 - y$.

11. Use your knowledge about linear inequations to sketch the region defined by
    $y \ge x^2 + 4x + 3$.

Reflection

Think of some real-life situations where inequalities could be used to help solve a problem.

4.9 Solving simultaneous linear inequalities

4.9.1 Solving simultaneous linear inequalities

- The graph of a linear inequality represents a region of the Cartesian plane.
- Two linear inequalities drawn on the same set of axes will represent two regions on the graph.
- If these regions intersect, they have an *infinite number* of points in common.
- To graph simultaneous inequalities:
  Step 1: Sketch each inequality on the same axes.
  Step 2: Identify the required region that satisfies both inequalities.
  Step 3: Select a point from the required region and check that the ordered pair satisfies all linear inequalities.

WORKED EXAMPLE 18

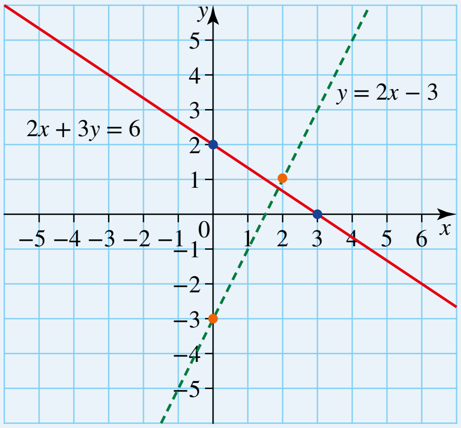
TI | CASIO

Identify the required region in the following pair of linear inequalities.
$2x + 3y \ge 6$, $y < 2x - 3$

THINK	**WRITE/DRAW**	
1 • To sketch each inequality, the boundary line needs to be drawn first. • To draw each line, identify two points on each line. – Use the intercepts method for $2x + 3y \ge 6$. – Use substitution of values for $y < 2x - 3$. • Write the coordinates. *Note:* The intercepts method could also have been used for the second equation.	$2x + 3y \ge 6$ For the line $2x + 3y = 6$, x-intercept: let $y = 0$ $2x + 0 = 6$ $x = 3$ y-intercept: Let $x = 0$ $0 + 3y = 6$ $y = 2$ $(3, 0), (0, 2)$	$y < 2x - 3$ For the line $y = 2x - 3$, let $x = 0$ $y = 2(0) - 3$ $y = -3$ Let $x = 2$ $y = 2(2) - 3$ $y = 4 - 3$ $y = 1$ $(0, -3), (2, 1)$

2 Plot the two points for each line.
- Plot the x- and y-intercepts for $2x + 3y = 6$, as shown in blue.
- Plot the two points for $y = 2x - 3$, as shown in orange.

3 Draw the boundary lines.
- For $2x + 3y \geq 6$, the points on the line are included. The boundary line is solid, as shown in red.
- For $y < 2x - 3$, the points on the line are not included. The boundary line is dashed, as shown in green.

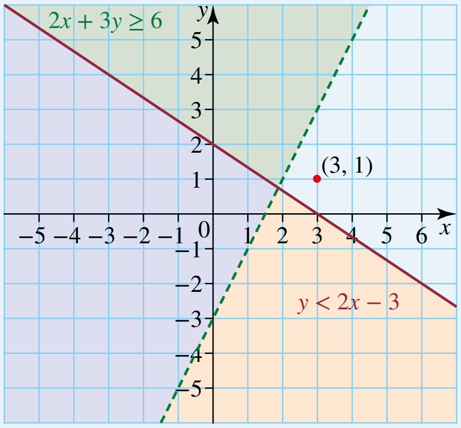


4 To determine which side of the line is the required region, select a point on one side of the line and check to see whether the point satisfies the equation. Choose the point (3, 1) to substitute into the equation.

Check the point (3, 1):
$x = 3, y = 1$

$2x + 3y \geq 6$	$y < 2x - 3$
$\text{LHS} = 2x + 3y$	$\text{LHS} = y$
$= 2(3) + 3(1)$	$= 1$
$= 6 + 3$	$\text{RHS} = 2(3) - 3$
$= 9$	$= 6 - 3$
$\text{RHS} = 6$	$= -3$
$\text{LHS} > \text{RHS}$	$\text{LHS} < \text{RHS}$
The point (3, 1) satisfies the inequality and is in the required region for $2x + 3y \geq 6$.	The point (3, 1) satisfies the inequality and is in the required region for $y < 2x - 3$.

5
- The region *not* required for:
  $2x + 3y \geq 6$ is shaded orange
  $y < 2x - 3$ is shaded green.
- Since the point (3, 1) satisfies both inequalities, it is in the required region. The required region is the unshaded section of the graph. Write a key.



The required region is ☐.

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Exercise 4.9 Solving simultaneous linear inequalities

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 5, 7	Questions: 1, 2, 4–7, 9	Questions: 1–10

🟩🟦🟪 **Individual pathway interactivity:** int-4584

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

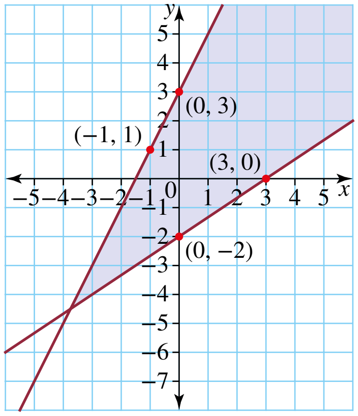
Fluency

1. **WE18** Identify the required region in the following pair of inequalities.

$4x + 7y \geq 21$

$10x - 2y \geq 16$

2. Given the graph shown, determine the inequalities that represent the shaded region.



3. Solve the following pairs of inequalities.

a. $y < 4$
$y \leq -x$

b. $y + 3x > 6$
$y - 2x < 9$

c. $5y - 3x \geq -10$
$6y + 4x \geq 12$

d. $\frac{1}{3}y + 2x \leq 4$
$y - 4x \geq -8$

Understanding

4. **MC** Which system of inequalities represents the required region on the graph at right?

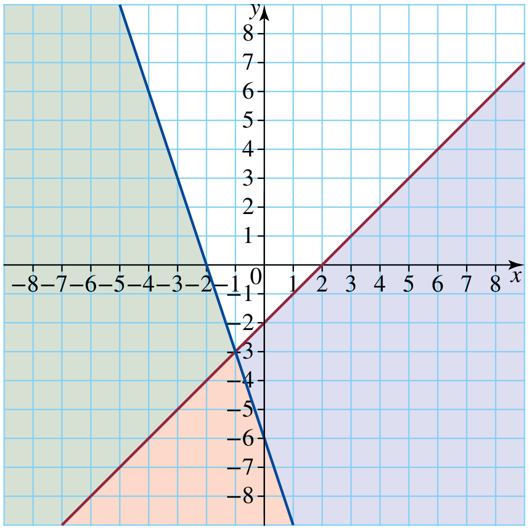
A. $y \leq x - 2$
$y > -3x - 6$

B. $y \geq x - 2$
$y \geq -3x - 6$

C. $y \leq x - 2$
$y \leq -3x - 6$

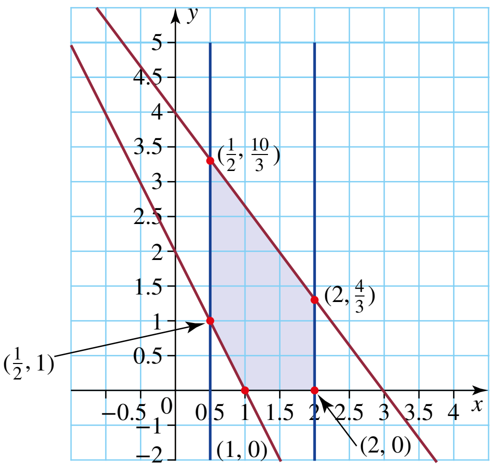
D. $y \geq x + 3$
$y \leq -3x - 6$

E. $y > x + 2$
$y < -3x + 6$



The required region is □.

5. Given the diagram at right, write the inequalities that created the shaded region.



6. a. Graph the following system of inequalities:

$y \ge -3, x + 2 \ge 0, 2y + 5x \le 7$

b. Calculate the coordinates of the vertices of the required region.

Reasoning

7. The sum of the lengths of any two sides of a triangle must be greater than the third side.

a. Given a triangle with sides x, 9 and 4, draw diagrams to show the possible triangles, using the above statement to establish inequalities.

b. Find the possible solutions for x and explain how you determined this.

8. Create a triangle with the points (0, 0), (0, 8) and (6, 0).

a. Calculate the equations of the lines for the three sides.

b. If you shade the interior of the triangle (including the boundary lines), what inequalities would create the shaded region?

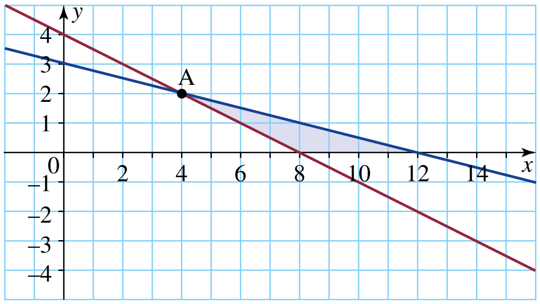
c. What are the side lengths of this triangle?

Problem solving

9. a. Find the equations of the two lines in the diagram shown at right.

a. Find the coordinates of the point A.

b. Write a system of inequations to represent the shaded region.



10. The Ecofriendly company manufactures two different detergents. Shine is specifically for dishwashers while Motherearth is a washing machine detergent. For the first week of June, the production manager has specified that the total amount of the two products produced should be at least 400 litres as one client has already pre-ordered 125 litres of Shine for that week. The time that is required to process one litre of Shine is 30 minutes while one litre of Motherearth requires 15 minutes. During the week mentioned, the factory can process the detergents for up to 175 hours. If x represents the number of litres of Shine produced and y represents the number of litres of Motherearth produced, formulate the constraints as linear inequations and show the feasible region. Also state the coordinates of the vertices of the region.

Reflection

How do the solutions for a system of equations differ from a system of inequations?

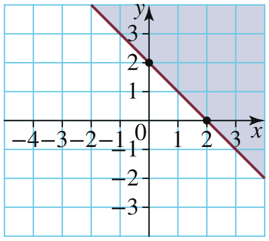
4.10 Review

4.10.1 Review questions

Fluency

1. **MC** The inequality that is represented by the following region is:

A. $y \geq 2 - x$ **B.** $y \geq x - 2$ **C.** $y \leq 2 - x$ **D.** $y \leq x - 2$ **E.** $y \geq 2x$



▨ Region required

2. **MC** The equation of a linear graph which passes through the origin with gradient −3 is:

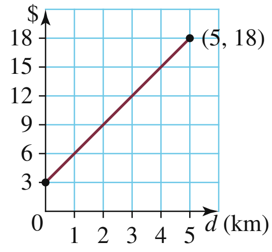
A. $y = -3$ **B.** $x = -3$ **C.** $y = -3x$ **D.** $y = 3 - 3x$ **E.** $y = 3x - 3$

3. **MC** An online music shop charges a flat rate of \$5 postage for 2 CDs and \$11 for 5 CDs. The equation that best represents this, if C is the cost and n is the number of CDs, is:

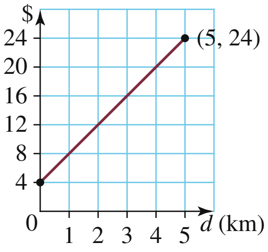
A. $C = 5n + 11$ **B.** $C = 6n + 5$ **C.** $C = n + 2$ **D.** $C = 5n + 1$ **E.** $C = 2n + 1$

4. **MC** During a walk-a-thon, Sarah receives \$4 plus \$3 per kilometre. The graph which best represents Sarah walking up to 5 kilometres is:

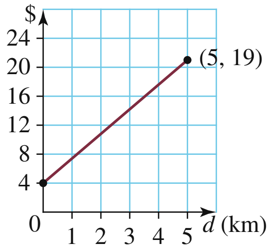
A.



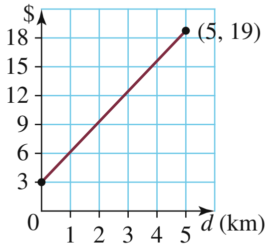
B.



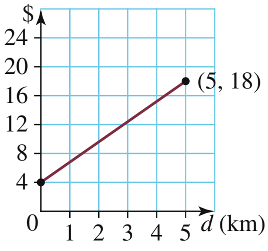
C.



D.



E.



5. **MC** Which of the following pairs of coordinates is the solution to the given simultaneous equations?

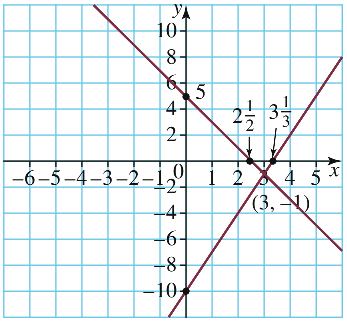
$$2x + 3y = 18$$
$$5x - y = 11$$

A. (6, 2) **B.** (3, −4) **C.** (3, 9) **D.** (3, 4) **E.** (5, 11)

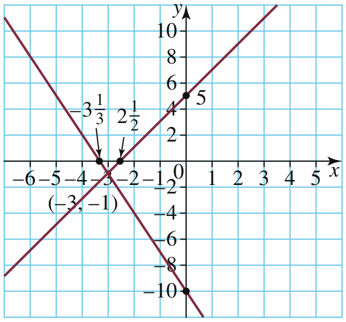
6. **MC** The graphical solution to the following pair of simultaneous equations is:

$$y = 5 - 2x$$
$$y = 3x - 10$$

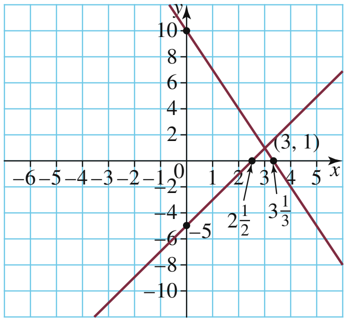
A.



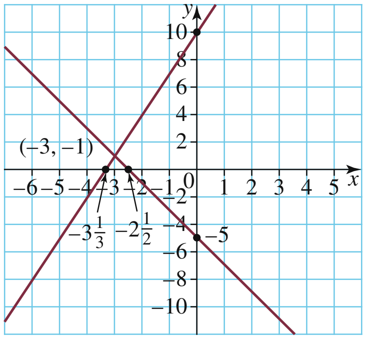
B.



C.



D.



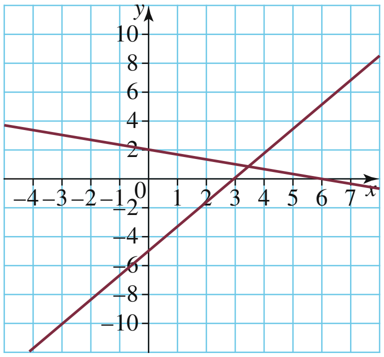
E. none of the above

7. Sketch the half plane given by each of the following inequalities.

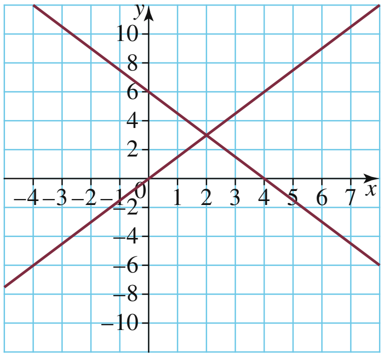
a. $y \leq x + 1$
b. $y \geq 2x + 10$
c. $y > 3x - 12$
d. $y < 5x$
e. $x \geq 7$
f. $y \leq \frac{1}{2}x + 1$
g. $2x + y\$9$
h. $4x - 3y \geq 48$
i. $y > -12$

8. Use the graphs below, showing the given simultaneous equations, to write the point of intersection of the graphs and, hence, the solution of the simultaneous equations.

a. $x + 3y = 6$
$y = 2x - 5$



a. $3x + 2y = 12$
$2y = 3x$



9. Use substitution to check if the given pair of coordinates is a solution to the given simultaneous equations.

a. (7, 1) $x - 2y = 5$
$5y + 2x = 18$

b. (4, 3) $y = 7 - x$
$5y - 2x = 7$

10. Solve each of the following pairs of simultaneous equations using a graphical method.

a. $4y - 2x = 8$
$x + 2y = 0$

b. $y = 2x - 2$
$x - 4y = 8$

c. $2x + 5y = 20$
$y = 7$

11. Solve the following simultaneous equations using the substitution method.

a. $y = 3x + 1$
$x + 2y = 16$

b. $y = 2x + 7$
$3y - 4x = 11$

c. $2x + 5y = 6$
$y = \frac{3}{2}x + 5$

d. $y = -x$
$y = 8x + 21$

e. $y = 3x - 11$
$y = 5x + 17$

f. $y = 4x - 17$
$y = 6x - 22$

12. Solve the following simultaneous equations using the elimination method.

a. $3x + y = 17$
$7x - y = 33$

b. $4x + 3y = 1$
$-4x + y = 11$

c. $3x - 7y = -2$
$-2x - 7y = 13$

d. $4y - 3x = 9$
$y + 3x = 6$

e. $5x + 2y = 6$
$4x + 3y = 2$

f. $x - 4y = -4$
$4x - 2y = 12$

13. Solve the following simultaneous equations using an appropriate method.

a. $3x + 2y = 6$
$3y + 5x = 9$

b. $6x - 4y = -6$
$7x + 3y = -30$

c. $6x + 2y = 14$
$x = -3 + 5y$

14. Solve the following simultaneous inequations.

a. $y \le x + 4$
$y \ge 3$

b. $2y - 3x \ge 12$
$y + 3x > 0$

c. $5x + y < 10$
$x + 2y < 11$

15. Find the point(s) of intersection for each of the following pairs of lines.

a. $y = x^2 - 6$
$y = 5x - 3$

b. $y = \dfrac{2}{x}$
$y = 5x - 3$

c. $x^2 + y^2 = 2$
$y = 5x - 3$

Problem solving

16. Write the following as a pair of simultaneous equations and solve.

a. Find two numbers whose difference is 5 and whose sum is 23.

b. A rectangular house has a total perimeter of 34 metres and the width is 5 metres less than the length. What are the dimensions of the house?

c. If two Chupa Chups and three Wizz Fizzes cost \$2.55, but five Chupa Chups and seven Wizz Fizzes cost \$6.10, find the price of each type of lolly.

17. Laurie buys milk and bread for his family on the way home from school each day, paying with a \$10 note. If he buys three cartons of milk and two loaves of bread, he receives 5 cents in change. If he buys two cartons of milk and one loaf of bread, he receives \$4.15 in change. How much does each item cost?

18. A paddock contains some cockatoos (2-legged) and kangaroos (4-legged). The total number of animals is 21 and they have 68 legs in total. Using simultaneous equations, determine how many cockatoos and kangaroos there are in the paddock.

19. Warwick was solving a pair of simultaneous equations using the elimination method and reached the result that $0 = -5$. Suggest a solution to the problem, giving a reason for your answer.

20. There are two sections to a concert hall. Seats in the 'Dress circle' are arranged in rows of 40 and cost \$140 each. Seats in the 'Bleachers' are arranged in rows of 70 and cost \$60 each. There are 10 more rows in the 'Dress circle' than in the 'Bleachers' and the capacity of the hall is 7000.

a. If *d* represents the number of rows in the 'Dress circle' and *b* represents the number of rows in the 'Bleachers' then write an equation in terms of these two variables based on the fact that there are 10 more rows in the 'Dress circle' than in the 'Bleachers'.

b. Write an equation in terms of these two variables based on the fact that the capacity of the hall is 7000 seats.

c. Solve the two equations from **a** and **b** simultaneously using the method of your choice to find the number of rows in each section.

d. Now that you have the number of rows in each section, calculate the number of seats in each section.

e. Hence, calculate the total receipts for a concert where all tickets are sold.

21. John is comparing two car rental companies, Golden Ace Rental Company and Silver Diamond Rental Company. Golden Ace Rental Company charges a flat rate of \$38 per day and \$0.20 per kilometre. The Silver Diamond Rental Company charges a flat rate of \$30 per day plus \$0.32 per kilometre.

a. Write an algebraic equation for the cost of renting a car for three days from the Golden Ace Rental Company in terms of the number of kilometres travelled, *k*.

b. Write an algebraic equation for the cost of renting a car for three days from the Silver Diamond Rental Company in terms of the number of kilometres travelled, *k*.

c. How many kilometres would John have to travel so that the cost of hiring from each company for three days is the same?

d. Write an inequation that, when solved, will tell you the number of kilometres for which it is cheaper to use Golden Ace Rental Company when renting for three days.

e. For what number of kilometres will it be cheaper to use Silver Diamond Rental Company for three days' hire?

22. Frederika has \$24 000 saved for a holiday and a new stereo. Her travel expenses are \$5400 and her daily expenses are \$260.

a. Write down an equation for the cost of her holiday if she stays for d days. Upon her return from holidays Frederika wants to purchase a new stereo system that will cost her \$2500.

b. How many days can she spend on her holiday if she wishes to purchase a new stereo upon her return?

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- **Try out this interactivity:** Sudoku: Topic 4 (int-3591)
- **Complete this digital doc:** Concept map: Topic 4 (doc-13805)

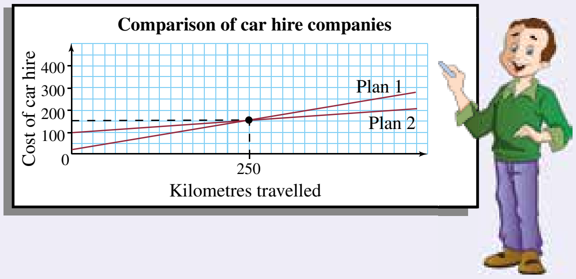
Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

collinear
elimination
equation
half-plane
inequality
infinite
intersection
number line
parallel
perpendicular
reverse
simultaneous
solution
substitution
tangent

Investigation | Rich task

Documenting business expenses



In business, expenses can be represented graphically, so that relevant features are clearly visible. The figure at right compares the costs of hiring cars from two different car rental companies. It will be cheaper to use Plan 1 when travelling distances less than 250 kilometres, and Plan 2 when travelling more than 250 kilometres. Both plans cost the same when you are travelling exactly 250 kilometres.

Jim works as a travelling sales representative. He needs to plan his next business trip to Port Hedland, which he anticipates will take him away from the office for 3 or 4 days. Due to other work commitments, he is not sure whether he can make the trip by the end of this month or early next month. He plans to fly to Port Hedland and use a hire car to travel when he arrives. Jim's boss has asked him to supply documentation detailing the anticipated costs for the hire car, based on the following quotes received.

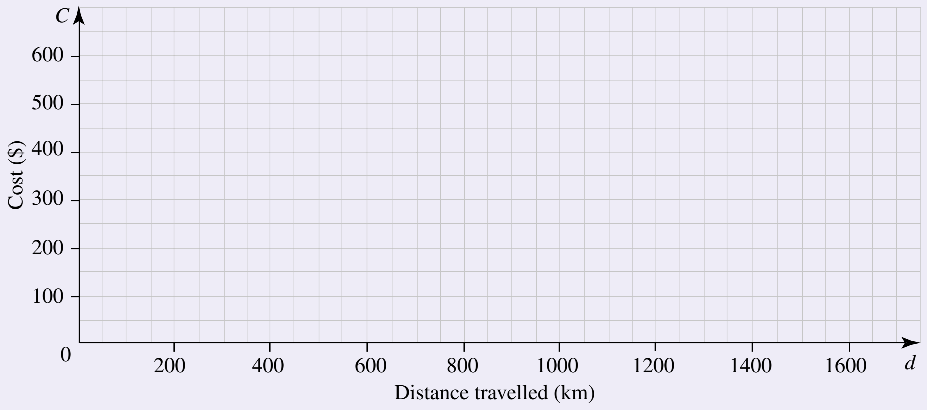
A1 Rentals	\$35 per day plus 28c per kilometre of travel
Cut Price Rentals	\$28 per day plus 30c per kilometre of travel

Jim is aware that, although the Cut Price Rentals deal looks cheaper, it could work out more expensive in the long run, because of the higher cost per kilometre of travel; he intends to travel a considerable distance. Jim is advised by both rental companies that their daily hire charges are due to rise by \$2 per day from the first day of next month.

Assuming that Jim is able to travel this month and his trip will last 3 days, use the information given to answer questions 1 to 3.

1. Write equations to represent the costs of hiring a car from A1 Rentals and Cut Price Rentals. Use the pronumeral C to represent the cost (in dollars) and d to represent the distance travelled (in kilometres).
2. Plot the two equations from question **1** on the set of axes provided to show how the costs compare over 1500 km.

Comparison of cost of hiring a car from A1 Rentals and Cut Price Rentals



3. Use the graph to determine how many kilometres Jim would have to travel to make the hire costs the same for both rental companies.

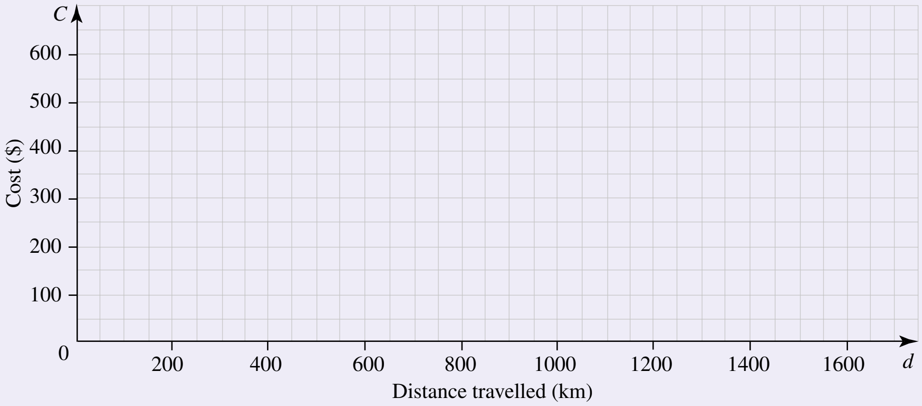
4. Assume Jim's trip is extended to four days. Use an appropriate method to show how this changes the answer found in question **3**.

For questions 5 to 7, assume that Jim has delayed his trip until next month when the hire charges have increased.

5. Write equations to show the cost of hiring a car from both car rental companies for a trip lasting:
   a. 3 days
   b. 4 days.

6. Plot the four equations from question **5** on the set of axes provided to show how the costs compare over 1500 km.

Comparison of cost of hiring a car from A1 Rentals and Cut Price Rentals



7. Comment on the results displayed in your graph.

8. Jim needs to provide his boss with documentation of the hire car costs, catering for all options. On a separate sheet of paper, prepare a document for Jim to hand to his boss.

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Complete this digital doc: Code puzzle: This world-first medical event took place in 1986 (doc-15923)

Answers

Topic 4 Simultaneous linear equations and inequalities

Exercise 4.2 Graphical solution of simultaneous linear equations

1. a. (2, 1) **b.** (1, 1) **c.** (0, 4) **d.** (2, −1) **e.** (−2, −4) **f.** (−0.5, 1.5)

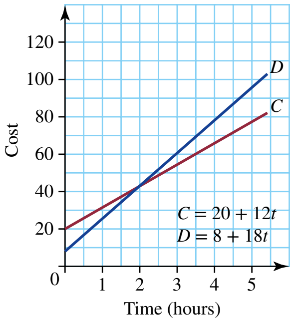
2. a. No **b.** Yes **c.** Yes **d.** No **e.** Yes
f. No **g.** No **h.** Yes **i.** No **j.** Yes

3. a.(3, 2) **b.** (4, 3) **c.** (−3, 4) **d.** (−2, 2) **e.** (2, 0) **f.** (3, 0)
g. (−2, 4) **h.** (3, 8) **i.** $(-\frac{1}{2}, 1\frac{1}{2})$ **j.** (2, 5) **k.** (5, 3) **l.** $(2, \frac{2}{3})$

4. a. No solution **b.** (2, −1) **c.** No solution **d.** (1, 9) **e.** (3, 1) **f.** No solution
g. No solution **h.** (2, 1)

5. $y = 4x - 16$

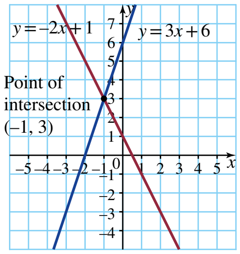
6. a. Northern beach
$C = 20 + 12t,\ 0 \le t \le 5$
Southern beach
$D = 8 + 18t,\ 0 \le t \le 5$
b. Northern beaches in red, southern beaches in blue
c. Time > 2 hours
d. Time = 2 hours, cost = $44



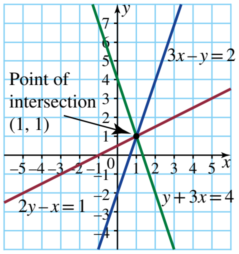
7. a. Same line **b.** Perpendicular **c.** Intersecting **d.** Parallel

8. a. 1 solution **b.** No solution (parallel lines) **c.** No solution (parallel lines)

9. a. $y = 3x + 6$ **b.** $y = -2x + 1$ **c.**



10.



Exercise 4.3 Solving simultaneous linear equations using substitution

1. a. (2, 3) **b.** (2, −1) **c.** (3, −2) **d.** (7, 6) **e.** (3, 6) **f.** (2, 1)
g. (−1, −2) **h.** (−4, 0) **i.** (−1, −2) **j.** (6, −2) **k.** $(3, 1\frac{1}{2})$ **l.** (−3, −5)

2. a. (−6, −23) **b.** (5, 23) **c.** (2, −6) **d.** $\left(\frac{3}{2}, -\frac{15}{2}\right)$ **e.** (1, −7) **f.** $(-\frac{1}{2}, -4)$
g. $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ **h.** $\left(-\frac{1}{5}, \frac{4}{5}\right)$ **i.** (−3, −1.5) **j.** (1, 0.3) **k.** $\left(-\frac{4}{5}, \frac{4}{5}\right)$ **l.** (1, −1)

3. 26 chickens

4. a. (3, 1) **b.** (−2, 3) **c.** (5, 1) **d.** (4, 2) **e.** (0, 3) **f.** (4, 0)

5. a. $x = \frac{n}{2m}, y = \frac{n}{2}$ **b.** $x = \frac{m}{n^2 + 1}, \ y = \frac{mn}{n^2 + 1}$ **c.** $x = \frac{n}{m - n}, y = \frac{n^2}{m - n}$

d. $x = \frac{n}{m - n}, y = \frac{n}{m - n}$ **e.** $x = \frac{-(m - n^2)}{m - n}, y = \frac{m(n - 1)}{m - n}$ **f.** $x = \frac{2m}{m + n}, y = \frac{m(n - m)}{m + n}$

6. $a = -1, b = 5$

7. $z = 24, m = 6, n = 9$

m	11	7
9	8	7
n	5	10

8. a. $x + 2y = 4$ **b.** $x = 2, y = 1$

9. Chemistry $21, physics $27

10. $x = 8, y = -7$

11. $x = 0, y = 1$

Exercise 4.4 Solving simultaneous linear equations using elimination

1. a. (3, 1) **b.** (−2, 3) **c.** (−2, 6)

2. a. (5, −1) **b.** (2, 3) **c.** (−3, 1)

3. a. (−3, 5) **b.** (−5, −8) **c.** (2, −2) **d.** $\left(1\frac{1}{2}, 3\frac{1}{2}\right)$ **e.** $\left(2, 1\frac{4}{5}\right)$ **f.** (1, 1)

4. a. (1, 3) **b.** (2, 4) **c.** (5, 2) **d.** (4, 2) **e.** (−3, 4) **f.** $\left(-3, -1\frac{1}{2}\right)$

g. (−6, −5) **h.** (−3, 5) **i.** (2, 1.8)

5. a. (6, 3) **b.** (2, −2) **c.** (1, 3) **d.** (−1.5, −3) **e.** (−8, 18) **f.** (−3, 5)

6. a. (1, 3) **b.** (4, 0) **c.** (−3, 5) **d.** (4, 3) **e.** (8, 5) **f.** $\left(\frac{1}{3}, -\frac{1}{3}\right)$

7. Ann 61 kg, Beth 58 kg, Celine 54 kg

8. a. i. $acx + bcy = ce \ \ (3)$ **ii.** $acx + ady = af \ \ (4)$ **iii.** $y = \frac{ce - af}{bc - ad}$

b. $x = \frac{de - bf}{ad - bc}$

c. i. $\left(\frac{106}{31}, \frac{1}{31}\right)$ **ii.** $\left(\frac{37}{14}, \frac{11}{14}\right)$

d. Because you cannot divide by 0

e. $ad - bc \neq 0$

9. $x = 7, y = -3$

10. $x = 4, y = 3, z = -6$

Challenge 4.1

$z = 6$

Exercise 4.5 Applications of simultaneous linear equations

1. Maths mark = 97, English mark = 66

2. 18 nuts, 12 bolts

3. 8 and 3

4. 9 and 7

5. 6 and 5

6. Length = 12 m and width = 8 m

7. Lemons cost 55 cents and oranges cost 25 cents.

8. Length 60 m and width 20 m

9. Eight 20-cent coins and three 50-cent coins

10. Twelve $1 coins and nine $2 coins

11. Paddlepops cost $1.20 and a Magnum costs $2.10.

12. Cost of the Golden Rough = 35 cents and cost of the Redskin = 25 cents

13. Fixed costs = $87, cost per person = $23.50

14. PE mark is 83 and Science mark is 71.

15. Mozzarella costs \$6.20, Swiss cheese costs \$5.80.

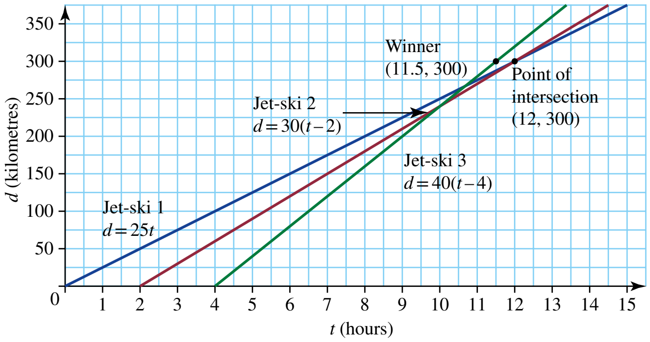
16. $x = 3$ and $y = 4$

17. Fixed costs = \$60, cost per person = \$25

18. \$4 each for DVDs and \$24 each for zip disks

19. 66 cups of hot chips, 33 meat pies and 22 hot dogs were sold during the half-hour period.

20. a.



b. Jet-ski 3 wins the race.

c. Jet-ski 1 and 2 reach the destination at the same time although jet-ski 2 started two hours after jet-ski 1. Jet-ski 3 overtakes jet-ski 1 6 hours and 40 minutes after its race begins or 10 hours and 40 minutes after jet-ski 1 starts the race. Jet-ski 3 overtakes jet-ski 2 6 hours after it starts the race or 8 hours after jet-ski 2 started the race.

Challenge 4.2

1. Rollercoaster = \$6, Ferris wheel = \$4, Gravitron = \$8

2. 89 246

Exercise 4.6 Solving simultaneous linear and non-linear equations

1. A parabola may intersect with a straight line twice, once or not at all.

2. a. $(-5, 4)$ and $(-1, 0)$ **b.** $(2, 3)$ **c.** $(1 - \sqrt{10}, -6)$ and $(1 + \sqrt{10}, -6)$

3. a. $\left(\frac{-1}{2} - \frac{3\sqrt{5}}{2}, \frac{-19}{2} - \frac{9\sqrt{5}}{2}\right)$ and $\left(\frac{-1}{2} + \frac{3\sqrt{5}}{2}, \frac{-19}{2} + \frac{9\sqrt{5}}{2}\right)$ **b.** $(-1, -2)$ and $(2, 1)$

c. $(-2.54, -8.17)$ and $(3.54, 16.17)$ **d.** $(-1.41, 4)$ and $(1.41, 4)$

e. $(-1, 2)$ and $\left(\frac{5}{2}, \frac{1}{4}\right)$ **f.** $(3, 37)$

4. B **5.** C

6. a. Yes **b.** No **c.** Yes **d.** No

7. a. $(1, 1), (-1, -1)$ **b.** $(1 + \sqrt{2}, -1 + \sqrt{2}), (1 - \sqrt{2}, -1 - \sqrt{2})$

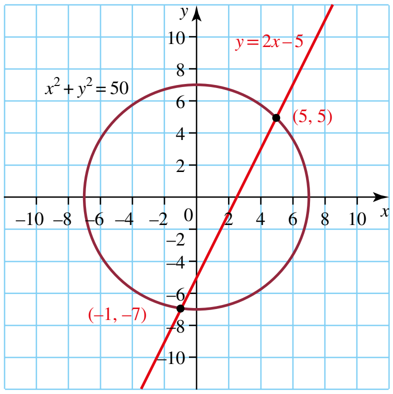
c. $\left(\frac{-\sqrt{15}}{3}, -\sqrt{15}), (\frac{\sqrt{15}}{3}, \sqrt{15}\right)$ **d.** $(-6, -1), (2, 3)$

8. a. $(-1, -3), (1, 3)$ **b.** $(-4, 3), (4, -3)$ **c.** $(-1, 7), (5, -5)$

d. $(\frac{(-1\sqrt{14} - 2)}{2}, \frac{(\sqrt{14} + 2)}{2}), (\frac{(\sqrt{14} + 2)}{2}, \frac{-(\sqrt{14} - 2)}{2})$

9. The straight line crosses the parabola at $(0, -7)$, so no matter what value m takes, there will be at least one intersection point.

10. a. $(5, 5), (-1, -7)$ **b.**



11. 9, 12

Exercise 4.7 Solving linear inequalities

1. a. $x > 2$ **b.** $a > -1$ **c.** $y \geq 7$ **d.** $m \geq 4$ **e.** $p < 1$ **f.** $x < 7$
g. $m \leq 9$ **h.** $a \leq 7$ **i.** $x > 3$ **j.** $m \geq 2$ **k.** $q \geq -4$ **l.** $a > -8$

2. a. $m > 3$ **b.** $p \leq 2$ **c.** $a < 4$ **d.** $x \geq 5$ **e.** $p > -5$ **f.** $x \leq -7$
g. $m \geq -0.5$ **h.** $b > -0.5$ **i.** $m > 18$ **j.** $x < 8$ **k.** $a \leq -14$ **l.** $m \geq 25$

3. a. $m < 4.5$ **b.** $x \geq 3$ **c.** $p > 4$ **d.** $n \leq 2$ **e.** $b < 5$ **f.** $y > 2$
g. $m \leq -1$ **h.** $a \geq -5$ **i.** $b < -4\frac{1}{3}$ **j.** $c \leq -1$ **k.** $p > -2$ **l.** $a \geq -7$

4. a. $m > 3$ **b.** $a \geq 2$ **c.** $a < -1$ **d.** $a \leq -3$ **e.** $x > 6$ **f.** $x \leq 2$
g. $b < 4$ **h.** $a > 5$ **i.** $m < 2$ **j.** $m \leq 3$ **k.** $b \leq -\frac{16}{11}$ **l.** $m \geq 1$

5. a. $x \leq 7$ **b.** $x \geq -18$ **c.** $x < -10$ **d.** $x > 10\frac{1}{2}$ **e.** $x \geq 5$ **f.** $x < -1\frac{4}{5}$

6. a. $m < -2$ **b.** $p \geq -3$ **c.** $a \leq 5$ **d.** $p \geq -5$ **e.** $y \leq -3$ **f.** $x > 7$
g. $p < 0$ **h.** $a \geq \frac{1}{5}$ **i.** $x > -3$ **j.** $a \leq -11$ **k.** $b \leq 3$ **l.** $x < -3$
m. $k > 8$ **n.** $x > -18\frac{1}{2}$ **o.** $a \leq 40$

7. B

8. a. $x < -1$ **b.** $m \leq -3$ **c.** $x > 17$ **d.** $a > \frac{5}{8}$ **e.** $m \geq 1\frac{1}{3}$ **f.** $m \geq -12$

9. a. $k > 2$ **b.** $a > -5$ **c.** $m \leq 1\frac{2}{3}$ **d.** $x > 5$ **e.** $y \geq 7$ **f.** $d < -2$
g. $p \geq \frac{-6}{7}$ **h.** $x \geq -5$ **i.** $m < -2$ **j.** $a < 9$ **k.** $p \geq 3$ **l.** $x > -4\frac{1}{2}$

10. a. $5x > 10$ **b.** $x - 3 \leq 5$ **c.** $7 + 3x < 42$

11. a. $-6.5 < x < -2$ **b.** $\dfrac{-c-b}{a} < x < \dfrac{-d-b}{a}$

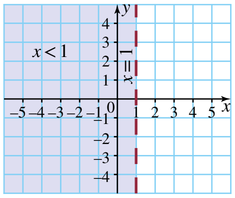
12. a. $S > 47$ **b.** No **c.** Answers will vary.

13. a. $n < 16\,800$ km **b.** Mick travelled less than 16 800 km for the year and his costs stayed below \$16 000.

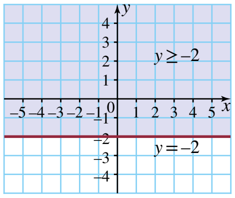
14. \$20 000

Exercise 4.8 Inequalities on the Cartesian plane

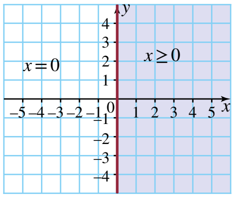
1. a.



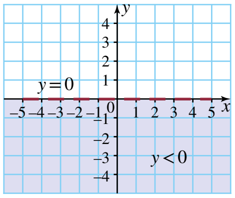
b.



c.



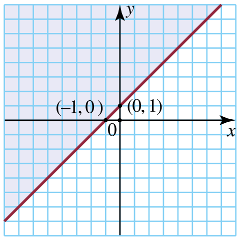
d.



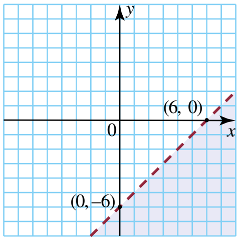
2. a. C **b.** A, C **c.** A, B **d.** A, B, C

3, 4

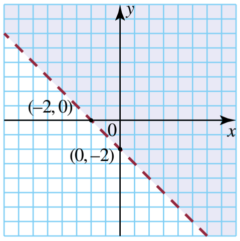
a. $y \geq x + 1$



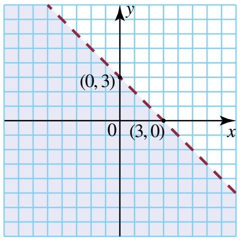
b. $y < x - 6$



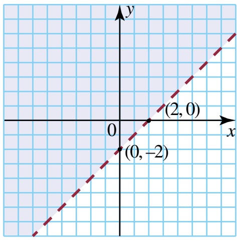
c. $y > -x - 2$



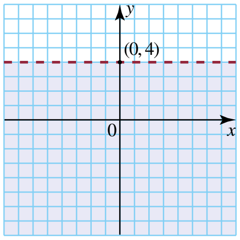
d. $y < 3 - x$



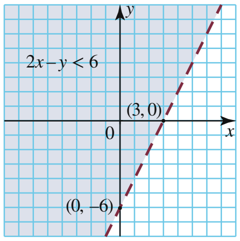
e. $y > x - 2$



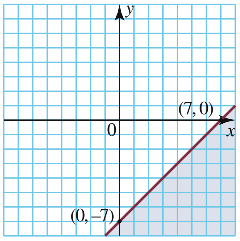
f. $y < 4$



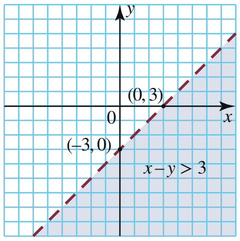
g. $2x - y < 6$



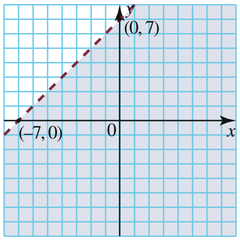
h. $y \leq x - 7$



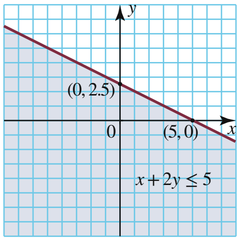
i. $x - y > 3$



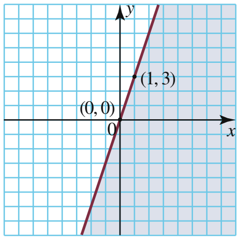
j. $y < x + 7$



k. $x + 2y \leq 5$



l. $y \leq 3x$

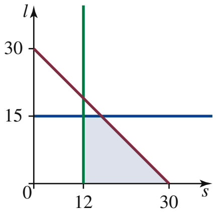


5. a. B **b.** D **c.** A

6. a. $y = \frac{1}{2}x + 3$ **b.** $y \geq \frac{1}{2}x + 3, x > 2, y \leq 7$

7. a. $l + s \leq 30$ **b.** At least 12 small dogs

c.

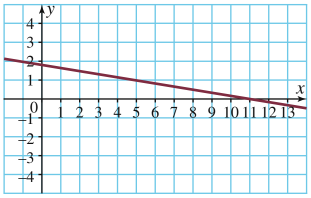


d. 15 large and 15 small dogs

8. a. $y \leq 2x - 2$ **b.** Answers will vary.

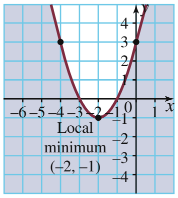
9. a. $y = -\frac{2}{3}x + 3$ **b.** $y > -\frac{2}{3}x + 3$ **c.** $y < -\frac{2}{3}x + 3$

10. a. $y = \frac{11}{6} - \frac{1}{6}x$



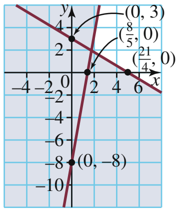
b. The unshaded region is the required region.

11. The unshaded region is the required region.



Exercise 4.9 Solving simultaneous linear inequalities

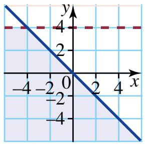
1.



Required region is ☐

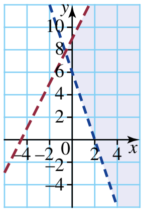
2. $y \leq 2x + 3$ and $y \geq \frac{2}{3}x - 2$

3. a.



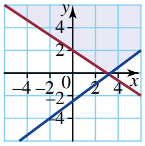
Required region is □

b.



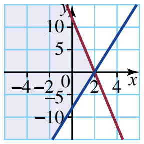
Required region is □

c.



Required region is □

d.

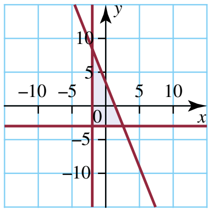


Required region is □

4. B

5. $\frac{1}{2} \leq x \leq 2, y \geq 0, 2x + y \geq 2, 4x + 3y \leq 12$

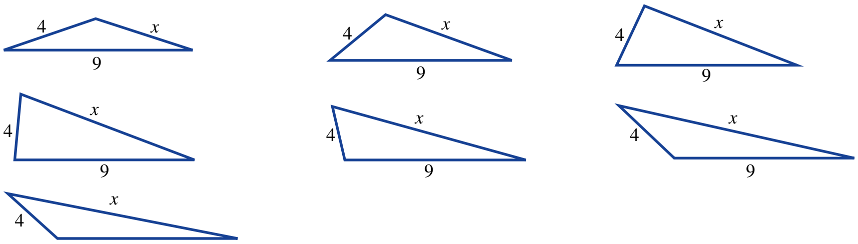
6. a.



Required region is □

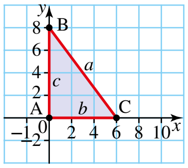
b. $(-2, 8.5), (-2, -3), (2.6, -3)$

7. a.



c. $5 < x < 13$

8.



a. AB: $x = 0$, AC: $y = 0$, BC: $3y + 4x = 24$

b. $x \geq 0, y \geq 0, 3y + 4x \leq 24$

c. $a = 10$ units, $b = 6$ units, $c = 8$ units

9. a. $y = -\frac{1}{4}x + 3$ or $x + 4y = 12$ and $y = -\frac{1}{2}x + 4$ or $x + 2y = 8$.

b. A(4, 2)

c. $y \leq -\frac{1}{4}x + 3$ or $x + 4y \leq 12$

$y \geq -\frac{1}{2}x + 4$ or $x + 2y \geq 8$

$x \geq 4$

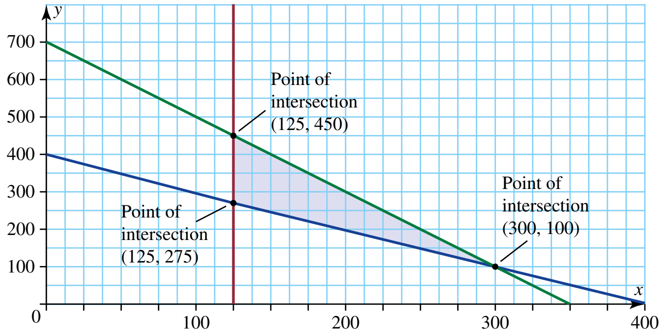
$y \geq 0$

10. $y \geq 0$

$x + y \geq 400$

$x \geq 125$

$\frac{1}{2}x + \frac{1}{4}y \leq 175$ or $2x + y \leq 700$



Vertices are (125, 275), (125, 450) and (300, 100).

4.10 Review

1. A

2. C

3. E

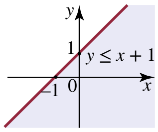
4. C

5. D

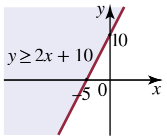
6. A

7. *Note:* The shaded region is the region required.

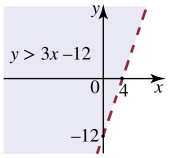
a.



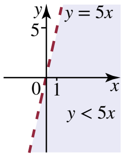
b.



c.



d.



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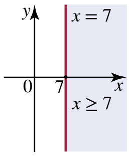

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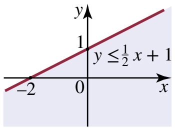

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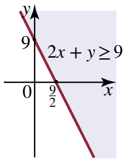

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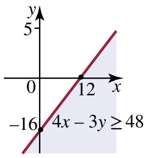
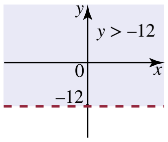


i.



8. a. (3, 1) **b.** (2, 3)

9. a. No **b.** Yes

10. a. $(-2,\ 1)$ **b.** $(0,\ -2)$ **c.** $(-7.5,\ 7)$

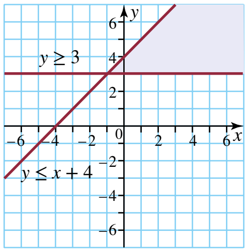
11. a. (2, 7) **b.** $(-5,\ -3)$ **c.** $(-2,\ 2)$ **d.** $\left(-\frac{7}{3},\ \frac{7}{3}\right)$ **e.** $(-14,\ -53)$ **f.** $\left(\frac{5}{2},\ -7\right)$

12. a. (5, 2) **b.** $(-2,\ 3)$ **c.** $(-3,\ -1)$ **d.** (1, 3) **e.** $(2,\ -2)$ **f.** (4, 2)

13. a. (0, 3) **b.** $(-3,\ -3)$ **c.** (2, 1)

14. *Note:* The shaded region is the region required.

a.



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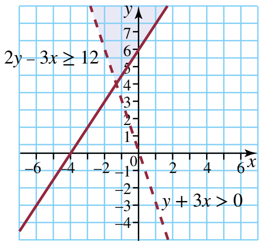

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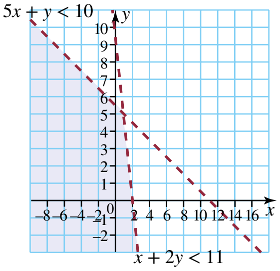


15. a. $\left(\frac{5-\sqrt{37}}{2}, \frac{19-5\sqrt{37}}{2}\right), \left(\frac{5+\sqrt{37}}{2}, \frac{19+5\sqrt{37}}{2}\right)$

b. $\left(-\frac{2}{5}, -5\right)$, $(1, 2)$

c. $\left(\frac{15+\sqrt{43}}{26}, \frac{-3+5\sqrt{43}}{26}\right), \left(\frac{15-\sqrt{43}}{26}, \frac{-3-5\sqrt{43}}{26}\right)$

16. a. Numbers are 9 and 14.

b. Length = 11 metres, width = 6 metres

c. Chupa-chups cost 45 cents and Whizz fizzes cost 55 cents.

17. Milk \$1.75, bread \$2.35

18. 13 kangaroos and 8 cockatoos

19. Any false statement that occurs during the solving of simultaneous equations indicates the lines are parallel, and have no points of intersection.

20. a. $d = b + 10$

b. $7000 = 70b + 40d$

c. $b = 60$ and $d = 70$

d. Number of seats in 'Bleachers' is 4200; the number of seats in the 'Dress circle' is 2800.

e. \$644 000

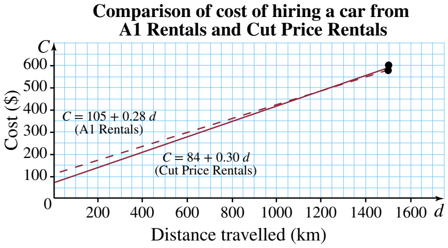
21. a. $C_G = 114 + 0.2k$ **b.** $C_S = 90 + 0.32k$ **c.** 200 km **d.** $114 + 0.2k < 90 + 0.32k$ **e.** $k < 200$

22. a. $5400 + 260d = C_H$ **b.** 61 days

Investigation — Rich task

1. A1 Rentals: $C = \$35 \times 3 + 0.28d$
Cut Price Rentals: $C = \$28 \times 3 + 0.3d$

2.



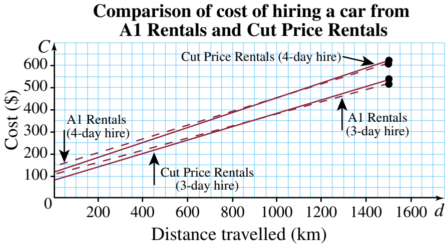
3. 1050 km

4. 1400 km

5. a. A1 Rentals: $C = \$37 \times 3 + 0.28d$
Cut Price Rentals: $C = \$30 \times 3 + 0.3d$

b. A1 Rentals: $C = \$37 \times 4 + 0.28d$
Cut Price Rentals: $C = \$30 \times 4 + 0.3d$

6.



7. The extra cost of \$2 per day for both rental companies has not affected the charges they make for the distances travelled. However, the overall costs have increased.

8. Presentation of the answers will vary. Answers will include:

Travelling 3 days this month:

- If Jim travels 1050 km, the cost will be the same for both rental companies; that is, \$399.
- If he travels less than 1050 km, Cut Price Rentals is cheaper.
- If he travels more than 1050 km, A1 Rentals is cheaper.

Travelling 4 days this month:

- If Jim travels 1400 km, the cost will be the same for both rental companies; that is, \$532.
- If he travels less than 1400 km, Cut Price Rentals is cheaper.
- If he travels more than 1400 km, A1 Rentals is cheaper.

Travelling 3 days next month:

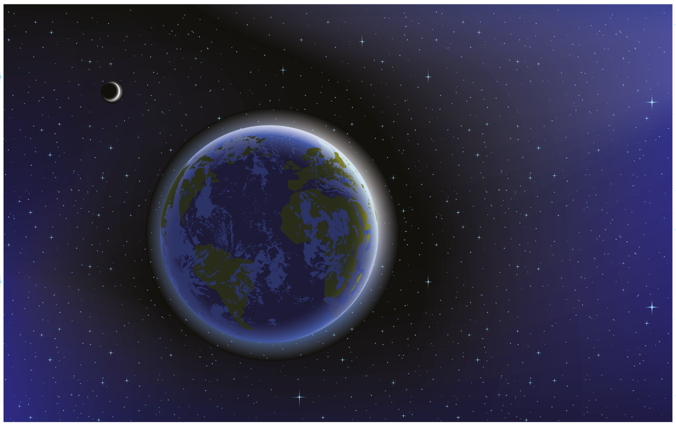
- If Jim travels 1050 km, the cost will be the same for both rental companies; that is, \$405.
- If he travels less than 1050 km, Cut Price Rentals is cheaper.
- If he travels more than 1050 km, A1 Rentals is cheaper.

Travelling 4 days next month:

- If Jim travels 1400 km, the cost will be the same for both rental companies; that is, \$540.
- If he travels less than 1400 km, Cut Price Rentals is cheaper.
- If he travels more than 1400 km, A1 Rentals is cheaper.

TOPIC 5
Trigonometry I

5.1 Overview



Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

5.1.1 Why learn this?

Nearly 2000 years ago, Ptolemy of Alexandria published the first book of trigonometric tables, which he used to chart the heavens and plot the courses of the Moon, stars and planets. He also created geographical charts and provided instructions on how to create maps. Trigonometry is the branch of mathematics that makes the whole universe more easily understood.

5.1.2 What do you know?

assesson

1. **THINK** List what you know about trigonometry. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of trigonometry.

LEARNING SEQUENCE

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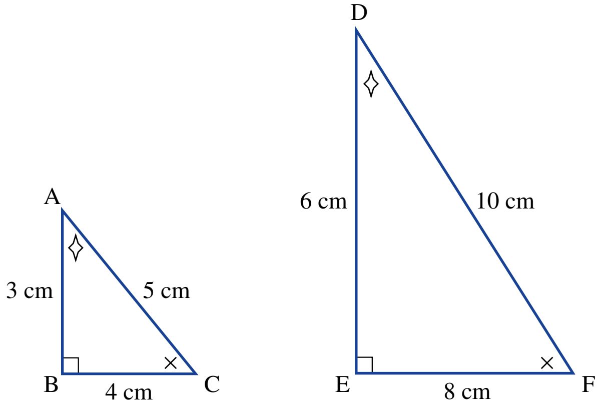
5.2 Pythagoras' theorem

5.2.1 Similar right-angled triangles

In the two similar right-angled triangles shown below, the angles are the same and the corresponding sides are in the same ratio.



The corresponding sides are in the same ratio.

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

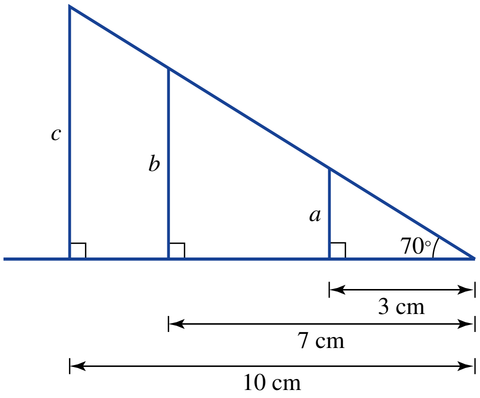
To write this using the side lengths of the triangles gives:

$$\frac{AB}{DE} = \frac{3}{6} = \frac{1}{2}$$
$$\frac{AC}{DF} = \frac{5}{10} = \frac{1}{2}$$
$$\frac{BC}{EF} = \frac{4}{8} = \frac{1}{2}$$

This means that for right-angled triangles, when the angles are fixed, the ratios of the sides in the triangle are constant.

We can examine this idea further by completing the following activity.

Using a protractor and ruler, draw an angle of 70°, measuring horizontal distances of 3 cm, 7 cm and 10 cm as demonstrated in the diagram below.



Note: Diagram not drawn to scale.

Measure the perpendicular heights a, b and c.

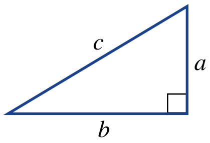
$a \approx 8.24$ cm $\qquad b \approx 19.23$ cm $\qquad c \approx 27.47$ cm

To test if the theory for right-angled triangles, that when the angles are fixed the ratios of the sides in the triangle are constant, is correct, calculate the ratios of the side lengths.

$\frac{a}{3} \approx \frac{8.24}{3} \approx 2.75 \qquad \frac{b}{7} \approx \frac{19.23}{7} \approx 2.75 \qquad \frac{c}{10} \approx \frac{27.47}{10} \approx 2.75$

The ratios are the same because the triangles are similar.
This important concept forms the basis of trigonometry.

5.2.2 Review of Pythagoras' theorem



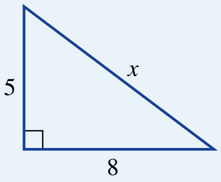
- The **hypotenuse** is the longest side of a right-angled triangle and is always the side that is opposite the right angle.
- **Pythagoras' theorem** states that in any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. The rule is written as $c^2 = a^2 + b^2$ where a and b are the two shorter sides and c is the hypotenuse.
- Pythagoras' theorem gives us a way of finding the length of the third side in a triangle, if we know the lengths of the two other sides.

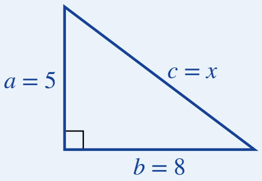
5.2.3 Finding the hypotenuse

- To calculate the length of the hypotenuse when given the length of the two shorter sides, substitute the known values into the formula for Pythagoras' theorem, $c^2 = a^2 + b^2$.

WORKED EXAMPLE 1

For the triangle at right, calculate the length of the hypotenuse, x, correct to 1 decimal place.



THINK	WRITE/DRAW
1 Copy the diagram and label the sides a, b and c. Remember to label the hypotenuse as c.	
2 Write Pythagoras' theorem.	$c^2 = a^2 + b^2$
3 Substitute the values of a, b and c into this rule and simplify.	$x^2 = 5^2 + 8^2$ $= 25 + 64$ $= 89$
4 Take the square root of both sides. Round the positive answer correct to 1 decimal place, since $x > 0$.	$x = \pm\sqrt{89}$ $x \approx 9.4$

5.2.4 Finding a shorter side

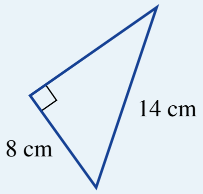
- Sometimes a question will give you the length of the hypotenuse and ask you to find one of the shorter sides. In such examples, we need to rearrange Pythagoras' formula. Given that $c^2 = a^2 + b^2$, we can rewrite this as:

$$a^2 = c^2 - b^2$$
$$\text{or} \quad b^2 = c^2 - a^2.$$

WORKED EXAMPLE 2

TI | CASIO

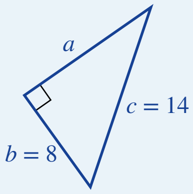
Calculate the length, correct to 1 decimal place, of the unmarked side of the triangle at right.



THINK

WRITE/DRAW

1 Copy the diagram and label the sides a, b and c. Remember to label the hypotenuse as c; it does not matter which side is a and which side is b.



2 Write Pythagoras' theorem.

$$c^2 = a^2 + b^2$$

3 Substitute the values of a, b and c into this rule and solve for a.

$$\begin{aligned} 14^2 &= a^2 + 8^2 \\ 196 &= a^2 + 64 \\ a^2 &= 196 - 64 \\ &= 132 \end{aligned}$$

4 Find a by taking the square root of both sides and round to 1 decimal place ($a > 0$).

$$\begin{aligned} a &= \pm\sqrt{132} \\ &\approx 11.5 \text{ cm} \end{aligned}$$

- Pythagoras' theorem can be used to solve many practical problems.

  First model the problem by drawing a diagram, then use Pythagoras' theorem to solve the right-angled triangle. Use the result to give a worded answer.

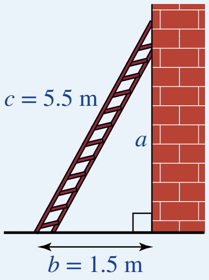
WORKED EXAMPLE 3

A ladder that is 5.5 m long leans up against a vertical wall. The foot of the ladder is 1.5 m from the wall. How far up the wall does the ladder reach? Give your answer correct to 1 decimal place.

THINK

WRITE/DRAW

1 Draw a diagram and label the sides a, b and c. Remember to label the hypotenuse as c.



2 Write Pythagoras' theorem.

$$c^2 = a^2 + b^2$$

3 Substitute the values of a, b and c into this rule and simplify.

$$\begin{aligned} 5.5^2 &= a^2 + 1.5^2 \\ 30.25 &= a^2 + 2.25 \\ a^2 &= 30.25 - 2.25 \\ &= 28 \end{aligned}$$

4 Find a by taking the square root of 28. Round to 1 decimal place, $a > 0$.

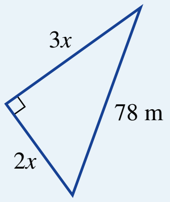
$$\begin{aligned} a &= \pm\sqrt{28} \\ &\approx 5.3 \end{aligned}$$

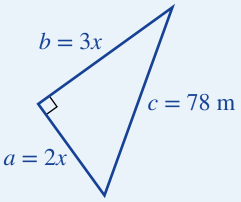
5 Answer the question in a sentence using words.

The ladder reaches 5.3 m up the wall.

WORKED EXAMPLE 4

Determine the unknown side lengths of the triangle, correct to 2 decimal places.



THINK	WRITE/DRAW
1 Copy the diagram and label the sides *a*, *b* and *c*.	
2 Write Pythagoras' theorem.	$c^2 = a^2 + b^2$
3 Substitute the values of *a*, *b* and *c* into this rule and simplify.	$78^2 = (3x)^2 + (2x)^2$ $6084 = 9x^2 + 4x^2$ $6084 = 13x^2$
4 Rearrange the equation so that the pronumeral is on the left-hand side of the equation.	$13x^2 = 6084$
5 Divide both sides of the equation by 13.	$\frac{13x^2}{13} = \frac{6084}{13}$ $x^2 = 468$
6 Find *x* by taking the square root of both sides. Round the answer correct to 2 decimal places.	$x = \pm\sqrt{468}$ ≈ 21.6333
7 Substitute the value of *x* into 2*x* and 3*x* to find the length of the unknown sides.	$2x \approx 43.27$ m $3x \approx 64.90$ m

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Exercise 5.2 Pythagoras' theorem

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Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 6, 12–15, 17, 20	Questions: 1–3, 5–8, 12, 15–18, 20, 22	Questions: 1, 2, 5, 7, 9–11, 19–23

🟩🟦🟪 **Individual pathway interactivity:** int-4585

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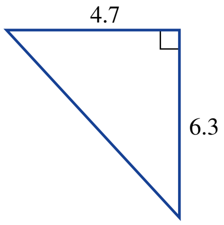
Fluency

1. WE1 For each of the following triangles, calculate the length of the hypotenuse, giving answers correct to 2 decimal places.

a.



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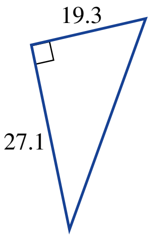
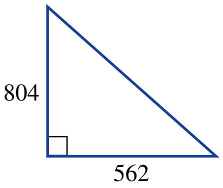


c.



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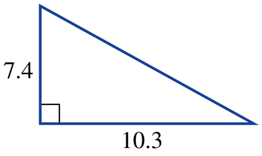

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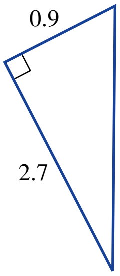

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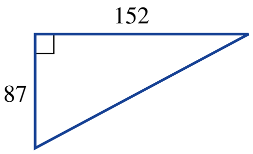
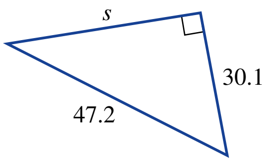
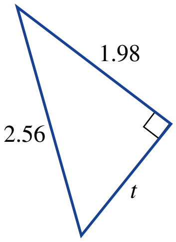


2. WE2 Find the value of the pronumeral, correct to 2 decimal places.

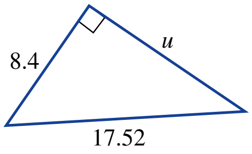
a.



b.



c.



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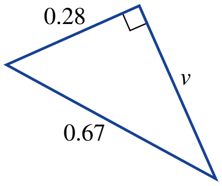

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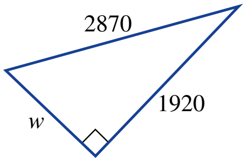

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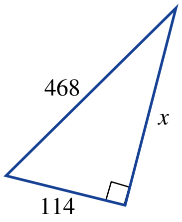


3. WE3 The diagonal of the rectangular sign at right is 34 cm. If the height of this sign is 25 cm, find the width.

4. A right-angled triangle has a base of 4 cm and a height of 12 cm. Calculate the length of the hypotenuse to 2 decimal places.

5. Calculate the lengths of the diagonals (to 2 decimal places) of squares that have side lengths of:

a. 10 cm **b.** 17 cm **c.** 3.2 cm.

6. The diagonal of a rectangle is 90 cm. One side has a length of 50 cm. Determine:

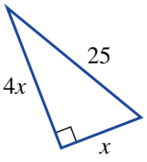
a. the length of the other side

b. the perimeter of the rectangle

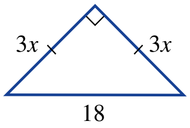
c. the area of the rectangle.

7. WE4 Find the value of the pronumeral, correct to 2 decimal places for each of the following.

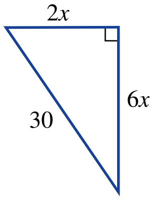
a.



b.



c.



Understanding

8. An isosceles triangle has a base of 25 cm and a height of 8 cm. Calculate the length of the two equal sides.

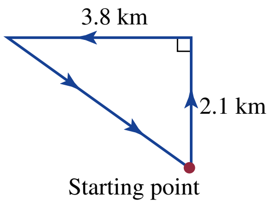
9. An equilateral triangle has sides of length 18 cm. Find the height of the triangle.

10. A right-angled triangle has a height of 17.2 cm, and a base that is half the height. Calculate the length of the hypotenuse, correct to 2 decimal places.

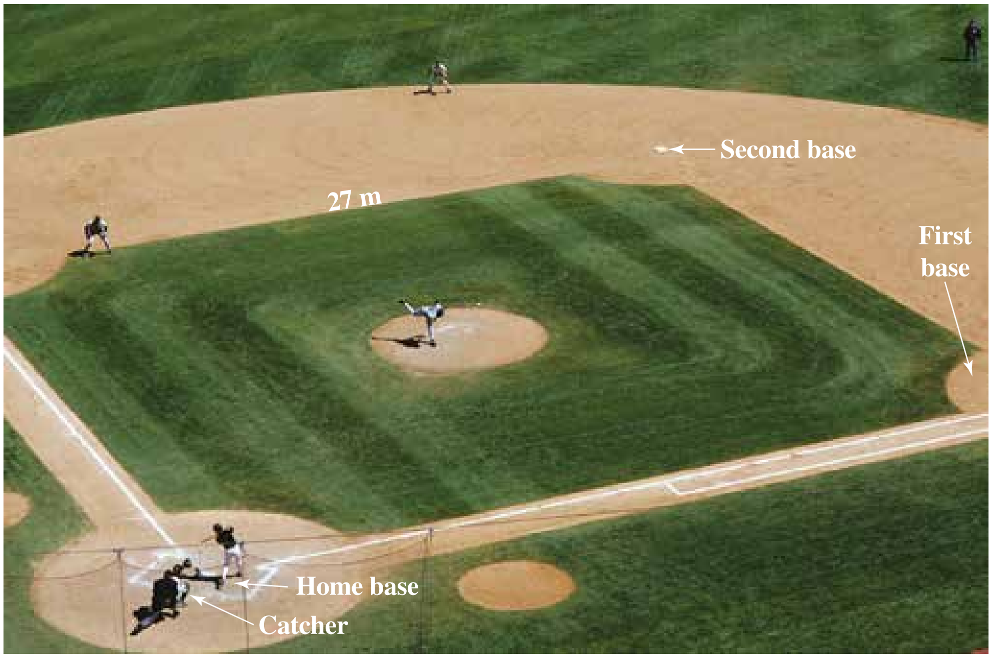
11. The road sign shown below is based on an equilateral triangle. Find the height of the sign and, hence, find its area.

12. A flagpole, 12 m high, is supported by three wires, attached from the top of the pole to the ground. Each wire is pegged into the ground 5 m from the pole. How much wire is needed to support the pole?

13. Sarah goes canoeing in a large lake. She paddles 2.1 km to the north, then 3.8 km to the west. Use the triangle at right to find out how far she must then paddle to get back to her starting point in the shortest possible way.



14. A baseball diamond is a square of side length 27 m. When a runner on first base tries to steal second base, the catcher has to throw the ball from home base to second base. How far is that throw?



15. A rectangle measures 56 mm by 2.9 cm. Calculate the length of its diagonal in millimetres to 2 decimal places.

16. A rectangular envelope has a length of 24 cm and a diagonal measuring 40 cm. Calculate:

- **a.** the width of the envelope
- **b.** the area of the envelope.

17. A swimming pool is 50 m by 25 m. Peter is bored by his usual training routine, and decides to swim the diagonal of the pool. How many diagonals must he swim to complete his normal distance of 1500 m? Give your answer to 2 decimal places.

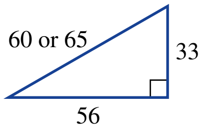
18. A hiker walks 2.9 km north, then 3.7 km east. How far in metres is she from her starting point? Give your answer to 2 decimal places.

19. A square has a diagonal of 14 cm. What is the length of each side?

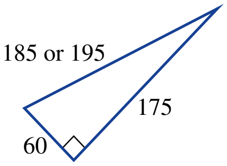
Reasoning

20. The triangles below are right-angled triangles. Two possible measurements have been suggested for the hypotenuse in each case. For each triangle, complete calculations to determine which of the lengths is correct for the hypotenuse in each case. Show your working.

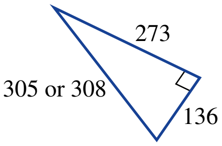
a.



b.



c.



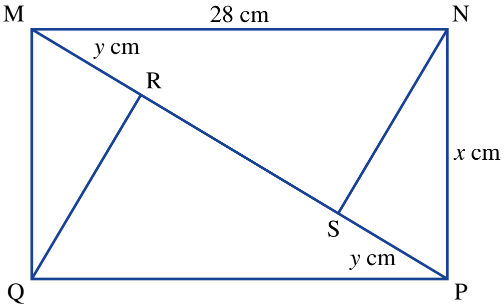
21. Four possible side length measurements are 105, 208, 230 and 233. Three of them together produce a right-angled triangle.

- **a.** Which of the measurements could not be the hypotenuse of the triangle? Explain.
- **b.** Complete as few calculations as possible to calculate which combination of side lengths will produce a right-angled triangle.

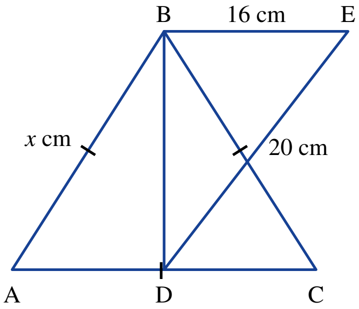
Problem solving

22. The area of the rectangle MNPQ is 588 cm^2. Angles MRQ and NSP are right angles.

- **a.** Find the integer value of x.
- **b.** Find the length of MP.
- **c.** Find the value of y and hence determine the length of RS.



23. Triangle ABC is an equilateral triangle of side length x cm. Angles ADB and DBE are right angles. Find the value of x, correct to 2 decimal places.



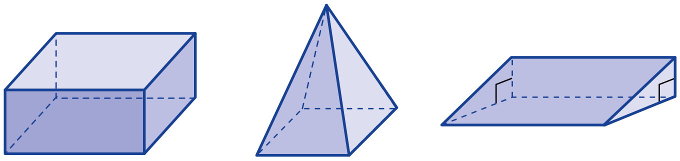
Reflection

The square root of a number usually gives us both a positive and negative answer. Why do we take only the positive answer when using Pythagoras' theorem?

5.3 Pythagoras' theorem in three dimensions

5.3.1 Pythagoras' theorem in three dimensions

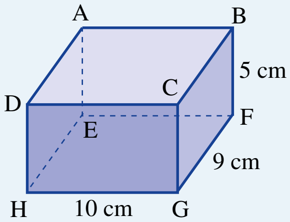
- Many real-life situations involve **3-dimensional** (3-D) objects: objects with length, width and height. Some common 3-D objects used in this section include cuboids, pyramids and right-angled wedges.

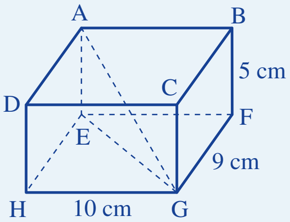
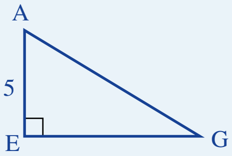
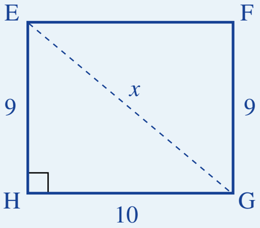


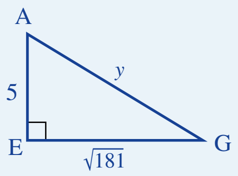
- In diagrams of 3-D objects, right angles may not look like right angles, so it is important to redraw sections of the diagram in two dimensions, where the right angles can be seen accurately.

WORKED EXAMPLE 5

Determine the length AG in this rectangular prism (cuboid), correct to two decimal places.

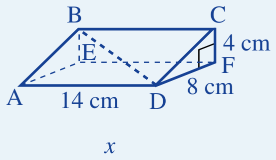
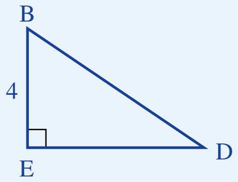
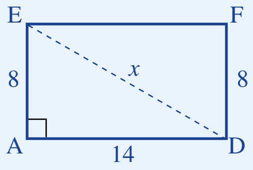
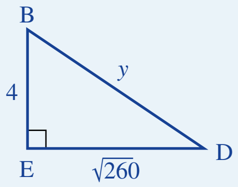


THINK	WRITE/DRAW
1 Draw the diagram in three dimensions. Draw the lines AG and EG. ∠AEG is a right angle.	
2 Draw ΔAEG, showing the right angle. Only 1 side is known, so EG must be found.	
3 Draw EFGH in two dimensions and label the diagonal EG as x.	
4 Use Pythagoras' theorem to calculate x. ($c^2 = a^2 + b^2$)	$x^2 = 9^2 + 10^2$ $= 81 + 100$ $= 181$ $x = \sqrt{181}$

5 Place this information on triangle AEG. Label the side AG as y.	
6 Use Pythagoras' theorem to find y. ($c^2 = a^2 + b^2$)	$y^2 = 5^2 + (\sqrt{181})^2$ $= 25 + 181$ $= 206$ $y = \sqrt{206}$ ≈ 14.35
7 Answer the question in a sentence.	The length of AG is 14.35 cm.

WORKED EXAMPLE 6

A piece of cheese in the shape of a right-angled wedge sits on a table. It has a rectangular base measuring 14 cm by 8 cm, and is 4 cm high at the thickest point. An ant crawls diagonally across the sloping face. How far, to the nearest millimetre, does the ant walk?

THINK	**WRITE/DRAW**
1 Draw a diagram in three dimensions and label the vertices. Mark BD, the path taken by the ant, with a dotted line. ∠BED is a right angle.	
2 Draw ΔBED, showing the right angle. Only one side is known, so ED must be found.	
3 Draw EFDA in two dimensions, and label the diagonal ED. Label the side ED as x in both diagrams.	
4 Use Pythagoras' theorem to calculate x.	$c^2 = a^2 + b^2$ $x^2 = 8^2 + 14^2$ $= 64 + 196$ $= 260$ $x = \sqrt{260}$
5 Place this information on triangle BED. Label the side BD as y.	

6	Solve this triangle for *y*.	$y^2 = 4^2 + (\sqrt{260})^2$ $= 16 + 260$ $= 276$ $y = \sqrt{276}$ ≈ 16.51 cm ≈ 166.1 mm
7	Answer the question in a sentence.	The ant walks 166 mm, correct to the nearest millimetre.

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Complete this digital doc: SkillSHEET: Drawing 3-D shapes (doc-5229)

Complete this digital doc: WorkSHEET: Pythagoras' theorem (doc-5230)

Exercise 5.3 Pythagoras' theorem in three dimensions

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–b, 2, 6, 7, 8a, 10	Questions: 1, 3, 4, 6, 7, 8, 10, 11, 13, 14	Questions: 1, 3–5, 8–16

🟩🟦🟪 **Individual pathway interactivity:** int-4586

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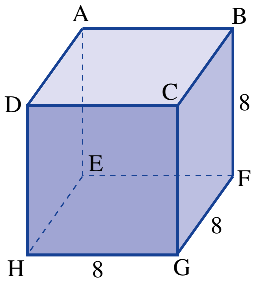
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Where appropriate in this exercise, give answers correct to 2 decimal places.

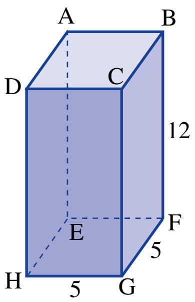
Fluency

1. **WE5** Calculate the length of AG in each of the following figures.

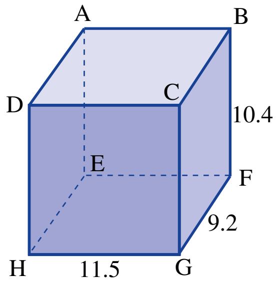
a.



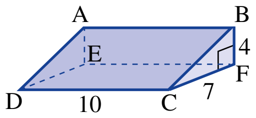
b.



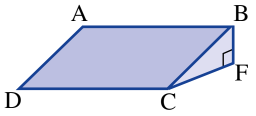
c.



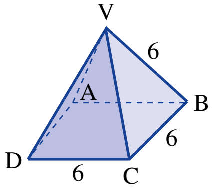
2. Calculate the length of CE in the wedge at right and, hence, obtain AC.



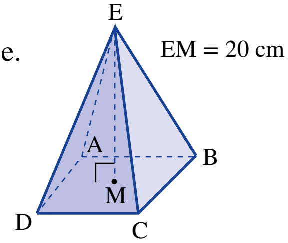
3. If DC = 3.2 m, AC = 5.8 m, and CF = 4.5 m in the figure at right, calculate the length of AD and BF.



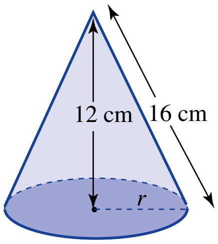
4. Calculate the length of BD and, hence, the height of the pyramid at right.



5. The pyramid ABCDE has a square base. The pyramid is 20 cm high. Each sloping edge measures 30 cm. Calculate the length of the sides of the base.

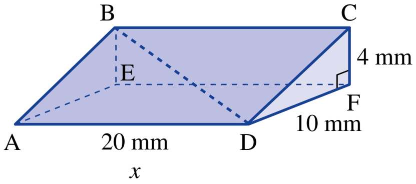


6. The sloping side of a cone is 16 cm and the height is 12 cm. What is the length of the radius of the base?



Understanding

7. **WE6** A piece of cheese in the shape of a right-angled wedge sits on a table. It has a base measuring 20 mm by 10 mm, and is 4 mm high at the thickest point, as shown in the figure. A fly crawls diagonally across the sloping face. How far, to the nearest millimetre, does the fly walk?



8. A 7 m high flagpole is in the corner of a rectangular park that measures 200 m by 120 m.

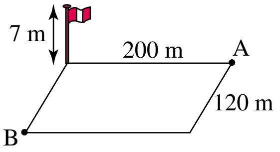
a. Calculate:

i. the length of the diagonal of the park

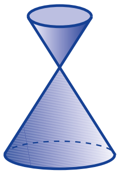
ii. the distance from A to the top of the pole

iii. the distance from B to the top of the pole.

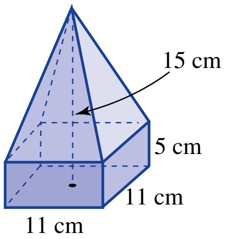
b. A bird flies from the top of the pole to the centre of the park. How far does it fly?



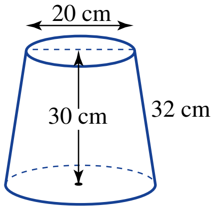
9. A candlestick is in the shape of two cones, joined at the vertices as shown. The smaller cone has a diameter and sloping side of 7 cm, and the larger one has a diameter and sloping side of 10 cm. How tall is the candlestick?



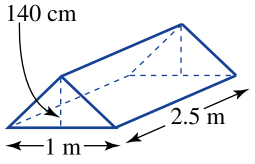
10. The total height of the shape below is 15 cm. Calculate the length of the sloping side of the pyramid.



11. A sandcastle is in the shape of a truncated cone as shown. Calculate the length of the diameter of the base.



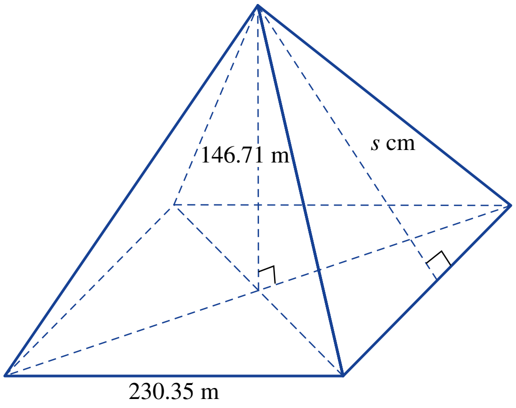
12. A tent is in the shape of a triangular prism, with a height of 140 cm as shown in the following diagram. The width across the base of the door is 1 m and the tent is 2.5 m long. Calculate the length of each sloping side, in metres. Then calculate the area of fabric used in the construction of the sloping rectangles which form the sides.



Reasoning

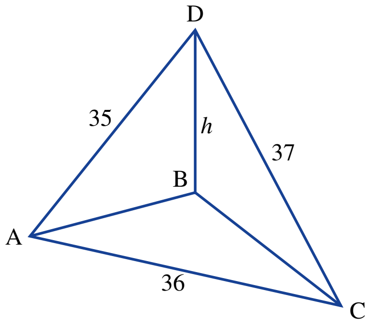
13. Stephano is renovating his apartment, which he accesses through two corridors. The corridors of the apartment building are 2 m wide with 2 m high ceilings, and the first corridor is at right angles to the second. Show that he can carry lengths of timber up to 6 m long to his apartment.

14. The Great Pyramid in Egypt is a square-based pyramid. The square base has a side length of 230.35 metres and the perpendicular height is 146.71 metres. Find the slant height, s cm, of the great pyramid. Give your answer correct to 1 decimal place.

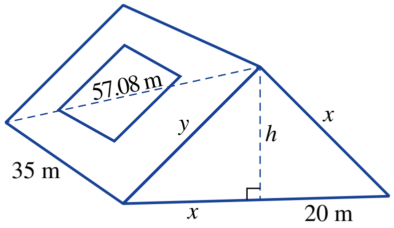


Problem solving

15. Angles ABD, CBD and ABC are right angles. Find the value of h, correct to 3 decimal places.



16. The roof of a squash centre is constructed to allow for maximum use of sunlight. Find the value of h, giving your answer correct to 1 decimal place.

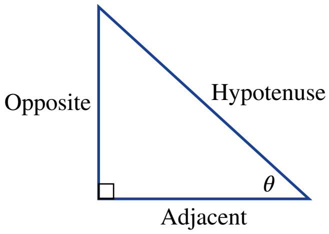


Reflection

The diagonal distance across a rectangle of dimensions x by y is $\sqrt{x^2 + y^2}$. What would be the rule to find the length of a diagonal across a cuboid of dimensions x by y by z? Use your rule to check your answers to question 1.

5.4 Trigonometric ratios

5.4.1 Naming the sides in a right-angled triangle



- In a right-angled triangle, the longest side is called the hypotenuse.
- If one of the two acute angles is named (say θ), then the other two sides can also be given names, as shown in the diagram.

5.4.2 Three basic definitions

- Using the diagram opposite, the following three **trigonometric ratios** can be defined:
  – the **sine ratio**, $\text{sine } \theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$
  – the **cosine ratio**, $\text{cosine } \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$
  – the **tangent ratio**, $\text{tangent } \theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- The names of the three ratios are usually shortened to sin θ, cos θ and tan θ.
- The three ratios are often remembered using the mnemonic SOHCAHTOA, where SOH means Sin θ = Opposite over Hypotenuse and so on.

5.4.3 Finding values using a calculator

- The sine, cosine and tangent of an angle have numerical values that can be found using a calculator.
- Traditionally angles were measured in **degrees**, minutes and seconds, where 60 seconds = 1 minute and 60 minutes = 1 degree.
  For example, $50°33'48''$ means 50 degrees, 33 minutes and 48 seconds.

WORKED EXAMPLE 7

TI | CASIO

Calculate the value of each of the following, correct to 4 decimal places, using a calculator. (Remember to first work to 5 decimal places before rounding.)

a **$\cos 65°57'$** **b** **$\tan 56°45'30''$**

THINK	WRITE
a Write your answer to the required number of decimal places.	**a** $\cos 65°57' \approx 0.40753$ ≈ 0.4075
b Write your answer to the correct number of decimal places.	**b** $\tan 56°45'30'' \approx 1.52573$ ≈ 1.5257

WORKED EXAMPLE 8

Calculate the size of angle θ, correct to the nearest degree, given $\sin\theta = 0.7854$.

THINK	WRITE
1 Write the given equation.	$\sin\theta = 0.7854$
2 To find the size of the angle, we need to undo sine with its inverse, $\sin^{-1}$. (Ensure your calculator is in degrees mode.)	$\theta = \sin^{-1} 0.7854$ $\approx 51.8°$
3 Write your answer to the nearest degree.	$\theta \approx 52°$

WORKED EXAMPLE 9

TI | CASIO

Calculate the value of θ:

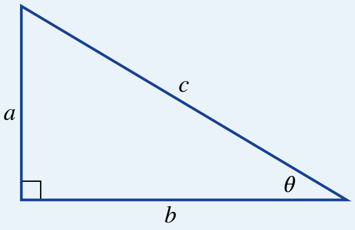
a correct to the nearest minute, given that $\cos\theta = 0.2547$

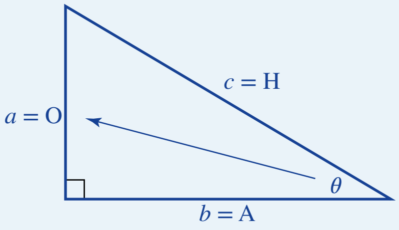
b correct to the nearest second, given that $\tan\theta = 2.364$

THINK	WRITE
a **1** Write the equation.	**a** $\cos\theta = 0.2547$
2 Write your answer, including seconds. There are 60 seconds in 1 minute. Round to the nearest minute. (Remember $60'' = 1'$, so $39''$ is rounded up.)	$\cos^{-1} 0.2547 \approx 75°14'39''$ $\approx 75°15'$
b **1** Write the equation.	**b** $\tan\theta = 2.364$
2 Write the answer, rounding to the nearest second.	$\tan^{-1} 2.364 \approx 67°4'\,15.8''$ $\approx 67°4'16''$

WORKED EXAMPLE 10

For the triangle shown, write the rules for the sine, cosine and tangent ratios of the given angle.

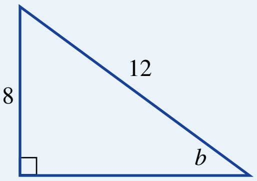


THINK	WRITE/DRAW
1 Label the diagram using the symbols O, A, H with respect to the given angle (angle θ).	
2 From the diagram, identify the values of O (opposite side), A (adjacent side) and H (the hypotenuse).	$\mathrm{O} = a,\ \mathrm{A} = b,\ \mathrm{H} = c$
3 Write the rule for each of the sine, cosine and tangent ratios.	$\sin\theta = \frac{\mathrm{O}}{\mathrm{H}},\ \cos\theta = \frac{\mathrm{A}}{\mathrm{H}},\ \tan\theta = \frac{\mathrm{O}}{\mathrm{A}}$
4 Substitute the values of A, O and H into each rule.	$\sin\theta = \frac{a}{c},\ \cos\theta = \frac{b}{c},\ \tan\theta = \frac{a}{b}$

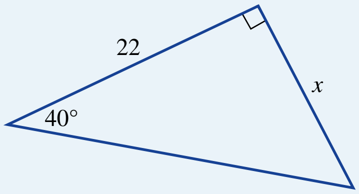
WORKED EXAMPLE 11

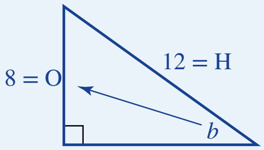
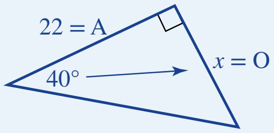
Write the equation that relates the two marked sides and the marked angle.

a



b



THINK	WRITE/DRAW
a **1** Label the given sides of the triangle.	**a** 
2 Write the ratio that contains O and H.	$\sin\theta = \frac{O}{H}$
3 Identify the values of the pronumerals.	$O = 8, H = 12$
4 Substitute the values of the pronumerals into the ratio. (Since the given angle is denoted with the letter *b*, replace θ with *b*.)	$\sin b = \frac{8}{12} = \frac{2}{3}$
b **1** Label the given sides of the triangle.	**b** 
2 Write the ratio that contains O and A.	$\tan\theta = \frac{O}{A}$
3 Identify the values of the pronumerals.	$O = x, A = 22, \theta = 40°$
4 Substitute the values of the pronumerals into the ratio.	$\tan 40° = \frac{x}{22}$

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Labelling the sides of a right-angled triangle (doc-5226)

Complete this digital doc: SkillSHEET: Selecting an appropriate trigonometric ratio based on the given information (doc-5231)

Exercise 5.4 Trigonometric ratios

assesson

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1, 3, 6a–f, 7, 8	Questions: 2–4, 6a–f, 7–9, 11	Questions: 2, 3, 4, 5, 6c–l, 7–12

■■■ **Individual pathway interactivity:** int-4587

learnon ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. Calculate each of the following, correct to 4 decimal places.

a. $\sin 30^\circ$ **b.** $\cos 45^\circ$ **c.** $\tan 25^\circ$

d. $\sin 57^\circ$ **e.** $\tan 83^\circ$ **f.** $\cos 44^\circ$

2. **WE7** Calculate each of the following, correct to 4 decimal places.

a. $\sin 40^\circ 30'$ **b.** $\cos 53^\circ 57'$ **c.** $\tan 27^\circ 34'$

d. $\tan 123^\circ 40'$ **e.** $\sin 92^\circ 32'$ **f.** $\sin 42^\circ 8'$

g. $\cos 35^\circ 42' 35''$ **h.** $\tan 27^\circ 42' 50''$ **i.** $\cos 143^\circ 25' 23''$

j. $\sin 23^\circ 58' 21''$ **k.** $\cos 8^\circ 54' 2''$ **l.** $\sin 286^\circ$

m. $\tan 420^\circ$ **n.** $\cos 845^\circ$ **o.** $\sin 367^\circ 35'$

3. **WE8** Find the size of angle θ, correct to the nearest degree, for each of the following.

a. $\sin \theta = 0.763$ **b.** $\cos \theta = 0.912$ **c.** $\tan \theta = 1.351$

d. $\cos \theta = 0.321$ **e.** $\tan \theta = 12.86$ **f.** $\cos \theta = 0.756$

4. **WE9a** Find the size of the angle θ, correct to the nearest minute.

a. $\sin \theta = 0.814$ **b.** $\sin \theta = 0.110$ **c.** $\tan \theta = 0.015$

d. $\cos \theta = 0.296$ **e.** $\tan \theta = 0.993$ **f.** $\sin \theta = 0.450$

5. **WE9b** Find the size of the angle θ, correct to the nearest second.

a. $\tan \theta = 0.5$ **b.** $\cos \theta = 0.438$ **c.** $\sin \theta = 0.9047$

d. $\tan \theta = 1.1141$ **e.** $\cos \theta = 0.8$ **f.** $\tan \theta = 43.76$

6. Find the value of each expression, correct to 3 decimal places.

a. $3.8 \cos 42^\circ$ **b.** $118 \sin 37^\circ$ **c.** $2.5 \tan 83^\circ$

d. $\dfrac{2}{\sin 45^\circ}$ **e.** $\dfrac{220}{\cos 14^\circ}$ **f.** $\dfrac{2 \cos 23^\circ}{5 \sin 18^\circ}$

g. $\dfrac{12.8}{\tan 60^\circ 32'}$ **h.** $\dfrac{18.7}{\sin 35^\circ 25' 42''}$ **i.** $\dfrac{55.7}{\cos 89^\circ 21'}$

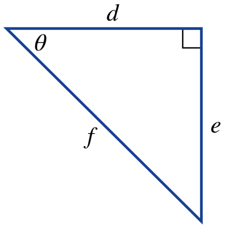
j. $\dfrac{3.8 \tan 1^\circ 51' 44''}{4.5 \sin 25^\circ 45'}$ **k.** $\dfrac{2.5 \sin 27^\circ 8'}{10.4 \cos 83^\circ 2'}$ **l.** $\dfrac{3.2 \cos 34^\circ 52'}{0.8 \sin 12^\circ 48'}$

Understanding

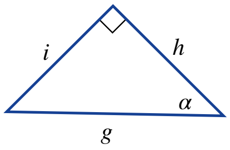
7. **WE10** For each labelled angle in the following triangles, write an expression for:

i. sine **ii.** cosine **iii.** tangent.

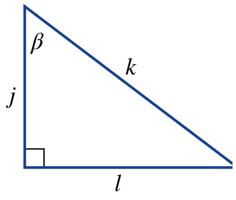
a.



b.



c.



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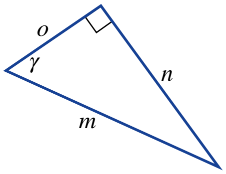
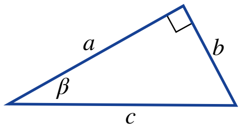
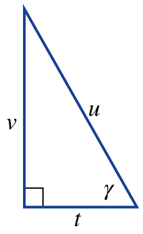


e.

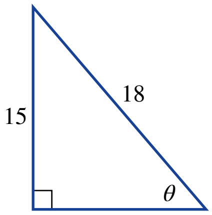


f.



8. **WE11** Write the equation that relates the two marked sides and the marked angle in each of the following triangles.

a.



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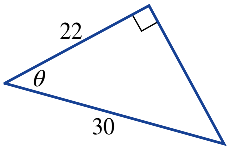
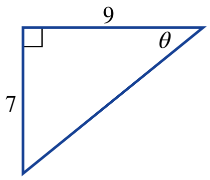


c.



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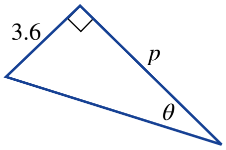
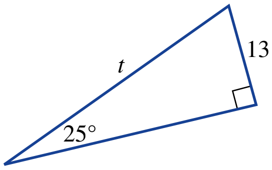
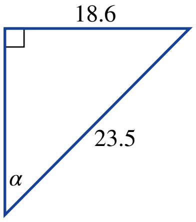


e.

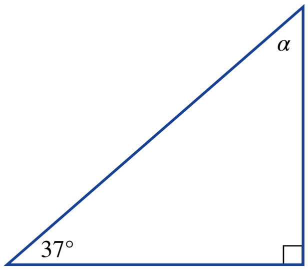


f.



Reasoning

9. Consider the right-angled triangle shown at right.



a. Label each of the sides using the letters O, A and H with respect to the 37° angle.

b. Determine the value of each trigonometric ratio. (Where applicable, answers should be given correct to 2 decimal places.)

i. sin 37°

ii. cos 37°

iii. tan 37°

c. What is the value of the unknown angle, α?

d. Determine the value of each of these trigonometric ratios, correct to 2 decimal places.

i. $\sin \alpha$

ii. $\cos \alpha$

iii. $\tan \alpha$

(*Hint:* First relabel the sides of the triangle with respect to angle α.)

e. What do you notice about the relationship between sin 37° and $\cos \alpha$?

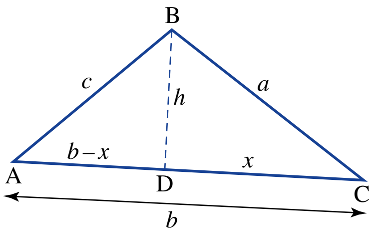
f. What do you notice about the relationship between $\sin \alpha$ and cos 37°?

g. Make a general statement about the two angles.

10. Using a triangle labelled with a, h and o and algebra, show that $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$.
(*Hint:* Write all the sides in terms of the hypotenuse.)

Problem solving

11. ABC is a scalene triangle with side lengths a, b and c as shown. Angles BDA and BDC are right angles.



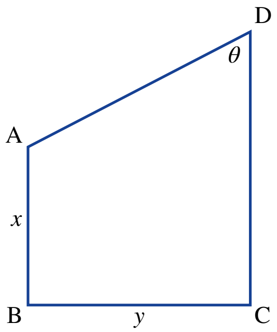
a. Express h^2 in terms of a and x.

b. Express h^2 in terms of b, c and x.

c. Equate the two equations for h^2 to show that $c^2 = a^2 + b^2 - 2bx$.

d. Use your knowledge of trigonometry to produce the equation $c^2 = a^2 + b^2 - 2ab\cos C$, which is known as the cosine rule for non-right-angled triangles.

12. Find the length of the side DC in terms of x, y and θ.



Reflection

How do we determine which of sin, cos or tan to use in a trigonometry question?

5.5 Using trigonometry to calculate side lengths

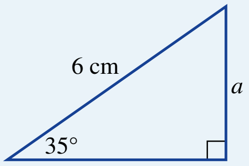
5.5.1 Using trigonometry to calculate side lengths

In a right-angled triangle if one side length and one acute angle are known, the lengths of the other sides can be found by applying trigonometric ratios.

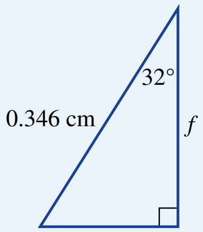
WORKED EXAMPLE 12

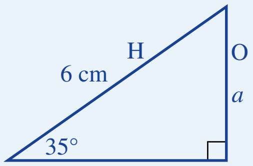
Find the value of each pronumeral giving answers correct to 3 decimal places.

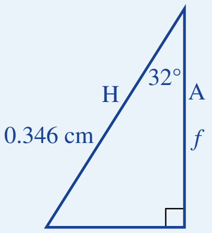
a



b



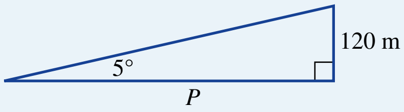
THINK	WRITE/DRAW
a **1** Label the marked sides of the triangle.	**a** 
2 Identify the appropriate trigonometric ratio to use.	$\sin \theta = \frac{O}{H}$
3 Substitute O = a, H = 6 and θ = 35°.	$\sin 35° = \frac{a}{6}$
4 Make a the subject of the equation.	$6 \sin 35° = a$ $a = 6 \sin 35°$
5 Calculate and round the answer, correct to 3 decimal places.	$a \approx 3.441$ cm

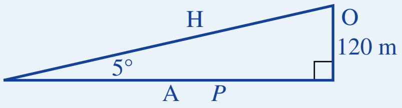
b **1** Label the marked sides of the triangle.	**b** 
2 Identify the appropriate trigonometric ratio to use.	$\cos\theta = \frac{\text{A}}{\text{H}}$
3 Substitute A $= f$, H $= 0.346$ and $\theta = 32°$.	$\cos 32° = \frac{f}{0.346}$
4 Make f the subject of the equation.	$0.346\cos 32° = f$ $f = 0.346\cos 32°$
5 Calculate and round the answer, correct to 3 decimal places.	≈ 0.293 cm

WORKED EXAMPLE 13

TI | CASIO

Find the value of the pronumeral in the triangle shown. Give the answer correct to 2 decimal places.



THINK	**WRITE/DRAW**
1 Label the marked sides of the triangle.	
2 Identify the appropriate trigonometric ratio to use.	$\tan\theta = \frac{\text{O}}{\text{A}}$
3 Substitute O $= 120$, A $= P$ and $\theta = 5°$.	$\tan 5° = \frac{120}{P}$
4 Make P the subject of the equation. **i.** Multiply both sides of the equation by P. **ii.** Divide both sides of the equation by tan 5°.	$P \times \tan 5° = 120$ $P = \frac{120}{\tan 5°}$
5 Calculate and round the answer, correct to 2 decimal places.	$P \approx 1371.61$ m

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Try out this interactivity: Using trigonometry (int-1146)

Exercise 5.5 Using trigonometry to calculate side lengths

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–5, 8	Questions: 1–6, 8, 9	Questions: 1–10

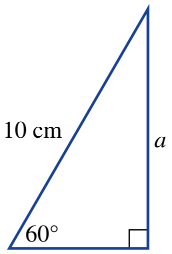
🟩🟦🟪 **Individual pathway interactivity:** int-4588 learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

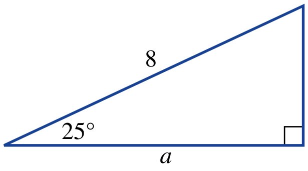
Fluency

1. **WE12** Find the length of the unknown side in each of the following, correct to 3 decimal places.

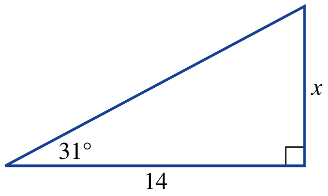
a.



b.

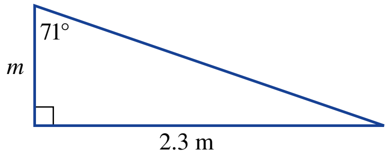


c.

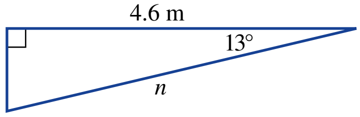


2. **WE13** Find the length of the unknown side in each of the following triangles, correct to 2 decimal places.

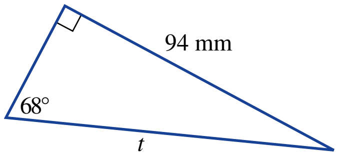
a.



b.

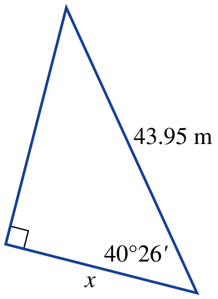


c.

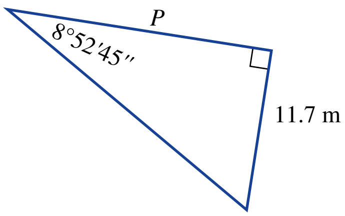


3. Find the length of the unknown side in each of the following, correct to 2 decimal places.

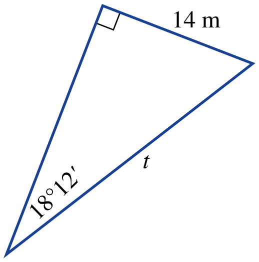
a.



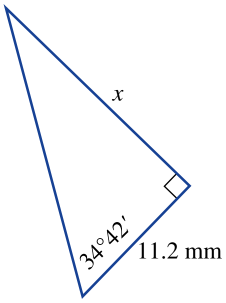
b.



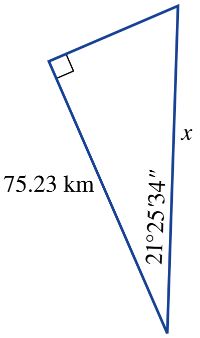
c.



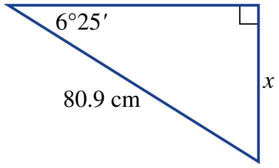
d.



e.

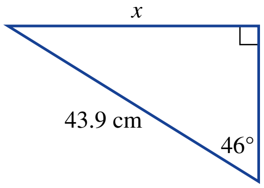


f.



4. Find the value of the pronumeral in each of the following, correct to 2 decimal places.

a.



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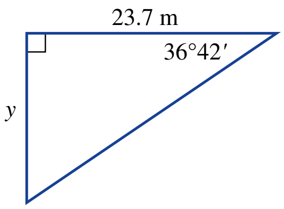
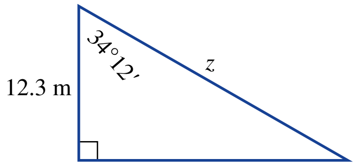


c.



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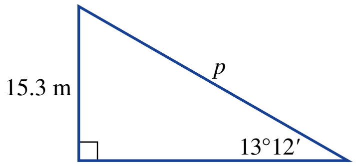

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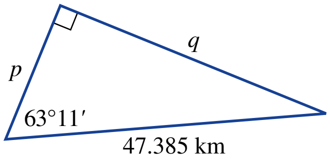
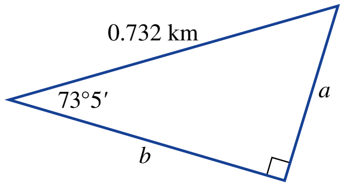


f.



Understanding

5. Given that the angle θ is 42° and the length of the hypotenuse is 8.95 m in a right-angled triangle, find the length of:

a. the opposite side

b. the adjacent side.

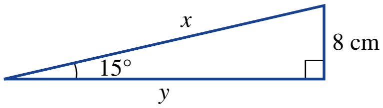
Give each answer correct to 1 decimal point.

6. A ladder rests against a wall. If the angle between the ladder and the ground is 35° and the foot of the ladder is 1.5 m from the wall, how high up the wall does the ladder reach?

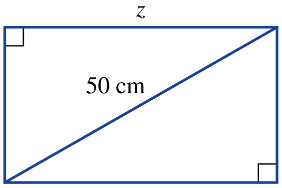
Reasoning

7. Tran is going to construct an enclosed rectangular desktop that is at an incline of 15°. The diagonal length of the desktop is 50 cm. At the high end, the desktop, including top, bottom and sides, will be raised 8 cm. The desktop will be made of wood. The diagram below represents this information.

Side view of the desktop



Top view of the desktop



a. Determine the values (in centimetres) of x, y and z of the desktop. Write your answers correct to 2 decimal places.

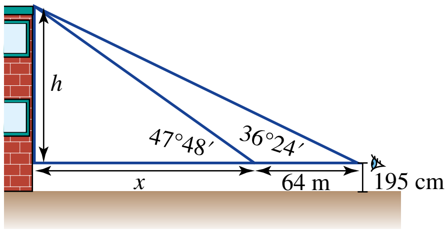
b. Using your answer from part **a** determine the minimum area of wood, in cm^2, Tran needs to construct his desktop including top, bottom and sides. Write your answer correct to 2 decimal places.

8. a. In a right-angled triangle, under what circumstances will the opposite side and the adjacent side have the same length?

b. In a right-angled triangle, for what values of θ (the reference angle) will the adjacent side be longer than the opposite side?

Problem solving

9. A surveyor needs to determine the height of a building. She measures the angle of elevation of the top of the building from two points, 64 m apart. The surveyor's eye level is 195 cm above the ground.

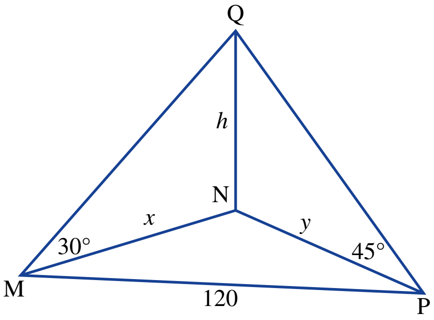


a. Find the expressions for the height of the building, h, in terms of x using the two angles.

b. Solve for x by equating the two expressions obtained in part **a**. Give your answer to 2 decimal places.

c. Find the height of the building correct to 2 decimal places.

10. If angles QNM, QNP and MNP are right angles, find the length of NQ.



Reflection

How does solving a trigonometric equation differ when we are finding the length of the hypotenuse side compared to when finding the length of a shorter side?

5.6 Using trigonometry to calculate angle size

5.6.1 Using trigonometry to calculate angle size

- Just as inverse operations are used to solve equations, inverse trigonometric ratios are used to solve trigonometric equations for the value of the angle.
  - Inverse sine ($\sin^{-1}$) is the inverse of sine.
  - Inverse cosine ($\cos^{-1}$) is the inverse of cosine.
  - Inverse tangent ($\tan^{-1}$) is the inverse of tangent.

  For example, since $\sin(30°) = 0.5$, then $\sin^{-1}(0.5) = 30°$; this is read as 'inverse sine of 0.5 is 30 degrees'.

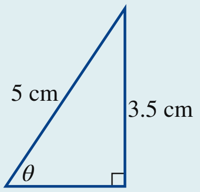
If $\sin\theta = a$, then $\sin^{-1} a = \theta$.
If $\cos\theta = a$, then $\cos^{-1} a = \theta$.
If $\tan\theta = a$, then $\tan^{-1} a = \theta$.

- A calculator can be used to calculate the values of inverse trigonometric ratios.
- The size of any angle in a right-angled triangle can be found if:
  - the lengths of any two sides are known
  - an appropriate trigonometric ratio is identified from the given lengths
  - a calculator is used to evaluate the inverse trigonometric ratio.

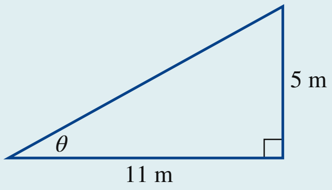
WORKED EXAMPLE 14

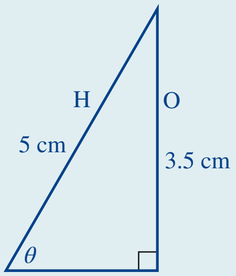
For each of the following, find the size of the angle, θ, correct to the nearest degree.

a



b



THINK	WRITE/DRAW
a 1 Label the given sides of the triangle.	**a** 
2 Identify the appropriate trigonometric ratio to use. We are given O and H.	$\sin\theta = \frac{O}{H}$

3 Substitute O = 3.5 and H = 5 and evaluate the expression.

$$\sin\theta = \frac{3.5}{5} = 0.7$$

4 Make θ the subject of the equation using inverse sine.

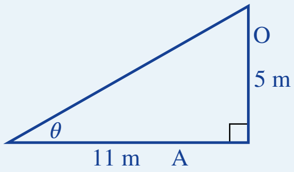
$$\theta = \sin^{-1} 0.7 = 44.427\,004°$$

5 Evaluate θ and round the answer, correct to the nearest degree.

$$\theta \approx 44°$$

b **1** Label the given sides of the triangle.

b



2 Identify the appropriate trigonometric ratio to use. Given O and A.

$$\tan\theta = \frac{O}{A}$$

3 Substitute O = 5 and A = 11.

$$\tan\theta = \frac{5}{11}$$

4 Make θ the subject of the equation using inverse tangent.

$$\theta = \tan^{-1}\left(\frac{5}{11}\right) = 24.443\,954\,78°$$

5 Evaluate θ and round the answer, correct to the nearest degree.

$$\theta \approx 24°$$

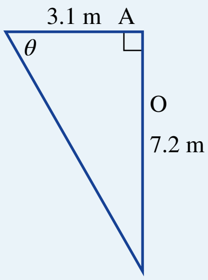
WORKED EXAMPLE 15

TI | CASIO

Find the size of angle θ:

a correct to the nearest second

b correct to the nearest minute.

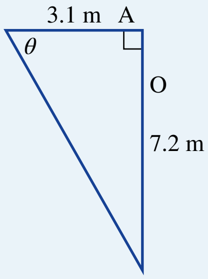


THINK

WRITE/DRAW

a **1** Label the given sides of the triangle.

a



2 Identify the appropriate trigonometric ratio to use.

$$\tan\theta = \frac{O}{A}$$

3 Substitute O = 7.2 and A = 3.1.

$$\tan\theta = \frac{7.2}{3.1}$$

4 Make θ the subject of the equation using inverse tangent.	$\theta = \tan^{-1}\left(\frac{7.2}{3.1}\right)$
5 Evaluate θ and write the calculator display.	$\theta = 66.705\,436\,75°$
6 Use the calculator to convert the answer to degrees, minutes and seconds.	$= 66°42'19.572''$
7 Round the answer to the nearest second.	$\theta \approx 66°\,42'20''$
b Round the answer to the nearest minute.	**b** $\theta \approx 66°\,42'$

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Complete this digital doc: SkillSHEET: Rounding angles to the nearest degree (doc-5232)

Complete this digital doc: WorkSHEET: Using trigonometry (doc-5233)

Exercise 5.6 Using trigonometry to calculate angle size

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–3–6, 8	Questions: 1–6, 8, 10	Questions: 1–11

🟩🟦🟪 **Individual pathway interactivity:** int-4589

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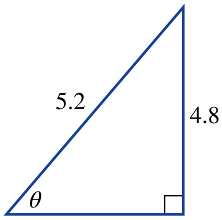
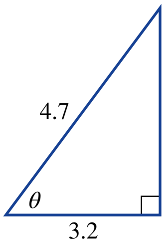
Fluency

1. **WE14** Find the size of the angle, θ, in each of the following. Give your answer correct to the nearest degree.

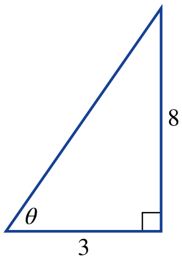
a.



b.

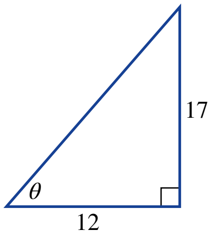


c.

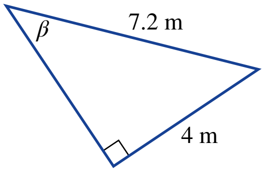


2. **WE15b** Find the size of the angle marked with the pronumeral in each of the following. Give your answer correct to the nearest minute.

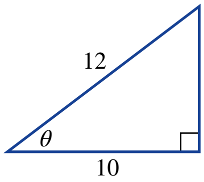
a.



b.

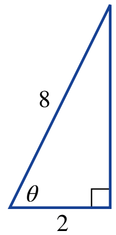


c.

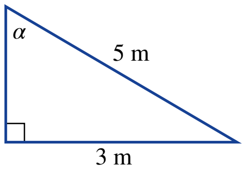


3. WE15a Find the size of the angle marked with the pronumeral in each of the following. Give your answer correct to the nearest second.

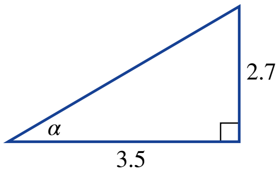
a.



b.

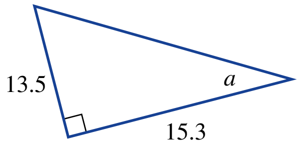


c.

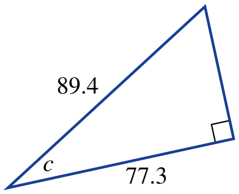


4. Find the size of the angle marked with the pronumeral in each of the following, giving your answer correct to the nearest degree.

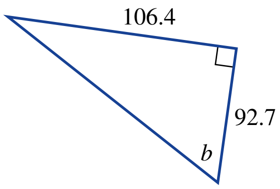
a.



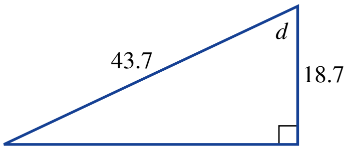
b.



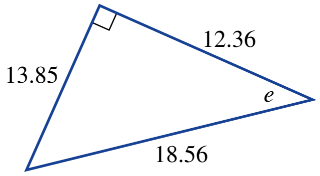
c.



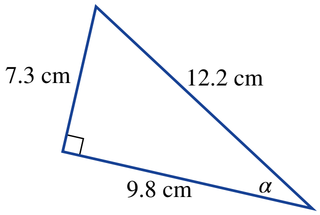
d.



e.

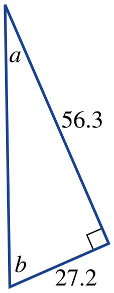


f.

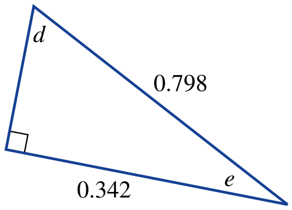


5. Find the size of each of the angles in the following, giving your answers correct to the nearest minute.

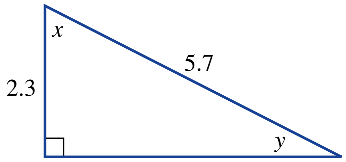
a.



b.

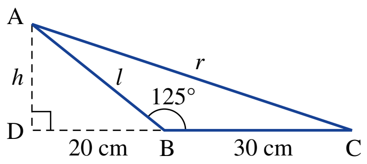


c.



Understanding

6. **a.** Calculate the length of the sides r, l and h. Write your answers correct to 2 decimal places.
   b. Calculate the area of ABC, correct to the nearest square centimetre.
   c. Calculate ∠BCA.



7. In the sport of air racing, small planes have to travel between two large towers (or pylons). The gap between a pair of pylons is smaller than the wing-span of the plane, so the plane has to go through on an angle with one wing 'above' the other. The wing-span of a competition airplane is 8 metres.

a. Determine the angle, correct to 1 decimal place, that the plane has to tilt if the gap between pylons is:

- **i.** 7 metres
- **ii.** 6 metres
- **iii.** 5 metres.

b. Because the plane has rolled away from the horizontal as it travels between the pylons it loses speed. If the plane's speed is below 96 km/h it will stall and possibly crash. For each degree of 'tilt' the speed of the plane is reduced by 0.98 km/h. What is the minimum speed the plane must go through each of the pylons in part **a**? Write your answer correct to 2 decimal places.

Reasoning

8. There are two important triangles commonly used in trigonometry. Complete the following steps and answer the questions to create these triangles.

Triangle 1

- Sketch an equilateral triangle with side length 2 units.
- Calculate the size of the internal angles.
- Bisect the triangle to form two right-angled triangles.
- Redraw one of the triangles formed.
- Calculate the side lengths of this right-angled triangle as exact values.
- Fully label your diagram showing all side lengths and angles.

Triangle 2

- Draw a right-angled isosceles triangle.
- Calculate the sizes of the internal angles.
- Let the sides of equal length be 1 unit long.
- Calculate the length of the third side.
- Fully label your diagram showing all side lengths and angles.

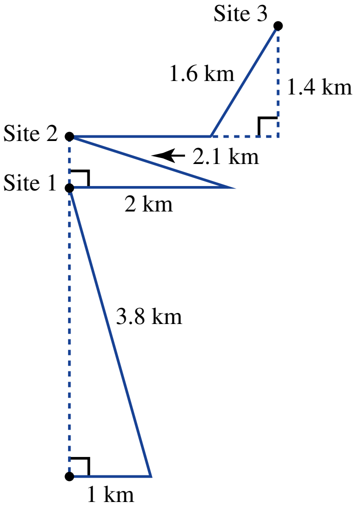
9. a. Use the triangles formed in question **8** to calculate exact values for sin 30°, cos 30° and tan 30°. Justify your answers.

b. Use the exact values for sin 30°, cos 30° and tan 30° to show that $\tan 30° = \dfrac{\sin 30°}{\cos 30°}$.

c. Use the formulas $\sin\theta = \dfrac{o}{h}$ and $\cos\theta = \dfrac{a}{h}$ to prove that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Problem solving

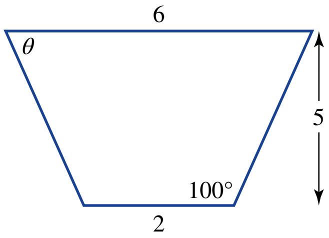
10. During a Science excursion, your class visited an underground cave to observe rock formations. You were required to walk along a series of paths and steps as shown in the diagram below.



a. Calculate the angle of the incline (slope) you have to travel down between each site. Give your answers to the nearest whole number.

b. Determine which path would have been the most challenging; that is, which path had the steepest slope.

11. Find the angle θ in degrees and minutes.



Reflection

How is finding the angle of a right-angled triangle different to finding a side length?

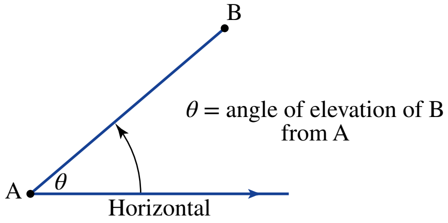
CHALLENGE 5.1

At midday, the hour hand and the minute hand on a standard clock are both pointing at the twelve. Calculate the angles the minute hand and the hour hand have moved 24.5 minutes later. Express both answers in degrees and minutes.

5.7 Angles of elevation and depression

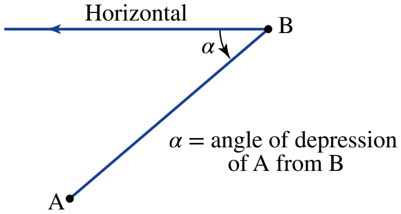
5.7.1 Angles of elevation and depression

- Consider the points A and B, where B is at a higher elevation than A.

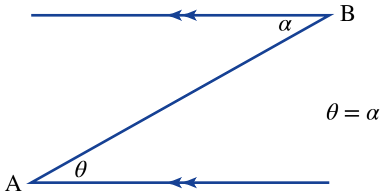


If a horizontal line is drawn from A as shown, forming the angle θ, then θ is called the **angle of elevation** of B *from* A.

- If a horizontal line is drawn from B, forming the angle α, then α is called the **angle of depression** of A *from* B.

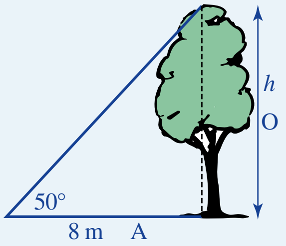


- Because the horizontal lines are parallel, θ and α have the same size (alternate angles).



WORKED EXAMPLE 16

From a point P, on the ground, the angle of elevation of the top of a tree is 50°. If P is 8 metres from the tree, find the height of the tree correct to 2 decimal places.

THINK	WRITE/DRAW
1 Let the height of the tree be h. Sketch a diagram and show the relevant information.	
2 Identify the appropriate trigonometric ratio.	$\tan\theta = \frac{O}{A}$
3 Substitute O = h, A = 8 and θ = 50°.	$\tan 50° = \frac{h}{8}$
4 Rearrange to make h the subject.	$h = 8\tan 50°$
5 Calculate and round the answer to 2 decimal places.	≈ 9.53
6 Give a worded answer.	The height of the tree is 9.53 m.

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Watch this eLesson: Height of a satellite (eles-0173)

Complete this digital doc: SkillSHEET: Drawing a diagram from given directions (doc-5228)

Complete this digital doc: WorkSHEET: Elevation and depression (doc-5234)

Exercise 5.7 Angles of elevation and depression

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–5, 8, 10	Questions: 1–6, 9, 10, 14, 15	Questions: 1–7, 9, 11–16

🟩🟦🟪 **Individual pathway interactivity:** int-4590

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Fluency

1. **WE16** From a point P on the ground the angle of elevation from an observer to the top of a tree is 54°22′. If the tree is known to be 12.19 m high, how far is P from the tree (measured horizontally)?

2. From the top of a cliff 112 m high, the angle of depression to a boat is 9°15′. How far is the boat from the foot of the cliff?

3. A person on a ship observes a lighthouse on the cliff, which is 830 metres away from the ship. The angle of elevation of the top of the lighthouse is 12°.

a. How far above sea level is the top of the lighthouse?

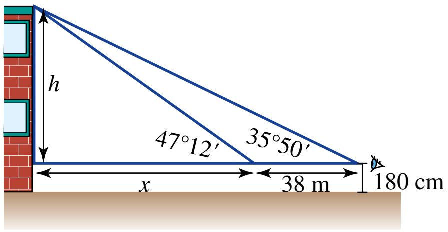
b. If the height of the lighthouse is 24 m, how high is the cliff?

4. At a certain time of the day a post, 4 m tall, casts a shadow of 1.8 m. What is the angle of elevation of the sun at that time?

5. An observer who is standing 47 m from a building measures the angle of elevation of the top of the building as 17°. If the observer's eye is 167 cm from the ground, what is the height of the building?

Understanding

6. A surveyor needs to determine the height of a building. She measures the angle of elevation of the top of the building from two points, 38 m apart. The surveyor's eye level is 180 cm above the ground.

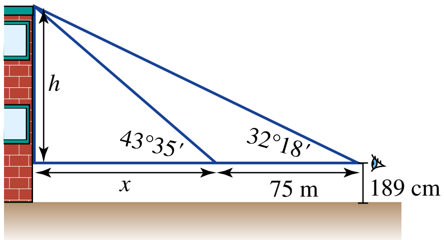


a. Find two expressions for the height of the building, h, in terms of x using the two angles.

b. Solve for x by equating the two expressions obtained in **a**.

c. Find the height of the building.

7. The height of another building needs to be determined but cannot be found directly. The surveyor decides to measure the angle of elevation of the top of the building from different sites, which are 75 m apart. The surveyor's eye level is 189 cm above the ground.

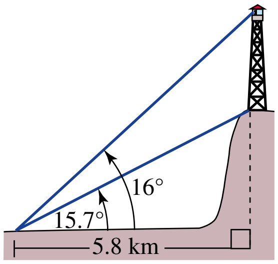


a. Find two expressions for the height of the building above the surveyor's eye level, h, in terms of x using the two angles.

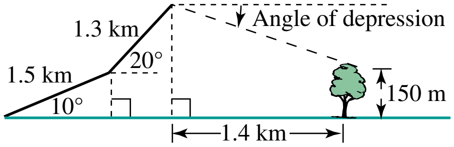
b. Solve for x.

c. Find the height of the building.

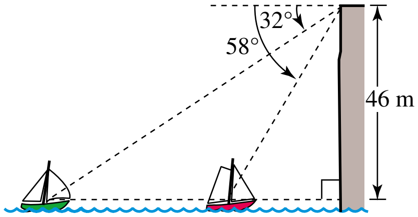
8. A lookout tower has been erected on top of a cliff. At a distance of 5.8 km from the foot of the cliff, the angle of elevation to the base of the tower is 15.7° and to the observation deck at the top of the tower is 16° respectively, as shown in the figure below. How high from the top of the cliff is the observation deck?



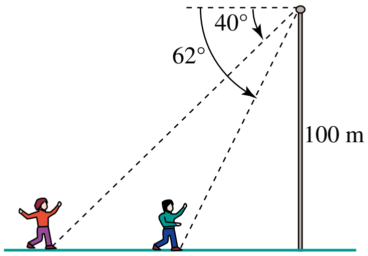
9. Elena and Sonja were on a camping trip to the Grampians, where they spent their first day hiking. They first walked 1.5 km along a path inclined at an angle of 10° to the horizontal. Then they had to follow another path, which was at an angle of 20° to the horizontal. They walked along this path for 1.3 km, which brought them to the edge of the cliff. Here Elena spotted a large gum tree 1.4 km away. If the gum tree is 150 m high, what is the angle of depression from the top of the cliff to the top of the gum tree?



10. From a point on top of a cliff, two boats are observed. If the angles of depression are 58° and 32° and the cliff is 46 m above sea level, how far apart are the boats?



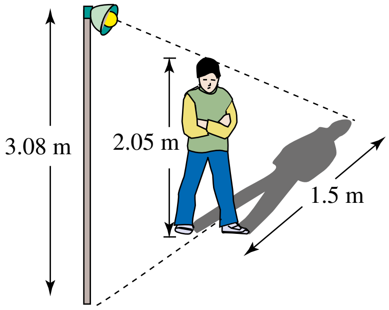
11. The competitors of a cross-country run are nearing the finish line. From a lookout 100 m above the track, the angles of depression to the two leaders, Nathan and Rachel, are 40° and 62° respectively. How far apart are the two competitors?



12. A 2.05 m tall man, standing in front of a street light 3.08 m high, casts a 1.5 m shadow.

a. What is the angle of elevation from the ground to the source of light?

b. How far is the man from the bottom of the light pole?

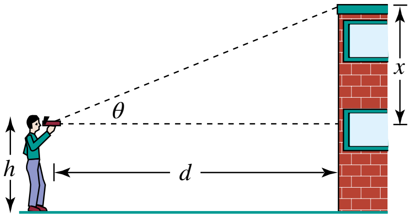


Reasoning

13. Joseph is asked to obtain an estimate of the height of his house using any mathematical technique. He decides to use an inclinometer and basic trigonometry. Using the inclinometer, Joseph determines the angle of elevation, θ, from his eye level to the top of his house to be 42°. The point from which Joseph measures the angle of elevation is 15 m away from his house and the distance from Joseph's eyes to the ground is 1.76 m.

a. Fill in the given information on the diagram provided (substitute values for the pronumerals).

b. Determine the height of Joseph's house.



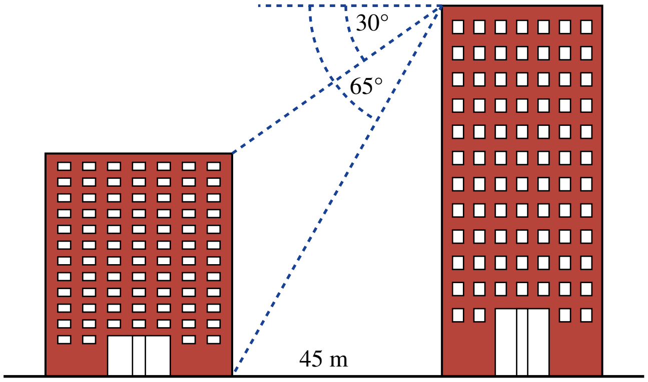
14. The angle of elevation of a vertically rising hot air balloon changes from 27° at 7.00 am to 61° at 7.03 am, according to an observer who is 300 m away from the take-off point.

a. Assuming a constant speed, calculate that speed (in m/s and km/h) at which the balloon is rising, correct to 2 decimal places.

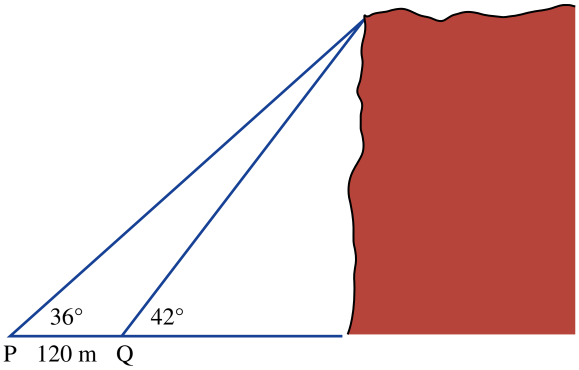
b. The balloon then falls 120 metres. What is the angle of elevation now? Write your answer correct to 1 decimal place.

Problem solving

15. The angle of depression from the top of one building to the foot of another building across the same street and 45 metres horizontally away is 65°. The angle of depression to the roof of the same building is 30°. Calculate the height of the shorter building.



16. P and Q are two points on a horizontal line that are 120 metres apart. The angles of elevation from P and Q to the top of a mountain are 36° and 42° respectively. Find the height of the mountain correct to 1 decimal place.



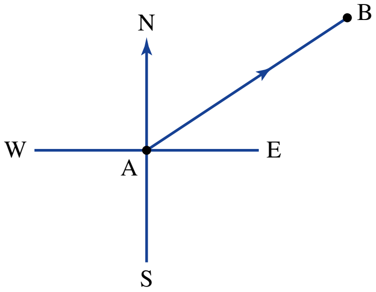
Reflection

What is the difference between an angle of elevation and an angle of depression?

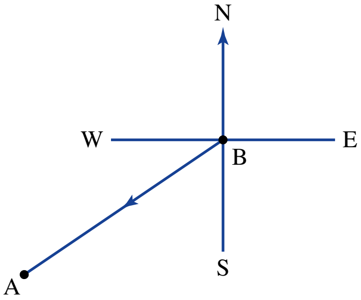
5.8 Bearings

5.8.1 Bearings

- A bearing gives the direction of travel from one point or object to another.
- The bearing of B from A tells how to get to B *from* A. A compass rose would be drawn at A.



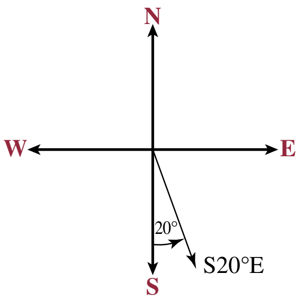
To illustrate the bearing of A *from* B, a compass rose would be drawn at B.

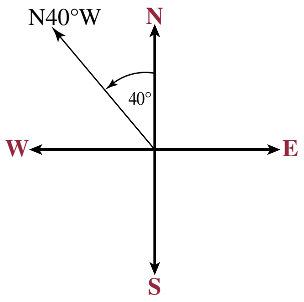


- There are two ways in which bearings are commonly written. They are compass bearings and true bearings.

5.8.2 Compass bearings

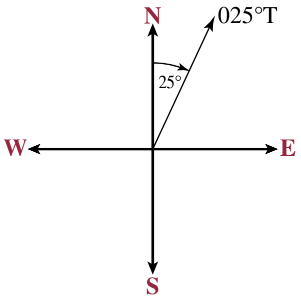
- A **compass bearing** (for example N40°E or S72°W) has three parts.
  - The first part is either N or S (for north or south).
  - The second part is an acute angle.
  - The third part is either E or W (for east or west).
- For example, the compass bearing S20°E means start by facing south and then turn 20° towards the east. This is the direction of travel.
  N40°W means start by facing north and then turn 40° towards the west.

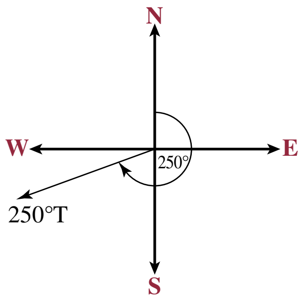




5.8.3 True bearings

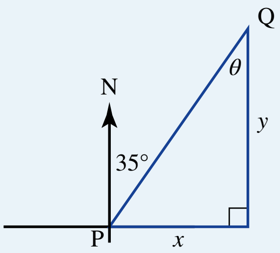
- **True bearings** are measured from north in a clockwise direction and are expressed in 3 digits.
- The diagrams below show the bearings of 025° true and 250° true respectively. (These true bearings are more commonly written as 025°T and 250°T.)

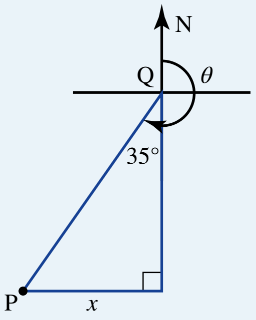




WORKED EXAMPLE 17

A boat travels a distance of 5 km from P to Q in a direction of 035°T.
a How far east of P is Q?
b How far north of P is Q?
c What is the true bearing of P from Q?

THINK	WRITE/DRAW
a 1 Draw a diagram showing the distance and bearing of Q from P. Complete a right-angled triangle travelling x km due east from P and then y km due north to Q.	**a** 
2 To determine how far Q is east of P, we need to find the value of x. We are given the length of the hypotenuse (H) and need to find the length of the opposite side (O). Write the sine ratio.	$\sin\theta = \dfrac{\text{O}}{\text{H}}$
3 Substitute O = x, H = 5 and $\theta = 35°$.	$\sin 35° = \dfrac{x}{5}$
4 Make x the subject of the equation.	$x = 5\sin 35°$
5 Evaluate and round the answer, correct to 2 decimal places.	≈ 2.87

6 Write the answer in words.	Point Q is 2.87 km east of P.
b 1 To determine how far Q is north of P, we need to find the value of y. This can be done in several ways, namely: using the cosine ratio, the tangent ratio, or Pythagoras' theorem. Write the cosine ratio.	**b** $\cos\ \theta = \frac{A}{H}$
2 Substitute $A = y$, $H = 5$ and $= 35°$.	$\cos 35° = \frac{y}{5}$
3 Make y the subject of the equation.	$y = 5 \cos 35°$
4 Evaluate and round the answer, correct to 2 decimal places.	≈ 4.10
5 Write the answer in words.	Point B is 4.10 km north of A.
c 1 To find the bearing of P from Q, draw a compass rose at Q. The true bearing is given by $\angle\theta$.	**c** 
2 The value of θ is the sum of 180° (from north to south) and 35°. Write the value of θ.	$\text{True bearing} = 180° + \alpha$ $\alpha = 35°$ $\text{True bearing} = 180° + 35°$ $= 215°$
3 Write the answer in words.	The bearing of P from Q is 215°T.

- Sometimes a journey includes a change in directions. In such cases, each section of the journey should be dealt with separately.

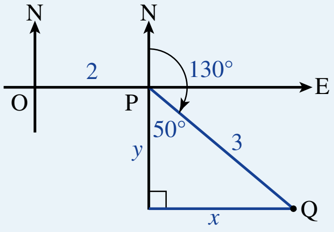
WORKED EXAMPLE 18

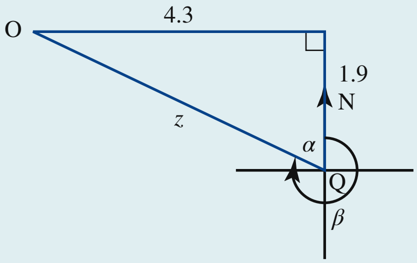
A boy walks 2 km on a true bearing of 090° and then 3 km on a true bearing of 130°.

a How far east of the starting point is the boy at the completion of his walk? (Answer correct to 1 decimal place.)

b How far south of the starting point is the boy at the completion of his walk? (Answer correct to 1 decimal place.)

c To return directly to his starting point, how far must the boy walk and on what bearing?

THINK	WRITE/DRAW
a 1 Draw a diagram of the boy's journey. The first leg of the journey is due east. Label the easterly component x and the southerly component y.	**a** 

2 Write the ratio to find x.	$\sin\theta = \dfrac{\text{O}}{\text{H}}$
3 Substitute O = x, H = 3 and $\theta = 50°$.	$\sin 50° = \dfrac{x}{3}$
4 Make x the subject of the equation.	$x = 3\sin 50°$
5 Evaluate and round correct to 1 decimal place.	≈ 2.3 km
6 Add to this the 2 km east that was walked in the first leg of the journey and give a worded answer.	Total distance east $= 2 + 2.3$ $= 4.3$ km The boy is 4.3 km east of the starting point.
b 1 To find y (see the diagram in part **a**) we can use Pythagoras' theorem, as we know the lengths of two out of three sides in the right-angled triangle. Round the answer correct to 1 decimal place. *Note:* Alternatively, the cosine ratio could have been used.	**b** Distance south $= y$ km $a^2 = c^2 - b^2$ $y^2 = 3^2 - 2.3^2$ $= 9 - 5.29$ $= 3.71$ $y = \sqrt{3.71}$ $= 1.9$ km
2 Write the answer in words.	The boy is 1.9 km south of the starting point.
c 1 Draw a diagram of the journey and write in the results found in parts **a** and **b**. Draw a compass rose at Q.	**c** 
2 Find z using Pythagoras' theorem.	$z^2 = 1.9^2 + 4.3^2$ $= 22.1$ $z = \sqrt{22.1}$ ≈ 4.70
3 Find α using trigonometry.	$\tan\alpha = \dfrac{4.3}{1.9}$
4 Make α the subject of the equation using the inverse tangent function.	$\alpha = \tan^{-1}\left(\dfrac{4.3}{1.9}\right)$
5 Evaluate and round to the nearest minute.	$= 66.161259\,82°$ $= 66°9'40.535''$ $= 66°10'$
6 The angle β gives the bearing.	$\beta = 360° - 66°10'$ $= 293°50'$
7 Write the answer in words.	The boy travels 4.70 km on a bearing of 293°50 T.

Exercise 5.8 Bearings

Individual pathways

🟩 **PRACTISE**
Questions:
1, 2, 3a–d, 4a–b, 5–7, 11

🟦 **CONSOLIDATE**
Questions:
1, 2, 3, 4a–c, 5–8, 11, 13

🟪 **MASTER**
Questions:
1–6, 8–14

🟩🟦🟪 **Individual pathway interactivity:** int-4591

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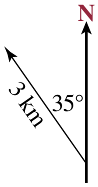
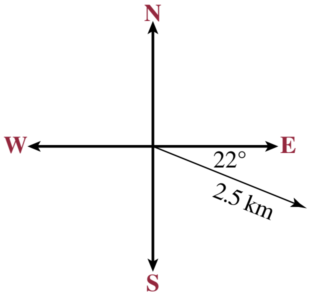
Fluency

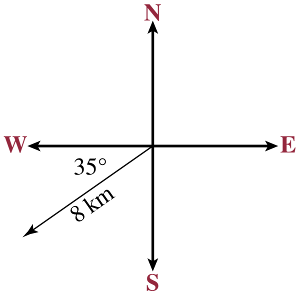
1. Change each of the following compass bearings to true bearings.
a. N20°E **b.** N20°W **c.** S35°W
d. S28°E **e.** N34°E **f.** S42°W

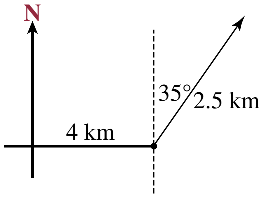
2. Change each of the following true bearings to compass bearings.
a. 049°T **b.** 132°T **c.** 267°T
d. 330°T **e.** 086°T **f.** 234°T

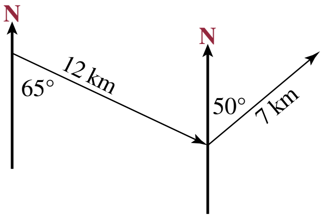
3. Describe the following paths using true bearings.

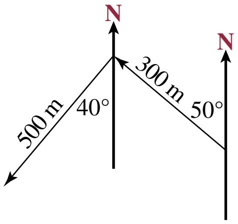
a.


b.


c.


d.


e.


f.


4. Show each of the following journeys as a diagram.

a. A ship travels 040°T for 40 km and then 100°T for 30 km.

b. A plane flies for 230 km in a direction 135°T and a further 140 km in a direction 240°T.

c. A bushwalker travels in a direction 260°T for 0.8 km, then changes direction to 120°T for 1.3 km, and finally travels in a direction of 32° for 2.1 km.

d. A boat travels N40°W for 8 km, then changes direction to S30°W for 5 km and then S50°E for 7 km.

e. A plane travels N20°E for 320 km, N70°E for 180 km and S30°E for 220 km.

5. **WE17** **a.** A yacht travels 20 km from A to B on a bearing of 042°T:

i. how far east of A is B?

ii. how far north of A is B?

iii. what is the bearing of A from B?

b. The yacht then sails 80 km from B to C on a bearing of 130°T.

i. Show the journey using a diagram.

ii. How far south of B is C?

iii. How far east of B is C?

iv. What is the bearing of B from C?

6. If a farmhouse is situated 220 m N35°E from a shed, what is the true bearing of the shed from the house?

Understanding

7. A pair of hikers travel 0.7 km on a true bearing of 240° and then 1.3 km on a true bearing of 300°. How far west have they travelled from their starting point?

8. **WE18** A boat travels 6 km on a true bearing of 120° and then 4 km on a true bearing of 080°.

a. How far east is the boat from the starting point on the completion of its journey?

b. How far south is the boat from the starting point on the completion of its journey?

c. What is the bearing of the boat from the starting point on the completion of its journey?

9. A plane flies on a true bearing of 320° for 450 km. It then flies on a true bearing of 350° for 130 km and finally on a true bearing of 050° for 330 km. How far north of its starting point is the plane?

Reasoning

10. A bushwalker leaves her tent and walks due east for 4.12 km, then walks a further 3.31 km on a bearing of N20°E. If she wishes to return directly to her tent, how far must she walk and what bearing should she take? (Answer to the nearest degree.)

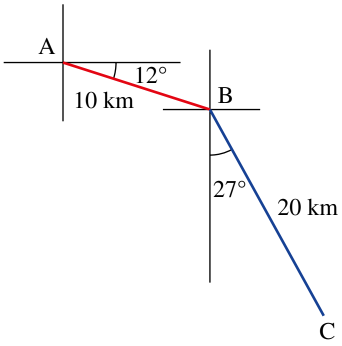
11. A car travels due south for 3 km and then due east for 8 km. What is the bearing of the car from its starting point? (Answer to the nearest degree.)

12. If the bearing of A from O is θ°T, then what is the bearing of O from A:

a. if $0° < \theta° < 180°$ **b.** if $180° < \theta° < 360°$?

Problem solving

13. A boat sails on a compass direction of E12°S for 10 km then changes direction to S27°E for another 20 km. The boat then decides to return to its starting point.



a. How far, correct to 2 decimal places, is the boat from its starting point?

b. On what bearing should the boat travel to return to its starting point? Write the angle correct to the nearest degree.

14. Samantha and Tim set off early from the car park of a national park to hike for the day. Initially they walk N60°E for 12 km to see a spectacular waterfall. They then change direction and walk in a south-easterly direction for 6 km, then stop for lunch. Give all answers correct to 2 decimal places.

a. Make a scale diagram of the hiking path they completed.

b. How far north of the car park are they at the lunch stop?

c. How far east of the car park are they at the lunch stop?

d. What is the bearing of the lunch stop from the car park?

Samantha and Tim then walk directly back to the car park.

e. Calculate the distance they have covered after lunch.

Reflection

What is the difference between true bearings and compass directions?

CHALLENGE 5.2

Starting from their base in the national park, a group of bushwalkers travel 1.5 km at a true bearing of 030°, then 3.5 km at a true bearing of 160°, and then 6.25 km at a true bearing of 300°. How far, and at what true bearing, should the group walk to return to its base?

5.9 Applications

5.9.1 Applications of trigonometry

- When applying trigonometry to practical situations, it is essential to draw good mathematical diagrams using points, lines and angles.
- Several diagrams may be required to show all the necessary right-angled triangles.

WORKED EXAMPLE 19

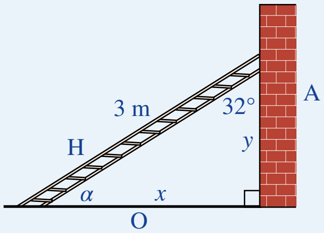
TI | CASIO

A ladder of length 3 m makes an angle of 32° with the wall.

a How far is the foot of the ladder from the wall?

b How far up the wall does the ladder reach?

c What angle does the ladder make with the ground?

THINK	**WRITE/DRAW**
Sketch a diagram and label the sides of the right-angled triangle with respect to the given angle.	
a 1 We need to find the distance of the foot of the ladder from the wall (O) and are given the length of the ladder (H). Write the sine ratio.	**a** $\sin\theta = \frac{O}{H}$
2 Substitute O = x, H = 3 and $\theta = 32°$.	$\sin 32° = \frac{x}{3}$
3 Make x the subject of the equation.	$x = 3\sin 32°$
4 Evaluate and round the answer to 2 decimal places.	≈ 1.59 m
5 Write the answer in words.	The foot of the ladder is 1.59 m from the wall.
b 1 We need to find the height the ladder reaches up the wall (A) and are given the hypotenuse (H). Write the cosine ratio.	**b** $\cos\theta = \frac{A}{H}$
2 Substitute A = y, H = 3 and $\theta = 32°$.	$\cos 32° = \frac{y}{3}$
3 Make y the subject of the equation.	$y = 3\cos\ 32°$
4 Evaluate and round the answer to 2 decimal places.	$y \approx 2.54$ m
5 Write the answer in words.	The ladder reaches 2.54 m up the wall.
c 1 To find the angle that the ladder makes with the ground, we could use any of the trigonometric ratios, as the lengths of all three sides are known. However, it is quicker to use the angle sum of a triangle.	**c** $\alpha + 90° + 32° = 180°$ $\alpha + 122° = 180°$ $\alpha = 180° - 122°$ $\alpha = 58°$
2 Write the answer in words.	The ladder makes a 58° angle with the ground.

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Exercise 5.9 Applications

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 8, 10, 15	Questions: 1–5, 8, 11, 13, 14, 16	Questions: 1, 3, 4, 6, 7, 9, 12–17

🟩🟦🟪 **Individual pathway interactivity:** int-4592 **learnON** ONLINE ONLY

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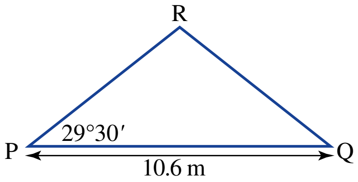
Fluency

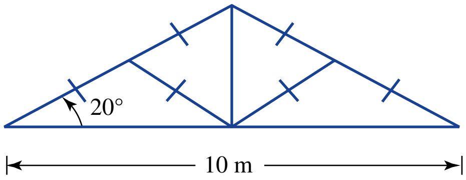
1. A carpenter wants to make a roof pitched at 29°30′, as shown in the diagram. How long should he cut the beam, PR?



2. The mast of a boat is 7.7 m high. A guy wire from the top of the mast is fixed to the deck 4 m from the base of the mast. Determine the angle the wire makes with the horizontal.

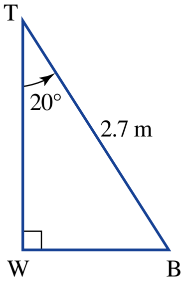
Understanding

3. A steel roof truss is to be made to the following design.



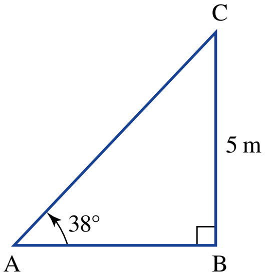
a. How high is the truss?
b. What is the total length of steel required to make the truss?

4. **WE19** A ladder that is 2.7 m long is leaning against a wall at an angle of 20° as shown.



If the base of the ladder is moved 50 cm further away from the wall, what angle will the ladder make with the wall?

5. A wooden framework is built as shown.



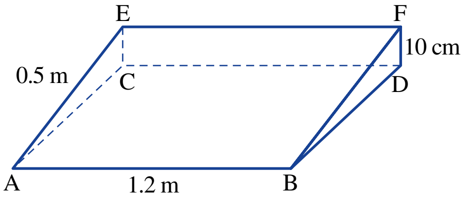
Bella plans to reinforce the framework by adding a strut from C to the midpoint of AB. What will be the length of the strut?

6. Atlanta is standing due south of a 20 m flagpole at a point where the angle of elevation of the top of the pole is 35°. Ginger is standing due east of the flagpole at a point where the angle of elevation of the top of the pole is 27°. How far is Ginger from Atlanta?

7. From a point at ground level, Henry measures the angle of elevation of the top of a tall building to be 41°. After walking directly towards the building, he finds the angle of elevation to be 75°. If the building is 220 m tall, how far did Henry walk between measurements?

8. Sailing towards a mountain peak of height 893 m, Imogen measured the angle of elevation to be 14°. A short time later the angle of elevation was 27°. How far had Imogen sailed in that time?

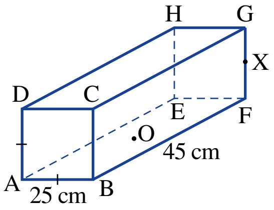
9. A desk top of length 1.2 m and width 0.5 m rises to 10 cm.



Calculate:

a. ∠DBF **b.** ∠CBE.

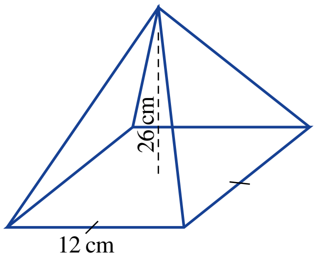
10. A cuboid has a square end.



a. If the length of the cuboid is 45 cm and its height and width are 25 cm each, calculate:
  i. the length of BD **ii.** the length of BG **iii.** the length of BE
  iv. the length of BH **v.** ∠FBG **vi.** ∠EBH.

b. If the midpoint of FG is X and the centre of the rectangle ABFE is O calculate:
  i. the length OF **ii.** the length FX
  iii. ∠FOX **iv.** the length OX.

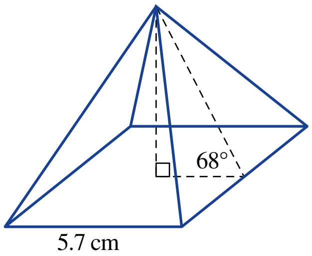
11. In a right square-based pyramid, the length of the side of the base is 12 cm and the height is 26 cm.



Determine:

a. the angle the triangular face makes with the base
b. the angle the sloping edge makes with the base
c. the length of the sloping edge.

12. In a right square-based pyramid, the length of the side of the square base is 5.7 cm.



If the angle between the triangular face and the base is 68°, determine:

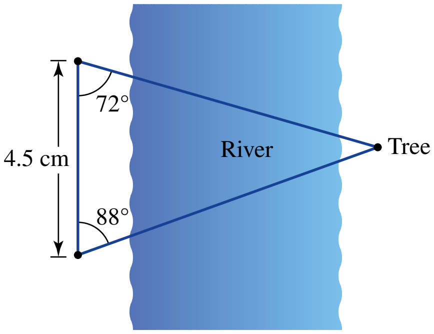
a. the height of the pyramid
b. the angle the sloping edge makes with the base
c. the length of the sloping edge.

13. In a right square-based pyramid, the height is 47 cm. If the angle between a triangular face and the base is 73°, calculate:

a. the length of the side of the square base
b. the length of the diagonal of the base
c. the angle the sloping edge makes with the base.

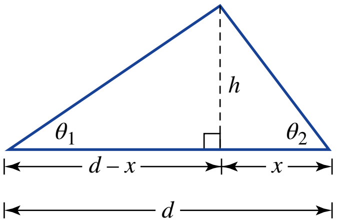
Reasoning

14. Aldo the carpenter is lost in a rainforest. He comes across a large river and he knows that he can not swim across it. Aldo intends to build a bridge across the river. He draws some plans to calculate the distance across the river as shown in the diagram below.



a. Aldo used a scale of 1 cm to represent 20 m. Find the real-life distance represented by 4.5 cm in Aldo's plans.

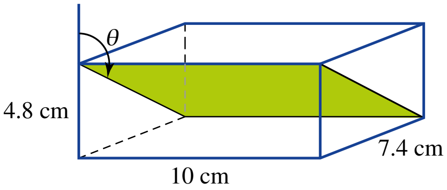
b. Use the diagram below to write an equation for h in terms of d and the two angles.



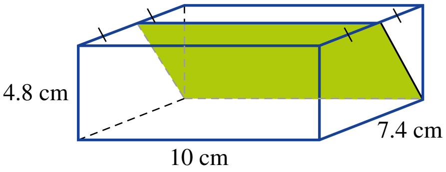
c. Use your equation from **b** to find the distance across the river, correct to the nearest metre.

15. A block of cheese is in the shape of a rectangular prism as shown. The cheese is to be sliced with a wide blade that can slice it in one go. Calculate the angle (to the vertical) that the blade must be inclined if:

a. the block is to be sliced diagonally into two identical triangular wedges

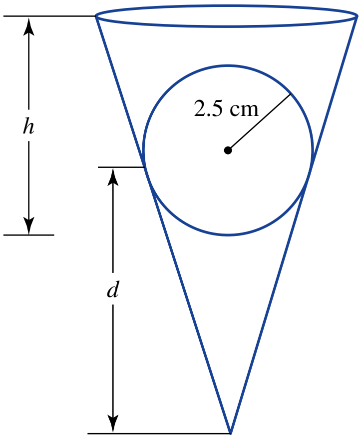


b. the blade is to be placed in the middle of the block and sliced through to the bottom corner, as shown.



Problem solving

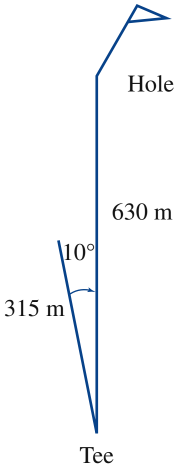
16. A sphere of radius length 2.5 cm rests in a hollow inverted cone as shown. The height of the cone is 12.5 cm and its vertical angle is equal to 36°.



a. Find the distance, d, from the tip of the cone to the point of contact with the sphere.

b. Find the distance, h, from the open end of the cone to the bottom of the ball.

17. The ninth hole on a municipal golf course is 630 m from the tee. A golfer drives a ball from the tee a distance of 315 m at a 10° angle off the direct line as shown.



Find how far the ball is from the hole and state the angle of the direct line that the ball must be hit along to go directly to the hole. Give your answers correct to 1 decimal place.

Reflection

What are some real-life applications of trigonometry?

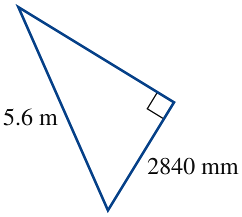
5.10 Review

5.10.1 Review questions

Fluency

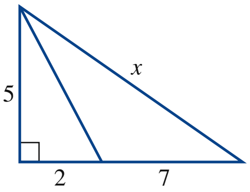
1. **MC** The most accurate measure for the length of the third side in the triangle below is:

A. 4.83 m **B.** 23.3 cm **C.** 3.94 m **D.** 2330 mm **E.** 4826 mm



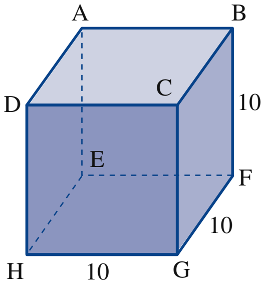
2. **MC** What is the value of x in this figure?

A. 5.4 **B.** 7.5 **C.** 10.1 **D.** 10.3 **E.** 4



3. **MC** What is the closest length of AG of the cube below?

A. 10 **B.** 30 **C.** 20 **D.** 14 **E.** 17



4. **MC** If sin 38° = 0.6157, which of the following will also give this result?

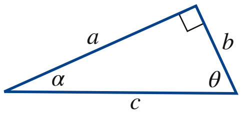
A. sin 218° **B.** sin 322° **C.** sin 578° **D.** sin 682° **E.** sin 142°

5. **MC** The angle 118°52′34″ is also equal to:

A. 118.5234° **B.** $118\frac{52}{34}°$ **C.** 118.861° **D.** 118.876° **E.** 118.786°

6. **MC** Which trigonometric ratio for the triangle shown below is incorrect?

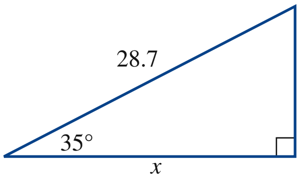
A. $\sin\alpha = \frac{b}{c}$ **B.** $\sin\alpha = \frac{a}{c}$ **C.** $\cos\alpha = \frac{a}{c}$ **D.** $\tan\alpha = \frac{b}{a}$ **E.** $\tan\theta = \frac{a}{b}$



7. **MC** Which of the following statements is correct?

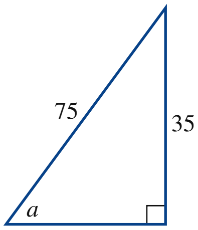
A. sin 55° = cos 55° **B.** sin 45° = cos 35° **C.** cos 15° = sin 85°
D. sin 30° = sin 60° **E.** sin 42° = cos 48°

8. **MC** Which of the following can be used to find the value of x in the diagram below?



A. 28.7 sin 35° **B.** 28.7 cos 35° **C.** 28.7 tan 35° **D.** $\frac{28.7}{\sin 35°}$ **E.** $\frac{28.7}{\cos 35°}$

9. **MC** Which of the following expressions can be used to find the value of a in the triangle shown?



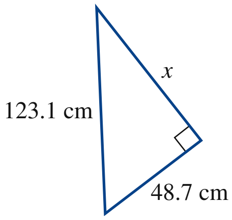
A. 35 sin 75° **B.** $\sin^{-1}\frac{35}{75}$ **C.** $\sin^{-1}\frac{75}{35}$ **D.** $\cos^{-1}\frac{35}{75}$

10. **MC** If a school is 320 m S42°W from the police station, what is the true bearing of the police station from the school?

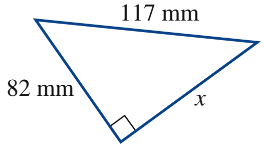
A. 042°T **B.** 048°T **C.** 222°T **D.** 228°T **E.** 312°T

11. Calculate x, correct to 2 decimal places.

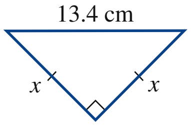
a.



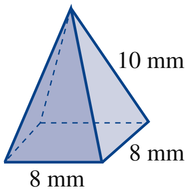
b.



12. Calculate the value of the pronumeral, correct to 2 decimal places.



13. Calculate the height of this pyramid.



14. A person standing 23 m away from a tree observes the top of the tree at an angle of elevation of 35°. If the person's eye level is 1.5 m from the ground, what is the height of the tree?

15. A man with an eye level height of 1.8 m stands at the window of a tall building. He observes his young daughter in the playground below. If the angle of depression from the man to the girl is 47° and the floor on which the man stands is 27 m above the ground, how far from the bottom of the building is the child?

16. A plane flies 780 km in a direction of 185°T. How far west has it travelled from the starting point?

17. A hiker travels 3.2 km on a bearing of 250°T and then 1.8 km on a bearing of 320°T. How far west has she travelled from the starting point?

18. If a 4 m ladder is placed against a wall and the foot of the ladder is 2.6 m from the wall, what angle (in degrees and minutes) does the ladder make with the wall?

Problem solving

19. The height of a right square-based pyramid is 13 cm. If the angle the face makes with the base is 67°, find:

a. the length of the edge of the square base

b. the length of the diagonal of the base

c. the angle the slanted edge makes with the base in degrees and minutes.

20. A car is travelling northwards on an elevated expressway 6 m above ground at a speed of 72 km/h. At noon another car passes under the expressway, at ground level, travelling west, at a speed of 90 km/h.

a. How far apart, in metres, are the two cars 40 seconds after noon?

b. At this time the first car stops, while the second car keeps going. At what time will they be 3.5 km apart? Write your answer correct to the nearest tenth of a second.

21. Two towers face each other separated by a distance, d, of 20 metres. As seen from the top of the first tower, the angle of depression of the second tower's base is 59° and that of the top is 31°. What is the height, in metres correct to 2 decimal places, of each of the towers?

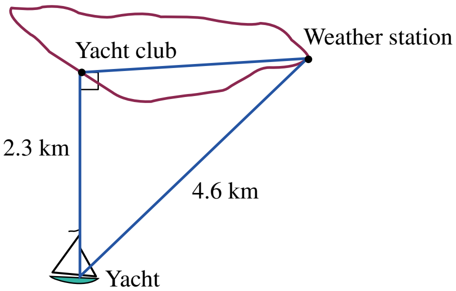
22. A piece of flat pastry is cut in the shape of a right-angled triangle. The longest side is $6b$ cm and the shortest is $2b$ cm.

a. Find the length of the third side. Give your answer in exact form.

b. Find the sizes of the angles in the triangle.

c. Prove that the area of the triangle is equal to $4\sqrt{2}b^2\,\text{cm}^2$.

23. A yacht is anchored off an island. It is 2.3 km from the yacht club and 4.6 km from a weather station. The three points form a right angled triangle at the yacht club.



a. Calculate the angle at the yacht between the yacht club and the weather station.
b. Calculate the distance between the yacht club and the weather station.
The next day the yacht travels directly towards the yacht club, but is prevented from reaching the club because of dense fog. The weather station notifies the yacht that it is now 4.2 km from the station.
c. Calculate the new angle at the yacht between the yacht club and the weather station.
d. Determine how far the yacht is now from the yacht club.

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- **Try out this interactivity:** Word search: Topic 5 (int-2838)
- **Try out this interactivity:** Crossword: Topic 5 (int-2839)
- **Try out this interactivity:** Sudoku: Topic 5 (int-3592)
- **Complete this digital doc:** Concept map: Topic 5 (doc-13720)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

adjacent
angle of depression
angle of elevation
bearing
compass rose
cosine
cuboid
degree
dimensions
equilateral
horizontal
hypotenuse
inverse
isosceles
minute
opposite
pyramid
Pythagoras' theorem
ratio
second
sine
tangent
true bearing
wedge

Investigation | Rich task

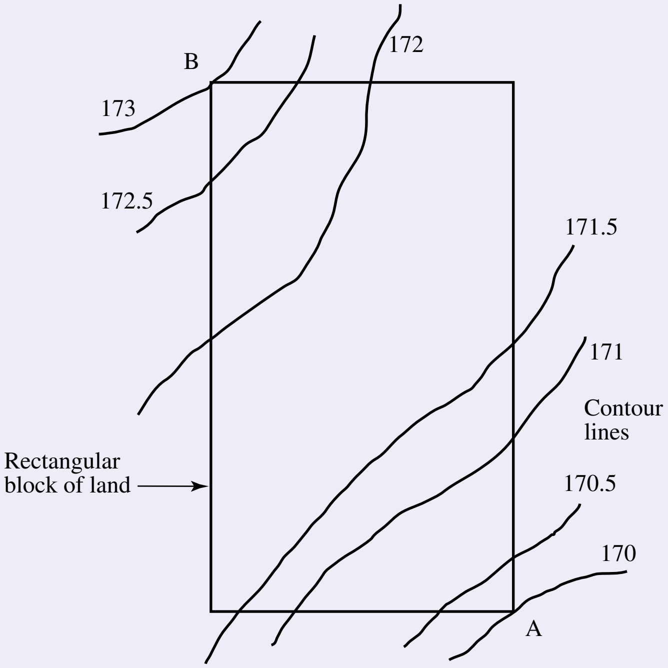
How steep is the land?

When buying a block of land on which to build a house, the slope of the land is often not very obvious. The slab of a house built on the ground must be level, so it is frequently necessary to remove or build up soil to obtain a flat area. The gradient of the land can be determined from a contour map of the area.

Consider the building block below right. The contour lines join points having the same height above sea level. Their measurements are in metres. The plan clearly shows that the land rises from A to B. The task is to determine the angle of this slope.

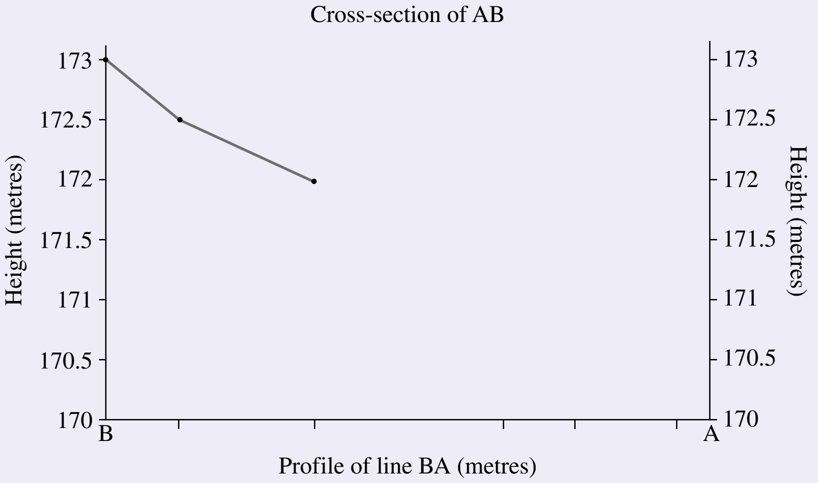
1. A cross-section shows a profile of the surface of the ground. Let us look at the cross-section of the ground between A and B. The technique used is as follows.

- Place the edge of a piece of paper on the line joining A and B.
- Mark the edge of the paper at the points where the contour lines intersect the paper.
- Transfer this paper edge to the horizontal scale of the profile and mark these points.
- Choose a vertical scale within the range of the heights of the contour lines.
- Plot the height at each point where a contour line crosses the paper.
- Join the points with a smooth curve.

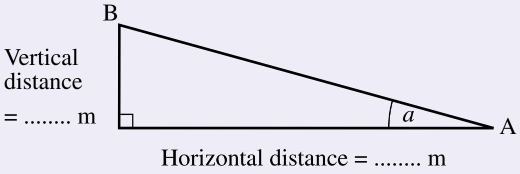


Scale 1 : 500

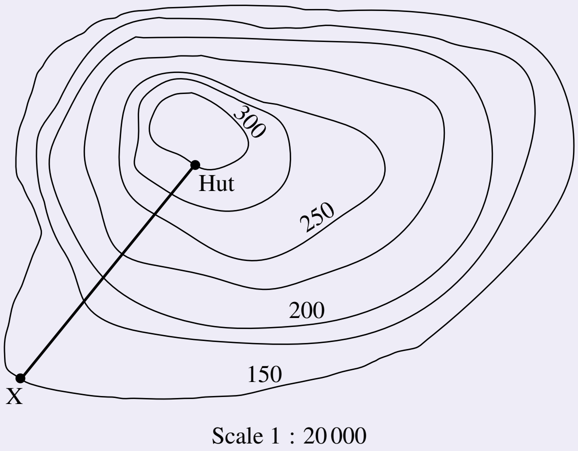
The cross-section has been started for you. Complete the profile of the line AB. You can now see a visual picture of the profile of the soil between A and B.



2. We now need to determine the horizontal distance between A and B.
   a. Measure the map distance between A and B using a ruler. What is the map length?
   b. Using the scale of 1:500, calculate the actual horizontal distance AB (in metres).
3. The vertical difference in height between A and B is indicated by the contour lines. What is this vertical distance?
4. Complete the measurements on this diagram.



5. The angle *a* represents the angle of the average slope of the land from A to B. Use the tangent ratio to calculate this angle (to the nearest minute).
6. In general terms, an angle less than 5° can be considered a gradual to moderate rise. An angle between 5° and 15° is regarded as moderate to steep while more than 15° is a steep rise. How would you describe this block of land?
7. Imagine that you are going on a bush walk this weekend with a group of friends. At right is a contour map of the area. Starting at *X*, the plan is to walk directly to the hut. Draw a cross-section profile of the walk and calculate the average slope of the land. How would you describe the walk?



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Complete this digital doc: Code puzzle: What will Sir have to follow the chicken? (doc-15925)

Answers

TOPIC 5 Trigonometry I

Exercise 5.2 Pythagoras' theorem

1. a. 7.86 **b.** 33.27 **c.** 980.95 **d.** 12.68 **e.** 2.85 **f.** 175.14

2. a. 36.36 **b.** 1.62 **c.** 15.37 **d.** 0.61 **e.** 2133.19 **f.** 453.90

3. 23.04 cm

4. 12.65 cm

5. a. 14.14 cm **b.** 24.04 cm **c.** 4.53 cm

6. a. 74.83 cm **b.** 249.67 cm **c.** 3741.66 cm^2

7. a. 6.06 **b.** 4.24 **c.** 4.74

8. 14.84 cm

9. 15.59 cm

10. 19.23 cm

11. 72.75 cm; 3055.34 cm^2

12. 39 m

13. 4.34 km

14. 38.2 m

15. 63.06 mm

16. a. 32 cm **b.** 768 cm^2

17. 26.83 diagonals, so would need to complete 27

18. 4701.06 m

19. 9.90 cm

20. a. 65 **b.** 185 **c.** 305

21. a. Neither 105 nor 208 can be the hypotenuse of the triangle, because they are the two smallest values. The other two values could be the hypotenuse if they enable the creation of a right-angled triangle.

b. 105, 208, 233

22. a. 21 cm **b.** 35 cm **c.** $y = 12.6$ cm and RS = 9.8 cm

23. 13.86 cm

Exercise 5.3 Pythagoras' theorem in three dimensions

1. a. 13.86 **b.** 13.93 **c.** 18.03

2. 12.21, 12.85

3. 4.84 m, 1.77 m

4. 8.49, 4.24

5. 31.62 cm

6. 10.58 cm

7. 23 mm

8. a. i. 233.24 m **ii.** 200.12 m **iii.** 120.20 m

b. 116.83 m

9. 14.72 cm

10. 12.67 cm

11. 42.27 cm

12. 1.49 m, 7.43 m^2

13. Students' own working

14. 186.5 m

15. 25.475

16. 28.6 m

Exercise 5.4 Trigonometric ratios

1. a. 0.5000 **b.** 0.7071 **c.** 0.4663 **d.** 0.8387 **e.** 8.1443 **f.** 0.7193

2. a. 0.6494 **b.** 0.5885 **c.** 0.5220 **d.** −1.5013 **e.** 0.9990 **f.** 0.6709
g. 0.8120 **h.** 0.5253 **i.** −0.8031 **j.** 0.4063 **k.** 0.9880 **l.** −0.9613
m. 1.7321 **n.** −0.5736 **o.** 0.1320

3. a. 50° **b.** 24° **c.** 53° **d.** 71° **e.** 86° **f.** 41°

4. a. 54°29′ **b.** 6°19′ **c.** 0°52′ **d.** 72°47′ **e.** 44°48′ **f.** 26°45′

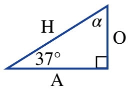
5. a. 26°33′54″ **b.** 64°1′25″ **c.** 64°46′59″ **d.** 48°5′22″ **e.** 36°52′12″ **f.** 88°41′27″

6. a. 2.824 **b.** 71.014 **c.** 20.361 **d.** 2.828 **e.** 226.735 **f.** 1.192
g. 7.232 **h.** 32.259 **i.** 4909.913 **j.** 0.063 **k.** 0.904 **l.** 14.814

7. a. i. $\sin(\theta) = \frac{e}{f}$ **ii.** $\cos(\theta) = \frac{d}{f}$ **iii.** $\tan(\theta) = \frac{e}{d}$
b. i. $\sin(\alpha) = \frac{i}{g}$ **ii.** $\cos(\alpha) = \frac{h}{g}$ **iii.** $\tan(\alpha) = \frac{i}{h}$
c. i. $\sin(\beta) = \frac{l}{k}$ **ii.** $\cos(\beta) = \frac{j}{k}$ **iii.** $\tan(\beta) = \frac{l}{j}$
d. i. $\sin(\gamma) = \frac{n}{m}$ **ii.** $\cos(\gamma) = \frac{o}{m}$ **iii.** $\tan(\gamma) = \frac{n}{o}$
e. i. $\sin(\beta) = \frac{b}{c}$ **ii.** $\cos(\beta) = \frac{a}{c}$ **iii.** $\tan(\beta) = \frac{b}{a}$
f. i. $\sin(\gamma) = \frac{v}{u}$ **ii.** $\cos(\gamma) = \frac{t}{u}$ **iii.** $\tan(\gamma) = \frac{v}{t}$

8. a. $\sin(\theta) = \frac{15}{18}$ **b.** $\cos(\theta) = \frac{22}{30}$ **c.** $\tan(\theta) = \frac{7}{9}$ **d.** $\tan(\theta) = \frac{3.6}{p}$ **e.** $\sin(25°) = \frac{13}{t}$ **f.** $\sin(\alpha) = \frac{18.6}{23.5}$

9. a.



b. i. $\sin(37°) = 0.60$ **ii.** $\cos(37°) = 0.75$ **iii.** $\tan(37°) = 0.80$
c. $\alpha = 53°$
d. i. $\sin(53°) = 0.80$ **ii.** $\cos(53°) = 0.60$ **iii.** $\tan(53°) = 1.33$
e. They are equal.
f. They are equal.
g. The sin of an angle is equal to the cos of its complement angle.

10. $\sin(\theta) = \frac{\text{opp}}{\text{hyp}}, \quad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{opp}}{\text{adj}} = \tan(\theta)$

11. a. $h^2 = a^2 - x^2$ **b.** $h^2 = c^2 - b^2 + 2bx - x^2$ **c.** Teacher to check **d.** Teacher to check

12. $\text{DC} = x + \dfrac{y}{\tan(\theta)}$

Exercise 5.5 Using trigonometry to calculate side lengths

1. a. 8.660 **b.** 7.250 **c.** 8.412

2. a. 0.79 **b.** 4.72 **c.** 101.38

3. a. 33.45 m **b.** 74.89 m **c.** 44.82 m **d.** 7.76 mm **e.** 80.82 km **f.** 9.04 cm

4. a. $x = 31.58$ cm **b.** $y = 17.67$ m **c.** $z = 14.87$ m
d. $p = 67.00$ m **e.** $p = 21.38$ km, $q = 42.29$ km **f.** $a = 0.70$ km, $b = 0.21$ km

5. a. 6.0 m **b.** 6.7 m

6. 1.05 m

7. a. $x = 30.91$ cm, $y = 29.86$ cm, $z = 39.30$ cm **b.** 2941.54 cm^2

8. a. In an isosceles right-angled triangle **b.** $\theta < 45°$

9. a. $h = \tan(47°48')x$ m
$h = \tan(36°24')\,(x + 64)$ m
b. 129.07 m **c.** 144.20 m

10. 60

Exercise 5.6 Using trigonometry to calculate angle size

1. **a.** 67° **b.** 47° **c.** 69°

2. **a.** 54°47′ **b.** 33°45′ **c.** 33°33′

3. **a.** 75°31′21″ **b.** 36°52′12″ **c.** 37°38′51″

4. **a.** 41° **b.** 30° **c.** 49° **d.** 65° **e.** 48° **f.** 37°

5. **a.** $a = 25°47′, b = 64°13′$ **b.** $d = 25°23′,\ e = 64°37′$ **c.** $x = 66°12′,\ y = 23°48′$

6. **a.** $r = 57.58,\ l = 34.87,\ h = 28.56$

b. 428 cm^2 **c.** 29.7°

7. **a.** **i.** 29.0° **ii.** 41.4° **iii.** 51.3°

b. **i.** 124.42 km/h **ii.** 136.57 km/h **iii.** 146.27 km/h

8. Answers will vary.

9. **a.** $\sin 30° = \frac{1}{2}, \cos 30° = \frac{\sqrt{3}}{2}, \tan 30° = \frac{\sqrt{3}}{3}$

b. Answers will vary. **c.** Answers will vary.

10. **a.** Between site 3 and site 2: 61°

Between site 2 and site 1: 18°

Between site 1 and bottom: 75°

b. Between site 1 and bottom: 75° slope

11. 31°57′

Challenge 5.1

147°0′; 12°15′

Exercise 5.7 Angles of elevation and depression

1. 8.74 m

2. 687.7 m

3. **a.** 176.42 m **b.** 152.42 m

4. 65°46′

5. 16.04 m

6. **a.** $h = x \tan(47°12′)$ m; $h = (x + 38) \tan(35°50′)$ m

b. $x = 76.69$ m **c.** 84.62 m

7. **a.** $h = x \tan(43°35′)$ m; $h = (x + 75) \tan(32°18′)$ m

b. 148.37 m **c.** 143.10 m

8. 0.033 km or 33 m

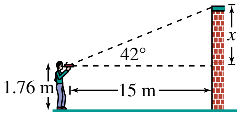
9. 21°

10. 44.88 m

11. 66 m

12. **a.** 54° **b.** 0.75 m

13. **a.**



b. 15.27 m

14. **a.** 2.16 m/s, 7.77 km/h **b.** 54.5°

15. 70.522 m

16. 451.5 m

Exercise 5.8 Bearings

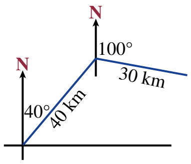
1. a. 020°T **b.** 340°T **c.** 215°T **d.** 152°T **e.** 034°T **f.** 222°T

2. a. N49°E **b.** S48°E **c.** S87°W **d.** N30°W **e.** N86°E **f.** S54°W

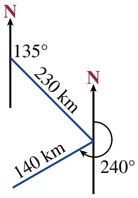
3. a. 3 km 325°T **b.** 2.5 km 112°T **c.** 8 km 235°T

d. 4 km 090°T, then 2.5 km 035°T **e.** 12 km115°T, then 7 km 050°T **f.** 300 m 310°T, then 500 m 220°T

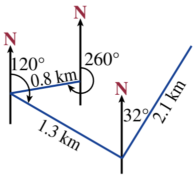
4. a.



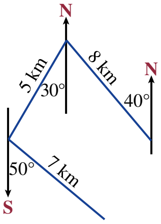
b.



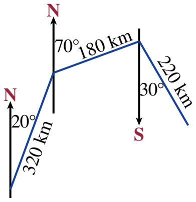
c.



d.

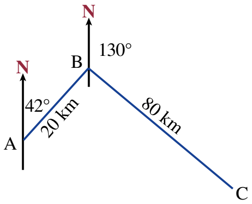


e.



5. a. i. 13.38 km **ii.** 14.86 km **iii.** 222°T

b. i



ii. 51.42 km **iii.** 61.28 km **iv.** 310°T

6. 215°T

7. 1.732 km

8. a. 9.135 km **b.** 2.305 km **c.** 104°10′T

9. 684.86 km

10. 6.10 km and 239°T

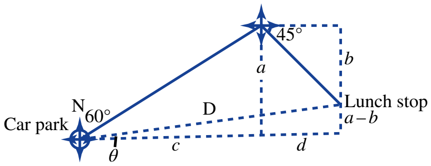
11. 111°T

12. a. $(180 - \theta)$°T **b.** $(\theta - 180)$°T

13. a. 27.42 km

b. N43°W or 317°T

14. a.



b. 1.76 km North **c.** 14.63 km East **d.** N83.15°E **e.** D = 14.74 km

Challenge 5.2

3.65 km on a bearing of 108°T

Exercise 5.9 Applications

1. 6.09 m

2. 62°33′

3. a. 1.82 m **b.** 27.78 m

4. 31°49′

5. 5.94 m

6. 49 m

7. 194 m

8. 1.829 km

9. a. 11°32′ **b.** 4°25′

10. a. i. 35.36 cm **ii.** 51.48 cm **iii.** 51.48 cm
iv. 57.23 cm **v.** 29°3′ **vi.** 25°54′

b. i. 25.74 cm **ii.** 12.5 cm **iii.** 25°54′ **iv.** 28.61 cm

11. a. 77° **b.** 71°56′ **c.** 27.35 cm

12. a. 7.05 cm **b.** 60°15′ **c.** 8.12 cm

13. a. 28.74 cm **b.** 40.64 cm **c.** 66°37′

14. a. 90 m **b.** $h = \dfrac{d\tan\theta_1}{\tan\theta_1 + \tan\theta_2} \times \tan\theta_2$
c. 250 m

15. a. 122.97° **b.** 142.37°

16. a. 8.09 cm **b.** 6.91 cm

17. Golfer must hit the ball 324.4 m at an angle of 9.7° off the direct line.

5.10 Review

1. e

2. d

3. e

4. e

5. d

6. b

7. e

8. b

9. b

10. a

11. a. $x = 113.06$ cm **b.** $x = 83.46$ mm

12. 9.48 cm

13. 8.25 mm

14. 17.6 m

15. 26.86 m

16. 67.98 km

17. 4.16 km

18. 40°32′

19. a. 11.04 cm **b.** 15.61 cm **c.** 59°1′

20. a. 1280.64 m **b.** 12:02:16.3 pm

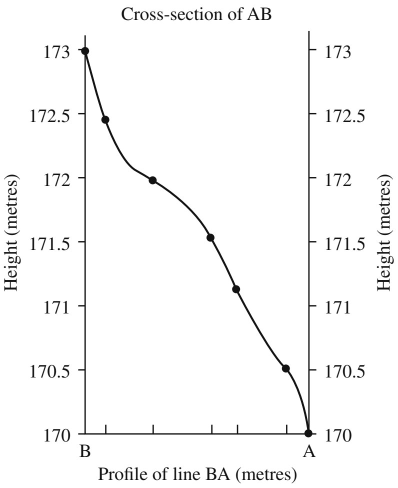
21. 33.29 m, 21.27 m

22. a. $4\sqrt{2}b$ **b.** 19.5°, 70.5°, 90°. **c.**
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\,\text{base} \times \text{height}\\ &= \tfrac{1}{2} \times 2b \times 4\sqrt{2}b\\ &= 4\sqrt{2}b^2\,\text{cm}^2.\end{aligned}$$

23. a. 60° **b.** 3.98 km **c.** 71.5° **d.** 1.33 km

Investigation — Rich task

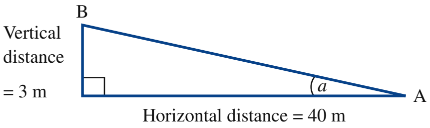
1.



2. a. 8 cm **b.** 40 m

3. 3 m

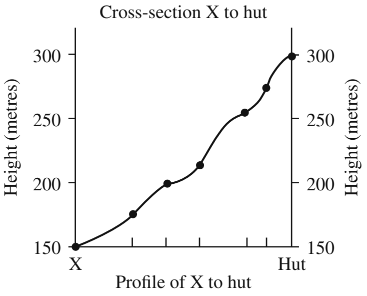
4.



5. $a = 4°17'$

6. Gradual to moderate

7.



The average slope is 11.46° — moderate to steep.

TOPIC 6
Surface area and volume

6.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

6.1.1 Why learn this?

Humans must measure! How much paint or carpet will you need to redecorate your bedroom? How many litres of water will it take to fill the new pool? How far is it to the end of the universe? These are just a few examples of where measurements skills are needed. Measuring tools have advanced significantly in their capacity to measure extremely small and extremely large amounts, leading to many breakthroughs in medicine, engineering, science, architecture and astronomy.

6.1.2 What do you know?

assesson

1. **THINK** List what you know about measurement. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map wheel that shows your class's knowledge of measurement.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Australian megafauna (eles-1845)

6.2 Area

6.2.1 Area

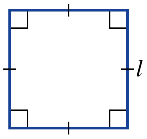
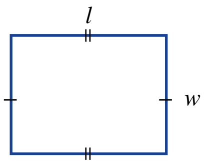
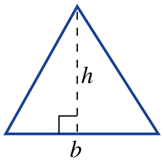
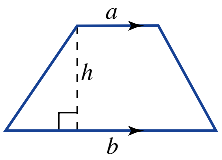
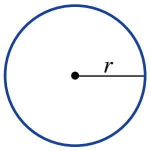
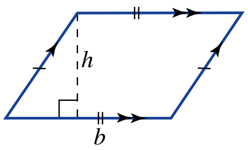
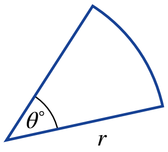
- The **area** of a figure is the amount of surface covered by the figure.
- The units used for area are mm^2, cm^2, m^2, km^2 or ha (hectares), depending upon the size of the figure.

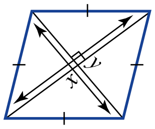
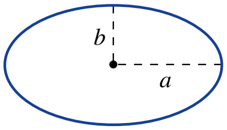
$$1 \text{ ha} = 10\,000 \text{ (or } 10^4\text{) m}^2$$

- There are many real-life situations that require an understanding of the area concept. Some are, 'the area to be painted', 'the floor area of a room or house', 'how much land one has' and 'how many tiles are needed for a wall'.
- It is important that you are familiar with converting units of area.

6.2.2 Using area formulas

- The area of many plane figures can be found by using a formula. The table below shows the formula for the area of some common shapes.

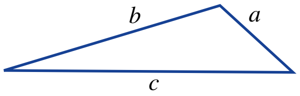
Shape	Formula
1. Square 	$A = l^2$
2. Rectangle 	$A = lw$
3. Triangle 	$A = \frac{1}{2}bh$
4. Trapezium 	$A = \frac{1}{2}(a + b)h$
5. Circle 	$A = \pi r^2$
6. Parallelogram 	$A = bh$
7. Sector 	$A = \frac{\theta°}{360°} \times \pi r^2$

Shape	Formula
8. Kite (including rhombus) 	$A = \frac{1}{2}xy$, where x and y are diagonals.
9. Ellipse 	$A = \pi ab$, where a and b are the lengths of the semi-major and semi-minor axes respectively.

Note: A calculator uses a stored value for π of approximately 3.141 592 654. Before calculators were in common usage, π was often taken to be approximately $\frac{22}{7}$ or 3.14. You are advised to use the π button on your calculator rather than $\frac{22}{7}$ or 3.14.

6.2.3 Heron's formula

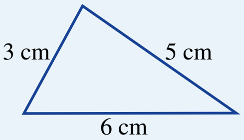
- If the lengths of all three sides of a triangle are known, its area, A, can be found by using **Heron's formula**: $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b and c are the lengths of the three sides and s is the semi-perimeter where $s = \dfrac{a+b+c}{2}$.



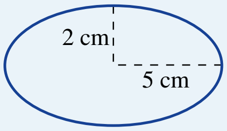
WORKED EXAMPLE 1 TI | CASIO

Find the areas of the following plane figures, correct to 2 decimal places.

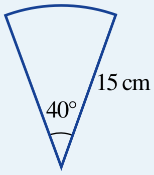
a



b



c



THINK	WRITE
a **1** Three side lengths are known, but not the height. In this case apply Heron's formula.	**a** $A = \sqrt{s(s-a)(s-b)(s-c)}$
2 Identify the values of a, b and c.	$a = 3, b = 5, c = 6$
3 Calculate the value of s, the semi-perimeter of the triangle.	$s = \dfrac{a+b+c}{2}$ $= \dfrac{3+5+6}{2}$ $= \dfrac{14}{2}$ $= 7$
4 Substitute the values of a, b, c and s into Heron's formula and evaluate, correct to 2 decimal places.	$A = \sqrt{7(7-3)(7-5)(7-6)}$ $= \sqrt{7 \times 4 \times 2 \times 1}$ $= \sqrt{56}$ $= 7.48\text{ cm}^2$
b **1** The shape shown is an ellipse. Write the appropriate area formula.	**b** $A = \pi ab$
2 Identify the values of a and b (the semi-major and semi-minor axes).	$a = 5, b = 2$

3 Substitute the values of *a* and *b* into the formula and evaluate, correct to 2 decimal places.	$A = \pi \times 5 \times 2$ $= 31.42\text{ cm}^2$
c **1** The shape shown is a sector. Write the formula for finding the area of a sector.	**c** $A = \dfrac{\theta}{360°} \times \pi r^2$
2 Write the value of θ and r.	$\theta = 40°,\ r = 15$
3 Substitute and evaluate the expression, correct to 2 decimal places.	$A = \dfrac{40°}{360°} \times \pi \times 15^2$ $= 78.54\text{ cm}^2$

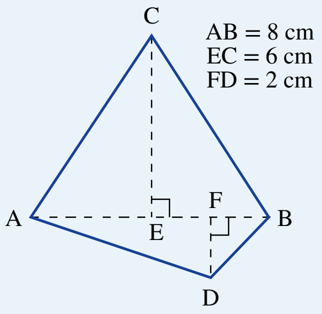
6.2.4 Areas of composite figures

- A **composite figure** is a figure made up of a combination of simple figures.
- The area of a composite figure can be calculated by:
  - calculating the sum of the areas of the simple figures that make up the composite figure
  - calculating the area of a larger shape and then subtracting the extra area involved.

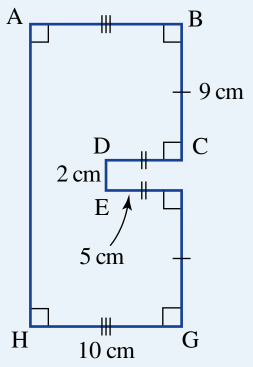
WORKED EXAMPLE 2

Find the area of each of the following composite shapes.

a



b



THINK	WRITE
a **1** ACBD is a quadrilateral that can be split into two triangles: ΔABC and ΔABD.	**a** Area ACBD = Area ΔABC + Area ΔABD
2 Write the formula for the area of a triangle containing base and height.	$A_{\text{triangle}} = \frac{1}{2}bh$
3 Identify the values of *b* and *h* for ΔABC.	$\Delta\text{ABC}: b = \text{AB} = 8,\ h = \text{EC} = 6$
4 Substitute the values of the pronumerals into the formula and, hence, calculate the area of ΔABC.	$\text{Area of } \Delta\text{ABC} = \frac{1}{2} \times \text{AB} \times \text{EC}$ $= \frac{1}{2} \times 8 \times 6$ $= 24\text{ cm}^2$
5 Identify the values of *b* and *h* for ΔABD.	$\Delta\text{ABD}: b = \text{AB} = 8,\ h = \text{FD} = 2$
6 Calculate the area of ΔABD.	$\text{Area of } \Delta\text{ABD} = \frac{1}{2}\text{AB} \times \text{FD}$ $= \frac{1}{2} \times 8 \times 2$ $= 8\text{ cm}^2$

	7 Add the areas of the two triangles together to find the area of the quadrilateral ACBD.	$\text{Area of ACBD} = 24\text{ cm}^2 + 8\text{ cm}^2$ $= 32\text{ cm}^2$
b	**1** One way to find the area of the shape shown is to find the total area of the rectangle ABGH and then subtract the area of the smaller rectangle DEFC.	**b** Area = Area ABGH − Area DEFC
	2 Write the formula for the area of a rectangle.	$A_{\text{rectangle}} = l \times w$
	3 Identify the values of the pronumerals for the rectangle ABGH.	$\text{Rectangle ABGH}: l = 9 + 2 + 9$ $= 20$ $w = 10$
	4 Substitute the values of the pronumerals into the formula to find the area of the rectangle ABGH.	$\text{Area of ABGH} = 20 \times 10$ $= 200\text{ cm}^2$
	5 Identify the values of the pronumerals for the rectangle DEFC.	$\text{Rectangle DEFC}: l = 5, w = 2$
	6 Substitute the values of the pronumerals into the formula to find the area of the rectangle DEFC.	$\text{Area of DEFC} = 5 \times 2$ $= 10\text{ cm}^2$
	7 Subtract the area of the rectangle DEFC from the area of the rectangle ABGH to find the area of the given shape.	$\text{Area} = 200 - 10$ $= 190\text{ cm}^2$

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: Heron's formula (eles-0177)

Complete this digital doc: SkillSHEET: Conversion of area units (doc-5236)

Complete this digital doc: SkillSHEET: Using a formula to find the area of a common shape (doc-5237)

Complete this digital doc: WorkSHEET: Area (doc-5241)

Exercise 6.2 Area

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 3–5, 8, 9, 11, 12, 14	Questions: 1–6, 8–10, 12, 14, 16 ,18	Questions: 1–9, 12–19

🟩🟦🟪 **Individual pathway interactivity:** int-4593

learnon ONLINE ONLY

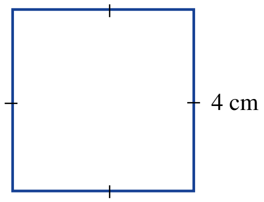
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Where appropriate, give answers correct to 2 decimal places.

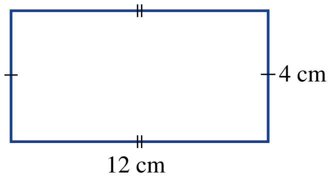
Fluency

1. Find the areas of the following shapes.

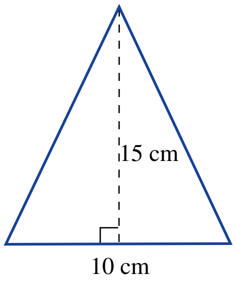
a.



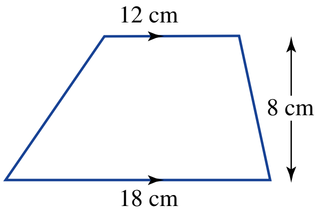
b.



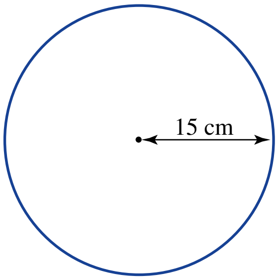
c.



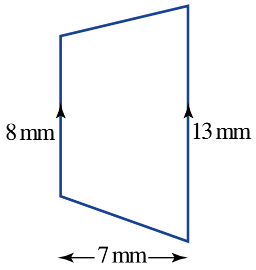
d.



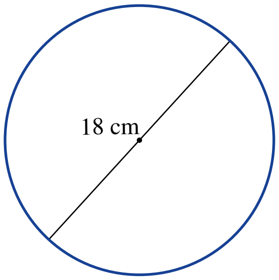
e.



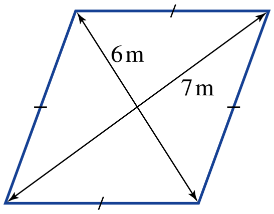
f.



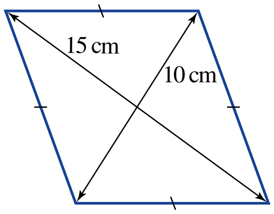
g.



h.



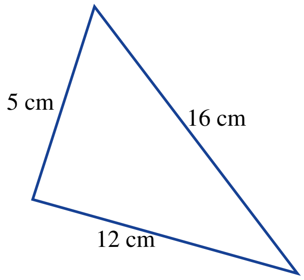
i.



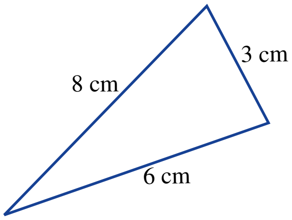
2. Express the area in questions **1e** and **1g** in terms of π.

3. **WE1a** Use Heron's formula to find the area of the following triangles correct to 2 decimal places.

a.

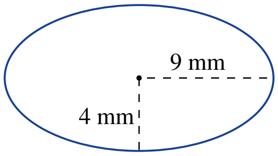


b.

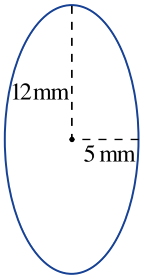


4. **WE1b** Find the area of the following ellipses. Answer correct to 1 decimal place.

a.

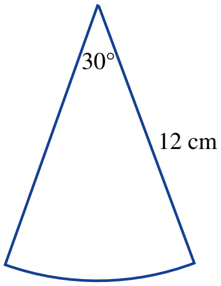


b.

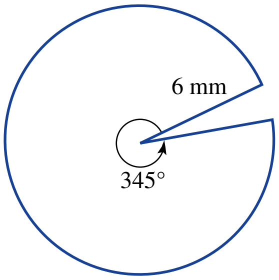


5. **WE1c** Find the area of the following shapes, **i** stating the answer exactly; that is, in terms of π and **ii** correct to 2 decimal places.

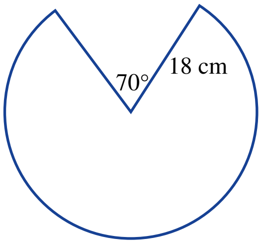
a.



b.



c.

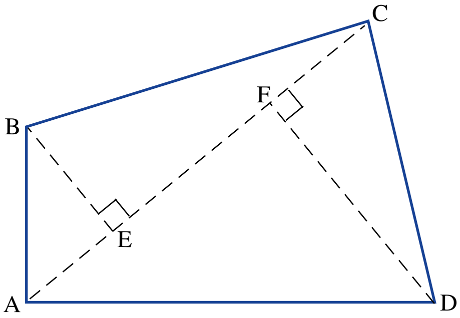


6. **MC** A figure has an area of about 64 cm^2. Which of the following *cannot* possibly represent the figure?

A. A triangle with base length 16 cm and height 8 cm
B. A circle with radius 4.51 cm
C. A rectangle with dimensions 16 cm and 4 cm
D. A square with side length 8 cm
E. A rhombus with diagonals 16 cm and 4 cm

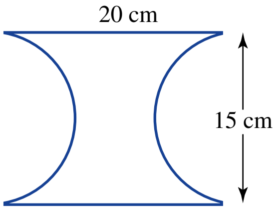
7. **MC** The area of the quadrilateral shown at right is to be calculated.
Which of the following lists all the lengths required to calculate the area?

A. AB, BC, CD and AD
B. AB, BE, AC and CD
C. BC, BE, AD and CD
D. AC, BE and FD
E. AC, CD and AB

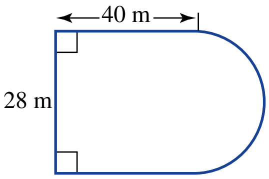


8. **WE2** Find the area of the following composite shapes.

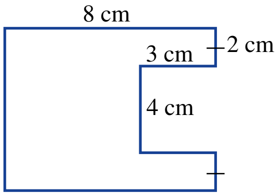
a.



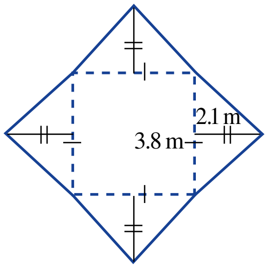
b.



c.



d.



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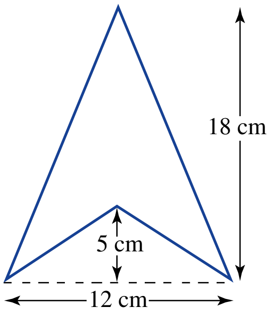

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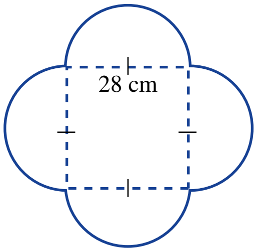
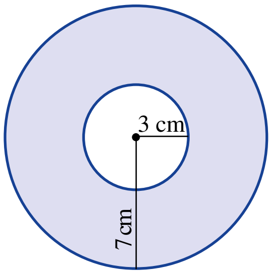


9. Find the shaded area in each of the following.

a.



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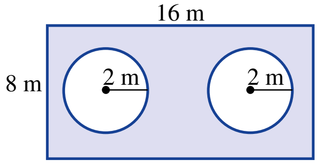

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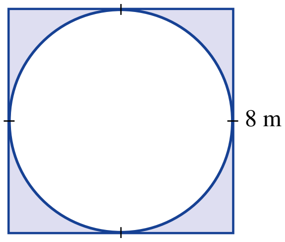

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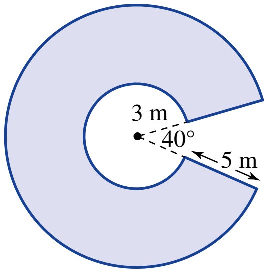

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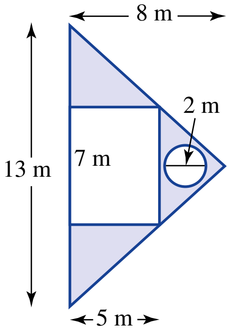

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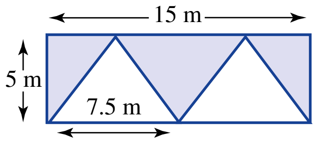


Understanding

10. A sheet of cardboard is 1.6 m by 0.8 m. The following shapes are cut from the cardboard:

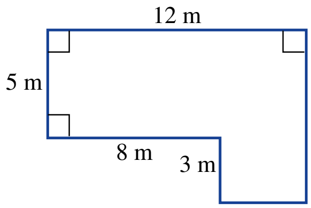
- a circular piece with radius 12 cm
- a rectangular piece 20 cm by 15 cm
- 2 triangular pieces with base 30 cm and height 10 cm
- a triangular piece with side length 12 cm, 10 cm and 8 cm.

What is the area of the remaining piece of cardboard?

11. A rectangular block of land, 12 m by 8 m, is surrounded by a concrete path 0.5 m wide. Find the area of the path.

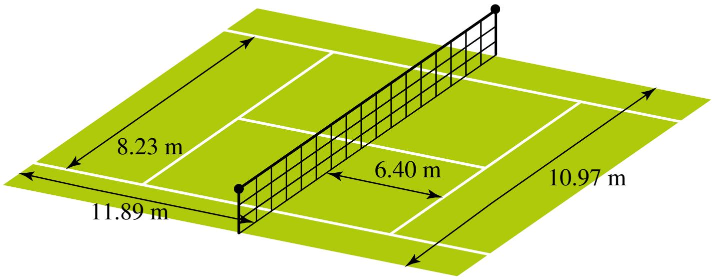
12. Concrete slabs 1 m by 0.5 m are used to cover a footpath 20 m by 1.5 m. How many slabs are needed?

13. A city council builds a 0.5 m wide concrete path around the garden as shown below.



Find the cost of the job if the workman charges \$40.00 per m^2.

14. A tennis court used for doubles is 10.97 m wide, but a singles court is only 8.23 m wide, as shown in the diagram.



a. What is the area of the doubles tennis court?
b. What is the area of the singles court?
c. What percentage of the doubles court is used for singles? Give your answer to the nearest whole number.

15. Ron the excavator operator has 100 metres of barricade mesh and needs to enclose an area to work in safely. He chooses to make a rectangular region with dimensions x and y.

a. Write an equation that connects x, y and the perimeter.
b. Write y in terms of x.
c. Write an equation for the area of the region in terms of x.
d. Fill in the table for different values of x.

x	0	5	10	15	20	25	30	35	40	45	50
Area (m^2)											

e. Can x have a value more than 50? Why?
f. Sketch a graph of area against x.
g. Determine the value of x that makes the area a maximum.
h. What is the value of y for maximum area?
i. What shape encloses the maximum area?
j. Calculate the maximum area.

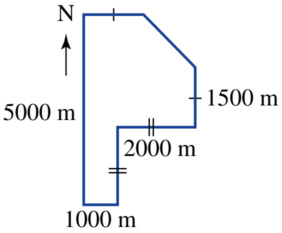
Ron decides to choose to make a circular area with the barricade mesh.

k. What is the radius of this circular region?
l. What area is enclosed in this circular region?
m. How much extra area does Ron now have compared to his rectangular region?

Reasoning

16. Dan has purchased a country property with layout and dimensions as shown in the diagram.

a. Show that the property has a total area of 987.5 ha.
b. Dan wants to split the property in half (in terms of area) by building a straight-lined fence running either north–south or east–west through the property. Assuming the cost of the fencing is a fixed amount per linear metre, justify where the fence should be built (that is, how many metres from the top left-hand corner and in which direction), to minimise the cost.

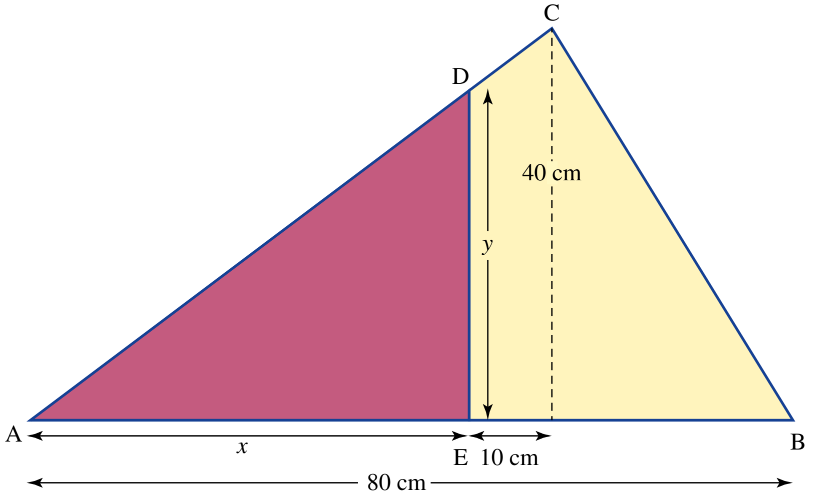


17. In question **15**, Ron the excavator operator could choose to enclose a rectangular or circular area with 100 m of barricade mesh. In this case, the circular region resulted in a larger safe work area.

a. Show that for 150 m of barricade mesh, a circular region again results in a larger safe work area as opposed to a rectangular region.
b. Show that for n metres of barricade mesh, a circular region will result in a larger safe work area as opposed to a rectangular region.

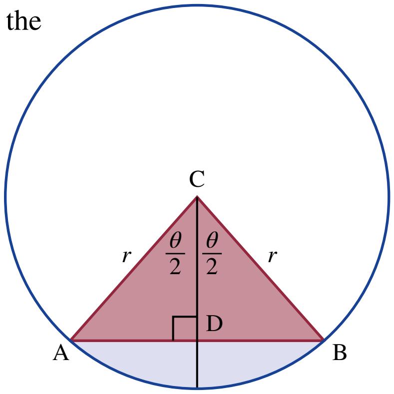
Problem solving

18. ABC is a scalene triangle with a base length of 80 cm and a perpendicular height of 40 cm. A right-angled triangle, AED, is nestled within ABC such that DE is 10 cm to the left of the perpendicular height, as shown. Find the lengths of the sides labelled x and y if the shorter side of the two is 20 cm less than the longer side and the areas of the two shaded regions are the same.



19. Proving the segment formula

Prove the formula for the area of a segment using the fact that area of the segment = area of sector ABC − 2 × area of triangle ACD.



a. Using trigonometry, show that $\dfrac{\text{AD}}{r} = \sin\left(\dfrac{\theta}{2}\right)$.

b. Show that $\dfrac{\text{CD}}{r} = \cos\left(\dfrac{\theta}{2}\right)$.

c. Show that the area of triangle ACD is $\dfrac{r^2 \sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right)}{2}$.

Note that this formula is the same if θ is in degrees or radians.

d. Finally, show that the area of the segment (in purple) is

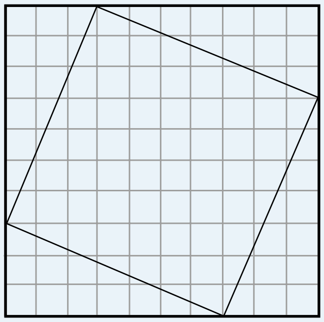
$r^2\left(\pi \times \dfrac{\theta}{360°} - \sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right)\right)$ if θ is in degrees.

Reflection

How are perimeter and area different but fundamentally related?

CHALLENGE 6.1

The diagram shows one smaller square drawn inside a larger square on grid paper. Represent the area of the smaller square as a fraction of the larger square.



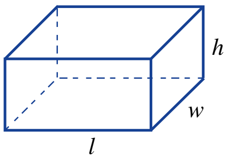
6.3 Total surface area

- The **total surface area (TSA)** of a solid is the sum of the areas of all the faces of that solid.

6.3.1 TSA of rectangular prisms and cubes

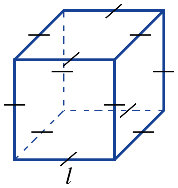
- The formula for finding the TSA of a **rectangular prism (cuboid)** is:

$$\text{TSA} = 2(lh + lw + wh)$$



- A special case of the rectangular prism is the **cube**, where all sides are equal ($l = w = h$).

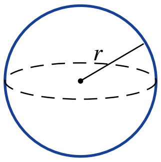
$$\text{TSA} = 6l^2$$



6.3.2 TSA of spheres and cylinders

Sphere:

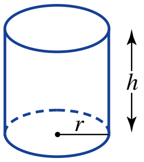
$\text{TSA} = 4\pi r^2$



Note: The mathematics required to prove the formula for the total surface area of a sphere is beyond the scope of Year 10.

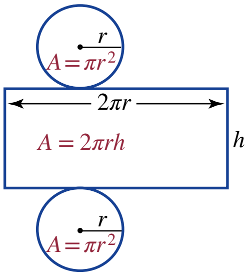
Cylinder:

$\text{TSA} + 2\pi r(r + h)$ or $2\pi r^2 + 2\pi rh$



- The formula for the TSA of a cylinder is found from the area of the net as shown.

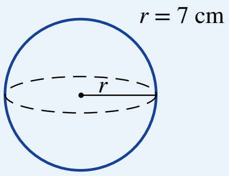
$$\begin{aligned}\text{TSA} &= \pi r^2 + \pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h)\end{aligned}$$



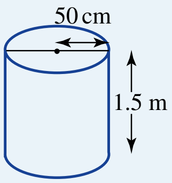
WORKED EXAMPLE 3

Find the total surface area of the solids, correct to the nearest cm^2.

a



b



THINK	**WRITE**
a 1 Write the formula for the TSA of a sphere.	**a** $\text{TSA} = 4\pi r^2$
2 Identify the value for r.	$r = 7$
3 Substitute and evaluate.	$\text{TSA} = 4 \times \pi \times 7^2$ $\approx 615.8\ \text{cm}^2$ $\approx 616\ \text{cm}^2$
b 1 Write the formula for the TSA of a cylinder.	**b** $\text{TSA} = 2\pi r(r + h)$
2 Identify the values for r and h. Note that the units will need to be the same.	$r = 50$ cm, h = 1.5 m $= 150$ cm
3 Substitute and evaluate.	$\text{TSA} = 2 \times \pi \times 50 \times (50 + 150)$ $\approx 62\,831.9\ \text{cm}^2$ $\approx 62\,832\ \text{cm}^2$

6.3.3 TSA of cones

- The total surface area of a cone can be found by considering its net, which is comprised of a small circle and a sector of a larger circle.

r = radius of the cone
l = slant height of the cone

- The sector is a fraction of the full circle of radius l with circumference $2\pi l$.
- The sector has an arc length equivalent to the circumference of the base of the cone, $2\pi r$.
- The fraction of the full circle represented by the sector can be found by writing the arc length as a fraction of the circumference of the full circle, $\dfrac{2\pi r}{2\pi l} = \dfrac{r}{l}$.

$$\begin{aligned}\text{Area of a sector} &= \text{fraction of the circle} \times \pi l^2\\ &= \frac{r}{l} \times \pi l^2\\ &= \pi r l\end{aligned}$$

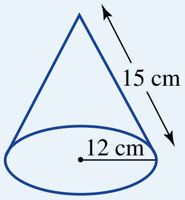
$$\begin{aligned}\text{SA} &= A_{\text{circular base}} + A_{\text{curved surface}}\\ \text{Therefore,}\quad &= \pi r^2 + \pi r l\\ &= \pi r(r + l)\end{aligned}$$

Cone: $\text{TSA} = \pi r(r + l)$ or $\pi r^2 + \pi r l$

WORKED EXAMPLE 4

TI | CASIO

Find the total surface area of the cone shown.



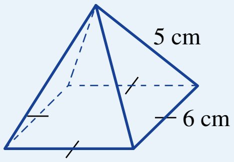
THINK	**WRITE**
1 Write the formula for the TSA of a cone.	$\text{TSA} = \pi r(r + l)$
2 State the values of r and l.	$r = 12, l = 15$
3 Substitute and evaluate.	$\text{TSA} = \pi \times 12 \times (12 + 15)$ $= 1017.9\ \text{cm}^2$

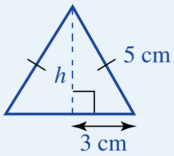
6.3.4 TSA of other solids

- TSA can be found by summing the areas of each face.
- The areas of each face may need to be calculated separately.
- Check the total number of faces to ensure that none are left out.

WORKED EXAMPLE 5

Find the total surface area of the square-based pyramid shown.



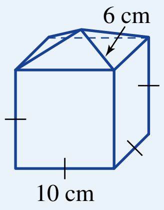
THINK	WRITE/DRAW
1 There are five faces: The square base and four identical triangles.	TSA = Area of square base + area of four triangular faces
2 Find the area of the square base.	Area of base = l^2, where $l = 6$ Area of base = 6^2 $= 36\text{ cm}^2$
3 Draw and label one triangular face and write the formula for finding its area.	 Area of a triangular face $= \frac{1}{2}bh;\ b = 6$
4 Find the height of the triangle, *h*, using Pythagoras' theorem.	$a^2 = c^2 - b^2$, where $a = h$, $b = 3$, $c = 5$ $h^2 = 5^2 - 3^2$ $h^2 = 25 - 9$ $h^2 = 16$ $h = 4\text{ cm}$
5 Calculate the area of the triangular face by substituting $b = 6$ and $h = 4$.	Area of triangular face $= \frac{1}{2} \times 6 \times 4$ $= 12\text{ cm}^2$
6 Calculate the TSA by adding the area of the square base and the area of four identical triangular faces together.	TSA $= 36 + 4 \times 12$ $= 36 + 48$ $= 84\text{ cm}^2$

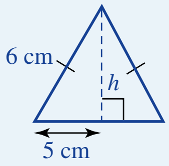
6.3.5 TSA of composite solids

- Composite solids are formed when two or more simple solids are joined together.
- The TSA of a composite solid is calculated by summing the areas of the solid's external faces.

WORKED EXAMPLE 6

Find the total surface area of the solid shown correct to 1 decimal place.

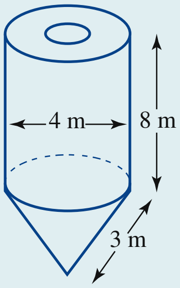


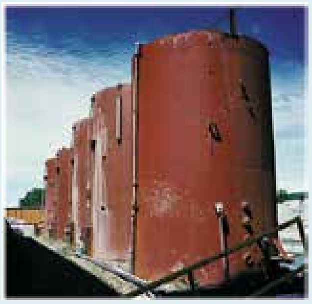
THINK	WRITE/DRAW
1 The solid shown has 9 faces — five identical squares and four identical triangles.	$\text{TSA} = 5 \times \text{area of a square}$ $+ 4 \times \text{area of a triangle}$
2 Find the area of one square face with the side length 10 cm.	$A_{\text{square}} = l^2$, where $l = 10$ $A = 10^2$ $A = 100\ \text{cm}^2$
3 Draw a triangular face and work out its height using Pythagoras' theorem.	 $a^2 = c^2 - b^2$, where $a = h$, $b = 5$, $c = 6$ $h^2 = 6^2 - 5^2$ $h^2 = 36 - 25$ $h^2 = 11$ $h = 3.316\,62\ldots$ cm (or with rounding, $h = 3.3$)
4 Find the area of one triangular face.	$A_{\text{triangle}} = \frac{1}{2}bh$, where $b = 10$, $h = 3.316\,62$ $= \frac{1}{2} \times 10 \times 3.31662\ldots$ $= 16.5831\ldots\ \text{cm}^2$ (or, with rounding, $A_{\text{triangle}} = 16.6\ \text{cm}^2$)
5 Find the TSA of the solid by adding the area of 5 squares and 4 triangles together.	$\text{TSA} = 5 \times 100 + 4 \times 16.5831\ldots$ $= 500 + 66.3324\ldots$ $\approx 566.3\ \text{cm}^2$ (or $= 566$ using the previously rounded value)

Note: Rounding is not done until the final step. If *h* had been rounded to 3.3 in step 3 and this value used in steps 4 and 5, the decimal place value of the TSA would have been lost. It is important to realise that rounding too early can affect the accuracy of results.

WORKED EXAMPLE 7

The silo shown below is to be built from metal. The top portion of the silo is a cylinder of diameter 4 m and height 8 m. The bottom part of the silo is a cone of slant height 3 m. The silo has a circular opening of radius 30 cm on the top.





a What area of metal (to the nearest m^2) is required to build the silo?
b If it costs \$12.50 per m^2 to cover the surface with an anti-rust material, how much will it cost to cover the silo completely?

THINK	WRITE
a 1 The surface area of the silo consists of an annulus, the curved part of the cylinder and the curved section of the cone.	**a** TSA = area of annulus + area of curved section of a cylinder + area of curved section of a cone
2 To find the area of the annulus, subtract the area of the small circle from the area of the larger circle. Let R = radius of small circle.	Area of annulus = $A_{\text{large scale}} - A_{\text{small circle}}$ $= \pi r^2 - \pi R^2$ where $r = \frac{4}{2} = 2$ m and $R = 30$ cm $= 0.3$ m. Area of annulus = $\pi \times 2^2 - \pi \times 0.3^2$ $= 12.28$ m
3 The middle part of the silo is the curved part of a cylinder. Find its area. (Note that in the formula $\text{TSA}_{\text{cylinder}} = 2\pi r^2 + 2\pi rh$, the curved part is represented by $2\pi rh$.)	Area of curved section of cylinder = $2\pi rh$ where $r = 2$, $h = 8$. Area of curved section of cylinder $= 2 \times \pi \times 2 \times 8$ $= 100.53$ m^2
4 The bottom part of the silo is the curved section of a cone. Find its area. (Note that in the formula $\text{TSA}_{\text{cone}} = \pi r^2 + \pi rl$, the curved part is given by πrl.)	Area of curved section of cone = πrl where $r = 2$, $l = 3$. Area of curved section of cone = $\pi \times 2 \times 3$ $= 18.85$ m^2
5 Find the total surface area of the silo by finding the sum of the surface areas calculated above.	TSA = $12.28 + 100.53 + 18.85$ $= 131.66$ m^2
6 Write the answer in words.	The area of metal required is 132 m^2, correct to the nearest square metre.
b To find the total cost, multiply the total surface area of the silo by the cost of the anti-rust material per m^2 (\$12.50).	**b** Cost = 132 × \$12.50 = \$1650.00

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Try out this interactivity: TSA—sphere (int-2782)

Complete this digital doc: SkillSHEET: Total surface area of cubes and rectangular prisms (doc-5238)

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Exercise 6.3 Total surface area

assess**on**

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 6a–e, 7, 10, 12	Questions: 1–4, 6, 7, 9–12, 15 ,18	Questions: 1–8, 10–18

🟩🟦🟪 **Individual pathway interactivity:** int-4594 learn**on** ONLINE ONLY

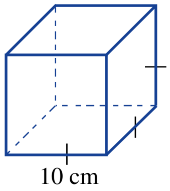
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

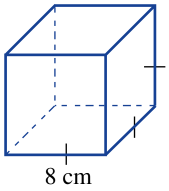
Note: Where appropriate, give the answers correct to 1 decimal place.

1. Find the total surface areas of the solids shown.

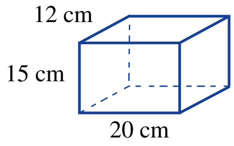
a.



b.



c.



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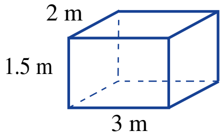
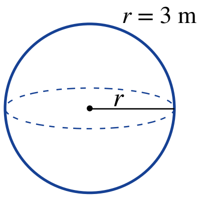
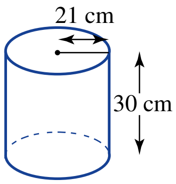


2. **WE3** Find the total surface area of the solids shown below.

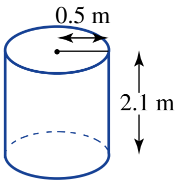
a.



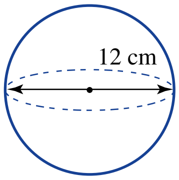
b.



c.

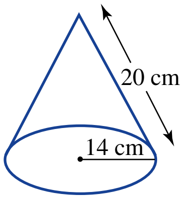


d.

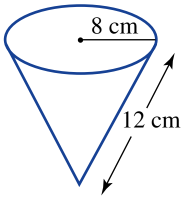


3. **WE4** Find the total surface area of the cones below.

a.

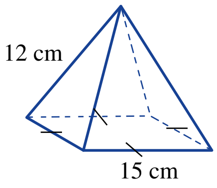


b.

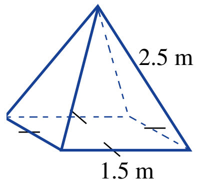


4. **WE5** Find the total surface area of the solids below.

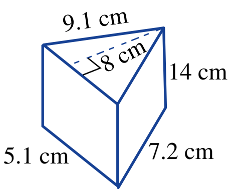
a.



b.



c.



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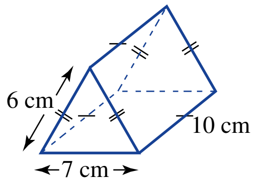


5. Find the surface areas of the following.

a. A cube of side length 1.5 m

b. A rectangular prism 6 m × 4 m × 2.1 m

c. A cylinder of radius 30 cm and height 45 cm, open at one end

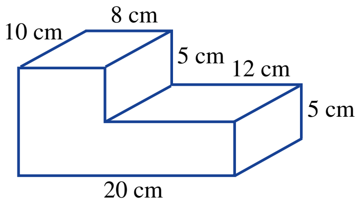
d. A sphere of radius 28 mm

e. An open cone of radius 4 cm and slant height 10 cm

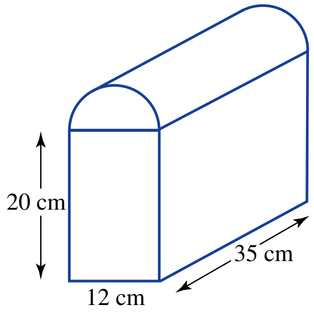
f. A square pyramid of base length 20 cm and slant edge 30 cm

6. **WE6** Find the total surface area of the objects shown.

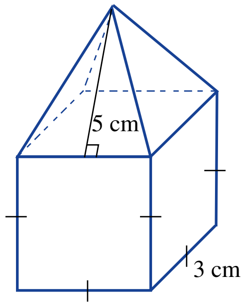
a.



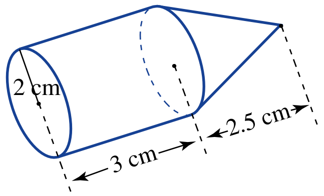
b.



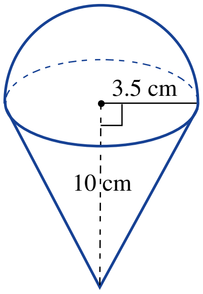
c.



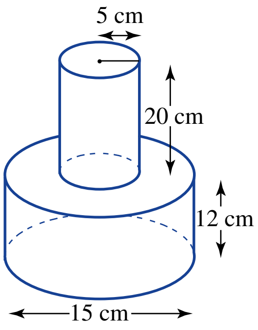
d.



e.



f.



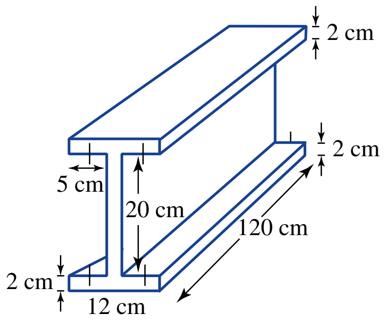
7. **MC** A cube has a total surface area of 384 cm^2. The length of the edge of the cube is:

A. 9 cm **B.** 8 cm **C.** 7 cm **D.** 6 cm **E.** 5 cm

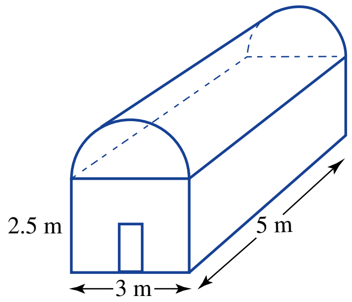
Understanding

8. Open cones are made from nets cut from a large sheet of paper 1.2 m × 1.0 m. If a cone has a radius of 6 cm and a slant height of 10 cm, how many cones can be made from the sheet? (Assume there is 5% wastage of paper.)

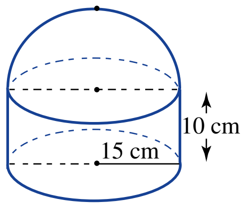
9. A steel girder is to be painted. Calculate the area of the surface to be painted.



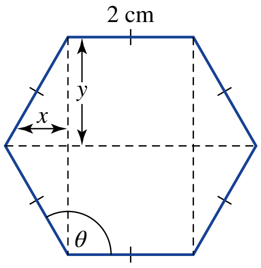
10. **WE7** The greenhouse shown at right is to be built using shade cloth. It has a wooden door of dimensions 1.2 m × 0.5 m.
 a. Find the total area of shade cloth needed to complete the greenhouse.
 b. Find the cost of the shade cloth at \$6.50 per m^2.



11. A cylinder is joined to a hemisphere to make a cake holder, as shown at right. The surface of the cake holder is to be chromed at 5.5 cents per cm^2.
 a. Find the total surface area to be chromed.
 b. Find the cost of chroming the cake holder.



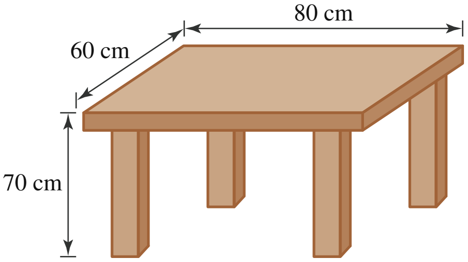
12. A soccer ball is made up of a number of hexagons sewn together on its surface. Each hexagon can be considered to have dimensions as shown in the diagram.
 a. Calculate $\theta°$.
 b. Calculate the values of x and y exactly.
 c. Calculate the area of the trapezium in the diagram.
 d. Hence, determine the area of the hexagon.
 e. If the total surface area of the soccer ball is $192\sqrt{3}$ cm^2, how many hexagons are on the surface of the soccer ball?



13. **a.** Determine the exact total surface area of a sphere with radius $\sqrt{2}$ metres.
 An inverted cone with side length 4 metres is placed on top of the sphere so that the centre of its base is 0.5 metres above the centre of the sphere.
 b. Find the radius of the cone exactly.
 c. Find the area of the curved surface of the cone exactly.
 d. What are the exact dimensions of a box that could precisely fit the cone connected to the sphere?

Reasoning

Complete the following question without the aid of a calculator.



14. The table shown at right is to be varnished (including the base of each leg). The table top has a thickness of 180 mm and the cross-sectional dimension of the legs is 50 mm by 50 mm.
A friend completes the calculation as shown. Assume there are no simple calculating errors. Analyse the working presented and justify if the TSA calculated is correct.

Table top (inc. leg bases)	0.96	$2 \times (0.8 \times 0.6)$
Legs	0.416	$16 \times (0.52 \times 0.05)$
Table top edging	0.504	$0.18 \times (2(0.8 + 0.6))$
TSA	1.88 m^2	

15. A shower recess with dimensions 1500 mm (back wall) by 900 mm (side wall) needs to have the back and two side walls tiled to a height of 2 m.

a. Calculate the area to be tiled in m^2.

b. Justify that 180 tiles (including those that need to be cut) of dimension 20 cm by 20 cm will be required. Disregard the grout and assume that once a tile is cut, only one piece of the tile can be used.

c. Evaluate the cheapest option of tiling; \$1.50/tile or \$39.50/box, where a box covers 1 m^2, or tiles of dimension 30 cm by 30 cm costing \$3.50/tile.

16. If the surface area of a sphere to that of a cylinder is in the ratio 4 : 3 and the sphere has a radius of $3a$, show that if the radius of the cylinder is equal to its height, then the radius of the cylinder is $\dfrac{3\sqrt{3}a}{2}$.

Problem solving

Frustum of a cone

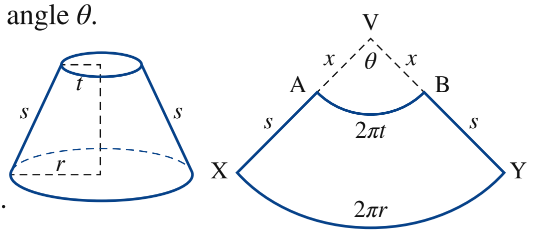
17. A frustum of a cone is a cone with the top sliced off (see the drawing on the right).
When the curved side is 'opened up', it creates a shape, ABYX, as shown in the diagram.

a. Write an expression for the arc length XY in terms of the angle θ.
Write another expression for the arc length AB in terms of the same angle θ. Show that, in radians,

$$\theta = \frac{2\pi(r - t)}{s}.$$

b. **i.** Using the above formula for θ, show that $x = \dfrac{st}{(r - t)}$.

ii. Use similar triangles to confirm this formula.



c. Determine the area of sectors AVB and XVY and hence determine the area of ABYX. Add the areas of the 2 circles to the area of ABYX to determine the TSA of a frustum.

18. Tina is re-covering a footstool in the shape of a cylinder with diameter 50 cm and height 30 cm. She also intends to cover the base of the cushion.

She has 1 m^2 of fabric to make this footstool. When calculating the area of fabric required, allow an extra 20% of the total surface area to cater for seams and pattern placings. Explain whether Tina has enough material to cover the footstool.

Reflection

Why is calculating the total surface area of a composite solid more difficult than for a simple solid such as a rectangular prism or cylinder?

6.4 Volume

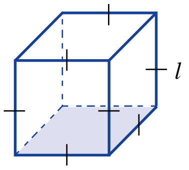
6.4.1 Volume

- The **volume** of a 3-dimensional object is the amount of space it takes up.
- The volume is measured in units of mm^3, cm^3 and m^3.

6.4.2 Volume of a prism

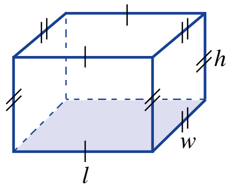
- The volume of any solid with a uniform cross-sectional area is given by the formula: $V = AH$, where A is the cross-sectional (or base) area and H is the height of the solid.

Cube



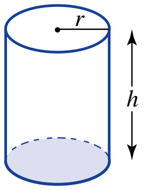
$$\begin{aligned}\text{Volume} &= AH\\ &= \text{area of a square} \times \text{height}\\ &= l^2 \times l\\ &= l^3\end{aligned}$$

Rectangular prism



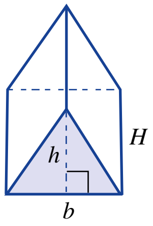
$$\begin{aligned}\text{Volume} &= AH\\ &= \text{area of a rectangle} \times \text{height}\\ &= lwh\end{aligned}$$

Cylinder



$$\begin{aligned}\text{Volume} &= AH\\ &= \text{area of a circle} \times \text{height}\\ &= \pi r^2 h\end{aligned}$$

Triangular prism



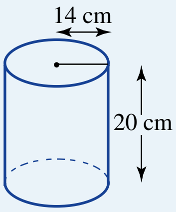
$$\begin{aligned}\text{Volume} &= AH\\ &= \text{area of a triangle} \times \text{height}\\ &= \tfrac{1}{2}bh \times H\end{aligned}$$

WORKED EXAMPLE 8

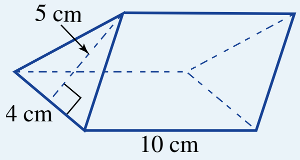
TI | CASIO

Find the volumes of the following shapes.

a



b



THINK	WRITE
a **1** Write the formula for the volume of the cylinder (prism).	**a** $V = AH$ $= \pi r^2 h$
2 Identify the value of the pronumerals.	$r = 14, \ h = 20$
3 Substitute and evaluate.	$V = \pi \times 14^2 \times 20$ $\approx 12\,315.04 \text{ cm}^3$
b **1** Write the formula for the volume of a triangular prism.	**b** $V = \frac{1}{2}bh \times H$
2 Identify the value of the pronumerals. (Note h is the height of the triangle and H is the depth of the prism.)	$b = 4, \ h = 5, \ H = 10$
3 Substitute and evaluate.	$V = \frac{1}{2} \times 4 \times 5 \times 10$ $= 100 \text{ cm}^3$

WORKED EXAMPLE 9

a What effect will doubling each of the side lengths of a cube have on its volume?
b What effect will halving the radius and doubling the height of a cylinder have on its volume?

THINK	WRITE
a **1** Write the formula for the volume of the cube.	**a** $V = l^3$
2 Identify the value of the pronumeral. *Note:* Doubling is the same as multiplying by 2.	$l_{\text{new}} = 2l$
3 Substitute and evaluate.	$V_{\text{new}} = (2l)^3$
4 Compare the answer obtained in step 3 with the volume of the original shape.	$= 8l^3$
5 Write your answer.	Doubling each side length of a cube increases the volume by a factor of 8; that is, the new volume will be 8 times as large as the original volume.
b **1** Write the formula for the volume of the cylinder.	**b** $V = \pi r^2 h$
2 Identify the value of the pronumerals. *Note:* Halving is the same as dividing by 2.	$r_{\text{new}} = \frac{r}{2}, \ h_{\text{new}} = 2h$
3 Substitute and evaluate.	$V_{\text{new}} = \pi \left(\frac{r}{2}\right)^2 2h$ $= \pi \times \frac{r^2}{{}_2\cancel{4}} \times \cancel{2}h$ $= \frac{\pi r^2 h}{2}$

4 Compare the answer obtained in step 3 with the volume of the original shape.

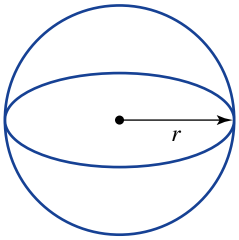
$= \frac{1}{2}\pi r^2 h$

5 Write your answer.

Halving the radius and doubling the height of a cylinder decreases the volume by a factor of 2; that is, the new volume will be half the original volume.

6.4.3 Volume of spheres

- The volume of a sphere of radius, r, can be calculated using the formula $V = \frac{4}{3}\pi r^3$.



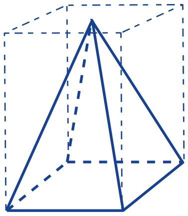
WORKED EXAMPLE 10

Find the volume of a sphere of radius 9 cm. Answer correct to 1 decimal place.

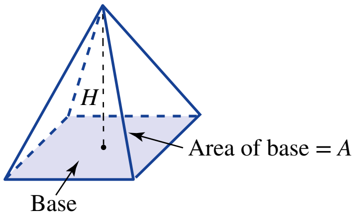
THINK	WRITE
1 Write the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$
2 Identify the value of r.	$r = 9$
3 Substitute and evaluate.	$V = \frac{4}{3} \times \pi \times 9^3$ $= 3053.6\ \text{cm}^3$

6.4.4 Volume of pyramids

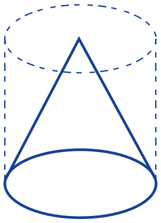
- Pyramids are not prisms as the cross-section changes from the base upwards.
- The volume of a pyramid is one-third the volume of an equivalent prism with the same base area and height.



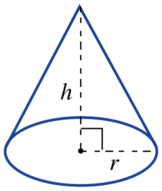
Volume of a pyramid $= \frac{1}{3}AH$



- Since a cone is a pyramid with a circular cross-section, the volume of a cone is one-third the volume of a cylinder with the same base area and height.



$$\text{Volume of a cone} = \frac{1}{3}AH$$
$$= \frac{1}{3}\pi r^2 h$$

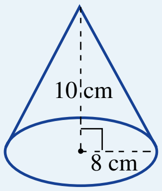


WORKED EXAMPLE 11

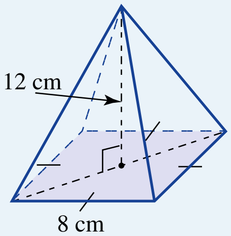
TI | CASIO

Find the volume of each of the following solids.

a



b



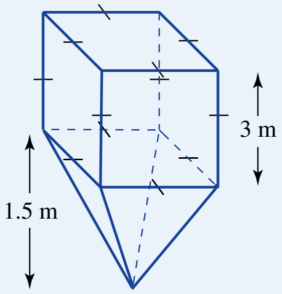
THINK	WRITE
a 1 Write the formula for the volume of a cone.	**a** $V = \frac{1}{3}\pi r^2 h$
2 Identify the values of *r* and *h*.	$r = 8,\ h = 10$
3 Substitute and evaluate.	$V = \frac{1}{3} \times \pi \times 8^2 \times 10$ $= 670.21\ \text{cm}^3$
b 1 Write the formula for the volume of a pyramid.	**b** $V = \frac{1}{3}AH$
2 Find the area of the square base.	$A = l^2$ where $l = 8$ $A = 8^2$ $= 64\ \text{cm}^2$
3 Identify the value of *H*.	$H = 12$
4 Substitute and evaluate.	$V = \frac{1}{3} \times 64 \times 12$ $= 256\ \text{cm}^3$

6.4.5 Volume of composite solids

- A composite solid is a combination of a number of solids.
- The volume of each smaller solid component can be calculated separately.
- The volume of a composite solid is calculated by summing the volumes of each of the smaller solid components.

WORKED EXAMPLE 12

Calculate the volume of the composite solid shown.



THINK	WRITE
1 The given solid is a composite figure, made up of a cube and a square-based pyramid.	$V = \text{Volume of cube} + \text{Volume of pyramid}$
2 Find the volume of the cube.	$V_{\text{cube}} = l^3$ where $l = 3$ $V_{\text{cube}} = 3^3$ $= 27\text{ m}^3$
3 Write the formula for finding the volume of a square-based pyramid.	$V_{\text{square-based pyramid}} = \frac{1}{3}AH$
4 Find the area of the square base.	$A = l^2$ $= 3^2$ $= 9\text{ m}^2$
5 Identify the value of H.	$H = 1.5$
6 Substitute and evaluate the volume of the pyramid.	$V_{\text{square-based pyramid}} = \frac{1}{3} \times 9 \times 1.5$ $= 4.5\text{ m}^3$
7 Find the total volume by adding the volume of the cube and pyramid.	$V = 27 + 4.5$ $= 31.5\text{ m}^3$

6.4.6 Capacity

- Some 3-dimensional objects are hollow and can be filled with liquid or some other substance.
- The amount of substance which a container can hold is called its capacity.
- **Capacity** is essentially the same as volume but is usually measured in mL, L and kL where

$$\begin{aligned} 1\text{ mL} &= 1\text{ cm}^3 \\ 1\text{L} &= 1000\text{ cm}^3 \\ 1\text{kL} &= 1\text{ m}^3. \end{aligned}$$

WORKED EXAMPLE 13

Find the capacity (in litres) of a cuboidal aquarium, which is 50 cm long, 30 cm wide and 40 cm high.

THINK	WRITE
1 Write the formula for the volume of a rectangular prism.	$V = lwh$
2 Identify the values of the pronumerals.	$l = 50,\ w = 30,\ h = 40$
3 Substitute and evaluate.	$V = 50 \times 30 \times 40$ $= 60\,000\text{ cm}^3$
4 State the capacity of the container in millilitres, using $1\text{ cm}^3 = 1\text{ mL}$.	$= 60\,000\text{ mL}$
5 Since $1\text{ L} = 1000\text{ mL}$, to convert millilitres to litres divide by 1000.	$= 60\text{ L}$
6 Give a worded answer.	The capacity of the fish tank is 60 L.

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Complete this digital doc: WorkSHEET: Volume (doc-6733)

Exercise 6.4 Volume

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 6–8, 9a, 10, 13, 14, 20	Questions: 1–8, 10–12, 14, 16, 19, 20, 22, 25	Questions: 1–18, 20–26

🟩🟦🟪 **Individual pathway interactivity:** int-4595

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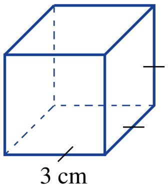
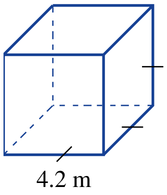
Fluency

1. Find the volumes of the following prisms.

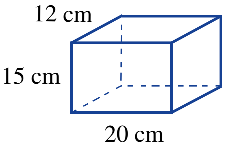
a.



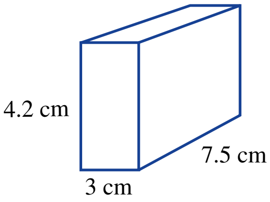
b.



c.

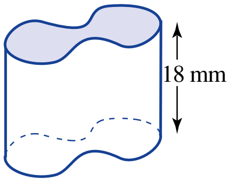


d.



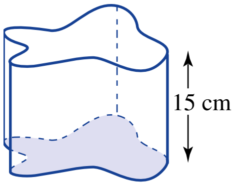
2. Calculate the volume of each of these solids.

a.



[Base area: 25 mm^2]

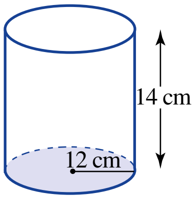
b.



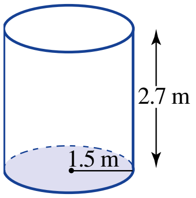
[Base area: 24 cm^2]

3. **WE8** Find the volume of each of the following. Give each answer correct to 1 decimal place where appropriate.

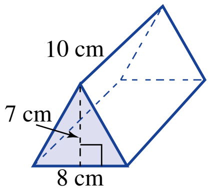
a.



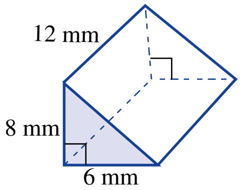
b.



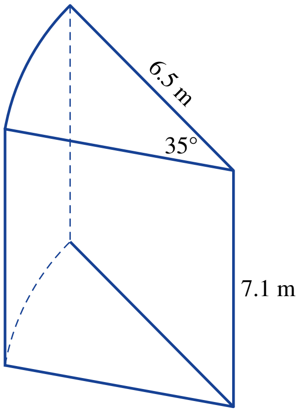
c.



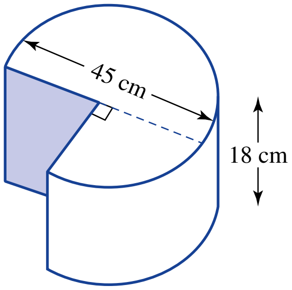
d.



e.



f.



4. **WE10** Find the volume of a sphere (correct to 1 decimal place) with a radius of:

a. 1.2 m

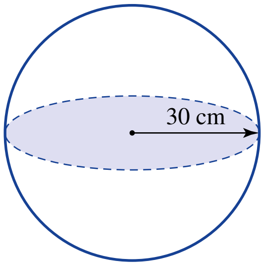
b. 15 cm

c. 7 mm

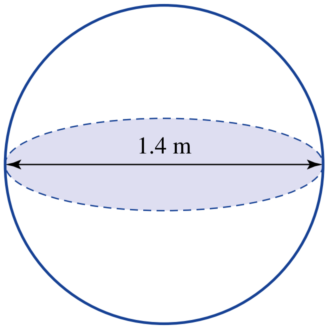
d. 50 cm.

5. Find the volume of each of these figures, correct to 2 decimal places.

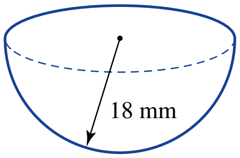
a.



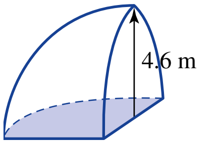
b.



c.

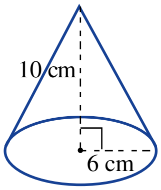


d.

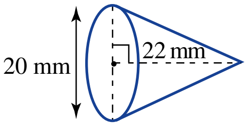


6. **WE11a** Find the volume of each of the following cones, correct to 1 decimal place.

a.

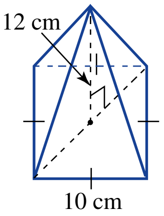


b.

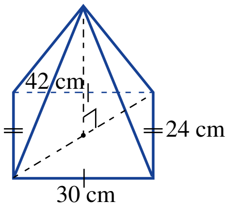


7. WE11b Find the volume of each of the following pyramids.

a.

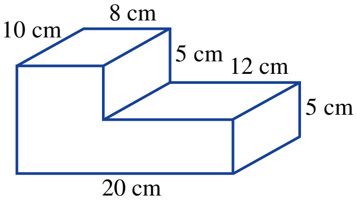


b.

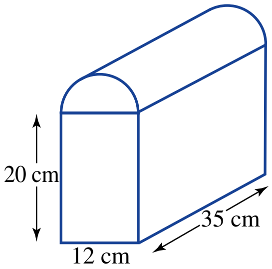


8. WE12 Calculate the volume of each of the following composite solids correct to 2 decimal places where appropriate.

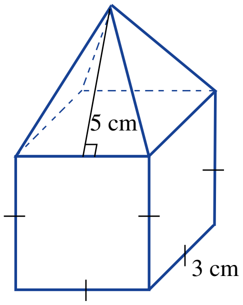
a.



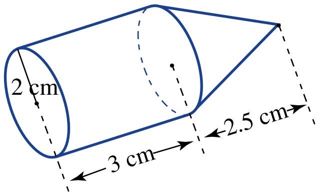
b.



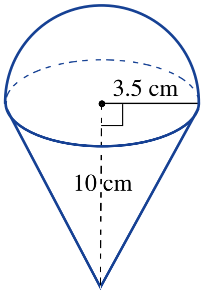
c.



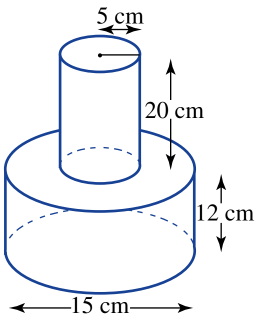
d.



e.



f.



Understanding

9. WE9 **a.** What effect will tripling each of the side lengths of a cube have on its volume?

b. What effect will halving each of the side lengths of a cube have on its volume?

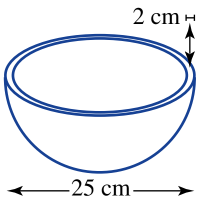
c. What effect will doubling the radius and halving the height of a cylinder have on its volume?

d. What effect will doubling the radius and dividing the height of a cylinder by 4 have on its volume?

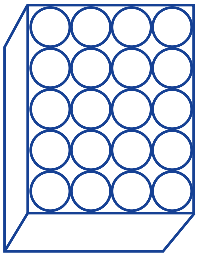
e. What effect will doubling the length, halving the width and tripling the height of a rectangular prism have on its volume?

10. MC A hemispherical bowl has a thickness of 2 cm and an outer diameter of 25 cm. If the bowl is filled with water the capacity of the water will be closest to:

A. 1.526 L
B. 1.308 33 L
C. 3.052 08 L
D. 2.616 66 L
E. 2.424 52 L



11. Tennis balls of diameter 8 cm are packed in a box 40 cm × 32 cm × 10 cm, as shown. How much space is left unfilled?



12. **WE13** A cylindrical water tank has a diameter of 1.5 m and a height of 2.5 m. What is the capacity (in litres) of the tank?

13. A monument in the shape of a rectangular pyramid (base length of 10 cm, base width of 6 cm, height of 8 cm), a spherical glass ball (diameter of 17 cm) and conical glassware (radius of 14 cm, height of 10 cm) are packed in a rectangular prism of dimensions 30 cm by 25 cm by 20 cm. The extra space in the box is filled up by a packing material. What volume of packing material is required?

14. A swimming pool is being constructed so that it is the upper part of an inverted square-based pyramid.

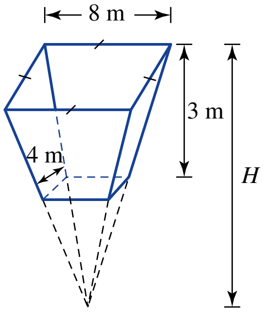
a. Calculate H.

b. Calculate the volume of the pool.

c. How many 6 m^3 bins will be required to take the dirt away?

d. How many litres of water are required to fill this pool?

e. How deep is the pool when it is half-filled?



15. A soft drink manufacturer is looking to repackage cans of soft drink to minimise the cost of packaging while keeping the volume constant.

Consider a can of soft drink with a capacity of 400 mL.

a. If the soft drink was packaged in a spherical can:

  i. find the radius of the sphere

  ii. find the total surface area of this can.

b. If the soft drink was packaged in a cylindrical can with a radius of 3 cm:

  i. find the height of the cylinder

  ii. find the total surface area of this can.

c. If the soft drink was packaged in a square-based pyramid with a base side length of 6 cm:

  i. find the height of the pyramid

  ii. find the total surface area of this can.

d. Which can would you recommend the soft drink manufacturer use for its repackaging? Why?

16. The volume of a cylinder is given by the formula $V = \pi r^2 h$.

a. Transpose the formula to make h the subject.

b. A given cylinder has a volume of 1600 cm^3. Find its height if it has a radius of:

  i. 4 cm

  ii. 8 cm.

c. Transpose the formula to make r the subject.

d. What restrictions must be placed on r? Why?

e. A given cylinder has a volume of 1800 cm^3. Find its radius if it has a height of:

  i. 10 cm

  ii. 15 cm.

17. A toy maker has enough rubber to make one super-ball of radius 30 cm. How many balls of radius 3 cm can he make from this rubber?

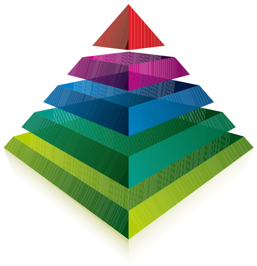
18. A manufacturer plans to make a cylindrical water tank to hold 2000 L of water.

a. What must the height be if he uses a radius of 500 cm?

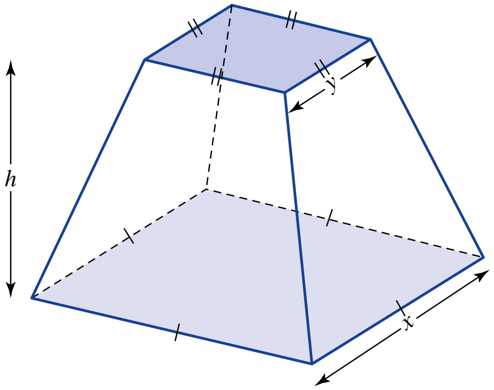
b. What must the radius be if he uses a height of 500 cm?

c. What will be the surface area of each of the two tanks? Assume the tank is a closed cylinder and give your answer in square metres.

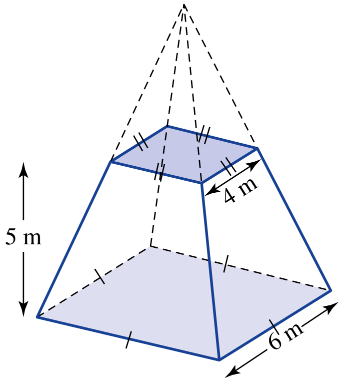
19. The ancient Egyptians knew that the volume of the frustum of a square-based pyramid was given by the formula $V = \frac{1}{3}h(x^2 + xy + y^2)$, although how they discovered this is unclear. (A frustum is the part of a cone or pyramid that is left when the top is cut off.)



a. Find the volume of the frustum shown below.

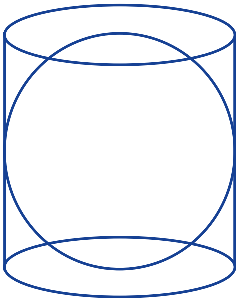


b. What would be the volume of the missing portion of the square-based pyramid shown?



Reasoning

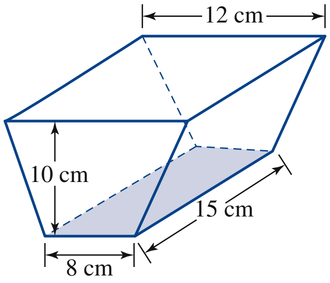
20. Archimedes is considered to be one of the three greatest mathematicians of all time (along with Newton and Gauss). He discovered several of the formulas used in this chapter. Inscribed on his tombstone was a diagram of his proudest discovery. It shows a sphere inscribed (fitting exactly) into a cylinder.

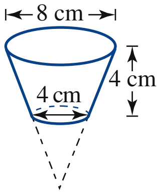


Show that $\dfrac{\text{volume of the cylinder}}{\text{volume of the sphere}} = \dfrac{\text{surface area of the cylinder}}{\text{surface area of the sphere}}$.

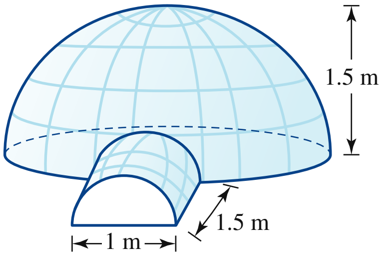
21. Marion has mixed together ingredients for a cake. The recipe requires a baking tin that is cylindrical in shape with a diameter of 20 cm and a height of 5 cm. Marion only has a tin as shown below left and a muffin tray consisting of 24 muffin cups. Each of the muffin cups in the tray is a portion of a cone as shown in the diagram below.

Should Marion use the tin or the muffin tray? Explain.





22. Nathaniel and Andrew are going to the snow for survival camp. They plan to construct an igloo, consisting of an entrance and hemispherical structure, as shown. Nathaniel and Andrew are asked to redraw their plans and increase the size of the liveable region (hemispherical structure) so that the total volume (including the entrance) is doubled. How can this be achieved?



23. Sam is having his 16th birthday party and wants to make an ice trough to keep drinks cold. He has found a square piece of sheet metal with a side length of 2 metres. He cuts squares of side length x metres from each corner, then bends the sides of the remaining sheet.

When four squares of the appropriate side length are cut from the corners the capacity of the trough can be maximised at 588 litres. Explain how Sam should proceed to maximise the capacity of the trough.

24. The Hastings family house has a rectangular roof with dimensions 17 m × 10 m providing water to three cylindrical water tanks, each with a radius of 1.25 m and a height of 2.1 m. Show that approximately 182 millimetres of rain must fall on the roof to fill the tanks.

Problem solving

25. Six tennis balls are just contained in a cylinder as the balls touch the sides and the end sections of the cylinder. Each tennis ball has a radius of R cm.

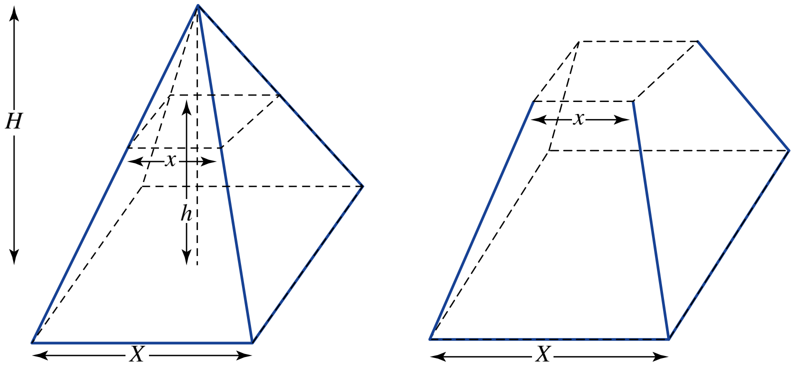
a. Express the height of the cylinder in terms of R.

b. Find the total volume of the tennis balls.

c. Find the volume of the cylinder in terms of R.

d. Show that the ratio of the volume of the tennis balls to the volume of the cylinder is 2 : 3.

26. A frustum of a square-based pyramid is a square pyramid with the top sliced off. H is the height of the full pyramid and h is the height of the frustum.



a. Find the volume of the large pyramid which has a square base side of X cm.
b. Find the volume of the small pyramid which has a square base side of x cm.
c. Show that the relationship between H and h is given by $H = \dfrac{Xh}{X - x}$.
d. Show that the volume of the frustum is given by $\frac{1}{3}h(X^2 + x^2 + Xx)$.

Reflection

Volume is measured in cubic units. How is this reflected in the volume formula?

CHALLENGE 6.2

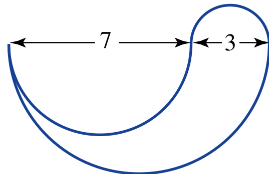
A large container is five-eighths full of ice-cream. After removing 27 identical scoops it is one-quarter full. How many scoops of ice-cream are left in the container?

6.5 Review

6.5.1 Review questions

Fluency

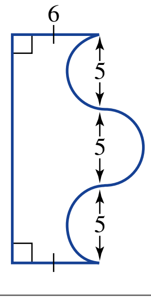
1. If all measurements are in cm, the area of the figure below is:



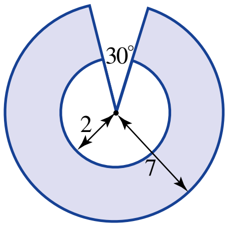
A. 16.49 cm^2 **B.** 39.25 cm^2 **C.** 9.81 cm^2 **D.** 23.56 cm^2 **E.** 30 cm^2

2. If all measurements are in centimetres, the area of the figure below is:

A. 50.73 cm^2 **B.** 99.82 cm^2 **C.** 80.18 cm^2 **D.** 90 cm^2 **E.** 119.45 cm^2

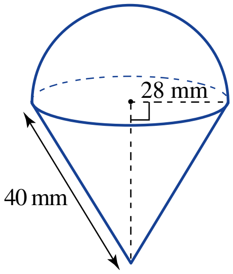


3. If all measurements are in centimetres, the shaded area of the figure below is:



A. 3.93 cm^2 **B.** 11.52 cm^2 **C.** 388.77 cm^2 **D.** 141.11 cm^2 **E.** 129.59 cm^2

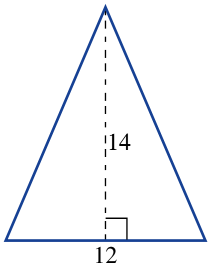
4. The total surface area of the solid below is:



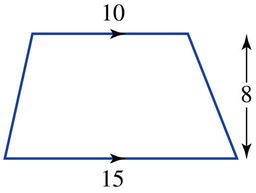
A. 8444.6 mm^2 **B.** 9221 mm^2 **C.** 14146.5 mm^2 **D.** 50271.1 mm^2 **E.** 16609.5 mm^2

5. Find the areas of the following plane figures. All measurements are in cm.

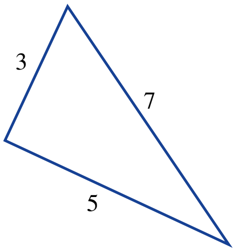
a.



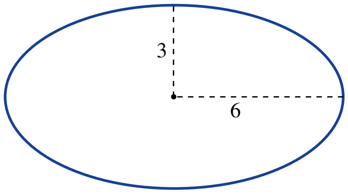
b.



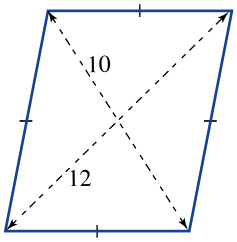
c.



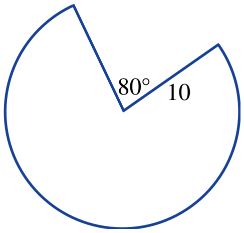
d.



e.

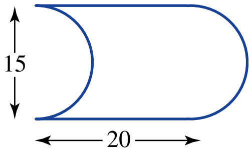


f.

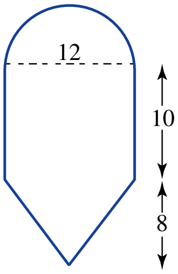


6. Find the areas of the following figures. All measurements are in cm.

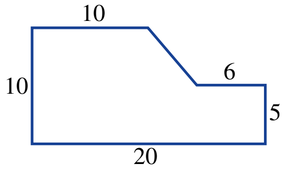
a.



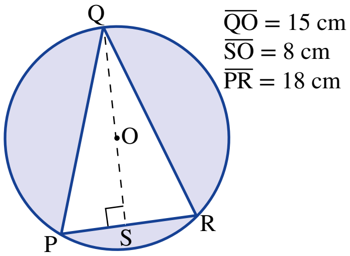
b.

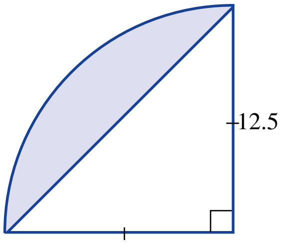


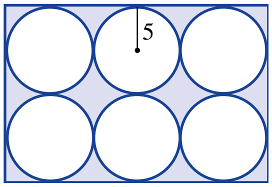
c.



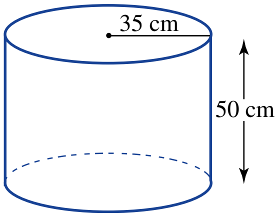
7. Find the shaded area in each of the following. All measurements are in cm.

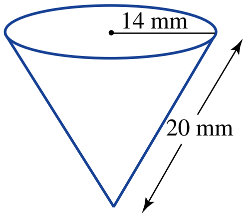
a.


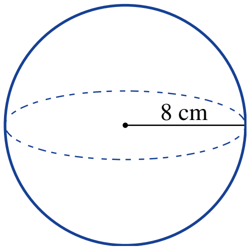
b.


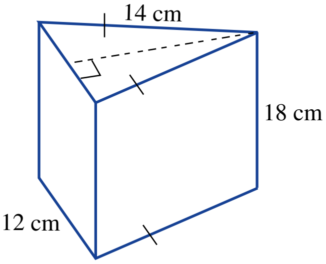
c.


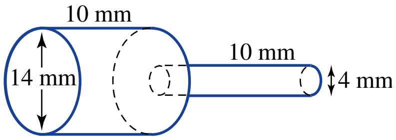
8. Find the total surface area of each of the following solids.

a.


b.


c.


d.


e.


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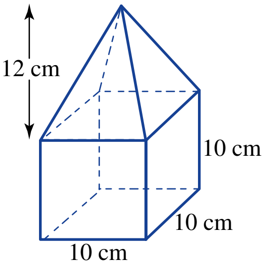
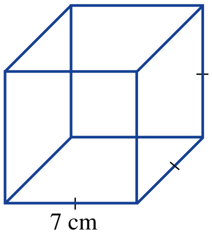
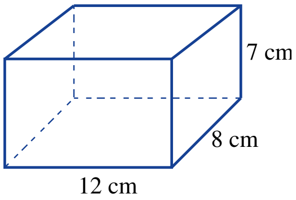
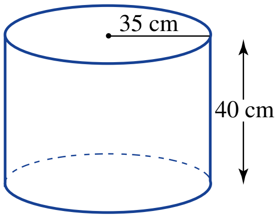
f.


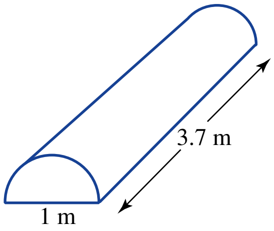
9. Find the volume of each of the following.

a.


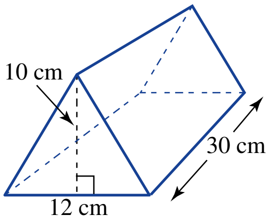
b.


c.


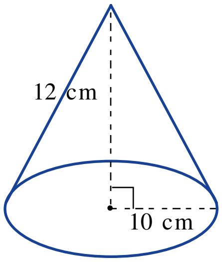
d.



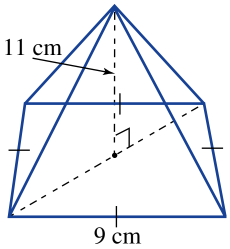
e.



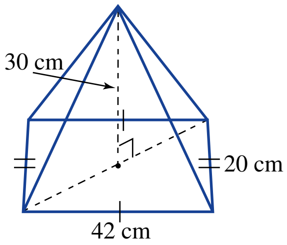
f.



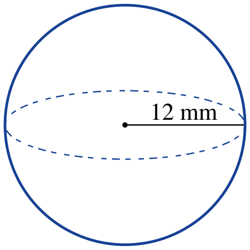
g.



h.



i.



Problem solving

10. A rectangular block of land 4 m × 25 m is surrounded by a concrete path 1 m wide.

a. Calculate the area of the path.

b. Calculate the cost of concreting at $45 per square metre.

11. What effect will tripling the radius and dividing the height of a cylinder by 6 have on its volume (in comparison with the original shape)?

12. What effect will halving the length, tripling the width and doubling the height of a rectangular prism have on its volume (in comparison with the original shape)?

13. A cylinder of radius 14 cm and height 20 cm is joined to a hemisphere of radius 14 cm to form a bread holder.

a. Find the total surface area.

b. Find the cost of chroming the bread holder on the outside at $0.05 per cm^2.

c. What is the storage volume of the bread holder?

d. How much more space is in this new bread holder than the one it is replacing, which had a quarter circle end with a radius of 18 cm and a length of 35 cm?

14. Bella Silos has two rows of silos for storing wheat. Each row has 16 silos and all the silos are identical, with a cylindrical base (height of 5 m, diameter of 1.5 m) and conical top (diameter of 1.5 m, height of 1.1 m).

a. What is the slant height of the conical tops?

b. What is the total surface area of all the silos?

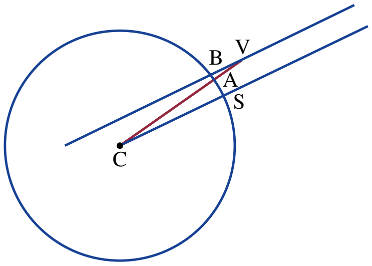
c. What will it cost to paint the silos if one litre of paint covers 40 m^2 at a bulk order price of $28.95 per litre?

d. How much wheat can be stored altogether in these silos?

e. Wheat is pumped from these silos into cartage trucks with rectangular containers 2.4 m wide, 5 m long and 2.5 m high. How many truckloads are necessary to empty all the silos?

f. If wheat is pumped out of the silos at 2.5 m^3/min, how long will it take to fill one truck?

15. The Greek mathematician Eratosthenes developed an accurate method for calculating the circumference of the Earth 2200 years ago! The figure at right illustrates how he did this. In this figure, A is the town of Alexandria and S is the town of Syene, exactly 787 km due south. When the sun's rays (blue lines) were vertical at Syene, they formed an angle of 7.2° at Alexandria (∠BVA = 7.2°), obtained by placing a stick at A and measuring the angle formed by the sun's shadow with the stick.



a. Assuming that the sun's rays are parallel, what is the angle $\angle SCA$?
b. Given that the arc AS = 787 km, determine the radius of the Earth, SC.
c. Given that the true radius is 6380 km, determine Eratosthenes' percentage error.

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- **Try out this interactivity:** Word search: Topic 6 (int-2841)
- **Try out this interactivity:** Crossword: Topic 6 (int-2842)
- **Try out this interactivity:** Sudoku: Topic 6 (int-3593)
- **Complete this digital doc:** Concept map: Topic 6 (doc-13722)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

area
capacity
circle
composite figure
cone
cross-section
cube
cylinder
ellipse
face
hemisphere
parallelogram
prism
pyramid
rectangle
rhombus
sector
semi-perimeter
sphere
square
surface
trapezium
triangle
volume

Investigation | Rich task

So close!

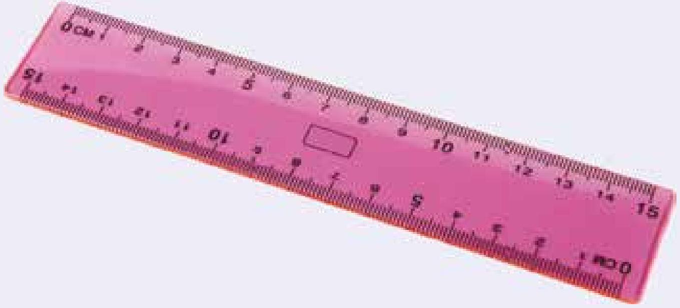
Humans must measure! Imagine what a chaotic world it would be if we didn't measure anything. Some of the things we measure are time, length, weight and temperature; we also use other measures derived from these such as area, volume, speed.

Accurate measurement is important. The accuracy of a measurement depends on the instrument being used to measure and the interpretation of the measurement. There is no such thing as a perfectly accurate measurement. The best we can do is learn how to make meaningful use of the numbers we read off our devices. It is also important to use appropriate units of measurement.

Measurement errors

When we measure a quantity by using a scale, the accuracy of our measurement depends on the markings on the scale. For example, the ruler shown can measure both in centimetres and millimetres.

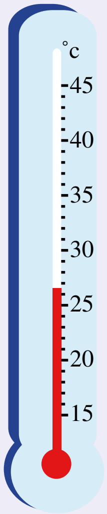
Measurements made with this ruler would have ± 0.5 mm added to the measurement. The quantity ± 0.5 is called the tolerance of measurement or measurement error.



Tolerance of measurement = $\frac{1}{2}$ × size of smallest marked unit

For a measurement of 5.6 ± 0.5 mm, the largest possible value is 5.6 cm + 0.5 mm = 5.65 cm, and the smallest value is 5.6 cm − 0.5 mm = 5.55 cm.

1. For the thermometer scale at right:
   - **a.** determine the temperature
   - **b.** state the measurement with its tolerance
   - **c.** determine the largest and smallest possible values.
2. Calculate the largest and smallest values for:
   - **a.** (56.2 ± 0.1) − (19.07 ± 0.05)
   - **b.** (78.4 ± 0.25) × (34 ± 0.1).



Significant figures in measurement

A significant figure is any non zero-digit, any zero appearing between two non-zero digits, any trailing zeros in a number containing a decimal point, and any digits in the decimal places. For example, the number 345.6054 has 7 significant figures, whereas 300 has 1 significant figure.

The number of significant figures is an expression of the accuracy of a measurement. The greater the number of significant figures, the more accurate the measurement. For example, a fast food chain claims it has sold 6 000 000 000 hamburgers, not 6 453 456 102. The first measurement has only 1 significant figure and is a very rough approximation of the actual number sold, which has 10 significant figures.

Reducing the number of significant figures is a process that is similar to rounding.

Rounding and measurement error in calculations

When you perform calculations, it is important to keep as many significant digits as practical, and to perform any rounding as the *final* step. For example, calculating 5.34×341 by rounding to 2 significant figures *before* multiplying gives $5.30 \times 340 = 1802$, compared with 1820 if the rounding is carried out after the multiplication.

Calculations that involve numbers from measurements containing errors can result in answers with even larger errors. The smaller the tolerances, the more accurate the answers will be.

3. a. Calculate $45\,943.4503 \times 86.765\,303$ by:
  - **i.** first rounding each number to 2 significant figures
  - **ii.** rounding only the answer to 2 significant figures.

b. Compare the two results.

Error in area and volume resulting from an error in a length measurement

The side length of a cube is measured and incorrectly recorded as 5 cm. The actual size is 6 cm. The effect of the length measurement error used on calculations of the surface area is shown below. Complete the calculations for volume.

Error used in length measurement = 1 cm

Surface area calculated with incorrectly recorded value $= 5^2 \times 6 = 150\text{ cm}^2$

Surface area calculated with actual value $= 6^2 \times 6 = 216\text{ cm}^2$

$$\textbf{Percentage error} = \frac{216 - 150}{6} \times 100\% \approx 30.5\%$$

4. a. Complete a similar calculation for the volume of the cube using the incorrectly recorded length. What conclusion can you make regarding errors when the number of dimensions increase?

b. Give three examples of a practical situation where an error in measuring or recording would have a potentially disastrous impact.

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Complete this digital doc: Code puzzle: Australian inventions! (doc-15927)

Answers

TOPIC 6 Surface area and volume

Exercise 6.2 Area

1. a. 16 cm^2 **b.** 48 cm^2 **c.** 75 cm^2 **d.** 120 cm^2 **e.** 706.86 cm^2 **f.** 73.5 mm^2

g. 254.47 cm^2 **h.** 21 m^2 **i.** 75 cm^2

2. Part **e** = 225π cm^2; part **g** = 81π cm^2

3. a. 20.66 cm^2 **b.** 7.64 cm^2

4. a. 113.1 mm^2 **b.** 188.5 mm^2

5. a. i. 12π cm^2 **ii.** 37.70 cm^2

b. i. $\frac{69\pi}{2}$ mm^2 **ii.** 108.38 mm^2

c. i. 261π cm^2 **ii.** 819.96 cm^2

6. E

7. D

8. a. 123.29 cm^2 **b.** 1427.88 m^2 **c.** 52 cm^2 **d.** 30.4 m^2 **e.** 78 cm^2 **f.** 2015.5 cm^2

9. a. 125.66 cm^2 **b.** 102.87 m^2 **c.** 13.73 m^2 **d.** 153.59 m^2 **e.** 13.86 m^2 **f.** 37.5 m^2

10. 11 707.92 cm^2

11. 21 m^2

12. 60

13. $840

14. a. 260.87 m^2 **b.** 195.71 m^2 **c.** 75%

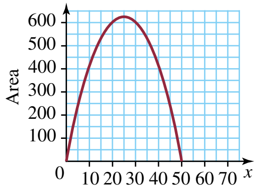
15. a. $50 = x + y$ **b.** $y = 50 - x$ **c.** Area $= 50x - x^2$

d.

x	0	5	10	15	20	25	30	35	40	45	50
Area(m^2)	0	225	400	525	600	625	600	525	400	225	0

e. No, impossible to make a rectangle.

f.



g. $x = 25$ **h.** $y = 25$ **i.** Square **j.** 625 m^2

k. $r = 15.92$ m **l.** 795.77 m^2 **m.** 170.77 m^2

16. a. Students' work **b.** 2020.83 m; horizontal

17. a. Circular area, 1790.49 m^2; rectangular area, 1406.25 m^2

b. Circular area, $\left(\frac{1}{4\pi}n^2\right)$ m^2; rectangular (square) area, $\left(\frac{1}{16}n^2\right)$ m^2. Circular area is always $\frac{4}{\pi}$ or 1.27 times larger.

18. $x = 50$ cm, $y = 30$ cm

19. Teacher to check

Challenge 6.1

$\frac{29}{50}$

Exercise 6.3 Total surface area

1. a. 600 cm^2 **b.** 384 cm^2 **c.** 1440 cm^2 **d.** 27 m^2

2. a. 113.1 m^2 **b.** 6729.3 cm^2 **c.** 8.2 m^2 **d.** 452.4 cm^2

3. a. 1495.4 cm^2 **b.** 502.7 cm^2

4. a. 506.0 cm^2 **b.** 9.4 m^2 **c.** 340.4 cm^2 **d.** 224.1 cm^2

5. a. 13.5 m^2 **b.** 90 m^2 **c.** 11 309.7 cm^2 **d.** 9852.0 mm^2 **e.** 125.7 cm^2 **f.** 1531.4 cm^2

6. a. 880 cm^2 **b.** 3072.8 cm^2 **c.** 75 cm^2 **d.** 70.4 cm^2 **e.** 193.5 cm^2 **f.** 1547.2 cm^2

7. B

8. 60

9. 11 216 cm^2

10. **a.** 70.0 m^2 **b.** \$455

11. **a.** 3063.1 cm^2 **b.** \$168.47

12. **a.** $\theta = 120°$ **b.** $x = 1; y = \sqrt{3}$ **c.** $3\sqrt{3}$ cm^2 **d.** $6\sqrt{3}$ cm^2 **e.** 32

13. **a.** 8π m^2 **b.** $\frac{\sqrt{7}}{2}$ m **c.** $2\sqrt{7}\pi$ m^2 **d.** $\sqrt{7} \times \sqrt{7} \times 1$

14. The calculation is correct.

15. **a.** 6.6 m^2

b. Back wall = 80 tiles
Side wall = 50 tiles
80 + 50 + 50 = 180 tiles

c. Cheapest: 30 cm by 30 cm, \$269.50; 20 cm by 20 cm (individually) \$270; 20 cm by 20 cm (boxed) \$276.50

16. $r = \dfrac{3\sqrt{3}a}{2}$

17. **a.** Arc length XY = $(x + s)\theta$

Arc length AB = $x\theta$

b. **i.** $x = \dfrac{2\pi t}{\theta} = \dfrac{st}{r - t}$

ii. $\dfrac{x}{x + s} = \dfrac{t}{r}$

c. Area of sector AVB = $\dfrac{x^2\theta}{2}$

Area of sector XVY = $\dfrac{(s + x)^2\theta}{2}$

Area of ABYX = $\dfrac{s\theta(s + 2x)}{2}$

TSA of frustum $= \pi(t^2 + r^2) + \dfrac{s\theta(s + 2x)}{2}$

18. The area of material required is 1.04 m^2. If Tina is careful in placing the pattern pieces, she may be able to cover the footstool.

Exercise 6.4 Volume

1. **a.** 27 cm^3 **b.** 74.088 m^3 **c.** 3600 cm^3 **d.** 94.5 cm^3

2. **a.** 450 mm^3 **b.** 360 cm^2

3. **a.** 6333.5 cm^3 **b.** 19.1 m^3 **c.** 280 cm^3 **d.** 288 mm^3 **e.** 91.6 m^3 **f.** 21 470.8 cm^3

4. **a.** 7.2 m^3 **b.** 14 137.2 cm^3 **c.** 1436.8 mm^3 **d.** 523 598.8 cm^3

5. **a.** 11 397.34 cm^3 **b.** 1.44 m^3 **c.** 12 214.51 mm^3 **d.** 101.93

6. **a.** 377.0 cm^3 **b.** 2303.8 mm^3

7. **a.** 400 cm^3 **b.** 10 080 cm^3

8. **a.** 1400 cm^3 **b.** 10 379.20 cm^3 **c.** 41.31 cm^3 **d.** 48.17 cm^3 **e.** 218.08 cm^3 **f.** 3691.37 cm^3

9. **a.** $V_{\text{new}} = 27l^3$, the volume will be 27 times as large as the original volume.

b. $V_{\text{new}} = \frac{1}{8} l^2$, the volume will be $\frac{1}{8}$ of the original volume.

c. $V_{\text{new}} = 2\pi r^2 h$, the volume will be twice as large as the original volume.

d. $V_{\text{new}} = \pi r^2 h$, the volume will remain the same.

e. $V_{\text{new}} = 3lwh$, the volume will be 3 times as large as the original value.

10. E

11. 7438.35 cm^3

12. 4417.9 L

13. 10 215.05 cm^3

14. **a.** $H = 6$ m **b.** 112 m^3 **c.** 19 bins **d.** 112 000 L **e.** 1.95 m from floor

15. **a.** **i.** 4.57 cm **ii.** 262.5 cm^2

b. **i.** 14.15 cm **ii.** 323.27 cm^2

c. **i.** 33.33 cm **ii.** 437.62 cm^2

d. Sphere. Costs less for a smaller surface area.

16. a. $h = \dfrac{V}{\pi r^2}$

b. i. 31.8 cm **ii.** 8.0 cm

c. $\sqrt{\dfrac{V}{\pi h}}$

d. $r \geq 0$, since r is a length

e. i. 7.6 cm **ii.** 6.2 cm

17. 1000

18. a. 2.55 cm **b.** 35.68 cm **c.** $A_a = 157.88\text{ m}^2$, $A_b = 12.01\text{ m}^2$

19. a. 126.67 m^3 **b.** 53.33 m^3

20. Answers will vary.

21. Required volume = 1570.80 cm^3; tin volume = 1500 cm^3; muffin tray volume = 2814.72 cm^3. Marion could fill the tin and have a small amount of mixture left over, or she could almost fill 14 of the muffin cups and leave the remaining cups empty.

22. Increase radius of hemispherical section to 1.92 m.

23. Cut squares of side length, $s = 0.3$ m or 0.368 m from the corners.

24. Volume of water needed; 30.9 m^3.

25. a. $H = 12R$ **b.** $8\pi R^3$ **c.** $12\pi R^3$ **d.** $8 : 12 = 2 : 3$

26. a. $\frac{1}{3}X^2H$ **b.** $\frac{1}{3}x^2(H - h)$ **c, d.** Check with your teacher.

Challenge 6.2

18 scoops

6.5 Review

1. D

2. C

3. E

4. A

5. a. 84 cm^2 **b.** 100 cm^2 **c.** 6.50 cm^2 **d.** 56.52 cm^2 **e.** 60 cm^2 **f.** 244.35 cm^2

6. a. 300 cm^2 **b.** 224.55 cm^2 **c.** 160 cm^2

7. a. 499.86 cm^2 **b.** 44.59 cm^2 **c.** 128.76 cm^2

8. a. $18\,692.48\text{ cm}^2$ **b.** 1495.40 cm^2 **c.** 804.25 cm^2 **d.** 871.79 cm^2 **e.** 873.36 mm^2 **f.** 760 cm^2

9. a. 343 cm^3 **b.** 672 cm^3 **c.** $153\,938.04\text{ cm}^3$ **d.** 1.45 m^3 **e.** 1800 cm^3 **f.** 1256.64 cm^3

g. 297 cm^3 **h.** 8400 cm^3 **i.** 7238.23 mm^3

10. a. 62 m^2 **b.** \$2790

11. $V = \frac{3}{2}\pi r^2 h$, the volume will be 1.5 times as large as the original volume.

12. $V = 3lwh$, the volume will be 3 times as large as (or triple) the original volume.

13. a. 3606.55 cm^2 **b.** \$180.33 **c.** 18062.06 cm^3 **d.** 9155.65 cm^3

14. a. 1.33 m **b.** 910.91 m^2 **c.** \$659.27 **d.** 303.48 m^3 **e.** 11 trucks **f.** 12 minutes

15. a. 7.2° **b.** 6263 km **c.** 1.8% error

Investigation — Rich task

1. a. The temperature reading is 26.5°C.

b. The smallest unit mark is 1°C, so the tolerance is 0.5.

c. Largest possible value = 27°C, smallest possible value = 26°C

2. a. Largest value = 37.28, smallest value = 36.98

b. Largest value = 2681.965, smallest value = 2649.285

3. a. i. 4002000 **ii.** 4000000

b. The result for **i** has 4 significant figures, whereas **ii** has only 1 significant figure after rounding. However, **ii** is closer to the actual value (3986297.3861449409).

4. a. Volume using the incorrectly recorded value = 125 cm^3

Volume using the actual value = 216 cm^3

The percentage error is 42.1%, which shows that the error compounds as the number of dimensions increases.

b. Check with your teacher.

TOPIC 7
Quadratic expressions

7.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

7.1.1 Why learn this?

How is your algebraic tool kit? Is there some room to expand your skills? As expressions become more complex, more power will be needed to manipulate them and to carry out basic skills such as adding, multiplying and factorising. Dealing with quadratic expressions is the first step to higher-level skills.

7.1.2 What do you know?

assesson

1. **THINK** List what you know about quadratic expressions. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of quadratic expressions.

LEARNING SEQUENCE

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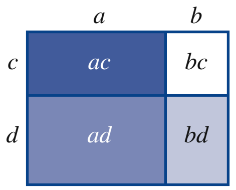
Watch this eLesson: The story of mathematics: Adelard of Bath (int-1846)

7.2 Expanding algebraic expressions

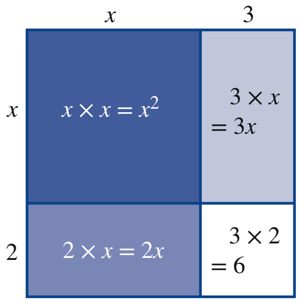
7.2.1 Binomial expansion

- Consider the rectangle of length $a + b$ and width $c + d$ shown below. Its area is equal to $(a + b)(c + d)$.



The diagram shows that $\underbrace{(a + b)(c + d)}_{\text{factorised form}} = \underbrace{ac + ad + bc + bd}_{\text{expanded form}}$.

- Expansion of the binomial expression $(x + 3)(x + 2)$ can be shown by this area model.



Expressed mathematically this is:

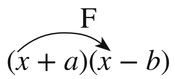
$$\begin{aligned}(x + 3)(x + 2) &= x^2 + 2x + 3x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

$$\underbrace{\text{factorised form}} \qquad \underbrace{\text{expanded form}}$$

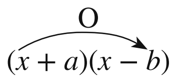
- There are several methods that can be used to expand binomial factors.

7.2.2 FOIL method

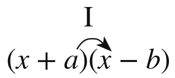
- The word **FOIL** provides us with an acronym for the expansion of a binomial product.
  - **First:** multiply the first terms in each bracket



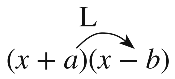
  - **Outer:** multiply the two outer terms



  - **Inner:** multiply the two inner terms



  - **Last:** multiply the last terms in each bracket



WORKED EXAMPLE 1

Expand each of the following.

a $(x+3)(x+2)$

b $(x-7)(6-x)$

THINK	WRITE
a 1 Write the expression.	**a** $(x+3)(x+2)$
2 Multiply the first terms in each bracket, then the outer terms, the inner terms and finally the last terms.	$= x \times x,\ x \times 2,\ x \times 3,\ 3 \times 2$
3 Collect like terms.	$= x^2 + 2x + 3x + 6$ $= x^2 + 5x + 6$
b 1 Write the expression.	**b** $(x-7)(6-x)$
2 Multiply the first terms in each bracket, then the outer terms, the inner terms and finally the last terms.	$= x \times 6,\ x \times -x,\ -7 \times 6,\ -7 \times -x$
3 Remove the brackets by multiplying each term in the brackets by the term outside the bracket. Remember to change the sign when the term outside the bracket is negative.	$= 6x - x^2 - 42 + 7x$
4 Collect like terms.	$= -x^2 + 13x - 42$

- If there is a term outside the pair of brackets, expand the brackets and then multiply each term of the expansion by that term.

WORKED EXAMPLE 2

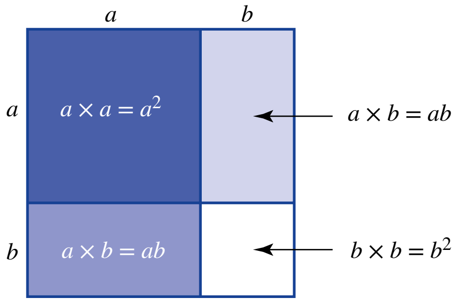
TI | CASIO

Expand $3(x+8)(x+2)$.

THINK	WRITE
1 Write the expression.	$3(x+8)(x+2)$
2 Use FOIL to expand the pair of brackets.	$= 3(x^2 + 2x + 8x + 16)$
3 Collect like terms within the brackets.	$= 3(x^2 + 10x + 16)$
4 Multiply each of the terms inside the brackets by the term outside the brackets.	$= 3x^2 + 30x + 48$

7.2.3 The square of a binomial

- The expansion of $(a + b)^2$ can be represented by this area model.



$$\begin{aligned}(a+b)^2 &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

- Similarly $(a - b)^2 = a^2 - 2ab + b^2$.
- This expansion is often memorised. To find the square of a binomial:
  - square the first term
  - multiply the two terms together and then double them
  - square the last term.

WORKED EXAMPLE 3 TI | CASIO

Expand and simplify each of the following.

a $(2x - 5)^2$ **b** $-3(2x + 7)^2$

THINK	WRITE
a 1 Write the expression.	**a** $(2x - 5)^2$
2 Expand using the rule $(a - b)^2 = a^2 - 2ab + b^2$.	$= (2x)^2 - 2 \times 2x \times 5 + (5)^2$ $= 4x^2 - 20x + 25$
b 1 Write the expression.	**b** $-3(2x + 7)^2$
2 Expand the brackets using the rule $(a + b)^2 = a^2 + 2ab + b^2$.	$= -3[(2x)^2 + 2 \times 2x \times 7 + (7)^2]$ $= -3(4x^2 + 28x + 49)$
3 Multiply every term inside the brackets by the term outside the brackets.	$= -12x^2 - 84x - 147$

7.2.4 The difference of two squares

- When $a + b$ is multiplied by $a - b$ (or vice-versa),

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

The expression is called the difference of two squares and is often referred to as DOTS. This result can be memorised as a short cut.

WORKED EXAMPLE 4

TI | CASIO

Expand and simplify each of the following.

a $(3x + 1)(3x - 1)$ **b** $4(2x - 7)(2x + 7)$

THINK	WRITE
a **1** Write the expression.	**a** $(3x + 1)(3x - 1)$
2 Expand using the rule $(a + b)(a - b) = a^2 - b^2$.	$= (3x)^2 - (1)^2$ $= 9x^2 - 1$
b **1** Write the expression.	**b** $4(2x - 7)(2x + 7)$
2 Expand using the difference of two squares rule.	$= 4[(2x)^2 - (7)^2]$ $= 4(4x^2 - 49)$
3 Multiply by 4.	$= 16x^2 - 196$

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Expanding brackets (doc-5244)

Complete this digital doc: SkillSHEET: Expanding a pair of brackets (doc-5245)

Exercise 7.2 Expanding algebraic expressions

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–f, 2a–h, 3a–d, 5a–d, 6a, 8a–d, 9a–d, 10a–f, 11–14	Questions: 1d–l, 2d–j, 3c–f, 4a–c, 5c–f, 6, 7, 8c–f, 9c–f, 10–17, 19	Questions: 1d–l, 2f–l, 3e–i, 4, 5e–h, 6, 7, 8g–l, 9e–i, 10–20

🟩🟦🟪 **Individual pathway interactivity:** int-4596

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Fluency

1. Expand each of the following.

a. $2(x + 3)$ **b.** $4(x - 5)$ **c.** $3(7 - x)$
d. $-(x + 3)$ **e.** $x(x + 2)$ **f.** $2x(x - 4)$
g. $3x(5x - 2)$ **h.** $5x(2 - 3x)$ **i.** $2x(4x + 1)$
j. $2x^2(2x - 3)$ **k.** $3x^2(2x - 1)$ **l.** $5x^2(3x + 4)$

2. **WE1** Expand each of the following.

a. $(x + 3)(x - 4)$ **b.** $(x + 1)(x - 3)$ **c.** $(x - 7)(x + 2)$
d. $(x - 1)(x - 5)$ **e.** $(2 - x)(x + 3)$ **f.** $(x - 4)(x - 2)$
g. $(2x - 3)(x - 7)$ **h.** $(x - 1)(3x + 2)$ **i.** $(3x - 1)(2x - 5)$
j. $(3 - 2x)(7 - x)$ **k.** $(5 - 2x)(3 + 4x)$ **l.** $(11 - 3x)(10 + 7x)$

3. **WE2** Expand each of the following.
   - a. $2(x+1)(x-3)$
   - b. $4(2x+1)(x-4)$
   - c. $-2(x+1)(x-7)$
   - d. $2x(x-1)(x+1)$
   - e. $3x(x-5)(x+5)$
   - f. $6x(x-3)(x+3)$
   - g. $-2x(3-x)(x-3)$
   - h. $-5x(2-x)(x-4)$
   - i. $6x(x+5)(4-x)$
4. Expand each of the following.
   - a. $(x-1)(x+1)(x+2)$
   - b. $(x-3)(x-1)(x+2)$
   - c. $(x-5)(x+1)(x-1)$
   - d. $(x-1)(x-2)(x-3)$
   - e. $(2x-1)(x+1)(x-4)$
   - f. $(3x+1)(2x-1)(x-1)$
5. Expand each of the following and simplify.
   - a. $(x+2)(x-1)-2x$
   - b. $3x-(2x-5)(x+2)$
   - c. $(2x-3)(x+1)+(3x+1)(x-2)$
   - d. $(3-2x)(2x-1)+(4x-5)(x+4)$
   - e. $(x+1)(x-7)-(x+2)(x-3)$
   - f. $(x-2)(x-5)-(x-1)(x-4)$
   - g. $(x-3)(x+1)+\sqrt{3}x$
   - h. $(\sqrt{2}-3x)(\sqrt{3}+2x)-\sqrt{5}x$
6. **MC** a. $(3x-1)(2x+4)$ expands to:
   - **A.** $6x^2+10x-4$
   - **B.** $5x^2-24x+3$
   - **C.** $3x^2+2x-4$
   - **D.** $6x^2-10x-4$
   - **E.** $6x^2-4$

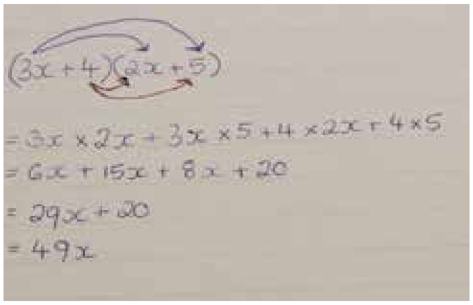
   b. $-2x(x-1)(x+3)$ expands to:
   - **A.** x^2+2x-3
   - **B.** $-2x^2-4x+6$
   - **C.** $-2x^3-4x^2+6x$
   - **D.** $-2x^3+4x^2-6x$
   - **E.** $-2x^3-3$
7. **MC** The expression $(x-1)(x-3)(x+2)$ is *not* the same as:
   - **A.** $(x-3)(x-1)(x+2)$
   - **B.** $(x+3)(x-1)(x-2)$
   - **C.** $(x-1)(x+2)(x-3)$
   - **D.** $(x+2)(x-1)(x-3)$
   - **E.** $(x-3)(x+2)(x-1)$
8. **WE3a** Expand and simplify each of the following.
   - a. $(x-1)^2$
   - b. $(x+2)^2$
   - c. $(x+5)^2$
   - d. $(4+x)^2$
   - e. $(7-x)^2$
   - f. $(12-x)^2$
   - g. $(3x-1)^2$
   - h. $(12x-3)^2$
   - i. $(5x+2)^2$
   - j. $(2-3x)^2$
   - k. $(5-4x)^2$
   - l. $(1-5x)^2$
9. **WE3b** Expand and simplify each of the following.
   - a. $2(x-3)^2$
   - b. $4(x-7)^2$
   - c. $3(x+1)^2$
   - d. $-(2x+3)^2$
   - e. $-(7x-1)^2$
   - f. $2(2x-3)^2$
   - g. $-3(2-9x)^2$
   - h. $-5(3-11x)^2$
   - i. $-4(2x+1)^2$
10. **WE4** Expand and simplify each of the following.
    - a. $(x+7)(x-7)$
    - b. $(x+9)(x-9)$
    - c. $(x-5)(x+5)$
    - d. $(x-1)(x+1)$
    - e. $(2x-3)(2x+3)$
    - f. $(3x-1)(3x+1)$
    - g. $(7-x)(7+x)$
    - h. $(8+x)(8-x)$
    - i. $(3-2x)(3+2x)$

Understanding

11. The length of the side of a rectangle is $(x+1)$ cm and the width is $(x-3)$ cm.
    - a. Find an expression for the area of the rectangle.
    - b. Simplify the expression by expanding.
    - c. If $x = 5$ cm, find the dimensions of the rectangle and, hence, its area.
12. Chickens are kept in a square enclosure with sides measuring x m. The number of chickens is increasing and so the size of the enclosure is to have 1 metre added to one side and 2 metres to the adjacent side.
    - a. Draw a diagram of the original enclosure.
    - b. Add to the first diagram or draw another one to show the new enclosure. Mark the lengths on each side on your diagram.
    - c. Write an expression for the area of the new enclosure in factorised form.
    - d. Expand and simplify the expression by removing the brackets.
    - e. If the original enclosure had sides of 2 metres, find the area of the original square and then the area of the new enclosure.

13. Shown below are three students' attempts at expanding $(3x + 4)(2x + 5)$.

STUDENT A



$(3x+4)(2x+5)$

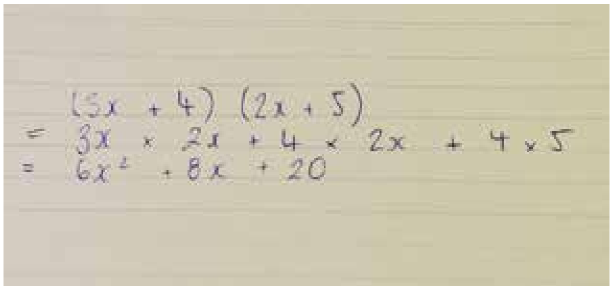
$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$

$= 6x + 15x + 8x + 20$

$= 29x + 20$

$= 49x$

STUDENT B

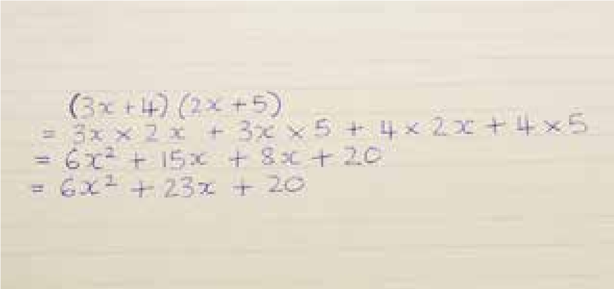


$(3x + 4)(2x + 5)$

$= 3x \times 2x + 4 \times 2x + 4 \times 5$

$= 6x^2 + 8x + 20$

STUDENT C



$(3x + 4)(2x + 5)$

$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$

$= 6x^2 + 15x + 8x + 20$

$= 6x^2 + 23x + 20$

- **a.** Which student's work was correct?
- **b.** Copy each of the incorrect answers into your workbook and correct the mistakes in each as though you were the teacher of these students.

14. If $a = 5$ and $b = 3$, show that $(a - b)(a + b) = a^2 - b^2$ by evaluating both expressions.

15. If $a = 5$ and $b = 3$, show that $(a + b)^2 = a^2 + 2ab + b^2$ by evaluating both expressions.

16. Write an expression in factorised and expanded form that is:

- **a.** a quadratic trinomial
- **b.** the square of a binomial
- **c.** the difference of two squares
- **d.** both **a** and **b**.

Reasoning

17. Explain the difference between ‘the square of a binomial’ and ‘the difference between two squares’.

18. Show that $(a + b)(c + d) = (c + d)(a + b)$.

Problem solving

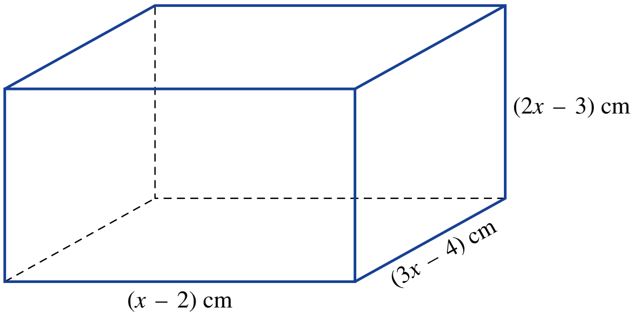
19. Expand:

a. $(2x + 3y - 5z)^2$

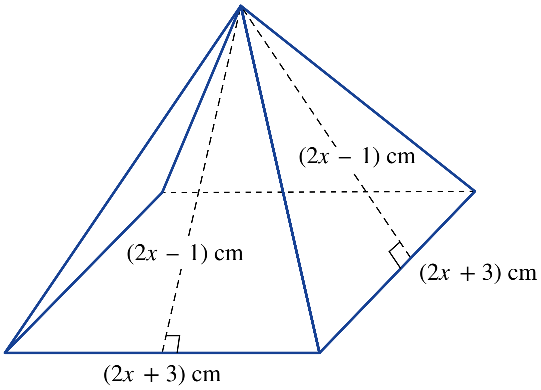
b. $\left(\left(1 + \frac{1}{2x}\right) - 2x\right)^2$

20. Find an expanded expression for:

a. the volume of the cuboid



b. the total surface area of the square-based pyramid.



Reflection

Why does the difference of two squares rule have that name?

CHALLENGE 7.1

Find all the positive integers, a, which make the expression $(a - 10)(a - 14)$ a perfect square. Consider 0 to be the first perfect square.

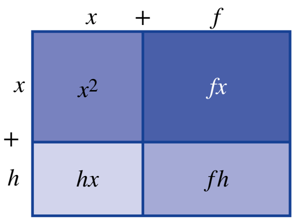
7.3 Factorising expressions with three terms

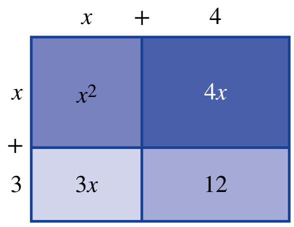
- A monic quadratic expression is an expression in the form $ax^2 + bx + c$ where $a = 1$.

7.3.1 Factorising monic quadratic trinomials

- The area model of binomial expansion can be used to find a pattern for factorising a general quadratic expression. For example,

$$\begin{aligned}(x + f)(x + h) &= x^2 + fx + hx + fh \\ &= x^2 + (f + h)x + fh\end{aligned} \qquad \begin{aligned}(x + 4)(x + 3) &= x^2 + 4x + 3x + 12 \\ &= x^2 + 7x + 12\end{aligned}$$





- To factorise a general quadratic, look for factors of c that add to b.

$$x^2 + bx + c = (x + f)\,(x + h)$$

Factors of c that add to b

$$3 + 4 = 7$$

For example, $x^2 + 7x + 12 = (x + 3)(x + 4)$.

$$3 \times 4 = 12$$

WORKED EXAMPLE 5

Factorise the following quadratic expressions.

a $x^2 + 5x + 6$ **b** $x^2 + 10x + 24$

THINK

a 1 Write the expression and:

  i check for a common factor. There is no common factor.

  ii check for a DOTS pattern. The expression is not in the form $a^2 - b^2$.

  iii check for a perfect squares pattern. 6 is not a perfect square.

  This must be a general quadratic expression.

2 i The general quadratic expression has the pattern $x^2 + 5x + 6 = (x + f)(x + h)$. f and h are a factor pair of 6 that add to 5.

  ii Calculate the sums of factor pairs of 6. The factors of 6 that add to 5 are 2 and 3, as shown in blue.

3 Substitute the values of f and h into the expression in its factorised form.

WRITE

a $x^2 + 5x + 6$

Factors of 6	Sum of factors
1 and 6	7
2 and 3	5

$x^2 + 5x + 6 = (x + 2)(x + 3)$

b 1 Check for patterns of common factors, DOTS and perfect squares patterns. None of these apply, so the expression is a general quadratic.

b $x^2 + 10x + 24$

2 **i** The general quadratic expression has the pattern $x^2 + 10x + 24 = (x + f)(x + h)$, where f and h are a factor pair of 24 that add to 10.

ii Calculate the sums of factor pairs of 24. The factors of 24 that add to 10 are 4 and 6, as shown in blue.

Factors of 24	Sum of factors
1 and 24	25
2 and 12	14
3 and 8	11
4 and 6	10

3 Substitute the values of f and h into the expression in its factorised form.

$x^2 + 10x + 24 = (x + 4)(x + 6)$

7.3.2 Factorising non-monic quadratic trinomials

- A non-monic quadratic expression is $ax^2 + bx + c$ where $a \neq 1$.
- When a quadratic trinomial in the form $ax^2 + bx + c$ is written as $ax^2 + mx + nx + c$, where $m + n = b$, the four terms can be factorised by grouping.

$$\begin{aligned}2x^2 + 11x + 12 &= 2x^2 + 8x + 3x + 12\\ &= 2x^2 + 8x + 3x + 12\\ &= 2x(x + 4) + 3(x + 4)\\ &= 2x(x + 4) + 3(x + 4)\\ &= (x + 4)(2x + 3)\end{aligned}$$

- There are many combinations of numbers that satisfy $m + n = b$; however, only one particular combination can be grouped and factorised. For example,

$$\begin{aligned}2x^2 + 11x + 12 &= 2x^2 + 7x + 4x + 12\\ &= 2x^2 + 7x + 4x + 12\\ &= x(2x + 7) + 4(x + 3)\end{aligned}$$

cannot be factorised further

or

$$\begin{aligned}2x^2 + 11x + 12 &= 2x^2 + 8x + 3x + 12\\ &= 2x^2 + 8x + 3x + 12\\ &= 2x(x + 4) + 3(x + 4)\\ &= (x + 4) + (2x + 3)\end{aligned}$$

- In examining the general binomial expansion, a pattern emerges that can be used to help identify which combination to use for $m + n = b$.

$$\begin{aligned}(dx + e)(fx + g) &= dfx^2 + dgx + efx + eg\\ &= dfx^2 + (dg + ef)x + eg\end{aligned}$$

$$\begin{aligned}m + n &= dg + ef \quad \text{and} \quad & m \times n &= dg \times ef\\ &= b & &= dgef\\ & & &= dfeg\\ & & &= ac\end{aligned}$$

Therefore, m and n are factors of ac that sum to b.

- To factorise a general quadratic where $a \neq 1$, look for factors of ac that sum to b. Then rewrite the quadratic trinomial with four terms that can then be grouped and factorised.

$$ax^2 + bx + c = ax^2 + \underset{\nwarrow}{m}x + \underset{\nearrow}{n}x + c$$

Factors of ac that sum to b

WORKED EXAMPLE 6

TI | CASIO

Factorise $6x^2 - 11x - 10$.

THINK / **WRITE**

1 Write the expression and look for common factors and special patterns. The expression is a general quadratic with $a = 6$, $b = -11$ and $c = -10$.

$6x^2 - 11x - 10 = 6x^2 + -11x + -10$

2 Since $a \neq 1$, rewrite $ax^2 + bx + c$ as $ax^2 + mx + nx + c$, where m and n are factors of ac (6×-10) that sum to b (-11). Calculate the sums of factor pairs of -60. As shown in blue, 4 and -15 are factors of -60 that add to -11.

Factors of −60 (6 × −10)	Sum of factors
−60, 1	−59
−20, 3	−17
−30, 2	−28
15, −4	11
− 15, 4	− 11

3 Rewrite the quadratic expression: $ax^2 + bx + c = ax^2 + mx + nx + c$ with $m = 4$ and $n = -15$.

$6x^2 - 11x - 10 = 6x^2 + 4x + -15x - 10$

4 Factorise using the grouping method: $6x^2 + 4x = 2x(3x + 2)$ and $-15x - 10 = -5\,(3x + 2)$
Write the answer.

$$\begin{aligned} 6x^2 - 11x - 10 &= 2x(3x + 2) + -5(3x + 2) \\ &= (3x + 2)(2x - 5) \end{aligned}$$

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Complete this digital doc: SkillSHEET: Finding a factor pair that adds to a given number (doc-5250)

Complete this digital doc: WorkSHEET: Factorising and expanding (doc-5251)

Exercise 7.3 Factorising expressions with three terms

assesson

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1a–e, 2a–e, 3a–e, 4–6, 7a–e, 8, 11, 15	Questions: 1f–j, 2f–i, 3f–i, 4, 5, 6, 7d–h, 8, 9a–d, 11, 12, 15, 16	Questions: 1k–o, 2k–l, 3j–l, 4, 5, 6, 7i–l, 8, 9, 10, 13–17

■■■ **Individual pathway interactivity:** int-4597

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE5 Factorise each of the following.

a. $x^2 + 3x + 2$
b. $x^2 + 4x + 3$
c. $x^2 + 10x + 16$
d. $x^2 + 8x + 16$
e. $x^2 - 2x - 3$
f. $x^2 - 3x - 4$
g. $x^2 - 11x - 12$
h. $x^2 - 4x - 12$
i. $x^2 + 3x - 4$
j. $x^2 + 4x - 5$
k. $x^2 + 6x - 7$
l. $x^2 + 3x - 10$
m. $x^2 - 4x + 3$
n. $x^2 - 9x + 20$
o. $x^2 + 9x - 70$

2. WE6 Factorise each of the following.

a. $-2x^2 - 20x - 18$
b. $-3x^2 - 9x - 6$
c. $-x^2 - 3x - 2$
d. $-x^2 - 11x - 10$
e. $-x^2 - 7x - 10$
f. $-x^2 - 13x - 12$
g. $-x^2 - 7x - 12$
h. $-x^2 - 8x - 12$
i. $2x^2 + 14x + 20$
j. $3x^2 + 33x + 30$
k. $5x^2 + 105x + 100$
l. $5x^2 + 45x + 100$

3. Factorise each of the following.

a. $a^2 - 6a - 7$
b. $t^2 - 6t + 8$
c. $b^2 + 5b + 4$
d. $m^2 + 2m - 15$
e. $p^2 - 13p - 48$
f. $c^2 + 13c - 48$
g. $k^2 + 22k + 57$
h. $s^2 - 16s - 57$
i. $g^2 - g - 72$
j. $v^2 - 28v + 75$
k. $x^2 + 14x - 32$
l. $x^2 - 19x + 60$

4. MC **a.** To factorise $-14x^2 - 49x + 21$, the first step is to:

A. find factors of 14 and 21 that will add to -49
B. take out 14 as a common factor
C. take out -7 as a common factor
D. find factors of 14 and -49 that will add to make 21
E. take out -14 as a common factor

b. The expression $42x^2 - 9x - 6$ can be completely factorised to:

A. $(6x - 3)(7x + 2)$
B. $3(2x - 1)(7x + 2)$
C. $(2x - 1)(21x + 6)$
D. $3(2x + 1)(7x - 2)$
E. $42(x - 3)(x + 2)$

5. MC When factorised, $(x + 2)^2 - (y + 3)^2$ equals:

A. $(x + y - 2)(x + y + 2)$
B. $(x - y - 1)(x + y - 1)$
C. $(x - y - 1)(x + y + 5)$
D. $(x - y + 1)(x + y + 5)$
E. $(x + y - 1)(x + y + 2)$

6. Factorise each of the following using an appropriate method.

a. $2x^2 + 5x + 2$
b. $2x^2 - 3x + 1$
c. $4x^2 - 17x - 15$
d. $4x^2 + 4x - 3$
e. $2x^2 - 9x - 35$
f. $3x^2 + 10x + 3$
g. $6x^2 - 17x + 7$
h. $12x^2 - 13x - 14$
i. $10x^2 - 9x - 9$
j. $20x^2 + 3x - 2$
k. $12x^2 + 5x - 2$
l. $15x^2 + x - 2$

7. Factorise each of the following, remembering to look for a common factor first.

a. $4x^2 + 2x - 6$
b. $9x^2 - 60x - 21$
c. $72x^2 + 12x - 12$
d. $-18x^2 + 3x + 3$
e. $-60x^2 + 150x + 90$
f. $24ax^2 + 18ax - 105a$
g. $-8x^2 + 22x - 12$
h. $-10x^2 + 31x + 14$
i. $-24x^2 + 35x - 4$
j. $-12x^2 - 2xy + 2y^2$
k. $-30x^2 + 85xy + 70y^2$
l. $-600x^2 - 780xy - 252y^2$

Understanding

8. Consider the expression $(x - 1)^2 + 5(x - 1) - 6$.

a. Substitute $w = x - 1$ in this expression.
b. Factorise the resulting quadratic.
c. Replace w with $x - 1$ and simplify each factor. This is the factorised form of the original expression.

9. Use the method outlined in question **8** to factorise each of the following expressions.

a. $(x + 1)^2 + 3(x + 1) - 4$ **b.** $(x + 2)^2 + (x + 2) - 6$
c. $(x - 3)^2 + 4(x - 3) + 4$ **d.** $(x + 3)^2 + 8(x + 3) + 12$
e. $(x - 7)^2 - 7(x - 7) - 8$ **f.** $(x - 5)^2 - 3(x - 5) - 10$

10. Factorise $x^2 + x - 0.75$.

11. Students decide to make Valentine's Day cards. The total area of each card is equal to $(x^2 - 4x - 5)$ cm^2.

a. Factorise the expression to find the dimensions of the cards in terms of x.
b. Write down the length of the shorter side in terms of x.
c. If the shorter sides of a card are 10 cm in length and the longer sides are 16 cm in length, find the value of x.
d. Find the area of the card proposed in part **c**.
e. If the students want to make 3000 Valentine's Day cards, how much cardboard will be required? Give your answer in terms of x.

12. The area of a rectangular playground is given by the general expression $(6x^2 + 11x + 3)$ m^2 where x is a positive whole number.

a. Find the length and width of the playground in terms of x.
b. Write an expression for the perimeter of the playground.
c. If the perimeter of a particular playground is 88 metres, find x.

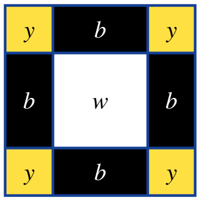
Reasoning

13. Cameron wants to build an in-ground 'endless' pool. Basic models have a depth of 2 metres and a length triple the width. A spa will also be attached to the end of the pool.

a. The pool needs to be tiled. Write an expression for the surface area of the empty pool (that is, the floor and walls only).
b. The spa needs an additional 16 m^2 of tiles. Write an expression for the total area of tiles needed for both the pool and the spa.
c. Factorise this expression.
d. Cameron decides to use tiles that are selling at a discount price, but there are only 280 m^2 of the tile available. Find the maximum dimensions of the pool he can build if the width is in whole metres. Assume the spa is to be included in the tiling.
e. What area of tiles is actually needed to construct the spa and pool?
f. What volume of water can the pool hold?

14. Fabric pieces comprising yellow squares, white squares and black rectangles are sewn together to make larger squares (patches) as shown in the diagram. The length of each black rectangle is twice its width. These patches are then sewn together to make a patchwork quilt. A finished square quilt, made from 100 patches, has an area of 1.44 m^2.

a. Determine the size of each yellow, black and white section in one fabric piece. Show your working.
b. How much (in m^2) of each of the coloured fabrics would be needed to construct the quilt? (Ignore seam allowances.)
c. Sketch a section of the finished product.



15. Each factorisation below contains an error. Identify the error in each statement.

a. $x^2 - 7x + 12 = (x + 3)(x - 4)$ **b.** $x^2 - x - 12 = (x - 3)(x + 4)$
c. $x^2 - x + 2 = (x - 1)(x + 1)$ **d.** $x^2 - 4x - 21 = (x - 3)(x - 7)$
e. $x^2 + 4x - 21 = (x + 3)(x - 7)$ **f.** $x^2 - x - 30 = (x - 5)(x + 6)$
g. $x^2 + 7x - 8 = (x + 1)(x - 8)$ **h.** $x^2 - 11x + 30 = (x - 5)(x + 6)$

Problem solving

16. Factorise:

a. $6(3a-1)^2 - 13(3a-1) - 5$ **b.** $3m^4 - 19m^2 - 14$ **c.** $2\sin^2(x) - 3\sin(x) + 1$.

17. Factorise:

a. $2x^2 + 3\sqrt{3}x - 6$ **b.** $(z+1)^3 + (z-1)^3$.

Reflection

In your own words, describe how you would factorise a quadratic trinomial.

7.4 Factorising expressions with two or four terms

7.4.1 Factorising expressions with two terms

- If the terms in an expanded expression have a common factor, the highest common factor is written at the front of the brackets and the remaining factor for each term in the expression is written in the brackets. For example, $4x^2 - 36 = 4(x^2 - 9)$.
- A Difference of Two Squares (DOTS) expression in expanded form has two squared terms separated by a subtraction symbol.

$$\underset{\substack{\uparrow \\ \text{Expanded form}}}{a^2 - b^2} = \underset{\substack{\uparrow \\ \text{Factorised form}}}{(a-b)(a+b)}$$

WORKED EXAMPLE 7 TI | CASIO

Factorise the following.

a $12k^2 + 18$ **b** $16a^2 - 25b^4$

THINK	WRITE
a 1 Write the expression and look for common factors. The terms have a highest common factor of 6. Write the 6 in front of a set of brackets, then determine what must go inside the brackets. $12k^2 = 6 \times 2k^2$, $18 = 6 \times 3$	**a** $12k^2 + 18 = 6(2k^2 + 3)$
2 Look for patterns in the expression inside the brackets to factorise further. The expression inside the brackets cannot be factorised further.	
b 1 Write the expression and look for common factors. The expression has no common factor.	**b** $16a^2 - 25b^4$
2 Look for the DOTS pattern in the expression. Write the equation showing squares.	$= 4^2a^2 - 5^2(b^2)^2$ $= (4a)^2 - (5b^2)^2$
3 Use the pattern for DOTS to write the factors. $a^2 - b^2 = (a+b)(a-b)$	$= (4a + 5b^2)(4a - 5b^2)$

7.4.2 Factorising expressions with four terms

- If there are four terms to be factorised, look for a common factor first.
- Then group the terms in pairs and look for a common factor in each pair. It may be that a new common factor emerges as a bracket (common binomial factor).
- If an expression has four terms, it may require grouping to factorise it.
- In the process known as grouping 'two and two', the terms of the expression are grouped into two pairs, then a common factor is removed from each pair.

- When selecting terms to place as pairs, each pair after factorising should result in a common binomial factor. For example:

$$2a - 6b + 3ac - 9bc = 2(a - 3b) + 3c(a - 3b)$$

WORKED EXAMPLE 8

Factorise each of the following.

a $x - 4y + mx - 4my$ **b** $x^2 + 3x - y^2 + 3y$

THINK	WRITE
a 1 Write the expression and look for a common factor. (There isn't one.)	**a** $x - 4y + mx - 4my$
2 Group the terms so that those with common factors are next to each other.	$= (x - 4y) + (mx - 4my)$
3 Take out a common factor from each group (it may be 1).	$= 1(x - 4y) + m(x - 4y)$
4 Factorise by taking out a common binomial factor. The factor $(x - 4y)$ is common to both groups.	$= (x - 4y)(1 + m)$
b 1 Write the expression and look for a common factor.	**b** $x^2 + 3x - y^2 + 3y$
2 Group the terms so that those with common factors are next to each other.	$= (x^2 - y^2) + (3x + 3y)$
3 Factorise each group.	$= (x + y)(x - y) + 3(x + y)$
4 Factorise by taking out a common binomial factor. The factor $(x + y)$ is common to both groups.	$= (x + y)(x - y + 3)$

- Now we will look at grouping a different combination, known as grouping 'three and one'.

WORKED EXAMPLE 9

TI | CASIO

Factorise the following expression: $x^2 + 12x + 36 - y^2$.

THINK	WRITE
1 Write the expression and look for a common factor.	$x^2 + 12x + 36 - y^2$
2 Group the terms so that those that can be factorised are next to each other.	$= (x^2 + 12x + 36) - y^2$
3 Factorise the quadratic trinomial. This is the form of a perfect square.	$= (x + 6)(x + 6) - y^2$ $= (x + 6)^2 - y^2$
4 Factorise the expression using $a^2 - b^2 = (a + b)(a - b)$.	$= (x + 6 + y)(x + 6 - y)$

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Complete this digital doc: SkillSHEET: Factorising by taking out the highest common factor (doc-5246)

Complete this digital doc: SkillSHEET: Factorising by taking out a common binomial factor (doc-5247)

Exercise 7.4 Factorising expressions with two or four terms

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 8a–h, 9a–d, 10a–c, 14, 15	Questions: 1–4, 5a, 7a–c, 8a–h, 9a–d, 10a–d, 11–13, 15, 16, 18	Questions: 1–7, 8e–l, 9–19

🟩🟦🟪 **Individual pathway interactivity** int-4598

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Fluency

1. Factorise each of the following by taking out a common factor.

a. $x^2 + 3x$ **b.** $x^2 - 4x$ **c.** $3x^2 - 6x$

d. $4x^2 + 16x$ **e.** $9x^2 - 3x$ **f.** $8x - 8x^2$

g. $12x - 3x^2$ **h.** $8x - 12x^2$ **i.** $8x^2 - 11x$

2. Factorise each of the following by taking out a common binomial factor.

a. $3x(x - 2) + 2(x - 2)$ **b.** $5(x + 3) - 2x(x + 3)$

c. $(x - 1)^2 + 6(x - 1)$ **d.** $(x + 1)^2 - 2(x + 1)$

e. $(x + 4)(x - 4) + 2(x + 4)$ **f.** $7(x - 3) - (x + 3)(x - 3)$

3. WE7 Factorise each of the following.

a. $x^2 - 1$ **b.** $x^2 - 9$ **c.** $x^2 - 25$

d. $x^2 - 100$ **e.** $y^2 - k^2$ **f.** $4x^2 - 9y^2$

g. $16a^2 - 49$ **h.** $25p^2 - 36q^2$ **i.** $1 - 100d^2$

4. Factorise each of the following.

a. $4x^2 - 4$ **b.** $5x^2 - 80$ **c.** $ax^2 - 9a$

d. $2b^2 - 8d^2$ **e.** $100x^2 - 1600$ **f.** $3ax^2 - 147a$

g. $4px^2 - 256p$ **h.** $36x^2 - 16$ **i.** $108 - 3x^2$

5. MC **a.** If the factorised expression is $(x + 7)(x - 7)$, then the expanded expression must have been:

A. $x^2 - 7$ **B.** $x^2 + 7$ **C.** $x^2 - 49$

D. $x^2 + 49$ **E.** $x^2 - 14x + 49$

b. If the factorised expression is $\left(\frac{x}{4} - \frac{3}{5}\right)\left(\frac{x}{4} + \frac{3}{5}\right)$, then the original expression must have been:

A. $\frac{x^2}{4} - \frac{3}{5}$ **B.** $\frac{x^2}{16} - \frac{9}{25}$ **C.** $\frac{x^2}{4} - \frac{(\sqrt{3})^2}{(\sqrt{5})^2}$

D. $\frac{x^2}{4} - \frac{9}{25}$ **E.** $\frac{x^2}{16} - \frac{(\sqrt{3})^2}{(\sqrt{5})^2}$

c. The factorised form of $64x^2 - 9y^2$ is:

A. $(64x + 9y)(64x - 9y)$ **B.** $(8x + 3y)(8x - 3y)$ **C.** $(8x - 3y)(8x - 3y)$

D. $(8x + 3y)(8x + 3y)$ **E.** $(16x + 3y)(16x - 3y)$

6. MC Which of the following expressions would be factorised by grouping 'two and two'?

A. $x^2 - a^2 + 12a - 36$ **B.** $x^2 - 7x - 10$ **C.** $2x^2 - 6x - xy + 3y$

D. $(s - 5)^2 - 25(s + 3)^2$ **E.** $(r + 5) - (r + 3)(r + 5)$

7. Factorise each of the following over the set of real numbers.

a. $x^2 - 11$ **b.** $x^2 - 7$ **c.** $x^2 - 15$

d. $4x^2 - 13$ **e.** $9x^2 - 19$ **f.** $3x^2 - 66$

g. $5x^2 - 15$ **h.** $2x^2 - 4$ **i.** $12x^2 - 36$

8. Factorise each of the following expressions.

a. $(x-1)^2-4$ **b.** $(x+1)^2-25$ **c.** $(x-2)^2-9$
d. $(x+3)^2-16$ **e.** $49-(x+1)^2$ **f.** $36-(x-4)^2$
g. $(x-1)^2-(x-5)^2$ **h.** $4(x+2)^2-9(x-1)^2$ **i.** $25(x-2)^2-16(x+3)^2$

9. **WE8a** Factorise each of the following.

a. $x-2y+ax-2ay$ **b.** $2x+ax+2y+ay$ **c.** $ax-ay+bx-by$
d. $4x+4y+xz+yz$ **e.** $ef-2e+3f-6$ **f.** $mn-7m+n-7$
g. $6rt-3st+6ru-3su$ **h.** $7mn-21n+35m-105$ **i.** $64-8j+16k-2jk$
j. $3a^2-a^2b+3ac-abc$ **k.** $5x^2+10x+x^2y+2xy$ **l.** $2m^2-m^2n+2mn-mn^2$

10. Factorise each of the following.

a. $xy+7x-2y-14$ **b.** $mn+2n-3m-6$ **c.** $pq+5p-3q-15$
d. $s^2+3s-4st-12t$ **e.** $a^2b-cd-bc+a^2d$ **f.** $xy-z-5z^2+5xyz$

11. **WE8b** Factorise each of the following.

a. $a^2-b^2+4a-4b$ **b.** $p^2-q^2-3p+3q$ **c.** $m^2-n^2+lm+ln$
d. $7x+7y+x^2-y^2$ **e.** $5p-10pq+1-4q^2$ **f.** $49g^2-36h^2-28g-24h$

12. **WE9** Factorise each of the following.

a. $x^2+14x+49-y^2$ **b.** $x^2+20x+100-y^2$ **c.** $a^2-22a+121-b^2$
d. $9a^2+12a+4-b^2$ **e.** $25p^2-40p+16-9t^2$ **f.** $36t^2-12t+1-5v^2$

13. **MC** **a.** In the expression $3(x-2)+4y(x-2)$, the common binomial factor is:

A. $3+4y$ **B.** $3-4y$ **C.** x **D.** $-x+2$ **E.** $x-2$

a. Which of the following terms is a perfect square?

A. 9 **B.** $(x+1)(x-1)$ **C.** $3x^2$
D. $5(a+b)^2$ **E.** $25x$

b. Which of the following expressions can be factorised using grouping?

A. x^2-y^2 **B.** $1+4y-2xy+4x^2$ **C.** $3a^2+8a+4$
D. $x^2+x+y-y^2$ **E.** $2a+4b-6ab+18$

14. **MC** When factorised, $6(a+b)-x(a+b)$ equals:

A. $6-x(a+b)$ **B.** $(6-x)(a+b)$ **C.** $6(a+b-x)$
D. $(6+x)(a-b)$ **E.** $(6+x)(a+b)$

Understanding

15. The area of a rectangle is $(x^2-25)\ \text{cm}^2$.

a. Factorise the expression.
b. Find the length of the rectangle if the width is $x+5$ cm.
c. If $x=7$ cm, find the dimensions of the rectangle.
d. Hence, find the area of the rectangle.
e. If $x=13$ cm, how much bigger would the area of this rectangle be?

Reasoning

16. A circular garden of diameter $2r$ m is to have a gravel path laid around it. The path is to be 1 m wide.

a. Find the area of the garden in terms of r.
b. Find the area of the garden and path together in terms of r, using the formula for the area of a circle.
c. Write an expression for the area of the path in fully factorised form.
d. If the radius of the garden is 5 m, then find the area of the path, correct to 2 decimal places. Show your working.

17. A roll of material is $(x + 2)$ metres wide. Annie buys $(x + 3)$ metres of the material and Bronwyn buys 5 metres of the material.

a. Write an expression, in terms of x, for the area of each piece of material purchased.
b. If Annie has bought more material than Bronwyn, write an expression for how much more she has than Bronwyn.
c. Factorise and simplify this expression.
d. Find the width of the material if Annie has 5 m^2 more than Bronwyn.
e. How much material does each person have? Explain your answer.

Problem solving

A polynomial in the form $a^3 - b^3$ is known as the difference of two cubes.
The difference of two cubes can be factorised as:
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Use either the difference of two squares or the difference of two cubes to answer these problem solving questions.

18. Factorise:
a. $x^2 - 4xy + 4y^2 - a^2 + 6ab - 9b^2$
b. $x^3 + 9x^2 + 27x - 37$.

19. Factorise:
a. $27x^3 - 1$
b. $12x^2 - 75y^2 - 9(4x - 3)$.

Reflection

What do you always check for first when factorising?

7.5 Factorising by completing the square

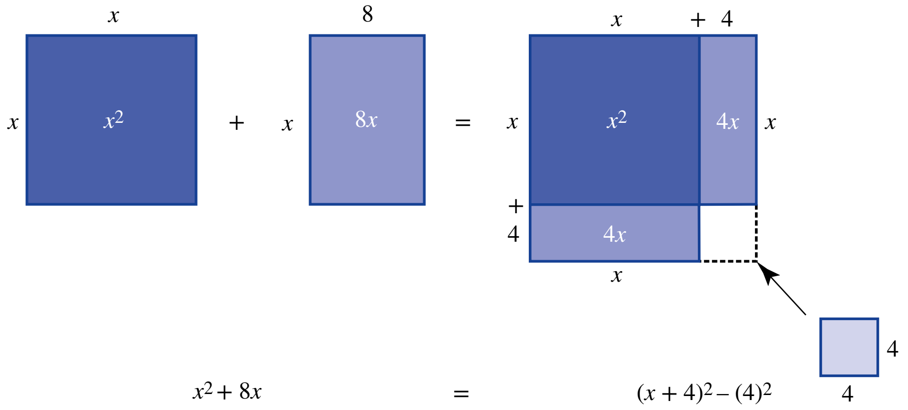
7.5.1 Completing the square

- **Completing the square** is the process of writing a general quadratic expression in turning point form.

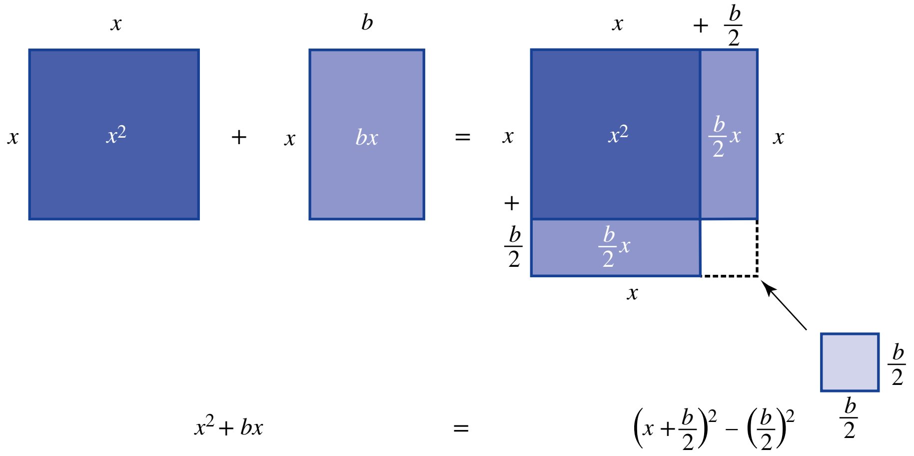
$$\underbrace{ax^2 + bx + c}_{\text{General form}} \xrightarrow{\text{Complete the square}} \underbrace{a(x-h)^2 + k}_{\text{Turning point form}}$$

$$\underbrace{ax^2 + bx + c}_{\text{General form}} = \underbrace{a(x-h)^2 + k}_{\text{Turning point form}}$$

The expression $x^2 + 8x$ can be modelled as a square with a smaller square missing from the corner, as shown below.



- In ‘completing the square’, the general equation is written as the area of the large square minus the area of the small square.
- In general, to complete the square for $x^2 + bx$, the small square has a side length equal to half of the coefficient of x; that is, the area of the small square is $\left(\frac{b}{2}\right)^2$.



WORKED EXAMPLE 10

Write the following in turning point form by completing the square.

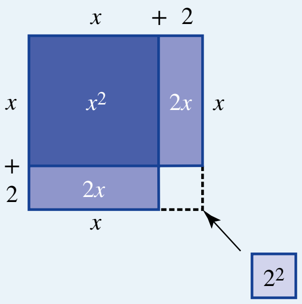
a $x^2 + 4x$ **b** $x^2 + 7x + 1$

THINK

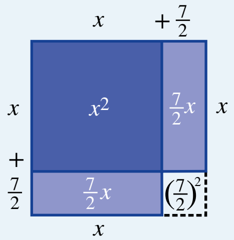
a
- The square will consist of a square that has an area of x^2 and two identical rectangles with a total area of $4x$.
- The length of the large square is $(x + 2)$ so its area is $(x + 2)^2$.
- The area of the smaller square is $(2)^2$.
- Write $x^2 + 4x$ in turning point form.

WRITE

a



$$\begin{aligned} x^2 + 4x &= (x + 2)^2 - (2)^2 \\ &= (x + 2)^2 - 4 \end{aligned}$$

b **1** • Complete the square with the terms containing x. • The square will consist of a square that has an area of x^2 and two identical rectangles with a total area of $7x$. • The length of the large square is $\left(x+\frac{7}{2}\right)$ so its area is $\left(x+\frac{7}{2}\right)^2$. • The area of the smaller square is $\left(\frac{7}{2}\right)^2$. • Write x^2+7x+1 in turning point form.	**b**  $x^2+7x+1=\left(x+\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2+1$
2 Simplify the last two terms.	$=\left(x+\frac{7}{2}\right)^2-\frac{49}{4}+\frac{4}{4}$ $=\left(x+\frac{7}{2}\right)^2-\frac{45}{4}$

- The process of completing the square is sometimes described as the process of adding the square of half of the coefficient of x then subtracting it, as shown in green below. The result of this process is a perfect square that is then factorised, as shown in blue.

$$\begin{aligned}x^2+bx &= x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2\\ &= x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2\\ &= \left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2\end{aligned}$$

- For example, factorise x^2+8x+2 by completing the square.

$$\begin{aligned}&x^2+8x+\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2+2\\ &= x^2+8x+(4)^2-(4)^2+2\\ &= x^2+8x+16-16+2\\ &= (x+4)^2-14\end{aligned}$$

7.5.2 Factorising by completing the square

- When an equation is written in turning point form, it can be factorised as a difference of two squares.

WORKED EXAMPLE 11

TI | CASIO

Factorise the following by completing the square.

a x^2+4x+2

b x^2-9x+1

THINK	WRITE
a **1** To complete the square, add the square of half of the coefficient of x and then subtract it.	**a** $x^2+4x+2=x^2+4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+2$
2 Write the perfect square created in its factorised form.	$=x^2+4x+(2)^2-(2)^2+2$ $=(x+2)^2-(2)^2+2$

3 Write the expression as a difference of two squares by: - simplifying the numerical terms - writing the numerical term as a square $(2 = (\sqrt{2})^2)$.	$= (x + 2)^2 - 4 + 2$ $= (x + 2)^2 - 2$ $= (x + 2)^2 - (\sqrt{2})^2$
4 Use the pattern for DOTS, $a^2 - b^2 = (a - b)(a + b)$, where $a = (x + 2)$ and $b = \sqrt{2}$.	$= (x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$
b 1 To complete the square, add the square of half of the coefficient of x, then subtract it.	**b** $x^2 - 9x + 1 = x^2 - 9x + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 1$
2 Write the perfect square created in its factorised form.	$= x^2 - 9x + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 1$ $= \left(x - \frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 1$
3 Write the expression as a difference of two squares by: - simplifying the numerical terms - writing the numerical term as a square. $\frac{77}{4} = \left(\sqrt{\frac{77}{4}}\right)^2 = \left(\frac{\sqrt{77}}{2}\right)^2$	$= \left(x - \frac{9}{2}\right)^2 - \frac{81}{4} + 1$ $= \left(x - \frac{9}{2}\right)^2 - \frac{77}{4}$ $= \left(x - \frac{9}{2}\right)^2 - \left(\frac{\sqrt{77}}{2}\right)^2$
4 Use the pattern for DOTS: $a^2 - b^2 = (a - b)(a + b)$, where $a = \left(x + \frac{9}{2}\right)$ and $b = \frac{\sqrt{77}}{2}$.	$= \left(x - \frac{9}{2} + \frac{\sqrt{77}}{2}\right)\left(x - \frac{9}{2} - \frac{\sqrt{77}}{2}\right)$

- Remember that you can expand the brackets to check your answer.
- If the coefficient of $x^2 \neq 1$, factorise the expression before completing the square.

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Try out this interactivity: Completing the square (int-2783)

Exercise 7.5 Factorising by completing the square

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2a–d, 3a–d, 4a–d, 5–7, 9	Questions: 1e–i, 2e–h, 3e–h, 4e–h, 5–8, 10	Questions: 1g–i, 2g–i, 3g–i, 4g–i, 5–11

🟩🟦🟪 **Individual pathway interactivity:** int-4599

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Fluency

1. WE10 Complete the square for each of the following expressions.

a. $x^2 + 10x$
b. $x^2 + 6x$
c. $x^2 - 4x$
d. $x^2 + 16x$
e. $x^2 - 20x$
f. $x^2 + 8x$
g. $x^2 - 14x$
h. $x^2 + 50x$
i. $x^2 + 7x$
j. $x^2 - x$

2. WE11a Factorise each of the following by first completing the square.

a. $x^2 - 4x - 7$
b. $x^2 + 2x - 2$
c. $x^2 - 10x + 12$
d. $x^2 + 6x - 10$
e. $x^2 + 16x - 1$
f. $x^2 - 14x + 43$
g. $x^2 + 8x + 9$
h. $x^2 - 4x - 13$
i. $x^2 - 12x + 25$

3. WE11b Factorise each of the following by first completing the square.

a. $x^2 - x - 1$
b. $x^2 - 3x - 3$
c. $x^2 + x - 5$
d. $x^2 + 3x - 1$
e. $x^2 + 5x + 2$
f. $x^2 + 5x - 2$
g. $x^2 - 7x - 1$
h. $x^2 - 9x + 13$
i. $x^2 - x - 3$

4. Factorise each of the following by first looking for a common factor and then completing the square.

a. $2x^2 + 4x - 4$
b. $4x^2 - 8x - 20$
c. $5x^2 + 30x + 5$
d. $3x^2 - 12x - 39$
e. $5x^2 - 30x + 10$
f. $6x^2 + 24x - 6$
g. $3x^2 + 30x + 39$
h. $2x^2 - 8x - 14$
i. $6x^2 + 36x - 30$

Understanding

5. Which method of factorising is the most appropriate for each of the following expressions?

a. Factorising using common factors
b. Factorising using the difference of two squares rule
c. Factorising by grouping
d. Factorising quadratic trinomials
e. Completing the square

i. $3x^2 - 8x - 3$
ii. $49m^2 - 16n^2$
iii. $x^2 + 8x + 4 - y^2$
iv. $7x^2 - 28x$
v. $6a - 6b + a^2 - b^2$
vi. $x^2 + x - 5$
vii. $(x - 3)^2 + 3(x - 3) - 10$
viii. $x^2 - 7x - 1$

6. MC **a.** To complete the square, the term which should be added to $x^2 + 4x$ is:

A. 16
B. 4
C. $4x$
D. 2
E. $2x$

b. To factorise the expression $x^2 - 3x + 1$, the term that must be both added and subtracted is:

A. 9
B. 3
C. $3x$
D. $\frac{3}{2}$
E. $\frac{9}{4}$

7. MC The factorised form of $x^2 - 6x + 2$ is:

A. $(x + 3 - \sqrt{7})(x + 3 + \sqrt{7})$
B. $(x + 3 - \sqrt{7})(x - 3 + \sqrt{7})$
C. $(x - 3 - \sqrt{7})(x - 3 - \sqrt{7})$
D. $(x - 3 - \sqrt{7})(x + 3 + \sqrt{7})$
E. $(x - 3 + \sqrt{7})(x - 3 - \sqrt{7})$

Reasoning

8. A square measuring x cm in side length has a cm added to its length and b cm added to its width. The resulting rectangle has an area of $(x^2 + 6x + 3)$ cm^2. Find the values of a and b, correct to 2 decimal places.

9. Show that $x^2 + 4x + 6$ cannot be factorised by completing the square.

Problem solving

10. For each of the following, complete the square to factorise the expression.

a. $2x^2 + 8x + 1$
b. $3x^2 - 7x + 5$

11. Use the technique of completion of the square to factorise $x^2 + 2(1 - p)x + p(p - 2)$.

Reflection

Why is this method called completing the square?

7.6 Mixed factorisation

7.6.1 Mixed factorisation

- Apply what has been covered in this chapter to the following exercise. Factorising monic and non-monic trinomials:
  - factorising by grouping
  - difference of two squares
  - completing the square.

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Complete this digital doc: SkillSHEET: Simplifying algebraic fractions (doc-5248)

Complete this digital doc: SkillSHEET: Simplifying surds (doc-5249)

Complete this digital doc: SkillSHEET: Factorising by grouping three and one (doc-5252)

Complete this digital doc: WorkSHEET: Mixed factorisation (doc-5254)

Exercise 7.6 Mixed factorisation

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–6, 8–10, 12–15, 18, 20, 21, 25, 26, 31–35, 46, 48	Questions: 1, 2, 4–9, 11, 13–16, 18–20, 22, 26–29, 30, 34, 36, 38, 40, 42, 44, 46, 47a–g, 48, 49, 52	Questions: 1–9, 13–15, 17, 18, 20, 23, 24, 26, 34–37, 39, 41, 43, 45, 47h–j, 48, 50–53

🟩🟦🟪 **Individual pathway interactivity:** int-4600

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

Factorise each of the following expressions in questions **1–45**.

1. $3x + 9$

2. $x^2 + 4x + 4 - 9y^2$

3. $x^2 - 36$

4. $x^2 - 49$

5. $5x^2 - 9x - 2$

6. $15x - 20y$

7. $5c + de + dc + 5e$

8. $5x^2 - 80$

9. $-x^2 - 6x - 5$

10. $x^2 + x - 12$

11. $mn + 1 + m + n$

12. $x^2 - 7$

13. $16x^2 - 4x$

14. $5x^2 + 60x + 100$

15. $18 + 9x - 6y - 3xy$

16. $x^2 - 8x + 16 - y^2$

17. $4x^2 + 8$

18. $fg + 2h + 2g + fh$

19. $x^2 - 5$

20. $10mn - 5n + 10m - 5$

21. $x^2 + 6x + 5$

22. $x^2 - 10x - 11$

23. $x^2 - 4$

24. $-5a + bc + ac - 5b$

25. $xy - 1 + x - y$

26. $3x^2 + 5x + 2$

27. $7x^2 - 28$

28. $-4x^2 - 28x - 24$

29. $2p - rs + pr - 2s$

30. $3x^2 - 27$

31. $-3u + tv + ut - 3v$

32. $x^2 - 11$

33. $12x^2 - 7x + 1$

34. $(x - 1)^2 - 4$

35. $(x + 2)^2 - 16$

36. $(2x + 3)^2 - 25$

37. $3(x + 5)^2 - 27$

38. $25 - (x - 2)^2$

39. $4(3 - x)^2 - 16y^2$

40. $(x + 2y)^2 - (2x + y)^2$

41. $(x + 3)^2 - (x + 1)^2$

42. $(2x - 3y)^2 - (x - y)^2$

43. $(x + 3)^2 + 5(x + 3) + 4$

44. $(x - 3)^2 + 3(x - 3) - 10$

45. $2(x + 1)^2 + 5(x + 1) + 2$

Understanding

46. Consider the following product of algebraic fractions.

$$\frac{x^2 + 3x - 10}{x^2 - 4} \times \frac{x^2 + 4x + 4}{x^2 - 2x - 8}$$

a. Factorise the expression in each numerator and denominator.

b. Cancel factors common to both the numerator and the denominator.

c. Simplify the expression as a single fraction.

47. Use the procedure in question **46** to factorise and simplify each of the following.

a. $\dfrac{x^2 - 4x + 3}{x^2 - 4x - 12} \times \dfrac{x^2 + 5x + 6}{x^2 - 9}$

b. $\dfrac{3x^2 - 17x + 10}{6x^2 + 5x - 6} \times \dfrac{x^2 - 1}{x^2 - 6x + 5}$

c. $\dfrac{6x - 12}{x^2 - 4} \times \dfrac{3x + 6}{x(x - 5)}$

d. $\dfrac{6x^2 - x - 2}{2x^2 + 3x + 1} \times \dfrac{2x^2 + x - 1}{3x^2 + 10x - 8}$

e. $\dfrac{x^2 + 4x - 5}{x^2 + x - 2} \div \dfrac{x^2 + 10x + 25}{x^2 + 4x + 4}$

f. $\dfrac{x^2 - 7x + 6}{x^2 + x - 2} \div \dfrac{x^2 - x - 12}{x^2 - 2x - 8}$

g. $\dfrac{4ab + 8a}{(c - 3)} \div \dfrac{5ac + 5a}{c^2 - 2c - 3}$

h. $\dfrac{p^2 - 7p}{p^2 - 49} \div \dfrac{p^2 + p - 6}{p^2 + 14p + 49}$

i. $\dfrac{m^2 + 4m + 4 - n^2}{4m^2 - 4m - 15} \div \dfrac{2m^2 + 4m - 2mn}{10m^2 + 15m}$

j. $\dfrac{d^2 - 6d + 9 - 25e^2}{4d^2 - 5d - 6} \div \dfrac{4d - 12 - 20e}{15d - 10}$

48. Find the original expression if the factorised expression is $\left(\dfrac{x}{4} + \dfrac{3}{5}\right)\left(\dfrac{x}{4} - \dfrac{3}{5}\right)$.

49. Factorise the following using grouping ‘three and one’ and DOTS.

a. $x^2 - 18x + 81 - y^2$

b. $4x^2 + 12x - 16y^2 + 9$

Reasoning

50. Expand the binomial factors $(x + a)(x - a)$ and $(-x + a)(-x - a)$. What do you notice? Use your findings to factorise each of the following, giving two possible answers.

a. $x^2 - 169$ **b.** $36b^2 - 144c^2$ **c.** $225x^4y^2 - 169x^2y^6$

51. Use grouping 'two and two' and DOTS to factorise the following. Show your working.

a. $x^2 + 3x - y^2 + 3y$
b. $7x + 7y + x^2 - y^2$
c. $5p - 10pq + 1 - 4q^2$

Problem solving

52. Simplify

$$\frac{2a^2 - 7a + 6}{a^3 + 8} \times \frac{5a^2 + 11a + 2}{a^3 - 8} \div \frac{10a^2 - 13a - 3}{a^2 - 2a + 4}.$$

Note: Use the difference of two cubes (page 282).

53. Factorise $x^2 + 12x + 40 - 4x^2y^2 - 4$.

Reflection

When an expression is fully factorised, what should it look like?

CHALLENGE 7.2

The expansion of perfect squares

$(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ can be used to simplify some arithmetic calculations. For example:

$$\begin{aligned} 97^2 &= (100 - 3)^2 \\ &= 100^2 - 2 \times 100 \times 3 + 3^2 \\ &= 9409 \end{aligned}$$

Use this method to calculate the following.

a. 103^2
b. 62^2
c. 997^2
d. 1012^2
e. 53^2
f. 98^2

7.7 Review

7.7.1 Review questions

Fluency

1. **MC** When expanded, $-3x(x + 4)(5 - x)$ becomes:

A. $-3x^3 - 3x^2 - 27x$ **B.** $-3x^3 + 3x - 27x$ **C.** $3x^3 + 3x^2 - 60x$
D. $-3x^3 + 3x^2 - 60x$ **E.** $3x^3 - 3x^2 - 60x$

2. **MC** When expanded, $(3x + 7)^2$ becomes:

A. $9x^2 + 49$ **B.** $3x^2 + 49$ **C.** $3x^2 + 21x + 49$
D. $9x^2 + 42x + 49$ **E.** $9x^2 + 21x + 49$

3. **MC** The factorised form of $-3d^2 - 9d + 30$ is:

A. $-3(d - 5)(d - 2)$ **B.** $-3(d + 5)(d - 6)$ **C.** $-(3d + 5)(d - 2)$
D. $-(3d + 5)(d - 6)$ **E.** $-3(d + 5)(d - 2)$

4. **MC** If the factorised expression is $(2x-5)(2x+5)$, then the original expression must have been:

A. $2x^2-5$ **B.** $4x^2-5$ **C.** $4x^2-25$

D. $4x^2-20x+25$ **E.** $2x^2+25$

5. **MC** To factorise $-5x^2-45x+100$, the first step is to:

A. find factors of 5 and 100 that sum to -45

B. take out 5 as a common factor

C. take out -5 as a common factor

D. find factors of 5 and -45 that will add to make 100

E. take out $-5x$ as a common factor

6. **MC** To complete the square, the term which should be added to x^2-12x is:

A. 36 **B.** -12 **C.** $-12x$ **D.** -6 **E.** $-6x$

7. **MC** Which of the following is equivalent to $5x^2-20x-5$?

A. $5(x-2)^2$ **B.** $5(x-2)^2-3$ **C.** $5(x-2)^2-15$

D. $5(x-2)^2-20$ **E.** $5(x-2)^2-25$

8. **MC** In the expanded form of $(x-3)(x+5)$, which of the following is incorrect?

A. The value of the constant is -15.

B. The coefficient of the x term is 2.

C. The coefficient of the x term is -8.

D. The coefficient of the x^2 term is 1.

E. The expansion shows this to be a trinomial expression.

9. Expand each of the following and simplify where necessary.

a. $3x(x-4)$ **b.** $-7x(3x+1)$

c. $(x-7)(x+1)$ **d.** $(2x-5)(x-3)$

e. $(4x-1)(3x-5)$ **f.** $3(x-4)(2x+7)$

g. $(2x-5)(x+3)(x+7)$ **h.** $(x+5)(x+7)+(2x-5)(x-6)$

i. $(x+3)(5x-1)-2x$

10. Expand and simplify each of the following.

a. $(x-7)^2$ **b.** $(2-x)^2$ **c.** $(3x+1)^2$

d. $-2(3x-2)^2$ **e.** $-7(2x+5)^2$ **f.** $-10(4x-5)^2$

g. $(x+9)(x-9)$ **h.** $(3x-1)(3x+1)$ **i.** $(5+2x)(5-2x)$

11. Factorise each of the following.

a. $2x^2-8x$ **b.** $-4x^2+12x$ **c.** $3ax-2ax^2$

d. $(x+1)^2+(x+1)$ **e.** $3(2x-5)-(2x-5)^2$ **f.** $(x-4)(x+2)-(x-4)$

12. Factorise each of the following.

a. x^2-16 **b.** x^2-25 **c.** $2x^2-72$

d. $3x^2-27y^2$ **e.** $4ax^2-16ay^2$ **f.** $(x-4)^2-9$

13. Factorise each of the following by grouping.

a. $ax-ay+bx-by$ **b.** $7x+ay+ax+7y$ **c.** $xy+2y+5x+10$

d. $mn-q-2q^2+2mnq$ **e.** $pq-5r^2-r+5pqr$ **f.** $uv-u+9v-9$

g. $a^2-b^2+5a-5b$ **h.** $d^2-4c^2-3d+6c$ **i.** $2+2m+1-m^2$

14. Factorise each of the following by grouping.

a. $4x^2+12x+9-y^2$ **b.** $49a^2-28a+4-4b^2$ **c.** $64s^2-16s+1-3t$

15. Factorise each of the following.

a. $x^2+10x+9$ **b.** $x^2-11x+18$ **c.** $x^2-4x-21$

d. $x^2+3x-28$ **e.** $-x^2+6x-9$ **f.** $3x^2+33x-78$

g. $-2x^2+8x+10$ **h.** $-3x^2+24x-36$ **i.** $8x^2+2x-1$

j. $6x^2+x-1$ **k.** $8x^2+4x-12$ **l.** $105x^2-10x-15$

m. $-12x^2+62x-70$ **n.** $-45x^2-3x+6$ **o.** $-60x^2-270x-270$

16. Factorise each of the following by completing the square.

a. $x^2 + 6x + 1$ **b.** $x^2 - 10x - 3$ **c.** $x^2 + 4x - 2$

d. $x^2 - 5x + 2$ **e.** $x^2 + 7x - 1$ **f.** $2x^2 + 18x - 2$

17. Factorise each of the following using the most appropriate method.

a. $3x^2 - 12x$ **b.** $x^2 + 6x + 2$ **c.** $4x^2 - 25$

d. $2x^2 + 9x + 10$ **e.** $2ax + 4x + 3a + 6$ **f.** $-3x^2 - 3x + 18$

18. First factorise then simplify each of the following.

a. $\dfrac{x+4}{5x-30} \times \dfrac{2x-12}{x+1}$ **b.** $\dfrac{3x+6}{4x-24} \times \dfrac{7x-42}{6x+12}$ **c.** $\dfrac{x^2-4}{x^2+5x} \times \dfrac{x^2+4x-5}{x^2-2x-8}$

Problem solving

19. A large storage box has a square base with sides measuring $(x + 2)$ cm and is 32 cm high.

a. Write an expression for the area of the base of the box.

b. Write an expression for the volume of the box ($V =$ area of base × height).

c. Simplify the expression by expanding the brackets.

d. If $x = 30$ cm, find the volume of the box in cm^3.

20. A section of garden is to have a circular pond of radius $2r$ with a 2 m path around its edge.

a. State the diameter of the pond.

b. State the radius of the pond and path.

c. State the area of the pond.

d. State the area of the pond and path.

e. Write an expression to find the area of the path only and write it in factorised form.

f. If the radius of the pond is 3 metres, find the area of the path.

21. In order to make the most of the space available for headlines and stories, the front page of a newspaper is given an area of $x^2 - 5x - 14$ cm^2.

a. If the length is $x + 2$ cm, what is the width?

b. Write down the length of the shorter side in terms of x.

c. If the shorter side of the front page is 28 cm, find the value of x.

d. Find the area of this particular paper.

22. Here is a well-known puzzle. Let $a = b = 1$.

Step 1: Write $a = b$. $a = b$

Step 2: Multiply both sides by a. $a^2 = ab$

Step 3: Subtract b^2 from both sides. $a^2 - b^2 = ab - b^2$

Step 4: Factorise. $(a + b)(a - b) = b(a - b)$

Step 5: Simplify by dividing by $(a - b)$. $(a + b) = b$

Step 6: Substitute $a = b = 1$. $1 + 1 = 1$

Where is the error? Show your thinking.

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Try out this interactivity: Word search: Topic 7 (int-2844)

Try out this interactivity: Crossword: Topic 7 (int-2845)

Try out this interactivity: Sudoku: Topic 7 (int-3594)

Complete this digital doc: Concept map: Topic 7 (doc-13807)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

binomial
coefficient
common factor
completing the square
difference of two squares
expand
expression
factorise
FOIL
general form
grouping three and one
grouping two and two
monic
non-monic
perfect squares
quadratic trinomial
term
turning point form

Investigation | Rich Task

Celebrity squares and doubles

In small groups or as a class, use the process of elimination to find your 'square and double pair' by playing 'Celebrity squares and doubles' as outlined below.

Equipment: roll of calculator paper, scissors, sticky tape, marker pen

1. Set-up
- Make a class set of headbands. Each headband will be part of a matching pair made by a number being squared and that same original number being doubled (16 and 8 would be a pair, because $4^2 = 16$ and $4 \times 2 = 8$). Your teacher will direct the class as to what number should be written on each headband.
- Place the headbands randomly on a table.

2. Beginning the game
- There is to be no communication between players at this time.
- Your teacher will randomly allocate a headband to each player by placing a headband on their head without the player seeing the number on their headband.

3. Playing the game

The object of the game

- The object of the game is to use the process of elimination for you to find your pair. A possible train of thought is illustrated at right.

Starting the game

- Once all headbands have been allocated, stand in a circle or walk around freely.
- Without speaking, determine who is a match; then, by a process of elimination, determine who might be your match.

Making a match

- When you think you have found your match, approach that person and say ‘I think I am your match.’
- The other player should now check to see if you have a match elsewhere and can reply by saying one of two things: ‘Yes, I think I am your match,’ or ‘I know your match is still out there.’
- If a match is agreed upon, the players should sit out for the remainder of the game. If a match is not agreed upon, players should continue looking.

Ending the game

- The class should continue until everyone is in a pair, at which time the class can check their results.
- The class should now discuss the different trains of thought they used to find their pair and how this relates to factorising quadratic trinomials.

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Complete this digital doc: Code puzzle: Isaac Newton demonstrated this in 1665 (doc-15929)

Answers

Topic 7 Quadratic expressions

Exercise 7.2 Expanding algebraic expressions

1. a. $2x + 6$ **b.** $4x - 20$ **c.** $21 - 3x$ **d.** $-x - 3$
e. $x^2 + 2x$ **f.** $2x^2 - 8x$ **g.** $15x^2 - 6x$ **h.** $10x - 15x^2$
i. $8x^2 + 2x$ **j.** $4x^3 - 6x^2$ **k.** $6x^3 - 3x^2$ **l.** $15x^3 + 20x^2$

2. a. $x^2 - x - 12$ **b.** $x^2 - 2x - 3$ **c.** $x^2 - 5x - 14$ **d.** $x^2 - 6x + 5$
e. $-x^2 - x + 6$ **f.** $x^2 - 6x + 8$ **g.** $2x^2 - 17x + 21$ **h.** $3x^2 - x - 2$
i. $6x^2 - 17x + 5$ **j.** $21 - 17x + 2x^2$ **k.** $15 + 14x - 8x^2$ **l.** $110 + 47x - 21x^2$

3. a. $2x^2 - 4x - 6$ **b.** $8x^2 - 28x - 16$ **c.** $-2x^2 + 12x + 14$ **d.** $2x^3 - 2x$ **e.** $3x^3 - 75x$
f. $6x^3 - 54x$ **g.** $2x^3 - 12x^2 + 18x$ **h.** $5x^3 - 30x^2 + 40x$ **i.** $-6x^3 - 6x^2 + 120x$

4. a. $x^3 + 2x^2 - x - 2$ **b.** $x^3 - 2x^2 - 5x + 6$ **c.** $x^3 - 5x^2 - x + 5$ **d.** $x^3 - 6x^2 + 11x - 6$
e. $2x^3 - 7x^2 - 5x + 4$ **f.** $6x^3 - 7x^2 + 1$

5. a. $x^2 - x - 2$ **b.** $-2x^2 + 4x + 10$ **c.** $5x^2 - 6x - 5$ **d.** $19x - 23$ **e.** $-5x - 1$
f. $-2x + 6$ **g.** $x^2 - 2x - 3 + \sqrt{3}x$ **h.** $\sqrt{6} + 2\sqrt{2}x - 3\sqrt{3}x - 6x^2 - \sqrt{5}x$

6. a. A **b.** C

7. B

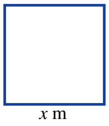
8. a. $x^2 - 2x + 1$ **b.** $x^2 + 4x + 4$ **c.** $x^2 + 10x + 25$ **d.** $16 + 8x + x^2$ **e.** $49 - 14x + x^2$
f. $144 - 24x + x^2$ **g.** $9x^2 - 6x + 1$ **h.** $144x^2 - 72x + 9$ **i.** $25x^2 + 20x + 4$ **j.** $4 - 12x + 9x^2$
k. $25 - 40x + 16x^2$ **l.** $1 - 10x + 25x^2$

9. a. $2x^2 - 12x + 18$ **b.** $4x^2 - 56x + 196$ **c.** $3x^2 + 6x + 3$ **d.** $-4x^2 - 12x - 9$ **e.** $-49x^2 + 14x - 1$
f. $8x^2 - 24x + 18$ **g.** $-12 + 108x - 243x^2$ **h.** $-45 + 330x - 605x^2$ **i.** $-16x^2 - 16x - 4$

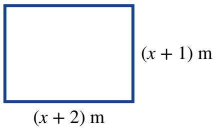
10. a. $x^2 - 49$ **b.** $x^2 - 81$ **c.** $x^2 - 25$ **d.** $x^2 - 1$ **e.** $4x^2 - 9$ **f.** $9x^2 - 1$
g. $49 - x^2$ **h.** $64 - x^2$ **i.** $9 - 4x^2$

11. a. $(x + 1)(x - 3)$ **b.** $x^2 - 2x - 3$ **c.** 6 cm, 2 cm, 12 cm^2

12. a.



b.



c. $(x + 1)(x + 2)$ **d.** $x^2 + 3x + 2$ **e.** 4 m^2, 12 m^2

13. a. Student C
b. Student B:
$(3x + 4)(2x + 5)$
$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$
$= 6x^2 + 23x + 20$
Student A:
$(3x + 4)(2x + 5)$
$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$
$= 6x^2 + 15x + 8x + 20$
$= 6x^2 + 23x + 20$

14. $(a - b)(a + b) = a^2 - b^2$
LHS
$(5 - 3)(5 + 3)$
$= 2 \times 8$
$= 16$
RHS:
$5^2 - 3^2$
$= 25 - 9$
$= 16$
LHS = RHS ⇒ True

15. $(a+b)^2 = a^2 + 2ab + b^2$
LHS:
$(5+3)^2$
$= 8^2$
$= 64$
RHS:
$5^2 + 2 \times 5 \times 3 + 3^2$
$= 25 + 30 + 9$
$= 64$
LHS = RHS ⇒ True

16. Answers will vary; examples are shown.

a. $(x+4)(x+3) = x^2 + 7x + 12$ **b.** $(x+4)^2 = x^2 + 8x + 16$

c. $(x+4)(x-4) = x^2 - 16$ **d.** $(x+4)^2 = x^2 + 8x + 16$

17. The square of a binomial is a trinomial; the difference of two squares has two terms.

18. $(a+b)(c+d) = (c+d)(a+b)$
LHS:
$(a+b)(c+d) = ac + ad + bc + bd$
RHS:
$(c+d)(a+b) = ca + cb + da + db$
LHS = RHS ⇒ True

19. a. $4x^2 + 12xy - 20xz + 9y^2 - 30yz + 25z^2$ **b.** $\frac{1}{x} + \frac{1}{4x^2} - 4x + 4x^2 - 1$

20. a. $V = 6x^3 - 29x^2 + 46x - 24$ **b.** $TSA = 12x^2 + 20x + 3$

Challenge 7.1

10, 11, 13, 18, 35

Exercise 7.3 Factorising expressions with three terms

1. a. $(x+2)(x+1)$ **b.** $(x+3)(x+1)$ **c.** $(x+8)(x+2)$ **d.** $(x+4)^2$ **e.** $(x-3)(x+1)$

f. $(x-4)(x+1)$ **g.** $(x-12)(x+1)$ **h.** $(x-6)(x+2)$ **i.** $(x+4)(x-1)$ **j.** $(x+5)(x-1)$

k. $(x+7)(x-1)$ **l.** $(x+5)(x-2)$ **m.** $(x-3)(x-1)$ **n.** $(x-4)(x-5)$ **o.** $(x+14)(x-5)$

2. a. $-2(x+9)(x+1)$ **b.** $-3(x+2)(x+1)$ **c.** $-(x+2)(x+1)$ **d.** $-(x+10)(x+1)$ **e.** $-(x+2)(x+5)$

f. $-(x+12)(x+1)$ **g.** $-(x+3)(x+4)$ **h.** $-(x+2)(x+6)$ **i.** $2(x+2)(x+5)$ **j.** $3(x+1)(x+10)$

k. $5(x+20)(x+1)$ **l.** $5(x+4)(x+5)$

3. a. $(a-7)(a+1)$ **b.** $(t-4)(t-2)$ **c.** $(b+4)(b+1)$ **d.** $(m+5)(m-3)$ **e.** $(p-16)(p+3)$

f. $(c+16)(c-3)$ **g.** $(k+19)(k+3)$ **h.** $(s-19)(s+3)$ **i.** $(g+8)(g-9)$ **j.** $(v-25)(v-3)$

k. $(x+16)(x-2)$ **l.** $(x-15)(x-4)$

4. a. C **b.** B

5. C

6. a. $(2x+1)(x+2)$ **b.** $(2x-1)(x-1)$ **c.** $(4x+3)(x-5)$ **d.** $(2x-1)(2x+3)$ **e.** $(x-7)(2x+5)$

f. $(3x+1)(x+3)$ **g.** $(3x-7)(2x-1)$ **h.** $(4x-7)(3x+2)$ **i.** $(5x+3)(2x-3)$ **j.** $(4x-1)(5x+2)$

k. $(3x+2)(4x-1)$ **l.** $(3x-1)(5x+2)$

7. a. $2(x-1)(2x+3)$ **b.** $3(3x+1)(x-7)$ **c.** $12(2x+1)(3x-1)$ **d.** $-3(3x+1)(2x-1)$

e. $-30(2x+1)(x-3)$ **f.** $3a(4x-7)(2x+5)$ **g.** $-2(4x-3)(x-2)$ **h.** $-(2x-7)(5x+2)$

i. $-(8x-1)(3x-4)$ **j.** $-2(3x-y)(2x+y)$ **k.** $-5(2x-7y)(3x+2y)$ **l.** $-12(5x+3y)(10x+7y)$

8. a. $w^2 + 5w - 6$ **b.** $(w+6)(w-1)$ **c.** $(x+5)(x-2)$

9. a. $x(x+5)$ **b.** $x(x+5)$ **c.** $(x-1)^2$

d. $(x+9)(x+5)$ **e.** $(x-15)(x-6)$ **f.** $(x-10)(x-3)$

10. $(x-0.5)(x+1.5)$

11. a. $(x-5)(x+1)$ **b.** $(x-5)$ cm **c.** $x = 15$ m **d.** 160 cm^2

e. $3000(x-5)(x+1)$ cm^2 or $(3000x^2 - 12\,000x - 15\,000)$ cm^2

12. a. $(2x+3)(3x+1)$ **b.** $P = 10x + 8$ **c.** $x = 8$ metres

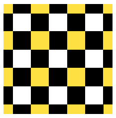
13. a. SA $= 3x^2 + 16x$ **b.** Total area $= 3x^2 + 16x + 16$ **c.** $(3x+4)(x+4)$

d. $l = 21$ m; $w = 7$ m; $d = 2$ m **e.** 275 m^2 **f.** 294 m^3

14. a. Yellow = 3 cm × 3 cm
Black = 3 cm × 6 cm
White = 6 cm × 6 cm

b. Yellow = 0.36 m^2
Black = 0.72 m^2
White = 0.36 m^2

c.



15. a. $x^2 - 7x + 12 = (x - 3)(x - 4)$ **b.** $x^2 + 7x - 12 = (x - 3)(x - 4)$ **c.** $x^2 - x + 2 = (x - 2)(x + 1)$
d. $x^2 - 4x - 21 = (x + 3)(x - 7)$ **e.** $x^2 + 4x - 21 = (x - 3)(x + 7)$ **f.** $x^2 - x - 30 = (x + 5)(x - 6)$
g. $x^2 + 7x - 8 = (x - 1)(x + 8)$ **h.** $x^2 - 11x + 30 = (x - 5)(x - 6)$

16. a. $(9a - 2)(6a - 7)$ **b.** $(3m^2 + 2)(m - \sqrt{7})(m + \sqrt{7})$ **c.** $(2\sin(x) - 1)(\sin(x) - 1)$

17. a. $(x + 2\sqrt{3})(2x - \sqrt{3})$ **b.** $2z(z^2 + 3)$

Exercise 7.4 Factorising expressions with two or four terms

1. a. $x(x + 3)$ **b.** $x(x - 4)$ **c.** $3x(x - 2)$ **d.** $4x(x + 4)$ **e.** $3x(3x - 1)$
f. $8x(1 - x)$ **g.** $3x(4 - x)$ **h.** $4x(2 - 3x)$ **i.** $x(8x - 11)$

2. a. $(x - 2)(3x + 2)$ **b.** $(x + 3)(5 - 2x)$ **c.** $(x - 1)(x + 5)$ **d.** $(x + 1)(x - 1)$
e. $(x + 4)(x - 2)$ **f.** $(x - 3)(4 - x)$

3. a. $(x + 1)(x - 1)$ **b.** $(x + 3)(x - 3)$ **c.** $(x + 5)(x - 5)$ **d.** $(x + 10)(x - 10)$ **e.** $(y + k)(y - k)$
f. $(2x + 3y)(2x - 3y)$ **g.** $(4a + 7)(4a - 7)$ **h.** $(5p + 6q)(5p - 6q)$ **i.** $(1 + 10d)(1 - 10d)$

4. a. $4(x + 1)(x - 1)$ **b.** $5(x + 4)(x - 4)$ **c.** $a(x + 3)(x - 3)$ **d.** $2(b + 2d)(b - 2d)$ **e.** $100(x + 4)(x - 4)$
f. $3a(x + 7)(x - 7)$ **g.** $4p(x + 8)(x - 8)$ **h.** $4(3x + 2)(3x - 2)$ **i.** $3(6 + x)(6 - x)$

5. a. C **b.** B **c.** B

6. C

7. a. $(x + \sqrt{11})(x - \sqrt{11})$ **b.** $(x + \sqrt{7})(x - \sqrt{7})$ **c.** $(x + \sqrt{15})(x - \sqrt{15})$ **d.** $(2x + \sqrt{13})(2x - \sqrt{13})$
e. $(3x + \sqrt{19})(3x - \sqrt{19})$ **f.** $3(x + \sqrt{22})(x - \sqrt{22})$ **g.** $5(x + \sqrt{3})(x - \sqrt{3})$ **h.** $2(x + \sqrt{2})(x - \sqrt{2})$
i. $12(x + \sqrt{3})(x - \sqrt{3})$

8. a. $(x - 3)(x + 1)$ **b.** $(x - 4)(x + 6)$ **c.** $(x - 5)(x + 1)$ **d.** $(x - 1)(x + 7)$ **e.** $(6 - x)(x + 8)$
f. $(10 - x)(x + 2)$ **g.** $8(x - 3)$ **h.** $(7 - x)(5x + 1)$ **i.** $(x - 22)(9x + 2)$

9. a. $(x - 2y)(1 + a)$ **b.** $(x + y)(2 + a)$ **c.** $(x - y)(a + b)$ **d.** $(x + y)(4 + z)$ **e.** $(f - 2)(e + 3)$
f. $(n - 7)(m + 1)$ **g.** $3(2r - s)(t + u)$ **h.** $7(m - 3)(n + 5)$ **i.** $2(8 - j)(4 + k)$ **j.** $a(3 - b)(a + c)$
k. $x(5 + y)(x + 2)$ **l.** $m(m + n)(2 - n)$

10. a. $(y + 7)(x - 2)$ **b.** $(m + 2)(n - 3)$ **c.** $(q + 5)(p - 3)$ **d.** $(s + 3)(s - 4t)$
e. $(b + d)(a^2 - c)$ **f.** $(1 + 5z)(xy - z)$

11. a. $(a - b)(a + b + 4)$ **b.** $(p - q)(p + q - 3)$ **c.** $(m + n)(m - n + l)$ **d.** $(x + y)(7 + x - y)$
e. $(1 - 2q)(5p + 1 + 2q)$ **f.** $(7g + 6h)(7g - 6h - 4)$

12. a. $(x + 7 + y)(x + 7 - y)$ **b.** $(x + 10 + y)(x + 10 - y)$ **c.** $(a - 11 + b)(a - 11 - b)$
d. $(3a + 2 + b)(3a + 2 - b)$ **e.** $(5p - 4 + 3t)(5p - 4 - 3t)$ **f.** $(6t - 1 + \sqrt{5}v)(6t - 1 - \sqrt{5}v)$

13. a. E **b.** A **c.** D

14. B

15. a. $(x - 5)(x + 5)$ **b.** $(x - 5)$ cm, $(x + 5)$ cm **c.** 2 cm, 12 cm **d.** 24 cm^2
e. 120 cm^2 or 6 times bigger

16. a. $A_1 = \pi r^2 \text{m}^2$ **b.** $A_2 = \pi(r + 1)^2 \text{m}^2$
c. $A = \pi(r + 1)^2 - \pi r^2 = \pi(2r + 1)\ \text{m}^2$ **d.** 34.56 m^2

17. a. Annie = $(x + 3)(x + 2)\ \text{m}^2$ Bronwyn = $5(x + 2)\text{m}^2$ **b.** $(x + 3)(x + 2) - 5(x + 2)$
c. $(x + 2)(x - 2) = x^2 - 4$ **d.** Width = 5 m
e. Annie has 30 m^2 and Bronwyn has 25 m^2

18. a. $(x - 2y - a + 3b)(x - 2y + a - 3b)$ **b.** $(x - 1)(x^2 + 10x + 37)$

19. a. $(3x - 1)(9x^2 + 3x + 1)$ **b.** $3(2x - 5y - 3)(2x + 5y - 3)$

Exercise 7.5 Factorising by completing the square

1. a. 25 **b.** 9 **c.** 4 **d.** 64 **e.** 100
f. 16 **g.** 49 **h.** 625 **i.** $\frac{49}{4}$ **j.** $\frac{1}{4}$

2. a. $(x-2+\sqrt{11})(x-2-\sqrt{11})$ **b.** $(x+1+\sqrt{3})(x+1-\sqrt{3})$ **c.** $(x-5+\sqrt{13})(x-5-\sqrt{13})$

d. $(x+3+\sqrt{19})(x+3-\sqrt{19})$ **e.** $(x+8+\sqrt{65})(x+8-\sqrt{65})$ **f.** $(x-7+\sqrt{6})(x-7-\sqrt{6})$

g. $(x+4+\sqrt{7})(x+4-\sqrt{7})$ **h.** $(x-2+\sqrt{17})(x-2-\sqrt{17})$ **i.** $(x-6+\sqrt{11})(x-6-\sqrt{11})$

3. a. $\left(x-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(x-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$ **b.** $\left(x-\frac{3}{2}+\frac{\sqrt{21}}{2}\right)\left(x-\frac{3}{2}-\frac{\sqrt{21}}{2}\right)$ **c.** $\left(x+\frac{1}{2}+\frac{\sqrt{21}}{2}\right)\left(x+\frac{1}{2}-\frac{\sqrt{21}}{2}\right)$

d. $\left(x+\frac{3}{2}+\frac{\sqrt{13}}{2}\right)\left(x+\frac{3}{2}-\frac{\sqrt{13}}{2}\right)$ **e.** $\left(x+\frac{5}{2}+\frac{\sqrt{17}}{2}\right)\left(x+\frac{5}{2}-\frac{\sqrt{17}}{2}\right)$ **f.** $\left(x+\frac{5}{2}+\frac{\sqrt{33}}{2}\right)\left(x+\frac{5}{2}-\frac{\sqrt{33}}{2}\right)$

g. $\left(x-\frac{7}{2}+\frac{\sqrt{53}}{2}\right)\left(x-\frac{7}{2}-\frac{\sqrt{53}}{2}\right)$ **h.** $\left(x-\frac{9}{2}+\frac{\sqrt{29}}{2}\right)\left(x-\frac{9}{2}-\frac{\sqrt{29}}{2}\right)$ **i.** $\left(x-\frac{1}{2}+\frac{\sqrt{13}}{2}\right)\left(x-\frac{1}{2}-\frac{\sqrt{13}}{2}\right)$

4. a. $2(x+1+\sqrt{3})(x+1-\sqrt{3})$ **b.** $4(x-1+\sqrt{6})(x-1-\sqrt{6})$ **c.** $5(x+3+2\sqrt{2})(x+3-2\sqrt{2})$

d. $3(x-2+\sqrt{17})(x-2-\sqrt{17})$ **e.** $5(x-3+\sqrt{7})(x-3-\sqrt{7})$ **f.** $6(x+2+\sqrt{5})(x+2-\sqrt{5})$

g. $3(x+5+2\sqrt{3})(x+5-2\sqrt{3})$ **h.** $2(x-2+\sqrt{11})(x-2-\sqrt{11})$ **i.** $6(x+3+\sqrt{14})(x+3-\sqrt{14})$

5. i. d **ii. b** **iii. c** **iv. a** **v. c** **vi. d** **vii. d** **h. e**

6. a. B **b.** E

7. E

8. $a=0.55$; $b=5.45$

9. Check with your teacher.

10. a. $2\left(x+2-\frac{\sqrt{14}}{2}\right)\left(x+2+\frac{\sqrt{14}}{2}\right)$

b. This expression cannot be factorised as there is no difference of two squares.

11. $(x-p)(x-p+2)$

Exercise 7.6 Mixed factorisation

1. $3(x+3)$
2. $(x+2+3y)(x+2-3y)$
3. $(x+6)(x-6)$
4. $(x+7)(x-7)$
5. $(5x+1)(x-2)$
6. $5(3x-4y)$
7. $(c+e)(5+d)$
8. $5(x+4)(x-4)$
9. $-(x+5)(x+1)$
10. $(x+4)(x-3)$
11. $(m+1)(n+1)$
12. $(x+\sqrt{7})(x-\sqrt{7})$
13. $4x(4x-1)$
14. $5(x+10)(x+2)$
15. $3(3-y)(x+2)$
16. $(x-4+y)(x-4-y)$
17. $4(x^2+2)$
18. $(g+h)(f+2)$
19. $(x+\sqrt{5})(x-\sqrt{5})$
20. $5(n+1)(2m-1)$
21. $(x+5)(x+1)$
22. $(x+1)(x-11)$
23. $(x+2)(x-2)$
24. $(a+b)(c-5)$
25. $(y+1)(x-1)$
26. $(3x+2)(x+1)$
27. $7(x+2)(x-2)$
28. $-4(x+6)(x+1)$
29. $(2+r)(p-s)$
30. $3(x+3)(x-3)$
31. $(u+v)(t-3)$
32. $(x+\sqrt{11})(x-\sqrt{11})$
33. $(4x-1)(3x-1)$
34. $(x+1)(x-3)$
35. $(x+6)(x-2)$
36. $4(x-1)(x+4)$
37. $3(x+2)(x+8)$
38. $(3+x)(7-x)$
39. $4(3-x+2y)(3-x-2y)$
40. $3(y+x)\,(y-x)$
41. $4(x+2)$
42. $(3x-4y)(x-2y)$
43. $(x+7)(x+4)$
44. $(x+2)(x-5)$
45. $(2x+3)(x+3)$

46. a. $\frac{(x+5)\,(x-2)}{(x+2)\,(x-2)}\times\frac{(x+2)\,(x+2)}{(x-4)\,(x+2)}$ **b.** $\frac{(x+5)\,\cancel{(x-2)}}{\cancel{(x+2)}\,\cancel{(x-2)}}\times\frac{\cancel{(x+2)}\,\cancel{(x+2)}}{(x-4)\,\cancel{(x+2)}}$ **c.** $\frac{x+5}{x-4}$

47. a. $\frac{x-1}{x-6}$ **b.** $\frac{x+1}{2x+3}$ **c.** $\frac{18}{x(x-5)}$ **d.** $\frac{2x-1}{x+4}$

e. $\frac{x+2}{x+5}$ **f.** $\frac{x-6}{x+3}$ **g.** $\frac{4(b+2)}{5}$ **h.** $\frac{p(p+7)}{(p+3)(p-2)}$

i. $\frac{5(m+2+n)}{2(2m-5)}$ **j.** $\frac{5(3d-2)(d-3+5e)}{4(d-2)(4d+3)}$

48. $\frac{x^2}{16}-\frac{9}{25}$

49. a. $(x-9-y)(x-9+y)$ **b.** $(2x+3-4y)(2x+3+4y)$

50. x^2-a^2; they are the same.

a. $(x-13)(x+13)$ or $(-x-13)(-x+13)$

b. $36(b-2c)(b+2c)$ or $36(-b-2c)(-b+2c)$

c. $x^2y^2(15x-13y^2)(15x+13y^2)$ or $x^2y^2(-15x-13y^2)(-15x+13y^2)$

51. a. $(x+y)(x-y+3)$ **b.** $(x+y)(7+x-y)$ **c.** $(1-2q)(5p+1+2q)$

52. $\frac{1}{a^2+2a+4}$

53. $(x+6-2xy)(x+6+2xy)$

Challenge 7.2

a. 10 609 **b.** 3844 **c.** 99 409 **d.** 1 024 144 **e.** 2809 **f.** 9604

7.7 Review

1. E
2. D
3. E
4. C
5. C
6. A
7. E
8. C
9. a. $3x^2 - 12x$ **b.** $-21x^2 - 7x$ **c.** $x^2 - 6x - 7$ **d.** $2x^2 - 11x + 15$ **e.** $12x^2 - 23x + 5$
f. $6x^2 - 3x - 84$ **g.** $2x^3 + 15x^2 - 8x - 105$ **h.** $3x^2 - 5x + 65$ **i.** $5x^2 + 12x - 3$
10. a. $x^2 - 14x + 49$ **b.** $4 - 4x + x^2$ **c.** $9x^2 + 6x + 1$ **d.** $-18x^2 + 24x - 8$ **e.** $-28x^2 - 140x - 175$
f. $-160x^2 + 400x - 250$ **g.** $x^2 - 81$ **h.** $9x^2 - 1$ **i.** $25 - 4x^2$
11. a. $2x(x - 4)$ **b.** $-4x(x - 3)$ **c.** $ax(3 - 2x)$ **d.** $(x + 1)(x + 2)$ **e.** $2(2x - 5)(4 - x)$ **f.** $(x - 4)(x + 1)$
12. a. $(x + 4)(x - 4)$ **b.** $(x + 5)(x - 5)$ **c.** $2(x + 6)(x - 6)$ **d.** $3(x + 3y)(x - 3y)$ **e.** $4a(x + 2y)(x - 2y)$ **f.** $(x - 1)(x - 7)$
13. a. $(x - y)(a + b)$ **b.** $(x + y)(7 + a)$ **c.** $(x + 2)(y + 5)$ **d.** $(1 + 2q)(mn - q)$ **e.** $(5r + 1)(pq - r)$
f. $(v - 1)(u + 9)$ **g.** $(a - b)(a + b + 5)$ **h.** $(d - 2c)(d + 2c - 3)$ **i.** $(1 + m)(3 - m)$
14. a. $(2x + 3 + y)(2x + 3 - y)$ **b.** $(7a - 2 + 2b)(7a - 2 - 2b)$ **c.** $(8s - 1 + \sqrt{3}t)(8s - 1 - \sqrt{3}t)$
15. a. $(x + 9)(x + 1)$ **b.** $(x - 9)(x - 2)$ **c.** $(x - 7)(x + 3)$ **d.** $(x + 7)(x - 4)$ **e.** $-(x - 3)^2$
f. $3(x + 13)(x - 2)$ **g.** $-2(x - 5)(x + 1)$ **h.** $-3(x - 6)(x - 2)$ **i.** $(4x - 1)(2x + 1)$ **j.** $(3x - 1)(2x + 1)$
k. $4(2x + 3)(x - 1)$ **l.** $5(7x - 3)(3x + 1)$ **m.** $-2(3x - 5)(2x - 7)$ **n.** $-3(3x - 1)(5x + 2)$ **o.** $-30(2x + 3)(x + 3)$
16. a. $(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$ **b.** $(x - 5 + 2\sqrt{7})(x - 5 - 2\sqrt{7})$ **c.** $(x + 2 + \sqrt{6})(x + 2 - \sqrt{6})$
d. $\left(x - \frac{5}{2} + \frac{\sqrt{17}}{2}\right)\left(x - \frac{5}{2} - \frac{\sqrt{17}}{2}\right)$ **e.** $\left(x + \frac{7}{2} + \frac{\sqrt{53}}{2}\right)\left(x + \frac{7}{2} - \frac{\sqrt{53}}{2}\right)$ **f.** $2\left(x + \frac{9}{2} + \frac{\sqrt{85}}{2}\right)\left(x + \frac{9}{2} - \frac{\sqrt{85}}{2}\right)$
17. a. $3x(x - 4)$ **b.** $(x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$ **c.** $(2x + 5)(2x - 5)$
d. $(2x + 5)(x + 2)$ **e.** $(a + 2)(2x + 3)$ **f.** $-3(x - 2)(x + 3)$
18. a. $\frac{2(x + 4)}{5(x + 1)}$ **b.** $\frac{7}{8}$ **c.** $\frac{(x - 2)(x - 1)}{x(x - 4)}$
19. a. $(x + 2)^2$ **b.** $32(x + 2)^2$ **c.** $32x^2 + 128x + 128$ **d.** 32 768 cm^3
20. a. $4r$ **b.** $2r + 2$ **c.** $4\pi r^2$ **d.** $(4r^2 + 8r + 4)\pi$ **e.** $4\pi(2r + 1)$ **f.** $28\pi\,\text{m}^2$
21. a. $(x - 7)$ **b.** $x - 7$ cm **c.** 35 **d.** 1036 cm^2
22. Division by zero in Step 5

Investigation — Rich task

Check with your teacher.

TOPIC 8
Quadratic equations

8.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

8.1.1 Why learn this?

The Guggenheim Museum in Bilbao (Spain) is covered with thin metal plates like the scales of a fish, each one designed and shaped by a computer. This project required the solving of thousands of non-linear equations. Parabolic shapes are widely used by engineers and architects.

8.1.2 What do you know?

assesson

1. **THINK** List what you know about quadratic equations. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of quadratic equations.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: The Chinese Golden Age of Mathematics (eles-1847)

8.2 Solving quadratic equations algebraically

8.2.1 Quadratic equations

- The general form of a quadratic equation is $ax^2 + bx + c = 0$.
- To solve an equation means to find the value of the pronumeral(s) or variables, which when substituted, will make the equation a true statement.

8.2.2 The Null Factor Law

- The **Null Factor Law** states that if the product of two numbers is zero then one or both of the numbers must equal zero.
- In other words, there are two solutions to the equation $pq = 0$; they are $p = 0$ and $q = 0$.
- When solving quadratic equations by applying the Null Factor Law, it is best to write the equations equal to zero.

WORKED EXAMPLE 1

Solve the equation $(x - 7)(x + 11) = 0$.

THINK	WRITE
1 Write the equation and check that the right-hand side equals zero. (The product of the two numbers is zero.)	$(x - 7)(x + 11) = 0$
2 The left-hand side is factorised, so apply the Null Factor Law.	$x - 7 = 0$ or $x + 11 = 0$
3 Solve for x.	$x = 7 \qquad x = -11$

WORKED EXAMPLE 2

Solve each of the following equations.

a $x^2 - 3x = 0$ **b** $3x^2 - 27 = 0$
c $x^2 - 13x + 42 = 0$ **d** $36x^2 - 21x = 2$

THINK	WRITE
a 1 Write the equation. Check that the right-hand side equals zero.	**a** $x^2 - 3x = 0$
2 Factorise by taking out the common factor of x^2 and $3x$, which is x.	$x(x - 3) = 0$
3 Apply the Null Factor Law.	$x = 0$ or $x - 3 = 0$
4 Solve for x.	$x = 0 \qquad x = 3$
b 1 Write the equation. Check that the right-hand side equals zero.	**b** $3x^2 - 27 = 0$
2 Factorise by taking out the common factor of $3x^2$ and 27, which is 3.	$3(x^2 - 9) = 0$
3 Factorise using the difference of two squares rule.	$3(x^2 - 3^2) = 0$ $3(x + 3)(x - 3) = 0$

4 Apply the Null Factor Law.

$x + 3 = 0$ or $x - 3 = 0$

5 Solve for x.

$x = -3 \quad x = 3$

(Alternatively, $x = \pm 3$.)

c **1** Write the equation. Check that the right-hand side equals zero.

c $x^2 - 13x + 42 = 0$

2 Factorise by finding a factor pair of 42 that adds to −13.

Factors of 42	Sum of factors
−6 and −7	−13

$(x - 6)(x - 7) = 0$

3 Use the Null Factor Law to write two linear equations.

$x - 6 = 0$ or $x - 7 = 0$

4 Solve for x.

$x = 6 \quad x = 7$

d **1** Write the equation. Check that the right-hand side equals zero. (It does not.)

d $36x^2 - 21x = 2$

2 Rearrange the equation so the right-hand side of the equation equals zero.

$36x^2 - 21x - 2 = 0$

3 Recognise that the expression to factorise is a quadratic trinomial.

Factors of −72	Sum of factors
3 and −24	−21

$36x^2 - 24x + 3x - 2 = 0$

4 Factorise the expression.

$12x(3x - 2) + (3x - 2) = 0$

$(3x - 2)(12x + 1) = 0$

5 Use the Null Factor Law to write two linear equations.

$3x - 2 = 0$ or $12x + 1 = 0$

$3x = 2 \qquad 12x = -1$

6 Solve for x.

$x = \frac{2}{3} \qquad x = -\frac{1}{12}$

8.2.3 Solving quadratic equations by completing the square

- Sometimes it is necessary to complete the square in order to factorise a quadratic trinomial.
- This is often necessary if the solutions are not rational numbers.

WORKED EXAMPLE 3

TI | CASIO

Find the solutions to the equation $x^2 + 2x - 4 = 0$. Give exact answers.

THINK	WRITE
1 Write the equation.	$x^2 + 2x - 4 = 0$
2 Identify the coefficient of x, halve it and square the result.	$\left(\frac{1}{2} \times 2\right)^2$

3 Add the result of step 2 to the equation, placing it after the x-term. To balance the equation, we need to subtract the same amount as we have added.	$x^2 + 2x + \left(\frac{1}{2} \times 2\right)^2 - 4 - \left(\frac{1}{2} \times 2\right)^2 = 0$ $x^2 + 2x + (1)^2 - 4 - (1)^2 = 0$ $x^2 + 2x + 1 - 4 - 1 = 0$
4 Insert brackets around the first three terms to group them and then simplify the remaining terms.	$(x^2 + 2x + 1) - 5 = 0$
5 Factorise the first three terms to produce a perfect square.	$(x + 1)^2 - 5 = 0$
6 Express as the difference of two squares and then factorise.	$(x + 1)^2 - (\sqrt{5})^2 = 0$ $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) = 0$
7 Apply the Null Factor Law to find linear equations.	$x + 1 + \sqrt{5} = 0$ or $x + 1 - \sqrt{5} = 0$
8 Solve for x. Keep the answer in surd form to provide an exact answer.	$x = -1 - \sqrt{5}$ or $x = -1 + \sqrt{5}$ (Alternatively, $x = -1 \pm \sqrt{5}$.)

8.2.4 Solving problems

- There are many problems that can be modelled by a quadratic equation. You should first form the quadratic equation that represents the situation before attempting to solve such problems.
- Recall that worded problems should always be answered with a sentence.

WORKED EXAMPLE 4

When two consecutive numbers are multiplied together, the result is 20. Determine the numbers.

THINK	WRITE
1 Define the unknowns. First number = x, second number = $x + 1$.	Let the two numbers be x and $(x + 1)$.
2 Write an equation using the information given in the question.	$x(x + 1) = 20$
3 Transpose the equation so that the right-hand side equals zero.	$x(x + 1) - 20 = 0$
4 Expand to remove the brackets.	$x^2 + x - 20 = 0$
5 Factorise.	$(x + 5)(x - 4) = 0$
6 Apply the Null Factor Law to solve for x.	$x + 5 = 0$ or $x - 4 = 0$ $x = -5$ $\quad$ $x = 4$
7 Use the answer to determine the second number.	If $x = -5$, $x + 1 = -4$. If $x = 4$, $x + 1 = 5$.
8 Check the solutions.	Check: $4 \times 5 = 20$ $\quad -5 \times -4 = 20$
9 Answer the question in a sentence.	The numbers are 4 and 5 or −5 and −4.

WORKED EXAMPLE 5

The height of a football after being kicked is determined by the formula $h = -0.1d^2 + 3d$, where d is the horizontal distance from the player.

a How far away is the ball from the player when it hits the ground?

b What horizontal distance has the ball travelled when it first reaches a height of 20 m?

THINK	WRITE
a 1 Write the formula.	**a** $h = -0.1\,d^2 + 3d$
2 The ball hits the ground when $h = 0$. Substitute $h = 0$ into the formula.	$-0.1\,d^2 + 3d = 0$
3 Factorise.	$-0.1d^2 + 3d = 0$ $d(-0.1d + 3) = 0$
4 Apply the Null Factor Law and simplify.	$d = 0$ or $-0.1d + 3 = 0$ $-0.1d = -3$ $d = \frac{-3}{-0.1}$ $= 30$
5 Interpret the solutions.	$d = 0$ is the origin of the kick. $d = 30$ is the distance from the origin that the ball has travelled when it lands.
6 Answer the question in a sentence.	The ball is 30 m from the player when it hits the ground.
b 1 The height of the ball is 20 m, so, substitute $h = 20$ into the formula.	**b** $h = -0.1d^2 + 3d$ $20 = -0.1d^2 + 3d$
2 Transpose the equation so that zero is on the right-hand side.	$0.1d^2 - 3d + 20 = 0$
3 Multiply both sides of the equation by 10 to remove the decimal from the coefficient.	$d^2 - 30d + 200 = 0$
4 Factorise.	$(d - 20)(d - 10) = 0$
5 Apply the Null Factor Law.	$d - 20 = 0$ or $d - 10 = 0$
6 Solve.	$d = 20$ $\qquad d = 10$
7 Interpret the solution. The ball reaches a height of 20 m on the way up and on the way down. The *first* time the ball reaches a height of 20 m is the smaller value of d. Answer in a sentence.	The ball first reaches a height of 20 m after it has travelled a distance of 10 m.

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- **Complete this digital doc:** SkillSHEET: Finding a factor pair that adds to a given number (doc-5257)
- **Complete this digital doc:** SkillSHEET: Simplifying surds (doc-5258)
- **Complete this digital doc:** SkillSHEET: Substituting into quadratic equations (doc-5259)
- **Complete this digital doc:** SkillSHEET: Equation of a vertical line (doc-5260)
- **Complete this digital doc:** WorkSHEET: Solving quadratic equations (doc-5261)

Exercise 8.2 Solving quadratic equations algebraically

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–f, 2a–c, 3a–d, 4a–c, 5a–h, 6–8, 10, 14, 16, 19, 20, 22	Questions: 1b–g, 2a–d, 3a–f, 4a–f, 5d–l, 6, 7, 8a–d, 9, 11, 14–16, 18, 20–22, 26	Questions: 1, 2, 3f–i, 4f–l, 5g–l, 6, 7, 8g–l, 9f–i, 10d–i, 11d–i, 12–27

🟩🟦🟪 **Individual pathway interactivity:** int-4601

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** Solve each of the following equations.

a. $(x+7)(x-9)=0$ **b.** $(x-3)(x+2)=0$ **c.** $(x-2)(x-3)=0$
d. $x(x-3)=0$ **e.** $x(x-1)=0$ **f.** $x(x+5)=0$
g. $2x(x-3)=0$ **h.** $9x(x+2)=0$ **i.** $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)=0$
j. $-(x+1.2)(x+0.5)=0$ **k.** $2(x-0.1)(2x-1.5)=0$ **l.** $(x+\sqrt{2})(x-\sqrt{3})=0$

2. Solve each of the following equations.

a. $(2x-1)(x-1)=0$ **b.** $(3x+2)(x+2)=0$ **c.** $(4x-1)(x-7)=0$
d. $(7x+6)(2x-3)=0$ **e.** $(5x-3)(3x-2)=0$ **f.** $(8x+5)(3x-2)=0$
g. $x(x-3)(2x-1)=0$ **h.** $x(2x-1)(5x+2)=0$ **i.** $x(x+3)(5x-2)=0$

3. **WE2a** Solve each of the following equations.

a. $x^2-2x=0$ **b.** $x^2+5x=0$ **c.** $x^2=7x$ **d.** $3x^2=-2x$ **e.** $4x^2-6x=0$
f. $6x^2-2x=0$ **g.** $4x^2-2\sqrt{7}x=0$ **h.** $3x^2+\sqrt{3}x=0$ **i.** $15x-12x^2=0$

4. **WE2b** Solve each of the following equations.

a. $x^2-4=0$ **b.** $x^2-25=0$ **c.** $3x^2-12=0$ **d.** $4x^2-196=0$ **e.** $9x^2-16=0$
f. $4x^2-25=0$ **g.** $9x^2=4$ **h.** $36x^2=9$ **i.** $x^2-\frac{1}{25}=0$ **j.** $\frac{1}{36}x^2-\frac{4}{9}=0$
k. $x^2-5=0$ **l.** $9x^2-11=0$

5. **WE2c** Solve each of the following equations.

a. $x^2-x-6=0$ **b.** $x^2+6x+8=0$ **c.** $x^2-6x-7=0$ **d.** $x^2-8x+15=0$
e. $x^2-2x+1=0$ **f.** $x^2-3x-4=0$ **g.** $x^2-10x+25=0$ **h.** $x^2-3x-10=0$
i. $x^2-8x+12=0$ **j.** $x^2-4x-21=0$ **k.** $x^2-x-30=0$ **l.** $x^2-7x+12=0$
m. $x^2-8x+16=0$ **n.** $x^2+10x+25=0$ **o.** $x^2-20x+100=0$

6. **MC** The solutions to the equation $x^2 + 9x - 10 = 0$ are:

A. $x = 1$ and $x = 10$ **B.** $x = 1$ and $x = -10$ **C.** $x = -1$ and $x = 10$
D. $x = -1$ and $x = -10$ **E.** $x = 1$ and $x = 9$

7. **MC** The solutions to the equation $x^2 - 100 = 0$ are:

A. $x = 0$ and $x = 10$ **B.** $x = 0$ and $x = -10$ **C.** $x = -10$ and $x = 10$
D. $x = 0$ and $x = 100$ **E.** $x = -100$ and $x = 100$

8. **WE2d** Solve each of the following equations.

a. $2x^2 - 5x = 3$ **b.** $3x^2 + x - 2 = 0$ **c.** $5x^2 + 9x = 2$
d. $6x^2 - 11x + 3 = 0$ **e.** $14x^2 - 11x = 3$ **f.** $12x^2 - 7x + 1 = 0$
g. $6x^2 - 7x = 20$ **h.** $12x^2 + 37x + 28 = 0$ **i.** $10x^2 - x = 2$
j. $6x^2 - 25x + 24 = 0$ **k.** $30x^2 + 7x - 2 = 0$ **l.** $3x^2 - 21x = -36$

9. **WE3** Find the solutions for each of the following equations. Give exact answers.

a. $x^2 - 4x + 2 = 0$ **b.** $x^2 + 2x - 2 = 0$ **c.** $x^2 + 6x - 1 = 0$
d. $x^2 - 8x + 4 = 0$ **e.** $x^2 - 10x + 1 = 0$ **f.** $x^2 - 2x - 2 = 0$
g. $x^2 + 2x - 5 = 0$ **h.** $x^2 + 4x - 6 = 0$ **i.** $x^2 + 4x - 11 = 0$

10. Find the solutions for each of the following equations. Give exact answers.

a. $x^2 - 3x + 1 = 0$ **b.** $x^2 + 5x - 1 = 0$ **c.** $x^2 - 7x + 4 = 0$ **d.** $x^2 - 5 = x$
e. $x^2 - 11x + 1 = 0$ **f.** $x^2 + x = 1$ **g.** $x^2 + 3x - 7 = 0$ **h.** $x^2 - 3 = 5x$
i. $x^2 - 9x + 4 = 0$

11. Solve each of the following equations, rounding answers to 2 decimal places.

a. $2x^2 + 4x - 6 = 0$ **b.** $3x^2 + 12x - 3 = 0$ **c.** $5x^2 - 10x - 15 = 0$ **d.** $4x^2 - 8x - 8 = 0$
e. $2x^2 - 6x + 2 = 0$ **f.** $3x^2 - 9x - 3 = 0$ **g.** $5x^2 - 15x - 25 = 0$ **h.** $7x^2 + 7x - 21 = 0$
i. $4x^2 + 8x - 2 = 0$

Understanding

12. **WE4** When two consecutive numbers are multiplied, the result is 72. Find the numbers.

13. When two consecutive even numbers are multiplied, the result is 48. Find the numbers.

14. When a number is added to its square the result is 90. Find the number.

15. Twice a number is added to three times its square. If the result is 16, find the number.

16. Five times a number is added to two times its square. If the result is 168, find the number

17. **WE5** A soccer ball is kicked. The height, h, in metres, of the soccer ball t seconds after it is kicked can be represented by the equation $h = -t(t - 6)$. Find how long it takes for the soccer ball to hit the ground again.

18. The length of an Australian flag is twice its width and the diagonal length is 45 cm.

a. If x cm is the width of the flag, find the length in terms of x.

b. Draw a diagram of the flag marking in the diagonal. Mark the length and the width in terms of x.

c. Use Pythagoras' theorem to write an equation relating the lengths of the sides to the length of the diagonal.

d. Solve the equation to find the dimensions of the Australian flag. Round your answer to the nearest cm.

19. If the length of a paddock is 2 m more than its width and the area is 48 m^2, find the length and width of the paddock.

20. Solve for x.

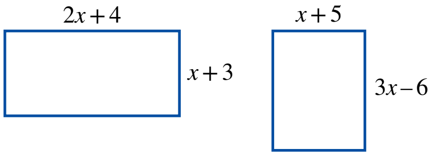
a. $x + 5 = \frac{6}{x}$ **b.** $x = \frac{24}{x - 5}$ **c.** $x = \frac{1}{x}$

Reasoning

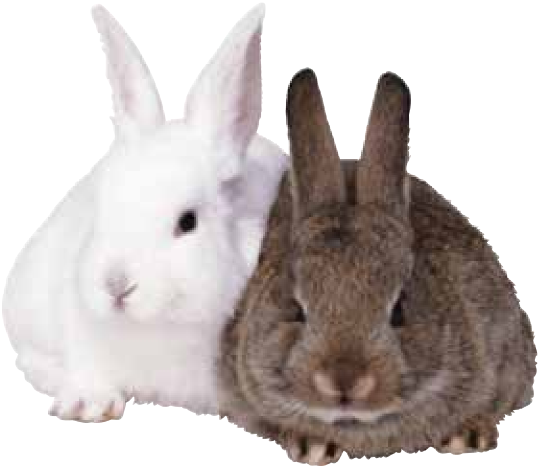
21. The sum of the first n numbers 1, 2, 3, 4 … n is given by the formula $S = \frac{n(n+1)}{2}$.

- **a.** Use the formula to find the sum of the first 6 counting numbers.
- **b.** How many numbers are added to give a sum of 153?

22. If these two rectangles have the same area, what is the value of x?



23. Henrietta is a pet rabbit who lives in an enclosure that is 2 m wide and 4 m long. Her human family has decided to purchase some more rabbits to keep her company and so the size of the enclosure must be increased.



- **a.** Draw a diagram of Henrietta's enclosure, clearly marking the lengths of the sides.
- **b.** If the length and width of the enclosure are increased by x m, find the new dimensions.
- **c.** If the new area is to be 24 m^2, write an equation relating the sides and the area of the enclosure (Area = length × width).
- **d.** Use the equation to find the value of x and, hence, the length of the sides of the new enclosure. Justify your answer.

24. The cost per hour, C, in thousands of dollars of running two cruise ships, *Annabel* and *Betty*, travelling at a speed of s knots is given by the following relationships.

$C_{Annabel} = 0.3s^2 + 4.2s + 12$ and $C_{Betty} = 0.4s^2 + 3.6s + 8$

- **a.** Determine the cost per hour for each ship if they are both travelling at 28 knots.
- **b.** Find the speed in knots at which both ships must travel for them to have the same cost.
- **c.** Explain why only one of the solutions obtained in your working for part **b** is valid.

25. Explain why the equation $x^2 + 4x + 10 = 0$ has no real solutions.

Problem solving

26. Solve $(x^2 - x)^2 - 32(x^2 - x) + 240 = 0$ for x.

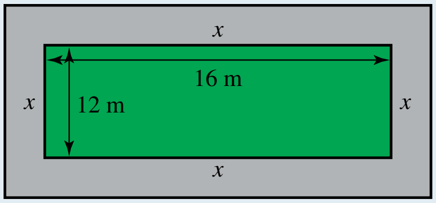
27. Solve $\dfrac{3z^2 - 35}{16} - z = 0$ for z.

Reflection

What does the Null Factor Law mean?

CHALLENGE 8.1

A garden measuring 12 metres by 16 metres is to have a pedestrian pathway installed all around it, increasing the total area to 285 square metres. What will be the width of the pathway?



8.3 The quadratic formula

8.3.1 The quadratic formula

- The method of solving quadratic equations by completing the square can be generalised to produce what is called the **quadratic formula**.
- The general equation $ax^2 + bx + c = 0$ can be solved by completing the square. We will first follow the steps involved in completing the square.

1. Divide both sides of the equation by a.
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2. Complete the square.
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

3. Factorise the first three terms as a perfect square.
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

4. Add the final two terms.
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

5. Write as the difference of two squares.
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$$

6. Factorise using the difference of two squares rule.
$$\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$$

7. Solve the two linear factors.
$$x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0 \text{ or } x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0$$
$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

- The solution can be summarised as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a is the coefficient of x^2, b is the coefficient of x and c is the constant or the term without an x.
- The quadratic formula can be used to solve any quadratic equation.
- If the value inside the square root sign is negative, then there are no solutions to the equation.

WORKED EXAMPLE 6

TI | CASIO

Use the quadratic formula to solve each of the following equations.

a $3x^2 + 4x + 1 = 0$ **(exact answer)**

b $-3x^2 - 6x - 1 = 0$ **(round to 2 decimal places)**

THINK	WRITE
a **1** Write the equation.	**a** $3x^2 + 4x + 1 = 0$
2 Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3 State the values for *a*, *b* and *c*.	where $a = 3, b = 4, c = 1$
4 Substitute the values into the formula.	$x = \frac{-4 \pm \sqrt{(4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$
5 Simplify and solve for *x*.	$= \frac{-4 \pm \sqrt{4}}{6}$ $= \frac{-4 \pm 2}{6}$ $x = \frac{-4 + 2}{6}$ or $x = \frac{-4 - 2}{6}$
6 Write the two solutions.	$x = -\frac{1}{3}$ $\quad$ $x = -1$
b **1** Write the equation.	**b** $-3x^2 - 6x - 1 = 0$
2 Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3 State the values for *a*, *b* and *c*.	where $a = -3, b = -6, c = -1$
4 Substitute the values into the formula.	$x = \frac{-(-6) \pm \sqrt{36 - 4 \times -3 \times -1}}{2 \times -3}$
5 Simplify the fraction.	$= \frac{6 \pm \sqrt{24}}{-6}$ $= \frac{6 \pm 2\sqrt{6}}{-6}$ $= \frac{3 \pm \sqrt{6}}{-3}$ $x = \frac{3 + \sqrt{6}}{-3}$ or $\frac{3 - \sqrt{6}}{-3}$
6 Write the two solutions correct to two decimal places.	$x \approx -1.82$ or $x \approx -0.18$

Note: When asked to give an answer in exact form, you should simplify any surds as necessary.

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Exercise 8.3 The quadratic formula

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Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2a–f, 3a–f, 4–7, 8a–g, 9	Questions: 1d–g, 2d–h, 3d–h, 4–7, 8d–i, 10, 12, 13	Questions: 1e–h, 2g–l, 3g–n, 4–7, 8i–o, 9–14

🟩🟦🟪 **Individual pathway interactivity:** int-4602 learnon ONLINE ONLY

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Fluency

1. State the values for a, b and c in each of the following equations of the form $ax^2 + bx + c = 0$.

a. $3x^2 - 4x + 1 = 0$ **b.** $7x^2 - 12x + 2 = 0$ **c.** $8x^2 - x - 3 = 0$
d. $x^2 - 5x + 7 = 0$ **e.** $5x^2 - 5x - 1 = 0$ **f.** $4x^2 - 9x - 3 = 0$
g. $12x^2 - 29x + 103 = 0$ **h.** $43x^2 - 81x - 24 = 0$

2. WE6a Use the quadratic formula to solve each of the following equations. Give exact answers.

a. $x^2 + 5x + 1 = 0$ **b.** $x^2 + 3x - 1 = 0$ **c.** $x^2 - 5x + 2 = 0$ **d.** $x^2 - 4x - 9 = 0$
e. $x^2 + 2x - 11 = 0$ **f.** $x^2 - 7x + 1 = 0$ **g.** $x^2 - 9x + 2 = 0$ **h.** $x^2 - 6x - 3 = 0$
i. $x^2 + 8x - 15 = 0$ **j.** $-x^2 + x + 5 = 0$ **k.** $-x^2 + 5x + 2 = 0$ **l.** $-x^2 - 2x + 7 = 0$

3. WE6b Use the quadratic formula to solve each of the following equations. Give approximate answers rounded to 2 decimal places.

a. $3x^2 - 4x - 3 = 0$ **b.** $4x^2 - x - 7 = 0$ **c.** $2x^2 + 7x - 5 = 0$ **d.** $7x^2 + x - 2 = 0$
e. $5x^2 - 8x + 1 = 0$ **f.** $2x^2 - 13x + 2 = 0$ **g.** $-3x^2 + 2x + 7 = 0$ **h.** $-7x^2 + x + 8 = 0$
i. $-12x^2 + x + 9 = 0$ **j.** $-6x^2 + 4x + 5 = 0$ **k.** $-11x^2 - x + 1 = 0$ **l.** $-4x^2 - x + 7 = 0$
m. $-2x^2 + 12x - 1 = 0$ **n.** $-5x^2 + x + 3 = 0$

4. MC The solutions of the equation $3x^2 - 7x - 2 = 0$ are:

A. 1, 2 **B.** 1, −2 **C.** −0.257, 2.59 **D.** −0.772, 7.772 **E.** −1.544, 15.544

5. MC In the expansion of $(6x - 5)(3x + 4)$, the coefficient of x is:

A. 18 **B.** −15 **C.** 9 **D.** 6 **E.** −2

6. MC In the expanded form of $(x - 2)(x + 4)$, which of the following is incorrect?

A. The value of the constant is −8. **B.** The coefficient of the x term is −6.
C. The coefficient of the x term is 2. **D.** The coefficient of the x^2 term is 1.
E. The expansion shows this to be a trinomial expression.

7. MC An exact solution to the equation $x^2 + 2x - 5 = 0$ is:

A. −3.449 **B.** $-1 + \sqrt{24}$ **C.** $-1 + \sqrt{6}$ **D.** $\dfrac{2 + \sqrt{-16}}{2}$ **E.** $\dfrac{2 + \sqrt{24}}{2}$

Understanding

8. Solve each of the following equations using any suitable method. Round to 3 decimal places where appropriate.

a. $2x^2 - 7x + 3 = 0$ **b.** $x^2 - 5x = 0$ **c.** $x^2 - 2x - 3 = 0$ **d.** $x^2 - 3x + 1 = 0$

e. $x^2 - 7x + 2 = 0$ **f.** $x^2 - 6x + 8 = 0$ **g.** $x^2 - 5x + 8 = 0$ **h.** $x^2 - 7x - 8 = 0$

i. $x^2 + 2x - 9 = 0$ **j.** $3x^2 + 3x - 6 = 0$ **k.** $2x^2 + 11x - 21 = 0$ **l.** $7x^2 - 2x + 1 = 0$

m. $-x^2 + 9x - 14 = 0$ **n.** $-6x^2 - x + 1 = 0$ **o.** $-6x^2 + x - 5 = 0$

9. The surface area of a closed cylinder is given by the formula $\text{SA} = 2\pi r(r + h)$, where r cm is the radius of the can and h cm is the height.

The height of a can of wood finish is 7 cm and its surface area is 231 cm^2.

a. Substitute values into the formula to form a quadratic equation using the pronumeral, r.

b. Use the quadratic formula to solve the equation and, hence, find the radius of the can correct to 1 decimal place.

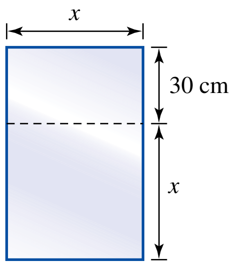
c. Calculate the area of the curved surface of the can, correct to the nearest square centimetre.

10. To satisfy lighting requirements, a window must have an area of 1500 cm^2.

a. Find an expression for the area of the window in terms of x.

b. Write an equation so that the window satisfies the lighting requirements.

c. Use the quadratic formula to solve the equation and find x to the nearest mm.



Reasoning

11. Two competitive neighbours build rectangular pools that cover the same area but are different shapes. Pool A has a width of $(x + 3)$ m and a length that is 3 m longer than its width. Pool B has a length that is double the width of Pool A. The width of Pool B is 4 m shorter than its length.

a. Find the exact dimensions of each pool if their areas are the same.

b. Verify that the areas are the same.

12. A block of land is in the shape of a right-angled triangle with a perimeter of 150 m and a hypotenuse of 65 m. Determine the lengths of the other two sides. Show your working.

Problem solving

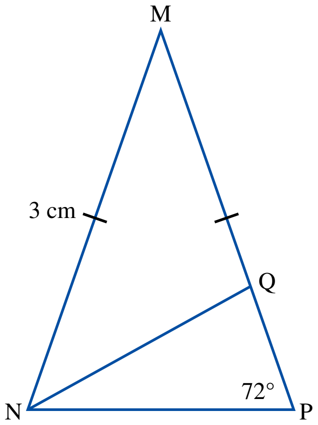
13. Solve $\left(x + \frac{1}{x}\right)^2 - 14\left(x + \frac{1}{x}\right) = 72$ for x.

14. Triangle MNP is an isosceles triangle with sides MN = MP = 3 cm. Angle MPN is equal to 72°. The line NQ bisects the angle MNP.

a. Prove that triangles MNP and NPQ are similar.

b. If NP = m cm and $PQ = 3 - m$, show that $m^2 + 3m - 9 = 0$.

c. Solve the equation $m^2 + 3m - 9 = 0$ and find the side length of NP, giving your answer correct to two decimal places.



Reflection

What kind of answer will you get if the value inside the square root sign in the quadratic formula is zero?

CHALLENGE 8.2

The equation $ax^4 + bx^2 + c = 0$ can be solved by applying substitution and the rules used to solve quadratics.
For example, $x^4 - 5x^2 + 4 = 0$ is solved for x as follows.
Notice that $x^4 - 5x^2 + 4 = (x^2)^2 - 5(x)^2 + 4$. Now let $x^2 = u$ and substitute.
$(x^2)^2 - 5(x)^2 + 4 = u^2 - 5u + 4$.
Solve for u. That is,

$$\begin{aligned} u^2 - 5u + 4 &= 0 \\ (u-4)(u-1) &= 0 \\ u - 4 = 0 \text{ or } u - 1 &= 0 \\ u = 4 \text{ or } u &= 1 \end{aligned}$$

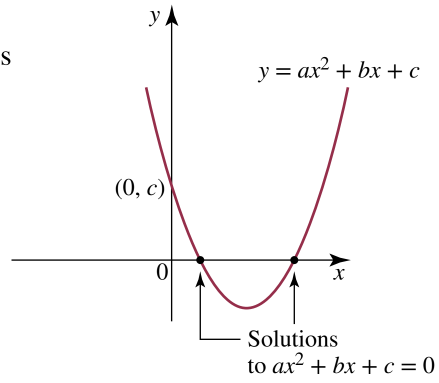
Since $x^2 = u$, that implies that
$x^2 = 4$ or $x^2 = 1$
$x = \pm 2$ or $x = \pm 1$
Using this or another method, solve the following for x.

1 $x^4 - 13x^2 + 36 = 0$
2 $4x^4 - 17x^2 = -4$

8.4 Solving quadratic equations graphically

8.4.1 Solving quadratic equations

- The graph of $y = ax^2 + bx + c$ is in the shape of a parabola.
- The graph can be used to locate the solutions to quadratic equations such as $ax^2 + bx + c = 0$.

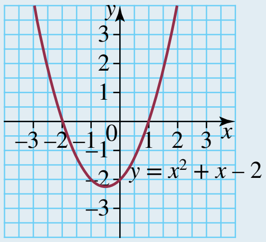


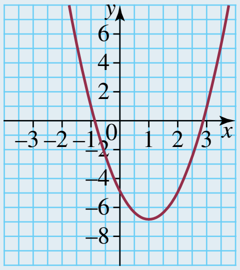
WORKED EXAMPLE 7

TI | CASIO

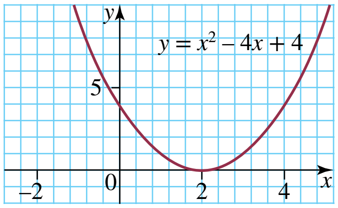
Determine the solutions of each of the following quadratic equations by inspecting their corresponding graphs. Give answers to 1 decimal place where appropriate.

a $x^2 + x - 2 = 0$ **b** $2x^2 + 4x - 5 = 0$

THINK	WRITE/DRAW
a 1 Examine the graph of $y = x^2 + x - 2$ and locate the points where $y = 0$, that is, where the graph intersects the x-axis.	**a** 
2 The graph cuts the x-axis ($y = 0$) at $x = 1$ and $x = -2$. Write the solutions.	$x^2 + x - 2 = 0$ From the graph, the solutions are $x = 1$ and $x = -2$.

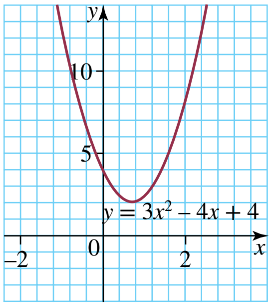
b **1** The graph of $y = 2x^2 - 4x - 5$ is equal to zero when $y = 0$. Look at the graph to see where $y = 0$; that is, where it intersects the x-axis. By sight, we can only give estimates of the solutions.	**b** 
2 The graph cuts the x-axis at approximately $x = -0.9$ and approximately $x = 2.9$. Write the solutions.	$2x^2 - 4x - 5 = 0$ From the graph, the solutions are $x \approx -0.9$ and $x \approx 2.9$.

8.4.2 Quadratic equations with only one solution



- Some quadratic equations have only one solution. For example, the graph of $y = x^2 - 4x + 4$ has only one solution at $x = 2$. That is, the graph of $y = x^2 - 4x + 4$ touches the x-axis at one point only at $x = 2$.

8.4.3 Quadratic equations with no solutions



- There are also quadratic equations that have no real solutions. For example, the graph of $y = 3x^2 - 4x + 4$ does not intersect the x-axis and so $3x^2 - 4x + 4 = 0$ has no real solutions (that is, no solutions that are real numbers).

8.4.4 Confirming solutions

- It is possible to confirm the solutions obtained by sight. This is achieved by substituting the solution or solutions into the original quadratic equation, and determining whether they make a true statement.

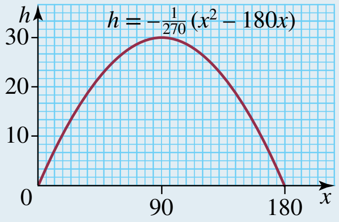
WORKED EXAMPLE 8

Confirm, by substitution, the solutions obtained in Worked example 7.
$x^2 + x - 2 = 0$; solutions: $x = 1$ and $x = -2$

THINK	**WRITE**
1 Write the left-hand side of the equation and substitute $x = 1$ into the expression.	When $x = 1$, LHS: $x^2 + x - 2 = 1^2 + 1 - 2$ $= 0$
2 Write the right-hand side.	RHS: $= 0$
3 Confirm the solution.	LHS = RHS ⇒ Solution is confirmed.
4 Write the left-hand side and substitute $x = -2$.	When $x = -2$, LHS: $x^2 + x - 2 = (-2)^2 + -2 - 2$
5 Write the right-hand side.	$= 4 - 2 - 2$ $= 0$ RHS: $= 0$
6 Confirm.	LHS = RHS ⇒ Solution is confirmed.

WORKED EXAMPLE 9

A golf ball hit along a fairway follows the path shown in the following graph. The height, h metres after it has travelled x metres horizontally, follows the rule $h = \frac{1}{270}(x^2 - 180x)$. Use the graph to find how far the ball landed from the golfer.



THINK

On the graph, the ground is represented by the x-axis since this is where $h = 0$. The golf ball lands when the graph intersects the x-axis.

WRITE

The golf ball lands 180 m from the golfer.

Exercise 8.4 Solving quadratic equations graphically

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2–4, 7	Questions: 1c–h, 2–5, 7	Questions: 1e–j, 2–8

🟩🟦🟪 **Individual pathway interactivity:** int-4603

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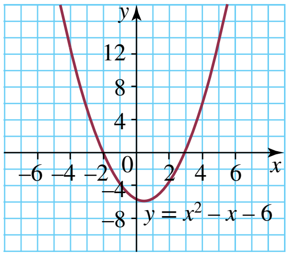
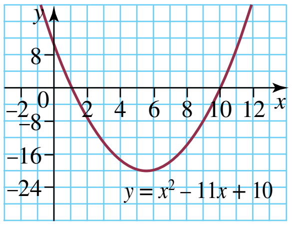
Fluency

1. **WE7** Determine the solutions of each of the following quadratic equations by inspecting the corresponding graphs. Give answers correct to 1 decimal place where appropriate.

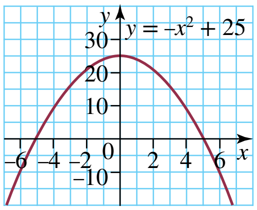
a. $x^2 - x - 6 = 0$



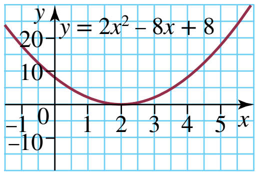
b. $x^2 - 11x + 10 = 0$



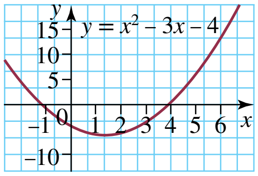
c. $-x^2 + 25 = 0$



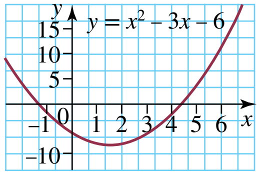
d. $2x^2 - 8x + 8 = 0$



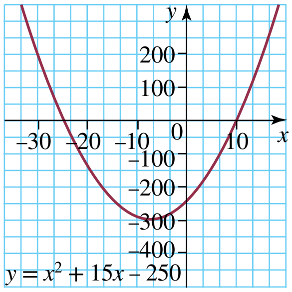
e. $x^2 - 3x - 4 = 0$



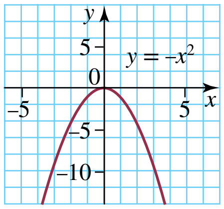
f. $x^2 - 3x - 6 = 0$



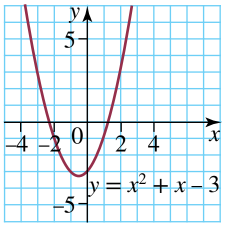
g. $x^2 + 15x - 250 = 0$



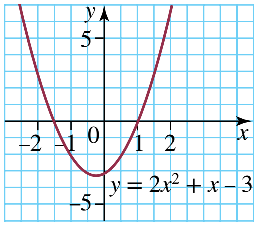
h. $-x^2 = 0$



i. $x^2 + x - 3 = 0$



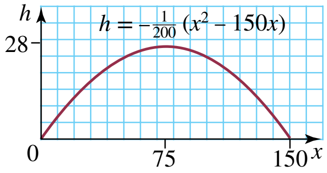
j. $2x^2 + x - 3 = 0$



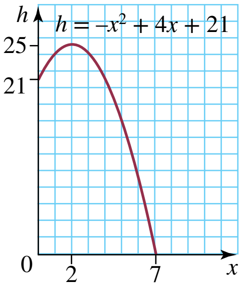
Understanding

2. **WE8** Confirm, by substitution, the solutions obtained in question **1**.

3. **WE9** A golf ball hit along a fairway follows the path shown in the graph.
The height, h metres after it has travelled x metres horizontally, follows the rule $h = -\frac{1}{200}(x^2 - 150x)$.
Use the graph to find how far the ball lands from the golfer.



4. A ball is thrown upwards from a building and follows the path shown in the graph until it lands on the ground.



The ball is h metres above the ground when it is a horizontal distance of x metres from the building. The path of the ball follows the rule $h = -x^2 + 4x + 21$. Use the graph to find how far from the building the ball lands.

Reasoning

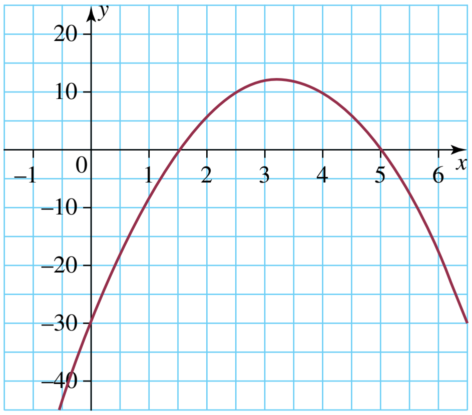
5. **a.** The x-intercepts of a particular equation are $x = 2$ and $x = 5$. Suggest a possible equation.
   b. If the y-intercept in part **a** is $(0, 4)$, give the exact equation.
6. **a.** The x-intercepts of a particular equation are $x = p, q$. Suggest a possible equation.
   b. If the y-intercept in part **a** is $(0, r)$, give the exact equation.

Problem solving

7. A platform diver follows a path determined by the equation $h = -0.5d^2 + 2d + 6$, where h represents the height of the diver above the water and d represents the distance from the diving board. Both pronumerals are measured in metres.

Use the graph to determine:
a. how far the diver landed from the edge of the diving board
b. how high the diving board is above the water.

8. Find the equation of the given parabola. Give your answer in the form $y = ax^2 + bx + c$.



Reflection

What does ‘the solution of a graph’ mean?

8.5 The discriminant

8.5.1 The discriminant

- Where $ax^2 + bx + c = 0$, the expression $\Delta = b^2 - 4ac$ is known as the **discriminant**.
- The symbol used for the discriminant, Δ, is the Greek capital letter delta.

Discriminant
$\Delta = b^2 - 4ac$

- The discriminant is found in the quadratic formula, as shown below.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

- The discriminant is the value that determines the number of solutions to the quadratic equation.
  – If $\Delta < 0$, there are *no* real solutions to the quadratic equation.
  – If $\Delta = 0$, there is *only one* solution to the quadratic equation.
  – If $\Delta > 0$, there are *two* solutions to the quadratic equation.

WORKED EXAMPLE 10

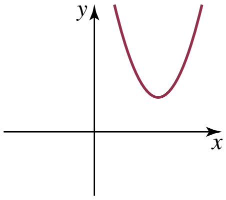
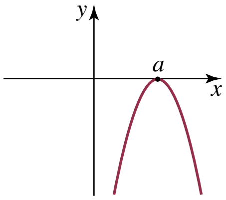
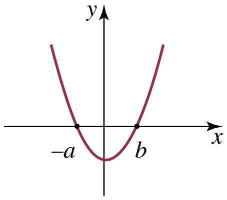
Calculate the value of the discriminant for each of the following and use it to determine how many solutions the equation will have.

a $2x^2 + 9x - 5 = 0$ **b** $x^2 + 10 = 0$

THINK	WRITE
a 1 Write the expression and determine the values of a, b and c given $ax^2 + bx + c = 0$.	**a** $2x^2 + 9x - 5 = 0$ $2x^2 + 9x + -5 = 0$ $a = 2, b = 9, c = -5$
2 Write the formula for the discriminant and substitute values of a, b and c.	$\Delta = b^2 - 4ac$ $= 9^2 - 4 \times 2 \times -5$
3 Simplify the equation and solve.	$= 81 - -40$ $= 121$
4 State the number of solutions. In this case $\Delta > 0$, which means there are two solutions.	$\Delta > 0$ There will be two solutions to the equation $2x^2 + 9x - 5 = 0$.
b 1 Write the expression and determine the values of a, b and c given $ax^2 + bx + c = 0$.	**b** $x^2 + 10 = 0$ $1x^2 + 0x + 10 = 0$ $a = 1, b = 0, c = 10$
2 Write the formula for the discriminant and substitute the values of a, b and c.	$\Delta = b^2 - 4ac$ $= (0)^2 - 4 \times 1 \times 10$ $= 0 - 40$ $= -40$
3 State the number of solutions. In this case $\Delta < 0$, which means there are no solutions.	$\Delta < 0$, so there will be no solutions to the equation $x^2 + 10 = 0$.

8.5.2 Types of solutions

- The table below summarises the types of solutions indicated by the discriminant.

	$\Delta < 0$ (negative)	$\Delta = 0$ (zero)	$\Delta > 0$ (positive)	
			Perfect square	**Not a perfect square**
Number of solutions	No real solutions	1 rational solution	2 rational solutions	2 irrational (surd) solutions
Description	Graph does not cross or touch the x-axis	Graph touches the x-axis	Graph intersects the x-axis twice	
Graph				

WORKED EXAMPLE 11

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By using the discriminant, determine whether the following equations have:

i two rational solutions
ii two irrational solutions
iii one rational solution (two equal solutions)
iv no real solutions.

a $x^2 - 9x - 10 = 0$

b $x^2 - 2x - 14 = 0$

c $x^2 - 2x + 14 = 0$

d $x^2 - 14x = -49$

THINK	WRITE
a 1 Write the equation.	**a** $x^2 - 9x - 10 = 0$
2 Identify the coefficients a, b and c.	$a = 1, b = -9, c = -10$
3 Find the discriminant.	$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-9)^2 - (4 \times 1 \times -10) \\ &= 121\end{aligned}$
4 Identify the number and type of solutions when $\Delta > 0$ and is a perfect square	The equation has two rational solutions.
b 1 Write the equation.	**b** $x^2 - 2x - 14 = 0$
2 Identify the coefficients a, b and c.	$a = 1, b = -2, c = -14$
3 Find the discriminant.	$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-2)^2 - 4 \times 1 \times -14 \\ &= 60\end{aligned}$
4 Identify the number and type of solutions when $\Delta > 0$ but not a perfect square.	The equation has two irrational solutions.

c **1** Write the equation.

c $x^2 - 2x + 14 = 0$

2 Identify the coefficients *a*, *b* and *c*.

$a = 1, b = -2, c = 14$

3 Find the discriminant.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-2)^2 - (4 \times 1 \times 14) \\ &= -52\end{aligned}$$

4 Identify the number and type of solutions when $\Delta < 0$.

The equation has no real solutions.

d **1** Write the equation, then rewrite it so the right side equals zero.

d $x^2 + 14x = -49$
$x^2 + 14x + 49 = 0$

2 Identify the coefficients *a*, *b* and *c*.

$a = 1, b = 14, c = 49$

3 Find the discriminant.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 14^2 - (4 \times 1 \times 49) \\ &= 0\end{aligned}$$

4 Identify the number and types of solutions when $\Delta = 0$.

The equation has 1 rational solution.

8.5.3 Using the discriminant to determine if graphs intersect

- The discriminant can be used to determine the number of points of intersection between graphs.

WORKED EXAMPLE 12

TI | CASIO

Determine whether the parabola $y = x^2 - 2$ and the line $y = x - 3$ intersect.

THINK	WRITE
1 If the parabola and the line intersect, there will be at least one solution to the simultaneous equations: let $y_1 = y_2$.	$y_1 = x^2 - 2$ $y_2 = x - 3$ $y_1 = y_2$ $x^2 - 2 = x - 3$
2 Collect all terms on one side and simplify.	$x^2 - 2 - x + 3 = x - 3 - x + 3$ $x^2 - 2 - x + 3 = 0$ $x^2 - x + 1 = 0$
3 • Use the discriminant to check if any solutions exist. • If $\Delta < 0$, then no solutions exist.	$\Delta = b^2 - 4ac$ $a = 1, b = -1$ $\Delta = (-1)^2 - 4(1)(1)$ $= 1 - 4$ $= -3$ $\Delta < 0 \therefore$ no solutions exist
4 Answer the question.	The parabola and the line do not intersect.

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Complete this digital doc: WorkSHEET: Using the discriminant (doc-13854)

Exercise 8.5 The discriminant

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–f, 2a–f, 3, 4, 6–8, 10, 13, 15	Questions: 1e–j, 2d–i, 3, 5–9, 11, 12, 14, 15, 17	Questions: 1h–n, 2f–l, 3, 5–18

🟩🟦🟪 **Individual pathway interactivity:** int-4604

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Fluency

1. **WE10** Calculate the value of the discriminant for each of the following and use it to determine how many solutions the equation will have.
   - **a.** $6x^2 + 13x - 5 = 0$
   - **b.** $x^2 + 9x - 90 = 0$
   - **c.** $x^2 + 4x - 2 = 0$
   - **d.** $36x^2 - 1 = 0$
   - **e.** $x^2 + 2x + 8 = 0$
   - **f.** $x^2 - 5x - 14 = 0$
   - **g.** $36x^2 + 24x + 4 = 0$
   - **h.** $x^2 - 19x + 88 = 0$
   - **i.** $x^2 - 10x + 17 = 0$
   - **j.** $30x^2 + 17x - 21 = 0$
   - **k.** $x^2 + 16x + 62 = 0$
   - **l.** $9x^2 - 36x + 36 = 0$
   - **m.** $2x^2 - 16x = 0$
   - **n.** $x^2 - 64 = 0$
2. **WE11** By using the discriminant, determine whether the equations below have:
   - **i.** two rational solutions
   - **ii.** two irrational solutions
   - **iii.** one rational solution (two equal solutions)
   - **iv.** no real solutions.
   - **a.** $x^2 - 3x + 5$
   - **b.** $4x^2 - 20x + 25 = 0$
   - **c.** $x^2 + 9x - 22 = 0$
   - **d.** $9x^2 + 12x + 4$
   - **e.** $x^2 + 3x - 7 = 0$
   - **f.** $25x^2 - 10x + 1 = 0$
   - **g.** $3x^2 - 2x - 4 = 0$
   - **h.** $2x^2 - 5x + 4 = 0$
   - **i.** $x^2 - 10x + 26 = 0$
   - **j.** $3x^2 + 5x - 7 = 0$
   - **k.** $2x^2 + 7x - 10 = 0$
   - **l.** $x^2 - 11x + 30 = 0$
3. **WE12** Determine whether the following graphs intersect.
   - **a.** $y = -x^2 + 3x + 4$ and $y = x - 4$
   - **b.** $y = -x^2 + 3x + 4$ and $y = 2x + 5$
   - **c.** $y = -(x + 1)^2 + 3$ and $y = -4x - 1$
   - **d.** $y = (x - 1)^2 + 5$ and $y = -4x - 1$
4. Consider the equation $3x^2 + 2x + 7 = 0$.
   - **a.** What are the values of a, b and c?
   - **b.** What is the value of $b^2 - 4ac$?
   - **c.** How many real solutions, and hence x-intercepts, are there for this equation?
5. Consider the equation $-6x^2 + x + 3 = 0$.
   - **a.** What are the values of a, b and c?
   - **b.** What is the value of $b^2 - 4ac$?
   - **c.** How many real solutions, and hence x-intercepts, are there for this equation?
   - **d.** With the information gained from the discriminant, use the most efficient method to solve the equation. Give an exact answer.

6. **MC** The discriminant of the equation $x^2 - 4x - 5 = 0$ is:

A. 36 **B.** 11 **C.** 4 **D.** 0 **E.** −4

7. **MC** Which of the following quadratic equations has two irrational solutions?

A. $x^2 - 8x + 16 = 0$ **B.** $2x^2 - 7x = 0$ **C.** $x^2 + 8x + 9 = 0$

D. $x^2 - 4 = 0$ **E.** $x^2 - 6x + 15 = 0$

8. **MC** The equation $x^2 = 2x - 3$ has:

A. two rational solutions **B.** exactly one solution

C. no solutions **D.** two irrational solutions

E. one rational and one irrational solution

Understanding

9. Find the value of k if $x^2 - 2x - k = 0$ has one solution.

10. Find the value of m for which $mx^2 - 6x + 5 = 0$ has one solution.

11. Find the values of n when $x^2 - 3x - n = 0$ has two solutions.

12. Show that $3x^2 + px - 2 = 0$ will have real solutions for all values of p.

13. The path of a dolphin as it leaps out of the water can be modelled by the equation $h = -0.4d^2 + d$, where h is the dolphin's height above water and d is the horizontal distance from its starting point. Both h and d are in metres.

a. How high above the water is the dolphin when it has travelled 2 m horizontally from its starting point?

b. What horizontal distance has the dolphin covered when it first reaches a height of 25 cm?

c. What horizontal distance has the dolphin covered when it next reaches a height of 25 cm? Explain your answer.

d. What horizontal distance does the dolphin cover in one leap? (*Hint:* What is the value of h when the dolphin has completed its leap?)

e. During a leap, can this dolphin reach a height of:

i. 0.5 m **ii.** 1 m?

How can you determine this without actually solving the equation?

f. Find the greatest height the dolphin reaches during a leap.

14. The parabolas $y = x^2 - 4$ and $y = 4 - x^2$ intersect in two places. Find the coordinates of their points of intersection.

Reasoning

15. **a.** For what values of a will the straight line $y = ax + 1$ have one intersection with the parabola $y = -x^2 - x - 8$?
    b. For what values of b will the straight line $y = 2x + b$ *not* intersect the parabola $y = x^2 + 3x - 5$?
16. **a.** Find how many points of intersection exist between the parabola $y = -2(x+1)^2 - 5$, where $y = f(x), x \in R$, and the straight line $y = mx - 7$, where $y = f(x), x \in R$.
    b. Find m $(m < 0)$ such that $y = mx - 7$ has one intersection point with $y = -m(x+1)^2 - 5$.

Problem solving

17. The parabola with the general equation $y = ax^2 + bx + 9$ where $0 < a < 10$ and $0 < b < 20$ touches the x-axis at one point only. The graph passes through the point $(1, 25)$. Find the values of a and b.
18. The line with equation $kx + y = 3$ is a tangent to the curve with equation $y = kx^2 + kx - 1$. Find the value of k.

Reflection

What does the discriminant tell us?

8.6 Review

8.6.1 Review questions

Fluency

1. **MC** The solutions to the equation $x^2 + 10x - 11 = 0$ are:
   A. $x = 1$ and $x = 11$ **B.** $x = 1$ and $x = -11$ **C.** $x = -1$ and $x = 11$
   D. $x = -1$ and $x = -11$ **E.** $x = 1$ and $x = 10$
2. **MC** The solutions to the equation $-5x^2 + x + 3 = 0$ are:
   A. $x = 1$ and $x = \frac{3}{5}$ **B.** $x = -0.68$ and $x = 0.88$ **C.** $x = 3$ and $x = -5$
   D. $x = 0.68$ and $x = -0.88$ **E.** $x = 1$ and $x = -\frac{3}{5}$
3. **MC** The discriminant of the equation $x^2 - 11x + 30 = 0$ are:
   A. 1 **B.** 241 **C.** 91
   D. 19 **E.** −11
4. **MC** Which of the following quadratic equations has two irrational solutions?
   A. $x^2 - 6x + 9 = 0$ **B.** $4x^2 - 11x = 0$ **C.** $x^2 - 25 = 0$
   D. $x^2 + 8x + 2 = 0$ **E.** $x^2 - 4x + 10 = 0$
5. The area of a pool is $(6x^2 + 11x + 4)\,\text{m}^2$. Find the length of the rectangular pool if its width is $(2x + 1)\,\text{m}$.
6. Solve each of the following quadratic equation by first factorising the left-hand side of the equation.
   a. $x^2 + 8x + 15 = 0$ **b.** $x^2 + 7x + 6 = 0$ **c.** $x^2 + 11x + 24 = 0$
   d. $x^2 + 4x - 12 = 0$ **e.** $x^2 - 3x - 10 = 0$ **f.** $x^2 + 3x - 28 = 0$
   g. $x^2 - 4x + 3 = 0$ **h.** $x^2 - 11x + 30 = 0$ **i.** $x^2 - 2x - 35 = 0$
7. Solve each of the following quadratic equations.
   a. $2x^2 + 16x + 24 = 0$ **b.** $3x^2 + 9x + 6 = 0$ **c.** $4x^2 + 10x - 6 = 0$
   d. $5x^2 + 25x - 70 = 0$ **e.** $2x^2 - 7x - 4 = 0$ **f.** $6x^2 - 8x - 8 = 0$
   g. $2x^2 - 6x + 4 = 0$ **h.** $6x^2 - 25x + 25 = 0$ **i.** $2x^2 + 13x - 7 = 0$
8. Solve each of the following by completing the square. Give an exact answer for each one.
   a. $x^2 + 8x - 1 = 0$ **b.** $3x^2 + 6x - 15 = 0$ **c.** $-4x^2 - 3x + 1 = 0$

9. Ten times an integer is added to seven times its square. If the result is 152, what was the original number?

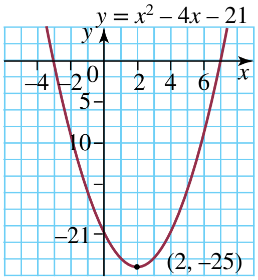
10. Solve each of the following by using the quadratic formula, rounding answers to 3 decimal places.

a. $4x^2 - 2x - 3 = 0$
b. $7x^2 + 4x - 1 = 0$
c. $-8x^2 - x + 2 = 0$

11. Solve each of the following equations, rounding answers to 3 decimal places.

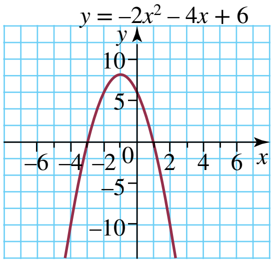
a. $18x^2 - 2x - 7 = 0$
b. $29x^2 - 105x - 24 = 0$
c. $-5x^2 + 2 = 0$

12. The graph of $y = x^2 - 4x - 21$ is shown.



Use the graph to find the solutions to the quadratic equation $x^2 - 4x - 21 = 0$.

13. Determine the roots of the quadratic graph shown.



14. Identify whether each of the equations below has no real solutions, one solution or two solutions. State whether the solutions are rational or irrational.

a. $x^2 + 11x + 9 = 0$
b. $3x^2 + 2x - 5 = 0$
c. $x^2 - 3x + 4 = 0$

15. Solve the following pairs of simultaneous equations.

a. $y = x^2 + 4x - 10$
$y = 6 - 2x$
b. $y = x^2 - 7x + 20$
$y = 3x - 5$
c. $y = x^2 + 7x + 11$
$y = x$

16. For each of the following pairs of equations:

i. solve simultaneously to find the points of intersection
ii. illustrate the solution (or lack of solution) using a sketch graph.

a. $y = x^2 + 6x + 5$ and $y = 11x - 1$
b. $y = x^2 + 5x - 6$ and $y = 8x - 8$
c. $y = x^2 + 9x + 14$ and $y = 3x + 5$
d. $y = x^2 - 7x + 10$ and $y = -11x + 6$
e. $y = -x^2 + 14x - 48$ and $y = 13x - 54$
f. $y = -x^2 + 4x + 12$ and $y = 9x + 16$

Problem solving

17. When a number is added to its square, the result is 56. Determine the number.

18. Leroy measures his bedroom and finds that its length is 3 metres more than its width. If the area of the bedroom is $18\,\text{m}^2$, calculate the length and width of the room.

19. The surface area of a cylinder is given by the formula $\text{SA} = 2\pi r(r + h)$, where r cm is the radius of the cylinder and h cm is the height.

The height of a can of soft drink is 10 cm and its surface area is 245 cm^2.

a. Substitute values into the formula to form a quadratic equation using the pronumeral r.

b. Use the quadratic formula to solve the equation and, hence, find the radius of the can. Round your answer to 1 decimal place.

c. Calculate the area of the label on the can. The label covers the entire curved surface. Round the answer to the nearest square centimetre.

20. Find the value of d when $2x^2 - 5x - d = 0$ has one solution.

21. For what values of k does $(k - 1)x^2 - (k - 1)x + 2 = 0$ have two distinct solutions?

22. Let m and n be the solutions to the quadratic equation $x^2 - 2\sqrt{5}x - 2 = 0$. Determine the value of $m^2 + n^2$.

23. Although it requires a minimum of 2 points to determine the graph of a line, it requires a minimum of 3 points to determine the shape of a parabola. The general equation of a parabola is $y = ax^2 + bx + c$, where a, b and c are the constants to be determined.

a. Determine the equation of the parabola that has a y-intercept of $(0, -2)$, and passes though the points $(1, -5)$ and $(-2, 16)$.

b. Determine the equation of a parabola that goes through the points $(0, 0)$, $(2, 2)$ and $(5, 5)$. Show full working to justify your answer.

24. When the radius of a circle increases by 6 cm, its area increases by 25%. Use the quadratic formula to find the exact radius of the original circle.

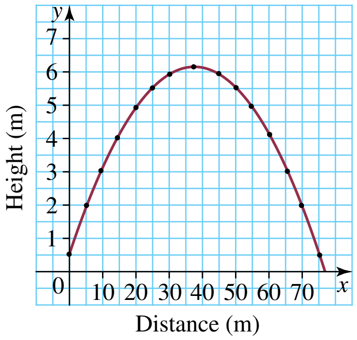
25. A football player received a hand pass and ran directly towards goal. Right on the 50-metre line he kicked the ball and scored a goal. The graph at right represents the path of the ball. Using the graph, answer the following questions.

a. At what height from the ground was the ball when it was kicked?

b. What was the greatest height the ball reached?

c. How long was the kick?

d. If there were defenders in the goal square, would it have been possible for one of them to mark the ball right on the goal line to prevent a goal? Explain your answer. (Hint: What was the height of the ball when it crossed the goal line?)



e. As the footballer kicked the ball, a defender rushed at him to try to smother the kick. If the defender can reach a height of 3 m when he jumps, how close to the player kicking the ball must he be to just touch the football as it passes over his outstretched hands?

26. The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

An alternative form of the quadratic formula is $x = \dfrac{2c}{-b \pm \sqrt{b^2 - 4ac}}$.

Choose a quadratic equation and show that the two formulas give the same answers.

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Complete this digital doc: Concept map: Topic 8 (doc-13809)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

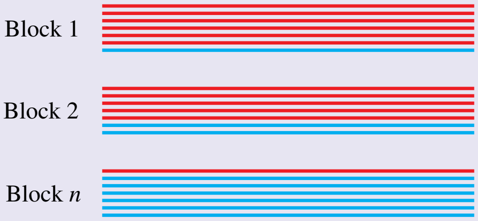
coefficient
completing the square
constant term
discriminant
factorise
intersection
irrational
linear equation
null factor law
parabola
perfect square
product
quadratic equation
quadratic formula
rational
real
solutions
substitution

Investigation | Rich task

Weaving

Many articles of clothing are sewn from materials that show designs and patterns made by weaving together threads of different colours. Intricate and complex designs can result. Let's investigate some very simple repetitive-type patterns. Knowledge of quadratic equations and the quadratic formula is helpful in creating these designs.

We need to understand the process of weaving. Weaving machines have parts called *warps*. Each warp is divided into a number of *blocks*. Consider a pattern that is made up of a series of blocks, where the first block is all one colour except for the last thread, which is a different colour. Let's say our pattern is red and blue. The first block contains all red threads, except for the last one, which is blue. The next block has all red threads, except for the last two threads, which are blue. The pattern continues in this manner. The last block has the first thread as red and the remainder as blue. The warp consists of a particular number of threads, let's say 42 threads. How many blocks and how many threads per block would be necessary to create a pattern of this type?



To produce this pattern, we need to divide the warp into equally sized blocks, if possible. What size block and how many threads per block would give us the 42-thread warp? We will need to look for a mathematical pattern. Look at the table (opposite), where we consider the smallest block consisting of 2 threads through to a block consisting of 7 threads.

Pattern	Number of threads per block	Number of blocks	Total threads in warp
RB	2	1	2
RRB RBB	3	2	6
RRRB RRBB RBBB	4		
	5		
	6		
	7		

1. Complete the entries in the table.

2. Consider a block consisting of *n* threads.
   a. How many blocks would be needed?
   b. What would be the total number of threads in the warp?

The 42-thread warp was chosen as a simple example to show the procedure involved in determining the number of blocks required and the number of threads per block. In this particular case, 6 blocks of 7 threads per block would give us our design for a 42-thread warp. In practice, you would not approach the problem by drawing up a table to determine the number of blocks and the size of each block.

3. Take your expression in question **2b** and let it equal 42. This should form a quadratic equation. Solve this equation to verify that you would need 6 blocks with 7 threads per block to fulfil the size of a 42-thread warp.

4. In reality, the size of each block is not always clearly defined. Also, the thread warp sizes are generally much larger, say, about 250. Let's determine the number of threads per block and the number of blocks required for a 250-thread warp.
   a. Form your quadratic equation with the thread warp size equal to 250.
   b. A solution to this equation can be found using the quadratic formula. Use the quadratic formula to determine a solution.
   c. The number of threads per block is represented by *n* and this obviously must be a whole number. Round your solution down to the nearest whole number.
   d. How many whole blocks are needed?
   e. Use your solutions to **c** and **d** to determine the total number of threads used for the pattern.
   f. How many more threads do you need to make the warp size equal to 250 threads?
   g. Distribute these threads by including them at the beginning of the first block and the end of the last block. Describe your overall pattern.

5. Investigate the number of blocks required and threads per block required for a 400-thread warp.

6. Investigate changing the pattern. Let the first block be all red. In the next block change the colour of the first and last threads to blue. With each progressive block, change the colour of an extra thread at the top and bottom to blue until the last block is all blue. On a separate sheet of paper, draw a table to determine the thread warp size for a block size of *n* threads. Draw the pattern and describe the result for a particular warp size.

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Answers

Topic 8 Quadratic equations

Exercise 8.2 Solving quadratic equations algebraically

1. a. $-7, 9$ **b.** $-2, 3$ **c.** $2, 3$ **d.** $0, 3$ **e.** $0, 1$ **f.** $-5, 0$
g. $0, 3$ **h.** $-2, 0$ **i.** $-\frac{1}{2}, \frac{1}{2}$ **j.** $-1.2, -0.5$ **k.** $0.1, 0.75$ **l.** $-\sqrt{2}, \sqrt{3}$

2. a. $\frac{1}{2}, 1$ **b.** $-2, -\frac{2}{3}$ **c.** $\frac{1}{4}, 7$ **d.** $-\frac{6}{7}, 1\frac{1}{2}$ **e.** $\frac{3}{5}, \frac{2}{3}$ **f.** $-\frac{5}{8}, \frac{2}{3}$
g. $0, \frac{1}{2}, 3$ **h.** $0, \frac{1}{2}, -\frac{2}{5}$ **i.** $0, -3, \frac{2}{5}$

3. a. $0, 2$ **b.** $-5, 0$ **c.** $0, 7$ **d.** $-\frac{2}{3}, 0$ **e.** $0, 1\frac{1}{2}$ **f.** $0, \frac{1}{3}$
g. $0, \frac{\sqrt{7}}{2}$ **h.** $-\frac{\sqrt{3}}{3}, 0$ **i.** $0, 1\frac{1}{4}$

4. a. $-2, 2$ **b.** $-5, 5$ **c.** $-2, 2$ **d.** $-7, 7$ **e.** $-1\frac{1}{3}, 1\frac{1}{3}$ **f.** $-2\frac{1}{2}, 2\frac{1}{2}$
g. $-\frac{2}{3}, \frac{2}{3}$ **h.** $-\frac{1}{2}, \frac{1}{2}$ **i.** $-\frac{1}{5}, \frac{1}{5}$ **j.** $-4, 4$ **k.** $-\sqrt{5}, \sqrt{5}$ **l.** $-\frac{\sqrt{11}}{3}, \frac{\sqrt{11}}{3}$

5. a. $-2, 3$ **b.** $-4, -2$ **c.** $-1, 7$ **d.** $3, 5$ **e.** 1 **f.** $-1, 4$
g. 5 **h.** $-2, 5$ **i.** $2, 6$ **j.** $-3, 7$ **k.** $-5, 6$ **l.** $3, 4$
m. 4 **n.** -5 **o.** 10

6. B

7. C

8. a. $-\frac{1}{2}, 3$ **b.** $\frac{2}{3}, -1$ **c.** $-2, \frac{1}{5}$ **d.** $\frac{1}{3}, 1\frac{1}{2}$ **e.** $-\frac{3}{14}, 1$ **f.** $\frac{1}{4}, \frac{1}{3}$
g. $-1\frac{1}{3}, 2\frac{1}{2}$ **h.** $-1\frac{3}{4}, -1\frac{1}{3}$ **i.** $-\frac{2}{5}, \frac{1}{2}$ **j.** $1\frac{1}{2}, 2\frac{2}{3}$ **k.** $-\frac{2}{5}, \frac{1}{6}$ **l.** $3, 4$

9. a. $2 + \sqrt{2}, 2 - \sqrt{2}$ **b.** $-1 + \sqrt{3}, -1 - \sqrt{3}$ **c.** $-3 + \sqrt{10}, -3 - \sqrt{10}$
d. $4 + 2\sqrt{3}, 4 - 2\sqrt{3}$ **e.** $5 + 2\sqrt{6}, 5 - 2\sqrt{6}$ **f.** $1 + \sqrt{3}, 1 - \sqrt{3}$
g. $-1 + \sqrt{6}, -1 - \sqrt{6}$ **h.** $-2 + \sqrt{10}, -2 - \sqrt{10}$ **i.** $-2 + \sqrt{15}, -2 - \sqrt{15}$

10. a. $\frac{3}{2} + \frac{\sqrt{5}}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}$ **b.** $-\frac{5}{2} + \frac{\sqrt{29}}{2}, -\frac{5}{2} - \frac{\sqrt{29}}{2}$ **c.** $\frac{7}{2} + \frac{\sqrt{33}}{2}, \frac{7}{2} - \frac{\sqrt{33}}{2}$
d. $\frac{1}{2} + \frac{\sqrt{21}}{2}, \frac{1}{2} - \frac{\sqrt{21}}{2}$ **e.** $\frac{11}{2} + \frac{\sqrt{117}}{2}, \frac{11}{2} - \frac{\sqrt{117}}{2}$ **f.** $-\frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{1}{2} - \frac{\sqrt{5}}{2}$
g. $-\frac{3}{2} + \frac{\sqrt{37}}{2}, -\frac{3}{2} - \frac{\sqrt{37}}{2}$ **h.** $\frac{5}{2} + \frac{\sqrt{37}}{2}, \frac{5}{2} - \frac{\sqrt{37}}{2}$ **i.** $\frac{9}{2} + \frac{\sqrt{65}}{2}, \frac{9}{2} - \frac{\sqrt{65}}{2}$

11. a. $-3, 1$ **b.** $-4.24, 0.24$ **c.** $-1, 3$ **d.** $-0.73, 2.73$ **e.** $0.38, 2.62$ **f.** $-0.30, 3.30$
g. $-1.19, 4.19$ **h.** $-2.30, 1.30$ **i.** $-2.22, 0.22$

12. 8 and 9 or -8 and -9

13. 6 and 8, -6 and -8

14. 9 or -10

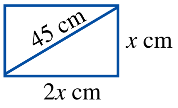
15. 2 or $-2\frac{2}{3}$

16. 8 or $-10\frac{1}{2}$

17. 6 seconds

18. a. $l = 2x$

b.



c. $x^2 + (2x)^2 = 45^2, 5x^2 = 2025$

d. Length 40 cm, width 20 cm

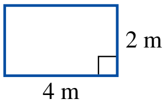
19. 8 m, 6 m

20. a. $-6, 1$ **b.** $8, -3$ **c.** $x = \pm 1$

21. a. 21 **b.** 17

22. 7

23. a.



b. $(2 + x)$ m, $(4 + x)$ m

c. $(2 + x)(4 + x) = 24$

d. $x = 2$, 4 m wide, 6 m long

24. a. $C_{Annabel}(28) = \$364\,800, C_{Betty}(28) = \$422\,400$

b. 10 knots

c. Speed can only be a positive quantity, so the negative solution is not valid.

25. No real solutions — when we complete the square we get the sum of two squares, not the difference of two squares and we cannot factorise the expression.

26. $x = 5, -4, 4, -3$

27. $z = -\frac{5}{3}, 7$

Challenge 8.1

The width of the pathway is 1.5 m.

Exercise 8.3 The quadratic formula

1. a. $a = 3, b = -4, c = 1$ **b.** $a = 7, b = -12, c = 2$ **c.** $a = 8, b = -1, c = -3$

d. $a = 1, b = -5, c = 7$ **e.** $a = 5, b = -5, c = -1$ **f.** $a = 4, b = -9, c = -3$

g. $a = 12, b = -29, c = 103$ **h.** $a = 43, b = -81, c = -24$

2. a. $\frac{-5 \pm \sqrt{21}}{2}$ **b.** $\frac{-3 \pm \sqrt{13}}{2}$ **c.** $\frac{5 \pm \sqrt{17}}{2}$ **d.** $2 \pm \sqrt{13}$ **e.** $-1 \pm 2\sqrt{3}$ **f.** $\frac{7 \pm 3\sqrt{5}}{2}$

g. $\frac{9 \pm \sqrt{73}}{2}$ **h.** $3 \pm 2\sqrt{3}$ **i.** $-4 \pm \sqrt{31}$ **j.** $\frac{1 \pm \sqrt{21}}{2}$ **k.** $\frac{5 \pm \sqrt{33}}{2}$ **l.** $-1 \pm 2\sqrt{2}$

3. a. $-0.54, 1.87$ **b.** $-1.20, 1.45$ **c.** $-4.11, 0.61$ **d.** $-0.61, 0.47$ **e.** $0.14, 1.46$ **f.** $0.16, 6.34$

g. $-1.23, 1.90$ **h.** $-1.00, 1.14$ **i.** $-0.83, 0.91$ **j.** $-0.64, 1.31$ **k.** $-0.35, 0.26$ **l.** $-1.45, 1.20$

m. $0.08, 5.92$ **n.** $-0.68, 0.88$

4. C

5. C

6. B

7. C

8. a. 0.5, 3 **b.** 0, 5 **c.** $-1, 3$ **d.** 0.382, 2.618 **e.** 0.298, 6.702

f. 2, 4 **g.** No real solution **h.** $-1, 8$ **i.** $-4.162, 2.162$ **j.** $-2, 1$

k. $-7, 1.5$ **l.** No real solution **m.** 2, 7 **n.** $-\frac{1}{2}, \frac{1}{3}$ **o.** No real solution

9. a. $2\pi r^2 + 14\pi r - 231 = 0$ **b.** 3.5 cm **c.** 154 cm^2

10. a. $x(x + 30)$ **b.** $x(x + 30) = 1500$ **c.** 265 mm

11. a. Pool A: $3\frac{2}{3}$ m by $6\frac{2}{3}$ m; Pool B: $3\frac{1}{3}$ m by $7\frac{1}{3}$ m

b. The area of each is $24\frac{4}{9}$ m^2.

12. 25 m, 60 m

13. $-2 \pm \sqrt{3},\ 9 \pm 4\sqrt{5}$

14. a. Teacher to check **b.** Teacher to check **c.** $m = 1.85$ so NP is 1.85 cm.

Challenge 8.2

1. $x = \pm 2$ or $x = \pm 3$
2. $x = \pm\frac{1}{2}$ or $x = \pm 2$

Exercise 8.4 Solving quadratic equations graphically

1. a. $x = -2, x = 3$ **b.** $x = 1, x = 10$ **c.** $x = -5, x = 5$ **d.** $x = 2$ **e.** $x = -1, x = 4$

f. $x \approx -1.4, x \approx 4.4$ **g.** $x = -25, x = 10$ **h.** $x = 0$ **i.** $x \approx -2.3, x \approx 1.3$ **j.** $x \approx -1.5, x = 1$

2. a–j. Confirm by substitution of above values into quadratic equations.

3. 150 m

4. 7 m

5. a. $y = a(x - 2)(x - 5)$ **b.** $y = \frac{2}{5}(x - 2)(x - 5)$

6. a. $y = a(x - p)(x - q)$ **b.** $y = \frac{r}{pq}(x - p)(x - q)$

7. a. 6 m **b.** 6 m

8. $y = -4x^2 + 26x - 30$

Exercise 8.5 The discriminant

1. a. $\Delta = 289$, 2 solutions **b.** $\Delta = 441$, 2 solutions **c.** $\Delta = 24$, 2 solutions **d.** $\Delta = 144$, 2 solutions

e. $\Delta = -28$, 0 solutions **f.** $\Delta = 81$, 2 solutions **g.** $\Delta = 0$, 1 solution **h.** $\Delta = 9$, 2 solutions

i. $\Delta = 32$, 2 solutions **j.** $\Delta = 2809$, 2 solutions **k.** $\Delta = 8$, 2 solutions **l.** $\Delta = 0$, 1 solution

m. $\Delta = 256$, 2 solutions **n.** $\Delta = 256$, 2 solutions

2. a. No real solutions **b.** 1 rational solution **c.** 2 rational solutions **d.** 1 rational solution

e. 2 irrational solutions **f.** 1 rational solution **g.** 2 irrational solutions **h.** No real solutions

i. No real solutions **j.** 2 irrational solutions **k.** 2 irrational solutions **l.** 2 rational solutions

3. a. Yes **b.** No **c.** Yes **d.** No

4. a. $a = 3, b = 2, c = 7$ **b.** -80 **c.** No real solutions

5. a. $a = -6, b = 1, c = 3$ **b.** 73 **c.** 2 real solutions **d.** $\dfrac{1 \pm \sqrt{73}}{12}$

6. A

7. C

8. C

9. $k = -1$

10. $m = 1.8$

11. $n > -\frac{9}{4}$

12. p^2 can only give a positive number, which, when added to 24, is always a positive solution.

13. a. 0.4 m **b.** 0.28 m **c.** 2.22 m **d.** 2.5 m

e. i. Yes **ii.** No

Find the halfway point between the beginning and the end of the leap, and substitute this value into the equation to find the maximum height.

f. 0.625 m

14. $(-2, 0)$, $(2, 0)$

15. a. $a = -7$ or 5 will give one intersection point.

b. For values of $< -\frac{21}{4}$, there will be no intersection points.

16. a. The straight line crosses the parabola at $(0, -7)$, so no matter what value m takes, there will be at least one intersection point and a maximum of two.

b. $m = -\frac{8}{5}$

17. $a = 4$, $b = 12$

18. $k = -4$

8.6 Review

1. B

2. B

3. A

4. D

5. $(3x + 4)$ m

6. a. $-5, -3$ **b.** $-6, -1$ **c.** $-8, -3$ **d.** $2, -6$ **e.** $5, -2$

f. $4, -7$ **g.** $3, 1$ **h.** $5, 6$ **i.** $7, -5$

7. a. $-2, -6$ **b.** $-2, -1$ **c.** $\frac{1}{2}, -3$ **d.** $2, -7$ **e.** $-\frac{1}{2}, 4$

f. $-\frac{2}{3}, 2$ **g.** $2, 1$ **h.** $\frac{5}{3}, \frac{5}{2}$ **i.** $-7, \frac{1}{2}$

8. a. $-4 \pm \sqrt{17}$ **b.** $-1 \pm \sqrt{6}$ **c.** $-1, \frac{1}{4}$

9. 4

10. a. $-0.651,\ 1.151$ **b.** $-0.760, 0.188$ **c.** $0.441,\ -0.566$

11. a. $-0.571,\ 0.682$ **b.** $-0.216,\ 3.836$ **c.** $-0.632,\ 0.632$

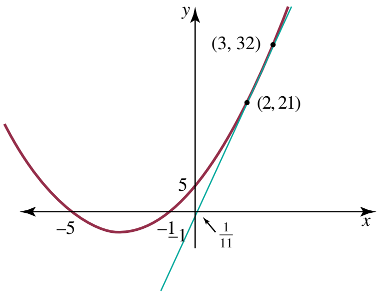
12. $-3, 7$

13. $-3, 1$

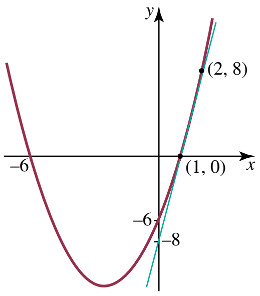
14. a. 2 irrational solutions **b.** 2 rational solutions **c.** No real solutions

15. a. $(-8, 22)$ and $(2, 2)$ **b.** $(5,\ 10)$ **c.** No solution

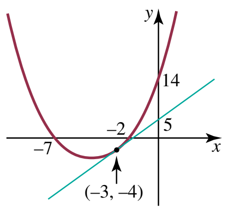
16. a.



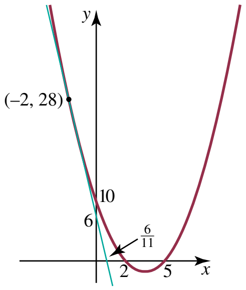
b.



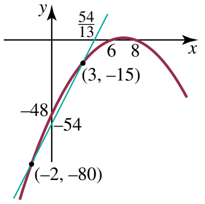
c.



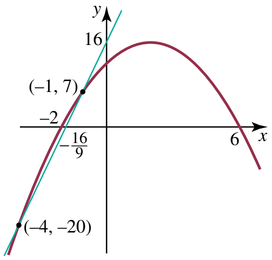
d.



e.



f.



17. −8 and 7

18. Length = 6 m, width = 3 m

19. a. $2\pi r(r + 10) = 245$ **b.** 3.0 cm **c.** 188 cm^2

20. $-\dfrac{25}{8}$

21. $k > 9$ and $k < 1$

22. 24

23. a. $y = 2x^2 - 5x - 2$

b. No parabola is possible. The points are on the same straight line.

24. $12(\sqrt{5} + 2)$ cm

25. a. 0.5 m **b.** 6.1 m **c.** 76.5 m

d. No, the ball is 5.5 m off the ground and nobody can reach it.

e. 9.5 m away

26. Check with your teacher.

Investigation — Rich task

1.

Pattern	Number of threads per block	Number of blocks	Total threads in warp
RB	2	1	2
RRB RRB	3	2	6
RRRB RRBB RBBB	4	3	12
RRRRB RRRBB RRBBB RBBBB	5	4	20
RRRRRB RRRRBB RRRBBB RRBBBB RBBBBB	6	5	30
RRRRRRB RRRRRBB RRRRBBB RRRBBBB RRBBBBB RBBBBBB	7	6	42

2. a. $n - 1$ **b.** $n^2 - n$

3. Teacher to check

4. a. $n^2 - n = 250$ **b.** $n = \dfrac{\sqrt{1001} + 1}{2}$ **c.** $n = 16$ **d.** 15

e. 240 **f.** 10 **g.** Answers will vary.

5. Answers will vary.

6. Answers will vary.

TOPIC 9
Non-linear relationships

9.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

9.1.1 Why learn this?

The world around us, both natural and artificial, is full of elegant curves and shapes that are not straight lines. Achieving the brilliance of modern and ancient architecture, or understanding the motion of planets and tennis balls requires an understanding of non-linear relationships.

9.1.2 What do you know?

assesson

1. **THINK** List what you know about non-linear relationships. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of non-linear relationships.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

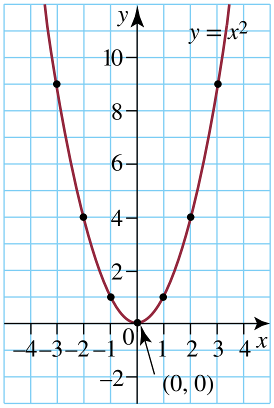
Watch this eLesson: The story of mathematics: Fibonacci (eles-1848)

9.2 Plotting parabolas

9.2.1 Plotting parabolas

- The graphs of all quadratic relationships are called **parabolas**.
- If the equation of the parabola is given, a table of values can be produced by substituting x-values into the equation to obtain the corresponding y-values. These x- and y-values provide the coordinates for points that can be plotted and joined to form the shape of the graph. When plotting graphs, use grid or graph paper for accuracy.
- The graph of $y = x^2$ shown has been produced by generating a table of values.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9



- Parabolas are symmetrical; in other words, they have an axis of symmetry. In the parabola above, the axis of symmetry is the y-axis, also called the line $x = 0$.
- A parabola has a vertex or turning point. In this case the vertex is at the origin and is called a 'minimum turning point'.
- Parabolas with the shape ∪ are said to be 'concave up' and have a minimum turning point. Parabolas with the shape ∩ are said to be 'concave down' and have a maximum turning point.

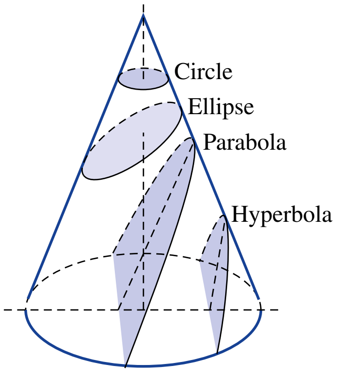
9.2.2 Parabolas in the world around us

- Parabolas abound in the world around us. Here are some examples.

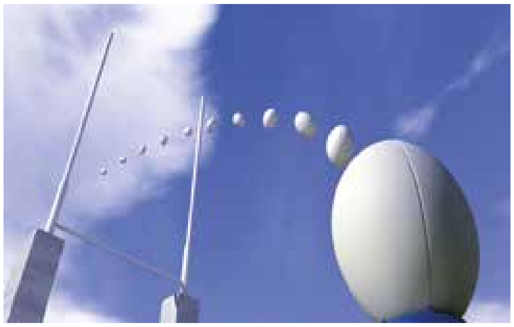

Satellite dishes


Water droplets from a hose

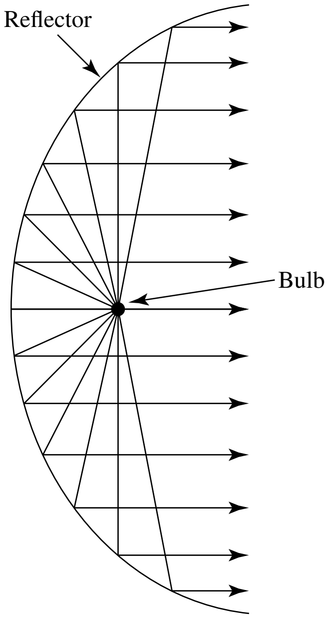

The cables from a suspension bridge



A cone when sliced parallel to its edge reveals a parabola.



The trajectory of a football when it is thrown or kicked



The reflectors in a car headlight

WORKED EXAMPLE 1

Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry and the coordinates of the turning point.

a $y = 2x^2$ **for** $-3 \le x \le 3$

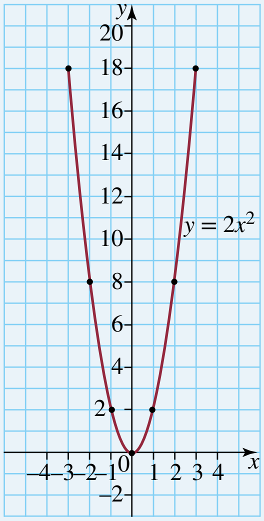
b $y = \frac{1}{2}x^2$ **for** $-3 \le x \le 3$

THINK	WRITE/DRAW
a 1 Write the equation.	**a** $y = 2x^2$
2 Produce a table of values using x-values from −3 to 3.	

x	−3	−2	−1	0	1	2	3
y	18	8	2	0	2	8	18

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. The scale would be from 20 to −2 on the y-axis and −4 to 4 on the x-axis.

4 Label the graph.



5 Write the equation of the axis of symmetry that divides the parabola exactly in half.

The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the turning point.

The turning point is (0, 0).

b 1 Write the equation.

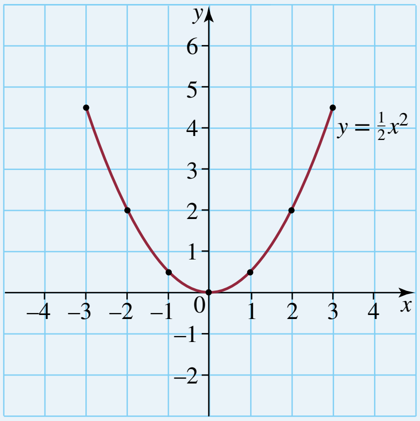
b $y = \frac{1}{2}x^2$

2 Produce a table of values using x-values from −3 to 3.

x	−3	−2	−1	0	1	2	3
y	4.5	2	0.5	0	0.5	2	4.5

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. The scale would be from 6 to −2 on the y-axis and −4 to 4 on the x-axis.

4 Label the graph.



5 Write the equation of the line that divides the parabola exactly in half.

The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the turning point.

The turning point is (0, 0).

WORKED EXAMPLE 2

TI | CASIO

Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry, the coordinates of the turning point and the y-intercept for each one.

a $y = x^2 + 2$ for $-3 \le x \le 3$ **b** $y = (x + 3)^2$ for $-6 \le x \le 0$ **c** $y = -x^2$ for $-3 \le x \le 3$

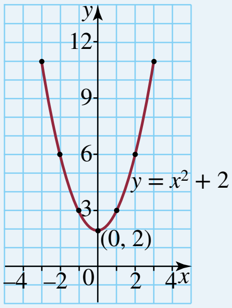
THINK	WRITE/DRAW
a 1 Write the equation.	**a** $y = x^2 + 2$

2 Produce a table of values.

x	−3	−2	−1	0	1	2	3
y	11	6	3	2	3	6	11

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. The scale on the y-axis would be from 0 to 12 and −4 to 4 on the x-axis.

4 Label the graph.



5 Write the equation of the line that divides the parabola exactly in half. — The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the turning point. — The turning point is $(0, 2)$.

7 Find the y-coordinate of the point where the graph crosses the y-axis. — The y-intercept is 2.

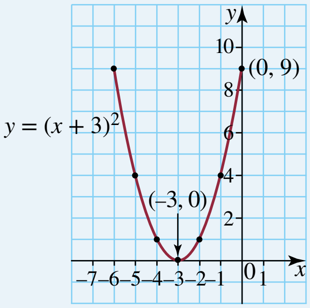
b 1 Write the equation. — **b** $y = (x + 3)^2$

2 Produce a table of values.

x	−6	−5	−4	−3	−2	−1	0
y	9	4	1	0	1	4	9

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. The scale on the y-axis would be from 0 to 10 and −7 to 1 on the x-axis.

4 Label the graph.



5 Write the equation of the line that divides the parabola exactly in half. — The equation of the axis of symmetry is $x = -3$.

6 Write the coordinates of the turning point. — The turning point is $(-3,\ 0)$.

7 Find the y-coordinate of the point where the graph crosses the y-axis.

The y-intercept is 9.

c **1** Write the equation.

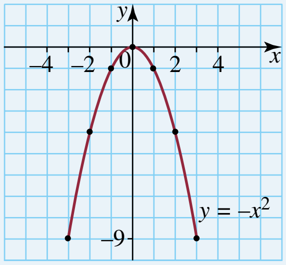
c $y = -x^2$

2 Produce a table of values.

x	−3	−2	−1	0	1	2	3
y	−9	−4	−1	0	−1	−4	−9

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. The scale on the y-axis would be from −10 to 1 and from −4 to 4 on the x-axis.

4 Label the graph.



5 Write the equation of the line that divides the parabola exactly in half.

The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the turning point.

The turning point is $(0, 0)$.

7 Find the y-coordinate of the point where the graph crosses the y-axis.

The y-intercept is 0.

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Complete this digital doc: SkillSHEET: Substitution into quadratic equations (doc-5266)

Complete this digital doc: SkillSHEET: Equation of a vertical line (doc-5267)

Exercise 9.2 Plotting parabolas

assesson

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–8, 9a–d, 10–15, 17	Questions: 1–8, 9a–d & g, 10–14, 16, 17, 18	Questions: 1–19

■■■ **Individual pathway interactivity:** int-4605

learnon ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

You may wish to use a graphing calculator for this exercise.

Fluency

1. **WE1** Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry and the coordinates of the turning point.
   a. $y = 3x^2$ for $-3 \le x \le 3$
   b. $y = \frac{1}{4}x^2$ for $-3 \le x \le 3$

2. Compare the graphs you have drawn for question **1** with that of $y = x^2$. Explain how placing a number in front of x^2 affects the graph obtained.

3. **WE2a** Plot the graph of each of the following for values of x between −3 and 3. State the equation of the axis of symmetry, the coordinates of the turning point and the y-intercept for each one.

a. $y = x^2 + 1$ **b.** $y = x^2 + 3$
c. $y = x^2 - 3$ **d.** $y = x^2 - 1$

4. Compare the graphs you have drawn for question **3** with the graph of $y = x^2$. Explain how adding to or subtracting from x^2 affects the graph obtained.

5. **WE2b** Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry, the coordinates of the turning point and the y-intercept for each one.

a. $y = (x + 1)^2 \quad -5 \le x \le 3$ **b.** $y = (x + 2)^2 \quad -6 \le x \le 2$
c. $y = (x - 2)^2 \quad -1 \le x \le 5$ **d.** $y = (x - 1)^2 \quad -2 \le x \le 4$

6. Compare the graphs you have drawn for question **5** with that for $y = x^2$. Explain how adding to or subtracting from x before squaring affects the graph obtained.

7. **WE2c** Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry, the coordinates of the turning point and the y-intercept for each one.

a. $y = -x^2 + 1 \quad -3 \le x \le 3$ **b.** $y = -(x - 1)^2 \quad -2 \le x \le 4$
c. $y = -(x + 2)^2 \quad -5 \le x \le 1$ **d.** $y = -x^2 - 3 \quad -3 \le x \le 3$

8. Compare the graphs you have drawn for question **7** with that for $y = x^2$. Explain how a negative sign in front of x^2 affects the graph obtained. Also compare the graphs obtained in question **7** with those in questions **3** and **5**. Which graphs have the same turning point? How are they different?

Understanding

9. Plot the graph of each of the following, and state:

i. the equation of the axis of symmetry
ii. the coordinates of the turning point and whether it is a maximum or a minimum
iii. the y-intercept.

a. $y = (x - 5)^2 + 1 \quad 0 \le x \le 6$ **b.** $y = 2(x + 2)^2 - 3 \quad -5 \le x \le 1$
c. $y = -(x - 3)^2 + 4 \quad 0 \le x \le 6$ **d.** $y = -3(x - 1)^2 + 2 \quad -2 \le x \le 4$
e. $y = x^2 + 4x - 5 \quad -6 \le x \le 2$ **f.** $y = -x^2 - 2x + 15 \quad -6 \le x \le 4$
g. $y = -3x^2 - 6x + 24 \quad -5 \le x \le 3$ **h.** $y = (x - 2)^2 + 1 \quad -2 \le x \le 4$

10. Use the equation $y = a(x - b)^2 + c$ to answer the following.

a. Explain how you can determine whether a parabola has a minimum or maximum turning point by looking only at its equation.

b. Explain how you can determine the coordinates of the turning point of a parabola by looking only at the equation.

c. Explain how you can obtain the equation of the axis of symmetry by looking only at the equation of the parabola.

11. **MC** For the graph of $y = (x - 2)^2 + 5$, the turning point is:

A. $(5, 2)$ **B.** $(2, -5)$ **C.** $(2, 5)$
D. $(-2, -5)$ **E.** $(-2, 5)$

12. **MC** For the graph of $y = 3(x - 1)^2 + 12$, the turning point is:

A. $(3, 12)$ **B.** $(1, 12)$ **C.** $(-1, 12)$
D. $(-3, 12)$ **E.** $(-1, -12)$

13. **MC** For the graph of $y = (x + 2)^2 - 7$, the y-intercept is:

A. −2 **B.** −7 **C.** −3
D. −11 **E.** 7

14. **MC** Which of the following is true for the graph of $y = -(x - 3)^2 + 4$?

A. Turning point $(3, 4)$, y-intercept -5

B. Turning point $(3, 4)$, y-intercept 5

C. Turning point $(-3, 4)$, y-intercept -5

D. Turning point $(-3, 4)$, y-intercept 5

E. Turning point $(3, -4)$, y-intercept 13

Reasoning

15. A ball is thrown into the air. The height, h metres, of the ball at any time, t seconds, can be found by using the equation $h = -(t - 4)^2 + 16$.

- **a.** Plot the graph for values of t between 0 and 8.
- **b.** Use the graph to find:
  - **i.** the maximum height of the ball
  - **ii.** how long it takes for the ball to fall back to the ground from the moment it is thrown.

16. From a crouching position in a ditch, an archer wants to fire an arrow over a horizontal tree branch, which is 15 metres above the ground. The height, in metres (h), of the arrow t seconds after it has been fired is given by the equation $h = -8t(t - 3)$.

- **a.** Plot the graph for $t = 0,\ 1,\ 1.5,\ 2,\ 3$.
- **b.** From the graph find:
  - **i.** the maximum height the arrow reaches
  - **ii.** whether the arrow clears the branch and the distance by which it clears or falls short of the branch
  - **iii.** the time it takes to reach maximum height
  - **iv.** how long it takes for the arrow to hit the ground after it has been fired.

17. There are 0, 1, 2 and infinite possible points of intersection for two parabolas.

- **a.** Illustrate these on separate graphs.
- **b.** Explain why infinite points of intersection are possible. Give an example.
- **c.** How many points of intersection are possible for a parabola and a straight line? Illustrate these.

Problem solving

18. The path taken by a netball thrown by a rising Australian player is given by the quadratic equation $y = -x^2 + 3.2x + 1.8$, where y is the height of the ball and x is the horizontal distance from the player's upstretched hand.

- **a.** Complete a table of values for $-1 \le x \le 4$.
- **b.** Plot the graph.
- **c.** What values of x are 'not reasonable' and why?
- **d.** What is the maximum height reached by the netball?
- **e.** Assuming that nothing hits the netball, how far away from the player will the netball strike the ground?

19. The values of a, b and c in the equation $y = ax^2 + bx + c$ can be calculated using three points that lie on the parabola. This requires solving triple simultaneous equations by algebra. This can also be done using a CAS calculator. If the points $(0, 1)$, $(1, 0)$ and $(2, 3)$ all lie on one parabola, find the equation of the parabola.

Reflection

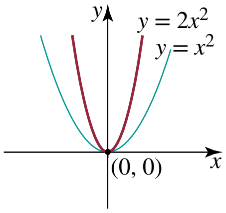
What x-values can a parabola have? What y-values can a parabola have?

9.3 Sketching parabolas

9.3.1 Sketching parabolas

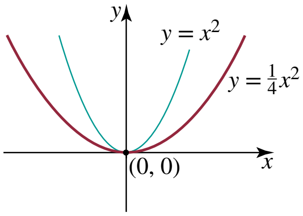
- A sketch graph of a parabola does not show a series of plotted points, but it does accurately locate important features such as x- and y-intercepts and turning points.
- The basic quadratic graph has the equation $y = x^2$. Transformations or changes in the features of the graph can be observed when the equation changes. These transformations include:
  - dilation
  - translation
  - reflection.

9.3.2 Dilation



- Compare the graph of $y = 2x^2$ with that of $y = x^2$. This graph is thinner or closer to the y-axis and has a dilation factor of 2. As the coefficient of x^2 increases, the graph becomes narrower and closer to the y-axis.
- The turning point has not changed under the transformation and is still $(0, 0)$.
- Compare the graph $y = \frac{1}{4}x^2$ with that of $y = x^2$.

  The graph is wider or closer to the x-axis and has a dilation factor of factor $\frac{1}{4}$. The turning point has not changed and is still $(0, 0)$. As the coefficient of x^2 decreases (but remains positive), the graph becomes wider or closer to the x-axis.



WORKED EXAMPLE 3

TI | CASIO

State whether each of the following graphs is wider or narrower than the graph of $y = x^2$ and state the coordinates of the turning point of each one.

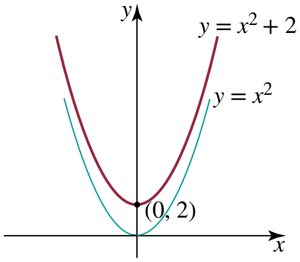
a $y = \frac{1}{5}x^2$ **b** $y = 4x^2$

THINK	**WRITE**
a 1 Write the equation.	**a** $y = \frac{1}{5}x^2$
2 Look at the coefficient of x^2 and decide whether it is greater than or less than 1.	$\frac{1}{5} < 1$, so the graph is wider than that of $y = x^2$.
3 The dilation doesn't change the turning point.	The turning point is $(0, 0)$.
b 1 Write the equation.	**b** $y = 4x^2$
2 Look at the coefficient of x^2 and decide whether it is greater than or less than 1.	$4 > 1$, so the graph is narrower than that of $y = x^2$.
3 The dilation doesn't change the turning point.	The turning point is $(0, 0)$.

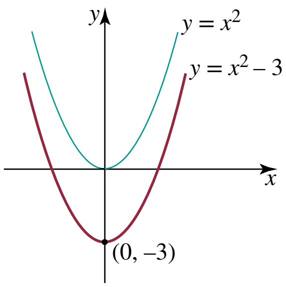
9.3.3 Vertical translation

- Compare the graph of $y = x^2 + 2$ with that of $y = x^2$.

  The whole graph has been moved or translated 2 units upwards. The turning point has become $(0, 2)$.



- Compare the graph of $y = x^2 - 3$ with that of $y = x^2$.
  The whole graph has been moved or translated 3 units downwards. The turning point has become $(0, -3)$.



WORKED EXAMPLE 4

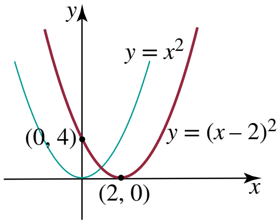
State the vertical translation and the coordinates of the turning point for the graphs of the following equations when compared to the graph of $y = x^2$.

a $y = x^2 + 5$ **b** $y = x^2 - 4$

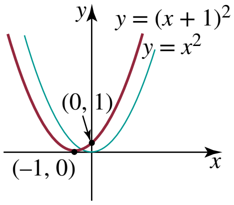
THINK	WRITE
a 1 Write the equation.	**a** $y = x^2 + 5$
2 +5 means the graph is translated upwards 5 units.	Vertical translation of 5 units up
3 Translate the turning point of $y = x^2$, which is $(0, 0)$. The x-coordinate of the turning point remains 0, and the y-coordinate has 5 added to it.	The turning point becomes $(0, 5)$.
b 1 Write the equation.	**b** $y = x^2 - 4$
2 −4 means the graph is translated downwards 4 units.	Vertical translation of 4 units down
3 Translate the turning point of $y = x^2$, which is $(0, 0)$. The x-coordinate of the turning point remains 0, and the y-coordinate has 4 subtracted from it.	The turning point becomes $(0, -4)$.

9.3.4 Horizontal translation

- Compare the graph of $y = (x - 2)^2$ with that of $y = x^2$.
  The whole graph has been moved or translated 2 units to the right. The turning point has become $(2, 0)$.



- Compare the graph of $y = (x + 1)^2$ with that of $y = x^2$.
  The whole graph has been moved or translated 1 unit left. The turning point has become $(-1, 0)$.



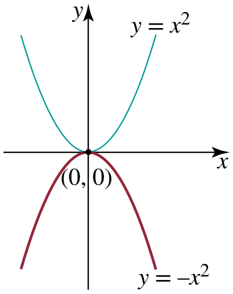
WORKED EXAMPLE 5

State the horizontal translation and the coordinates of the turning point for the graphs of the following equations when compared to the graph of $y = x^2$.

a $y = (x - 3)^2$ **b** $y = (x + 2)^2$

THINK	**WRITE**
a 1 Write the equation.	**a** $y = (x - 3)^2$
2 −3 means the graph is translated to the right 3 units.	Horizontal translation of 3 units to the right
3 Translate the turning point of $y = x^2$, which is $(0, 0)$. The y-coordinate of the turning point remains 0, and the x-coordinate has 3 added to it.	The turning point becomes $(3, 0)$.
b 1 Write the equation.	**b** $y = (x + 2)^2$
2 +2 means the graph is translated to the left 2 units.	Horizontal translation of 2 units to the left
3 Translate the turning point of $y = x^2$, which is $(0, 0)$. The y-coordinate of the turning point remains 0, and the x-coordinate has 2 subtracted from it.	The turning point becomes $(-2, 0)$.

9.3.5 Reflection



- Compare the graph of $y = -x^2$ with that of $y = x^2$.

  In each case the axis of symmetry is the line $x = 0$ and the turning point is $(0, 0)$. The only difference between the equations is the negative sign in $y = -x^2$, and the difference between the graphs is that $y = x^2$ 'sits' on the x-axis and $y = -x^2$ 'hangs' from the x-axis. (One is a reflection or mirror image of the other.) The graph of $y = x^2$ has a minimum turning point, and the graph of $y = -x^2$ has a maximum turning point.
- Any quadratic graph where x^2 is positive has a ∪ shape and is said to be upright. Conversely, if x^2 is negative the graph has a ∩ shape and is said to be inverted.

WORKED EXAMPLE 6

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For each of the following graphs, give the coordinates of the turning point and state whether it is a maximum or a minimum.

a $y = -(x - 7)^2$ **b** $y = 5 - x^2$

THINK	**WRITE**
a 1 Write the equation.	**a** $y = -(x - 7)^2$
2 It is a horizontal translation of 7 units to the right, so 7 units is added to the x-coordinate of $(0, 0)$.	The turning point is $(7, 0)$.
3 The sign in front of the x^2 term is negative, so it is inverted.	Maximum turning point

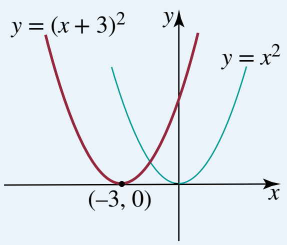
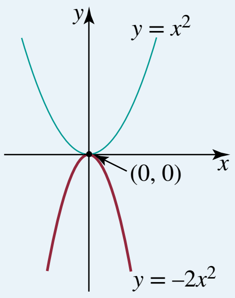
b **1** Write the equation.	**b** $y = 5 - x^2$
2 Rewrite the equation so that the x^2 term is first.	$y = -x^2 + 5$
3 The vertical translation is 5 units up, so 5 units is added to the y-coordinate of $(0, 0)$.	The turning point is $(0, 5)$.
4 The sign in front of the x^2 term is negative, so the graph is inverted.	Maximum turning point

WORKED EXAMPLE 7

For each of the following quadratic equations:

- **i** **state the appropriate dilation, reflection and translation of the graph of $y = x^2$ needed to obtain the graph**
- **ii** **state the coordinates of the turning point**
- **iii** **hence, sketch the graph.**

a $y = (x + 3)^2$ **b** $y = -2x^2$

THINK	**WRITE/DRAW**
a **1** Write the quadratic equation.	**a** $y = (x + 3)^2$
2 Identify the transformation needed — horizontal translation only, no dilation or reflection.	**i** Horizontal translation of 3 units to the left
3 State the turning point.	**ii** The turning point is $(-3, 0)$.
4 Sketch the graph of $y = (x + 3)^2$. You may find it helpful to lightly sketch the graph of $y = x^2$ on the same set of axes first.	**iii** 
b **1** Write the quadratic equation.	**b** $y = -2x^2$
2 Identify the transformations needed — dilation (2 in front of x^2) and reflection (negative in front of x^2 term), no translation.	**i** This is a reflection, so the graph is inverted. As $2 > 1$, the graph is narrower than that of $y = x^2$.
3 The turning point remains the same as there is no translation.	**ii** The turning point is $(0, 0)$.
4 Sketch the graph of $y = -2x^2$. You may find it helpful to lightly sketch the graph of $y = x^2$ on the same set of axes first.	**iii** 

learnON RESOURCES — ONLINE ONLY

Try out this interactivity: Dilation of $y = ax^2$ (int-1148)

Try out this interactivity: Vertical translation of $y = x^2 + c$ (int-1192)

Try out this interactivity: Horizontal translation of $y = (x - h)^2$ (int-1193)

Complete this digital doc: WorkSHEET: Quadratic graphs (doc-5272)

Exercise 9.3 Sketching parabolas

assessON

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1a–d, 2a–d, 3a–d, 4a–d, 5a–d, 6a–d, 7	Questions: 1c–f, 2c–f, 3c–f, 4c–f, 5c–f, 6c–f, 7, 8, 9	Questions: 1c–h, 2c–h, 3c–h, 4e–h, 5e–h, 6i–p, 7–10

🟩🟦🟪 **Individual pathway interactivity:** int-4606 **learnON** ONLINE ONLY

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Fluency

1. **WE3** State whether each of the following graphs is wider or narrower than the graph of $y = x^2$ and state the coordinates of the turning point of each one.

a. $y = 5x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = 7x^2$ **d.** $y = 10x^2$

e. $y = \frac{2}{5}x^2$ **f.** $y = 0.25x^2$ **g.** $y = 1.3x^2$ **h.** $y = \sqrt{3}x^2$

2. **WE4** State the vertical translation and the coordinates of the turning point for the graphs of each of the following equations when compared to the graph of $y = x^2$.

a. $y = x^2 + 3$ **b.** $y = x^2 - 1$ **c.** $y = x^2 - 7$ **d.** $y = x^2 + \frac{1}{4}$

e. $y = x^2 - \frac{1}{2}$ **f.** $y = x^2 - 0.14$ **g.** $y = x^2 + 2.37$ **h.** $y = x^2 + \sqrt{3}$

3. **WE5** State the horizontal translation and the coordinates of the turning point for the graphs of the following equations when compared to the graph of $y = x^2$.

a. $y = (x - 1)^2$ **b.** $y = (x - 2)^2$ **c.** $y = (x + 10)^2$ **d.** $y = (x + 4)^2$

e. $y = (x - \frac{1}{2})^2$ **f.** $y = (x + \frac{1}{5})^2$ **g.** $y = (x + 0.25)^2$ **h.** $y = (x + \sqrt{3})^2$

4. **WE6** For each of the following graphs give the coordinates of the turning point and state whether it is a maximum or a minimum.

a. $y = -x^2 + 1$ **b.** $y = x^2 - 3$ **c.** $y = -(x + 2)^2$ **d.** $y = 3x^2$

e. $y = 4 - x^2$ **f.** $y = -2x^2$ **g.** $y = (x - 5)^2$ **h.** $y = 1 + x^2$

5. In each of the following state whether the graph is wider or narrower than $y = x^2$ and whether it has a maximum or a minimum turning point.

a. $y = 3x^2$ **b.** $y = -3x^2$ **c.** $y = \frac{1}{2}x^2$ **d.** $y = -\frac{1}{5}x^2$

e. $y = -\frac{4}{3}x^2$ **f.** $y = 0.25x^2$ **g.** $y = \sqrt{3}x^2$ **h.** $y = -0.16x^2$

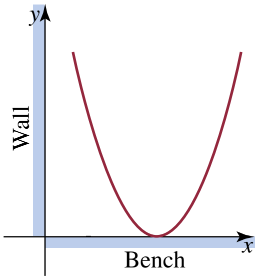
Understanding

6. **WE7** For each of the following quadratic equations:
   - i. state the appropriate dilation, reflection and translation of the graph of $y = x^2$ needed to obtain the graph
   - ii. state the coordinates of the turning point
   - iii. hence, sketch the graph.

a. $y = (x + 1)^2$
b. $y = -3x^2$
c. $y = x^2 + 1$
d. $y = \frac{1}{3}x^2$
e. $y = x^2 - 3$
f. $y = (x - 4)^2$
g. $y = -\frac{2}{5}x^2$
h. $y = 5x^2$
i. $y = -x^2 + 2$
j. $y = -(x - 6)^2$
k. $y = -x^2 - 4$
l. $y = -(x + 1)^2$
m. $y = 2(x + 1)^2 - 4$
n. $y = \frac{1}{2}(x - 3)^2 + 2$
o. $y = -\frac{1}{3}(x + 2)^2 + \frac{1}{4}$
p. $y = -\frac{7}{4}(x - 1)^2 - \frac{3}{2}$

Reasoning

7. A vase 25 cm tall is positioned on a bench near a wall as shown. The shape of the vase follows the curve $y = (x - 10)^2$, where y cm is the height of the vase and x cm is the distance of the vase from the wall.
   a. How far is the base of the vase from the wall?
   b. What is the shortest distance from the top of the vase to the wall?
   c. If the vase is moved so that the top just touches the wall, find the new distance from the wall to the base.
   d. Find the new equation that follows the shape of the vase.



8. A ball is thrown vertically upwards. Its height in metres after t seconds is given by $h = 7t - t^2$.
   a. Sketch the path of the ball.
   b. What is the highest point reached by the ball?

   A second ball is thrown vertically upwards. Its total time in flight lasts 3 seconds longer than the first ball.

   c. State the equation for the flight of the second ball.
   d. On the same set of axes used for part **a**, sketch the path of the second ball.
   e. State the difference in the highest point reached by the two balls.

Problem solving

9. Consider the quadratic equation $y = x^2 - 4x + 7$.
   a. Determine the equivalent inverted equation of the quadratic that just touches the one above at the turning point.
   b. Confirm your result graphically.

10. A parabola has the equation $y = -\frac{1}{2}(x - 3)^2 + 4$. A second parabola has an equation defined by $Y = 2(y - 1) - 3$.
    a. Find the equation relating Y to x.
    b. State the appropriate dilation, reflection and translation of the graph of $Y = x^2$ required to obtain the graph of $Y = 2(y - 1) - 3$.
    c. State the coordinates of the turning point.
    d. Sketch the graph of $Y = 2(y - 1) - 3$.

Reflection

What are the turning points of the graphs $y = x^2 + k$ and $y = (x - h)^2$?

CHALLENGE 9.1

A ball blasted upwards follows a parabolic path. It reaches a maximum height of 200 m when its horizontal distance from its starting point is 10 m. When the ball's horizontal distance from the starting point was 1 m, the ball had reached a height of 38 m. Suggest an equation to model the ball's flight, clearly defining your chosen pronumerals.

9.4 Sketching parabolas in turning point form

9.4.1 Turning point form

- When a quadratic equation is expressed in the form $y = a(x - h)^2 + k$:
  - the turning point is the point (h, k)
  - the axis of symmetry is $x = h$
  - the x-intercepts are calculated by solving $a(x - h)^2 + k = 0$.
- Changing the values of a, h and k in the equation transforms the shape and position of the parabola when compared with the parabola $y = x^2$.

$$y = \underset{\substack{\nearrow\\ \text{Reflects}\\ \text{and}\\ \text{dilates}}}{a}(x - \underset{\substack{\uparrow\\ \text{Translates}\\ \text{left and}\\ \text{right}}}{h})^2 + \underset{\substack{\nwarrow\\ \text{Translates}\\ \text{up and}\\ \text{down}}}{k}$$

WORKED EXAMPLE 8

For each of the following equations, state the coordinates of the turning point of the graph and whether it is a maximum or a minimum.

a $y = (x - 6)^2 - 4$ **b** $y = -(x + 3)^2 + 2$

	THINK	WRITE
a	**1** Write the equation.	**a** $y = (x - 6)^2 - 4$
	2 Identify the transformations — horizontal translation of 6 units to the right and a vertical translation of 4 units down. State the turning point.	The turning point is $(6, -4)$.
	3 As a is positive $(a = 1)$, the graph is upright with a minimum turning point.	Minimum turning point
b	**1** Write the equation	**b** $y = -(x + 3)^2 + 2$
	2 Identify the transformations — horizontal translation of 3 units to the left and a vertical translation of 2 units up. State the turning point.	The turning point is $(-3, 2)$.
	3 As a is negative $(a = -1)$, the graph is inverted with a maximum turning point.	Maximum turning point

9.4.2 *x*- and *y*-intercepts of quadratic graphs

- Other key features such as the x- and y-intercepts can also be determined from the equation of a parabola.
- The point(s) where the graph cuts or touches the x-axis are called the x-intercept(s). At these points, $y = 0$.
- The point where the graph cuts the y-axis is called the y-intercept. At this point, $x = 0$.

WORKED EXAMPLE 9

TI | CASIO

Determine i the *y*-intercept and ii the *x*-intercepts (where they exist) for the parabolas with equations:

a $y = (x + 3)^2 - 4$

b $y = 2(x - 1)^2$

c $y = -(x + 2)^2 - 1.$

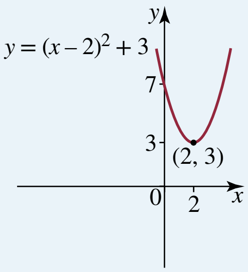
THINK	WRITE
a **1** Write the equation.	**a** $y = (x + 3)^2 - 4$
2 Find the *y*-intercept by substituting $x = 0$ into the equation.	**i** *y*-intercept: when $x = 0$, $y = (0 + 3)^2 - 4$ $= 9 - 4$ $= 5$ The *y*-intercept is 5.
3 Find the *x*-intercepts by substituting $y = 0$ into the equation and solving for *x*. Add 4 to both sides of the equation. Take the square root of both sides of the equation. Subtract 3 from both sides of the equation. Solve for *x*.	**ii** *x*-intercepts: when $y = 0$, $(x + 3)^2 - 4 = 0$ $(x + 3)^2 = 4$ $(x + 3) = +2 \text{ or } -2$ $x = 2 - 3 \text{ or } x = -2 - 3$ $x = -1 \qquad x = -5$ The *x*-intercepts are −5 and −1.
b **1** Write the equation.	**b** $y = 2(x - 1)^2$
2 Find the *y*-intercept by substituting $x = 0$ into the equation.	**i** *y*-intercept: when $x = 0$, $y = 2(0 - 1)^2$ $= 2 \times 1$ $= 2$ The *y*-intercept is 2.
3 Find the *x*-intercepts by substituting $y = 0$ into the equation and solving for *x*. Note that there is only one solution for *x* and so there is only one *x*-intercept. (The graph touches the *x*-axis.)	**ii** *x*-intercepts: when, $y = 0$, $2(x - 1)^2 = 0$ $(x - 1)^2 = 0$ $x - 1 = 0$ $x = 0 + 1$ $x = 1$ The *x*-intercept is 1.
c **1** Write the equation.	**c** $y = -(x + 2)^2 - 1$
2 Find the *y*-intercept by substituting $x = 0$ into the equation.	**i** *y*-intercept: when $x = 0$, $y = -(0 + 2)^2 - 1$ $= -4 - 1$ $= -5$ The *y*-intercept is −5.
3 Find the *x*-intercepts by substituting $y = 0$ into the equation and solving for *x*. We cannot take the square root of −1 to obtain real solutions; therefore, there are no *x*-intercepts.	**ii** *x*-intercepts: when $y = 0$, $-(x + 2)^2 - 1 = 0$ $(x + 2)^2 = -1$ There are no real solutions, so there are no *x*-intercepts.

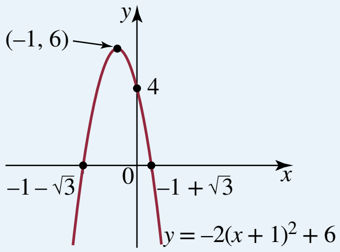
WORKED EXAMPLE 10

For each of the following:

- **i write the coordinates of the turning point**
- **ii state whether the graph has a maximum or a minimum turning point**
- **iii state whether the graph is wider, narrower or the same width as the graph of $y = x^2$**
- **iv find the y-intercept**
- **v find the x-intercepts**
- **vi sketch the graph.**

a $y = (x - 2)^2 + 3$ **b** $y = -2(x + 1)^2 + 6$

THINK	WRITE/DRAW
a 1 Write the equation.	**a** $y = (x - 2)^2 + 3$
2 State the coordinates of the turning point from the equation. Use (h, k) as the equation is in the turning point form of $y = a(x - h)^2 + k$ where $a = 1,\ h = 2$ and $k = 3$.	**i** The turning point is $(2, 3)$.
3 State the nature of the turning point by considering the sign of a.	**ii** The graph has a minimum turning point as the sign of a is positive.
4 Specify the width of the graph by considering the magnitude of a.	**iii** The graph has the same width as $y = x^2$ since $a = 1$.
5 Find the y-intercept by substituting $x = 0$ into the equation.	**iv** y-intercept: when $x = 0$, $y = (0 - 2)^2 + 3$ $= 4 + 3$ $= 7$ y-intercept is 7.
6 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x. As we have to take the square root of a negative number, we cannot solve for x.	**v** x-intercepts: when $y = 0$, $(x - 2)^2 + 3 = 0$ $(x - 2)^2 = -3$ There are no real solutions, and hence no x-intercepts.
7 Sketch the graph, clearly showing the turning point and the y-intercept.	**vi** 
8 Label the graph.	
b 1 Write the equation.	**b** $y = -2(x + 1)^2 + 6$
2 State the coordinates of the turning point from the equation. Use (h, k) as the equation is in the turning point form of $y = a(x - h)^2 + k$ where $a = -2,\ h = -1$ and $k = 6$.	**i** The turning point is $(-1, 6)$.
3 State the nature of the turning point by considering the sign of a.	**ii** The graph has a maximum turning point as the sign of a is negative.

4 Specify the width of the graph by considering the magnitude of a.	**iii** The graph is narrower than $y = x^2$ since $\lvert a \rvert > 1$.
5 Find the y-intercept by substituting $x = 0$ into the equation.	**iv** y-intercept: when $x = 0$, $y = -2(0 + 1)^2 + 6$ $= -2 \times 1 + 6$ $= 4$ The y-intercept is 4.
6 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x.	**v** x-intercepts: when $y = 0$, $-2(x + 1)^2 + 6 = 0$ $2(x + 1)^2 = 6$ $(x + 1)^2 = 3$ $x + 1 = \sqrt{3}$ or $x + 1 = -\sqrt{3}$ $x = -1 + \sqrt{3}$ $x = -1 - \sqrt{3}$ The x-intercepts are $-1 - \sqrt{3}$ and $-1 + \sqrt{3}$ (or approximately -2.73 and 0.73).
7 Sketch the graph, clearly showing the turning point and the x- and y-intercepts.	**vi** 
8 Label the graph.	

Note: Unless otherwise stated, exact values for the intercepts should be shown on sketch graphs.

Exercise 9.4 Sketching parabolas in turning point form

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1a–d, 2a–c, 3a–c, 4, 5, 6a–d, 7, 10, 12	Questions: 1c–f, 2c–e, 3c–e, 4, 5, 6c–f, 7, 11, 12, 14	Questions: 1c–i, 2c–f, 3c–f, 4, 5, 6e–i, 7–15

■■■ **Individual pathway interactivity:** int-4607

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Fluency

1. **WE8** For each of the following equations, state the coordinates of the turning point of the graph and whether it is a maximum or a minimum.

a. $y = (x - 1)^2 + 2$
b. $y = (x + 2)^2 - 1$
c. $y = (x + 1)^2 + 1$
d. $y = -(x - 2)^2 + 3$
e. $y = -(x - 5)^2 + 3$
f. $y = (x + 2)^2 - 6$
g. $y = \left(x - \frac{1}{2}\right)^2 - \frac{3}{4}$
h. $y = \left(x - \frac{1}{3}\right)^2 + \frac{2}{3}$
i. $y = (x + 0.3)^2 - 0.4$

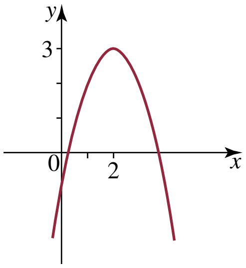
2. For each of the following, state:

i. the coordinates of the turning point
ii. whether the graph has a maximum or a minimum turning point
iii. whether the graph is wider, narrower or the same width as that of $y = x^2$.

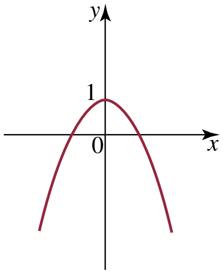
a. $y = 2(x + 3)^2 - 5$ **b.** $y = -(x - 1)^2 + 1$ **c.** $y = -5(x + 2)^2 - 4$
d. $y = \frac{1}{4}(x - 3)^2 + 2$ **e.** $y = -\frac{1}{2}(x + 1)^2 + 7$ **f.** $y = 0.2\left(x - \frac{1}{5}\right)^2 - \frac{1}{2}$

3. Select the equation that best suits each of the following graphs.

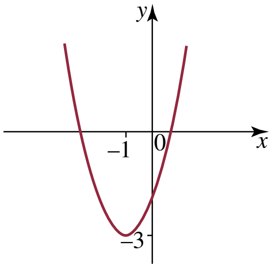
i.



ii.



iii.



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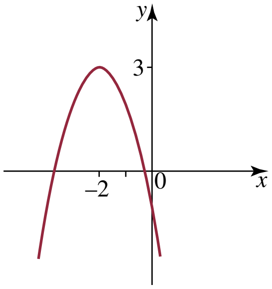

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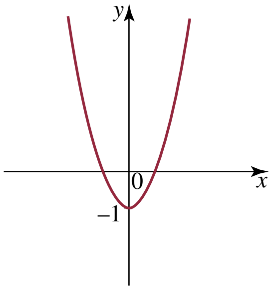

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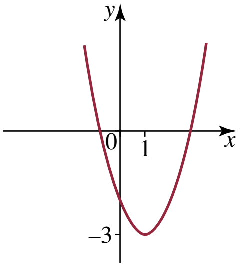


a. $y = (x - 1)^2 - 3$ **b.** $y = -(x - 2)^2 + 3$ **c.** $y = x^2 - 1$
d. $y = -(x + 2)^2 + 3$ **e.** $y = -x^2 + 1$ **f.** $y = (x + 1)^2 - 3$

4. MC **a.** The translations required to change $y = x^2$ into $y = \left(x - \frac{1}{2}\right)^2 + \frac{1}{3}$ are:

A. right $\frac{1}{2}$, up $\frac{1}{3}$ **B.** left $\frac{1}{2}$, down $\frac{1}{3}$
C. right $\frac{1}{2}$, down $\frac{1}{3}$ **D.** left $\frac{1}{2}$, up $\frac{1}{3}$
E. right $\frac{1}{3}$, up $\frac{1}{2}$

b. For the graph $\frac{1}{4}\left(x - \frac{1}{2}\right)^2 + \frac{1}{3}$, the effect of the $\frac{1}{4}$ on the graph is:

A. no effect **B.** to make the graph narrower
C. to make the graph wider **D.** to invert the graph
E. to translate the graph up $\frac{1}{4}$ of a unit

c. Compared to the graph of $y = x^2$, $y = -2(x + 1)^2 - 4$ is:

A. inverted and wider **B.** inverted and narrower
C. upright and wider **D.** upright and narrower
E. inverted and the same width

d. A graph that has a minimum turning point $(1, 5)$ and that is narrower than the graph of $y = x^2$ is:

A. $y = (x - 1)^2 + 5$ **B.** $y = \frac{1}{2}(x + 1)^2 + 5$
C. $y = 2(x - 1)^2 + 5$ **D.** $y = 2(x + 1)^2 + 5$
E. $y = \frac{1}{2}(x - 1)^2 + 5$

e. Compared to the graph of $y = x^2$, the graph of $y = -3(x - 1)^2 - 2$ has the following features.

A. Maximum TP at $(-1, -2)$, narrower **B.** Maximum TP at $(1, -2)$, narrower
C. Maximum TP at $(1, 2)$, wider **D.** Minimum TP at $(1, -2)$, narrower
E. Minimum TP at $(-1, -2)$, wider

5. WE9 Determine **i** the y-intercept and **ii** the x-intercepts (where they exist) for the parabolas with the following equations.

a. $y = (x + 1)^2 - 4$	**b.** $y = 3(x - 2)^2$	**c.** $y = -(x + 4)^2 - 2$
d. $y = (x - 2)^2 - 9$	**e.** $y = 2x^2 + 4$	**f.** $y = (x + 3)^2 - 5$

Understanding

6. WE10 For each of the following:
   - **i.** write the coordinates of the turning point
   - **ii.** state whether the graph has a maximum or a minimum turning point
   - **iii.** state whether the graph is wider, narrower or the same width as the graph of $y = x^2$
   - **iv.** find the y-intercept
   - **v.** find the x-intercepts
   - **vi.** sketch the graph.

a. $y = (x - 4)^2 + 2$	**b.** $y = (x - 3)^2 - 4$	**c.** $y = (x + 1)^2 + 2$
d. $y = (x + 5)^2 - 3$	**e.** $y = -(x - 1)^2 + 2$	**f.** $y = -(x + 2)^2 - 3$
g. $y = -(x + 3)^2 - 2$	**h.** $y = 2(x - 1)^2 + 3$	**i.** $y = -3(x + 2)^2 + 1$

7. Consider the equation $2x^2 - 3x - 8 = 0$.
   - **a.** Complete the square.
   - **b.** Use the result to determine the exact solutions to the original equation.
   - **c.** Determine the turning point of $y = 2x^2 - 3x - 8$ and indicate its type.

8. **a.** Find the equation of a quadratic that has a turning point of $(-4, 6)$ and has an x-intercept at $(-1, 0)$.
   - **b.** State the other x-intercept (if any).

9. Write the new equation for the parabola $y = x^2$ that has been:
   - **a.** reflected in the x-axis
   - **b.** dilated by a factor of 7 away from the x-axis
   - **c.** translated 3 units in the negative direction of the x-axis
   - **d.** translated 6 units in the positive direction of the y-axis
   - **e.** dilated by a factor of $\frac{1}{4}$ from the x-axis, reflected in the x-axis, and translated 5 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Reasoning

10. The price of shares in fledgling company ‘Lollies’r’us’ plunged dramatically one afternoon, following the breakout of a small fire on the premises. However, Ms Sarah Sayva of Lollies Anonymous agreed to back the company, and share prices began to rise.

    Sarah noted at the close of trade that afternoon that the company’s share price followed the curve: $P = 0.1(t - 3)^2 + 1$ where $\$P$ is the price of shares t hours after noon.

a. Sketch a graph of the relationship between time and share price to represent the situation.
b. What was the initial share price?
c. What was the lowest price of shares that afternoon?
d. At what time was the price at its lowest?
e. What was the final price of 'Lollies'r'us' shares as trade closed at 5 pm?

11. Rocky is practising for a football kicking competition. After being kicked, the path that the ball follows can be modelled by the quadratic relationship:

$$h = -\tfrac{1}{30}(d - 15)^2 + 8$$

where h is the vertical distance the ball reaches (in metres), and d is the horizontal distance (in metres).

a. Determine the initial vertical height of the ball.
b. Determine the exact maximum horizontal distance the ball travels.
c. Write down both the maximum height and the horizontal distance when the maximum height is reached.

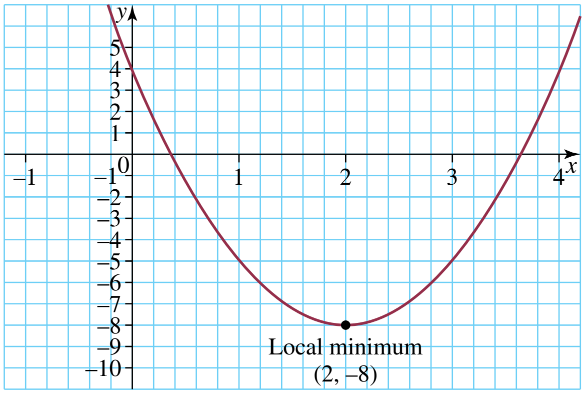
12. a. If the turning point of a particular parabola is $(2, 6)$, suggest a possible equation for the parabola.
b. If the y-intercept in part **a** is $(0, 4)$, give the exact equation for the parabola.

13. a. If the turning point of a particular parabola is (p, q), suggest a possible equation for the parabola.
b. If the y-intercept in part **a** is $(0,\ r)$, give the exact equation for the parabola.

Problem solving

14. Use the completing the square method to write each of the following in turning point form and sketch the parabola for each part.
a. $y = x^2 - 8x + 1$
b. $y = x^2 + 4x - 5$
c. $y = x^2 + 3x + 2$

15. a. Find the equation of the parabola shown.



b. State the dilation and translation transformations that have been applied to $y = x^2$ to achieve this parabola.
c. This graph is reflected in the x-axis. State the equation of the reflected graph.
d. Sketch the graph of the reflected parabola.

Reflection

Does a in the equation $y = a(x - h)^2 + k$ have any impact on the turning point?

9.5 Sketching parabolas of the form $y = ax^2 + bx + c$

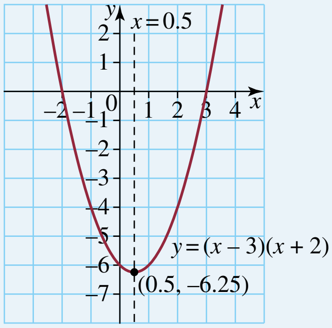
9.5 Parabolas of the form $y = ax^2 + bx$

- The general form of a quadratic equation is $y = ax^2 + bx + c$ where a, b and c are constants.
- A sketch of a parabola usually shows x- and y-intercepts and the turning point.
- The x-coordinate of the turning point lies midway between the x-intercepts.
- The x-coordinate of the turning point can also be found using the formula $x = \dfrac{-b}{2a}$ from the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For example, for the equation $y = x^2 + 4x - 6$, the x-coordinate of the turning point will be at $x = \dfrac{-4}{2 \times 1} = -2$.

  The y-coordinate of the turning point can be found by substitution. Continuing the example, given $x = -2$, $y = (-2)^2 + 4(-2) - 6 = -10$. The turning point is $(-2, -10)$.
- If the equation is not written in turning point form, the coordinates of the turning point may be found by:
  - finding the midpoint between the x-intercepts
  - using the formula $x = \dfrac{-b}{2a}$
  - writing the equation in turning point form by completing the square.

WORKED EXAMPLE 11

Sketch the graph of $y = (x - 3)(x + 2)$.

THINK	WRITE/DRAW
1 The equation is in factorised form. To find the x-intercepts, let $y = 0$ and use the Null Factor Law.	$y = (x - 3)(x + 2)$ $0 = (x - 3)(x + 2)$ $x - 3 = 0$ or $x + 2 = 0$ (NFL) $x = 3$ or $x = -2$ x-intercepts: $(3, 0)(-2, 0)$
2 The x-coordinate of the turning point is midway between the x-intercepts. Find the average of the two x-intercepts to find the midpoint between them.	$x_{TP} = \dfrac{3 + (-2)}{2}$ $= 0.5$
3 • To find the y-coordinate of the turning point, substitute x_{TP} into the equation.	$y = (x - 3)(x + 2)$ $y_{TP} = (0.5 - 3)(0.5 + 2)$ $= -6.25$
• State the turning point.	Turning point: $(0.5, -6.25)$
4 • To find the y-intercept, let $x = 0$ and substitute.	$y = (0 - 3)(0 + 2)$ $= -6$
• State the y-intercept.	y-intercept: $(0, -6)$
5 • Sketch the graph, showing all the important features. • Label the graph.	

WORKED EXAMPLE 12

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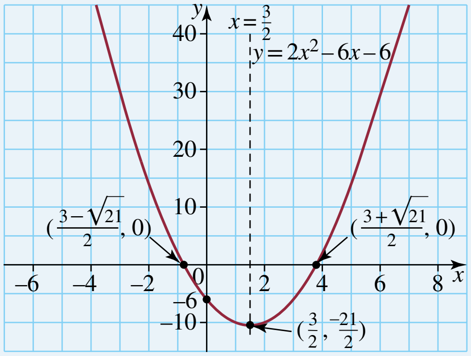
Sketch the graph of $y = 2x^2 - 6x - 6$.

THINK	WRITE/DRAW
1 The equation is not in factorised form, but there is a common factor of 2. Take out the common factor of 2.	$y = 2x^2 - 6x - 6$ $= 2(x^2 - 3x - 3)$
2 The equation cannot be factorised (no factors of −3 add to −3), so use completing the square to write the equation in turning point form. • Halve and then square the coefficient of *x*. • Add this and then subtract it from the equation. • Collect the terms for and write the perfect square. • Simplify the brackets to write the equation in turning point form. • Identify the coordinates of the turning point (h, k).	$y = 2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 3\right)$ $= 2\left(\left[x - \frac{3}{2}\right]^2 - \left(\frac{3}{2}\right)^2 - 3\right)$ $= 2\left(\left[x - \frac{3}{2}\right]^2 - \frac{9}{4} - 3\right)$ $= 2\left(\left[x - \frac{3}{2}\right]^2 - \frac{21}{4}\right)$ $= 2\left(x - \frac{3}{2}\right)^2 - 2 \times \frac{21}{4}$ $= 2\left(x - \frac{3}{2}\right)^2 - \frac{21}{2}$ Turning point: $\left(\frac{3}{2}, \frac{-21}{2}\right)$
3 • To find the *x*-intercepts, let $y = 0$. No factors of −3 add to −3, so use the quadratic formula to find the *x*-intercepts.	*x*-intercepts: let $y = 0$. $0 = 2x^2 - 6x - 6$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a = 2, b = -6, c = -6$ $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(-6)}}{2(2)}$ $= \frac{6 \pm \sqrt{36 + 48}}{4}$ $= \frac{6 \pm \sqrt{84}}{4} = \frac{6 \pm 2\sqrt{21}}{4}$
• State the *x*-intercepts.	The *x*-intercepts are: $x = \frac{3 + \sqrt{21}}{2}$ and $x = \frac{3 - \sqrt{21}}{2}$ $x \approx 3.79$ $\quad x \approx -0.79$
4 • To find the *y*-intercepts, let $x = 0$ and substitute. • State the *y*-intercept.	$y = 2x^2 - 6x - 6$ $y = 2(0)^2 - 6(0) - 6$ $= -6$ *y*-intercept: $(0, -6)$
5 • Sketch the graph, showing all the important features. • Label the graph and show the exact values of the *x*-intercepts.	

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Exercise 9.5 Sketching parabolas of the form $y = ax^2 + bx + c$

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Questions: 1, 2a–c, 3a–c, 4, 5, 7	Questions: 1, 2b–d, 3c–d, 4, 6–8, 10	Questions: 1, 2c–f, 3c–g, 5, 7–9, 11

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Fluency

1. What information is necessary to be able to sketch a parabola?

2. WE11 Sketch the graph of each of the following.

a. $y = (x - 5)(x - 2)$
b. $y = (x + 4)(x - 7)$
c. $y = (x + 3)(x + 5)$
d. $y = (2x + 3)(x + 5)$
e. $y = (4 - x)(x + 2)$
f. $y = \left(\frac{x}{2} + 3\right)(5 - x)$

3. WE12 Sketch the graph of each of the following.

a. $y = x^2 + 4x + 2$
b. $y = x^2 - 4x - 5$
c. $y = 2x^2 - 4x - 3$
d. $y = -2x^2 + 11x + 5$
e. $y = -2x^2 + 12x$
f. $y = 3x^2 + 6x + 1$
g. $y = -3x^2 - 5x + 2$

Understanding

4. The path of a soccer ball kicked by the goal keeper can be modelled by the equation $y = -\frac{1}{144}(x^2 - 24x)$ where y is the height of the soccer ball and x is the horizontal distance from the goalie, both in metres.

a. Sketch the graph.
b. How far away from the player does the ball first bounce?
c. What is the maximum height of the ball?

5. The monthly profit or loss, p, (in thousands of dollars) for a new brand of chicken loaf is given by $p = 3x^2 - 15x - 18$ where x is the number of months after its introduction (when $x = 0$).

a. Sketch the graph.
b. During which month was a profit first made?
c. In which month is the profit \$54 000?

6. The height, h metres, of a model rocket above the ground t seconds after launch is given by the equation $h = 4t(50 - t)$ for $0 < t < 50$.
   - **a.** Sketch the graph of the rocket's flight.
   - **b.** Find the height of the rocket above the ground when it is launched.
   - **c.** What is the greatest height reached by the rocket?
   - **d.** How long does the rocket take to reach its greatest height?
   - **e.** For how long is the rocket in the air?

Reasoning

7. The equation $y = x^2 + bx + 7500$ has x-intercepts of $(-150, 0)$ and $(-50, 0)$. What is the value of b in the equation? Justify your answer.
8. The equation $y = x^2 + bx + c$ has x-intercepts of m and n. What is the value of b in the equation? Justify your answer.
9. **a.** What path does a spaceship take to get to the Moon?
   - **b.** What is a transfer orbit?
   - **c.** Is any part of the flight path parabolic?

Problem solving

10. A ball is thrown upwards from a building and follows the path given by the formula $h = -x^2 + 4x + 21$. The ball is h metres above the ground when it is a horizontal distance of x metres from the building.
    - **a.** Sketch the graph of the path of the ball.
    - **b.** What is the maximum height the ball reaches?
    - **c.** How far is the ball from the wall when it reaches the maximum height?
    - **d.** How far from the building does the ball land?
11. During an 8-hour period, an experiment is done in which the temperature of a room follows the relationship $T = h^2 - 8h + 21$, where T is the temperature in degrees Celsius h hours after starting the experiment.
    - **a.** Sketch the graph of this quadratic.
    - **b.** What is the initial temperature?
    - **c.** After 3 hours, is the temperature increasing or decreasing?
    - **d.** After 5 hours, is the temperature increasing or decreasing?
    - **e.** State the minimum temperature and when it occurred.
    - **f.** What is the temperature after 8 hours?

Reflection

What strategy can you use to remember all of the information necessary to sketch a parabola?

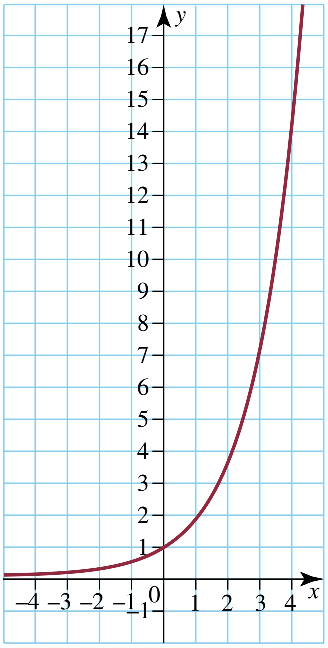
9.6 Exponential functions and graphs

9.6.1 Exponential functions

- Relationships of the form $y = a^x$ are called **exponential functions** with base a, where a is a real number not equal to 1, and x is the index power or exponent.
- The term 'exponential' is used, as x is an exponent (or index).
- For example, the graph of the exponential function $y = 2^x$ can be plotted by completing a table of values.

  Remember that $2^{-3} = \frac{1}{2^3}$
  $= \frac{1}{8}$, and so on.

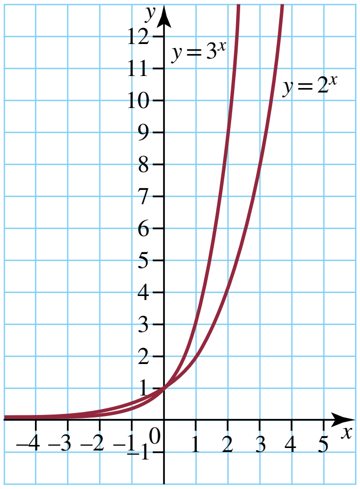
x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



- The graph has many significant features.
  - The y-intercept is 1.
  - The value of y is always greater than zero.
  - As x decreases, y gets closer to but never reaches zero. So the graph gets closer to but never reaches the x-axis. The x-axis (or the line $y = 0$) is called an **asymptote**.
  - As x increases, y becomes very large.

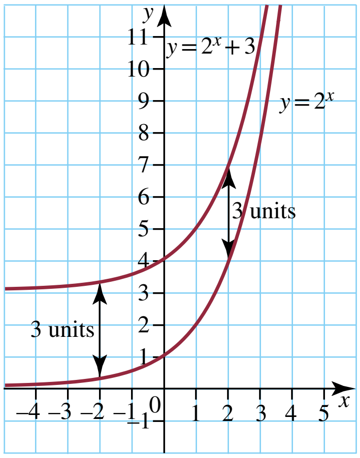
9.6.2 Comparing exponential graphs

- The diagram at right shows the graphs of $y = 2^x$ and $y = 3^x$.
- The graphs both pass through the point (0, 1).
- The graph of $y = 3^x$ climbs more steeply than the graph of $y = 2^x$.
- $y = 0$ is an asymptote for both graphs.



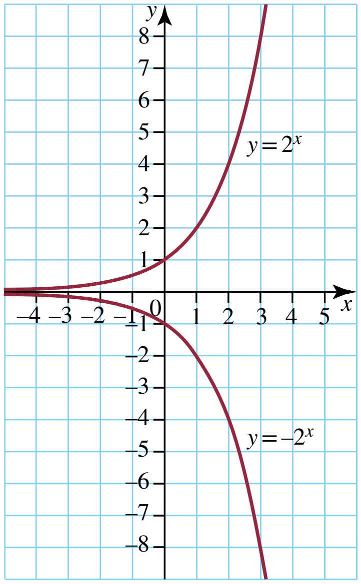
9.6.3 Vertical translation

- The diagram at right shows the graphs of $y = 2^x$ and $y = 2^x + 3$.
- The graphs have identical shape.
- Although they appear to get closer to each other, the graphs are constantly 3 units apart.
- As x becomes very small, the graph of $y = 2^x + 3$ approaches but never reaches the line $y = 3$, so $y = 3$ is the horizontal asymptote.
- When the graph of $y = 2^x$ is translated 3 units upward, it becomes the graph of $y = 2^x + 3$.



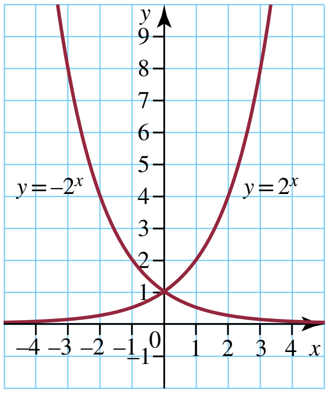
9.6.4 Reflection about the x-axis

- The diagram at right shows the graphs of $y = 2^x$ and $y = -2^x$.
- The graphs have identical shape.
- The graph of $y = -2^x$ is a reflection about the x-axis of the graph of $y = 2^x$.
- The x-axis ($y = 0$) is an asymptote for both graphs.
- In general, the graph of $y = -a^x$ is a reflection about the x-axis of the graph of $y = a^x$.



9.6.5 Reflection about the *y*-axis

- The diagram at right shows the graphs of $y = 2^x$ and $y = 2^{-x}$.
- The graphs have identical shape.
- The graph of $y = 2^{-x}$ is a reflection about the y-axis of the graph of $y = 2^x$.
- Both graphs pass through the point (0, 1).
- The x-axis ($y = 0$) is an asymptote for both graphs.
- In general, the graph of $y = a^{-x}$ is a reflection about the y-axis of the graph of $y = a^x$.



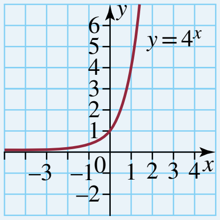
WORKED EXAMPLE 13

Given the graph of $y = 4^x$, sketch on the same axes the graphs of:

a $y = 4^x - 2$

b $y = -4^x$

c $y = 4^{-x}$.

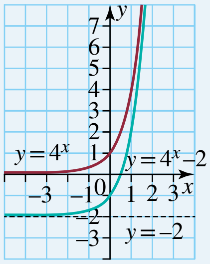


THINK

DRAW

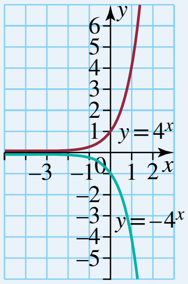
a The graph of $y = 4^x$ has already been drawn. It has a y-intercept of 1 and a horizontal asymptote at $y = 0$. The graph of $y = 4^x - 2$ has the same shape as $y = 4^x$ but is translated 2 units vertically down. It has a y-intercept of −1 and a horizontal asymptote at $y = -2$.

a



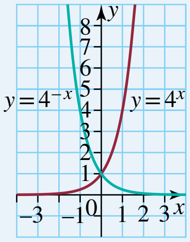
b $y = -4^x$ has the same shape as $y = 4^x$ but is reflected about the x-axis. It has a y-intercept of −1 and a horizontal asymptote at $y = 0$.

b



c $y = 4^{-x}$ has the same shape as $y = 4^x$ but is reflected about the y-axis. The graphs have the same y-intercept and the same horizontal asymptote ($y = 0$).

c



9.6.6 Combining transformations

- It is possible to combine translations, dilations and reflections in one graph.

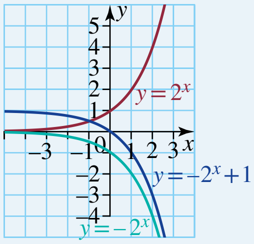
WORKED EXAMPLE 14 TI | CASIO

By considering transformations to the graph of $y = 2^x$, sketch the graph of $y = -2^x + 1$.

THINK

Start by sketching $y = 2^x$.
It has a y-intercept of 1 and a horizontal asymptote at $y = 0$.
Sketch $y = -2^x$ by reflecting $y = 2^x$ about the x-axis.
It has a y-intercept of -1 and a horizontal asymptote at $y = 0$.
Sketch $y = -2^x + 1$ by translating $y = -2^x$ upwards by 1 unit.
The graph has a y-intercept of 0 and a horizontal asymptote at $y = 1$.

DRAW



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Try out this interactivity: Exponential graphs (int-1149)

Exercise 9.6 Exponential functions and graphs

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–16	Questions: 1–17	Questions: 1–18

Individual pathway interactivity: int-4609

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. Complete the table below and use the table to plot the graph of $y = 3^x$ for $-3 \le x \le +3$.

x	-3	-2	-1	0	1	2	3
y							

2. If $x = 1$, find the value of y when:

a. $y = 2^x$ **b.** $y = 3^x$ **c.** $y = 4^x$ **d.** $y = 10^x$ **e.** $y = a^x$.

3. Using a calculator or graphing program, sketch the graphs of $y = 2^x$, $y = 3^x$ and $y = 4^x$ on the same set of axes.

a. What do the graphs have in common?

b. How does the value of the base $(2, 3, 4)$ affect the graph?

c. Predict where the graph $y = 8^x$ would lie and sketch it in.

4. Using graphing technology, sketch the following graphs on one set of axes.

$y = 3^x$, $y = 3^x + 2$, $y = 3^x + 5$, $y = 3^x - 3$

a. What remains the same in all of these graphs?

b. What is changed?

c. For the graph of $y = 3^x + 10$, write down:

- **i.** the y-intercept
- **ii.** the equation of the horizontal asymptote.

5. a. Using graphing technology, sketch the graphs of:

i. $y = 2^x$ and $y = -2^x$ **ii.** $y = 3^x$ and $y = -3^x$ **iii.** $y = 6^x$ and $y = -6^x$.

b. What is the relationship between these pairs of graphs?

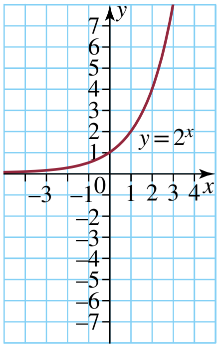
6. a. Using graphing technology, sketch the graphs of:

i. $y = 2^x$ and $y = 2^{-x}$ **ii.** $y = 3^x$ and $y = 3^{-x}$ **iii.** $y = 6^x$ and $y = 6^{-x}$.

b. What is the relationship between these pairs of graphs?

7. **WE13** Given the graph of $y = 2^x$, sketch on the same axes the graphs of:

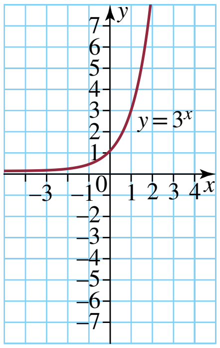
a. $y = 2^x + 6$ **b.** $y = -2^x$ **c.** $y = 2^{-x}$.



8. Given the graph of $y = 3^x$, sketch on the same axes the graphs of:

a. $y = 3^x + 2$

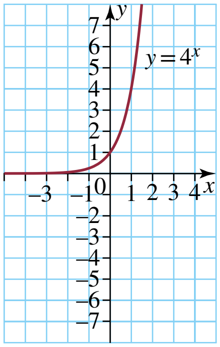
b. $y = -3^x$.



9. Given the graph of $y = 4^x$, sketch on the same axes the graphs of:

a. $y = 4^x - 3$

b. $y = 4^{-x}$.



10. **WE14** By considering transformations of the graph of $y = 2^x$, sketch the following graphs on the same set of axes.

a. $y = 2^{-x} + 2$

b. $y = -2^x + 3$

11. By considering transformations of the graph of $y = 5^x$, sketch the following graphs on the same set of axes.

a. $y = -5^x + 10$

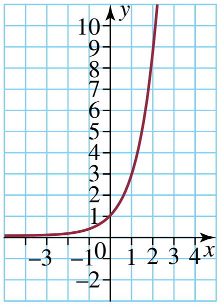
b. $y = 5^{-x} + 10$

Understanding

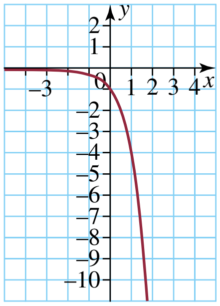
12. Match each graph with its correct label.

A. $y = 2^x$ **B.** $y = 3^x$ **C.** $y = -4^x$ **D.** $y = 5^{-x}$

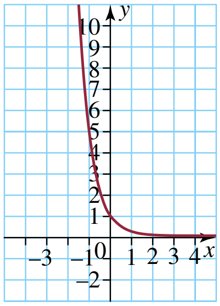
a.



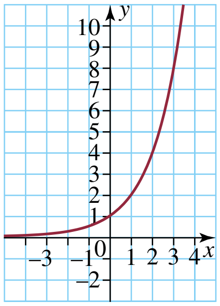
b.



c.



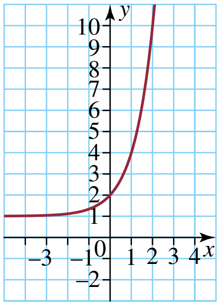
d.



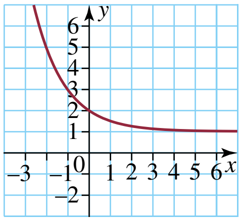
13. Match each graph with its correct label. Explain your answer.

A. $y = 2^x + 1$ **B.** $y = 3^x + 1$ **C.** $y = -2^x + 1$ **D.** $y = 2^{-x} + 1$

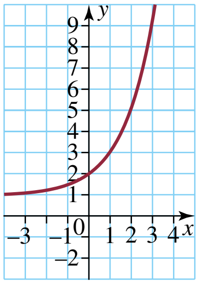
a.



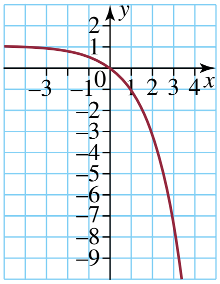
b.



c.



d.



Reasoning

14. By considering transformations of the graph of $y = 3^x$, sketch the graph of $y = -3^{-x} - 3$.

15. The graph of $f(x) = 16^x$ can be used to solve for x in the exponential equation $16^x = 32$. Draw a graph of $f(x) = 16^x$ and use it to solve $16^x = 32$.

16. The graph of $f(x) = 6^{x-1}$ can be used to solve for x in the exponential equation $6^{x-1} = 36\sqrt{6}$. Draw a graph of $f(x) = 6^{x-1}$ and use it to solve $6^{x-1} = 36\sqrt{6}$.

Problem solving

17. The number of bacteria, N, in a certain culture is reduced by a third every hour so

$$N = N_0 \times \left(\frac{1}{3}\right)^t$$

where t is the time in hours after 12 noon on a particular day. Initially there are 10 000 bacteria present.

a. Find the value of N_0.

b. Find the number of bacteria, correct to the nearest whole number, in the culture when:

i. $t = 2$ **ii.** $t = 5$ **iii.** $t = 10$.

18. a. The table below shows the population of a city between 1850 and 1930. Is the population growth exponential?

Year	1850	1860	1870	1880	1890	1900	1910	1920	1930
Population (million)	1.0	1.3	1.69	2.197	2.856	3.713	4.827	6.275	8.157

b. What is the common ratio in part **a**?

c. What is the annual percentage increase?

d. Estimate the population in 1895.

e. Estimate the population in 1980.

Reflection

Will the graph of an exponential function always have a horizontal asymptote? Why?

9.7 The hyperbola

9.7.1 Hyperbolas

- A **hyperbola** is a function of the form $xy = k$ or $y = \frac{k}{x}$.

WORKED EXAMPLE 15

Complete the table of values below and use it to plot the graph of $y = \frac{1}{x}$.

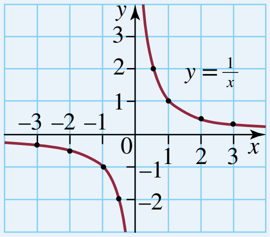
x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y									

THINK | **WRITE/DRAW**

1 Substitute each x-value into the function $y = \frac{1}{x}$ to obtain the corresponding y-value.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	Undefined	2	1	$\frac{1}{2}$	$\frac{1}{3}$

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.



- The graph in Worked example 15 has several important features.
  1. There is no function value (y-value) when $x = 0$. At this point the hyperbola is undefined. When this occurs, the line that the graph approaches ($x = 0$) is called a vertical asymptote.
  2. As x becomes larger and larger, the graph gets very close to but will never touch the x-axis. The same is true as x becomes smaller and smaller. The hyperbola also has a horizontal asymptote at $y = 0$.
  3. The hyperbola has two separate branches. It cannot be drawn without lifting your pen from the page and is an example of a discontinuous graph.
- Graphs of the form $y = \frac{k}{x}$ are the same basic shape as $y = \frac{1}{x}$ with y-values dilated by a factor of k.

WORKED EXAMPLE 16

a Plot the graph of $y = \frac{4}{x}$ for $-2 \le x \le 2$.

b Write down the equation of each asymptote.

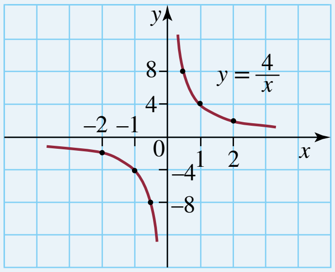
THINK | **WRITE/DRAW**

a 1 Prepare a table of values taking x-values from -2 to 2. Fill in the table by substituting each x-value into the given equation to find the corresponding y-value.

a

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-2	-4	-8	Undefined	8	4	2

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.



b Consider any lines that the curve approaches but does not cross.

b Vertical asymptote is $x = 0$.
Horizontal asymptote is $y = 0$.

WORKED EXAMPLE 17

TI | CASIO

Plot the graph of $y = \frac{-3}{x}$ for $-3 \le x \le 3$.

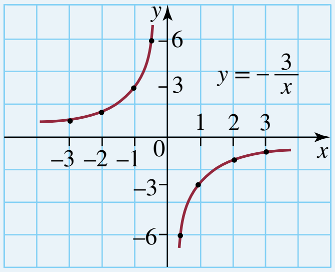
THINK

1 Draw a table of values and substitute each x-value into the given equation to find the corresponding y-value.

WRITE/DRAW

x	−3	−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	1	1.5	3	6	Undefined	−6	−3	−1.5	−1

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.



Exercise 9.7 The hyperbola

assess**on**

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–12	Questions: 1–14	Questions: 1–15

🟩🟦🟪 **Individual pathway interactivity:** int-4610

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Fluency

1. **WE15** Complete the table of values below and use it to plot the graph of $y = \frac{10}{x}$.

x	−5	−4	−3	−2	−1	**0**	**1**	**2**	**3**	**4**	**5**
y											

2. **WE16** **a.** Plot the graph of each hyperbola.

b. Write down the equation of each asymptote.

i. $y = \frac{5}{x}$ **ii.** $y = \frac{20}{x}$ **iii.** $y = \frac{100}{x}$

3. On the same set of axes, draw the graphs of $y = \frac{2}{x}$, $y = \frac{3}{x}$ and $y = \frac{4}{x}$.

4. Use your answer to question **3** to describe the effect of increasing the value of k on the graph of $y = \frac{k}{x}$.

5. **WE17** Plot the graph of $y = \frac{-10}{x}$ for $-5 \le x \le 5$.

6. On the same set of axes, draw the graphs of $y = \frac{6}{x}$ and $y = \frac{-6}{x}$.

7. Use your answer to question **6** to describe the effect of the negative in $y = \frac{-k}{x}$.

8. Complete the table of values below and use the points to plot $y = \frac{1}{x-1}$. State the equation of the vertical asymptote.

x	−3	−2	−1	0	1	2	3	4
y								

9. Plot the graph of each hyperbola and label the vertical asymptote.

a. $y = \frac{1}{x-2}$ **b.** $y = \frac{1}{x-3}$ **c.** $y = \frac{1}{x+1}$

Understanding

10. Use your answers to question **9** to describe the effect of a in $y = \frac{1}{x-a}$.

11. Sketch each of the following, showing the position of the vertical asymptote.

a. $y = \frac{-4}{x+1}$ **b.** $y = \frac{2}{x-1}$ **c.** $y = \frac{5}{x+2}$

Reasoning

12. Give an example of the equation of a hyperbola that has a vertical asymptote of:

a. $x = 3$ **b.** $x = -10$.

13. The graph of $y = \frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 2 parallel to the y-axis or from the x-axis and translated 3 units to the left and down 1 unit. Find the equation of the resultant hyperbola and give the equations of any asymptotes.

Problem solving

14. **a.** Complete the following table in order to graph the hyperbola defined by $y = \frac{1}{x^2}$.

x	−2	−1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2

This hyperbola is also known as a truncus. Give the equations of any asymptotes.

b. Find the equation of the truncus which results when $y = \frac{1}{x^2}$ is reflected in the x-axis.

c. Find the equation of the truncus which results when $y = \frac{1}{x^2}$ is reflected in the y-axis.

15. Consider again the truncus defined by $y = \frac{1}{x^2}$. This hyperbola is reflected in the x-axis, dilated by a factor of 3 parallel to the y-axis or from the x-axis and translated 1 unit to the left and up 2 units. Find the equation of the resulting hyperbola and give the equations of any asymptotes.

Reflection

How could you summarise the effect of the transformations dealt with in this exercise on the shape of the basic hyperbola $y = \frac{1}{x}$?

9.8 The circle

9.8.1 Circles

- A circle is the path traced out by a point at a constant distance (the radius) from a fixed point (the centre).
- Consider the circles shown below right. The first circle has its centre at the origin and radius r.

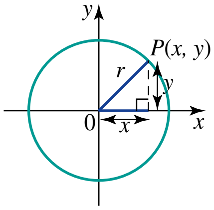
  Let $P(x, y)$ be a point on the circle.

  By Pythagoras: $x^2 + y^2 = r^2$.

  This relationship is true for all points, P, on the circle.

  The equation of a circle, with centre $(0, 0)$ and radius r, is:

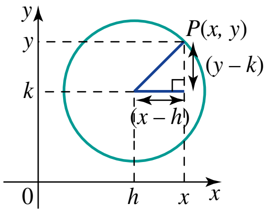
$$x^2 + y^2 = r^2$$



- If the circle is translated h units to the right, parallel to the x-axis, and k units upwards, parallel to the y-axis, then:

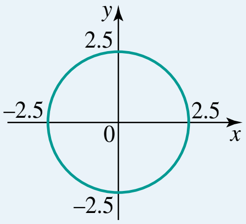
  The equation of a circle, with centre (h, k) and radius r, is:

$$(x - h)^2 + (y - k)^2 = r^2$$



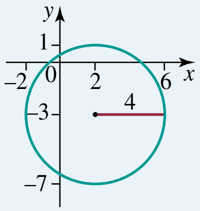
WORKED EXAMPLE 18

Sketch the graph of $4x^2 + 4y^2 = 25$, stating the centre and radius.

THINK	WRITE/DRAW
1 Express the equation in general form by dividing both sides by 4.	$x^2 + y^2 = r^2$ $4x^2 + 4y^2 = 25$ $x^2 + y^2 = \frac{25}{4}$
2 State the coordinates of the centre.	Centre $(0, 0)$
3 Find the length of the radius by taking the square root of both sides. (Ignore the negative results.)	$r^2 = \frac{25}{4}$ $r = \frac{5}{2}$ Radius = 2.5 units
4 Sketch the graph.	

WORKED EXAMPLE 19

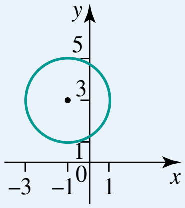
Sketch the graph of $(x - 2)^2 + (y + 3)^2 = 16$, clearly showing the centre and radius.

THINK	WRITE/DRAW
1 Express the equation in general form.	$(x - h)^2 + (y - k)^2 = r^2$ $(x - 2)^2 + (y + 3)^2 = 16$
2 State the coordinates of the centre.	Centre $(2, -3)$
3 State the length of the radius.	$r^2 = 16$ $r = 4$ Radius = 4 units
4 Sketch the graph.	

WORKED EXAMPLE 20

TI | CASIO

Sketch the graph of the circle $x^2 + 2x + y^2 - 6y + 6 = 0$.

THINK	WRITE/DRAW
1 Express the equation in general form by completing the square on the x terms and again on the y terms.	$(x - h)^2 + (y - k)^2 = r^2$ $x^2 + 2x + y^2 - 6y + 6 = 0$ $(x^2 + 2x + 1) - 1 + (y^2 - 6y + 9) - 9 + 6 = 0$ $(x + 1)^2 + (y - 3)^2 - 4 = 0$ $(x + 1)^2 + (y - 3)^2 = 4$
2 State the coordinates of the centre.	Centre $(-1, 3)$
3 State the length of the radius.	$r^2 = 4$ $r = 2$ Radius = 2 units
4 Sketch the graph.	

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Try out this interactivity: Compare and contrast types of graphs (int-3920)

Exercise 9.8 The circle

assess on

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions:	Questions:	Questions:
1a–c, 2a–c, 3a–c, 4, 5, 7	1c–e, 2c–e, 3c–e, 4–7, 10	1d–f, 2d–f, 3d–f, 4–11

■■■ **Individual pathway interactivity:** int-4611

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Fluency

1. **WE18** Sketch the graphs of the following, stating the centre and radius of each.

a. $x^2 + y^2 = 49$
b. $x^2 + y^2 = 4^2$
c. $x^2 + y^2 = 36$
d. $x^2 + y^2 = 81$
e. $2x^2 + 2y^2 = 50$
f. $9x^2 + 9y^2 = 100$

2. **WE19** Sketch the graphs of the following, clearly showing the centre and the radius.

a. $(x - 1)^2 + (y - 2)^2 = 5^2$
b. $(x + 2)^2 + (y + 3)^2 = 6^2$
c. $(x + 3)^2 + (y - 1)^2 = 49$
d. $(x - 4)^2 + (y + 5)^2 = 64$
e. $x^2 + (y + 3)^2 = 4$
f. $(x - 5)^2 + y^2 = 100$

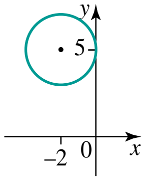
3. **WE20** Sketch the graphs of the following circles.

a. $x^2 + 4x + y^2 + 8y + 16 = 0$
b. $x^2 - 10x + y^2 - 2y + 10 = 0$
c. $x^2 - 14x + y^2 + 6y + 9 = 0$
d. $x^2 + 8x + y^2 - 12y - 12 = 0$
e. $x^2 + y^2 - 18y - 19 = 0$
f. $2x^2 - 4x + 2y^2 + 8y - 8 = 0$

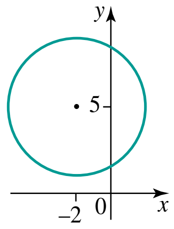
Understanding

4. **MC** The graph of $(x - 2)^2 + (y + 5)^2 = 4$ is:

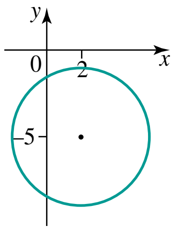
A.



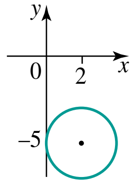
B.



C.



D.

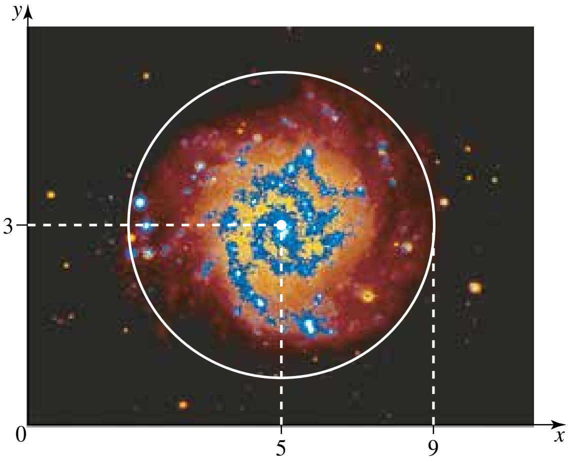


5. **MC** The centre and radius of the circle $(x + 1)^2 + (y - 3)^2 = 4$ is:

A. $(1, -3), 4$
B. $(-1, 3), 2$
C. $(3, -1), 4$
D. $(1, -3), 2$

Reasoning

6. Find the equation representing the outer edge of the galaxy as shown in the photo below, using the astronomical units provided.



7. Circular ripples are formed when a water drop hits the surface of a pond.

If one ripple is represented by the equation $x^2 + y^2 = 4$ and then 3 seconds later by $x^2 + y^2 = 190$, where the length of measurements are in centimetres:

a. find the radius (in cm) of the ripple in each case

b. calculate how fast the ripple is moving outwards.

(State your answers to 1 decimal place.)

8. Two circles with equations $x^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 9$ intersect. Determine the point(s) of intersection. Show your working.

9. a. Graph the line $y = x$, the parabola $y = x^2$ and the circle $x^2 + y^2 = 1$ on the one set of axes.

b. Find algebraically the points of intersection of:

- **i.** the line and the circle
- **ii.** the line and the parabola
- **iii.** the parabola and the circle.

Problem solving

10. Find the point(s) of intersection of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 8x - 2y + 16 = 0$ both algebraically and graphically.

11. The general equation of a circle is given by $x^2 + y^2 + ax + by + c = 0$. Find the equation of the circle which passes through the points $(4, 5)$, $(2, 3)$ and $(0, 5)$. State the centre of the circle and its radius.

Reflection

How could you write equations representing a set of concentric circles (circles with the same centre, but different radii)?

CHALLENGE 9.2

Does the point (1, 1.5) lie on, inside or outside the circle with equation $x^2 + 2x + y^2 - 6y + 6 = 0$?
Hint: It is important to know the length of the radius and the location of the centre of the circle.

9.9 Review

9.9.1 Review questions

Fluency

1. **MC** The turning point for the graph $y = 3x^2 - 4x + 9$ is:

A. $\left(\frac{1}{3}, 1\frac{2}{3}\right)$ **B.** $\left(\frac{1}{3}, \frac{2}{3}\right)$ **C.** $\left(\frac{1}{6}, 1\frac{1}{6}\right)$ **D.** $\left(\frac{2}{3}, 7\frac{2}{3}\right)$ **E.** $\left(\frac{2}{3}, 6\frac{2}{3}\right)$

2. **MC** Which graph of the following equations has the x-intercepts closest together?

A. $y = x^2 + 3x + 2$ **B.** $y = x^2 + x - 2$ **C.** $y = 2x^2 + x - 15$
D. $y = 4x^2 + 27x - 7$ **E.** $y = x^2 - 2x - 8$

3. **MC** Which graph of the equations below has the largest y-intercept?

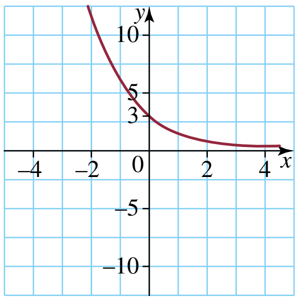
A. $y = 3(x - 2)^2 + 9$ **B.** $y = 5(x - 1)^2 + 8$ **C.** $y = 2(x - 1)^2 + 19$
D. $y = 2(x - 5)^2 + 4$ **E.** $y = 12(x - 1)^2 + 10$

4. **MC** The translation required to change $y = x^2$ into $y = (x - 3)^2 + \frac{1}{4}$ is:

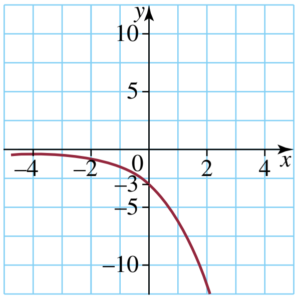
A. right 3, up $\frac{1}{4}$ **B.** right 3, down $\frac{1}{4}$ **C.** left 3, down $\frac{1}{4}$
D. left 3, up $\frac{1}{4}$ **E.** right $\frac{1}{4}$, up 3

5. **MC** The graph of $y = -3 \times 2^x$ is best represented by:

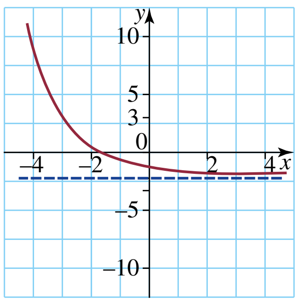
A.



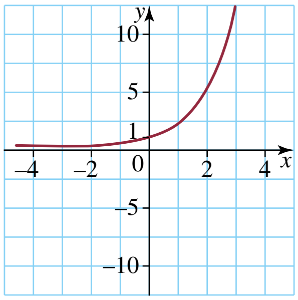
B.



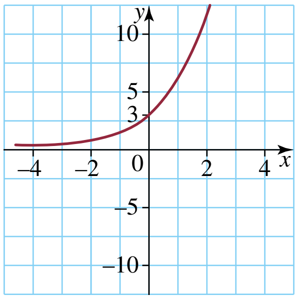
C.



D.



E.



6. Use the completing the square method to find the turning point for each of the following graphs.

a. $y = x^2 - 8x + 1$ **b.** $y = x^2 + 4x - 5$

7. For the graph of the equation $y = x^2 + 8x + 7$, produce a table of values for the x-values between -9 and 1, and then plot the graph. Show the y-intercept and turning point. From your graph, state the x-intercepts.

8. For each of the following, find the coordinates of the turning point and the x- and y-intercepts and sketch the graph.

a. $y = (x - 3)^2 + 1$ **b.** $y = 2(x + 1)^2 - 5$

9. For the equation $y = -x^2 - 2x + 15$, sketch the graph and determine the x- and y-intercepts and the coordinates of the turning point.

10. For the exponential function $y = 5^x$:

a. complete the table of values below

x	y
-3	
-2	
-1	
0	
1	
2	
3	

b. plot the graph.

11. Draw the graph of $y = 10 \times 3^x$ for $-4 \leq x \leq 4$.

12. Draw the graph of $y = 10^{-x}$ for $-4 \leq x \leq 4$.

13. a. On the same axes draw the graphs of $y = (1.2)^x$ and $y = (1.5)^x$.

b. Use your answer to part **a** to explain the effect of changing the value of a in the equation of $y = a^x$.

14. a. On the one set of axes draw the graphs of $y = 2 \times 3^x$, $y = 5 \times 3^x$ and $y = \frac{1}{2} \times 3^x$

b. Use your answer to part **a** to explain the effect of changing the value of k in the equation of $y = ka^x$.

15. a. On the same set of axes sketch the graphs of $y = (2.5)^x$ and $y = (2.5)^{-x}$.

b. Use your answer to part **a** to explain the effect of a negative index on the equation $y = a^x$.

16. Sketch each of the following.

a. $y = \frac{4}{x}$ **b.** $y = -\frac{2}{x}$

17. Sketch $y = \frac{-3}{x - 2}$.

18. Give an example of an equation of a hyperbola that has a vertical asymptote at $x = -3$.

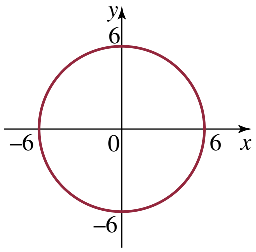
19. Sketch each of these circles. Clearly show the centre and the radius.

a. $x^2 + y^2 = 16$ **b.** $(x - 5)^2 + (y + 3)^2 = 64$

20. Sketch the following circles. Remember to first complete the square.

a. $x^2 + 4x + y^2 - 2y = 4$ **b.** $x^2 + 8x + y^2 + 8y = 32$

21. Find the equation of this circle.



Problem solving

22. The height, h, in metres of a golf ball t seconds after it is hit is given by the formula $h = 4t - t^2$.

a. Sketch the graph of the path of the ball.

b. What is the maximum height the golf ball reaches?

c. How long does it take for the ball to reach the maximum height?

d. How long is it before the ball lands on the ground after it has been hit?

23. A soccer ball is kicked upwards in the air. The height, h, in metres, t seconds after the kick is modelled by the quadratic equation $h = -5t^2 + 20t$.

a. Sketch the graph of this relationship.

b. For how many seconds is the ball in the air?

c. For how many seconds is the ball above a height of 15 m? That is, solve the quadratic inequation $-5t^2 + 20t > 15$.

d. For how many seconds is the ball above a height of 20 m?

24. The height of the water level in a cave is determined by the tides. At any time, t, in hours after 9 am, the height, $h(t)$, in metres, can be modelled by the function $h(t) = t^2 - 12t + 32,\ 0 \le t \le 12$.

a. What values of t is the model valid for? Write your answer in interval notation.

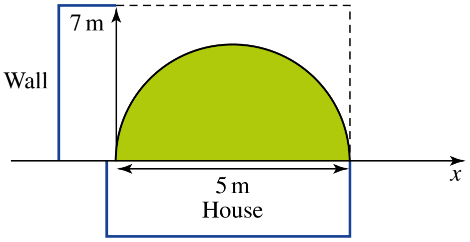
b. Determine the initial height of the water.

c. Bertha has dropped her keys onto a ledge which is 7 metres from the bottom of the cave. By using a graphics calculator, determine the times in which she would be able to climb down to retrieve her keys. Write your answers correct to the nearest minute.

25. A grassed area is planted in a courtyard that has a width of 5 metres and length of 7 metres. The shape of the grassed area is described by the function $P = -x^2 + 5x$, where P is the distance, in metres, from the house and x is the distance, in metres from the side wall. The diagram below represents this information on a Cartesian plane.

a. In terms of P, write down an inequality that describes the region where the grass has been planted.

b. Determine the maximum distance the grass area is planted from the house.



c. The owners of the house have decided that they would prefer all of the grass to be within a maximum distance of 3.5 metres from the house. The shape of the lawn following this design can be described by the equation $N(x) = ax^2 + bx + c$

i. Using algebra, show that this new design can be described by the function $N(x) = -0.56x(x - 5)$.

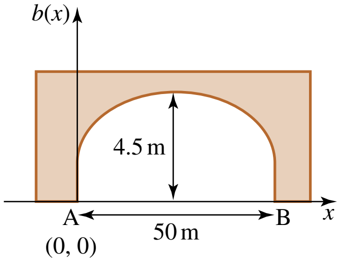
ii. Describe the transformation that maps $P(x)$ to $N(x)$.

d. If the owners decide on the first design, $P(x)$, the percentage of area within the courtyard without grass is 40.5%. By using any method, find the approximate percentage of area of courtyard without lawn with the new design, $N(x)$.

26. A stone arch bridge has a span of 50 metres. The shape of the curve AB can be modelled using a quadratic equation.

a. Taking A as the origin (0, 0) and given that the maximum height of the arch above the water level is 4.5 metres, show using algebra, that the shape of the arch can be modelled using the equation $b(x) = -0.0072x^2 + 0.36x$, where $b(x)$ is the vertical height of the bridge, in metres, and x is the horizontal distance, in metres.

b. A floating platform p metres high is towed under the bridge. Given that the platform needs to have a clearance of at least 30 centimetres on each side, explain why the maximum value of p is 10.7 centimetres.



27. When a drop of water hits the flat surface of a pool, circular ripples are made. One ripple is represented by the equation $x^2 + y^2 = 9$ and 5 seconds later, the ripple is represented by the equation $x^2 + y^2 = 225$, where the lengths of the radii are in cm.

a. State the radius of each of the ripples.
b. Sketch these graphs.
c. How fast is the ripple moving outwards?
d. If the ripple continues to move at the same rate, when will it hit the edge of the pool which is 2 m from its centre?

28. During an 8-hour period, an experiment is done in which the temperature of a room follows the relationship $T = h^2 - 8h + 21$, where T is the temperature in degrees Celsius h hours after starting the experiment.

a. Change the equation into turning point form and hence sketch the graph of this quadratic.
b. What is the initial temperature?
c. After three hours, is the temperature increasing or decreasing?
d. After five hours is the temperature increasing or decreasing?
e. State the minimum temperature and when it occurred.
f. What is the temperature after 8 hours?

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Complete this digital doc: Concept map: Topic 9 (doc-13811)

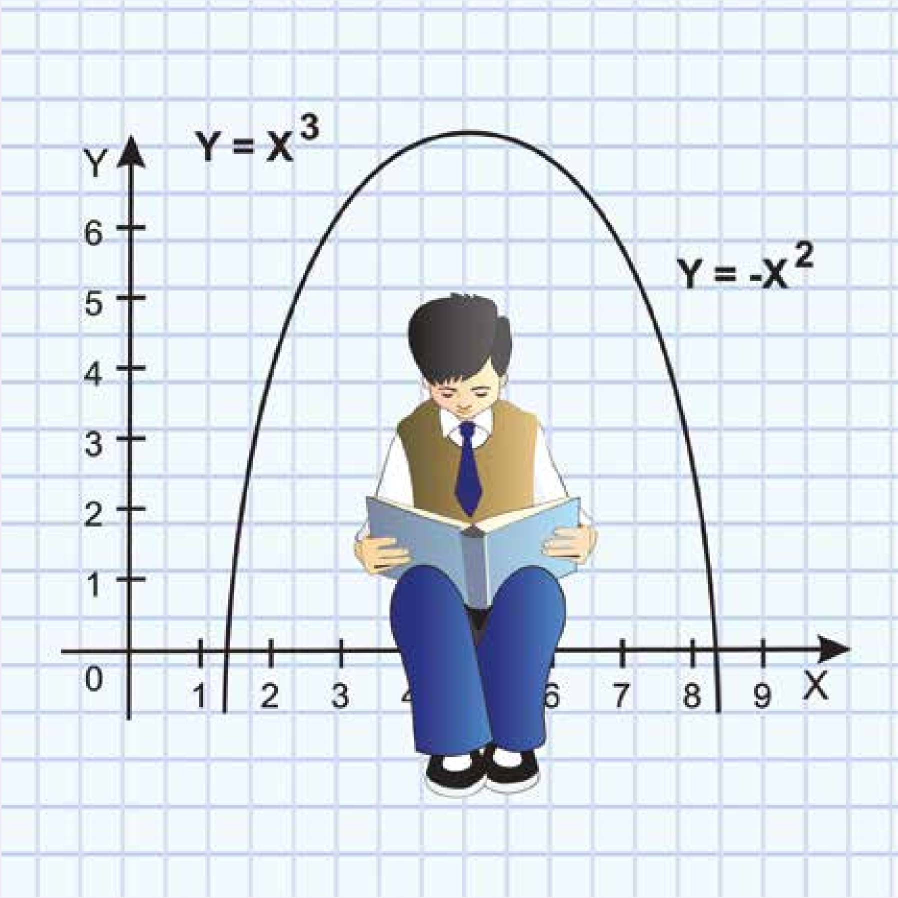
Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

algebraically
asymptote
axes
centre
circle
concave down
concave up
dilation
dilation factor
exponent
exponential
horizontal
hyperbola
linear
maximum
minimum
non-linear
origin
parabola
plot
radius
reflection
sketch
substitute
symmetrical
symmetry
transformation
translation
turning point
undefined
vertex
vertical
***y*-intercept**

Investigation | Rich task

Parametric equations



You are familiar with the quadratic equation $y = x^2$ and its resulting graph. Let us consider an application of this equation by forming a relationship between x and y through a third variable, say, t.

$$x = t \text{ and } y = t^2$$

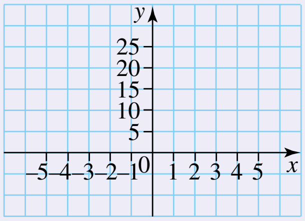
It is obvious that these two equations are equivalent to the equation $y = x^2$. This third variable t is known as a parameter, and the two equations are now called parametric equations. We cannot automatically assume that the resulting graph of these two parametric equations is the same as that of $y = x^2$ for all real values of x. It is dependent on the range of values of t.

Consider the parametric equations $x = t$ and $y = t^2$ for values of the parameter $t \geq 0$ for questions 1 to 3.

1. Complete the following table by calculating x- and y-values from the corresponding t-value.

t	x	y
0		
1		
2		
3		
4		
5		

Parametric equations



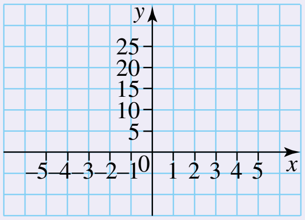
2. Graph the x-values and corresponding y-values on this Cartesian plane. Join the points with a smooth curve and place an arrow on the curve to indicate the direction of increasing t-values.

3. Is there any difference between this graph and that of $y = x^2$? Explain your answer.

4. Consider now the parametric equations $x = 1 - t$ and $y = (1 - t)^2$. These clearly are also equivalent to the equation $y = x^2$. Complete the table and draw the graph of these two equations for values of the parameter $t \geq 0$. Draw an arrow on the curve in the direction of increasing t-values.

t	x	y
0		
1		
2		
3		
4		
5		

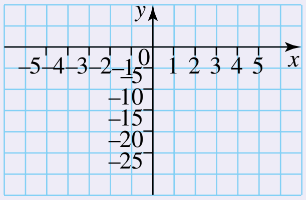
Parametric equations



Describe the shape of your resulting graph. What values of the parameter t would produce the same curve as that obtained in question **2**?

5. The graph of $y = -x^2$ is a reflection of $y = x^2$ in the x-axis. Construct a table and draw the graph of the parametric equations $x = t$ and $y = -t^2$ for parameter values $t \geq 0$. Remember to place an arrow on the curve in the direction of increasing t-values.

Parametric equations
$x = t$ and $y = -t^2$



6. Without constructing a table, predict the shape of the graph of the parametric equations $x = 1 - t$ and $y = -(1 - t)^2$ for parameter values $t \geq 0$. Draw a sketch of the shape.

7. This task requires you to produce the shape of the parabola $y = x^2$ in the range $-2 \leq x \leq 2$ by considering two different parametric equations to those already considered. Answer this question on a separate sheet of paper.

a. State your two equations and the range of the parameter values.

b. Construct a table showing calculated values.

c. Draw a sketch of the graph.

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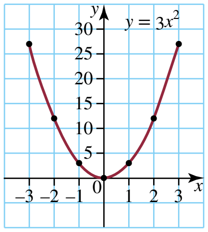
Answers

Topic 9 Non-linear relationships

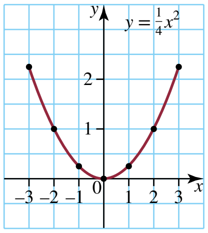
Exercise 9.2 Plotting parabolas

1. a.



$x = 0,\ (0,\ 0)$

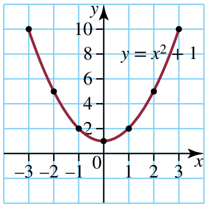
b.



$x = 0,\ (0,\ 0)$

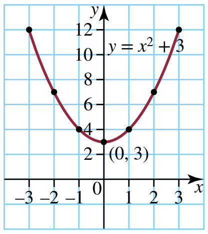
2. Placing a number greater than 1 in front of x^2 makes the graph thinner. Placing a number greater than 0 but less than 1 in front of x^2 makes the graph wider.

3. a.



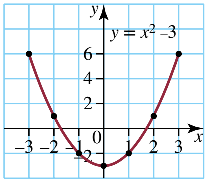
$x = 0,\ (0,\ 1),\ 1$

b.



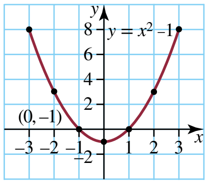
$x = 0,\ (0,\ 3),\ 3$

c.



$x = 0,\ (0, -3), -3$

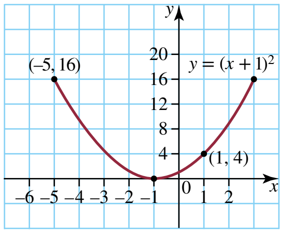
d.



$x = 0, (0, -1), -1$

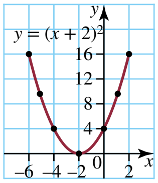
4. Adding a number raises the graph of $y = x^2$ vertically that number of units. Subtracting a number lowers the graph of $y = x^2$ vertically that number of units.

5. a.



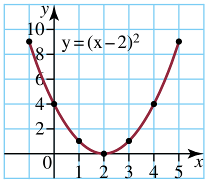
$x = -1,\ (-1,\ 0),\ 1$

b.



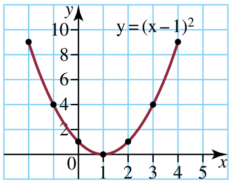
$x = -2,\ (-2,\ 0),\ 4$

c.



$x = 2,\ (2, 0),\ 4$

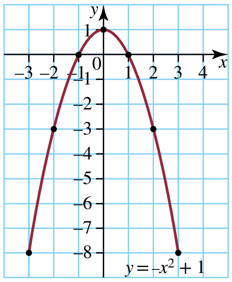
d.



$x = 1,\ (1, 0),\ 1$

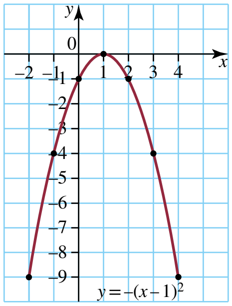
6. Adding a number moves the graph of $y = x^2$ horizontally to the left by that number of units. Subtracting a number moves the graph of $y = x^2$ horizontally to the right by that number of units.

7. a.



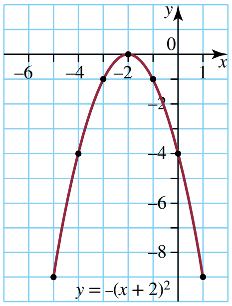
$x = 0,\ (0,\ 1),\ 1$

b.



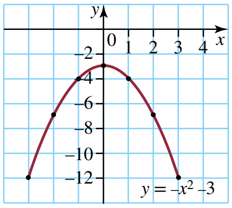
$x = 1,\ (1,\ 0),\ -1$

c.



$x = -2,\ (-2,\ 0), -4$

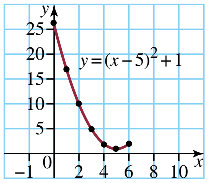
d.



$x = 0,\ (0,\ -3), -3$

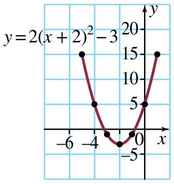
8. The negative sign inverts the graph of $y = x^2$. The graphs with the same turning points are: $y = x^2 + 1$ and $y = -x^2 + 1$; $y = (x - 1)^2$ and $y = -(x - 1)^2$; $y = (x + 2)$ and $y = -(x + 2)^2$; $y = x^2 - 3$ and $y = -x^2 - 3$. They differ in that the first graph is upright while the second graph is inverted.

9. a.



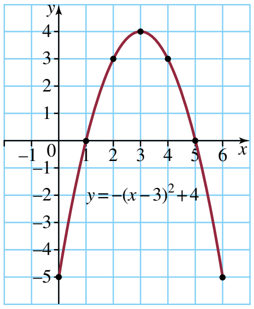
i. $x = 5$ **ii.** $(5, 1)$, min **iii.** 26

b.



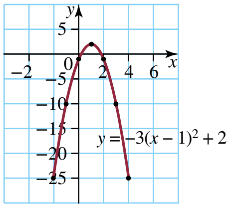
i. $x = -2$ **ii.** $(-2, -3)$, min **iii.** 5

c.



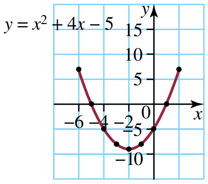
i. $x = 3$ **ii.** $(3, 4)$, max **iii.** -5

d.



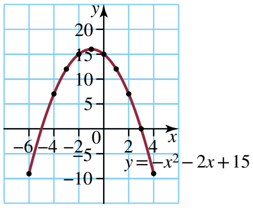
i. $x = 1$ **ii.** $(1, 2)$, max **iii.** -1

e.



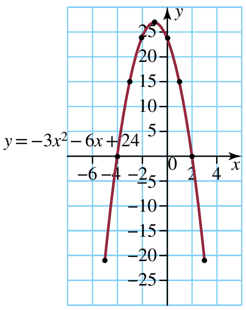
i. $x = -2$ **ii.** $(-2, -9)$, min **iii.** -5

f.



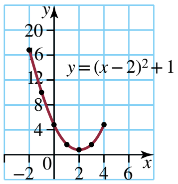
i. $x = -1$ **ii.** $(-1, 16)$, max **iii.** 15

g.



i. $x = 1$ **ii.** $(-1, 27)$, max **iii.** 24

h.



i. $x = 2$ **ii.** $(2, 1)$, min **iii.** 5

10. a. If the x^2 term is positive, the parabola has a minimum turning point. If the x^2 term is negative, the parabola has a maximum turning point.

b. If the equation is of the form $y = a(x - b)^2 + c$, the turning point has coordinates (b, c).

c. The equation of the axis of symmetry can be found from the x-coordinate of the turning point. That is, $x = b$.

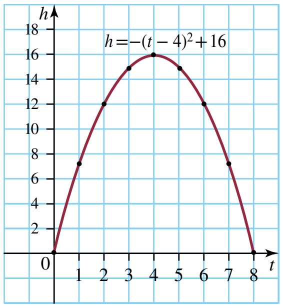
11. C

12. B

13. C

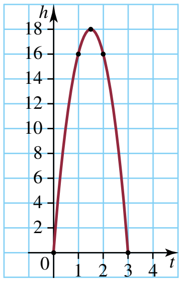
14. A

15. a.



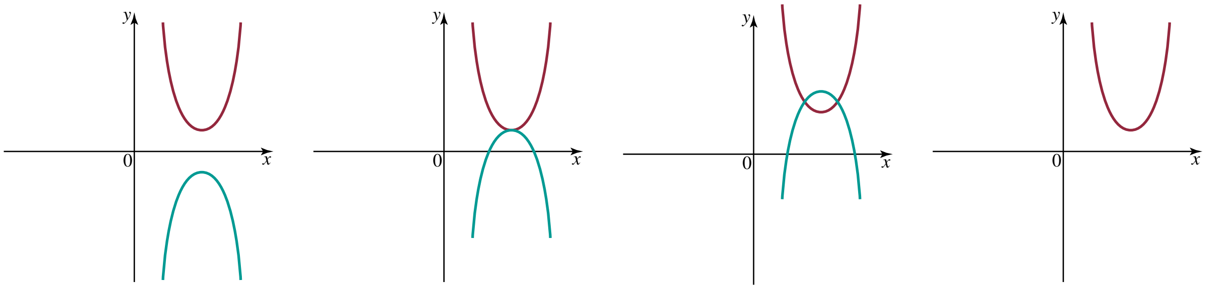
b. i. 16 m **ii.** 8 s

15. a.



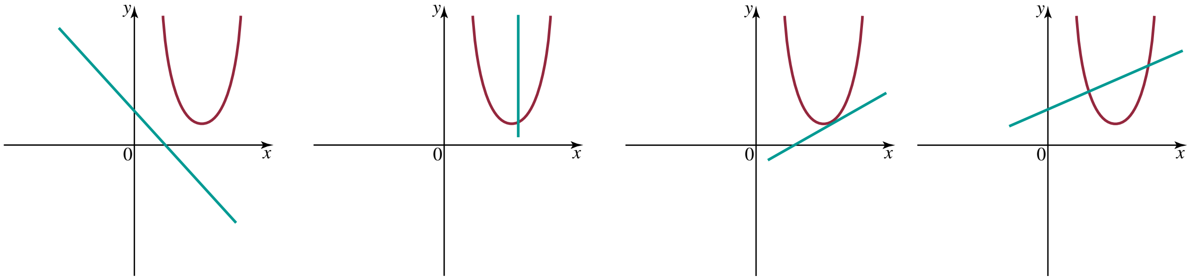
b. i. 18 m **ii.** Yes, by 3 m **iii.** 1.5 s **iv.** 3 s

16. a.



b. An infinite number of points of intersection occur when the two equations represent the same parabola, with the effect that the two parabolas superimpose. For example $y = x^2 + 4x + 3$ and $2y = 2x^2 + 8x + 6$.

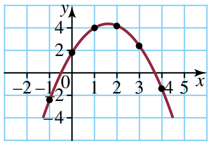
c. It is possible to have 0, 1 or 2 points of intersection.



17. a.

x	-1	0	1	2	3	4
y	-2.4	1.8	4	4.2	2.4	-1.4

b.



c. x cannot equal -1 as this would put the ball behind her; at $x = 4$, the ball is under ground level.

d. The maximum height reached is 4.36 m.

e. The ball will hit the ground 3.688 m from the player.

18. $y = 2x^2 - 3x + 1$

Exercise 9.3 Sketching parabolas

1. a. Narrower, TP $(0, 0)$ **b.** Wider, $(0, 0)$ **c.** Narrower, TP $(0, 0)$ **d.** Narrower, TP $(0, 0)$

e. Wider, TP $(0, 0)$ **f.** Wider, TP $(0, 0)$ **g.** Narrower, TP $(0, 0)$ **h.** Narrower, TP $(0, 0)$

2. a. Vertical 3 up, TP $(0, 3)$ **b.** Vertical 1 down, TP $(0, -1)$ **c.** Vertical 7 down, TP $(0, -7)$

d. Vertical up, TP $(0, \frac{1}{4})$ **e.** Vertical down, TP $(0, -\frac{1}{2})$ **f.** Vertical 0.14 down, TP $(0, -0.14)$

g. Vertical 2.37 up, TP $(0, 2.37)$ **h.** Vertical $\sqrt{3}$ up, TP $(0, \sqrt{3})$

3. a. Horizontal 1 right, $(1, 0)$ **b.** Horizontal 2 right, $(2, 0)$ **c.** Horizontal 10 left, $(-10, 0)$

d. Horizontal 4 left, $(-4, 0)$ **e.** Horizontal $\frac{1}{2}$ right, $(\frac{1}{2}, 0)$ **f.** Horizontal $\frac{1}{5}$ left, $(-\frac{1}{5}, 0)$

g. Horizontal 0.25 left, $(-0.25, 0)$ **h.** Horizontal $\sqrt{3}$ left, $(-\sqrt{3}, 0)$

4. a. $(0, 1)$, max **b.** $(0, -3)$, min **c.** $(-2, 0)$, max **d.** $(0, 0)$, min

e. $(0, 4)$, max **f.** $(0, 0)$, max **g.** $(5, 0)$, min **h.** $(0, 1)$, min

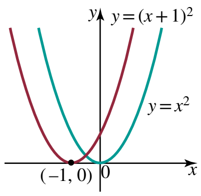
5. a. Narrower, min **b.** Narrower, max **c.** Wider, min **d.** Wider, max

e. Narrower, max **f.** Wider, min **g.** Narrower, min **h.** Wider, max

6. a. i. Horizontal translation 1 left

ii. $(-1, 0)$

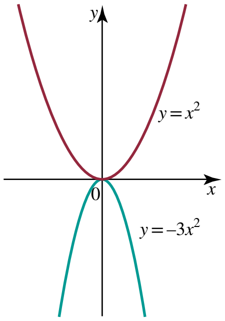
iii.



b. i. Reflected, narrower (dilation)

ii. $(0, 0)$

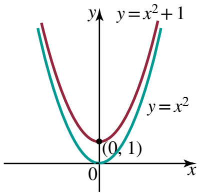
iii.



c. i. Vertical translation 1 up

ii. $(0, 1)$

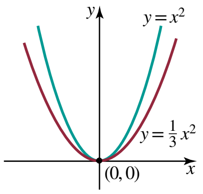
iii.



d. i. Wider (dilation)

ii. (0, 0)

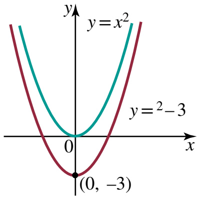
iii.



e. i Vertical translation 3 down

ii. (0, −3)

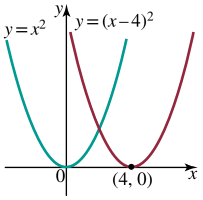
iii.



f. i. Horizontal translation 4 right

ii. (4, 0)

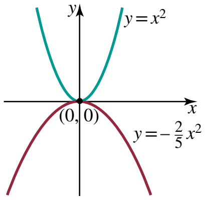
iii.



g. i. Reflected, wider (dilation)

ii. (0, 0)

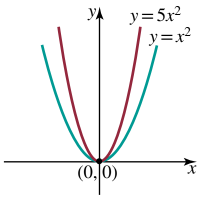
iii.



h. i. Narrower (dilation)

ii. (0, 0)

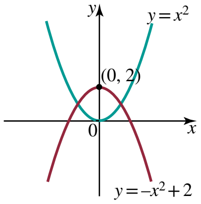
iii.



i. i. Reflected, vertical translation 2 up

ii. (0, 2)

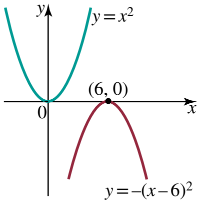
iii.



j. i. Reflected, horizontal translation 6 right

ii. (6, 0)

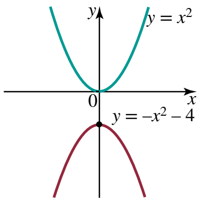
iii.



k. i. Reflected, vertical translation 4 down

ii. (0, −4)

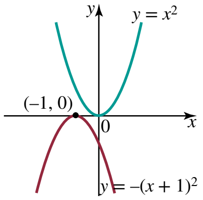
iii.



l. i. Reflected, horizontal translation 1 left

ii. (−1, 0)

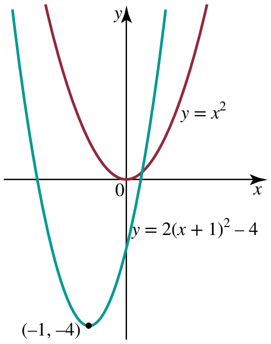
iii.



m. i. Narrower (dilation), horizontal translation 1 left, vertical translation 4 down

ii. (−1, −4)

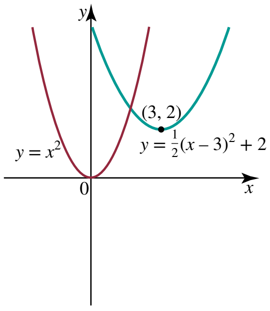
iii.



n. i. Wider (dilation), horizontal translation 3 right, vertical translation 2 up

ii. (3, 2)

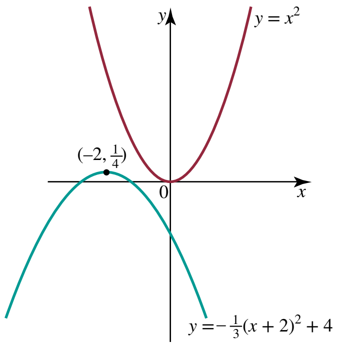
iii.



o. i. Wider (dilation), reflected, horizontal translation 2 left, vertical translation $\frac{1}{4}$ up

ii. $(-2, \frac{1}{4})$

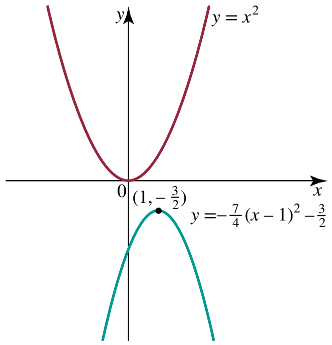
iii.



p. i. Narrower (dilation), reflected, horizontal translation 1 right, vertical translation $\frac{3}{2}$ down

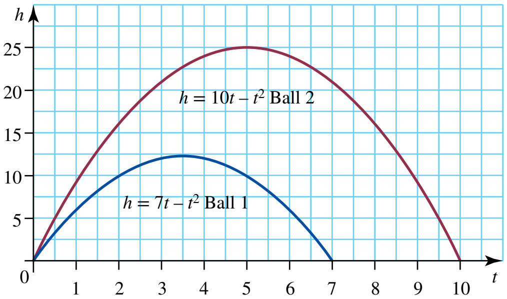
ii. $(1, -\frac{3}{2})$

iii.



7. a. 10 cm **b.** 5 cm **c.** 5 cm **d.** $y = (x - 5)^2$

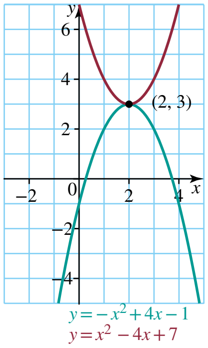
8. a. and **d.**



b. 12.25 m **c.** $h = t(10 - t)$ **e.** 12.75 m

9. a. $y = -(x - 2)^2 + 3 = -x^2 + 4x - 1$

b.

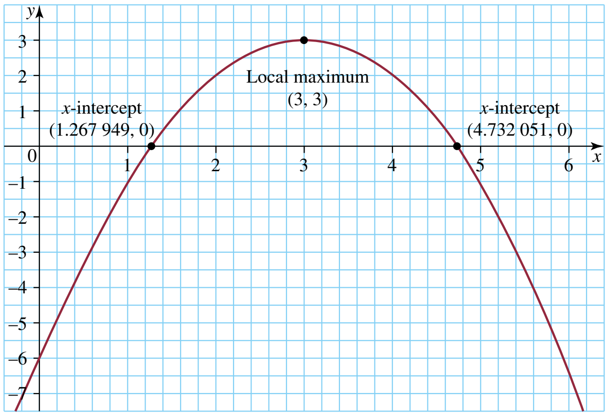


10. a. $Y = -(x - 3)^2 + 3$

b. Reflected in x-axis, translated 3 units to the right and up 3 units. No dilation.

c. (3, 3)

d.



Challenge 9.1

$v = -2h^2 + 40h$, where v is the vertical distance h is the horizontal distance.

Exercise 9.4 Sketching parabolas in turning point form

1. a. $(1, 2)$, min **b.** $(-2, -1)$, min **c.** $(-1, 1)$, min
d. $(2, 3)$, max **e.** $(5, 3)$, max **f.** $(-2, -6)$, min
g. $(\frac{1}{2}, -\frac{3}{4})$, min **h.** $(\frac{1}{3}, \frac{2}{3})$, min **i.** $(-0.3, -0.4)$, min

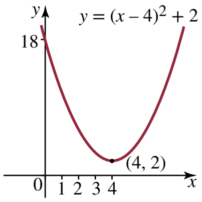
2. a. i. $(-3, -5)$ **ii.** Min **iii.** Narrower
b. i. $(1, 1)$ **ii.** Max **iii.** Same
c. i. $(-2, -4)$ **ii.** Max **iii.** Narrower
d. i. $(3, 2)$ **ii.** Min **iii.** Wider
e. i. $(-1, 7)$ **ii.** Max **iii.** Wider
f. i. $(\frac{1}{5}, -\frac{1}{2})$ **ii.** Min **iii.** Wider

3. a. vi $y = (x - 1)^2 - 3$ **b. i** $y = -(x - 2)^2 + 3$
c. v $y = x^2 - 1$ **d. iv** $y = -(x + 2)^2 + 3$
e. ii $y = -x^2 + 1$ **f. iii** $y = (x + 1)^2 - 3$

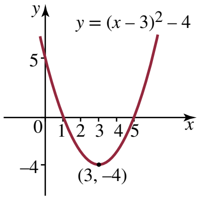
4. a. A **b.** C **c.** B **d.** C **e.** B

5. a. i. -3 **ii.** $-3, 1$ **b. i.** 12 **ii.** 2
c. i. -18 **ii.** No x-intercepts **d. i.** -5 **ii.** $-1, 5$
e. i. 4 **ii.** No x-intercepts
f. i. 4 **ii.** $-3 - \sqrt{5}, -3 + \sqrt{5}$ (approx. $-5.24, -0.76$)

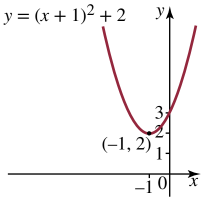
6. a. i. $(4, 2)$ **ii.** Min **iii.** Same width **iv.** 18
v. No x-intercepts **vi.**



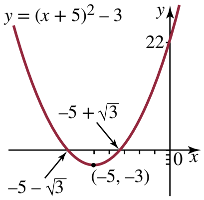
b. i. $(3, -4)$ **ii.** Min **iii.** Same width **iv.** 5
v. 1, 5 **vi.**



c. i. $(-1, 2)$ **ii.** Min **iii.** Same width **iv.** 3
v. No x-intercepts **vi.**

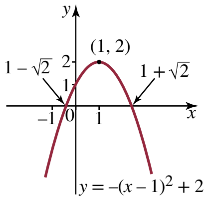


d. i. $(-5, -3)$ **ii.** Min **iii.** Same width **iv.** 22
v. $-5 - \sqrt{3}, -5 + \sqrt{3}$ (approx. $-6.73, -3.27$) **vi.**



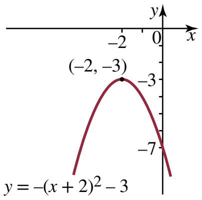
e. i. (1, 2) **ii.** Max **iii.** Same width **iv.** 1

v. $1-\sqrt{2}, 1+\sqrt{2}$ (approx. −0.41, 2.41) **vi.**



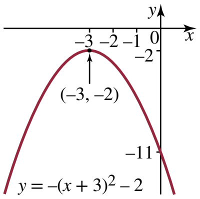
f. i. (−2, −3) **ii.** Max **iii.** Same width **iv.** −7

v. No x-intercepts **vi.**



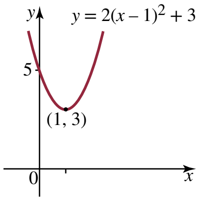
g. i. (−3, −2) **ii.** Max **iii.** Same width **iv.** −11

v. No x-intercepts **vi.**



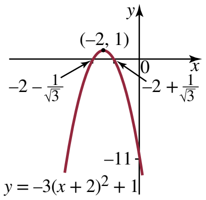
h. i. (1, 3) **ii.** Min **iii.** Narrower **iv.** 5

v. No x-intercepts **vi.**



e. i. (−2, 1) **ii.** Max **iii.** Narrower **iv.** −11

v. $-2-\frac{1}{\sqrt{3}}, -2+\frac{1}{\sqrt{3}}$ (approx. −2.58, −1.42) **vi.**

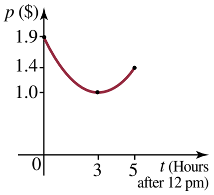


7. a. $2(x-\frac{3}{4})^2-\frac{73}{8}=0$ **b.** $x=\frac{3}{4}\pm\frac{\sqrt{73}}{4}$ **c.** $(\frac{3}{4}, -\frac{73}{8})$, minimum

8. a. $y=-\frac{2}{3}(x+4)^2+6$ **b.** (−7, 0)

9. a. $y=-x^2$ **b.** $y=7x^2$ **c.** $y=(x+3)^2$ **d.** $y=x^2+3$ **e.** $y=-\frac{1}{4}(x-5)^2-3$

10. a.



b. \$1.90 **c.** \$1 **d.** 3 pm **e.** \$1.40

11. a. 0.5 m **b.** $(15+4\sqrt{15})$ m

c. Maximum height is 8 metres when horizontal distance is 15 metres.

12. a. Answer will vary.

An example is $y = (x - 2)^2 + 6$.

b. $y = -\frac{1}{2}(x - 2)^2 + 6$

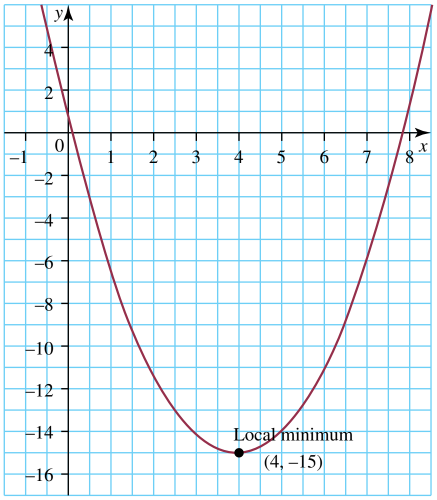
13. a. Answer will vary.

An example is $y = (x - p)^2 + q$.

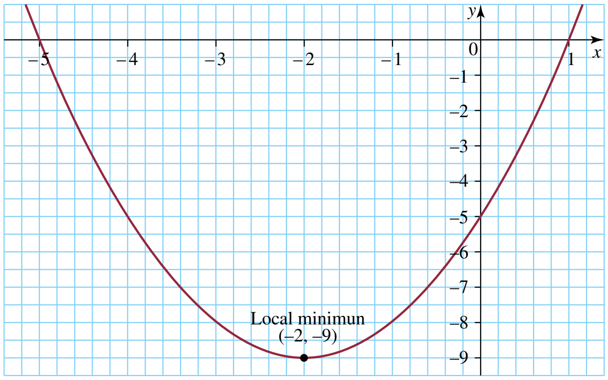
b. $y = \left(\dfrac{r - q}{p^2}\right)(x - p)^2 + q$

14. a. $y = (x - 4)^2 - 15$ **b.** $y = (x + 2)^2 - 9$ **c.** $y = (x + \frac{3}{2})^2 - \frac{1}{4}$

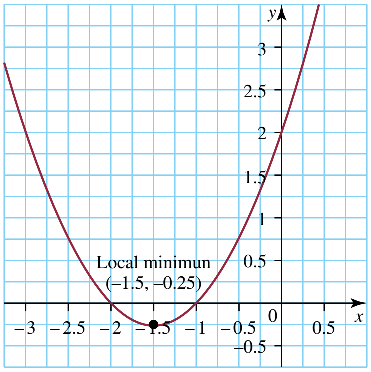
a.



b.



c.

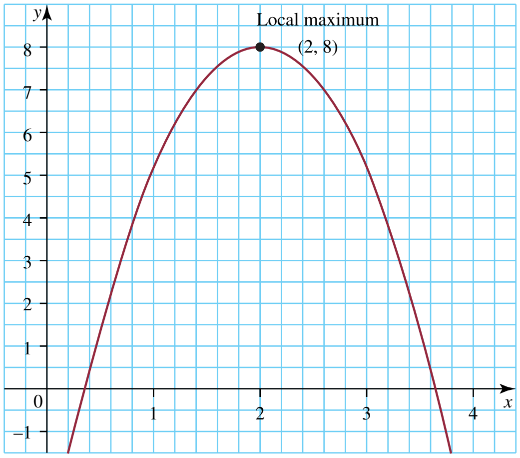


15. a. $y = 3(x - 2)^2 - 8$

b. Dilated by a factor of 3 parallel to the y-axis or from the x-axis as well as being translated 2 units to the right and down 8 units.

c. $y = -3(x - 2)^2 + 8$

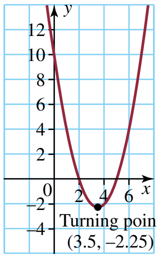
d.



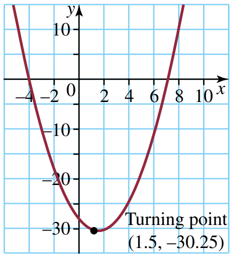
Exercise 9.5 Sketching parabolas of the form $y = ax^2 + bx + c$

1. You need the x-intercepts, the y-intercept and the turning point to sketch a parabola.

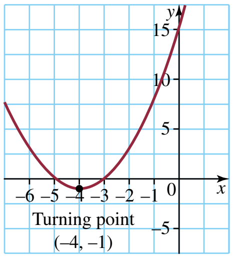
2. a.



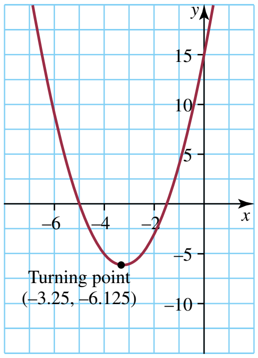
b.



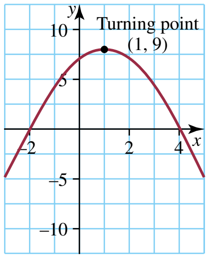
c.



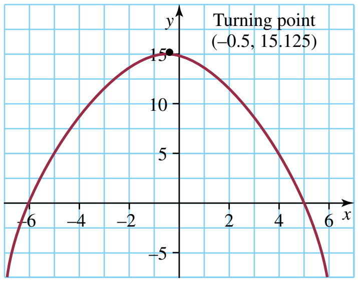
d.



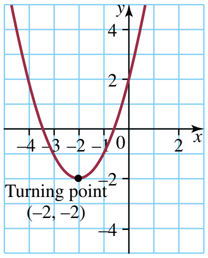
e.



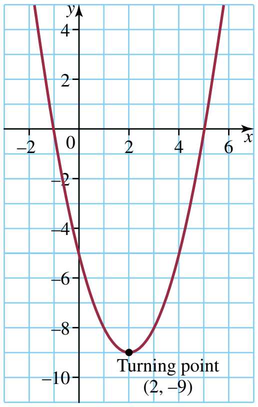
f.



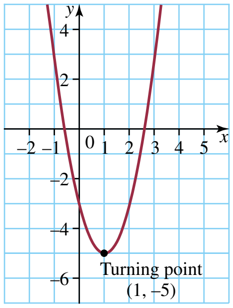
3. a.



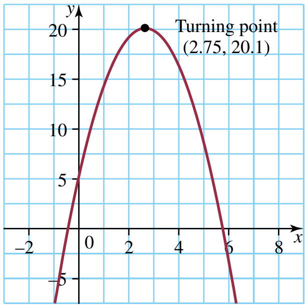
b.



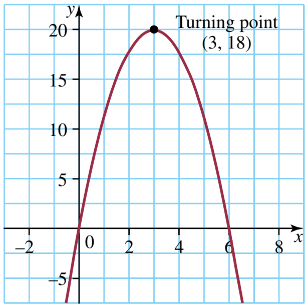
c.



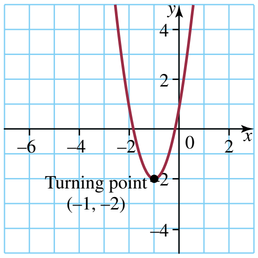
d.



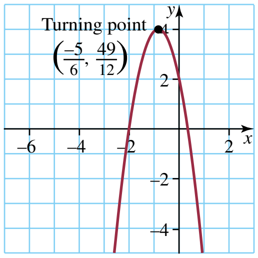
e.



f.



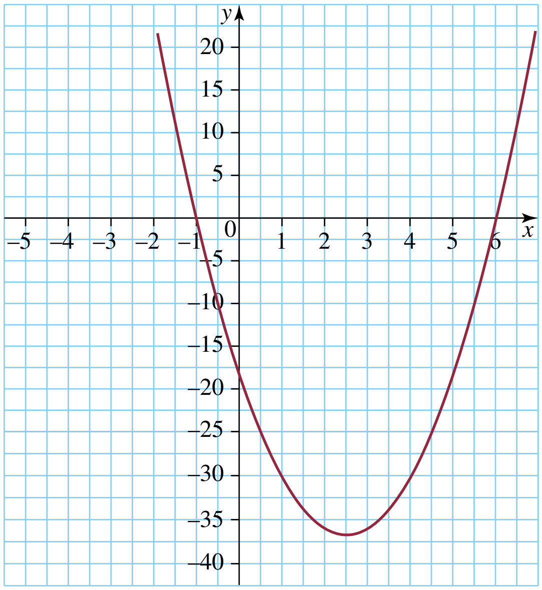
g.



4. a. 

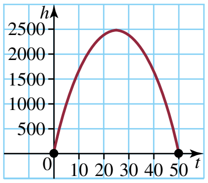
b. 24 m

c. 1 m

5. a. 

b. 6th month

c. 8th month

6. a. 

b. 0

c. 2500 m

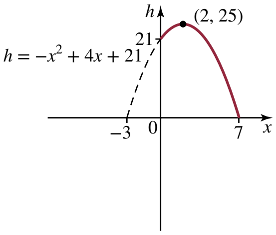
d. 25 seconds

e. 50 seconds

7. 200

8. $-(m + n)$

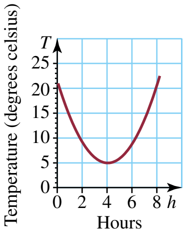
9. Answers will vary.

10. a. 

b. 25 m

c. 2 m

d. 7 m

11. a. 

b. 21°C

c. Decreasing

d. Increasing

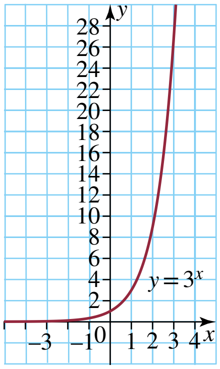
e. 5°C after 4 hours

f. 21°C

Exercise 9.6 Exponential functions and graphs

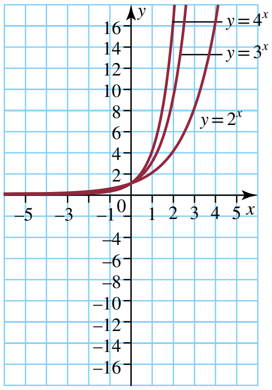
1.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



2. a. 2 **e.** 3 **f.** 4 **g.** 10 **h.** *a*

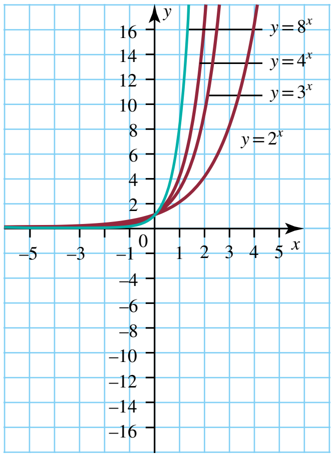
3.



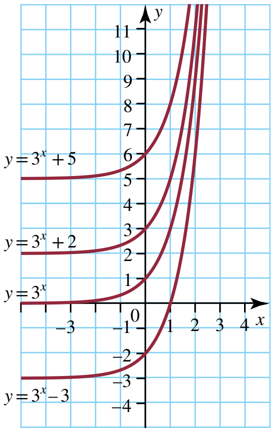
a. The graphs all pass through (0, 1). The graphs have the same horizontal asymptote, ($y = 0$). The graphs are all very steep.

b. As the base grows larger, the graphs become steeper.

c.



4.

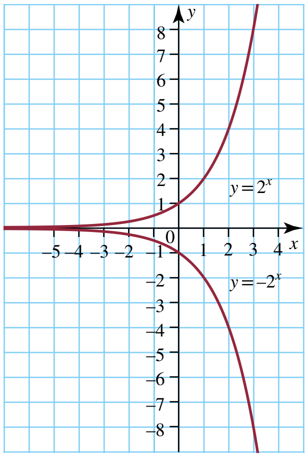


a. The shape of each graph is the same.

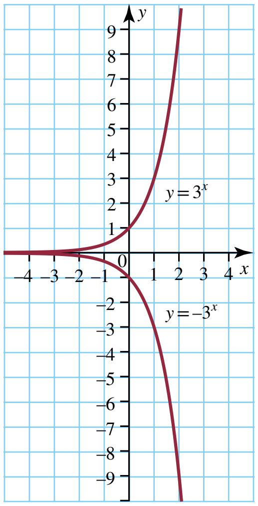
b. Each graph has a different y-intercept and a different horizontal asymptote.

c. i. $(0, 11)$ **ii.** $y = 10$

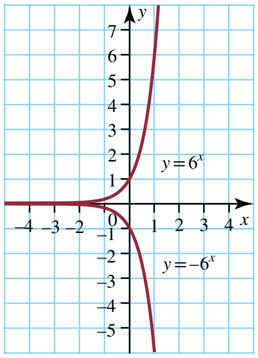
5. a. i.



ii.

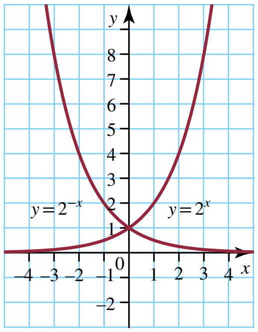


iii.

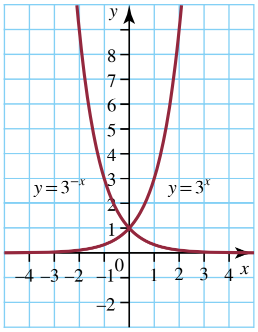


b. In each case the graphs are symmetric about the x-axis.

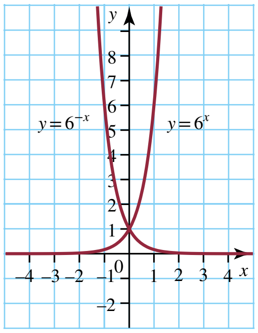
6. a. i.



ii.

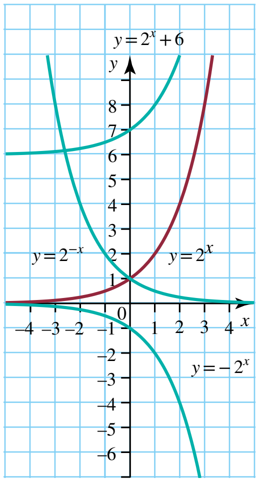


iii.

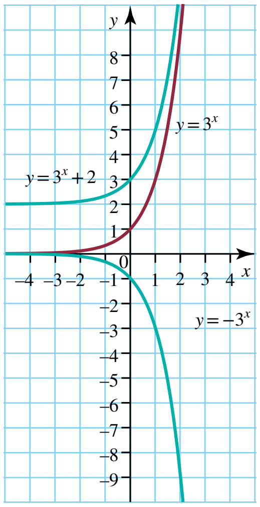


b. In each case the graphs are symmetric about the y-axis.

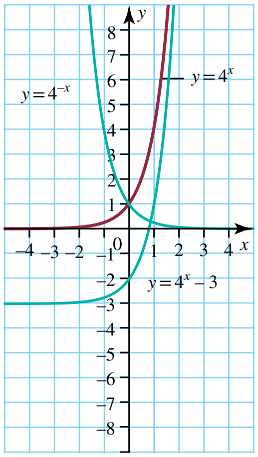
7. a–c.



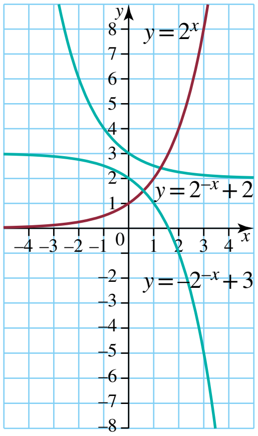
8. a, b.



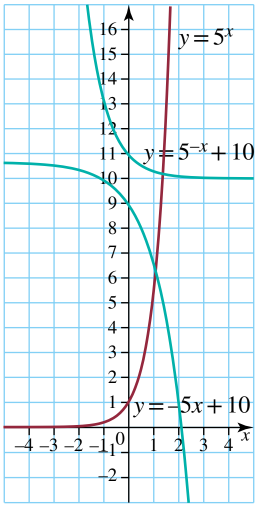
9. a, b.



10. a, b.



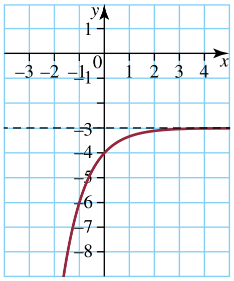
11. a, b.



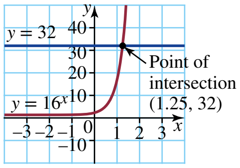
12. a. B **b.** C **c.** D **d.** A

13. a. B **b.** D **c.** A **d.** C

14.

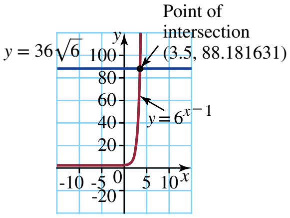


15.



$x = 1.25$

16.



$x = 3.5$

17. a. 10 000

b. i. 1111 **ii.** 41 **iii.** 0

18. a. Yes

b. There is a constant ratio of 1.3.

c. 30%

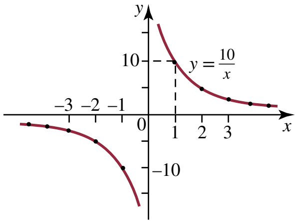
d. 3.26 million

e. 30 million

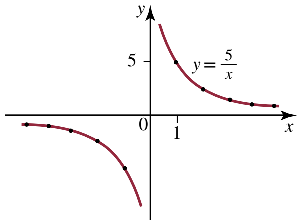
Exercise 9.7 The hyperbola

1.

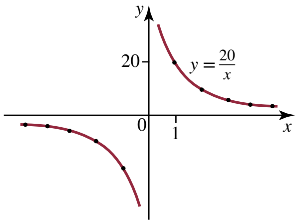
x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	2	−2.5	−3.3	−5	−10	Undefined	10	5	3.3	2.5	2



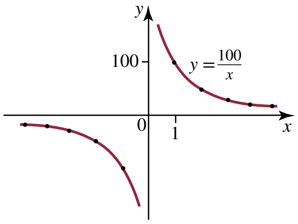
2. a. i.



ii.



iii.

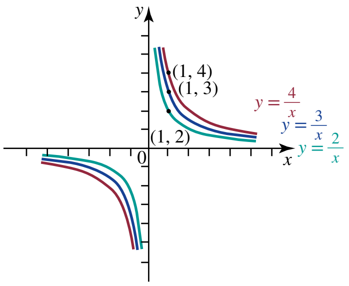


b. i. $x = 0, \; y = 0$

ii. $x = 0, \; y = 0$

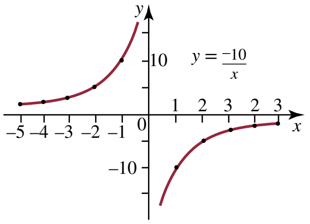
iii. $x = 0, \; y = 0$

3.

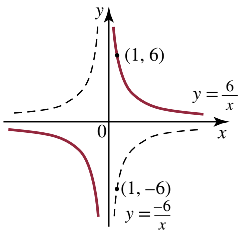


4. It increases the y-values by a factor of k and hence dilates the curve by a factor of k.

5.



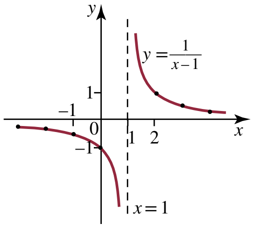
6.



7. The negative reflects the curve $y = \frac{k}{x}$ in the x-axis.

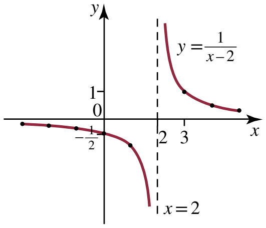
8.

x	−3	−2	−1	0	1	2	3	4
y	−0.25	−0.33	−0.5	−1	Undefined	1	0.5	0.33

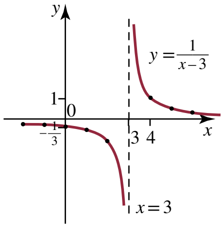


Equation of vertical asymptote is $x = 1$.

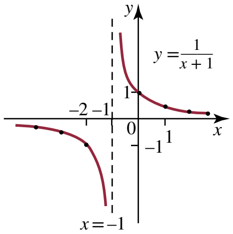
9. a.



b.

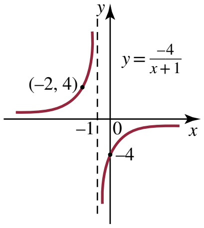


c.

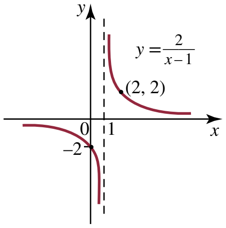


10. The a translates the graph left or right, and $x = a$ becomes the vertical asymptote.

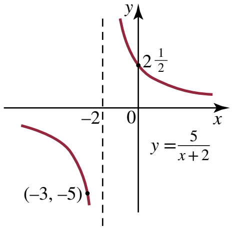
11. a.



b.



c.



12. Check with your teacher. Possible answers:

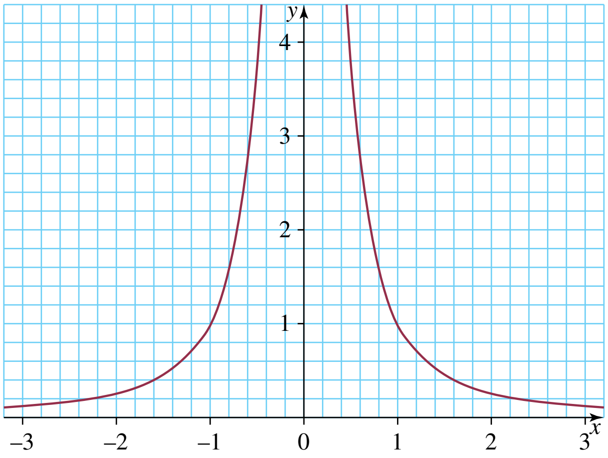
a. $y = \dfrac{1}{x - 3}$

b. $y = \dfrac{1}{x + 10}$

13. $y = \dfrac{-2}{x + 3} - 1,\ x = -3,\ y = -1$

14. a.

x	−2	−1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{1}{4}$	1	4	4	1	$\frac{1}{4}$



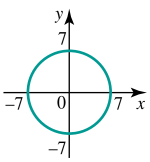
$x = 0,\ y = 0$

b. $y = -\dfrac{1}{x^2}$ **c.** $y = -\dfrac{1}{x^2}$

2. $y = -\dfrac{3}{(x+1)^2} + 2, x = -1, y = 2$

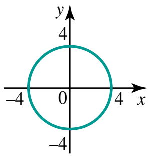
Exercise 9.8 The circle

1. a.



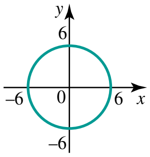
Centre $(0, 0)$, radius 7

b.



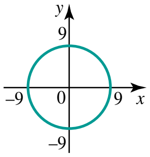
Centre $(0, 0)$, radius 4

c.



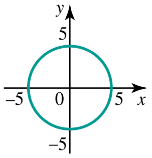
Centre $(0, 0)$, radius 6

d.



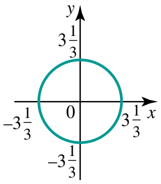
Centre $(0, 0)$, radius 9

e.



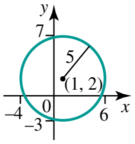
Centre $(0, 0)$, radius 5

f.

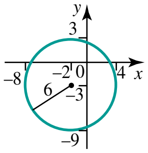


Centre $(0, 0)$, radius $\frac{10}{3}$

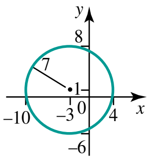
2. a.



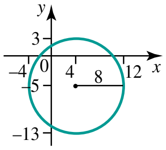
b.



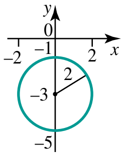
c.



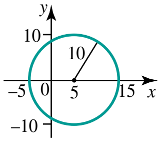
d.



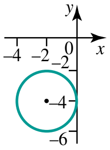
e.



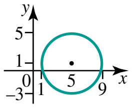
f.



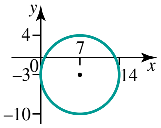
3. a. $(x + 2)^2 + (y + 4)^2 = 2^2$



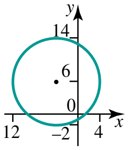
b. $(x - 5)^2 + (y - 1)^2 = 4^2$



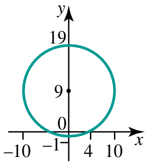
c. $(x - 7)^2 + (y + 3)^2 = 7^2$



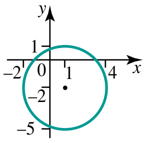
d. $(x + 4)^2 + (y - 6)^2 = 8^2$



e. $x^2 + (y - 9)^2 = 10^2$



f. $(x - 1)^2 + (y + 2)^2 = 3^2$



4. D

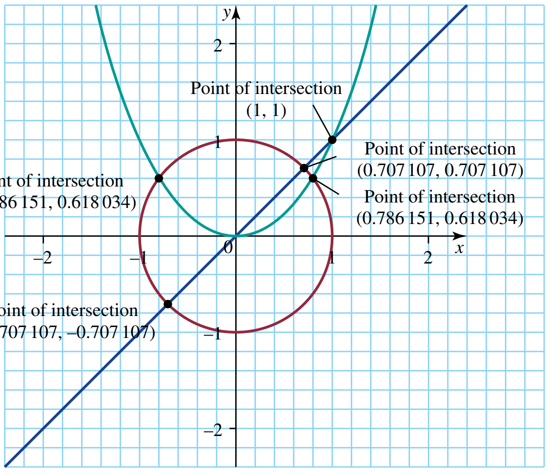
5. B

6. $(x - 5)^2 + (y - 3)^2 = 16$

7. a. 2 cm, 13.8 cm **b.** 3.9 cm/s

8. $(-2, 0)$

9. a.



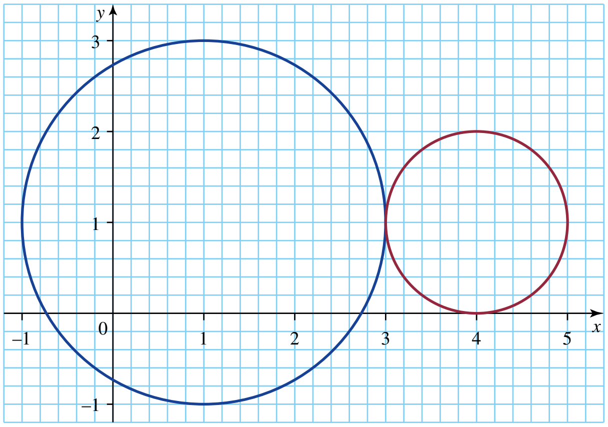
b. i. (0.707, 0.707) and (−0.707, −0.707)

ii. (0, 0) and (1, 1)

iii. (0.786, 0.618) and (−0.786, 0.618)

10. $(x-1)^2 + (y-1)^2 = 4$ centre at (1, 1) and radius of 2 units.
$(x-4)^2 + (y-1)^2 = 1$ centre at (4, 1) and radius of 1 unit.

The circles intersect (touch) at (3, 1).



11. $(x-2)^2 + (y-5)^2 = 4$ centre at (2, 5) and radius of 2 units.

Challenge 9.2

The distance between the centre and the point (1, 1.5) is 2.5, which is greater than the length of the radius. Therefore, the point (1, 1.5) lies outside the circle with equation $x^2 + 2x + y^2 - 6y + 6 = 0$.

9.9 Review

1. d

2. a

3. d

4. a

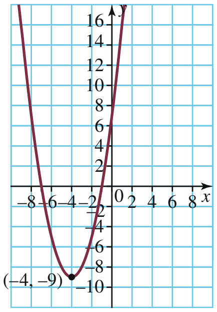
5. b

6. a. (4, −15)

b. (−2, −9)

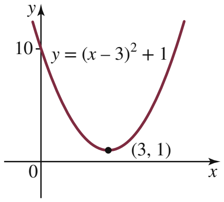
7.

x	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
y	16	7	0	−5	−8	−9	−8	−5	0	7	16

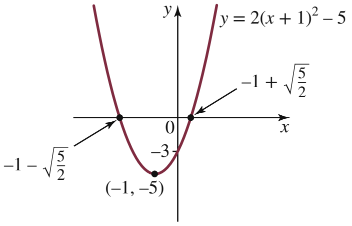


TP (−4, −9); x-intercepts: −7 and −1

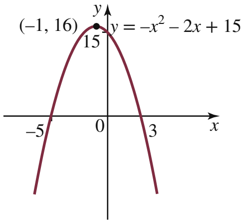
8. a. TP (3, 1); no x-intercepts; y-intercept: (0, 10)



b. TP (−1, −5); x-intercepts: $-1-\sqrt{\frac{5}{2}}, -1+\sqrt{\frac{5}{2}}$; y-intercept: (0, −3)



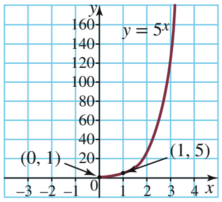
9. TP (−1, 16); x-intercepts: −5 and 3; y-intercept: (0, 15)



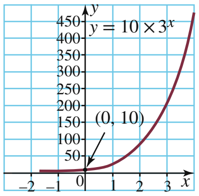
10. a.

x	−3	−2	−1	0	1	2	3
y	0.008	0.04	0.2	1	5	25	125

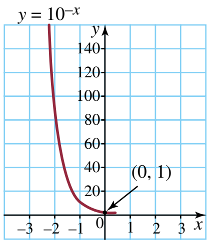
b.



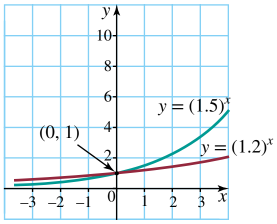
11.



12.

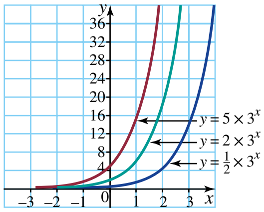


13. a.



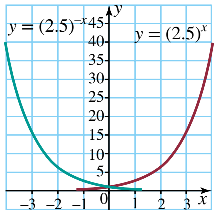
a. Increasing the value of a makes the graph steeper for positive x-values and flatter for negative x-values.

14. a.



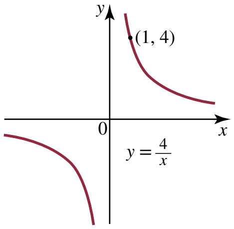
b. Increasing the value of k makes the graph steeper.

15. a.

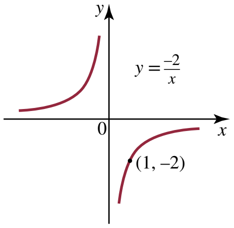


b. Changing the sign of the index reflects the graph in the y-axis.

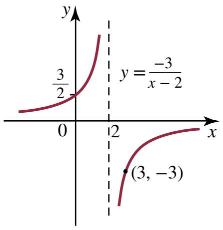
16. a.



b.

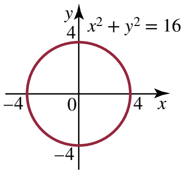


17.

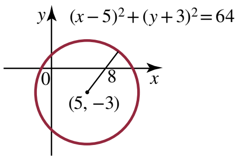


18. Check with your teacher. Possible answer is $y = \frac{1}{x + 3}$.

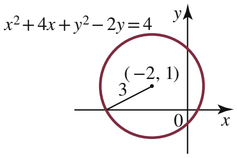
19. a.



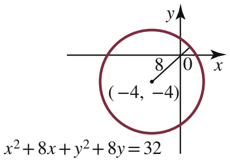
b.



20. a.

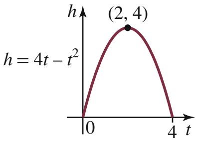


b.



21. $x^2 + y^2 = 36$

22. a.

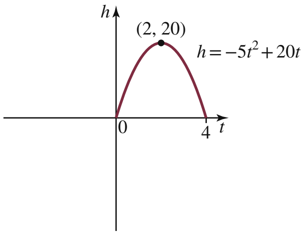


b. 4 m

c. 2 s

d. 4 s

23. a.



b. 4 s

c. 2 s ($1 > t > 3$)

d. The ball is never above a height of 20 m.

24. a. [0, 12]

b. 32 m

c. 11:41 am to 6:19 pm

25. a. $P \le -x^2 + 5x$, $0 \le x \le 5$

b. 6.25 m

c. i. Check with your teacher.

ii. Dilation by a factor of 0.56 parallel to the y-axis

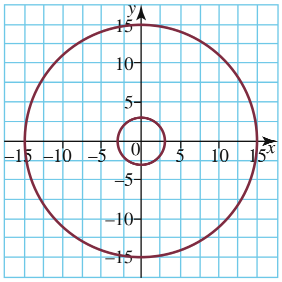
d. 66.7%

26. a. Check with your teacher.

b. When $x = 0.3$, $b = 10.7$. Therefore if p is greater than 10.7 cm the platform would hit the bridge.

27. a. First ripple's radius is 3 cm, second ripple's radius is 15 cm.

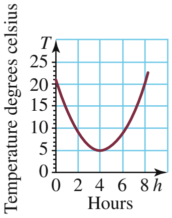
b.



c. 2.4 cm/s

d. 1 minute 22.1 seconds after it is dropped

28. a.



b. 21°C

c. Decreasing

d. Increasing

e. 5°C after 4 hours

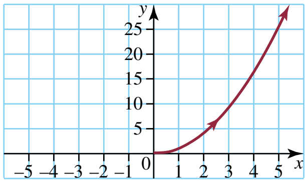
f. 21°C

Investigation — Rich task

1.

t	*x*	*y*
0	0	0
1	1	1
2	2	4
3	3	9
4	4	16
5	5	25

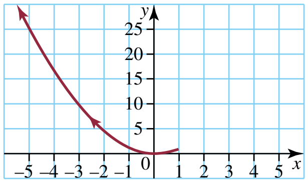
2.


Parametric equations
$x = t$ and $y = t^2$

3. Answers will vary.

4.

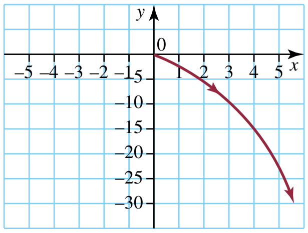
t	*x*	*y*
0	1	1
1	0	0
2	–1	1
3	–2	4
4	–3	9
5	–4	16


Parametric equations
$x = 1 - t$ and $y = (1 - t)^2$

Answers will vary; $t = 1, 0, -1, -2, -3, -4$.

5.

t	*x*	*y*
0	0	0
1	1	–1
2	2	–4
3	3	–9
4	4	–16
5	5	–25


Parametric equations
$x = t$ and $y = -t^2$

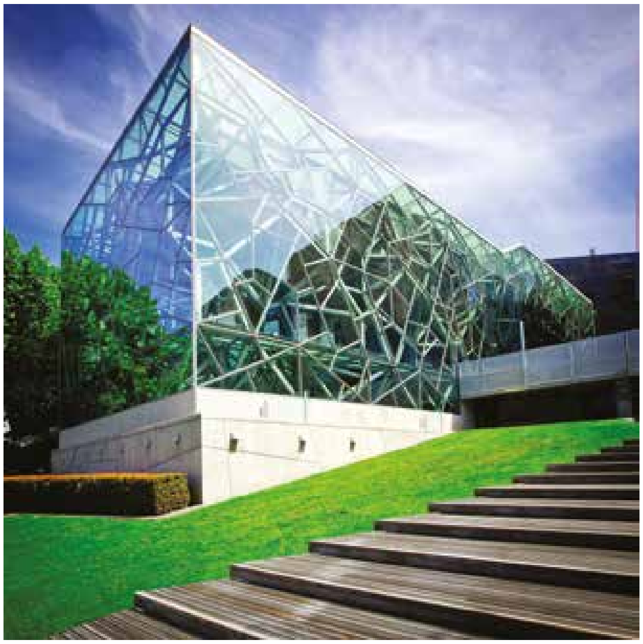
6. Answers will vary.

7. Answers will vary.

TOPIC 10
Deductive geometry

10.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.



10.1.1 Why learn this?

Learning about geometry includes being able to reason deductively and to prove logically that certain mathematical statements are true. It is important to be able to prove theories meticulously and step by step in order to show that the conclusions reached are soundly based. Mathematicians spend most of their time trying to prove new theories, and they rely heavily on all the proofs that have gone before. Reasoning skills, and hence the ability to prove theories, can be developed and learned through practice and application.

10.1.2 What do you know?

assesson

1. **THINK** List what you know about geometry. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of geometry.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this video: The story of mathematics: Euclid (eles-1849)

10.2 Angles, triangles and congruence

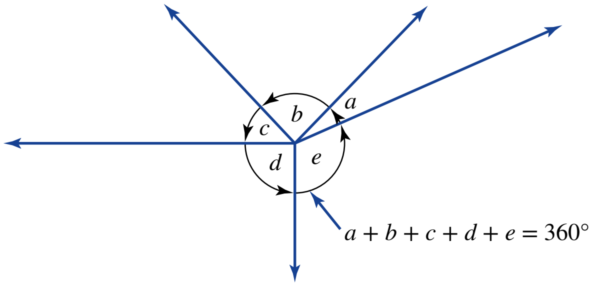
10.2.1 Proofs and theorems

- Euclid (c. 300 BC) was the mathematician who developed a systematic approach to geometry, now referred to as Euclidean geometry, that relied on mathematical proofs.
- A **proof** is an argument that shows why a statement is true.
- A **theorem** is a statement that can be demonstrated to be true. To demonstrate that a statement is proven, formal language needs to be used. It is conventional to use the following structure when setting out a theorem.
  - **Given**: a summary of the information given
  - **To prove**: a statement that needs to be proven
  - **Construction**: a description of any additions to the diagram given
  - **Proof**: a sequence of steps that can be justified and form part of a formal mathematical proof.

10.2.2 Angles at a point

- The sum of the angles at a point is 360°.

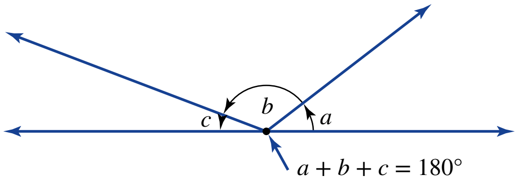
$$a + b + c + d + e = 360°$$



10.2.3 Supplementary angles

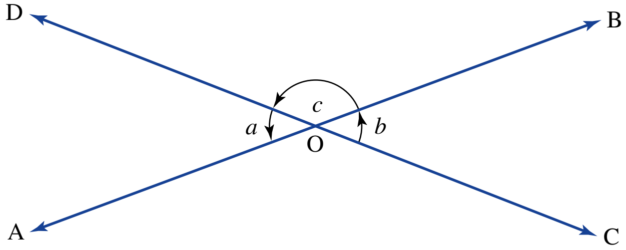
- The sum of the angles on a straight line is 180°.
- Angles a, b and c are **supplementary angles**.

$$a + b + c = 180°$$



10.2.4 Vertically opposite angles

- **Theorem 1: Vertically opposite angles** are equal.



Given: Straight lines AB and CD intersect at O.

To prove: $\angle AOD = \angle BOC$ and $\angle BOD = \angle AOC$

Construction: Label $\angle AOD$ as a, $\angle BOC$ as b and $\angle BOD$ as c.

Proof: Let $\angle AOD = a°$, $\angle BOC = b°$ and $\angle BOD = c°$.

$a + c = 180°$ (supplementary angles)

$b + c = 180°$ (supplementary angles)

$\therefore\ a + c = b + c$

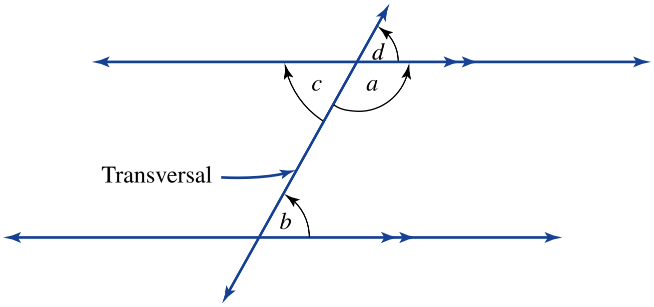
$\therefore\ a = b$

So, $\angle AOD = \angle BOC$.

Similarly, $\angle BOD = \angle AOC$.

10.2.5 Parallel lines

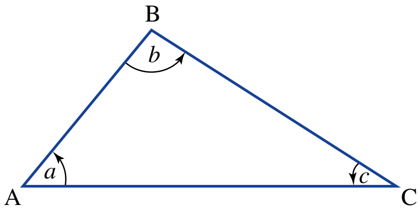
- If two lines are parallel and cut by a **transversal**, then:
  - co-interior angles are supplementary. For example, $a + b = 180°$.
  - corresponding angles are equal. For example, $b = d$.
  - alternate angles are equal. For example, $b = c$.
  - opposite angles are equal. For example, $c = d$.



10.2.6 Angle properties of triangles

Theorem 2

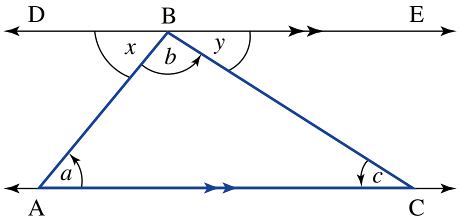
- **Theorem 2:** The sum of the interior angles of a triangle is 180°.



Given: ΔABC with interior angles a, b and c

To prove: $a + b + c = 180°$

Construction: Draw a line parallel to AC, passing through B and label it DE as shown. Label $\angle ABD$ as x and $\angle CBE$ as y.



Proof:

$a = x$ (alternate angles)

$c = y$ (alternate angles)

$x + b + y = 180°$ (supplementary angles)

$\therefore\ a + b + c = 180°$

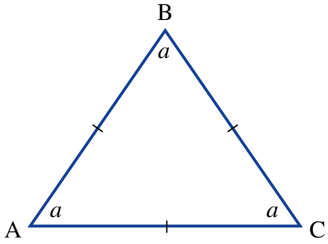
10.2.7 Equilateral triangles

- It follows from Theorem 2 that each interior angle of an **equilateral triangle** is 60°, and, conversely, if the three angles of a triangle are equal, then the triangle is equiangular.

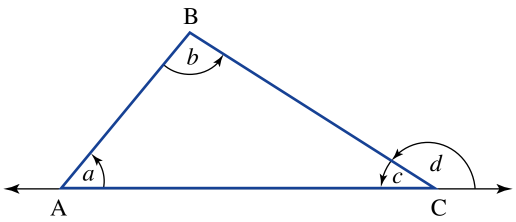
$a + a + a = 180°$ (sum of interior angles in a triangle is 180°)

$3a = 180°$

$a = 60°$



- **Theorem 3:** The exterior angle of a triangle is equal to the sum of the opposite interior angles.



Given: ΔABC with the exterior angle labelled d

To prove: $d = a + b$

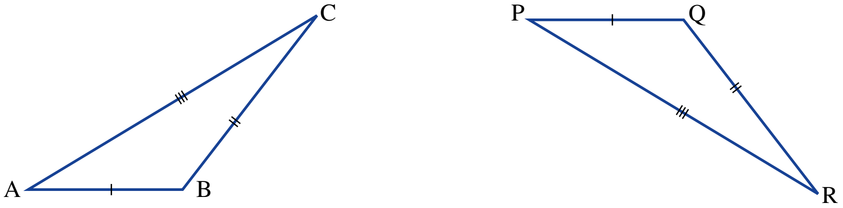
Proof: $c + d = 180°$ (supplementary angles)

$a + b + c = 180°$ (sum of interior angles in a triangle is 180°)

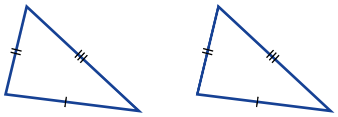
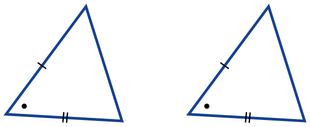
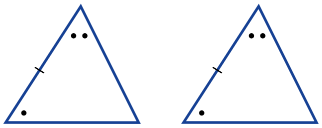
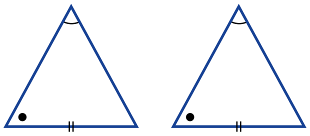
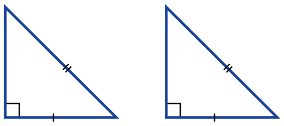
$\therefore d = a + b$

10.2.8 Congruent triangles

- **Congruent triangles** have the same size and the same shape; that is, they are identical in all respects.
- The symbol used for congruency is $\cong$.
- For example, ΔABC in the diagram below is congruent to ΔPQR. This is written as $\Delta ABC \cong \Delta PQR$.



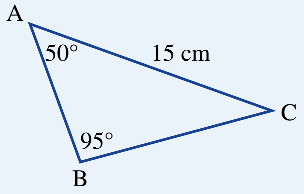
- Note that the vertices of the two triangles are written in corresponding order.
- There are five tests designed to check whether triangles are congruent. The tests are summarised in the table below.

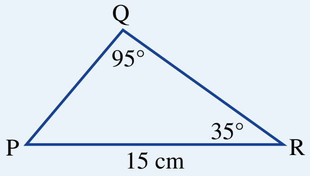
Test	Diagram	Abbreviation
All three sides in one triangle are equal in length to the corresponding sides in the other triangle.		SSS
Two corresponding sides and the included angle are the same in both triangles.		SAS
Two corresponding angles and a pair of corresponding sides are the same in both triangles.		ASA
A pair of corresponding angles and a non-included side are equal in both triangles.		AAS
The hypotenuse and one pair of the other corresponding sides in two right-angled triangles are the same in two right-angled triangles.		RHS

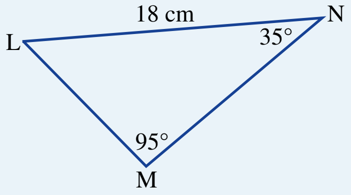
- In each of the tests we need to show three equal measurements about a pair of triangles in order to show they are congruent.

WORKED EXAMPLE 1

Select a pair of congruent triangles from the diagrams below, giving a reason for your answer.



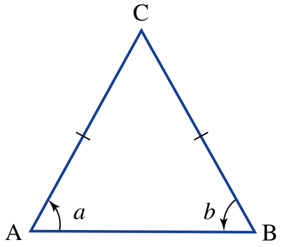




THINK	WRITE
1 In each triangle the length of the side opposite the 95° angle is given. If triangles are to be congruent, the sides opposite the angles of equal size must be equal in length. Draw your conclusion.	All three triangles have equal angles, but the sides opposite the angle 95° are not equal. $AC = PR = 15$ and $LN = 18$ cm
2 To test whether ΔABC is congruent to ΔPQR, first find the angle C.	ΔABC: $\angle A = 50°$, $\angle B = 95°$, $\angle C = 180° - 50° - 95°$ $= 35°$
3 Apply a test for congruence. Triangles ABC and PQR have a pair of corresponding sides equal in length and 2 pairs of angles the same, so draw your conclusion.	A pair of corresponding angles ($\angle B = \angle Q$ and $\angle C = \angle R$) and a non-included side ($AP = PR$) are equal. $\Delta ABC \cong \Delta PQR$ (AAS)

10.2.9 Isosceles triangles

- A triangle is isosceles if the lengths of two sides are equal but the third side is not equal.



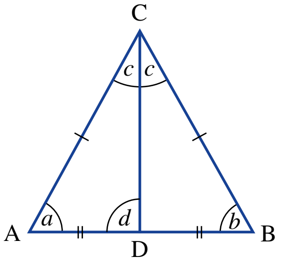
Theorem 4

- **Theorem 4**: The angles at the base of an **isosceles triangle** are equal.

Given: $AC = CB$

To prove: $\angle BAC = \angle CBA$

Construction: Draw a line from the vertex C to the midpoint of the base AB and label the midpoint D. CD is the bisector of $\angle ACB$.



Proof: In ΔACD and ΔBCD,

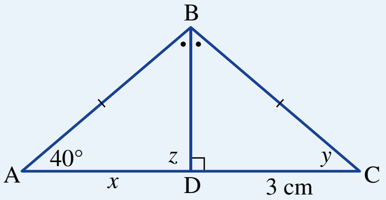
$CD = CD$	(common side)
$AD = DB$	(construction, D is the midpoint of AB)
$AC = CB$	(given)
$\Rightarrow \Delta ACD \cong \Delta BCD$	(SSS)
$\therefore \angle BAC = \angle CBA$	

- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.
- It also follows that $\angle ADC = \angle BDC = d$

  and that $$\begin{aligned} 2d &= 180^\circ \\ \Rightarrow d &= 90^\circ \end{aligned} \quad \text{(supplementary)}$$

WORKED EXAMPLE 2

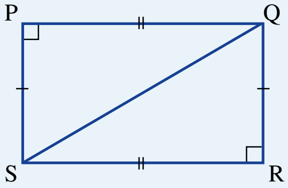
Given that $\Delta ABD \cong \Delta CBD$, find the values of the pronumerals in the figure below.



THINK	WRITE
1 In congruent triangles corresponding sides are equal in length. Side AD (marked x) corresponds to side DC, so state the value of x.	$\Delta ABD \cong \Delta CBD$ $AD = CD, \ AD = x, \ CD = 3$ So $x = 3$ cm.
2 Since the triangles are congruent, corresponding angles are equal. State the angles corresponding to y and z and hence find the values of these pronumerals.	$\angle BAD = \angle BCD$ $\angle BAD = 40^\circ, \ \angle BCD = y$ So $y = 40^\circ$ $\angle BDA = \angle BDC$ $\angle BDA = z, \ \angle BDC = 90^\circ$ So $z = 90^\circ$.

WORKED EXAMPLE 3

Prove that ΔPQS is congruent to ΔRSQ.



THINK	WRITE
1 Write the information given.	Given: Rectangle PQRS with diagonal QS.
2 Write what needs to be proved.	To prove: that ΔPQS is congruent to ΔRSQ. $QP = SR$ (given) $\angle SPQ = \angle SRQ = 90^\circ$ (given) QS is common.
3 Select the appropriate congruency test for proof. (In this case it is RHS because the triangles have an equal side, a right angle and a common hypotenuse.)	So $\Delta PQS \cong \Delta RSQ$ (RHS)

learnON RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Naming angles, lines and figures (doc-5276)

Complete this digital doc: SkillSHEET: Corresponding sides and angles of congruent triangles (doc-5277)

Complete this digital doc: SkillSHEET: Angles and parallel lines (doc-5280)

Exercise 10.2 Angles, triangles and congruence

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–5, 7, 9, 11	Questions: 1–5, 6, 8–10, 12, 13	Questions: 1–14

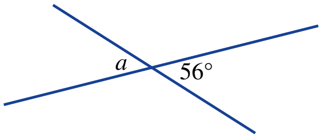
🟩🟦🟪 **Individual pathway interactivity:** int-4612 **learnON** ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

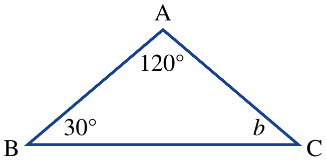
Fluency

1. Determine the values of the unknown in each of the following.

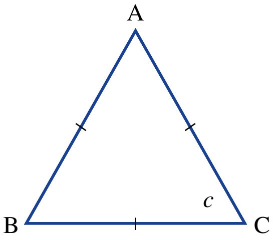
a.



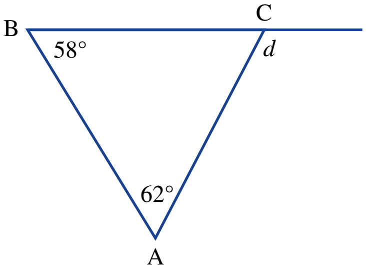
b.



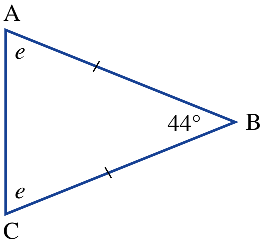
c.



d.

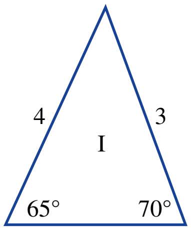


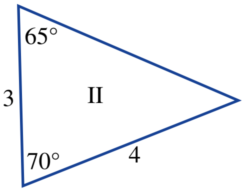
e.

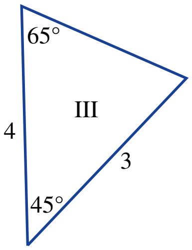


2. **WE1** Select a pair of congruent triangles in each of the following, giving a reason for your answer. All side lengths are in cm.

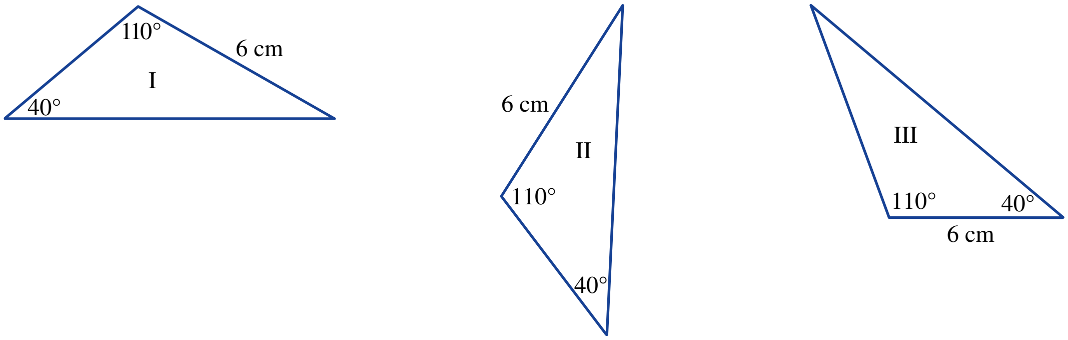
a.



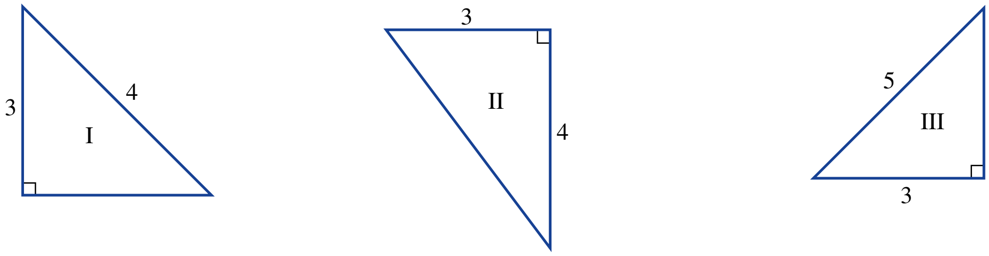




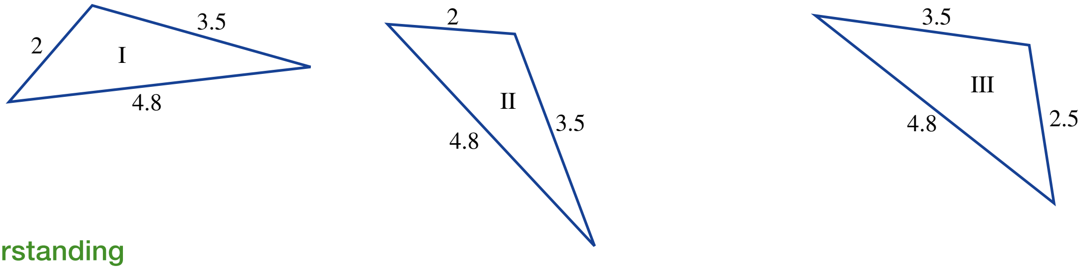
b.



c.



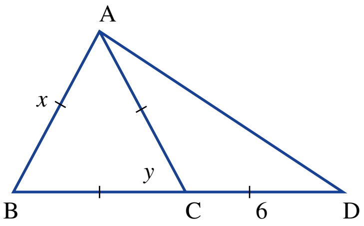
d.



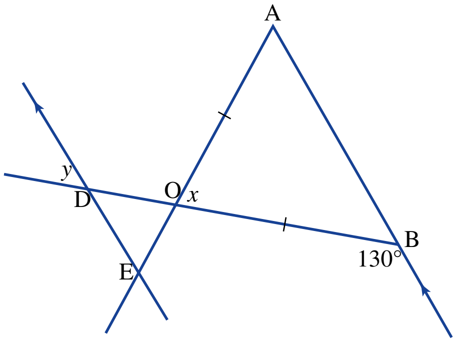
Understanding

3. Find the missing values of x and y in each of the following diagrams. Give reasons for your answers.

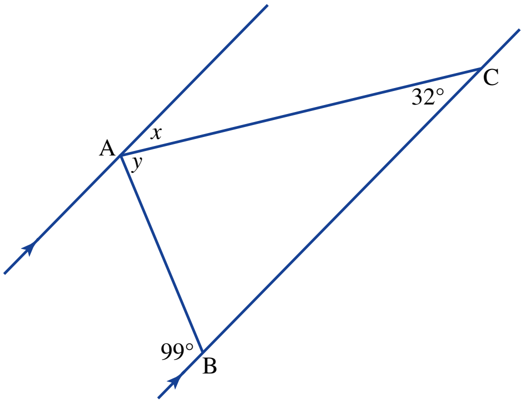
a.



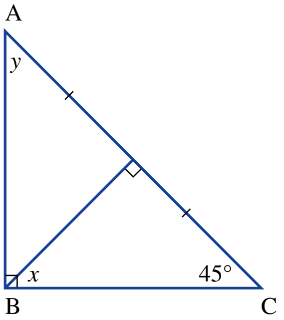
b.



c.

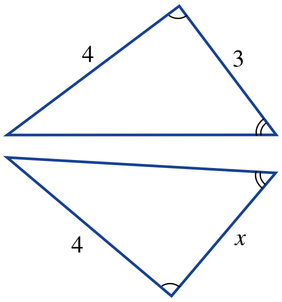


d.

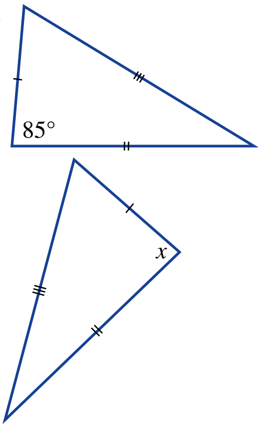


4. **WE2** Find the value of the pronumeral in each of the following pairs of congruent triangles. All side lengths are in cm.

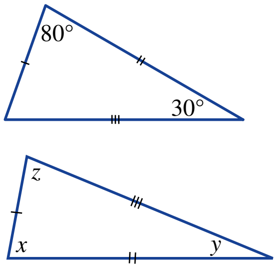
a.



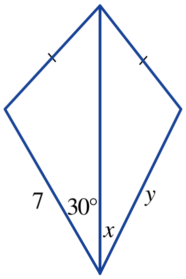
b.



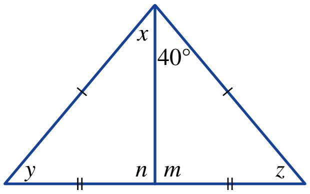
c.



d.



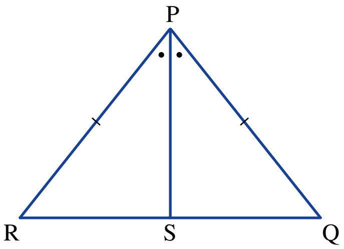
e.



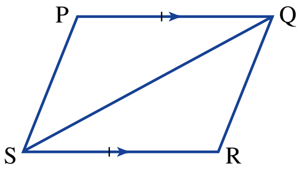
Reasoning

5. **WE3** Prove that each of the following pairs of triangles are congruent.

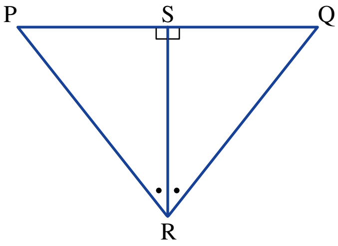
a.



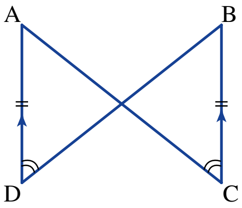
b.



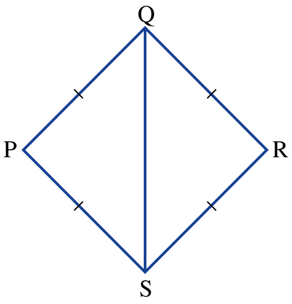
c.



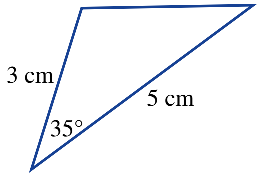
d.



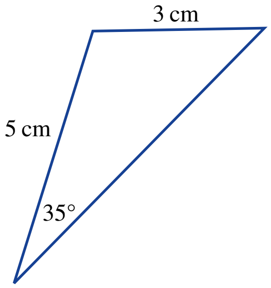
e.



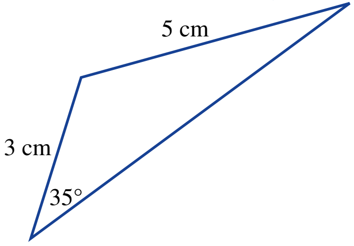
6. **MC** *Note:* There may be more than one correct answer.
Which of the following is congruent to the triangle shown at right?



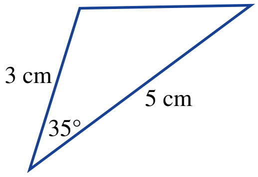
A.



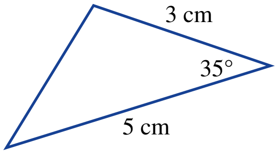
B.



C.

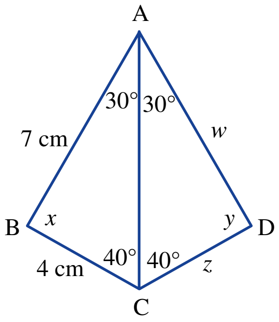


D.

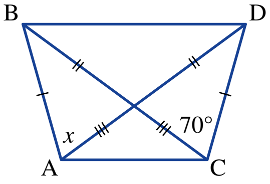


7. Prove that ΔABC ≅ ΔADC and hence find the values of the pronumerals in each of the following.

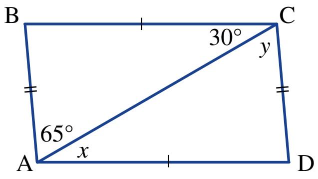
a.



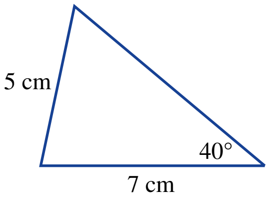
b.

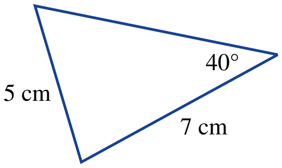


c.

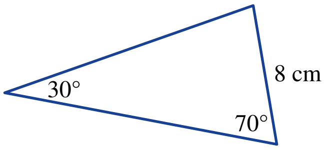


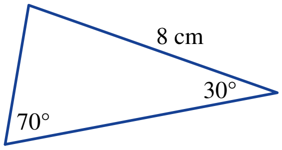
8. Explain why the triangles shown below are not necessarily congruent.



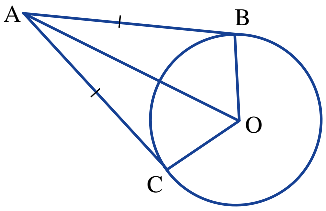


9. Explain why the triangles shown below are not congruent.

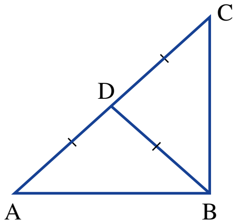




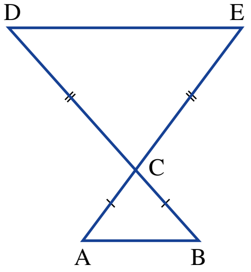
10. Show that ΔABO ≅ ΔACO, if O is the centre of the circle.



11. If DA = DB = DC, prove that ∠ABC is a right angle.

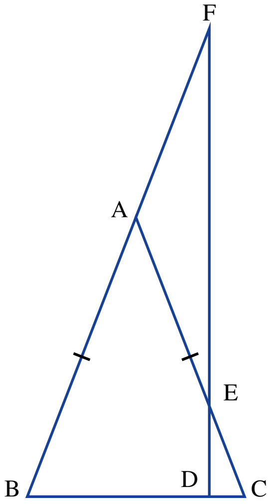


12. If AC = CB and DC = CE in the diagram shown, prove that AB ∥ DE.

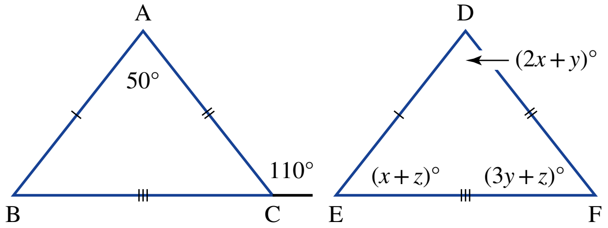


Problem solving

13. ABC is an isosceles triangle in which AB and AC are equal in length. BDF is a right-angled triangle. Show that triangle AEF is an isosceles triangle.



14. Triangles ABC and DEF are congruent.



Find the values of x, y and z.

Reflection

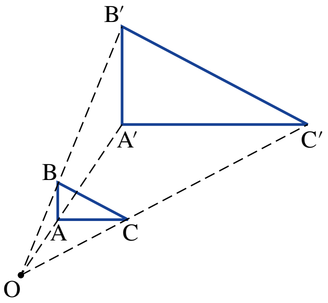
How can you be certain that two figures are congruent?

10.3 Similar triangles

10.3.1 Similar figures

- Two geometric shapes are **similar** when one is an **enlargement** or reduction of the other shape.

  - An enlargement increases the length of each side of a figure in all directions by the same factor. For example, in the diagram shown, triangle A′B′C′ is an enlargement of triangle ABC by a factor of 3 from its **centre of enlargement** at O.



- The symbol for similarity is ~ and is read as 'is similar to'.
- The **image** of the original object is the enlarged or reduced shape.
- To create a similar shape, use a **scale factor** to enlarge or reduce the original shape.
- The scale factor can be found using the formula below and the lengths of a pair of corresponding sides.

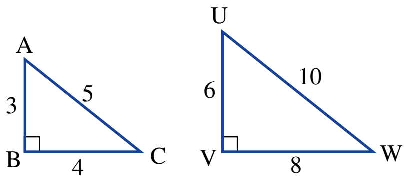
$$\text{Scale factor} = \frac{\text{image side length}}{\text{object side length}}$$

- If the scale factor is less than 1, the image is a reduced version of the original shape. If the scale factor is greater than 1, the image is an enlarged version of the original shape.

10.3.2 Similar triangles

- Two triangles are similar if:
  - the angles are equal, or
  - the corresponding sides are proportional.

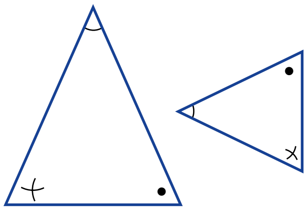
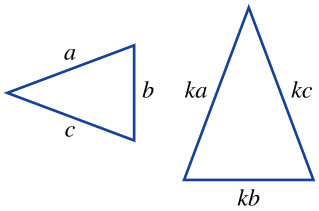
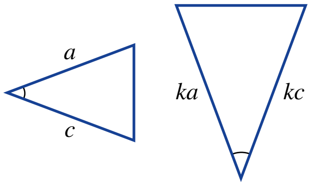
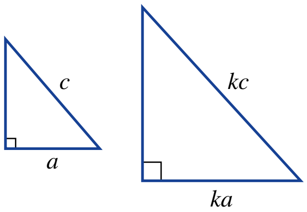
- Consider the pair of **similar triangles** below.



- The following statements are true for these triangles.
  - Triangle UVW is similar to triangle ABC or, using symbols, $\Delta UVW \sim \Delta ABC$.
  - The corresponding angles of the two triangles are equal in size: $\angle CAB = \angle WUV$, $\angle ABC = \angle UVW$ and $\angle ACB = \angle UWV$.
  - The corresponding sides of the two triangles are in the same ratio. $\frac{UV}{AB} = \frac{VW}{BC} = \frac{UW}{AC} = 2$; that is, ΔUVW has each of its sides twice as long as the corresponding sides in ΔABC.
  - The scale factor is 2.

10.3.3 Testing triangles for similarity

- Triangles can be checked for similarity using one of the tests described in the table below.

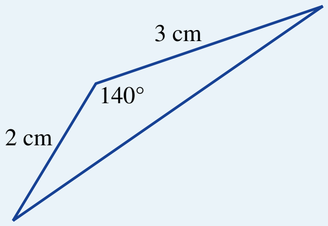
Test	Diagram	Abbreviation
Two angles of a triangle are equal to two angles of another triangle. This implies that the third angles are equal, as the sum of angles in a triangle is 180°.		AAA
The three sides of a triangle are proportional to the three sides of another triangle.		SSS
Two sides of a triangle are proportional to two sides of another triangle, and the included angles are equal.		SAS
The hypotenuse and a second side of a right-angled triangle are proportional to the hypotenuse and a second side of another right-angled triangle.		RHS

- *Note:* When using the equiangular test, only two corresponding angles have to be checked. Since the sum of the interior angles in any triangle is a constant number (180°), the third pair of corresponding angles will automatically be equal, provided that the first two pairs match exactly.

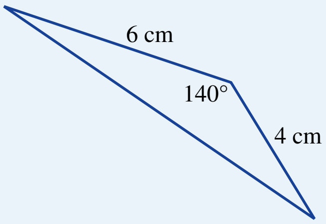
WORKED EXAMPLE 4

Find a pair of similar triangles among those shown. Give a reason for your answer.

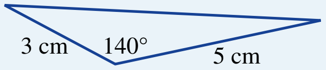
a



b



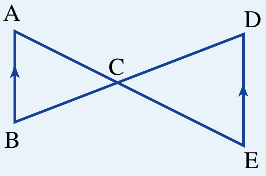
c



THINK	WRITE
1 In each triangle the lengths of two sides and the included angle are known, so the SAS test can be applied. Since all included angles are equal (140°), we need to the find ratios of corresponding sides, taking two triangles at a time.	For triangles **a** and **b**: $\frac{6}{3} = \frac{4}{2} = 2$ For triangles **a** and **c**: $\frac{5}{3} = 1.6, \frac{3}{2} = 1.5$ For triangles **b** and **c**: $\frac{5}{6} = 0.83, \frac{3}{4} = 0.75$
2 Only triangles **a** and **b** have corresponding sides in the same ratio (and included angles are equal). State your conclusion, specifying the similarity test that has been used.	Triangle **a** ~ triangle **b** (SAS)

WORKED EXAMPLE 5

Prove that ΔABC is similar to ΔEDC.



THINK	WRITE
1 Write the information given. AB is parallel to DE. Transversal BD forms two alternate angles: ∠ABC and ∠EDC.	**1** Given: ΔABC and ΔDCE AB ∥ DE C is common.
2 Write what is to be proved.	**2** To prove: ΔABC ~ ΔEDC
3 Write the proof.	**3** Proof: ∠ABC = ∠EDC (alternate angles) ∠BAC = ∠DEC (alternate angles) ∠BCA = ∠DCE (vertically opposite angles) ∴ ΔABC ~ ΔEDC (equiangular, AAA)

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Writing similarity statements (doc-5278)

Complete this digital doc: SkillSHEET: Calculating unknown side lengths in a pair of similar triangles (doc-5281)

Complete this digital doc: WorkSHEET: Deductive geometry I (doc-5282)

Exercise 10.3 Similar triangles

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–10	Questions: 1–11, 15	Questions: 1–15

🟩🟦🟪 **Individual pathway interactivity:** int-4613

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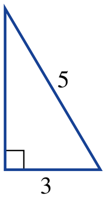
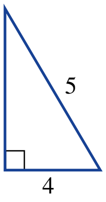
Fluency

1. **WE4** Find a pair of similar triangles among those shown in each part. Give a reason for your answer.

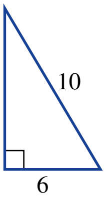
a. i.



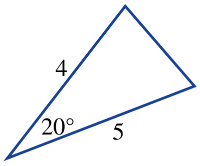
ii.



iii.



b. i.



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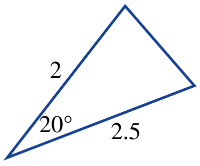

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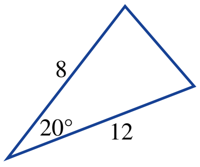
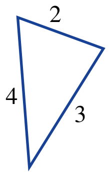


c. i.



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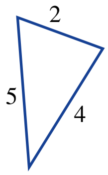

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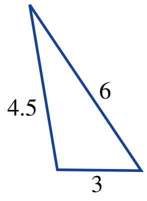
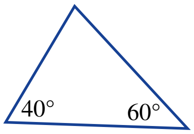


d. i.



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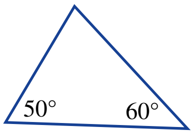
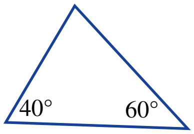
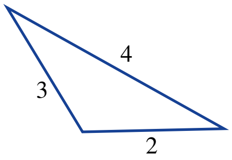


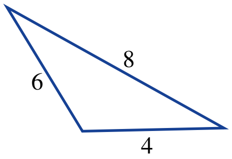
iii.



e. i.



ii.



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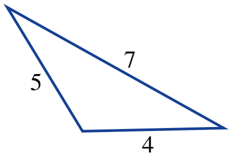
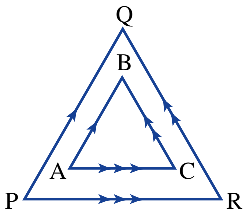


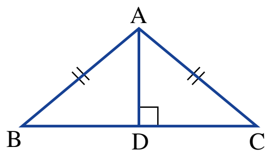
Understanding

2. Name two similar triangles in each of the following figures.

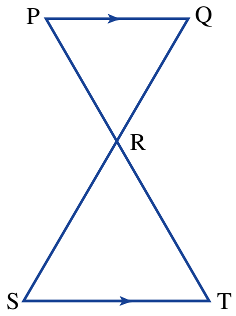
a.



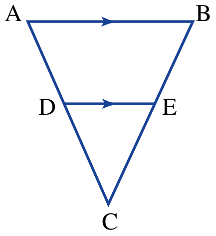
b.



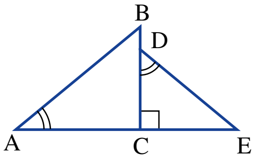
c.



d.

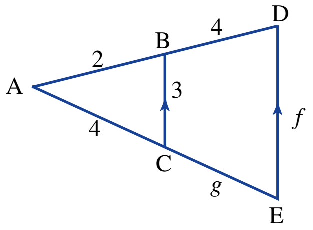


e.

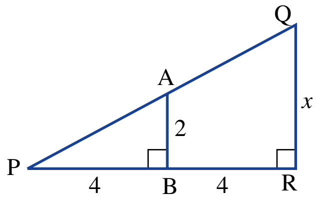


3. a. Complete this statement: $\frac{AB}{AD} = \frac{BC}{\_\_\_} = \frac{\_\_\_}{AE}$.

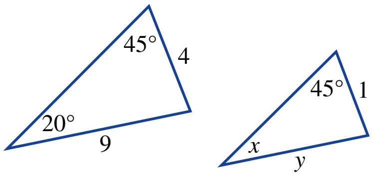
b. Find the value of the pronumerals.



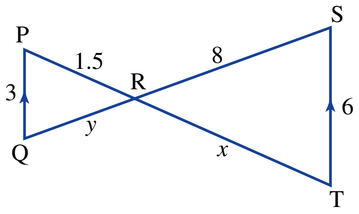
4. Find the value of the pronumeral in the diagram at right.



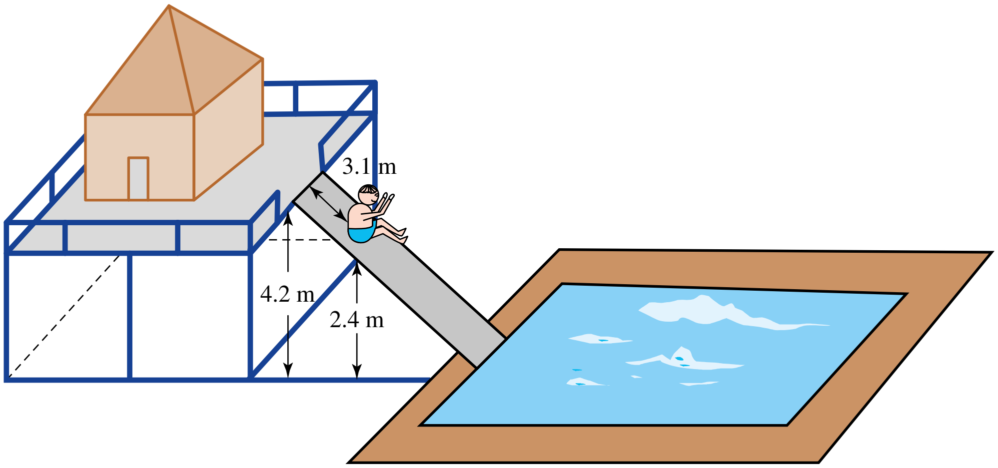
5. The triangles shown at right are similar.
Find the value of x and y.



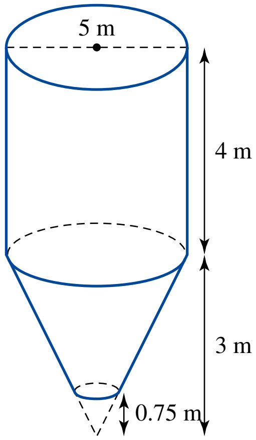
6. a. State why these two triangles shown at right are similar.
b. Find the values of x and y in the diagram.



7. A waterslide is 4.2 m high and has a support 2.4 m tall. If a student reaches this support when she is 3.1 m down the slide, how long is the slide?

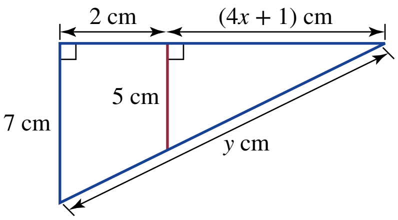


8. A storage tank as shown in the diagram is made of a 4-m-tall cylinder joined by a 3-m-tall cone. If the diameter of the cylinder is 5 m, what is the radius of the end of the cone if 0.75 m has been cut off the tip?

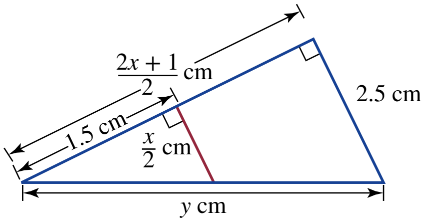


9. Calculate the values of the pronumerals.

a.



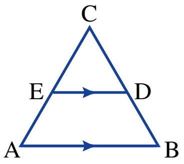
b.



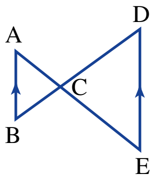
Reasoning

10. **WE5** Prove that ΔABC is similar to ΔEDC in each of the following.

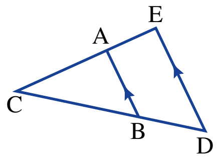
a.



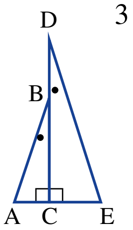
b.



c.



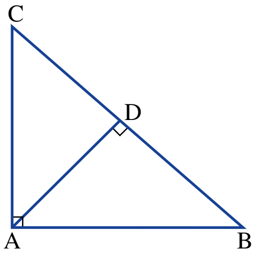
d.



11. ΔABC is a right-angled triangle. A line is drawn from A to D as shown so that AD ⊥ BC. Prove that:

a. ΔABD ~ ΔACB

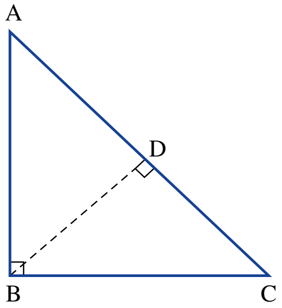
b. ΔACD ~ ΔACB.



12. Explain why the AAA test cannot be used to prove congruence but can be used to prove similarity.

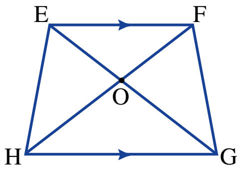
13. a. Prove Pythagoras' theorem, $AC^2 = AB^2 + BC^2$, using similar triangles.

b. Show that the converse of Pythagoras' theorem holds true; that is, if the square on one side of a triangle equals the sum of the squares on the other two sides, then the angle between these other two sides is a right angle.

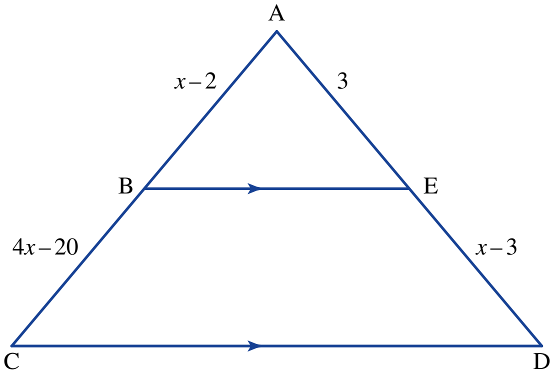


Problem solving

14. Prove that $\Delta EFO \sim \Delta GHO$.



15. Solve for x.

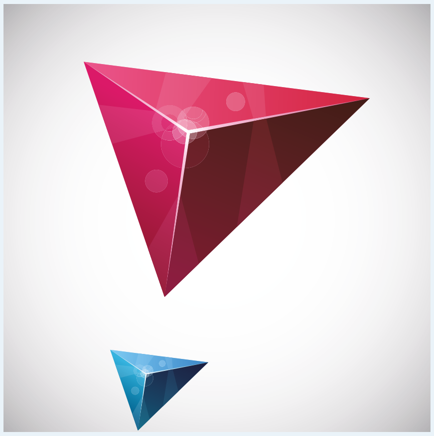


Reflection

How can you be certain that two figures are similar?

CHALLENGE 10.1

A tetrahedron (regular triangular-based pyramid) has an edge length of 2 cm. A similar tetrahedron has a total surfce area of $36\sqrt{3}$ cm^2. What is the scale factor relationship between the side lengths of the two tetrahedra?



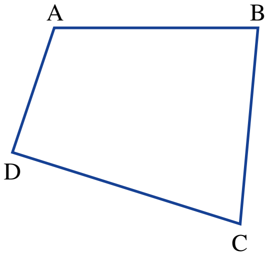
10.4 Quadrilaterals

10.4.1 Quadrilaterals

- Quadrilaterals are four-sided plane shapes whose interior angles sum to 360°.

Theorem 5

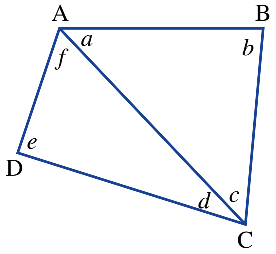
- **Theorem 5:** The sum of the interior angles in a quadrilateral is 360°.



Given: A quadrilateral ABCD

To prove: $\angle ABC + \angle BCD + \angle ADC + \angle BAD = 360°$

Construction: Draw a line joining vertex A to vertex C. Label the interior angles of the triangles formed.



Proof: $a + b + c = 180°$ (sum of interior angles in a triangle is 180°)

$d + e + f = 180°$ (sum of interior angles in a triangle is 180°)

$\Rightarrow a + b + c + d + e + f = 360°$

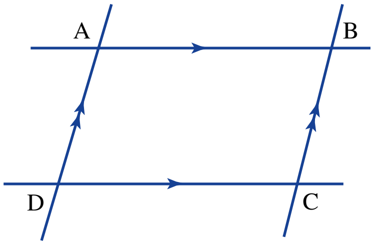
$\therefore \angle ABC + \angle BCD + \angle ADC + \angle BAD = 360°$

10.4.2 Parallelograms

- A **parallelogram** is a quadrilateral with two pairs of parallel sides.

Theorem 6

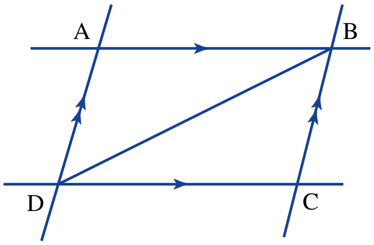
- **Theorem 6:** Opposite angles of a parallelogram are equal.



Given: $AB \parallel DC$ and $AD \parallel BC$

To prove: $\angle ABC = \angle ADC$

Construction: Draw a diagonal from B to D.



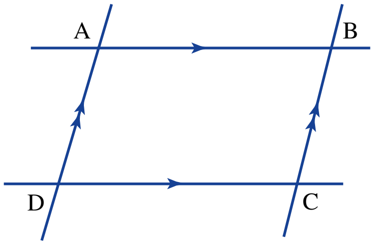
Proof:

$\angle ABD = \angle BDC$ (alternate angles)
$\angle ADB = \angle CBD$ (alternate angles)
$\angle ABC = \angle ABD + \angle CBD$ (by construction)
$\angle ADC = \angle BDC + \angle ADB$ (by construction)
$\therefore \angle ABC = \angle ADC$

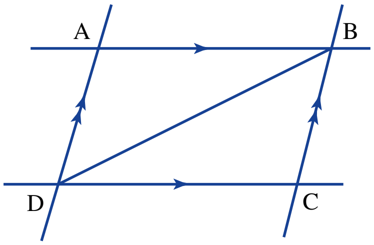
- Conversely, if each pair of opposite angles of a quadrilateral is equal then it is a parallelogram.

10.4.3 Theorem 7

- **Theorem 7:** Opposite sides of a parallelogram are equal.



Given: $AB \parallel DC$ and $AD \parallel BC$
To prove: $AB = DC$
Construction: Draw a diagonal from B to D.



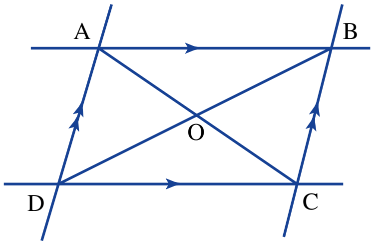
Proof:

$\angle ABD = \angle BDC$ (alternate angles)
$\angle ADB = \angle CBD$ (alternate angles)
BD is common to ΔABD and ΔBCD.
$\Rightarrow \Delta ABD \cong \Delta BCD$ (ASA)
$\therefore AB = DC$

- Conversely, if each pair of opposite sides of a quadrilateral is equal then it is a parallelogram.

10.4.4 Theorem 8

- **Theorem 8:** The diagonals of a parallelogram bisect each other.



Given: $AB \parallel DC$ and $AD \parallel BC$ with diagonals AC and BD

To prove: $AO = OC$ and $BO = OD$

Proof: In ΔAOB and ΔCOD,

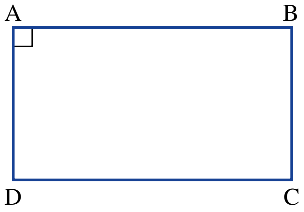
$\angle OAB = \angle OCD$ (alternate angles)
$\angle OBA = \angle ODC$ (alternate angles)
$AB = CD$ (opposite sides of a parallelogram)
$\Rightarrow \Delta AOB \cong \Delta COD$ (ASA)
$\Rightarrow AO = OC$ (corresponding sides in congruent triangles)
and $BO = OD$ (corresponding sides in congruent triangles)

10.4.5 Rectangles

- A rectangle is a parallelogram with four right angles.

Theorem 9

- **Theorem 9:** A parallelogram with a right angle is a rectangle.



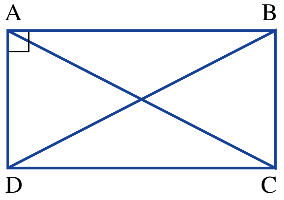
Given: Parallelogram ABCD with $\angle BAD = 90°$

To prove: $\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90°$

Proof: $AB \parallel CD$ (properties of a parallelogram)
$\Rightarrow \angle BAD + \angle ADC = 180°$ (co-interior angles)
But $\angle BAD = 90°$ (given)
$\Rightarrow \angle ADC = 90°$
Similarly, $\angle BCD = \angle ADC = 90°$
$\therefore \angle BAD = \angle ABC = \angle BCD = \angle ADC = 90°$

10.4.6 Theorem 10

- **Theorem 10:** The diagonals of a rectangle are equal.



Given: Rectangle ABCD with diagonals AC and BD

To prove: $AC = BD$

Proof: In ΔADC and ΔBCD,

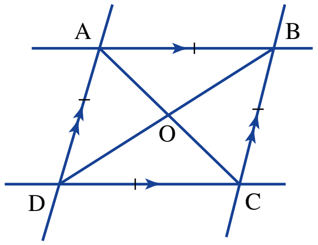
$AD = BC$ (opposite sides equal in a rectangle)
$DC = CD$ (common)
$\angle ADC = \angle BCD = 90°$ (right angles in a rectangle)
$\Rightarrow \Delta ADC \cong \Delta BCD$ (SAS)
$\therefore AC = BD$

10.4.7 Rhombuses

- A **rhombus** is a parallelogram with four equal sides.

Theorem 11

- **Theorem 11:** The diagonals of a rhombus are perpendicular.



Given: Rhombus ABCD with diagonals AC and BD

To prove: $AC \perp BD$

Proof: In ΔAOB and ΔBOC,

$AO = OC$ (property of parallelogram)
$AB = BC$ (property of rhombus)
$BO = OB$ (common)
$\Rightarrow \Delta AOB \cong \Delta BOC$ (SSS)
$\Rightarrow \angle AOB = \angle BOC$

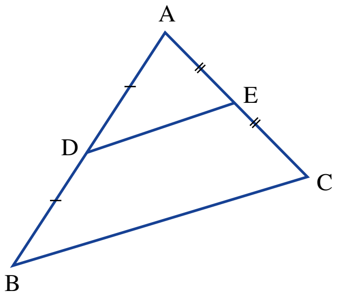
But $\angle AOB + \angle BOC = 180°$ (supplementary angles)
$\Rightarrow \angle AOB = \angle BOC = 90°$
Similarly, $\angle AOD = \angle DOC = 90°$.
Hence, $AC \perp BD$

10.4.8 The midpoint theorem

- Now that the properties of quadrilaterals have been explored, the midpoint theorem can be tackled.

Theorem 12

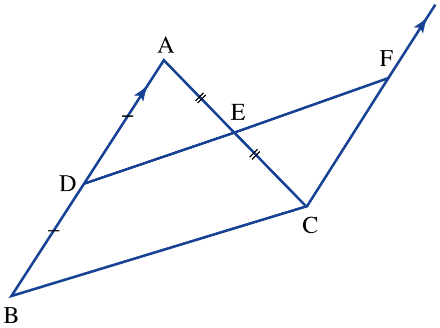
- **Theorem 12:** The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.



Given: ΔABC in which $AD = DB$ and $AE = EC$

To prove: $DE \parallel BC$ and $DE = \frac{1}{2}BC$

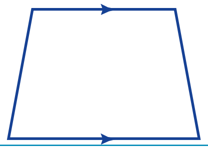
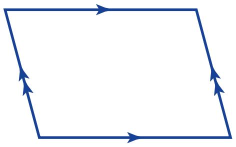
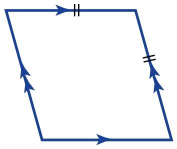
Construction: Draw a line through C parallel to AB. Extend DE to F on the parallel line.

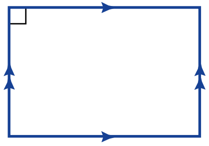
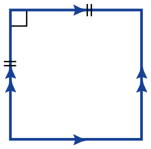


Proof:

In ΔADE and ΔCEF,	
$AE = EC$	(E is the midpoint of AC, given)
$\angle AED = \angle CEF$	(vertically opposite angles)
$\angle EAD = \angle ECF$	(alternate angles)
$\Rightarrow \Delta ADE \cong \Delta CEF$	(ASA)
$\therefore AD = CF$ and $DE = EF$	(corresponding sides in congruent triangles)
So, $AD = DB = CF$.	
We have $AB \parallel CF$	(by construction)
So BDFC is a parallelogram.	
$\Rightarrow DE \parallel BC$	
Also, $BC = DF$	(opposite sides in parallelogram)
But $DE = DF$	(sides in congruent triangles)
$\Rightarrow DE = \frac{1}{2}BC$	
Therefore, $DE \parallel BC$ and $DE = \frac{1}{2}BC$.	

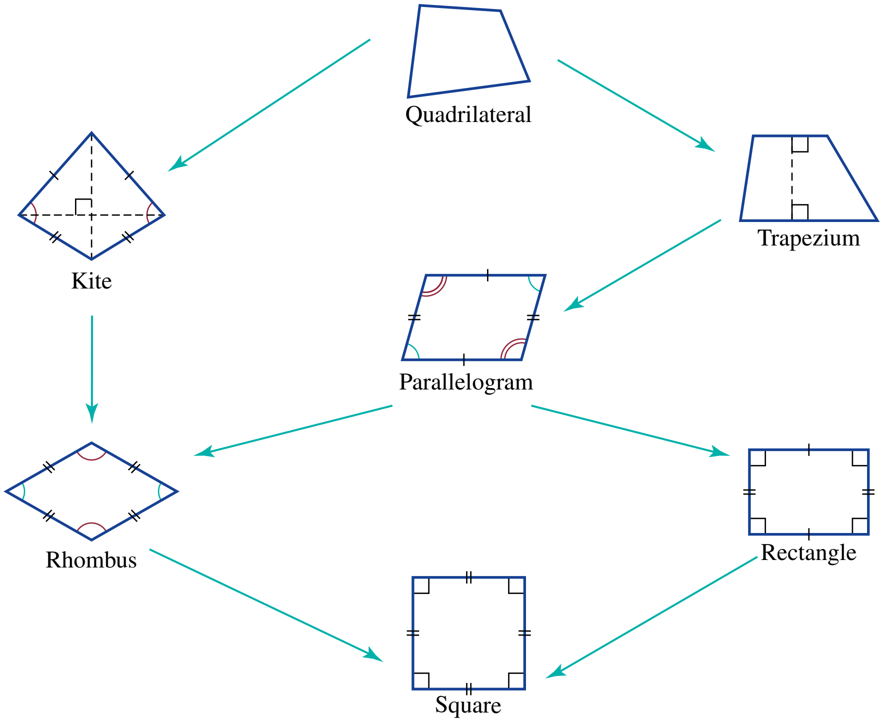
- Conversely, if a line interval is drawn parallel to a side of a triangle and half the length of that side, then the line interval bisects each of the other two sides of the triangle.
- A summary of the definitions and properties of quadrilaterals is shown in the table.

Shape	Definition	Properties
Trapezium 	A trapezium is a quadrilateral with one pair of opposite sides parallel.	One pair of opposite sides is parallel but not equal in length.
Parallelogram 	A parallelogram is a quadrilateral with both pairs of opposite sides parallel.	• Opposite angles are equal. • Opposite sides are equal. • Diagonals bisect each other.
Rhombus 	A rhombus is a parallelogram with four equal sides.	• Diagonals bisect each other at right angles. • Diagonals bisect the angles at the vertex through which they pass.

Shape	Definition	Properties
Rectangle 	A rectangle is a parallelogram whose interior angles are right angles.	Diagonals are equal. Diagonals bisect each other.
Square 	A square is a parallelogram whose interior angles are right angles with four equal sides.	• All angles are right angles. • All side lengths are equal. • Diagonals are equal in length and bisect each other at right angles. • Diagonals bisect the vertex through which they pass (45°).

10.4.9 Relationships between quadrilaterals

- The flowchart below shows the relationships between quadrilaterals.



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Try out this interactivity: Quadrilateral definitions (int-2786)

Complete this digital doc: SkillSHEET: Identifying quadrilaterals (doc-5279)

Complete this digital doc: WorkSHEET: Deductive geometry II (doc-5283)

Exercise 10.4 Quadrilaterals

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–10	Questions: 1–14, 16	Questions: 1–19

🟩🟦🟪 **Individual pathway interactivity:** int-4614

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. Use the definitions of the five special quadrilaterals to decide if the following statements are true or false.

a. A square is a rectangle.

b. A rhombus is a parallelogram.

c. A square is a rhombus.

d. A rhombus is a square.

e. A square is a trapezium.

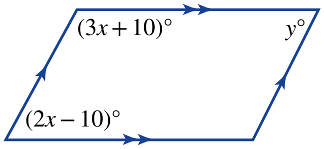
f. A parallelogram is a rectangle.

g. A trapezium is a rhombus.

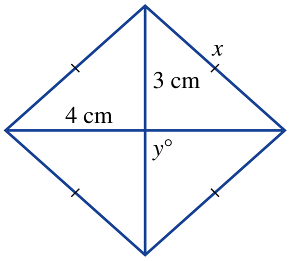
h. A rectangle is a square.

2. Determine the values of x and y in each of the following figures.

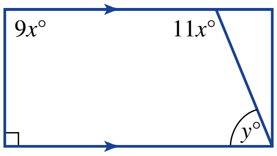
a.



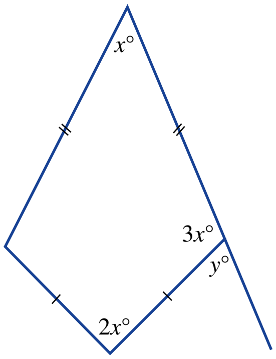
b.



c.



d.



Understanding

3. Draw three different trapeziums. Using your ruler, compass and protractor, decide which of the following properties are true in a trapezium.

a. Opposite sides are equal.

b. All sides are equal.

c. Opposite angles are equal.

d. All angles are equal.

e. Diagonals are equal in length.

f. Diagonals bisect each other.

g. Diagonals are perpendicular.

h. Diagonals bisect the angles they pass through.

4. Draw three different parallelograms. Using your ruler and protractor to measure, decide which of the following properties are true in a parallelogram.

a. Opposite sides are equal.

b. All sides are equal.

c. Opposite angles are equal.

d. All angles are equal.

e. Diagonals are equal in length.

f. Diagonals bisect each other.

g. Diagonals are perpendicular.

h. Diagonals bisect the angles they pass through.

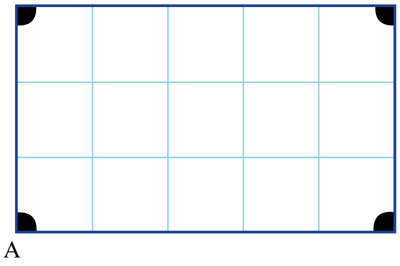
5. Name four quadrilaterals that have at least one pair of opposite sides that are parallel and equal.

6. Name a quadrilateral that has equal diagonals that bisect each other and bisect the angles they pass through.

7. Pool is played on a rectangular table. Balls are hit with a cue and bounce off the sides of the table until they land in one of the holes or pockets.

a. Draw a rectangular pool table measuring 5 cm by 3 cm on graph paper. Mark the four holes, one in each corner.

b. A ball starts at A. It is hit so that it travels at a 45° diagonal across the grid. When it hits the side of the table, it bounces off at a 45° diagonal as well. How many sides does the ball bounce off before it goes in a hole?



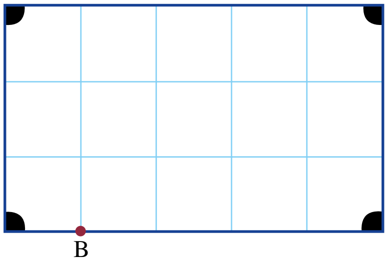
c. A different size table is 7 cm by 2 cm. How many sides does a ball bounce off before it goes in a hole when hit from A?

d. Complete the following table.

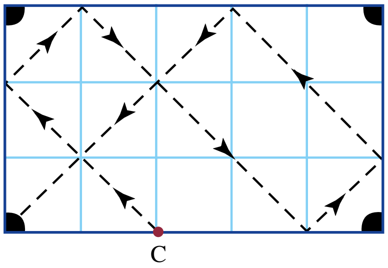
Table size	Number of sides hit
5 cm × 3 cm	
7 cm × 2 cm	
4 cm × 3 cm	
4 cm × 2 cm	
6 cm × 3 cm	
9 cm × 3 cm	
12 cm × 4 cm	

e. Can you see a pattern? How many sides would a ball bounce off before going in a hole when hit from A on an $m \times n$ table?

f. The ball is now hit from B on a 5 cm × 3 cm pool table. How many *different* paths can a ball take when hit along 45° diagonals? Do these paths all hit the same number of sides before going in a hole? Does the ball end up in the same hole each time? Justify your answer.



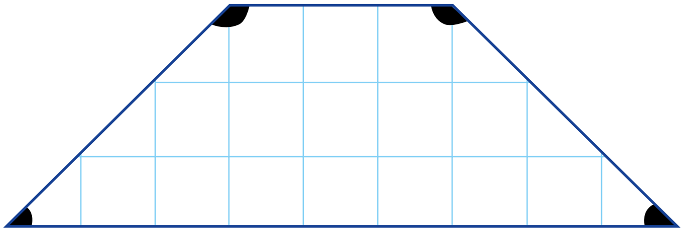
g. The ball is now hit from C along the path shown. What type of triangles and quadrilaterals are formed by the path of the ball with itself and the sides of the table? Are any of the triangles congruent?



h. A ball is hit from C on a 6 cm by 3 cm table. What shapes are formed by the path of the ball with itself and the sides of the table? Is there only one path possible?

i. *Challenge:* A ball is hit from A along 45° diagonals. The table is $m \times n$. Can you find a formula to predict which hole the ball will go in?

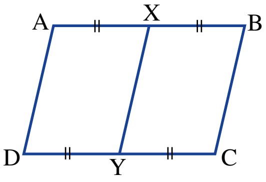
j. *Challenge:* What would happen if the game was played on a trapezoidal table?



Reasoning

8. Prove that the diagonals of a rhombus bisect each other.

9. ABCD is a parallelogram. X is the midpoint of AB and Y is the midpoint of DC. Prove that AXYD is also a parallelogram.

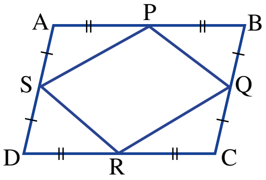


10. ABCD is a parallelogram. P, Q, R and S are all midpoints of their respective sides of ABCD.

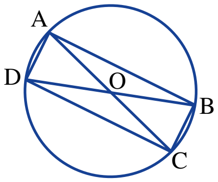
a. Prove $\Delta PAS \cong \Delta RCQ$.

b. Prove $\Delta SDR \cong \Delta PBQ$.

c. Hence, prove that PQRS is also a parallelogram.

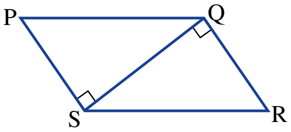


11. AC and BD are diameters of a circle with centre O. Prove that ABCD is a rectangle.

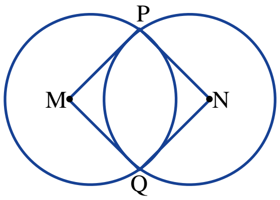


12. The diagonals of a parallelogram meet at right angles. Prove that the parallelogram is a rhombus.

13. Two congruent right-angled triangles are arranged as shown. Show that PQRS is a parallelogram.

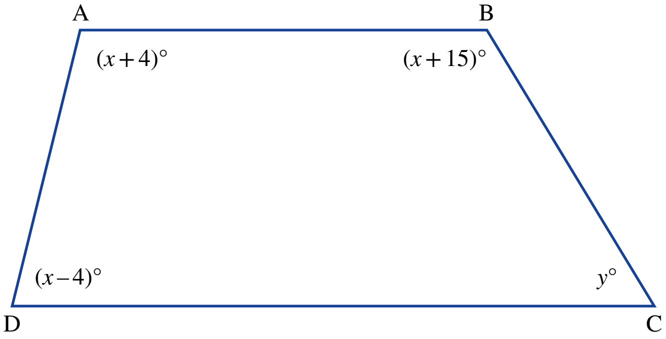


14. Two circles, centred at M and N, have equal radii and intersect at P and Q. Prove that PNQM is a rhombus.



15. Give reasons why a square is a rhombus, but a rhombus is not necessarily a square.

16. ABCD is a trapezium.



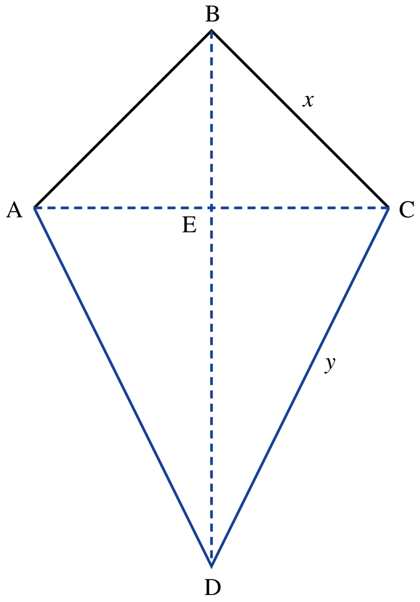
a. What fact do you know about a trapezium?

b. Find the values of x and y.

17. ABCD is a kite where AC = 8 cm, BE = 5 cm and ED = 9 cm. Find the exact values of:

a. i. x **ii.** y.

b. Find angle BAD and hence angle BCD.



Problem solving

18. ABCDE is a regular pentagon whose side lengths are 2 cm. Each diagonal is x cm long.

a. What kind of shape is AEDF and what is the length of FD?

b. What kind of shape is ABCD?

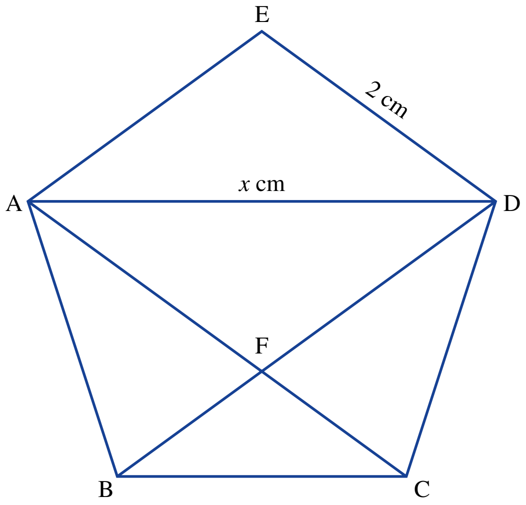
c. If ∠EDA is 40°, find the value of ∠ACB, giving reasons for your findings.

d. Which triangle is similar to AED?

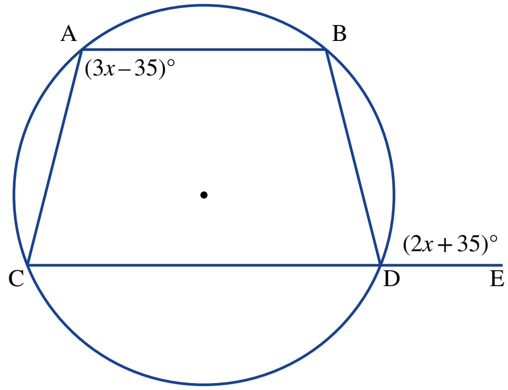
e. Explain why FB = $(x - 2)$ cm.

f. Show that $x^2 - 2x - 4 = 0$.

g. Solve the equation $x^2 - 2x - 4 = 0$, giving your answer as an exact value.



19. ABCD is called a cyclic quadrilateral because it is inscribed inside a circle.

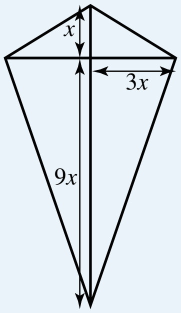


A characteristic of a cyclic quadrilateral is that the opposite angles are supplementary. Find the value of x.

Reflection

How do you know if a quadrilateral is a rhombus?

CHALLENGE 10.2

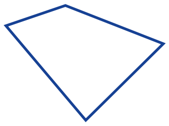


The perimeter of this kite is 80 cm. Determine the exact value of x.

10.5 Polygons

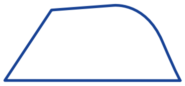
10.5.1 Polygons

- **Polygons** are closed shapes that have three or more straight sides.



Irregular polygon

Not a polygon



Not a polygon

- **Regular polygons** are polygons with sides of the same length and interior angles of the same size, like the pentagon shown in the centre of the photo above.
- **Convex polygons** are polygons with no interior reflex angles.
- **Concave polygons** are polygons with at least one reflex interior angle. For example, the pentagon shown above is a concave polygon as well as a regular polygon.

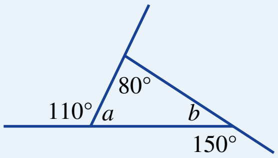
10.5.2 Interior angles of a polygon

- The interior angles of a polygon are the angles inside the polygon at each vertex.
- The sum of the interior angles of a polygon is given by the formula:

Angle sum $= 180° \times (n - 2)$
where $n =$ the number of sides of the polygon

WORKED EXAMPLE 6

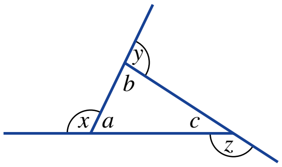
Calculate the value of the pronumerals in the figure below.

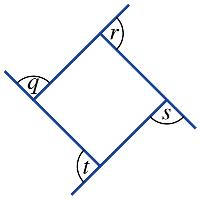


THINK	WRITE
1 Angles *a* and 110° form a straight line and so are supplementary (add to 180°).	$a + 110° = 180°$ $a + 110° - 110° = 180° - 110°$ $a = 70°$
2 The interior angles of a triangle sum to 180°.	$b + a + 80° = 180$
3 Substitute 70° for *a* and solve for *b*.	$b + 70° + 80° = 180°$ $b + 150° = 180°$ $b = 30°$

10.5.3 Exterior angles of a polygon

- The exterior angles of a polygon are formed by the side of the polygon and an extension of its adjacent side. For example, x, y and z are external angles for the polygon (triangle) below.





- The exterior angle and interior angle at that vertex are supplementary (add to 180°). For example, $x + a = 180°$.
- Exterior angles of polygons can be measured in a clockwise or anticlockwise direction.
- In a regular polygon, the size of the exterior angle can be found by dividing 360° by the number of sides.

$$\text{Exterior angle} = \frac{360°}{n}$$

- The sum of the exterior angles of a polygon equals 360°.
- The exterior angle of a triangle is equal to the sum of the opposite interior angles.

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Try out this interactivity: Angle sum of a polygon (int-0818)

Try out this interactivity: Exterior angles of a polygon (int-0819)

Complete this digital doc: WorkSHEET: Deductive geometry III (doc-5284)

Exercise 10.5 Polygons

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 9	Questions: 1–9, 12	Questions: 1–12

🟩🟦🟪 **Individual pathway interactivity:** int-4615

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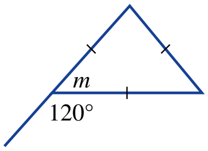
Fluency

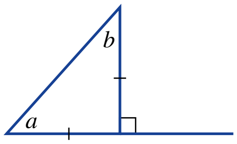
1. How are the internal and external angles of a polygon related to the number of sides in a polygon?

2. **WE6** Calculate the values of the pronumerals in the diagrams below.

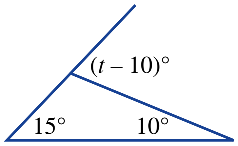
a.



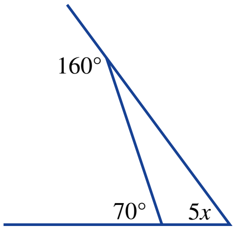
b.



c.

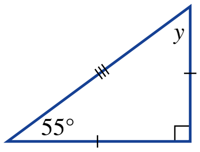


d.

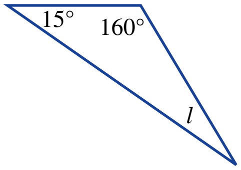


3. For the five triangles below, evaluate the pronumerals and determine the size of the interior angles.

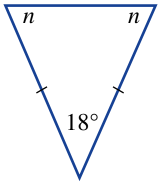
a.



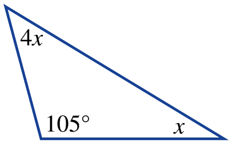
b.



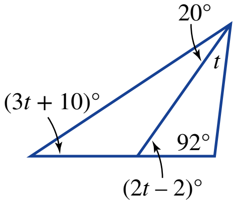
c.



d.



e.

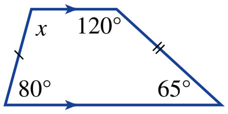


4. For the five quadrilaterals below:

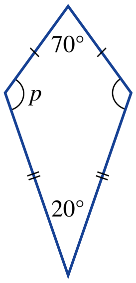
i. label the quadrilaterals as regular or irregular

ii. determine the value of the pronumeral for each shape.

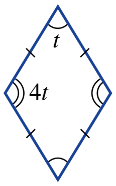
a.



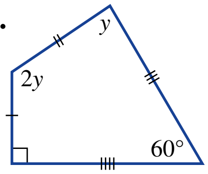
b.



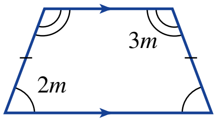
c.



d.



e.



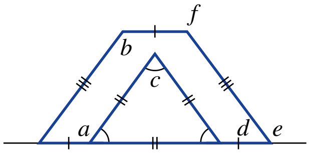
Understanding

5. The photograph below shows a house built on the side of a hill. Use your knowledge of angles to calculate the values of the pronumerals.

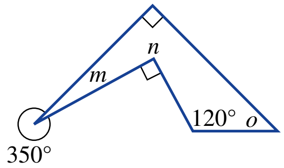
6. Calculate the values of the four interior angles of the front face of the building in the photograph below.

7. Calculate the values of the pronumerals for the irregular polygons below.

a.



b.



8. Calculate the size of the exterior angle of a regular hexagon (6 sides).

Reasoning

9. A diagonal of a polygon joins two vertices.

a. Calculate the number of diagonals in a regular polygon with:

i. 4 sides **ii.** 5 sides **iii.** 6 sides **iv.** 7 sides.

b. Write a formula that relates the number of diagonals for an *n*-sided polygon.

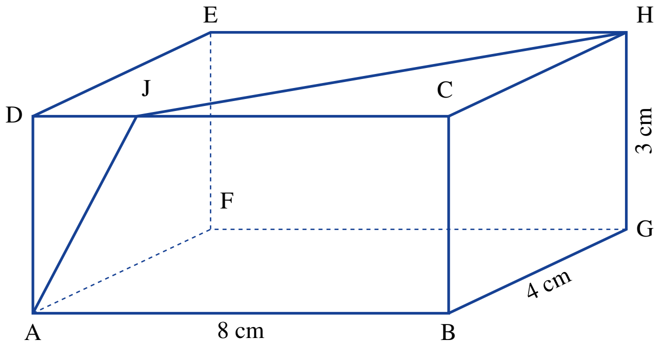
10. The external angle of a polygon can be calculated using the formula:

$$\text{exterior angle } = \frac{360°}{n}$$

Use the relationship between internal and external angles of a polygon to write a formula for the internal angle of a regular polygon.

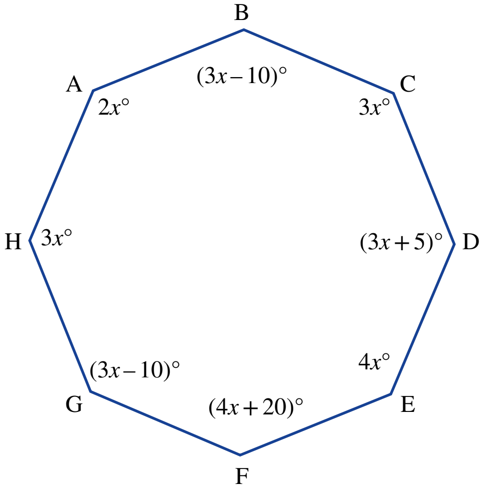
Problem solving

11.



A piece of string is fixed at A and H as shown. The string is tight and fixed to the surface of the cuboid. Locate the exact position of J on the edge CD.

12. ABCDEFGH is an octagon.



a. What is the sum of the interior angles of an octagon?

b. Find the value of x.

Reflection

How are the angles associated with polygons related to each other and the polygon?

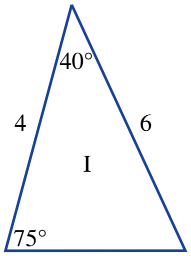
10.6 Review

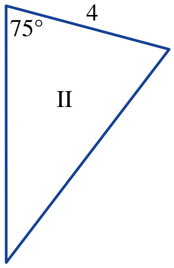
10.6.1 Review questions

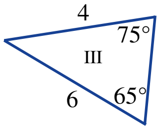
Fluency

1. Select a pair of congruent triangles in each of the following sets of triangles, giving a reason for your answer. All angles are in degrees and side lengths in cm. (The figures are not drawn to scale.)

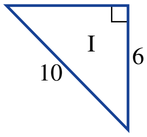
a.

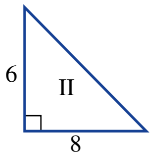


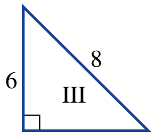




b.

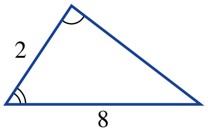


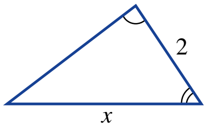




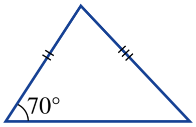
2. Find the value of the pronumeral in each pair of congruent triangles. All angles are given in degrees and side lengths in cm.

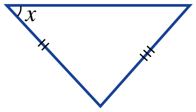
a.



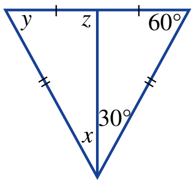


b.

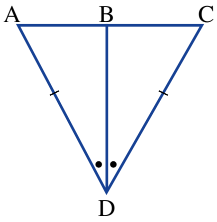




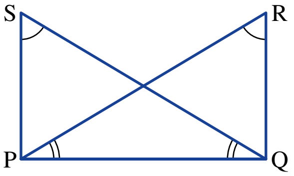
c.



3. a. Prove that the two triangles shown in the diagram below are congruent.

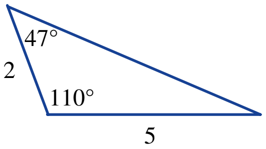


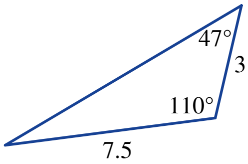
b. Prove that ΔPQR is congruent to ΔQPS.



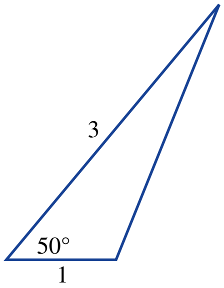
4. Test whether the following pairs of triangles are similar. For similar triangles find the scale factor. All angles are in degrees and side lengths in cm.

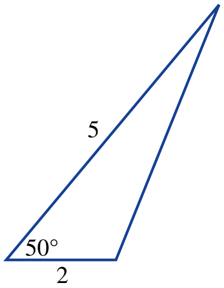
a.



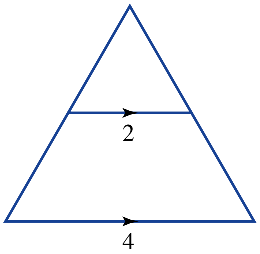


b.



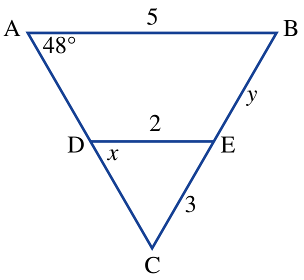


c.

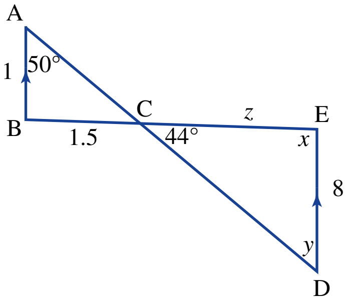


5. Find the value of the pronumeral in each pair of similar triangles. All angles are given in degrees and side lengths in cm.

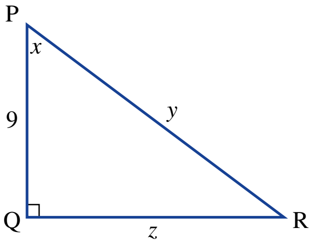
a.

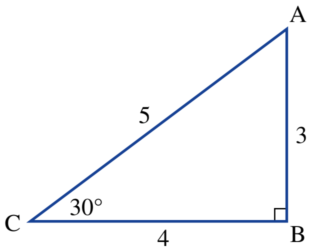


b.

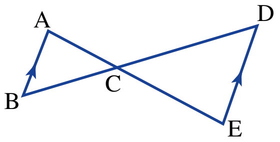


c.

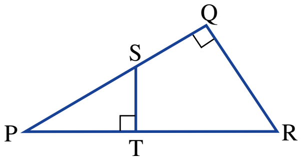




6. Prove that $\Delta ABC \sim \Delta EDC$.



7. Prove that $\Delta PST \sim \Delta PRQ$.



8. Prove that the angles opposite the equal sides in an isosceles triangle are equal.

9. Two corresponding sides in a pair of similar octagons have lengths of 4 cm and 60 mm. The respective scale factor in length is:

a. 4 : 60 **b.** 6 : 40 **c.** 40 : 60 **d.** 60 : 40

10. A regular nonagon has side length x cm. Use a scale factor of $\frac{x+1}{x}$ to find the side length of a similar nonagon.

Problem solving

11. ABC is a triangle. D is the midpoint of AB, E is the midpoint of AC and F is the midpoint of BC. $DG \perp AB$, $EG \perp AC$ and $FG \perp BC$.

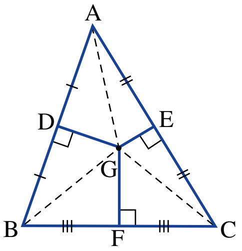
a. Prove that $\Delta GDA \cong \Delta GDB$.

b. Prove that $\Delta GDE \cong \Delta GCE$.

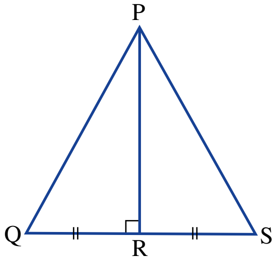
c. Prove that $\Delta GBF \cong \Delta GCF$.

d. What does this mean about AG, BG and CG?

e. A circle centred at G is drawn through A. What other points must it pass through?



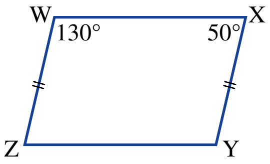
12. PR is the perpendicular bisector of QS. Prove that ΔPQS is isosceles.



13. Name any quadrilaterals that have diagonals that bisect the angles they pass through.

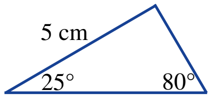
14. State three tests that can be used to show that a quadrilateral is a rhombus.

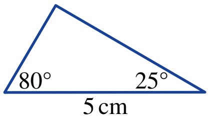
15. Prove that WXYZ is a parallelogram.



16. Prove that the diagonals in a rhombus bisect the angles they pass through.

17. Explain why the triangles shown below are not congruent.

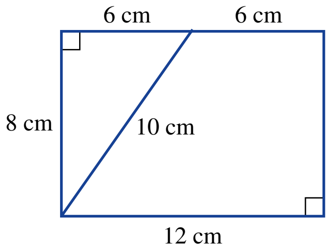




18. State the definition of a rhombus.

19. Name any quadrilaterals that have equal diagonals.

20. This 8 cm by 12 cm rectangle is cut into two sections as shown.



a. Draw labelled diagrams to show how the two sections can be rearranged to form:
  - **i.** a parallelogram
  - **ii.** a right-angled triangle
  - **iii.** a trapezium.

b. Comment on the perimeters of the figures.

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Try out this interactivity: Word search: Topic 10 (int-2853)

Try out this interactivity: Crossword: Topic 10 (int-2854)

Try out this interactivity: Sudoku: Topic 10 (int-3597)

Complete this digital doc: Concept map: Topic 10 (doc-13762)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

bisect
concave
congruency
convex
corresponding angles
corresponding sides
diagonal
exterior angle
interior angle
parallelogram
polygon
quadrilateral
rectangle
reflex
rhombus
similarity
square
trapezium

Investigation | Rich task

Enlargement activity

Enlargement is the construction of a bigger picture from a small one. The picture is identical to the other except that it is bigger. The new picture is often called the image. This can also be called creating a similar figure.

The geometrical properties shared by a shape and its image under enlargement can be listed as:

- lines are enlarged as lines
- sides are enlarged to corresponding sides by the same factor
- matching angles on the two shapes are equal.

In this activity, we will start with a small cartoon character, and then 'blow it up' to almost life-size.

Equipment: ruler, pencil, cartoon print, butcher's paper or some other large piece of paper.

1. Do some research on the internet and select a cartoon character or any character of your choice.
2. Draw a grid of 2-cm squares over the small cartoon character.
   Example: The small Casper is 9 squares wide and 7 squares tall.
3. Label the grids with letters across the top row and numbers down the first column.
4. Get a large piece of paper and draw the same number of squares. You will have to work out the ratio of similitude (e.g. 2 cm : 8 cm).

5. If your small cartoon character stretches from one side of the 'small' paper (the paper the image is printed on) to the other, your 'large' Casper must stretch from one side of the 'big' paper to the other. Your large grid squares may have to be 8 cm by 8 cm or larger, depending on the paper size.
6. Draw this enlarged grid on your large paper. Use a metre ruler or some other long straight-edged tool. Be sure to keep all of your squares the same size.
   - At this point, you are ready to DRAW. Remember, you do NOT have to be an artist to produce an impressive enlargement.
   - All you do is draw EXACTLY what you see in each small cell into its corresponding large cell.
   - For example, in cell B3 of the Casper enlargement, you see the tip of his finger, so draw this in the big grid.
   - If you take your time and are very careful, you will produce an extremely impressive enlargement.
   - What you have used is called a RATIO OF SIMILITUDE. This ratio controls how large the new picture will be.

A 2 : 5 ratio will give you a smaller enlargement than a 2 : 7 ratio, because for every two units on the original you are generating only 5 units of enlargement instead of 7.

If Casper's ratio is 1 : 4, it produces a figure that has a linear measure that is four times bigger.

Big Casper's overall **area**, however, will be **16 times larger** than small Casper's. This is because area is found by taking length times width.

The length is 4 times longer and the width is 4 times longer.

Thus the **area** is $4 \times 4 = 16$ times **larger** than the original Casper.

His overall **volume** will be $4 \times 4 \times 4$ or **64 times larger!** This means that big Casper will weigh 64 times more than small Casper.

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Complete this digital doc: Code puzzle: Why was the archaeologist upset? (doc-15935)

Answers

Topic 10 Deductive geometry

Exercise 10.2 Angles, triangles and congruence

1. a. $a = 56^\circ$ **b.** $b = 30^\circ$ **c.** $c = 60^\circ$ **d.** $d = 120^\circ$ **e.** $e = 68^\circ$

2. a. I and III, SAS **b.** I and II, AAS **c.** II and III, RHS **d.** I and II, SSS

3. a. $x = 6,\ y = 60^\circ$ **b.** $x = 80^\circ,\ y = 50^\circ$ **c.** $x = 32^\circ,\ y = 67^\circ$ **d.** $x = 45^\circ,\ y = 45^\circ$

4. a. $x = 3$ cm **b.** $x = 85^\circ$ **c.** $x = 80^\circ,\ y = 30^\circ,\ z = 70^\circ$
d. $x = 30^\circ,\ y = 7$ cm **e.** $x = 40^\circ,\ y = 50^\circ,\ z = 50^\circ,\ m = 90^\circ,\ n = 90^\circ$

5. a. Use SAS **b.** Use SAS. **c.** Use ASA. **d.** Use ASA. **e.** Use SSS.

6. C, D

7. a. $x = 110^\circ,\ y = 110^\circ,\ z = 4$ cm, $w = 7$ cm **b.** $x = 70^\circ$
c. $x = 30^\circ,\ y = 65^\circ$

8. The third sides are not necessarily the same.

9. Corresponding sides are not the same.

10. Use SSS.

11, **12** Check with your teacher.

13. Check with your teacher.

14. $x = 20^\circ,\ y = 10^\circ$ and $z = 40^\circ$

Exercise 10.3 Similar triangles

1. a. i and **iii**, RHS **b. i** and **ii**, SAS **c. i** and **iii**, SSS **d. i** and **iii**, AAA **e. i** and **ii**, SSS

2. a. Triangles PQR and ABC **b.** Triangles ADB and ADC **c.** Triangles PQR and TSR
d. Triangles ABC and DEC **e.** Triangles ABC and DEC

3. a. $\dfrac{\text{AB}}{\text{AD}} = \dfrac{\text{BC}}{\text{DE}} = \dfrac{\text{AC}}{\text{AE}}$ **b.** $f = 9,\ g = 8$

4. $x = 4$

5. $x = 20^\circ,\ y = 2\frac{1}{4}$

6. a. AAA **b.** $x = 3,\ y = 4$

7. The slide is 7.23 m long.

8. Radius = 0.625 m

9. a. $x = 1,\ y = 7\sqrt{2}$ **b.** $x = 2.5,\ y = 3.91$

10. Check with your teacher. One method would be to use Pythagorean triples.

11. a. $\angle\text{ABD} = \angle\text{ABC}$ (common angle)
$\angle\text{ADB} = \angle\text{BAC} = 90^\circ$
$\Delta\text{ABD} \sim \Delta\text{ACB}$ (AAA)

b. $\angle\text{ACD} = \angle\text{BCA}$ (common angle)
$\angle\text{ADC} = \angle\text{CAB} = 90^\circ$
$\Delta\text{ACD} \sim \Delta\text{ACB}$ (AAA)

12. Congruent triangles must be identical, that is, the angles must be equal and the side lengths must be equal. Therefore, it is not enough just to prove that the angles are equal.

13. Check with your teacher.

14. $\angle\text{FEO} = \angle\text{OGH}$ (alternate angles equal as EF || HG)
$\angle\text{EFO} = \angle\text{OHG}$ (alternate angles equal as EF || HG)
$\angle\text{EOF} = \angle\text{HOG}$ (vertically opposite angles equal)
$\therefore\ \Delta\,\text{EFO} \sim \Delta\,\text{GHO}$ (equiangular)

15. $x = 6$ or 11

Challenge 10.1

3

Exercise 10.4 Quadrilaterals

1. a. True **b.** True **c.** True **d.** False **e.** False **f.** False **g.** False **h.** False

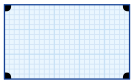
2. a. $x = 36^\circ,\ y = 62^\circ$ **b.** $x = 5$ cm, $y = 90^\circ$ **c.** $x = 10^\circ,\ y = 70^\circ$ **d.** $x = 40^\circ,\ y = 60^\circ$

3. None are true, unless the trapezium is a regular trapezium, then **e** is true.

4. a, c, f

5. Parallelogram, rhombus, rectangle, square

6. Square

7. a.  **b.** 6 sides **c.** 7 sides

d.

Table size	Number of sides hit
5 cm × 3 cm	6
7 cm × 2 cm	7
4 cm × 3 cm	5
4 cm × 2 cm	1
6 cm × 3 cm	1
9 cm × 3 cm	2
12 cm × 4 cm	2

e. If the ratio of the sides is written in simplest form then the pattern is $m + n - 2$.

f. There are two routes for the ball when hit from B. Either 2 or 3 sides are hit. The ball does not end up in the same hole each time.

A suitable justification would be a diagram — student to draw.

g. Isosceles triangles and parallelograms. The triangles are congruent.

h. The shapes formed are parallelograms. There is only one possible path although the ball could be hit in either of two directions initially.

i. Given m:n is the ratio length to width in simplest form. When m is even and n is odd the destination pocket will be the upper left. When m and n are both odd, the destination pocket will be the upper right. When m is odd and n is even the destination pocket will be the lower right.

j. Students to investigate.

8. Check with your teacher.

9. AX || DY because ABCD is a parallelogram
AX = DY (given)
∴ AXYD is a parallelogram since opposite sides are equal and parallel.

10. a. Use SAS. **b.** Use SAS. **c.** Opposite sides are equal.

11. AC = DB (diameters of the same circle are equal)
AO = OC and OD = OB (radii of the same circle are equal)
∴ ABCD is a rectangle. (Diagonals are equal and bisect each other.)

12. Check with your teacher.

13. PS = QR (corresponding sides in congruent triangles are equal)
PS || QR (alternate angles are equal)
∴ PQRS is a parallelogram since one pair of opposite sides are parallel and equal.

14. MP = MQ (radii of same circle)
PN = QN (radii of same circle) and circles have equal radii.
∴ All sides are equal.
∴ PNQM is a rhombus.

15. Check with your teacher.

16. a. One pair of opposite sides are parallel. **b.** $x = 90°$, $y = 75°$

17. a. i. $x = \sqrt{41}$ **ii.** $y = \sqrt{97}$ **b.** ∠BAD = ∠BCD = 117°23′

18. a. Rhombus, 2 cm **b.** Trapezium **c.** 40° **d.** Triangle BFC
e. Check with your teacher. **f.** Check with your teacher. **g.** $x = (1 + \sqrt{5})$ cm

19. 70°

Challenge 10.2

$x = \sqrt{10}$ cm

Exercise 10.5 Polygons

1. The sum of the interior angles is based on the number of sides of the polygon.
The size of the exterior angle can be found by dividing 360° by the number of sides.

2. a. $m = 60°$ **b.** $a = 45°$, $b = 45°$ **c.** $t = 35°$ **d.** $x = 10°$

3. a. $y = 35°$ **b.** $t = 5°$ **c.** $n = 81°$ **d.** $x = 15°$ **e.** $t = 30°$

4. a. i. Irregular **ii.** $x = 95°$ **b. i.** Irregular **ii.** $p = 135°$

c. i. Irregular **ii.** $t = 36°$ **d. i.** Irregular **ii.** $y = 70°$

e. i. Irregular **ii.** $p = 36°$

5. $w = 75°,\ x = 105°,\ y = 94°,\ z = 133°$

6. 82.5°, 82.5°, 97.5°, 97.5°

7. a. $a = 120°,\ b = 120°,\ c = 60°,\ d = 60°,\ e = 120°,\ f = 240°$

b. $m = 10°,\ n = 270°,\ o = 50°$

8. 60°

9. a. i. 2 **ii.** 5 **iii.** 9 **iv.** 14

b. Number of diagonals $= \frac{1}{2}n(n - 3)$

10. Internal angle $= 180° - \frac{360°}{n}$

11. J is $\frac{24}{7}$ cm from D.

12. a. 1080° **b.** 43°

10.6 Review

1. a. a I and III, ASA or SAS **b.** I and II, RHS

2. a. $x = 8$ cm **b.** $x = 70°$ **c.** $x = 30°,\ y = 60°,\ z = 90°$

3. a. Use SAS. **b.** Use ASA.

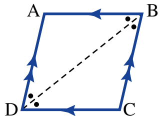
4. a. Similar, scale factor = 1.5 **b.** Not similar **c.** Similar, scale factor = 2

5. a. $x = 48°,\ y = 4.5$ cm **b.** $x = 86°,\ y = 50°,\ z = 12$ cm **c.** $x = 60°,\ y = 15$ cm, $z = 12$ cm

6. Use the equiangular test.

7. Use the equiangular test.

8.



Bisect ∠BAC
AB = AC (given)
∠BAD = ∠DAC
AD is common.
∴ ΔABD ≅ ΔACD (SAS)
∴ ∠ABD = ∠ACD (corresponding sides in congruent triangles are equal)

9. C

10. $x + 1$

11. a. Use SAS. **b.** Use SAS. **c.** Use SAS.

d. They are all the same length. **e.** B and C

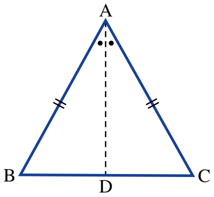
12. Use SAS.
PQ = PS (corresponding sides in congruent triangles are equal)

13. Rhombus, square

14. A quadrilateral is a rhombus if:
a. all sides are equal
b. the diagonals bisect each other at right angles
c. the diagonals bisect the angles they pass through.

15. WZ ∥ XY (co-interior angles are supplementary) and WZ = XY (given)
∴ WXYZ is a parallelogram since one pair of sides is parallel and equal.

16.



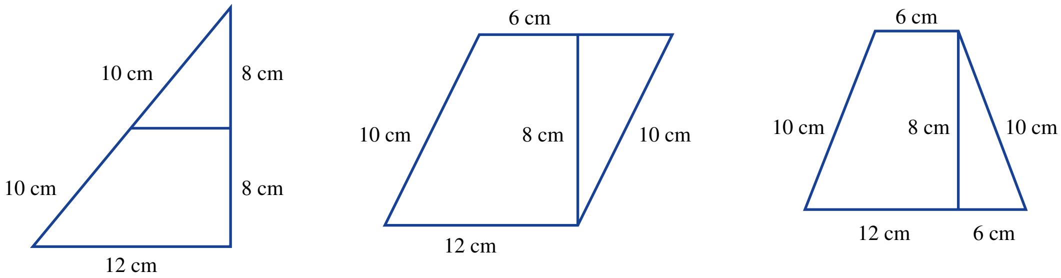
∠ABD = ∠ADB (angles opposite the equal sides in an isosceles triangle are equal)
∠ABD = ∠BDC (alternate angles equal as AB || DC)
∴ ∠ADB = ∠BDC
∴ Diagonals bisect the angles they pass through.

17. Corresponding sides are not the same.

18. A rhombus is a parallelogram with two adjacent sides equal in length.

19. Rectangle, square

20. a. i.



b. Perimeter of rectangle = 40 cm
Perimeter of parallelogram = 44 cm
Perimeter of triangle = 48 cm
Perimeter of trapezium = 44 cm
The triangle has the largest perimeter, while the rectangle has the smallest.

Investigation — Rich task

Check with your teacher.

learnon

Project: Getting the budget in order

Scenario

Budgets are of great importance to everyone. From the federal budget down to the local council budget, they all impact on our lives. On a smaller scale, but of no less importance, is the family budget.

As a financial planner, you have been asked to prepare a budget for the Thompson family. The Thompson family lives in a major city in Australia and comprises Mr and Mrs Thompson and their two teenage children. They want to go on a holiday and they plan to save for a year. Taking their ongoing expenses into account, they want to see if and how they can afford this, following your recommendations.

You are to form a financial-planning group of three people. Each group member is to produce a budget plan that addresses the following holiday and ongoing expenses.

- Holiday: They want to be able to spend $5000 on a holiday within Australia. Where can they go and for how long, taking flights, accommodation, meals and sightseeing expenses into account?
- Phone plans: Their two children have phone plans that each cost $29 per month, currently paid by their parents. However, to help fund the holiday, their parents need them to pay back the $29 per month, with either simple interest or compound interest charged. What is the best option for the children? How much do they need to work at their $9.75-per-hour part-time jobs to cover the cost?
- Rent: The Thompson family lives in a rental house within 50 km from a major city. How much rent would they be paying?
- Taxes
- Utilities (electricity, water, gas, phone)
- Food
- Other expenses

Within the Thompson family, Mr Thompson takes care of the household duties and Mrs Thompson works in a salaried position, earning $85 000 per annum. Their children have part-time jobs to repay the cost of their phone plans and to try and save for new clothes.

Task

You will produce an oral presentation and a written budget plan with recommendations for the Thompsons' family holiday. The budget plan will include the preferred method of interest charged to the children for the cost of their phone plan and the hours per week the children need to work at their part-time jobs to make this repayment. Your budget plan should also include value-for-money rentals located within 50 km of a major city, which will help the family assess how much they should be spending on rent. You also need to recommend a package holiday that includes flights and accommodation. Your presentation should explain why you have made each of your recommendations and give evidence to back up your decisions, taking all the expenses into account.

Process

- Watch the introductory video lesson.
- Navigate to your Research Forum. Here you will find a series of topics that will help you complete your task. Select the expenses you are researching or add new expenses you wish to include.
- Research. Make notes of important information and ideas that you discovered during your research. Enter your findings as articles under your topics in the Research Forum. You should each find at least three sources of information (including off-line resources such as books and newspapers). You can view and comment on other group members' articles and rate the information they have entered. When your research is complete, print your Research Report to hand in to your teacher.
- Visit your Media Centre and download the budget template and PowerPoint sample to help you prepare your presentation. Your Media Centre also includes images to help liven up your presentation.
- Use the budget template to give a clear overview of all the expenses taken into account and to give an overall summary of the whole family budget. Make sure you remember to address all the expenses that the Thompson family has requested you take into account.
- Use the PowerPoint template to develop your presentation. Remember that you are making recommendations that you believe are best for the Thompson family. Make sure you cover all the details they that have requested, and that your presentation will grab their attention.

SUGGESTED SOFTWARE

- ProjectsPLUS
- Microsoft Word
- PowerPoint, Prezi, Keynote or other presentation software
- Microsoft Excel
- CAS calculator (optional)

TOPIC 11
Probability

11.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

11.1.1 Why learn this?

Probability is the mathematics of chance. Our entire lives are affected by events that are based on chance. Learning about probability will help you understand how chance is involved in everyday events and in many decisions that you will make in your life.

11.1.2 What do you know?

assesson

1. **THINK** List what you know about chance. Use a thinking tool such as a concept map to show your list
2. **PAIR** Share what you know with a partner and then with a small group
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of chance.

LEARNING SEQUENCE

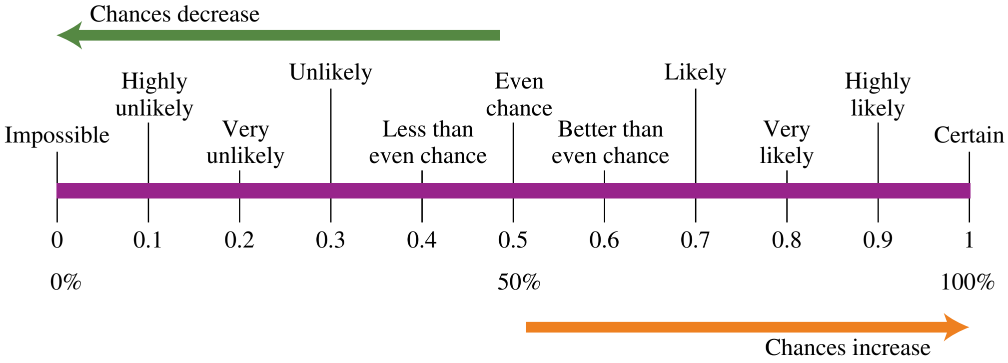
learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: The mathematics of chance (eles-1851)

11.2 Review of probability

11.2.1 The language of probability

- **Probability** measures the chance of an event taking place and ranges from zero (0) for an impossible event to one (1) for a certain event.

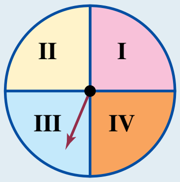


- The **experimental probability** of an event is based on the outcomes of experiments, simulations or surveys.
- A **trial** is a single experiment; for example, a single flip of a coin.

$$\text{Experimental probability} = \frac{\text{number of successful trials}}{\text{total number of trials}}$$

- The **relative frequency** of an event is the same as the experimental probability of that event.
- The list of all possible outcomes of an experiment is known as the **event space** or **sample space**. For example, when flipping a coin there are two possible outcomes: Heads or Tails. The event space can be written, using set notation, as {H, T}.

WORKED EXAMPLE 1



The spinner shown here is made up of 4 equal-sized segments. It is known that the probability that the spinner will land on any one of the four segments from one spin is $\frac{1}{4}$. To test if the spinner shown here is fair, a student spun the spinner 20 times and each time recorded the segment in which the spinner stopped. The spinner landed as follows.

Segment	I	II	III	IV
Tally	5	4	8	3

a List the event space.
b Given the experimental results, determine the relative frequency for each segment.
c Compare the results from the experiment with the known probabilities and suggest how the experiment could be changed to ensure that the results give a better estimate of the true probability.

THINK	WRITE
a The event space lists all possible outcomes from one spin of the spinner. There are four possible outcomes.	**a** Event space = {I, II, III, IV}

b 1 For segment I there were 5 successful trials out of the 20. Substitute these values into the relative frequency formula.

b $\text{Relative frequency}_{\text{I}} = \dfrac{\text{number of successful trials}}{\text{total number of trials}}$

$$= \frac{5}{20}$$

$$= 0.25$$

2 Repeat for segments:

- II (4 successes)
- III (8 successes)
- IV (3 successes).

$$\text{Relative frequency}_{\text{II}} = \frac{4}{20} = 0.2$$

$$\text{Relative frequency}_{\text{III}} = \frac{8}{20} = 0.4$$

$$\text{Relative frequency}_{\text{IV}} = \frac{3}{20} = 0.15$$

c Compare the relative frequency values with the known value of $\frac{1}{4}$ (0.25). Answer the question.

c The relative frequency of segment I was the only segment that mirrored the known value. To ensure that relative frequency gives a better estimate of the true probability, the spinner should be spun many more times.

11.2.2 Two-way tables

- The sample space can be displayed using a **two-way table**.
- A two-way table represents two of the outcomes of events in a two-dimensional table. A two-way table for the experiment of tossing a coin and rolling a die simultaneously is shown below.

		Die outcomes					
Coin outcomes		**1**	**2**	**3**	**4**	**5**	**6**
	H	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
	T	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

WORKED EXAMPLE 2

Two dice are rolled, and the values on the two uppermost faces are multiplied together. Draw a diagram to illustrate the sample space.

THINK

The sample space for rolling 1 die is {1, 2, 3, 4, 5, 6}. When two dice are rolled and the two uppermost faces are multiplied, the sample space is made up of 36 products. This is best represented with the use of a two-way table.

- Draw a 7 × 7 grid.
- In the first row and column list the outcomes of each die.
- At the intersection of a column and row write the product of the relevant die outcomes, as shown in green.

DRAW

FIRST DIE

SECOND DIE

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

11.2.3 Theoretical probability

- **Theoretical probability** is the probability of an event occurring, based on the number of possible favourable outcomes, $n(A)$, and the total number of possible outcomes, $n(\xi)$.
- When all outcomes are equally likely, the theoretical probability of an event can be calculated using the formula:

$$\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} \quad \text{or} \quad \text{P(event)} = \frac{n(E)}{n(\xi)}$$

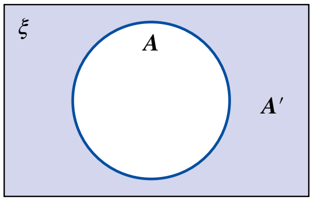
where $n(E)$ is the number of favourable events and $n(\xi)$ is the total number of possible outcomes.

WORKED EXAMPLE 3

A fair die is rolled and the value of the uppermost side is recorded. Calculate the theoretical probability that a 4 is uppermost.

THINK	WRITE
1 Write the number of favourable outcomes and the total number of possible outcomes. The number of fours on a fair die is 1. There are 6 possible outcomes.	$n(E) = 1$ $n(\xi) = 6$
2 Substitute the values found in part **1** to calculate the probability of the event that a four is uppermost when a die is rolled.	$\text{P(a four)} = \frac{n(E)}{n(\xi)}$ $= \frac{1}{6}$
3 Write the answer.	The probability that a 4 is uppermost when a fair die is rolled is $\frac{1}{6}$.

11.2.4 Complementary events



- The **complement** of the set A is the set of all elements that belong to the universal set (ξ) but that do *not* belong to A. For example, the complement of {blue socks in a drawer} is {all socks in the drawer that are not blue}.
- The complement of A is written as A' and is read as 'A dashed' or 'A prime'.
- On a Venn diagram, **complementary events** appear as separate regions that together occupy the whole universal set.

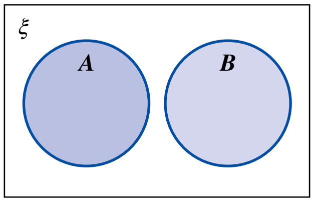
$$\begin{aligned}\text{Since } n(A) + n(A') &= n(\xi)\\ \text{P}(A) + \text{P}(A') &= 1\end{aligned}$$

WORKED EXAMPLE 4

A player is chosen from a cricket team. Are the events 'selecting a batsman' and 'selecting a bowler' complementary events if a player can have more than one role? Give a reason for your answer.

THINK	**WRITE**
Explain the composition of a cricket team. Players who can bat and bowl are not necessarily the only players in a cricket team. There is a wicket-keeper as well. Some players (all rounders) can bat and bowl.	No, the events ‘selecting a batsman’ and ‘selecting a bowler’ are not complementary events. These events may have common elements; that is, the all rounders in the team who can bat and bowl. The cricket team also includes a wicket-keeper.

11.2.5 Mutually exclusive events (*A* or *B*)

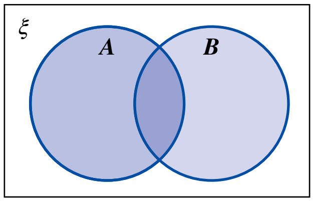


- Two events are **mutually exclusive** if one event happening excludes the other from happening. These events may not encompass all possible events. For example, when selecting a card from a deck of cards, selecting a black card excludes the possibility that the card is a Heart.
- For events that are mutually exclusive: $P(A \cap B) = 0$.
- On a Venn diagram, mutually exclusive events appear as disjointed sets within the universal set.

A and *B* are mutually exclusive events.

$$P(A \cup B) = P(A) + P(B)$$

11.2.6 Probability of intersecting events (*A* or *B* or both)



- When two events have outcomes in common, $P(A \cup B) \neq P(A) + P(B)$, since this would count the outcomes they have in common twice.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- This is known as the **Addition Law of probability**.

WORKED EXAMPLE 5

A card is drawn from a pack of 52 playing cards. What is the probability that the card is a heart or a club?

THINK	**WRITE**
1 Determine whether the given events are mutually exclusive.	The two events are mutually exclusive as they have no common elements.
2 Determine the probability of drawing a heart and of drawing a club.	$\text{P(heart)} = \frac{13}{52} = \frac{1}{4}$ $\text{P(club)} = \frac{13}{52} = \frac{1}{4}$
3 Write the Addition Law for two mutually exclusive events.	$P(A \text{ or } B) = P(A) + P(B)$ where A = drawing a heart and B = drawing a club
4 Substitute the known values into the rule.	$\text{P(heart or club)} = \text{P(heart)} + \text{P(club)}$ $= \frac{1}{4} + \frac{1}{4}$ $= \frac{2}{4}$
5 Evaluate and simplify.	$= \frac{1}{2}$

6 Write your answer.	The probability of drawing a heart or a club is $\frac{1}{2}$.
Note: Alternatively, we can use the formula for theoretical probability.	$\text{P(heart or club)} = \dfrac{n(\text{heart or club})}{n(\xi)}$ $= \frac{26}{52}$ $= \frac{1}{2}$

WORKED EXAMPLE 6

A die is rolled. Determine:

a P(an odd number)

b P(a number less than 4)

c P(an odd number or a number less than 4).

THINK	WRITE
a 1 Determine the probability of obtaining an odd number; that is, {1, 3, 5}.	**a** $\text{P(odd)} = \frac{3}{6}$ $= \frac{1}{2}$
2 Write your answer.	The probability of obtaining an odd number is $\frac{1}{2}$.
b 1 Determine the probability of obtaining a number less than 4; that is, {1, 2, 3}.	**b** $\text{P(less than 4)} = \frac{3}{6}$ $= \frac{1}{2}$
2 Write your answer.	The probability of obtaining a number less than 4 is $\frac{1}{2}$.
c 1 Determine whether the given events are mutually exclusive.	**c** The two events are not mutually exclusive as they have common elements; that is, 1 and 3.
2 Write the Addition Law for two non-mutually exclusive events.	$\text{P}(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ where A = selecting an odd number and B = selecting a number less than 4.
3 Substitute the known values into the rule. *Note:* $\text{P}(A \text{ and } B) = \frac{2}{6}\left(= \frac{1}{3}\right)$ since the events have two elements in common.	$\text{P[odd number} \cup \text{(number} < 4)]$ $= \text{P(odd number)} + \text{P[(number} < 4)]$ $- \text{P[odd number} \cap \text{(number} < 4)]$ $= \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$
4 Evaluate and simplify.	$= \frac{2}{3}$
5 Write your answer.	The probability of obtaining an odd number or a number less than 4 is $\frac{2}{3}$.

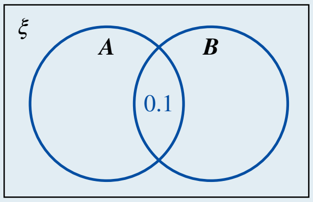
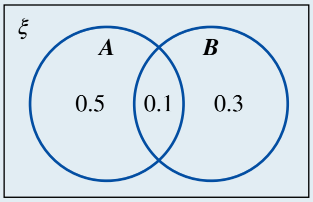
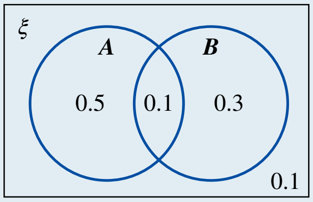
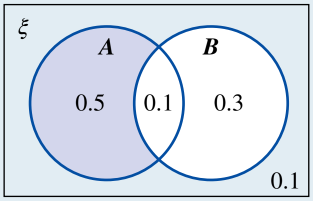
WORKED EXAMPLE 7

Given $\text{P}(A) = 0.6$, $\text{P}(B) = 0.4$ and $\text{P}(A \cup B) = 0.9$:

a use the Addition Law of probability to calculate the value of $\text{P}(A \cap B)$

b draw a Venn diagram to represent the universal set

c calculate $\text{P}(A \cap B')$.

THINK	WRITE
a 1 Write the Addition Law of probability and substitute given values.	**a** $P(A\cup B) = P(A) + P(B) - P(A\cap B)$ $0.9 = 0.6 + 0.4 - P(A\cap B)$
2 Collect like terms and rearrange to make $P(A\cap B)$ the subject. Solve the equation.	$0.9 = 0.6 + 0.4 - P(A\cap B)$ $0.9 = 1.0 - P(A\cap B)$ $P(A\cap B) = 1.0 - 0.9$ $= 0.1$
b 1 Draw intersecting sets A and B within the universal set and write $P(A\cap B) = 0.1$ inside the overlapping section, as shown in blue.	**b** 
2 • As $P(A) = 0.6$, 0.1 of this belongs in the overlap, the remainder of set A is 0.5 $(0.6 - 0.1)$. • Since $P(B) = 0.4$, 0.1 of this belongs in the overlap, the remainder of set B is 0.3 $(0.4 - 0.1)$.	
3 The total probability for events A and B is 1. That means $P(A\cup B)' = 0.1$. Write 0.1 outside sets A and B to form the remainder of the universal set.	
c $P(A\cap B')$ is the overlapping region of $P(A)$ and $P(B')$. Shade the region and write down the corresponding probability value for this area.	**c**  $P(A\cap B') = 0.5$

WORKED EXAMPLE 8

a Draw a Venn diagram representing the relationship between the following sets. Show the position of all the elements in the Venn diagram.

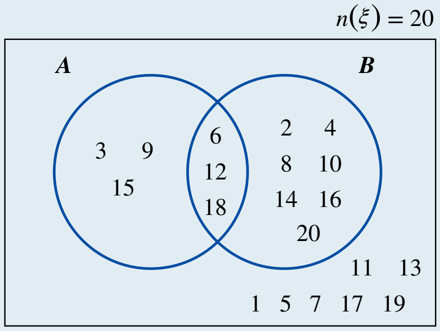
$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

$A = \{3, 6, 9, 12, 15, 18\}$

$B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

b Determine:

i $P(A)$
ii $P(B)$
iii $P(A\cap B)$
iv $P(A\cup B)$
v $P(A'\cap B')$

THINK	WRITE/DRAW
a **1** Draw a rectangle with two partly intersecting circles labelled *A* and *B*.	**a** 
2 Analyse sets *A* and *B* and place any common elements in the central overlap.	
3 Place the remaining elements of set *A* in circle A.	
4 Place the remaining elements of set *B* in circle B.	
5 Place the remaining elements of the universal set ξ in the rectangle.	
b **i** **1** Write the number of elements that belong to set *A* and the total number of elements.	**b** **i** $n(A) = 6, n(\xi) = 20$
2 Write the rule for probability.	$\text{P}(A) = \dfrac{n(A)}{n(\xi)}$
3 Substitute the known values into the rule.	$\text{P}(A) = \frac{6}{20}$
4 Evaluate and simplify.	$= \frac{3}{10}$
ii **1** Write the number of elements that belong to set *B* and the total number of elements.	**ii** $n(B) = 10, n(\xi) = 20$
2 Repeat steps 2 to 4 of part **b i**.	$\text{P}(B) = \dfrac{n(B)}{n(\xi)}$ $\text{P}(B) = \frac{10}{20}$ $= \frac{1}{2}$
iii **1** Write the number of elements that belong to set $A \cap B$ and the total number of elements.	**iii** $n(A \cap B) = 3, n(\xi) = 20$
2 Repeat steps 2 to 4 of part **b i**.	$\text{P}(A \cap B) = \dfrac{n(A \cap B)}{n(\xi)}$ $\text{P}(A \cap B) = \frac{3}{20}$
iv **1** Write the number of elements that belong to set $A \cup B$ and the total number of elements.	**iv** $n(A \cup B) = 13, n(\xi) = 20$
2 Repeat steps 2 to 4 of part **b i**.	$\text{P}(A \cup B) = \dfrac{n(A \cup B)}{n(\xi)}$ $\text{P}(A \cup B) = \frac{13}{20}$
v **1** Write the number of elements that belong to set $A' \cap B'$ and the total number of elements.	**v** $n(A' \cap B') = 7, n(\xi) = 20$
2 Repeat steps 2 to 4 of part **b i**.	$\text{P}(A' \cap B') = \dfrac{n(A' \cap B')}{n(\xi)}$ $\text{P}(A' \cap B') = \frac{7}{20}$

WORKED EXAMPLE 9

In a class of 35 students, 6 students like all three subjects: PE, Science and Music. Eight of the students like PE and Science, 10 students like PE and Music, and 12 students like Science and Music. Also, 22 students like PE, 18 students like Science and 17 like Music. Two students don't like any of the subjects.

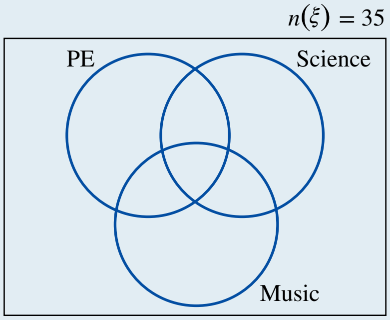
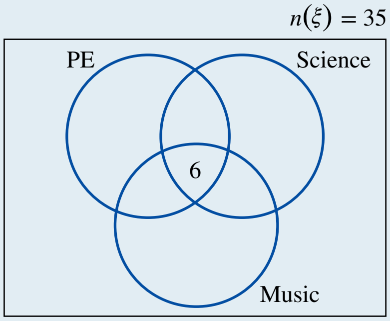
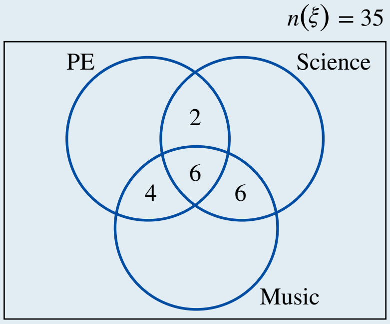
a Display this information on a Venn diagram.

b Determine the probability of selecting a student who:

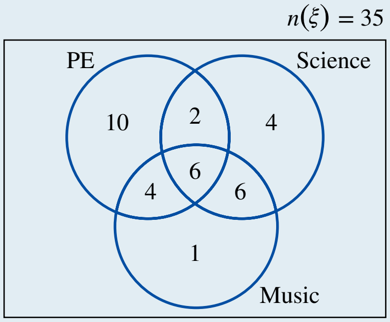
i likes PE only

ii does not like Music.

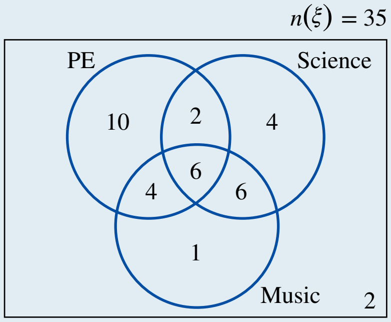
c Find P[(Science ∪ Music) ∩ PE′].

THINK	WRITE/DRAW
a 1 Draw a rectangle with three partly intersecting circles, labelled PE, Science and Music.	**a** 
2 Extract the information relating to students liking all three subjects. *Note:* The central overlap is the key to solving these problems. Six students like all three subjects, so place the number 6 into the section corresponding to the intersection of the three circles.	
3 Extract the relevant information from the second sentence and place it into the appropriate position. *Note:* Eight students like PE and Science; however, 6 of these students have already been accounted for in step 2. Therefore, 2 will fill the intersection of only PE and Science. Similarly, 4 of the 10 who like PE and Music will fill the intersection of only PE and Music, and 6 of the 12 students will fill the intersection of only Science and Music.	

4 Extract the relevant information from the third sentence and place it into the appropriate position.
Note: Twenty-two students like PE and 12 have already been accounted for in the set. Therefore, 10 students are needed to fill the circle corresponding to PE only. Similarly, 4 students are needed to fill the circle corresponding to Science only to make a total of 18 for Science. One student is needed to fill the circle corresponding to Music only to make a total of 17 for Music.



5 Extract the relevant information from the final sentence and place it into the appropriate position.
Note: Two students do not like any of the subjects, so they are placed in the rectangle outside the three circles.



6 Check that the total number in all positions is equal to the number in the universal set.
$10 + 2 + 4 + 4 + 6 + 6 + 1 + 2 = 35$

b i 1 Write the number of students who like PE only and the total number of students in the class.

b i $n(\text{students who like PE only}) = 10$
$n(\xi) = 35$

2 Write the rule for probability.

$$\text{P(likes PE only)} = \frac{n(\text{likes PE only})}{n(\xi)}$$

3 Substitute the known values into the rule.

$$\text{P(likes PE only)} = \frac{10}{35}$$

4 Evaluate and simplify.

$$= \frac{2}{7}$$

5 Write your answer.

The probability of selecting a student who likes PE only is $\frac{2}{7}$.

ii 1 Write the number of students who do not like Music and the total number of students in the class.
Note: Add all the values that do not appear in the Music circle as well as the two that sit in the rectangle outside the circles.

ii $n(\text{students who do not like Music}) = 18$
$n(\xi) = 35$

2 Write the rule for probability.

$$\text{P(does not like Music)} = \frac{n(\text{does not like Music})}{n(\xi)}$$

3 Substitute the known values into the rule.	P(does not like Music) $= \frac{18}{35}$
4 Write your answer.	The probability of selecting a student who does not like Music is $\frac{18}{35}$.
c 1 Write the number of students who like Science and Music but not PE. *Note:* Add the values that appear in the Science and Music circles but do not overlap with the PE circle.	**c** $n[(\text{Science} \cup \text{Music}) \cap \text{PE}'] = 11$ $n(\xi) = 35$
2 Repeat steps 2 to 4 of part **b ii**.	$\text{P}[(\text{Science} \cup \text{Music}) \cap \text{PE}'] = \frac{n[(\text{Science} \cup \text{Music}) \cap \text{PE}']}{n(\xi)}$ $\text{P}[(\text{Science} \cup \text{Music}) \cap \text{PE}'] = \frac{11}{35}$. The probability of selecting a student who likes Science or Music but not PE is $\frac{11}{35}$.

11.2.7 Odds

- Probabilities in gambling can be expressed as **odds**.
- Odds are common in racing, where they are given as ratios; for example 5–1 (or $\frac{5}{1}$ or 5 : 1).
- In the odds of a–b,

$$\begin{tikzcd} & a\text{–}b \arrow[dl] \arrow[dr] & \\ a \text{ is the chance against the event} & & b \text{ is the chance for the event} \end{tikzcd}$$

- If the odds for a horse to win are given as 5–1, then from 6 races the horse is expected to lose 5 and win 1. The probability that the horse wins or loses can be calculated from the odds given. These calculations are shown below.

$$\text{P(win)} = \frac{n(\text{expected wins})}{n(\text{races})} = \frac{1}{6} \qquad \text{P(lose)} = \frac{n(\text{expected losses})}{n(\text{races})} = \frac{5}{6}$$

- If given odds of a–b, then:
  - $\text{P(win)} = \dfrac{b}{a+b}$
  - $\text{P(loss)} = \dfrac{a}{a+b}$

11.2.8 Payouts

- The payout in races is based on the odds given.
- If the odds are a–b, you can win \$$\frac{a}{b}$ for every \$1 bet or \$$a$ for every \$$b$ bet. The bookmaker will pay out your win plus the initial amount wagered.
- The TAB quotes a whole payout figure for a horse, made up of the winnings and the initial wager.

- For example:

Odds	Bet	Winnings	Payout figure
5–1	\$10	\$5 for every \$1 bet: $\frac{5}{1} \times \$10 = \$50$	\$60 (\$50 + \$10)
7–2	\$14	\$7 for every \$2 bet: $\frac{7}{2} \times \$14 = \$49$	\$63 (\$49 + \$14)

WORKED EXAMPLE 10

The odds given for the horse Gunnawin to win the Melbourne Cup are 9–4.

a Determine the probability of Gunnawin winning the Melbourne Cup.

b Tony decides to bet \$12 on Gunnawin to win. If the horse does win, what is Tony's payout?

c In the same race, the probability that the horse 'Can't Lose' wins is given as $\frac{5}{17}$. What are the odds that this horse will win?

THINK	WRITE
a 1 Write the number of ways Gunnawin can win (4) and the total number of outcomes (9 + 4 = 13).	**a** $n(\text{Gunnawin wins}) = 4$ $n(\xi) = 13$
2 Write the rule for probability.	$\text{P(Gunnawin wins)} = \dfrac{n(\text{Gunnawin wins})}{n(\xi)}$
3 Substitute the known values into the rule.	$\text{P(Gunnawin wins)} = \frac{4}{13}$
4 Write your answer.	The probability of Gunnawin winning the Melbourne Cup is $\frac{4}{13}$.

b **1** Explain what the ratio means and relate it to the bet.	**b** In the odds 9–4 the punter can win $\$\frac{9}{4}$ for every \$1 that is bet (or for every \$4 bet the punter will win \$9). Therefore, if Tony bets \$12 he will win $\frac{9}{4} \times \$12 = \$27$.
2 Add the original amount invested to the amount returned.	$\text{Payout} = \$27 + \$12$ $= \$39$
3 Write your answer.	Tony's payout will be \$39.
c **1** Look at the given fraction. The numerator corresponds to the 'win' component (second number) of the ratio.	**c** This horse has been given the chance of winning as $\frac{5}{17}$. Therefore its chance of losing is $\frac{12}{17}$.
2 The lose component of the ratio is always the first number.	Therefore the lose–win ratio is 12–5.
3 Write your answer.	The odds of Can't Lose winning the Melbourne Cup are 12–5.

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- **Try out this interactivity:** Random number generator (int-0089)
- **Complete this digital doc:** SkillSHEET: Set notation (doc-5286)
- **Complete this digital doc:** SkillSHEET: Simplifying fractions (doc-5287)
- **Complete this digital doc:** SkillSHEET: Determining complementary events (doc-5288)
- **Complete this digital doc:** SkillSHEET: Addition and subtraction of fractions (doc-5289)
- **Complete this digital doc:** SkillSHEET: Working with Venn diagrams (doc-5291)
- **Complete this digital doc:** SkillSHEET: Writing odds as probabilities (doc-5292)
- **Complete this digital doc:** SkillSHEET: Writing probabilities as odds (doc-5293)
- **Complete this digital doc:** SkillSHEET: Distinguishing between complementary and mutually exclusive events (doc-5294)
- **Complete this digital doc:** WorkSHEET: Introducing probability (doc-14590)

Exercise 11.2 Review of probability

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–8, 10, 12, 14, 16, 18, 20, 22, 23	Questions: 1, 3–6, 8, 9, 11, 13, 15, 17, 19, 21, 22, 23	Questions: 1, 2, 4, 5, 7, 9–13, 15, 17, 19–24

🟩🟦🟪 **Individual pathway interactivity:** int-4616

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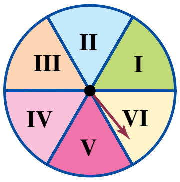
Fluency

1. Explain the difference between experimental and theoretical probability.

2. **WE1** The spinner shown at right was spun 50 times and the outcome each time was recorded in the table below.



Segment	I	II	III	IV	V	VI
Tally	10	6	8	7	12	7

a. List the event space.
b. Given the experimental results, determine the relative frequency for each segment.
c. The theoretical probability of the spinner landing on any particular segment with one spin is $\frac{1}{6}$. How could the experiment be changed to give a better estimate of the true probabilities?

3. A laptop company conducted a survey to see what were the most appealing colours for laptop computers among 15–18-year-old students. The results were as follows:

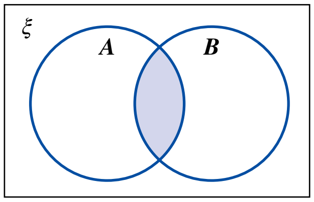
Colour	Black Black	Sizzling Silver	Power Pink	Blazing Blue	Gooey Green	Glamour Gold
Number	102	80	52	140	56	70

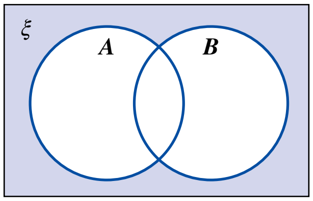
a. How many students were surveyed?
b. What is the relative frequency of students who found silver the most appealing laptop colour?
c. What is the relative frequency of students who found black and green to be their most appealing colours?
d. Which colour was found to be most appealing?

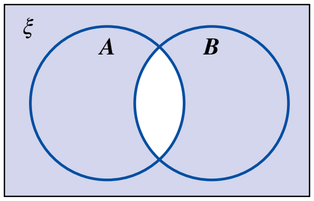
4. **WE2** Two dice are rolled and the values on the two uppermost faces are added together.
a. Construct a table to illustrate the sample space.
b. What is the most likely outcome?
c. What is the least likely outcome?

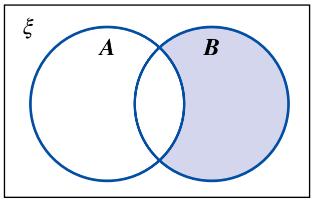
5. **WE7** Given $\text{P}(A) = 0.5$, $\text{P}(B) = 0.4$ and $\text{P}(A \cup B) = 0.8$:
a. use the Addition Law of probability to calculate the value of $\text{P}(A \cap B)$
b. draw a Venn diagram to represent the universal set
c. calculate $\text{P}(A \cap B')$.

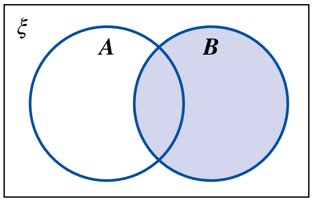
6. Let $\text{P}(A) = 0.25$, $\text{P}(B) = 0.65$ and $\text{P}(A \cap B) = 0.05$.
a. Calculate:
  i. $\text{P}(A \cup B)$
  ii. $\text{P}(A \cap B)'$.
b. **MC** Which Venn diagram below best illustrates $\text{P}(A \cap B)'$?

A.


B.


C.


D.


E.


7. **WE3** A die is rolled. What is the probability that the outcome is an even number or a 5?

8. **WE6** A card is drawn from a well-shuffled pack of 52 playing cards. Calculate:

a. P(a king is drawn)

b. P(a heart is drawn)

c. P(a king or a heart is drawn).

9. **WE4** For each of the following pairs of events:

i. state, giving justification, if the pair are complementary events

ii. alter the statements, where applicable, so that the events become complementary events.

a. Having Weet Bix or having Strawberry Pops for breakfast

b. Walking to a friend's place or driving there

c. Watching TV or reading as a leisure activity

d. Rolling a number less than 5 or rolling a number greater than 5 with a ten-sided die with faces numbered 1 to 10

e. Passing a maths test or failing a maths test

10. a. **WE8** Draw a Venn diagram representing the relationship between the following sets. Show the position of all the elements in the Venn diagram.

$$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$
$$A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$$
$$B = \{1, 4, 9, 16\}$$

b. Calculate:

i. $P(A)$ **ii.** $P(B)$ **iii.** $P(A \cap B)$

iv. $P(A \cup B)$ **v.** $P(A' \cap B')$.

11. Write the following odds as probabilities.

a. 5 – 1 **b.** 13 – 4 **c.** 7 – 1

12. **MC** The probability of $\frac{2}{9}$ written as odds is:

A. 7–2 **B.** 2–7 **C.** 2–9

D. 9–2 **E.** 11–9

Understanding

13. You and a friend are playing a dice game. You have an eight-sided die (with faces numbered 1 to 8 inclusive) and your friend has a six-sided die (with faces numbered 1 to 6 inclusive). You each roll your own die.

a. The person who rolls the number 4 wins. Is this game fair?

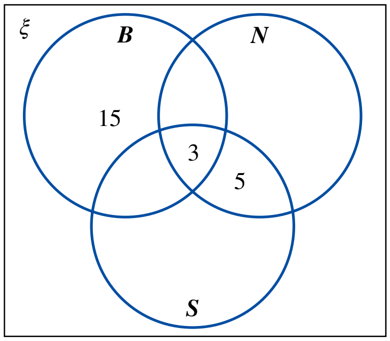
b. The person who rolls an odd number wins. Is this game fair?

14. A six-sided die has three faces numbered 5; the other faces are numbered 6. Are the events 'rolling a 5' and 'rolling a 6' equally likely?

15. 90 students were asked which lunchtime sports on offer, of basketball, netball and soccer, they had participated in, on at least one occasion in the last week. The results are shown in the following table.

Sport	Basketball	Netball	Soccer	Basketball and netball	Basketball and soccer	Netball and soccer	All three
Number of students	35	25	39	5	18	8	3

a. Copy and complete the Venn diagram shown below to illustrate the sample space.



b. How many students did not play basketball, netball or soccer at lunchtime?
c. How many students played basketball and/or netball but not soccer?
d. How many students are represented by the region (basketball ∩ not netball ∩ soccer)?
e. Calculate the relative frequency of the region described in part **d** above.
f. Estimate the probability that a student will play 3 of the sports offered.

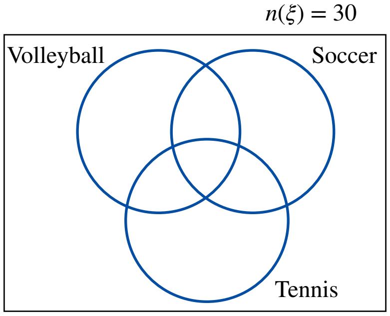
16. **WE5** A card is drawn from a shuffled pack of 52 cards. Find the probability that the card drawn is:

a. an ace
b. a club
c. a red card
d. not a jack
e. a green card
f. not a red card.

17. A bag contains 4 blue marbles, 7 red marbles and 9 yellow marbles. All marbles are of the same size. A marble is selected at random. What is the probability that the marble is:

a. blue
b. red
c. not yellow
d. black?

18. **WE9** Thirty students were asked which lunchtime sports they enjoyed — volleyball, soccer or tennis. Five students chose all three sports. Six students chose volleyball and soccer, 7 students chose volleyball and tennis, and 9 chose soccer and tennis. Fifteen students chose volleyball, 14 students chose soccer and 18 students chose tennis.



a. Copy the Venn diagram shown and enter the given information.
b. If a student is selected at random, determine the probability of selecting a student who:
  i. chose volleyball
  ii. chose all three sports
  iii. chose both volleyball and soccer but not tennis
  iv. did not choose tennis
  v. chose soccer.
c. Determine:
  i. P[(soccer ∪ tennis) ∩ volleyball′]
  ii. P[(volleyball ∪ tennis) ∩ soccer′].

19. **WE10** The odds given for the greyhound ‘Dog’s Breakfast’ to win its race are 7–3.

a. Determine the probability of Dog’s Breakfast winning its race.
b. Maria decides to bet \$15 on Dog’s Breakfast to win the race. If Dog’s Breakfast wins, calculate Maria’s payout.
c. The dog ‘Zoom Top’ is also in the race. If the probability of Zoom Top winning is $\frac{4}{13}$, what odds should be given for Zoom Top?

Reasoning

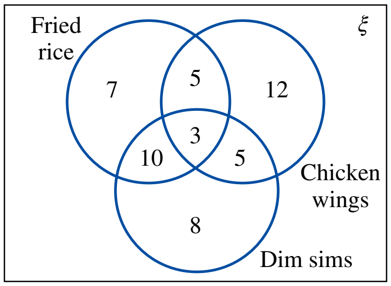
20. A six-sided die has three faces numbered 1 and the other three faces numbered 2. Are the events 'rolling a 1' and 'rolling a 2' equally likely?

21. Are the odds 10–6 the same as 5–3? Explain.

22. With the use of diagrams, show that $P(A' \cap B') = P(A \cup B)'$.

Problem solving

23. The Venn diagram at right shows the results of a survey completed by a Chinese restaurateur to find out the food preferences of his regular customers.



a. Determine the number of customers:
  i. surveyed
  ii. showing a preference for fried rice only
  iii. showing a preference for fried rice
  iv. showing a preference for chicken wings and dim sims.

b. A customer from this group won the draw for a lucky door prize. Determine the probability that this customer:
  i. likes fried rice
  ii. likes all three — fried rice, chicken wings and dim sims
  iii. prefers chicken wings only.

c. A similar survey was conducted a month later with another group of 50 customers. This survey yielded the following results: 2 customers liked all three foods; 6 preferred fried rice and chicken wings only; 7 preferred chicken wings and dim sims only; 8 preferred fried rice and dim sims only; 22 preferred fried rice; 23 preferred chicken wings; and 24 preferred dim sims.
  i. Display this information on a Venn diagram.
  ii. What is the probability of selecting a customer who prefers all three foods, if a random selection is made?

24. A pair of dice is rolled and the sum of the numbers shown is noted.

a. Show the sample space in a two-way table.

b. In how many different ways can the sum of 7 be obtained?

c. Are all outcomes equally likely?

d. Complete the given table.

Sum	2	3	4	5	6	7	8	9	11	12
Frequency										

e. What are the relative frequencies of the following sums?
  i. 2 **ii.** 7 **iii.** 11

f. What is the probability of obtaining the following sums?
  i. 2 **ii.** 7 **iii.** 11

g. If a pair of dice is rolled 300 times, how many times do you expect to obtain the sum of 7?

Reflection

What basic formula must be remembered in order to calculate simple probabilities?

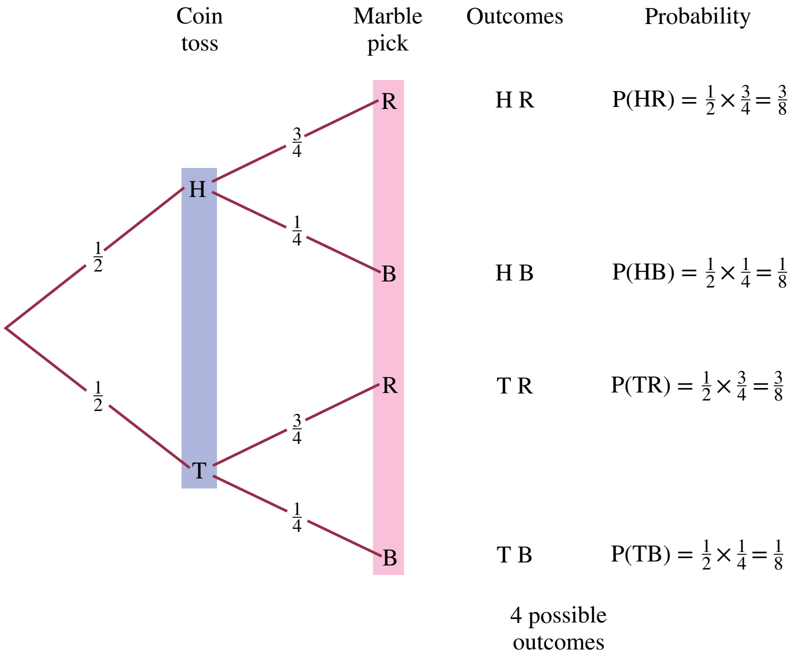
CHALLENGE 11.1

A drawer contains purple socks and red socks. The chance of obtaining a red sock is 2 in 9. There are 10 red socks in the drawer. What is the smallest number of socks that need to be added to the drawer so that the probability of drawing a red sock increases to 3 in 7?

11.3 Tree diagrams

11.3.1 Two-step chance experiments

- In **two-step chance experiments** the result is obtained after performing two trials. Two-step chance experiments are often represented using tree diagrams.
- **Tree diagrams** are used to list all possible outcomes of two or more events that are not necessarily equally likely.
- The probability of obtaining the result for a particular event is listed on the branches.
- The probability for each outcome in the sample space is the product of the probabilities associated with the respective branches. For example, the tree diagram shown here represents the sample space for flipping a coin, then choosing a marble from a bag containing three red marbles and one black marble.



- When added together, all the probabilities for the outcomes should sum to 1. They are complementary events. For example,

$$\begin{aligned} P(HR) + P(HB) + P(TR) + P(TB) &= \tfrac{3}{8} + \tfrac{1}{8} + \tfrac{3}{8} + \tfrac{1}{8} \\ &= 1 \end{aligned}$$

- Other probabilities can also be calculated from the tree diagram. For example, the probability of getting an outcome that contains a red marble can be calculated by summing the probabilities of each of the possible outcomes that include a red marble. Outcomes that contain a red marble are HR and TR, therefore:

$$\begin{aligned}\text{P(red marble)} &= \text{P(HR)} + \text{P(TR)}\\ &= \tfrac{3}{8} + \tfrac{3}{8}\\ &= \tfrac{6}{8}\\ &= \tfrac{3}{4}\end{aligned}$$

WORKED EXAMPLE 11

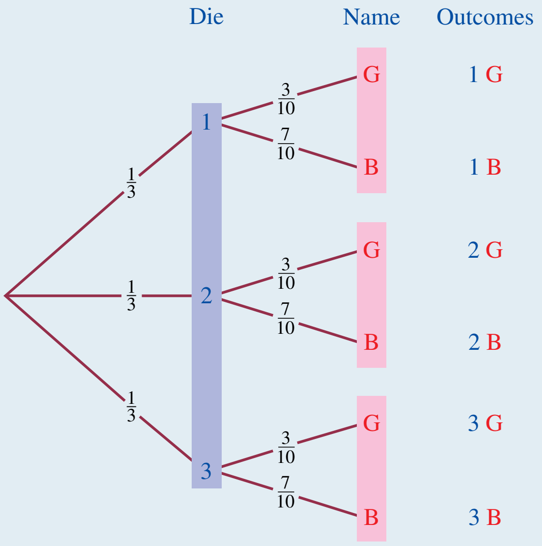
A three-sided die is rolled and a name is picked out of a hat that contains 3 girls' names and 7 boys' names.

a Use a tree diagram to display the sample space.

b Calculate the probability of:

i rolling a 3, then choosing a boy's name

ii choosing a boy's name after rolling an odd number.

THINK	WRITE
a 1 Draw 3 branches from the starting point to show the 3 possible outcomes of rolling a three-sided die (shown in blue), and then draw 2 branches off each of these to show the 2 possible outcomes of choosing a name out of a hat (shown in red).	**a** 
2 Write probabilities on the branches to show the individual probabilities of rolling a 1, 2 or 3 on a three-sided die. As these are equally likely outcomes, $P(1) = P(2) = P(3) = \frac{1}{3}$.	
3 Write probabilities on the branches to show the individual probabilities of choosing a name. Since there are 3 girls' names and 7 boys' names in the hat, $P(G) = \frac{3}{10}$ and $P(B) = \frac{7}{10}$.	
b i 1 Follow the pathway of rolling a 3 $\left[P(3) = \frac{1}{3}\right]$ and choosing a boy's name $\left[P(B) = \frac{7}{10}\right]$, and multiply the probabilities.	**b i** $\begin{aligned}P(3B) &= P(3) \times P(B)\\ &= \frac{1}{3} \times \frac{7}{10}\\ &= \frac{7}{30}\end{aligned}$
2 Write the answer.	The probability of rolling a 3, then choosing a boy's name is $\frac{7}{10}$.

ii 1 To roll an odd number (1 or 3) then choose a boy's name:

- roll a 1, then choose a boy's name *or*
- roll a 3, then choose a boy's name.

Find the probability of each of these and add them together to find the total probability.

ii $$\begin{aligned}\text{P(odd B)} &= \text{P(1B)} + \text{P(3B)}\\ &= \text{P(1)} \times \text{P(B)} + \text{P(3)} \times \text{P(B)}\\ &= \frac{1}{3} \times \frac{7}{10} + \frac{1}{3} \times \frac{7}{10}\\ &= \frac{7}{30} + \frac{7}{30}\\ &= \frac{14}{30}\\ &= \frac{7}{15}\end{aligned}$$

2 Write the answer.

The probability of choosing a boy's name after rolling an odd number is $\frac{7}{15}$.

11.3.2 Three-step chance experiments

- Outcomes are often made up of combinations of events. For example, when a coin is flipped three times, three of the possible outcomes are HHT, HTH and THH. These outcomes all contain 2 Heads and 1 Tail.
- The probability of an outcome with a particular order is written such that the order required is shown. For example, P(HHT) is the probability of H on the first die, H on the second die and T on the third die.
- The probability of an outcome with a particular combination of events in which the order is not important is written describing the particular combination required. For example, P(2 Heads and 1 Tail).

WORKED EXAMPLE 12

A coin is biased so that the chance of it falling as a Head when flipped is 0.75.

a Draw a tree diagram to represent the coin being flipped three times.

b Calculate the following probabilities:

i P(HTT) **ii P(1H and 2T)** **iii P(at least 2 Tails).**

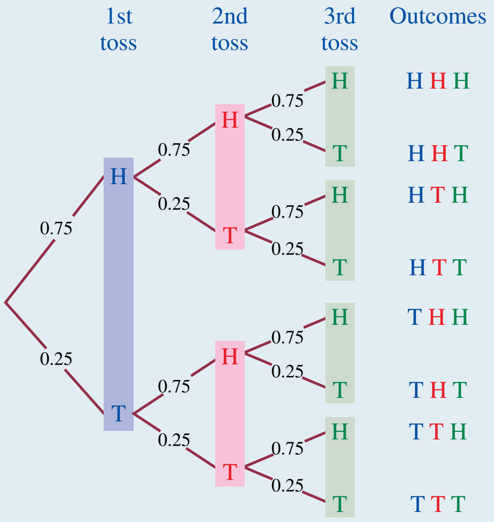
THINK

a 1 Tossing a coin has two outcomes. Draw 2 branches from the starting point to show the first toss, 2 branches off each of these to show the second toss and then 2 branches off each of these to show the third toss.

2 Write probabilities on the branches to show the individual probabilities of tossing a Head (0.75) and a Tail. Because tossing a Head and tossing a Tail are mutually exclusive, P(T) = 1 − P(H) = 0.25.

WRITE

a



b i • P(HTT) implies the order: H (0.75), T (0.25), T (0.25).

• Multiply the probabilities.

b i $$\begin{aligned}\text{P(HTT)} &= \text{P(H)} \times \text{P(T)} \times \text{P(T)}\\ &= (0.75) \times (0.25)^2\\ &= 0.047\end{aligned}$$

ii • P(1H and 2T) implies:
P(HTT), P(THT), P(TTH).
• Add these probabilities.

ii P(1H and 2T)
$= \text{P(HTT)} + \text{P(THT)} + \text{P(TTH)}$
$= 3(0.75 \times 0.25^2)$
$= 0.141$

iii • P(at least 2 Tails) implies: P(HTT), P(THT), P(TTH) and P(TTT).
• Add these probabilities.

iii P(at least 2T)
$= \text{P(HTT)} + \text{P(THT)} + \text{P(TTH)} + \text{P(TTT)}$
$= 3(0.75 \times 0.25^2) + 0.25^3$
$= 0.156$

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Complete this digital doc: SkillSHEET: Multiplying fractions for calculating probabilities (doc-5290)

Complete this digital doc: WorkSHEET: Tree diagrams (doc-14591)

Exercise 11.3 Tree diagrams

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9, 11	Questions: 1, 3–6, 8, 9, 11	Questions: 1–12

Individual pathway interactivity: int-4617

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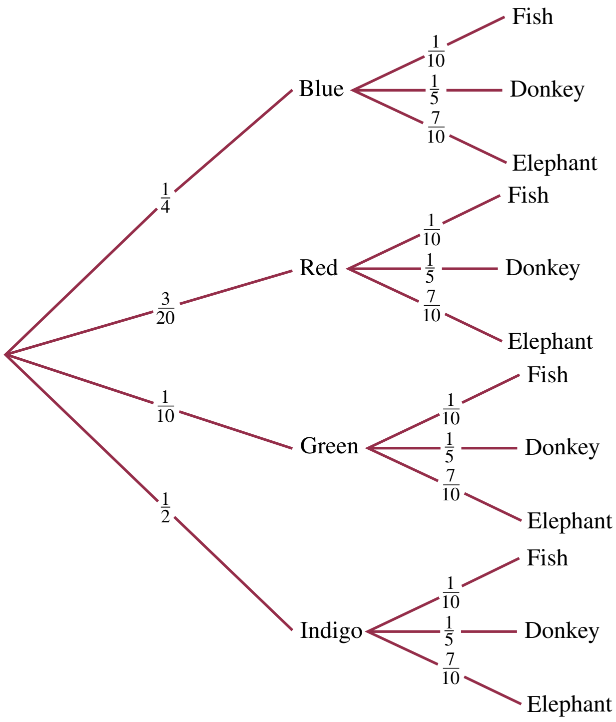
Fluency

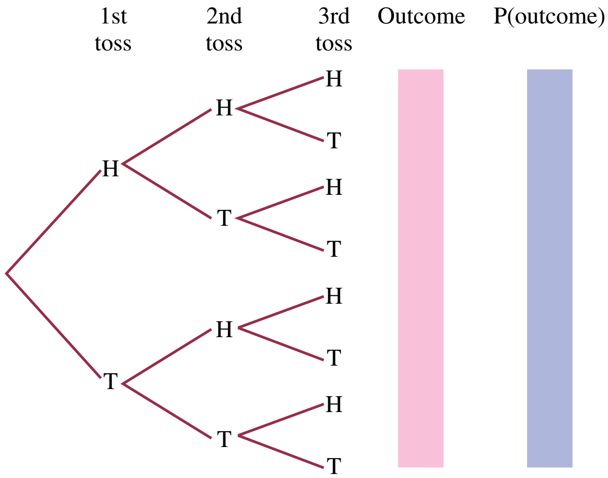
1. Explain how a tree diagram can be used to calculate probabilities of events that are not equally likely.

2. Use this tree diagram to answer the following questions.



a. How many different outcomes are there?
b. Are all outcomes equally likely? Explain.
c. Is getting a red fish more, less or equally likely than getting a green elephant?
d. What is the most likely outcome?
e. Calculate the following probabilities.
  i. P(blue elephant)
  ii. P(indigo elephant)
  iii. P(donkey)

3. a. Copy the tree diagram shown here and complete the labelling for tossing a biased coin three times when the chance of tossing one Head in one toss is 0.7.



b. What is the probability of tossing three Heads?

c. What is the probability of getting at least one Tail?

d. What is the probability of getting exactly two Tails?

4. The questions below relate to rolling a fair die.

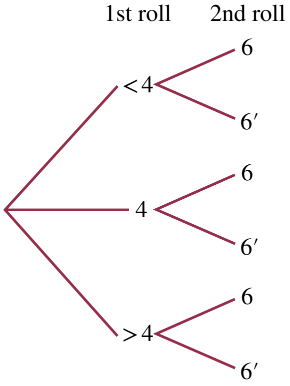
a. What is the probability of each of the following outcomes from one roll of a die?

i. P(rolling number < 4)

ii. P(rolling a 4)

iii. P(rolling a number other than a 6)

b. The tree diagram shown below has been condensed to depict rolling a die twice, noting the number relative to 4 on the first roll and 6 on the second. Complete a labelled tree diagram, showing probabilities on the branches and all outcomes, similar to that shown.



c. What is the probability of rolling the following with 2 rolls of the die?

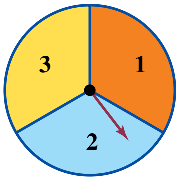
i. P(a 4 then a 6)

ii. P(a number less than 4 then a 6)

iii. P(a 4 then 6′)

iv. P(a number > 4 and then a number < 6)

5. **WE11** The spinner shown at right is divided into 3 equal-sized wedges labelled 1, 2 and 3. It is spun three times, and it is noted whether the spinner lands on a prime number, P = {2, 3} = 'prime', or not a prime number, $P' = \{1\}$ = 'not prime'.



a. Construct a labelled tree diagram for 3 spins of the spinner, showing probabilities on the branches and all possible outcomes.

b. Find the following probabilities.

i. P(3 prime numbers)

ii. P(PPP′ in this order)

iii. P(PPP′ in any order)

Understanding

6. **WE12** A coin is biased so that the chance of it falling as a Tail when tossed is 0.2.

a. Draw a tree diagram to represent the coin being tossed three times.

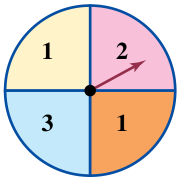
b. What is the probability of getting the same outcome on each toss?

7. A die is tossed twice and each time it is recorded whether or not the number is a multiple of 3. If M = the event of getting a multiple of 3 on any one toss and M' = the event of not getting a multiple of 3 on any one toss:

a. draw a tree diagram to represent the 2 tosses

b. what is the probability of getting two multiples of 3?

8. The biased spinner illustrated is spun three times.



a. Draw a completely labelled tree diagram for 3 spins of the spinner, showing probabilities on the branches and all possible outcomes and associated probabilities.

b. What is the probability of getting exactly 2 *ones*?

c. What is the probability of getting at most 2 *ones*?

Reasoning

9. A restaurant offers its customers a three-course dinner, where they choose between two entrées, three main meals and two desserts. The managers find that 30% choose soup and 70% choose prawn cocktail for the entrée; 20% choose vegetarian, 50% chicken, and the rest have beef for their main meal; and 75% have sticky date pudding while the rest have apple crumble for dessert.

a. Draw a fully labelled tree diagram showing all possible choices.

b. What is the probability that a customer will choose the soup, chicken and sticky date pudding?

c. If there are 210 people booked for the following week at the restaurant, how many would you expect to have the meal combination referred to in part **b**?

10. A bag contains 7 red and 3 white balls. A ball is taken at random, its colour noted and it is then placed back in the bag before a second ball is chosen at random and its colour noted.

a. **i.** Show the possible outcomes with a fully labelled tree diagram.

ii. As the first ball was chosen, how many balls were in the bag?

iii. As the second ball was chosen, how many balls were in the bag?

iv. Does the probability of choosing a red or white ball change from the first selection to the second? Explain.

v. Calculate the probability of choosing a red ball twice.

b. Suppose that after the first ball had been chosen it was not placed back in the bag.

i. As the second ball is chosen, how many balls are in the bag?

ii. Does the probability of choosing a red or white ball change from the first selection to the second? Explain.

iii. Construct a fully labelled tree diagram to show all possible outcomes.

iv. Calculate the probability of choosing two red balls.

Problem solving

11. An eight-sided die is rolled three times to see whether 5 occurs.
- **a.** Draw a tree diagram to show the sample space.
- **b.** Calculate:
  - **i.** P(three 5s)
  - **ii.** P(no 5s)
  - **iii.** P(two 5s)
  - **iv.** P(at least two 5s).

12. A tetrahedral die (four faces labelled 1, 2, 3 and 4) is rolled and a coin is tossed simultaneously.
- **a.** Show all the outcomes on a two-way table.
- **b.** Draw a tree diagram and list all outcomes and their respective probabilities.
- **c.** Calculate the probability of getting a Head on the coin and an even number on the die.

Reflection

WHOLE CLASS

What strategies would you use to remember how to construct tree diagrams?

11.4 Independent and dependent events

11.4.1 Independent events

- If a coin is tossed the outcome is a Head or a Tail. The outcome of the first toss does not affect the outcome of the next toss of the coin. The second toss will still yield a Head or a Tail irrespective of the outcome of the first toss. Similarly, the outcome on the roll of a die will not affect the outcome of the next roll.
- If successive events have no effect on each other, they are called **independent events**.
- If events A and B are independent then the **Multiplication Law of probability** states that:
  $P(A \text{ and } B) = P(A) \times P(B)$ or $P(A \cap B) = P(A) \times P(B)$
- The reverse is also true. If:
  $P(A \text{ and } B) = P(A) \times P(B)$ or $P(A \cap B) = P(A) \times P(B)$
  is true then event A and event B are independent events.

WORKED EXAMPLE 13

Adam is one of the 10 young golfers to represent his state. Paz is one of the 12 netball players to represent her state. All the players in their respective teams have an equal chance of being nominated as captains.

a Are the events 'Adam is nominated as captain' and 'Paz is nominated as captain' independent?

b Determine:

i P(Adam is nominated as captain)

ii P(Paz is nominated as captain).

c What is the probability that both Adam and Paz are nominated as captains of their respective teams?

THINK	WRITE
a Determine whether the given events are independent and write your answer.	a Adam's nomination has nothing to do with Paz's nomination and vice versa. Therefore, the events are independent.
b i 1 Determine the probability of Adam being nominated as captain. He is one of 10 players.	b i $\text{P(Adam is nominated)} = P(A)$ $= \dfrac{n(\text{Adam is nominated})}{n(\xi)}$ $\text{P(Adam is nominated)} = \dfrac{1}{10}$

2 Write your answer.	The probability that Adam is nominated as captain is $\frac{1}{10}$.
ii 1 Determine the probability of Paz being nominated as captain. She is one of 12 players.	**ii** $\text{P(Paz is nominated)} = \text{P}(A)$ $= \frac{n(\text{Paz is nominated})}{n(\xi)}$ $\text{P(Paz is nominated)} = \frac{1}{12}$
2 Write your answer.	The probability that Paz is nominated as captain is $\frac{1}{12}$.
c 1 Write the Multiplication Law of probability for independent events.	**c** $\text{P}(A \text{ and } P)$ $= \text{P}(A \cap P)$ $= \text{P}(A) \times \text{P}(P)$ P(Adam and Paz are nominated) = P(Adam is nominated) × P(Paz is nominated)
2 Substitute the known values into the rule.	$= \frac{1}{10} \times \frac{1}{12}$
3 Evaluate.	$= \frac{1}{120}$
4 Write your answer.	The probability that both Adam and Paz are nominated as captains is $\frac{1}{120}$.

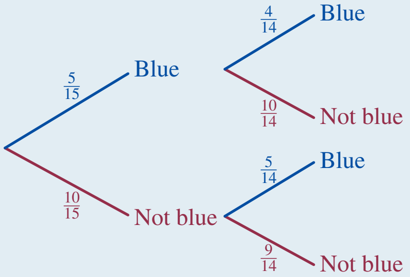
11.4.2 Dependent events

- Sometimes one event affects the outcome of another. For example, if a card is drawn from a pack of playing cards, the probability that its suit is hearts, P(hearts), is $\frac{13}{52}$ (or $\frac{1}{4}$). If this card is a heart and is not replaced, then this will affect the probability of subsequent draws. The probability that the second card drawn is a heart will be $\frac{12}{51}$, while the probability that the second card is not a heart will be $\frac{39}{51}$.
- When one event affects the occurrence of another, the events are called **dependent events**.
- If two events are dependent, then the probability of occurrence of one event affects that of the subsequent event.

WORKED EXAMPLE 14

A bag contains 5 blue, 6 green and 4 yellow marbles. The marbles are identical in all respects except in their colours. Two marbles are picked in succession without replacement. Determine the probability of picking 2 blue marbles.

THINK	WRITE
1 Determine the probability of picking the first blue marble.	$\text{P(picking a blue marble)} = \frac{n(B)}{n(\xi)}$ $\text{P(picking a blue marble)} = \frac{5}{15}$ $= \frac{1}{3}$

2 Determine the probability of picking the second blue marble. *Note:* The two events are dependent since marbles are not being replaced. Since we have picked a blue marble this leaves 4 blue marbles remaining out of a total of 14 marbles.	$\text{P(picking second blue marble)} = \frac{n(B)}{n(\xi)}$ $\text{P(picking second blue marble)} = \frac{4}{14}$ $= \frac{2}{7}$
3 Calculate the probability of obtaining 2 blue marbles.	$\text{P(2 blue marbles)} = \text{P(1st blue)} \times \text{P(2nd blue)}$ $= \frac{1}{3} \times \frac{2}{7}$ $= \frac{2}{21}$
4 Write your answer.	The probability of obtaining 2 blue marbles is $\frac{2}{21}$.
Note: Alternatively, a tree diagram could be used to solve this question. The probability of selecting 2 blue marbles successively can be read directly from the first branch of the tree diagram.	 $\text{P(2 blue marbles)} = \frac{5}{15} \times \frac{4}{14}$ $= \frac{1}{3} \times \frac{2}{7}$ $= \frac{2}{21}$

learnon RESOURCES — ONLINE ONLY

- **Try out this interactivity:** Independent and dependent events (int-2787)
- **Try out this interactivity:** Random numbers (int-0085)

Exercise 11.4 Independent and dependent events

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 9, 11, 13, 16	Questions: 1, 3–6, 8, 9, 11, 12, 14, 16	Questions: 1–17

🟩🟦🟪 **Individual pathway interactivity:** int-4618

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Fluency

1. If A and B are independent events and $\mathrm{P}(A) = 0.7$ and $\mathrm{P}(B) = 0.4$, calculate:

- **a.** $\mathrm{P}(A \text{ and } B)$
- **b.** $\mathrm{P}(A' \text{and } B)$ where A' is the complement of A
- **c.** $\mathrm{P}(A \text{ and } B')$ where B' is the complement of B
- **d.** $\mathrm{P}(A' \text{and } B')$.

Understanding

2. **WE13** A die is rolled and a coin is tossed.

- **a.** Are the outcomes independent?
- **b.** Determine:
  - **i.** P(Head) on the coin
  - **ii.** P(6) on the die.
- **c.** Determine P(6 on the die and Head on the coin).

3. A tetrahedron (4-faced) die and a 10-sided die are rolled simultaneously. What is the probability of getting a 3 on the tetrahedral die and an 8 on the 10-sided die?

4. A blue die and a green die are rolled. What is the probability of getting a 5 on the blue die and not a 5 on the green die?

5. Dean is an archer. The experimental probability that Dean will hit the target is $\frac{4}{5}$.

- **a.** What is the probability that Dean will hit the target on two successive attempts?
- **b.** What is the probability that Dean will hit the target on three successive attempts?
- **c.** What is the probability that Dean will not hit the target on two successive attempts?
- **d.** What is the probability that Dean will hit the target on the first attempt but miss on the second attempt?

6. **MC** A bag contains 20 apples, of which 5 are bruised. Peter picks an apple and realises that it is bruised. He puts the apple back in the bag and picks another one.

- **a.** The probability that Peter picks 2 bruised apples is:
  - **A.** $\frac{1}{4}$
  - **B.** $\frac{1}{2}$
  - **C.** $\frac{1}{16}$
  - **D.** $\frac{3}{4}$
  - **E.** $\frac{15}{16}$
- **b.** The probability that Peter picks a bruised apple first but a good one on his second attempt is:
  - **A.** $\frac{1}{4}$
  - **B.** $\frac{1}{2}$
  - **C.** $\frac{3}{4}$
  - **D.** $\frac{3}{16}$
  - **E.** $\frac{1}{16}$

7. The probability that John will be late for a meeting is $\frac{1}{7}$ and the probability that Phil will be late for a meeting is $\frac{3}{11}$. What is the probability that:

a. John and Phil are both late
b. neither of them is late
c. John is late but Phil is not late
d. Phil is late but John is not late?

8. On the roulette wheel at the casino there are 37 numbers, 0 to 36 inclusive. Bidesi puts his chip on number 8 in game 20 and on number 13 in game 21.

a. What is the probability that he will win in game 20?
b. What is the probability that he will win in both games?
c. What is the probability that he wins at least one of the games?

9. Based on her progress through the year, Karen was given a probability of 0.8 of passing the Physics exam. If the probability of passing both Maths and Physics is 0.72, what is her probability of passing the Maths exam?

10. Suresh found that, on average, he is delayed 2 times out of 7 at Melbourne airport. Rakesh made similar observations at Brisbane airport, but found he was delayed 1 out of every 4 times. Find the probability that both Suresh and Rakesh will be delayed if they are flying out of their respective airports.

11. Bronwyn has 3 pairs of Reebok and 2 pairs of Adidas running shoes. She has 2 pairs of Reebok, 3 pairs of Rio and a pair of Red Robin socks. Preparing for an early morning run, she grabs at random for a pair of socks and a pair of shoes. What is the probability that she chooses:

a. Reebok shoes and Reebok socks
b. Rio socks and Adidas shoes
c. Reebok shoes and Red Robin socks
d. Adidas shoes and socks that are not Red Robin?

12. **WE14** Two cards are drawn successively and without replacement from a pack of playing cards. Determine the probability of drawing:

a. 2 hearts
b. 2 kings
c. 2 red cards.

13. In a class of 30 students there are 17 girls. Two students are picked randomly to represent the class in the Student Representative Council. Determine the probability that:

a. both students are boys
b. both students are girls
c. one of the students is a boy.

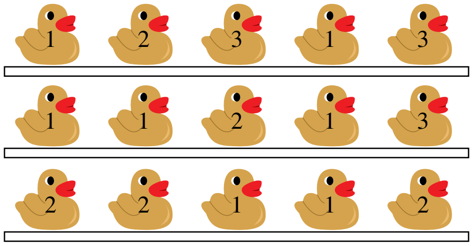
Reasoning

14. Greg has tossed a tail on each of 9 successive coin tosses. He believes that his chances of tossing a Head on his next toss must be very high. Is Greg correct? Justify your answer.

15. The multiplication law of probability relates to *independent events*. Tree diagrams can illustrate the sample space of successive *dependent events* and the probability of any one combination of events can be calculated by *multiplying* the stated probabilities along the branches. Is this a contradiction to the multiplication law of probability? Explain.

Problem solving

16. A game at a carnival requires blindfolded contestants to throw balls at numbered ducks sitting on 3 shelves. The game ends when 3 ducks have been knocked off the shelves. Assume that the probability of hitting each duck is equal.



a. Are the events described in the game dependent or independent?
b. What are the initial probabilities of hitting each number?
c. Draw a labelled tree diagram to show the possible outcomes for a contestant.
d. Calculate the probabilities of hitting the following:
  i. P(1, 1, 1)
  ii. P(2, 2, 2)
  iii. P(3, 3, 3)
  iv. P(at least one 3).

17. Question **16** described a game at a carnival. A contestant pays $3 to play and must make 3 direct hits to be eligible for a prize. The numbers on the ducks hit are then summed and the contestant wins a prize according to the winners table.

WINNERS' TABLE	
TOTAL SCORE	**PRIZE**
9	Major prize ($30 value)
7–8	Minor prize ($10 value)
5–6	$2 prize
3–4	No prize

a. Calculate the probability of winning each prize listed.
b. Suppose 1000 games are played on an average show day. What profit could be expected to be made by the sideshow owner on any average show day?

Reflection

How are dependent events, independent events and the multiplication law of probability reflected on a tree diagram?

11.5 Conditional probability

11.5.1 Conditional probability

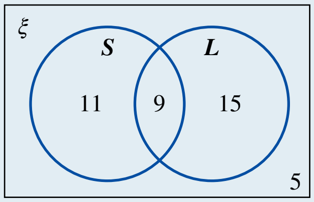
- **Conditional probability** is when the probability of an event is conditional (depends) on another event occurring first.
- The effect of conditional probability is to reduce the event space and thus increase the probability of the desired outcome.
- For two events, A and B, the conditional probability of event B, *given* that event A occurs, is denoted by $\mathrm{P}(B|A)$ and can be calculated using the formula:

$$\mathbf{P}(B|A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)}, \quad \mathbf{P}(A) \neq 0$$

- Conditional probability can be expressed using a variety of language. Some examples of conditional probability statements follow. The key words to look for in a conditional probability statement have been highlighted in each instance.
  - **If** a student receives a B+ or better in their first Maths test, **then** the chance of them receiving a B+ or better in their second Maths test is 75%.
  - **Given that** a red marble was picked out of the bag with the first pick, the probability of a blue marble being picked out with the second pick is 0.35.
  - **Knowing that** the favourite food of a student is hot chips, the probability of their favourite condiment being tomato sauce is 68%.

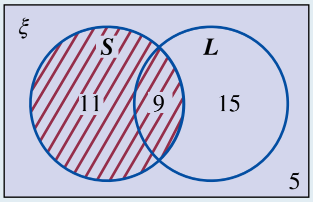
WORKED EXAMPLE 15

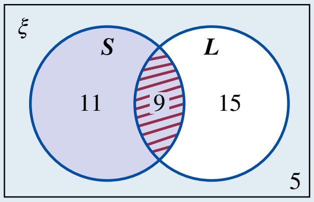
A group of students was asked to nominate their favourite food, spaghetti (S) or lasagne (L). The results are illustrated in the Venn diagram at right. Use the Venn diagram to calculate the following probabilities relating to a student's favourite food.



a What is the probability that a randomly selected student prefers spaghetti?

b What is the probability that a randomly selected student likes lasagne given that they also like spaghetti?

THINK	WRITE/DRAW
a 1 From 40 students surveyed, shown in blue, 20 nominated their favourite food as 'spaghetti' or 'spaghetti and lasagne' as shown in red.	**a** 
2 The probability that a randomly selected student prefers spaghetti is found by substituting these values into the probability formula.	$\mathrm{P(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$ $\mathrm{P(spaghetti)} = \frac{20}{40}$ $= \frac{1}{2}$

b **1** The condition imposed ‘given they also like spaghetti’ alters the sample space to the 20 students described in part **a**, as shaded in blue. Of these 20 students, 9 stated their favourite foods as lasagne and spaghetti, as shown in red.	**b** 
2 The probability that a randomly selected student likes lasagne, given that they like spaghetti, is found by substituting these values into the probability formula for conditional probability.	$\mathrm{P}(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ $\mathrm{P}(L \mid S) = \dfrac{\frac{9}{40}}{\frac{1}{2}}$ $= \frac{9}{20}$

WORKED EXAMPLE 16

If $\mathrm{P}(A) = 0.3$, $\mathrm{P}(B) = 0.5$ and $\mathrm{P}(A \cup B) = 0.6$, calculate:

a $\mathrm{P}(A \cap B)$

b $\mathrm{P}(B \mid A)$

THINK	WRITE
a **1** State the addition law for probability to determine $\mathrm{P}(A \cup B)$.	**a** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
2 Substitute the values given in the question into this formula and simplify.	$0.6 = 0.3 + 0.5 - \mathrm{P}(A \cap B)$ $\mathrm{P}(A \cap B) = 0.3 + 0.5 - 0.6$ $= 0.2$
b **1** State the formula for conditional probability.	**b** $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}, \mathrm{P}(A) \neq 0$
2 Substitute the values given in the question into this formula and simplify.	$\mathrm{P}(B \mid A) = \dfrac{0.2}{0.3}$ $= \frac{2}{3}$

- It is possible to transpose the formula for conditional probability to calculate $\mathrm{P}(A \cap B)$:

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}, \mathrm{P}(A) \neq 0$$
$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A)$$

This is called the multiplication rule for probability.

Exercise 11.5 Conditional probability

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 9, 11, 14	Questions: 1, 3–6, 8, 9, 11, 12, 14	Questions: 1–15

🟩🟦🟪 **Individual pathway interactivity:** int-4619

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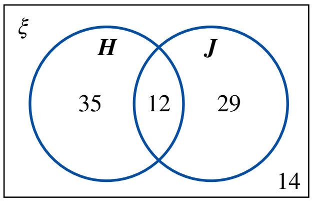
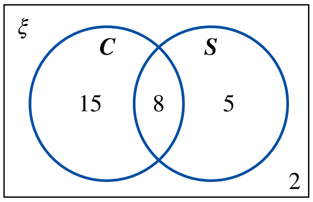
Fluency

1. **WE15** A group of students was asked to nominate their favourite form of dance, hip hop (H) or jazz (J). The results are illustrated in the Venn diagram below. Use the Venn diagram given to calculate the following probabilities relating to a student's favourite form of dance.



   a. What is the probability that a randomly selected student prefers jazz?
   b. What is the probability that a randomly selected student prefers hip hop, given that they prefer jazz?

2. A group of students was asked which seats they found most comfortable, the seats in the computer lab or the science lab. The results are illustrated in the Venn diagram below. Use the Venn diagram given to calculate the following probabilities relating to the most comfortable seats.



   a. What is the probability that a randomly selected student prefers the science lab?
   b. What is the probability that a randomly selected student prefers the science lab, given that they might prefer the computer lab or the science lab?

3. **WE16** If $P(A) = 0.7,\ P(B) = 0.5$ and $P(A \cup B) = 0.9$, calculate:
   a. $P(A \cap B)$
   b. $P(B|A)$.

4. If $P(A) = 0.65,\ P(B) = 0.75$ and $P(A \cap B) = 0.45$, calculate:
   a. $P(B|A)$
   b. $P(A|B)$.

Understanding

5. A medical degree requires applicants to participate in two tests, an aptitude test and an emotional maturity test. This year 52% passed the aptitude test and 30% passed both tests. Use the conditional probability formula to calculate the probability that a student who passed the aptitude test also passed the emotional maturity test.

6. At a school classified as a 'Music school for excellence', the probability that a student elects to study Music and Physics is 0.2. The probability that a student takes Music is 0.92. What is the probability that a student takes Physics, given that the student is taking Music?

7. The probability that a student is well and misses a work shift the night before an exam is 0.045, and the probability that a student misses a work shift is 0.05. What is the probability that a student is well, given they miss a work shift the night before an exam?

8. Two marbles are chosen, without replacement, from a jar containing only red and green marbles. The probability of selecting a green marble and then a red marble is 0.67. The probability of selecting a green marble on the first draw is 0.8. What is the probability of selecting a red marble on the second draw, given the first marble drawn was green?

9. Consider rolling a red and a black die and the probabilities of the following events:

Event A	the red die lands on 5
Event B	the black die lands on 2
Event C	the sum of the dice is 10.

a. **MC** The initial probability of each event described is:

A. $P(A) = \frac{1}{6}$
$P(B) = \frac{1}{6}$
$P(C) = \frac{1}{6}$

B. $P(A) = \frac{5}{6}$
$P(B) = \frac{2}{6}$
$P(C) = \frac{7}{36}$

C. $P(A) = \frac{5}{6}$
$P(B) = \frac{2}{6}$
$P(C) = \frac{5}{18}$

D. $P(A) = \frac{1}{6}$
$P(B) = \frac{1}{6}$
$P(C) = \frac{1}{12}$

E. $P(A) = \frac{1}{6}$
$P(B) = \frac{2}{6}$
$P(C) = \frac{1}{12}$

b. Calculate the following probabilities.

i. $P(A|B)$ **ii.** $P(B|A)$ **iii.** $P(C|A)$ **iv.** $P(C|B)$

10. **MC** A group of 80 schoolgirls consists of 54 dancers and 35 singers. Each member of the group is either a dancer or a singer, or both. The probability that a randomly selected student is a singer given that she is a dancer is:

A. 0.17 **B.** 0.44 **C.** 0.68 **D.** 0.11 **E.** 0.78

Reasoning

11. Explain how imposing a condition alters probability calculations.

12. At your neighbouring school, 65% of the students are male and 35% are female. Of the male students, 10% report that dancing is their favourite activity; of the female students, 25% report that dancing is their favourite activity.

Find the probability that:

a. a student selected at random prefers dancing and is female

b. a student selected at random prefers dancing and is male.

13. Using the information presented in question **12**, construct a tree diagram. From your diagram, calculate:

a. the probability that a student is male and does not prefer dancing

b. the overall percentage of students who prefer dancing.

Problem solving

14. Two marbles are chosen, without replacement, from a jar containing only red and green marbles. The probability of selecting a green marble and then a red marble is 0.72. The probability of selecting a green marble on the first draw is 0.85. What is the probability of selecting a red marble on the second draw if the first marble drawn was green?

15. When walking home from school during the summer months, Harold buys either an ice-cream or a drink from the corner shop. If Harold bought an ice-cream the previous day, there is a 30% chance that he will buy a drink the next day. If he bought a drink the previous day, there is a 40% chance that he will buy an ice-cream the next day. On Monday, Harold bought an ice-cream. Determine the probability that he buys an ice-cream on Wednesday.

Reflection

How does imposing a condition alter the probability of an event?

CHALLENGE 11.2

Four letters, A, B, C and D, are arranged in a row. What is the probability that A and B will always be together?

11.6 Subjective probability

11.6.1 Subjective probability

- Consider the following claims:
  ‘I feel the Australian cricket team will win this year’s Test cricket series because, in my opinion, they have a stronger side than the opposition.’ Claims like this are often made by people who may not have all the facts, and may also be biased.

  ‘I think this summer will be a cold one.’ A statement like this will have merit if it comes from an individual with relevant knowledge, such as a meteorologist or a scientist. However, often people make these remarks with limited observation.
- **Subjective probability** is usually based on limited mathematical evidence and may involve one or more of the following: judgements, opinions, assessments, estimations and conjectures by individuals. It can also involve beliefs, sentiments and emotions that may result in a certain amount of bias.

WORKED EXAMPLE 17

On Anzac Day Peter plays two-up, which involves tossing two coins. Heads win if both coins land Heads, while Tails win if both coins land Tails. If the coins land with one Head and one Tail they are called 'odd', and the coins are tossed again until either Heads or Tails wins.

After observing for a while, Peter notices that the last five tosses had either Tails winning or were odd. This leads Peter to believe that Heads will win the next game, so he places $50 on Heads and loses. Peter questions the fairness of the game and states that the game is biased and favours Tails. Discuss the accuracy of Peter's statement.

THINK	WRITE
Discuss the statement made and comment on the probability of obtaining Heads or Tails in this particular game.	Each game is independent, so five Tails or odd outcomes in the previous games have no effect on the outcome of the current game. The game is not biased. Peter took a risk and paid for it. He is wrong in suggesting that the game is not fair.

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Exercise 11.6 Subjective probability

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–6	Questions: 1–7	Questions: 1–7

🟩🟦🟪 **Individual pathway interactivity:** int-4620

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Understanding

1. WE17 Discuss the accuracy of these statements.

- **a.** The team batting last can never win a cricket match at the MCG.
- **b.** The Australian cricket team is so good that not even bad weather can stop it from winning.
- **c.** Two children in John's family are girls so the third one will be a girl, too.
- **d.** The Wallabies defeated the All Blacks three times last year so they will win the first game this year.
- **e.** It rained heavily on the last three consecutive Fridays so do not organise sport on a Friday.
- **f.** According to the weather report only three in every twenty houses were damaged by the cyclone, so my house will not be damaged.

g. New Zealand lost its cricket match against Australia because their team uniform looked boring.
h. This coin is biased because we obtained six Heads in a row.
i. The USA topped the medal tally in the last Olympics so they will do the same again in the next Olympics.
j. Australian Rules football is the best sport in the world.

2. Comment on the accuracy of these statements.
a. I have bought only one ticket for the raffle, therefore I cannot win.
b. This particular horse has odds of 1–2. It is certain to win.
c. If you keep on betting on Heads, you cannot lose.
d. If you want to win at all times, bet on the favourites.
e. It is no use betting on the favourites as you cannot win a great deal of money, therefore you should bet on the outsiders.

3. Assign a probability to each of the following, based on your experience or judgement.
a. The probability that you will be late for a class this week
b. The probability that your favourite sporting team will win its next match
c. The probability that two traffic lights in a row will be red when you approach successive intersections
d. The probability that you will see a dog some time today

Reasoning

4. Comment on the contradictions involved in the following statements.
a. That job was hers but she did not do well in the interview.
b. The team had won the match but they became a little complacent towards the end.
c. 'Makybe Diva' was certain to win. I cannot believe she lost the race.

5. Explain why each of the following examples involves subjective probability.
a. A business must decide whether or not to market a new type of product. The decision will be based on the fact that prior information suggests that there is an 80% chance of the product having market acceptance.
b. I estimate that I have a 60% chance of being married by the time I am 30 years old.
c. The probability that the housing market bubble will burst in the next 12 months is estimated to be 0.65.

Problem solving

6. A school is staging a musical and must have the scenery and programs ready for the opening night. The following probabilities are estimated for the duration of the activities.

Task	5 days	6 days	7 days
Build scenery	0.1	0.45	0.45
Paint scenery	0.3	0.4	0.3
Print programs	0.5	0.3	0.2

a. What is the probability that the painting and building of the scenery takes exactly 13 days?
b. What is the probability that the three tasks together take exactly 19 days?

7. It has been estimated that there is a 60% chance of a major earthquake (7.5–8 on the Richter scale) occuring in Southern California along the southern portion of the San Andreas fault within the next thirty years. Based on this prediction, calculate the probability:
a. that Alaska, Chile and California, all of which lie on the San Andreas fault, will each suffer a major earthquake in the next thirty years
b. that two of the three regions will suffer a major earthquake in the next thirty years.

11.7 Review

11.7.1 Review questions

Fluency

1. **MC** Which of the following is always true for an event, M, and its complementary event, M'?
A. $\mathrm{P}(M) + \mathrm{P}(M') = 1$ **B.** $\mathrm{P}(M) - \mathrm{P}(M') = 1$ **C.** $\mathrm{P}(M) + \mathrm{P}(M') = 0$
D. $\mathrm{P}(M) - \mathrm{P}(M') = 0$ **E.** $\mathrm{P}(M) \times \mathrm{P}(M') = 1$

2. **MC** A number is chosen from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Which of the following pairs of events is mutually exclusive?
A. $\{2, 4, 6\}$ and $\{1, 2, 3\}$ **B.** $\{1, 2, 3, 5\}$ and $\{4, 6, 7, 8\}$
C. $\{0, 1, 2, 3\}$ and $\{3, 4, 5, 6\}$ **D.** {multiples of 2} and {factors of 8}
E. {even numbers} and {multiples of 3}

3. **MC** Which of the following states the Multiplication Law of probability correctly?
A. $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B)$ **B.** $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$ **C.** $\mathrm{P}(A \cup B) = \mathrm{P}(A) \times \mathrm{P}(B)$
D. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ **E.** $\mathrm{P}(A) = \mathrm{P}(A \cup B) + \mathrm{P}(B)$

4. **MC** The odds 3–2 expressed as a probability are:
A. $\frac{1}{5}$ **B.** $\frac{3}{5}$ **C.** $\frac{1}{2}$ **D.** $\frac{2}{5}$ **E.** $\frac{1}{3}$

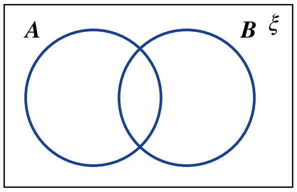
The following information relates to questions **5** and **6**.
$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 8\}$

5. **MC** $A \cap B$ equals:
A. $\{2, 3, 3, 4, 4, 5, 8\}$ **B.** $\{3, 4\}$ **C.** $\{2, 3, 4\}$
D. $\{2, 3, 4, 5, 8\}$ **E.** $\{2, 5, 8\}$

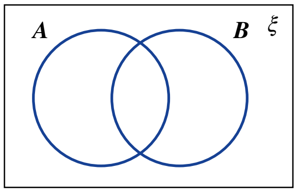
6. **MC** $A \cap B'$ equals:
A. $\{3, 4\}$ **B.** $\{2\}$ **C.** $\{2, 3, 4, 5, 8\}$
D. $\{2, 3, 4\}$ **E.** $\{1, 2, 6, 7, 9, 10\}$

7. Shade the region stated for each of the following Venn diagrams.

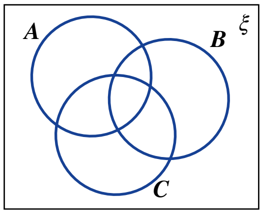
a. $A' \cup B$



b. $A' \cap B'$



c. $A' \cap B' \cap C$



8. Convert the following odds to probabilities.

a. 3–7 **b.** 5–2 **c.** 12–5

9. Convert the following probabilities to odds.

a. $\frac{7}{11}$ **b.** $\frac{6}{7}$ **c.** $\frac{25}{33}$

Problem solving

10. **MC** From past experience, it is concluded that there is a 99% probability that July will be a wet month in Launceston (it has an average rainfall of approximately 80 mm). The probability that July will not be a wet month next year in Launceston is:

A. 99% **B.** 0.99 **C.** $\frac{1}{100}$ **D.** 1 **E.** 0

11. **MC** A card is drawn from a well-shuffled deck of 52 cards. What is the theoretical probability of not selecting a red card?

A. $\frac{3}{4}$ **B.** $\frac{1}{4}$ **C.** $\frac{1}{13}$ **D.** $\frac{1}{2}$ **E.** 0

12. **MC** Which of the following events is not equally likely?

A. Obtaining a 5 or obtaining a 1 when a die is rolled
B. Obtaining a club or obtaining a diamond when a card is drawn from a pack of cards
C. Obtaining 2 Heads or obtaining 2 Tails when a coin is tossed twice
D. Obtaining 2 Heads or obtaining 1 Head when a coin is tossed twice
E. Obtaining a 3 or obtaining a 6 when a die is rolled

13. **MC** The Australian cricket team has won 12 of the last 15 Test matches. What is the experimental probability of Australia not winning its next Test match?

A. $\frac{4}{5}$ **B.** $\frac{1}{5}$ **C.** $\frac{1}{4}$ **D.** $\frac{3}{4}$ **E.** 1

14. A card is drawn from a well-shuffled pack of 52 cards. What is the theoretical probability of drawing:

a. an ace **b.** a spade
c. a queen or a king **d.** not a heart?

15. The odds for a horse to win a race are 4–3.

a. What is the probability that this horse will win the race?
b. What is the probability that this horse will not win the race?
c. Charlie bets $12 that this horse will win. If the horse wins, what is Charlie's payout?

16. A die is rolled five times.

a. What is the probability of rolling five 6s?
b. What is the probability of not rolling five 6s?

17. Alan and Mary own 3 of the 8 dogs in a race. What is the probability that:

a. one of Alan's or Mary's dogs will win?
b. none of Alan's or Mary's dogs will win?

18. A die is rolled. Event A is obtaining an even number. Event B is obtaining a 3.

a. Are events A and B mutually exclusive?
b. Calculate $\mathrm{P}(A)$ and $\mathrm{P}(B)$.
c. Calculate $\mathrm{P}(A \cup B)$.

19. A card is drawn from a shuffled pack of 52 playing cards. Event A is drawing a club and event B is drawing an ace.

a. Are events A and B mutually exclusive?

b. Calculate $\mathrm{P}(A)$, $\mathrm{P}(B)$ and $\mathrm{P}(A \cap B)$

c. Calculate $\mathrm{P}(A \cup B)$.

20. Discuss the accuracy of the following statements.

a. It did not rain on Monday, Tuesday or Wednesday, so it will not rain on Thursday.

b. A cricket team lost because two of its batsmen scored ducks.

c. The Rams family had a boy, then a girl and then another boy. They must have a girl next.

21. Comment on the contradictions involved in these statements.

a. I was defeated by a loser.

b. The slowest motocross racer in the competition won the race.

c. The most popular person did not get elected.

22. A tetrahedral die is numbered 0, 1, 2 and 3. Two of these dice are rolled and the sum of the numbers (the number on the face that the die sits on) is taken.

a. Show the possible outcomes in a two-way table.

b. Are all the outcomes equally likely?

c. Which total has the least chance of being rolled?

d. Which total has the best chance of being rolled?

e. Which sums have the same chance of being rolled?

23. A bag contains 20 pears, of which 5 are bad. Cathy picks 2 pears (without replacement) from the bag. What is the probability that:

a. both pears are bad?

b. both pears are good?

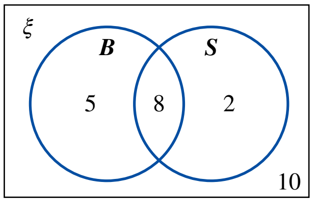
c. one of the two pears is good?

24. Determine the probability of drawing 2 aces from a pack of cards if:

a. the first card is replaced before the second one is drawn

b. the first card drawn is not replaced.

25. On grandparents day at a school a group of grandparents was asked where they most like to take their grandchildren — the beach (B) or shopping (S). The results are illustrated in the Venn diagram below. Use the Venn diagram given to calculate the following probabilities relating to the place grandparents most like to take their grandchildren.



a. What is the probability that a randomly selected grandparent preferred to take their grandchildren to the beach or shopping?

b. What is the probability that a randomly selected grandparent preferred to take their grandchildren to the beach, given that they preferred to take their grandchildren shopping?

26. When all of Saphron's team players turn up for their twice weekly netball training the chance that they then win their Saturday game is 0.65. If not all players are at the training session then the chance of winning their Saturday game is 0.40. Over a four week period, Saphron's players all turn up for training three times.

a. Using a tree diagram, with T to represent all players training and W to represent a win, represent the winning chance of Saphron's netball team.

b. Using the tree diagram constructed in part **a**, determine the probability of Saphron's team winning their Saturday game. Write your answer correct to 4 decimal places.

c. Determine the exact probability that Saphron's team did not train given that they won their Saturday game.

27. Andrew does not know the answer to two questions on a multiple choice exam. The first question has four choices and the second question he does not know has five choices.

a. What is the probability that he will get both questions wrong?

b. If he is certain that one of the choices cannot be the answer in the first question, how will this change the probability that he will get both questions wrong?

28. Mariah the Mathematics teacher wanted to give her students a chance to win a reward at the end of the term. She placed 20 cards into a box, and wrote the word ON on 16 cards, and OFF on 4 cards. After a student chooses a card, that card is replaced into the box for the next student to draw. If a student chooses an OFF card, then they do not have to attend school on a specified day. If they choose an ON card, then they do not receive a day off.

a. Mick, a student, chose a random card from the box. What is the probability he received a day off?

b. Juanita, a student, chose a random card from the box after Mick. What is the probability that she did not receive a day off?

c. What is the probability that Mick and Juanita both received a day off?

29. In the game of draw poker, a player is dealt 5 cards from a deck of 52. To obtain a flush, all 5 cards must be of the same suit.

a. Determine the probability of getting a diamond flush.

b. Determine the probability of getting any flush.

30. a. A Year 10 boy is talking with a Year 10 girl and asks her if she has any brothers or sisters. She says, ‘Yes, I have one’. What is the probability that she has a sister?

b. A Year 10 boy is talking with a Year 10 girl and asks her if she has any brothers or sisters. She says, ‘Yes, I have an older one’. What is the probability that she has a sister?

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Try out this interactivity: Crossword: Topic 11 (int-2857)

Try out this interactivity: Sudoku: Topic 11 (int-3598)

Complete this digital doc: Concept map: Topic 11 (doc-14594)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

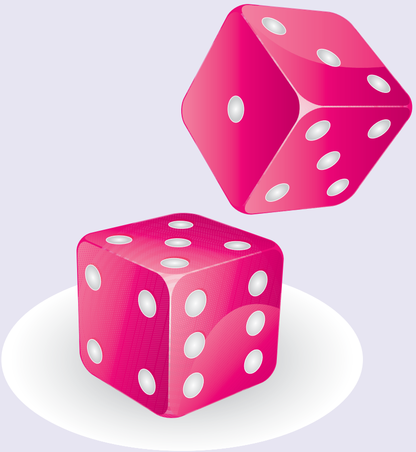
array
certain
chance
chance experiment
complement
complementary events
conditional
dependent
equally likely
even chance
event
experimental probability
favourable outcome
impossible
independent
intersection
likely
mutually exclusive
odds
outcome
payout
probability
random
relative frequency
sample space
scale
subjective probability
theoretical probability
tree diagram
trial
two-way table
Venn diagrams

Investigation | Rich task

Tricky dice

Dice games have been played throughout the world for many years. Professional gamblers resort to all types of devious measures in order to win. Often the other players are unaware of the tricks employed.

Imagine you are playing a game that involves rolling two dice. Instead of having each die marked with the numbers 1 to 6, let the first die have only the numbers 1, 2 and 3 (two of each) and the second die the numbers 4, 5 and 6 (two of each). If you were an observer to this game, you would see the numbers 1 to 6 occurring and probably not realise that the dice were not the regular type.



1. Complete the grid at right to show the sample space on rolling these two dice.
2. How many different outcomes are there? Compare this with the number of different outcomes using two regular dice.

Die 1

		1	2	3	1	2	3
Die 2	4						
	5						
	6						
	4						
	5						
	6						

3. What is the chance of rolling a double using these dice?
4. The numbers on the two dice are added after rolling. Complete the table at right to show the totals possible.

Die 1

		1	2	3	1	2	3
Die 2	4						
	5						
	6						
	4						
	5						
	6						

5. How many different totals are possible? What are they?

6. Which total do you have the greatest chance of rolling? Which total do you have the least chance of rolling?

7. If you played a game in which you had to bet on rolling a total of less than 7, equal to 7 or greater than 7, which option would you be best to take? Explain why.

8. If you had to bet on an even-number outcome or an odd-number outcome, which would be the better option? Explain your answer.

9. The rules are changed to subtracting the numbers on the two dice instead of adding them. Complete the following table to show the outcomes possible.

Die 1

Die 2	1	2	3	1	2	3
4						
5						
6						
4						
5						
6						

10. How many different outcomes are possible in this case? What are they?

11. What is the most frequently occurring outcome? How many times does it occur?

12. Devise a game of your own using these dice. On a separate sheet of paper, write out rules for your game and provide a solution, indicating the best options for winning.

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Complete this digital doc: Code puzzle: The study of . . . (doc-15937)

Answers

Topic 11 Probability

Exercise 11.2 Review of probability

1. Experimental probability is based on the outcomes of experiments, simulations or surveys. Theoretical probability is based on the number of possible favourable outcomes and the total possible outcomes.

2. a. {I, II, III, IV, V, VI}

b. Frequency for I = 0.2
Frequency for II = 0.12
Frequency for III = 0.16
Frequency for IV = 0.14
Frequency for V = 0.24
Frequency for VI = 0.14

c. The spinner should be spun a larger number of times.

3. a. 500 students

b. Frequency for silver = 0.16

c. Frequency for Black and Green = 0.316

d. Blazing Blue

4. a.

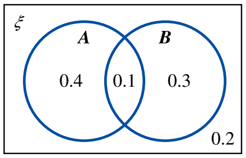
+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b. The sum of 7

c. The sum of 2 or 12

5. a. $P(A \cap B) = 0.1$

b.



c. $P(A \cap B') = 0.4$

6. a. i. $P(A \cup B) = 0.85$

ii. $P(A \cap B)' = 0.95$

b. C

7. $\frac{2}{3}$

8. a. $\frac{1}{13}$

b. $\frac{1}{4}$

c. $\frac{4}{13}$

9. Answers may vary; check with your teacher.

a. i. No. There are many others foods one could have.

ii. Having Weet Bix and not having Weet Bix

b. i. No. There are other means of transport; for example, catching a bus.

ii. Walking to a friend's place and not walking to a friend's place

c. i. No. There are other possible leisure activities.

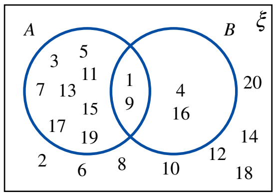
ii. Watching TV and not watching TV

d. i. No. The number 5 can be rolled too.

ii. Rolling a number less than 5 and rolling a number 5 or greater

e. Yes. There are only two possible outcomes; passing or failing.

10. a.



b. i. $\frac{10}{20} = \frac{1}{2}$

ii. $\frac{4}{20} = \frac{1}{5}$

iii. $\frac{2}{20} = \frac{1}{10}$

iv. $\frac{12}{20} = \frac{3}{5}$

v. $\frac{8}{20} = \frac{2}{5}$

11. a. $\frac{1}{6}$ **b.** $\frac{4}{17}$ **c.** $\frac{1}{8}$

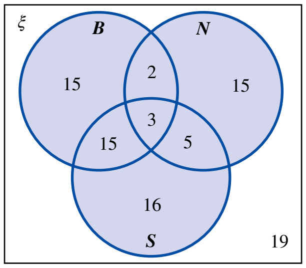
12. A

13. a. No. For a 6-sided die, $P(4) = \frac{1}{6}$; for an 8-sided die, $P(4) = \frac{1}{8}$.

b. Yes; $P(\text{odd}) = \frac{1}{2}$.

14. Yes; $P(5) = \frac{1}{2}$, $P(6) = \frac{1}{2}$.

15. a.



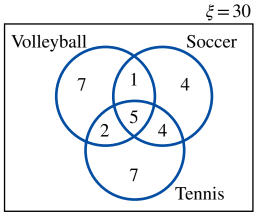
b. 19 students **c.** 32 students **d.** 15 students

e. Frequency = 0.1667 **f.** Probability = $\frac{3}{90}$

16. a. $\frac{1}{13}$ **b.** $\frac{1}{4}$ **c.** $\frac{1}{2}$ **d.** $\frac{12}{13}$ **e.** 0 **f.** $\frac{1}{2}$

17. a. $\frac{1}{5}$ **b.** $\frac{7}{20}$ **c.** $\frac{11}{20}$ **d.** 0

18. a.



b. i. $\frac{1}{2}$ **ii.** $\frac{1}{6}$ **iii.** $\frac{1}{30}$ **iv.** $\frac{2}{5}$ **v.** $\frac{7}{15}$

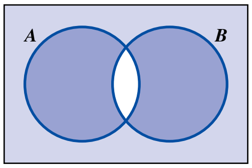
c. i. $\frac{1}{2}$ **ii.** $\frac{8}{15}$

19. a. $\frac{3}{10}$ **b.** $50 **c.** 9–4

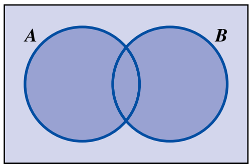
20. Yes. Both have a probability of $\frac{1}{2}$.

21. Yes, equivalent fractions; $\frac{6}{16} = \frac{3}{8}$

22.



Overlaying A' and B' shows $A' \cap B'$ as the area surrounding A and B.

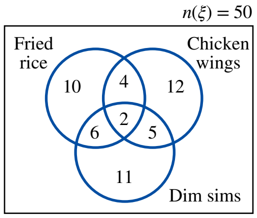


The union of A and B is shown in brown, leaving the surrounding area as $(A \cup B)'$.

23. a. i. 50 **ii.** 7 **iii.** 25 **iv.** 8

b. i. $\frac{1}{2}$ **ii.** $\frac{3}{50}$ **iii.** $\frac{6}{25}$

c. i.



ii. $\frac{1}{25}$

24. a.

Die 2 outcomes

Die 1 outcomes	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

b. 6

c. No. The frequency of the numbers is different.

d.

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	2	3	4	5	6	5	4	3	2	1

e. i. $\frac{1}{36}$ **ii.** $\frac{1}{6}$ **iii.** $\frac{1}{18}$

f. i. $\frac{1}{36}$ **ii.** $\frac{1}{6}$ **iii.** $\frac{1}{18}$

g. 50

Challenge 11.1

11 red and 4 purple, i.e. 15 socks more

Exercise 11.3 Tree diagrams

1. If the probabilities of 2 events are different, the first column of branches indicates the probabilities for the first event and the second column of branches indicates the probabilities for the second event. The product of each branch gives the probability. All probabilities add to 1.

2. a. 12 different outcomes

b. No. Each branch is a product of different probabilities.

c. Less likely

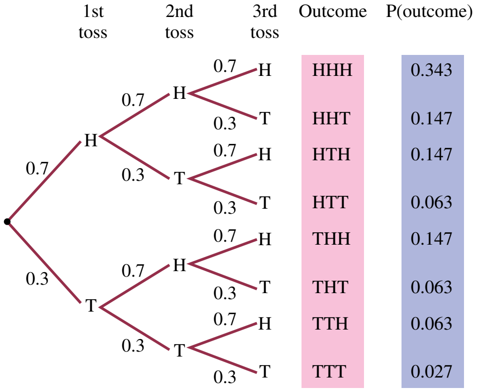
d. Indigo elephant

e. i. P(Blue elephant) $= \frac{7}{40}$

ii. P(Indigo elephant) $= \frac{7}{20}$

iii. P(Donkey) $= \frac{1}{5}$

3. a.



b. P(HHH) = 0.343

c. P(at least 1 Tail) = 0.657

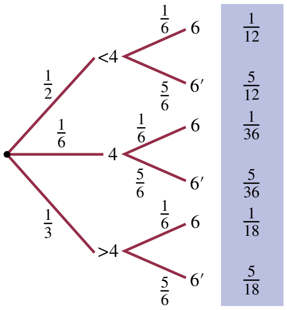
d. P(exactly 2 Tails) = 0.189

4. a. i. $\frac{1}{2}$

ii. $\frac{1}{6}$

iii. $\frac{5}{6}$

b.



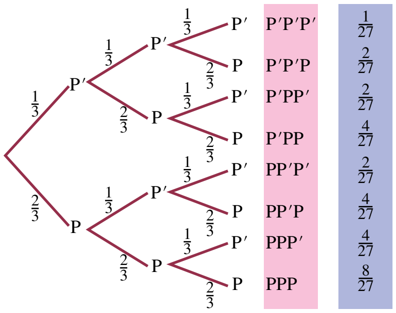
c. i. $\frac{1}{36}$

ii. $\frac{1}{12}$

iii. $\frac{5}{36}$

iv. $\frac{5}{18}$

5. a.

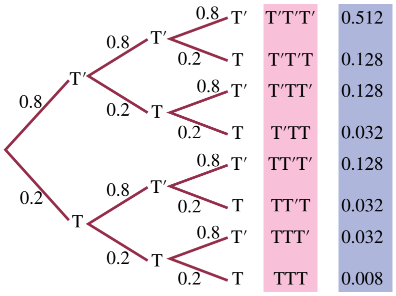


b. i. $\frac{8}{27}$

ii. $\frac{4}{27}$

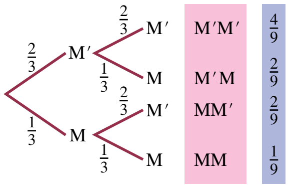
iii. $\frac{12}{27}$

6. a.



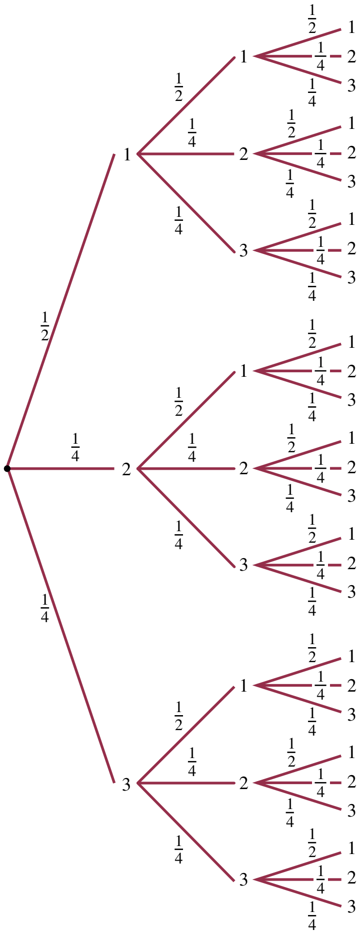
b. 0.520

7. a.



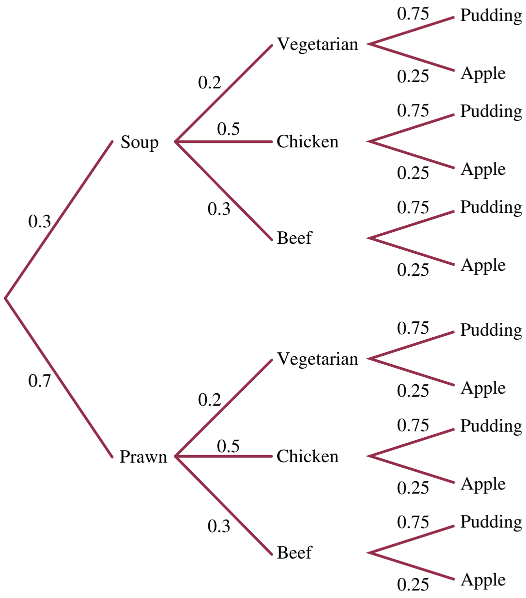
b. $\frac{1}{9}$

8. a.



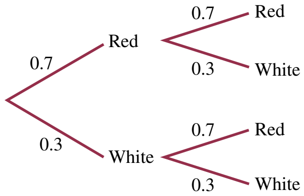
b. $\frac{3}{8}$ **c.** $\frac{7}{8}$

9. a.



c. 0.1125 **d.** 24 people

10. a. i.



ii. 10 balls

iii. 10 balls

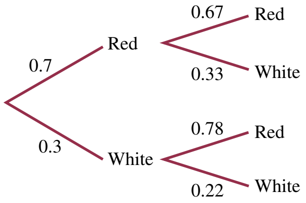
iv. No; the events are independent.

v. P(RR) = 0.49

b. i. 9 balls

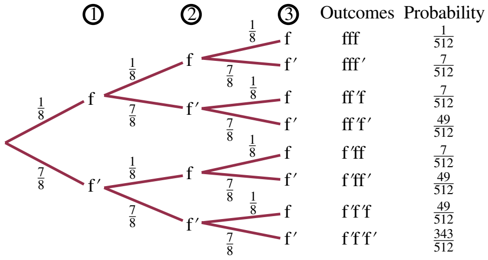
ii. Yes. One ball has been removed from the bag.

iii.



iv. P(RR) = $\frac{7}{15}$ or 0.469 using the rounded values from **iii**

11. a.



f = outcome of 5

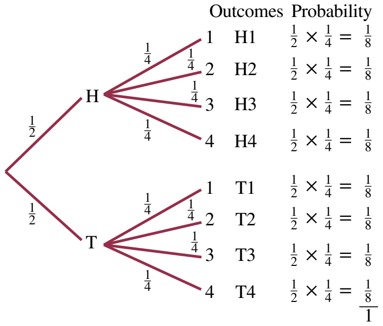
b. i. $\frac{1}{512}$ **ii.** $\frac{343}{512}$ **iii.** $\frac{21}{512}$ **iv.** $\frac{11}{256}$

12. a.

Die outcomes

Coin outcomes	**1**	**2**	**3**	**4**
H	(H, 1)	(H, 2)	(H, 3)	(H, 4)
T	(T, 1)	(T, 2)	(T, 3)	(T, 4)

b.



c. $\frac{1}{4}$

Exercise 11.4 Independent and dependent events

1. a. 0.28 **b.** 0.12 **c.** 0.42 **d.** 0.18

2. a. Yes **b. i.** $\frac{1}{2}$ **ii.** $\frac{1}{6}$ **c.** $\frac{1}{12}$

3. $\frac{1}{40}$

4. $\frac{5}{36}$

5. a. $\frac{16}{25}$ **b.** $\frac{64}{125}$ **c.** $\frac{1}{25}$ **d.** $\frac{4}{25}$

6. a. C **b.** D

7. a. $\frac{3}{77}$ **b.** $\frac{48}{77}$ **c.** $\frac{8}{77}$ **d.** $\frac{18}{77}$

8. a. $\frac{1}{37}$ **b.** $\frac{1}{1369}$ **c.** $\frac{73}{1369}$

9. 0.9

10. $\frac{1}{14}$

11. a. $\frac{1}{5}$ **b.** $\frac{1}{5}$ **c.** $\frac{1}{10}$ **d.** $\frac{1}{3}$

12. a. $\frac{1}{17}$ **b.** $\frac{1}{221}$ **c.** $\frac{25}{102}$

13. a. $\frac{26}{145}$ **b.** $\frac{136}{435}$ **c.** $\frac{221}{435}$

14. No. Coin tosses are independent events. No one toss affects the outcome of the next. The probability of a Head or Tail on a fair coin is always 0.5. Greg has a 50% chance of tossing a Head on the next coin toss as was the chance in each of the previous 9 tosses.

15. No. As events are illustrated on a tree diagram, the individual probability of each outcome is recorded. The probability of a dependent event is calculated (altered according to the previous event) and can be considered as if it was an independent event. As such, the multiplication law of probability can be applied along the branches to calculate the probability of successive events.

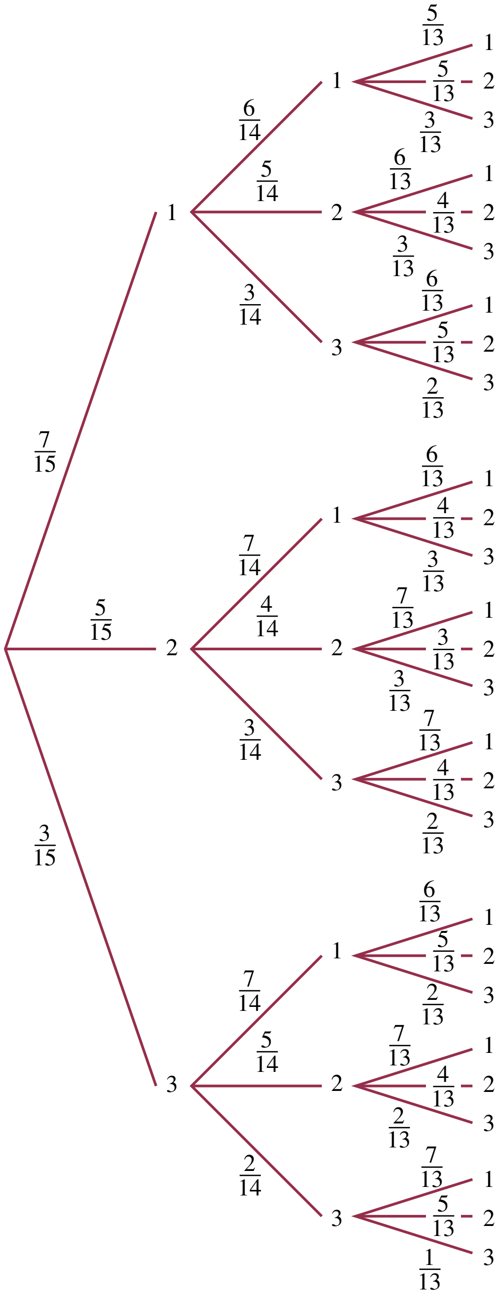
16. a. Dependent

b. $P(1) = \frac{7}{15}$

$P(2) = \frac{5}{15}$

$P(3) = \frac{3}{15}$

c.



d. i. $P(1, 1, 1) = \frac{1}{13}$ **ii.** $P(2, 2, 2) = \frac{2}{91}$ **iii.** $P(3, 3, 3) = \frac{1}{455}$ **iv.** $P(\text{at least one } 3) = \frac{47}{91}$

17. a. $P(9) = \frac{1}{455}$ $P(7–8) = \frac{66}{455}$ $P(5–6) = \frac{248}{455}$ $P(3–4) = \frac{4}{13}$

b. $393.40

Exercise 11.5 Conditional probability

1. a. $P(J) = \frac{41}{90}$ **b.** $P(H|J) = \frac{12}{41}$

2. a. $P(S) = \frac{13}{30}$ **b.** $P(S|(C \cup S)) = \frac{13}{28}$

3. a. 0.3 **b.** $\frac{3}{7}$

4. a. $\frac{9}{13}$ **b.** $\frac{3}{5}$

5. 0.58 or $\frac{15}{26}$

6. 0.22 or $\frac{5}{23}$

7. 0.9

8. 0.8375

9. a. D

b. i. $P(A|B) = \frac{1}{6}$ **ii.** $P(B|A) = \frac{1}{6}$ **iii.** $P(C|A) = \frac{1}{6}$ **iv.** $P(C|B) = 0$

10. A

11. Conditional probability is when the probability of one event depends on the outcome of another event.

12. a. 0.0875 **b.** 0.065

13. a. 0.585 **b.** 0.1525 or 15.25%.

14. 0.847

15. 0.61

Challenge 11.2

Probability $= \frac{1}{2}$

Exercise 11.6 Subjective probability

1. a. The outcome depends upon whether it is a Test match or a one-day game and how effective the bowlers and batsmen are; not forgetting the pitch usually favours spin bowling.

b. The outcome depends on which team is better on the day and which team can adjust to the conditions.

c. No. The third one has an equal chance of being a girl or a boy.

d. This is not necessarily true. Current position and form of both teams should be used as a gauge.

e. It does not mean it will rain again on Friday.

f. There is no certainty about that. It depends upon the condition and location of your house.

g. Cricket games are not won or lost by the attractiveness of the uniform.

h. It is possible to get 6 Heads in a row on a normal coin.

i. They will have a good chance but there is no certainty. The country with the best competitors on the day of each event will win.

j. This is dependent on the person's own interests.

2. a. You still have a chance.

b. No horse is certain to win. Lots of problems can occur on the track.

c. This is not true. Even though Heads and Tails have equal chances, it does not mean half the results will show Heads.

d. Favourites do not always win.

e. Sometimes outsiders pay well, if you back the right one! You can lose more money than you win.

3. Answers will vary. Class discussion is required as there are many factors to consider.

4. a. There is a contradiction. The job was never hers. She had to do well to win the position.

b. The team may have had a lead but a match is only won when finished.

c. No horse is certain to win.

5. a. It is only an estimation of success because the actual probability of success is unknown.

b. Being married by the age of 30 depends on many events occurring so this is a guess.

c. Nobody knows when and if the housing market bubble will burst so this is a guess.

6. In these answers we are making the assumption that the tasks cannot be completed concurrently.

a. 0.315 **b.** 0.231

7. In these answers we are making the assumption that the probability of an earthquake occurring at the different locations is the same as in Southern California and that major earthquakes are independent events.
a. 0.216 **b.** 0.432

11.7 Review

1. a

2. b

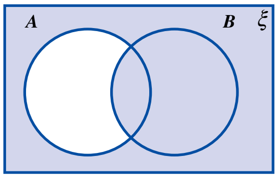
3. b

4. d

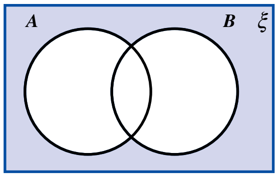
5. b

6. b

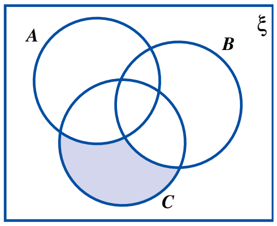
7. a.



b.



c.



8. a. $\frac{7}{10}$ **b.** $\frac{2}{7}$ **c.** $\frac{5}{17}$

9. a. 4–7 **b.** 1–6 **c.** 8–25

10. c

11. d

12. d

13. b

14. a. $\frac{1}{13}$ **b.** $\frac{1}{4}$ **c.** $\frac{2}{13}$ **d.** $\frac{3}{4}$

15. a. $\frac{3}{7}$ **b.** $\frac{4}{7}$ **c.** $28

16. a. $\frac{1}{7776}$ **b.** $\frac{7775}{7776}$

17. a. $\frac{3}{8}$ **b.** $\frac{5}{8}$

18. a. Yes **b.** $\mathrm{P}(A) = \frac{1}{2}$ and $\mathrm{P}(B) = \frac{1}{6}$ **c.** $\frac{2}{3}$

19. a. No

b. $\mathrm{P}(A) = \frac{1}{4}, P(B) = \frac{1}{13}, P(A \cap B) = \frac{1}{52}$

c. $\frac{4}{13}$

20. a. Whether it rains or not on Thursday is not determined by what happened on Monday, Tuesday or Wednesday. It can still rain on Thursday.

b. The team's win or loss depends upon how other players bat and bowl or how the other team plays.

c. There is an equal chance of having a boy or a girl.

21. a. If you were defeated, the opponent was the winner.

b. The motorcycle rider who crossed the line first and won the race would not be the slowest rider.

c. The person elected was the most popular choice for the position.

22. a.

	Die 2 outcomes			
Die 1 outcomes	**0**	**1**	**2**	**3**
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)
2	(2, 0)	(2, 1)	(2, 2)	(2, 3)
3	(3, 0)	(3, 1)	(3, 2)	(3, 3)

a. No **b.** 0 and 6

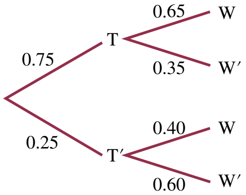
c. 3 **d.** 0 and 6, 1 and 5, 2 and 4

23. a. $\frac{1}{19}$ **b.** $\frac{21}{38}$ **c.** $\frac{15}{38}$

24. a. $\frac{1}{169}$ **b.** $\frac{1}{221}$

25. a. $\frac{15}{25} = \frac{3}{5}$ **b.** $\frac{8}{10} = \frac{4}{5}$

26. a.



b. 0.5875 **c.** $\frac{8}{47}$

27. a. $\frac{3}{5}$ **b.** $\frac{8}{15}$

28. a. $\frac{1}{5}$ **b.** $\frac{4}{5}$ **c.** $\frac{1}{25}$

29. a. 0.000 495 **b.** 0.001 981

30. a. $\frac{1}{3}$ **b.** $\frac{1}{2}$

Investigation — Rich task

1.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**	(1, 4)	(2, 4)	(3, 4)	(1, 4)	(2, 4)	(3, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(1, 5)	(2, 5)	(3, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(1, 6)	(2, 6)	(3, 6)
	4	(1, 4)	(2, 4)	(3, 4)	(1, 4)	(2, 4)	(3, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(1, 5)	(2, 5)	(3, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(1, 6)	(2, 6)	(3, 6)

2. 9

3. 0

4.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**	5	6	7	5	6	7
	5	6	7	8	6	7	8
	6	7	8	9	7	8	9
	4	5	6	7	5	6	7
	5	6	7	8	6	7	8
	6	7	8	9	7	8	9

5. 5; 5, 6, 7, 8, 9

6. 7; 5, 9

7. Equal to 7; probability is the highest.

8. Odd-number outcome; probability is higher.

		Die							
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Coin	**Head**	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)	(H, 7)	(H, 8)
	Tail	(T, 1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)	(T, 7)	(T, 8)

9.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**	3	2	1	3	2	1
	5	4	3	2	4	3	2
	6	5	4	3	5	4	3
	4	3	2	1	3	2	1
	5	4	3	2	4	3	2
	6	5	4	3	5	4	3

10. 5; 1, 2, 3, 4, 5,

11. 3;12

12. Answers will vary.

TOPIC 12
Univariate data

12.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

12.1.1 Why learn this?

According to the novelist Mark Twain, 'There are three kinds of lies: lies, damned lies and statistics.' There is so much information in our lives, increasingly so with the World Wide Web, smart phones and social media tracking our every move and accumulating vast amounts of data about us. The data are used to gather information about our likes and dislikes, our buying habits, our voting preferences and so on. Statistics can easily be used to manipulate people unless they have an understanding of the basic concepts involved.

12.1.2 What do you know?

assesson

1. **THINK** List what you know about data. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of data.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Koby's bid to make the Olympic athletics team (eles-1852)

12.2 Measures of central tendency

12.2.1 Univariate data

- In this chapter you will learn how to measure and analyse **univariate data**. Univariate data are **data** with one variable; for example, the heights of Year 10 students.
- **Measures of central tendency** are summary statistics that measure the middle (or centre) of the data. These are known as the mean, median and mode.
  - The **mean** is the average of all observations in a set of data.
  - The **median** is the middle observation in an ordered set of data.
  - The **mode** is the most frequent observation in a data set.

12.2.2 Ungrouped data

Mean, median and mode of ungrouped data

Mean

- To obtain the mean of a set of ungrouped data, all numbers (scores) in the set are added together and then the total is divided by the number of scores in that set.

$$\text{Mean} = \frac{\text{sum of all scores}}{\text{number of scores}}$$

- Symbolically this is written $\bar{x} = \frac{\Sigma x}{n}$.

Median

- The median is the middle value of any set of data arranged in numerical order. In the set of n numbers, the median is located at the $\frac{n+1}{2}$th score. The median is:
  - the middle score for an odd number of scores arranged in numerical order
  - the average of the two middle scores for an even number of scores arranged in numerical order.

Mode

- The mode is the score that occurs most often in a set of data.
- A set of data may contain:
  1. no mode; that is, each score occurs once only
  2. one mode
  3. more than one mode.

WORKED EXAMPLE 1 TI | CASIO

For the data 6, 2, 4, 3, 4, 5, 4, 5, find the:

a. mean **b. median** **c. mode.**

THINK	WRITE
a **1** Calculate the sum of the scores; that is, Σx.	**a** $\Sigma x = 6 + 2 + 4 + 3 + 4 + 5 + 4 + 5$ $= 33$
2 Count the number of scores; that is, n.	$n = 8$
3 Write the rule for the mean.	$\bar{x} = \frac{\Sigma x}{n}$
4 Substitute the known values into the rule.	$= \frac{33}{8}$

5 Evaluate.	$= 4.125$
6 Answer the question.	The mean is 4.125.
b **1** Write the median scores in ascending numerical order.	**b** 2 3 4 4 4 5 5 6
2 Locate the position of the median using the rule $\frac{n + 1}{2}$, where $n = 8$. This places the median as the 4.5th score; that is, between the 4th and 5th score.	$\text{Median} = \frac{n + 1}{2}\text{th score}$ $= \frac{8 + 1}{2}\text{th score}$ $= 4.5\text{th score}$ 2 3 4 **4 4** 5 5 6
3 Obtain the average of the two middle scores.	$\text{Median} = \frac{4 + 4}{2}$ $= \frac{8}{2}$ $= 4$
4 Answer the question.	The median is 4.
c **1** Systematically work through the set and make note of any repeated values (scores).	$\downarrow \downarrow$ **c** 2 3 4 4 4 5 5 6 $\uparrow\uparrow \uparrow$
2 Answer the question.	The mode is 4.

12.2.3 Calculating mean, median and mode from a frequency distribution table

- If data are presented in a frequency distribution table, the formula used to calculate the mean is $\bar{x} = \frac{\Sigma(f \times x)}{n}$.
- Here, each value (score) in the table is multiplied by its corresponding frequency; then all the $f \times x$ products are added together and the total sum is divided by the number of observations in the set.
- To find the median, find the position of each score from the cumulative frequency column.
- The mode is the score with the highest **frequency**.

WORKED EXAMPLE 2

TI | CASIO

For the table at right, find the:

a mean
b median
c mode.

Score (x)	**Frequency (f)**
4	**1**
5	**2**
6	**5**
7	**4**
8	**3**
Total	**15**

THINK	WRITE
1 Rule up a table with four columns titled Score (x), Frequency (f), Frequency × score ($f \times x$) and Cumulative frequency (cf).	
2 Enter the data and complete both the $f \times x$ and cumulative frequency columns.	

Score (x)	**Frequency (f)**	**Frequency × score ($f \times x$)**	**Cumulative frequency (cf)**
4	1	4	1
5	2	10	$1 + 2 = 3$
6	5	30	$3 + 5 = 8$
7	4	28	$8 + 4 = 12$
8	3	24	$12 + 3 = 15$
	$n = 15$	$\Sigma(f \times x) = 96$	

THINK	WRITE
a **1** Write the rule for the mean.	**a** $\bar{x} = \dfrac{\Sigma(f \times x)}{n}$
2 Substitute the known values into the rule and evaluate.	$\bar{x} = \dfrac{96}{15}$ $= 6.4$
3 Answer the question.	The mean of the data set is 6.4.
b **1** Locate the position of the median using the rule $\dfrac{n+1}{2}$, where $n = 15$. This places the median as the 8th score.	**b** The median is the $\dfrac{15+1}{2}$th or 8th score.
2 Use the cumulative frequency column to find the 8th score and answer the question.	The median of the data set is 6.
c **1** The mode is the score with the highest frequency.	**c** The score with the highest frequency is 6.
2 Answer the question.	The mode of the data set is 6.

12.2.4 Mean, median and mode of grouped data

Mean

- When the data are grouped into class intervals, the actual values (or data) are lost. In such cases we have to approximate the real values with the midpoints of the intervals into which these values fall. For example, when measuring heights of students in a class, if we found that 4 students had a height between 180 and 185 cm, we have to assume that each of those 4 students is 182.5 cm tall. The formula for calculating the mean:

$$\bar{x} = \frac{\Sigma(f \times x)}{n}$$

Here x represents the midpoint (or class centre) of each class interval, f is the corresponding frequency and n is the total number of observations in a set.

Median

- The median is found by drawing a **cumulative frequency** curve (ogive) of the data and estimating the median from the 50th percentile.

Modal class

- The modal class is the class interval that has the highest frequency.

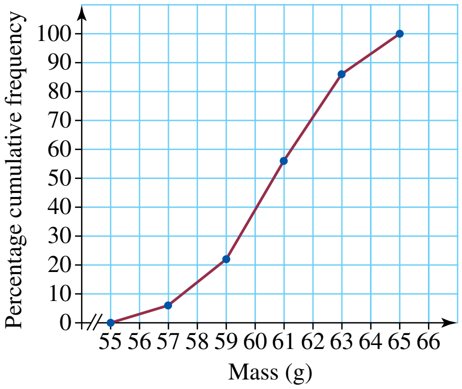
12.2.5 Cumulative frequency curves (ogives)

Ogives

- Data from a cumulative frequency table can be plotted to form a **cumulative frequency curve** (sometimes referred to as cumulative frequency polgons), which is also called an **ogive** (pronounced '*oh-jive*').
- To plot an ogive for data that is in class intervals, the maximum value for the class interval is used as the value against which the cumulative frequency is plotted.

For example, the following table and graph show the mass of cartons of eggs ranging from 55 g to 65 g.

Mass (g)	**Frequency (*f*)**	**Cumulative frequency (*cf*)**	**Percentage cumulative frequency (%*cf*)**
55–<57	2	2	6%
57–<59	6	2 + 6 = 8	22%
59–<61	12	8 + 12 = 20	56%
61–<63	11	20 + 11 = 31	86%
63–<65	5	31 + 5 = 36	100%



Quantiles

- An ogive can be used to divide the data into any given number of equal parts called **quantiles**.
- Quantiles are named after the number of parts that the data are divided into.
  - **Percentiles** divide the data into 100 equal-sized parts.
  - **Quartiles** divide the data into 4 equal-sized parts. For example, 25% of the data values lie at or below the first quartile.

Percentile	**Quartile and symbol**	**Common name**
25th percentile	First quartile, Q_1	Lower quartile
50th percentile	Second quartile, Q_2	Median
75th percentile	Third quartile, Q_3	Upper quartile
100th percentile	Fourth quartile, Q_4	Maximum

- A percentile is named after the percentage of data that lies at or below that value. For example, 60% of the data values lie at or below the 60th percentile.
- Percentiles can be read off a percentage cumulative frequency curve.

- A percentage cumulative frequency curve is created by:
  - writing the cumulative frequencies as a percentage of the total number of data values
  - plotting the percentage cumulative frequencies against the maximum value for each interval.

WORKED EXAMPLE 3

For the given data:

a estimate the mean

b estimate the median

c find the modal class.

Class interval	Frequency
60–<70	**5**
70–<80	**7**
80–<90	**10**
90–<100	**12**
100–<110	**8**
110–<120	**3**
Total	**45**

THINK

WRITE

1 Draw up a table with 5 columns headed Class interval, Class centre (x), Frequency (f), Frequency × class centre ($f \times x$) and Cumulative frequency (cf).

2 Complete the x, $f \times x$ and cf columns.

Class interval	Class centre (x)	Freq. (f)	Frequency × class centre ($f \times x$)	Cumulative frequency (cf)
60–<70	65	5	325	5
70–<80	75	7	525	12
80–<90	85	10	850	22
90–<100	95	12	1140	34
100–<110	105	8	840	42
110–<120	115	3	345	45
		$n = 45$	$\Sigma(f \times x) = 4025$	

a **1** Write the rule for the mean.

a $\bar{x} = \dfrac{\Sigma(f \times x)}{n}$

2 Substitute the known values into the rule and evaluate.

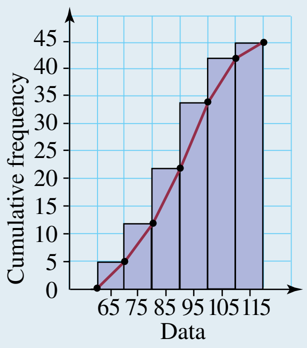
$$\bar{x} = \frac{4025}{45} \simeq 89.4$$

3 Answer the question.

The mean for the given data is approximately 89.4.

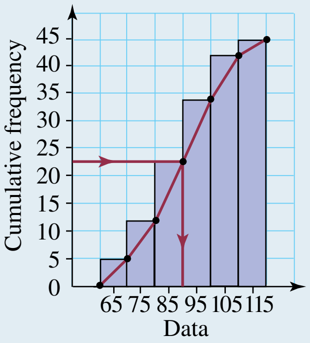
b **1** Draw a combined cumulative frequency histogram and ogive, labelling class centres on the horizontal axis and cumulative frequency on the vertical axis. Join the end-points of each class interval with a straight line to form the ogive.

b



2 Locate the middle of the cumulative frequency axis, which is 22.5.

3 Draw a horizontal line from this point to the ogive and a vertical line to the horizontal axis.



4 Read off the value of the median from the *x*-axis and answer the question.

The median for the given data is approximately 90.

c **1** The modal class is the class interval with the highest frequency.

c The class internal 90–100 occurs twelve times, which is the highest frequency.

2 Answer the question.

The modal class is the 90–100 class interval.

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Complete this digital doc: SkillSHEET: Finding the median of a small data set (doc-5300)

Complete this digital doc: SkillSHEET: Finding the mode of a small data set (doc-5301)

Complete this digital doc: SkillSHEET: Finding the mean, median and mode from a stem-and-leaf plot (doc-5302)

Complete this digital doc: SkillSHEET: Presenting data in a frequency distribution table (doc-5303)

Complete this digital doc: SkillSHEET: Drawing statistical graphs (doc-5304)

Exercise 12.2 Measures of central tendency

assesson

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–10, 12, 15, 16, 18	Questions: 1–10, 12, 14, 16, 18	Questions: 1–19

■■■ **Individual pathway interactivity:** int-4621

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Fluency

1. **WE1** For each of the following sets of data find the:

i. mean **ii.** median **iii.** mode.

a. 3, 5, 6, 8, 8, 9, 10

b. 4, 6, 7, 4, 8, 9, 7, 10

c. 17, 15, 48, 23, 41, 56, 61, 52

d. 4.5, 4.7, 4.8, 4.8, 4.9, 5.0, 5.3

e. $7\frac{1}{2}$, $10\frac{1}{4}$, 12, $12\frac{1}{4}$, 13, $13\frac{1}{2}$, $13\frac{1}{2}$, 14

2. The back-to-back stem-and-leaf plot below shows the test results of 25 Year 10 students in Mathematics and Science. Find the mean, median and mode for each of the two subjects.

Key: $3|2 = 32$

Leaf: Science	*Stem*	*Leaf: Mathematics*
8 7 3	3	2 9
9 6 2 2 1	4	0 6 8
8 7 6 1 1 0	5	1 3 5
9 7 4 3 2	6	2 6 7 9
8 5 1 0	7	3 6 7 8
7 3	8	0 4 4 6 8 9
	9	2 5 8

3. WE2 For the data shown in each of the following frequency distribution tables, find the:

i. mean **ii.** median **iii.** mode.

a.

Score (x)	Frequency (f)
4	3
5	6
6	9
7	4
8	2
Total	24

b.

Score (x)	Frequency (f)
12	4
13	5
14	10
15	12
16	9
Total	40

4. The following data show the number of bedrooms in each of the 10 houses in a particular neighbourhood: 2, 1, 3, 4, 2, 3, 2, 2, 3, 3.

a. Calculate the mean and median number of bedrooms.

b. A local motel contains 20 rooms. Add this observation to the set of data and recalculate the values of the mean and median.

c. Compare the answers obtained in parts **a** and **b** and complete the following statement: When the set of data contains an unusually large value(s), called an outlier, the _________ (mean/median) is the better measure of central tendency, as it is less affected by this extreme value.

5. WE3 For the given data:

a. estimate the mean **b.** estimate the median **c.** find the modal class.

Class interval	Frequency
40–<50	2
50–<60	4
60–<70	6
70–<80	9
80–<90	5
90–<100	4
Total	30

6. Calculate the mean of the grouped data shown in the table below.

Class interval	Frequency
100–109	3
110–119	7
120–129	10
130–139	6
140–149	4
Total	30

7. Find the modal class of the data shown in the table below.

Class interval	Frequency
50–< 55	1
55–< 60	3
60–< 65	4
65–< 70	5
70–< 75	3
75–< 80	2
Total	18

8. **MC** The number of textbooks sold by various bookshops during the second week of December was recorded. The results are summarised in the table below.

Number of books sold	Frequency
220–229	2
230–239	2
240–249	3
250–259	5
260–269	4
270–279	4
Total	20

a. The modal class of the data is given by the class interval(s):

A. 220–229 and 230–239
B. 250–259
C. 260–269 and 270–279
D. of both A and C

b. The class centre of the first class interval is:

A. 224 **B.** 224.5 **C.** 224.75 **D.** 225

c. The median of the data is in the interval:

A. 230–239 **B.** 240–249 **C.** 250–259 **D.** 260–269

d. The estimated mean of the data is:

A. 251 **B.** 252 **C.** 253 **D.** 254

Understanding

9. A random sample was taken, composed of 30 people shopping at a Coles supermarket on a Tuesday night. The amount of money (to the nearest dollar) spent by each person was recorded as follows:

6, 32, 66, 17, 45, 1, 19, 52, 36, 23, 28, 20, 7, 47, 39
6, 68, 28, 54, 9, 10, 58, 40, 12, 25, 49, 74, 63, 41, 13

a. Find the mean and median amount of money spent at the checkout by the people in this sample.

b. Group the data into class intervals of 10 and complete the frequency distribution table. Use this table to estimate the mean amount of money spent.

c. Add the cumulative frequency column to your table and fill it in. Hence, construct the ogive. Use the ogive to estimate the median.

d. Compare the mean and the median of the original data from part **a** with the mean and the median obtained for grouped data in parts **b** and **c**. Were the estimates obtained in parts **b** and **c** good enough? Explain your answer.

10. a. Add one more number to the set of data 3, 4, 4, 6 so that the mean of a new set is equal to its median.

b. Design a set of five numbers so that mean = median = mode = 5.

c. In the set of numbers 2, 5, 8, 10, 15, change one number so that the median remains unchanged while the mean increases by 1.

11. Thirty men were asked to reveal the number of hours they spent doing housework each week. The results are detailed below.

1	5	2	12	2	6	2	8	14	18
0	1	1	8	20	25	3	0	1	2
7	10	12	1	5	1	18	0	2	2

a. Present the data in a frequency distribution table. (Use class intervals of 0–4, 5–9 etc.)

b. Use your table to estimate the mean number of hours that the men spent doing housework.

c. Find the median class for hours spent by the men at housework.

d. Find the modal class for hours spent by the men at housework.

Reasoning

12. The data at right give the age of 25 patients admitted to the emergency ward of a hospital.

18	16	6	75	24
23	82	75	25	21
43	19	84	76	31
78	24	20	63	79
80	20	23	17	19

a. Present the data in a frequency distribution table. (Use class intervals of 0−<15, 15−<30 and so on.)

b. Draw a histogram of the data.

c. What word could you use to describe the pattern of the data in this distribution?

d. Use your table to estimate the mean age of patients admitted.

e. Find the median class for age of patients admitted.

f. Find the modal class for age of patients admitted.

g. Draw an ogive of the data.

h. Use the ogive to determine the median age.

i. Do any of your statistics (mean, median or mode) give a clear representation of the typical age of an emergency ward patient?

j. Give some reasons which could explain the pattern of the distribution of data in this question.

13. The batting scores for two cricket players over 6 innings are as follows:

Player A 31, 34, 42, 28, 30, 41
Player B 0, 0, 1, 0, 250, 0

a. Find the mean score for each player.

b. Which player appears to be better, based upon mean result?

c. Find the median score for each player.

d. Which player appears to be better when the decision is based on the median result?

e. Which player do you think would be the most useful to have in a cricket team and why? How can the mean result sometimes lead to a misleading conclusion?

14. The resting pulse rate of 20 female athletes was measured. The results are detailed below.

50	52	48	52	71	61	30	45	42	48
43	47	51	62	34	61	44	54	38	40

a. Construct a frequency distribution table. (Use class sizes of 1–<10, 10–<20 etc.)

b. Use your table to estimate the mean of the data.

c. Find the median class of the data.

d. Find the modal class of the data.

e. Draw an ogive of the data. (You may like to use a graphics calculator for this.)

f. Use the ogive to determine the median pulse rate.

15. **MC** In a set of data there is one score that is extremely small when compared to all the others. This outlying value is most likely to:

A. have greatest effect upon the mean of the data.

B. have greatest effect upon the median of the data.

C. have greatest effect upon the mode of the data.

D. have very little effect on any of the statistics as we are told that the number is extremely small.

16. The following frequency table gives the number of employees in different salary brackets for a small manufacturing plant.

Position	Salary ($)	Number of employees
Machine operator	18 000	50
Machine mechanic	20 000	15
Floor steward	24 000	10
Manager	62 000	4
Chief executive officer	80 000	1

a. Workers are arguing for a pay rise but the management of the factory claims that workers are well paid because the mean salary of the factory is $22 100. Are they being honest?

b. Suppose that you were representing the factory workers and had to write a short submission in support of the pay rise. How could you explain the management's claim? Quote some other statistics in favour of your case.

17. Design a set of five numbers with:

a. mean = median = mode

b. mean > median > mode

c. mean < median = mode.

Problem solving

18. The numbers 15, a, 17, b, 22, c, 10 and d have a mean of 14. Find the mean of a, b, c and d.

19. The numbers m, n, p, q, r, and s have a mean of a while x, y and z have a mean of b. Find the mean of all nine numbers.

Reflection

Under what circumstances might the median be a more reliable measure of centre than the mean?

CHALLENGE 12.1

The mean and median of six two-digit prime numbers is 39 and the mode is 31. The smallest number is 13. What are the six numbers?

12.3 Measures of spread

12.3.1 Measures of spread

- **Measures of spread** describe how far data values are spread from the centre or from each other.
- A music store proprietor has stores in Newcastle and Wollongong. The number of CDs sold each day over one week is recorded below.

Newcastle:	45, 60, 50, 55, 48, 40, 52
Wollongong:	20, 85, 50, 15, 30, 60, 90

In each of these data sets consider the measures of central tendency.

Newcastle:	Mean = 50	Wollongong:	Mean = 50
	Median = 50		Median = 50
	No mode		No mode

With these measures being the same for both data sets we could come to the conclusion that both data sets are very similar; however, if we look at the data sets, they are very different. We can see that the data for Newcastle are very clustered around the mean, whereas the Wollongong data are spread out more.

- The data from Newcastle are between 45 and 60, whereas the Wollongong data are between 15 and 90.
- **Range** and **interquartile range (IQR)** are both measures of spread.

12.3.2 Range

- The most basic measure of spread is the range. It is defined as the difference between the highest and the lowest values in the set of data.

$$\text{Range} = \text{highest score} - \text{lowest score}$$
$$\Rightarrow \quad \text{Range} = X_{\max} - X_{\min}$$

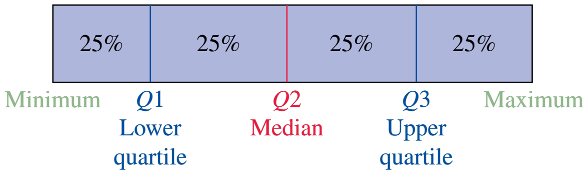
WORKED EXAMPLE 4

Find the range of the given data set: 2.1, 3.5, 3.9, 4.0, 4.7, 4.8, 5.2.

THINK	WRITE
1 Identify the lowest score ($X_{\min}$) of the data set.	Lowest score = 2.1
2 Identify the highest score ($X_{\max}$) of the data set.	Highest score = 5.2
3 Write the rule for the range.	Range $= X_{\max} - X_{\min}$
4 Substitute the known values into the rule.	$= 5.2 - 2.1$
5 Evaluate.	$= 3.1$

12.3.3 Interquartile range

- The interquartile range (IQR) is the range of the middle 50% of all the scores in an ordered set. When calculating the interquartile range, the data are first organised into quartiles, each containing 25% of the data. The word ‘quartile’ comes from the word ‘quarter’.



Interquartile range = upper quartile − lower quartile
This can be written as:

$$IQR = Q_{\text{upper}} - Q_{\text{lower}}$$

or

$$IQR = Q_3 - Q_1$$

- The IQR is not affected by extremely large or extremely small data values (**outliers**), so in some circumstances the IQR is a better indicator of the spread of data than the range.

WORKED EXAMPLE 5

TI | CASIO

Calculate the interquartile range (IQR) of the following set of data: 3, 2, 8, 6, 1, 5, 3, 7, 6.

THINK	WRITE
1 Arrange the scores in order.	1 2 3 3 5 6 6 7 8
2 Locate the median and use it to divide the data into two halves. *Note:* The median is the 5th score in this data set and should not be included in the lower or upper ends of the data.	1 2 3 3 5 6 6 7 8
3 Find Q_1, the median of the lower half of the data.	$Q_1 = \frac{2+3}{2}$ $= \frac{5}{2}$ $= 2.5$
4 Find Q_3, the median of the upper half of the data.	$Q_3 = \frac{6+7}{2}$ $= \frac{13}{2}$ $= 6.5$
5 Calculate the interquartile range.	$\text{IQR} = Q_3 - Q_1$ $= 6.5 - 2.5$ $= 4$

12.3.4 Determining the IQR from a graph

- When data are presented in a frequency distribution table, either ungrouped or grouped, the interquartile range is found by drawing an ogive.

WORKED EXAMPLE 6

The following frequency distribution table gives the number of customers who order different volumes of concrete from a readymix concrete company during the course of a day. Find the interquartile range of the data.

Volume (m^3)	Frequency
0.0–<0.5	15
0.5–<1.0	12
1.0–<1.5	10

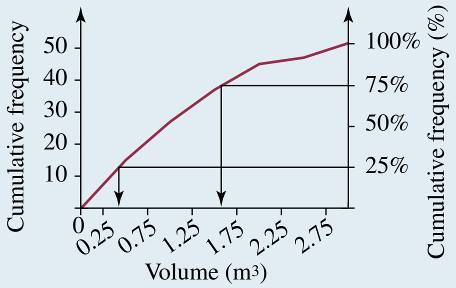
Volume (m^3)	Frequency
1.5–<2.0	8
2.0–<2.5	2
2.5–<3.0	4

THINK | **WRITE/DRAW**

1 To find the 25th and 75th percentiles from the ogive, first add a class centre column and a cumulative frequency column to the frequency distribution table and fill them in.

Volume	Class centre	f	cf
0.0–<0.5	0.25	15	15
0.5–<1.0	0.75	12	27
1.0–<1.5	1.25	10	37
1.5–<2.0	1.75	8	45
2.0–<2.5	2.25	2	47
2.5–<3.0	2.75	4	51

2 Draw the ogive. A percentage axis will be useful.



3 Find the upper quartile (75th percentile) and lower quartile (25th percentile) from the ogive.

$Q_3 = 1.6\text{ m}^3$
$Q_1 = 0.4\text{ m}^3$

4 The interquartile range is the difference between the upper and lower quartiles.

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 1.6 - 0.4\\ &= 1.2\text{ m}^3\end{aligned}$$

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Exercise 12.3 Measures of spread

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 10, 12	Questions: 1–8, 10, 11, 12	Questions: 1–13

🟩🟦🟪 **Individual pathway interactivity:** int-4622

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Fluency

1. WE4 Find the range for each of the following sets of data.

a. 4, 3, 9, 12, 8, 17, 2, 16

b. 49.5, 13.7, 12.3, 36.5, 89.4, 27.8, 53.4, 66.8

c. $7\frac{1}{2}, 12\frac{3}{4}, 5\frac{1}{4}, 8\frac{2}{3}, 9\frac{1}{6}, 3\frac{3}{4}$

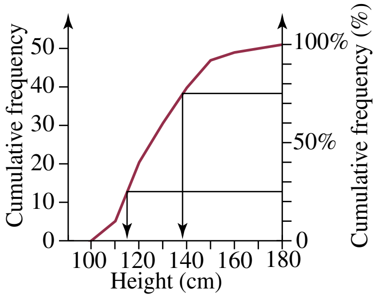
2. WE5 Calculate the interquartile range (IQR) for the following sets of data.
   a. 3, 5, 8, 9, 12, 14
   b. 7, 10, 11, 14, 17, 23
   c. 66, 68, 68, 70, 71, 74, 79, 80
   d. 19, 25, 72, 44, 68, 24, 51, 59, 36

3. The following stem-and-leaf plot shows the mass of newborn babies (rounded to the nearest 100 g). Find the:
   a. range of the data
   b. IQR of the data.

Key: $1^* | 9 = 1.9\ kg$

Stem	*Leaf*
1*	9
2	2 4
2*	6 7 8 9
3	0 0 1 2 3 4
3*	5 5 6 7 8 8 8 9
4	0 1 3 4 4
4*	5 6 6 8 9
5	0 1 2 2

4. Use the ogive at right to determine the interquartile range of the data.



5. WE6 The following frequency distribution table gives the amount of time spent by 50 people on shopping for Christmas presents. Estimate the IQR of the data.

Time (h)	0–<0.5	0.5–<1	1–<1.5	1.5–<2	2–<2.5	2.5–<3	3–<3.5	3.5–<4
Frequency	1	2	7	15	13	8	2	2

6. MC Calculate the interquartile range of the following data:

17, 18, 18, 19, 20, 21, 21, 23, 25

A. 8 **B.** 18 **C.** 4 **D.** 20

Understanding

7. The following frequency distribution table shows the life expectancy in hours of 40 household batteries.

Life (h)	50–<55	55–<60	60–<65	65–<70	70–<75	75–<80
Frequency	4	10	12	8	5	1

   a. Draw an ogive curve that represents the data in the table above.

b. Use the ogive to answer the following questions.
  i. What is the median score?
  ii. What are the upper and lower quartiles?
  iii. What is the interquartile range?
  iv. How many batteries lasted less than 60 hours?
  v. How many batteries lasted 70 hours or more?

8. Calculate the IQR for the following data.

Class interval	Frequency
120–<130	2
130–<140	3
140–<150	9
150–<160	14
160–<170	10
170–<180	8
180–<190	6
190–<200	3

9. For each of the following sets of data, state: **i** the range and **ii** the IQR of each set.
  a. 6, 9, 12, 13, 20, 22, 26, 29
  b. 7, 15, 2, 26, 47, 19, 9, 33, 38
  c. 120, 99, 101, 136, 119, 87, 123, 115, 107, 100

Reasoning

10. As newly appointed coach of Terrorolo's Meteors netball team, Kate decided to record each player's statistics for the previous season. The number of goals scored by the leading goal shooter was:

1, 3, 8, 18, 19, 23, 25, 25, 25, 26, 27, 28,
28, 28, 28, 29, 29, 30, 30, 33, 35, 36, 37, 40.

  a. Find the mean of the data.
  b. Find the median of the data.
  c. Find the range of the data.
  d. Find the interquartile range of the data.
  e. There are three scores that are much lower than most. Explain the effect these scores have on the summary statistics.

11. The following back-to-back stem-and-leaf plot shows the ages of 30 pairs of men and women when entering their first marriage.

Key: 1 | 6 = 16 years old

Leaf: Men	*Stem*	*Leaf: Women*
9 9 8	1	6 7 7 8 9
9 9 8 8 7 6 4 4 3 2 0	2	0 0 1 2 3 4 5 6 7 7 8 9
9 8 8 8 6 5 5 4 3 2	3	0 1 2 2 3 4 7 9
6 3 0 0	4	1 2 4 8
6 0	5	2

  a. Find the mean, median, range and interquartile range of each set.
  b. Write a short paragraph comparing the two distributions.

Problem solving

12. Find the mean, median, mode, range and IQR of the following data collected when the temperature of the soil around 25 germinating seedlings was recorded: 28.9, 27.4, 23.6, 25.6, 21.1, 22.9, 29.6, 25.7, 27.4, 23.6, 22.4, 24.6, 21.8, 26.4, 24.9, 25.0, 23.5, 26.1, 23.6, 25.3, 29.5, 23.5, 22.0, 27.9, 23.6.

13. Four positive numbers *a*, *b*, *c* and *d* have a mean of 12, a median and mode of 9 and a range of 14. Find the values of *a*, *b*, *c* and *d*.

Reflection

What do measures of spread tell us about a set of data?

12.4 Box-and-whisker plots

12.4.1 Five-number summary

- A five-number summary is a list consisting of the lowest score, lower quartile, median, upper quartile and greatest score of a set of data.

$X_{\min}$	Q_1	Median (Q_2)	Q_3	$X_{\max}$

WORKED EXAMPLE 7

From the following five-number summary, find:
a the interquartile range
b the range.

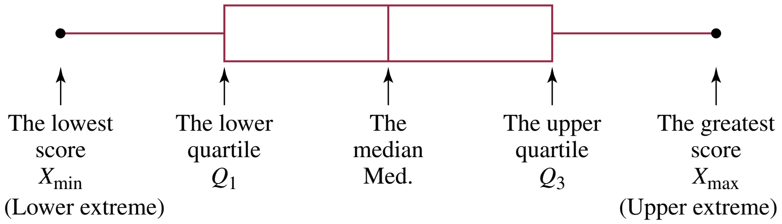
$X_{\min}$	Q_1	Median (Q_2)	Q_3	$X_{\max}$
29	37	39	44	48

THINK	WRITE
	$Q_3 = 44, X_{\max} = 48$
a The interquartile range is the difference between the upper and lower quartiles.	**a** $\text{IQR} = Q_3 - Q_1$ $= 44 - 37$ $= 7$
b The range is the difference between the greatest score and the lowest score.	**b** $\text{Range} = X_{\max} - X_{\min}$ $= 48 - 29$ $= 19$

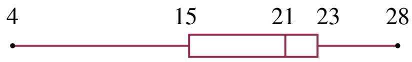
12.4.2 Box-and-whisker plots

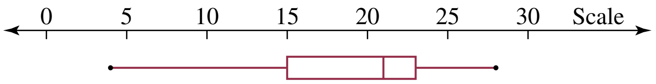
- A **box-and-whisker plot** (or **boxplot**) is a graph of the five-number summary.
- Box-and-whisker plots consist of a central divided box with attached whiskers.
- The box spans the interquartile range.
- The median is marked by a vertical line drawn inside the box.

- The whiskers indicate the range of scores:



- Box-and-whisker plots are *always drawn to scale*.
- They are presented either with the five-number summary figures attached as labels (diagram at right) or with a scale presented alongside the box-and-whisker plot like the diagram below. They can also be drawn vertically.

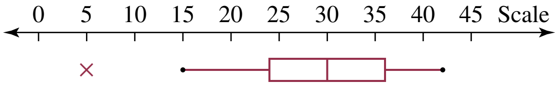




12.4.3 Identification of extreme values

- If an extreme value or outlier occurs in a set of data, it can be denoted by a small cross on the box-and-whisker plot. The whisker is then shortened to the next largest (or smallest) figure.

  The box-and-whisker plot below shows that the lowest score was 5. This was an extreme value as the rest of the scores were located within the range 15 to 42.

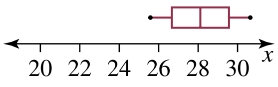


12.4.4 Describing distributions

Symmetry and skewness

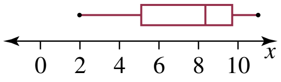
- A **symmetrical** plot has data that are evenly spaced around a central point. Examples of a stem-and-leaf plot and a symmetrical boxplot are shown below.

Stem	*Leaf*
26*	6
27	0 1 3
27*	5 6 8 9
28	0 1 1 1 2 4
28*	5 7 8 8
29	2 2 2
29*	5



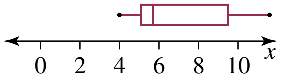
- A **negatively skewed** plot has larger amounts of data at the higher end. This is illustrated by the stem-and-leaf plot below where the leaves increase in length as the data increase in value. It is illustrated on the boxplot when the median is to the right within the box.

Stem	*Leaf*
5	1
6	2 9
7	1 1 2 2
8	1 4 4 5 6 6
9	5 3 4 4 5 6 7 7 7



- A **positively skewed** plot has larger amounts of data at the lower end. This is illustrated on the stem-and-leaf plot below where the leaves increase in length as the data decrease in value. It is illustrated on the boxplot when the median is to the left within the box.

Stem	*Leaf*
5	1 3 4 4 5 6 7 7 7
6	2 4 4 5 6 6
7	1 1 2 2
8	1 6
9	5



WORKED EXAMPLE 8

TI | CASIO

The following stem-and-leaf plot gives the speed of 25 cars caught by a roadside speed camera.

Key: 8 | 2 = 82 km/h, 8* | 6 = 86 km/h

Stem	***Leaf***
8	**2 2 4 4 4 4**
8*	**5 5 6 6 7 9 9 9**
9	**0 1 1 2 4**
9*	**5 6 9**
10	**0 2**
10*	
11	**4**

a Prepare a five-number summary of the data.
b Draw a box-and-whisker plot of the data. (Identify any extreme values.)
c Describe the distribution of the data.

THINK	WRITE
1 First identify the positions of the median and upper and lower quartiles. There are 25 data values. The median is the $\frac{n+1}{2}$th score. The lower quartile is the median of the lower half of the data. The upper quartile is the median of the upper half of the data (each half contains 12 scores).	The median is the $\frac{25+1}{2}$th score — that is, the 13th score. Q_1 is the $\frac{12+1}{2}$th score in the lower half — that is, the 6.5th score. That is, halfway between the 6th and 7th scores. Q_3 is halfway between the 6th and 7th scores in the upper half of the data.
2 Mark the positions of the median and upper and lower quartiles on the stem-and-leaf plot.	Key: 8 | 2 = 82 km/h 8* | 6 = 86 km/h

Stem	*Leaf*
8	2 2 4 4 4 4|
8*	5 5 6 6 7 9 ⑨ 9
9	0 1 1 2 4|
9*	5 6 9
10	0 2
10*	
11	4

$Q1$
Median
$Q3$

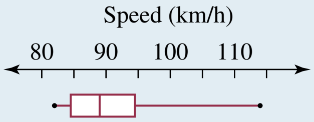
a Write the five-number summary:
The lowest score is 82.
The lower quartile is between 84 and 85; that is, 84.5.
The median is 89.
The upper quartile is between 94 and 95; that is, 94.5.
The greatest score is 114.

a Five-number summary:

X_{min}	Q_1	Median	Q_3	X_{max}
82	84.5	89	94.5	114

b Draw a labelled axis using an appropriate scale. Plot the points from the five-number summary.

b



c Describe the distribution.

c The data are skewed (positively).

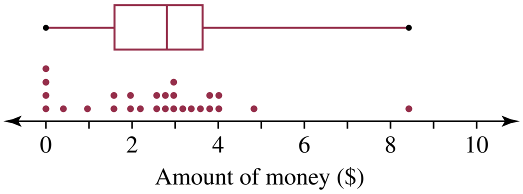
12.4.5 Shapes of graphs

Boxplots and dot plots

- Boxplots are a concise summary of data. A boxplot can be directly related to a dot plot.
- **Dot plots** display each data value represented by a dot placed on a number line.
  The following data are the amount of money (in $) that a group of 27 five-year-olds had with them on a day visiting the zoo with their parents.

0	1.65	0	2.60	3	8.45	4	0.55	4.10	3.35	3.25
2	2.85	2.90	1.70	3.65	1	0	0	2.25	2.05	3
3.80	2.65	4.75	3.90	2.95						

- The dot plot below and its comparative boxplot show the distribution of these data.



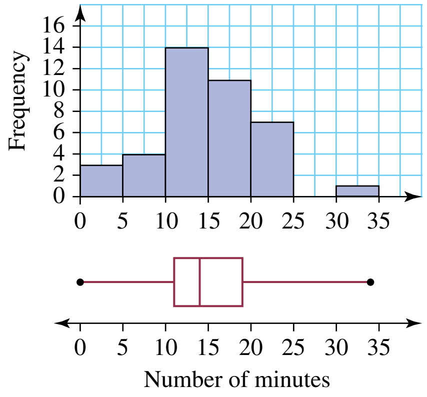
Both graphs indicate that the data are positively skewed. The dot plot clearly shows the presence of the outlier. This is less obvious with the boxplot. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

Boxplots and histograms

- **Histograms** are graphs that display continuous numerical variables and do not retain all original data.
- The following data are the number of minutes, rounded to the nearest minute, that forty Year 10 students take to travel to their school on a particular day.

15	22	14	12	21	34	19	11	13	0	16
4	23	8	12	18	24	17	14	3	10	12
9	15	20	5	19	13	17	11	16	19	24
12	7	14	17	10	14	23				

The data are displayed in the histogram and boxplot shown.



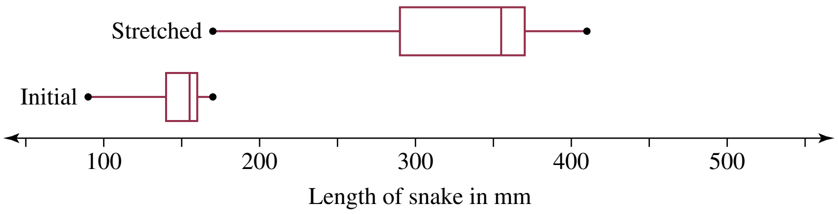
Both graphs indicate that the data are slightly positively skewed. The histogram clearly shows the frequencies of each class interval. Neither graph displays the original values. The histogram does not give precise information about the centre, but the distribution of the data is visible. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

12.4.6 Parallel boxplots

- A major reason for developing statistical skills is to be able to make comparisons between sets of data.
- Consider the following scenario.
  - Each member of a class was given a jelly snake to stretch. They each measured the initial length of their snake to the nearest centimetre and then slowly stretched the snake to make it as long as possible. They then measured the maximum length of the snake by recording how far it had stretched at the time it broke. The results were recorded in the following table.

Initial length (cm)	Stretched length (cm)	Initial length (cm)	Stretched length (cm)
13	29	14	27
14	28	13	27
17	36	15	36
10	24	16	36
14	35	15	36
16	36	16	34
15	37	17	35
16	37	12	27
14	30	9	17
16	33	16	41
17	36	17	38
16	38	16	36
17	38	17	41
14	31	16	33
17	40	11	21

- The data were then displayed on **parallel boxplots**, with the axis displaying in millimetres.
- By drawing the two boxplots on a single axis, it is easy to compare them.



The change in the length of the snake when stretched is evidenced by the increased median and spread shown on the boxplots. The median snake length before being stretched was 150 mm, but the median snake length after being stretched was 350 mm. The range increased after stretching, as did the IQR.

Exercise 12.4 Box-and-whisker plots

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 10, 13, 16, 19	Questions: 1–8, 10–12, 14, 16, 19	Questions: 1–20

🟩🟦🟪 **Individual pathway interactivity:** int-4623 learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE7** From the following five-number summary find:

$X_{\min}$	Q_1	Median	Q_3	$X_{\max}$
6	11	13	16	32

   a. the interquartile range
   b. the range.

2. From the following five-number summary find:

$X_{\min}$	Q_1	Median	Q_3	$X_{\max}$
101	119	122	125	128

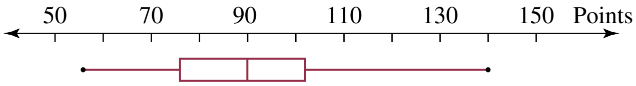
   a. the interquartile range
   b. the range.

3. From the following five-number summary find:

$X_{\min}$	Q_1	Median	Q_3	$X_{\max}$
39.2	46.5	49.0	52.3	57.8

   a. the interquartile range
   b. the range.

4. The box-and-whisker plot at right shows the distribution of final points scored by a football team over a season's roster.



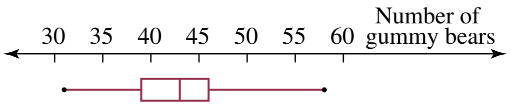
   a. What was the team's greatest points score?
   b. What was the team's least points score?

c. What was the team's median points score?

d. What was the range of points scored?

e. What was the interquartile range of points scored?

5. The box-and-whisker plot at right shows the distribution of data formed by counting the number of gummy bears in each of a large sample of packs.



a. What was the largest number of gummy bears in any pack?

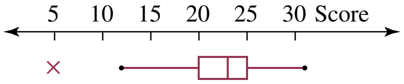
b. What was the smallest number of gummy bears in any pack?

c. What was the median number of gummy bears in any pack?

d. What was the range of numbers of gummy bears per pack?

e. What was the interquartile range of gummy bears per pack?

Questions **6** to **8** refer to the following box-and-whisker plot.



6. **MC** The median of the data is:

A. 20 **B.** 23 **C.** 25 **D.** 31

7. **MC** The interquartile range of the data is:

A. 23 **B.** 26 **C.** 5 **D.** 20 to 25

8. **MC** Which of the following is not true of the data represented by the box-and-whisker plot?

A. One-quarter of the scores are between 5 and 20.

B. Half of the scores are between 20 and 25.

C. The lowest quarter of the data is spread over a wide range.

D. Most of the data are contained between the scores of 5 and 20.

Understanding

9. The number of sales made each day by a salesperson is recorded over a 2-week period:

25, 31, 28, 43, 37, 43, 22, 45, 48, 33

a. Prepare a five-number summary of the data. (There is no need to draw a stem-and-leaf plot of the data. Just arrange them in order of size.)

b. Draw a box-and-whisker plot of the data.

10. The data below show monthly rainfall in millimetres.

J	F	M	A	M	J	J	A	S	O	N	D
10	12	21	23	39	22	15	11	22	37	45	30

a. Prepare a five-number summary of the data.

b. Draw a box-and-whisker plot of the data.

11. **WE8** The stem-and-leaf plot at right details the age of 25 offenders who were caught during random breath testing.

a. Prepare a five-number summary of the data.
b. Draw a box-and-whisker plot of the data.
c. Describe the distribution of the data.

Key: 1 | 8 = 18 years

Stem	*Leaf*
1	8 8 9 9 9
2	0 0 0 1 1 3 4 6 9
3	0 1 2 7
4	2 5
5	3 6 8
6	6
7	4

12. The following stem-and-leaf plot details the price at which 30 blocks of land in a particular suburb sold for.

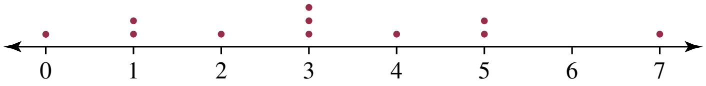
Key: 12 | 4 = $124 000

Stem	*Leaf*
12	4 7 9
13	0 0 2 5 5
14	0 0 2 3 5 5 7 9 9
15	0 0 2 3 7 7 8
16	0 2 2 5 8
17	5

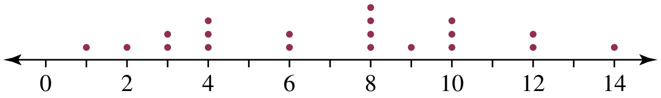
a. Prepare a five-number summary of the data.
b. Draw a box-and-whisker plot of the data.

13. Prepare comparative boxplots for the following dot plots (using the same axis) and describe what each plot reveals about the data.

a. Number of sick days taken by workers last year at factory A



b. Number of sick days taken by workers last year at factory B



14. An investigation into the transport needs of an outer suburb community recorded the number of passengers boarding a bus during each of its journeys, as follows.

12, 43, 76, 24, 46, 24, 21, 46, 54, 109, 87, 23, 78, 37, 22, 139, 65, 78, 89, 52, 23, 30, 54, 56, 32, 66, 49

Display the data by constructing a histogram using class intervals of 20 and a comparative boxplot on the same axis.

15. At a weight-loss clinic, the following weights (in kilograms) were recorded before and after treatment.

Before	75	80	75	140	77	89	97	123	128	95	152	92
After	69	66	72	118	74	83	89	117	105	81	134	85

Before	85	90	95	132	87	109	87	129	135	85	137	102
After	79	84	90	124	83	102	84	115	125	81	123	94

a. Prepare a five-number summary for weight before and after treatment.
b. Draw parallel boxplots for weight before and after treatment.
c. Comment on the comparison of weights before and after treatment.

Reasoning

16. The following data detail the number of hamburgers sold by a fast food outlet every day over a 4-week period.

M	T	W	T	F	S	S
125	144	132	148	187	172	181
134	157	152	126	155	183	188
131	121	165	129	143	182	181
152	163	150	148	152	179	181

a. Prepare a stem-and-leaf plot of the data. (Use a class size of 10.)
b. Draw a box-and-whisker plot of the data.
c. What do these graphs tell you about hamburger sales?

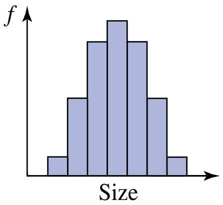
17. The following data show the ages of 30 mothers upon the birth of their first baby.

22	21	18	33	17	23	22	24	24	20
25	29	32	18	19	22	23	24	28	20
31	22	19	17	23	48	25	18	23	20

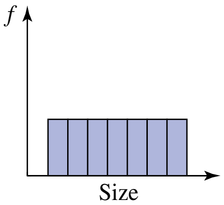
a. Prepare a stem-and-leaf plot of the data. (Use a class size of 5.)
b. Draw a box-and-whisker plot of the data. Indicate any extreme values appropriately.
c. Describe the distribution in words. What does the distribution say about the age that mothers have their first baby?

18. **MC** Match the box-and-whisker plot with its most likely histogram.

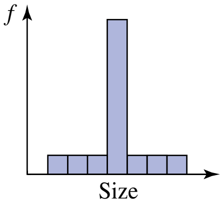
A.



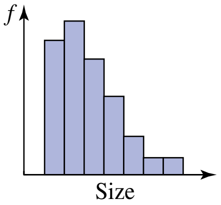
B.



C.

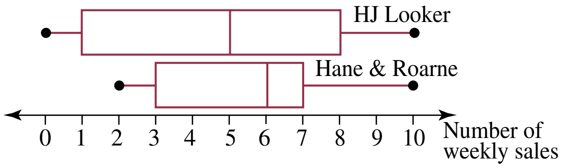


D.



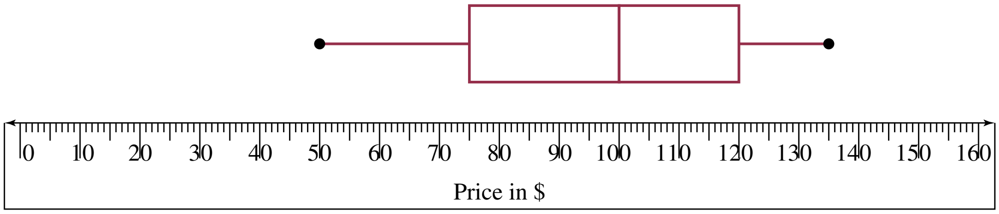
Problem solving

19. Consider the box-and-whisker plot below which shows the number of weekly sales of houses by two real estate agencies.



a. What is the median number of weekly sales for each real estate agency?
b. Which agency had the greater range of sales?
c. Which agency had the greater interquartile range of sales?
d. Which agency performed better? Explain your answer.

20. Fifteen French restaurants were visited by three newspaper restaurant reviewers. The average price of a meal for a single person was investigated. The following box-and-whisker plot shows the results.



a. What was the price of the cheapest meal?
b. What was the price of the most expensive meal?
c. What is the median cost of a meal?
d. What is the interquartile range for the price of a meal?
e. What percentage of the prices were below the median?

Reflection

What advantages and disadvantages do box-and-whisker plots have as a visual form of representing data?

12.5 The standard deviation

12.5.1 Standard deviation

- The **standard deviation** for a set of data is a measure of how far the data values are spread out (deviate) from the mean.
- **Deviation** is the difference between each data value and the mean $(x - \bar{x})$. The standard deviation is calculated from the square of the deviations.
- Standard deviation is denoted by the Greek letter sigma, σ, and can be calculated by using the formula

$$\sigma = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}$$

  where $\bar{x}$ is the mean of the data values and n is the number of data values.
- A low standard deviation indicates that the data values tend to be close to the mean.
- A high standard deviation indicates that the data values tend to be spread out over a large range, away from the mean.
- Standard deviation can be calculated using a scientific or graphics calculator, or it can be calculated from a frequency table by following the steps below.

Step 1 Calculate the mean.
Step 2 Calculate the deviations.
Step 3 Square each deviation.
Step 4 Sum the squares.
Step 5 Divide the sum of the squares by the number of data values.
Step 6 Take the square root of the result.

WORKED EXAMPLE 9

TI | CASIO

The number of lollies in each of 8 packets is 11, 12, 13, 14, 16, 17, 18, 19.
Calculate the mean and standard deviation correct to 2 decimal places.

THINK	WRITE
1 Calculate the mean.	$\bar{x} = \frac{11 + 12 + 13 + 14 + 16 + 17 + 18 + 19}{8}$ $= \frac{120}{8}$ $= 15$

2 To calculate the deviations $(x - \bar{x})$, set up a frequency table as shown and complete.

No. of lollies (x)	$(x - \bar{x})$
11	$11 - 15 = -4$
12	-3
13	-2
14	-1
16	1
17	2
18	3
19	4
Total	

3 Add another column to the table to calculate the square of the deviations, $(x - x)^2$. Then sum the results: $\sum (x - \bar{x})^2$.

No. of lollies (x)	$(x - \bar{x})$	$(x - \bar{x})^2$
11	$11 - 15 = -4$	16
12	-3	9
13	-2	4
14	-1	1
16	1	1
17	2	4
18	3	9
19	4	16
Total		$\sum (x - \bar{x})^2 = 60$

4 To calculate the standard deviation, divide the sum of the squares by the number of data values, then take the square root of the result.

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{60}{8}}$$

≈ 2.74 (correct to 2 decimal places)

5 Check the result using a calculator.

The calculator returns an answer of $\sigma_n = 2.738\,61$. Answer confirmed.

6 Interpret the result.

The average (mean) number of lollies in each pack is 15 with a standard deviation of 2.74, which means that the number of lollies in each pack differs from the mean by an average of 2.74.

- When calculating the standard deviation from a frequency table, the frequencies must be taken into account. Therefore, the following formula is used.

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}}$$

WORKED EXAMPLE 10

TI | CASIO

Lucy's scores in her last 12 games of golf were 87, 88, 88, 89, 90, 90, 90, 92, 93, 93, 95 and 97. Calculate the mean score and the standard deviation correct to 2 decimal places.

THINK

WRITE

1 To calculate the mean, first set up a frequency table.

Golf score (x)	Frequency (f)	fx
87	1	87
88	2	176
89	1	89
90	3	270
92	1	92
93	2	186
95	1	95
97	1	97
Total	$\sum f = 12$	$\sum fx = 1092$

2 Calculate the mean.

$$\bar{x} = \frac{\sum fx}{\sum f}$$
$$= \frac{1092}{12}$$
$$= 91$$

3 To calculate the deviations $(x - \bar{x})$, add another column to the frequency table and complete.

Golf score (x)	Frequency (f)	fx	$(x - \bar{x})$
87	1	87	$87 - 91 = -4$
88	2	176	-3
89	1	89	-2
90	3	270	-1
92	1	92	1
93	2	186	2
95	1	95	4
97	1	97	6
Total	$\sum f = 12$	$\sum fx = 1092$	

4 Add another column to the table and multiply the square of the deviations, $(x - \bar{x})^2$, by the frequency $f(x - \bar{x})^2$.

Then sum the results:

$\sum f(x - \bar{x})^2$.

Golf score (x)	Frequency (f)	fx	$(x - \bar{x})$	$f(x - \bar{x})^2$
87	1	87	$87 - 91 = -4$	$1 \times (-4)^2 = 16$
88	2	176	-3	18
89	1	89	-2	4
90	3	270	-1	3
92	1	92	1	1
93	2	186	2	8
95	1	95	4	16
97	1	97	6	36
Total	$\sum f = 12$	$\sum fx = 1092$		$\sum f(x - \bar{x})^2 = 102$

▶

5 Calculate the standard deviation using the formula.	$\sigma = \sqrt{\dfrac{\sum f(x - \bar{x})^2}{n}}$ $= \sqrt{\dfrac{120}{12}}$ ≈ 2.92 (correct to 2 decimal places)
6 Check the result using a calculator.	The calculator returns an answer of $\sigma_n = 2.915\,48$. The answer is confirmed.
7 Interpret the result.	The average (mean) score for Lucy is 91 with a standard deviation of 2.92, which means that her score differs from the mean by an average of 2.92.

12.5.2 Why the deviations are squared

- For large data sets that are symmetrically distributed, the sum of the deviations is usually zero, that is, $\sum (x - \bar{x}) = 0$. When the mean is greater than the data value ($\bar{x} > x$), the deviation is negative. When the mean is smaller than the data value ($\bar{x} < x$), the deviation is positive. The negative and positive deviations cancel each other out; therefore, calculating the sum and average of the deviations is not useful. This explains why the standard deviation is calculated using the squares of the deviations, $(x - \bar{x})^2$, for all data values.

12.5.3 Standard deviations of populations and samples

- So far we have calculated the standard deviation for a population of data, that is, for complete sets of data. There is another formula for calculating standard deviation for samples of data, that is, data that have been randomly selected from a larger population.
- For example, a sample of 100 Year 10 students from New South Wales is taken to determine the amount of time they spend on their mobile phones. In this case, the standard deviation formula, denoted by s, that would apply is

$$s = \sqrt{\frac{\sum f(x - \bar{x})^2}{n - 1}}$$

- The calculator usually displays both values for the standard deviation, so it is important to understand the difference between them. However, in this course we will use the formula for populations, σ.

12.5.4 Effects on standard deviation

- The standard deviation is affected by extreme values.

WORKED EXAMPLE 11

On a particular day Lucy played golf brilliantly and scored 60. The scores in her previous 12 games of golf were 87, 88, 88, 89, 90, 90, 90, 92, 93, 93, 95 and 97 (see Worked example 10). Comment on the effect this latest score has on the standard deviation.

THINK	WRITE
1 Use a calculator to calculate the mean and the standard deviation.	$\bar{x} = 88.6154 \approx 88.62$ $\quad \sigma = 8.7225 \approx 8.72$
2 Interpret the result and compare it to the results found in Worked example 10.	In the first 12 games Lucy's mean score was 91 with a standard deviation of 2.92. This implied that Lucy's scores on average were 2.92 either side of her average of 91. Lucy's latest performance resulted in a mean score of 88.62 with a standard deviation of 8.72. This indicates a slightly lower mean score, but the much higher standard deviation indicates that the data are now much more spread out and that the extremely good score of 60 is an anomaly.

12.5.5 Properties of standard deviation

- If a constant c is added to all data values in a set, the deviations $(x - \bar{x})$ will remain unchanged and consequently the standard deviation remains unchanged.
- If all data values in a set are multiplied by a constant k, the deviations $(x - \bar{x})$ will be multiplied by k, that is $k(x - \bar{x})$; consequently the standard deviation is increased by a factor of k.
- Standard deviation can be used to measure consistency.
- When the standard deviation is low we are able to say that the scores in the data set are more consistent with each other.

WORKED EXAMPLE 12

For the data 5, 9, 6, 11, 10, 7:

a calculate the standard deviation

b calculate the standard deviation if 4 is added to each data value. Comment on the effect.

c calculate the standard deviation if all data values are multiplied by 2. Comment on the effect.

THINK

a 1 Calculate the mean.

2 Set up a frequency table and enter the squares of the deviations.

WRITE

a $$\bar{x} = \frac{5 + 9 + 6 + 11 + 10 + 7}{6} = 8$$

(x)	$(x - \bar{x})$	$(x - \bar{x})^2$
5	$5 - 8 = -3$	9
6	-2	4
7	-1	1
9	1	1
10	2	4
11	3	9
Total		$\sum (x - \bar{x})^2 = 28$

▶

3 To calculate the standard deviation, apply the formula for standard deviation.

$$\sigma = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{28}{6}}$$

$$\approx 2.16$$

(correct to 2 decimal places)

b 1 Add 4 to each data value in the set.

b 9, 13, 10, 15, 14, 11

2 Calculate the mean.

$$\bar{x} = \frac{9 + 13 + 10 + 15 + 14 + 11}{6}$$

$$= 12$$

3 Set up a frequency table and enter the squares of the deviations.

(x)	$(x-\bar{x})$	$(x-\bar{x})^2$
9	$9-12=-3$	9
10	-2	4
11	-1	1
13	1	1
14	2	4
15	3	9
Total		$\sum (x-\bar{x})^2 = 28$

4 To calculate the standard deviation, apply the formula for standard deviation.

$$\sigma = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{28}{6}}$$

$$\approx 2.16$$

(correct to 2 decimal places)

5 Comment on the effect of adding of 4 to each data value.

Adding 4 to each data value increased the mean but had no effect on the standard deviation, which remained at 2.16.

c 1 Multiply each data value in the set by 2.

c 10, 18, 12, 22, 20, 14

2 Calculate the mean.

$$\bar{x} = \frac{10 + 18 + 12 + 22 + 20 + 14}{6}$$

$$= 16$$

3 Set up a frequency table and enter the squares of the deviations.

(x)	$(x-\bar{x})$	$(x-\bar{x})^2$
10	$10-16=-6$	36
12	-4	16
14	-2	4
18	2	4
20	4	16
22	6	36
Total		$\sum (x-\bar{x})^2 = 112$

4 To calculate the standard deviation, apply the formula for standard deviation.

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\frac{112}{6}}$$

≈ 4.32

(correct to 2 decimal places)

5 Comment on the effect of multiplying each data value by 2.

Multiplying each data value by 2 doubled the mean and doubled the standard deviation, which changed from 2.16 to 4.32.

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Complete this digital doc: WorkSHEET: Univariate data II (doc-14596)

Exercise 12.5 The standard deviation

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 9, 10, 13	Questions: 1–11, 13	Questions: 1–14

🟩🟦🟪 **Individual pathway interactivity:** int-4624

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Fluency

1. **WE9** Calculate the standard deviation of each of the following data sets, correct to 2 decimal places.

a. 3, 5, 8, 2, 7, 1, 6, 5

b. 11, 8, 7, 12, 10, 11, 14

c. 25, 15, 78, 35, 56, 41, 17, 24

d. 5.2, 4.7, 5.1, 12.6, 4.8

2. **WE10** Calculate the standard deviation of each of the following data sets, correct to 2 decimal places.

a.

Score (x)	Frequency (f)
1	1
2	5
3	9
4	7
5	3

b.

Score (x)	Frequency (f)
16	15
17	24
18	26
19	28
20	27

c.

Score (x)	Frequency (f)
8	15
10	19
12	18
14	7
16	6
18	2

d.

Score (x)	Frequency (f)
65	15
66	15
67	16
68	17
69	16
70	15
71	15
72	12

3. Complete the following frequency distribution table and use the table to calculate the standard deviation of the data set, correct to 2 decimal places.

Class	Class centre (x)	Frequency (f)
1–10		6
11–20		15
21–30		25
31–40		8
41–50		6

4. First-quarter profit increases for 8 leading companies are given below as percentages.

2.3 0.8 1.6 2.1 1.7 1.3 1.4 1.9

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

5. The heights in metres of a group of army recruits are given below.

1.8 1.95 1.87 1.77 1.75 1.79 1.81 1.83 1.76 1.80 1.92 1.87 1.85 1.83

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

6. Times (to the nearest tenth of a second) for the heats in the open 100 m sprint at the school sports are given at right.

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

Key: 11 | 0 = 11.0 s

Stem	*Leaf*
11	0
11	2 3
11	4 4 5
11	6 6
11	8 8 9
12	0 1
12	2 2 3
12	4 4
12	6
12	9

7. The number of outgoing phone calls from an office each day over a 4-week period is shown on the stem plot at right.

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

Key: 1 | 3 = 13 calls

Stem	*Leaf*
0	8 9
1	3 4 7 9
2	0 1 3 7 7
3	3 4
4	1 5 6 7 8
5	3 8

8. MC A new legal aid service has been operational for only 5 weeks. The number of people who have made use of the service each day during this period is set out below.

Key: 1 | 6 = 16 people

Stem	*Leaf*
0	2 4
0	7 7 9
1	0 1 4 4 4 4
1	5 6 6 7 8 8 9
2	1 2 2 3 3 3
2	7

The standard deviation (to 2 decimal places) of these data is:

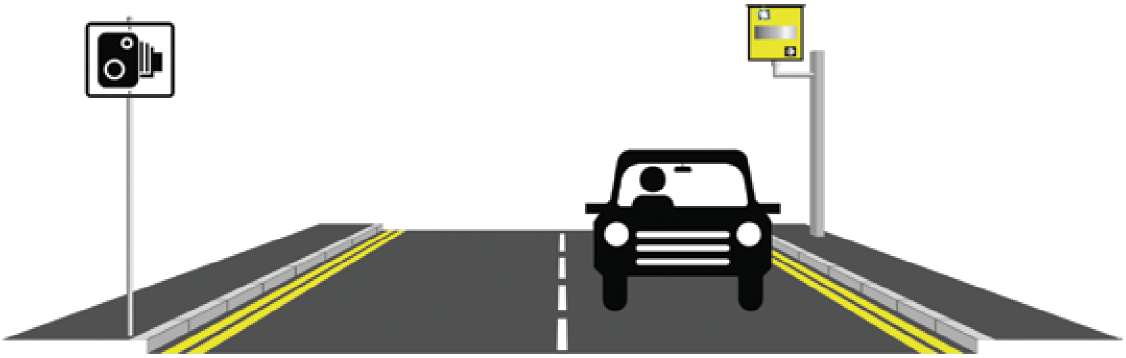
A. 6.00 **B.** 6.34 **C.** 6.47 **D.** 15.44

Understanding

9. **WE11** The speeds, in km/h, of the first 25 cars caught by a roadside speed camera on a particular day were:

82, 82, 84, 84, 84, 84, 85, 85, 85, 86, 86, 87, 89, 89, 89, 90, 91, 91, 92, 94, 95, 96, 99, 100, 102

The next car that passed the speed camera was travelling at 140 km/h. Comment on the effect of the speed of this last car on the standard deviation for the data.



Reasoning

10. **WE12** For the data 1, 4, 5, 9, 11:

a. calculate the standard deviation

b. calculate the standard deviation if 7 is added to each data value. Comment on the effect.

c. calculate the standard deviation if all data values are multiplied by 3. Comment on the effect.

11. Show using an example the effect, if any, on the standard deviation of adding a data value to a set of data that is equivalent to the mean.

12. If the mean for a set of data is 45 and the standard deviation is 6, how many standard deviations above the mean is a data value of 57?

Problem solving

13. Five numbers a, b, c, d and e have a mean of 12 and a standard deviation of 4.

a. If each number is increased by 3, find the new mean and standard deviation in terms of the original mean and standard deviation.

b. If each number is multiplied by 3, find the new mean and standard deviation in terms of the original mean and standard deviation.

14. Twenty-five students sat a test and the results for 24 of the students are given in the following stem-and-leaf plot.

Stem	*Leaf*
0	8 9
1	1 2 3 7 8 9
2	2 3 5 6 8
3	0 1 2 4 6 8
4	0 2 5 6 8

a. If the average mark for the test was 27.84, determine the mark obtained by the 25th student.

b. How many students scored higher than the median score?

c. Find the standard deviation of the marks, giving your answer correct to 2 decimal places.

Reflection

What does the standard deviation tell us about a set of data?

12.6 Comparing data sets

12.6.1 Comparing data sets

- Besides locating the centre of the data (the mean, median or mode), any analysis of data must measure the extent of the spread of the data (range, interquartile range and standard deviation). Two data sets may have centres that are very similar but be quite differently distributed.
- Decisions need to be made about which measure of centre and which measure of spread to use when analysing and comparing data.
- The mean is calculated using every data value in the set. The median is the middle score of an ordered set of data, so it does not include every individual data value in its calculation. The mode is the most frequently occurring data value, so it also does not include every individual data value in its calculation.
- The range is calculated by finding the difference between the maximum and minimum data values, so it includes outliers. It provides only a rough idea about the spread of the data and is inadequate in providing sufficient detail for analysis. It is useful, however, when we are interested in extreme values such as high and low tides or maximum and minimum temperatures.

  The interquartile range is the difference between the upper and lower quartiles, so it does not include every data value in its calculation, but it will overcome the problem of outliers skewing data.

  The standard deviation is calculated using every data value in the set.

WORKED EXAMPLE 13

For the two sets of data 6, 7, 8, 9, 10 and 12, 4, 10, 11, 3:

a calculate the mean

b calculate the standard deviation

c comment on the similarities and differences.

THINK	WRITE
a 1 Calculate the mean of the first set of data.	**a** $\bar{x}_1 = \dfrac{6+7+8+9+10}{5}$ $= 8$
2 Calculate the mean of the second set of data.	$\bar{x}_2 = \dfrac{12+4+10+11+3}{5}$ $= 8$
b 1 Calculate the standard deviation of the first set of data.	**b** $\sigma_1 = \sqrt{\dfrac{(6-8)^2+(7-8)^2+(8-8)^2+(9-8)^2+(10-8)^2}{5}}$ ≈ 1.41
2 Calculate the standard deviation of the second set of data.	$\sigma_2 = \sqrt{\dfrac{(12-8)^2+(4-8)^2+(10-8)^2+(11-8)^2+(3-8)^2}{5}}$ ≈ 3.74
c Comment on the findings.	**c** For both sets of data the mean was the same, 8. However, the standard deviation for the second set (3.74) was much higher than the standard deviation of the first set (1.41), implying that the second set is more widely distributed than the first. This is confirmed by the range, which is 10 − 6 = 4 for the first set and 12 − 3 = 9 for the second.

- When multiple data displays are used to display similar sets of data, comparisons and conclusions can then be drawn about the data.
- We can use **back-to-back stem-and-leaf plots** and multiple or parallel box-and-whisker plots to help compare statistics such as the median, range and interquartile range.

WORKED EXAMPLE 14

TI | CASIO

Below are the scores achieved by two students in eight Mathematics tests throughout the year.
John: 45, 62, 64, 55, 58, 51, 59, 62
Penny: 84, 37, 45, 80, 74, 44, 46, 50

a Determine the most appropriate measure of centre and measure of spread to compare the performance of the students.
b Which student had the better overall performance on the eight tests?
c Which student was more consistent over the eight tests?

THINK	WRITE
a In order to include all data values in the calculation of measures of centre and spread, calculate the mean and standard deviation.	**a** John: $\bar{x} = 57$, $\sigma = 6$ Penny: $\bar{x} = 57.5$, $\sigma = 17.4$
b Compare the mean for each student. The student with the higher mean performed better overall.	**b** Penny performed slightly better on average as her mean mark was higher than John's.
c Compare the standard deviation for each student. The student with the lower standard deviation performed more consistently.	**c** John was the more consistent student because his standard deviation was much lower than Penny's. This means that his test results were closes to his mean score than Penny's were to hers.

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Exercise 12.6 Comparing data sets

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Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 9, 10, 12, 14	Questions: 1–12, 15	Questions: 1–18

🟩🟦🟪 **Individual pathway interactivity:** int-4625

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Fluency

1. **WE13** For the two sets of data 65, 67, 61, 63, 62, 60 and 56, 70, 65, 72, 60, 55:
   - **a.** calculate the mean
   - **b.** calculate the standard deviation
   - **c.** comment on the similarities and differences.
2. A bank surveys the average morning and afternoon waiting times for customers. The figures were taken each Monday to Friday in the morning and afternoon for one month. The stem-and-leaf plot below shows the results.

Key: $1|2 = 1.2$ minutes

Leaf: Morning	*Stem*	*Leaf: Afternoon*
7	0	7 8 8
8 6 3 1 1	1	1 1 2 4 4 5 6 6 6 7
9 6 6 6 5 5 4 3 3 1	2	2 5 5 8
9 5 2	3	1 6
5	4	
	5	7

   - **a.** Find the median morning waiting time and the median afternoon waiting time.
   - **b.** Calculate the range for morning waiting times and the range for afternoon waiting times.
   - **c.** What conclusions can be made from the display about the average waiting time at the bank in the morning compared with the afternoon?
3. In a class of 30 students there are 15 boys and 15 girls. Their heights are measured (in metres) and are listed below.

Boys: 1.65, 1.71, 1.59, 1.74, 1.66, 1.69, 1.72, 1.66,
1.65, 1.64, 1.68, 1.74, 1.57, 1.59, 1.60

Girls: 1.66, 1.69, 1.58, 1.55, 1.51, 1.56, 1.64, 1.69,
1.70, 1.57, 1.52, 1.58, 1.64, 1.68, 1.67

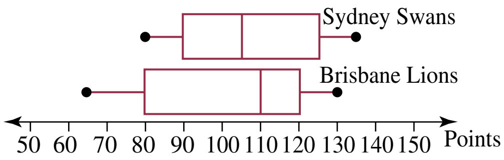
Display this information in a back-to-back stem-and-leaf plot.

4. The stem-and-leaf plot at right is used to display the number of vehicles sold by the Ford and Holden dealerships in a Sydney suburb each week for a three-month period.
   - **a.** State the median of both distributions.
   - **b.** Calculate the range of both distributions.
   - **c.** Calculate the interquartile range of both distributions.
   - **d.** Show both distributions on a box-and-whisker plot.

Key: $1|5 = 15$ vehicles

Leaf: Ford	*Stem*	*Leaf: Holden*
7 4	0	3 9
9 5 2 2 1 0	1	1 1 1 6 6 8
8 5 4 4	2	2 2 7 9
0	3	5

5. The box-and-whisker plot drawn below displays statistical data of two AFL teams over a season.



   - **a.** Which team had the higher median score?
   - **b.** What was the range of scores for each team?
   - **c.** For each team calculate the interquartile range.

Understanding

6. Tanya measures the heights (in m) of a group of Year 10 boys and girls and produces the following five-point summaries for each data set.

Boys:	1.45, 1.56, 1.62, 1.70, 1.81
Girls:	1.50, 1.55, 1.62, 1.66, 1.73

a. Draw a box-and-whisker plot for both sets of data and display them on the same scale.

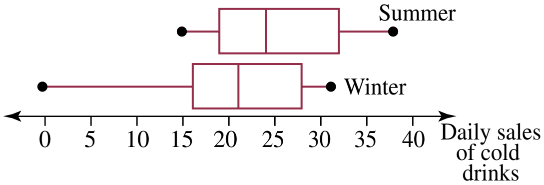
b. What is the median of each distribution?

c. What is the range of each distribution?

d. What is the interquartile range for each distribution?

e. Comment on the spread of the heights among the boys and the girls.

7. The box-and-whisker plots at right show the average daily sales of cold drinks at the school canteen in summer and winter.

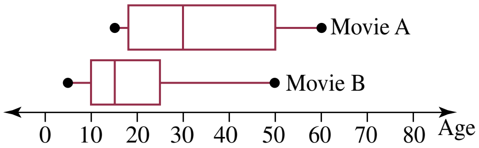


a. Calculate the range of sales in both summer and winter.

b. Calculate the interquartile range of the sales in both summer and winter.

c. Comment on the relationship between the two data sets, both in terms of measures of centre and measures of spread.

8. MC Andrea surveys the age of people at two movies being shown at a local cinema. The box-and-whisker plot at right shows the results.



Which of the following conclusions could be drawn based on the above information?

A. Movie A attracts an older audience than Movie B.

B. Movie B attracts an older audience than Movie A.

C. Movie A appeals to a wider age group than Movie B.

D. Movie B appeals to a wider age group than Movie A.

9. MC *Note:* There may be more than one correct answer.

The figures below show the age of the first 10 men and women to finish a marathon.

Men:	28, 34, 25, 36, 25, 35, 22, 23, 40, 24
Women:	19, 27, 20, 26, 30, 18, 28, 25, 28, 22

Which of the following statements is correct?

A. The mean age of the men is greater than the mean age of the women.

B. The range is greater among the men than among the women.

C. The interquartile range is greater among the men than among the women.

D. The standard deviation is greater among the men than among the women.

Reasoning

10. WE14 Cory recorded his marks for each test that he did in English and Science throughout the year.

English:	55, 64, 59, 56, 62, 54, 65, 50
Science:	35, 75, 81, 32, 37, 62, 77, 75

a. In which subject did Cory achieve the better average mark?

b. In which subject was Cory more consistent? Explain your answer.

11. The police set up two radar speed checks on a back street of Sydney and on a main road. In both places the speed limit is 60 km/h. The results of the first 10 cars that have their speed checked are given below.

Back street:	60, 62, 58, 55, 59, 56, 65, 70, 61, 64
Main road:	55, 58, 59, 50, 40, 90, 54, 62, 60, 60

a. Calculate the mean and standard deviation of the readings taken at each point.

b. On which road are drivers generally driving faster?

c. On which road is the spread of the reading taken greater? Justify your answer.

12. Nathan and Timana are wingers in their local rugby league team. The number of tries they have scored in each season are listed below.

Nathan:	25, 23, 13, 36, 1, 8, 0, 9, 16, 20
Timana:	5, 10, 12, 14, 18, 11, 8, 14, 12, 19

a. Calculate the mean number of tries scored by each player.

b. What is the range of tries scored by each player?

c. What is the interquartile range of tries scored by each player?

d. Which player would you consider to be the more consistent player? Justify your answer.

13. In boxes of Smarties it is advertised that there are 50 Smarties in each box. Two machines are used to distribute the Smarties into the boxes. The results from a sample taken from each machine are shown in the stem-and-leaf plot below.

Key: $5|1 = 51 \quad 5^*|6 = 56$

Leaf: Machine A	*Stem*	*Leaf: Machine B*
4	4	
9 9 8 7 7 6 6 5	4*	5 7 8 9 9 9 9 9 9 9 9
4 3 2 2 2 1 1 1 0 0 0 0 0 0	5	0 0 0 0 0 1 1 1 1 1 2 2 3
5 5	5*	9

a. Display the data from both machines on parallel box-and-whisker plots.

b. Calculate the mean and standard deviation of the number of Smarties distributed from both machines.

c. Which machine is the more dependable? Justify your answer.

14. Year 10 students at Merrigong High School sit exams in Science and Maths. The results are shown in the table below.

Mark	Number of students in Science	Number of students in Maths
51–60	7	6
61–70	10	7
71–80	8	12
81–90	8	9
91–100	2	6

a. Is either distribution symmetrical?

b. If either distribution is not symmetrical, state whether it is positively or negatively skewed.

c. Discuss the possible reasons for any skewness.

d. State the modal class of each distribution.

e. In which subject is the standard deviation greater? Explain your answer.

15. Draw an example of a graph that is:

a. symmetrical

b. positively skewed with one mode

c. negatively skewed with two modes.

16. A new drug for the relief of cold symptoms has been developed. To test the drug, 40 people were exposed to a cold virus. Twenty patients were then given a dose of the drug while another 20 patients were given a placebo. (In medical tests a control group is often given a *placebo* drug. The subjects in this group believe that they have been given the real drug but in fact their dose contains no drug at all.) All participants were then asked to indicate the time when they first felt relief of symptoms. The number of hours from the time the dose was administered to the time when the patients first felt relief of symptoms are detailed below.

Group A (drug)

25	29	32	45	18	21	37	42	62	13
42	38	44	42	35	47	62	17	34	32

Group B (placebo)

25	17	35	42	35	28	20	32	38	35
34	32	25	18	22	28	21	24	32	36

a. Detail the data on a back-to-back stem-and-leaf plot.

b. Display the data for both groups on a box-and-whisker plot.

c. Make comparisons of the data. Use statistics in your answer.

d. Does the drug work? Justify your answer.

e. What other considerations should be taken into account when trying to draw conclusions from an experiment of this type?

Problem solving

17. The heights of Year 10 and Year 12 students (to the nearest centimetre) are being investigated. The results of some sample data are shown below.

Year 10	160	154	157	170	167	164	172	158	177	180	175	168	159	155	163	163	169	173	172	170
Year 12	160	172	185	163	177	190	183	181	176	188	168	167	166	177	173	172	179	175	174	180

a. Draw a back-to-back stem-and-leaf plot.

b. Draw a parallel boxplot.

c. Comment on what the plots tell you about the heights of Year 10 and Year 12 students.

18. Kloe compares her English and Maths marks. The results of eight tests in each subject are shown below.

English:	76, 64, 90, 67, 83, 60, 85, 37
Maths:	80, 56, 92, 84, 65, 58, 55, 62

a. Calculate Kloe's mean mark in each subject.
b. Calculate the range of marks in each subject.
c. Calculate the standard deviation of marks in each subject.
d. Based on the above data, in which subject would you say that Kloe has performed more consistently?

Reflection

Which data display is best for comparing data sets?

CHALLENGE 12.2

A sample of 50 students was surveyed on whether they owned an iPod or a mobile phone. The results showed that 38 per cent of the students owned both. Sixty per cent of the students owned a mobile phone and there were four students who had an iPod only. What percentage of students did not own a mobile phone or an iPod?

12.7 Review

12.7.1 Review questions

Fluency

1. Find the mean, median and mode for each of the following sets of data:

a. 7, 15, 8, 8, 20, 14, 8, 10, 12, 6, 19

b. Key: $1|2 = 12$

Stem	*Leaf*
1	2 6
2	1 7 8
3	0 3 3 4 6 8
4	0 1 1 5 9
5	1 3 6

c.

Score (x)	Frequency (f)
70	2
71	6
72	9
73	7
74	4

2. For each of the following data sets, find the range.

a. 4, 3, 6, 7, 2, 5, 8, 4, 3

b.

x	13	14	15	16	17	18	19
f	3	6	7	12	6	7	8

c. Key: $1|8 = 18$

Stem	*Leaf*
1	7 8 8 9
2	1 2 4 4 5 7 7 7 8 9 9
3	0 0 0 1 3 4 7

3. For each of the following data sets, find the interquartile range.

a. 18, 14, 15, 19, 20, 11, 16, 19, 18, 19

b. Key: 9|8 = 9.8

Stem	*Leaf*
8	7 8 8 9
9	0 2 4 4 5 7 7 7 8 9 9
10	0 1 1 1 3

4. The following back-to-back stem-and-leaf plot shows the typing speed in words per minute (wpm) of 30 Year 8 and Year 10 students.

Key: 2|6 = 26 wpm

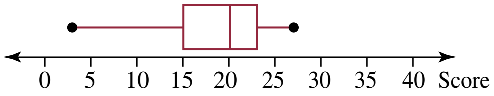
Leaf Year 8	*Stem*	*Leaf Year* 10
9 9	0	
9 8 6 5 4 2 0	1	7 9
9 8 8 6 4 2 1 0 0	2	2 3 6 8 9
9 7 7 6 4 1 0	3	0 2 4 5 5 7 8 8
8 6 5 2 0	4	1 2 5 8 8 9 9
	5	0 3 5 7 8
	6	0 0 3

a. Using a calculator or otherwise, construct a pair of parallel box-and-whisker plots to represent the two sets of data.

b. Find the mean, median, range, interquartile range and standard deviation of each set.

c. Compare the two distributions, using your answers to parts **a** and **b**.

5. Consider the box-and-whisker plot drawn below.



a. Find the median.

b. Find the range.

c. Find the interquartile range.

6. The following data give the amount of cut meat (in kg) obtained from each of 20 lambs.

4.5	6.2	5.8	4.7	4.0	3.9	6.2	6.8	5.5	6.1
5.9	5.8	5.0	4.3	4.0	4.6	4.8	5.3	4.2	4.8

a. Detail the data on a stem-and-leaf plot. (Use a class size of 0.5 kg.)

b. Prepare a five-point summary of the data.

c. Draw a box-and-whisker plot of the data.

7. Find the standard deviation of each of the following data sets correct to one decimal place.

a. 58, 12, 98, 45, 60, 34, 42, 71, 90, 66

b.

x	1	2	3	4	5
f	2	6	12	8	5

c. Key: 1|4 = 14

Stem	*Leaf*
0	1 3 4 4 5 7 8
1	0 0 0 1 2 2 4 5 7 8 9
2	0 2 2 3 5 7

8. **MC** The Millers obtained a number of quotes on the price of having their home painted. The quotes, to the nearest hundred dollars, were:

4200 5100 4700 4600
4800 5000 4700 4900

The standard deviation for this set of data, to the nearest whole dollar, is:

A. 260 **B.** 278 **C.** 324 **D.** 325

9. **MC** The number of Year 12 students who, during semester 2, spent all their spare periods studying in the resource centre is shown on the stem-and-leaf plot below.

Key: $2|5 = 25$ students

Stem	*Leaf*
0	8
1	
2	5 6 6 7
3	0 2 3 6 9
4	7 9
5	6
6	1

The standard deviation for this set of data, to the nearest whole number is:

A. 12 **B.** 14 **C.** 17 **D.** 35

10. Each week, varying amounts of a chemical are added to a filtering system. The amounts required (in mL) over the past 20 weeks are shown in the stem-and-leaf plot below.

Key: $3|8$ represents 0.38 mL

Stem	*Leaf*
2	1
2	2 2
2	4 4 4 5
2	6 6
2	8 8 9 9
3	0
3	2 2
3	4 5
3	6
3	8

Calculate to 2 decimal places the standard deviation of the amounts used.

Problem solving

11. A sample of 30 people was selected at random from those attending a local swimming pool. Their ages (in years) were recorded as follows:

19 7 58 41 17 23 62 55
40 37 32 29 21 18 16 10
40 36 33 59 65 68 15 9
20 29 38 24 10 30

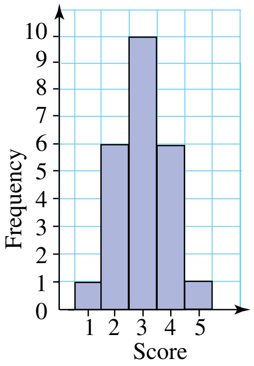
a. Find the mean and the median age of the people in this sample.
b. Group the data into class intervals of 10 (0–9 etc.) and complete the frequency distribution table.
c. Use the frequency distribution table to estimate the mean age.
d. Calculate the cumulative frequency and, hence, plot the ogive.

e. Estimate the median age from the ogive.

f. Compare the mean and median of the original data in part **a** with the estimates of the mean and the median obtained for the grouped data in parts **c** and **e**.

g. Were the estimates good enough? Explain your answer.

12. Consider the data set represented by the frequency histogram at right.



a. Are the data symmetrical?

b. Can the mean and median of the data be seen? If so, what are their values?

c. What is the mode of the data?

13. The table below shows the number of cars that are garaged at each house in a certain street each night.

Number of cars	Frequency
1	9
2	6
3	2
4	1
5	1

a. Show these data in a frequency histogram.

b. Are the data positively or negatively skewed? Justify your answer.

14. Find the mean, median and mode of this data set: 2, 5, 6, 2, 5, 7, 8. Comment on the shape of the distribution.

15. A data set has a mean of 75 and a standard deviation of 5. Another score of 50 is added to the data set. Which of the following will occur?

A. The mean will increase and the standard deviation will increase.

B. The mean will increase and the standard deviation will decrease.

C. The mean will decrease and the standard deviation will increase.

D. The mean will decrease and the standard deviation will decrease.

16. *Note:* There may be more than one correct answer.

A data set has a mean of 60 and a standard deviation of 10. A score of 100 is added to the data set. This score becomes the highest score in the data set. Which of the following will increase?

A. Mean **B.** Standard deviation **C.** Range **D.** Interquartile range

17. There are $3m$ values in a data set for which $\bar{x} = m$ and $\sigma = \frac{m}{2}$.

a. Comment on the changes to the mean and standard deviation if each value of the data set is multiplied by m.

b. An additional value is added to the original data set, giving a new mean of $m + 2$. Find the additional value.

18. The following data show the number of pets in each of the 12 houses in Coral Avenue, Rosebud.

2, 3, 3, 2, 2, 3, 2, 4, 3, 1, 1, 0

a. Calculate the mean and median number of pets.

b. The empty block of land at the end of the street was bought by a Cattery and now houses 20 cats. Recalculate the mean and median.

c. Explain why the answers are so different, and which measure of central tendency is best used for certain data.

19. The number of Year 10 students in all the 40 schools in the Northern District of the Education Department was recorded as follows:

56, 134, 93, 67, 123, 107, 167, 124, 108, 78, 89, 99, 103, 107, 110, 45, 112, 127, 106, 111, 127, 145, 87, 75, 90, 123, 100, 87, 116, 128, 131, 106, 123, 87, 105, 112, 145, 115, 126, 92

a. Using an interval of 10, produce a table showing the frequency for each interval.
b. Use the table to estimate the mean.
c. Calculate the mean of the ungrouped data.
d. Compare the results from parts **b** and **c** and explain any differences.

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- **Try out this interactivity:** Word search: Topic 12 (int-2859)
- **Try out this interactivity:** Crossword: Topic 12 (int-2860)
- **Try out this interactivity:** Sudoku: Topic 12 (int-3599)
- **Complete this digital doc:** Concept map: Topic 12 (doc-14598)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

box-and-whisker plot
cumulative frequency curve
data
data sets
dot plot
extreme values
five-number summary
frequency
frequency distribution
frequency distribution table
grouped data
histogram
interquartile range
interval
mean
measures of central tendency
median
modal class
mode
negatively skewed
ogive
outlier
percentile
positively skewed
quartile
range
score
skewed
skewness
spread
stem-and-leaf plot
symmetrical

Investigation | Rich task

Cricket scores

Data are used to predict, analyse, compare and measure many aspects of the game of cricket. Attendance is tallied at every match. Players' scores are analysed to see if they should be kept on the team. Comparisons of bowling and batting averages are used to select winners for awards. Runs made, wickets taken, no-balls bowled, the number of ducks scored in a game as well as the number of 4s and 6s are all counted and analysed after the game. Data of all sorts are gathered and recorded, and measures of central tendency and spread are then calculated and interpreted.

Sets of data have been made available for you to analyse, and decisions based on the resultant measures can be made.

Batting averages

The following table shows the runs scored by four cricketers who are vying for selection to the state team.

Player	Runs in the last 25 matches	Mean	Median	Range	IQR
Allan	13, 18, 23, 21, 9, 12, 31, 21, 20, 18, 14, 16, 28, 17, 10, 14, 9, 23, 12, 24, 0, 18, 14, 14, 20				
Shane	2, 0, 112, 11, 0, 0, 8, 0, 10, 0, 56, 4, 8, 164, 6, 12, 2, 0, 5, 0, 0, 0, 8, 18, 0				
Glenn	12, 0, 45, 23, 0, 8, 21, 32, 6, 0, 8, 14, 1, 27, 23, 43, 7, 45, 2, 32, 0, 6, 11, 21, 32				
Rod	2, 0, 3, 12, 0, 2, 5, 8, 42, 0, 12, 8, 9, 17, 31, 28, 21, 42, 31, 24, 30, 22, 18, 20, 31				

1. Find the mean, median, range and IQR scored for each cricketer.

2. You need to recommend the selection of two of the four cricketers. For each player, write two points as to why you would or would not select them. Use statistics in your comments.

a. Allan ____________________

____

b. Shane ____________________

____

c. Glenn ____________________

____

d. Rod ____________________

____

Bowling averages

The bowling average is the number of runs per wicket taken

$$\textbf{Bowling average} = \frac{\textbf{no. of runs scored}}{\textbf{no. of wickets taken}}$$

The smaller the average, the better the bowler has performed.
Brad and Dennis were competing for three bowling awards:

- **Best in semifinal**
- **Best in final**
- **Best overall**

The following table gives their scores.

	Semifinal		**Final**	
	Runs scored	Wickets taken	Runs scored	Wickets taken
Brad	12	5	28	6
Dennis	10	4	15	3

3. Calculate the bowling averages for the following and fill in the table below.
- Semifinal
- Final
- Overall

	Semifinal average	**Final average**	**Overall average**
Brad			
Dennis			

4. Explain how Dennis can have the better overall average when Brad has the better average in both the semifinal and final.

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Answers

Topic 12 Univariate data

Exercise 12.2 Measures of central tendency

1. a. i. 7 **ii.** 8 **iii.** 8

b. i. 6.875 **ii.** 7 **iii.** 4, 7

c. i. 39.125 **ii.** 44.5 **iii.** No mode

d. i. 4.857 **ii.** 4.8 **iii.** 4.8

e. i. 12 **ii.** 12.625 **iii.** 13.5

2. Science: mean = 57.6, median = 57, mode = 42, 51
Maths: mean = 69.12, median = 73, mode = 84

3. a. i. 5.83 **ii.** 6 **iii.** 6

b. i. 14.425 **ii.** 15 **iii.** 15

4. a. Mean = 2.5, median = 2.5 **b.** Mean = 4.09, median = 3 **c.** Median

5. a. $72\frac{2}{3}$ **b.** 73 **c.** 70–<80

6. 124.83

7. 65–<70

8. a. B **b.** B **c.** C **d.** D

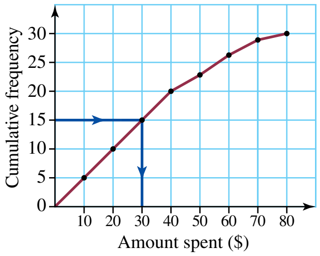
9. a. Mean = \$32.93, median = \$30

b.

Class interval	Frequency	Cumulative frequency
0–9	5	5
10–19	5	10
20–29	5	15
30–39	3	18
40–49	5	23
50–59	3	26
60–69	3	29
70–79	1	30
Total	30	

Mean = \$32.50, median = \$30

c.



d. The mean is slightly underestimated; the median is exact. The estimate is good enough as it provides a guide only to the amount that may be spent by future customers.

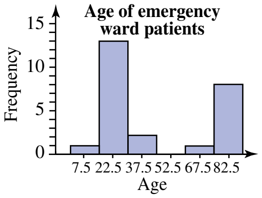
10. a. 3 **b.** 4, 5, 5, 5, 6 (one possible solution)

c. One possible solution is to exchange 15 with 20.

11. a. Frequency column: 16, 6, 4, 2, 1, 1 **b.** 6.8

c. 0–4 hours **d.** 0–4 hours

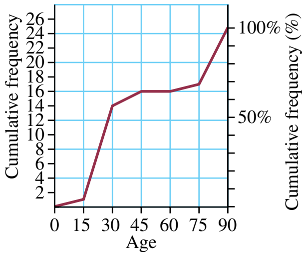
12. a. Frequency column: 1, 13, 2, 0, 1, 8 **b.**



c. Asymmetrical or bimodal (as if the data come from two separate graphs).

d. 44.1 **e.** 15–<30 **f.** 15–<30

g.



h. 28 **i.** No

j. Class discussion

13. a. Player A median = 34.33, Player B median = 41.83

b. Player B

c. Player A median = 32.5, Player B median = 0

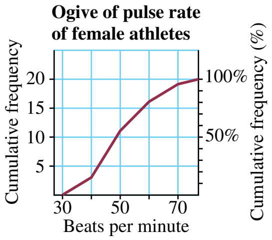
d. Player A

e. Player A is more consistent. One large score can distort the mean.

14. a. Frequency column: 3, 8, 5, 3, 1

b. 50.5 **c.** 40–<50 **d.** 40–<50

e.



f. Approximately 48 beats/min

15. A

16. Check with your teacher.

17. Answers will vary. Examples given.

a. 3, 4, 5, 5, 8 **b.** 4, 4, 5, 10, 16 **c.** 2, 3, 6, 6, 12

18. 12

19. $\frac{2a + b}{3}$

Challenge 12.1

13, 31, 31, 47, 53, 59

Exercise 12.3 Measures of spread

1. a. 15 **b.** 77.1 **c.** 9

2. a. 7 **b.** 7 **c.** 8.5 **d.** 39

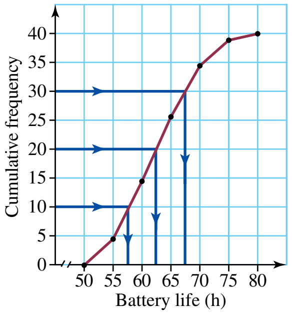
3. a. 3.3 kg **b.** 1.5 kg

4. 22 cm

5. 0.8

6. C

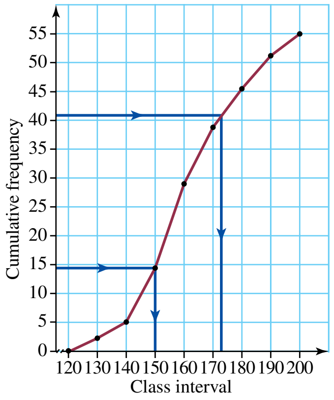
7. a.



b. **i.** 62.5 **ii.** $Q_1 = 58$, $Q_3 = 67$ **iii.** 9

iv. 14 **v.** 6

8.



IQR = 27

9. a. **i.** Range = 23 **ii.** IQR = 13.5

b. **i.** Range = 45 **ii.** IQR = 27.5

c. **i.** Range = 49 **ii.** IQR = 20

10. a. 25.5 **b.** 28 **c.** 39 **d.** 6

e. The three lower scores affect the mean but not the median or mode.

11. a. Men: mean = 32.3; median = 32.5; range = 38; IQR = 14

Women: mean = 29.13; median = 27.5; range = 36; IQR = 13

b. Typically, women marry younger than men, although the spread of ages is similar.

12. Mean = 25.036, median = 24.9, mode = 23.6

range = 8.5, IQR =3.4

13. $a = 22$, $b = 9$, $c = 9$ and $d = 8$

Exercise 12.4 Box-and-whisker plots

1. a. 5 **b.** 26

2. a. 6 **b.** 27

3. a. 5.8 **b.** 18.6

4. a. 140 **b.** 56 **c.** 90 **d.** 84 **e.** 26

5. a. 58 **b.** 31 **c.** 43 **d.** 27 **e.** 7

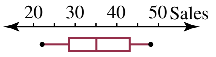
6. B

7. C

8. D

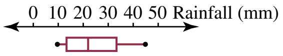
9. a. (22, 28, 35, 43, 48)

b.



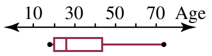
10. a. (10, 13.5, 22, 33.5, 45)

b.



11. a. (18, 20, 26, 43.5, 74)

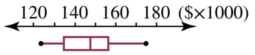
b.



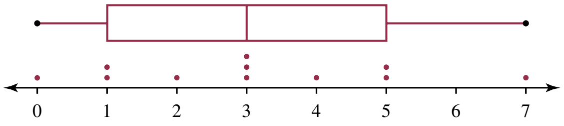
c. The distribution is positively skewed, with most of the offenders being young drivers.

12. a. (124 000, 135 000, 148 000, 157 000, 175 000)

b.

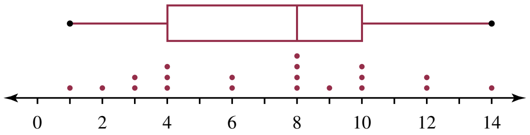


13. a.



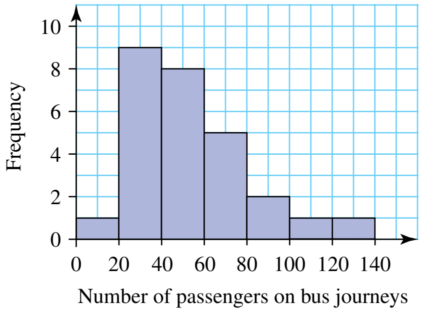
Both graphs indicate that the data is slightly positively skewed. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

b.



Both graphs indicate that the data is slightly negatively skewed. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

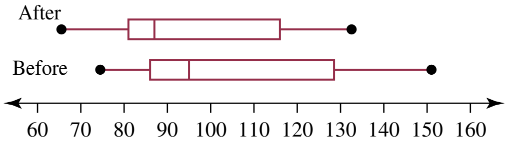
14.



15. a.

	X_{min}	Q_1	Median	Q_3	X_{max}
Before	75	86	95	128.5	152
After	66	81	87	116	134

b.

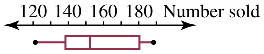


c. As a whole, the program was effective. The median weight dropped from 95 kg to 87 kg, a loss of 8 kg. A noticeable shift in the graph shows that after the program 50% of participants weighed between 66 and 87 kg, compared to 25% of participants weighing between 75 and 86 kg before they started. Before the program the range of weights was 77 kg (from 75 kg to 152 kg); after the program the range had decreased to 68 kg. The IQR also diminished from 42.5 kg to 35 kg.

16. a. Key: 12|1 = 121

Stem	*Leaf*
12	1 5 6 9
13	1 2 4
14	3 4 8 8
15	0 2 2 2 5 7
16	3 5
17	2 9
18	1 1 1 2 3 7 8

b.

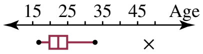


c. On most days the hamburger sales are less than 160. Over the weekend the sales figures spike beyond this.

17. a. Key: 1* | 7 = 17 years

Stem	*Leaf*
1*	7 7 8 8 8 9 9
2	0 0 0 1 2 2 2 2 3 3 3 3 4 4 4
2*	5 5 8 9
3	1 2 3
3*	
4	
4*	8

b.



c. The distribution is positively skewed, with first-time mothers being under the age of 30. There is one outlier (48) in this group.

18. C

19. a. HJ Looker: median = 5;
Hane and Roarne: median = 6

b. HJ Looker

c. HJ Looker

d. Hane and Roarne had a higher median and a lower spread and so they appear to have performed better.

20. a. \$50 **b.** \$135 **c.** \$100 **d.** \$45 **e.** 50%

Exercise 12.5 The standard deviation

1. a. 2.29 **b.** 2.19 **c.** 20.17 **d.** 3.07

2. a. 1.03 **b.** 1.33 **c.** 2.67 **d.** 2.22

3. 10.82

4. 0.45%

5. 0.06 m

6. 0.49 s

7. 15.10 calls

8. B

9. The mean of the first 25 cars is 89.24 km/h with a standard deviation of 5.60. The mean of the first 26 cars is 91.19 with a standard deviation of 11.20, indicating that the extreme speed of 140 km/h is an anomaly.

10. a. $\sigma \approx 3.58$

b. The mean is increased by 7 but the standard deviation remains at $\sigma \approx 3.58$.

c. The mean is tripled and the standard deviation is tripled to $\sigma \approx 10.74$.

11. The standard deviation will decrease because the average distance to the mean has decreased.

12. 57 is two standard deviations above the mean.

13. a. New mean is the old mean increased by 3 but no change to the standard deviation.

b. New mean is 3 times the old mean and new standard deviation is 3 times the old standard deviation.

14. a. 43 **b.** 12 **c.** 12.19

Exercise 12.6 Comparing data sets

1. a. The mean of the first set is 63. The mean of the second set is 63.

b. The standard deviation of the first set is 2.38. The standard deviation of the second set is 6.53.

c. For both sets of data the mean is the same, 63. However, the standard deviation for the second set (6.53) is much higher than the standard deviation of the first set (2.38), implying that the second set is more widely distributed than the first. This is confirmed by the range, which is 67 − 60 = 7 for the first set and 72 − 55 = 17 for the second.

2. a. Morning: median = 2.45; afternoon: median = 1.6

b. Morning: range = 3.8; afternoon: range = 5

c. The waiting time is generally shorter in the afternoon. One outlier in the afternoon data causes the range to be larger. Otherwise the afternoon data are far less spread out.

3. Key: 16 | 1 = 1.61 m

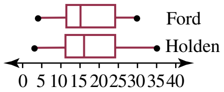
Leaf: Boys	*Stem*	*Leaf: Girls*
9 9 7	15	1 2 5 6 7 8 8
9 8 6 6 5 5 4 0	16	4 4 6 7 8 9 9
4 4 2 1	17	0

4. a. Ford: median = 15; Holden: median = 16

b. Ford: range = 26; Holden: range = 32

c. Ford: QR = 14; Holden: IQR = 13.5

d.

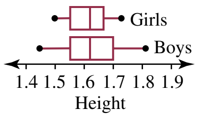


5. a. Brisbane Lions

b. Brisbane Lions: range = 63;
Sydney Swans: range = 55

c. Brisbane Lions: IQR = 40;
Sydney Swans: IQR = 35

6. a.



b. Boys: median = 1.62; girls: median = 1.62

c. Boys: range = 0.36; girls: range = 0.23

d. Boys: IQR = 0.14; girls: IQR = 0.11

e. Although boys and girls have the same median height, the spread of heights is greater among boys as shown by the greater range and interquartile range.

7. a. Summer: range = 23; winter: range = 31

b. Summer: IQR = 14; winter: IQR = 11

c. There are generally more cold drinks sold in summer as shown by the higher median. The spread of data is similar as shown by the IQR although the range in winter is greater.

8. A

9. A, B, C, D

10. a. Cory achieved a better average mark in Science (59.25) than he did in English (58.125).

b. Cory was more consistent in English ($\sigma = 4.9$) than he was in Science ($\sigma = 19.7$)

11. a. Back street: $\bar{x} = 61$, $\sigma = 4.3$;

main road: $\bar{x} = 58.8$, $\sigma = 12.1$

b. The drivers are generally driving faster on the back street.

c. The spread of speeds is greater on the main road as indicated by the higher standard deviation.

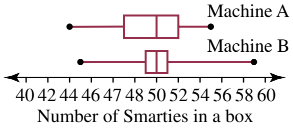
12. a. Nathan: mean = 15.1; Timana: mean = 12.3

b. Nathan: range = 36; Timana: range = 14

c. Nathan: IQR = 15; Timana: IQR = 4

d. Timana's lower range and IQR shows that he is the more consistent player.

13. a.



b. Machine A: mean = 49.88,
standard deviation = 2.87;
Machine B: mean = 50.12,
standard deviation = 2.44

c. Machine B is more reliable, as shown by the lower standard deviation and IQR. The range is greater on machine B only because of a single outlier.

14. a. Yes — Maths

b. Science: positively skewed

c. The Science test may have been more difficult.

d. Science: 61–70, Maths: 71–80

e. Maths has a greater standard deviation (12.6) compared to Science (11.9).

15. Answers will vary. Check with your teacher.

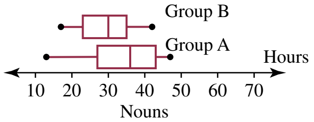
16. a. Key: 2 | 3 = 2.3 hours

Leaf: Group A	*Stem*	*Leaf: Group B*
8 7 3	1	7 8
9 5 1	2	0 1 2 4 5 5 8 8
8 7 5 4 2 2	3	2 2 2 4 5 5 5 6 8
7 5 4 2 2 2	4	2
	5	
2 2	6	

b. Five-point summary

Group A: 13 27 36 43 62

Group B: 17 23 30 35 42



c. Student comparison

Statistics	Group A	Group B
Five-point summary	13 27 36 43 62	17 23 30 35 42
x	35.85 hours	28.95 hours
Range	49 hours	25 hours
IQR	16 hours	12 hours
σ	13 hours	7 hours

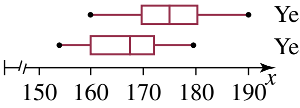
d. Student decision, justifying answer

e. Class discussion

17. a.

Leaf: Year 10	Stem	*Leaf: Year 11*
9 8 7 5 4	15	
9 8 7 4 3 3 0	16	0 3 6 7 8
7 5 3 2 2 0 0	17	2 2 3 4 5 6 7 7 9
0	18	0 1 3 5 8
	19	0

b.



c. On average, the Year 12 students are about 6–10 cm taller than the Year 10 students. The heights of the majority of Year 12 students are between 170 cm and 180 cm, whereas the majority of the Year 10 students are between 160 and 172 cm in height.

18. a. English: mean = 70.25; Maths: mean = 69

b. English: range = 53; Maths: range = 37

c. English: $\sigma = 16.1$; Maths: $\sigma = 13.4$

d. Kloe has performed more consistently in Maths as the range and standard deviation are both lower.

Challenge 12.2

32%

12.7 Review

1. a. Mean = 11.55; median = 10; mode = 8

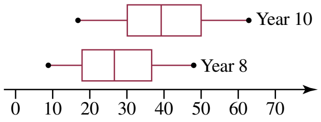
b. Mean = 36; median = 36; mode = 33, 41

c. Mean = 72.18; median = 72; mode = 72

2. a. 6 **b.** 6 **c.** 20

3. a. 4 **b.** 8.5

4. a.



b. Year 8: mean = 26.83, median = 27, range = 39, IQR = 19, sd = 11.45

Year 10: mean = 40.7, median = 39.5, range = 46, IQR = 20, sd = 12.98

c. The typing speed of Year 10 students is about 13 to 14 wpm faster than that of Year 8 students. The spread of data in Year 8 is slightly less than in Year 10.

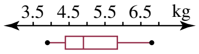
5. a. 20 **b.** 24 **c.** 8

6. a. Key: 3* | 9 = 3.9 kg

Stem	*Leaf*
3*	9
4	0 0 2 3
4*	5 6 7 8 8
5	0 3
5*	5 8 8 9
6	1 2 2
6*	8

b. (3.9, 4.4, 4.9, 5.85, 6.8)

c.



7. a. 24.4 **b.** 1.1 **c.** 7.3

8. A

9. B

10. 0.05 mL

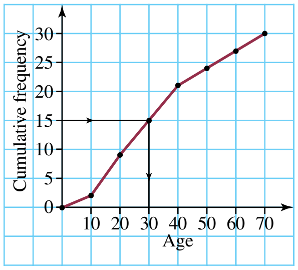
11. a. Mean = 32.03; median = 29.5

b.

Class interval	Frequency
0–9	2
10–19	7
20–29	6
30–39	6
40–49	3
50–59	3
60–69	3
Total	30

c. Mean = 31.83

d.



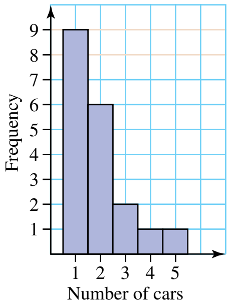
e. Median = 30

f. Estimates from parts **c** and **e** were fairly accurate.

g. Yes, they were fairly close to the mean and median of the raw data.

12. a. Yes **b.** Yes. Both are 3. **c.** 3

13. a.



b. Positively skewed — a greater number of scores is distributed at the lower end of the distribution.

14. Mean = 5, median = 5, mode = 2 and 5. The distribution is positively skewed and bimodal.

15. C

16. A, B and C

17. a. $\bar{x} = m^2 \quad \sigma = \dfrac{m^2}{2}$ **b.** $7m + 2$

18. a. Mean = 2.17, median = 2

b. Mean = 3.54, median = 2

c. The median relies on the middle value of the data and won't change much if an extra value is added. The mean however has increased because this large value will change the average of the numbers. The mean is used as a measure of central tendency if there are no outliers or if the data are symmetrical. The median is used as a measure of central tendency if there are outliers or the data are skewed.

19. a.

Interval	Frequency (f)	Midpoint × (f)
40–49	1	$44.5 \times 1 = 44.5$
50–59	1	$54.5 \times 1 = 54.5$
60–69	1	$64.5 \times 1 = 64.5$
70–79	2	$74.5 \times 2 = 149$
80–89	4	$84.5 \times 4 = 338$
90–99	4	$94.5 \times 4 = 378$
100–109	8	$104.5 \times 8 = 836$
110–119	6	$114.5 \times 6 = 687$
120–129	8	$124.5 \times 8 = 996$
130–139	2	$134.5 \times 2 = 269$
140–149	2	$144.5 \times 2 = 289$
150–159	0	$154.5 \times 0 = 0$
160–169	1	$164.5 \times 1 = 164.5$
Total	40	4270

b. 106.75

c. 107.15

d. The differences in this case were minimal; however, the grouped data mean is not based on the actual data but on the frequency in each interval and the interval midpoint. It is unlikely to yield an identical value to the actual mean. The spread of the scores within the class interval has a great effect on the grouped data mean.

Investigation — Rich task

1.

Player	Runs in the last 25 matches	Mean	Median	Range	IQR
Allan	13, 18, 23, 21, 9, 12, 31, 21, 20, 18, 14, 16, 28, 17, 10, 14, 9, 23, 12, 24, 0, 18, 14, 14, 20	16.76	17	31	8.5
Shane	2, 0, 112, 11, 0, 0, 8, 0, 10, 0, 56, 4, 8, 164, 6, 12, 2, 0, 5, 0, 0, 0, 8, 18, 0	17.04	4	164	10.5
Glenn	12, 0, 45, 23, 0, 8, 21, 32, 6, 0, 8, 14, 1, 27, 23, 43, 7, 45, 2, 32, 0, 6, 11, 21, 32	16.76	12	45	25.5
Rod	2, 0, 3, 12, 0, 2, 5, 8, 42, 0, 12, 8, 9, 17, 31, 28, 21, 42, 31, 24, 30, 22, 18, 20, 31	16.72	17	42	25

2. a. Allan: has a similar mean and median, which shows he was fairly consistent. The range and IQR values are low indicating that his scores remain at the lower end with not much deviation for the middle 50%.

b. Shane: has the best average but a very low median indicating his scores are not consistent. The range is extremely high and the IQR very low in comparison showing he can score very well at times but is not a consistent scorer.

c. Glenn: has a similar mean to Allan and Rod but a lower median, indicating his scores are sometimes high but generally are lower than the average. The range and IQR show a consistent batting average and spread with only a few higher scores and some lower ones.

d. Rod: has a similar mean and median which shows he was a consistent player. The range and IQR show a consistent batting average and spread.

Players to be selected:

Would recommend **Allan** if the team needs someone with very consistent batting scores every game but no outstanding runs.

Would recommend **Shane** if the team needs someone who might score very high occasionally but in general fails to score many runs.

Would recommend **Glenn** if the team needs someone who is fairly consistent but can score quite well at times and the rest of the time does OK.

Would recommend **Rod** if the team needs someone who is fairly consistent but can score quite well at times and the rest of the time has a better median than Glenn.

3.

	Semifinal average	**Final average**	**Overall average**
Brad	2.4	4.67	3.64
Dennis	2.5	5	3.57

a. Brad was best in the semifinal.

b. Brad was best in the final.

c. Dennis was best overall.

4. In the final, wickets were more costly than in the semifinal. Brad therefore conceded many runs in getting his six wickets.

This affected the overall mean. In reality Brad was the most valuable player overall, but this method of combining the data of the two matches led to this unexpected result.

TOPIC 13
Bivariate data

13.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

13.1.1 Why learn this?

Observations of two or more variables are often recorded, for example, the heights and weights of individuals. Studying the data allows us to investigate whether there is any relationship between the variables, how strong the relationship is, and whether one variable can be effectively predicted from information about another variable. Statistics can be applied to medical research, sport, agriculture, sustainability, weather forecasting and fashion trends, to name but a few fields. The capacity to analyse data and draw conclusions is an essential skill in a world where information is readily available and often manipulated.

13.1.2 What do you know?

assesson

1. **THINK** List what you know about interpreting bivariate data. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of interpreting bivariate data.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Florence Nightingale (eles-1853)

13.2 Bivariate data

13.2.1 Bivariate data

- **Bivariate data** are data with two variables.
- The list of bivariate data can be considered as numerical pairs of the type:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

- Bivariate data are usually represented graphically on scatterplots.

13.2.2 Scatterplots

- A **scatterplot** is a graph that shows whether there is a relationship between two variables.
- Each data value on a scatterplot is shown by a point on a Cartesian plane.

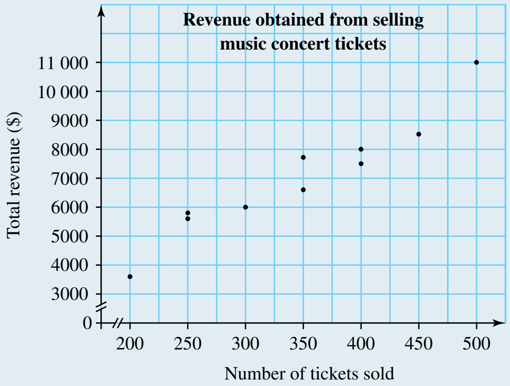
13.2.3 Dependent and independent variables

- One variable is generally the **dependent variable**, and the other the **independent variable**.
- The dependent variable, as the name suggests, is the one whose value depends on the other variable. The independent variable takes on values that do *not* depend on the value of the other variable.
- When data are expressed in the form of a table, generally the independent variable is written in the first row or the first column.
- The independent variable is placed on the x-axis and the dependent variable on the y-axis.

WORKED EXAMPLE 1

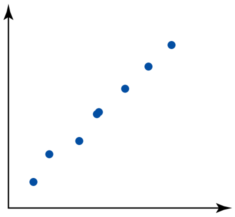
The table shows the total revenue from selling tickets for a number of different chamber music concerts. Represent these data on a scatterplot.

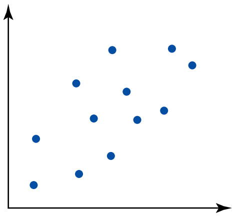
Number of tickets sold	**400**	**200**	**450**	**350**	**250**	**300**	**500**	**400**	**350**	**250**
Total revenue ($)	**8000**	**3600**	**8500**	**7700**	**5800**	**6000**	**11 000**	**7500**	**6600**	**5600**

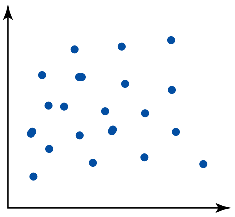
THINK	WRITE/DRAW
1 Determine which is the dependent variable and which is the independent variable.	The total revenue depends on the number of tickets being sold, so the number of tickets is the independent variable and the total revenue is the dependent variable.
2 Rule up a set of axes. Label the title of the graph. Label the horizontal axis 'Number of tickets sold' and the vertical axis 'Total revenue ($)'. **3** Use an appropriate scale on the horizontal and vertical axes. **4** Plot the points on the scatterplot.	

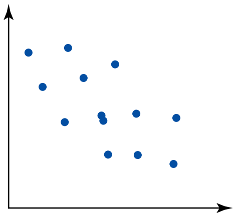
13.2.4 Correlation

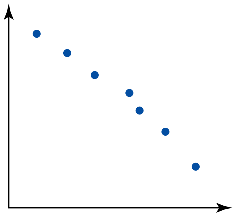
- **Correlation** is a descriptive measure of the relationship between two variables.
- Correlation describes the **strength**, the **direction** and the **form** of the relationship between the two variables.
- The closer the points are to a straight line form, the stronger the correlation between the two variables. The strength is described as **weak**, **moderate** or **strong**.
- If the points on a scatterplot have a generally positive slope, the relationship has a positive direction. If the slope is negative, the direction is negative.


Strong positive linear correlation


Weak positive linear correlation

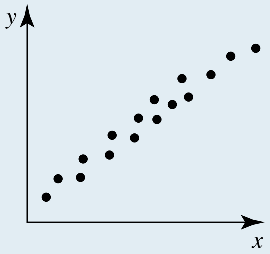

No correlation


Weak negative linear correlation


Strong negative linear correlation

WORKED EXAMPLE 2

State the type of correlation between the variables *x* and *y*, shown on the scatterplot.



THINK	WRITE
Carefully analyse the scatterplot and comment on its form, direction and strength.	The points on the scatterplot are close together and constantly increasing therefore the relationship is linear. The path is directed from the bottom left corner to the top right corner and the value of y increases as x increases. Therefore the correlation is positive. The points are close together so the correlation can be classified as strong. There is a strong, positive, linear relationship between x and y.

13.2.5 Correlation and causation

- Even a strong correlation does not necessarily mean that the increase or decrease in the level of one variable *causes* an increase or decrease in the level of the other. It is best to avoid statements such as: ‘An increase in rainfall *causes* an increase in the wheat growth.’
- The following guidelines should be closely followed in order to draw a conclusion about the relationship between the two variables based on the scatterplot.
  - If the correlation between x and y is *weak*, we can conclude that *there is little evidence to show* that the larger x is, the larger (positive correlation) or smaller (negative correlation) y is.
  - If the correlation between x and y is *moderate*, we can conclude that *there is evidence to show* that the larger x is, the larger (positive correlation) or smaller (negative correlation) y is.
  - If the correlation between x and y is *strong*, we can conclude that the larger x is, the larger (positive correlation) or smaller (negative correlation) y is.

WORKED EXAMPLE 3

TI | CASIO

Mary sells business shirts in a department store. She always records the number of different styles of shirt sold during the day. The table below shows her sales over one week.

Price ($)	**14**	**18**	**20**	**21**	**24**	**25**	**28**	**30**	**32**	**35**
Number of shirts sold	**21**	**22**	**18**	**19**	**17**	**17**	**15**	**16**	**14**	**11**

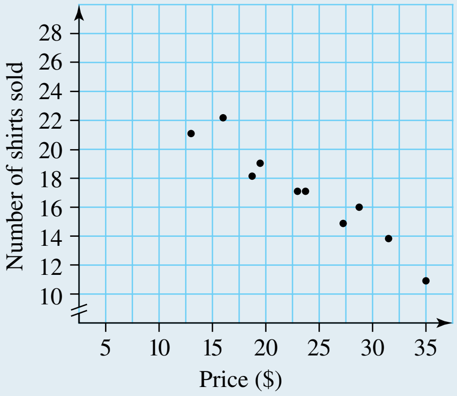
a Construct a scatterplot of the data.
b State the type of correlation between the two variables and, hence, draw a corresponding conclusion.

THINK

a Draw the scatterplot showing 'Price ($)' (independent variable) on the horizontal axis and 'Number of shirts sold' (dependent variable) on the vertical axis.

WRITE/DRAW

a



THINK

b 1 Carefully analyse the scatterplot and comment on its form, direction and strength.

2 Draw a conclusion corresponding to the analysis of the scatterplot.

WRITE/DRAW

b The points on the plot form a path that resembles a straight, narrow band, directed from the top left corner to the bottom right corner. The points are close to forming a straight line. There is a strong, negative, linear correlation between the two variables.

The price of the shirt appears to affect the number sold; that is, the more expensive the shirt the fewer sold.

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Complete this digital doc: SkillSHEET: Solving linear equations that arise when finding *x*- and *y*-intercepts (doc-5406)

Complete this digital doc: SkillSHEET: Transposing linear equations to standard form (doc-5407)

Complete this digital doc: SkillSHEET: Measuring the rise and the run (doc-5408)

Complete this digital doc: SkillSHEET: Finding the gradient given two points (doc-5409)

Complete this digital doc: SkillSHEET: Graphing linear equations using the *x*- and *y*-intercept method (doc-5410)

Complete this digital doc: SkillSHEET: Determining independent and dependent variables (doc-5411)

Complete this digital doc: SkillSHEET: Determining the type of correlation (doc-5413)

Exercise 13.2 Bivariate data

assess**on**

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2–6, 8, 12	Questions: 1d–g, 2–5, 7, 9, 12	Questions: 1e–h, 2–4, 7–13

🟩🟦🟪 **Individual pathway interactivity:** int-4626

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

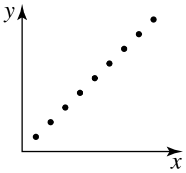
1. For each of the following pairs, decide which of the variables is independent and which is dependent.
   a. Number of hours spent studying for a Mathematics test and the score on that test
   b. Daily amount of rainfall (in mm) and daily attendance at the Botanical Gardens
   c. Number of hours per week spent in a gym and the annual number of visits to the doctor
   d. The amount of computer memory taken by an essay and the length of the essay (in words)
   e. The cost of care in a childcare centre and attendance at the childcare centre
   f. The cost of the property (real estate) and the age of the property
   g. The entry requirements for a certain tertiary course and the number of applications for that course
   h. The heart rate of a runner and the running speed

2. **WE1** The following table shows the cost of a wedding reception at 10 different venues. Represent the data on a scatterplot.

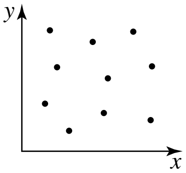
No of guests	30	40	50	60	70	80	90	100	110	120
Total cost (× $1000)	1.5	1.8	2.4	2.3	2.9	4	4.3	4.5	4.6	4.6

3. **WE2** State the type of relationship between x and y for each of the following scatterplots.

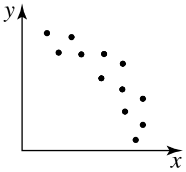
a.



b.



c.



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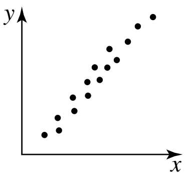
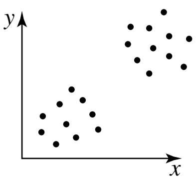


e.


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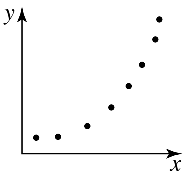

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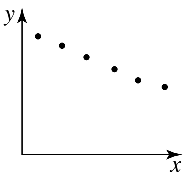

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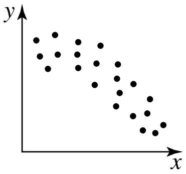
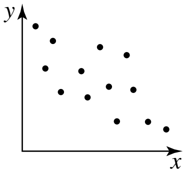


i.


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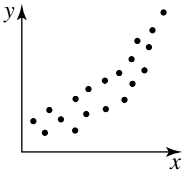

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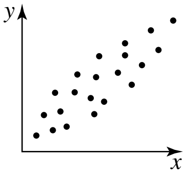

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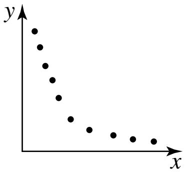

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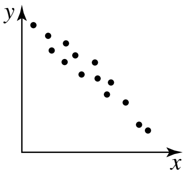
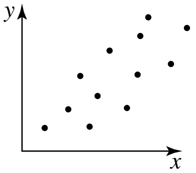
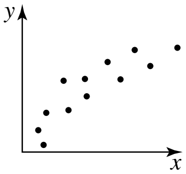


n.


o.


4. **WE3** Eugene is selling leather bags at the local market. During the day he keeps records of his sales. The table below shows the number of bags sold over one weekend and their corresponding prices (to the nearest dollar).

Price ($) of a bag	30	35	40	45	50	55	60	65	70	75	80
Number of bags sold	10	12	8	6	4	3	4	2	2	1	1

a. Construct a scatterplot of the data.
b. State the type of correlation between the two variables and, hence, draw a corresponding conclusion.

Understanding

5. The table below shows the number of bedrooms and the price of each of 30 houses.

Number of bedrooms	Price (× $1000)	Number of bedrooms	Price (× $1000)	Number of bedrooms	Price (× $1000)
2	180	3	279	3	243
2	160	2	195	3	198
3	240	6	408	3	237
2	200	4	362	2	226
2	155	2	205	4	359

Number of bedrooms	Price (× $1000)	Number of bedrooms	Price (× $1000)	Number of bedrooms	Price (× $1000)
4	306	7	420	4	316
3	297	5	369	2	200
5	383	1	195	2	158
2	212	3	265	1	149
4	349	2	174	3	286

a. Construct a scatterplot of the data.

b. State the type of correlation between the number of bedrooms and the price of the house and, hence, draw a corresponding conclusion.

c. Suggest other factors that could contribute to the price of the house.

6. The table below shows the number of questions solved by each student on a test, and the corresponding total score on that test.

Number of questions	2	4	7	10	5	2	6	3	9	4	8	3	6
Total score (%)	22	39	69	100	56	18	60	36	87	45	84	32	63

a. Construct a scatterplot of the data.

b. What type of correlation does the scatterplot suggest?

c. Give a possible explanation as to why the scatterplot is not perfectly linear.

7. A sample of 25 drivers who had obtained a full licence within the last month was asked to recall the approximate number of driving lessons they had taken (to the nearest 5), and the number of accidents they had while being on P plates. The results are summarised in the table that follows.

Number of lessons	Number of accidents	Number of lessons	Number of accidents
5	6	5	5
20	2	20	3
15	3	40	0
25	3	25	4
10	4	30	1
35	0	15	4
5	5	35	1
15	1	5	4
10	3	30	0
20	1	15	2
40	2	20	3
25	2	10	4
10	5		

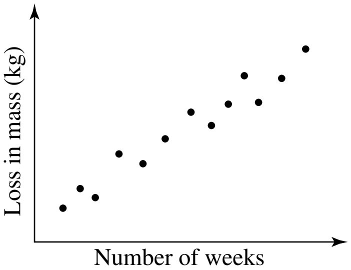
a. Represent these data on a scatterplot.

b. Specify the relationship suggested by the scatterplot.

c. Suggest some reasons why this scatterplot is not perfectly linear.

Reasoning

8. Each point on the scatterplot below shows the time (in weeks) spent by a person on a healthy diet and the corresponding mass lost (in kg).

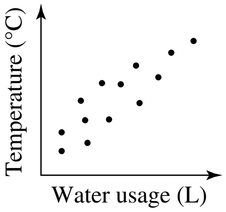


Study the scatterplot and state whether each of the following statements is true or false.

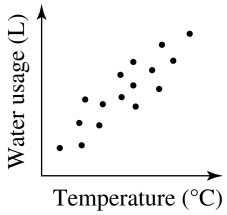
a. The number of weeks that the person stays on a diet is the independent variable.
b. The y-coordinates of the points represent the time spent by a person on a diet.
c. There is evidence to suggest that the longer the person stays on a diet, the greater the loss in mass.
d. The time spent on a diet is the only factor that contributes to the loss in mass.
e. The correlation between the number of weeks on a diet and the number of kilograms lost is positive.

9. **MC** The scatterplot that best represents the relationship between the amount of water consumed daily by a certain household for a number of days in summer and the daily temperature is:

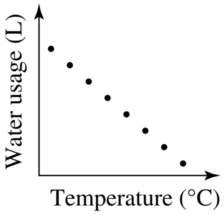
A.



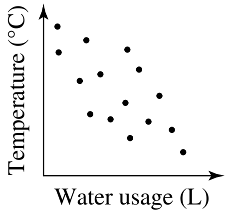
B.



C.



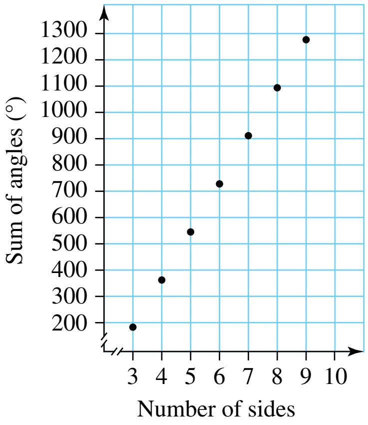
D.



10. **MC** The scatterplot at right shows the number of sides and the sum of interior angles for a number of polygons.

Which of the following statements is not true?

A. The correlation between the number of sides and the angle sum of the polygon is perfectly linear.
B. The increase in the number of sides causes the increase in the size of the angle sum.
C. The number of sides depends on the sum of the angles.
D. The correlation between the two variables is positive.



11. **MC** After studying a scatterplot, it was concluded that there was evidence that the greater the level of one variable, the smaller the level of the other variable. The scatterplot must have shown a:

A. strong, positive correlation
B. strong, negative correlation
C. moderate, positive correlation
D. moderate, negative correlation

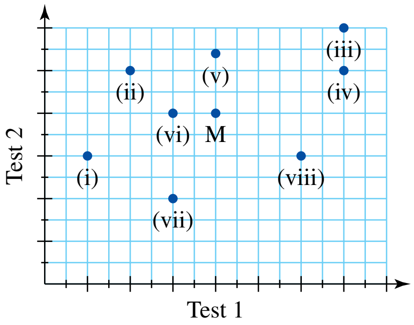
Problem solving

12. The table below gives the number of kicks and handballs obtained by the top 8 players in an AFL game.

Player	A	B	C	D	E	F	G	H
Number of kicks	20	27	21	19	17	18	21	22
Number of handballs	11	3	11	6	5	1	9	7

a. Represent this information on a scatterplot by using the x-axis as the number of kicks and the y-axis as the number of handballs.
b. Does the scatterplot support the claim that the more kicks a player obtains, the more handballs he gives?

13. The scatterplot shown gives the marks obtained by students in two mathematics tests. Mardi's score in the tests is represented by M. Which point represents each of the following students?



a. Mandy, who got the highest mark in both tests
b. William, who got the top mark in test 1 but not in test 2
c. Charlotte, who did better on test 1 than Mardi but not as well in test 2
d. Dario, who did not do as well as Charlotte in both tests
e. Edward, who got the same mark as Mardi in test 2 but did not do so well in test 1
f. Cindy, who got the same mark as Mardi for test 1 but did better than her for test 2
g. Georgina, who was the lowest in test 1
h. Harrison, who had the greatest discrepancy between his two marks

Reflection

How could you determine whether the change in one variable *causes* the change in another variable?

13.3 Lines of best fit

13.3.1 Lines of best fit

- If the points on a scatterplot appear to lie fairly closely distributed in a linear pattern, a straight line can be drawn through the data. The line can then be used to make predictions about the data.
- A **line of best fit** is a line on a scatterplot that is positioned so that it is as *close as possible to all the data points*.
- A line of best fit is used to generalise the relationship between two variables.

13.3.2 Lines of best fit by eye

- A line of best fit can be drawn on a scatterplot **by eye**. This means that a line is positioned so that there is an equal number of points above and below the line.
- Once a line of best fit has been placed on the scatterplot, an equation for this line can be established, using the coordinates of any two points on the line. These two points do not necessarily have to be actual data points, but if any data points do lie on the line, these are chosen as their values are known accurately.
- The equation for the line passing through the two selected points can then be calculated.

  The equation through the two points (x_1, y_1) and (x_2, y_2) is given by:

$$y = mx + c \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}.$$

WORKED EXAMPLE 4

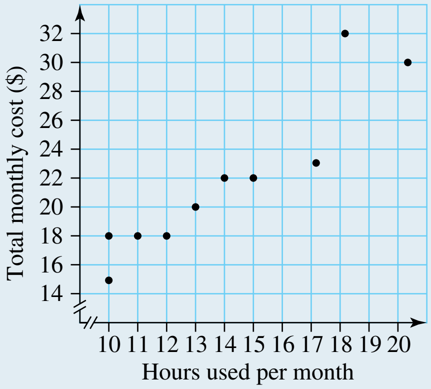
The data in the table show the cost of using the internet at a number of different internet cafes based on hours used per month.

Hours used per month	**10**	**12**	**20**	**18**	**10**	**13**	**15**	**17**	**14**	**11**
Total monthly cost ($)	**15**	**18**	**30**	**32**	**18**	**20**	**22**	**23**	**22**	**18**

a Construct a scatterplot of the data.

b Draw in the line of best fit by eye.

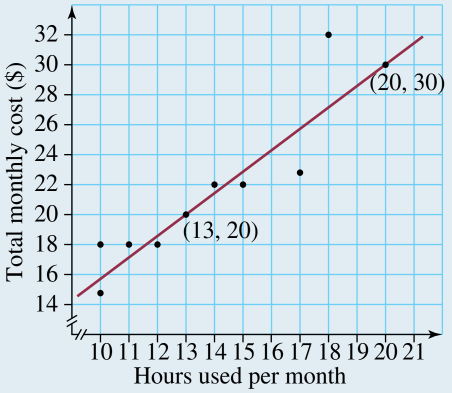
c Find the equation of the line of best fit in terms of the variables *n* (number of hours) and *C* (monthly cost).

THINK	**WRITE/DRAW**
a Draw the scatterplot placing the independent variable (hours used per month) on the horizontal axis and the dependent variable (total monthly cost) on the vertical axis. Label the axes.	**a** 

b **1** Carefully analyse the scatterplot.

2 Position the line of best fit so there is approximately an equal number of data points on either side of the line and so that all points are close to the line.
Note: With the line of best fit, there is no single definite solution.

b



c **1** Select two points on the line that are not too close to each other.

c Let $(x_1, y_1) = (13, 20)$ and $(x_2, y_2) = (20, 30)$.

2 Calculate the gradient of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{30 - 20}{20 - 13}$$

$$= \frac{10}{7}$$

3 Write the rule for the equation of a straight line.

$$y = mx + c$$

4 Substitute the known values into the equation.

$$y = \frac{10}{7}x + c$$

5 Substitute one pair of coordinates (say, 13, 20) into the equation to calculate *c*.

$$20 = \frac{10}{7}(13) + c$$

$$c = 20 - \frac{130}{7}$$

$$= \frac{140 - 130}{7}$$

$$= \frac{10}{7}$$

6 Write the equation.

$$y = \frac{10}{7}x + \frac{10}{7}$$

7 Replace *x* with *n* (number of hours used) and *y* with *C* (the total monthly cost) as required.

$$C = \frac{10}{7}n + \frac{10}{7}$$

13.3.3 Making predictions

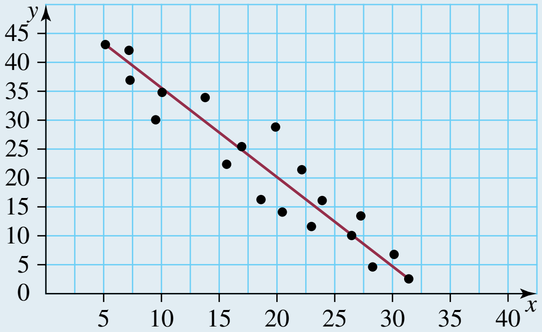
- The line of best fit can be used to predict the value of one variable from that of another. Because of the subjective nature of the line, it should be noted that predictions are not accurate values, but rough estimates. Although this is the case, predictions using this method are considered valuable when no other methods are available.
- If the equation of the line of best fit is known, or can be derived, predictions can be made by substituting known values into the equation of the line of best fit.

WORKED EXAMPLE 5

Use the given scatterplot and line of best fit to predict:

a the value of y when $x = 10$

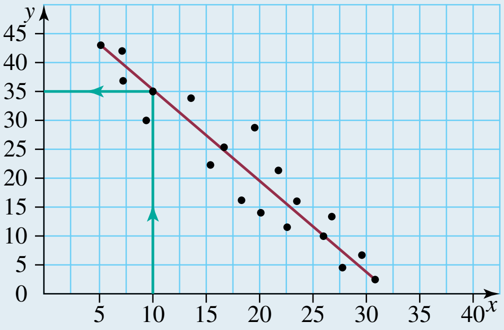
b the value of x when $y = 10$.



THINK / **WRITE/DRAW**

a 1 Locate 10 on the x-axis and draw a vertical line until it meets with the line of best fit. From that point, draw a horizontal line to the y-axis. Read the value of y indicated by the horizontal line.

a

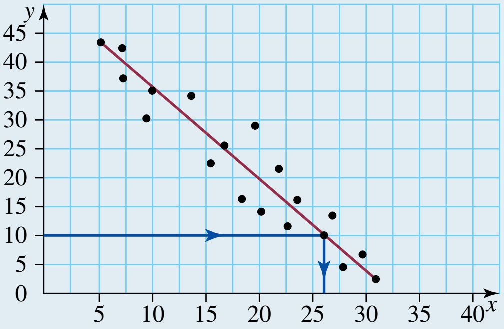


2 Write your answer.

When $x = 10$, y is predicted to be 35.

b 1 Locate 10 on the y-axis and draw a horizontal line until it meets with the line of best fit. From that point draw a vertical line to the x-axis. Read the value of x indicated by the vertical line.

b



2 Write your answer.

When $y = 10$, x is predicted to be 27.

WORKED EXAMPLE 6

The table below shows the number of boxes of tissues purchased by hayfever sufferers and the number of days affected by hay fever during the blooming season in spring.

Number of days affected by hay fever (d)	**3**	**12**	**14**	**7**	**9**	**5**	**6**	**4**	**10**	**8**
Total number of boxes of tissues purchased (T)	**1**	**4**	**5**	**2**	**3**	**2**	**2**	**2**	**3**	**3**

a Construct a scatterplot of the data and draw a line of best fit.

b Determine the equation of the line of best fit.

c Interpret the meaning of the gradient.

d **Use the equation of the line of best fit to predict the number of boxes of tissues purchased by people suffering from hay fever over a period of:**

i **11 days**

ii **15 days.**

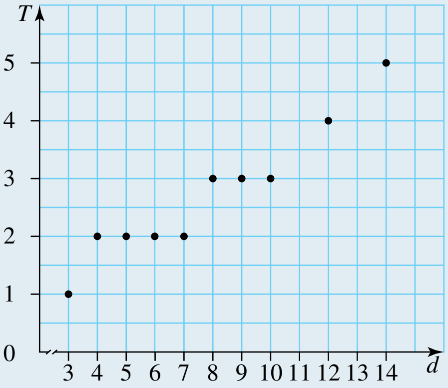
THINK

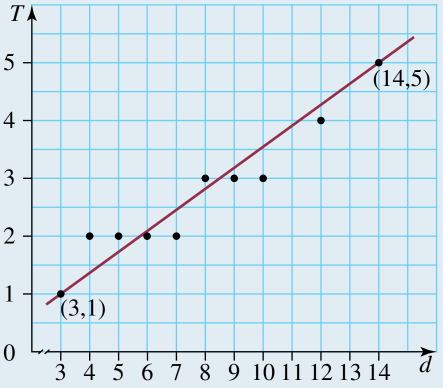
a **1** Draw the scatterplot showing the independent variable (number of days affected by hay fever) on the horizontal axis and the dependent variable (total number of boxes of tissues purchased) on the vertical axis.

2 Position the line of best fit on the scatterplot so there is approximately an equal number of data points on either side of the line.

WRITE/DRAW

a





THINK

b **1** Select two points on the line that are not too close to each other.

2 Calculate the gradient of the line.

3 Write the rule for the equation of a straight line.

4 Substitute the known values into the equation, say, 3, 1, into the equation to calculate *C*.

5 Replace *x* with *d* (number of days with hay fever) and *y* with *T* (total number of boxes of tissues used) as required.

WRITE/DRAW

b Let $(x_1, y_1) = (3, 1)$ and $(x_2, y_2) = (14, 5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{5 - 1}{14 - 3} = \frac{4}{11}$$

$$y = mx + c$$

$$y = \tfrac{4}{11}x + c$$

$$1 = \tfrac{4}{11}(3) + c$$

$$c = 1 - \tfrac{12}{11}$$

$$= \tfrac{-1}{11}$$

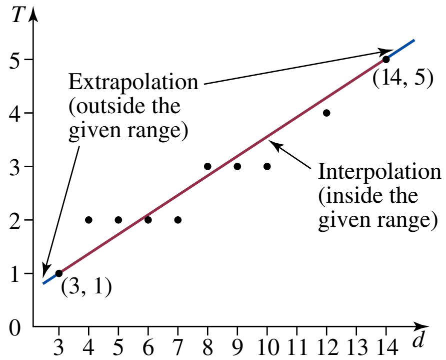
$$y = \tfrac{4}{11}x - \tfrac{1}{11}$$

$$T = \tfrac{4}{11}d - \tfrac{1}{11}$$

c Interpret the meaning of the gradient of the line of best fit.	**c** The gradient indicates an increase in sales of tissues as the number of days affected by hay fever increases. A hay fever sufferer is using on average $\frac{4}{11}$ (or about 0.36) of a box of tissues per day.
d **i** **1** Substitute the value $d = 11$ into the equation and evaluate.	**d** **i** When $d = 11$, $T = \frac{4}{11} \times 11 - \frac{1}{11}$ $= 4 - \frac{1}{11}$ $= 3\frac{10}{11}$
2 Interpret and write your answer.	In 11 days the hay fever sufferer will need about 4 boxes of tissues.
ii **1** Substitute the value $d = 15$ into the equation of the line of best fit and evaluate.	**ii** When $d = 15$, $T = \frac{4}{11} \times 15 - \frac{1}{11}$ $= \frac{60}{11} - \frac{1}{11}$ $= 5\frac{4}{11}$
2 Interpret and write your answer.	In 15 days the hay fever sufferer will need 6 boxes of tissues.

13.3.4 Interpolation and extrapolation

- **Interpolation** is the term used for predicting a value of a variable from within the range of the given data.
- **Extrapolation** occurs when the value of the variable being predicted is outside the range of the given data.
- In Worked example 6, the number of days ranged from 3 days to 14 days. Making a prediction for 11 days is an example of interpolation, whereas making a prediction for 15 days is an example of extrapolation.
- Predictions involving interpolation are considered to be quite reliable. Those involving extrapolation should be treated with caution, as they rely on the **trend** of the line remaining unchanged beyond the range of the data.

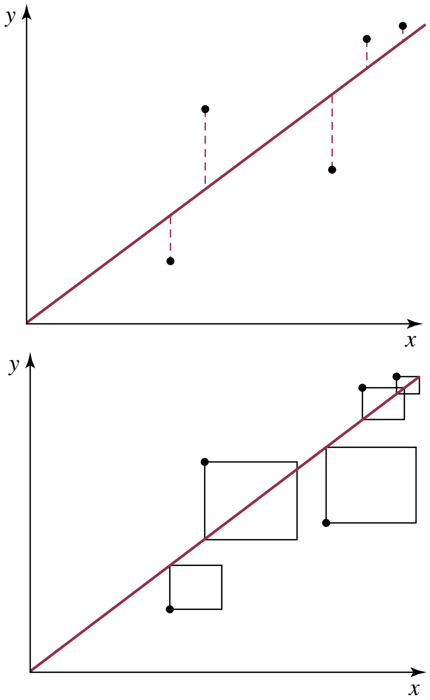


13.3.5 Reliability of predictions

- When predictions of any type are made, it is useful to know whether they are reliable or not.
- If the line of best fit is used to make predictions, they can be considered to be reliable if each of the following is observed.
  - The number of data values is large.
  - The scatterplot indicates reasonably strong correlation between the variables.
  - The predictions are made using interpolation.

13.3.6 Least squares regression

- Least squares regression involves an exact mathematical approach to fitting a line of best fit to bivariate data that show a strong linear correlation.
- Consider the regression line shown at right. The vertical lines give an indication of how well the line best ‘fits’ the data. The line of best fit is placed so that these ‘error’ lines are minimised, by balancing the errors above and below the line.
- Least-squares regression takes these error lines, forms squares, and minimises the sum of the areas of the squares.
- The actual calculation of the equation of a least squares regression line is complicated; however, a calculator can generate the equation with ease.



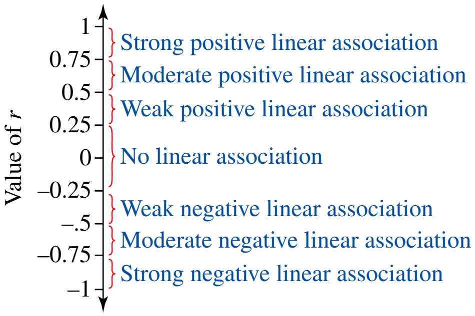
13.3.7 Correlation coefficient

- Once a relationship between two variables has been established, it is helpful to develop a quantitative value to measure the strength of the relationship. One method is to calculate a **correlation coefficient (*r*)**. This is easily done using a calculator, but a manual method is shown below.
- The formula for the correlation coefficient r is:

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}$$

where x and y are the two sets of scores
$\bar{x}$ and $\bar{y}$ are the means of those scores
the symbol $\sum$ representing the sum of the expressions indicated.

- The correlation coefficient is a value in the range -1 to $+1$. The value of -1 indicates a perfect negative relationship between the two variables, while the value of $+1$ indicates a perfect positive relationship. For values within this range, a variety of descriptors are used.



WORKED EXAMPLE 7

TI | CASIO

The percentages from two tests (English and Maths) for a group of 5 students are as shown.

a Calculate the correlation coefficient between the two sets of results.

b Based on this value, describe the relationship between the English and Maths results for this group of students.

Student	English (%)	Maths (%)
1	95	85
2	85	95
3	80	70
4	70	65
5	60	70

THINK

a **1** Draw up a table to calculate all the necessary data: $\bar{x}, \bar{y}, (x-\bar{x}), (y-\bar{y}), (x-\bar{x})^2, (y-\bar{y})^2$

WRITE

a

	x	y	$(x-\bar{x})$	$(x-\bar{x})^2$	$(y-\bar{y})$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
	95	85	17	289	8	64	136
	85	95	7	49	18	324	126
	80	70	2	4	−7	49	−14
	70	65	−8	64	−12	144	96
	60	70	−18	324	−7	49	126
Σ	390	385		730		630	470
Mean	78	77					

2 Substitute into the formula to calculate the correlation coefficient r.

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}}$$

$$= \frac{470}{\sqrt{730 \times 630}}$$

$$= 0.69$$

b Describe the relationship.

b A correlation coefficient of 0.69 indicates that the relationship between English and Maths marks for this group of students is only moderate. This seems to indicate that students who are good at English are not necessarily good at Maths, and vice versa.

learnON RESOURCES — ONLINE ONLY

Try out this interactivity: Applying lines of best fit (int-2798)

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Complete this digital doc: WorkSHEET: Lines of best fit (doc-14599)

Exercise 13.3 Lines of best fit

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 11	Questions: 1–8, 10, 11	Questions: 1–12

Individual pathway interactivity: int-4627

learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE4** The data in the table below show the distances travelled by 10 cars and the amount of petrol used for their journeys (to the nearest litre).
   a. Construct a scatterplot of the data.
   b. Draw in the line of best fit.
   c. Determine the equation of the line of best fit in terms of the variables *d* (distance travelled) and *P* (petrol used).

Distance travelled (km), *d*	52	36	83	12	44	67	74	23	56	95
Petrol used (L), *P*	7	5	9	2	7	9	12	3	8	14

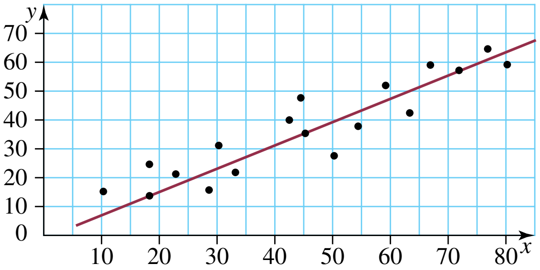
2. A random sample of ten Year 10 students who have part-time jobs was selected. Each student was asked to state the average number of hours they work per week and their average weekly earnings (to the nearest dollar). The results are summarised in the table below.

Hours worked, *h*	4	8	15	18	10	5	12	16	14	6
Weekly earnings ($), *E*	23	47	93	122	56	33	74	110	78	35

   a. Construct a scatterplot of the data using technology.
   b. Draw in the line of best fit using technology.
   c. Write the equation of the line of best fit, in terms of variables *h* (hours worked) and *E* (weekly earnings).
   d. Interpret the meaning of the gradient.

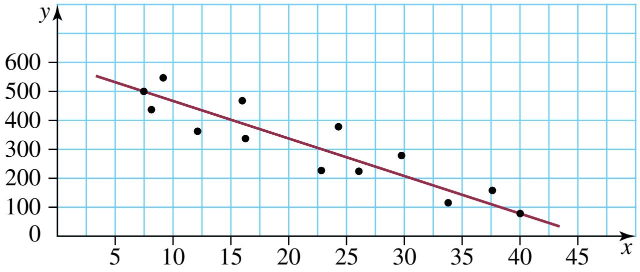
Understanding

3. **WE5** Use the given scatterplot and line of best fit to predict:



   a. the value of y when $x = 45$
   b. the value of x when $y = 15$.

4. Analyse the following graph.



   a. Use the line of best fit to predict the value of y when the value of x is:
   i. 7 **ii.** 22 **iii.** 36.
   b. Use the line of best fit to predict the value of x when the value of y is:
   i. 120 **ii.** 260 **iii.** 480.
   c. Determine the equation of the line of best fit, if it is known that it passes through the points $(5, 490)$ and $(40, 80)$.
   d. Use the equation of the line to verify the values obtained from the graph in parts **a** and **b**.

5. **WE6** The following table shows the average weekly expenditure on food for households of various sizes.

Number of people in a household	1	2	4	7	5	4	3	5
Cost of food ($ per week)	70	100	150	165	150	140	120	155
Number of people in a household	2	4	6	5	3	1	4	
Cost of food ($ per week)	90	160	160	160	125	75	135	

a. Construct a scatterplot of the data and draw in the line of best fit.

b. Determine the equation of the line of best fit. Write it in terms of variables *n* (for the number of people in a household) and *C* (weekly cost of food).

c. Interpret the meaning of the gradient.

d. Use the equation of the line of best fit to predict the weekly food expenditure for a family of:

i. 8 **ii.** 9 **iii.** 10.

6. The following table shows the gestation time and the birth mass of 10 babies.

Gestation time (weeks)	31	32	33	34	35	36	37	38	39	40
Birth mass (kg)	1.080	1.470	1.820	2.060	2.230	2.540	2.750	3.110	3.080	3.370

Use technology to answer the following questions.

a. Construct a scatterplot of the data. What type of correlation does the scatterplot suggest?

b. Draw in the line of best fit and determine its equation. Write it in terms of the variables *t* (gestation time) and *M* (birth mass).

c. What does the value of the gradient represent?

d. Although full term of gestation is considered to be 40 weeks, some pregnancies last longer. Use the equation obtained in part **b** to predict the birth mass of babies born after 41 and 42 weeks of gestation.

e. Many babies are born prematurely. Using the equation obtained in part **b**, predict the birth mass of a baby whose gestation time was 30 weeks.

f. If the birth mass of the baby was 2.390 kg, what was his or her gestation time (to the nearest week)?

7. **WE7** The number of hours spent studying, and the marks obtained by a group of students on a test are shown in this table.

Hours spent studying	45	30	90	60	105	65	90	80	55	75
Marks obtained	40	35	75	65	90	50	90	80	45	65

a. Calculate the correlation coefficient between the two sets of data.

b. Based on this value, describe the relationship between the number of hours spent studying, and the mark obtained.

Reasoning

8. As a part of her project, Rachel is growing a crystal. Every day she measures the crystal's mass using special laboratory scales and records it. The table below shows the results of her experiment.

Day number	1	2	3	4	5	8	9	10	11	12	15	16
Mass (g)	2.5	3.7	4.2	5.0	6.1	8.4	9.9	11.2	11.6	12.8	16.1	17.3

Measurements on days 6, 7, 13 and 14 are missing, since these were 2 consecutive weekends and, hence, Rachel did not have a chance to measure her crystal, which is kept in the school laboratory.

a. Construct the scatterplot of the data and draw in the line of best fit.

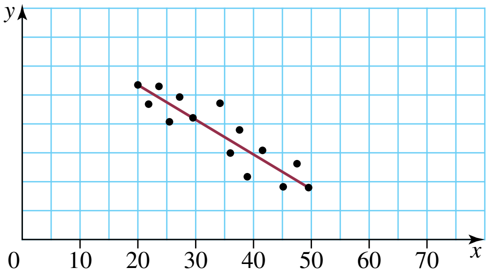
b. Determine the equation of the line of best fit. Write the equation, using variables d (day of the experiment) and M (mass of the crystal).

c. Interpret the meaning of the gradient.

d. For her report, Rachel would like to fill in the missing measurements (that is, the mass of the crystal on days 6, 7, 13 and 14). Use the equation of the line of best fit to help Rachel find these measurements. Is this an example of interpolation or extrapolation? Explain your answer.

e. Rachel needed to continue her experiment for 2 more days, but she fell ill and had to miss school. Help Rachel to predict the mass of the crystal on those two days (that is, days 17 and 18), using the equation of the line of best fit. Are these predictions reliable? Explain your answer.

9. **MC** Consider the figure shown below.



The line of best fit on the scatterplot is used to predict the values of y when $x = 15$, $x = 40$ and $x = 60$.

a. Interpolation would be used to predict the value of y when the value of x is:

A. 15 and 40 **B.** 15 and 60 **C.** 15 only **D.** 40 only **E.** 60 only

b. The prediction of the y-value(s) can be considered reliable when:

A. $x = 15$ and $x = 40$

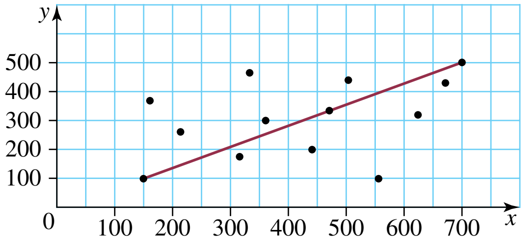
B. $x = 15$, $x = 40$ and $x = 60$

C. $x = 40$

D. $x = 40$ and $x = 60$

E. $x = 60$

10. **MC** The scatterplot below is used to predict the value of y when $x = 300$. This prediction is:



A. reliable, because it is obtained using interpolation

B. not reliable, because it is obtained using extrapolation

C. not reliable, because only x-values can be predicted with confidence

D. reliable because the scatterplot contains a large number of points

E. not reliable, because there is no correlation between x and y

Problem solving

11. For his birthday, Ari was given a small white rabbit. To monitor the rabbit's growth, Ari decided to measure it once a week. The table below shows the length of the rabbit for various weeks.

Week number, n	1	2	3	4	6	8	10	13	14	17	20
Length (cm), l	20	21	23	24	25	30	32	35	36	37	39

a. Construct a scatterplot of the data.

b. Draw a line of best fit and determine its equation.

c. As can be seen from the table, Ari did not measure his rabbit on weeks 5, 7, 9, 11, 12, 15, 16, 18 and 19. Use the equation of the line of best fit to predict the length of the rabbit for those weeks.

d. Were the predictions made in part **c** an example of interpolation or extrapolation? Explain.

e. Predict the length of the rabbit in the next three weeks (that is, weeks 21–23), using the line of best fit from part **c**.

f. Are the predictions that have been made in part **e** reliable? Explain.

g. Check your results using technology.

12. Laurie is training for the long jump, hoping to make the Australian Olympic team. His best jump each year is shown in the table below.

Age (a)	**Best jump (B) (metres)**
8	4.31
9	4.85
10	5.29
11	5.74
12	6.05
13	6.21
14	—
15	6.88
16	7.24
17	7.35
18	7.57

a. Plot the points generated by the table on a scatterplot.

b. Join the points generated with straight line segments.

c. Draw a line of best fit and determine its equation.

d. The next Olympic Games will occur when Laurie is 20 years old. Use the equation of the line of best fit to estimate Laurie's best jump that year and whether it will pass the qualifying mark of 8.1 metres.

e. Is a line of best fit a good way to predict future improvement in this situation? What problems are there with using a line of best fit?

f. Olympic Games will also be held when Laurie is 24 years old and 28 years old. Using extrapolation, what length would you predict Laurie could jump at these two ages? Is this realistic?

g. When Laurie was 14, he twisted a knee in training and did not compete for the whole season. In that year, a national junior championship was held. The winner of that championship jumped 6.5 metres. Use your line of best fit to predict whether Laurie would have won that championship.

Reflection

Why is extrapolation considered to be not reliable?

CHALLENGE 13.1

Sam has a mean score of 88 per cent for his first nine tests of the semester. In order to receive an A^+ his score must be 90 per cent or higher. There is one test remaining for the semester. Is it possible for him to receive an A^+? Why or why not?

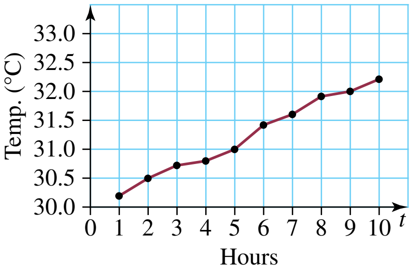
13.4 Time series

13.4.1 Time series

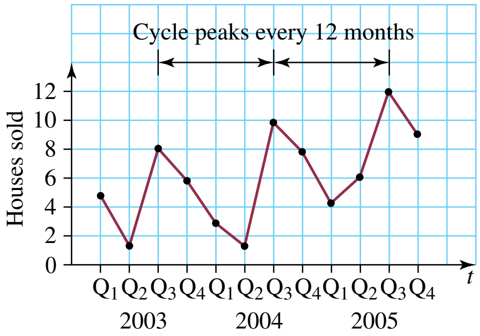
- **Time series** data are any data that have time as the independent variable.
- The data are graphed and the graphs are used to determine if a trend is present in the data.
- Identifying a trend can help when making predictions about the future.

13.4.2 Types of trends

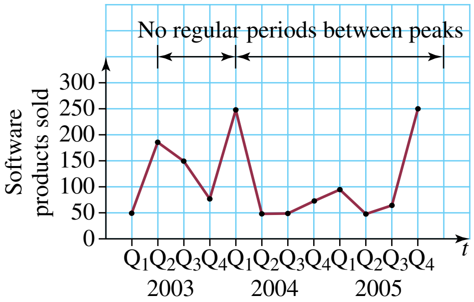
- A **general upward or downward trend** is a graph that overall goes up or down as illustrated in the graph shown below.



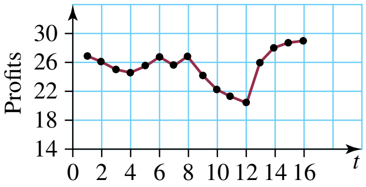
- A **seasonal pattern** displays fluctuations that repeat at the same time each week, month or quarter and usually last less than one year. The graph below illustrates that the peak selling time for houses is in the spring.



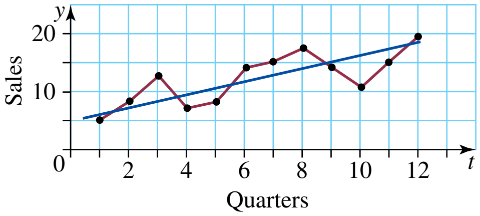
- A **cyclical pattern** displays fluctuations that repeat but will usually take longer than a year to repeat. An example of this is shown in the graph below, which depicts software products sold.



- **Random patterns** do not show any regular fluctuation. They are usually caused by unpredictable events such as the economic recession illustrated in the graph below.

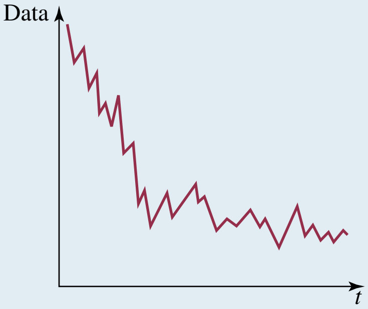


- Trends can work in combinations; for example, you can have a seasonal pattern with an upward trend.



WORKED EXAMPLE 8

Classify the trend suggested by the time series graph at right as being linear or non-linear, and upward, downward or no trend.



THINK

Carefully analyse the given graph and comment on whether the graph resembles a straight line or not and whether the values of y increase or decrease over time.

WRITE

The time series graph does not resemble a straight line and overall the level of the variable, y, decreases over time. The time series graph suggests a non-linear downward trend.

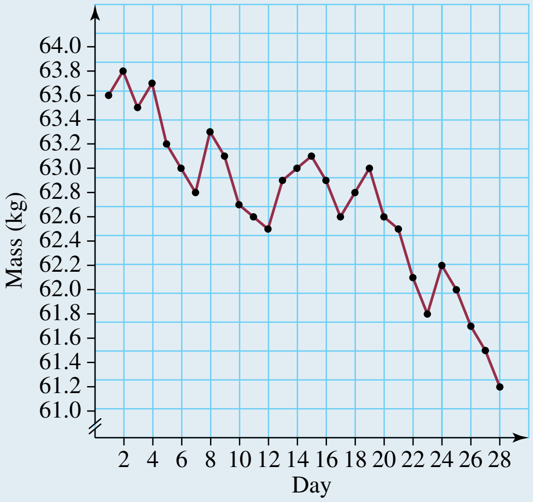
WORKED EXAMPLE 9

The data below show the average daily mass of a person (to the nearest 100 g), recorded over a period of 4 weeks.

63.6, 63.8, 63.5, 63.7, 63.2, 63.0, 62.8, 63.3, 63.1, 62.7, 62.6, 62.5, 62.9, 63.0, 63.1, 62.9, 62.6, 62.8, 63.0, 62.6, 62.5, 62.1, 61.8, 62.2, 62.0, 61.7, 61.5, 61.2

a Plot these masses as a time series graph.

b Comment on the trend.

THINK	WRITE/DRAW
a 1 Draw the points on a scatterplot with day on the horizontal axis and mass on the vertical axis.	**a** 
2 Join the points with straight line segments to create a time series plot.	
b Carefully analyse the given graph and comment on whether the graph resembles a straight line or not and whether the values of *y* (in this case, mass) increase or decrease over time.	**b** The graph resembles a straight line that slopes downwards from left to right (that is, mass decreases with increase in time). Although a person's mass fluctuates daily, the time series graph suggests a downward trend. That is, overall, the person's mass has decreased over the 28-day period.

13.4.3 Trend lines

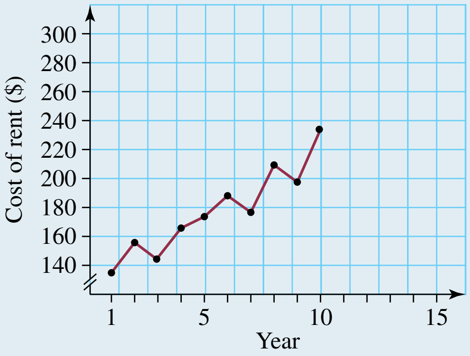
- A **trend line** is a type of line of best fit. Trend lines indicate the general trend of the data.
- Trend lines are useful in forecasting, or making predictions about the future, by extrapolation. Extrapolation can have limited reliability, as predictions are based on the assumption that the current trend will continue into the future.

WORKED EXAMPLE 10

The graph at right shows the average cost of renting a one-bedroom flat, as recorded over a 10-year period.

a If appropriate, draw in a line of best fit and comment on the type of the trend.

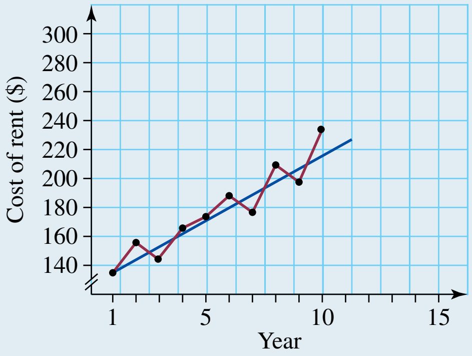
b Assuming that the current trend will continue, use the line of best fit to predict the cost of rent in 5 years' time.



THINK / **WRITE/DRAW**

a 1 Analyse the graph and observe what occurs over a period of time. Draw a line of best fit.

a



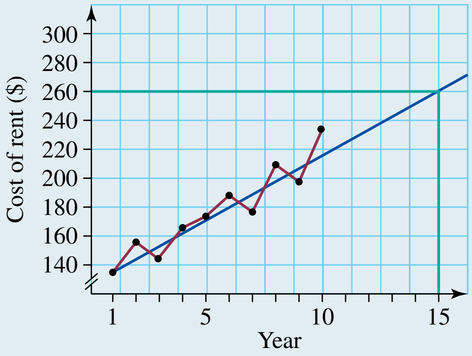
2 Comment on the type of trend observed.

The graph illustrates that the cost of rent increases steadily over the years. The time series graph indicates an upward linear trend.

b 1 Extend the line of best fit drawn in part **a**. The last entry corresponds to the 10th year and we need to predict the cost of rent in 5 years' time; that is, in the 15th year.

b

2 Locate the 15th year on the time axis and draw a vertical line until it meets with the line of best fit. From the trend line (line of best fit) draw a horizontal line to the cost axis.



3 Read the cost from the vertical axis.

Cost of rent = $260

4 Write your answer.

Assuming that the cost of rent will continue to increase at the present rate, in 5 years we can expect the cost of rent to reach $260 per week.

13.4.4 Using spreadsheets to determine the correlation

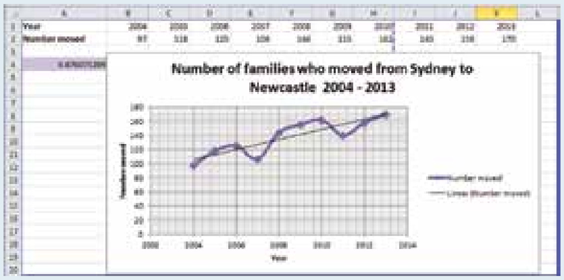
It is possible to draw scatterplots and time series graphs using a variety of digital technologies, including spreadsheets.

WORKED EXAMPLE 11 TI | CASIO

Data were recorded about the number of families who moved from Sydney to Newcastle over the past 10 years.

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
Number moved	97	118	125	106	144	155	162	140	158	170

a Use technology to construct a time series graph, with a line of best fit, that represents the data.
b Describe the trend.
c Measure the correlation.
d Comment on the results.

THINK	**WRITE**
a Enter the data into a spreadsheet. Select the type of graph — a scatter-plot with the points joined and a trend line fitted.	**a** 
b Describe the trend.	**b** There appears to be an upward trend over the 10 years.
c To measure the correlation, place the cursor in a cell — see the purple cell. From the menu ribbon, select 'More formulas', then 'Statistics', then 'CORREL'. Complete the values for the relevant cells and press Enter. The correlation value will be shown.	**c** The correlation is 0.8761.
d Interpret the results.	**d** Over the last 10 years, an increasing number of families have decided to make the move from Sydney to Newcastle. The correlation is strong and positive (0.8761), making it possible to predict that this trend is likely to continue.

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: Fluctuations and cycles (eles-0181)

Complete this digital doc: WorkSHEET: Time series (doc-14600)

Exercise 13.4 Time series

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–3, 5–7, 9	Questions: 1–7, 9, 10	Questions: 1–11

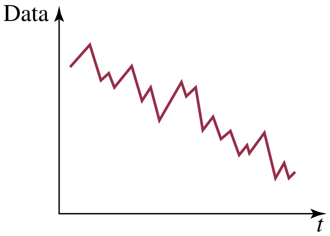
🟩🟦🟪 **Individual pathway interactivity:** int-4628

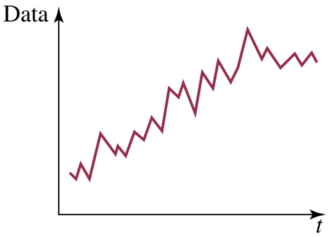
learnON ONLINE ONLY

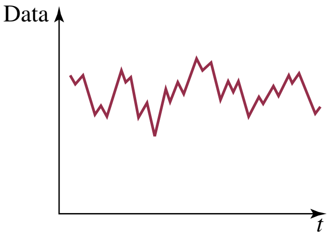
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

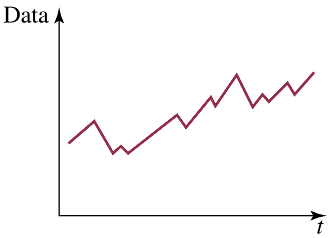
Fluency

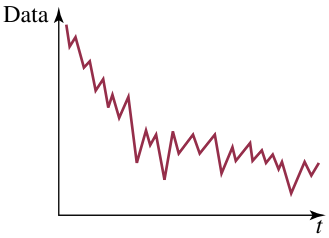
1. **WE8** Classify the trend suggested by each time series graph as being linear or non-linear, and upward, downward or stationary in the mean (no trend).

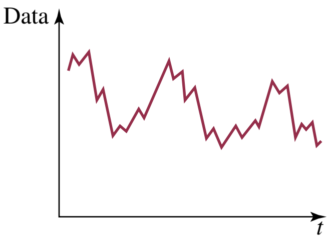
a.


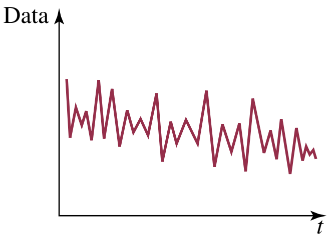
b.


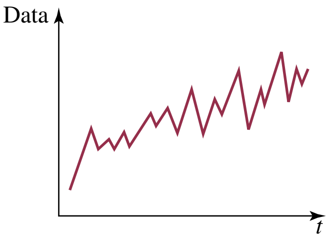
c.


d.


e.


f.


g.


h.


2. **WE9** The data below show the average daily temperatures recorded in June.
17.6, 17.4, 18.0, 17.2, 17.5, 16.9, 16.3, 17.1, 16.9, 16.2, 16.0, 16.6, 16.1, 15.4, 15.1 15.5, 16.0, 16.0, 15.4, 15.2, 15.0, 15.5, 15.1, 14.8, 15.3, 14.9, 14.6, 14.4, 15.0, 14.2
a. Plot these temperatures as a time series graph.
b. Comment on the trend.

Understanding

3. The data below show the quarterly sales (in thousands of dollars) recorded by the owner of a sheepskin product store over a period of 4 years.

Quarter	2006	2007	2008	2009
1	57	59	50	52
2	100	102	98	100
3	125	127	120	124
4	74	70	72	73

   a. Plot the time series.

   b. The time series plot displays seasonal fluctuations of period 4 (since there are four quarters). Explain in your own words what this means. Also write one or two possible reasons for the occurrence of these fluctuations.

   c. Overall, does the time series plot indicate upward, downward or no trend?

4. The table below shows the total monthly revenue (in thousands of dollars) obtained by the owners of a large reception hall. The revenue comes from rent and catering for various functions over a period of 3 years.

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
2007	60	65	40	45	40	50	45	50	55	50	55	70
2008	70	65	60	65	55	60	60	65	70	75	80	85
2009	80	70	65	70	60	65	70	75	80	85	90	100

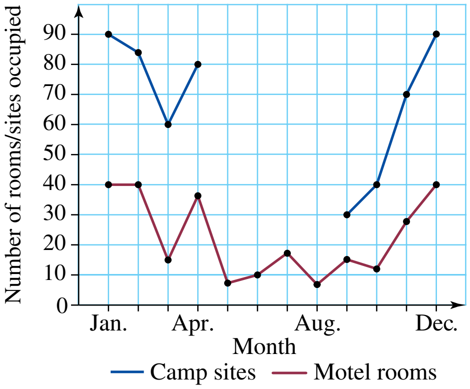
   a. Construct a time series plot for these data.

   b. Describe the graph (peaks and troughs, long-term trend, any other patterns).

   c. Try to give possible reasons for monthly fluctuations.

   d. Does the graph show seasonal fluctuations of period 12? Are there any patterns that repeat from year to year?

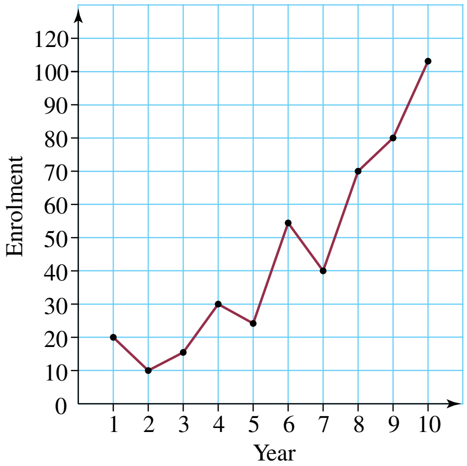
5. The owner of a motel and caravan park in a small town keeps records of the total number of rooms and total number of camp sites occupied per month. The time series plots based on his records are shown below.



   a. Describe each graph, discussing general trend, peaks and troughs and so on. Explain particular features of the graphs and give possible reasons.

   b. Compare the two graphs and write a short paragraph commenting on any similarities and differences between them.

6. **WE10** The graph below shows enrolments in the Health and Nutrition course at a local college over a 10-year period.



a. If appropriate, draw in a line of best fit and comment on the type of the trend.
b. Assuming that the trend will continue, use the line of best fit to predict the enrolment for the course in 5 years' time; that is, in the 15th year.

Reasoning

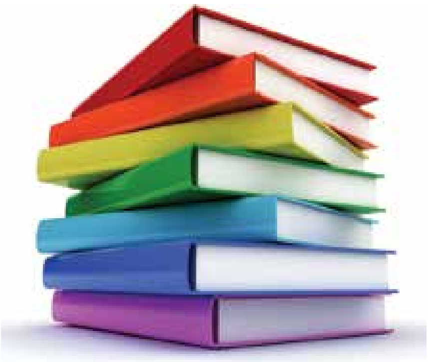
7. **WE11** In June a new childcare centre was opened. The number of children attending full time (according to the enrolment at the beginning of each month) during the first year of operation is shown in the table below.

June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May
6	8	7	9	10	9	12	10	11	13	12	14

a. Plot this time series using a digital technology. (*Hint*: Let June = 1, July = 2 etc.)
b. Is the child care business going well? Justify your answer.
c. Draw a line of best fit and find its equation, using coordinates of any two points on the line.
d. Use your equation of the line of best fit to predict the enrolment in the centre during the second year of operation at the beginning of:
  i. August
  ii. January.

What assumptions have you made?

8. The graph below shows the monthly sales of a certain book since its publication. Explain in your own words why linear trend forecasting of the future sales of this book is not appropriate.



9. **a.** Choose an object or subject that is of interest to you and that can be observed and measured during one day. For example, you might decide to measure your own pulse rate.

b. Prepare a table where you will record your results every hour within the school day. For example, for the pulse rate the table might look like this.

Time	8 am	9 am	10 am	11 am	12 pm	1 pm	2 pm	3 pm
Pulse rate								

c. Take your measurements at the regular time intervals you have decided on and record them in the table.

d. Plot the time series obtained as a result of your experiment.

e. Describe the graph and comment on the trend.

f. If appropriate, draw in a line of best fit and predict the values (that is, your pulse rate) for the next 2–3 hours.

g. Take the actual measurements during the hours you have made predictions for. Compare your predictions with the actual measurements. Were your predictions accurate? Why or why not?

Problem solving

10. Running a small business is difficult when your business is affected by seasonal patterns. It can make it difficult to do a budget when you have a fluctuating income. For example, if you owned a cafe at Mt Buller you would have a lot of business during the ski season but very little business over summer. If you wish to remain open over the summer, you need to find a way to remove the impact of the seasons from your sales data to see if it is viable. Investigate a way of removing the effects of the seasons from the data.

11. The table below gives the quarterly sales figures for a second-hand car dealer over a three-year period.

Year	Q1	Q2	Q3	Q4
2012	75	65	92	99
2013	91	79	115	114
2014	93	85	136	118

a. Represent this data on a time series plot.

b. Briefly describe how the car sales have altered over the time period.

c. Does it appear that the car dealer can sell more cars in a particular period each year?

Reflection

Why are predictions in the future appropriate for time series even though they involve extrapolation?

CHALLENGE 13.2

Mr MacDonald recorded the test marks for his Year 10 class of 25 students. He calculated the average mark to be 72. Sandra's mark of 86 was incorrectly marked as 36. What was the correct average mark for the test?

13.5 Review

13.5.1 Review questions

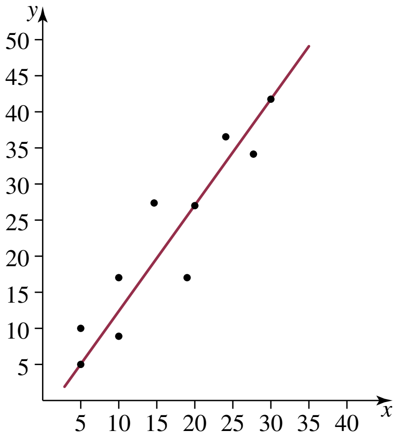
Fluency

1. As preparation for a Mathematics test, a group of 20 students was given a revision sheet containing 60 questions. The table below shows the number of questions from the revision sheet successfully completed by each student and the mark, out of 100, of that student on the test.

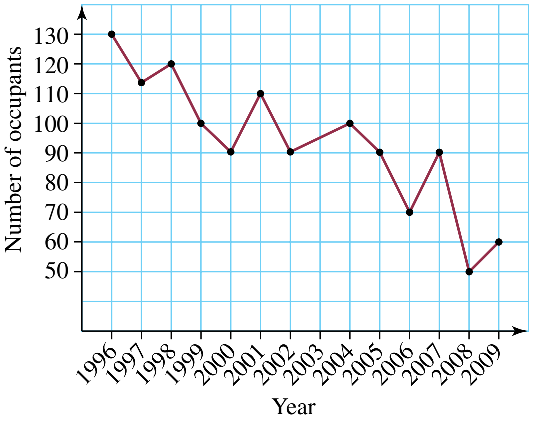
Number of questions	9	12	37	60	55	40	10	25	50	48	60
Test result	18	21	52	95	100	67	15	50	97	85	89
Number of questions	50	48	35	29	19	44	49	20	16	58	52
Test result	97	85	62	54	30	70	82	37	28	99	80

a. State which of the variables is dependent and which is independent.
b. Construct a scatterplot of the data.
c. State the type of correlation between the two variables suggested by the scatterplot and draw a corresponding conclusion.
d. Suggest why the relationship is not perfectly linear.

2. a. Use the line of best fit shown on the graph at right to predict the value of y, when the value of x is:
i. 10
ii. 35.
b. Use the line of best fit to predict the value of x, when the value of y is:
i. 15
ii. 30.
c. Find the equation of a line of best fit if it is known that it passes through the points (5, 5) and (20, 27).
d. Use the equation of the line to algebraically verify the values obtained from the graph in parts **a** and **b**.



3. The graph shows the number of occupants of a large nursing home over the last 14 years.
a. Comment on the type of trend displayed.
b. Explain why it is appropriate to draw in a line of best fit.
c. Draw a line of best fit and use it to predict the number of occupants in the nursing home in 3 years' time.
d. What assumption has been made when predicting figures for part **c**?



4. The table below shows the advertised sale price ('000s dollars) and the land size (m^2) for ten vacant blocks of land.

Land size (m^2)	Sale price (×\$1000)
632	36
1560	58
800	40
1190	44
770	41
1250	52
1090	43
1780	75
1740	72
920	43

a. Construct a scatterplot and determine the equation of the line of best fit.
b. What does the gradient represent?
c. Using the line of best fit, predict the approximate sale price, to the nearest thousand dollars for a block of land with an area of 1600 m^2.
d. Using the line of best fit, predict the approximate land size, to the nearest 10 square metres, you could purchase with \$500 000.

5. The table below shows, for fifteen students, the amount of pocket money they receive and spend at the school canteen in an average week.

Pocket money (\$)	Canteen spending (\$)
30	16
40	17
15	12
25	14
40	16
15	14
30	16
30	17

Pocket money (\$)	Canteen spending (\$)
25	15
15	13
50	19
20	14
35	17
20	15
10	13

a. Construct a scatterplot and determine the equation of the line of best fit.
b. What does the gradient represent?
c. Using your line of best fit, predict the amount of money spent at the canteen for a student receiving \$45 pocket money a week.
d. Using your line of best fit, predict the amount of money spent at the canteen by a student who receives \$100 pocket money each week? Does this seem reasonable? Explain.

6. The table below shows, for 10 ballet students, the number of hours a week spent training and the number of pirouettes in a row they can complete.

Training (h)	11	11	2	8	4	16	11	16	5	3
Number of pirouettes	15	13	3	12	7	17	13	16	8	5

a. Construct a scatterplot and determine the equation of the line of best fit.
b. What does the gradient represent?

c. Using your line of best fit, predict the number of pirouettes that could be complete if a student undertakes 14 hours of training.

d. Professional ballet dancers may undertake up to 30 hours of training a week. Using your line best fit, predict the number of pirouettes they should be able to do in a row. Comment on your findings.

7. a. Use the data given below to draw a scatterplot and determine the equation of the line of best fit.

Age in years	7	11	8	16	9	8	14	19	17	10	20	15
Hours of television watched in a week	20	19	25	55	46	50	53	67	59	25	70	58

b. Use the line of best fit to predict the value of the number of hours of television watched by a person aged 15.

c. Determine the age when the number of hours of television watched is 60.

Problem solving

8. Describe the trends present in the following time series data that shows the mean monthly daily hours of sunshine in Melbourne from January to December.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Daily hours of sunshine	8.7	8.0	7.5	6.4	4.8	4.0	4.5	5.5	6.3	7.3	7.5	8.3

9. The existence of the following situations is often considered an obstacle to making estimates from data.

a. Outlier

b. Extrapolation

c. Small range of data

d. Small number of data points

Explain why each of these situations is considered an obstacle to making estimates of data and how each might be overcome.

10. The table below shows the heights of 10 students and the distances along the ground between their feet as they attempt to do the splits.

Height (cm)	Distance stretched (cm)
134.5	150
156	160
133.5	147
145	160
160	162
135	149
163	163
138	149
152	158
159	160

Using the data, estimate the distance a person 1.8 m tall can achieve when attempting the splits. Write a detailed analysis of your result. Include:

- an explanation of the method(s) used
- any plots or formula generated
- comments on validity of the estimate
- any ways the validity of the estimate could be improved.

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- **Try out this interactivity:** Word search: Topic 13 (int-2886)
- **Try out this interactivity:** Crossword: Topic 13 (int-2887)
- **Try out this interactivity:** Sudoku: Topic 13 (int-3600)
- **Complete this digital doc:** Concept map: Topic 13 (doc-14602)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

bivariate
continuous
denominator
dependent variable
direction
discrete
form
independent
independent variable
line of best fit
moderate
predictions
relationship
scatterplot
strength
strong
time series
trend
univariate
variable
weak

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assessON provides sets of questions for every topic in your course, as well as giving instant feedback and worked solutions to help improve your mathematical skills.

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Investigation | Rich Task

Collecting, recording and analysing data over time

A time series is a sequence of measurements taken at regular intervals (daily, weekly, monthly and so on) over a certain period of time. Time series are best represented using time-series plots, which are line graphs with the time plotted on the horizontal axis. Examples of time series include daily temperature, monthly unemployment rates and daily share prices.

When data are recorded on a regular basis, the value of the variable may go up and down in what seems to be an erratic pattern. These are called fluctuations. However, over a long period of time, the time series usually suggests a certain trend. These trends can be classified as being linear or non-linear, and upward, downward or stationary (no trend).

Time series are often used for forecasting, that is, making predictions about the future. The predictions made with the help of time series are always based on the assumption that the observed trend will continue in the future.

1. Choose a subject that is of interest to you and that can be observed and measured during one day or over the period of a week or more. (Suitable subjects are shown in the list below.)

2. Prepare a table for recording your results. Select appropriate regular time intervals. An example is shown below.

Time	8 am	9 am	10 am	11 am	12 pm	1 pm	2 pm	3 pm	4 pm	5 pm
Pulse rate										

3. Take your measurements at the selected time intervals and record them in the table.

4. Use your data to plot the time series. You can use software such as Excel or draw the scatterplot by hand.

5. Describe the graph and comment on its trend.

6. If appropriate, draw a line of best fit and predict the next few data values.

7. Take the actual measurements during the hours you have made predictions for. Compare the predictions with the actual measurements. Were your predictions good? Give reasons.

Here are some suitable subjects for data observation and recording:

- minimum and maximum temperatures each day for 2 weeks (use the TV news or newspaper as resources)
- the value of a stock on the share market (e.g. Telstra, Wesfarmers and Rio Tinto)
- your pulse over 12 hours (ask your teacher how to do this or check on the internet)
- the value of sales each day at the school canteen
- the number of students absent each day
- the position of a song in the Top 40 over a number of weeks
- petrol prices each day for 2 weeks
- other measurements (check with your teacher)
- world population statistics over time.

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: Code puzzle: What did the chewing gum say to the shoe? (doc-15941)

Answers

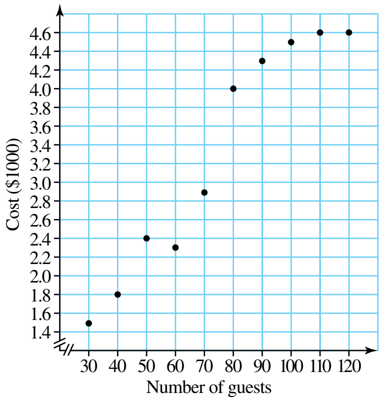
Topic 13 Bivariate data

Exercise 13.2 Bivariate data

1.

	Independent	**Dependent**
a.	Number of hours	Test results
b.	Rainfall	Attendance
c.	Hours in gym	Visits to the doctor
d.	Lengths of essay	Memory taken
e.	Cost of care	Attendance
f.	Age of property	Cost of property
g.	Number of applicants	Cut-off OP score
h.	Running speed	Heart rate

2.



3. a. Perfectly linear, positive

b. No correlation

c. Non-linear, negative, moderate

d. Strong, positive, linear

e. No correlation

f. Non-linear, positive, strong

g. Strong, negative, negative

h. Non-linear, moderate, negative

i. Weak, negative, linear

j. Non-linear, moderate, positive

k. Positive, moderate, linear

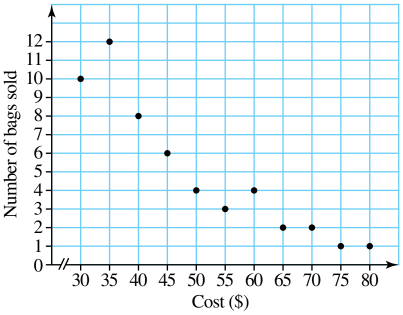
l. Non-linear, strong, negative

m. Strong, negative, linear

n. Weak, positive, linear

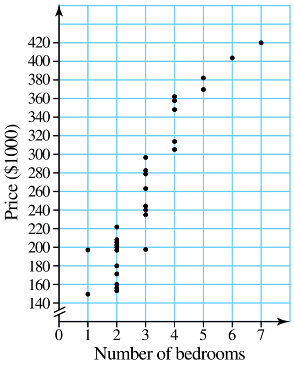
o. Non-linear, moderate, positive

4. a.



b. Negative, linear, moderate. The price of the bag appeared to affect the numbers sold; that is, the more expensive the bag, the fewer sold.

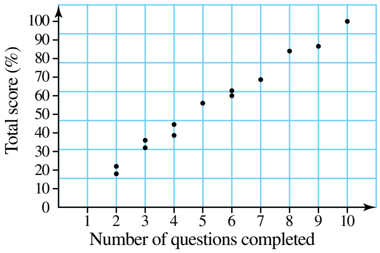
5. a.



b. Moderate positive linear correlation. There is evidence to show that the larger the number of bedrooms, the higher the price of the house.

c. Various answers; location, age, number of people interested in the house, and so on.

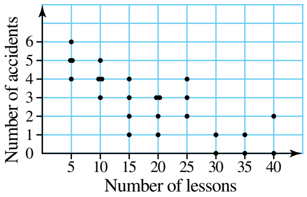
6. a.



b. Strong, positive, linear correlation

c. Various answers — some students are of different ability levels and they may have attempted the questions but had incorrect answers.

7. a.



b. Weak, negative, linear relation

c. Various answers, such as some drivers are better than others, live in lower traffic areas, traffic conditions etc.

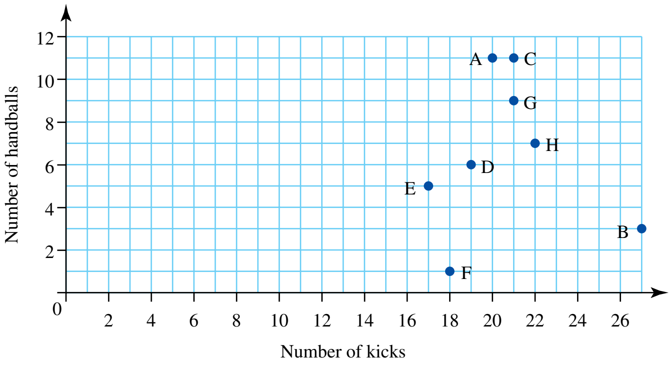
8. a. T
b. F
c. T
d. F
e. T

9. B

10. C

11. D

12. a.



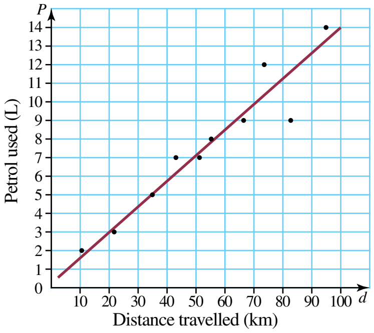
b. This scatterplot does not support the claim.

13. a. Mandy (iii)
b. William (iv)
c. Charlotte (viii)
d. Dario (vii)
e. Edward (vi)
f. Cindy (v)
g. Georgina (i)
h. Harrison (ii)

Exercise 13.3 Lines of best fit

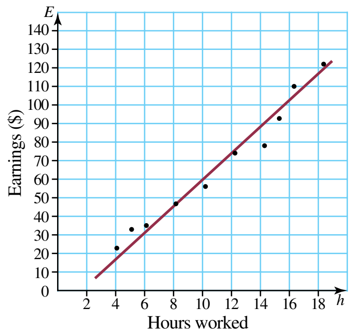
Note: Answers may vary depending on the line of best fit drawn.

1. a, b.



c. Using (23, 3) and (56, 8), the equation is $P = \frac{5}{33}d - \frac{16}{33}$.

2. a, b.



c. Using (8, 47) and (12, 74), the equation is $E = 6.75h - 7$.

d. On average, students were paid $6.75 per hour.

3. a. 38 **b.** 18

4. a. i. 460 **ii.** 290 **iii.** 130

b. i. 39 **ii.** 24 **iii.** 6

c. $y = -11.71x + 548.57$

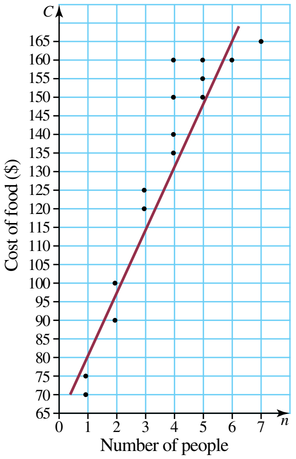
d. y-values:

i. 466.60 **ii.** 290.95 **iii.** 127.01

x-values:

i. 36.60 **ii.** 24.64 **iii.** 5.86

5. a.

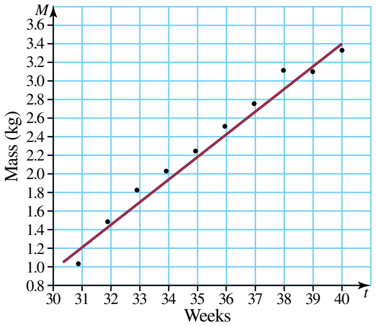


b. Using (1, 75) and (5, 150), the equation is $C = 18.75n + 56.25$.

c. On average, weekly cost of food increases by $18.75 for every extra person.

d. i. $206.25 **ii.** $225.00 **iii.** $243.75

6. a.



Positive, strong, linear correlation

b. Using (32, 1.470) and (35, 2.230), $M = 0.25t - 6.5$.

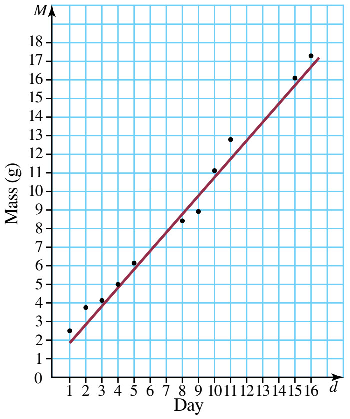
c. With every week of gestation the mass of the baby increases by approximately 250 g.

d. 3.75 kg; 4 kg **e.** Approximately 1 kg **f.** Between 35 and 36 weeks

7. a. $r = 0.9$

b. There is a strong positive relationship between the number of hours spent studying and the marks obtained. This seems to indicate that greater dedication to studying will produce better results.

8. a.



b. Using (2, 3.7) and (10, 11.2), $M = 0.973d + 1.285$.

c. Each day Rachel's crystal gains 0.973 g in mass.

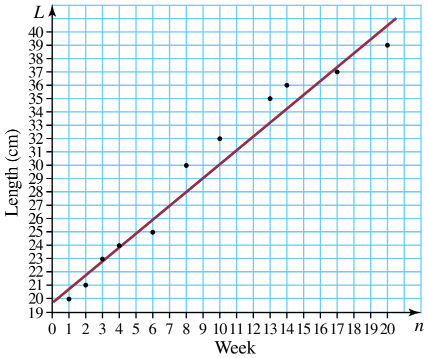
d. 7.123 g; 8.096 g; 13.934 g; 14.907 g; interpolation (within the given range of 1–16)

e. 17.826 g; 18.799 g; predictions are not reliable, since they were obtained using extrapolation.

9. a. D **b.** C

10. E

11. a.



b. Using (2, 21) and (17, 37), $L = 1.07n + 18.9$.

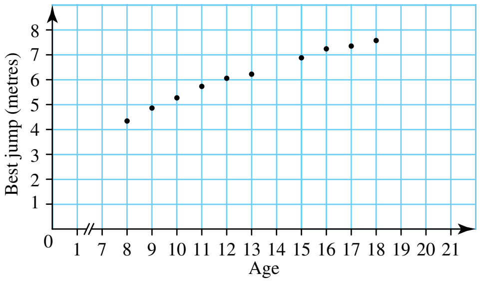
c. 24.25 cm; 26.39 cm; 28.53 cm; 30.67 cm; 31.74 cm; 34.95 cm; 36.02 cm; 38.16 cm; 39.23 cm

d. Interpolation (within the given range of 1–20)

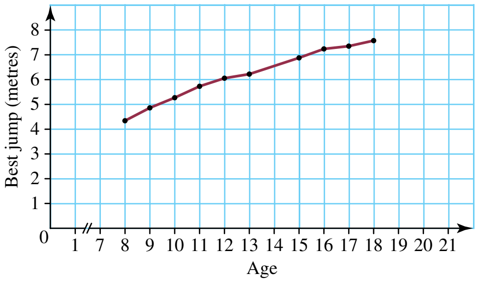
e. 41.37 cm; 42.44 cm; 43.51 cm

f. Not reliable, because extrapolation has been used.

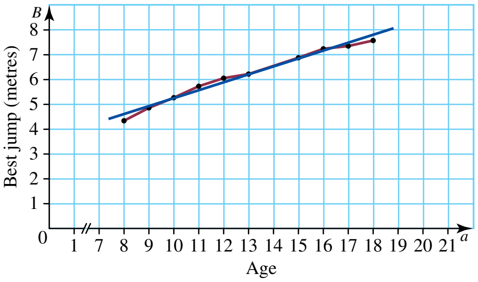
12. a.



b.



c.



d. Yes. Using points (9, 4.85) and (16, 7.24), $B = 0.34a + 1.8$; estimated best jump = 8.6 m.

e. No, trends work well over the short term but long term are affected by other variables.

f. 24 years old: 9.97 m; 28 years old: 11.33 m. It is unrealistic to expect his jumping distance to increase indefinitely.

g. Equal first

Challenge 13.1

No. He would have to get 108%, which would be impossible on a test.

Exercise 13.4 Time series

1. a. Linear, downward

b. Non-linear, upward

c. Non-linear, stationary in the mean

d. Linear, upward

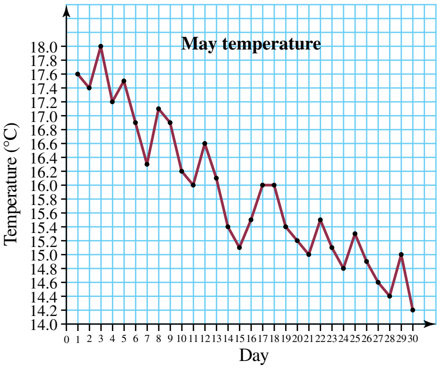
e. Non-linear, downward

f. Non-linear, stationary in the mean

g. Non-linear, stationary in the mean

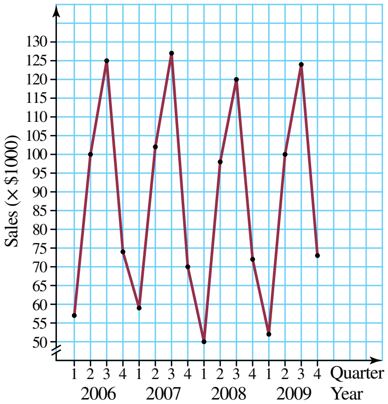
h. Linear, upward

2. a.



b. Linear downward trend

3. a.



b. Sheepskin products more popular in the third quarter (presumably winter) — discount sales, increase in sales, and so on.

c. No trend

4. a.

b. General upward trend with peaks around December and troughs around April.

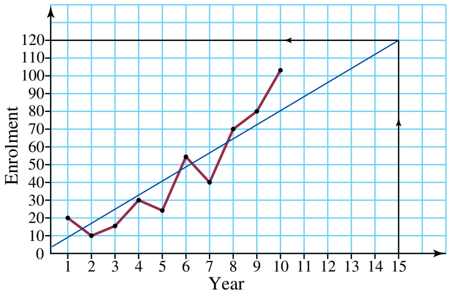
c. Peaks around Christmas where people have lots of parties, troughs around April where weather gets colder and people less inclined to go out.

d. Yes. Peaks in December, troughs in April.

5. a. Peaks around Christmas holidays and a minor peak at Easter. No camping in colder months.

b. Check with your teacher.

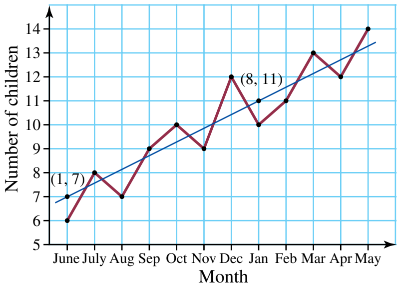
6. a.



Upward linear

b. In 15th year the expected amount = 122

7. a.



b. Yes, the graph shows an upward trend.

c. $y = \frac{4}{7}x + \frac{45}{7}$

d. i. 15

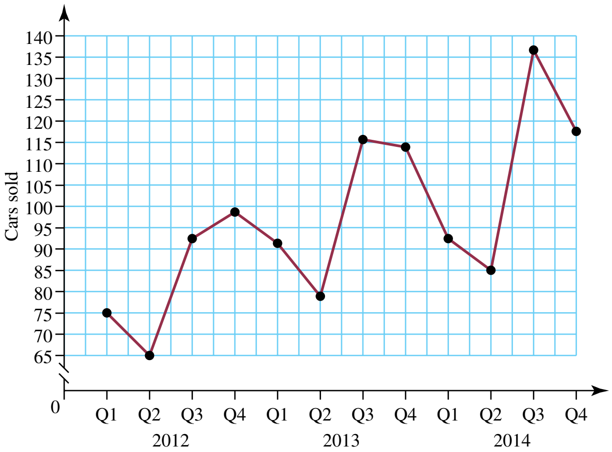
ii. 18 (The assumption made was that business will continue on a linear upward trend.)

8. The trend is non-linear, therefore unable to forecast future sales.

9. Answers will vary.

10. Check with your teacher. Some methods could involve 'smoothing' and 'seasonal adjustments'.

11. a.



b. Secondhand car sales per quarter have shown a general upward trend but with some major fluctuations.

c. More cars are sold in the third and fourth quarters compared to the first and second quarters.

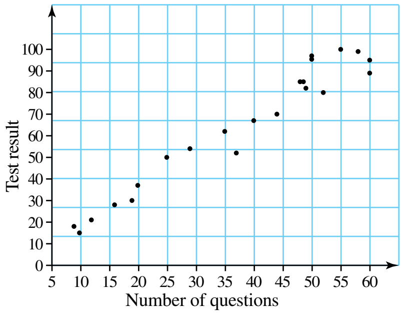
Challenge 13.2

74

13.5 Review

1. **a.** Number of questions — independent; test result — dependent

   b.



   c. Strong, positive, linear correlation; the larger the number of completed revision questions, the higher the mark on the test.

   d. Different abilities of the students

2. **a. i.** 12.5

   ii. 49

   b. i. 12

   ii. 22.5

   c. $y = \frac{22}{15}x - \frac{7}{3}$

   d. i. 12.33

   ii. 49

   and

   i. 11.82

   ii. 22.05

3. **a.** Linear downwards

   b. The trend is linear.

   c. About 65 occupants

   d. Assumes that the current trend will continue.

4. **a.** $P = 31.82a + 13070.4$, where P is the sale price and a is the land area.

   b. The price of land is approximately $31.82 per square metre.

   c. $64 000

   d. $1160\,\text{m}^2$

5. **a.** $C = 0.15p + 11.09$, where C is the money spent at the canteen and p represents the pocket money received.

   b. Students spend 15 cents at the canteen per dollar received for pocket money.

   c. $18

   d. $26. This involves extrapolation, which is considered unreliable. It does not seem reasonable that, if a student receives more money, they will eat more or have to purchase more than any other student.

6. **a.** $P = 0.91t + 2.95$, where P is the number of pirouettes and t is the number of hours of training.

   b. Ballet students can do approximately 0.91 pirouettes for each hour of training.

   c. Approximately 15 pirouettes

   d. Approximately 30 pirouettes. This estimate is based on extrapolation, which is considered unreliable. To model this data linearly as the number of hours of training becomes large is unrealistic.

7. **a.** Hours of TV watched = 3.31 × age + 3.05

   b. Approximately 53 hours

   c. Approximately 17 years old

8. Overall the data appears to be following a seasonal trend, with peaks at either end of the year and a trough in the middle.

9. **a.** Outliers can unfairly skew data and as such dramatically alter the line of best fit. Identify and remove any outliers from the data before determining the line of best fit.

   b. Extrapolation involves making estimates outside the data range and this is considered unreliable. When extrapolation is required, consider the data and the likelihood that the data would remain linear if extended. When giving results, make comment on the validity of the estimation.

c. A small range may not give a fair indication if a data set shows a strong linear correlation. Try to increase the range of the data set by taking more measurements or undertaking more research.

d. A small number of data points may not be able to establish with confidence the existence of a strong linear correlation. Try to increase the number of data points by taking more measurements or undertaking more research.

10. About 170 cm; data was first plotted as a scatter plot. (145, 160) was identified as an outlier and removed from the data set. A line of best fit was then fitted to the remaining data and its equation determined as $d = 0.5h + 80$, where d is the distance stretched and h is the height. Substitution was used to obtain the estimate.

The estimation requires extrapolation and cannot be considered reliable. The presence of the outlier may indicate variation in flexibility rather than a strong linear correlation between the data. Estimate is based on a small set of data. More data should be collected in order to determine the suitability of least squares regression.

Investigation — Rich task

Answers will vary; teacher to check.

TOPIC 14
Statistics in the media

14.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

14.1.1 Why learn this?

Statistics, when used properly, can be an invaluable aid to good decision making. However, deliberate distortion of the data or meaningless pictures can be used to support almost any claim or point of view. Whenever you read an advertisement, hear a news report or are given some data by a friend, you need to have a healthy degree of scepticism about the reliability of the source and nature of the data presented.

14.1.2 What do you know?

assesson

1. **THINK** List what you know about how the media reports data. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of how the media reports data.

LEARNING SEQUENCE

- **14.1** Overview
- **14.2** Populations and samples
- **14.3** Primary and secondary data
- **14.4** Evaluating inquiry methods and statistical reports
- **14.5** Statistical investigations
- **14.6** Review

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Behind the advertising (eles-1854)

14.2 Populations and samples

14.2.1 Populations

- The term **population** refers to a complete set of individuals, objects or events belonging to some category.
- When data are collected from a *whole* population, the process is known as a **census**.
  - It is often not possible, nor cost-effective, to conduct a census.
  - For this reason, **samples** have to be selected carefully from the population. A sample is a subset of its population.

WORKED EXAMPLE 1

List some of the problems you might encounter in trying to collect data on the following populations.

a The life of a mobile phone battery
b The number of possums in a local area
c The number of males in Australia
d The average cost of a loaf of white bread

THINK	WRITE
For each of these scenarios, consider how the data might be collected, and the problems in obtaining these data.	
a The life of a mobile phone battery	**a** The life of a mobile phone battery cannot be measured until it is dead. The battery life also depends on how the phone is used, and how many times it has been recharged.
b The number of possums in a local area	**b** It would be almost impossible to find all the possums in a local area in order to count them. The possums also may stray into other areas.
c The number of males in Australia	**c** The number of males in Australia is constantly changing. There are births and deaths every second.
d The average cost of a loaf of white bread	**d** The price of one particular loaf of white bread varies widely from one location to another. Sometimes the bread is on ‘Special’ and this would affect the calculations.

14.2.2 Samples

- Surveys are conducted using samples. Ideally the sample should reveal generalisations about the population.
- A random sample is generally accepted as being an ideal representation of the population from which it was drawn. However, it must be remembered that different random samples from the same population can produce different results. This means that we must be cautious about making predictions about a population, as results of surveys conducted using random samples may vary.
- A sample size must be sufficiently large. As a general rule, the sample size should be about $\sqrt{N}$, where N is the size of the population. It is a misconception that a larger sample will produce a more reliable prediction of the characteristics of its population.

WORKED EXAMPLE 2

TI | CASIO

A die was rolled 50 times and the following results were obtained.

6 5 3 1 6 2 3 6 2 5 3 4 1 3 2 6 4 5 5 4 3 1 2 1 6 4 5 2
3 6 1 5 3 3 2 4 1 4 2 3 2 6 3 4 6 2 1 2 4 2.

a Determine the mean of the population (to 1 decimal place).

b A suitable sample size for this population would be 7 ($\sqrt{50} \approx 7.1$).

  i Select a random sample of 7 scores, and determine the mean of these scores.

  ii Select a second random sample of 7 scores, and determine the mean of these.

  iii Select a third random sample of 20 scores, and determine the mean of these.

c Comment on your answers to parts a and b.

THINK	WRITE
a Calculate the mean by first finding the sum of all the scores, then dividing by the number of scores (50).	**a** Population mean $= \frac{\Sigma x}{n}$ $= \frac{169}{50}$ $= 3.4$
b i Use a calculator to randomly generate 7 scores from 1 to 50. Relate these numbers back to the scores, then calculate the mean.	**b i** The 7 scores randomly selected are numbers 17, 50, 11, 40, 48, 12, 19 in the set of 50 scores. These correspond to the scores 4, 2, 3, 3, 2, 4, 5. The mean of these scores $= \frac{23}{7} = 3.3$.
ii Repeat **bi** to obtain a second set of 7 randomly selected scores. This second set of random numbers produced the number 1 twice. Try again. Another attempt produced the number 14 twice. Try again. A third attempt produced 7 different numbers. This set of 7 random numbers will then be used to, again, calculate the mean of the scores.	**ii** Ignore the second and third attempts to select 7 random numbers because of repeated numbers. The second set of 7 scores randomly selected is numbers 16, 49, 2, 42, 31, 11, 50 of the set of 50. These correspond to the scores 6, 4, 5, 6, 1, 3, 2. The mean of these scores $= \frac{27}{7} = 3.9$.
iii Repeat for a randomly selected 20 scores.	**iii** The set of 20 randomly selected numbers produced a total of 68. Mean of 20 random scores $= \frac{68}{20} = 3.4$
c Comment on the results.	**c** The population mean is 3.4. The means of the two samples of 7 are 3.3 and 3.9. This shows that, even though the samples are randomly selected, their calculated means may be different. The mean of the sample of 20 scores is 3.4. This indicates that by using a bigger sample the result is more accurate than those obtained with the smaller samples.

14.2.3 To sample or to conduct a census?

- The particular circumstances determine whether data are collected from a population, or from a sample of the population. For example, suppose you collected data on the height of every Year 10 student in your class. If your class was the only Year 10 class in the school, your class would be the population. If, however, there were several Year 10 classes in your school, your class would be a sample of the Year 10 population.
- Worked example 2 showed that different random samples can produce different results. For this reason, it is important to acknowledge that there could be some uncertainty when using sample results to make predictions about the population.

WORKED EXAMPLE 3

For each of the following situations, state whether the information was obtained by census or survey. Justify why that particular method was used.

a A roll call is conducted each morning at school to determine which students are absent.
b TV ratings are collected from a selection of viewers to discover the popular TV shows.
c Every hundredth light bulb off an assembly production line is tested to determine the life of that type of light bulb.
d A teacher records the examination results of her class.

THINK	WRITE
a Every student is recorded as being present or absent at the roll call.	**a** This is a census. If the roll call only applied to a sample of the students, there would not be an accurate record of attendance at school. A census is essential in this case.
b Only a selection of the TV audience contributed to these data.	**b** This is a survey. To collect data from the whole viewer population would be time-consuming and expensive. For this reason, it is appropriate to select a sample to conduct the survey.
c Only 1 bulb in every 100 is tested.	**c** This is a survey. Light bulbs are tested to destruction (burn-out) to determine their life. If every bulb was tested in this way, there would be none left to sell! A survey on a sample is essential.
d Every student's result is recorded.	**d** This is a census. It is essential to record the result of every student.

learnon RESOURCES — ONLINE ONLY

- **Complete this digital doc:** SkillSHEET: Determining suitability of questions for a survey (doc-5337)
- **Complete this digital doc:** SkillSHEET: Finding proportions (doc-5338)
- **Complete this digital doc:** SkillSHEET: Distinguishing between types of data (doc-5339)
- **Complete this digital doc:** SkillSHEET: Reading bar graphs (doc-5340)
- **Complete this digital doc:** SkillSHEET: Determining independent and dependent variables (doc-5341)
- **Complete this digital doc:** WorkSHEET: Populations and samples (doc-14603)

Exercise 14.2 Populations and samples

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–3, 5, 7, 10	Questions: 1–4, 6, 8, 11, 12	Questions: 1–3, 5, 7, 9–13

🟩🟦🟪 **Individual pathway interactivity:** int-4629 learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** List some of the problems you might encounter in trying to collect data from the following populations.
   - **a.** The life of a laptop computer battery
   - **b.** The number of dogs in your neighbourhood
   - **c.** The number of fish for sale at the fish markets
   - **d.** The average number of pieces of popcorn in a bag of popcorn

2. **WE2** The data below show the results of the rolled die from Worked example 2.

6 5 3 1 6 2 3 6 2 5 3 4 1 3 2 6 4 5 5 4 3 1 2 1 6 4 5
2 3 6 1 5 3 3 2 4 1 4 2 3 2 6 3 4 6 2 1 2 4 2

The mean of the population is 3.4. Select your own samples for the following questions.
   - **a.** Select a random sample of 7 scores, and determine the mean of these scores.
   - **b.** Select a second random sample of 7 scores, and determine the mean of these.
   - **c.** Select a third random sample of 20 scores, and determine the mean of these.
   - **d.** Comment on your answers to parts **a**, **b** and **c**.

3. **WE3** In each of the following scenarios, state whether the information was obtained by census or survey. Justify why that particular method was used.
   - **a.** Seating for all passengers is recorded for each aeroplane flight.
   - **b.** Movie ratings are collected from a selection of viewers to discover the best movies for the week.
   - **c.** Every hundredth soft drink bottle off an assembly production line is measured to determine the volume of its contents.
   - **d.** A car driving instructor records the number of hours each learner driver has spent driving.

4. For each of the following, state whether a census or a survey has been used.
   - **a.** Two hundred people in a shopping centre are asked to nominate the supermarket where they do most of their grocery shopping.
   - **b.** To find the most popular new car on the road, 500 new car buyers are asked what make and model they purchased.
   - **c.** To find the most popular new car on the road, data are obtained from the transport department.
   - **d.** Your Year 10 Maths class completed a series of questions on the amount of maths homework for Year 10 students.

Understanding

5. To conduct a statistical investigation, Gloria needs to obtain information from 630 students.
   - **a.** What size sample would be appropriate?
   - **b.** Describe a method of generating a set of random numbers for this sample.

6. A local council wants the opinions of its residents regarding its endeavours to establish a new sporting facility for the community. It has specifically requested all residents over 10 years of age to respond to a set of on-line questions.

a. Is this a census or a survey?

b. What problems could you encounter collecting data this way?

7. A poll was conducted at a school a few days before the election for Head Boy and Head Girl. After the election, it was discovered that the polls were completely misleading. Explain how this could have happened.

8. A sampling error is said to occur when results of a sample are different from those of the population from which the sample was drawn. Discuss some factors which could introduce sampling errors.

Reasoning

9. Since 1961, a census has been conducted in Australia every 5 years. Some people object to the census on the basis that their privacy is being invaded. Others say that the expense involved could be directed to a better cause. Others say that a sample could obtain statistics which are just as accurate. What are your views on this? Justify your statements.

10. Australia has a very small population compared with other countries like China and India. These are the world's most populous nations, so the problems we encounter in conducting a census in Australia would be insignificant compared with those encountered in those countries. What different problems would authorities come across when conducting a census there?

11. The game of Lotto involves picking the same 6 numbers in the range 1 to 45 as have been randomly selected by a machine containing 45 numbered balls. The balls are mixed thoroughly, then 8 balls are selected representing the 6 main numbers, plus 2 extra numbers, called supplementary numbers.

Here is a list of the number of times each number had been drawn over a period of time, and also the number of weeks since each particular number has been drawn.

NUMBER OF WEEKS SINCE EACH NUMBER DRAWN

1	2	3	4	5	6	7	8
1	**5**	**2**	**1**	**1**	**7**	**-**	**4**
9	10	11	12	13	14	15	16
3	**3**	**1**	**5**	**5**	**7**	**-**	**4**
17	18	19	20	21	22	23	24
9	**-**	**9**	**2**	**2**	**12**	**10**	**8**
25	26	27	28	29	30	31	32
5	**11**	**17**	**2**	**3**	**3**	**-**	**22**
33	34	35	36	37	38	39	40
4	**3**	**-**	**1**	**12**	**-**	**6**	**-**
41	42	43	44	45			
6	**1**	**7**	**-**	**31**			

NUMBER OF TIMES EACH NUMBER DRAWN SINCE DRAW 413

1	2	3	4	5	6	7	8
246	**238**	**244**	**227**	**249**	**241**	**253**	**266**
9	10	11	12	13	14	15	16
228	**213**	**250**	**233**	**224**	**221**	**240**	**223**
17	18	19	20	21	22	23	24
217	**233**	**240**	**226**	**238**	**240**	**253**	**228**
25	26	27	28	29	30	31	32
252	**239**	**198**	**229**	**227**	**204**	**230**	**226**
33	34	35	36	37	38	39	40
246	**233**	**232**	**251**	**222**	**221**	**219**	**259**
41	42	43	44	45			
245	**242**	**237**	**221**	**224**			

If these numbers are randomly chosen, explain the differences shown in the tables.

Problem solving

12. A sample of 30 people was selected at random from those attending a local swimming pool. Their ages (in years) were recorded as follows:

19, 7, 58, 41, 17, 23, 62, 55, 40, 37, 32, 29, 21, 18, 16, 10, 40, 36, 33, 59, 65, 68, 15, 9, 20, 29, 38, 24, 10, 30.

a. Find the mean and the median age of the people in this sample.

b. Group the data into class intervals of 10 (0–9 etc.) and complete the frequency distribution table.

c. Use the frequency distribution table to estimate the mean age.

d. Calculate the cumulative frequency and, hence, plot the ogive.

e. Estimate the median age from the ogive.

f. Compare the mean and median of the original data in part **a** with the estimates of the mean and the median obtained for the grouped data in parts **c** and **e**.

g. Were the estimates good enough? Explain your answer.

13. The typing speed (words per minute) was recorded for a group of Year 8 and Year 10 students. The results are displayed in this back-to-back stem plot.

Key: 2 | 6 = 26 wpm

Leaf: *Year* 8	*Stem*	*Leaf:* *Year* 10
9 9	0	
9 8 6 5 4 2 0	1	7 9
9 8 8 6 4 2 1 0 0	2	2 3 6 8 9
9 7 7 6 4 1 0	3	0 2 4 5 5 7 8 8
8 6 5 2 0	4	1 2 5 8 8 9 9
	5	0 3 5 7 8
	6	0 0 3

Write a report comparing the typing speeds of the two groups.

Reflection

A well-known saying about statistics is: *Statistics means never having to say you're certain.* What does this saying mean?

14.3 Primary and secondary data

14.3.1 Primary data

- **Primary data** collection involves collecting **data** yourself.
- This means that you have ownership of the data, and no one else has access to the data until it is released or published.
- A variety of methods for collecting primary data are used. These include observation, measurement, survey, experiment or simulation.

14.3.2 Observation

- Observation involves recording the behavioural patterns of people, objects and events in a systematic manner.
- The data are collected as a disguised observation (respondents are unaware they are being observed) or undisguised observation (the respondent is aware). Video surveillance cameras are an example of people knowing that their movements are being recorded, but are not always aware of where the recording takes place.
- Observations can be in a natural environment (for example, in a food hall), or a contrived environment (a food-tasting session for a food company).
- Mechanical devices (video cameras, closed circuit television, counting devices across a road) can also be used.

14.3.3 Measurement

- Measurement involves using some measuring device to collect data.
- This generally involves conducting an experiment of some type.
  - The height of everyone in your class can be measured.
  - The mass of all new-born babies can be collected.
  - A pedometer measures the number of steps the wearer takes.

14.3.4 Surveys

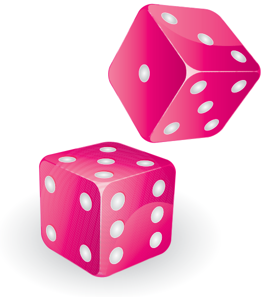
- Surveys involve designing a questionnaire to interview people. Often the questionnaire requires many rewrites to obtain one which is clear and unbiased.
- The interview can be in person — face to face or by telephone. The advantage of this method is that you are able to gauge the reactions of those you are interviewing, and explain particular questions, if necessary.
- Most frequently these days, email is used to survey participants; however, there are advantages and disadvantages to using this type of survey.
  - Advantages:
    - Can cover a large number of people or organisations
    - A wide geographic coverage is possible.
    - It avoids embarrassment on the part of the respondent.
    - There is no interviewer bias.
    - The respondent has time to consider responses.
    - It's relatively cheap.
  - Disadvantages:
    - The questions have to be relatively simple.
    - The response rate is often quite low (inducements often given as an incentive to return).
    - The reliability of the answers is questionable.
    - No control over who actually completes the questionnaire
    - Problems with incomplete questionnaires

14.3.5 Experiment

- Generally, when conducting an experiment the data collected are quantitative.
- Particular care should be taken to ensure that the experiment is conducted in a manner which would produce similar results if repeated.
- Care must be taken with the recording of results.
- The results must be in a form which can readily be analysed.
- All results need to be recorded, including the *weird* or unexpected outcomes.

14.3.6 Simulation

- Experiments such as rolling a die, tossing a coin or drawing a card from a deck may be conducted to model some real-life situation.
- Simulations occur in areas such as business, engineering, medical and scientific research.
- They are often used to imitate real-life situations which may be dangerous, impractical or too expensive to explore by other means.
- Before collecting any primary data, it must be clear what data are to be collected.



WORKED EXAMPLE 4

It is widely believed that there is equal chance of having a boy or girl with each birth. Genetics and the history of births in a family sometimes have a great influence on the sex of the child as well. Ignore those factors in this question.

a Design an experiment to simulate the chance of giving birth to a boy or a girl.

b Describe how your experiment could be conducted to determine the number of children a couple should have, on average, to ensure they have offspring of both sexes.

THINK	WRITE
a Use a device that can simulate two outcomes which are equally likely. This could be a random number generator to generate two integers, say a 0 (representing a boy) and 1 (representing a girl). A fair coin could be tossed, such that, a Head represents a boy, and a Tail represents a girl.	**a** A fair coin will be tossed with a Head representing a boy (B), and a Tail representing a girl (G).
b 1 Describe how the experiment will be conducted.	**b** The experiment will be conducted 50 times, and a record kept of each experiment. For each experiment, the coin will be tossed until both sexes result. This may mean that there could be for example 7 trials in an experiment (GGGGGGB) before both sexes are represented.

2 Display the table of results.

The table below shows the results of the 50 experiments.

Exp. no.	Results	No. of trials	Exp. no.	Results	No. of trials
1	BG	2	26	GGGB	4
2	GGB	3	27	GGGGB	5
3	BG	2	28	GGGB	4
4	GGGGB	5	29	BG	2
5	BBBBBBG	7	30	BBBG	4
6	GGGB	4	31	BG	2
7	BBG	3	32	GB	2
8	BBG	3	33	GGGB	4
9	BBBBG	5	34	BG	2
10	GB	2	35	GGGGGGB	7
11	BG	2	36	BBBBBBG	7
12	GGGB	4	37	GB	2
13	BBG	3	38	BG	2
14	BBG	3	39	GGB	3
15	GB	2	40	GGGGB	5
16	BG	2	41	BBG	3
17	GGB	3	42	BBBBBG	6
18	GB	2	43	GGB	3
19	GGB	3	44	GGB	3
20	BBBG	4	45	BBBG	4
21	BG	2	46	BBG	3
22	GB	2	47	GGGGGGB	7
23	GGGGB	5	48	BG	2
24	BG	2	49	BBG	3
25	GGGGB	5	50	GGGGGB	6
				Total	175

This table shows that 175 trials were undertaken in 50 experiments where each experiment resulted in both sexes.

3 Determine the average number of children required to produce offspring of both sexes.

$$\text{Average number of children } = \frac{175}{50} = 3.5$$

4 Write a conclusion.

The average number of children a couple should have to reach the goal of having both sexes is 4.

WORKED EXAMPLE 5

You have been asked to obtain primary data to determine the methods of transport used to travel to school by the students at your school. The data collected are to provide support for the Student Council's proposal for a school bus.

a What data should be collected?

b Outline possible methods which could be used to collect the data.

c Decide which method you consider to be the best option, and discuss its advantages and disadvantages.

THINK	WRITE
a Outline the various forms of transport available to the students.	**a** The modes of transport available to students at the school are: car, bus, train, bicycle and walking
b Consider all the alternatives for collecting the data.	**b** Several methods could be used to collect the data. • Could stand at the school gate one morning and ask students as they arrive. • A questionnaire could be designed. • Students could be asked to write their mode of transport on a piece of paper and place in a collection tin.
c 1 Decide on the best option.	**c** The first option of standing at the school gate is very time-consuming, and students could arrive at the back gate. The third option does not seem reliable, as some students may not comply, and other students may place multiple pieces of paper in the collection tin. The second option seems the best of the three.
2 Discuss advantages and disadvantages.	The advantages of a questionnaire include: • There is a permanent record on paper. • It is not time-consuming to distribute or collect. • Students can complete it at their leisure. Disadvantages include: • Students may not return it. • Expense involved in producing copies.

- A decision must be made as to the method of collection.
- The advantages and disadvantages of the collection method must be acknowledged.
- The reason for the data collection should be clear from the outset.

Note: This example does not represent the views of all those collecting such data. It merely serves to challenge students to explore and discuss available options.

14.3.7 Secondary data

- **Secondary data** are data that have already been collected by someone else.
- The data can come from a variety of sources:
  - Paper — books, journals, magazines, company reports
  - Electronic — online databases, internet, broadcasts, DVDs
  - Government sources — ABS provides a wealth of statistical data.
  - General business sources — academic institutions, stockbroking firms, sporting clubs
  - Media — newspapers, TV reports.
- Secondary data sources often provide data that would not be possible for an individual to collect.
- The data can be qualitative or quantitative.

- The accuracy and reliability of the data sometimes needs to be questioned, depending on its source.
- The age of the data should always be considered.
- Often the data that surrounds us passes by unnoticed.
- It is important to learn the skills to be able to critically analyse secondary data.

WORKED EXAMPLE 6

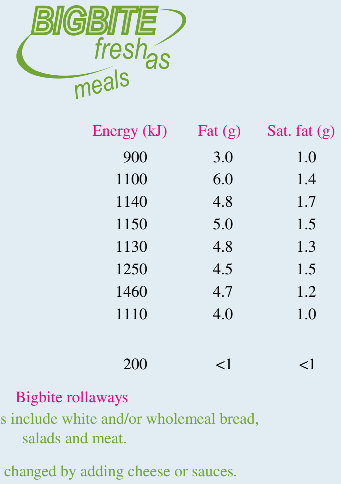
Bigbite advertises the energy and fat content of some of their rollaways on their placemats.

a What information can you gain from these data?

b Bigbite advertise that they have a range of rollaways with less than 6 grams of fat. Comment on this claim.

c This could be the starting point of a statistical investigation. How could you proceed from here?

d Investigations are not conducted simply for the sake of investigating. Suggest some aims for investigating further.



BIGBITE fresh as meals

Bigbite fresh as rollaways	Energy (kJ)	Fat (g)	Sat. fat (g)
Roasted vegetable	900	3.0	1.0
HAM	1100	6.0	1.4
TURKEY	1140	4.8	1.7
BBQ BEEF	1150	5.0	1.5
Bigbite ribbon	1130	4.8	1.3
TURKEY AND HAM	1250	4.5	1.5
BBQ CHICKEN	1460	4.7	1.2
CHICKEN Tandoori	1110	4.0	1.0
Fresh as Dessert			
Fruit Slices	200	<1	<1

Bigbite rollaways

Regular rollaways include white and/or wholemeal bread, salads and meat.

Nutritional value is changed by adding cheese or sauces.

THINK	WRITE
a Look at the data on the placemat to gain as much information as possible.	**a** The placemat reveals the following information: • A higher energy content of a rollaway does not necessarily mean that its fat content is higher. • As the fat content of a rollaway increases, generally the saturated fat content also increases. • The addition of some types of protein (ham, turkey, beef, chicken) increases the energy content of the rollaway. • These data are only for those rollaways on white or wholemeal bread with salads and meat. • The addition of condiments (sauces) or cheese will alter these figures. • A fruit slice has much less energy and fat than a rollaway.
b Examine the data to discover if there is evidence to support the claim. Make further comment.	**b** All the rollaways displayed have less than 6 grams of fat, so Bigbite's claim is true. It must be remembered that the addition of cheese and sauce to these rollaways would increase their fat content. Also, if the rollaway was on any bread other than white or wholemeal, the fat content could go beyond 6 grams.
c What would be the next step in the investigation?	**c** The placemat displays a toll-free phone number for further information. Their website also contains additional detailed information.
d What are some interesting facts which could be revealed through a deeper investigation?	**d** Suggested aims for investigating further could be: • How much extra fat is added to a rollaway by the addition of cheese and/or sauce? • What difference does a different type of bread make to the fat content of the rollaway? • Which rollaway contains the highest fat content? • What is the sugar content of the rollaways?

Exercise 14.3 Primary and secondary data

assess**on**

Individual pathways

🟩 **PRACTISE**	🟦 **CONSOLIDATE**	🟪 **MASTER**
Questions: 1–8, 11, 14	Questions: 1–9, 11, 14	Questions: 1, 3, 6, 7, 10–14

🟩🟦🟪 **Individual pathway interactivity:** int-4630 learn**on** ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE4** Devise an experiment to simulate each of the following situations and specify the device used to represent the outcomes.
   - **a.** A true/false test in which answers are randomly distributed.
   - **b.** A casino game with outcomes grouped in colours of either red or black.
   - **c.** Breakfast cereal boxes containing 4 different types of plastic toys.
   - **d.** In a group of six people, one person is to be chosen as the leader.
   - **e.** A choice of three main meals on a restaurant's menu, all of which are equally popular.
   - **f.** Five possible holiday destinations offered by a travel agent; such that all destinations are equally available and equally priced.
   - **g.** Five types of takeaway fast foods available in one area, where one pizza is twice as popular as each of the other types of takeaway food (the other 4 are equally popular).
2. **WE5** You have been asked to obtain primary data from students at your school to determine internet access students have at home. The data collected are to provide support for opening the computer room for student use at night.
   - **a.** What data should be collected?
   - **b.** Outline possible methods which could be used to collect these data.
   - **c.** Decide which method you consider to be the best option, and discuss its advantages and disadvantages.
3. **WE6** This label shows the nutrition information of Brand X rolled oats.

Nutrition information			
Servings per package: 25			Serving size 30 g
	Per serving 30 g	%DI* per serving	Per 100 g
Energy	486 kJ	6%	1620 kJ
Protein	4.3 g	9%	14.3 g
Fat - Total	2.8 g	4%	9.3 g
- Saturated	0.5 g	2%	1.7 g
- Trans	Less than 0.1 g	-	Less than 0.1 g
- Polyunsaturated	1.0 g	-	3.2 g
- Monounsaturated	1.3 g	-	4.4 g
Carbohydrate	16.8 g	5%	56 g
- Sugars	0.9 g	1%	3.0 g
Dietary fibre	3.1 g	10%	10.4 g
Sodium	0.7 mg	0.1%	2 mg

* % DI = Percentage daily intake

   - **a.** What information can you gain from the data?
   - **b.** This could be the starting point of a statistical investigation. How could you proceed from here?
   - **c.** Suggest some aims for investigating further.

4. **a.** Provide a list of methods you could use to collect primary data.
   b. Describe which method you would use to collect the following primary data.
      i. Heights of trees along the footpaths of a tree-lined street
      ii. Number of buses that transport students to your school in the morning
      iii. Sunrise times during summer
      iv. Student opinion regarding length of lessons

Understanding

For questions **5** and **6**, design an experiment to simulate the situation, carry out the experiment and give the results of the experiment.

5. A mouse in a maze can make left or right turns at each junction. Assuming each turn is equally likely, how many junctions on average must the mouse go through before each type of turn will have been made?
6. A restaurant menu features 4 desserts which are assumed to be equally popular. How many dessert orders must be filled (on average) before the owner can be sure all types will have been ordered?
7. This label shows the nutrition information of Brand Y rolled oats.

Nutrition information

Servings per package: 30
Serving size: 30 g

	Avg. quantity per serving 30 g	Avg. quantity per 100 g
Energy	480 kJ (115 Cal)	1600 kJ (383 Cal)
Protein	3.2 g	10.5 g
Fat, total	2.4 g	8.0 g
- saturated	LESS THAN 1 g	1.5 g
Carbohydrate	18.3 g	61.0 g
- sugars	0.0 g	0.0 g
Dietary Fibre, total	3.3 g	10.0 g
Sodium	LESS THAN 5 mg	LESS THAN 5 mg

Ingredients

Oats (100%)

Attention

THIS PRODUCT CONTAINS GLUTEN.

Storage

Store in a cool, dry place.

Compare the nutrition information with that on the Brand X label in question **3**.

8. Comment on this claim.

> We did a survey on 100 people regarding eating chocolate.
> 60 of these people said they regularly ate chocolate.
> We then measured the heights of all 100 people.
> **** The result ****
> Eating chocolate makes you taller!!

9. Russel operates a computer software sales outlet. He keeps a log of all complaints from customers. Suggest how he could organise his log.

10. The following claim has been made regarding secondary data.
There's a lot more secondary data than primary data, it's a lot cheaper and it's easier to acquire.
Comment on this statement.

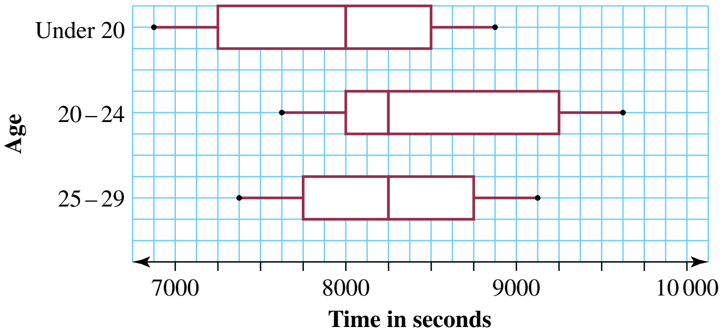
Reasoning

11. The local Bed Barn was having a sale on selected beds by Sealy and Sleepmaker. Four of the beds on sale were:

Sealy Posturepremier	on sale for \$1499	a saving of \$1000
Sealy Posturepedic	on sale for \$2299	a saving of \$1600
Sleepmaker Casablanca	on sale for \$1199	a saving of \$800
Sleepmaker Umbria	on sale for \$2499	a saving of \$1800

The store claimed that all these beds had been discounted by 40%. Comment on whether this statement is true, supporting your comments with sound mathematical reasoning.

12. The following data give the boxplots for three different age groups in a triathlon for under thirties.



a. What was the slowest time for the 20–24 year olds?

b. Estimate the difference in time between the fastest triathlete in:

- **i.** the under 20s and the 20–24 group
- **ii.** the under 20s and the 25–29 group
- **iii.** the 20–24 group and the 25–29 group.

c. What conclusion could you draw about the overall performance between the three groups?

Problem solving

13. This taste test on corn chips appeared in a newspaper.

TASTE TEST: CORN CHIPS

Byron Bay Chilli Co. Corn chips

230 g \$3.40 (\$1.48 per 100 g)

- Made in Australia
- Fat 24.9 g/100 g
- Saturated fat 11.8 g/100 g
- Sodium 44 mg/100 g

Verdict: Crisp, thick chips with fresh corn flavour and low sodium content. No preservatives, no GM corn. Put Byron Bay in the title and things cost more!

IGA Black and Gold Plain Corn chips

230 g \$1.99 (87 cents per 100 g)

- Made in Australia
- Fat 24.9 g/100 g
- Saturated fat 11.8 g/100 g
- Sodium 415 mg/100 g

Verdict: Cheap in comparison and quite nice. A good budget option as they are as good as the popular branded ones.

CCs Corn Chips

200 g $2.69 ($1.34 per 100 g)

- Made in Australia
- Fat 24.4 g/100 g
- Saturated fat 10.6 g/100 g
- Sodium 550 mg/100 g

Verdict: Chips are quite thin and very salty to taste. Very high sodium content.

Doritos Corn Chips

200 g $2.99 ($1.50 per 100 g)

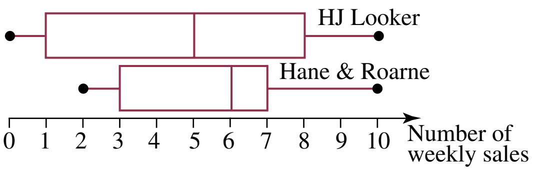
- Made in Australia
- Fat 23.2 g/100 g
- Saturated Fat 11.2 g/100 g
- Sodium 435 mg/100 g

Verdict: Very crisp and fresh, but really no better than the home brand which is $1 cheaper.

Source: *The Sunday Mail*, 4 Apr. 2010, p. 26.

Comment on the information displayed.

14. These parallel boxplots show the number of weekly house sales by two real estate agencies over a 3-month period.



Prepare a report to compare the performance of the two agencies.

Reflection

When using secondary data from other countries, what different unit classifications could you encounter?

CHALLENGE 14.1

A fisheries and wildlife officer released 200 tagged trout into a lake. A week later, the officer took a sample of 50 trout and found that 8 of them were tagged. The officer can use this information to estimate the population of trout in the lake. How many trout are now in the lake?

14.4 Evaluating inquiry methods and statistical reports

- Statistical investigations involve collecting data, recording the data, analysing the data then reporting the results.

14.4.1 Data collection methods

- Collection methods involve gathering primary data, or using secondary data from stored records.
- Primary data can be collected by observation, measurement, survey, experiment or simulation as discussed earlier.

- Secondary data can be collected electronically or via a hard copy.
- It is important to be able to justify the particular method chosen for each of these processes.
- Sometimes alternative methods are just as appropriate.

WORKED EXAMPLE 7

You have been given an assignment to investigate which year level uses the school library, after school, the most.

a Explain whether it is more appropriate to use primary or secondary data in this case. Justify your choice.
b Describe how the data could be collected. Discuss any problems which might be encountered.
c Explain whether an alternative method would be just as appropriate.

THINK	WRITE
a No records have been kept on library use.	**a** Since records are not kept on the library use, secondary data is not an option. Primary data collection could be either sampling or census. A sufficiently large sample size could be chosen; this would take less time than conducting a census, although it would not be as accurate. Sampling would be considered appropriate in this case.
b The data can be collected via a questionnaire or in person.	**b** A questionnaire could be designed and distributed to a randomly-chosen sample. The problem here would be the non-return of the forms. Observation could be used to personally interview students as they entered the library. This would take more time, but random interview times could be selected.
c A census is the other option.	**c** A census could be conducted, either by questionnaire or observation. This should yield a more accurate outcome.

WORKED EXAMPLE 8

Which method would be the most appropriate to collect the following data? Suggest an alternative method in each case.

a The number of cars parked in the staff car park each day
b The mass of books students carry to school each day
c The length a spring stretches when weights are added to it
d The cost of mobile phone plans with various network providers

THINK	WRITE
a Observation	**a** The best way would probably be observation by visiting the staff car park to count the number of cars there. An alternative method would be to conduct a census of all workers to ask if they parked in the staff car park. This is probably not as good.

b Measurement	**b** The mass of the books could be measured by weighing each student's pack on scales. A random sample would probably yield a reasonably accurate result.
c Experiment	**c** Conduct an experiment and measure the extension of the spring with various weights. There is probably no alternative to this method.
d Internet search	**d** An internet search would enable data to be collected. Alternatively, a visit to mobile phone outlets would yield similar results.

14.4.2 Analysing the data

- Once the data have been collected and collated, a decision must be made with regard to the best methods for analysing the data.
  - a measure of *central tendency* — mean, median or mode
  - a measure of spread — range, interquartile range or standard deviation
  - an appropriate graph.

Statistical graphs

- Data can be graphed in a variety of ways — line graphs, bar graphs, histograms, stem plots, box plots, etc. These have all been discussed in detail previously.
- In media reports it is common to see line and bar graphs.
- Because graphs give a quick visual impression, the temptation is to not look at them in great detail. Often graphs can be quite misleading.
- It is easy to manipulate a graph to give an impression which is supported by the creator of the graph. This is achieved by careful choice of scale on the horizontal and vertical axes.
  - Shortening the horizontal axis tends to highlight the increasing/decreasing nature of the trend of the graph. Lengthening the vertical axis tends to have the same effect.
  - Lengthening the horizontal and shortening the vertical axes tends to level out the trends.

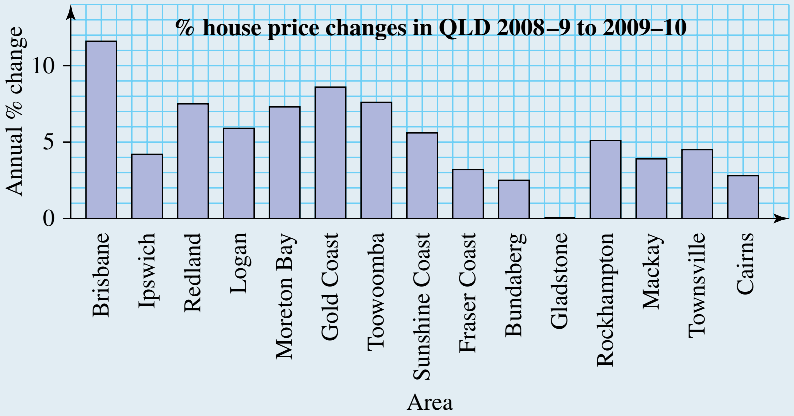
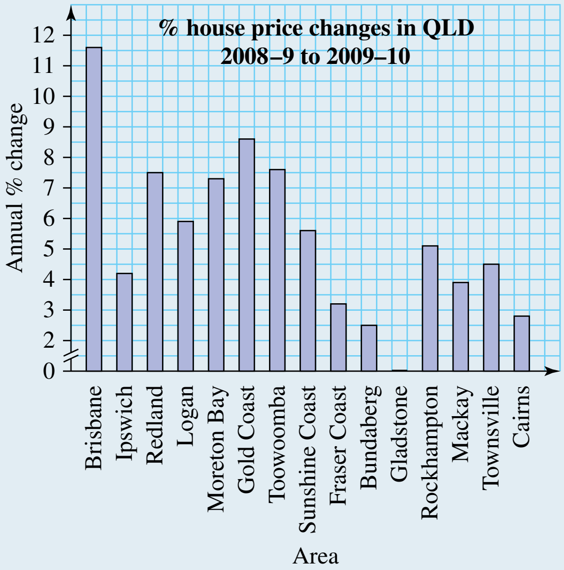
WORKED EXAMPLE 9

This report shows the annual change in median house prices in the local government areas (LGA) of Queensland from 2008–09 to 2009–10.

a Draw a bar graph which would give the impression that the percentage annual change was much the same throughout the whole state.

b Construct a bar graph to give the impression that the percentage annual change in Brisbane was far greater than that in the other local government areas.

HOUSES **Suburb/locality**	**Median house price**		**Annual change**
	2009–10	**2008–09**	
Brisbane (LGA)	\$530,000	\$475,000	11.6%
Ipswich City (LGA)	\$323,000	\$310,000	4.2%
Redland City (LGA)	\$467,500	\$435,000	7.5%
Logan City (LGA)	\$360,000	\$340,000	5.9%
Moreton Bay (LGA)	\$399,000	\$372,000	7.3%
Gold Coast City (LGA)	\$505,000	\$465,000	8.6%
Toowoomba (LGA)	\$289,500	\$269,000	7.6%
Sunshine Coast (LGA)	\$470,000	\$445,000	5.6%
Fraser Coast (LGA)	\$307,400	\$297,750	3.2%
Bundaberg (LGA)	\$282,000	\$275,000	2.5%
Gladstone (LGA)	\$370,000	\$370,000	0.0%
Rockhampton (LGA)	\$315,250	\$300,000	5.1%
Mackay (LGA)	\$398,000	\$383,000	3.9%
Townsville City (LGA)	\$375,000	\$359,000	4.5%
Cairns (LGA)	\$365,000	\$355,000	2.8%

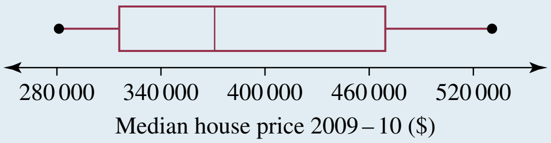
THINK	WRITE/DRAW
a To flatten out trends, lengthen the horizontal axis and shorten the vertical axis.	**a** 
b To accentuate trends, shorten the horizontal axis and lengthen the vertical axis.	**b** 

WORKED EXAMPLE 10

Consider the data displayed in the table of Worked example 9. Use the data collected for the median house prices in 2009–10.

a Explain whether these data would be classed as primary or secondary data.
b Why do the data show median house prices rather than the mean or modal house price?
c Calculate a measure of central tendency for the data. Explain the reason for this choice.
d Give a measure of spread of the data, giving a reason for the particular choice.
e Display the data in a graphical form, explaining why this particular form was chosen.

THINK	WRITE
a These are data which have been collected by someone else.	**a** These are secondary data because they have been collected by someone else.
b Median is the middle price, mean is the average price, and mode is the most frequently-occurring price.	**b** The median price is the middle one. It is not affected by outliers as the mean is. The modal house price may only occur for two house sales with the same value. On the other hand, there may not be any mode. The median price is the most appropriate in this case.

c Which measure of central tendency is the most appropriate one?	**c** The measures of central tendency are the mean, median and mode. The mean is affected by high values (i.e. \$530 000) and low values (i.e. \$282 000). These are not typical values, so the mean would not be appropriate. There is no modal value, as all the house prices are different. The median house price is the most suitable measure of central tendency to represent the house prices in the Queensland local government areas. The median value is \$370 000.
d Consider the range and the interquartile range as measures of spread.	**d** The five-number summary values are: Lowest score = \$282 000 Lowest quartile = \$315 250 Median = \$370 000 Upper quartile = \$467 500 Highest score = \$530 000 Range = \$530 000 − \$282 000 = \$248 000 Interquartile range = \$467 500 − \$315 250 = \$152 250 The interquartile range is a better measure for the range as the house prices form a cluster in this region.
e Consider the graphing options.	**e** Of all the graphing options, the boxplot seems the most appropriate as it shows the spread of the prices as well as how they are grouped around the median price. 

WORKED EXAMPLE 11

TI | CASIO

The Australian women's national basketball team, the Opals, competed at the 2008 Olympic Games in Beijing, winning a silver medal. These are the heights (in metres) of the 12 team members:

1.73, 1.65, 1.8, 1.83, 1.96, 1.88, 1.63, 1.88, 1.83, 1.88, 1.8, 1.96

Provide calculations and explanations as evidence to verify or refute the following statements.

a The mean height of the team is greater than their median height.

b The range of the heights of the 12 players is almost 3 times their interquartile range.

c Only 5 players are on the court at any one time. A team of 5 players can be chosen such that their mean, median and modal heights are all the same.

THINK	WRITE
a 1 Calculate the mean height of the 12 players.	**a** Mean $= \frac{\Sigma x}{n} = \frac{21.83}{12} = 1.82$ m

2	Order the heights to determine the median.	The heights of the players, in order, are: 1.63, 1.65, 1.73, 1.8, 1.8, 1.83, 1.83, 1.88, 1.88, 1.88, 1.96, 1.96 There are 12 scores, so the median is the average of the 6th and 7th scores. $\text{Median} = \frac{1.83 + 1.83}{2} = 1.83 \text{ m}$
3	Comment on the statement.	The mean is 1.82 m, while the median is 1.83 m. This means that the mean is less than the median, so the statement is not true.
b 1	Determine the range and the interquartile range of the 12 heights.	**b** $\text{Range} = 1.96 - 1.63 = 0.33 \text{ m}$ Lower quartile is the average of 3rd and 4th scores. $\text{Lower quartile} = \frac{1.73 + 1.8}{2} = 1.765 \text{ m}$ Upper quartile is average of 3rd and 4th scores from the end. $\text{Upper quartile} = \frac{1.88 + 1.88}{2} = 1.88 \text{ m}$ $\text{Interquartile range} = 1.88 - 1.765 = 0.115 \text{ m}$
2	Compare the two values.	$\text{Range} = 0.33 \text{ m}$ $\text{Interquartile range} = 0.115 \text{ m}$ $\frac{\text{Range}}{\text{Interquartile range}} = \frac{0.33}{0.115} = 2.9$
3	Comment on the statement.	$\text{Range} = 2.9 \times \text{interquartile range}$ This is almost 3 times, so the statement is true.
c 1	Choose 5 players whose mean, median and modal heights are all equal. Trial and error is appropriate here. There may be more than one answer.	**c** Three players have a height of 1.88 m. If a player shorter and one taller are chosen, both the same measurement from 1.88 m, this would make the mean, median and mode all the same. Choose players with heights: 1.8, 1.88, 1.88, 1.88, 1.96 $\text{Mean} = \frac{9.4}{5} = 1.88 \text{ m}$ $\text{Median} = \text{3rd score} = 1.88 \text{ m}$ $\text{Mode} = \text{Most frequent score} = 1.88 \text{ m}$
2	Comment on the statement.	The 5 players with heights 1.8 m, 1.88 m, 1.88 m, 1.88 m, 1.96 m have a mean, median and modal height of 1.88 m. It is true that a team of 5 such players can be chosen.

14.4.3 Statistical reports

- Reported data must not be simply taken at face value; all reports should be examined with a critical eye.

WORKED EXAMPLE 12

This is an excerpt from an article that appeared in a newspaper on Father's Day. It was reported to be a national survey findings of a *Gallup Poll* of data from 1255 fathers of children aged 17 and under.

THE GREAT AUSSIE DADS SURVEY

Thinking about all aspects of your life, how happy would you say you are?

	%
I am very happy	**26**
I am fairly happy	**49**
Totally happy	**75**
Some days I'm happy and some days I'm not	**21**
I am fairly unhappy	**3**
I am very unhappy	**1**
Totally unhappy	**4**

How often, if ever, do you regret having children?

Every day	**1**
Most days	**2**
Some days	**18**
Never	**79**

Which one of these best describes the impact of having children on your relationship with your partner?

We're closer than ever	**29**
We don't spend as much time together as we should	**40**
We're more like friends now than lovers	**21**
We have drifted apart	**6**
None of the above	**4**

Which one of these best describes the allocation of cooking and cleaning duties in your household?

My partner does nothing/I do everything	**1**
I do most of it	**11**
We share the cooking and cleaning	**42**
My partner does most of it	**41**
I do nothing/my partner does everything	**4**
None of the above	**1**

Which of these aspects of your children's future do you have concerns about?

	%
Their safety	**70**
Being exposed to drugs	**67**
Their health	**54**
Bullying or cyber-bullying	**50**
Teenage violence	**50**
Their ability to afford a home	**50**
Alcohol consumption and binge drinking	**47**
Achieving academic success	**47**
Feeling pressured into sex	**41**
Being able to afford the lifestyle they expect to have	**38**
Climate change	**23**
Having them living with you in their mid 20s	**14**
None of the above	**3**

What is the best thing about being a dad?

The simple pleasures of family life	**61**
Enjoying the successes of your kids	**24**
The unpredictability it brings	**9**
The comfort of knowing that you will be looked after in later life	**3**
None of the above	**3**

Key findings

75%	of Aussie dads are totally happy
79%	have never regretted having children
67%	are worried about their children being exposed to drugs
57%	would like more intimacy with their partner

"Work–life balance is definitely an issue for dads in 2010."

David Briggs
Galaxy principal

Source: *The Sunday Mail,* 5 Sept. 2010, pp. 14–15.

a **Comment on the sample chosen.**
b **Discuss the percentages displayed.**
c **Comment on the claim that 57% of dads would like more intimacy with their partner.**

THINK	WRITE
a How is the sample chosen? Is it truly representative of the population of Australian dads?	**a** The results of a national survey such as this should reveal the outlook of the whole nation's dads. There is no indication of how the sample was chosen, so without further knowledge we tend to accept that it is representative of the population. A sample of 1255 is probably large enough.
b Look at the percentages in each of the categories.	**b** For the first question regarding happiness, the percentages total more than 100%. It seems logical that, in a question such as this, the respondents would tick only one box, but obviously this has not been the case. In the question regarding aspects of concern of 'your children's future', these percentages also total more than 100%. It seems appropriate here that dads would have more than one concerning area, so it is possible for the percentages to total more than 100%. In each of the other three questions, the percentages total 100%, which is appropriate.
c Look at the tables to try to find the source of this figure.	**c** Examining the reported percentages in the question regarding 'relationship with your partner', there is no indication how a figure of 57% was determined.

Note: Frequently media reports make claims where the reader has no hope of confirming their truth.

WORKED EXAMPLE 13

This article appeared in a newspaper. Read the article, then answer the following questions.

SPONGES ARE TOXIC

Washing dishes can pose a serious health risk, with more than half of all kitchen sponges containing high levels of dangerous bacteria, research shows.

A new survey dishing the dirt on washing up shows more than 50 per cent of kitchen sponges have high levels of *E.coli*, which can cause severe cramps and diarrhoea, and *staphylococcus aureus*, which releases toxins that can lead to food poisoning or toxic shock syndrome.

Microbiologist Craig Andrew-Kabilafkas of Australian Food Microbiology said the Westinghouse study of more than 1000 households revealed germs can spread easily to freshly washed dishes.

The only way to safeguard homes from sickness was to wash utensils at very high temperatures in a dishwasher.

a **Comment on the sample used in this survey.**
b **Comment on the claims of the survey.**
c **Is the heading of the article appropriate?**

Source: *The Sunday Mail,* 5 Sept. 2010, p. 36.

THINK	WRITE
a Look at sample size and selection of sample.	**a** The report claims that the sample size was more than 1000. There is no indication how the sample was selected. The point to keep in mind is whether this sample is truly representative of the population consisting of all households. We have no way of knowing.
b What are the results of the survey?	**b** The survey claims that 50% of kitchen sponges have high levels of *E. coli* which can cause severe medical problems. The study was conducted by Westinghouse, so it is not surprising they recommend using a dishwasher.
c Examine the heading in the light of the contents of the article.	**c** The heading is sensational, designed to catch the attention of readers.

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Exercise 14.4 Evaluating inquiry methods and statistical reports

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Individual pathways

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Questions: 1–4, 6, 8–10, 12	Questions: 1–4, 7–10, 12	Questions: 1–13

🟩🟦🟪 **Individual pathway interactivity:** int-4631

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Fluency

1. WE7, 8 You have been given an assignment to investigate which Year level has the greatest number of students who are driven to school each day by car.
   a. Explain whether it is more appropriate to use primary or secondary data in this case. Justify your choice.
   b. Describe how the data could be collected. Discuss any problems which might be encountered.
   c. Explain whether an alternative method would be just as appropriate.

2. WE9 You run a small company that is listed on the Australian Stock Exchange (ASX). During the past year you have given substantial rises in salary to all your staff. However, profits have not been as spectacular as in the year before. This table gives the figures for the salary and profits for each quarter.

	1st quarter	2nd quarter	3rd quarter	4th quarter
Profits ($’000 000)	6	5.9	6	6.5
Salaries ($’000 000)	4	5	6	7

Draw two graphs, one showing profits, the other showing salaries, which will show you in the best possible light to your shareholders.

3. **WE10** The data below were collected from a real estate agent and show the sale prices of ten blocks of land in a new estate.
$150 000, $190 000, $175 000, $150 000, $650 000, $150 000, $165 000, $180 000, $160 000, $180 000
   a. Calculate a measure of central tendency for the data. Explain the reason for this choice.
   b. Give a measure of spread of the data, giving a reason for the particular choice.
   c. Display the data in a graphical form, explaining why this particular form was chosen.
   d. The real estate agent advertises the new estate land as:
   Own one of these amazing blocks of land for only $150 000 *(average)!*
   Comment on the agent's claims.

4. **WE11** Use the data for the heights of the Opal players in Worked example 11 (page 611) to answer the following question.
Provide calculations and explanations as evidence to verify or refute the following statements.
   a. The mean height of the team is closer to the lower quartile than it is to the median.
   b. Half the players have a height within the interquartile range.
   c. Which 5 players could be chosen to have the minimum range in heights?

5. The table below shows the number of shoes of each size that were sold over a week at a shoe store.

Size	Number sold
4	5
5	7
6	19
7	24
8	16
9	8
10	7

   a. Calculate the mean shoe size sold.
   b. Determine the median shoe size sold.
   c. Determine the modal shoe size sold.
   d. Explain which measure of central tendency has the most meaning to the store proprietor.

6. The resting pulse of 20 female athletes was measured and is shown below.
50 62 48 52 71 61 30 45 42 48 43 47 51 52 34 61 44 54 38 40
   a. Represent the data in a distribution table using appropriate groupings.
   b. Find the mean, median and mode of the data.
   c. Comment on the similarities and differences between the three values.

Understanding

7. The batting scores for two cricket players over six innings were recorded as follows.

Player A 31, 34, 42, 28, 30, 41

Player B 0, 0, 1, 0, 250, 0

Player B was hailed as a hero for his score of 250.

Comment on the performance of the two players.

8. A small manufacturing plant employs 80 workers. The table below shows the structure of the plant.

Position	Salary ($)	Number of employees
Machine operator	18 000	50
Machine mechanic	20 000	15
Floor steward	24 000	10
Manager	62 000	4
Chief Executive Officer	80 000	1

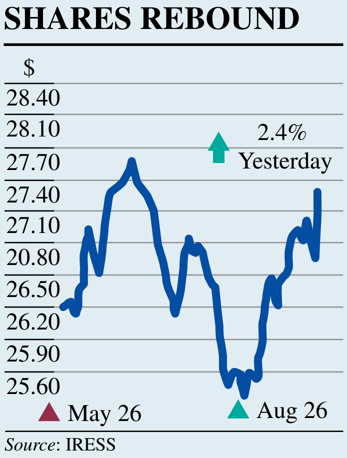
a. Workers are arguing for a pay rise, but the management of the factory claims that workers are well paid because the mean salary of the factory is $22 100. Explain whether this is a sound argument.

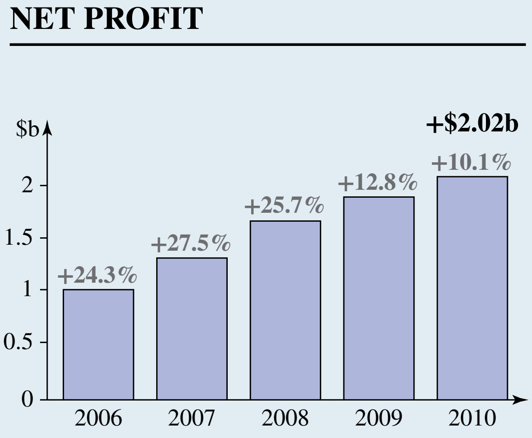
b. Suppose that you were representing the factory workers and had to write a short submission in support of the pay rise. How could you explain the management's claim? Provide some other statistics to support your case.

9. **WE12, 13** This report from Woolworths appeared in a newspaper.

IT'S A RECORD

- Woolworths posted **10.1**% gain in annual profit to $**2.02b**
- **11**th consecutive year of double-digit growth
- Flags **8**% to **11**% growth in the current financial year
- Sales rose **4.8**% to $**51.2b**
- Wants to increase its share of the fresh food market
- Announced $**700m** off-market share buyback
- Final fully franked dividend **62**% a share





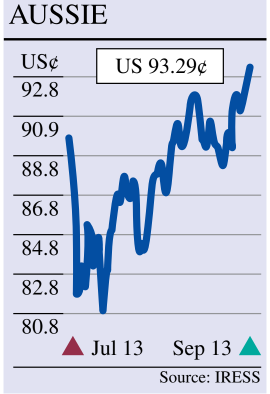
Source: *The Courier Mail*, 27 Aug. 2010, pp. 40–1.

Comment on the report.

Reasoning

10. The graph at right shows the fluctuation in the Australian dollar in terms of the US dollar during the period 13 July to 13 September 2010. The higher the Australian dollar, the cheaper it is for Australian companies to import goods from overseas, and the cheaper they should be able to sell their goods to the Australian public.

    The manager of Company XYZ produced a graph to support his claim that, because there hasn't been much change in the Aussie dollar over that period, there hasn't been any change in the price he sells his imported goods to the Australian public. Draw a graph that would support his claim. Explain how you were able to achieve this effect.



Source: The Courier Mail, 14 Sept. 2010, p. 25.

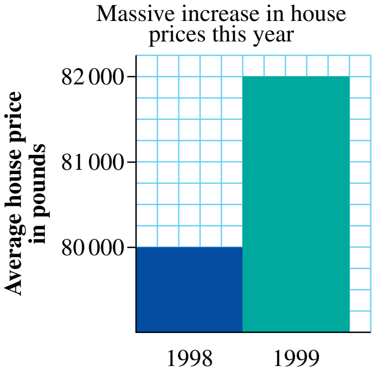
11. Two brands of light globes were tested by a consumer organisation. They obtained the following results.

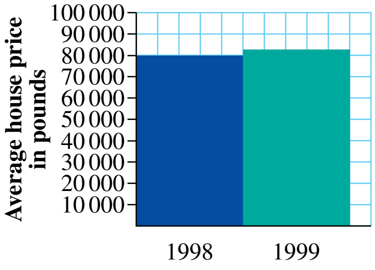
Brand A (Hours lasted)	**Brand B (Hours lasted)**
385 390 425 426 570	**500 555 560 630 720**
640 645 730 735 760	**735 742 770 820 860**

    a. Complete a back-to-back stem plot for the data.
    b. Which brand had the shortest lifetime?
    c. Which brand had the longest lifetime?
    d. If you wanted to be certain that a globe you bought would last at least 500 hours, which brand would you buy?

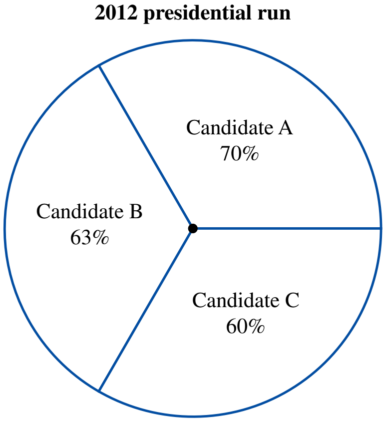
Problem solving

12. Look at the following bar charts and discuss why the one on the left is misleading and what characteristics the one on the right possesses that makes it acceptable.





13. a. What is wrong with this pie graph?
    b. Why is the following information misleading?
       Did scientists falsify research to support their own theories on global warming?
       59% somewhat likely
       35% very likely
       26% not very likely
    c. Discuss the implications of this falsification by statistics.



Reflection

What is the point of drawing a misleading graph in a report?

14.5 Statistical investigations

14.5.1 Using primary data

- This section deals with the steps involved in carrying out a statistical investigation with primary data collection.
- For this exercise we will assume you have been given this task.

> *Which pizza on the market is the best value for money?*

- This is a very broad investigation, and each stage of the investigation must be carefully planned.
  - Collecting the data
  - Organising the data
  - Performing calculations
  - Analysing the data
  - Reporting the results

Collecting the data

- At this initial stage, questions should be posed with regard to the data.
  - What data should be collected?
    - Best value for money involves the price and size of the pizza. Data on both of these need to be collected.
    - Stores have different prices for different sizes.
    - Would size best be measured as area or mass?
    - Not all pizzas are round; some are rectangular.
    - What about the variety of toppings? A standard one should be chosen.
    - Should frozen pizzas be included?
  - How should the data be collected?
    - It is not possible to buy every pizza on the market, so what alternatives are there?
    - A store is probably not willing to allow their pizzas to be weighed, so mass is most likely out of the question.
    - Will the store allow their pizzas to be measured?
  - What problems are likely to be encountered?
    - How many different companies market pizza?

Organising the data

- The data should be organised into some sort of table format.
  - What format is appropriate for this investigation?
  - A table with column headings *Price* and *Measurements* will organise the data.

Price	Measurements	Area	Value for money

- Take time to design the table so figures required for calculation are readily visible.
  - What calculations are required at this stage?
  - Measurements are required to calculate the area of each pizza.
- Think forward and add extra columns for future calculations.
  - What further calculations are needed?
  - *Area* and *Value for money* need to be calculated. Provide two extra columns for these.

Performing calculations

- What calculations need to be performed?
- The *Area* and *Value for money* are required in this case. How should these be calculated? Dividing price by area gives ($\$/\text{cm}^2$), while dividing area by price gives ($\text{cm}^2/\$$).

Analysing the data

- Are there any anomalies, or obvious calculation errors?
- Do the calculated results 'make sense'?
- In this case, if *Value for money* is calculated in units of $\$/\text{cm}^2$, the pizza with the smallest of these values is the best value for money. Using units of $\text{cm}^2/\$$, the pizza with the highest of these values gives the best value for money.
- Would the inclusion of graphs be appropriate?

Reporting the results

- The results should be reported in a clear, concise manner.
- Justify any conclusions.
- Are there any anomalies or exceptions to mention?

14.5.2 Using secondary data

- The procedure for undertaking a statistical investigation using secondary data is similar to that for primary data, the difference being that you sometimes have to search for data in several areas before you find the appropriate source.
  Suppose you were given this assignment.

There have been __________ prime ministers of Australia since 1901 until this day.
There have been __________ elections.
__________ prime ministers have been defeated at a general election.
There have been __________ changes of prime ministers without an election.
The average length these prime ministers served in office is __________.
Undertake a statistical investigation to complete the details.

Collecting the data

- What data should be collected?
- Where can the data be found? The internet is probably a good starting point, but not all sites are reliable.
- If there are multiple sources for the data, are they all in agreement?
- How many of these statements require calculations?

Organising the data

- Design a table to record all the data.
- Consider how many columns are necessary.
- Leave columns for calculations.

Performing calculations

- There is at least one calculation here — to determine the average length of time served in office. Are there any more?

Analysing the data

- Do all the calculated values make sense?
- Would a graph be appropriate?

Reporting the results

- Complete the details.
- Acknowledge the source of the secondary data.

Investigating media reports

- Frequently reports in the media provide a good starting point for an interesting investigation. Here are a few suggestions.

Media report 1

Here is an article on an analysis of the speech habits of two high-profile parliamentarians — Julia Gillard (who was Prime Minister when the article was written) and Tony Abbott (who was then the Leader of the Opposition). It was written by Roly Sussex, a professor of English.

WORD LIMIT

Roly Sussex
sussex@uq.edu.au

We all have individual features in the way we speak. Our tone of voice, an intonation, a rhythm, a favourite word or phrase — the things that make us quickly recognised even on bad telephone connections. But if you are a public figure, and especially a political leader, your speech habits will be picked up, criticised, satirised and caricatured. As I discussed in previous weeks, our current political leaders show a wide variety of habits and idiosyncrasies.

Our new PM, Julia Gillard is a very consecutive speaker. Like Kevin Rudd before her, she speaks in long complete sentences. But unlike Rudd, her sentences contain a fair number of pre-programmed mantras and phrases: “happy to be judged”, “enhanced the capacity”, “regional neighbours”. Her pronunciation is also distinctive. She grew up in South Australia, and so says W instead of L at the end of a syllable. Her “milk” is MIWK, and her “football” is FOOTBAW. And she is our first Prime Minister to have high rising tone, the rising intonation at the end of a clause (rising pitch).

Her favourite word is “obviously” and she has also quickly assumed the leader’s “I”. The closer you get to the top job, the more the ego asserts itself in grammar. Especially in answer to a probing question: “I’m not going to be …”. Former PM John Howard perfected this technique, and it is piously observed by our current leaders. Treasurer Wayne Swan is acquiring it. Aha.

Compared to the PM, Opposition Leader Tony Abbott is a less fluent speaker. He is an “um”, “look” and “ah” man. His sentences contain pauses, sometimes for reflection, sometimes for emphasis, sometimes both. He is somewhat less given to mantra, and greetings-wise he is more a man of the people: he says “G’day”. But Gillard and Abbott share three features which are now so ingrained under the fingernails of our pollies that they won’t scrub off.

One feature is repetition. “As I said in my speech …” says the PM, “… as I said in my speech”. Well, yes, we know that. We heard the speech. Abbott, on the other hand, repeats repeated negatives about the Government: “Spin … contradiction … incompetent …. disarray …”. It’s like swearing — the more you use these words, the less meaning they convey. The second shared feature is the pre-programmed response. A trigger in the question presses a specific answer-button. “Asylum seekers”, “deficit”, “mining super-tax”, “health” and similar key issues prompt the automatic rehearsed rejoinder. You know it’s pre-programmed because you’ll always hear the same words, whenever the trigger is pressed.

The third thing they share is that they won't say "yes" or "no". Both respond to a "can you tell us, yes or no?" with streams of verbal flimflam. Interviewers should give up trying to prise a clear yea/nay out of either of them. But the public does have a right to know, yes or no, where they stand on issues, and we aren't getting what we crave.

Source: *The Courier Mail*, 14–15 Aug. 2010, p. 25.

There is no doubt that these comments are true. How could you find evidence of this?

Media report 2

SINGLE WOMEN EARN MORE

WASHINGTON: The income of one group of US women is catching up to and even overtaking men, a study shows.

They are single women in their 20s without children, who live in large cities and work full-time, according to a study of census data by Reach Advisors, a New York–based strategy and research firm focused on emerging shifts in the consumer landscape.

These young women earn on average 8 per cent more than men in their age group, but in some cities, such as Atlanta in Georgia and Memphis, Tennessee, women earn about one-fifth more than men. On average, American women who work full-time earn about 80 per cent of what men earn.

The report says that one reason for the finding is that girls are "going to college in droves".

Nearly three-quarters of girls who complete high school go on to university, compared with two-thirds of boys.

Women are $1\frac{1}{2}$ times more likely than men to graduate from university and to obtain a masters degree or higher.

Census data released in April showed that 58 per cent of all US masters degrees or PhDs were awarded to women. As women go further in their education, they are also delaying getting married and starting a family.

Source: *The Weekend Australian*, 4–5 Sept. 2010, p. 20.

Is this report really true? Is it perhaps only true in America? (The heading seems to suggest that it is universally true.) What is the status of women in other parts of the world? Further investigation could reveal interesting comparisons.

Media report 3

EGG SHORTAGE IS NO YOLK

Producers lay plan to meet need

Peddy Hintz

Blame MasterChef or the Heart Foundation, but it's getting harder to find the right kind of eggs at the supermarket — and it's likely to stay that way until Christmas.

Queensland egg producers are struggling to keep up with demand but the boom in sales has also been matched by the recent interest in keeping backyard chickens.

The winter shortage of eggs on Australian supermarket shelves will mean that instead of sitting in a coolroom for a week, eggs are being transferred to shelves almost straight from the supplier.

"It's currently taking only about 48 hours from being laid to getting onto the shelves so the eggs that people do buy will be a really good, fresh product," chief executive of Sunny Queen, one of the country's biggest suppliers, John O'Hara said.

The Australian Egg Corporation has put the increased demand for eggs down to revised Heart Foundation guidelines raising the number of eggs recommended for a healthy diet from two a week to six.

But, Mr O'Hara said, cooking shows such as *MasterChef* had also led to a rise in demand.

Current estimates have Australians eating 203–205 eggs per person a year, compared with 195 last year, 156 the year before and a low of 132 10 years ago.

▶

UNSCRAMBLING EGGS

Annual egg consumption (per person)

Australia: 205

Japan: 320

US: 230

UK: 230

NZ: 214

Egg sales

65% caged

24–25% free range

10% cage free

Fresh is best

The most effective way to test if an egg is fresh is to put it in water. The more it sinks, the fresher the egg. If it floats, it's nearly off.

Source: *The Courier Mail*, 28–9 Aug. 2010, p. 13.

Note the catchy heading on this article. Does the advice from the Heart Foundation or cooking shows like *MasterChef* really have that much effect on egg sales? How does egg consumption in Australia compare with that in the other countries mentioned? This is worthy of further investigation.

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Complete this digital doc: WorkSHEET: Statistics in the media (doc-14604)

Exercise 14.5 Statistical investigations

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 9	Questions: 1–9	Questions: 1–10

🟩🟦🟪 **Individual pathway interactivity:** int-4632 learnON ONLINE ONLY

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding

1. a. Write a plan detailing how you would collect primary data to undertake an investigation to determine which pizza on the market is the best value for money.

b. Undertake your investigation.

c. Report on the results of your findings.

2. Undertake the investigation on the history of prime ministers in Australia. Report your findings.
3. Find evidence from speeches of Julia Gillard and Tony Abbott to support Roly Sussex's report on the speech habits of these two politicians.
4. Do single women really earn more? Investigate.
5. What's the story on egg consumption in Australia?
6. Search for a media article you would like to investigate. Provide a full report on your findings.

Reasoning

7. Below are a few statistics on Facebook users. These figures are those reported in the Year 2010.

- There are more than 400 million active users.
- 70% of Facebook users are outside the US.
- 50% of active users log on to Facebook in any given day.
- More than 60 million updates are posted each day.
- More than 3 billion photos are uploaded to the site each month.
- The average user has 130 friends on the site.
- The average user spends more than 55 minutes per day on Facebook.

The top 10 countries on Facebook represent just a little over half of the Facebook users. China (population 1.3 billion) and India (1.2 billion) do not appear in the top 10 list. Write a report summarising the usage of Facebook throughout the world.

These are the top 10 countries on Facebook.

	Country	Population (millions)	% of world population	Users (millions)
1	USA	310.3	4.5	111.2
2	UK	62	0.9	23.5
3	Indonesia	237.6	3.5	19.5
4	Turkey	72.6	1.1	18.7
5	France	65.4	1.0	15.9
6	Italy	60.4	0.9	14.9
7	Canada	34.3	0.5	13.4
8	Philippines	94	1.4	10.6
9	Spain	46.1	0.7	8.9
10	Mexico	108.4	1.6	8.2

8. An investigation is to be conducted to find the two most popular television programs in Australia. For each of the following samples, explain why they would be biased.
a. A sample of 100 students from a city secondary school.
b. A sample of 100 people passing a certain point in a busy city street at lunchtime.
c. How would you go about selecting a sample of people?

Problem solving

9. There has been a rise in supermarket-own brands in Australia. These are commonly available in supermarkets like Woolworths, Coles and Aldi. It has been said that these brands account for almost one-quarter of all grocery sales. It has also been claimed that the quality of supermarket-own brands is comparable with the equivalent market-leading brand, at a much reduced cost.

   Assume you are planning to undertake a study of a particular grocery line (e.g., baked beans or breakfast cereal). Write a plan of how you would undertake this study.

10. You wish to assess the opinion of a local population on the possible closure of their hospital.
    - **a.** What target population would you use?
    - **b.** What resources would you use?
    - **c.** What are the possible biases in conducting such an experiment?

Reflection

What would you consider to be the most important factor in reporting the results of a statistical investigation?

CHALLENGE 14.2

List five positive even integers that have a mean of 8 and a median of 10. How many possible solutions are there?

14.6 Review

14.6.1 Review questions

Fluency

1. List some problems you might encounter in trying to collect data from the following populations.
   - **a.** The average number of mL in a can of soft drink.
   - **b.** The number of fish in a dam.
   - **c.** The number of workers who catch public transport to work each weekday morning.
2. **a.** Calculate the mean of the integers 1 to 100.
   - **b.** **i.** Randomly select 10 numbers in the range 1 to 100.
     ii. Calculate the mean of these numbers.
   - **c.** **i.** Randomly select 20 numbers in the range 1 to 100.
     ii. Calculate the mean of these numbers.
   - **d.** Comment on the similarities/differences between your means calculated in parts **a**, **b** and **c**.

3. For each of the following investigations, state whether a census or a survey has been used.
  a. The average price of petrol in Canberra was estimated by averaging the price at 30 petrol stations in the area.
  b. The performance of a cricketer is measured by looking at his performance in every match he has played.
  c. Public opinion on an issue is sought by a telephone poll of 2000 homes.

4. Traffic lights (red, amber, green) are set so that each colour shows for a set amount of time. Describe how you could use a spinner to simulate the situation so that you could determine (on average) how many sets of lights you must encounter in order to get two green lights in succession.

5. John and Bill play squash each week. In any given game they are evenly matched. A device that could not be used to represent the outcomes of the situation is:
  A. a die
  B. a coin
  C. a circular spinner divided into 2 equal sectors
  D. a circular spinner divided into 5 equal sectors
  E. a circular spinner divided into 4 equal sectors

6. The table below shows the number of students in each year level from Years 7 to 12.

Year	Number of students
7	230
8	200
9	189
10	175
11	133
12	124

Draw two separate graphs to illustrate the following.
  a. The principal of the school claims a high retention rate in Years 11 and 12 (that is, most of the students from Year 10 continue on to complete Years 11 and 12).
  b. The parents claim that the retention rate of students in Years 11 and 12 is low (that is, a large number of students leave at the end of Year 10).

7. Records from a school were examined to determine the number of absent days of both boys and girls over the two years of Year 9 and Year 10. The result is shown in this stem-and-leaf plot.

Key: 2 | 1 = 21 days

Leaf Boys	*Stem*	*Leaf Girls*
	0	1 7
7 4 1 0	1	2 4 7 9 9
9 9 7 6 6 5 3 1 1 0	2	1 3 3 4 6 6
8 7 7 5 2	3	4 4 4 8
2	4	3 6
	5	4

  a. Calculate the median number of days absent for both boys and girls.
  b. Calculate the range for both boys and girls.
  c. Comment on the distribution of days absent for each group.

8. 15 boys and 15 girls were randomly chosen from a group of 900 students. Their heights (in metres) were measured as shown below.

Boys: 1.65, 1.71, 1.59, 1.74, 1.66, 1.69, 1.72, 1.66, 1.65, 1.64, 1.68, 1.74, 1.57, 1.59, 1.60

Girls: 1.66, 1.69, 1.58, 1.55, 1.51, 1.56, 1.64, 1.69, 1.70, 1.57, 1.52, 1.58, 1.64, 1.68, 1.67

a. Comment on the size of the sample.

b. Display the data as a back-to-back stem plot.

c. Compare the heights of the boys and girls.

9. The stem plot below is used to display the number of vehicles sold by the Ford and Holden dealerships in a Sydney suburb each week for a three-month period.

Key: 1 | 5 = 15 vehicles

Leaf Ford	*Stem*	*Leaf Holden*
7 4	0	3 9
9 5 2 2 1 0	1	1 1 1 6 6 8
8 5 4 4	2	2 2 7 9
0	3	5

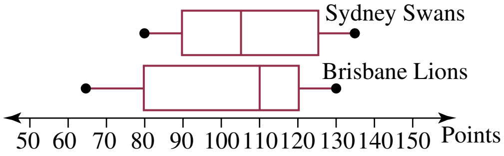
a. State the median of both distributions.

b. Calculate the range of both distributions.

c. Calculate the interquartile range of both distributions.

d. Show both distributions on a parallel box plot.

10. The box plots drawn below display statistical data for two AFL teams over a season.



a. Which team had the higher median score?

b. What was the range of scores for each team?

c. For each team calculate the interquartile range.

11. Tanya measures the heights (in m) of a group of Year 10 boys and girls and produces the following five-point summaries for each data set.

Boys: 1.45, 1.56, 1.62, 1.70, 1.81

Girls: 1.50, 1.55, 1.62, 1.66, 1.73

a. Draw a box plot for both sets of data and display them on the same scale.

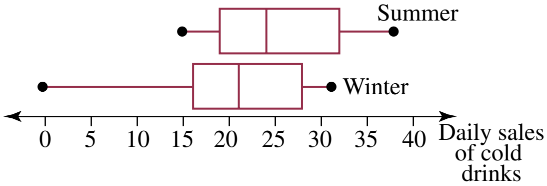
b. What is the median of each distribution?

c. What is the range of each distribution?

d. What is the interquartile range for each distribution?

e. Comment on the spread of the heights among the boys and the girls.

12. The box plots below show the average daily sales of cold drinks at the school canteen in summer and winter.

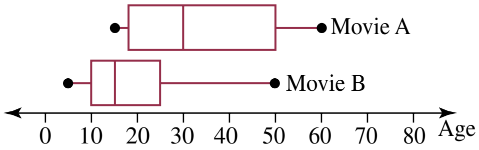


a. Calculate the range of sales in both summer and winter.

b. Calculate the interquartile range of the sales in both summer and winter.

c. Comment on the relationship between the two data sets, both in terms of measures of location and measures of spread.

13. A movie theatre has taken a survey of the ages of people at a showing of two of their movies. The results are shown in these box plots.



Which of the following conclusions could be drawn based on the preceding information?

a. Movie A attracts an older audience than Movie B.
b. Movie B attracts an older audience than Movie A.
c. Movie A appeals to a wider age group than Movie B.
d. Movie B appeals to a wider age group than Movie A.
e. More people went to Movie A.

Problem solving

14. The following data show the ages of a group of 30 males and 30 females as they enter hospital for the first time.

Male		*Female*
9 8	0	5
9 9 8 8 8 6 3 2 1	1	7 7 8 9 9
8 7 7 6 4 3 2 0	2	0 0 1 2 4 5 5 6 7 9
8 6 3 1 0	3	0 1 3 3 5 8
7 5 2	4	2 3 6 8
5 3	5	1 3 4
	6	2
8	7	

a. Construct a pair of parallel boxplots to represent the two sets of data, showing working out for the median and 1st and 3rd quartiles.
b. Calculate the mean, range and IQR for both sets of data.
c. Determine any outliers if they exist.
d. Write a short paragraph comparing the data.

15. The times, in seconds, of the duration of 20 TV advertisements shown in the 6–8 pm time slot are recorded below.

16 60 35 23 45 15 25 55 33 20 22 30 28 38 40 18 29 19 35 75

a. From the data, determine the:
- **i.** mode
- **ii.** median
- **iii.** mean, write your answer correct to 2 decimal places
- **iv.** range
- **v.** lower quartile
- **vi.** upper quartile
- **vii.** interquartile range.

b. Using your results from part **a**, construct a boxplot for the time, in seconds, for the 20 TV advertisements in the 6–8 pm time slot.

c. From your boxplot, determine:

  i. the percentage of advertisements that are more than 39 seconds in length

  ii. the percentage of advertisements that last between 21 and 39 seconds

  iii. the percentage of advertisements that are more than 21 seconds in length

The types of TV advertisements during the 6–8 pm time slot were categorised as Fast Food, Supermarkets, Program information, Retail (clothing, sporting goods, furniture). A frequency table for the frequency of these advertisements being shown during this time slot is shown below.

Type	**Frequency**
Fast food	7
Supermarkets	5
Program information	3
Retail	5

d. What type of data has been collected in the table?

e. What percentage of advertisements are advertisements for fast food outlets?

f. What would be good options for a graphical representation of this type of data?

16. The test scores, out of a total score of 50, for two classes A and B are shown in the stem plot below.

Class A		*Class B*
5	0	1 2 4
9 7 5 3	1	1 4 5
9 7 7 5 4	2	0 0 5
8 8 6 5 5 1	3	1 5 5
3 2 0	4	1 5 7 7 8 9
0	5	0 0

a. Ms Vinculum teaches both classes and made the statement that 'Class A's performance on the test showed that the students' ability was more closely matched than the students' ability in Class B'. By finding the measure of centre, first and third quartiles, and the measure of spread for the test scores for each class, explain if Ms Vinculum's statement was correct.

b. Would it be correct to say that Class A performed better on the test than Class B? Justify your answer by comparing the quartiles and median for each class.

17. The speeds, in km/h, of 55 cars travelling along a major road are recorded below.

Speed	**Frequency**
60–64	1
65–69	1
70–74	10
75–79	13
80–84	9
85–89	8
90–94	6

Speed	**Frequency**
95–99	3
100–104	2
105–109	1
110–114	1
Total	55

a. By finding the midpoint for each class interval, determine the mean speed, in km/h, of the cars travelling along the road. Write your answer correct to two decimal places.

b. The speed limit along the road is 75 km/h. A speed camera is set to photograph the license plates of cars travelling 7% more than the speed limit. A speeding fine is automatically sent to the owners of the cars photographed. Based on the 55 cars recorded, how many speeding fines were issued?

c. Drivers of cars travelling 5 km/h up to 15 km/h over the speed limit are fined $135. Drivers of cars travelling more than 15 km/h and up to 25 km/h over the speed limit are fined $165 and drivers of cars recorded travelling more than 25 km/h and up to 35 km/h are fined $250. Drivers travelling more than 35 km/h pay a $250 fine in addition to having their driver's license suspended. If it is assumed that this data is representative of the speeding habits of drivers along a major road and there are 30 000 cars travelling along this road on any given month. Determine:

  i. The amount, in dollars, collected in fines throughout the month. Write your answer correct to the nearest cent.

  ii. How many drivers would expect to have their licenses suspended throughout the month?

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Try out this interactivity: Word search: Topic 14 (int-2865)

Try out this interactivity: Crossword: Topic 14 (int-2866)

Try out this interactivity: Sudoku: Topic 14 (int-3601)

Complete this digital doc: Concept map: Topic 14 (doc-14606)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

biased
census
data collection methods
experiment
observation
population
primary data
sample
secondary data
simulation
statistical data
statistical investigations
statistical reports
surveys
trends

Investigation | Rich task

Populations and samples

The following table gives information about literacy rates for the entire population in 100 countries (percentage for each country) and for males and females in each country. In this task we will see how closely a sample resembles the population.

Use a digital technology to work with a large set of data.

This task will only use the first column of literacy rates for the country. Other conclusions may be drawn from the male and female literacy rates and possibly compared to the overall literacy rates in this investigation.

1. Find the five-figure summary for the 100 countries' literacy rates and draw a boxplot.
2. Find the mean and standard deviation.
3. Take a sample of 20 from the set of 100 countries using the random number generator and make a list of the country and the literacy rates.
4. Find the five-number summary for the sample.
5. Find the mean and standard deviation for the sample.
6. Draw a boxplot that compares your sample with the whole population.
7. Describe the sample data obtained in terms of the population as a whole; in particular, concentrate on the middle 50 per cent of the data.
8. Now take a further two random samples of 20 countries and draw boxplots of all three on the same scale. Comment on these three samples.
9. Find the mean and standard deviation for the three samples. How do they compare?
10. From the results you have obtained, how would you describe the reliability of a sample compared with using the whole population?

	Country	Literacy (%)	Male literacy (%)	Female literacy (%)		Country	Literacy (%)	Male literacy (%)	Female literacy (%)
1	Afghanistan	29	44	14	8	Barbados	99	99	99
2	Argentina	98	99	98	9	Belarus	99	100	100
3	Armenia	98	100	100	10	Bolivia	78	85	71
4	Australia	100	100	100	11	Botswana	72	32	16
5	Azerbaijan	98	100	100	12	Brazil	81	82	80
6	Bahrain	77	55	55	13	Bulgaria	98	99	98
7	Bangladesh	35	47	22	14	Burkina Faso	24	34	14

	Country	Literacy (%)	Male literacy (%)	Female literacy (%)		Country	Literacy (%)	Male literacy (%)	Female literacy (%)
15	Burundi	50	61	40	46	Iran	54	64	43
16	Cambodia	35	48	22	47	Iraq	60	70	49
17	Cameroon	54	66	45	48	Ireland	98	99	97
18	Canada	97	97	97	49	Israel	92	95	89
19	Cent. Afri. R	27	33	15	50	Italy	97	98	96
20	Chile	93	94	93	51	Japan	99	99	99
21	China	78	87	68	52	Jordan	80	89	70
22	Colombia	87	88	86	53	Kenya	69	80	58
23	Costa Rica	93	93	93	54	Kuwait	73	77	67
24	Croatia	98	99	97	55	Latvia	99	100	100
25	Cuba	94	95	93	56	Lebanon	80	88	73
26	Dominican R.	83	85	82	57	Liberia	40	50	29
27	Ecuador	88	90	86	58	Libya	64	75	50
28	Egypt	48	63	34	59	Lithuania	99	99	98
29	El Salvador	73	76	70	60	Malaysia	78	86	70
30	Estonia	99	100	100	61	Mexico	87	90	85
31	Ethiopia	24	32	16	62	Morocco	50	61	38
32	Finland	100	100	100	63	N. Korea	99	99	99
33	France	99	99	98	64	Netherlands	100	100	100
34	Gabon	61	74	48	65	New Zealand	99	99	99
35	Gambia	27	39	16	66	Nicaragua	57	57	57
36	Georgia	99	100	100	67	Nigeria	51	62	40
37	Germany	99	99	98	68	Norway	99	99	99
38	Greece	93	98	89	69	Oman	71	80	62
39	Guatemala	55	63	47	70	Pakistan	35	47	21
40	Haiti	53	59	47	71	Panama	88	88	88
41	Honduras	73	76	71	72	Paraguay	90	92	88
42	Hungary	99	99	98	73	Peru	85	92	79
43	Iceland	100	100	100	74	Philippines	90	90	90
44	India	52	64	39	75	Poland	99	99	98
45	Indonesia	77	84	68	76	Portugal	85	89	82

	Country	Literacy (%)	Male literacy (%)	Female literacy (%)		Country	Literacy (%)	Male literacy (%)	Female literacy (%)
77	Russia	99	100	100	89	Thailand	93	96	90
78	Rwanda	50	64	37	90	Turkey	81	90	71
79	S. Korea	96	99	99	91	U. Arab Em.	68	70	63
80	Saudi Arabia	62	73	48	92	UK	99	99	99
81	Senegal	38	52	25	93	USA	97	97	97
82	Singapore	88	93	84	94	Uganda	48	62	35
83	Somalia	24	36	14	95	Ukraine	97	100	100
84	South Africa	85	86	86	96	Uruguay	96	97	96
85	Spain	95	97	93	97	Uzbekistan	97	100	100
86	Sweden	99	99	99	98	Venezuela	88	90	87
87	Switzerland	99	99	99	99	Vietnam	88	93	83
88	Syria	64	78	51	100	Zambia	73	81	65

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Complete this digital doc: Code puzzle: Animals that originated in Australia include... (doc-15943)

Answers

TOPIC 14 Statistics in the media

Exercise 14.2 Populations and samples

1. a. When was it first put into the machine? How old was the battery before being purchased? How frequently has the computer been used on battery?

b. Can't always see if a residence has a dog; a census is very time-consuming; perhaps could approach council for dog registrations.

c. This number is never constant with ongoing purchases, and continuously replenishing stock.

d. Would have to sample in this case as a census would involve opening every packet.

2. These answers will vary with the samples chosen.

3. a. Census. The airline must have a record of every passenger on every flight.

b. Survey. It would be impossible to interview everyone.

c. Survey. A census would involve opening every bottle.

d. Census. The instructor must have an accurate record of each learner driver's progress.

4. a. Survey **b.** Survey **c.** Census **d.** Survey

5. a. About 25

b. Drawing numbers from a hat, using a calculator, …..

6. a. The council is probably hoping it is a census, but it will probably be a survey because not all those over 10 will respond.

b. Residents may not all have internet access. Only those who are highly motivated are likely to respond.

7. The sample could have been biased. The questionnaire may have been unclear.

8. Sample size, randomness of sample

9. Answers will vary. Check with your teacher.

10. Populations growing very rapidly, large number of expatriate workers in China have a different background and forms need to be modified for them, people from Hong Kong working on mainland China, large migrant population in New Delhi, often migrants don't have residency permits (so the truth of their answers is questionable), many people live in inaccessible areas, some families in China have more than 1 child and do not disclose this.

11. There is quite a variation in the frequency of particular numbers drawn. For example, the number 45 has not been drawn for 31 weeks, while most have been drawn within the last 10 weeks. In the long term, one should find the frequency of drawing each number is roughly the same. It may take a long time for this to happen, as only 8 numbers are drawn each week.

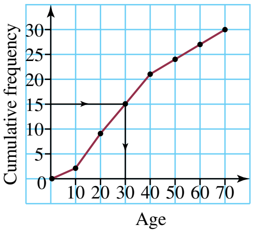
12. a. Mean = 32.03; median = 29.5

b.

Class interval	Frequency
0–9	2
10–19	7
20–29	6
30–39	6
40–49	3
50–59	3
60–69	3
Total	30

c. Mean = 31.83

d.



e. Median = 30

f. Estimates from parts **c** and **e** were fairly accurate.

g. Yes, they were fairly close to the mean and median of the raw data.

13. Year 8: mean = 26.83, median = 27, range = 39, IQR = 19 Year 10: mean = 40.7, median = 39.5, range = 46, IQR = 20 The typing speed of Year 10 students is about 13 to 14 wpm faster than that of Year 8 students. The spread of data in Year 8 is slightly less than the spread in Year 10.

Exercise 14.3 Primary and secondary data

1. These are simply examples of simulations which could be conducted.
   - **a.** Coin could be flipped (Heads represents 'True', while Tails represents 'False')
   - **b.** Coin could be flipped (Heads represents 'red', while Tails represents 'black')
   - **c.** Spinner with 4 equal sectors (each sector representing a different toy)
   - **d.** Roll a die (each face represents a particular person)
   - **e.** Spinner with 3 equal sectors (each one representing a particular meal)
   - **f.** Spinner with 5 equal sectors (each one representing a particular destination)
   - **g.** Spinner with 5 sectors, one which will have an angle size of 120°, while the other 4 each have an angle size of 60° (each one representing a particular fast food)
2. Answers will vary, however some possible suggestions include:

   Which students have internet access at home?

   Do the students need access at night?

   What hours would be suitable?

   How many would make use of this facility?
3. Answers will vary. Check with your teacher.
4. Answers will vary, however some possible suggestions include:
   - **a.** Census, survey, questionnaire, interview, observation, experiment, on-line response, …
   - **b.** **i.** Measurement **ii.** Observation

     iii. Newspaper recordings **iv.** Survey

5–7 Student's own response

8. The claim is false. It is not a logical deduction.
9. Student's own response
10. Student's own response
11. Sealy Posturepremier 40% off $\left(\frac{1000}{2499} \times 100\right)$,

    Sealy Posturepedic 41% off $\left(\frac{1600}{3899} \times 100\right)$,

    Sleepmaker Casablanca 40% off $\left(\frac{800}{1999} \times 100\right)$,

    Sleepmaker Umbria 42% off $\left(\frac{1800}{4299} \times 100\right)$.

    There is at least 40% off these beds.
12. **a.** 7750 seconds
    - **b.** **i.** Under 20 − (20–24): 1000 seconds difference
      - **ii.** Under 20 − (25–29): 500 seconds difference
      - **iii.** (20–24) − (25–29): 500 seconds
    - **c.** Median time for 20–24 year olds same as for 25–29 year olds, which was about 250 seconds greater than the under 20s. The 20–24 year olds had and the under 20s both had a range of 2500 seconds, which was slightly higher than the range for the older group.
13. They are all made in Australia and have comparable fat and saturated fat contents. The Byron Bay Chilli corn chips have a much lower salt content than the other three varieties. The verdict comments require a mention.
14. Hane and Roarne had a higher median and a lower spread, so they appear to have performed better.

Challenge 14.1

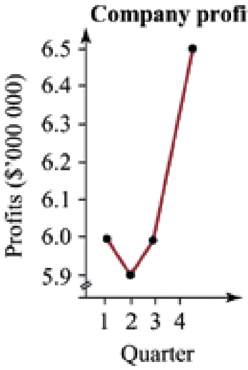
1250 trout

Exercise 14.4 Evaluating inquiry methods and statistical reports

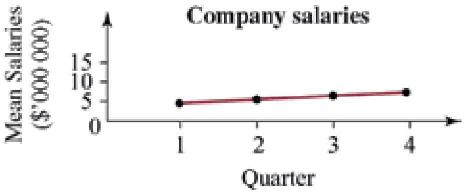
1. **a.** Primary. There is probably no secondary data available.

   b, **c** Answers will vary. Check with your teacher.

2. Company profits



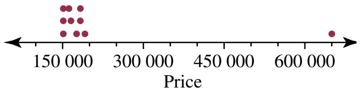
Mean salaries



3. a. Mean = \$21 5000, median = \$170 000, mode = \$150 000. The median best represents these land prices. The mean is inflated by one large score, and the mode is the lowest price.

b. Range = \$500 000, interquartile range = \$30 000. The interquartile range is the better measure of spread.

c.



This dot plot shows how 9 of the scores are grouped close together, while the score of \$650 000 is an outlier.

d. The agent is quoting the modal price, which is the lowest price. This is not a true reflection of the average price of these blocks of land.

4. a. False. Mean = 1.82 m, lower quartile = 1.765 m, median = 1.83 m

b. True. This is the definition of interquartile range.

c. Players with heights 1.83 m, 1.83 m, 1.88 m, 1.88 m, 1.88 m

5. a. 7.1 **b.** 7 **c.** 7

d. The mode has the most meaning as this size sells the most.

6. Check with your teacher. Answers depend on groupings used.

7. Player B appears to be the better player if the mean result is used. However, Player A is the more consistent player.

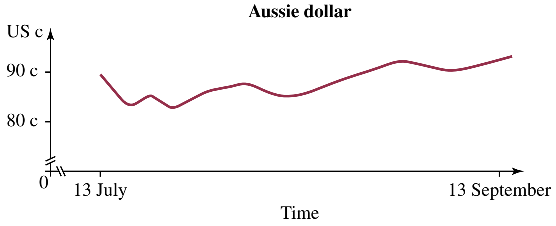
8. a. The statement is true, but misleading as most of the employees earn \$18 000.

b. The median and modal salary is \$18 000 and only 15 out of 80 (less than 20%) earn more than the mean.

9. Points which could be mentioned include:

- 10.1% is only just 'double digit' growth.
- 2006–08 showed mid to low 20% growth. Growth has been declining since 2008.
- The share price has rebounded, but not to its previous high.
- The share price scale is not consistent. Most increments are 30c, except for \$27.70 to \$28.10 (40c increment). Note also the figure of 20.80 — probably a typo instead of 26.80.

10. Shorten the y-axis and expand the x-axis.



11. a. Key: 3 | 85 = 385 hours

Brand B	Stem	Brand A
	3	85 90
	4	25 26
60 55 00	5	70
30	6	40 45
70 42 35 20	7	30 35
60 20	8	60

Brand A: mean = 570.6, median = 605

Brand B: mean = 689.2, median = 727.5

b. Brand A had the shortest mean lifetime.

c. Brand B had the longest mean lifetime.

d. Brand B

12. Left bar chart suggests prices have tripled in one year due to fact vertical axis does not start at zero. Bar chart on right is truly indicative of situation.

13. a and **b** Percentages do not add to 100%.

c Such representation allows multiple choices to have closer percentages than really exist.

Exercise 14.5 Statistical investigations

The questions in this exercise relate to student investigations, so there will be a variety of answers.

2. There have been 27 prime ministers of Australia since 1901 until this day.

There have been 42 elections.

10 prime ministers have been defeated at a general election.

There have been 21 changes of prime minister without an election.

3–7 Discuss with your teacher.

8. a. Sample would be biased since respondents would all be from the same area, same age group and not representative of the total population.

b. Sample would be biased since it is from only one city, so for example the respondents might not be representative of different socio-economic groups and ethnic backgrounds. The survey should include people from the country as well as the city.

9. Student's plan for an investigation — check with teacher.

10. Answers will vary but might include:

a. All households in the area serviced by the hospital

b. Contacts obtained from the local electoral rolls

c. Bias could be introduced in the wording of the questions, only people who feel passionately about the issue might respond.

Challenge 14.2

2, 4, 10, 10, 14; 6 solutions

14.6 Review

1. a. You would need to open every can to determine this.

b. Fish are continuously dying, being born, being caught.

c. Approaching work places and public transport offices

2. a. 50.5 **b.** Answers will vary. **c.** Answers will vary. **d.** Answers will vary.

3. a. Survey **b.** Census **c.** Survey

4. Use a spinner of 3 equal sectors, each sector having an angle size of 120° and representing a particular colour. Twirl the spinner until a green-green combination has been obtained. This is defined as one experiment. Count the number of trials required for this experiment. Repeat this procedure a number of times and determine an average.

5. D

6. a. Check with your teacher.

b. This graph should look relatively flat, with little decline in the Years 11 and 12 region.

c. This graph should show a sharp decline in the Years 11 and 12 region.

7. a. Boys: median = 26; girls: median = 23.5

b. Boys: range = 32; girls: range = 53

c. Both sets have similar medians, but the girls have a greater range of absenteeism than the boys.

8. a. The sample is an appropriate size as $\sqrt{900} = 30$.

b. Key: 16 | 1 = 1.61

Leaf Boys	*Stem*	*Leaf Girls*
9 9 7	15	1 2 5 6 7 8 8
9 8 6 6 5 5 4 0	16	4 4 6 7 8 9 9
4 4 2 1	17	0

c. The boys are generally taller than the girls, with the mean of the boys being 1.66 m and that of the girls being 1.62 m. The five-number summaries are:

Boys: 1.57 m, 1.60 m, 1.66 m, 1.71 m, 1.74 m

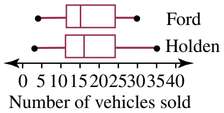
Girls: 1.51 m, 1.56 m, 1.64 m, 1.68 m, 1.70 m

9. a. Ford: median = 15; Holden: median = 16

b. Ford: range = 26; Holden: range = 32

c. Ford: IQR = 14; Holden: IQR = 13.5

d.



10. a. Brisbane Lions

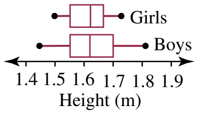
b. Brisbane Lions: range = 65;

Sydney Swans: range = 55

c. Brisbane Lions: IQR = 40;

Sydney Swans: IQR = 35

11. a.



b. Boys: median = 1.62 m; girls: median = 1.62 m

c. Boys: range = 0.36 m; girls: range = 0.23 m

d. Boys: IQR = 0.14 m; girls: IQR = 0.11 m

e. Although the boys and girls have the same median height, the spread of heights is greater among boys as shown by the greater range and interquartile range.

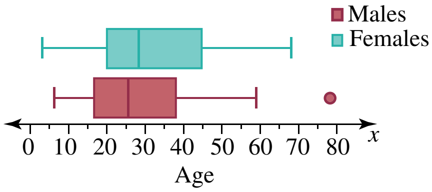
12. a. Summer: range = 23; winter: range = 31

b. Summer: IQR = 13; winter: IQR = 12

c. There are generally more cold drinks sold in summer as shown by the higher median. The spread of data is similar as shown by the IQR although the range in winter is greater.

13. A

14. a.



b.

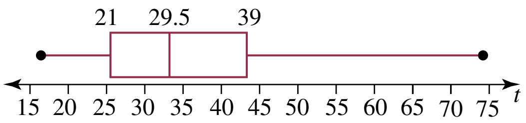
	Males	**Females**
Mean	28.2	31.1
Range	70	57
IQR	18	22

c. There is one outlier — a male aged 78.

d. Typically males seem to enter hospital for the first time at a younger age than females.

15. a. i. 35 s **ii.** 29.5 s **iii.** 33.05 s **iv.** 60 s **v.** 21 s **vi.** 39 s **vii.** 18 s

b.



c. **i.** 25% **ii.** 50% **iii.** 75%

d. Categorical **e.** 35% **f.** Pictogram, pie chart or bar chart

16. a. Class A: Q_1 – 21.5, Median – 30, Q_3 – 38, IQR – 16.5

Class B: Q_1 – 14.5, Median – 33, Q_3 – 47, IQR – 32.5

Based on the comparison between Class A's IQR (16.5) and Class B's IQR (32.5), Ms Vinculum was correct in her statement.

b. No, Class B has a higher median and upper quartile score than Class A, while Class A has a higher lower quartile. You can't confidently say that either class did better in the test than the other.

17. a. 82.73 km/h

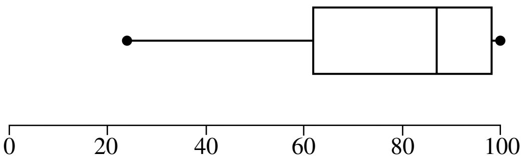
b. 30 cars

c. i. \$2 607 272.73 **ii.** About 545

Investigation — Rich task

1. Data for the whole population

Min	24
Q_1	61.75
Median	87
Q_3	98
Max	100



2. Mean = 77.85; standard deviation = 23

3–10. Answers will vary.

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Project: Climate change

Scenario

Climate change is upon us and has become one of the great challenges facing humanity. Our fossil fuel–driven economies are producing large volumes of greenhouse gases (water vapour, carbon dioxide, methane and ozone) that are warming the planet. As our planet heats up, the ice sheets at the poles slowly melt, causing sea levels to rise. Islands in the Pacific Ocean are already being overcome by water, leading the inhabitants of islands, such as Tuvalu Island in the Pacific Ocean, to announce that they are abandoning their homeland due to rising sea levels. In Sydney, many well known suburbs could be threatened by rising sea levels in the future, including Caringbah, Kurnell, Cromer, Manly Vale, Newcastle, the central coast, Homebush Bay, Newington Silverwater, Arncliffe, Marrickville and Sydney Airport. Climate change could cause the extinction of many species as ecosystems are damaged by rising temperatures.

In order to address these apocalyptic issues, we need to understand the role of human activity in climate change. It will be your job to investigate and understand the relationships that underpin global warming. You can then make recommendations to our political leaders and take action yourself to help save our planet.

Task

You will need to analyse **real data sets** to develop a mathematical understanding of climate change issues. The analysis will involve the use of scatterplots, box-and-whisker plots and five number summaries. Scatterplots will be used to investigate and comment on relationships between two climate change variables. Data sets will be compared using box-and-whisker plots, dot plots and histograms. Environmental data will be graphed, such that the independent variable is time. At the end of your project, your improved mathematical understanding of climate change will allow you to make key recommendations on how we can meet the environmental challenges of the future.

Process

You will use **Microsoft Excel, Google Fusion Tables** and **Google Public Data Explorer** to investigate global environmental data.

- You must have a **gmail** account and internet access to use the Google data tools.
- You will need Microsoft Excel and GeoGebra installed on your computer. Go to projectsPLUS in your learnON title, set up a group and then open the Media Centre to locate everything you need.
- Open the Word documents titled Lesson 1, Lesson 2 etc. Follow the instructions in each document to complete your project.
- At various stages of your project, you will need to access data sets in Microsoft Excel files.

SUGGESTED SOFTWARE

- Microsoft Excel
- GeoGebra
- Internet connection
- Internet browser with Adobe Flash player installed.
- Use the **World Bank** weblink in your Resources section to locate banks of data in Excel form.

TOPIC 15
Financial mathematics

15.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

15.1.1 Why learn this?

Everyone requires food, housing, clothing and transport, and a fulfilling social life. Money allows us to purchase the things we need and desire. The ability to manage money is key to a financially secure future and a reasonable retirement with some fun along the way. Each individual is responsible for managing his or her own finances; therefore, it is imperative that everyone is financially literate.

15.1.2 What do you know?

assesson

1. **THINK** List what you know about financial maths. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of financial maths.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Money, money, money! (eles-1855)

15.2 Purchasing goods

15.2.1 Purchasing goods

- There are many different payment options when purchasing major goods, such as flat screen televisions and computers. Payment options include:
  - cash
  - credit card
  - lay-by
  - deferred payment
  - buying on terms
  - loan.
- The cost of purchasing an item can vary depending on the method of payment used.
- Some methods of payment involve borrowing money and, as such, mean that interest is charged on the money borrowed.
- The **simple interest** formula can be used to calculate the interest charged on borrowed money,

$$I = \frac{P \times r \times T}{100}$$

where: I is the simple interest ($)

P is the principal or amount borrowed or invested ($)

r is the rate of interest per time period

T is the time for which the money is invested or borrowed.

If T is in years, then r is the rate of interest per annum (% p.a.).

WORKED EXAMPLE 1

Find the simple interest on $4000 invested at 4.75% p.a. for 4 years.

THINK	WRITE
1 Write the formula and the known values of the variables.	$I = \frac{P \times r \times T}{100}$, where $P = \$4000,\ r = 4.75\%,\ T = 4$
2 Substitute known values to find I.	$I = \frac{\$4000 \times 4.75 \times 4}{100}$
3 Calculate the value of I.	$= \$760$

- What are the ways of purchasing the item shown in the advertisement below?

120 cm HD TV

5 year warranty

- High definition
- HDMI ports
- 16 : 9 aspect ratio
- 1080i

$800

15.2.2 Payment options

Cash

- With cash, the marked price is paid on the day of purchase with nothing more to pay.
- A cash-paying customer can often negotiate, with the retailer, to obtain a lower price for the item.

Lay-by

- With **lay-by**, the item is held by the retailer while the customer makes regular payments towards paying off the marked price.
- In some cases a small administration fee may be charged.

Credit cards

- With a credit card, the retailer is paid by the credit card provider, generally a financial lender.
- The customer takes immediate possession of the goods.
- The financial lender later bills the customer — collating all purchases over a monthly period and billing the customer accordingly. The entire balance shown on the bill can often be paid with no extra charge, but if the balance is not paid in full, interest is charged on the outstanding amount, generally at a very high rate.

WORKED EXAMPLE 2

TI | CASIO

The ticketed price of a mobile phone is \$600. Andrew decides to purchase the phone using his credit card. At the end of 1 month the credit card company charges interest at a rate of 15% p.a. Calculate the amount of interest that Andrew must pay on his credit card after 1 month.

THINK	WRITE
1 Write the formula and the known values of the variables. Remember that 1 month $= \frac{1}{12}$ year.	$I = \frac{P \times r \times t}{100}$ $P = \$600, r = 15\%, T = \frac{1}{12}$
2 Substitute known values to find I.	$I = \frac{600 \times 15 \times 1}{100 \times 12}$
3 Calculate the value of I.	$= \$7.50$

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Complete this digital doc: SkillSHEET: Converting a percentage to a decimal (doc-5345)

Complete this digital doc: SkillSHEET: Finding simple interest (doc-5346)

Exercise 15.2 Purchasing goods

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–3, 5, 7, 10	Questions: 1–4, 6, 8, 10, 11	Questions: 1–3, 5, 7, 9–12

🟩🟦🟪 **Individual pathway interactivity:** int-4633

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** Find the simple interest payable on a loan of \$8000 at 6% p.a. for 5 years.
2. Find the simple interest on each of the following loans.
   - **a.** \$5000 at 9% p.a. for 4 years
   - **b.** \$4000 at 7.5% p.a. for 3 years
   - **c.** \$12 000 at 6.4% p.a. for $2\frac{1}{2}$ years
   - **d.** \$6000 at 8% p.a. for $1\frac{1}{2}$ years
3. Find the simple interest on each of the following investments.
   - **a.** \$50 000 at 6% p.a. for 6 months
   - **b.** \$12 500 at 12% p.a. for 1 month
   - **c.** \$7500 at 15% p.a. for 3 months
   - **d.** \$4000 at 18% p.a. for 18 months
4. Calculate the monthly interest charged on each of the following outstanding credit card balances.
   - **a.** \$1500 at 15% p.a.
   - **b.** \$4000 at 16.5% p.a.
   - **c.** \$2750 at 18% p.a.
   - **d.** \$8594 at 17.5% p.a.
   - **e.** \$5690 at 21% p.a.

Understanding

5. **WE2** The ticketed price of a mobile phone is \$800. Elena decides to purchase the phone using her credit card. After 1 month the credit card company charges interest at a rate of 15% p.a. Calculate the amount of interest that Elena must pay on her credit card after 1 month.
6. Arup decides to purchase a new sound system using her credit card. The ticketed price of the sound system is \$900. When Arup's credit card statement arrives, it shows that she will pay no interest if she pays the full amount by the due date.

   - **a.** If Arup pays \$200 by the due date, what is the balance owing?
   - **b.** If the interest rate on the credit card is 18% p.a., how much interest will Arup be charged in the month?
   - **c.** What will be the balance that Arup owes at the end of the month?
   - **d.** At this time Arup pays another \$500 off her credit card. How much interest is Arup then charged for the next month?
   - **e.** Arup then pays off the entire remaining balance of her card. What was the true cost of the sound system including all the interest payments?
7. Carly has an outstanding balance of \$3000 on her credit card for June and is charged interest at a rate of 21% p.a.
   - **a.** Calculate the amount of interest that Carly is charged for June.
   - **b.** Carly makes the minimum repayment of \$150 and makes no other purchases using the credit card in the next month. Calculate the amount of interest that Carly will be charged for July.

c. If Carly had made a repayment of $1000 at the end of June, calculate the amount of interest that Carly would then have been charged for July.

d. How much would Carly save in July had she made the higher repayment at the end of June?

8. Shane buys a new home theatre system using his credit card. The ticketed price of the bundle is $7500. The interest rate that Shane is charged on his credit card is 18% p.a. Shane pays off the credit card at a rate of $1000 each month.

a. Complete the table below.

Month	Balance owing	Interest	Payment	Closing balance
January	$7500.00	$112.50	$1000.00	$6612.50
February	$6612.50	$99.19	$1000.00	
March			$1000.00	
April			$1000.00	
May			$1000.00	
June			$1000.00	
July			$1000.00	
August			$1015.86	$0

b. What is the total amount of interest that Shane pays?

c. What is the total cost of purchasing the home theatre system using his credit card?

Reasoning

9. Design a table that compares the features of each method of payment: cash, lay-by and credit card.

10. Choose the most appropriate method of payment for each of the described scenarios below. Explain your choice.

Scenario 1: Andy has no savings and will not be paid for another two weeks. Andy would like to purchase an HD television and watch tomorrow's football final.

Scenario 2: In September Lena spots on special a home theatre system which she would like to purchase for her family for Christmas.

Problem solving

11. Merchant banks offer simple interest on all investments. Merchant bank A had an investor invest $10 000 for 5 years. Merchant bank B had a different investor invest $15 000 for 3 years. Investor B obtained $2500 more in interest than investor A because the rate of interest per annum she received was 6% greater than the interest obtained by investor A. Find the simple interest and rate of interest for each investor.

12. Compare the following two investments where simple interest is paid.

	Rate	Principal	Time	Interest
Investment A:	r_A	$8000	4 years	SI_A
Investment B:	r_B	$7000	5 years	SI_B

It is known that $r_A : r_B = 2 : 3$ and that investment B earned $2000 more interest than investment A. Find the values of r_A, r_B, SI_A and SI_B.

Reflection

What can you do to remember the simple interest formula?

15.3 Buying on terms

15.3.1 Buying on terms

- When buying an item on terms:
  - a deposit is paid
  - the balance is paid off over an agreed period of time with set payments
  - the set payments may be calculated as a stated arbitrary amount or interest rate
  - total monies paid will exceed the initial cash price.

WORKED EXAMPLE 3

The cash price of a computer is \$2400. It can also be purchased on the following terms: 25% deposit and payments of \$16.73 per week for 3 years. Calculate the total cost of the computer purchased on terms as described.

THINK	WRITE
1 Calculate the deposit.	$\text{Deposit} = 25\% \text{ of } \$2400$ $= 0.25 \times \$2400$ $= \$600$
2 Calculate the total of the weekly repayments.	$\text{Total repayment} = \$16.73 \times 52 \times 3$ $= \$2609.88$
3 Add these two amounts together to find the total cost.	$\text{Total cost} = \$600 + \$2609.88$ $= \$3209.88$

WORKED EXAMPLE 4

A diamond engagement ring has a purchase price of \$2500. Michael buys the ring on the following terms: 10% deposit with the balance plus simple interest paid monthly at 12% p.a. over 3 years.

a Calculate the amount of the deposit.

b What is the balance owing after the initial deposit?

c Calculate the interest payable.

d What is the total amount to be repaid?

e Find the amount of each monthly repayment.

THINK	WRITE
a Calculate the deposit by finding 10% of \$2500.	**a** $\text{Deposit} = 10\% \text{ of } \$2500$ $= 0.1 \times \$2500$ $= \$250$
b Find the balance owing by subtracting the deposit from the purchase price.	**b** $\text{Balance} = \$2500 - \$250$ $= \$2250$

c Find the simple interest on \$2250 at 12% p.a. for 3 years.	**c** $I = \dfrac{P \times r \times T}{100},$ where $P = \$2250,\ r = 12\%,\ T = 3$ $I = \$2250 \times 0.12 \times 3$ $= \$810$
d Find the total repayment by adding the balance owing with the interest payable.	**d** Total repayment $= \$2250 + \$810$ $= \$3060$
e Find the monthly repayment by dividing the total repayment by the number of months over which the ring is to be repaid.	**e** Monthly repayment $= \$3060 \div 36$ $= \$85$

15.3.2 Loans

- Money can be borrowed from a bank or other financial institution.
- Interest is charged on the amount of money borrowed.
- Both the money borrowed and the interest charged must be paid back.
- The interest rate on a loan is generally lower than the interest rate offered on a credit card or when buying on terms.
- The calculation of loan payments is done in the same way as for buying on terms; that is, calculate the interest and add it to the **principal** before dividing into equal monthly repayments.

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Complete this digital doc: SkillSHEET: Finding a percentage of a quantity (money) (doc-5347)

Complete this digital doc: WorkSHEET: Buying on terms (doc-14607)

Exercise 15.3 Buying on terms

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 8, 10, 13–15	Questions: 1–4, 6, 8, 10, 12–15	Questions: 1–5, 8–16

🟩🟦🟪 **Individual pathway interactivity:** int-4634 **learnon** ONLINE ONLY

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Fluency

1. Calculate the total cost of a \$3000 purchase given the terms described below.

- **a.** **i.** 12% deposit and monthly payments of \$60 over 5 years
  - **ii.** 20% deposit and weekly payments of \$20 over 3 years
  - **iii.** 15% deposit and annual payments of \$700 over 5 years
- **b.** Which of these options is the best deal for a purchaser?

2. Calculate the amount of each repayment for a \$5000 purchase given the terms described below.
   a. 10% deposit with the balance plus simple interest paid monthly at 15% p.a. over 5 years
   b. 10% deposit with the balance plus simple interest paid fortnightly at 12% over 5 years
   c. 20% deposit with the balance plus simple interest paid monthly at 10% over 3 years
3. Calculate the total repayment and the amount of each monthly repayment for each of the following loans.
   a. \$10 000 at 9% p.a. repaid over 4 years
   b. \$25 000 at 12% p.a. repaid over 5 years
   c. \$4500 at 7.5% p.a. repaid over 18 months
   d. \$50 000 at 6% p.a. repaid over 10 years
   e. \$200 000 at 7.2% p.a. repaid over 20 years

Understanding

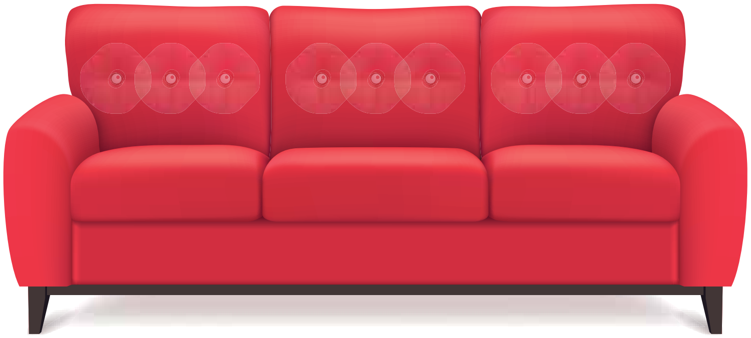
4. **WE3** The cash price of a bedroom suite is \$4200. The bedroom suite can be purchased on the following terms: 20% deposit and weekly repayments of \$43.94 for 2 years. Calculate the total cost of the bedroom suite if you bought it on terms.

5. Guy purchases a computer that has a cash price of \$3750 on the following terms: \$500 deposit with the balance plus interest paid over 2 years at \$167.92 per month. What is the total amount that Guy pays for the computer?

6. Dmitry wants to buy a used car with a cash price of \$12 600. The dealer offers terms of 10% deposit and monthly repayments of \$812.70 for 2 years.
   a. Calculate the amount of the deposit.
   b. Calculate the total amount to be paid in monthly repayments.
   c. What is the total amount Dmitry pays for the car?
   d. How much more than the cash price of the car does Dmitry pay? (This is the interest charged by the dealer.)
7. Alja wants to purchase an entertainment system that has a cash price of \$5800. She purchases the entertainment system on terms of no deposit and monthly repayments of \$233.61 for 3 years.
   a. Calculate the total amount that Alja pays for the entertainment system.
   b. Calculate the amount that Alja pays in interest.
   c. Calculate the amount of interest that Alja pays each year.
   d. Calculate this amount as a percentage of the cash price of the entertainment system.
8. **WE4** A used car has a purchase price of \$9500. Dayna buys the car on the following terms: 25% deposit with balance plus interest paid at 12% p.a. interest over 3 years.
   a. Calculate the amount of the deposit.
   b. What is the balance owing?
   c. Calculate the interest payable.
   d. What is the total amount to be repaid?
   e. Find the amount of each monthly repayment.

9. A department store offers the following terms: one-third deposit with the balance plus interest paid in equal, monthly instalments over 18 months. The interest rate charged is 9% p.a. Ming buys a lounge suite with a ticketed price of $6000.

a. Calculate the amount of the deposit.
b. What is the balance owing?
c. Calculate the interest payable.
d. What is the total amount to be repaid?
e. Find the amount of each monthly repayment.



10. Calculate the monthly payment on each of the following items bought on terms.
(*Hint:* Use the steps shown in question **8**.)

a. Dining suite: cash price $2700, deposit 10%, interest rate 12% p.a., term 1 year
b. Video camera: cash price $990, deposit 20%, interest rate 15% p.a., term 6 months
c. Car: cash price $16 500, deposit 25%, interest rate 15% p.a., term 5 years
d. Mountain bike: cash price $3200, one-third deposit, interest rate 9% p.a., term $2\frac{1}{2}$ years
e. Watch: cash price $675, no deposit, interest rate 18% p.a., term 9 months

11. Samin wants to purchase his first car. He has saved $1000 as a deposit but the cost of the car is $5000. Samin takes out a loan from the bank to cover the balance of the car plus $600 worth of on-road costs.

a. How much will Samin need to borrow from the bank?
b. Samin takes the loan out over 4 years at 9% p.a. interest. How much interest will Samin need to pay?
c. What will be the amount of each monthly payment that Samin makes?
d. What is the total cost of the car after paying off the loan, including the on-road costs? Give your answer to the nearest $.

12. **MC** Kelly wants to borrow $12 000 for some home improvements. Which of the following loans will lead to Kelly making the lowest total repayment?

A. Interest rate 6% p.a. over 4 years
B. Interest rate 7% p.a. over 3 years
C. Interest rate 5.5% p.a. over $3\frac{1}{2}$ years
D. Interest rate 6.5% p.a. over 5 years
E. Interest rate 7.5% p.a. over 3 years

Reasoning

13. **MC** Without completing any calculations explain which of the following loans will be the best value for the borrower.
    - **A.** Interest rate 8.2% p.a. over 5 years
    - **B.** Interest rate 8.2% over 4 years
    - **C.** Interest rate 8% over 6 years
    - **D.** Interest rate 8% over 5 years
    - **E.** Interest rate 8% over 4 years
14. Explain how, when purchasing an item, making a deposit using existing savings and taking out a loan for the balance can be an advantage.

Problem solving

15. Gavin borrows \$18 000 over 5 years from the bank. The loan is charged at 8.4% p.a. flat-rate interest. The loan is to be repaid in equal monthly instalments. Calculate the amount of each monthly repayment.
16. Andrew purchased a new car valued at \$32 000. He paid a 10% deposit and was told he could have 4 years to pay off the balance of the car price plus interest. An alternate scheme was also offered to him. It involved paying off the balance of the car price plus interest in 8 years. If he chose the latter scheme, he would end up paying \$19 584 more. The interest rate for the 8-year scheme was 1% more than for the 4-year scheme.
    - **a.** How much deposit did he pay?
    - **b.** What was the balance to be paid on the car?
    - **c.** Find the interest rate for each of the two schemes.
    - **d.** Find the total amount paid for the car for each of the schemes.
    - **e.** What were the monthly repayments for each of the schemes?

Reflection

When buying on terms, what arrangements are the most beneficial to the buyer?

CHALLENGE 15.1

Ingrid offered to pay her brother \$2 for doing her share of the housework each day, but fined him \$5 if he forgot to do it. After 4 weeks, Ingrid discovered that she did not owe her brother any money. For how many days did Ingrid's brother do her share of the housework?

15.4 Successive discounts

15.4.1 Successive discounts

- Consider the case of Ziggy, who is a mechanic. Ziggy purchases his hardware from Tradeways hardware store, which is having a 10%-off sale. Tradeways also offers a 5% discount to tradespeople. Ziggy purchases hardware that has a total value of $800. What price does Ziggy pay for these supplies?

  After the 10% discount, the price of the supplies is

$$\begin{aligned}90\% \text{ of } \$800 &= 0.90 \times \$800\\ &= \$720\end{aligned}$$

  The 5% trades discount is then applied.

$$\begin{aligned}95\% \text{ of } \$720 &= 0.95 \times \$720\\ &= \$684\end{aligned}$$

  So the price Ziggy pays is $684.
  Now let us consider what single discount Ziggy has actually received.

$$\begin{aligned}\text{Amount of discount} &= \$800 - \$684\\ &= \$116\\ \text{Percentage discount} &= \frac{\$116}{\$800} \times 100\%\\ &= 14.50\%\end{aligned}$$

  So we can conclude that the **successive discounts** of 10% followed by a further 5% is equivalent to receiving a single discount of 14.50%.
- When two discounts are applied one after the other, the total discount is *not* the same as a single discount found by adding the two percentages together.
- The order of calculating successive discounts does not affect the final answer.

WORKED EXAMPLE 5 TI | CASIO

A furniture store offers a discount of 15% during a sale. A further 5% discount is then offered to customers who pay cash.

a Find the price paid by Lily, who pays cash for a bedroom suite originally priced at $2500.

b What single percentage discount does Lily receive on the price of the bedroom suite?

THINK	WRITE
a 1 Subtract 15% from 100% to find the percentage paid.	**a** 100% − 15% = 85%

	2 Calculate 85% of the price.	85% of \$2500 $= 0.85 \times \$2500$ $= \$2125$
	3 Subtract 5% from 100% to find the next percentage paid.	$100\% - 5\% = 95\%$
	4 Calculate 95% of \$2125.	95% of \$2125 $= 0.95 \times \$2125$ $= \$2018.75$
b	**1** Calculate the amount of discount received.	**b** Discount $= \$2500 - \$2018.75$ $= \$481.25$
	2 Express the discount as a percentage of the original marked price.	Percentage discount $= \frac{\$481.25}{\$2500} \times 100\%$ $= 19.25\%$

- The single discount that is equivalent to successive discounts can also be worked out by working out a percentage of a percentage, as shown in Worked example 6.

WORKED EXAMPLE 6

Find the single percentage discount that is equivalent to successive discounts of 15% and 5%.

THINK	WRITE
1 Subtract 15% from 100% to find the percentage paid after the first discount.	$100\% - 15\% = 85\%$
2 Subtract 5% from 100% to find the percentage paid after the second discount.	$100\% - 5\% = 95\%$
3 Find 95% of 85%. This is actually the percentage of the marked price that the customer pays.	95% of 85% $= 0.95 \times 0.85$ $= 0.8075$ $= 80.75\%$
4 Subtract the percentage from 100% to find the single percentage discount. This answer should be less than 15% + 5%.	Discount $= 100\% - 80.75\%$ $= 19.25\%$

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Complete this digital doc: SkillSHEET: Decreasing a quantity by a percentage (doc-5349)

Complete this digital doc: SkillSHEET: Expressing one quantity as a percentage of another (doc-5351)

Exercise 15.4 Successive discounts

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–4, 6, 8, 10, 11, 13	Questions: 2–7, 9–11, 13	Questions: 2–6, 8–14

🟩🟦🟪 **Individual pathway interactivity:** int-4635

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Fluency

1. In each of the following, an item is reduced in price. Calculate the percentage discount, correct to 1 decimal place.

- **a.** A jumper, usually $29.95, is reduced to $24.95.
- **b.** A video game, usually $60, is reduced to $53.90.
- **c.** A child's bike, usually $158, is reduced to $89.
- **d.** A new car, usually $29 500, is reduced to $24 950.
- **e.** A plot of land, priced at $192 000, is reduced to $177 500 for a quick sale.

2. **WE6** Calculate the single percentage discount that is equivalent to successive discounts of 15% and 10%.

3. **MC** The single percentage discount that is equivalent to successive discounts of 10% and 20% is:

A. 10% **B.** 18% **C.** 28% **D.** 30% **E.** 35%

4. Find the single percentage discount that is equivalent to each of the following successive discounts.

a. 15% and 20% **b.** 12% and 8% **c.** 10% and 7.5% **d.** 50% and 15%

5. Calculate the single percentage discount that is equivalent to two successive 10% discounts.

Understanding

6. **WE5** A supplier of electrical parts offers tradespeople a 20% trade discount. If accounts are settled within 7 days, a further 5% discount is given.

- **a.** Calculate the price paid by an electrician for parts to the value of $4000 if the account is settled within 7 days.
- **b.** What single percentage discount does the electrician receive on the price of the electrical parts?

7. At a confectionary wholesaler, customers have their accounts reduced by 10% if they are paid within 7 days.

- **a.** Jacinta pays her $100 account within 7 days. How much does she actually pay?
- **b.** If customers pay cash, they receive a further 5% discount. How much would Jacinta pay if she pays cash?
- **c.** By how much in total has her account been reduced?
- **d.** What is the single percentage discount equivalent to these successive discounts?

8. A fabric supplier offers discounts to fashion stores and a further discount if the store's account is paid with 14 days. 'David's Fashion Stores' have ordered fabric to the value of $2000 from the fabric supplier.

a. If fashion stores receive a reduction of 8%, how much does 'David's Fashion Stores' owe on its account?

b. This amount is reduced by a further 5% for payment within 14 days. How much needs to be paid now?

c. What has been the total reduction in the cost?

d. What do the successive discounts of 8% and 5% equal as a single percentage discount?

9. Tony is a mechanic who wants to buy equipment worth $250 at a hardware store. Tony receives 15% off the marked price of all items and then a further 5% trade discount.

a. Calculate the amount that is due after Tony is given the first 15% discount.

b. From this amount, apply the trade discount of 5% to find the amount due.

c. How much is the cash discount that Tony receives?

d. Calculate the amount that would have been due had Tony received a single discount of 20%. Is this the same answer?

e. Calculate the amount of cash discount that Tony receives as a percentage of the original bill.

f. Would the discount have been the same had the 5% discount been applied before the 15% discount?

g. Calculate the single percentage discount that is equivalent to successive discounts of 10% and 20%.

10. A car has a marked price of $25 000.

a. Find the price paid for the car after successive discounts of 15%, 10% and 5%.

b. What single percentage discount is equivalent to successive discounts of 15%, 10% and 5%?

Reasoning

11. Is a *12.5% discount followed by a 2.5% discount* the same single discount as a *2.5% discount followed by a 12.5% discount?* Investigate and explain your answer giving mathematical evidence.

12. Derive a mathematical formula to calculate the single discount (expressed as a decimal) generated by two successive discounts, a and b (expressed as decimals).

Problem solving

13. The Fruitz fruit and vegetable shop is selling grapes at a price which is 10% cheaper than the Happy Fruiterer fruit and vegetable shop. A customer bought n kilograms of grapes from the Fruitz shop for $50.

a. How much would this $50 worth of grapes cost had he bought the grapes from the Happy Fruiterer?

b. The Happy Fruiterer wants to be competitive so for the coming week he discounts the grapes by 5% and the following week discounts them a further 5%. How much would the grapes originally bought for $50 from Fruitz cost during this second week of discounting at the Happy Fruiterer?

c. Which shop is the cheapest during this second week of discounting?

14. The Big Rabbit Easter eggs were reduced from $3.00 to $2.35 each just before Easter so they would clear. There were still eggs remaining after Easter so they were further reduced to $1.95 each. Under similar circumstances the Hoppity Hop Easter eggs were reduced from $4.75 to $3.85 and then to $3.25 each.

a. What was the total percentage discount for each egg?
b. What was the difference in the total percentage discounts and which egg was discounted by the larger amount?

Reflection

In what situations might a successive discount be applied?

15.5 Compound interest

15.5.1 Compound interest

- Interest on the principal in a savings account, or short or long term deposit, is generally calculated using compound interest rather than simple interest.
- When interest is added to the principal at regular intervals, increasing the balance of the account, and each successive interest payment is calculated on the new balance, it is called **compound interest**.
- Compound interest can be calculated by calculating simple interest one period at a time.
- The amount to which the initial investment grows is called the **compounded value** or **future value**.

WORKED EXAMPLE 7

Kyna invests \$8000 at 8% p.a. for 3 years with interest paid at the end of each year. Find the compounded value of the investment by calculating the simple interest on each year separately.

THINK	WRITE
1 Write the initial (first year) principal.	Initial principal = \$8000
2 Calculate the interest for the first year.	Interest for year 1 = 8% of \$8000 = \$640
3 Calculate the principal for the second year by adding the first year's interest to the initial principal.	Principal for year 2 = \$8000 + \$640 = \$8640
4 Calculate the interest for the second year.	Interest for year 2 = 8% of \$8640 = \$691.20
5 Calculate the principal for the third year by adding the second year's interest to the second year's principal.	Principal for year 3 = \$8640 + \$691.20 = \$9331.20
6 Calculate the interest for the third year.	Interest for year 3 = 8% of \$9331.20 = \$746.50
7 Calculate the future value of the investment by adding the third year's interest to the third year's principal.	Compounded value after 3 years = \$9331.20 + \$746.50 = \$10 077.70

- To calculate the actual amount of interest received, we subtract the initial principal from the future value.
- In the example above, compound interest = \$10077.70 − \$8000
  = \$2077.70

  We can compare this with the simple interest earned at the same rate.

$$\begin{aligned} I &= \frac{p \times r \times T}{100} \\ &= \frac{8000 \times 8 \times 3}{100} \\ &= \$1920 \end{aligned}$$

- The table below shows a comparison between the total interest earned on an investment of \$8000 earning 8% p.a. at both simple interest (I) and compound interest (CI) over an eight year period.

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Total (I)	\$640.00	\$1280.00	\$1920.00	\$2560.00	\$3200.00	\$3840.00	\$4480.00	\$5120.00
Total (CI)	\$640.00	\$1331.20	\$2077.70	\$2883.91	\$3754.62	\$4694.99	\$5710.59	\$6807.44

- We can develop a formula for the future value of an investment rather than do each example by repeated use of simple interest. Consider Worked example 7. Let the compounded value after each year, n, be A_n.

$$\begin{aligned}
\text{After 1 year, } A_1 &= 8000 \times 1.08 \text{ (increasing \$8000 by 8\%)}\\
\text{After 2 years, } A_2 &= A_1 \times (1.08)\\
&= 8000 \times 1.08 \times 1.08 \text{ (substituting the value of } A_1\text{)}\\
&= 8000 \times 1.08^2\\
\text{After 3 years, } A_3 &= A_2 \times 1.08\\
&= 8000 \times 1.08^2 \times 1.08 \text{ (substituting the value of } A_2\text{)}\\
&= 8000 \times 1.08^3
\end{aligned}$$

The pattern then continues such that the value of the investment after n years equals:

$$\$8000 \times 1.08^n.$$

This can be generalised for any investment:

$$\boxed{A = P(1 + R)^n}$$

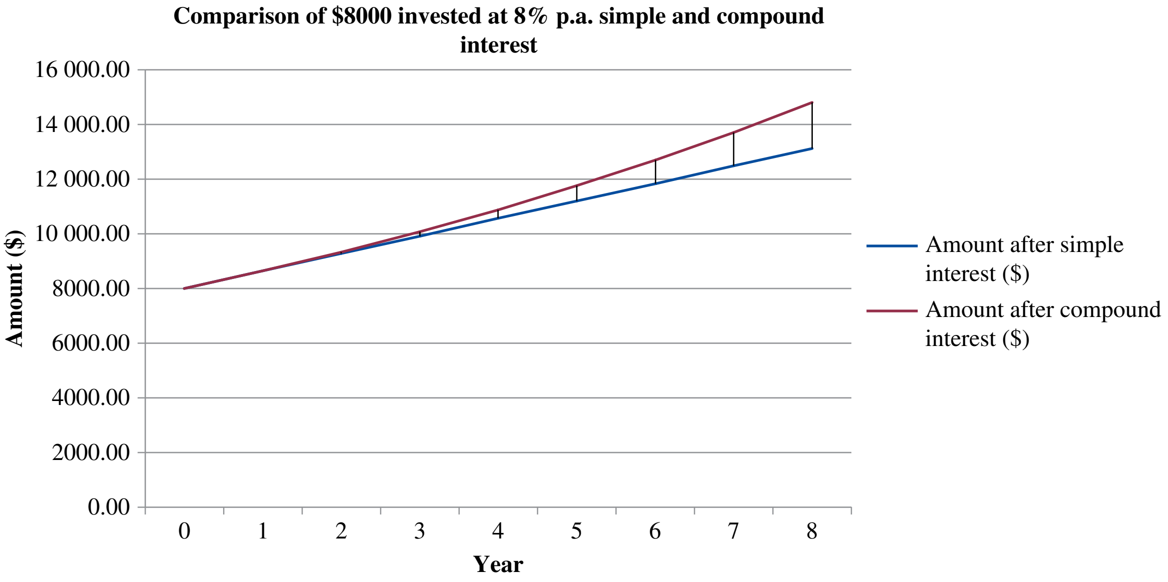
where A = amount (or future value) of the investment
P = principal (or present value)
R = interest rate per compounding period expressed as a decimal
number of compounding periods.

- To calculate the amount of compound interest (CI) we then use the formula

$$\boxed{CI = A - P}$$

15.5.2 Using technology

- Digital technologies such as spreadsheets can be used to draw graphs in order to compare interest accrued through simple interest and compound interest.



WORKED EXAMPLE 8 TI | CASIO

William has \$14 000 to invest. He invests the money at 9% p.a. for 5 years with interest compounded annually.

a Use the formula $A = P(1 + R)^n$ to calculate the amount to which this investment will grow.

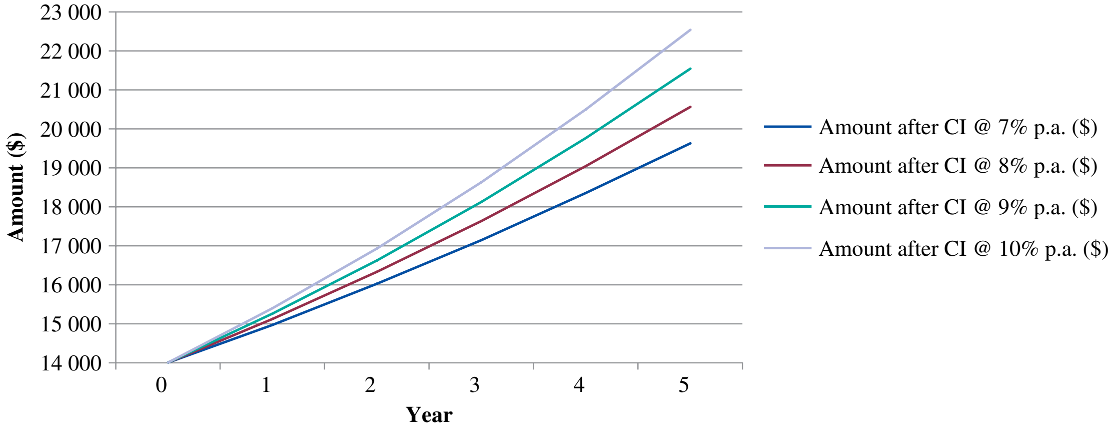
b Calculate the compound interest earned on the investment.

THINK	WRITE
a 1 Write the compound interest formula.	**a** $A = P(1 + R)^n$
2 Write down the values of P, R and n.	$P = \$14000, R = 0.09, n = 5$
3 Substitute the values into the formula.	$A = \$14000 \times 1.09^5$
4 Calculate.	$= \$21540.74$ The investment will grow to \$21 540.74.
b Calculate the compound interest earned.	**b** $CI = A - P$ $= \$21540.74 - \$14000$ $= \$7540.74$ The compound interest earned is \$7540.74.

15.5.3 Comparison of fixed principal at various interest rates over a period of time

- It is often helpful to compare the future value (\$A) of the principal at different compounding interest rates over a fixed period of time.
- Spreadsheets are very useful tools for making comparisons. The graph below, generated from a spreadsheet, shows the comparisons for \$14 000 invested for 5 years at 7%, 8%, 9% and 10% compounding annually.
- There is a significant difference in the future value depending on which interest rate is applied.

\$14 000 invested at 7%, 8%, 9% and 10% p.a. over five years



15.5.4 Compounding period

- In Worked example 8, interest is paid annually.
- Interest can be paid more regularly — it may be paid six-monthly (twice a year), quarterly (4 times a year), monthly or even daily. This is called the **compounding period**.
- The time and interest rate on an investment must reflect the compounding period. For example, an investment over 5 years at 6% p.a. compounding *quarterly* will have:

$$n = 20\ (5 \times 4) \text{ and } R = 0.015\ (6\% \div 4).$$

- To find n:

$$n = \text{number of years} \times \text{compounding periods per year}$$

- To find R:

$$R = \text{interest rate per annum} \div \text{compounding periods per year}$$

WORKED EXAMPLE 9 TI | CASIO

Calculate the future value of an investment of \$4000 at 6% p.a. for 2 years with interest compounded quarterly.

THINK	**WRITE**
1 Write the compound interest formula.	$A = P(1 + R)^n$
2 Write the values of P, R and n.	$P = \$4000, R = 0.015, n = 8$
3 Substitute the values into the formula.	$A = \$4000 \times 1.015^8$
4 Calculate.	$= \$4505.97$ The future value of the investment is \$4505.97.

15.5.5 Guess and refine

- Sometimes it is useful to know approximately how long it will take to reach a particular future value once an investment has been made. Mathematical formulas can be applied to determine when a particular future value will be reached. In this section, a 'guess and refine' method will be shown.
- For example, to determine the number of years required for an investment of \$1800 at 9% compounded quarterly to reach a future value of \$2500, the following method can be used.
- Let n = the number of compounding periods (quarters) and A = the future value in \$.

n	$A = P\left(1 + \frac{R}{4}\right)^n$	**Comment**
1	\$1840.50	It is useful to know how the principal is growing after 1 quarter, but the amount is quite far from \$2500.
3	\$1924.25	The amount is closer to \$2500 but still a long way off, so jump to a higher value for n.
10	\$2248.57	The amount is much closer to \$2500.
12	\$2350.89	The amount is much closer to \$2500.
14	\$2457.87	The amount is just below \$2500.
15	\$2513.17	The amount is just over \$2500.

Therefore, it will take approximately 15 quarters, or 3 years and 9 months, to reach the desired amount.

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Try out this interactivity: Compound interest (int-2791)

Complete this digital doc: WorkSHEET: Compound interest (doc-14608)

Exercise 15.5 Compound interest

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–3, 5, 6, 11, 13, 15, 18	Questions: 1–6, 7, 9, 10, 12–16, 18	Questions: 1–8, 11, 12, 14–19

🟩🟦🟪 **Individual pathway interactivity:** int-4636

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Fluency

1. Use the formula $A = P(1 + R)^n$ to calculate the amount to which each of the following investments will grow with interest compounded annually.

a. \$3000 at 4% p.a. for 2 years

b. \$9000 at 5% p.a. for 4 years

c. \$16 000 at 9% p.a. for 5 years

d. \$12 500 at 5.5% p.a. for 3 years

e. \$9750 at 7.25% p.a. for 6 years

f. \$100 000 at 3.75% p.a. for 7 years

2. Calculate the compounded value of each of the following investments.

a. \$870 for 2 years at 3.50% p.a. with interest compounded six-monthly

b. \$9500 for $2\frac{1}{2}$ years at 4.6% p.a. with interest compounded quarterly

c. \$148 000 for $3\frac{1}{2}$ years at 9.2% p.a. with interest compounded six-monthly

d. \$16 000 for 6 years at 8% p.a. with interest compounded monthly

e. \$130 000 for 25 years at 12.95% p.a. with interest compounded quarterly

Understanding

3. **WE7** Danielle invests \$6000 at 10% p.a. for 4 years with interest paid at the end of each year. Find the compounded value of the investment by calculating the simple interest on each year separately.

4. Ben is to invest \$13 000 for 3 years at 8% p.a. with interest paid annually. Find the amount of interest earned by calculating the simple interest for each year separately.

5. **WE8** Simon has \$2000 to invest. He invests the money at 6% p.a. for 6 years with interest compounded annually.

a. Use the formula $A = P(1 + R)^n$ to calculate the amount to which this investment will grow.

b. Calculate the compound interest earned on the investment.

6. **WE9** Calculate the future value of an investment of \$14 000 at 7% p.a. for 3 years with interest compounded quarterly.

7. A passbook savings account pays interest of 0.3% p.a. Jill has \$600 in such an account. Calculate the amount in Jill's account after 3 years, if interest is compounded quarterly.

8. Damien is to invest \$35 000 at 7.2% p.a. for 6 years with interest compounded six-monthly. Calculate the compound interest earned on the investment.

9. Sam invests \$40 000 in a one-year fixed deposit at an interest rate of 7% p.a. with interest compounding monthly.

a. Convert the interest rate of 7% p.a. to a rate per month.

b. Calculate the value of the investment upon maturity.

10. **MC** A sum of \$7000 is invested for 3 years at the rate of 5.75% p.a., compounded quarterly. The interest paid on this investment, to the nearest dollar, is:

A. \$1208 **B.** \$1308 **C.** \$8208 **D.** \$8308 **E.** \$8508

11. **MC** After selling their house and paying off their mortgage, Mr and Mrs Fong have $73 600. They plan to invest it at 7% p.a. with interest compounded annually. The value of their investment will first exceed $110 000 after:

A. 5 years
B. 6 years
C. 8 years
D. 10 years
E. 15 years

12. **MC** Maureen wishes to invest $15 000 for a period of 7 years. The following investment alternatives are suggested to her. The best investment would be:

A. simple interest at 8% p.a.
B. compound interest at 6.7% p.a. with interest compounded annually
C. compound interest at 6.6% p.a. with interest compounded six-monthly
D. compound interest at 6.5% p.a. with interest compounded quarterly
E. compound interest at 6.4% p.a. with interest compounded monthly

13. **MC** An amount is to be invested for 5 years and compounded semi-annually at 7% p.a. Which of the following investments will have a future value closest to $10 000?

A. $700 **B.** $6500 **C.** $7400 **D.** $9000 **E.** $9900

14. Jake invests $120 000 at 9% p.a. for a 1-year term. For such large investments interest is compounded daily.

a. Calculate the daily percentage interest rate, correct to 4 decimal places. Use 1 year = 365 days.
b. Hence, calculate the compounded value of Jake's investment on maturity.
c. Calculate the amount of interest paid on this investment.
d. Calculate the extra amount of interest earned compared with the case where the interest is calculated only at the end of the year.

Reasoning

15. Daniel has $15 500 to invest. An investment over a 2-year term will pay interest of 7% p.a.

a. Calculate the compounded value of Daniel's investment if the compounding period is:
  i. 1 year **ii.** 6 months **iii.** 3 months **iv.** monthly.
b. Explain why it is advantageous to have interest compounded on a more frequent basis.

16. Jasmine invests $6000 for 4 years at 8% p.a. simple interest. David also invests $6000 for 4 years, but his interest rate is 7.6% p.a. with interest compounded quarterly.

a. Calculate the value of Jasmine's investment on maturity.
b. Show that the compounded value of David's investment is greater than Jasmine's investment.
c. Explain why David's investment is worth more than Jasmine's investment despite receiving a lower rate of interest.

17. Quan has $20 000 to invest over the next 3 years. He has the choice of investing his money at 6.25% p.a. simple interest or 6% p.a. compound interest.

a. Calculate the amount of interest that Quan will earn if he selects the simple interest option.
b. Calculate the amount of interest that Quan will earn if the interest is compounded:
  i. annually **ii.** six monthly **iii.** quarterly.
c. Clearly Quan's decision will depend on the compounding period. Under what conditions should Quan accept the lower interest rate on the compound interest investment?
d. Consider an investment of $10 000 at 8% p.a. simple interest over 5 years. Use a trial-and-error method to find an equivalent rate of compound interest over the same period.
e. Will this equivalent rate be the same if we change:
  i. the amount of the investment
  ii. the period of the investment?

Problem solving

18. A building society advertises investment accounts at the following rates:

- **a.** 3.875% p.a. compounding daily
- **b.** 3.895% p.a. compounding monthly
- **c.** 3.9% p.a. compounding quarterly.

Peter thinks the first account is the best one because the interest is calculated more frequently. Paul thinks the last account is the best one because it has the highest interest rate. Explain whether either is correct.

19. Two banks offer the following investment packages.

Bankwest: 7.5% p. a. compounded annually fixed for 7 years.

Bankeast: 5.8% p. a. compounded annually fixed for 9 years.

- **a.** Which bank's package will yield the greatest interest?
- **b.** If a customer invests \$20 000 with Bankwest, how much would she have to invest with Bankeast to produce the same amount as Bankwest at the end of the investment period?

Reflection

How is compound interest calculated differently to simple interest?

CHALLENGE 15.2

How long will it take for a sum of money to double if it is invested at a rate of 15% p.a. compounded monthly?

15.6 Depreciation

15.6.1 Depreciation

- **Depreciation** is the reduction in the value of an item as it ages over a period of time. For example, a car that is purchased new for \$45 000 will be worth less than that amount 1 year later and less again each year.
- Depreciation is usually calculated as a percentage of the yearly value of the item.
- To calculate the depreciated value of an item use the formula
$$A = P(1 - R)^n$$
where A is the depreciated value of the item, P is the initial value of the item, R is the percentage that the item depreciates each year expressed as a decimal and n is the number of years that the item has been depreciating for.
- This formula is almost the same as the compound interest formula except that it subtracts a percentage of the value each year instead of adding.
- In many cases, depreciation can be a tax deduction.
- When the value of an item falls below a certain value it is said to be *written off*. That is to say that, for tax purposes, the item is considered to be worthless.
- Trial-and-error methods can be used to calculate the length of time that the item will take to reduce to this value.

WORKED EXAMPLE 10

A farmer purchases a tractor for \$115 000. The value of the tractor depreciates by 12% p.a. Find the value of the tractor after 5 years.

THINK	WRITE
1 Write the depreciation formula.	$A = P(1 - R)^n$
2 Write the values of P, R and n.	$P = \$115\,000, R = 0.12, n = 5$
3 Substitute the values into the formula.	$A = \$115\,000 \times (0.88)^5$
4 Calculate.	$= \$60\,689.17$ The value of the tractor after 5 years is \$60 689.17.

WORKED EXAMPLE 11

TI | CASIO

A truck driver buys a new prime mover for \$500 000. The prime mover depreciates at the rate of 15% p.a. and is written off when its value falls below \$100 000. How long will it take for the prime mover to be written off?

THINK	WRITE
1 Make an estimate of, say, $n = 5$. Use the depreciation formula to find the value of the prime mover after 5 years.	Consider $n = 5$. $A = P(1 - R)^n$ $= \$500\,000 \times (0.85)^5$ $= \$221\,852.66$
2 Because the value will still be greater than \$100 000, try a larger estimate, say, $n = 10$.	Consider $n = 10$. $A = P(1 - R)^n$ $= \$500\,000 \times (0.85)^{10}$ $= \$98\,437.20$
3 As the value is below \$100 000, check $n = 9$.	Consider $n = 9$. $A = P(1 - R)^n$ $= \$500\,000 \times (0.85)^9$ $= \$115\,808.47$
4 Because $n = 10$ is the first time that the value falls below \$100 000, conclude that it takes 10 years to be written off.	The prime mover will be written off in 10 years.

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Watch this eLesson: What is depreciation? (eles-0182)

Try out this interactivity: Different rates of depreciation (int-1155)

Exercise 15.6 Depreciation

assesson

Individual Pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 7, 9, 12, 14, 16	Questions: 1–3, 5, 7, 9, 11, 12, 14, 16	Questions: 1–8, 11, 12, 14–17

Individual pathway interactivity: int-4637

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Fluency

1. Calculate the depreciated value of an item for the initial value, depreciation rate and time, in years, given below.

a. Initial value of \$30 000 depreciating at 16% p.a. over 4 years
b. Initial value of \$5000 depreciating at 10.5% p.a. over 3 years
c. Initial value of \$12 500 depreciating at 12% p.a. over 5 years

Understanding

2. **WE10** A laundromat installs washing machines and clothes dryers to the value of \$54 000. If the value of the equipment depreciates at a rate of 20% p.a., find the value of the equipment after 5 years.

3. A drycleaner purchases a new machine for \$38 400. The machine depreciates at 16% p.a.

a. Calculate the value of the machine after 4 years.
b. Find the amount by which the machine has depreciated over this period of time.

4. A tradesman values his new tools at \$10 200. For tax purposes, their value depreciates at a rate of 15% p.a.

a. Calculate the value of the tools after 6 years.
b. Find the amount by which the value of the tools has depreciated over these 6 years.
c. Calculate the percentage of the initial value that the tools are worth after 6 years.

5. A taxi is purchased for \$52 500 with its value depreciating at 18% p.a.

a. Find the value of the taxi after 10 years.
b. Calculate the accumulated depreciation over this period.

6. A printer depreciates the value of its printing presses by 25% p.a. Printing presses are purchased new for \$2.4 million. What is the value of the printing presses after:

a. 1 year **b.** 5 years **c.** 10 years?

7. **MC** A new computer workstation costs \$5490. With 26% p.a. reducing-value depreciation, the workstation's value at the end of the third year will be close to:

A. \$1684 **B.** \$2225 **C.** \$2811
D. \$3082 **E.** \$3213

8. **MC** The value of a new photocopier is \$8894. Its value depreciates by 26% in the first year, 21% in the second year and 16% reducing balance in the remaining 7 years. The value of the photocopier after this time, to the nearest dollar, is:

A. \$1534 **B.** \$1851 **C.** \$2624 **D.** \$3000 **E.** \$3504

9. **MC** A company was purchased 8 years ago for \$2.6 million. With a depreciation rate of 12% p.a., the total amount by which the company has depreciated is closest to:

A. \$0.6 million **B.** \$1.0 million **C.** \$1.7 million
D. \$2.0 million **E.** \$2.3 million

10. **MC** Equipment is purchased by a company and is depreciated at the rate of 14% p.a. The number of years that it will take for the equipment to reduce to half of its initial value is:

A. 4 years **B.** 5 years **C.** 6 years **D.** 7 years **E.** 8 years

11. **MC** An asset, bought for \$12 300, has a value of \$6920 after 5 years. The depreciation rate is close to:

A. 10.87% **B.** 16.76% **C.** 18.67% **D.** 21.33% **E.** 27.34%

12. WE11 A farmer buys a light aeroplane for crop-dusting. The aeroplane costs \$900 000. The aeroplane depreciates at the rate of 18% p.a. and is written off when its value falls below \$150 000. How long will it take for the aeroplane to be written off?
13. A commercial airline buys a jumbo jet for \$750 million. The value of this aircraft depreciates at a rate of 12.5% p.a.
    - **a.** Find the value of the plane after 5 years, correct to the nearest million dollars.
    - **b.** How many years will it take for the value of the jumbo jet to fall below \$100 million?

Reasoning

14. A machine purchased for \$48 000 will have a value of \$3000 in 9 years.
    - **a.** Use a trial-and-error method to find the rate at which the machine is depreciating per annum.
    - **b.** Consider the equation $x = a^n$, $a = \sqrt[n]{x}$. Verify your answer to part **a** using this relationship.
15. Camera equipment purchased for \$150 000 will have a value of \$9000 in 5 years.
    - **a.** Find the rate of annual depreciation using trial and error first and then algebraically with the relationship ‘if $x = a^n$ then $a = \sqrt[n]{x}$’.
    - **b.** Compare and contrast each method.

Problem solving

16. The value of a new tractor is \$175 000. The value of the tractor depreciates by 22.5% p.a.
    - **a.** Find the value of the tractor after 8 years.
    - **b.** What percentage of its initial value is the tractor worth after 8 years?
17. Anthony has a home theatre valued at \$$P$. The value of the home theatre depreciates by r% annually over a period of 5 years. At the end of the 5 years, the value of the home theatre has been reduced by \$$\frac{P}{12}$. Find the value of r correct to 3 decimal places.

Reflection

How and why is the formula for depreciation different to compound interest?

15.7 Loan repayments

15.7.1 Loan repayments

- The simple interest formula is used to calculate the interest on a **flat-rate loan**.

WORKED EXAMPLE 12 TI | CASIO

Calculate the interest payable on a loan of \$5000 to be repaid at 12% p.a. flat interest over 4 years.

THINK	WRITE
1 Write the simple interest formula.	$I = \frac{P \times r \times T}{100}$
2 List the known values.	$P = \$5000, r = 12\%, T = 4$
3 Substitute the values into the formula.	$I = \frac{5000 \times 12 \times 4}{100}$
4 Calculate the interest.	$= \$2400$ The interest payable is \$2400.

- The total amount that would have to be repaid under the loan in Worked example 12 is \$7400, and this could be made in 4 equal payments of \$1850. With a flat-rate loan, the interest is calculated on the initial amount borrowed regardless of the amount of any repayments made.
- In contrast, taking a **reducible-interest-rate loan** means that each annual amount of interest is based on the amount owing at the time.
- Consider the same loan of \$5000, this time at 12% p.a. reducible interest and an agreed annual repayment of \$1850. At the end of each year, the outstanding balance is found by adding the amount of interest payable and then subtracting the amount of each repayment.

$$\begin{aligned}
\text{Interest for year 1} &= 12 \text{ of } \$5000 \\
&= 0.12 \times \$5000 \\
&= \$600 \\
\text{Balance for year 2} &= \$5000 + \$600 - \$1850 \\
&= \$3750 \\
\text{Interest for year 2} &= 12 \times \$3750 \\
&= 0.12 \times \$3750 \\
&= \$450 \\
\text{Balance for year 3} &= \$3750 + \$450 - \$1850 \\
&= \$2350 \\
\text{Interest for year 3} &= 12 \text{ of } \$2350 \\
&= 0.12 \times \$2350 \\
&= \$282 \\
\text{Balance for year 4} &= \$2350 + \$282 - \$1850 \\
&= \$782 \\
\text{Interest for year 4} &= 12 \text{ of } \$782 \\
&= 0.12 \times \$782 \\
&= \$93.84
\end{aligned}$$

In the fourth year, a payment of only \$875.84 is required to fully repay the loan. The total amount of interest charged on this loan is \$1425.84, which is \$974.16 less than the same loan calculated using flat-rate interest.

WORKED EXAMPLE 13

Calculate the amount of interest paid on a loan of \$10 000 that is charged at 9% p.a. reducible interest over 3 years. The loan is repaid in two annual instalments of \$4200 and the balance at the end of the third year.

THINK	WRITE
1 Calculate the interest for the first year.	$\text{Interest for year 1} = 9\% \text{ of } \$10\,000$ $= 0.09 \times \$10\,000$ $= \$900$
2 Calculate the balance at the start of the second year.	$\text{Balance for year 2} = \$10\,000 + \$900 - \$4200$ $= \$6700$
3 Calculate the interest for the second year.	$\text{Interest for year 2} = 9\% \text{ of } \$6700$ $= 0.09 \times \$6700$ $= \$603$
4 Calculate the balance at the start of the third year.	$\text{Balance for year 3} = \$6700 + \$603 - \$4200$ $= \$3103$
5 Calculate the interest for the third year.	$\text{Interest for year 3} = 9\% \text{ of } \$3103$ $= 0.09 \times \$3103$ $= \$279.27$

6 Calculate the amount of the final repayment and ensure that the loan is fully repaid.	Balance remaining at end of year 3 = \$3103 + \$279.27 = \$3382.27
7 Find the total amount of interest paid by adding each year's amount.	Interest charged = \$900 + \$603 + \$279.27 = \$1782.27

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Complete this digital doc: WorkSHEET: Loan repayments (doc-14609)

Exercise 15.7 Loan repayments

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 2, 4, 7, 10, 12	Questions: 1–4, 6, 8, 9, 11, 12	Questions: 1, 2c–e, 3–5, 8–13

🟩🟦🟪 **Individual pathway interactivity:** int-4638 learnon ONLINE ONLY

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Fluency

1. **WE12** Calculate the interest payable on a loan of \$10 000 to be repaid at 15% p.a. flat-rate interest over 3 years.

2. Calculate the interest payable on each of the following loans.
   a. \$20 000 at 8% p.a. flat-rate interest over 5 years
   b. \$15 000 at 11% p.a. flat-rate interest over 3 years
   c. \$7500 at 12.5% p.a. flat-rate interest over 2 years
   d. \$6000 at 9.6% p.a. flat-rate interest over 18 months
   e. \$4000 at 21% p.a. flat-rate interest over 6 months

Understanding

3. Larry borrows \$12 000 to be repaid at 12% p.a. flat rate of interest over 4 years.
   a. Calculate the interest that Larry must pay.
   b. What is the total amount that Larry must repay?
   c. If Larry repays the loan in equal annual instalments, calculate the amount of each repayment.

4. **WE13** Calculate the amount of interest paid on a loan of \$12 000 that is charged at 10% p.a. reducible interest over 3 years. The loan is repaid in two annual instalments of \$5000 and the balance at the end of the third year.

5. Calculate the total amount that is to be repaid on a loan of \$7500 at 12% p.a. reducible interest over 3 years with two annual repayments of \$3400 and the balance repaid at the end of the third year.

6. Brian needs to borrow \$20 000. He finds a loan that charges 15% p.a. flat-rate interest over 4 years.
   a. Calculate the amount of interest that Brian must pay on this loan.
   b. Calculate the total amount that Brian must repay on this loan.
   c. Brian repays the loan in 4 equal annual instalments. Calculate the amount of each instalment.
   d. Brian can borrow the \$20 000 at 15% p.a. reducible interest instead of flat-rate interest. If Brian makes the same annual repayment at the end of the first three years and the balance in the fourth, calculate the amount of money that Brian will save.

7. Farrah borrows \$12 000 at 10% p.a. reducible interest over 3 years. Farrah repays the loan in two equal annual payments of \$4900 and the balance at the end of the third year.
   - **a.** Calculate the amount of interest that Farrah must pay on this loan.
   - **b.** Farrah finds that she can afford to repay \$5200 each year. How much does Farrah save by making this higher repayment?

8. Aamir borrows \$25 000 at 12% p.a. reducible interest over 3 years with two annual repayments of \$11 000 and the balance repaid at the end of the third year.
   - **a.** Find the total amount of interest that Aamir pays on this loan.
   - **b.** What is the average amount of interest charged on this loan per year?
   - **c.** By writing your answer to part **b** as a percentage of the initial amount borrowed, find the equivalent flat rate of interest on the loan.

9. Felicity borrows \$8000 at 8% p.a. reducible interest over 3 years, repaying the loan in two annual payments of \$3200 and the balance repaid at the end of the third year.
   - **a.** Using the method described in question **8**, find the equivalent flat rate of interest.
   - **b.** Find the equivalent flat rate of interest charged if Felicity increases the amount of each annual repayment to \$4000.

Reasoning

10. Natalie has the choice of two loans of \$15 000. Each loan is to be taken over a three-year term with annual repayments of \$6350. Loan A is charged at 9% flat-rate interest; Loan B is charged at 10% reducible interest. As Natalie's financial planner, construct a detailed report to advise Natalie which loan would be better for her to take.

11. Barack borrows \$13 500 at 10% p.a. reducible interest over 2 years, making an annual repayment of \$7800 and the balance repaid at the end of the second year. Show that if interest is added every six months, at which time a repayment of \$3900 is made, a saving of approximately \$350 is made.

Problem solving

12. Erin borrows \$12 000 for a new car at 9% p.a. over 4 years.
   - **a.** Calculate the total amount to be repaid if the interest is compounded monthly.
   - **b.** How much will be paid in interest for this loan?
   - **c.** How much would each repayment be in order to repay the loan in equal monthly instalments?

13. A loan of \$15 000 is charged at 12% p. a. interest, which is reduced over 3 years. The loan is repaid with 2 annual instalments of \$$x$ and the balance of \$12 763.52 is paid at the end of the 3 years. Determine the value of x to the nearest dollar.

Reflection

How does a loan at reducible interest compare with the same loan at flat-rate interest?

15.8 Review

15.8.1 Review questions

Fluency

1. Calculate the simple interest that is earned on \$5000 at 5% p.a. for 4 years.

2. **MC** Jim invests a sum of money at 9% p.a. Which one of the following statements is true?
   - **A.** Simple interest will earn Jim more money than if compound interest is paid annually.
   - **B.** Jim will earn more money if interest is compounded annually rather than monthly.
   - **C.** Jim will earn more money if interest is compounded quarterly rather than six-monthly.
   - **D.** Jim will earn more money if interest is compounded annually rather than six-monthly.
   - **E.** It does not matter whether simple interest or compound interest is used to calculate the growth of Jim's investment.

3. Find the single discount that is equivalent to successive discounts of 12.5% and 5%.

4. **MC** Which one of the following statements is correct?

A. Successive discounts of 10% and 15% are less than a single discount of 25%.
B. Successive discounts of 10% and 15% are equal to a single discount of 25%.
C. Successive discounts of 10% and 15% are greater than a single discount of 25%.
D. Successive discounts of 10% and 15% are equal to successive discounts of 12% and 13%.
E. Successive discounts of 10% and 15% are equal to successive discounts of 13% and 12%.

5. Benito has a credit card with an outstanding balance of \$3600. The interest rate charged on the loan is 18% p.a. Calculate the amount of interest that Benito will be charged on the credit card for the next month.

6. An LCD television has a cash price of \$5750. It can be purchased on terms of 20% deposit plus weekly repayments of \$42.75 for 3 years. Calculate the total cost of the television if it is purchased on terms.

7. Erin purchases a new entertainment unit that has a cash price of \$6400. Erin buys the unit on the following terms: 10% deposit with the balance plus interest to be repaid in equal monthly repayments over 4 years. The simple interest rate charged is 12% p.a.

a. Calculate the amount of the deposit.
b. Calculate the balance owing after the deposit has been paid.
c. Calculate the interest that will be charged.
d. What is the total amount that Erin has to repay?
e. Calculate the amount of each monthly repayment.

8. A new car has a marked price of \$40 000. The car can be purchased on terms of 10% deposit and monthly repayments of \$1050 for 5 years.

a. Find the total cost of the car if it is purchased on terms.
b. Calculate the amount of interest paid.
c. Calculate the amount of interest paid per year.
d. Calculate the interest rate charged.

9. **MC** The single discount that is equivalent to successive discounts of 15% and 20% is:

A. 10% **B.** 18% **C.** 28% **D.** 30% **E.** 32%

10. A car dealership offers a 10% discount on the price of service of a car purchased at the dealership.

a. Calculate the price Callum would expect to pay for a service valued at \$290.00 if he purchased his car at the dealership.
b. During November, the dealership offers an extra 15% discount on all services and mechanical repairs. Calculate the price Callum, who purchased his car at the dealership, pays for a service in November.
c. What is the total discount given on this service?
d. Determine the single percentage discount that would be equivalent to the successive discounts of 10% and 15% that Callum receives.

11. Ryan invests \$12 500 for 3 years at 8% p.a. with interest paid annually. By calculating the amount of simple interest earned each year separately, determine the amount to which the investment will grow.

12. Calculate the compound interest earned on \$45 000 at 12% p.a. over 4 years if interest is compounded:

a. annually **b.** six-monthly **c.** quarterly **d.** monthly.

13. **MC** A new computer server costs \$7290. With 22% p.a. reducing-value depreciation, the server's value at the end of the third year will be close to:

A. \$1486 **B.** \$2257 **C.** \$2721 **D.** \$3023 **E.** \$3460

14. **MC** An asset, bought for \$34 100, has a value of \$13 430 after 5 years. The depreciation rate is close to:

A. 11% **B.** 17% **C.** 18% **D.** 21% **E.** 22%

15. The value of a new car depreciates by 15% p.a. Find the value of the car after 5 years if it was purchased for \$55 000.

Problem solving

16. Asuka sells musical instruments at discount prices. She had a drum kit on sale for 15% off the retail price of \$5000. After two months the drum kit did not sell, and Asuka decided to apply an extra 10% discount to the existing sale price.

a. What is the total amount saved by the customer?

b. What is the final price of the drum kit?

c. Explain how a 25% discount on the retail price would compare with the successive discounts.

17. Virgin Blue buys a new plane so that extra flights can be arranged between Sydney, Australia and Wellington, New Zealand. The plane costs \$1 200 000. It depreciates at a rate of 16.5% p.a. and is written off when its value falls below \$150 000. How long can Virgin Blue use this plane before it is written off?

18. An electronics store is having trouble selling the latest mp3 player. The original price was \$99 but on October 1 it was reduced 10%. On October 8 it was reduced a further 10%. On October 12 the regional manager decided to increase all prices by 5%. On October 15 the local manager decided to reduce the price by another 10% anyway.

a. Calculate the prices on all 4 dates after the discounts/increases have been applied.

b. What is the final percentage discount after October 15?

19. Thomas went to an electronics store to buy a flat screen HD TV together with some accessories. The store offered him two different loans to buy the television and equipment.
The following agreement was struck with the store.

- Thomas will not be penalised for paying off the loans early.
- Thomas does not have to pay the principal and interest until the end of the loan period.

Loan 1 \$7000 for 3 years at 10.5% p.a. compounding yearly
Loan 2 \$7000 for 5 years at 8% p.a. compounding yearly

a. Explain which loan Thomas should choose if he decides to pay off the loan at the end of the first, second or third year.

b. Explain which loan Thomas should choose for these two options.
Paying off Loan 1 at term
Paying off Loan 2 at the end of four years

c. Thomas considers the option to pay off the loans at the end of their terms. Explain how you can determine the better option without further calculations.

d. Why would Thomas decide to choose Loan 2 instead of Loan 1 (paying over its full term), even if it cost him more money?

20. Jan bought a computer for her business at a cost of \$2500. She elected to use the diminishing value method (compound depreciation), instead of the straight-line method of depreciation. Her accountant told her that she was entitled to depreciate the cost of the computer over 5 years at 40% per year.

a. How much was the computer worth at the end of the first year?

b. By how much could Jan reduce her taxable income at the end of the first year? (The amount Jan can reduce her taxable income is equal to how much value the asset lost from one year to the next.)

c. Explain whether the amount she can deduct from her taxable income will increase or decrease at the end of the second year.

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Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

compound interest
compounding period
deposit
depreciation
discount
flat rate
future value
interest
interest rate
investment
lay-by
loan
principal
repayments
simple interest
successive discounts
terms
time period

Investigation | Rich task

Consumer price index

The Consumer Price Index (CPI) measures price movements in Australia. Let's investigate this further to gain an understanding of how this index is calculated.

A collection of goods and services is selected as representative of a high proportion of household expenditure. The prices of these goods are recorded each quarter. The collection on which the CPI is based is divided into eight groups, which are further divided into subgroups. The groups are: food, clothing, tobacco/alcohol, housing, health/personal care, household equipment, transportation and recreation/education. Weights are attached to each of these subgroups to reflect the importance of each in relation to the total household expenditure. The table below shows the weights of the eight groups.

The weights indicate that a typical Australian household spends 19% of its income on food purchases, 7% on clothing and so on. The CPI is regarded as an indication of the cost of living as it records changes in the level of retail prices from one period to another.

CPI group	Weight (% of total)
Food	19
Clothing	7
Tobacco/alcohol	8.2
Housing	14.1
Health/personal care	5.6
Household equipment	18.3
Transportation	17
Recreation/education	10.8

Consider a simplified example showing how this CPI is calculated and how we are able to compare prices between one period and another. Take three items with prices as follows: a pair of jeans costing $75, a hamburger costing $3.90 and a CD costing $25. Let us say that during the next period of time, the jeans sell for $76, the hamburger for $4.20 and the CD for $29. This can be summarised in the following table.

		Period 1		Period 2	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Jeans	7	$75	525		
Hamburger	19	$3.90	74.1		
CD	10.8	$25	270		
Total			869.1		

In order to calculate the CPI for Period 2, we regard the first period as the base and allocate it an index number of 100 (it is classed as 100%). We compare the second period with the first by expressing it as a percentage of the first period.

$$\text{CPI} = \frac{\text{weighted expenditure for Period 2}}{\text{weighted expenditure for Period 2}} \times 100\%$$

1. Complete the table to determine the total weighted price for Period 2.

2. a. Calculate the CPI for the above example, correct to 1 decimal place.

b. This figure is over 100%. The amount over 100% is known as the inflation factor. What is the inflation factor in this case?

3. Now apply this procedure to a more varied basket of goods. Complete the following table then calculate the CPI and inflation factor for the second period.

		Period 1		Period 2	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Bus fare		$4.80		$4.95	
Rent		$220		$240	
Movie ticket		$10.50		$10.80	
Air conditioner		$1200		$1240	
Haircut		$18.50		$21.40	
Bread		$2.95		$3.20	
Shirt		$32.40		$35	
Bottle of scotch		$19.95		$21	
Total					

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Answers

TOPIC 15 Financial mathematics

Exercise 15.2 Purchasing goods

1. $2400

2. a. $1800 **b.** $900 **c.** $1920 **d.** $720

3. a. $1500 **b.** $125 **c.** $281.25 **d.** $1080

4. a. $18.75 **b.** $55.00 **c.** $41.25 **d.** $125.33 **e.** $99.58

5. $10

6. a. $700 **b.** $10.50 **c.** $710.50 **d.** $3.16 **e.** $913.66

7. a. $52.50 **b.** $50.79 **c.** $35.92 **d.** $14.87

8. a.

Month	Balance owing	Interest	Payment	Closing balance
January	$7500.00	$112.50	$1000.00	$6612.50
February	$6612.50	$99.19	$1000.00	$5711.69
March	$5711.69	$85.68	$1000.00	$4797.37
April	$4797.37	$71.96	$1000.00	$3869.33
May	$3869.33	$58.04	$1000.00	$2927.37
June	$2927.37	$43.91	$1000.00	$1971.28
July	$1971.28	$29.57	$1000.00	$1000.85
August	$1000.85	$15.01	$1015.86	$0

b. $515.86 **c.** $8015.86

9.

Payment option	Immediate payment	Immediate possession	Possible extra cost	Possible price negotiation
Cash	✓	✓		✓
Lay-by	Possible deposit		✓	
Credit card		✓	✓	
Payment option	**Payment**	**Possession**	**Extra cost**	**Price**
Cash	Immediate	Immediate	Nil	Negotiable
Lay-by	Intervals	Delayed	Limited	–
Credit card	Delayed	Immediate	Possible	–

10. S1: Credit card — payment is delayed, but possession is immediate
S2: Lay-by, or cash if she has savings, would like to negotiate a lower price and has somewhere to store it.

11. $r_A = 4\%$, $r_B = 10\%$, $SI_A = \$2000$ and $SI_B = \$4500$

12. $r_A = 9.76\%$, $r_B = 14.63\%$, $SI_A = \$3121.95$ and $SI_B = \$5121.95$

Exercise 15.3 Buying on terms

1. a. i. $3960 **ii.** $3720 **iii.** $3950

b. The best deal is the one with the lowest cost — 20% deposit and weekly payments of $20 over 3 years.

2. a. $131.25 **b.** $55.38 **c.** $144.44

3. a. $13 600, $283.33 **b.** $40000, $666.67 **c.** $5006.25, $278.13

d. $80 000, $666.67 **e.** $488000, $2033.33

4. $5409.76

5. $4530.08

6. a. $1260 **b.** $19 504.80 **c.** $20 764.80 **d.** $8164.80

7. a. $8409.96 **b.** $2609.96 **c.** $869.99 **d.** 15%

8. a. $2375 **b.** $7125 **c.** $2565 **d.** $9690 **e.** $269.17

9. a. $2000 **b.** $4000 **c.** $540 **d.** $4540 **e.** $252.22

10. a. $226.80 **b.** $141.90 **c.** $360.94 **d.** $87.11 **e.** $85.13

11. a. $4600 **b.** $1656 **c.** $130.33 **d.** $7256

12. C

13. E. This option has the lowest interest rate and time frame when compared to all others.

14. The larger the deposit the smaller the loan and hence the interest charged. Loans generally offer a lower rate than buying on terms.

15. $426

16. a. $3200 **b.** $28 800 **c.** $r_4 = 15\%, r_8 = 16\%$

d. 4 years — $49 280, 8 years — $68 864

e. 4 years — $960, 8 years — $684

Challenge 15.1

20 days

Exercise 15.4 Successive discounts

1. a. 16.7% **b.** 10.2% **c.** 43.7% **d.** 15.4% **e.** 7.6%

2. 23.5%

3. C

4. a. 32% **b.** 19.04% **c.** 16.75% **d.** 57.5%

5. 19%

6. a. $3040 **b.** 24%

7. a. $90 **b.** $85.50 **c.** $14.50 **d.** 14.5%

8. a. $1840 **b.** $1748 **c.** $252 **d.** 12.6%

9. a. $212.50 **b.** $201.88 **c.** $48.12 **d.** $200, no

e. 19.25% **f.** Yes **g.** 28%

10. a. $18 168.75 **b.** 27.325%

11. Yes. Both lead to a single discount of 14.69%.

12. Single discount $= 1 - (1 - a) \times (1 - b)$, where a and b are successive discounts (as decimals).

13. a. $55.56 **b.** $50.14 **c.** Fruitz by 14 cents

14. a. Big Rabbit 38.69%, Hoppity Hop 34.53%

b. Hoppity Hop by 4.16%

Exercise 15.5 Compound interest

1. a. $3244.80 **b.** $10 939.56 **c.** $24 617.98

d. $14 678.02 **e.** $14 838.45 **f.** $129 394.77

2. a. $932.52 **b.** $10 650.81 **c.** $202 760.57 **d.** $25 816.04 **e.** $3 145 511.41

3. $8784.60

4. $3376.26

5. a. $2837.04 **b.** $837.04

6. $17 240.15

7. $605.42

8. $18 503.86

9. a. 0.5833% **b.** $42 891.60

10. B **11.** B **12.** C **13.** C

14. a. 0.0247% **b.** ≈ $131 319.80 **c.** ≈ $11 319.80 **d.** ≈ $519.80

15. a. i. $17 745.95 **ii.** $17 786.61 **iii.** $17 807.67 **iv.** $17 821.99

b. The interest added to the principal also earns interest.

16. a. $7920

b. David’s investment = $8108.46

c. Because David’s interest is compounded, the interest is added to the principal each quarter and earns interest itself.

17. a. $3750 interest

b. i. $3820.32 interest **ii.** $3881.05 **iii.** $3912.36

c. Compound quarterly gives the best return.

d. If we assume that interest is compounded annually, an equivalent return of $R = 7\%$ would be achieved.

e. i. Yes **ii.** No

18. Neither is correct. The best option is to choose 3.895% p.a. compounding monthly.

19. a. Bankeast **b.** $19 976.45

Challenge 15.2

4 years, 8 months

Exercise 15.6 Depreciation

1. a. $14 936.14 **b.** $3584.59 **c.** $6596.65

2. $17 694.72

3. a. $19 118.26 **b.** $19 281.74

4. a. $3846.93 **b.** $6353.07 **c.** 38%

5. a. $7216.02 **b.** $45 283.98

6. a. $1.8 million **b.** $569 531.25 **c.** $135 152.44

7. B **8.** A **9.** C **10.** B **11.** A

12. 10 years

13. a. $385 million **b.** 16 years

14. a. 27%

b. $A = P^{(1-R)^n}$

$\frac{A}{P} = {}^{(1-R)^n}$

$\sqrt[n]{\frac{A}{P}} = (1 - R)$

$R = 1 - \sqrt[n]{\frac{A}{P}}$

15. a. Approx. 43%

b. Trial and error: can be time consuming, answer is often an estimate; algebraic solution: correct answer calculated immediately from equation

16. a. $22 774.65 **b.** 13%

17. 1.725%

Exercise 15.7 Loan repayments

1. $4500

2. a. $8000 **b.** $4950 **c.** $1875 **d.** $864 **e.** $420

3. a. $5760 **b.** $17 760 **c.** $4440

4. $2422

5. $9264

6. a. $12 000 **b.** $32 000 **c.** $8000 **d.** $4966.87

7. a. $2453 **b.** $93

8. a. $6004.80 **b.** $2001.60 **c.** 8%

9. a. 5.4% **b.** 4.6%

10. Loan B better (total savings $1053.50)

11. Actual savings $355.15

12. a. $17 176.86 **b.** $5176.86 **c.** $357.85

13. $3500

15.8 Review

1. $1000 **2.** C **3.** 16.875% **4.** A **5.** $54 **6.** $7819

7. a. $640 **b.** $5760 **c.** $2764.80 **d.** $8524.80 **e.** $177.60

8. a. $67 000 **b.** $27 000 **c.** $5400 **d.** 13.5% p.a.

9. E

10. a. $261 **b.** $221.85 **c.** $68.15 **d.** $23.5%

11. $15 746.40

12. a. $25 808.37 **b.** $26 723.16 **c.** $27 211.79 **d.** $27 550.17

13. E

14. B

15. $24 403.79

16. a. $1175 **b.** $3825

c. 25% discount gives a final price of $3750. The customer would be $75 better off.

17. 12 years

18. a. October 1: $89.10, October 8: $80.19, October 12: $84.20, October 15: $75.78

b. 23.5%

19. a. Since the interest rate is lower for Loan 2 than for Loan 1, Thomas should choose Loan 2 if he decides to pay the loan off at the end of the first, second or third year.

b. Loan 1 at term amounts to $9444.63. Loan 2 at the end of 4 years amounts to $9523.42. Thomas should choose Loan 1.

c. Thomas should choose Loan 1. At the end of its term (3 years), it amounts to less than Loan 2 at 4 years, 1 year before its term is finished.

d. Thomas may not have the money to pay off Loan 1 in 3 years. He may need the extra 2 years to accumulate his funds.

20. a. $1500 **b.** $1000

c. Since the depreciation of 40% is on a lower value each year, the amount Jan can deduct from her taxable income decreases every year.

Investigation — Rich task

1.

		Period 1		**Period 2**	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Jeans	7	$75	525	$76	532
Hamburger	19	$3.90	74.1	$4.20	79.8
CD	10.8	$25	270	$29	313.2
Total	36.8	$103.90	869.1	$109.20	925

2. a. 106.4% **b.** 6.4%

3. 104%;4%

		Period 1		**Period 2**	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Bus fare	17	$4.80	81.6	$4.95	84.15
Rent	14.1	$220	3102	$240	3384
Movie ticket	10.8	$10.50	113.4	$10.80	116.64
Air conditioner	18.3	$1200	21 960	$1240	22 692
Haircut	5.6	$18.50	103.6	$21.40	119.84
Bread	19	$2.95	56.05	$3.20	60.8
Shirt	7	$32.40	226.8	$35	245
Bottle of Scotch	8.2	$19.95	163.59	$21	172.2
Total	100	$1509.10	25 807.04	$1576.35	26 874.63

TOPIC 16
Real numbers

16.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

16.1.1 Why learn this?

A knowledge of number is crucial if we are to understand the world around us. Over time, you have been building your knowledge of the concept of number, starting with the counting numbers, also known as natural numbers. Moving on, you needed to include zero. You then had to learn about integers and fractions, which are also called rational numbers. But even the rational numbers do not include all of the numbers on the number line, as they do not include numbers that cannot be written as fractions. That brings us to the concept of real numbers, the set of numbers that includes both rational and irrational numbers.

16.1.2 What do you know?

assesson

1. **THINK** List what you know about real numbers. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of real numbers.

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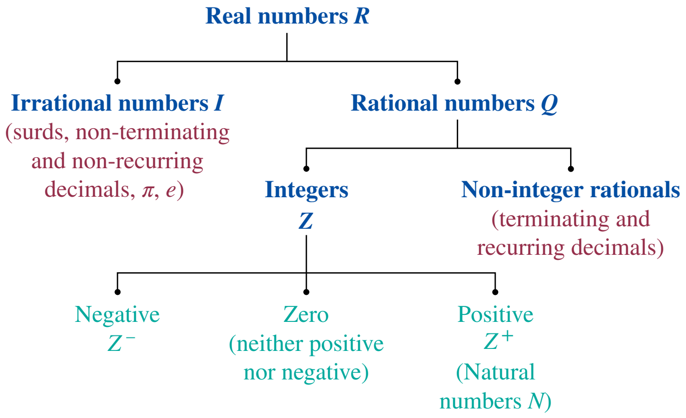
16.2 Number classification review

16.2.1 The number system

- The number systems used today evolved from a basic and practical need of primitive people to count and measure magnitudes and quantities such as livestock, people, possessions, time and so on.
- As societies grew and architecture and engineering developed, number systems became more sophisticated. Number use developed from solely whole numbers to fractions, decimals and irrational numbers.

- The real number system contains the set of rational and irrational numbers. It is denoted by the symbol R. The set of real numbers contains a number of subsets which can be classified as shown in the chart below.



16.2.2 Rational numbers (Q)

- A **rational number** (*ratio*-nal) is a number that can be expressed as a ratio of two whole numbers in the form $\frac{a}{b}$, where $b \neq 0$.
  – Rational numbers are given the symbol Q. Examples are:

$$\frac{1}{5}, \frac{2}{7}, \frac{3}{10}, \frac{9}{4}, 7, -6, 0.35, 1,\dot{4}$$

16.2.3 Integers (Z)

- Rational numbers may be expressed as **integers**. Examples are:

$$\tfrac{5}{1} = 5, -\tfrac{4}{1} = -4, \tfrac{27}{1} = 27, -\tfrac{15}{1} = -15$$

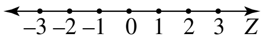
- The set of integers consists of positive and negative whole numbers and 0 (which is neither positive nor negative). They are denoted by the letter Z and can be further divided into subsets. That is:

$$\begin{aligned} Z &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\ Z^+ &= \{1, 2, 3, 4, 5, 6, \ldots\} \\ Z^- &= \{\ldots, -5, -4, -3, -2, -1\} \end{aligned}$$

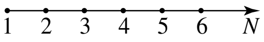
- Positive integers are also known as **natural numbers** (or counting numbers) and are denoted by the letter N. That is:

$$N = \{1, 2, 3, 4, 5, 6, \ldots\}$$

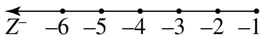
- Integers may be represented on the number line as illustrated below.



The set of integers



The set of positive integers
or natural numbers



The set of negative integers

Note: Integers on the number line are marked with a solid dot to indicate that they are the only points in which we are interested.

16.2.4 Non-integer rational numbers

- Rational numbers may be expressed as **terminating decimals**. Examples are:

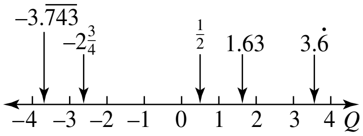
$$\tfrac{7}{10} = 0.7, \tfrac{1}{4} = 0.25, \tfrac{5}{8} = 0.625, \tfrac{9}{5} = 1.8$$

These decimal numbers terminate after a specific number of digits.

- Rational numbers may be expressed as **recurring decimals** (non-terminating or periodic decimals). For example:

$$\begin{aligned} \tfrac{1}{3} &= 0.333\ 333 \ldots \text{ or } 0.\dot{3} \\ \tfrac{9}{11} &= 0.818\ 181 \ldots \text{ or } 0.\dot{8}\dot{1} \left(\text{or } 0.\overline{81}\right) \\ \tfrac{5}{6} &= 0.833\ 333 \ldots \text{ or } 0.8\dot{3} \\ \tfrac{3}{13} &= 0.230\ 769\ 230\ 769 \ldots \text{ or } 0.\dot{2}30\ 76\dot{9} \left(\text{or } 0.\overline{230\ 769}\right) \end{aligned}$$

- These decimal numbers do not terminate, and the specific digit (or number of digits) is repeated in a pattern. Recurring decimals are represented by placing a dot or line above the repeating digit or pattern.



- Rational numbers are defined in set notation as:

Q = rational numbers

$$Q = \left\{\frac{a}{b}, a, b \in Z, b \neq 0\right\} \text{ where } \in \text{ means 'an element of'.}$$

16.2.5 Irrational numbers (I)

- An **irrational number** (ir-*ratio*-nal) is a number that cannot be expressed as a ratio of two whole numbers in the form $\frac{a}{b}$, where $b \neq 0$.
- Irrational numbers are given the symbol I. Examples are:

$$\sqrt{7}, \sqrt{13}, 5\sqrt{21}, \frac{\sqrt{7}}{9}, \pi, e$$

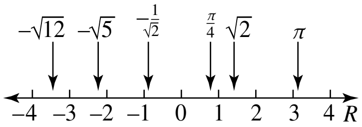
- Irrational numbers may be expressed as decimals. For example:

$$\begin{aligned} \sqrt{5} &= 2.236\,067\,977\,5\ \ldots & \sqrt{0.03} &= 0.173\,205\,080\,757\ \ldots \\ \sqrt{18} &= 4.242\,640\,687\,12\ \ldots & 2\sqrt{7} &= 5.291\,502\,622\,13\ \ldots \\ \pi &= 3.141\,592\,653\,59\ \ldots & e &= 2.718\,281\,828\,46\ \ldots \end{aligned}$$

- These decimal numbers do not terminate, and the digits do not repeat themselves in any particular pattern or order (that is, they are non-terminating and non-recurring).

16.2.6 Real numbers

- Rational and irrational numbers belong to the set of **real numbers** (denoted by the symbol R). They can be positive, negative or 0. The real numbers may be represented on a number line as shown at right (irrational numbers above the line; rational numbers below it).



- To classify a number as either rational or irrational:
  1. Determine whether it can be expressed as a whole number, a fraction or a terminating or recurring decimal.
  2. If the answer is yes, the number is rational; if the answer is no, the number is irrational.

16.2.7 π (pi)

- The symbol **π (pi)** is used for a particular number; that is, the circumference of a circle whose diameter length is 1 unit.
- It can be approximated as a decimal that is non-terminating and non-recurring. Therefore, π is classified as an irrational number. (It is also called a **transcendental number** and cannot be expressed as a surd.)
- In decimal form, $\pi = 3.141\,592\,653\,589\,793\,23\ \ldots$ It has been calculated to 29 000 000 (29 million) decimal places with the aid of a computer.

WORKED EXAMPLE 1

Specify whether the following numbers are rational or irrational.

a $\frac{1}{5}$
b $\sqrt{25}$
c $\sqrt{13}$
d 3π
e 0.54
f $\sqrt[3]{64}$
g $\sqrt[3]{32}$
h $\sqrt[3]{\frac{1}{27}}$

THINK	WRITE
a $\frac{1}{5}$ is already a rational number.	**a** $\frac{1}{5}$ is rational.
b **1** Evaluate $\sqrt{25}$.	**b** $\sqrt{25} = 5$
2 The answer is an integer, so classify $\sqrt{25}$.	$\sqrt{25}$ is rational.
c **1** Evaluate $\sqrt{13}$.	**c** $\sqrt{13} = 3.605\,551\,275\,46\ \ldots$
2 The answer is a non-terminating and non-recurring decimal; classify $\sqrt{13}$.	$\sqrt{13}$ is irrational.
d **1** Use your calculator to find the value of 3π.	**d** $3\pi = 9.424\,777\,960\,77\ \ldots$
2 The answer is a non-terminating and non-recurring decimal; classify 3π.	3π is irrational.

e 0.54 is a terminating decimal; classify it accordingly.	**e** 0.54 is rational.
f **1** Evaluate $\sqrt[3]{64}$.	**f** $\sqrt[3]{64} = 4$
2 The answer is a whole number, so classify $\sqrt[3]{64}$.	$\sqrt[3]{64}$ is rational.
g **1** Evaluate $\sqrt[3]{32}$.	**g** $\sqrt[3]{32} = 3.174\,802\,103\,94\ \ldots$
2 The result is a non-terminating and non-recurring decimal; classify $\sqrt[3]{32}$.	$\sqrt[3]{32}$ is irrational.
h **1** Evaluate $\sqrt[3]{\frac{1}{27}}$.	**h** $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$.
2 The result is a number in a rational form.	$\sqrt[3]{\frac{1}{27}}$ is rational.

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Try out this interactivity: Classifying numbers (int-2792)

Exercise 16.2 Number classification review

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6, 8, 10	Questions: 1–8, 10, 12	Questions: 1–13

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** Specify whether the following numbers are rational (Q) or irrational (I).

a. $\sqrt{4}$ **b.** $\frac{4}{5}$ **c.** $\frac{7}{9}$ **d.** $\sqrt{2}$

e. $\sqrt{7}$ **f.** $\sqrt{0.04}$ **g.** $2\frac{1}{2}$ **h.** $\sqrt{5}$

i. $\frac{9}{4}$ **j.** 0.15 **k.** -2.4 **l.** $\sqrt{100}$

m. $\sqrt{14.4}$ **n.** $\sqrt{1.44}$ **o.** π **p.** $\sqrt{\frac{25}{9}}$

q. 7.32 **r.** $-\sqrt{21}$ **s.** $\sqrt{1000}$ **t.** 7.216 349 157 …

u. $-\sqrt{81}$ **v.** 3π **w.** $\sqrt[3]{62}$ **x.** $\sqrt{\frac{1}{16}}$

y. $\sqrt[3]{0.0001}$

2. Specify whether the following numbers are rational (Q), irrational (I) or neither.

a. $\frac{1}{8}$ **b.** $\sqrt{625}$ **c.** $\frac{11}{4}$ **d.** $\frac{0}{8}$ **e.** $-6\frac{1}{7}$

f. $\sqrt[3]{81}$ **g.** $-\sqrt{11}$ **h.** $\sqrt{\frac{1.44}{4}}$ **i.** $\sqrt{\pi}$ **j.** $\frac{8}{0}$

k. $\sqrt[3]{21}$ **l.** $\frac{\pi}{7}$ **m.** $\sqrt[3]{(-5)^2}$ **n.** $-\frac{3}{11}$ **o.** $\sqrt{\frac{1}{100}}$

p. $\frac{64}{16}$ **q.** $\sqrt{\frac{2}{25}}$ **r.** $\frac{\sqrt{6}}{2}$ **s.** $\sqrt[3]{27}$ **t.** $\frac{1}{\sqrt{4}}$

u. $\frac{22\pi}{7}$ **v.** $\sqrt[3]{-1.728}$ **w.** $6\sqrt{4}$ **x.** $4\sqrt{6}$ **y.** $(\sqrt{2})^4$

3. **MC** Which of the following best represents a rational number?

A. π **B.** $\sqrt{\frac{4}{9}}$ **C.** $\sqrt{\frac{9}{12}}$
D. $\sqrt[3]{3}$ **E.** $\sqrt{5}$

4. **MC** Which of the following best represents an irrational number?

A. $-\sqrt{81}$ **B.** $\frac{6}{5}$ **C.** $\sqrt[3]{343}$
D. $\sqrt{22}$ **E.** $\sqrt{144}$

5. **MC** Which of the following statements regarding the numbers -0.69, $\sqrt{7}$, $\frac{\pi}{3}$, $\sqrt{49}$ is correct?

A. $\frac{\pi}{3}$ is the only rational number.
B. $\sqrt{7}$ and $\sqrt{49}$ are both irrational numbers.
C. -0.69 and $\sqrt{49}$ are the only rational numbers.
D. -0.69 is the only rational number.
E. $\sqrt{7}$ is the only rational number.

6. **MC** Which of the following statements regarding the numbers $2\frac{1}{2}$, $-\frac{11}{3}$, $\sqrt{624}$, $\sqrt[3]{99}$ is correct?

A. $-\frac{11}{3}$ and $\sqrt{624}$ are both irrational numbers.
B. $\sqrt{624}$ is an irrational number and $\sqrt[3]{99}$ is a rational number.
C. $\sqrt{624}$ and $\sqrt[3]{99}$ are both irrational numbers.
D. $2\frac{1}{2}$ is a rational number and $-\frac{11}{3}$ is an irrational number.
E. $\sqrt[3]{99}$ is the only rational number.

Understanding

7. Simplify $\sqrt{\dfrac{a^2}{b^2}}$.

8. **MC** If $p < 0$, then $\sqrt{p}$ is:

A. positive **B.** negative **C.** rational **D.** none of the above

9. **MC** If $p < 0$, then $\sqrt{p^2}$ must be:

A. positive **B.** negative **C.** rational **D.** any of the above

Reasoning

10. Simplify $(\sqrt{p} - \sqrt{q}) \times (\sqrt{p} + \sqrt{q})$.

11. Prove that if $c^2 = a^2 + b^2$, it does not follow that $a = b + c$.

Problem solving

12. Find the value of m and n if $\dfrac{36}{11}$ is written as:

a. $3 + \dfrac{1}{\dfrac{m}{n}}$ **b.** $3 + \dfrac{1}{3 + \dfrac{m}{n}}$ **c.** $3 + \dfrac{1}{3 + \dfrac{1}{\dfrac{m}{n}}}$ **d.** $3 + \dfrac{1}{3 + \dfrac{1}{1 + \dfrac{m}{n}}}$.

13. If x^{-1} means $\dfrac{1}{x}$, what is the value of $\dfrac{3^{-1} - 4^{-1}}{3^{-1} + 4^{-1}}$?

Reflection

Why is it important to understand the real number system?

16.3 Surds

16.3.1 Surds

- A **surd** is an irrational number that is represented by a root sign or a radical sign, for example: $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$.
  Examples of surds include: $\sqrt{7}$, $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[4]{15}$.
  Examples that are not surds include:
  $$\sqrt{9},\ \sqrt{16},\ \sqrt[3]{125},\ \sqrt[4]{81}.$$
- Numbers that are not surds can be simplified to rational numbers, that is:
  $$\sqrt{9} = 3, \sqrt{16} = 4, \sqrt[3]{125} = 5, \sqrt[4]{81} = 3.$$

WORKED EXAMPLE 2

Which of the following numbers are surds?

a $\sqrt{16}$	**b** $\sqrt{13}$	**c** $\sqrt{\frac{1}{16}}$
d $\sqrt[3]{17}$	**e** $\sqrt[4]{63}$	**f** $\sqrt[3]{1728}$

THINK	WRITE
a 1 Evaluate $\sqrt{16}$.	**a** $\sqrt{16} = 4$
2 The answer is rational (since it is a whole number), so state your conclusion.	$\sqrt{16}$ is not a surd.
b 1 Evaluate $\sqrt{13}$.	**b** $\sqrt{13} = 3.605\,551\,275\,46\ldots$
2 The answer is irrational (since it is a non-recurring and non-terminating decimal), so state your conclusion.	$\sqrt{13}$ is a surd.
c 1 Evaluate $\sqrt{\frac{1}{16}}$.	**c** $\sqrt{\frac{1}{16}} = \frac{1}{4}$
2 The answer is rational (a fraction); state your conclusion.	$\sqrt{\frac{1}{16}}$ is not a surd.
d 1 Evaluate $\sqrt[3]{17}$.	**d** $\sqrt[3]{17} = 2.571\,281\,590\,66\ldots$
2 The answer is irrational (a non-terminating and non-recurring decimal), so state your conclusion.	$\sqrt[3]{17}$ is a surd.
e 1 Evaluate $\sqrt[4]{63}$.	**e** $\sqrt[4]{63} = 2.817\,313\,247\,26\ldots$
2 The answer is irrational, so classify $\sqrt[4]{63}$ accordingly.	$\sqrt[4]{63}$ is a surd.
f 1 Evaluate $\sqrt[3]{1728}$.	**f** $\sqrt[3]{1728} = 12$
2 The answer is rational; state your conclusion.	$\sqrt[3]{1728}$ is not a surd. So **b**, **d** and **e** are surds.

16.3.2 Proof that a number is irrational

- In Mathematics you are required to study a variety of types of proofs. One such method is called proof by contradiction.
- This proof is so named because the logical argument of the proof is based on an assumption that leads to contradiction within the proof. Therefore the original assumption must be false.
- An irrational number is one that cannot be expressed in the form $\frac{a}{b}$ (where a and b are integers). The next worked example sets out to prove that $\sqrt{2}$ is irrational.

WORKED EXAMPLE 3

Prove that $\sqrt{2}$ is irrational.

THINK	WRITE
1 Assume that $\sqrt{2}$ is rational; that is, it can be written as $\frac{a}{b}$ in simplest form. We need to show that a and b have no common factors.	Let $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$.
2 Square both sides of the equation.	$2 = \frac{a^2}{b^2}$
3 Rearrange the equation to make a^2 the subject of the formula.	$a^2 = 2b^2$ [1]
4 $2b^2$ is an even number and $2b^2 = a^2$.	$\therefore a^2$ is an even number and a must also be even; that is, a has a factor of 2.
5 Since a is even it can be written as $a = 2r$.	$\therefore a = 2r$
6 Square both sides.	$a^2 = 4r^2$ But $a^2 = 2b^2$ from [1] [2]
7 Equate [1] and [2].	$\therefore 2b^2 = 4r^2$ $b^2 = \frac{4r^2}{2}$ $= 2r^2$ $\therefore b^2$ is an even number and b must also be even; that is, b has a factor of 2.
8 Repeat the steps for b as previously done for a.	Both a and b have a common factor of 2. This contradicts the original assumption that $\sqrt{2} = \frac{a}{b}$, where a and b have no common factor. $\therefore \sqrt{2}$ is not rational. $\therefore$ It must be irrational.

- *Note:* An irrational number written in surd form gives an exact value of the number; whereas the same number written in decimal form (for example, to 4 decimal places) gives an approximate value.

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Exercise 16.3 Surds

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–8, 10	Questions: 1–10	Questions: 1–11

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Fluency

1. **WE2** Which of the numbers below are surds?

a. $\sqrt{81}$
b. $\sqrt{48}$
c. $\sqrt{16}$
d. $\sqrt{1.6}$
e. $\sqrt{0.16}$
f. $\sqrt{11}$
g. $\sqrt{\frac{3}{4}}$
h. $\sqrt[3]{\frac{3}{27}}$
i. $\sqrt{1000}$
j. $\sqrt{1.44}$
k. $4\sqrt{100}$
l. $2 + \sqrt{10}$
m. $\sqrt[3]{32}$
n. $\sqrt{361}$
o. $\sqrt[3]{100}$
p. $\sqrt[3]{125}$
q. $\sqrt{6} + \sqrt{6}$
r. 2π
s. $\sqrt[3]{169}$
t. $\sqrt{\frac{7}{8}}$
u. $\sqrt[4]{16}$
v. $(\sqrt{7})^2$
w. $\sqrt[3]{33}$
x. $\sqrt{0.0001}$
y. $\sqrt[5]{32}$
z. $\sqrt{80}$

2. **MC** The correct statement regarding the set of numbers $\left\{\sqrt{\frac{6}{9}}, \sqrt{20}, \sqrt{54}, \sqrt[3]{27}, \sqrt{9}\right\}$ is:

A. $\sqrt[3]{27}$ and $\sqrt{9}$ are the only rational numbers of the set.
B. $\sqrt{\frac{6}{9}}$ is the only surd of the set.
C. $\sqrt{\frac{6}{9}}$ and $\sqrt{20}$ are the only surds of the set.
D. $\sqrt{20}$ and $\sqrt{54}$ are the only surds of the set.
E. $\sqrt{9}$ and $\sqrt{20}$ are the only surds of the set.

3. **MC** Which of the numbers of the set $\left\{\sqrt{\frac{1}{4}}, \sqrt[3]{\frac{1}{27}}, \sqrt{\frac{1}{8}}, \sqrt{21}, \sqrt[3]{8}\right\}$ are surds?

A. $\sqrt{21}$ only
B. $\sqrt{\frac{1}{8}}$ only
C. $\sqrt{\frac{1}{8}}$ and $\sqrt[3]{8}$
D. $\sqrt{\frac{1}{8}}$ and $\sqrt{21}$ only
E. $\sqrt{\frac{1}{4}}$ and $\sqrt{21}$ only

4. **MC** Which statement regarding the set of numbers $\left\{\pi, \sqrt{\frac{1}{49}}, \sqrt{12}, \sqrt{16}, \sqrt{3}, +1\right\}$ is *not* true?

A. $\sqrt{12}$ is a surd.
B. $\sqrt{12}$ and $\sqrt{16}$ are surds.
C. π is irrational but not a surd.
D. $\sqrt{12}$ and $\sqrt{3} + 1$ are not rational.
E. π is not a surd.

5. **MC** Which statement regarding the set of numbers $\left\{6\sqrt{7}, \sqrt{\frac{144}{16}}, 7\sqrt{6}, 9\sqrt{2}, \sqrt{18}, \sqrt{25}\right\}$ is *not* true?

A. $\sqrt{\frac{144}{16}}$ when simplified is an integer.
B. $\sqrt{\frac{144}{16}}$ and $\sqrt{25}$ are not surds.
C. $7\sqrt{6}$ is smaller than $9\sqrt{2}$.
D. $9\sqrt{2}$ is smaller than $6\sqrt{7}$.
E. $\sqrt{18}$ is a surd.

Understanding

6. Complete the following statement by selecting appropriate words, suggested in brackets: $\sqrt[6]{a}$ is definitely not a surd, if a is… (any multiple of 4; a perfect square and cube).

7. Find the smallest value of m, where m is a positive integer, so that $\sqrt[3]{16m}$ is not a surd.

Reasoning

8. **WE3** Prove that the following numbers are irrational, using a proof by contradiction:

a. $\sqrt{3}$
b. $\sqrt{5}$
c. $\sqrt{7}$.

9. π is an irrational number and so is $\sqrt{3}$. Therefore, determine whether

$$(\pi - \sqrt{3})(\pi + \sqrt{3})$$

is an irrational number.

Problem solving

10. Many composite numbers have a variety of factor pairs. For example, factor pairs of 24 are 1 and 24, 2 and 12, 3 and 8, 4 and 6.

a. Use each pair of possible factors to simplify the following surds.

i. $\sqrt{48}$

ii. $\sqrt{72}$

b. Does the factor pair chosen when simplifying a surd affect the way the surd is written in simplified form?

c. Does the factor pair chosen when simplifying a surd affect the value of the surd when it is written in simplified form? Explain.

11. Solve $\sqrt{3}x - \sqrt{12} = \sqrt{3}$ and indicate whether the result is rational or irrational and integral or not integral.

Reflection

How can you be certain that $\sqrt{a}$ is a surd?

16.4 Operations with surds

16.4.1 Simplifying surds

- To simplify a surd means to make a number (or an expression) under the radical sign ($\sqrt{\ }$) as small as possible.
- To simplify a surd (if it is possible), it should be rewritten as a product of two factors, one of which is a perfect square, that is, 4, 9, 16, 25, 36, 49, 64, 81, 100 and so on.
- We must always aim to obtain the largest perfect square when simplifying surds so that there are fewer steps involved in obtaining the answer. For example $\sqrt{32}$ could be written as $\sqrt{4 \times 8} = 2\sqrt{8}$; however, $\sqrt{8}$ can be further simplified to $2\sqrt{2}$, so $\sqrt{32} = 2 \times 2\sqrt{2}$; that is $\sqrt{32} = 4\sqrt{2}$. If, however, the largest perfect square had been selected and $\sqrt{32}$ had been written as $\sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$, the same answer would be obtained in fewer steps.

WORKED EXAMPLE 4

Simplify the following surds. Assume that x and y are positive real numbers.

a $\sqrt{384}$ **b** $3\sqrt{405}$ **c** $-\frac{1}{8}\sqrt{175}$ **d** $5\sqrt{180x^3y^5}$

THINK	WRITE
a 1 Express 384 as a product of two factors where one factor is the largest possible perfect square.	**a** $\sqrt{384} = \sqrt{64 \times 6}$
2 Express $\sqrt{64 \times 6}$ as the product of two surds.	$= \sqrt{64} \times \sqrt{6}$
3 Simplify the square root from the perfect square (that is, $\sqrt{64} = 8$).	$= 8\sqrt{6}$
b 1 Express 405 as a product of two factors, one of which is the largest possible perfect square.	**b** $3\sqrt{405} = 3\sqrt{81 \times 5}$
2 Express $\sqrt{81 \times 5}$ as a product of two surds.	$= 3\sqrt{81} \times \sqrt{5}$

3 Simplify $\sqrt{81}$.	$= 3 \times 9\sqrt{5}$
4 Multiply together the whole numbers outside the square root sign (3 and 9).	$= 27\sqrt{5}$
c **1** Express 175 as a product of two factors in which one factor is the largest possible perfect square.	**c** $-\frac{1}{8}\sqrt{175} = -\frac{1}{8}\sqrt{25 \times 7}$
2 Express $\sqrt{25 \times 7}$ as a product of 2 surds.	$= -\frac{1}{8} \times \sqrt{25} \times \sqrt{7}$
3 Simplify $\sqrt{25}$.	$= -\frac{1}{8} \times 5\sqrt{7}$
4 Multiply together the numbers outside the square root sign.	$= -\frac{5}{8}\sqrt{7}$
d **1** Express each of 180, x^3 and y^5 as a product of two factors where one factor is the largest possible perfect square.	**d** $5\sqrt{180x^3y^5} = 5\sqrt{36 \times 5 \times x^2 \times x \times y^4 \times y}$
2 Separate all perfect squares into one surd and all other factors into the other surd.	$= 5 \times \sqrt{36x^2y^4} \times \sqrt{5xy}$
3 Simplify $\sqrt{36x^2y^4}$.	$= 5 \times 6 \times x \times y^2 \times \sqrt{5xy}$
4 Multiply together the numbers and the pronumerals outside the square root sign.	$= 30xy^2\sqrt{5xy}$

16.4.2 Addition and subtraction of surds

- Surds may be added or subtracted only if they are *alike*.
  Examples of *like* surds include $\sqrt{7}$, $3\sqrt{7}$ and $-5\sqrt{7}$. Examples of *unlike* surds include $\sqrt{11}$, $\sqrt{5}$, $2\sqrt{13}$ and $-2\sqrt{3}$.
- In some cases surds will need to be simplified before you decide whether they are like or unlike, and then addition and subtraction can take place. The concept of adding and subtracting surds is similar to adding and subtracting like terms in algebra.

WORKED EXAMPLE 5

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Simplify each of the following expressions containing surds. Assume that *a* and *b* are positive real numbers.

a $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6}$

b $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$

c $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$

THINK	WRITE
a All 3 terms are alike because they contain the same surd ($\sqrt{6}$). Simplify.	**a** $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6} = (3 + 17 - 2)\sqrt{6}$ $= 18\sqrt{6}$

b **1** Simplify surds where possible.	**b** $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$ $= 5\sqrt{3} + 2\sqrt{4 \times 3} - 5\sqrt{2} + 3\sqrt{4 \times 2}$ $= 5\sqrt{3} + 2 \times 2\sqrt{3} - 5\sqrt{2} + 3 \times 2\sqrt{2}$ $= 5\sqrt{3} + 4\sqrt{3} - 5\sqrt{2} + 6\sqrt{2}$
2 Add like terms to obtain the simplified answer.	$= 9\sqrt{3} + \sqrt{2}$
c **1** Simplify surds where possible.	**c** $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$ $= \frac{1}{2} \times 10\sqrt{a^2 \times a \times b^2} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$ $= \frac{1}{2} \times 10 \times a \times b\sqrt{a} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$ $= 5ab\sqrt{a} + 6ab\sqrt{a} - 10a\sqrt{b}$ $= 11ab\sqrt{a} - 10a\sqrt{b}$
2 Add like terms to obtain the simplified answer.	

16.4.3 Multiplication and division of surds

Multiplying surds

- To multiply surds, multiply together the expressions under the radical signs. For example, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, where a and b are positive real numbers.
- When multiplying surds it is best to first simplify them (if possible). Once this has been done and a mixed surd has been obtained, the coefficients are multiplied with each other and then the surds are multiplied together. For example,

$$m\sqrt{a} \times n\sqrt{b} = mn\sqrt{ab}.$$

WORKED EXAMPLE 6

TI | CASIO

Multiply the following surds, expressing answers in the simplest form. Assume that x and y are positive real numbers.

a $\sqrt{11} \times \sqrt{7}$ **b** $5\sqrt{3} \times 8\sqrt{5}$ **c** $6\sqrt{12} \times 2\sqrt{6}$ **d** $\sqrt{15x^5y^2} \times \sqrt{12x^2y}$

THINK	WRITE
a Multiply the surds together, using $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (that is, multiply expressions under the square root sign). *Note:* This expression cannot be simplified any further.	**a** $\sqrt{11} \times \sqrt{7} = \sqrt{11 \times 7}$ $= \sqrt{77}$
b Multiply the coefficients together and then multiply the surds together.	**b** $5\sqrt{3} \times 8\sqrt{5} = 5 \times 8 \times \sqrt{3} \times \sqrt{5}$ $= 40 \times \sqrt{3 \times 5}$ $= 40\sqrt{15}$
c **1** Simplify $\sqrt{12}$.	**c** $6\sqrt{12} \times 2\sqrt{6} = 6\sqrt{4 \times 3} \times 2\sqrt{6}$ $= 6 \times 2\sqrt{3} \times 2\sqrt{6}$ $= 12\sqrt{3} \times 2\sqrt{6}$
2 Multiply the coefficients together and multiply the surds together.	$= 24\sqrt{18}$

3 Simplify the surd.	$= 24\sqrt{9 \times 2}$ $= 24 \times 3\sqrt{2}$ $= 72\sqrt{2}$
d 1 Simplify each of the surds.	**d** $\sqrt{15x^5y^2} \times \sqrt{12x^2y}$ $= \sqrt{15 \times x^4 \times x \times y^2} \times \sqrt{4 \times 3 \times x^2 \times y}$ $= x^2 \times y \times \sqrt{15 \times x} \times 2 \times x \times \sqrt{3 \times y}$ $= x^2y\sqrt{15x} \times 2x\sqrt{3y}$
2 Multiply the coefficients together and the surds together.	$= x^2y \times 2x\sqrt{15x \times 3y}$ $= 2x^3y\sqrt{45xy}$ $= 2x^3y\sqrt{9 \times 5xy}$
3 Simplify the surd.	$= 2x^3y \times 3\sqrt{5xy}$ $= 6x^3y\sqrt{5xy}$

- When working with surds, it is sometimes necessary to multiply surds by themselves; that is, square them. Consider the following examples:

$$(\sqrt{2})^2 = \sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$$
$$(\sqrt{5})^2 = \sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$$

- Observe that squaring a surd produces the number under the radical sign. This is not surprising, because squaring and taking the square root are *inverse operations* and, when applied together, leave the original unchanged.
- When a surd is squared, the result is the number (or expression) under the radical sign; that is, $(\sqrt{a})^2 = a$, where a is a positive real number.

WORKED EXAMPLE 7

Simplify each of the following.

a $(\sqrt{6})^2$ **b** $(3\sqrt{5})^2$

THINK	WRITE
a Use $(\sqrt{a})^2 = a$, where $a = 6$.	**a** $(\sqrt{6})^2 = 6$
b 1 Square 3 and apply $(\sqrt{a})^2 = a$ to square $\sqrt{5}$.	**b** $(3\sqrt{5})^2 = 3^2 \times (\sqrt{5})^2$ $= 9 \times 5$
2 Simplify.	$= 45$

16.4.4 Dividing surds

- To divide surds, divide the expressions under the radical signs; that is, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$, where a and b are whole numbers.
- When dividing surds it is best to simplify them (if possible) first. Once this has been done, the coefficients are divided next and then the surds are divided.

WORKED EXAMPLE 8

TI | CASIO

Divide the following surds, expressing answers in the simplest form. Assume that *x* and *y* are positive real numbers.

a $\dfrac{\sqrt{55}}{\sqrt{5}}$ **b** $\dfrac{\sqrt{48}}{\sqrt{3}}$ **c** $\dfrac{9\sqrt{88}}{6\sqrt{99}}$ **d** $\dfrac{\sqrt{36xy}}{\sqrt{25x^{99}y^{11}}}$

THINK	WRITE
a **1** Rewrite the fraction, using $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.	**a** $\dfrac{\sqrt{55}}{\sqrt{5}} = \sqrt{\dfrac{55}{5}}$
2 Divide the numerator by the denominator (that is, 55 by 5).	$= \sqrt{11}$
3 Check if the surd can be simplified any further.	
b **1** Rewrite the fraction, using $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.	**b** $\dfrac{\sqrt{48}}{\sqrt{3}} = \sqrt{\dfrac{48}{3}}$
2 Divide 48 by 3.	$= \sqrt{16}$
3 Evaluate $\sqrt{16}$.	$= 4$
c **1** Rewrite surds, using $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.	**c** $\dfrac{9\sqrt{88}}{6\sqrt{99}} = \dfrac{9}{6}\sqrt{\dfrac{88}{99}}$
2 Simplify the fraction under the radical by dividing both numerator and denominator by 11.	$= \dfrac{9}{6}\sqrt{\dfrac{8}{9}}$
3 Simplify surds.	$= \dfrac{9 \times 2\sqrt{2}}{6 \times 3}$
4 Multiply the whole numbers in the numerator together and those in the denominator together.	$= \dfrac{18\sqrt{2}}{18}$
5 Cancel the common factor of 18.	$= \sqrt{2}$
d **1** Simplify each surd.	**d** $\dfrac{\sqrt{36xy}}{\sqrt{25x^9y^{11}}} = \dfrac{6\sqrt{xy}}{5\sqrt{x^8 \times x \times y^{10} \times y}}$ $= \dfrac{6\sqrt{xy}}{5x^4y^5\sqrt{xy}}$
2 Cancel any common factors — in this case $\sqrt{xy}$.	$= \dfrac{6}{5x^4y^5}$

16.4.5 Rationalising denominators

- If the **denominator** of a fraction is a surd, it can be changed into a rational number through multiplication. In other words, it can be rationalised.
- As discussed earlier in this chapter, squaring a simple surd (that is, multiplying it by itself) results in a rational number. This fact can be used to rationalise denominators as follows.

$$\frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}, \text{ where } \frac{\sqrt{b}}{\sqrt{b}} = 1$$

- If both numerator and denominator of a fraction are multiplied by the surd contained in the denominator, the denominator becomes a rational number. The fraction takes on a different appearance, but its numerical value is unchanged, because multiplying the numerator and denominator by the same number is equivalent to multiplying by 1.

WORKED EXAMPLE 9

TI | CASIO

Express the following in their simplest form with a rational denominator.

a $\frac{\sqrt{6}}{\sqrt{13}}$ **b** $\frac{2\sqrt{12}}{3\sqrt{54}}$ **c** $\frac{\sqrt{17} - 3\sqrt{14}}{\sqrt{7}}$

THINK	**WRITE**
a 1 Write the fraction.	**a** $\frac{\sqrt{6}}{\sqrt{13}}$
2 Multiply both the numerator and denominator by the surd contained in the denominator (in this case $\sqrt{13}$). This has the same effect as multiplying the fraction by 1, because $\frac{\sqrt{13}}{\sqrt{13}} = 1$.	$= \frac{\sqrt{6}}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$ $= \frac{\sqrt{78}}{13}$
b 1 Write the fraction.	**b** $\frac{2\sqrt{12}}{3\sqrt{54}}$
2 Simplify the surds. (This avoids dealing with large numbers.)	$\frac{2\sqrt{12}}{3\sqrt{54}} = \frac{2\sqrt{4 \times 3}}{3\sqrt{9 \times 6}}$ $= \frac{2 \times 2\sqrt{3}}{3 \times 3\sqrt{6}}$ $= \frac{4\sqrt{3}}{9\sqrt{6}}$
3 Multiply both the numerator and denominator by $\sqrt{6}$. (This has the same effect as multiplying the fraction by 1, because $\frac{\sqrt{6}}{\sqrt{6}} = 1$.) *Note:* We need to multiply only by the surd part of the denominator (that is, by $\sqrt{6}$ rather than by $9\sqrt{6}$.)	$= \frac{4\sqrt{3}}{9\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$ $= \frac{4\sqrt{18}}{9 \times 6}$

4 Simplify $\sqrt{18}$.	$= \dfrac{4\sqrt{9 \times 2}}{9 \times 6}$ $= \dfrac{4 \times 3\sqrt{2}}{54}$ $= \dfrac{12\sqrt{2}}{54}$
5 Divide both the numerator and denominator by 6 (cancel down).	$= \dfrac{2\sqrt{2}}{9}$
c 1 Write the fraction.	**c** $\dfrac{\sqrt{17} - 3\sqrt{14}}{\sqrt{7}}$
2 Multiply both the numerator and denominator by $\sqrt{7}$. Use grouping symbols (brackets) to make it clear that the whole numerator must be multiplied by $\sqrt{7}$.	$= \dfrac{(\sqrt{17} - 3\sqrt{14})}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}}$
3 Apply the Distributive Law in the numerator. $a(b + c) = ab + ac$	$= \dfrac{\sqrt{17} \times \sqrt{7} - 3\sqrt{14} \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}}$ $= \dfrac{\sqrt{119} - 3\sqrt{98}}{7}$
4 Simplify $\sqrt{98}$.	$= \dfrac{\sqrt{119} - 3\sqrt{49 \times 2}}{7}$ $= \dfrac{\sqrt{119} - 3 \times 7\sqrt{2}}{7}$ $= \dfrac{\sqrt{119} - 21\sqrt{2}}{7}$

16.4.6 Rationalising denominators using conjugate surds

- The product of pairs of **conjugate surds** results in a rational number. (Examples of pairs of conjugate surds include $\sqrt{6} + 11$ and $\sqrt{6} - 11$, $\sqrt{a} + b$ and $\sqrt{a} - b$, $2\sqrt{5} - \sqrt{7}$ and $2\sqrt{5} + \sqrt{7}$.)

  This fact is used to rationalise denominators containing a sum or a difference of surds.
- To rationalise the denominator that contains a sum or a difference of surds, multiply both numerator and denominator by the conjugate of the denominator.

  Two examples are given below:

  1. To rationalise the denominator of the fraction $\dfrac{1}{\sqrt{a} + \sqrt{b}}$. multiply it by $\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.
  2. To rationalise the denominator of the fraction $\dfrac{1}{\sqrt{a} - \sqrt{b}}$, multiply it by $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$.

  A quick way to simplify the denominator is to use the difference of two squares identity:

$$\begin{aligned}(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) &= (\sqrt{a})^2 - (\sqrt{b})^2 \\ &= a - b\end{aligned}$$

WORKED EXAMPLE 10

TI | CASIO

Rationalise the denominator and simplify the following.

a $\dfrac{1}{4-\sqrt{3}}$

b $\dfrac{\sqrt{6}+3\sqrt{2}}{3+\sqrt{3}}$

THINK	WRITE
a 1 Write the fraction.	**a** $\dfrac{1}{4-\sqrt{3}}$
2 Multiply the numerator and denominator by the conjugate of the denominator. (Note that $\dfrac{(4+\sqrt{3})}{(4+\sqrt{3})}=1$.)	$=\dfrac{1}{(4-\sqrt{3})}\times\dfrac{(4+\sqrt{3})}{(4+\sqrt{3})}$
3 Apply the Distributive Law in the numerator and the difference of two squares identity in the denominator.	$=\dfrac{4+\sqrt{3}}{(4)^2-(\sqrt{3})^2}$
4 Simplify.	$=\dfrac{4+\sqrt{3}}{16-3}$ $=\dfrac{4+\sqrt{3}}{13}$
b 1 Write the fraction.	**b** $\dfrac{\sqrt{6}+3\sqrt{2}}{3+\sqrt{3}}$
2 Multiply the numerator and denominator by the conjugate of the denominator. (Note that $\dfrac{(3-\sqrt{3})}{(3-\sqrt{3})}=1$.)	$=\dfrac{(\sqrt{6}+3\sqrt{2})}{(3+\sqrt{3})}\times\dfrac{(3-\sqrt{3})}{(3-\sqrt{3})}$
3 Multiply the expressions in grouping symbols in the numerator, and apply the difference of two squares identity in the denominator.	$=\dfrac{\sqrt{6}\times 3+\sqrt{6}\times -\sqrt{3}+3\sqrt{2}\times 3+3\sqrt{2}\times -\sqrt{3}}{(3)^2-(\sqrt{3})^2}$
4 Simplify.	$=\dfrac{3\sqrt{6}-\sqrt{18}+9\sqrt{2}-3\sqrt{6}}{9-3}$ $=\dfrac{-\sqrt{18}+9\sqrt{2}}{6}$ $=\dfrac{-\sqrt{9\times 2}+9\sqrt{2}}{6}$ $=\dfrac{-3\sqrt{2}+9\sqrt{2}}{6}$ $=\dfrac{6\sqrt{2}}{6}$ $=\sqrt{2}$

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Exercise 16.4 Operations with surds

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–h, 2a–h, 3a–h, 4a–d, 5a–h, 6a–h, 7a–h, 8a–d, 9a–d, 10a–h, 11a–f, 12a–c, 13, 15	Questions: 1e–j, 2e–j, 3e–k, 4c–f, 5c–i, 6e–j, 7g–l, 8d–f, 9g–k, 10f–j, 11e–h, 12d–f, 13–15	Questions: 1g–l, 2g–l, 3g–l, 4e–h, 5g–l, 6g–l, 7j–r, 8e–h, 9i–n, 10k–o, 11i–l, 12g–i, 13–16

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE4a** Simplify the following surds.

a. $\sqrt{12}$ **b.** $\sqrt{24}$ **c.** $\sqrt{27}$ **d.** $\sqrt{125}$
e. $\sqrt{54}$ **f.** $\sqrt{112}$ **g.** $\sqrt{68}$ **h.** $\sqrt{180}$
i. $\sqrt{88}$ **j.** $\sqrt{162}$ **k.** $\sqrt{245}$ **l.** $\sqrt{448}$

2. **WE4b, c** Simplify the following surds.

a. $2\sqrt{8}$ **b.** $8\sqrt{90}$ **c.** $9\sqrt{80}$
d. $7\sqrt{54}$ **e.** $-6\sqrt{75}$ **f.** $-7\sqrt{80}$
g. $16\sqrt{48}$ **h.** $\frac{1}{7}\sqrt{392}$ **i.** $\frac{1}{9}\sqrt{162}$
j. $\frac{1}{4}\sqrt{192}$ **k.** $\frac{1}{9}\sqrt{135}$ **l.** $\frac{3}{10}\sqrt{175}$

3. **WE4d** Simplify the following surds. Assume that a, b, c, d, e, f, x and y are positive real numbers.

a. $\sqrt{16a^2}$ **b.** $\sqrt{72a^2}$ **c.** $\sqrt{90a^2b}$ **d.** $\sqrt{338a^4}$
e. $\sqrt{338a^3b^3}$ **f.** $\sqrt{68a^3b^5}$ **g.** $\sqrt{125x^6y^4}$ **h.** $5\sqrt{80x^3y^2}$
i. $6\sqrt{162c^7d^5}$ **j.** $2\sqrt{405c^7d^9}$ **k.** $\frac{1}{2}\sqrt{88ef}$ **l.** $\frac{1}{2}\sqrt{392e^{11}f^{11}}$

4. **WE5a** Simplify the following expressions containing surds. Assume that x and y are positive real numbers.

a. $3\sqrt{5} + 4\sqrt{5}$ **b.** $2\sqrt{3} + 5\sqrt{3} + \sqrt{3}$
c. $8\sqrt{5} + 3\sqrt{3} + 7\sqrt{5} + 2\sqrt{3}$ **d.** $6\sqrt{11} - 2\sqrt{11}$
e. $7\sqrt{2} + 9\sqrt{2} - 3\sqrt{2}$ **f.** $9\sqrt{6} + 12\sqrt{6} - 17\sqrt{6} - 7\sqrt{6}$
g. $12\sqrt{3} - 8\sqrt{7} + 5\sqrt{3} - 10\sqrt{7}$ **h.** $2\sqrt{x} + 5\sqrt{y} + 6\sqrt{x} - 2\sqrt{y}$

5. **WE5b** Simplify the following expressions containing surds. Assume that a and b are positive real numbers.

a. $\sqrt{200} - \sqrt{300}$

b. $\sqrt{125} - \sqrt{150} + \sqrt{600}$

c. $\sqrt{27} - \sqrt{3} + \sqrt{75}$

d. $2\sqrt{20} - 3\sqrt{5} + \sqrt{45}$

e. $6\sqrt{12} + 3\sqrt{27} - 7\sqrt{3} + \sqrt{18}$

f. $\sqrt{150} + \sqrt{24} - \sqrt{96} + \sqrt{108}$

g. $3\sqrt{90} - 5\sqrt{60} + 3\sqrt{40} + \sqrt{100}$

h. $5\sqrt{11} + 7\sqrt{44} - 9\sqrt{99} + 2\sqrt{121}$

i. $2\sqrt{30} + 5\sqrt{120} + \sqrt{60} - 6\sqrt{135}$

j. $6\sqrt{ab} - \sqrt{12ab} + 2\sqrt{9ab} + 3\sqrt{27ab}$

k. $\frac{1}{2}\sqrt{98} + \frac{1}{3}\sqrt{48} + \frac{1}{3}\sqrt{12}$

l. $\frac{1}{8}\sqrt{32} - \frac{7}{6}\sqrt{18} + 3\sqrt{72}$

6. **WE5c** Simplify the following expressions containing surds. Assume that a and b are positive real numbers.

a. $7\sqrt{a} - \sqrt{8a} + 8\sqrt{9a} - \sqrt{32a}$

b. $10\sqrt{a} - 15\sqrt{27a} + 8\sqrt{12a} + 14\sqrt{9a}$

c. $\sqrt{150ab} + \sqrt{96ab} - \sqrt{54ab}$

d. $16\sqrt{4a^2} - \sqrt{24a} + 4\sqrt{8a^2} + \sqrt{96a}$

e. $\sqrt{8a^3} + \sqrt{72a^3} - \sqrt{98a^3}$

f. $\frac{1}{2}\sqrt{36a} + \frac{1}{4}\sqrt{128a} - \frac{1}{6}\sqrt{144a}$

g. $\sqrt{9a^3} + \sqrt{3a^5}$

h. $6\sqrt{a^5b} + \sqrt{a^3b} - 5\sqrt{a^5b}$

i. $ab\sqrt{ab} + 3ab\sqrt{a^2b} + \sqrt{9a^3b^3}$

j. $\sqrt{a^3b} + 5\sqrt{ab} - 2\sqrt{ab} + 5\sqrt{a^3b}$

k. $\sqrt{32a^3b^2} - 5ab\sqrt{8a} + \sqrt{48a^5b^6}$

l. $\sqrt{4a^2b} + 5\sqrt{a^2b} - 3\sqrt{9a^2b}$

7. **WE6** Multiply the following surds, expressing answers in the simplest form. Assume that a, b, x and y are positive real numbers.

a. $\sqrt{2} \times \sqrt{7}$

b. $\sqrt{6} \times \sqrt{7}$

c. $\sqrt{8} \times \sqrt{6}$

d. $\sqrt{10} \times \sqrt{10}$

e. $\sqrt{21} \times \sqrt{3}$

f. $\sqrt{27} \times 3\sqrt{3}$

g. $5\sqrt{3} \times 2\sqrt{11}$

h. $10\sqrt{15} \times 6\sqrt{3}$

i. $4\sqrt{20} \times 3\sqrt{5}$

j. $10\sqrt{6} \times 3\sqrt{8}$

k. $\frac{1}{4}\sqrt{48} \times 2\sqrt{2}$

l. $\frac{1}{9}\sqrt{48} \times 2\sqrt{3}$

m. $\frac{1}{10}\sqrt{60} \times \frac{1}{5}\sqrt{40}$

n. $\sqrt{xy} \times \sqrt{x^3y^2}$

o. $\sqrt{3a^4b^2} \times \sqrt{6a^5b^3}$

p. $\sqrt{12a^7b} \times \sqrt{6a^3b^4}$

q. $\sqrt{15x^3y^2} \times \sqrt{6x^2y^3}$

r. $\frac{1}{2}\sqrt{15a^3b^3} \times 3\sqrt{3a^2b^6}$

8. **WE7** Simplify each of the following.

a. $(\sqrt{2})^2$

b. $(\sqrt{5})^2$

c. $(\sqrt{12})^2$

d. $(\sqrt{15})^2$

e. $(3\sqrt{2})^2$

f. $(4\sqrt{5})^2$

g. $(2\sqrt{7})^2$

h. $(5\sqrt{8})^2$

9. **WE8** Simplify the following surds, expressing answers in the simplest form. Assume that a, b, x and y are positive real numbers.

a. $\dfrac{\sqrt{15}}{\sqrt{3}}$

b. $\dfrac{\sqrt{8}}{\sqrt{2}}$

c. $\dfrac{\sqrt{60}}{\sqrt{10}}$

d. $\dfrac{\sqrt{128}}{\sqrt{8}}$

e. $\dfrac{\sqrt{18}}{4\sqrt{6}}$

f. $\dfrac{\sqrt{65}}{2\sqrt{13}}$

g. $\dfrac{\sqrt{96}}{\sqrt{8}}$

h. $\dfrac{7\sqrt{44}}{14\sqrt{11}}$

i. $\dfrac{9\sqrt{63}}{15\sqrt{7}}$

j. $\dfrac{\sqrt{2040}}{\sqrt{30}}$

k. $\dfrac{\sqrt{x^4y^3}}{\sqrt{x^2y^5}}$

l. $\dfrac{\sqrt{16xy}}{\sqrt{8x^7y^9}}$

m. $\dfrac{\sqrt{xy}}{\sqrt{x^5y^7}} \times \dfrac{\sqrt{12x^8y^{12}}}{\sqrt{x^2y^3}}$

n. $\dfrac{2\sqrt{2a^2b^4}}{\sqrt{5a^3b^6}} \times \dfrac{\sqrt{10a^9b^3}}{3\sqrt{a^7b}}$

10. **WE9a, b** Express the following in their simplest form with a rational denominator.

a. $\dfrac{5}{\sqrt{2}}$

b. $\dfrac{7}{\sqrt{3}}$

c. $\dfrac{4}{\sqrt{11}}$

d. $\dfrac{8}{\sqrt{6}}$

e. $\dfrac{\sqrt{12}}{\sqrt{7}}$

f. $\frac{\sqrt{15}}{\sqrt{6}}$ **g.** $\frac{2\sqrt{3}}{\sqrt{5}}$ **h.** $\frac{3\sqrt{7}}{\sqrt{5}}$ **i.** $\frac{5\sqrt{2}}{2\sqrt{3}}$ **j.** $\frac{4\sqrt{3}}{3\sqrt{5}}$

k. $\frac{5\sqrt{14}}{7\sqrt{8}}$ **l.** $\frac{16\sqrt{3}}{6\sqrt{5}}$ **m.** $\frac{8\sqrt{3}}{7\sqrt{7}}$ **n.** $\frac{8\sqrt{60}}{\sqrt{28}}$ **o.** $\frac{2\sqrt{35}}{3\sqrt{14}}$

Understanding

11. **WE9c** Express the following in their simplest form with a rational denominator.

a. $\frac{\sqrt{6}+\sqrt{12}}{\sqrt{3}}$ **b.** $\frac{\sqrt{15}-\sqrt{22}}{\sqrt{6}}$ **c.** $\frac{6\sqrt{2}-\sqrt{15}}{\sqrt{10}}$

d. $\frac{2\sqrt{18}+3\sqrt{2}}{\sqrt{5}}$ **e.** $\frac{3\sqrt{5}+6\sqrt{7}}{\sqrt{8}}$ **f.** $\frac{4\sqrt{2}+3\sqrt{8}}{2\sqrt{3}}$

g. $\frac{3\sqrt{11}-4\sqrt{5}}{\sqrt{18}}$ **h.** $\frac{2\sqrt{7}-2\sqrt{5}}{\sqrt{12}}$ **i.** $\frac{7\sqrt{12}-5\sqrt{6}}{6\sqrt{3}}$

j. $\frac{6\sqrt{2}-\sqrt{5}}{4\sqrt{8}}$ **k.** $\frac{6\sqrt{3}-5\sqrt{5}}{7\sqrt{20}}$ **l.** $\frac{3\sqrt{5}+7\sqrt{3}}{5\sqrt{24}}$

12. **WE10** Rationalise the denominator and simplify.

a. $\frac{1}{\sqrt{5}+2}$ **b.** $\frac{1}{\sqrt{8}-\sqrt{5}}$ **c.** $\frac{4}{2\sqrt{11}-\sqrt{13}}$

d. $\frac{5\sqrt{3}}{3\sqrt{5}+4\sqrt{2}}$ **e.** $\frac{\sqrt{8}-3}{\sqrt{8}+3}$ **f.** $\frac{\sqrt{12}-\sqrt{7}}{\sqrt{12}+\sqrt{7}}$

g. $\frac{\sqrt{3}-1}{\sqrt{5}+1}$ **h.** $\frac{3\sqrt{6}-\sqrt{15}}{\sqrt{6}+2\sqrt{3}}$ **i.** $\frac{\sqrt{5}-\sqrt{3}}{4\sqrt{2}-\sqrt{3}}$

Reasoning

13. Express the average of $\frac{1}{2\sqrt{x}}$ and $\frac{1}{3-2\sqrt{x}}$, writing your answer with a rational denominator.

14. a. Show that $(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$.

b. Use this result to find:

i. $\sqrt{8+2\sqrt{15}}$

ii. $\sqrt{8-2\sqrt{15}}$

iii. $\sqrt{7+4\sqrt{3}}$.

Problem solving

15. Simplify $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{3}+\sqrt{3}+\sqrt{5}} - \frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{3}-\sqrt{5}}$.

16. Solve for x.

a. $\sqrt{9+x}-\sqrt{x} = \frac{5}{\sqrt{9+x}}$

b. $\frac{9\sqrt{x}-7}{3\sqrt{x}} = \frac{3\sqrt{x}+1}{\sqrt{x}+5}$

Reflection

Under what circumstance might you need to rationalise the denominator of a fraction?

16.5 Fractional indices

16.5.1 Fractional indices

- Consider the expression $a^{\frac{1}{2}}$. Now consider what happens if we square that expression.

$$\left(a^{\frac{1}{2}}\right)^2 = a \text{ (using the Fourth Index Law, } (a^m)^n = a^{m \times n}\text{)}$$

- Now, from our work on surds we know that $(\sqrt{a})^2 = a$.
- From this we can conclude that $\left(a^{\frac{1}{2}}\right)^2 = (\sqrt{a})^2$ and further conclude that $a^{\frac{1}{2}} = \sqrt{a}$.
- We can similarly show that $a^{\frac{1}{3}} = \sqrt[3]{a}$.
- This pattern can be continued and generalised to produce $a^{\frac{1}{n}} = \sqrt[n]{a}$.

WORKED EXAMPLE 11

Evaluate each of the following without using a calculator.

a $9^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$

THINK	WRITE
a 1 Write $9^{\frac{1}{2}}$ as $\sqrt{9}$.	**a** $9^{\frac{1}{2}} = \sqrt{9}$
2 Evaluate.	$= 3$
b 1 Write $64^{\frac{1}{3}}$ as $\sqrt[3]{64}$.	**b** $64^{\frac{1}{3}} = \sqrt[3]{64}$
2 Evaluate.	$= 4$

WORKED EXAMPLE 12 TI | CASIO

Use a calculator to find the value of the following, correct to 1 decimal place.

a $10^{\frac{1}{4}}$ **b** $200^{\frac{1}{5}}$

THINK	WRITE
a Use a calculator to produce the answer.	**a** $10^{\frac{1}{4}} = 1.778\,279\,41$ ≈ 1.8
b Use a calculator to produce the answer.	**b** $200^{\frac{1}{5}} = 2.885\,399\,812$ ≈ 2.9

- Consider the expression $(a^m)^{\frac{1}{n}}$. From earlier, we know that $(a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$.

  We also know $(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$ using the index laws.

  We can therefore conclude that $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
- Such expressions can be evaluated on a calculator either by using the index function, which is usually either ^ or x^y and entering the fractional index, or by separating the two functions for power and root.

WORKED EXAMPLE 13

Evaluate $3^{\frac{2}{7}}$, correct to 1 decimal place.

THINK	WRITE
Use a calculator to evaluate $3^{\frac{2}{7}}$.	$3^{\frac{2}{7}} \approx 1.4$

- The index law $a^{\frac{1}{2}} = \sqrt{a}$ can be applied to convert between expressions that involve fractional indices and surds.

WORKED EXAMPLE 14

Write each of the following expressions in simplest surd form.

a $\mathbf{10^{\frac{1}{2}}}$ **b** $\mathbf{5^{\frac{3}{2}}}$

THINK	WRITE
a Since an index of $\frac{1}{2}$ is equivalent to taking the square root, this term can be written as the square root of 10.	**a** $10^{\frac{1}{2}} = \sqrt{10}$
b 1 A power of $\frac{3}{2}$ means the square root of the number cubed.	**b** $5^{\frac{3}{2}} = \sqrt{5^3}$
2 Evaluate 5^3.	$= \sqrt{125}$
3 Simplify $\sqrt{125}$.	$= 5\sqrt{5}$

WORKED EXAMPLE 15

Simplify each of the following.

a $\mathbf{m^{\frac{1}{5}} \times m^{\frac{2}{5}}}$ **b** $\mathbf{(a^2b^3)^{\frac{1}{6}}}$ **c** $\mathbf{\left(\frac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}}$

THINK	WRITE
a 1 Write the expression.	**a** $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$
2 Multiply numbers with the same base by adding the indices.	$= m^{\frac{3}{5}}$
b 1 Write the expression.	**b** $(a^2b^3)^{\frac{1}{6}}$
2 Multiply each index inside the grouping symbols (brackets) by the index on the outside.	$= a^{\frac{2}{6}}b^{\frac{3}{6}}$
3 Simplify the fractions.	$= a^{\frac{1}{3}}b^{\frac{1}{2}}$
c 1 Write the expression.	**c** $\left(\frac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$
2 Multiply the index in both the numerator and denominator by the index outside the grouping symbols.	$= \frac{x^{\frac{1}{3}}}{y^{\frac{3}{8}}}$

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Exercise 16.5 Fractional indices

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–5, 6a–f, 7a–c, 8a–f, 9a–d, 10a–d, 11a–d, 12–14, 16	Questions: 1–5, 6d–g, 7b–d, 8d–f, 9b–d, 10c–f, 11c–f, 12–16	Questions: 1–5, 6g–i, 7d–f, 8f–i, 9c–f, 10e–i, 11e–i, 12–17

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Fluency

1. **WE11** Evaluate each of the following without using a calculator.

a. $16^{\frac{1}{2}}$ **b.** $25^{\frac{1}{2}}$ **c.** $81^{\frac{1}{2}}$ **d.** $8^{\frac{1}{3}}$ **e.** $27^{\frac{1}{3}}$ **f.** $125^{\frac{1}{3}}$

2. **WE12** Use a calculator to evaluate each of the following, correct to 1 decimal place.

a. $81^{\frac{1}{4}}$ **b.** $16^{\frac{1}{4}}$ **c.** $3^{\frac{1}{3}}$ **d.** $5^{\frac{1}{2}}$ **e.** $7^{\frac{1}{5}}$ **f.** $8^{\frac{1}{9}}$

3. **WE13** Use a calculator to find the value of each of the following, correct to 1 decimal place.

a. $12^{\frac{3}{8}}$ **b.** $100^{\frac{5}{9}}$ **c.** $50^{\frac{2}{3}}$ **d.** $(0.6)^{\frac{4}{5}}$ **e.** $\left(\frac{3}{4}\right)^{\frac{3}{4}}$ **f.** $\left(\frac{4}{5}\right)^{\frac{2}{3}}$

4. **WE14** Write each of the following expressions in simplest surd form.

a. $7^{\frac{1}{2}}$ **b.** $12^{\frac{1}{2}}$ **c.** $72^{\frac{1}{2}}$ **d.** $2^{\frac{5}{2}}$ **e.** $3^{\frac{3}{2}}$ **f.** $10^{\frac{5}{2}}$

5. Write each of the following expressions with a fractional index.

a. $\sqrt{5}$ **b.** $\sqrt{10}$ **c.** $\sqrt{x}$ **d.** $\sqrt{m^3}$ **e.** $2\sqrt{t}$ **f.** $\sqrt[3]{6}$

6. **WE15a** Simplify each of the following. Leave your answer in index form.

a. $4^{\frac{3}{5}} \times 4^{\frac{1}{5}}$ **b.** $2^{\frac{1}{8}} \times 2^{\frac{3}{8}}$ **c.** $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$

d. $x^{\frac{3}{4}} \times x^{\frac{2}{5}}$ **e.** $5m^{\frac{1}{3}} \times 2m^{\frac{1}{5}}$ **f.** $\frac{1}{2}b^{\frac{3}{7}} \times 4b^{\frac{2}{7}}$

g. $-4y^2 \times y^{\frac{2}{9}}$ **h.** $\frac{2}{5}a^{\frac{3}{8}} \times 0.05a^{\frac{3}{4}}$ **i.** $5x^3 \times x^{\frac{1}{2}}$

7. Simplify each of the following.

a. $a^{\frac{2}{3}}b^{\frac{3}{4}} \times a^{\frac{1}{3}}b^{\frac{3}{4}}$ **b.** $x^{\frac{3}{5}}y^{\frac{2}{9}} \times x^{\frac{1}{5}}y^{\frac{1}{3}}$ **c.** $2ab^{\frac{1}{3}} \times 3a^{\frac{3}{5}}b^{\frac{4}{5}}$

d. $6m^{\frac{3}{7}} \times \frac{1}{3}m^{\frac{1}{4}}n^{\frac{2}{5}}$ **e.** $x^3y^{\frac{1}{2}}z^{\frac{1}{3}} \times x^{\frac{1}{6}}y^{\frac{1}{3}}z^{\frac{1}{2}}$ **f.** $2a^{\frac{2}{5}}b^{\frac{3}{8}}c^{\frac{1}{4}} \times 4b^{\frac{3}{4}}c^{\frac{3}{4}}$

8. Simplify each of the following.

a. $3^{\frac{1}{2}} \div 3^{\frac{1}{3}}$ **b.** $5^{\frac{2}{3}} \div 5^{\frac{1}{4}}$ **c.** $12^2 \div 12^{\frac{3}{2}}$

d. $a^{\frac{6}{7}} \div a^{\frac{3}{7}}$ **e.** $x^{\frac{3}{2}} \div x^{\frac{1}{4}}$ **f.** $\dfrac{m^{\frac{4}{5}}}{m^{\frac{5}{9}}}$

g. $\dfrac{2x^{\frac{3}{4}}}{4x^{\frac{3}{5}}}$ **h.** $\dfrac{7n^2}{21n^{\frac{4}{3}}}$ **i.** $\dfrac{25b^{\frac{3}{5}}}{20b^{\frac{1}{4}}}$

9. Simplify each of the following.

a. $x^3y^2 \div x^{\frac{4}{3}}y^{\frac{3}{5}}$ **b.** $a^{\frac{5}{9}}b^{\frac{2}{3}} \div a^{\frac{2}{5}}b^{\frac{2}{5}}$ **c.** $m^{\frac{3}{8}}n^{\frac{4}{7}} \div 3n^{\frac{3}{8}}$

d. $10x^{\frac{4}{5}}y \div 5x^{\frac{2}{3}}y^{\frac{1}{4}}$ **e.** $\dfrac{5a^{\frac{3}{4}}b^{\frac{3}{5}}}{20a^{\frac{1}{5}}b^{\frac{1}{4}}}$ **f.** $\dfrac{p^{\frac{7}{8}}q^{\frac{1}{4}}}{7p^{\frac{2}{3}}q^{\frac{1}{6}}}$

10. Simplify each of the following.

a. $\left(2^{\frac{3}{4}}\right)^{\frac{3}{5}}$
b. $\left(5^{\frac{2}{3}}\right)^{\frac{1}{4}}$
c. $\left(7^{\frac{1}{5}}\right)^{6}$
d. $(a^{3})^{\frac{1}{10}}$
e. $\left(m^{\frac{4}{9}}\right)^{\frac{3}{8}}$
f. $\left(2b^{\frac{1}{2}}\right)^{\frac{1}{3}}$
g. $4\left(p^{\frac{3}{7}}\right)^{\frac{15}{14}}$
h. $\left(x^{\frac{m}{n}}\right)^{\frac{n}{p}}$
i. $\left(3m^{\frac{a}{b}}\right)^{\frac{b}{c}}$

Understanding

11. **WE15b, c** Simplify each of the following.

a. $\left(a^{\frac{1}{2}}b^{\frac{1}{3}}\right)^{\frac{1}{2}}$
b. $(a^{4}b)^{\frac{3}{4}}$
c. $\left(x^{\frac{3}{5}}y^{\frac{7}{8}}\right)^{2}$
d. $\left(3a^{\frac{1}{3}}b^{\frac{3}{5}}c^{\frac{3}{4}}\right)^{\frac{1}{3}}$
e. $\left(x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{2}{5}}\right)^{\frac{1}{2}}$
f. $\left(\dfrac{a^{\frac{3}{4}}}{b}\right)^{\frac{2}{3}}$
g. $\left(\dfrac{m^{\frac{4}{5}}}{n^{\frac{7}{8}}}\right)^{2}$
h. $\left(\dfrac{b^{\frac{3}{5}}}{c^{\frac{4}{9}}}\right)^{\frac{2}{3}}$
i. $\left(\dfrac{4x^{7}}{2y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

12. **MC** *Note:* There may be more than one correct answer.

If $\left(a^{\frac{3}{4}}\right)^{\frac{m}{n}}$ is equal to $a^{\frac{1}{4}}$, then m and n could not be:

A. 1 and 3
B. 2 and 6
C. 3 and 8
D. 4 and 9

13. Simplify each of the following.

a. $\sqrt{a^{8}}$
b. $\sqrt[3]{b^{9}}$
c. $\sqrt[4]{m^{16}}$
d. $\sqrt{16x^{4}}$
e. $\sqrt[3]{8y^{9}}$
f. $\sqrt[4]{16x^{8}y^{12}}$
g. $\sqrt[3]{27m^{9}n^{15}}$
h. $\sqrt[5]{32p^{5}q^{10}}$
i. $\sqrt[3]{216a^{6}b^{18}}$

Reasoning

14. Manning's formula is used to calculate the flow of water in a river during a flood situation. Manning's formula is $v = \dfrac{R^{\frac{2}{3}}S^{\frac{1}{2}}}{n}$, where R is the hydraulic radius, S is the slope of the river and n is the roughness coefficient. This formula is used by meteorologists and civil engineers to analyse potential flood situations.

a. Find the flow of water in metres per second in the river if $R = 8$, $S = 0.0025$ and $n = 0.625$.

b. To find the volume of water flowing through the river, we multiply the flow rate by the average cross-sectional area of the river. If the average cross-sectional area is 52 m^2, find the volume of water (in L) flowing through the river each second. (Remember 1 m^3 = 1000 L.)

c. If water continues to flow at this rate, what will be the total amount of water to flow through in one hour? Justify your answer.

d. Use the internet to find the meaning of the terms 'hydraulic radius' and 'roughness coefficient'.

15. Find x if $m^{x} = \dfrac{\sqrt{m^{10}}}{\left(\sqrt{m^{4}}\right)^{2}}$.

Problem solving

16. Simplify:

a. $\dfrac{x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y - z}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}\right)}$

b. $\sqrt[5]{\dfrac{t^2}{\sqrt{t^3}}}$.

17. Expand $\left(m^{\frac{3}{4}} + m^{\frac{1}{2}}n^{\frac{1}{2}} + m^{\frac{1}{4}}n + n^{\frac{3}{2}}\right)\left(m^{\frac{1}{4}} - n^{\frac{1}{2}}\right)$.

Reflection

How will you remember the rule for fractional indices?

16.6 Negative indices

16.6.1 Negative indices

- Consider the following division $\dfrac{2^3}{2^4} = 2^{-1}$ (using the Second Index Law).

  Alternatively, $\dfrac{2^3}{2^4} = \dfrac{8}{16} = \dfrac{1}{2}$.

  We can conclude that $2^{-1} = \dfrac{1}{2}$.
- In general form:

$$a^{-1} = \frac{1}{a} \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

WORKED EXAMPLE 16 TI | CASIO

Evaluate each of the following using a calculator.

a 4^{-1} **b** 2^{-4}

THINK	WRITE
a Use a calculator to evaluate 4^{-1}.	**a** $4^{-1} = 0.25$
b Use a calculator to evaluate 2^{-4}.	**b** $2^{-4} = 0.0625$

- Consider the index law $a^{-1} = \dfrac{1}{a}$. Now consider the case in which a is fractional.

  Consider the expression $\left(\dfrac{a}{b}\right)^{-1}$.

$$\begin{aligned}\left(\frac{a}{b}\right)^{-1} &= \frac{1}{\frac{a}{b}}\\ &= 1 \times \frac{b}{a}\\ &= \frac{b}{a}\end{aligned}$$

We can therefore consider an index of -1 to be a reciprocal function.

WORKED EXAMPLE 17

Write down the value of each of the following without the use of a calculator.

a $\left(\frac{2}{3}\right)^{-1}$ **b** $\left(\frac{1}{5}\right)^{-1}$ **c** $\left(1\frac{1}{4}\right)^{-1}$

THINK	WRITE
a To evaluate $\left(\frac{2}{3}\right)^{-1}$ take the reciprocal of $\frac{2}{3}$.	**a** $\left(\frac{2}{3}\right)^{-1}=\frac{3}{2}$
b 1 To evaluate $\left(\frac{1}{5}\right)^{-1}$ take the reciprocal of $\frac{1}{5}$.	**b** $\left(\frac{1}{5}\right)^{-1}=\frac{5}{1}$
2 Write $\frac{5}{1}$ as a whole number.	$=5$
c 1 Write $1\frac{1}{4}$ as an improper fraction.	**c** $\left(1\frac{1}{4}\right)^{-1}=\left(\frac{5}{4}\right)^{-1}$
2 Take the reciprocal of $\frac{5}{4}$.	$=\frac{4}{5}$

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Exercise 16.6 Negative indices

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2a–e, 3a–e, 4a–e, 5a–e, 6a–d, 7a–d, 8–12	Questions: 1d–f, 2d–f, 3d–f, 4d–f, 5e–h, 6c–f, 7c–f, 8–12	Questions: 1e–h, 2e–h, 3e–h, 4e–h, 5g–l, 6e–h, 7e–h, 8–13

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE16 Evaluate each of the following using a calculator.

a. 5^{-1} **b.** 3^{-1} **c.** 8^{-1} **d.** 10^{-1}
e. 2^{-3} **f.** 3^{-2} **g.** 5^{-2} **h.** 10^{-4}

2. Find the value of each of the following, correct to 3 significant figures.

a. 6^{-1} **b.** 7^{-1} **c.** 6^{-2} **d.** 9^{-3}
e. 6^{-3} **f.** 15^{-2} **g.** 16^{-2} **h.** 5^{-4}

3. Find the value of each of the following, correct to 2 significant figures.

a. $(2.5)^{-1}$ **b.** $(0.4)^{-1}$ **c.** $(1.5)^{-2}$ **d.** $(0.5)^{-2}$
e. $(2.1)^{-3}$ **f.** $(10.6)^{-4}$ **g.** $(0.45)^{-3}$ **h.** $(0.125)^{-4}$

4. Find the value of each of the following, correct to 2 significant figures.

a. $(-3)^{-1}$ **b.** $(-5)^{-1}$ **c.** $(-2)^{-2}$ **d.** $(-4)^{-2}$
e. $(-1.5)^{-1}$ **f.** $(-2.2)^{-1}$ **g.** $(-0.6)^{-1}$ **h.** $(-0.85)^{-2}$

5. **WE17** Write down the value of each of the following without the use of a calculator.

a. $\left(\frac{4}{5}\right)^{-1}$ **b.** $\left(\frac{3}{10}\right)^{-1}$ **c.** $\left(\frac{7}{8}\right)^{-1}$ **d.** $\left(\frac{13}{20}\right)^{-1}$

e. $\left(\frac{1}{2}\right)^{-1}$ **f.** $\left(\frac{1}{4}\right)^{-1}$ **g.** $\left(\frac{1}{8}\right)^{-1}$ **h.** $\left(\frac{1}{10}\right)^{-1}$

i. $\left(1\frac{1}{2}\right)^{-1}$ **j.** $\left(2\frac{1}{4}\right)^{-1}$ **k.** $\left(1\frac{1}{10}\right)^{-1}$ **l.** $\left(5\frac{1}{2}\right)^{-1}$

6. Find the value of each of the following, leaving your answer in fraction form if necessary.

a. $\left(\frac{1}{2}\right)^{-2}$ **b.** $\left(\frac{2}{5}\right)^{-2}$ **c.** $\left(\frac{2}{3}\right)^{-3}$

d. $\left(\frac{1}{4}\right)^{-2}$ **e.** $\left(1\frac{1}{2}\right)^{-2}$ **f.** $\left(2\frac{1}{4}\right)^{-2}$

g. $\left(1\frac{1}{3}\right)^{-3}$ **h.** $\left(2\frac{1}{5}\right)^{-3}$

7. Find the value of each of the following.

a. $\left(-\frac{2}{3}\right)^{-1}$ **b.** $\left(-\frac{3}{5}\right)^{-1}$ **c.** $\left(-\frac{1}{4}\right)^{-1}$

d. $\left(-\frac{1}{10}\right)^{-1}$ **e.** $\left(-\frac{2}{3}\right)^{-2}$ **f.** $\left(-\frac{1}{5}\right)^{-2}$

g. $\left(-1\frac{1}{2}\right)^{-1}$ **h.** $\left(-2\frac{3}{4}\right)^{-2}$

Understanding

8. Without using a calculator, evaluate $\dfrac{\left(\dfrac{2^{-1}}{3}\right)^{-1}}{\left(\dfrac{4}{5^{-1}}\right)}$.

9. Simplify $\left(\sqrt{\dfrac{a^2}{b^2}}\right)^{-1}$.

Reasoning

10. Consider the equation $y = \dfrac{6}{x}$. Clearly $x \neq 0$, as $\dfrac{6}{x}$ would be undefined.

What happens to the value of y as x gets closer to zero coming from:

a. the positive direction

b. the negative direction?

11. Consider the expression 2^{-n}. Explain what happens to the value of this expression as n increases.

Problem solving

12. Solve the following pair of simultaneous equations.

$$3^{y+1} = \frac{1}{9} \text{ and } \frac{5^y}{125^x} = 125$$

13. Simplify $\dfrac{x^{n+2} + x^{n-2}}{x^{n-4} + x^n}$.

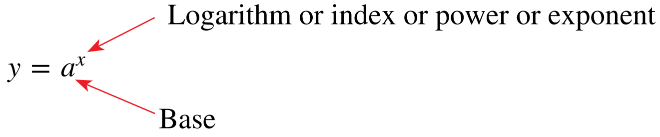
Reflection

How can division be used to explain negative indices?

16.7 Logarithms

16.7.1 Logarithms

- The index, power or exponent in the statement $y = a^x$ is also known as a **logarithm** (or log for short).



- This statement $y = a^x$ can be written in an alternative form as $\log_a y = x$, which is read as 'the logarithm of y to the base a is equal to x'. These two statements are equivalent.

$$\boxed{\begin{array}{ll} a^x = y \;\Leftrightarrow & \log_a y = x \\ \text{Index form} & \text{Logarithmic form} \end{array}}$$

- For example, $3^2 = 9$ can be written as $\log_3 9 = 2$. The log form would be read as 'the logarithm of 9, to the base of 3, is 2'. In both forms, the base is 3 and the logarithm is 2.

WORKED EXAMPLE 18

Write the following in logarithmic form.

a $\mathbf{10^4 = 10\,000}$ **b** $\mathbf{6^x = 216}$

THINK	WRITE
a 1 Write the given statement.	**a** $10^4 = 10\,000$
2 Identify the base (10) and the logarithm (4) and write the equivalent statement in logarithmic form. (Use $a^x = y \Leftrightarrow \log_a y = x$, where the base is a and the log is x.)	$\log_{10} 10\,000 = 4$
b 1 Write the given statement.	**b** $6^x = 216$
2 Identify the base (6) and the logarithm (x) and write the equivalent statement in logarithmic form.	$\log_6 216 = x$

WORKED EXAMPLE 19

Write the following in index form.

a $\mathbf{\log_2 8 = 3}$ **b** $\mathbf{\log_{25} 5 = \frac{1}{2}}$

THINK	WRITE
a 1 Write the statement.	**a** $\log_2 8 = 3$
2 Identify the base (2) and the log (3), and write the equivalent statement in index form. Remember that the log is the same as the index.	$2^3 = 8$
b 1 Write the statement.	**b** $\log_{25} 5 = \frac{1}{2}$
2 Identify the base (25) and the log $\left(\frac{1}{2}\right)$, and write the equivalent statement in index form.	$25^{\frac{1}{2}} = 5$

- In the previous examples, we found that:

$$\log_2 8 = 3 \Leftrightarrow 2^3 = 8 \text{ and } \log_{10} 10\,000 = 4 \Leftrightarrow 10^4 = 10\,000.$$

We could also write $\log_2 8 = 3$ as $\log_2 2^3 = 3$ and $\log_{10} 10\,000 = 4$ as $\log_{10} 10^4 = 4$.

- Can this pattern be used to work out the value of $\log_3 81$? We need to find the power when the base of 3 is raised to that power to give 81.

WORKED EXAMPLE 20 TI | CASIO

Evaluate $\log_3 81$.

THINK	WRITE
1 Write the log expression.	$\log_3 81$
2 Express 81 in index form with a base of 3.	$= \log_3 3^4$
3 Write the value of the logarithm.	$= 4$

16.7.2 Using logarithmic scales in measurement

- Logarithms can also be used to display data sets that cover a range of values which vary greatly in size. For example, when measuring the amplitude of earthquake waves, some earthquakes will have amplitudes of 10 000, whereas other earthquakes may have amplitudes of 10 000 000 (1000 times greater). Rather than trying to display this data on a linear scale, we can take the logarithm of the amplitude, which gives us the magnitude of each earthquake. The Richter scale uses the magnitudes of earthquakes to display the difference in their power.

WORKED EXAMPLE 21

Convert the following amplitudes of earthquakes into values on the Richter scale, correct to 1 decimal place.

a 1989 Newcastle earthquake: amplitude 398 000

b 2010 Canterbury earthquake: amplitude 12 600 000

c 2010 Chile earthquake: amplitude 631 000 000

THINK	WRITE
a Use a calculator to calculate the logarithmic value of the amplitude. Round the answer to 1 decimal place. Write the answer in words.	**a** $\log 398\,000 = 5.599\ldots$ $= 5.6$ The 1989 Newcastle earthquake rated 5.6 on the Richter scale.
b Use a calculator to calculate the logarithmic value of the amplitude. Round the answer to 1 decimal place. Write the answer in words.	**b** $\log 12\,600\,000 = 7.100\ldots$ $= 7.1$ The 2010 Canterbury earthquake rated 7.1 on the Richter scale.
c Use a calculator to calculate the logarithmic value of the amplitude. Round the answer to 1 decimal place. Write the answer in words.	**c** $\log 631\,000\,000 = 8.800\ldots$ $= 8.8$ The 2010 Chile earthquake rated 8.8 on the Richter scale.

Displaying logarithmic data in histograms

- If we are given a data set in which the data vary greatly in size, we can use logarithms to transform the data into more manageable figures, and then group the data into intervals to provide an indication of the spread of the data.

WORKED EXAMPLE 22

The following table displays the population of 10 different towns and cities in Victoria (using data from the 2011 census).

Town or city	Population
Benalla	9 328
Bendigo	76 051
Castlemaine	9 124
Echuca	12 613
Geelong	143 921
Kilmore	6 142
Melbourne	3 707 530
Stawell	5 734
Wangaratta	17 377
Warrnambool	29 284

a Convert the populations into logarithmic form, correct to 2 decimal places.
b Group the data into a frequency table.
c Draw a histogram to represent the data.

THINK	WRITE/DRAW
a Use a calculator to calculate the logarithmic values of all of the populations. Round the answers to 2 decimal places.	**a**

Town or city	Log (population)
Benalla	3.97
Bendigo	4.88
Castlemaine	3.96
Echuca	4.10
Geelong	5.16
Kilmore	3.79
Melbourne	6.57
Stawell	3.76
Wangaratta	4.24
Warrnambool	4.67

THINK	WRITE/DRAW
b Group the logarithmic values into class intervals and create a frequency table.	**b**

Log (population)	Frequency
3–<4	4
4–<5	4
5–<6	1
6–<7	1

c Construct a histogram of the data set.

c

Exercise 16.7 Logarithms

assess on

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2, 3a–e, 4, 5a–e, 6–8, 10	Questions: 1e–k, 2, 3d–i, 4, 5e–h, 6–10	Questions: 1i–p, 2, 3g–l, 4, 5g–l, 6–11

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Fluency

1. **WE18** Write the following in logarithmic form.

a. $4^2 = 16$ **b.** $2^5 = 32$ **c.** $3^4 = 81$ **d.** $6^2 = 36$

e. $1000 = 10^3$ **f.** $25 = 5^2$ **g.** $4^3 = x$ **h.** $5^x = 125$

i. $7^x = 49$ **j.** $p^4 = 16$ **k.** $9^{\frac{1}{2}} = 3$ **l.** $0.1 = 10^{-1}$

m. $2 = 8^{\frac{1}{3}}$ **n.** $2^{-1} = \frac{1}{2}$ **o.** $a^0 = 1$ **p.** $4^{\frac{3}{2}} = 8$

2. **MC** The statement $w = h^t$ is equivalent to:

A. $w = \log_t h$

B. $h = \log_t w$

C. $t = \log_w h$

D. $t = \log_h w$

3. **WE19** Write the following in index form.

a. $\log_2 16 = 4$ **b.** $\log_3 27 = 3$ **c.** $\log_{10} 1\,000\,000 = 6$

d. $\log_5 125 = 3$ **e.** $\log_{16} 4 = \frac{1}{2}$ **f.** $\log_4 64 = x$

g. $\frac{1}{2} = \log_{49} 7$ **h.** $\log_3 x = 5$ **i.** $\log_{81} 9 = \frac{1}{2}$

j. $\log_{10} 0.01 = -2$ **k.** $\log_8 8 = 1$ **l.** $\log_{64} 4 = \frac{1}{3}$

4. **MC** The statement $q = \log_r p$ is equivalent to:

A. $q = r^p$ **B.** $p = r^q$ **C.** $r = p^q$ **D.** $r = q^p$

5. **WE20** Evaluate the following logarithms.

a. $\log_2 16$ **b.** $\log_4 16$ **c.** $\log_{11} 121$

d. $\log_{10} 100\,000$ **e.** $\log_3 243$ **f.** $\log_2 128$

g. $\log_5 1$ **h.** $\log_9 3$ **i.** $\log_3 \left(\frac{1}{3}\right)$

j. $\log_6 6$ **k.** $\log_{10} \left(\frac{1}{100}\right)$ **l.** $\log_{125} 5$

6. Write the value of each of the following.

a. $\log_{10} 1$ **b.** $\log_{10} 10$ **c.** $\log_{10} 100$

d. $\log_{10} 1000$ **e.** $\log_{10} 10\,000$ **f.** $\log_{10} 100\,000$

Understanding

7. Use your results to question **6** to answer the following.

a. Between which two whole numbers would $\log_{10} 7$ lie?
b. Between which two whole numbers would $\log_{10} 4600$ lie?
c. Between which two whole numbers would $\log_{10} 85$ lie?
d. Between which two whole numbers would $\log_{10} 12\,750$ lie?
e. Between which two whole numbers would $\log_{10} 110$ lie?
f. Between which two whole numbers would $\log_{10} 81\,000$ lie?

8. **WE21** Convert the following amplitudes of earthquakes into values on the Richter scale, correct to 1 decimal place.

a. 2016 Northern Territory earthquake: amplitude 1 260 000
b. 2011 Christchurch earthquake: amplitude 2 000 000
c. 1979 Tumaco earthquake: amplitude 158 000 000

Reasoning

9. a. If $\log_{10} g = k$, find the value of $\log_{10} g^2$. Justify your answer.
b. If $\log_x y = 2$, find the value of $\log_y x$. Justify your answer.
c. By referring to the equivalent index statement, explain why x must be a positive number given $\log_4 x = y$, for all values of y.

10. Calculate each of the following logarithms.

a. $\log_2(64)$ **b.** $\log_3\left(\frac{1}{81}\right)$ **c.** $\log_{10}(0.00001)$

Problem solving

11. Find the value of x.

a. $\log_x\left(\frac{1}{243}\right) = -5$ **b.** $\log_x(343) = 3$ **c.** $\log_{64}(x) = -\frac{1}{2}$

12. Simplify $10^{\log_{10}(x)}$.

Reflection

How are indices and logarithms related?

16.8 Logarithm laws

16.8.1 Logarithm laws

- The index laws are:

1. $a^m \times a^n = a^{m+n}$ 2. $\frac{a^m}{a^n} = a^{m-n}$ 3. $(a^m)^n = a^{mn}$
4. $a^0 = 1$ 5. $a^1 = a$ 6. $a^{-1} = \frac{1}{a}$

- The index laws can be used to produce equivalent logarithm laws.

16.8.2 Law 1

- If $x = a^m$ and $y = a^n$, then $\log_a x = m$ and $\log_a y = n$ (equivalent log form).

Now $xy = a^m \times a^n$
or $xy = a^{m+n}$ (First Index law)
So $\log_a(xy) = m + n$ (equivalent log form)
or $\log_a(xy) = \log_a x + \log_a y$ (substituting for m and n)

$$\boxed{\log_a x + \log_a y = \log_a(xy)}$$

- This means that the sum of two logarithms with the same base is equal to the logarithm of the product of the numbers.

WORKED EXAMPLE 23

TI | CASIO

Evaluate $\log_{10}20 + \log_{10}5$.

THINK	WRITE
1 Since the same base of 10 is used in each log term, use $\log_a x + \log_a y = \log_a (xy)$ and simplify.	$\log_{10}20 + \log_{10}5 = \log_{10}(20 \times 5)$ $= \log_{10}100$
2 Evaluate. (Remember that $100 = 10^2$.)	$= 2$

16.8.3 Law 2

- If $x = a^m$ and $y = a^n$, then $\log_a x = m$ and $\log_a y = n$ (equivalent log form).

$$\text{Now} \qquad \frac{x}{y} = \frac{a^m}{a^n}$$

$$\text{or} \qquad \frac{x}{y} = a^{m-n} \qquad \text{(Second Index Law).}$$

$$\text{So} \qquad \log_a\left(\frac{x}{y}\right) = m - n \qquad \text{(equivalent log form)}$$

$$\text{or} \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y \qquad \text{(substituting for } m \text{ and } n\text{).}$$

$$\boxed{\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)}$$

- This means that the difference of two logarithms with the same base is equal to the logarithm of the quotient of the numbers.

WORKED EXAMPLE 24

Evaluate $\log_4 20 - \log_4 5$.

THINK	WRITE
1 Since the same base of 4 is used in each log term, use $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ and simplify.	$\log_4 20 - \log_4 5 = \log_4\left(\frac{20}{5}\right)$ $= \log_4 4$
2 Evaluate. (Remember that $4 = 4^1$.)	$= 1$

WORKED EXAMPLE 25

TI | CASIO

Evaluate $\log_5 35 + \log_5 15 - \log_5 21$.

THINK	WRITE
1 Since the first two log terms are being added, use $\log_a x + \log_a y = \log_a (xy)$ and simplify.	$\log_5 35 + \log_5 15 - \log_5 21$ $= \log_5 (35 \times 15) - \log_5 21$ $= \log_5 525 - \log_5 21$
2 To find the difference between the two remaining log terms, use $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ and simplify.	$= \log_5\left(\frac{525}{21}\right)$ $= \log_5 25$
3 Evaluate. (Remember that $25 = 5^2$.)	$= 2$

- Once you have gained confidence in using the first two laws, you can reduce the number of steps of working by combining the application of the laws. In Worked example 25, we could write:

$$\begin{aligned}\log_5 35 + \log_5 15 - \log_5 21 &= \log_5\left(\frac{35 \times 15}{21}\right)\\ &= \log_5 25\\ &= 2\end{aligned}$$

16.8.4 Law 3

- If $x = a^m$, then $\log_a x = m$ (equivalent log form).

$$\begin{array}{lll}\text{Now} & x^n = (a^m)^n & \\ \text{or} & x^n = a^{mn} & \text{(Third Index Law).} \\ \text{So} & \log_a x^n = mn & \text{(equivalent log form)} \\ \text{or} & \log_a x^n = (\log_a x) \times n & \text{(substituting for } m\text{)} \\ \text{or} & \log_a x^n = n \log_a x & \end{array}$$

$$\boxed{\log_a x^n = n \log_a x}$$

- This means that the logarithm of a number raised to a power is equal to the product of the power and the logarithm of the number.

WORKED EXAMPLE 26

Evaluate $2 \log_6 3 + \log_6 4$.

THINK	WRITE
1 The first log term is not in the required form to use the log law relating to sums. Use $\log_a x^n = n \log_a x$ to rewrite the first term in preparation for applying the first log law.	$2 \log_6 3 + \log_6 4 = \log_6 3^2 + \log_6 4$ $= \log_6 9 + \log_6 4$
2 Use $\log_a x + \log_a y = \log_a (xy)$ to simplify the two log terms to one.	$= \log_6 (9 \times 4)$ $= \log_6 36$
3 Evaluate. (Remember that $36 = 6^2$.)	$= 2$

16.8.5 Law 4

- As $a^0 = 1$ (Fourth Index Law), $\log_a 1 = 0$ (equivalent log form).

$$\boxed{\log_a 1 = 0}$$

- This means that the logarithm of 1 with any base is equal to 0.

16.8.6 Law 5

- As $a^1 = a$ (Fifth Index Law), $\log_a a = 1$ (equivalent log form).

$$\boxed{\log_a a = 1}$$

- This means that the logarithm of any number a with base a is equal to 1.

16.8.7 Law 6

- Now $\log_a\left(\frac{1}{x}\right) = \log_a x^{-1}$ (Sixth Index Law)

  or $\log_a\left(\frac{1}{x}\right) = -1 \times \log_a x$ (using the fourth log law)

  or $\log_a\left(\frac{1}{x}\right) = -\log_a x.$

$$\boxed{\log_a\left(\frac{1}{x}\right) = -\log_a x}$$

16.8.8 Law 7

- Now $\log_a a^x = x \log_a a$ (using the third log law)

  or $\log_a a^x = x \times 1$ (using the fifth log law)

  or $\log_a a^x = x.$

$$\boxed{\log_a a^x = x}$$

Exercise 16.8 Logarithm laws

assess**on**

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 8a–f, 9a–f, 10, 11a–g, 12, 13, 15	Questions: 1–7, 8d–i, 9e–j, 10, 11e–i, 12–15	Questions: 1–7, 8g–l, 9g–l, 10, 11g–l, 12–16

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. Use a calculator to evaluate the following, correct to 5 decimal places.

a. $\log_{10} 50$ **b.** $\log_{10} 25$ **c.** $\log_{10} 5$ **d.** $\log_{10} 2$

2. Use your answers to question **1** to show that each of the following statements is true.

a. $\log_{10} 25 + \log_{10} 2 = \log_{10} 50$

b. $\log_{10} 50 - \log_{10} 2 = \log_{10} 25$

c. $\log_{10} 25 = 2\log_{10} 5$

d. $\log_{10} 50 - \log_{10} 25 - \log_{10} 2 = \log_{10} 1$

3. **WE23** Evaluate the following.

a. $\log_6 3 + \log_6 2$ **b.** $\log_4 8 + \log_4 8$

c. $\log_{10} 25 + \log_{10} 4$ **d.** $\log_8 32 + \log_8 16$

e. $\log_6 108 + \log_6 12$ **f.** $\log_{14} 2 + \log_{14} 7$

4. **WE24** Evaluate the following.

a. $\log_2 20 - \log_2 5$ **b.** $\log_3 54 - \log_3 2$

c. $\log_4 24 - \log_4 6$ **d.** $\log_{10} 30\,000 - \log_{10} 3$

e. $\log_6 648 - \log_6 3$ **f.** $\log_2 224 - \log_2 7$

5. **WE25** Evaluate the following.

a. $\log_3 27 + \log_3 2 - \log_3 6$ **b.** $\log_4 24 - \log_4 2 - \log_4 6$

c. $\log_6 78 - \log_6 13 + \log_6 1$ **d.** $\log_2 120 - \log_2 3 - \log_2 5$

6. Evaluate $2\log_4 8$.

7. **WE26** Evaluate the following.

a. $2\log_{10}5 + \log_{10}4$

b. $\log_3 648 - 3\log_3 2$

c. $4\log_5 10 - \log_5 80$

d. $\log_2 50 + \frac{1}{2}\log_2 16 - 2\log_2 5$

8. Evaluate the following.

a. $\log_8 8$

b. $\log_5 1$

c. $\log_2\left(\frac{1}{2}\right)$

d. $\log_4 4^5$

e. $\log_6 6^{-2}$

f. $\log_{20} 20$

g. $\log_2 1$

h. $\log_3\left(\frac{1}{9}\right)$

i. $\log_4\left(\frac{1}{2}\right)$

j. $\log_5 \sqrt{5}$

k. $\log_3\left(\frac{1}{\sqrt{3}}\right)$

l. $\log_2 8\sqrt{2}$

Understanding

9. Use the logarithm laws to simplify each of the following.

a. $\log_a 5 + \log_a 8$

b. $\log_a 12 + \log_a 3 - \log_a 2$

c. $4\log_x 2 + \log_x 3$

d. $\log_x 100 - 2\log_x 5$

e. $3\log_a x - \log_a x^2$

f. $5\log_a a - \log_a a^4$

g. $\log_x 6 - \log_x 6x$

h. $\log_a a^7 + \log_a 1$

i. $\log_p \sqrt{p}$

j. $\log_k k\sqrt{k}$

k. $6\log_a\left(\frac{1}{a}\right)$

l. $\log_a\left(\frac{1}{\sqrt[3]{a}}\right)$

10. **MC** *Note:* There may be more than one correct answer.

a. The equation $y = 10^x$ is equivalent to:

A. $x = 10^y$

B. $x = \log_{10} y$

C. $x = \log_x 10$

D. $x = \log_y 10$

b. The equation $y = 10^{4x}$ is equivalent to:

A. $x = \log_{10}\sqrt{4y}$

B. $x = \log_{10}\sqrt[4]{y}$

C. $x = 10^{\frac{1}{4}y}$

D. $x = \frac{1}{4}\log_{10} y$

c. The equation $y = 10^{3x}$ is equivalent to:

A. $x = \frac{1}{3}\log_{10} y$

B. $x = \log_{10} y^{\frac{1}{3}}$

C. $x = \log_{10} y - 3$

D. $x = 10^{y-3}$

d. The equation $y = ma^{nx}$ is equivalent to:

A. $x = \frac{1}{n}a^{my}$

B. $x = \log_a\left(\frac{m}{y}\right)^n$

C. $x = \frac{1}{n}(\log_a y - \log_a m)$

D. $x = \frac{1}{n}\log_a\left(\frac{y}{m}\right)$

11. Simplify, and evaluate where possible, each of the following without a calculator.

a. $\log_2 8 + \log_2 10$

b. $\log_3 7 + \log_3 15$

c. $\log_{10} 20 + \log_{10} 5$

d. $\log_6 8 + \log_6 7$

e. $\log_2 20 - \log_2 5$

f. $\log_3 36 - \log_3 12$

g. $\log_5 100 - \log_5 8$

h. $\log_2 \frac{1}{3} + \log_2 9$

i. $\log_4 25 + \log_4 \frac{1}{5}$

j. $\log_{10} 5 - \log_{10} 20$

k. $\log_3 \frac{4}{5} - \log_3 \frac{1}{5}$

l. $\log_2 9 + \log_2 4 - \log_2 12$

m. $\log_3 8 - \log_3 2 + \log_3 5$

n. $\log_4 24 - \log_4 2 - \log_4 6$

12. **MC** **a.** The expression $\log_{10} xy$ is equal to:

A. $\log_{10} x \times \log_{10} y$ **B.** $\log_{10} x - \log_{10} y$ **C.** $\log_{10} x + \log_{10} y$ **D.** $y \log_{10} x$

b. The expression $\log_{10} x^y$ is equal to:

A. $x \log_{10} y$ **B.** $y \log_{10} x$ **C.** $10 \log_x y$ **D.** $\log_{10} x + \log_{10} y$

c. The expression $\frac{1}{3} \log_2 64 + \log_2 10$ is equal to:

A. $\log_2 40$ **B.** $\log_2 80$ **C.** $\log_2 \frac{64}{10}$ **D.** 1

Reasoning

13. For each of the following, write the possible strategy you intend to use.

a. Evaluate $(\log_3 81)(\log_3 27)$.

b. Evaluate $\dfrac{\log_a 81}{\log_a 3}$.

c. Evaluate $5^{\log_5 7}$.

In each case, explain how you obtained your final answer.

14. Simplify $\log_2\left(\frac{8}{125}\right) - 3\log_2\left(\frac{3}{5}\right) - 4\log_2\left(\frac{1}{2}\right)$.

Problem solving

15. Simplify $\log_a (a^5 + a^3) - \log_a (a^4 + a^2)$.

16. If $2 \log_a (x) = 1 + \log_a (8x - 15a)$, find x in terms of a where a is a positive constant and x is positive.

Reflection

What technique will you use to remember the log laws?

CHALLENGE 16.1

Evaluate $\dfrac{\log_2 8 \times \log_2 16}{4^{\log_4 8}}$.

16.9 Solving equations

16.9.1 Solving equations with logarithms

- The equation $\log_a y = x$ is an example of a general **logarithmic equation**. Laws of logarithms and indices are used to solve these equations.

WORKED EXAMPLE 27

Solve for x in the following equations.

a $\log_2 x = 3$ **b** $\log_6 x = -2$ **c** $\log_3 x^4 = -16$ **d** $\log_5 (x - 1) = 2$

THINK	WRITE
a 1 Write the equation.	**a** $\log_2 x = 3$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = x$
3 Rearrange and simplify.	$x = 8$

b **1** Write the equation.	**b** $\log_6 x = -2$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$6^{-2} = x$
3 Rearrange and simplify.	$x = \frac{1}{6^2}$ $= \frac{1}{36}$
c **1** Write the equation.	**c** $\log_3 x^4 = -16$
2 Rewrite using $\log_a x^n = n \log_a x$.	$4\log_3 x = -16$
3 Divide both sides by 4.	$\log_3 x = -4$
4 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$3^{-4} = x$
5 Rearrange and simplify.	$x = \frac{1}{3^4}$ $= \frac{1}{81}$
d **1** Write the equation.	**d** $\log_5 (x - 1) = 2$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$5^2 = x - 1$
3 Solve for x.	$x - 1 = 25$ $x = 26$

WORKED EXAMPLE 28

TI | CASIO

Solve for x in $\log_x 25 = 2$, given that $x > 0$.

THINK	WRITE
1 Write the equation.	$\log_x 25 = 2$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$x^2 = 25$
3 Solve for x. *Note:* $x = -5$ is rejected as a solution because $x > 0$.	$x = 5$ (because $x > 0$)

WORKED EXAMPLE 29

Solve for x in the following.

a $\log_2 16 = x$ **b** $\log_3 \left(\frac{1}{3}\right) = x$ **c** $\log_9 3 = x$

THINK	WRITE
a **1** Write the equation.	**a** $\log_2 16 = x$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^x = 16$
3 Write 16 with base 2.	$= 2^4$
4 Equate the indices.	$x = 4$
b **1** Write the equation.	**b** $\log_3 \left(\frac{1}{3}\right) = x$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$3^x = \frac{1}{3}$ $= \frac{1}{3^1}$
3 Write $\frac{1}{3}$ with base 3.	$3^x = 3^{-1}$
4 Equate the indices.	$x = -1$
c 1 Write the equation.	**c** $\log_9 3 = x$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$9^x = 3$
3 Write 9 with base 3.	$(3^2)^x = 3$
4 Remove the grouping symbols.	$3^{2x} = 3^1$
5 Equate the indices.	$2x = 1$
6 Solve for x.	$x = \frac{1}{2}$

WORKED EXAMPLE 30

TI | CASIO

Solve for x in the equation $\log_2 4 + \log_2 x - \log_2 8 = 3$.

THINK	**WRITE**
1 Write the equation.	$\log_2 4 + \log_2 x - \log_2 8 = 3$
2 Simplify the left-hand side. Use $\log_a x + \log_a y = \log_a (xy)$ and $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$.	$\log_2 \left(\frac{4 \times x}{8}\right) = 3$
3 Simplify.	$\log_2 \left(\frac{x}{2}\right) = 3$
4 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = \frac{x}{2}$
5 Solve for x.	$x = 2 \times 2^3$ $= 2 \times 8$ $= 16$

- When solving an equation like $\log_2 8 = x$, it could be rewritten in index form as $2^x = 8$. This can be written with the same base of 2 to produce $2^x = 2^3$. Equating the indices gives us a solution of $x = 3$.
- Can we do this to solve the equation $2^x = 7$? Consider the method shown in the next worked example. It involves the use of logarithms and the $\log_{10}$ function on a calculator.

WORKED EXAMPLE 31

TI | CASIO

Solve for x, correct to 3 decimal places, if:

a $2^x = 7$

b $3^{-x} = 0.4$

THINK	**WRITE**
a 1 Write the equation.	**a** $2^x = 7$
2 Take $\log_{10}$ of both sides.	$\log_{10} 2^x = \log_{10} 7$
3 Use the logarithm-of-a-power law to bring the power, x, to the front of the logarithmic equation.	$x \log_{10} 2 = \log_{10} 7$

	4 Divide both sides by $\log_{10} 2$ to get x by itself.	Therefore, $x = \dfrac{\log_{10} 7}{\log_{10} 2}$
	5 Use a calculator to evaluate the logarithms and write the answer correct to 3 decimal places.	$= 2.807$
b	1 Write the equation.	**b** $3^{-x} = 0.4$
	2 Take $\log_{10}$ of both sides.	$\log_{10} 3^{-x} = \log_{10} 0.4$
	3 Use the logarithm of a power law to bring the power, x, to the front of the logarithmic equation.	$-x \log_{10} 3 = \log_{10} 0.4$
	4 Divide both sides by $\log_{10} 3$ to get the $-x$ by itself.	$-x = \dfrac{\log_{10} 0.4}{\log_{10} 3}$
	5 Use a calculator to evaluate the logarithms and write the answer correct to 3 decimal places.	$-x = -0.834$
	6 Divide both sides by -1 to get x by itself.	$x = 0.834$

- Therefore, we can state the following rule:

$$\textbf{If } a^x = b\textbf{, then } x = \frac{\log_{10} b}{\log_{10} a}.$$

This rule applies to any base, but since your calculator has base 10, this is the most commonly used for this solution technique.

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Exercise 16.9 Solving equations

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–h, 2a–e, 3a–f, 4a–h, 5, 6a–h, 7a–f, 8, 9, 11	Questions: 1d–k, 2d–f, 3c–f, 4e–j, 5, 6e–l, 7d–i, 8–11	Questions: 1g–l, 2d–h, 3e–j, 4i–n, 5, 6i–o, 7g–l, 8–12

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Fluency

1. **WE27** Solve for x in the following.

a. $\log_5 x = 2$
b. $\log_3 x = 4$
c. $\log_2 x = -3$
d. $\log_4 x = -2$
e. $\log_{10} x^2 = 4$
f. $\log_2 x^3 = 12$
g. $\log_3 (x + 1) = 3$
h. $\log_5 (x - 2) = 3$
i. $\log_4 (2x - 3) = 0$
j. $\log_{10} (2x + 1) = 0$
k. $\log_2 (-x) = -5$
l. $\log_3 (-x) = -2$
m. $\log_5 (1 - x) = 4$
n. $\log_{10} (5 - 2x) = 1$

2. **WE28** Solve for x in the following, given that $x > 0$.

a. $\log_x 9 = 2$
b. $\log_x 16 = 4$
c. $\log_x 25 = \dfrac{2}{3}$
d. $\log_x 125 = \frac{3}{4}$
e. $\log_x \left(\frac{1}{8}\right) = -3$
f. $\log_x \left(\frac{1}{64}\right) = -2$
g. $\log_x 6^2 = 2$
h. $\log_x 4^3 = 3$

3. **WE29** Solve for x in the following.

a. $\log_2 8 = x$
b. $\log_3 9 = x$
c. $\log_5 \left(\frac{1}{5}\right) = x$
d. $\log_4 \left(\frac{1}{16}\right) = x$
e. $\log_4 2 = x$
f. $\log_8 2 = x$
g. $\log_6 1 = x$
h. $\log_8 1 = x$
i. $\log_{\frac{1}{2}} 2 = x$
j. $\log_{\frac{1}{3}} 9 = x$

4. **WE30** Solve for x in the following.

a. $\log_2 x + \log_2 4 = \log_2 20$
b. $\log_5 3 + \log_5 x = \log_5 18$
c. $\log_3 x - \log_3 2 = \log_3 5$
d. $\log_{10} x - \log_{10} 4 = \log_{10} 2$
e. $\log_4 8 - \log_4 x = \log_4 2$
f. $\log_3 10 - \log_3 x = \log_3 5$
g. $\log_6 4 + \log_6 x = 2$
h. $\log_2 x + \log_2 5 = 1$
i. $3 - \log_{10} x = \log_{10} 2$
j. $5 - \log_4 8 = \log_4 x$
k. $\log_2 x + \log_2 6 - \log_2 3 = \log_2 10$
l. $\log_2 x + \log_2 5 - \log_2 10 = \log_2 3$
m. $\log_3 5 - \log_3 x + \log_3 2 = \log_3 10$
n. $\log_5 4 - \log_5 x + \log_5 3 = \log_5 6$

5. **MC** **a.** The solution to the equation $\log_7 343 = x$ is:

A. $x = 2$
B. $x = 3$
C. $x = 1$
D. $x = 0$

b. If $\log_8 x = 4$, then x is equal to:

A. 4096
B. 512
C. 64
D. 2

c. Given that $\log_x 3 = \frac{1}{2}$, x must be equal to:

A. 3
B. 6
C. 81
D. 9

d. If $\log_a x = 0.7$, then $\log_a x^2$ is equal to:

A. 0.49
B. 1.4
C. 0.35
D. 0.837

6. Solve for x in the following equations.

a. $2^x = 128$
b. $3^x = 9$
c. $7^x = \frac{1}{49}$
d. $9^x = 1$
e. $5^x = 625$
f. $64^x = 8$
g. $6^x = \sqrt{6}$
h. $2^x = 2\sqrt{2}$
i. $3^x = \dfrac{1}{\sqrt{3}}$
j. $4^x = 8$
k. $9^x = 3\sqrt{3}$
l. $2^x = \dfrac{1}{4\sqrt{2}}$
m. $3^{x+1} = 27\sqrt{3}$
n. $2^{x-1} = \dfrac{1}{32\sqrt{2}}$
o. $4^{x+1} = \dfrac{1}{8\sqrt{2}}$

Understanding

7. **WE31** Solve the following equations, correct to 3 decimal places.

a. $2^x = 11$
b. $2^x = 0.6$
c. $3^x = 20$
d. $3^x = 1.7$
e. $5^x = 8$
f. $0.7^x = 3$
g. $0.4^x = 5$
h. $3^{x+2} = 12$
i. $7^{-x} = 0.2$
j. $8^{-x} = 0.3$
k. $10^{-2x} = 7$
l. $8^{2-x} = 0.75$

8. The decibel (dB) scale for measuring loudness, d, is given by the formula $d = 10\log_{10}(I \times 10^{12})$, where I is the intensity of sound in watts per square metre.

a. Find the number of decibels of sound if the intensity is 1.

b. Find the number of decibels of sound produced by a jet engine at a distance of 50 metres if the intensity is 10 watts per square metre.

c. Find the intensity of sound if the sound level of a pneumatic drill 10 metres away is 90 decibels.

d. Find how the value of d changes if the intensity is doubled. Give your answer to the nearest decibel.
e. Find how the value of d changes if the intensity is 10 times as great.
f. By what factor does the intensity of sound have to be multiplied in order to add 20 decibels to the sound level?

Reasoning

9. The Richter scale is used to describe the energy of earthquakes. A formula for the Richter scale is: $R = \frac{2}{3}\log_{10}K - 0.9$, where R is the Richter scale value for an earthquake that releases K kilojoules (kJ) of energy.

a. Find the Richter scale value for an earthquake that releases the following amounts of energy:

i. 1000 kJ
ii. 2000 kJ
iii. 3000 kJ
iv. 10 000 kJ
v. 100 000 kJ
vi. 1 000 000 kJ

b. Does doubling the energy released double the Richter scale value? Justify your answer.
c. Find the energy released by an earthquake of:
  i. magnitude 4 on the Richter scale
  ii. magnitude 5 on the Richter scale
  iii. magnitude 6 on the Richter scale.
d. What is the effect (on the amount of energy released) of increasing the Richter scale value by 1?
e. Why is an earthquake measuring 8 on the Richter scale so much more devastating than one that measures 5?

10. Solve for x.
a. $3^{x+1} = 7$
b. $3^{x+1} = 7^x$

Problem solving

11. Solve for x.
$(27 \times 3^x)^3 = 81^x \times 3^2$
12. Solve $\{x: (3^x)^2 = 30 \times 3^x - 81\}$.

Reflection

Tables of logarithms were used in classrooms before calculators were used there. Would using logarithms have any effect on the accuracy of calculations?

CHALLENGE 16.2

This challenge explores an interesting association between logarithms and quadratics.
Consider solving the logarithmic equation $\log_{10}(x+1) + \log_{10}x - \log_{10}6 = 0$.

This first step in the solution could be: $\log_{10}\left[\frac{x(x+1)}{6}\right] = 0$.

Continue the solution by converting the logarithmic equation into a quadratic equation, then solving for x.

16.10 Review

16.10.1 Review questions

Fluency

1. Which of the given numbers are rational?

$$\sqrt{\frac{6}{12}}, \sqrt{0.81}, 5, -3.26, 0.5, \frac{\pi}{5}, \sqrt{\frac{3}{12}}$$

a. $\sqrt{0.81},\ 5,\ -3.26,\ 0.5$ and $\sqrt{\dfrac{3}{12}}$

b. $\sqrt{\dfrac{6}{12}}$ and $\dfrac{\pi}{5}$

c. $\sqrt{\dfrac{6}{12}},\ \sqrt{0.81}$ and $\sqrt{\dfrac{3}{12}}$

d. $5,\ -3.26$ and $\sqrt{\dfrac{6}{12}}$

2. For each of the following, state whether the number is rational or irrational and give the reason for your answer:

a. $\sqrt{12}$ **b.** $\sqrt{121}$ **c.** $\dfrac{2}{9}$ **d.** $0.\dot{6}$ **e.** $\sqrt[3]{0.08}$

3. Which of the numbers of the given set are surds?

$\{3\sqrt{2}, 5\sqrt{7}, 9\sqrt{4}, 6\sqrt{10}, 7\sqrt{12}, 12\sqrt{64}\}$

a. $9\sqrt{4}, 12\sqrt{64}$ **b.** $3\sqrt{2}$ and $7\sqrt{12}$ only

c. $3\sqrt{2}, 5\sqrt{7}$ and $6\sqrt{10}$ only **d.** $3\sqrt{2}, 5\sqrt{7}, 6\sqrt{10}$ and $7\sqrt{12}$

4. Which of $\sqrt{2m}, \sqrt{25m}, \sqrt{\frac{m}{16}}, \sqrt{\frac{20}{m}}, \sqrt[3]{m}, \sqrt[3]{8m}$ are surds:

a. if $m = 4$? **b.** if $m = 8$?

5. Simplify each of the following.

a. $\sqrt{50}$ **b.** $\sqrt{180}$ **c.** $2\sqrt{32}$ **d.** $5\sqrt{80}$

6. The expression $\sqrt{392x^8y^7}$ may be simplified to:

a. $196x^4y^3\sqrt{2y}$ **b.** $2x^4y^3\sqrt{14y}$ **c.** $14x^4y^3\sqrt{2y}$ **d.** $14x^4y^3\sqrt{2}$

7. Simplify the following surds. Give the answers in the simplest form.

a. $4\sqrt{648x^7y^9}$ **b.** $-\dfrac{2}{5}\sqrt{\dfrac{25}{64}x^5y^{11}}$

8. Simplify the following, giving answers in the simplest form.

a. $7\sqrt{12} + 8\sqrt{147} - 15\sqrt{27}$ **b.** $\dfrac{1}{2}\sqrt{64a^3b^3} - \dfrac{3}{4}ab\sqrt{16ab} + \dfrac{1}{5ab}\sqrt{100a^5b^5}$

9. Simplify each of the following.

a. $\sqrt{3} \times \sqrt{5}$ **b.** $2\sqrt{6} \times 3\sqrt{7}$ **c.** $3\sqrt{10} \times 5\sqrt{6}$ **d.** $(\sqrt{5})^2$

10. Simplify the following, giving answers in the simplest form.

a. $\dfrac{1}{5}\sqrt{675} \times \sqrt{27}$ **b.** $10\sqrt{24} \times 6\sqrt{12}$

11. Simplify the following.

a. $\dfrac{\sqrt{30}}{\sqrt{10}}$ **b.** $\dfrac{6\sqrt{45}}{3\sqrt{5}}$ **c.** $\dfrac{3\sqrt{20}}{12\sqrt{6}}$ **d.** $\dfrac{\left(\sqrt{7}\right)^2}{14}$

12. Rationalise the denominator of each of the following.

a. $\dfrac{2}{\sqrt{6}}$ **b.** $\dfrac{\sqrt{3}}{2\sqrt{6}}$ **c.** $\dfrac{2}{\sqrt{5} - 2}$ **d.** $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

13. Evaluate each of the following, correct to 1 decimal place if necessary.

a. $64^{\frac{1}{3}}$ **b.** $20^{\frac{1}{2}}$ **c.** $10^{\frac{1}{3}}$ **d.** $50^{\frac{1}{4}}$

14. Evaluate each of the following, correct to 1 decimal place.

a. $20^{\frac{2}{3}}$ **b.** $2^{\frac{3}{4}}$ **c.** $(0.7)^{\frac{3}{5}}$ **d.** $\left(\dfrac{2}{3}\right)^{\frac{2}{3}}$

15. Write each of the following in simplest surd form.

a. $2^{\frac{1}{2}}$ **b.** $18^{\frac{1}{2}}$ **c.** $5^{\frac{3}{2}}$ **d.** $8^{\frac{4}{3}}$

16. Evaluate each of the following, without using a calculator. Show all working.

a. $\dfrac{16^{\frac{3}{4}} \times 81^{\frac{1}{4}}}{6 \times 16^{\frac{1}{2}}}$ **b.** $\left(125^{\frac{2}{3}} - 27^{\frac{2}{3}}\right)^{\frac{1}{2}}$

17. Evaluate each of the following, giving your answer as a fraction.

a. 4^{-1} **b.** 9^{-1} **c.** 4^{-2} **d.** 10^{-3}

18. Find the value of each of the following, correct to 3 significant figures.

a. 12^{-1} **b.** 7^{-2} **c.** $(1.25)^{-1}$ **d.** $(0.2)^{-4}$

19. Write down the value of each of the following.

a. $\left(\frac{2}{3}\right)^{-1}$ **b.** $\left(\frac{7}{10}\right)^{-1}$ **c.** $\left(\frac{1}{5}\right)^{-1}$ **d.** $\left(3\frac{1}{4}\right)^{-1}$

20. **a.** The expression $\sqrt{250}$ may be simplified to:

A. $25\sqrt{10}$ **B.** $5\sqrt{10}$ **C.** $10\sqrt{5}$ **D.** $5\sqrt{50}$

b. When expressed in its simplest form, $2\sqrt{98} - 3\sqrt{72}$ is equal to:

A. $-4\sqrt{2}$ **B.** -4 **C.** $-2\sqrt{4}$ **D.** $4\sqrt{2}$

c. When expressed in its simplest form, $\sqrt{\dfrac{8x^3}{32}}$ is equal to:

A. $\dfrac{x\sqrt{x}}{2}$ **B.** $\dfrac{\sqrt{x^3}}{4}$ **C.** $\dfrac{\sqrt{x^3}}{2}$ **D.** $\dfrac{x\sqrt{x}}{4}$

21. Find the value of the following, giving your answer in fraction form.

a. $\left(\frac{2}{5}\right)^{-1}$ **b.** $\left(\frac{2}{3}\right)^{-2}$

22. Find the value of each of the following, leaving your answer in fraction form.

a. 2^{-1} **b.** 3^{-1} **c.** 4^{-3} **d.** $\left(\frac{1}{2}\right)^{-1}$

23. Evaluate the following.

a. $\log_{12} 18 + \log_{12} 8$ **b.** $\log_4 60 - \log_4 15$ **c.** $\log_9 9^8$ **d.** $2\log_3 6 - \log_3 4$

24. Use the logarithm laws to simplify each of the following.

a. $\log_a 16 + \log_a 3 - \log_a 2$ **b.** $\log_x x\sqrt{x}$ **c.** $4\log_a x - \log_a x^2$ **d.** $5\log_x\left(\frac{1}{x}\right)$

25. Solve for x in the following, given that $x > 0$.

a. $\log_2 x = 9$ **b.** $\log_5 x = -2$ **c.** $\log_x 25 = 2$

d. $\log_x 2^6 = 6$ **e.** $\log_3 729 = x$ **f.** $\log_7 1 = x$

26. Solve for x in the following.

a. $\log_5 4 + \log_5 x = \log_5 24$ **b.** $\log_3 x - \log_3 5 = \log_3 7$

27. Solve for x in the following equations.

a. $6^x = \dfrac{1}{36}$ **b.** $7^x = \dfrac{1}{\sqrt{7}}$ **c.** $2^{x+1} = 8\sqrt{2}$

28. Solve for x in the following equations, correct to 3 decimal places.

a. $2^x = 25$ **b.** $0.6^x = 7$ **c.** $9^{-x} = 0.84$

Problem solving

29. Answer the following. Explain how you reached your answer.

a. What is the hundred's digit in 3^{3^3}?

b. What is the one's digit in 6^{704}?

c. What is the thousand's digit in 9^{1000}?

30. **a.** Plot a graph of $y = 4^x$ by first producing a table of values. Label the y-intercept and the equation of any asymptotes.

b. Draw the line $y = x$ on the same set of axes.

c. Use the property of inverse graphs to draw the graph of $y = \log_4 x$. Label any intercepts and the equation of any asymptotes.

d. Use a graphics calculator or graphing software to check your graphs.

31. Solve for x: $\left(\frac{6}{x}\right)^{-1}+\frac{1}{6}=x^{-1}$

32. Simplify $\left(\left(\frac{(a^2)^{-1}}{b\frac{1}{2}}\right)^{-1}\right)^{-1}$.

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Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

base
conjugate
contradiction
exponent
fractional power
index
indices
integer
irrational
laws of logarithms
logarithm
logarithmic equation
negative index
number base
pi
power
rational
rational denominator
real
surd
transcendental number

Investigation | Rich task

Other number systems

Throughout history, different systems have been used to aid with counting. Ancient tribes are known to have used stones, bones and knots in rope to help keep count. The counting system that is used around the world today is called the Hindu-Arabic system. This system had its origin in India around 300–200 BC. The Arabs brought this method of counting to Europe in the Middle Ages.

The Hindu–Arabic method is known as the decimal or base 10 system, as it is based on counting in lots of ten. This system uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Notice that the largest digit is one less than the base number, that is, the largest digit in base 10 is 9. To make larger numbers, digits are grouped together. The position of the digit tells us about its value. We call this *place value*. For example, in the number 325, the 3 has a value of 'three lots of a hundred', the 2 has a value of 'two lots of ten' and the 5 has a value of 'five lots of units'. Another way to write this is:

$$3 \times 100 + 2 \times 10 + 5 \times 1 \text{ or } 3 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$$

In a decimal system, every place value is based on the number 10 raised to a power. The smallest place value (units) is described by 10^0, the tens place value by 10^1, the hundreds place value by 10^2, the thousands by 10^3 and so on.

Computers do not use a decimal system. The system for computer languages is based on the number 2 and is known as the binary system. The only digits needed in the binary system are the digits 0 and 1. Can you see why?

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Binary number	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101

Consider the decimal number 7. From the table above, you can see that its binary equivalent is 111. How can you be sure this is correct?

$$111 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 4 + 2 + 1 = 7$$

Notice that this time each place value is based on the number 2 raised to a power. You can use this technique to change any binary number into a decimal number. (The same pattern applies to other bases, for example, in base 6 the place values are based on the number 6 raised to a power.)

Binary operations

When adding in the decimal system, each time the addition is greater than 9, we need to 'carry over' into the next place value. In the example below, the units column adds to more than 9, so we need to carry over into the next place value.

$$\begin{array}{r} {}^{1}17 \\ +\ 13 \\ \hline 30 \end{array}$$

The same is true when adding in binary, except we need to 'carry over' every time the addition is greater than 1.

$$\begin{array}{r} {}^{1}01 \\ +\ 01 \\ \hline 10 \end{array}$$

1. Perform the following binary additions.

a. $\begin{array}{r} 11_2 \\ +\ 01_2 \\ \hline \\ \hline \end{array}$ **b.** $\begin{array}{r} 111_2 \\ +\ 110_2 \\ \hline \\ \hline \end{array}$ **c.** $\begin{array}{r} 1011_2 \\ +\ 101_2 \\ \hline \\ \hline \end{array}$

2. Perform the following binary subtractions. Remember that if you need to borrow a number from a column on the left-hand side, you will actually be borrowing a 2 (not a 10).

a. $\begin{array}{r} 11_2 \\ -\ 01_2 \\ \hline \\ \hline \end{array}$ **b.** $\begin{array}{r} 111_2 \\ -\ 110_2 \\ \hline \\ \hline \end{array}$ **c.** $\begin{array}{r} 1011_2 \\ -\ 101_2 \\ \hline \\ \hline \end{array}$

3. Try some multiplication. Remllember to carry over lots of 2.

a. $\begin{array}{r} 11_2 \\ \times\ 01_2 \\ \hline \\ \hline \end{array}$ **b.** $\begin{array}{r} 111_2 \\ \times\ 110_2 \\ \hline \\ \hline \end{array}$ **c.** $\begin{array}{r} 1011_2 \\ \times\ 101_2 \\ \hline \\ \hline \end{array}$

4. What if our number system had an 8 as its basis (that is, we counted in lots of 8)? The only digits available for use would be 0, 1, 2, 3, 4, 5, 6 and 7. (Remember the maximum digit is 1 less than the base value.) Give examples to show how numbers would be added, subtracted and multiplied using this base system. Remember that you would 'carry over' or 'borrow' lots of 8.

5. The hexadecimal system has 16 as its basis. Investigate this system. Explain how it would be possible to have 15, for example, in a single place position. Give examples to show how the system would add, subtract and multiply.

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Complete this digital doc: Code puzzle: Who is Jørn Utzon? (doc-15947)

Answers

Topic 16 Real numbers

Exercise 16.2 Number classification review

1. a. Q **b.** Q **c.** Q **d.** I **e.** I
f. Q **g.** Q **h.** I **i.** Q **j.** Q
k. Q **l.** Q **m.** I **n.** Q **o.** I
p. Q **q.** Q **r.** I **s.** I **t.** I
u. Q **v.** I **w.** I **x.** Q **y.** I

2. a. Q **b.** Q **c.** Q **d.** Q **e.** Q
f. I **g.** I **h.** Q **i.** I **j.** Undefined
k. I **l.** I **m.** I **n.** Q **o.** Q
p. Q **q.** I **r.** I **s.** Q **t.** Q
u. I **v.** Q **w.** Q **x.** I **y.** Q

3. B

4. D

5. C

6. C

7. $\frac{a}{b}$

8. D

9. A

10. $p - q$

11. Check with your teacher.

12. a. $m = 11, n = 3$ **b.** $m = 2, n = 3$ **c.** $m = 3, n = 2$ **d.** $m = 1, n = 2$

13. $\frac{1}{7}$ or 7^{-1}

Exercise 16.3 Surds

1. b d f g h i l m o q s t w z

2. A

3. D

4. B

5. C

6. Any perfect square

7. $m = 4$

8. Check with your teacher.

9. Irrational

10. a. i. $4\sqrt{3}$ **ii.** $6\sqrt{2}$

b. Yes. If you don't choose the largest perfect square, then you will need to simplify again.

c. No

11. Integral and rational

Exercise 16.4 Operations with surds

1. a. $2\sqrt{3}$ **b.** $2\sqrt{6}$ **c.** $3\sqrt{3}$ **d.** $5\sqrt{5}$ **e.** $3\sqrt{6}$ **f.** $4\sqrt{7}$
g. $2\sqrt{17}$ **h.** $6\sqrt{5}$ **i.** $2\sqrt{22}$ **j.** $9\sqrt{2}$ **k.** $7\sqrt{5}$ **l.** $8\sqrt{7}$

2. a. $4\sqrt{2}$ **b.** $24\sqrt{10}$ **c.** $36\sqrt{5}$ **d.** $21\sqrt{6}$ **e.** $-30\sqrt{3}$ **f.** $-28\sqrt{5}$
g. $64\sqrt{3}$ **h.** $2\sqrt{2}$ **i.** $\sqrt{2}$ **j.** $2\sqrt{3}$ **k.** $\frac{1}{3}\sqrt{15}$ **l.** $\frac{3}{2}\sqrt{7}$

3. a. $4a$ **b.** $6a\sqrt{2}$ **c.** $3a\sqrt{10b}$ **d.** $13a^2\sqrt{2}$ **e.** $13ab\sqrt{2ab}$ **f.** $2ab^2\sqrt{17ab}$
g. $5x^3y^2\sqrt{5}$ **h.** $20xy\sqrt{5x}$ **i.** $54c^3d^2\sqrt{2cd}$ **j.** $18c^3d^4\sqrt{5cd}$ **k.** $\sqrt{22ef}$ **l.** $7e^5f^5\sqrt{2ef}$

4. a. $7\sqrt{5}$ **b.** $8\sqrt{3}$ **c.** $15\sqrt{5} + 5\sqrt{3}$ **d.** $4\sqrt{11}$
e. $13\sqrt{2}$ **f.** $-3\sqrt{6}$ **g.** $17\sqrt{3} - 18\sqrt{7}$ **h.** $8\sqrt{x} + 3\sqrt{y}$

5. a. $10(\sqrt{2} - \sqrt{3})$ **b.** $5(\sqrt{5} + \sqrt{6})$ **c.** $7\sqrt{3}$ **d.** $4\sqrt{5}$
e. $14\sqrt{3} + 3\sqrt{2}$ **f.** $3\sqrt{6} + 6\sqrt{3}$ **g.** $15\sqrt{10} - 10\sqrt{15} + 10$ **h.** $-8\sqrt{11} + 22$
i. $12\sqrt{30} - 16\sqrt{15}$ **j.** $12\sqrt{ab} + 7\sqrt{3ab}$ **k.** $\frac{7}{2}\sqrt{2} + 2\sqrt{3}$ **l.** $15\sqrt{2}$

6. a. $31\sqrt{a} - 6\sqrt{2a}$ **b.** $52\sqrt{a} - 29\sqrt{3a}$ **c.** $6\sqrt{6ab}$ **d.** $32a + 2\sqrt{6a} + 8a\sqrt{2}$
e. $a\sqrt{2a}$ **f.** $\sqrt{a} + 2\sqrt{2a}$ **g.** $3a\sqrt{a} + a^2\sqrt{3a}$ **h.** $(a^2 + a)\sqrt{ab}$
i. $4ab\sqrt{ab} + 3a^2b\sqrt{b}$ **j.** $3\sqrt{ab}(2a + 1)$ **k.** $-6ab\sqrt{2a} + 4a^2b^3\sqrt{3a}$ **l.** $-2a\sqrt{b}$

7. a. $\sqrt{14}$ **b.** $\sqrt{42}$ **c.** $4\sqrt{3}$ **d.** 10 **e.** $3\sqrt{7}$ **f.** 27
g. $10\sqrt{33}$ **h.** $180\sqrt{5}$ **i.** 120 **j.** $120\sqrt{3}$ **k.** $2\sqrt{6}$ **l.** $2\frac{2}{3}$
m. $\frac{2}{5}\sqrt{6}$ **n.** $x^2y\sqrt{y}$ **o.** $3a^4b^2\sqrt{2ab}$ **p.** $6a^5b^2\sqrt{2b}$ **q.** $3x^2y^2\sqrt{10xy}$ **r.** $\frac{9}{2}a^2b^4\sqrt{5ab}$

8. a. 2 **b.** 5 **c.** 12 **d.** 15
e. 18 **f.** 80 **g.** 28 **h.** 200

9. a. $\sqrt{5}$ **b.** 2 **c.** $\sqrt{6}$ **d.** 4 **e.** $\frac{\sqrt{3}}{4}$
f. $\frac{\sqrt{5}}{2}$ **g.** $2\sqrt{3}$ **h.** 1 **i.** $1\frac{4}{5}$ **j.** $2\sqrt{17}$
k. $\frac{x}{y}$ **l.** $\frac{\sqrt{2}}{x^3y^4}$ **m.** $2xy\sqrt{3y}$ **n.** $\frac{4\sqrt{a}}{3}$

10. a. $\frac{5\sqrt{2}}{2}$ **b.** $\frac{7\sqrt{3}}{3}$ **c.** $\frac{4\sqrt{11}}{11}$ **d.** $\frac{4\sqrt{6}}{3}$ **e.** $\frac{2\sqrt{21}}{7}$
f. $\frac{\sqrt{10}}{2}$ **g.** $\frac{2\sqrt{15}}{5}$ **h.** $\frac{3\sqrt{35}}{5}$ **i.** $\frac{5\sqrt{6}}{6}$ **j.** $\frac{4\sqrt{15}}{15}$
k. $\frac{5\sqrt{7}}{14}$ **l.** $\frac{8\sqrt{15}}{15}$ **m.** $\frac{8\sqrt{21}}{49}$ **n.** $\frac{8\sqrt{105}}{7}$ **o.** $\frac{\sqrt{10}}{3}$

11. a. $\sqrt{2} + 2$ **b.** $\frac{3\sqrt{10} - 2\sqrt{33}}{6}$ **c.** $\frac{12\sqrt{5} - 5\sqrt{6}}{10}$ **d.** $\frac{9\sqrt{10}}{5}$ **e.** $\frac{3\sqrt{10} + 6\sqrt{14}}{4}$ **f.** $\frac{5\sqrt{6}}{3}$
g. $\frac{3\sqrt{22} - 4\sqrt{10}}{6}$ **h.** $\frac{\sqrt{21} - \sqrt{15}}{3}$ **i.** $\frac{14 - 5\sqrt{2}}{6}$ **j.** $\frac{12 - \sqrt{10}}{16}$ **k.** $\frac{6\sqrt{15} - 25}{70}$ **l.** $\frac{\sqrt{30} + 7\sqrt{2}}{20}$

12. a. $\sqrt{5} - 2$ **b.** $\frac{2\sqrt{2} + \sqrt{5}}{3}$ **c.** $\frac{8\sqrt{11} + 4\sqrt{13}}{31}$
d. $\frac{15\sqrt{15} - 20\sqrt{6}}{13}$ **e.** $12\sqrt{2} - 17$ **f.** $\frac{19 - 4\sqrt{21}}{5}$
g. $\frac{\sqrt{15} - \sqrt{3} - \sqrt{5} + 1}{4}$ **h.** $\frac{-6 + 6\sqrt{2} + \sqrt{10} - 2\sqrt{5}}{2}$ **i.** $\frac{4\sqrt{10} + \sqrt{15} - 4\sqrt{6} - 3}{29}$

13. $\frac{9\sqrt{x} + 6x}{36x - 16x^2}$

14. a. Check with your teacher.
b. i. $\sqrt{5} + \sqrt{3}$ **ii.** $\sqrt{5} - \sqrt{3}$ **iii.** $\sqrt{3} + 2$

15. $\frac{2}{7}$

16. a. $x = 16$ **b.** $x = 1$

Exercise 16.5 Fractional indices

1. a. 4 **b.** 5 **c.** 9 **d.** 2 **e.** 3 **f.** 5
2. a. 3 **b.** 2 **c.** 1.4 **d.** 2.2 **e.** 1.5 **f.** 1.3
3. a. 2.5 **b.** 12.9 **c.** 13.6 **d.** 0.7 **e.** 0.8 **f.** 0.9
4. a. $\sqrt{7}$ **b.** $2\sqrt{3}$ **c.** $6\sqrt{2}$ **d.** $4\sqrt{2}$ **e.** $3\sqrt{3}$ **f.** $100\sqrt{10}$
5. a. $5^{\frac{1}{2}}$ **b.** $10^{\frac{1}{2}}$ **c.** $x^{\frac{1}{2}}$ **d.** $m^{\frac{3}{2}}$ **e.** $2t^{\frac{1}{2}}$ **f.** $6^{\frac{1}{3}}$

6. a. $4^{\frac{4}{5}}$ **b.** $2^{\frac{1}{2}}$ **c.** $a^{\frac{5}{6}}$ **d.** $x^{\frac{23}{20}}$ **e.** $10m^{\frac{8}{15}}$
f. $2b^{\frac{5}{7}}$ **g.** $-4y^{\frac{20}{9}}$ **h.** $0.02a^{\frac{9}{8}}$ **i.** $5x^{\frac{7}{2}}$

7. a. $ab^{\frac{3}{2}}$ **b.** $x^{\frac{4}{5}}y^{\frac{5}{9}}$ **c.** $6a^{\frac{8}{5}}b^{\frac{17}{15}}$ **d.** $2m^{\frac{19}{28}}n^{\frac{2}{5}}$ **e.** $x^{\frac{19}{6}}y^{\frac{5}{6}}z^{\frac{5}{6}}$ **f.** $8a^{\frac{2}{5}}b^{\frac{9}{8}}c$

8. a. $3^{\frac{1}{6}}$ **b.** $5^{\frac{5}{12}}$ **c.** $12^{\frac{1}{2}}$ **d.** $a^{\frac{3}{7}}$ **e.** $x^{\frac{5}{4}}$
f. $m^{\frac{11}{45}}$ **g.** $\frac{1}{2}x^{\frac{3}{20}}$ **h.** $\frac{1}{3}n^{\frac{2}{3}}$ **i.** $\frac{5}{4}b^{\frac{7}{20}}$

9. a. $x^{\frac{5}{3}}y^{\frac{7}{5}}$ **b.** $a^{\frac{7}{45}}b^{\frac{4}{15}}$ **c.** $\frac{1}{3}m^{\frac{3}{8}}n^{\frac{11}{56}}$ **d.** $2x^{\frac{2}{15}}y^{\frac{3}{4}}$ **e.** $\frac{1}{4}a^{\frac{11}{20}}b^{\frac{7}{20}}$ **f.** $\frac{1}{7}p^{\frac{5}{24}}q^{\frac{1}{12}}$

10. a. $2^{\frac{9}{20}}$ **b.** $5^{\frac{1}{6}}$ **c.** $7^{\frac{6}{5}}$ **d.** $a^{\frac{3}{10}}$ **e.** $m^{\frac{1}{6}}$
f. $2^{\frac{1}{3}}b^{\frac{1}{6}}$ **g.** $4p^{\frac{2}{5}}$ **h.** $x^{\frac{m}{p}}$ **i.** $3^{\frac{b}{c}}m^{\frac{a}{c}}$

11. a. $a^{\frac{1}{4}}b^{\frac{1}{6}}$ **b.** $a^3b^{\frac{3}{4}}$ **c.** $x^{\frac{6}{5}}y^{\frac{7}{4}}$ **d.** $3^{\frac{1}{3}}a^{\frac{1}{9}}b^{\frac{1}{5}}c^{\frac{1}{4}}$ **e.** $x^{\frac{1}{4}}y^{\frac{1}{3}}z^{\frac{1}{5}}$ **f.** $\dfrac{a^{\frac{1}{2}}}{b^{\frac{2}{3}}}$
g. $\dfrac{m^{\frac{8}{5}}}{n^{\frac{7}{4}}}$ **h.** $\dfrac{b^{\frac{2}{5}}}{c^{\frac{8}{27}}}$ **i.** $\dfrac{2^{\frac{1}{2}}x^{\frac{7}{2}}}{y^{\frac{3}{8}}}$

12. C, D

13. a. a^4 **b.** b^3 **c.** m^4 **d.** $4x^2$ **e.** $2y^3$
f. $2x^2y^3$ **g.** $3m^3n^5$ **h.** $2pq^2$ **i.** $6a^2b^6$

14. a. 0.32 m/s **b.** 16 640 L/s
c. 59 904 000 L/hr
That is 16 640 × 60 × 60.
d. The hydraulic radius is the measure of a channel flow efficiency. The roughness coefficient is the resistance of the bed of a channel to the flow of water in it.

15. $x = 1$

16. a. $x^{\frac{1}{2}} + y^{\frac{1}{2}} - z^{\frac{1}{2}}$ **b.** $t^{\frac{1}{10}}$

17. $m - n^2$

Exercise 16.6 Negative indices

1. a. $\frac{1}{5} = 0.2$ **b.** $\frac{1}{3} = 0.\dot{3}$ **c.** $\frac{1}{8} = 0.125$ **d.** $\frac{1}{10} = 0.1$
e. $\frac{1}{8} = 0.125$ **f.** $\frac{1}{9} = 0.\dot{1}$ **g.** $\frac{1}{25} = 0.04$ **h.** $\frac{1}{10\,000} = 0.0001$

2. a. 0.167 **b.** 0.143 **c.** 0.0278 **d.** 0.001 37
e. 0.004 63 **f.** 0.004 44 **g.** 0.003 91 **h.** 0.001 60

3. a. 0.40 **b.** 2.5 **c.** 0.44 **d.** 4.0
e. 0.11 **f.** 0.000 079 **g.** 11 **h.** 4100

4. a. −0.33 **b.** −0.20 **c.** 0.25 **d.** 0.063
e. −0.67 **f.** −0.45 **g.** −1.7 **h.** 1.4

5. a. $\frac{5}{4}$ or $1\frac{1}{4}$ **b.** $\frac{10}{3}$ or $3\frac{1}{3}$ **c.** $\frac{8}{7}$ or $1\frac{1}{7}$ **d.** $\frac{20}{13}$ or $1\frac{7}{13}$ **e.** 2 **f.** 4
g. 8 **h.** 10 **i.** $\frac{2}{3}$ **j.** $\frac{4}{9}$ **k.** $\frac{10}{11}$ **l.** $\frac{2}{11}$

6. a. 4 **b.** $6\frac{1}{4}$ **c.** $3\frac{3}{8}$ **d.** 16 **e.** $\frac{4}{9}$ **f.** $\frac{16}{81}$
g. $\frac{27}{64}$ **h.** $\frac{125}{1331}$

7. a. $-\frac{3}{2}$ **b.** $-\frac{5}{3}$ **c.** −4 **d.** −10
e. $\frac{9}{4}$ **f.** 25 **g.** $-\frac{2}{3}$ **h.** $\frac{16}{121}$

8. $\frac{3}{10}$

9. $\frac{b}{a}$

10. a. $y \to \infty$ **b.** $y \to -\infty$

11. As the value of n increases, the value of 2^{-n} gets closer to 0.

12. $x = -2$, $y = -3$

13. x^2

Exercise 16.7 Logarithms

1. a. $\log_4 16 = 2$ **b.** $\log_2 32 = 5$ **c.** $\log_3 81 = 4$ **d.** $\log_6 36 = 2$

e. $\log_{10} 1000 = 3$ **f.** $\log_5 25 = 2$ **g.** $\log_4 x = 3$ **h.** $\log_5 125 = x$

i. $\log_7 49 = x$ **j.** $\log_p 16 = 4$ **k.** $\log_9 3 = \frac{1}{2}$ **l.** $\log_{10} 0.1 = -1$

m. $\log_8 2 = \frac{1}{3}$ **n.** $\log_2 \frac{1}{2} = -1$ **o.** $\log_a 1 = 0$ **p.** $\log_4 8 = \frac{3}{2}$

2. D

3. a. $2^4 = 16$ **b.** $3^3 = 27$ **c.** $10^6 = 1\,000\,000$ **d.** $5^3 = 125$ **e.** $16^{\frac{1}{2}} = 4$ **f.** $4^x = 64$

g. $49^{\frac{1}{2}} = 7$ **h.** $3^5 = x$ **i.** $81^{\frac{1}{2}} = 9$ **j.** $10^{-2} = 0.01$ **k.** $8^1 = 8$ **l.** $64^{\frac{1}{3}} = 4$

4. B

5. a. 4 **b.** 2 **c.** 2 **d.** 5 **e.** 5 **f.** 7

g. 0 **h.** $\frac{1}{2}$ **i.** -1 **j.** 1 **k.** -2 **l.** $\frac{1}{3}$

6. a. 0 **b.** 1 **c.** 2 **d.** 3 **e.** 4 **f.** 5

7. a. 0 and 1 **b.** 3 and 4 **c.** 1 and 2 **d.** 4 and 5 **e.** 2 and 3 **f.** 4 and 5

8. a. 6.1 **b.** 6.3 **c.** 8.2

9. a. $\log_{10} g = k$ implies that $g = 10^k$ so $g^2 = (10^k)^2$. That is, $g^2 = 10^{2k}$; therefore, $\log_{10} g^2 = 2k$.

b. $\log_x y = 2$ implies that $y = x^2$, so $x = y^{\frac{1}{2}}$ and therefore $\log_y x = \frac{1}{2}$.

c. The equivalent exponential statement is $x = 4^y$, and we know that 4^y is greater than zero for all values of y. Therefore, x is a positive number.

10. a. 6 **b.** -4 **c.** -5

11. a. 3 **b.** 7 **c.** $\frac{1}{8}$

12. x

Exercise 16.8 Logarithm laws

1. a. 1.698 97 **b.** 1.397 94 **c.** 0.698 97 **d.** 0.301 03

2. Teacher to check.

3. a. 1 **b.** 3 **c.** 2 **d.** 3 **e.** 4 **f.** 1

4. a. 2 **b.** 3 **c.** 1 **d.** 4 **e.** 3 **f.** 5

5. a. 2 **b.** $\frac{1}{2}$ **c.** 1 **d.** 3

6. 3

7. a. 2 **b.** 4 **c.** 3 **d.** 3

8. a. 1 **b.** 0 **c.** -1 **d.** 5 **e.** -2 **f.** 1

g. 0 **h.** -2 **i.** $-\frac{1}{2}$ **j.** $\frac{1}{2}$ **k.** $-\frac{1}{2}$ **l.** $\frac{7}{2}$

9. a. $\log_a 40$ **b.** $\log_a 18$ **c.** $\log_x 48$ **d.** $\log_x 4$ **e.** $\log_a x$ **f.** 1

g. -1 **h.** 7 **i.** $\frac{1}{2}$ **j.** $\frac{3}{2}$ **k.** -6 **l.** $-\frac{1}{3}$

10. a. B **b.** B, D **c.** A, B **d.** C, D

11. a. $\log_2 80$ **b.** $\log_3 105$ **c.** $\log_{10} 100 = 2$ **d.** $\log_6 56$ **e.** $\log_2 4 = 2$

f. $\log_3 3 = 1$ **g.** $\log_5 12.5$ **h.** $\log_2 3$ **i.** $\log_4 5$ **j.** $\log_{10} \frac{1}{4}$

k. $\log_3 4$ **l.** $\log_2 3$ **m.** $\log_3 20$ **n.** $\log_4 2 = \frac{1}{2}$

12. a. C **b.** B **c.** A

13. a. 12 (Evaluate each logarithm separately and then find the product.)

b. 4 (First simplify the numerator by expressing 81 as a power of 3.)

c. 7 (Let $y = 5^{\log_5 7}$ and write an equivalent statement in logarithmic form.)

14. $7 - 3\log_2(3)$ **15.** 1 **16.** $x = 3a, 5a$

Challenge 16.1

$\frac{3}{2}$

Exercise 16.9 Solving equations

1. a. 25 **b.** 81 **c.** $\frac{1}{8}$ **d.** $\frac{1}{16}$ **e.** 100, −100
f. 16 **g.** 26 **h.** 127 **i.** 2 **j.** 0
k. $-\frac{1}{32}$ **l.** $-\frac{1}{9}$ **m.** −624 **n.** −2.5
2. a. 3 **b.** 2 **c.** 125 **d.** 625
e. 2 **f.** 8 **g.** 6 **h.** 4
3. a. 3 **b.** 2 **c.** −1 **d.** −2 **e.** $\frac{1}{2}$
f. $\frac{1}{3}$ **g.** 0 **h.** 0 **i.** −1 **j.** −2
4. a. 5 **b.** 6 **c.** 10 **d.** 8 **e.** 4
f. 2 **g.** 9 **h.** $\frac{2}{5}$ **i.** 500 **j.** 128
k. 5 **l.** 6 **m.** 1 **n.** 2
5. a. B **b.** A **c.** D **d.** B
6. a. 7 **b.** 2 **c.** −2 **d.** 0 **e.** 4
f. $\frac{1}{2}$ **g.** $\frac{1}{2}$ **h.** $\frac{3}{2}$ **i.** $-\frac{1}{2}$ **j.** $\frac{3}{2}$
k. $\frac{3}{4}$ **l.** $-\frac{5}{2}$ **m.** $\frac{5}{2}$ **n.** $-\frac{9}{2}$ **o.** $-\frac{11}{4}$
7. a. 3.459 **b.** −0.737 **c.** 2.727 **d.** 0.483 **e.** 1.292 **f.** −3.080
g. −1.756 **h.** 0.262 **i.** 0.827 **j.** 0.579 **k.** −0.423 **l.** 2.138
8. a. 120 **b.** 130 **c.** 0.001
d. 3 dB are added.
e. 10 dB are added.
f. 100
9. a. i. 1.1 **ii.** 1.3 **iii.** 1.418 **iv.** 1.77 **v.** 2.43 **vi.** 3.1
b. No; see answers to **9a i** and **ii** above.
c. i. 22 387 211 kJ **ii.** 707 945 784 kJ **iii.** 22 387 211 385 kJ.
d. The energy is increased by a factor of 31.62.
e. It releases 31.62^3 times more energy.
10. a. $x = 0.7712$ **b.** $x = 1.2966$
11. $x = 7$ **12.** $x = 1, 3$

Challenge 16.2

The remaining steps of the solution are $\frac{x(x+1)}{6}$, $x^2 + x - 6 = 0$, $x = -3$ or 2.

16.10 Review

1. a
2. a. Irrational, since equal to non-recurring and non-terminating decimal
b. Rational, since can be expressed as a whole number
c. Rational, since given in a rational form
d. Rational, since it is a recurring decimal
e. Irrational, since equal to non-recurring and non-terminating decimal
3. d
4. a. $\sqrt{2m}$, $\sqrt{\frac{20}{m}}$, $\sqrt[3]{m}$, $\sqrt[3]{8m}$ **b.** $\sqrt{25m}$, $\sqrt{\frac{m}{16}}$, $\sqrt{\frac{20}{m}}$
5. a. $5\sqrt{2}$ **b.** $6\sqrt{5}$ **c.** $8\sqrt{2}$ **d.** $20\sqrt{5}$
6. c
7. a. $72x^3y^4\sqrt{2xy}$ **b.** $-\frac{1}{4}x^2y^5\sqrt{xy}$
8. a. $25\sqrt{3}$ **b.** $3ab\sqrt{ab}$
9. a. $\sqrt{15}$ **b.** $6\sqrt{42}$ **c.** $30\sqrt{15}$ **d.** 5

10. a. 27 **b.** $720\sqrt{2}$

11. a. $\sqrt{3}$ **b.** 6 **c.** $\frac{\sqrt{10}}{4\sqrt{3}}$ or $\frac{\sqrt{30}}{12}$ **d.** $\frac{1}{2}$

12. a. $\frac{\sqrt{6}}{3}$ **b.** $\frac{\sqrt{2}}{4}$ **c.** $2\sqrt{5}+4$ **d.** $2-\sqrt{3}$

13. a. 4 **b.** 4.5 **c.** 2.2 **d.** 2.7

14. a. 7.4 **b.** 1.7 **c.** 0.8 **d.** 0.8

15. a. $\sqrt{2}$ **b.** $3\sqrt{2}$ **c.** $5\sqrt{5}$ **d.** 16

16. a. 1 **b.** 4

17. a. $\frac{1}{4}$ **b.** $\frac{1}{9}$ **c.** $\frac{1}{16}$ **d.** $\frac{1}{1000}$

18. a. 0.0833 **b.** 0.0204 **c.** 0.800 **d.** 625

19. a. $1\frac{1}{2}$ **b.** $1\frac{3}{7}$ **c.** 5 **d.** $\frac{4}{13}$

20. a. B **b.** A **c.** A

21. a. $2\frac{1}{2}$ **b.** $2\frac{1}{4}$

22. a. $\frac{1}{2}$ **b.** $\frac{1}{9}$ **c.** $\frac{1}{64}$ **d.** $\frac{2}{1}$

23. a. 2 **b.** 1 **c.** 8 **d.** 2

24. a. $\log_a 24$ **b.** $\frac{3}{2}$ **c.** $\log_a x^2$ or $2\log_a x$ **d.** -5

25. a. 512 **b.** $\frac{1}{25}$ **c.** 5 **d.** 2 **e.** 6 **f.** 0

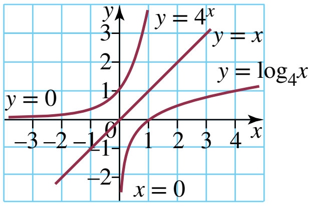
26. a. 6 **b.** 35

27. a. -2 **b.** $-\frac{1}{2}$ **c.** $\frac{5}{2}$

28. a. 4.644 **b.** -3.809 **c.** 0.079

29. a. 9 **b.** 6 **c.** 0

30. a, b, c



31. $x = 2, -3$

32. $\frac{1}{a^2 b^{\frac{1}{2}}}$

Investigation — Rich task

1. a. 100_2 **b.** 1101_2 **c.** 10000_2

2. a. 10_2 **b.** 101_2 **c.** 110_2

3. a. 11_2 **b.** 1001_2 **c.** 10101_2

4. Answers will vary; teacher to check.

5. Answers will vary; teacher to check. The numbers 10, 11, 12, 13, 14 and 15 are allocated the letters A, B,C, D, E and F respectively.

TOPIC 17
Polynomials

17.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

17.1.1 Why learn this?

Just as number is learned in stages, so too are graphs. You have been building your knowledge of graphs and functions over time. First you encountered linear functions, then quadratic and hyperbolic functions. Polynomials are higher-order functions represented by smooth and continuous curves. They can be used to model situations in many fields, such as business, science, architecture, design and engineering.

17.1.2 What do you know?

assesson

1. **THINK** List what you know about polynomials. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of polynomials.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Mary Somerville (eles-2020)

17.2 Polynomials

17.2.1 Polynomials

- A **polynomial** *in* x, sometimes denoted $P(x)$, is an expression containing only non-negative integer powers of x.
- The **degree** of a polynomial in x is the highest power of x in the expression. For example:

$3x + 1$	is a polynomial of degree 1, or linear polynomial.
$x^2 + 4x - 7$	is a polynomial of degree 2, or quadratic polynomial.
$-5x^3 + \frac{x}{2}$	is a polynomial of degree 3, or cubic polynomial.
10	is a polynomial of degree 0 (think of 10 as $10x^0$).

- Expressions containing a term similar to any of the following terms are not polynomials:

$$\frac{1}{x}, \quad x^{-2}, \quad \sqrt{x}, \quad 2^x, \quad \sin x, \quad \text{etc.}$$

For example, the following are not polynomials.

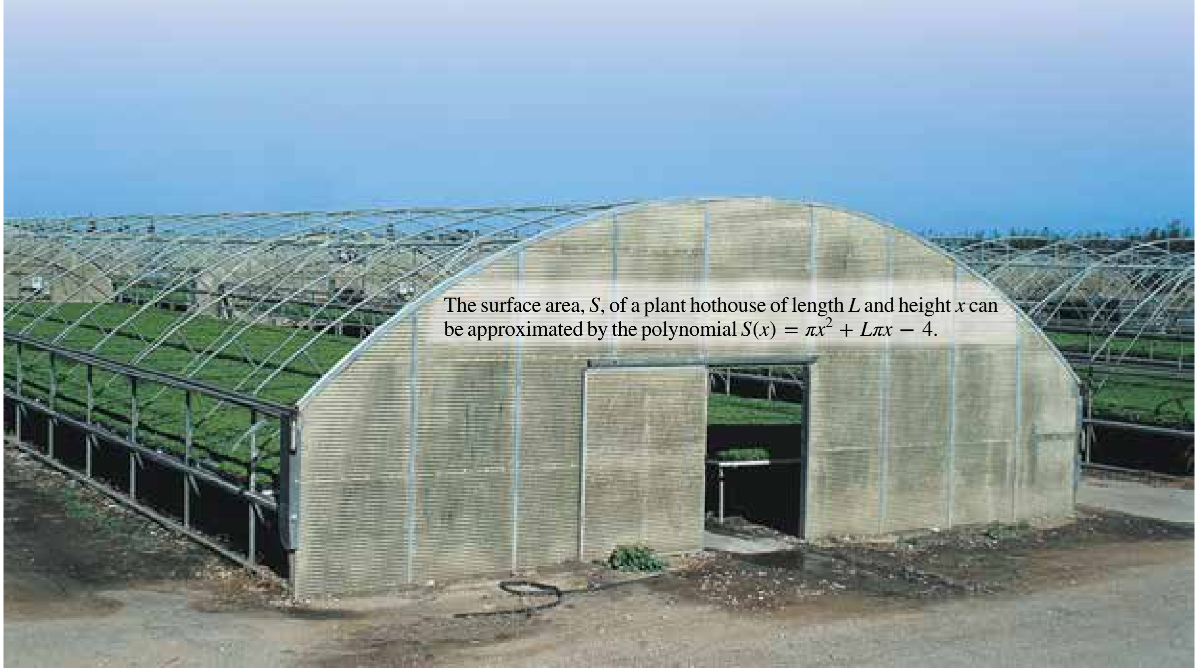
$$3x^2 - 4x + \frac{2}{x} \qquad -5x^4 + x^3 - 2\sqrt{x} \qquad x^2 + \sin x + 1$$

- In the expression $6x^3 + 13x^2 - x + 1$

x is the *variable*.
6 is the *coefficient* of x^3.
13 is the *coefficient* of x^2.
-1 is the *coefficient* of x.
$6x^3$, $13x^2$, $-x$ and $+1$ are all *terms*.
The *constant* term is $+1$.
The *degree* of the polynomial is 3.

- The leading term is $6x^3$ because it is the term that contains the highest power of x.
- The leading coefficient is 6.
- Any polynomial with a leading coefficient of 1 is called *monic*.

An example of where polynomials are useful is shown below.



The surface area, S, of a plant hothouse of length L and height x can be approximated by the polynomial $S(x) = \pi x^2 + L\pi x - 4$.

Exercise 17.2 Polynomials

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a, b, f, 2–5, 7, 8, 11	Questions: 1c–e, g, i, 2–4, 6, 8, 10, 11	Questions: 1a, c, f–i, 2–4, 6–12

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. State the degree of each of the following polynomials.

a. $x^3 - 9x^2 + 19x + 7$
b. $65 + 2x^7$
c. $3x^2 - 8 + 2x$
d. $x^6 - 3x^5 + 2x^4 + 6x + 1$
e. $y^8 + 7y^3 - 5$
f. $\frac{1}{2}u^5 - \frac{u^4}{3} + 2u - 6$
g. $18 - \frac{e^5}{6}$
h. $2g - 3$
i. $1.5f^6 - 800f$

2. State the variable for each polynomial in question **1**.

3. Which polynomials in question **1** are:

a. linear
b. quadratic
c. cubic
d. monic?

4. State whether each of the following is a polynomial (P) or not (N).

a. $7x + 6x^2 + \frac{5}{x}$
b. $33 - 4p$
c. $\frac{x^2}{9} + x$
d. $3x^4 - 2x^3 - 3\sqrt{x} - 4$
e. $k^{-2} + k - 3k^3 + 7$
f. $5r - r^9 + \frac{1}{3}$
g. $\frac{4c^6 - 3c^3 + 1}{2}$
h. $2^x - 8x + 1$
i. $\sin x + x^2$

5. Consider the polynomial $P(x) = -2x^3 + 4x^2 + 3x + 5$.

a. What is the degree of the polynomial?
b. What is the variable?
c. What is the coefficient of x^2?
d. What is the value of the constant term?
e. Which term has a coefficient of 3?
f. Which is the leading term?

6. Consider the polynomial $P(w) = 6w^7 + 7w^6 - 9$.

a. What is the degree of the polynomial?
b. What is the variable?
c. What is the coefficient of w^6?
d. What is the coefficient of w?
e. What is the value of the constant term?
f. Which term has a coefficient of 6?

7. Consider the polynomial $f(x) = 4 - x^2 + x^4$.

a. What is the degree of the polynomial?
b. What is the coefficient of x^4?
c. What is the leading term?
d. What is the leading coefficient?

Understanding

8. A sports scientist determines the following equation for the velocity of a breaststroke swimmer during one complete stroke:

$$v(t) = 63.876t^6 - 247.65t^5 + 360.39t^4 - 219.41t^3 + 53.816t^2 + 0.4746t.$$

- **a.** What is the degree of the polynomial?
- **b.** What is the variable?
- **c.** How many terms are there?
- **d.** Use a graphics calculator or graphing software to draw the graph of this polynomial.
- **e.** Match what happens during one complete stroke with points on the graph.

Reasoning

9. The distance travelled by a body after t seconds is given by $d(t) = t^3 + 2t^2 - 4t + 5$. Using a graphing calculator or suitable computer software, draw a graph of the above motion for $0 \leq t \leq 3$. Use the graph to help you answer the following:

- **a.** What information does the constant term give?
- **b.** What is the position of the body after 1 second?
- **c.** Describe in words the motion in the first 2 seconds.

10. Write the following polynomials as simply as possible, arranging terms in descending powers of x.

- **a.** $7x + 2x^2 - 8x + 15 + 4x^3 - 9x + 3$
- **b.** $x^2 - 8x^3 + 3x^4 - 2x^2 + 7x + 5x^3 - 7$
- **c.** $x^3 - 5x^2 - 11x - 1 + 4x^3 - 2x + x^2 - 5$

Problem solving

11. If $x^2 + 2x - 1 \equiv (x - 1)^2 + a(x + 1) + b$, find the values of a and b.

12. If $x^3 + 9x^2 + 12x + 7 \equiv x^3 + (ax + b)^2 + 3$, find the values of a and b.

Reflection

How can you tell what the degree of a polynomial is?

17.3 Adding, subtracting and multiplying polynomials

17.3.1 Operations with polynomials

- To add or subtract polynomials, we simply add or subtract any like terms in the expressions.

WORKED EXAMPLE 1

TI | CASIO

Simplify each of the following.

a $(5x^3 + 3x^2 - 2x - 1) + (x^4 + 5x^2 - 4)$

b $(5x^3 + 3x^2 - 2x - 1) - (x^4 + 5x^2 - 4)$

THINK	WRITE
a 1 Write the expression.	**a** $(5x^3 + 3x^2 - 2x - 1) + (x^4 + 5x^2 - 4)$
2 Remove any grouping symbols, watching any signs.	$= 5x^3 + 3x^2 - 2x - 1 + x^4 + 5x^2 - 4$
3 Identify any like terms and change the order.	$= x^4 + 5x^3 + 3x^2 + 5x^2 - 2x - 1 - 4$
4 Simplify by collecting like terms.	$= x^4 + 5x^3 + 8x^2 - 2x - 5$
b 1 Write the expression.	**b** $(5x^3 + 3x^2 - 2x - 1) - (x^4 + 5x^2 - 4)$
2 Remove any grouping symbols, watching any signs.	$= 5x^3 + 3x^2 - 2x - 1 - x^4 - 5x^2 + 4$
3 Identify any like terms and change the order.	$= -x^4 + 5x^3 + 3x^2 - 5x^2 - 2x - 1 + 4$
4 Simplify by collecting like terms.	$= -x^4 + 5x^3 - 2x^2 - 2x + 3$

- If we expand linear factors, for example $(x+1)(x+2)(x-7)$, we may also get a polynomial as the following worked example shows.

WORKED EXAMPLE 2

TI | CASIO

Expand and simplify:

a $\mathbf{x(x+2)(x-3)}$ **b** $\mathbf{(x-1)(x+5)(x+2)}$.

THINK	WRITE
a 1 Write the expression.	**a** $x(x+2)(x-3)$
2 Expand the last two linear factors.	$=x(x^2-3x+2x-6)$ $=x(x^2-x-6)$
3 Multiply the expression in the grouping symbols by x.	$=x^3-x^2-6x$
b 1 Write the expression.	**b** $(x-1)(x+5)(x+2)$
2 Expand the last two linear factors.	$=(x-1)(x^2+2x+5x+10)$
3 Multiply the expression in the second pair of grouping symbols by x and then by -1.	$=(x-1)(x^2+7x+10)$ $=x^3+7x^2+10x-x^2-7x-10$
4 Collect like terms.	$=x^3+6x^2+3x-10$

learnON RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Expanding the product of two linear factors (doc-5366)

Exercise 17.3 Adding, subtracting and multiplying polynomials

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–c, 3a–c, 4, 5a–c, 6, 7, 9, 12	Questions: 1b–d, 2b–d, 3b–d, 4, 5b, d, f, h, i, 6, 8, 10, 12	Questions: 1c–e, 2c–e, 3–5, 6b, d, f, 7–13

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1a** Simplify each of the following.

a. $(x^4+x^3-x^2+4)+(x^3-14)$ **b.** $(x^6+x^4-3x^3+6x^2)+(x^4+3x^2+5)$
c. $(x^3+x^2+2x-4)+(4x^3-6x^2+5x-9)$ **d.** $(2x^4-3x^3+7x^2+9)+(6x^3+5x^2-4x+5)$
e. $(15x^4-3x^2+4x-7)+(x^5-2x^4+3x^2-4x-3)$

2. **WE1b** Simplify each of the following.

a. $(x^4+x^3+4x^2+5x+5)-(x^3+2x^2+3x+1)$
b. $(x^6+x^3+1)-(x^5-x^2-1)$
c. $(5x^7+6x^5-4x^3+8x^2+5x-3)-(6x^5+8x^2-3)$
d. $(10x^4-5x^2+16x+11)-(2x^2-4x+6)$
e. $(6x^3+5x^2-7x+12)-(4x^3-x^2+3x-3)$

3. **WE2a** Expand and simplify each of the following.

   a. $x(x+6)(x+1)$ **b.** $x(x-9)(x+2)$
   c. $x(x-3)(x+11)$ **d.** $2x(x+2)(x+3)$
   e. $-3x(x-4)(x+4)$ **f.** $5x(x+8)(x+2)$
   g. $x^2(x+4)$ **h.** $-2x^2(7-x)$
   i. $(5x)(-6x)(x+9)$ **j.** $-7x(x+4)^2$

4. **WE2b** Expand and simplify each of the following.

   a. $(x+7)(x+2)(x+3)$ **b.** $(x-2)(x+4)(x-5)$
   c. $(x-1)(x-4)(x+8)$ **d.** $(x-1)(x-2)(x-3)$
   e. $(x+6)(x-1)(x+1)$ **f.** $(x-7)(x+7)(x+5)$
   g. $(x+11)(x+5)(x-12)$ **h.** $(x+5)(x-1)^2$
   i. $(x+2)(x-7)^2$ **j.** $(x+1)(x-1)(x+1)$

5. Expand and simplify each of the following.

   a. $(x-2)(x+7)(x+8)$ **b.** $(x+5)(3x-1)(x+4)$
   c. $(4x-1)(x+3)(x-3)(x+1)$ **d.** $(5x+3)(2x-3)(x-4)$
   e. $(1-6x)(x+7)(x+5)$ **f.** $3x(7x-4)(x-4)(x+2)$
   g. $-9x(1-2x)(3x+8)$ **h.** $(6x+5)(2x-7)^2$
   i. $(3-4x)(2-x)(5x+9)(x-1)$ **j.** $2(7+2x)(x+3)(x+4)$

6. Expand and simplify each of the following.

   a. $(x+2)^3$ **b.** $(x+5)^3$ **c.** $(x-1)^3$
   d. $(x-3)^4$ **e.** $(2x-6)^3$ **f.** $(3x+4)^4$

Understanding

7. Simplify $2(ax+b)-5(c-bx)$.
8. Expand and simplify $(x+a)(x-b)(x^2-3bx+2a)$.

Reasoning

9. If $(x-3)^4 = ax^4+bx^3+cx^2+dx+e$, find a, b, c, d and e.
10. Simplify $(2x-3)^3-(4-3x)^2$.
11. Find the difference in volume between a cube of side $\frac{3(x-1)}{2}$ and a cuboid whose sides are x, $(x+1)$ and $(2x+1)$.

Problem solving

12. Find the constants a, b and c if

$$\frac{5x-7}{(x-1)(x+1)(x-2)} \equiv \frac{a}{(x-1)} + \frac{b}{(x+1)} + \frac{c}{(x-2)}.$$

13. Write $\frac{3x-5}{(x^2+1)(x-1)}$ in the form $\frac{ax+b}{(x^2+1)} + \frac{c}{(x-1)}$ and hence find the values of a, b and c.

Reflection

How do you add or subtract polynomials?

17.4 Long division of polynomials

17.4.1 Long division of polynomials

- The reverse of expanding is factorising (expressing a polynomial as a product of its linear factors).
- Before learning how to factorise, you must be familiar with long division of polynomials. You will remember in earlier levels doing long division questions.
- The same process can be used to divide polynomials by polynomial factors.

Consider $(x^3 + 2x^2 - 13x + 10) \div (x - 3)$ or

$$x-3\overline{)x^3+2x^2-13x+10}$$

x into x^3 goes x^2 times (consider only the leading terms). Write x^2 at the top.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-13x+10} \end{array}$$

$x^2 \times (x - 3) = x^3 - 3x^2$
Write the $x^3 - 3x^2$.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-13x+10} \\ x^3-3x^2 \end{array}$$

Subtract.
$(x^3 - x^3 = 0, 2x^2 - -3x^2 = 5x^2)$
Note: Subtracting a negative is the same as changing the sign and adding.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2 \end{array}$$

Bring down the $-13x$.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2}\downarrow \\ 5x^2-13x \end{array}$$

x into $5x^2$ goes $5x$. Write $+5x$ at the top.

$$\begin{array}{r} x^2+5x \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2-13x \end{array}$$

$5x \times (x - 3) = 5x^2 - 15x$

Write the $5x^2 - 15x$.

$$\begin{array}{r} x^2+5x \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2-13x \\ 5x^2-15x \end{array}$$

Subtract.

Note: $5x^2 - 5x^2 = 0, \ -13x - -15x = +2x$

$$\begin{array}{r} x^2+5x \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2-13x \\ \underline{5x^2-15x} \\ 2x \end{array}$$

Bring down the 10.

$$\begin{array}{r} x^2+\ \ 5x \\ x-3\overline{)x^3+2x^3-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2-13x\downarrow \\ \underline{5x^2-15x} \\ 2x+10 \end{array}$$

x into $2x$ goes 2. Write $+2$ at the top.

$$\begin{array}{r} x^2+\ \ 5x+2 \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2-13x \\ \underline{5x^2-15x} \\ 2x+10 \end{array}$$

Write the $2x - 6$.

$$\begin{array}{r} x^2+\ \ 5x+2 \\ x-3\overline{)x^3+2x^2-13x+10} \\ \underline{x^3-3x^2} \\ 5x^2-13x \\ \underline{5x^2-15x} \\ 2x+10 \\ 2x-6 \end{array}$$

Subtract to get 16.

$$\begin{array}{r rrrrl}
 & & x^2 & +5x & +2 & \leftarrow \text{Quotient}\\
\cline{2-5}
x-3\,\big) & x^3 & +2x^2 & -13x & +10 & \\
 & x^3 & -3x^2 & & & \\
\cline{2-3}
 & & 5x^2 & -13x & & \\
 & & 5x^2 & -15x & & \\
\cline{3-4}
 & & & 2x & +10 & \\
 & & & 2x & -6 & \\
\cline{4-5}
 & & & & 16 & \leftarrow \text{Remainder}
\end{array}$$

Answer: $x^2 + 5x + 2$ remainder 16

WORKED EXAMPLE 3

Perform the following long divisions and state the quotient and remainder.

a $(x^3 + 3x^2 + x + 9) \div (x + 2)$

b $(x^3 - 4x^2 - 7x - 5) \div (x - 1)$

c $(2x^3 + 6x^2 - 3x + 2) \div (x - 6)$

THINK

WRITE

a **1** Write the question in long division format.

2 Perform the long division process.

$$\begin{array}{r rrrrl}
 & & x^2 & +x & -1 & \leftarrow Q\\
\cline{2-5}
x+2\,\big) & x^3 & +3x^2 & +x & +9 & \\
 & x^3 & +2x^2 & & & \\
\cline{2-3}
 & & x^2 & +x & & \\
 & & x^2 & +2x & & \\
\cline{3-4}
 & & & -x & +9 & \\
 & & & -x & -2 & \\
\cline{4-5}
 & & & & 11 & \leftarrow R
\end{array}$$

3 Write the quotient and remainder.

Quotient is $x^2 + x - 1$; remainder is 11.

b **1** Write the question in long division format.

2 Perform the long division process.

$$\begin{array}{r rrrrl}
 & & x^2 & -3x & -10 & \leftarrow Q\\
\cline{2-5}
x-1\,\big) & x^3 & -4x^2 & -7x & -5 & \\
 & x^3 & -x^2 & \downarrow & \downarrow & \\
\cline{2-3}
 & & -3x^2 & -7x & & \\
 & & -3x^2 & +3x & & \\
\cline{2-4}
 & & & -10x & -5 & \\
 & & & -10x & +10 & \\
\cline{4-5}
 & & & & -15 & \leftarrow R
\end{array}$$

3 Write the quotient and remainder.

Quotient is $x^2 - 3x - 10$; remainder is -15.

c **1** Write the question in long division format.

2 Perform the long division process.

$$\begin{array}{r rrrrl}
 & & 2x^2 & +18x & +105 & \leftarrow Q\\
\cline{2-5}
x-6\,\big) & 2x^3 & +6x^2 & -3x & +2 & \\
 & 2x^3 & -12x^2 & \downarrow & \downarrow & \\
\cline{2-3}
 & & 18x^2 & -3x & & \\
 & & 18x^2 & -108x & & \\
\cline{3-4}
 & & & 105x & +2 & \\
 & & & 105x & -630 & \\
\cline{4-5}
 & & & & 632 & \leftarrow R
\end{array}$$

3 Write the quotient and remainder.

Quotient is $2x^2 + 18x + 105$; remainder is 632.

WORKED EXAMPLE 4

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State the quotient and remainder for $(x^3 - 7x + 1) \div (x + 5)$.

THINK	WRITE
1 Write the question in long division format. Note that there is no x^2 term in this equation. Include $0x^2$ as a 'placeholder'. **2** Perform the long division process.	$\begin{array}{rl} & x^2 - 5x + 18 \longleftarrow Q \\ x + 5 \big) & \overline{x^3 + 0x^2 - 7x + 1} \\ & \underline{x^3 + 5x^2} \quad\;\; \downarrow \qquad \downarrow \\ & \quad -5x^2 - 7x \qquad \downarrow \\ & \quad \underline{-5x^2 - 25x} \quad \downarrow \\ & \qquad\qquad 18x + 1 \\ & \qquad\qquad \underline{18x + 90} \\ & \qquad\qquad\quad -89 \longleftarrow R \end{array}$
3 Write the quotient and remainder.	Quotient is $x^2 - 5x + 18$; remainder is -89.

WORKED EXAMPLE 5

Find the quotient and the remainder when $x^4 - 3x^3 + 2x^2 - 8$ is divided by the linear expression $x + 2$.

THINK	WRITE
1 Set out the long division with each polynomial in descending powers of x. If one of the powers of x is missing, include it with 0 as the coefficient.	$x + 2\big)\overline{x^4 - 3x^3 + 2x^2 + 0x - 8}$
2 Divide x into x^4 and write the result above.	$\begin{array}{rl} & x^3 \\ x + 2\big) & \overline{x^4 - 3x^3 + 2x^2 + 0x - 8} \end{array}$
3 Multiply the result x^3 by $x + 2$ and write the result underneath.	$\begin{array}{rl} & x^3 \\ x + 2\big) & \overline{x^4 - 3x^3 + 2x^2 + 0x - 8} \\ & x^4 + 2x^3 \end{array}$
4 Subtract and bring down the remaining terms to complete the expression.	$\begin{array}{rl} & x^3 - 5x^2 \\ x + 2\big) & \overline{x^4 - 3x^3 + 2x^2 + 0x - 8} \\ & \underline{-(x^4 + 2x^3)} \\ & \qquad -5x^3 + 2x^2 + 0x - 8 \end{array}$
5 Divide x into $-5x^3$ and write the result above. **6** Continue this process to complete the long division.	$\begin{array}{rl} & x^3 - 5x^2 \\ x + 2\big) & \overline{x^4 - 3x^3 + 2x^2 + 0x - 8} \\ & \underline{-(x^4 + 2x^3)} \\ & \qquad -5x^3 + 2x^2 + 0x - 8 \\ & \qquad \underline{-(-5x^3 - 10x^2)} \\ & \qquad\qquad 12x^2 + 0x - 8 \\ & \qquad\qquad \underline{-(12x^2 + 24x)} \\ & \qquad\qquad\qquad -24x - 8 \\ & \qquad\qquad\qquad \underline{-(-24x - 48)} \\ & \qquad\qquad\qquad\qquad 40 \end{array}$
7 The polynomial $x^3 - 5x^2 + 12x - 24$, at the top, is the quotient.	The quotient is $x^3 - 5x^2 + 12x - 24$.
8 The result of the final subtraction, 40, is the remainder.	The remainder is 40.

learnON RESOURCES — ONLINE ONLY

Try out this interactivity: Long division of polynomials (int-2793)

Exercise 17.4 Long division of polynomials

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3, 4a, b, 5, 6a–c, 7a–c, 8, 10	Questions: 1e–h, 2e–h, 3a, c, e, 4c, d, 5, 6d–f, 7d–f, 8, 10	Questions: 1g–j, 2e–h, 3b, d, f, 4e, f, 5, 6e–h, 7d–f, 8–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE3a Perform the following long divisions and state the quotient and remainder.

a. $(x^3 + 4x^2 + 4x + 9) \div (x + 2)$
b. $(x^3 + 2x^2 + 4x + 1) \div (x + 1)$
c. $(x^3 + 6x^2 + 3x + 1) \div (x + 3)$
d. $(x^3 + 3x^2 + x + 3) \div (x + 4)$
e. $(x^3 + 4x^2 + 3x + 4) \div (x + 2)$
f. $(x^3 + 6x^2 + 2x + 2) \div (x + 2)$
g. $(x^3 + x^2 + x + 3) \div (x + 1)$
h. $(x^3 + 8x^2 + 5x + 4) \div (x + 8)$
i. $(x^3 + x^2 + 4x + 1) \div (x + 2)$
j. $(x^3 + 9x^2 + 3x + 2) \div (x + 5)$

2. WE3b State the quotient and remainder for each of the following.

a. $(x^3 + 2x^2 - 5x - 9) \div (x - 2)$
b. $(x^3 + x^2 + x + 9) \div (x - 3)$
c. $(x^3 + x^2 - 9x - 5) \div (x - 2)$
d. $(x^3 - 4x^2 + 10x - 2) \div (x - 1)$
e. $(x^3 - 5x^2 + 3x - 8) \div (x - 3)$
f. $(x^3 - 7x^2 + 9x - 7) \div (x - 1)$
g. $(x^3 + 9x^2 + 2x - 1) \div (x - 5)$
h. $(x^3 + 4x^2 - 5x - 4) \div (x - 4)$

3. WE3c Divide the first polynomial by the second and state the quotient and remainder.

a. $3x^3 - x^2 + 6x + 5, x + 2$
b. $4x^3 - 4x^2 + 10x - 4, x + 1$
c. $2x^3 - 7x^2 + 9x + 1, x - 2$
d. $2x^3 + 8x^2 - 9x - 1, x + 4$
e. $4x^3 - 10x^2 - 9x + 8, x - 3$
f. $3x^3 + 16x^2 + 4x - 7, x + 5$

4. Divide the first polynomial by the second and state the quotient and remainder.

a. $6x^3 - 7x^2 + 4x + 4, 2x - 1$
b. $6x^3 + 23x^2 + 2x - 31, 3x + 4$
c. $8x^3 + 6x^2 - 39x - 13, 2x + 5$
d. $2x^3 - 15x^2 + 34x - 13, 2x - 7$
e. $3x^3 + 5x^2 - 16x - 23, 3x + 2$
f. $9x^3 - 6x^2 - 5x + 9, 3x - 4$

Understanding

5. State the quotient and remainder for each of the following.

a. $\dfrac{-x^3 - 6x^2 - 7x - 16}{x + 1}$
b. $\dfrac{-3x^3 + 7x^2 + 10x - 15}{x - 3}$
c. $\dfrac{-2x^3 + 9x^2 + 17x + 15}{2x + 1}$
d. $\dfrac{4x^3 - 20x^2 + 23x - 2}{-2x + 3}$

6. WE4 State the quotient and remainder for each of the following.

a. $(x^3 - 3x + 1) \div (x + 1)$
b. $(x^3 + 2x^2 - 7) \div (x + 2)$
c. $(x^3 - 5x^2 + 2x) \div (x - 4)$
d. $(-x^3 - 7x + 8) \div (x - 1)$
e. $(5x^2 + 13x + 1) \div (x + 3)$
f. $(2x^3 + 8x^2 - 4) \div (x + 5)$
g. $(-2x^3 - x + 2) \div (x - 2)$
h. $(-4x^3 + 6x^2 + 2x) \div (2x + 1)$

7. **WE5** Find the quotient and the remainder when each polynomial is divided by the linear expression given.
   - **a.** $x^4 + x^3 + 3x^2 - 7x, x - 1$
   - **b.** $x^4 - 13x^2 + 36, x - 2$
   - **c.** $x^5 - 3x^3 + 4x + 3, x + 3$
   - **d.** $2x^6 - x^4 + x^3 + 6x^2 - 5x, x + 2$
   - **e.** $6x^4 - x^3 + 2x^2 - 4x, x - 3$
   - **f.** $3x^4 - 6x^3 + 12x, 3x + 1$

Reasoning

8. Find the quotient and remainder when $ax^2 + bx + c$ is divided by $(x - d)$.
9. A birthday cake in the shape of a cube had side length $(x + p)$ cm. The cake was divided between $(x - p)$ guests. The left-over cake was used for lunch the next day. There were q^3 guests for lunch the next day and each received $c^3\text{cm}^3$ of cake, which was then all finished.

   Find q in terms of p and c.

Problem solving

10. When $x^3 - 2x^2 + 4x + a$ is divided by $x - 1$ the remainder is zero. Use long division to determine the value of a.
11. When $2x^2 + ax + b$ is divided by $x - 1$ the remainder is zero but when $2x^2 + ax + b$ is divided by $x - 2$ the remainder is 9. Use long division to determine the value of a and b.

Reflection

Can you think of an alternative way to divide polynomials?

17.5 Polynomial values

17.5.1 Polynomial values

- Consider the polynomial $P(x) = x^3 - 5x^2 + x + 1$.
- The value of the polynomial when $x = 3$ is denoted by $P(3)$ and is found by substituting $x = 3$ into the equation in place of x. That is:

$$P(3) = (3)^3 - 5(3)^2 + (3) + 1$$
$$P(3) = 27 - 5(9) + 3 + 1$$
$$P(3) = 27 - 45 + 4$$
$$P(3) = -14$$

WORKED EXAMPLE 6

TI | CASIO

If $P(x) = 2x^3 + x^2 - 3x - 4$, find:

a $P(1)$ **b** $P(-2)$ **c** $P(a)$ **d** $P(2b)$ **e** $P(x + 1)$.

THINK	WRITE
a 1 Write the expression.	**a** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with 1.	$P(1) = 2(1)^3 + (1)^2 - 3(1) - 4$
3 Simplify.	$= 2 + 1 - 3 - 4$ $= -4$

b **1** Write the expression.

b $P(x) = 2x^3 + x^2 - 3x - 4$

2 Replace each occurrence of x with -2.

$P(-2) = 2(-2)^3 + (-2)^2 - 3(-2) - 4$

3 Simplify.

$$
\begin{aligned}
&= 2(-8) + (4) + 6 - 4\\
&= -16 + 4 + 6 - 4\\
&= -10
\end{aligned}
$$

c **1** Write the expression.

c $P(x) = 2x^3 + x^2 - 3x - 4$

2 Replace each occurrence of x with a.

$P(a) = 2a^3 + a^2 - 3a - 4$

3 No further simplification is possible, so stop here.

d **1** Write the expression.

d $P(x) = 2x^3 + x^2 - 3x - 4$

2 Replace each occurrence of x with $2b$.

$P(2b) = 2(2b)^3 + (2b)^2 - 3(2b) - 4$

3 Simplify.

$$
\begin{aligned}
&= 2(8b^3) + 4b^2 - 6b + 4\\
&= 16b^3 + 4b^2 - 6b + 4
\end{aligned}
$$

e **1** Write the expression.

e $P(x) = 2x^3 + x^2 - 3x - 4$

2 Replace each occurrence of x with $(x + 1)$.

$P(x + 1) = 2(x + 1)^3 + (x + 1)^2 - 3(x + 1) - 4$

3 Expand the right-hand side and collect like terms.

$$
\begin{aligned}
&= 2(x + 1)(x + 1)(x + 1) + (x + 1)(x + 1) - 3(x + 1) - 4\\
&= 2(x + 1)(x^2 + 2x + 1) + x^2 + 2x + 1 - 3x - 3 - 4\\
&= 2(x^3 + 2x^2 + x + x^2 + 2x + 1) + x^2 - x - 6\\
&= 2(x^3 + 3x^2 + 3x + 1) + x^2 - x - 6\\
&= 2x^3 + 6x^2 + 6x + 2 + x^2 - x - 6\\
&= 2x^3 + 7x^2 + 5x - 4
\end{aligned}
$$

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Complete this digital doc: SkillSHEET: Substitution into quadratic equations (doc-5367)

Complete this digital doc: WorkSHEET: Polynomials I (doc-14618)

Exercise 17.5 Polynomial values

assesson

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2–8, 11	Questions: 1a, e–h, 2–7, 9, 11	Questions: 1a, i–l, 2–12

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE6** If $P(x) = 2x^3 - 3x^2 + 2x + 10$, find the following.

   a. $P(0)$ **b.** $P(1)$ **c.** $P(2)$ **d.** $P(3)$
   e. $P(-1)$ **f.** $P(-2)$ **g.** $P(-3)$ **h.** $P(a)$
   i. $P(2b)$ **j.** $P(x + 2)$ **k.** $P(x - 3)$ **l.** $P(-4y)$

2. Copy the following table.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem when divided by $(x - 1)$	Rem when divided by $(x - 2)$	Rem when divided by $(x + 1)$	Rem when divided by $(x + 2)$
a								
b								
c								
d								

Complete columns 2 to 5 of the table for each of the following polynomials.

a. $P(x) = x^3 + x^2 + x + 1$ **b.** $P(x) = x^3 + 2x^2 + 5x + 2$
c. $P(x) = x^3 - x^2 + 4x - 1$ **d.** $P(x) = x^3 - 4x^2 - 7x + 3$

Understanding

3. Find the remainder when each polynomial in question **2** is divided by $(x - 1)$ and complete column 6 of the table.
4. Find the remainder when each polynomial in question **2** is divided by $(x - 2)$ and complete column 7 of the table.
5. Find the remainder when each polynomial in question **2** is divided by $(x + 1)$ and complete column 8 of the table.
6. Find the remainder when each polynomial in question **2** is divided by $(x + 2)$ and complete column 9 of the table.
7. Copy and complete:

   a. A quick way of finding the remainder when $P(x)$ is divided by $(x + 8)$ is to calculate ____________.
   b. A quick way of finding the remainder when $P(x)$ is divided by $(x - 7)$ is to calculate ____________.
   c. A quick way of finding the remainder when $P(x)$ is divided by $(x - a)$ is to calculate ____________.

Reasoning

8. If $P(x) = 2(x - 3)^5 + 1$, find:

   a. $P(2)$ **b.** $P(-2)$ **c.** $P(a)$ **d.** $P(-2a)$.

9. When $x^2 + bx + 2$ is divided by $(x - 1)$, the remainder is $b^2 - 4b + 7$. Find the possible values of b.
10. If $P(x) = -2x^3 - 3x^2 + x + 3$, find:

    a. $P(a) + 1$ **b.** $P(a + 1)$.

Problem solving

11. If $P(x) = 3x^3 - 2x^2 - x + c$ and $P(2) = 8P(1)$, find the value of c.
12. If $P(x) = 5x^2 + bx + c$ and $P(-1) = 12$ while $P(2) = 21$, find the values of b and c.

Reflection

Is there a quick way to find a remainder when dividing polynomials?

17.6 The remainder and factor theorems

17.6.1 The remainder theorem

- In the previous exercise, you may have noticed that:

  the remainder when $P(x)$ is divided by $(x - a)$ is equal to $P(a)$.

  That is, $R = P(a)$.

  This is called the **remainder theorem**.
- If $P(x) = x^3 + x^2 + x + 1$ is divided by $(x - 2)$, the quotient is $x^2 + 3x + 7$ and the remainder is $P(2)$, which equals 15. That is:

$$(x^3 + x^2 + x + 1) \div (x - 2) = x^2 + 3x + 7 + \frac{15}{x - 2} \qquad \text{and}$$
$$(x^3 + x^2 + x + 1) = (x^2 + 3x + 7)(x - 2) + 15$$

- In general, if $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, we can write:

$$P(x) \div (x - a) = Q(x) + \frac{R}{(x - a)} \quad \text{where } R = P(a)$$
$$\Rightarrow P(x) = (x - a)Q(x) + R$$

17.6.2 The factor theorem

- The remainder when 12 is divided by 4 is zero, since 4 is a factor of 12.
- Similarly, if the remainder (R) when $P(x)$ is divided by $(x - a)$ is zero, then $(x - a)$ is a factor of $P(x)$.
- Since $R = P(a)$, find a value of a that makes $P(a) = 0$, then $(x - a)$ is a factor.

  If $P(a) = 0$, then $(x-a)$ is a factor of $P(x)$.

  This is called the **factor theorem**.
- Imagine $P(x)$ could be factorised as follows:

  $P(x) = (x - a)Q(x)$, where $Q(x)$ is 'the other' factor of $P(x)$.
- If $P(a) = 0$, $(x - a)$ is a factor.

WORKED EXAMPLE 7

Without actually dividing, find the remainder when $x^3 - 7x^2 - 2x + 4$ is divided by:

a $x - 3$

b $x + 6$.

THINK	WRITE
a **1** Name the polynomial.	**a** Let $P(x) = x^3 - 7x^2 - 2x + 4$.
2 The remainder when $P(x)$ is divided by $(x - 3)$ is equal to $P(3)$.	$R = P(3)$ $= 3^3 - 7(3)^2 - 2(3) + 4$ $= 27 - 7(9) - 6 + 4$ $= 27 - 63 - 6 + 4$ $= -38$
b The remainder when $P(x)$ is divided by $(x + 6)$ is equal to $P(-6)$.	**b** $R = P(-6)$ $= (-6)^3 - 7(-6)^2 - 2(-6) + 4$ $= -216 - 7(36) + 12 + 4$ $= -216 - 252 + 12 + 4$ $= -452$

WORKED EXAMPLE 8

TI | CASIO

$(x - 2)$ is a factor of $x^3 + kx^2 + x - 2$. Find the value of k.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 + kx^2 + x - 2$.
2 The remainder when $P(x)$ is divided by $(x - 2)$ is equal to $P(2) = 0$.	$0 = P(2)$ $= 2^3 + k(2)^2 + 2 - 2$ $0 = 8 + 4k$
3 Solve for k.	$4k = -8$ $k = -2$

Exercise 17.6 The remainder and factor theorems

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 2a–d, 3a–d, 4, 7a–d	Questions: 1, 2e–h, 3e–h, 4–6, 7d–g, 8	Questions: 1, 2g–j, 3f–h, 4–9

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE7** Without actually dividing, find the remainder when $x^3 + 3x^2 - 10x - 24$ is divided by:

a. $x - 1$
b. $x + 2$
c. $x - 3$
d. $x + 5$
e. $x - 0$
f. $x - k$
g. $x + n$
h. $x + 3c$.

2. Find the remainder when the first polynomial is divided by the second without performing long division.

a. $x^3 + 2x^2 + 3x + 4, x - 3$
b. $x^3 - 4x^2 + 2x - 1, x + 1$
c. $x^3 + 3x^2 - 3x + 1, x + 2$
d. $x^3 - x^2 - 4x - 5, x - 1$
e. $2x^3 + 3x^2 + 6x + 3, x + 5$
f. $-3x^3 - 2x^2 + x + 6, x + 1$
g. $x^3 + x^2 + 8, x - 5$
h. $x^3 - 3x^2 - 2, x - 2$
i. $-x^3 + 8, x + 3$
j. $x^3 + 2x^2, x - 7$

Understanding

3. **WE8** **a.** The remainder when $x^3 + kx + 1$ is divided by $(x + 2)$ is -19. Find the value of k.

b. The remainder when $x^3 + 2x^2 + mx + 5$ is divided by $(x - 2)$ is 27. Find the value of m.

c. The remainder when $x^3 - 3x^2 + 2x + n$ is divided by $(x - 1)$ is 1. Find the value of n.

d. The remainder when $ax^3 + 4x^2 - 2x + 1$ is divided by $(x - 3)$ is -23. Find the value of a.

e. The remainder when $x^3 - bx^2 - 2x + 1$ is divided by $(x + 1)$ is 0. Find the value of b.

f. The remainder when $-4x^2 + 2x + 7$ is divided by $(x - c)$ is -5. Find a possible whole number value of c.

g. The remainder when $x^2 - 3x + 1$ is divided by $(x + d)$ is 11. Find the possible values of d.

h. The remainder when $x^3 + ax^2 + bx + 1$ is divided by $(x - 5)$ is -14. When the cubic polynomial is divided by $(x + 1)$, the remainder is -2. Find a and b.

4. **MC** *Note:* There may be more than one correct answer.

a. When $x^3 + 2x^2 - 5x - 5$ is divided by $(x + 2)$, the remainder is:

A. -5
B. -2
C. 2
D. 5

b. Which of the following is a factor of $2x^3 + 15x^2 + 22x - 15$?

A. $(x - 1)$ **B.** $(x - 2)$ **C.** $(x + 3)$ **D.** $(x + 5)$

c. When $x^3 - 13x^2 + 48x - 36$ is divided by $(x - 1)$, the remainder is:

A. -3 **B.** -2 **C.** -1 **D.** 0

d. Which of the following is a factor of $x^3 - 5x^2 - 22x + 56$?

A. $(x - 2)$ **B.** $(x + 2)$ **C.** $(x - 7)$ **D.** $(x + 4)$

5. Find one factor of each of the following cubic polynomials.

a. $x^3 - 3x^2 + 3x - 1$ **b.** $x^3 - 7x^2 + 16x - 12$

c. $x^3 + x^2 - 8x - 12$ **d.** $x^3 + 3x^2 - 34x - 120$

Reasoning

6. Prove that each of the following is a linear factor of $x^3 + 4x^2 - 11x - 30$ by substituting values into the cubic function: $(x + 2)$, $(x - 3)$, $(x + 5)$.

7. Without actually dividing, show that the first polynomial is exactly divisible by the second (that is, the second polynomial is a factor of the first).

a. $x^3 + 5x^2 + 2x - 8, x - 1$ **b.** $x^3 - 7x^2 - x + 7, x - 7$

c. $x^3 - 7x^2 + 4x + 12, x - 2$ **d.** $x^3 + 2x^2 - 9x - 18, x + 2$

e. $x^3 + 3x^2 - 9x - 27, x + 3$ **f.** $-x^3 + x^2 + 9x - 9, x - 1$

g. $-2x^3 + 9x^2 - x - 12, x - 4$ **h.** $3x^3 + 22x^2 + 37x + 10, x + 5$

Problem solving

8. When $x^4 + ax^3 - 4x^2 + b$ and $x^3 - ax^2 - 7x + b$ are each divided by $(x - 2)$, the remainders are 26 and 8 respectively. Find the values of a and b.

9. Both $(x - 1)$ and $(x - 2)$ are factors of $P(x) = x^4 + ax^3 - 7x^2 + bx - 30$. Find the values of a and b and the remaining two linear factors.

Reflection

How are the remainder and factor theorems related?

CHALLENGE 17.1

The remainder when $2x - 1$ is divided into $6x^3 - x^2 + 3x + k$ is the same as when it is divided into $4x^3 - 8x^2 - 5x + 2$. What is the value of k?

17.7 Factorising polynomials

17.7.1 Using long division

- Once one factor of a polynomial has been found (using the factor theorem as in the previous section), long division may be used to find other factors. In the case of a cubic polynomial, one — possibly two — other factors may be found.

WORKED EXAMPLE 9

Use long division to factorise the following.

a $x^3 - 5x^2 - 2x + 24$ **b** $x^3 - 19x + 30$ **c** $-2x^3 - 8x^2 + 6x + 4$

THINK	WRITE
a **1** Name the polynomial.	**a** $P(x) = x^3 - 5x^2 - 2x + 24$

2 Look for a value of x such that $P(x) = 0$. For cubics containing a single x^3, try a factor of the constant term (24 in this case).
Try $P(1)$.

$$\begin{aligned} P(1) &= 1^3 - 5 \times 1^2 - 2 \times 1 + 24 \\ &= 1 - 5 - 2 + 24 \\ &= 18 \\ &\neq 0 \end{aligned}$$

$P(1) \neq 0$, so $(x - 1)$ is not a factor.
Try $P(2)$.

$$\begin{aligned} P(2) &= 2^3 - 5 \times 2^2 - 2 \times 2 + 24 \\ &= 8 - 20 - 4 + 24 \\ &\neq 0 \end{aligned}$$

$P(2) \neq 0$, so $(x - 2)$ is not a factor.
Try $P(-2)$.

$$\begin{aligned} P(-2) &= (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24 \\ &= -8 - 20 + 4 + 24 \\ &= -28 + 28 \\ &= 0 \end{aligned}$$

$P(-2)$ does equal 0, so $(x + 2)$ is a factor.

$(x + 2)$ is a factor.

3 Divide $(x + 2)$ into $P(x)$ using long division to find a quadratic factor.

$$\begin{array}{rllll}
 & & x^2 & -\ 7x & +\ 12 \\ \cline{2-5}
x+2\,\big) & x^3 & -\ 5x^2 & -\ 2x & +\ 24 \\
 & x^3 & +\ 2x^2 & \downarrow & \downarrow \\ \cline{2-3}
 & & -7x^2 & -\ 2x & \\
 & & -7x^2 & -\ 14x & \\ \cline{3-4}
 & & & 12x & +\ 24 \\
 & & & 12x & +\ 24 \\ \cline{4-5}
 & & & & 0
\end{array}$$

4 Write $P(x)$ as a product of the two factors found so far.

$$P(x) = (x + 2)(x^2 - 7x + 12)$$

5 Factorise the quadratic factor if possible.

$$P(x) = (x + 2)(x - 3)(x - 4)$$

b 1 Name the polynomial.
Note: There is no x^2 term, so include $0x^2$.

b
$$\begin{aligned} P(x) &= x^3 - 19x + 30 \\ P(x) &= x^3 + 0x^2 - 19x + 30 \end{aligned}$$

2 Look at the last term in $P(x)$, which is 30. This suggests it is worth trying $P(5)$ or $P(-5)$. Try $P(-5)$. $P(-5) = 0$ so $(x + 5)$ is a factor.

$$\begin{aligned} P(-5) &= (-5)^3 - 19 \times (-5) + 30 \\ &= -125 + 95 + 30 \\ &= 0 \end{aligned}$$

So $(x + 5)$ is a factor.

3 Divide $(x + 5)$ into $P(x)$ using long division to find a quadratic factor.

$$\begin{array}{rllll}
 & & x^2 & -\ 5x & +\ 6 \\ \cline{2-5}
x+5\,\big) & x^3 & +\ 0x^2 & -\ 19x & +\ 30 \\
 & x^3 & +\ 5x^2 & \downarrow & \downarrow \\ \cline{2-3}
 & & -5x^2 & -\ 19x & \\
 & & -5x^2 & -\ 25x & \\ \cline{3-4}
 & & & 6x & +\ 30 \\
 & & & 6x & +\ 30 \\ \cline{4-5}
 & & & & 0
\end{array}$$

4 Write $P(x)$ as a product of the two factors found so far.

$$P(x) = (x + 5)(x^2 - 5x + 6)$$

5 Factorise the quadratic factor if possible.

$$P(x) = (x + 5)(x - 2)(x - 3)$$

c **1** Write the given polynomial.	**c** Let $P(x) = -2x^3 - 8x^2 + 6x + 4$
2 Take out a common factor of −2. (We could take out +2 as the common factor, but taking out −2 results in a positive leading term in the part still to be factorised.)	$= -2(x^3 + 4x^2 - 3x - 2)$
3 Let $Q(x) = (x^3 + 4x^2 - 3x - 2)$. (We have already used P earlier.)	Let $Q(x) = (x^3 + 4x^2 - 3x - 2)$.
4 Evaluate $Q(1)$. $Q(1) = 0$, so $(x - 1)$ is a factor.	$Q(1) = 1 + 4 - 3 - 2$ $= 0$ So $(x - 1)$ is a factor.
5 Divide $(x - 1)$ into $Q(x)$ using long division to find a quadratic factor.	$\begin{array}{r} x^2 + 5x + 2 \\ x - 1\,\overline{)\,x^3 + 4x^2 - 3x - 2} \\ \underline{x^3 - x^2}\phantom{{}-3x-2} \\ 5x^2 - 3x\phantom{{}-2} \\ \underline{5x^2 - 5x}\phantom{{}-2} \\ 2x - 2 \\ \underline{2x - 2} \\ 0 \end{array}$
6 Write the original polynomial $P(x)$ as a product of the factors found so far.	$P(x) = -2(x - 1)(x^2 + 5x + 2)$
7 In this case, it is not possible to further factorise $P(x)$.	

- *Note:* In these examples, $P(x)$ may have been factorised without long division by finding all three values of x that make $P(x) = 0$ (and hence three factors) and then checking that the three factors multiply to give $P(x)$.

17.7.2 Using short division

- The process of long division can be quite time (and space) consuming. An alternative is short division, which may take a little longer to understand, but is quicker once mastered.
- Consider $P(x) = x^3 + 2x^2 - 13x + 10$. Using the factor theorem, we can find that $(x - 1)$ is a factor of $P(x)$. So, $P(x) = (x - 1)(\ ?\ )$.

  Actually, we know more than this: as $P(x)$ begins with x^3 and ends with +10, we could write
  $$P(x) = (x - 1)(x^2 + ? - 10)$$
  The x^2 in the second pair of grouping symbols produces the desired x^3 (the leading term in $P(x)$) when the expressions are multiplied. The −10 in the second pair of grouping symbols produces +10 (the last term in $P(x)$) when the expressions are multiplied.
- Imagine expanding this version of $P(x)$. Multiplying x in the first pair of grouping symbols by x^2 in the second would produce x^3, which is what we want, but multiplying −1 in the first pair of grouping symbols by x^2 in the second gives $-1x^2$.

  Since $P(x) = x^3 + 2x^2 - 13x + 10$, we really need $+2x^2$, not $-1x^2$. That is, we need $+3x^2$ more. To get this, the? must be $3x$, because when x in the first pair of grouping symbols is multiplied by $3x$ in the second pair, $+ 3x^2$ results. That is, we have deduced $P(x) = (x - 1)(x^2 + 3x - 10)$.

  Factorising the expression in the second pair of grouping symbols gives
  $$P(x) = (x - 1)(x + 5)(x - 2).$$
- This procedure, which we will call *short division*, can be confusing at first, but with persistence it can be a quick and easy method for factorising polynomials.
- The following worked example is a repeat of a previous one, but explains the use of short, rather than long, division.

WORKED EXAMPLE 10

TI | CASIO

Use short division to factorise $x^3 - 5x^2 - 2x + 24$.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 - 5x^2 - 2x + 24$.
2 Look for a value of x such that $P(x) = 0$. Try $P(-2)$.	$P(-2) = (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24$ $= -8 - 20 + 4 + 24$ $= -28 + 28$ $= 0$
$P(-2)$ does equal 0, so $(x + 2)$ is a factor.	So $(x + 2)$ is a factor.
3 Look again at the original $P(x) = x^3 - 5x^2 - 2x + 24$. The first term in the grouping symbols must be x^2, and the last term must be 12.	$P(x) = (x + 2)(x^2 + ax + 12)$
4 Imagine the expansion of the expression in step 3. We have x^3 and $2x^2$, but require $-5x^2$. We need an extra $-7x^2$. We get this by inserting a $-7x$ term in the second pair of grouping symbols.	$P(x) = (x + 2)(x^2 - 7x + 12)$
5 Factorise the expression in the second pair of grouping symbols if possible.	$P(x) = (x + 2)(x - 3)(x - 4)$

learnON RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Factorising quadratic trinomials (doc-5368)

Exercise 17.7 Factorising polynomials

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–d, 3a–d, 4a–c, 5, 6	Questions: 1d–f, 2e–h, 3e–h, 4d–g, 5, 6, 8	Questions: 1d–f, 2i–n, 3g–j, 4h–k, 5–9

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE9** Use long division to factorise each dividend.

a. $x + 1\overline{)x^3 + 10x^2 + 27x + 18}$

b. $x + 2\overline{)x^3 + 8x^2 + 17x + 10}$

c. $x + 9\overline{)x^3 + 12x^2 + 29x + 18}$

d. $x + 1\overline{)x^3 + 8x^2 + 19x + 12}$

e. $x + 3\overline{)x^3 + 14x^2 + 61x + 84}$

f. $x + 7\overline{)x^3 + 12x^2 + 41x + 42}$

g. $x + 2\overline{)x^3 + 4x^2 + 5x + 2}$

h. $x + 3\overline{)x^3 + 7x^2 + 16x + 12}$

i. $x + 5\overline{)x^3 + 14x^2 + 65x + 100}$

j. $x\overline{)x^3 + 13x^2 + 40x}$

k. $x\overline{)x^3 + 7x^2 + 12x}$

l. $x + 5\overline{)x^3 + 10x^2 + 25x}$

m. $x + 1\overline{)x^3 + 6x^2 + 5x}$

n. $x + 6\overline{)x^3 + 6x^2}$

2. **WE9, 10** Factorise the following as fully as possible.

a. $x^3 + x^2 - x - 1$
b. $x^3 - 2x^2 - x + 2$
c. $x^3 + 7x^2 + 11x + 5$
d. $x^3 + x^2 - 8x - 12$
e. $x^3 + 9x^2 + 24x + 16$
f. $x^3 - 5x^2 - 4x + 20$
g. $x^3 + 2x^2 - x - 2$
h. $x^3 - 7x - 6$
i. $x^3 + 3x^2 - 4$
j. $x^3 + x^2 + x + 6$
k. $x^3 + 8x^2 + 17x + 10$
l. $x^3 + x^2 - 9x - 9$
m. $x^3 - x^2 - 8x + 12$
n. $x^3 + 9x^2 - 12x - 160$

Understanding

3. Factorise as fully as possible.

a. $2x^3 + 5x^2 - x - 6$
b. $3x^3 + 14x^2 + 7x - 4$
c. $3x^3 + 2x^2 - 12x - 8$
d. $4x^3 + 35x^2 + 84x + 45$
e. $5x^3 + 9x^2 + 3x - 1$
f. $x^3 + x^2 + x + 1$
g. $4x^3 + 16x^2 + 21x + 9$
h. $6x^3 - 23x^2 + 26x - 8$
i. $10x^3 + 19x^2 - 94x - 40$
j. $7x^3 + 12x^2 - 60x + 16$

4. Factorise as fully as possible.

a. $3x^3 - x^2 - 10x$
b. $4x^3 + 2x^2 - 2x$
c. $3x^3 - 6x^2 - 24x$
d. $-2x^3 - 12x^2 - 18x$
e. $6x^3 - 6x^2$
f. $-x^3 - 7x^2 - 12x$
g. $-x^3 - 3x^2 + x + 3$
h. $-2x^3 + 10x^2 - 12x$
i. $-6x^3 - 5x^2 + 12x - 4$
j. $-5x^3 + 24x^2 - 36x + 16$
k. $-x^5 - x^4 + 21x^3 + 49x^2 - 8x - 60$

Reasoning

5. Factorise $x^4 - 9x^2 - 4x + 12$.

6. Factorise $-x^5 + 6x^4 + 11x^3 - 84x^2 - 28x + 240$.

7. Two of the factors of $x^3 + px^2 + qx + r$ are $(x + a)$ and $(x + b)$. Find the third factor.

Problem solving

8. $(x - 1)$ and $(x - 2)$ are known to be factors of $x^5 + ax^4 - 2x^3 + bx^2 + x - 2$. Find the values of a and b and hence fully factorise this fifth-degree polynomial.

9. Factorise $x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$.

Reflection

Explain the steps in factorising polynomials.

CHALLENGE 17.2

The polynomial $x^4 - 6x^3 + 13x^2 - 12x - 32$ has three factors, one of which is $x^2 - 3x + 8$. What are the other two factors?

17.8 Solving polynomial equations

17.8.1 Solving polynomial equations

- A polynomial equation of the form $P(x) = 0$ may be solved by factorising $P(x)$ and applying the Null Factor Law. The solutions are also called zeros. They are the x-intercepts on the graph of $P(x)$. If $P(x)$ is of degree n, expect n zeros.
- The Null Factor Law applies to polynomial equations just as it does for quadratics.
- If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions can be found as follows.
  Let each factor = 0:

$$x - a = 0 \qquad x - b = 0 \qquad x - c = 0$$

Solving each of these equations produces the solutions (roots)

$$x = a \qquad x = b \qquad x = c.$$

- If $P(x) = k(lx - a)(mx - b)(nx - c) = 0$, then the solutions can be found as follows. Let each factor $= 0$:

  $$lx - a = 0 \qquad mx - b = 0 \qquad nx - c = 0$$

  Solving each of these equations produces the solutions

  $$x = \frac{a}{l} \qquad x = \frac{b}{m} \qquad x = \frac{c}{n}.$$

  Note: The coefficient k used in this example does not produce a solution because $k \neq 0$.

WORKED EXAMPLE 11

TI | CASIO

Solve:

a $x^3 = 9x$ **b** $-2x^3 + 4x^2 + 70x = 0$ **c** $2x^3 - 11x^2 + 18x - 9 = 0.$

THINK	WRITE
a 1 Write the equation.	**a** $x^3 = 9x$
2 Rearrange so all terms are on the left.	$x^3 - 9x = 0$
3 Take out a common factor of x.	$x(x^2 - 9) = 0$
4 Factorise the expression in the grouping symbols using the difference of squares rule.	$x(x + 3)(x - 3) = 0$
5 Use the Null Factor Law to solve.	$x = 0, x + 3 = 0$ or $x - 3 = 0$ $x = 0, x = -3$ or $x = 3$
b 1 Write the equation.	**b** $-2x^3 + 4x^2 + 70x = 0$
2 Take out a common factor of $-2x$.	$-2x(x^2 - 2x - 35) = 0$
3 Factorise the expression in the grouping symbols.	$-2x(x - 7)(x + 5) = 0$
4 Use the Null Factor Law to solve.	$-2x = 0, x - 7 = 0$ or $x + 5 = 0$ $x = 0, x = 7$ or $x = -5$
c 1 Name the polynomial.	**c** Let $P(x) = 2x^3 - 11x^2 + 18x - 9$.
2 Use the factor theorem to find a factor (search for a value a such that $P(a) = 0$). Consider factors of the constant term (that is, factors of 9 such as 1, 3). The simplest value to try is 1.	$P(1) = 2 - 11 + 18 - 9$ $= 0$ So $(x - 1)$ is a factor.
3 Use long or short division to find another factor of $P(x)$.	$\begin{array}{r} 2x^2 - 9x + 9 \\ x - 1\overline{)2x^3 - 11x^2 + 18x - 9} \\ \underline{2x^3 - 2x^2} \qquad\qquad \\ -9x^2 + 18x \qquad \\ \underline{-9x^2 + 9x} \qquad \\ 9x - 9 \\ \underline{9x - 9} \\ 0 \end{array}$ $P(x) = (x - 1)(2x^2 + 9x - 9)$
4 Factorise the quadratic factor.	$P(x) = (x - 1)(2x - 3)(x - 3)$
5 Consider the factorised equation to solve.	For $(x - 1)(2x - 3)(x - 3) = 0$
6 Use the Null Factor Law to solve.	$x - 1 = 0, 2x - 3 = 0$ or $x - 3 = 0$ $x = 1, x = \frac{3}{2}$ or $x = 3$

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Complete this digital doc: SkillSHEET: Factorising difference of two squares expressions (doc-5369)

Complete this digital doc: SkillSHEET: Solving quadratic equations (doc-5370)

Complete this digital doc: WorkSHEET: Polynomials II (doc-14619)

Exercise 17.8 Solving polynomial equations

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1a–d, 2a–d, 3, 4, 5a–d, 6a–c, 7, 10	Questions: 1e–h, 2e–h, 3, 4, 5e–h, 6d–f, 8, 10	Questions: 1i–n, 2i–n, 3, 4, 5e–h, 6d–f, 7–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE11a, b** Solve the following.

a. $x^3 - 4x = 0$
b. $x^3 - 16x = 0$
c. $2x^3 - 50x = 0$
d. $-3x^3 + 81 = 0$
e. $x^3 + 5x^2 = 0$
f. $x^3 - 2x^2 = 0$
g. $-4x^3 + 8x = 0$
h. $12x^3 + 3x^2 = 0$
i. $4x^2 - 20x^3 = 0$
j. $x^3 - 5x^2 + 6x = 0$
k. $x^3 - 8x^2 + 16x = 0$
l. $x^3 + 6x^2 = 7x$
m. $9x^2 = 20x + x^3$
n. $x^3 + 6x = 4x^2$

2. **WE11c** Use the factor theorem to solve the following.

a. $x^3 - x^2 - 16x + 16 = 0$
b. $x^3 - 6x^2 - x + 30 = 0$
c. $x^3 - x^2 - 25x + 25 = 0$
d. $x^3 + 4x^2 - 4x - 16 = 0$
e. $x^3 - 4x^2 + x + 6 = 0$
f. $x^3 - 4x^2 - 7x + 10 = 0$
g. $x^3 + 6x^2 + 11x + 6 = 0$
h. $x^3 - 6x^2 - 15x + 100 = 0$
i. $x^3 - 3x^2 - 6x + 8 = 0$
j. $x^3 + 2x^2 - 29x + 42 = 0$
k. $2x^3 + 15x^2 + 19x + 6 = 0$
l. $-4x^3 + 16x^2 - 9x - 9 = 0$
m. $-2x^3 - 9x^2 - 7x + 6 = 0$
n. $2x^3 + 4x^2 - 2x - 4 = 0$

3. **MC** *Note:* There may be more than one correct answer.
Which of the following is a solution to $x^3 - 7x^2 + 2x + 40 = 0$?
A. 5
B. −4
C. −2
D. 1

4. **MC** A solution of $x^3 - 9x^2 + 15x + 25 = 0$ is $x = 5$. How many other (distinct) solutions are there?
A. 0
B. 1
C. 2
D. 3

Understanding

5. Solve $P(x) = 0$.

a. $P(x) = x^3 + 4x^2 - 3x - 18$
b. $P(x) = 3x^3 - 13x^2 - 32x + 12$
c. $P(x) = -x^3 + 12x - 16$
d. $P(x) = 8x^3 - 4x^2 - 32x - 20$
e. $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$
f. $P(x) = -72 - 42x + 19x^2 + 7x^3 - 2x^4$
g. $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$
h. $P(x) = 4x^4 + 12x^3 - 24x^2 - 32x$

6. Solve each of the following equations.

a. $x^3 - 3x^2 - 6x + 8 = 0$

b. $x^3 + x^2 - 9x - 9 = 0$

c. $3x^3 + 3x^2 - 18x = 0$

d. $2x^4 + 10x^3 - 4x^2 - 48x = 0$

e. $2x^4 + x^3 - 14x^2 - 4x + 24 = 0$

f. $x^4 - 2x^2 + 1 = 0$

Reasoning

7. Solve for a if $x = 2$ is a solution of $ax^3 - 6x^2 + 3x - 4 = 0$.

8. Solve for p if $x = \frac{p}{2}$ is a solution of $x^3 - 5x^2 + 2x + 8 = 0$.

9. Show that it is possible for a cuboid of side lengths x cm, $(x - 1)$ cm and $(x + 2)$ cm to have a volume that is 4 cm^3 less than twice the volume of a cube of side length x cm. Comment on the shape of such a cuboid.

Problem solving

10. Solve for x.

$x^3 + 8 = x(5x - 2)$

11. Solve for z.

$z(z - 1)^3 = -2(z^3 - 5z^2 + z + 3)$

Reflection

Can you predict the number of solutions a polynomial might have?

17.9 Review

17.9.1 Review questions

Fluency

1. **MC** Which of the following is *not* a polynomial?

A. $x^3 - \frac{x^2}{3} + 7x - 1$

B. $a^4 + 4a^3 + 2a + 2$

C. $\sqrt{x^2 + 3x + 2}$

D. 5

2. Consider the polynomial $f(x) = -\frac{1}{7}x^4 + x^5 + 3$.

a. What is the degree of $f(x)$?

b. What is the coefficient of x^4?

c. What is the constant term?

d. What is the leading term?

3. **MC** The expansion of $(x + 5)(x + 1)(x - 6)$ is:

A. $x^3 - 30$

B. $x^3 + 12x^2 - 31x + 30$

C. $x^3 - 31x - 30$

D. $x^3 + 5x^2 - 36x - 30$

4. **MC** $x^3 + 5x^2 + 3x - 9$ is the expansion of:

A. $(x + 3)^3$

B. $x(x + 3)(x - 3)$

C. $(x - 1)(x + 3)^2$

D. $(x - 1)(x + 1)(x + 3)$

5. Expand each of the following.

a. $(x - 2)^2(x + 10)$

b. $(x + 6)(x - 1)(x + 5)$

c. $(x - 7)^3$

d. $(5 - 2x)(1 + x)(x + 2)$

6. **MC** Consider the following long division.

$$\begin{array}{rl}
 & \qquad\qquad x^2 + x + 2 \\
x - 4 & \overline{)\,x^3 + 5x^2 + 6x - 1} \\
 & \;\;\underline{x^3 + 4x^2}\;\downarrow \qquad\;\; \downarrow \\
 & \qquad\quad x^2 + 6x \\
 & \qquad\quad \underline{x^2 + 4x}\;\;\; \downarrow \\
 & \qquad\qquad\quad 2x - 1 \\
 & \qquad\qquad\quad \underline{2x + 8} \\
 & \qquad\qquad\qquad -9
\end{array}$$

a. The quotient is:

A. -9 **B.** 9 **C.** $x + 4$ **D.** $x^2 + x + 2$

b. The remainder is:

A. -9 **B.** 2 **C.** 4 **D.** $2x - 1$

7. Find the quotient and remainder when the first polynomial is divided by the second in each case.

a. $x^3 + 2x^2 - 16x - 3,\ x + 2$ **b.** $x^3 + 3x^2 - 13x - 7,\ x - 3$ **c.** $-x^3 + x^2 + 4x - 7,\ x + 1$

8. **MC** If $P(x) = x^3 - 3x^2 + 7x + 1$, then $P(-2)$ equals:

A. -34 **B.** -33 **C.** -9 **D.** 9

9. If $P(x) = -3x^3 + 2x^2 + x - 4$, find:

a. $P(1)$ **b.** $P(-4)$ **c.** $P(2a)$

10. Without dividing, find the remainder when $x^3 + 3x^2 - 16x + 5$ is divided by $x - 1$.

11. Show that $x + 3$ is a factor of $x^3 - 2x^2 - 29x - 42$.

12. Factorise $x^3 + 4x^2 - 100x - 400$.

13. Solve:

a. $(2x + 1)(x - 3)^2 = 0$ **b.** $x^3 - 9x^2 + 26x - 24 = 0$

c. $x^4 - 4x^3 - x^2 + 16x - 12 = 0$

Problem solving

14. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ \dots + a_1x + a_0$ be a polynomial where the coefficients are integers. Also let $P(w) = 0$ where w is an integer. Show that w is a factor of a_0.

15. Find the area of a square whose sides are $(2x - 3)$ cm. Expand and simplify your answer. If the area is 16 cm^2, find x.

16. A window is in the shape of a semicircle above a rectangle. The height of the window is $(6x + 1)$ cm and its width is $(2x + 2)$ cm.

a. Find the total area of the window.

b. Expand and simplify your answer.

c. What is the perimeter of the window?

17. a. Find the volume of a cube of side $(x + 4)$ cm.

b. Find the surface area of the cube.

c. Find the value of x for which the volume and surface are numerically equal.

d. Find x if the numerical value of the volume is 5 less than the numerical value of the surface area.

18. Find the quotient and remainder when $mx^2 + nx + q$ is divided by $(x - p)$.

19. When $P(x)$ is divided by $(x - n)$, the quotient is $x^2 - 2x + n$ and the remainder is $(n + 1)$. Find $P(x)$.

learnon RESOURCES — ONLINE ONLY

- **Try out this interactivity:** Word search: Topic 17 (int-2874)
- **Try out this interactivity:** Crossword: Topic 17 (int-2875)
- **Try out this interactivity:** Sudoku: Topic 17 (int-3892)
- **Complete this digital doc:** Concept map: Topic 17 (doc-14621)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

coefficient
cubic
degree of a polynomial
factor
factor theorem
leading term
long division
monic
polynomial
quadratic
quartic
quotient
remainder
remainder theorem
short division

Investigation | Rich task

Investigating polynomials

A polynomial is a function involving the sum of integer powers of a variable (for example, $y = -4x^3 + 3x^2 - 4$). The highest power of the variable determines the degree of the polynomial. In the case of the given example, the degree is 3. A polynomial of the first degree is a linear function (for example, $y = 3x - 8$, and a second-degree function is a quadratic (for example, $y = 5x^2 - 6x + 7$). Let us investigate how the degree of a polynomial affects the shape of its graph.

In order to simplify the graphing of these functions, the polynomials will be expressed in factor form. A graphics calculator or some other digital technology will make the graphing process less tedious. It will be necessary to adjust the window of the calculator from time to time in order to capture the relevant features of the graph.

1. Consider the following polynomials.
   - **a.** $y_1 = (x + 1)$
   - **b.** $y_2 = (x + 1)(x - 2)$
   - **c.** $y_3 = (x + 1)(x - 2)(x + 3)$
   - **d.** $y_4 = (x + 1)(x - 2)(x + 3)(x - 4)$
   - **e.** $y_5 = (x + 1)(x - 2)(x + 3)(x - 4)(x + 5)$
   - **f.** $y_6 = (x + 1)(x - 2)(x + 3)(x - 4)(x + 5)(x - 6)$

   For each of the functions:
   - **i.** give the degree of the polynomial
   - **ii.** sketch the graph, marking in the x-intercepts
   - **iii.** describe how the degree of the polynomial affects the shape of the graph.

   Complete question **1** on a separate sheet of paper.

2. Let us now look at the effect that the exponent of each factor has on the shape of the graph of the polynomial. Consider the following functions.

a. $y_1 = (x + 1)(x - 2)(x + 3)$ **b.** $y_2 = (x + 1)^2(x - 2)(x + 3)$
c. $y_3 = (x + 1)^2(x - 2)^2(x + 3)$ **d.** $y_4 = (x + 1)^2(x - 2)(x + 3)^3$
e. $y_5 = (x + 1)^3(x - 2)(x + 3)^4$ **f.** $y_6 = (x + 1)^5(x - 2)^3(x + 3)^2$

i. On a separate sheet of paper, draw a sketch of each of the polynomials, marking in the x-intercepts.

ii. Explain how the power of the factor affects the behaviour of the graph at the x-intercept.

3. Create and draw a sketch of polynomials with the following given characteristics. Complete your graphs on a separate sheet of paper.

a. A first-degree polynomial that:
i. crosses the x-axis **ii.** does not cross the x-axis.

b. A second-degree polynomial that:
i. crosses the x-axis twice
ii. touches the x-axis at one and only one point.

c. A third-degree polynomial that crosses the x-axis:
i. three times **ii.** twice **iii.** once.

d. A fourth-degree polynomial that crosses the x-axis:
i. four times **ii.** three times **iii.** twice **iv.** once.

4. Considering the powers of factors of polynomials, write a general statement outlining the conditions under which the graph of a polynomial will pass through the x-axis or just touch the x-axis.

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: Code puzzle: Who were Australia's first three Prime Ministers? (doc-15949)

Answers

Topic 17 Polynomials

Exercise 17.2 Polynomials

1. a. 3 **b.** 7 **c.** 2 **d.** 6 **e.** 8 **f.** 5 **g.** 5 **h.** 1 **i.** 6

2. a. x **b.** x **c.** x **d.** x **e.** y **f.** u **g.** e **h.** g **i.** f

3. a. Polynomial 1h **b.** Polynomial 1c

c. Polynomial 1a **d.** Polynomials 1a, 1d and 1e

4. a. N **b.** P **c.** P **d.** N **e.** N **f.** P **g.** P **h.** N **i.** N

5. a. 3 **b.** x **c.** 4 **d.** 5 **e.** $3x$ **f.** $-2x^3$

6. a. 7 **b.** w **c.** 7 **d.** 0 **e.** -9 **f.** $6w^7$

7. a. 4 **b.** 1 **c.** x^4 **d.** 1

8. a. 6 **b.** t **c.** 6

d, e. Check with your teacher.

9. a. 5 units to the right of the origin **b.** 4 units to the right of the origin

c. The body moves towards the origin, then away.

10. a. $4x^3 + 2x^2 - 10x + 18$ **b.** $3x^4 - 3x^3 - x^2 + 7x - 7$ **c.** $5x^3 - 4x^2 - 13x - 6$

11. $a = 4, b = -6$

12. $a = \pm 3, b = \pm 2$

Exercise 17.3 Adding, subtracting and multiplying polynomials

1. a. $x^4 + 2x^3 - x^2 - 10$ **b.** $x^6 + 2x^4 - 3x^3 + 9x^2 + 5$ **c.** $5x^3 - 5x^2 + 7x - 13$

d. $2x^4 + 3x^3 + 12x^2 - 4x + 14$ **e.** $x^5 + 13x^4 - 10$

2. a. $x^4 + 2x^2 + 2x + 4$ **b.** $x^6 - x^5 + x^3 + x^2 + 2$ **c.** $5x^7 - 4x^3 + 5x$

d. $10x^4 - 7x^2 + 20x + 5$ **e.** $2x^3 + 6x^2 - 10x + 15$

3. a. $x^3 + 7x^2 + 6x$ **b.** $x^3 - 7x^2 - 18x$ **c.** $x^3 + 8x^2 - 33x$ **d.** $2x^3 + 10x^2 + 12x$

e. $48x - 3x^3$ **f.** $5x^3 + 50x^2 + 80x$ **g.** $x^3 + 4x^2$ **h.** $2x^3 - 14x^2$

i. $-30x^3 - 270x^2$ **j.** $-7x^3 - 56x^2 - 112x$

4. a. $x^3 + 12x^2 + 41x + 42$ **b.** $x^3 - 3x^2 - 18x + 40$ **c.** $x^3 + 3x^2 - 36x + 32$ **d.** $x^3 - 6x^2 + 11x - 6$

e. $x^3 + 6x^2 - x - 6$ **f.** $x^3 + 5x^2 - 49x - 245$ **g.** $x^3 + 4x^2 - 137x - 660$ **h.** $x^3 + 3x^2 - 9x + 5$

i. $x^3 - 12x^2 + 21x + 98$ **j.** $x^3 + x^2 - x - 1$

5. a. $x^3 + 13x^2 + 26x - 112$ **b.** $3x^3 + 26x^2 + 51x - 20$ **c.** $4x^4 + 3x^3 - 37x^2 - 27x + 9$

d. $10x^3 - 49x^2 + 27x + 36$ **e.** $-6x^3 - 71x^2 - 198x + 35$ **f.** $21x^4 - 54x^3 - 144x^2 + 96x$

g. $54x^3 + 117x^2 - 72x$ **h.** $24x^3 - 148x^2 + 154x + 245$ **i.** $20x^4 - 39x^3 - 50x^2 + 123x - 54$

j. $4x^3 + 42x^2 + 146x + 168$

6. a. $x^3 + 6x^2 + 12x + 8$ **b.** $x^3 + 15x^2 + 75x + 125$

c. $x^3 - 3x^2 + 3x - 1$ **d.** $x^4 - 12x^3 + 54x^2 - 108x + 81$

e. $8x^3 - 72x^2 + 216x - 216$ **f.** $81x^4 + 432x^3 + 864x^2 + 768x + 256$

7. $(2a + 5b)x + (2b - 5c)$

8. $x^4 + (a - 4b)x^3 + (2a - 4ab + 3b^2)x^2 + (2a^2 - 2ab + 3ab^2)x - 2a^2b$

9. $a = 1, b = -12, c = 54, d = -108, e = 81$

10. $8x^3 - 45x^2 + 78x - 43$

11. $\frac{1}{8}(11x^3 - 105x^2 + 73x - 27)$

12. $a = 1$, $b = -2$ and $c = 1$

13. $a = 1$, $b = 4$ and $c = -1$

Exercise 17.4 Long division of polynomials

1. a. $x^2 + 2x,\ 9$ **b.** $x^2 + x + 3, -2$ **c.** $x^2 + 3x - 6, 19$ **d.** $x^2 - x + 5, -17$

e. $x^2 + 2x - 1, 6$ **f.** $x^2 + 4x - 6, 14$ **g.** $x^2 + 1, 2$ **h.** $x^2 + 5, -36$

i. $x^2 - x + 6, -11$ **j.** $x^2 + 4x - 17, 87$

2. a. $x^2 + 4x + 3, -3$ **b.** $x^2 + 4x + 13, 48$ **c.** $x^2 + 3x - 3, -11$ **d.** $x^2 - 3x + 7, 5$
e. $x^2 - 2x - 3, -17$ **f.** $x^2 - 6x + 3, -4$ **g.** $x^2 + 14x + 72, 359$ **h.** $x^2 + 8x + 27, 104$

3. a. $3x^2 - 7x + 20, -35$ **b.** $4x^2 - 8x + 18, -22$ **c.** $2x^2 - 3x + 3, 7$ **d.** $2x^2 - 9, 35$
e. $4x^2 + 2x - 3, -1$ **f.** $3x^2 + x - 1, -2$

4. a. $3x^2 - 2x + 1, 5$ **b.** $2x^2 + 5x - 6, -7$ **c.** $4x^2 - 7x - 2, -3$ **d.** $x^2 - 4x + 3, 8$
e. $x^2 + x - 6, -11$ **f.** $3x^2 + 2x + 1, 13$

5. a. $-x^2 - 5x - 2, -14$ **b.** $-3x^2 - 2x + 4, -3$ **c.** $-x^2 + 5x + 6, 9$ **d.** $-2x^2 + 7x - 1, 1$

6. a. $x^2 - x - 2, 3$ **b.** $x^2, -7$ **c.** $x^2 - x - 2, -8$ **d.** $-x^2 - x - 8, 0$
e. $5x - 2, 7$ **f.** $2x^2 - 2x + 10, -54$ **g.** $-2x^2 - 4x - 9, -16$ **h.** $-2x^2 + 4x - 1, 1$

7. a. $x^3 + 2x^2 + 5x - 2, -2$ **b.** $x^3 + 2x^2 - 9x - 18, 0$
c. $x^4 - 3x^3 + 6x^2 - 18x + 58, -171$ **d.** $2x^5 - 4x^4 + 7x^3 - 13x^2 + 32x - 69, 138$
e. $6x^3 + 17x^2 + 53x + 155, 465$ **f.** $x^3 - \frac{7}{3}x^2 + \frac{7}{9}x + 3\frac{20}{27}, -3\frac{20}{27}$

8. Quotient $= ax + (b + ad)$
Remainder $= Rc + d(b + ad)$

9. $q = \dfrac{2p}{c}$

10. $a = -3$

11. $a = 3, b = -5$

Exercise 17.5 Polynomial values

1. a. 10 **b.** 11 **c.** 18
d. 43 **e.** 3 **f.** −22
g. −77 **h.** $2a^3 - 3a^2 + 2a + 10$ **i.** $16b^3 - 12b^2 + 4b + 10$
j. $2x^3 + 9x^2 + 14x + 18$ **k.** $2x^3 - 21x^2 + 74x - 77$ **l.** $-128y^3 - 48y^2 - 8y + 10$

2. to **6.**

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem when divided by $(x - 1)$	Rem when divided by $(x - 2)$	Rem when divided by $(x + 1)$	Rem when divided by $(x + 2)$
a	4	15	0	−5	4	15	0	−5
b	10	28	−2	−8	10	28	−2	−8
c	3	11	−7	−21	3	11	−7	−21
d	−7	−19	5	−7	−7	−19	5	−7

7. a. $P(-8)$ **b.** $P(7)$ **c.** $P(a)$

8. a. −1 **b.** −6249 **c.** $2(a - 5)^5 + 1$ **d.** $-2(2a + 5)^5 + 1$

9. $b = 1, 4$

10. a. $-2a^3 - 3a^2 + a + 4$ **b.** $-2a^3 - 9a^2 - 11a - 1$

11. $c = 2$

12. $b = -2, c = 5$

Exercise 17.6 The remainder and factor theorems

1. a. −30 **b.** 0 **c.** 0
d. −24 **e.** −24 **f.** $k^3 + 3k^2 - 10k - 24$
g. $-n^3 + 3n^2 + 10n - 24$ **h.** $-27c^3 + 27c^2 + 30c - 24$

2. a. 58 **b.** −8 **c.** 11 **d.** −9 **e.** −202 **f.** 6 **g.** 158 **h.** −6 **i.** 35 **j.** 441

3. a. 6 **b.** 3 **c.** 1 **d.** −2 **e.** 2 **f.** 2 **g.** −5, 2 **h.** $a = -5, b = -3$

4. a. D **b.** C, D **c.** D **d.** A, C, D

5. a. $(x - 1)$ **b.** $(x - 3)$ or $(x - 2)$ **c.** $(x - 3)$ or $(x + 2)$
d. $(x - 6)$ or $(x + 4)$ or $(x + 5)$

6. Show $P(-2) = 0, P(3) = 0$ and $P(-5) = 0$.

7. a. Show $P(1) = 0$ **b.** Show $P(7) = 0$ **c.** Show $P(2) = 0$ **d.** Show $P(-2) = 0$
e. Show $P(-3) = 0$ **f.** Show $P(1) = 0$ **g.** Show $P(4) = 0$ **h.** Show $P(-5) = 0$

8. $a = 3, b = 2$

9. $a = -5, b = 41, (x + 3)$ and $(x - 5)$

Challenge 17.1

$k = -4$

Exercise 17.7 Factorising polynomials

1. a. $(x + 1)(x + 3)(x + 6)$ **b.** $(x + 1)(x + 2)(x + 5)$ **c.** $(x + 1)(x + 2)(x + 9)$ **d.** $(x + 1)(x + 3)(x + 4)$
e. $(x + 3)(x + 4)(x + 7)$ **f.** $(x + 2)(x + 3)(x + 7)$ **g.** $(x + 1)^2(x + 2)$ **h.** $(x + 2)^2(x + 3)$
i. $(x + 4)(x + 5)^2$ **j.** $x(x + 5)(x + 8)$ **k.** $x(x + 3)(x + 4)$ **l.** $x(x + 5)^2$
m. $x(x + 1)(x + 5)$ **n.** $x^2(x + 6)$

2. a. $(x - 1)(x + 1)^2$ **b.** $(x - 2)(x - 1)(x + 1)$ **c.** $(x + 1)^2(x + 5)$ **d.** $(x - 3)(x + 2)^2$
e. $(x + 1)(x + 4)^2$ **f.** $(x - 5)(x - 2)(x + 2)$ **g.** $(x - 1)(x + 1)(x + 2)$ **h.** $(x - 3)(x + 1)(x + 2)$
i. $(x - 1)(x + 2)^2$ **j.** $(x + 2)(x^2 - x + 3)$ **k.** $(x + 1)(x + 2)(x + 5)$ **l.** $(x - 3)(x + 1)(x + 3)$
m. $(x - 2)^2(x + 3)$ **n.** $(x - 4)(x + 5)(x + 8)$

3. a. $(2x + 3)(x - 1)(x + 2)$ **b.** $(3x - 1)(x + 1)(x + 4)$ **c.** $(3x + 2)(x - 2)(x + 2)$ **d.** $(4x + 3)(x + 3)(x + 5)$
e. $(5x - 1)(x + 1)^2$ **f.** $(x + 1)(x^2 + 1)$ **g.** $(x + 1)(2x + 3)^2$ **h.** $(x - 2)(2x - 1)(3x - 4)$
i. $(x + 4)(2x - 5)(5x + 2)$ **j.** $(7x - 2)(x - 2)(x + 4)$

4. a. $x(x - 2)(3x + 5)$ **b.** $2x(x + 1)(2x - 1)$ **c.** $3x(x - 4)(x + 2)$ **d.** $-2x(x + 3)^2$
e. $6x^2(x - 1)$ **f.** $-x(x + 4)(x + 3)$ **g.** $-(x - 1)(x + 1)(x + 3)$ **h.** $-2x(x - 3)(x - 2)$
i. $-(x + 2)(2x - 1)(3x - 2)$ **j.** $-(x - 2)^2(5x - 4)$ **k.** $-(x - 1)(x + 3)(x - 5)(x + 2)^2$

5. $(x - 1)(x + 2)(x + 2)(x - 3)$

6. $-(x - 2)(x + 2)(x + 3)(x - 4)(x - 5)$

7. $(x - p + (a + b))$

8. $a = -2, b = 4, (x - 1)^2\ (x + 1)^2\ (x - 2)$

9. $x(x - 1)(x + 1)(x - 2)(x - 3)$

Challenge 17.2

The other two factors are $(x - 4)$ and $(x + 1)$.

Exercise 17.8 Solving polynomial equations

1. a. $-2, 0, 2$ **b.** $-4, 0, 4$ **c.** $-5, 0, 5$ **d.** 3 **e.** $-5, 0$ **f.** $0, 2$
g. $-\sqrt{2}, 0, \sqrt{2}$ **h.** $-\frac{1}{4}, 0$ **i.** $0, \frac{1}{5}$ **j.** $0, 2, 3$ **k.** $0, 4$ **l.** $-7, 0, 1$
m. $0, 4, 5$ **n.** 0

2. a. $-4, 1, 4$ **b.** $-2, 3, 5$ **c.** $-5, 1, 5$ **d.** $-4, -2, 2$ **e.** $-1, 2, 3$ **f.** $-2, 1, 5$
g. $-3, -2, -1$ **h.** $-4, 5$ **i.** $-2, 1, 4$ **j.** $-7, 2, 3$ **k.** $-6, -\frac{1}{2}, -1$ **l.** $-\frac{1}{2}, \frac{3}{2}, 3$
m. $-3, -2, \frac{1}{2}$ **n.** $-2, -1, 1$

3. A, C

4. B

5. a. $-3, 2$ **b.** $-2, \frac{1}{3}, 6$ **c.** $-4, 2$ **d.** $-1, \frac{5}{2}$ **e.** $-4, -2, 1, 3$ **f.** $-2, -\frac{3}{2}, 3, 4$
g. $-3, -2, 1, 2$ **h.** $-4, -1, 0, 2$

6. a. $-2, 1, 4$ **b.** $-3, -1, 3$ **c.** $-3, 0, 2$ **d.** $-4, -3, 0, 2$ **e.** $-2, \frac{3}{2}, 2$ **f.** $-1, 1$

7. 3.75

8. $-2, 4, 8$

9. Proof — check with your teacher.

10. $x = -1, 4$ and 2

11. $z = -1, 1, -2$ and 3

17.9 Review

1. C

2. a. 5 **b.** $-\frac{1}{7}$ **c.** 3 **d.** x^5

3. C

4. C

5. **a.** $x^3 + 6x^2 - 36x + 40$ **b.** $x^3 + 10x^2 + 19x - 30$ **c.** $x^3 - 21x^2 + 147x - 343$ **d.** $-2x^3 - x^2 + 11x + 10$

6. **a.** D **b.** A

7. **a.** $x^2 - 16,\ 29$ **b.** $x^2 + 6x + 5,\ 8$ **c.** $-x^2 + 2x + 2,\ -9$

8. B

9. **a.** -4 **b.** 216 **c.** $-24a^3 + 8a^2 + 2a - 4$

10. -7

11. Show $P(-3) = 0$.

12. $(x - 10)(x + 4)(x + 10)$

13. **a.** $-\frac{1}{2},\ 3$ **b.** 2, 3, 4 **c.** $-2,\ 1,\ 2,\ 3$

14. Teacher to check.

For example, given $P(x) = x^3 - x^2 - 34x - 56$ and $P(7) = 0 \Rightarrow (x - 7)$ is a factor and 7 is a factor of 56.

15. $4x^2 - 12x + 9;\ x = -\frac{1}{2},\ \frac{7}{2}$

16. **a, b** Area $= (\frac{1}{2}\pi + 10)x^2 + (\pi + 10)x + \frac{\pi}{2}$

c Perimeter $= (12 + \pi)x + (2 + \pi)$

17. **a.** $(x + 4)^3$

b. $6(x + 4)^2$

c. $x = 2$

d. $-3,\ \dfrac{-3 + 3\sqrt{5}}{2},\ \dfrac{-3 - 3\sqrt{5}}{2}$

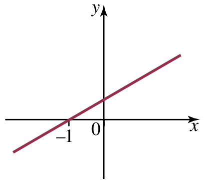
18. $mx + (n + mp);\ q + p(n + mp)$

19. $x^3 - (2 + n)x^2 + 3nx - (n^2 - n - 1)$

Investigation — Rich task

1. **a.** **i.** 1

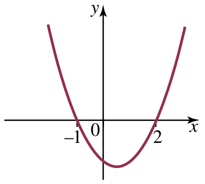
ii.



iii. The graph is linear and crosses the x-axis once (at $x = -1$).

b. **i.** 2

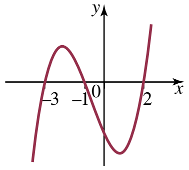
ii.



iii. The graph is quadratic and crosses the x-axis twice (at $x = -1$ and $x = 2$).

c. **i.** 3

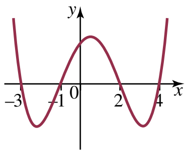
ii.



iii. The graph is a curve and crosses the x-axis 3 times (at $x = -1$, $x = 2$ and $x = -3$).

d. **i.** 4

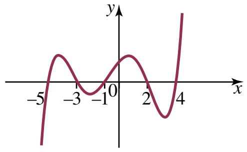
ii.



iii. The graph is a curve and crosses the x-axis 4 times (at $x = -1$, $x = 2$, $x = -3$ and $x = 4$).

e. **i.** 5

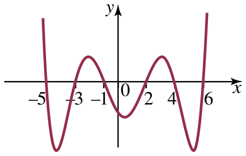
ii.



iii. The graph is a curve and crosses the x-axis 5 times (at $x = -1$, $x = 2$, $x = -3$, $x = 4$ and $x = -5$).

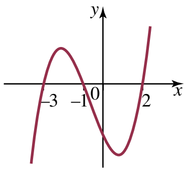
f. **i.** 6

ii.



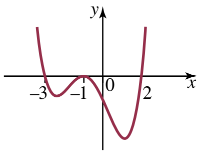
iii. The graph is a curve and crosses the x-axis 6 times (at $x = -1$, $x = 2$, $x = -3$, $x = 4$, $x = -5$ and $x = 6$).

2. a. i.



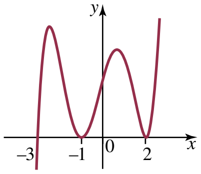
ii. Each factor is raised to the power 1. The polynomial is of degree 3 and the graph crosses the x-axis in 3 places (-3, -1 and 2).

b. i.



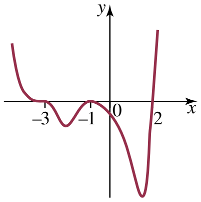
ii. The factor $(x + 1)$ is raised to the power 2 while the other two factors are raised to the power 1. The power 2 causes the curve not to cross the x-axis at $x = -1$ but to be curved back on itself.

c. i.



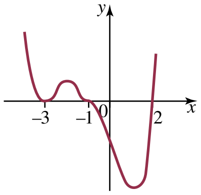
ii. The power 2 on the two factors $(x + 1)$ and $(x - 2)$ causes the curve to be directed back on itself and not to cross the x-axis at those two points ($x = -1$ and $x = 2$).

d. i.



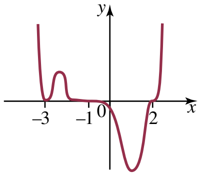
ii. The power 3 on the factor $(x + 3)$ causes the curve to run along the axis at that point then to cross the axis (at $x = -3$).

e. i.



ii. The power 3 on the factor $(x + 1)$ causes the curve to run along the axis at $x = -1$, then cross the axis. The power 4 on the factor $(x + 3)$ causes the curve to be directed back on itself without crossing the axis at $x = -3$.

f. i.



ii. The power 5 on the factor $(x + 1)$ causes the curve to run along the axis at $x = -1$, then cross the axis.

3. Answers will vary. Teacher to check. Possible answers could be as follows.

a. i. $y = 3x + 2$ **ii.** $y = 4$

b. i. $y = (x + 1)(x + 2)$ **ii.** $y = (x + 1)^2$

c. i. $y = (x + 1)(x + 2)(x + 3)$ **ii.** Not possible **iii.** $y = (x + 1)^2(x + 2)$

d. i. $y = (x + 1)(x + 2)(x + 3)(x + 4)$ **ii.** Not possible

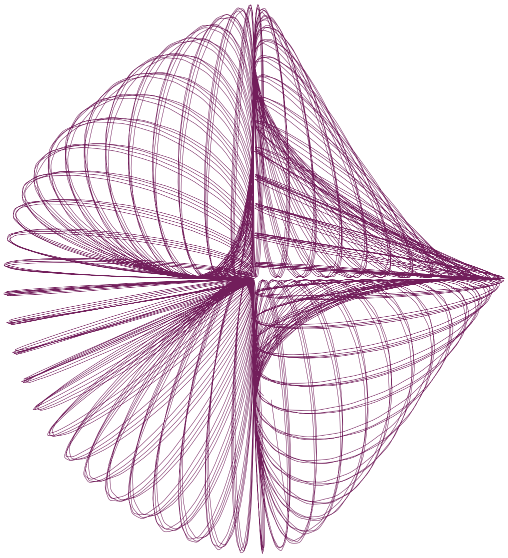
iii. $y = (x + 1)^2(x + 2)(x + 3)$, $y = (x + 1)^3(x + 2)$ **iv.** Not possible

4. If the power of the factor of a polynomial is an odd integer, the curve will pass through the x-axis. If the power is 1, the curve passes straight through. If the power is 3, 5 . . ., the curve will run along the x-axis before passing through it. On the other hand, an even power of a factor causes the curve to just touch the x-axis then move back on the same side of the x-axis.

TOPIC 18
Functions and relations

18.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.



18.1.1 Why learn this?

A relation is a set of ordered pairs; functions are special types of relations. It is important to understand relationships between variables in order to be able to model the relationships. Many different functions can be used to model events in the world. You have already studied lines of best fit, where straight lines are fitted to data in order to make predictions. Exponential functions are used to model growth and decay. There are many examples of functions used in engineering, science, finance, architecture and medicine.

18.1.2 What do you know?

assesson

1. **THINK** List what you know about functions and relations. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of functions and relations.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Amalie Noether (eles-2021)

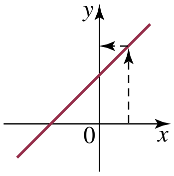
18.2 Functions and relations

18.2.1 Relations

- A **relation** is a set of ordered pairs of values such as all the points on the circle $x^2 + y^2 = 4$ or all the points on the exponential $y = 2^x$. Relations can be grouped into the following four categories.

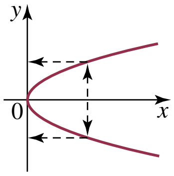
One-to-one relations

- A **one-to-one relation** exists if for any x-value there is only one corresponding y-value and vice versa. For example:



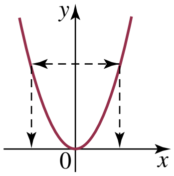
One-to-many relations

- A **one-to-many relation** exists if for any x-value there is more than one y-value, but for any y-value there is only one x-value. For example:



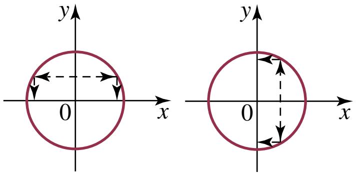
Many-to-one relations

- A **many-to-one relation** exists if there is more than one x-value for any y-value but for any x-value there is only one y-value. For example:



Many-to-many relations

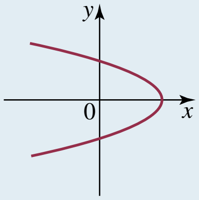
- A **many-to-many relation** exists if there is more than one x-value for any y-value and vice versa. For example:



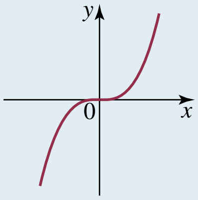
WORKED EXAMPLE 1

What type of relation does each graph represent?

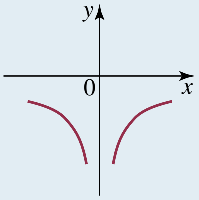
a



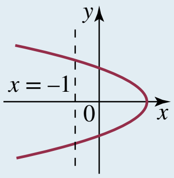
b



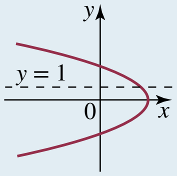
c



THINK	WRITE
a **1** For some x-values there is more than one y-value. A line through some x-values shows that 2 y-values are available.	**a** One-to-many relation



2 For any y-value there is only one x-value. A line through any y-value shows that only one x-value is available.



b **1** For any x-value there is only one y-value.

b One-to-one relation

2 For any y-value there is only one x-value.

c **1** For any x-value there is only one y-value.

c Many-to-one relation

2 For some y-values there is more than one x-value.

18.2.2 Functions

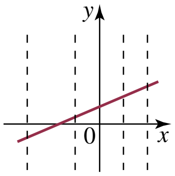
- Relations that are one-to-one or many-to-one are called **functions**. That is, a function is a relation where for any x-value there is at most one y-value.

Vertical line test

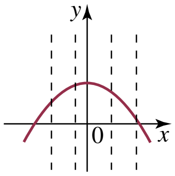
- To determine if a graph is a function, a vertical line is drawn anywhere on the graph. If it does not intersect with the curve more than once, then the graph is a function.

  For example, in each of the two graphs below, each vertical line intersects the graph only once.

1.



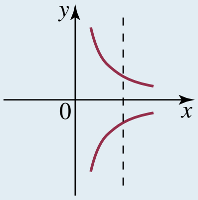
2.



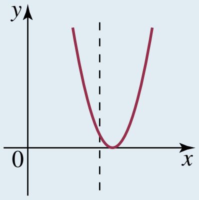
WORKED EXAMPLE 2

State whether or not each of the following relations are functions.

a



b



THINK	WRITE
a It is possible for a vertical line to intersect with the curve more than once.	**a** Not a function
b It is not possible for any vertical line to intersect with the curve more than once.	**b** Function

18.2.3 Function notation

- Consider the relation $y = 2x$, which is a function.

  The y-values are determined from the x-values, so we say 'y is a function of x', which is abbreviated to $y = f(x)$.

  So, the rule $y = 2x$ can also be written as $f(x) = 2x$.

  $$\text{If } x = 1, \text{ then } y = f(1) \\ = 2 \times 1 \\ = 2.$$

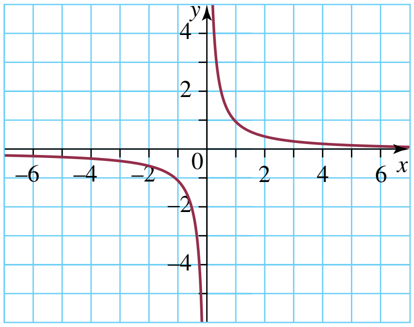
  $$\text{If } x = 2, \text{ then } y = f(2) \\ = 2 \times 2 \\ = 4, \text{ and so on.}$$

Domain and range

- The **domain** of a function is the set of all allowable values of x. It is sometimes referred to as the **maximal domain**.
- The **range** of a function is the set of y-values produced by the function.

  For example, the domain of the function $f(x) = 2x + 3$ is the set of all real numbers ($x \in R$), and the range is the set of all real numbers ($y \in R$).

  The domain of the function $f(x) = \dfrac{1}{x}$ is the set of all real numbers apart from 0 ($x \in R\backslash 0$), and the range is the set of all real numbers apart from 0 ($y \in R\backslash 0$).



Evaluating functions

- For a given function $y = f(x)$, the value of y when $x = 1$ is written as $f(1)$, the value of y when $x = 5$ is written as $f(5)$, the value of y when $x = a$ as $f(a)$, etc.

WORKED EXAMPLE 3 TI | CASIO

If $f(x) = x^2 - 3$, find:

a $f(1)$ **b** $f(a)$ **c** $3f(2a)$ **d** $f(a) + f(b)$ **e** $f(a + b)$.

THINK	**WRITE**
a 1 Write the rule.	**a** $f(x) = x^2 - 3$
2 Substitute $x = 1$ into the rule.	$f(1) = 1^2 - 3$
3 Simplify.	$= 1 - 3$ $= -2$
b 1 Write the rule.	**b** $f(x) = x^2 - 3$
2 Substitute $x = a$ into the rule.	$f(a) = a^2 - 3$

c **1** Write the rule.	**c** $f(x) = x^2 - 3$
2 Substitute $x = 2a$ into the rule and simplify.	$f(2a) = (2a)^2 - 3$
3 Multiply the answer by 3 and simplify.	$= 2^2a^2 - 3$ $= 4a^2 - 3$ $3f(2a) = 3(4a^2 - 3)$ $= 12a^2 - 9$
d **1** Write the rule.	**d** $f(x) = x^2 - 3$
2 Evaluate $f(a)$.	$f(a) = a^2 - 3$
3 Evaluate $f(b)$.	$f(b) = b^2 - 3$
4 Evaluate $f(a) + f(b)$.	$f(a) + f(b) = a^2 - 3 + b^2 - 3$ $= a^2 + b^2 - 6$
e **1** Write the rule.	**e** $f(x) = x^2 - 3$
2 Evaluate $f(a + b)$.	$f(a + b) = (a + b)^2 - 3$ $= (a + b)(a + b) - 3$ $= a^2 + 2ab + b^{2-}3$

18.2.4 Identifying features of functions

- We can identify features of certain functions by observing what happens to the function value (y value) when x approaches a very small value such as 0 ($x \rightarrow 0$) or a very large value such as $\infty (x \rightarrow \infty)$.

WORKED EXAMPLE 4

Describe what happens to these functions as the value of x increases, that is, as $x \rightarrow \infty$.

a $f(x) = x^2$ **b** $f(x) = 2^{-x}$ **c** $f(x) = \dfrac{1}{x} + 1$

THINK	WRITE
a **1** Write the function.	**a** $f(x) = x^2$
2 Substitute large x values into the function, such as $x = 10000$ and $x = 1000000$.	$f(10000) = 100000000$ $f(1000000) = 1 \times 10^{12}$
3 Write a conclusion.	As $x \rightarrow \infty$, $f(x)$ also increases; that is, $f(x) \rightarrow \infty$.
b **1** Write the function.	**b** $f(x) = 2^{-x}$
2 Substitute large x values into the function, such as $x = 10000$ and $x = 1000000$.	$f(10000) \approx 0$ $f(1000000) \approx 0$
3 Write a conclusion.	As $x \rightarrow \infty$, $f(x) \rightarrow 0$.
c **1** Write the function.	**c** $f(x) = \dfrac{1}{x} + 1$
2 Substitute large x values into the function, such as $x = 10000$ and $x = 1000000$.	$f(10000) = 1.0001$ $f(1000000) = 1.000001$
3 Write a conclusion.	As $x \rightarrow \infty$, $f(x) \rightarrow 1$.

Points of intersection

- If two functions are drawn on the one set of axes, there may be a point or points where the curves intersect. The function equations can be solved simultaneously to find the coordinates of these points of intersection.

WORKED EXAMPLE 5

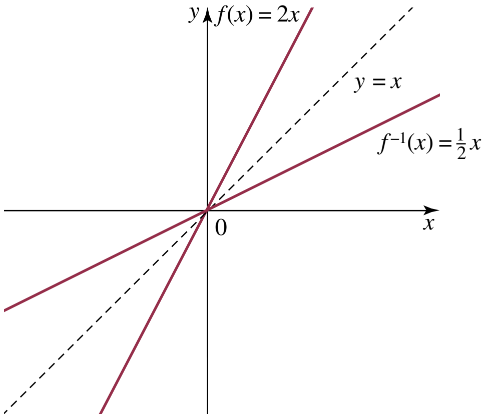
TI | CASIO

Find any points of intersection between $f(x) = 2x + 1$ and $g(x) = \frac{1}{x}$.

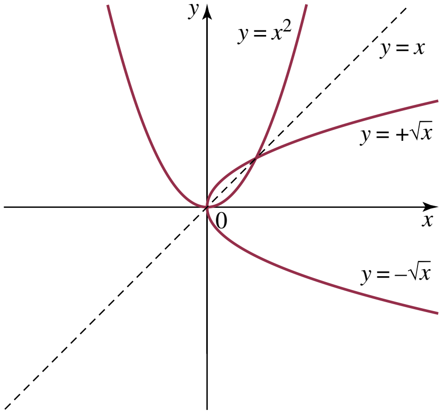
THINK	WRITE
1 Write the two equations.	$f(x) = 2x + 1$ $g(x) = \frac{1}{x}$
2 Points of intersection are common values between the two curves. To solve the equations simultaneously, equate both functions.	For points of intersection: $2x + 1 = \frac{1}{x}$
3 Rearrange the resulting equation and solve for x.	$2x^2 + x = 1$ $2x^2 + x - 1 = 0$ $(2x - 1)(x + 1) = 0$ $x = \frac{1}{2}$ or -1
4 Substitute the x values into either function to find the y values.	$f(\frac{1}{2}) = 2 \times \frac{1}{2} + 1 = 2$ $f(-1) = 2 \times -1 + 1 = -1$
5 Write the coordinates of the two points of intersection.	The points of intersection are $\left(\frac{1}{2}, 2\right)$ and $(-1, -1)$.

18.2.5 Inverse functions

- An inverse graph is created when a graph is reflected in the line $y = x$, the 45° line. Algebraically the resulting equation is created by interchanging x and y in the original equation.
- For example, take the function $f(x) = 2x$.
  Let $y = 2x$.
  Interchange x and y: $x = 2y$
  Make y the subject: $y = \frac{1}{2}x$
  Write the inverse function using function notation.
  The inverse of the function $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.



- Not all functions have inverses that are functions. For example, the inverse of the function $y = x^2$ is $x = y^2$ or $y = \pm\sqrt{x}$.

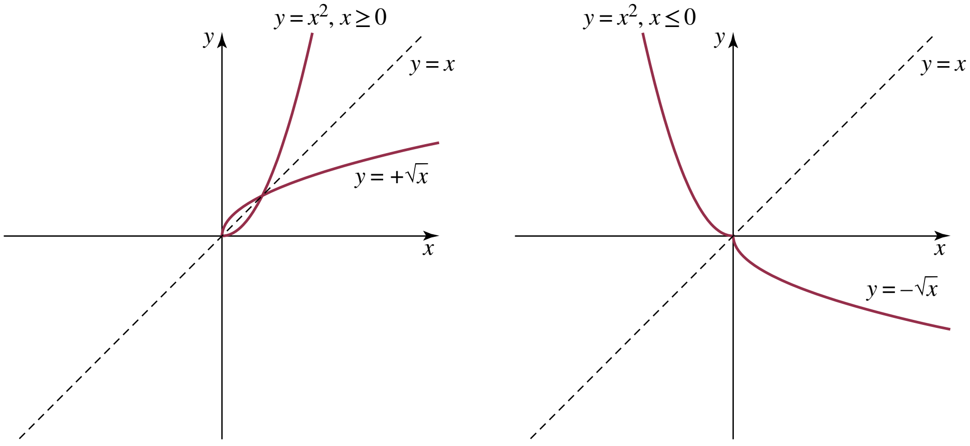


  - Notice that $y = \pm\sqrt{x}$ is not a function, as it is a one-to-many relation.
  - For this inverse to be a function, a restriction must be placed on the domain of the original quadratic function. Normally, either the part of the graph to the left of the turning point or the part to the right is chosen. The original function is re-stated as two functions:

$$y = x^2,\ x \geq 0 \text{ and } y = x^2,\ x \leq 0.$$

Thus the inverse functions are

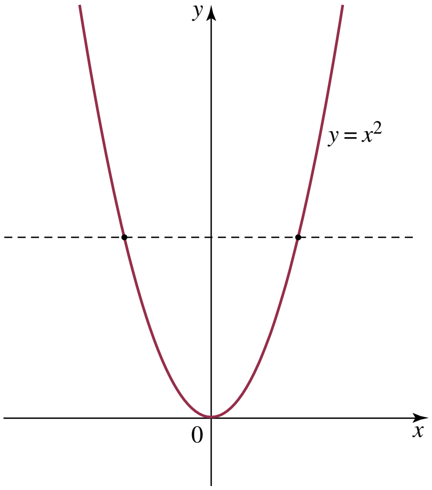
$$y = +\sqrt{x},\ x \geq 0 \text{ and } y = -\sqrt{x},\ x \geq 0.$$



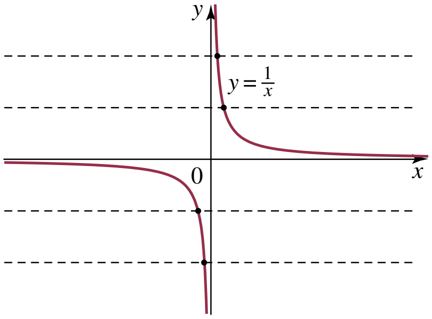
(a) **(b)**

The horizontal line test

- Not all functions have inverses that are functions. For a function to have an inverse function, it must be a one-to-one function. It was shown above that $f(x) = x^2$ does not have an inverse function unless restrictions are placed.
- A function has an inverse function when a horizontal line cannot be drawn that cuts through the graph more than once.
- The inverse of the function $f(x) = x^2$ is not a function, as its graph does not satisfy the horizontal line test.

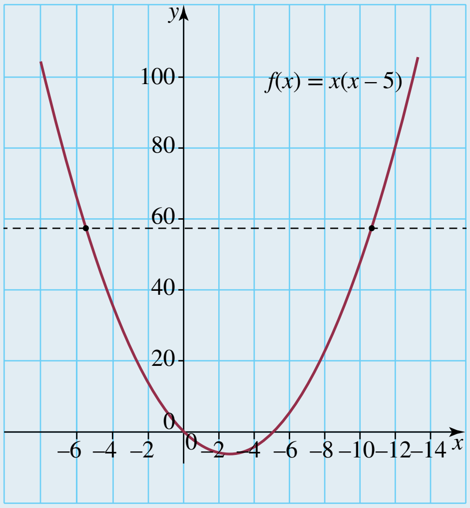


- The inverse of the function $f(x) = \dfrac{1}{x}$ is a function, as its graph satisfies the horizontal line test.



WORKED EXAMPLE 6

a i Show that the function $f(x) = x(x - 5)$ will have not have an inverse function.
ii Suggest a restriction that would result in an inverse function.
b i Show that the function $f(x) = x^2 + 4, x \geq 0$ will have an inverse function.
ii Determine the equation of the inverse function.

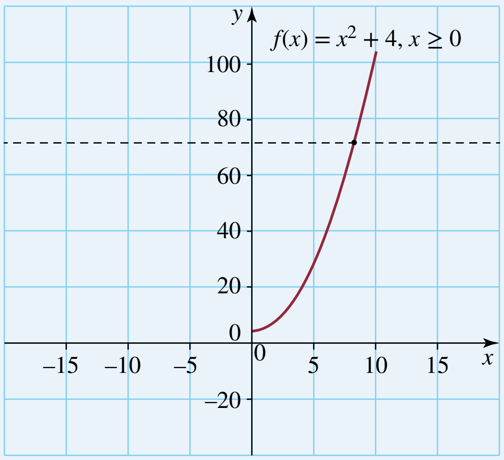
THINK	WRITE/DRAW
a i 1 Sketch the graph of $f(x) = x(x - 5)$.	**a i** 
2 Draw a dotted horizontal line(s) through the graph.	The graph does not satisfy the horizontal line test, so the function $f(x) = x(x - 5)$ will not have an inverse function.

ii Apply a restriction to the function so that it will have an inverse.

ii An inverse function will exist if
$f(x) = x(x - 5), x \leq 2.5$ or
$f(x) = x(x - 5), x \geq 2.5$.

b i 1 Sketch the graph of $f(x) = x^2 + 4, x \geq 0$.

b i



2 Draw a dotted horizontal line through the graph.

The graph satisfies the horizontal line test, so the function $f(x) = x^2 + 4, x \geq 0$ has an inverse function.

ii 1 Determine the equation of the inverse function by interchanging x and y and simplifying.

ii Let $y = x^2 + 4, x \geq 0$.
Interchange x and y.
$x = y^2 + 4$
Make y the subject.

$$\begin{aligned} x &= y^2 + 4 \\ x - 4 &= y^2 \\ \sqrt{x - 4} &= y \\ y &= \sqrt{x - 4} \end{aligned}$$

2 Write the answer in correct form, noting the domain.

The inverse of $f(x) = x^2 + 4$ is $f^{-1}(x) = \sqrt{x - 4}, x \geq 4$.

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- **Complete this digital doc:** SkillSHEET: Completing the square (doc-5381)
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- **Complete this digital doc:** WorkSHEET: Functions and relations (doc-14622)

Exercise 18.2 Functions and relations

Individual pathways

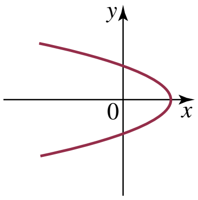
■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–9, 11, 12	Questions: 1–3, 6, 8, 10, 11, 12	Questions: 1–13

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** What type of relation does each graph represent?

a.



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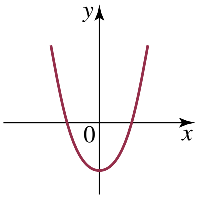

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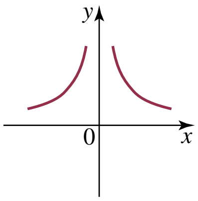

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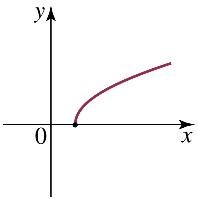
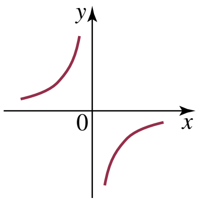
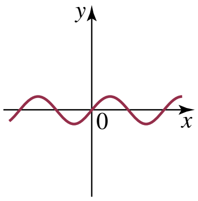


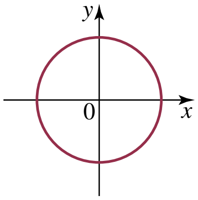
e.



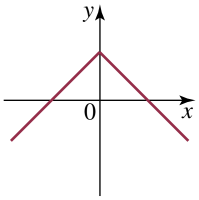
f.



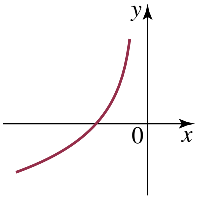
g.



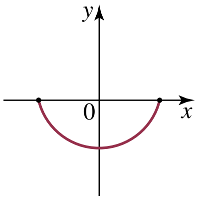
h.



i.



j.



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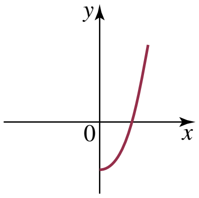

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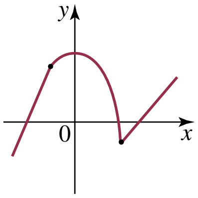


2. **WE2** **a.** Use the vertical line test to determine which of the relations in question **1** are functions.
   b. Which of these functions have inverses that are also functions?

3. **WE3** **a.** If $f(x) = 3x + 1$, find:
   i. $f(0)$ **ii.** $f(2)$ **iii.** $f(-2)$ **iv.** $f(5)$.
   b. If $g\,(x) = \sqrt{x + 4}$, find:
   i. $g(0)$ **ii.** $g(-3)$ **iii.** $g(5)$ **iv.** $g(-4)$.
   c. If $g(x) = 4 - \dfrac{1}{x}$, find:
   i. $g(1)$ **ii.** $g\left(\frac{1}{2}\right)$ **iii.** $g\left(-\frac{1}{2}\right)$ **iv.** $g\left(-\frac{1}{5}\right)$.

d. If $f(x) = (x + 3)^2$, find:

i. $f(0)$ **ii.** $f(-2)$ **iii.** $f(1)$ **iv.** $f(a)$.

e. If $h(x) = \dfrac{24}{x}$, find:

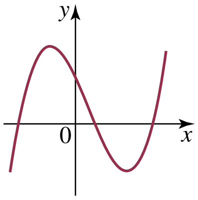
i. $h(2)$ **ii.** $h(4)$ **iii.** $h(-6)$ **iv.** $h(12)$.

Understanding

4. **MC** *Note:* There may be more than one correct answer.

Which of the following relations is a function?

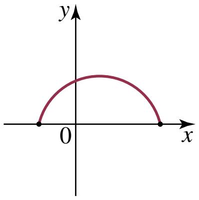
A.



B. $x^2 + y^2 = 9$

C. $y = 8x - 3$

D.



5. Which of the following relations are functions?

a. $y = 2x + 1$ **b.** $y = x^2 + 2$ **c.** $y = 2^x$

d. $x^2 + y^2 = 25$ **e.** $x^2 + 4x + y^2 + 6y = 14$ **f.** $y = -4x$

6. Given that $f(x) = \dfrac{10}{x} - x$, find:

a. $f(2)$ **b.** $f(-5)$ **c.** $f(2x)$

d. $f(x^2)$ **e.** $f(x + 3)$ **f.** $f(x - 1)$.

7. Find the value (or values) of x for which each function has the value given.

a. $f(x) = 3x - 4$, $f(x) = 5$ **b.** $g(x) = x^2 - 2$, $g(x) = 7$ **c.** $f(x) = \dfrac{1}{x}$, $f(x) = 3$

d. $h(x) = x^2 - 5x + 6$, $h(x) = 0$ **e.** $g(x) = x^2 + 3x$, $g(x) = 4$ **f.** $f(x) = \sqrt{8 - x}$, $f(x) = 3$

Reasoning

8. **WE4** Describe what happens to:

a. $f(x) = x^2 + 3$ as $x \to \infty$ **b.** $f(x) = 2^x$ as $x \to -\infty$ **c.** $f(x) = \dfrac{1}{x}$ as $x \to \infty$

d. $f(x) = x^3$ as $x \to -\infty$ **e.** $f(x) = -5^x$ as $x \to -\infty$.

9. **WE5** Find any points of intersection between the following curves.

a. $f(x) = 2x - 4$ and $g(x) = x^2 - 4$ **b.** $f(x) = -3x + 1$ and $g(x) = -\dfrac{2}{x}$

c. $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$ **d.** $f(x) = \frac{3}{4}x - 6\frac{1}{4}$ and $x^2 + y^2 = 25$

10. Find the equation of the inverse function of each of the following, placing restrictions on the original x values as required.

a. $f(x) = 2x - 1$ **b.** $f(x) = x^2 - 3$ **c.** $f(x) = (x - 2)^2 + 4$

11. **WE6** **a.** **i.** Show that the function $f(x) = x(x + 2)$ will not have an inverse function.

ii. Suggest a restriction that would result in an inverse function.

b. **i.** Show that the function $f(x) = -x^2 + 4$, $x \le 0$ will have an inverse function.

ii. Determine the equation of the inverse function.

Problem solving

12. Find the value(s) of x for which:

a. $f(x) = x^2 + 7$ and $f(x) = 16$

b. $g(x) = \dfrac{1}{x - 2}$ and $g(x) = 3$

c. $h(x) = \sqrt{8 + x}$ and $h(x) = 6$.

13. Consider the function defined by the rule $f: R \rightarrow R, f(x) = (x-1)^2 + 2$.

a. State the range of the function.
b. Determine the type of mapping for the function.
c. Sketch the graph of the function stating where it cuts the y-axis and its turning point.
d. Select a domain where x is positive such that f is a one-to-one function.
e. Determine the inverse function. Give the domain and range of the inverse function.
f. Sketch the graph of the inverse function on the same set of axes used for part **c**.
g. Find where f and the function $g(x) = x + 3$ intersect each other.

Reflection

How do you determine the difference between functions and relations?

CHALLENGE 18.1

Famous inverses

Draw and compare the graphs of the inverse functions $y = a^x$ and $y = \log_a x$, choosing various values for a. Explain why these graphs are inverses.

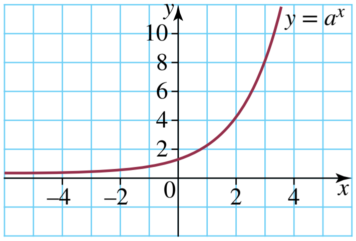
18.3 Exponential functions

18.3.1 Exponential functions

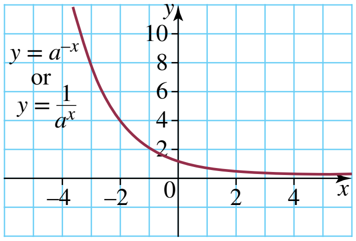
- **Exponential functions** can be used to model many real situations involving natural growth and decay.
- **Exponential growth** is when a quantity grows by a constant percentage in each fixed period of time. Examples of exponential growth include growth of investment at a certain rate of compound interest and growth in the number of cells in a bacterial colony.
- **Exponential decay** is when a quantity decreases by a constant percentage in each fixed period of time. Examples of exponential decay include yearly loss of value of an item (called depreciation) and radioactive decay.
- Both exponential growth and decay can be modelled by exponential functions of the type $y = ka^x (y = k \times a^x)$. The difference is in the value of the base a. When $a > 1$, there is exponential growth and when $0 < a < 1$ there is exponential decay.

  The value of k corresponds to the initial quantity that is growing or decaying.

Exponential growth



Exponential decay

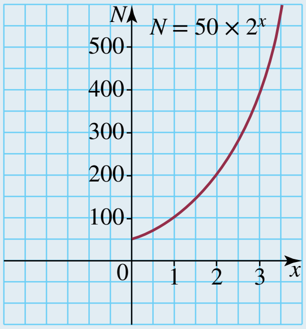
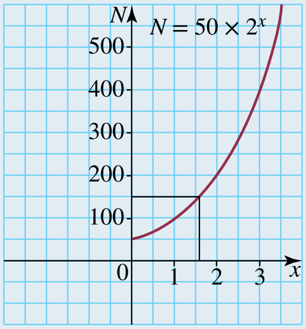


WORKED EXAMPLE 7

TI | CASIO

The number of bacteria, N, in a Petri dish after x hours is given by the equation $N = 50 \times 2^x$.

a **Determine the initial number of bacteria in the Petri dish.**
b **Determine the number of bacteria in the Petri dish after 3 hours.**
c **Draw the graph of the function of N against x.**
d **Use the graph to estimate the length of time it will take for the initial number of bacteria to treble.**

THINK	**WRITE/DRAW**
a **1** Write the equation.	**a** $N = 50 \times 2^x$
2 Substitute $x = 0$ into the given formula and evaluate. (Notice that this is the value of k for equations of the form $y = k \times a^x$.)	When $x = 0$, $N = 50 \times 2^0$ $= 50 \times 1$ $= 50$
3 Write the answer in a sentence.	The initial number of bacteria in the Petri dish is 50.
b **1** Substitute $x = 3$ into the formula and evaluate.	**b** When $x = 3$, $N = 50 \times 2^3$ $= 50 \times 8$ $= 400$
2 Write the answer in a sentence.	After 3 hours there are 400 bacteria in the Petri dish.
c **1** Draw a set of axes, labelling the horizontal axis as x and the vertical axis as N.	**c**
2 Plot the points generated by the answers to parts **a** and **b**.	
3 Calculate the value of N when $x = 1$ and $x = 2$ and plot the points generated.	At $x = 1$, $N = 50 \times 2^1$ $= 50 \times 2$ $= 100$ At $x = 2$, $N = 50 \times 2^2$ $= 50 \times 4$ $= 200$
4 Join the points plotted with a smooth curve.	
5 Label the graph.	
d **1** Determine the number of bacteria required.	**d** Number of bacteria $= 3 \times 50$ $= 150$
2 Draw a horizontal line from $N = 150$ to the curve and from this point draw a vertical line to the x-axis.	

3 The point on the x-axis will be the estimate of the time taken for the number of bacteria to treble.	
4 Write the answer in a sentence.	The time taken will be approximately 1.6 hours.

WORKED EXAMPLE 8

A new computer costs $3000. It is estimated that each year it will be losing 12% of the previous year's value.

a Determine the value, $\$V$, of the computer after the first year.

b Determine the value of the computer after the second year.

c Determine the equation that relates the value of the computer to the number of years, n, it has been used.

d Use your equation to determine the value of the computer in 10 years' time.

THINK	**WRITE**
a **1** State the original value of the computer.	**a** $V_0 = 3000$
2 Since 12% of the value is being lost each year, the value of the computer will be 88% or (100 − 12)% of the previous year's value. Therefore, the value after the first year (V_1) is 88% of the original cost.	$V_1 = 88\%$ of 3000 $= 0.88 \times 3000$ $= 2640$
3 Write the answer in a sentence.	The value of the computer after 1 year is \$2640.
b **1** The value of the computer after the second year, V_2, is 88% of the value after the first year.	**b** $V_2 = 88\%$ of 2640 $= 0.88 \times 2640$ $= 2323.2$
2 Write the answer in a sentence.	The value of the computer after the second year is \$2323.20.
c **1** The original value is V_0.	**c** $V_0 = 3000$
2 The value after the first year, V_1, is obtained by multiplying the original value by 0.88.	$V_1 = 3000 \times 0.88$
3 The value after the second year, V_2, is obtained by multiplying V_1 by 0.88, or by multiplying the original value, V_0, by $(0.88)^2$.	$V_2 = (3000 \times 0.88) \times 0.88$ $= 3000 \times (0.88)^2$
4 The value after the third year, V_3, is obtained by multiplying V_2 by 0.88, or V_0 by $(0.88)^3$.	$V_3 = (3000 \times 0.88)^2 \times 0.88$ $= 3000 \times (0.88)^3$
5 By observing the pattern we can generalise as follows: the value after the nth year, V_n, can be obtained by multiplying the original value, V_0, by 0.88 n times; that is, by $(0.88)^n$.	$V_n = 3000 \times (0.88)^n$
d **1** Substitute $n = 10$ into the equation obtained in part **c** to find the value of the computer after 10 years.	**d** When $n = 10$, $V_{10} = 3000 \times (0.88)^{10}$ $= 835.50$
2 Write the answer in a sentence.	The value of the computer after 10 years is \$835.50.

- Sometimes the relationship between the two variables closely resembles an exponential pattern, but cannot be described exactly by an exponential function. In such cases, part of the data are used to model the relationship with exponential growth or the decay function.

WORKED EXAMPLE 9

The population of a certain city is shown in the table below.

Year	**1985**	**1990**	**1995**	**2000**	**2005**	**2010**
Population ($\times 1000$)	**128**	**170**	**232**	**316**	**412**	**549**

Assume that the relationship between the population, P, and the year, x, can be modelled by the function $P = ka^x$, where x is the number of years after 1985. The value of P must be multiplied by 1000 in order to find the actual population.

a State the value of k, which is the population, in thousands, at the start of the period.

b Use a middle point in the data set to find the value of a, correct to 2 decimal places. Hence, write the formula, connecting the population, P, with the number of years, x, since 1985.

c For the years given, find the size of the population using the formula obtained in part b. Compare it with the actual size of the population in those years.

d Predict the population of the city in the years 2015 and 2020.

THINK	WRITE
a From the given table, state the value of k that corresponds to the population of the city in the year 1985.	**a** $k = 128$
b 1 Write the given formula for the population of the city.	**b** $P = ka^x$
2 Replace the value of k with the value found in **a**.	$P = 128 \times a^x$
3 Using a middle point of the data, replace x with the number of years since 1985 and P with the corresponding value.	Middle point is (1995, 232). When $x = 10$, $P = 232$, so $232 = 128 \times a^{10}$.
4 Solve the equation for a.	$a^{10} = \frac{232}{128}$ $a^{10} = 1.8125$ $a = \sqrt[10]{1.8125}$ $a = 1.0613...$
5 Round the answer to 2 decimal places.	$a \approx 1.06$
6 Rewrite the formula with this value of a.	So $P = 128 \times (1.06)^x$.
c 1 Draw a table of values and enter the given years, the number of years since 1985, x, and the population for each year, P. Round values of P to the nearest whole number.	**c** (table below)
2 Comment on the closeness of the fit.	The values for the population obtained using the formula closely resemble the actual data.

Year	1985	1990	1995	2000	2005	2010
x	0	5	10	15	20	25
P	128	171	229	307	411	549

d	**1** Find the value of x, the number of years after 1985.	**d** For the year 2015, $x = 30$.
	2 Substitute this value of x into the formula and evaluate.	$P = 128 \times (1.06)^{30}$ $= 735.166\,87...$
	3 Round to the nearest whole number.	$P \approx 735$
	4 Answer the question in a sentence.	The predicted population for 2015 is 735 000.
	5 Repeat for the year 2020.	For the year 2020, $x = 35$. $P = 128 \times (1.06)^{35}$ $= 983.819...$ $P \approx 984$ The predicted population for 2020 is 984 000.

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Exercise 18.3 Exponential functions

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3, 5, 6, 8–10, 13, 15, 16	Questions: 1, 2, 4, 6, 7, 9, 11, 12, 14–16	Questions: 1, 4–6, 9–17

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE7** The number of micro-organisms, N, in a culture dish after x hours is given by the equation $N = 2000 \times 3^x$.

a. Determine the initial number of micro-organisms in the dish.

b. Determine the number of micro-organisms in a dish after 5 hours.

c. Draw the graph of N against x.

d. Use the graph to estimate the number of hours needed for the initial number of micro-organisms to quadruple.

2. The value of an investment (in dollars) after n years is given by $A = 5000 \times (1.075)^n$.

a. Determine the size of the initial investment.

b. Determine the value of the investment (to the nearest dollar) after 6 years.

c. Draw the graph of A against n.

d. Use the graph to estimate the number of years needed for the initial investment to double.

3. **MC** **a.** The function $P = 300 \times (0.89)^n$ represents an:

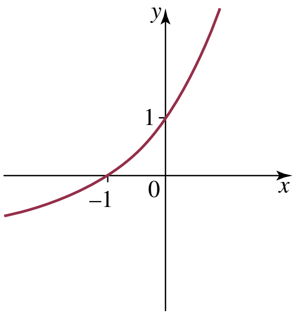
- **A.** exponential growth with the initial amount of 300
- **B.** exponential growth with the initial amount of 0.89
- **C.** exponential decay with the initial amount of 300
- **D.** exponential decay with the initial amount of 0.89
- **E.** exponential decay with the initial amount of 300×0.89

b. The relationship between two variables, A and t, is described by the function $A = 45 \times (1.095)^t$, where t is the time, in months, and A is the amount, in dollars. This function indicates:

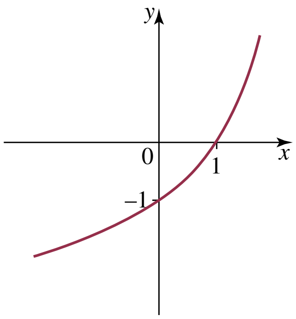
- **A.** a monthly growth of \$45
- **B.** a monthly growth of 9.5 cents
- **C.** a monthly growth of 1.095%
- **D.** a monthly growth of 9.5%
- **E.** a yearly growth of 9.5%

4. **MC** The graph of $y = 2^{x+1} - 1$ is best represented by:

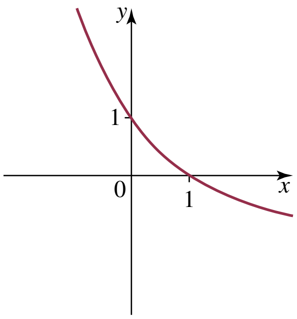
A.



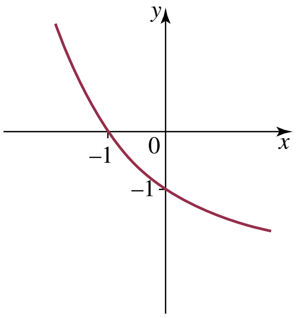
B.



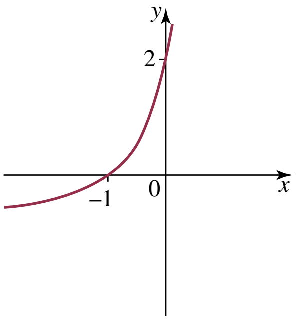
C.



D.



E.



5. **MC** The graph of $y = 3^{x-2} + 2$ has an asymptote and y-intercept respectively at:

- **A.** $y = 0,\ 2\frac{1}{9}$
- **B.** $y = 2,\ 2\frac{1}{9}$
- **C.** $y = 2,\ 2$
- **D.** $y = 2,\ 1\frac{8}{9}$
- **E.** $y = 0, 2$

Understanding

6. **WE8** A new washing machine costs \$950. It is estimated that each year it will be losing 7% of the previous year's value.

a. Calculate the value of the machine after the first year.

b. Calculate the value of the machine after the second year.

c. Determine the equation that relates the value of the machine, $\$V$, to the number of years, n, that it has been used.

d. Use your equation to find the value of the machine in 12 years' time.

7. A certain radioactive element decays in such a way that every 50 years the amount present decreases by 15%. In 1900, 120 mg of the element was present.

a. Calculate the amount present in 1950.

b. Calculate the amount present in the year 2000.

c. Determine the rule that connects the amount of the element present, A, with the number of 50-year intervals, t, since 1900.

d. Calculate the amount present in the year 2010. Round your answer to 3 decimal places.

e. Graph the function of A against t.

f. Use the graph to estimate the half-life of this element (that is, the number of years needed for half the initial amount to decay).

8. When a shirt made of a certain fabric is washed, it loses 2% of its colour.

a. Determine the percentage of colour that remains after:

i. two washes **ii.** five washes.

b. Write a function for the percentage of colour, C, remaining after w washings.

c. Draw the graph of C against w.

d. Use the graph to estimate the number of washes after which there is only 85% of the original colour left.

9. **WE9** The population of a certain country is shown in the table below.

Year	Population (in millions)
1990	118
1995	130
2000	144
2005	160
2010	178

Assume that the relationship between the population, P, and the year, n, can be modelled by the formula $P = ka^n$, where n is the number of years since 1990.

a. State the value of k.

b. Use the middle point of the data set to find the value of a rounded to 2 decimal places. Hence, write the formula that connects the two variables, P and n.

c. For the years given in the table, find the size of the population, using your formula. Compare the numbers obtained with the actual size of the population.

d. Predict the population of the country in the year 2035.

10. The temperature in a room (in degrees Celsius), recorded at 10-minute intervals after the air conditioner was turned on, is shown in the table below.

Time (min)	0	10	20	30	40
Temperature (°C)	32	26	21	18	17

Assume that the relationship between the temperature, T, and the time, t, can be modelled by the formula $T = ca^t$, where t is the time, in minutes, since the air conditioner was turned on.

a. State the value of c.

b. Use the middle point in the data set to find the value of a to 2 decimal places.

c. Write the rule connecting T and t.

d. Using the rule, find the temperature in the room 10, 20, 30 and 40 minutes after the air conditioner was turned on and compare your numbers with the recorded temperature. Comment on your findings. (Give answers correct to 1 decimal place.)

11. The population of a species of dogs (D) increases exponentially and is described by the equation $D = 60(1 - 0.6^t) + 3$, where t represents the time in years.

  a. Calculate the initial number of dogs.
  b. Calculate the number of dogs after 1 year.
  c. Determine the time taken for the population to reach 50 dogs.

12. Carbon-14 decomposes in such a way that the amount present can be calculated using the equation, $Q = Q_0(1 - 0.038)^t$, where Q is measured in milligrams and t in centuries.

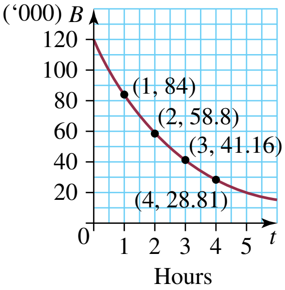
  a. If there is 40 mg present initially, how much is present in:
    i. 10 years' time **ii.** 2000 years' time?
  b. How many years will it take for there to be less than 10 mg?

Reasoning

13. Fiona is investing \$20 000 in a fixed term deposit earning 6% p.a. interest. When Fiona has \$30 000 she intends to put a deposit on a house.

  a. Determine an exponential function that will model the growth of Fiona's investment.
  b. Graph this function.
  c. Determine the length of time (correct to the nearest year) that it will take for Fiona's investment to grow to \$30 000.
  d. Suppose Fiona had been able to invest at 8% p.a. How much quicker would Fiona's investment have grown to the \$30 000 she needs?
  e. Alvin has \$15 000 to invest. Find the interest rate at which Alvin must invest his money, if his investment is to grow to \$30 000 in less than 8 years.

14. A Petri dish containing a bacteria colony was exposed to an antiseptic. The number of bacteria within the colony, B, over time, t, in hours is shown in the graph at right.



  a. Using the graph, predict the number of bacteria in the Petri dish after 5 hours.
  b. Using the points from the graph, show that if B can be modelled by the function B (in thousands) $= ab^t$, then $a = 120$ and $b = 0.7$.
  c. After 8 hours, another type of antiseptic was added to the Petri dish. Within three hours, the number of bacteria in the Petri dish had decreased to 50. If the number of bacteria decreased at a constant rate, show that the total of number of bacteria that had decreased within two hours was approximately 6700.

15. One hundred people were watching a fireworks display at a local park. As the fireworks were set off, more people started to arrive to see the show. The number of people, P, at time, t minutes, after the start of the fireworks display, can be modelled by the function, $P = ab^t$.

a. If after 5 minutes there were approximately 249 people, show that the number of people arriving at the park to watch the fireworks increased by 20% each minute.

b. The fireworks display lasted for 40 minutes. After 40 minutes, people started to leave the park. The number of people leaving the park could be modelled by an exponential function. 15 minutes after the fireworks ceased there were only 700 people in the park.

Derive an exponential function that can determine the number of people, N, remaining in the park after the fireworks had finished at any time, m, in minutes.

Problem solving

16. A hot plate used as a camping stove is cooling down. The formula that describes this cooling pattern is $T = 500 \times 0.5^t$ where T is the temperature in degrees Celsius and t is the time in hours.

a. What is the initial temperature of the stove?

b. What is the temperature of the stove after 2 hours?

c. Decide when the stove will be cool enough to touch and give reasons.

17. The temperature in a greenhouse is monitored when the door is left open. The following measurements are taken.

Time (min)	0	5	10	15	20
Temperature (°C)	45	35	27	21	16

a. State the initial temperature of the greenhouse.

b. Determine an exponential equation to fit the collected data.

c. What will the temperature be after 30 minutes?

d. Will the temperature ever reach 0 °C? Explain.

Reflection

What are the main differences between a graph modelling exponential growth compared with one showing decay?

18.4 Cubic functions

18.4.1 Cubic functions

- **Cubic functions** are polynomials where the highest power of x is 3. These include functions such as $y = x^3$ or $y = (x + 1)(x - 2)(x + 3)$.

WORKED EXAMPLE 10

Plot the graph of $y = x^3 - 1$ by completing a table of values.

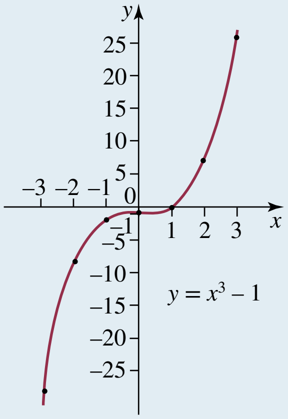
THINK

WRITE/DRAW

1 Prepare a table of values, taking x-values from -3 to 3. Fill in the table by substituting each x-value into the given equation to find the corresponding y-value.

x	-3	-2	-1	0	1	2	3
y	-28	-9	-2	-1	0	7	26

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.



WORKED EXAMPLE 11

Plot the curve of $y = x(x - 2)(x + 2)$ by completing a table of values.

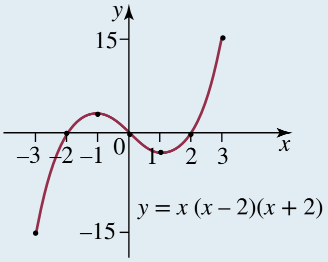
THINK

WRITE/DRAW

1 Prepare a table of values, taking x-values from -3 to 3. Fill in the table by substituting each x-value into the given equation.

x	-3	-2	-1	0	1	2	3
y	-15	0	3	0	-3	0	15

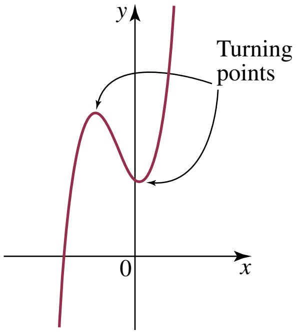
2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.

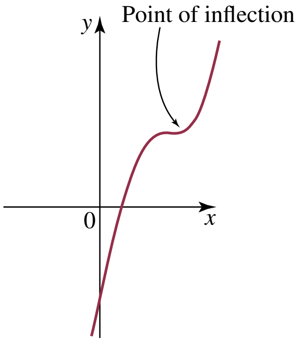


- A good sketch of a cubic function shows:
  1. x- and y-intercepts
  2. the behaviour of the function at extreme values of x, that is, as x approaches infinity ($x \to +\infty$) and as x approaches negative infinity ($x \to -\infty$)
  3. the general location of turning points.

Note that for cubic functions, ‘humps’ are not symmetrical as they are for parabolas, but are skewed to one side.

The graphs below show the two main types of cubic graph.



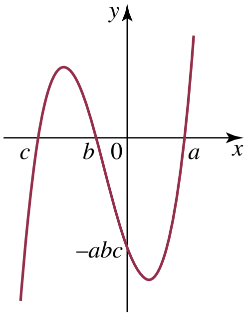


Consider the general factorised cubic $y = (x - a)\,(x - b)\,(x - c)$.

The x-intercepts occur when $y = 0$, that is, when $x = a$ or $x = b$ or $x = c$.

The y-intercept occurs when $x = 0$, that is, the y-intercept is

$$\begin{aligned} y &= (0 - a)(0 - b)(0 - c) \\ &= -abc \end{aligned}$$



WORKED EXAMPLE 12

TI | CASIO

Sketch the following, showing all intercepts.

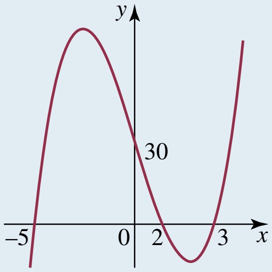
a $\boldsymbol{y = (x - 2)(x - 3)(x + 5)}$

b $\boldsymbol{y = (x - 6)^2(4 - x)}$

c $\boldsymbol{y = (x - 2)^3}$

THINK	WRITE/DRAW
a **1** Write the equation.	**a** $y = (x - 2)(x - 3)(x + 5)$
2 The y-intercept occurs where $x = 0$. Substitute $x = 0$ into the equation.	y-intercept: if $x = 0$, $y = (-2)(-3)(5)$ $= 30$ Point: $(0,\ 30)$
3 Solve $y = 0$ to find the x-intercepts.	x-intercepts: if $y = 0$, $x - 2 = 0,\ x - 3 = 0$ or $x + 5 = 0$ $x = 2,\ x = 3$ or $x = -5$ Points: $(2, 0), (3, 0), (-5, 0)$

4 Combine the above steps to sketch.



b **1** Write the equation.

b $y = (x - 6)^2(4 - x)$

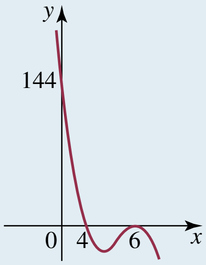
2 Substitute $x = 0$ to find the y-intercept.

y-intercept: if $x = 0$,
$y = (-6)^2(4)$
$\quad = 144$
Point: $(0, 144)$

3 Solve $y = 0$ to find the x-intercepts.

x-intercepts: if $y = 0$,
$x - 6 = 0$ or $4 - x = 0$
$x = 6$ or $x = 4$
Points: $(6, 0)$, $(4, 0)$

4 Combine all information and sketch the graph.
Note: The curve just touches the x-axis at $x = 6$. This occurs with a double factor such as $(x - 6)^2$.



c **1** Write the equation.

c $y = (x - 2)^3$

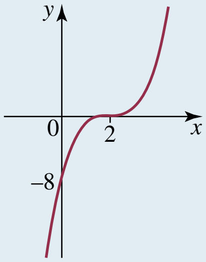
2 Substitute $x = 0$ to find the y-intercept.

y-intercept: if $x = 0$,
$y = (-2)^3$
$\quad = -8$

3 Solve $y = 0$ to find the x-intercepts.

x-intercept: if $y = 0$,
$x - 2 = 0$
$\quad x = 2$

4 Combine all information and sketch the graph.
Note: The point of inflection is at $x = 2$. This occurs with a triple factor such as $(x - 2)^3$.



Exercise 18.4 Cubic functions

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1a–f, 2a–f, 3–8	Questions: 1e–h, 2e–h, 3–8, 10	Questions: 1i–l, 2i–l, 3–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE10, 11 , 12 Sketch the following, showing all intercepts.

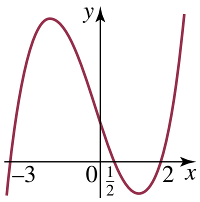
a. $y = (x - 1)(x - 2)(x - 3)$ **b.** $y = (x - 3)(x - 5)(x + 2)$
c. $y = (x + 6)(x + 1)(x - 7)$ **d.** $y = (x + 4)(x + 9)(x + 3)$
e. $y = (x + 8)(x - 11)(x + 1)$ **f.** $y = (2x - 6)(x - 2)(x + 1)$
g. $y = (2x - 5)(x + 4)(x - 3)$ **h.** $y = (3x + 7)(x - 5)(x + 6)$
i. $y = (4x - 3)(2x + 1)(x - 4)$ **j.** $y = (2x + 1)(2x - 1)(x + 2)$
k. $y = (x - 3)^2(x - 6)$ **l.** $y = (x + 2)(x + 5)^2$

2. Sketch the following (a mixture of positive and negative cubics).

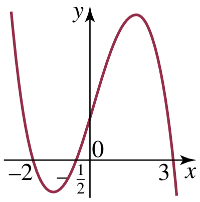
a. $y = (2 - x)(x + 5)(x + 3)$ **b.** $y = (1 - x)(x + 7)(x - 2)$
c. $y = (x + 8)(x - 8)(2x + 3)$ **d.** $y = (x - 2)(2 - x)(x + 6)$
e. $y = x(x + 1)(x - 2)$ **f.** $y = -2(x + 3)(x - 1)(x + 2)$
g. $y = 3(x + 1)(x + 10)(x + 5)$ **h.** $y = -3x(x - 4)^2$
i. $y = 4x^2(x + 8)$ **j.** $y = (5 - 3x)(x - 1)(2x + 9)$
k. $y = (6x - 1)^2(x + 7)$ **l.** $y = -2x^2(7x + 3)$

3. MC Which of the following is a reasonable sketch of $y = (x + 2)(x - 3)(2x + 1)$?

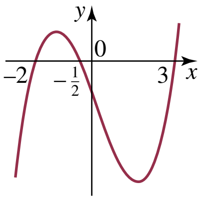
A.



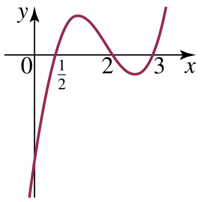
B.



C.

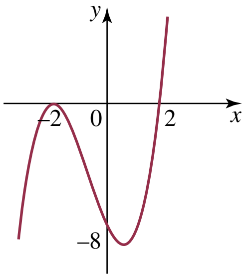


D.



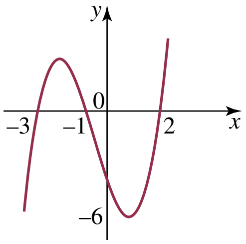
4. MC The graph shown could be that of:

A. $y = x^2(x + 2)$
B. $y = (x + 2)^3$
C. $y = (x - 2)(x + 2)^2$
D. $y = (x - 2)^2(x + 2)$

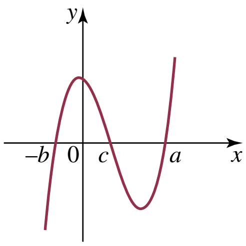


5. MC The graph at right has the equation:

A. $y = (x + 1)(x + 2)(x + 3)$
B. $y = (x + 1)(x - 2)(x + 3)$
C. $y = (x - 1)(x + 2)(x + 3)$
D. $y = (x - 1)(x + 2)(x - 3)$



6. **MC** If a, b and c are positive numbers, the equation of the graph shown at right could be:



A. $y = (x - a)(x - b)(x - c)$

B. $y = (x + a)(x - b)(x + c)$

C. $y = (x + a)(x + b)(x - c)$

D. $y = (x - a)(x + b)(x - c)$

Understanding

7. Sketch the graph of each of the following.

a. $y = x(x - 1)^2$

b. $y = -(x + 1)^2(x - 1)$

c. $y = (2 - x)(x^2 - 9)$

d. $y = -x(1 - x^2)$

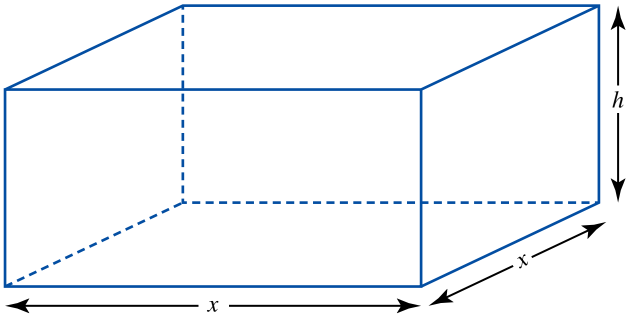
Reasoning

8. The function $f(x) = x^3 + ax^2 + bx + 4$ has x-intercepts at $(1, 0)$ and $(-4, 0)$. Find the values of a and b.

9. The graphs of the functions $f(x) = x^3 + (a + b)x^2 + 3x - 4$ and $g(x) = (x - 3)^3 + 1$ touch. Express a in terms of b.

Problem solving

10. A girl uses 140 cm of wire to make a frame of a cuboid with a square base as shown.



The base length of the cuboid is x cm and the height is h cm.

a. Explain why the volume cm^3 is given by $V = 35x^2 - 2x^3$.

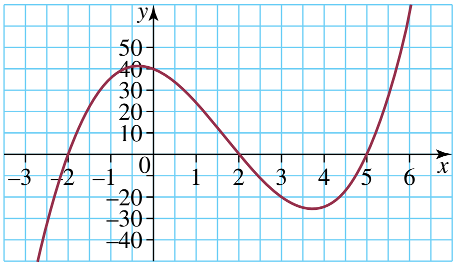
b. What possible values can x assume?

c. Find the volume of the cuboid when the base area is 81cm^2.

d. Sketch the graph of V versus x.

e. Use technology to determine the coordinates of the maximum turning point. Explain what these coordinates mean.

11. Find the rule for the cubic function shown.



Reflection

Is it possible to get symmetrical 'humps' for the graphs of cubic functions?

18.5 Quartic functions

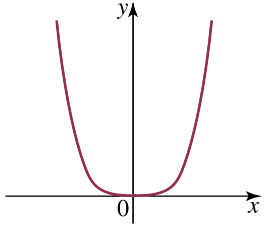
- **Quartic functions** are polynomials where the highest power of x is 4. These include functions such as $y = x^4$ or $y = (x + 1)(x - 2)(x + 3)(x - 5)$.
- It is necessary, when sketching the graphs of quartic functions, to find all the intercepts on both the x- and y-axes.

18.5.1 Basic shapes of quartic graphs

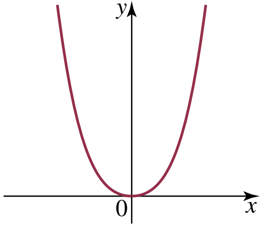
- The direction of a quartic graph is determined by the coefficient of the x^4 term. This is similar to the effect the coefficient of x^2 has on the shape of a parabola. Consider the coefficient of x^4 to be a.

When *a* is positive ($a > 0$)

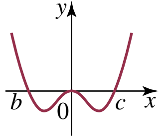
1. $y = ax^4$



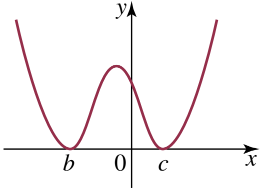
2. $y = ax^4 + cx^2,\ c \geq 0$



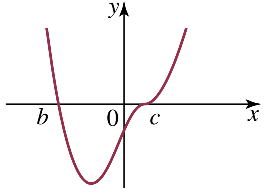
3. $y = ax^2(x - b)(x - c)$



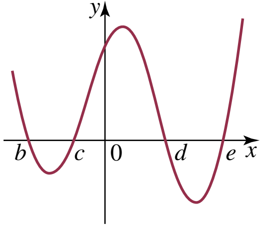
4. $y = a(x - b)^2(x - c)^2$



5. $y = a(x - b)(x - c)^3$

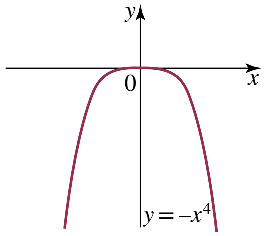


6. $y = a(x - b)(x - c)(x - d)(x - e)$



When *a* is negative ($a < 0$)

If a was negative in each of the previous graphs, they would be reflected in the x-axis.



WORKED EXAMPLE 13

TI | CASIO

Sketch the graph of $y = x^4 - 2x^3 - 7x^2 + 8x + 12$, showing all intercepts.

THINK	WRITE/DRAW
1 Find the y-intercept.	When $x = 0, y = 12$. The y-intercept is 12.
2 Let $P(x) = y$.	Let $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$.

3 Find two linear factors of the quartic expressions, if possible, using the factor theorem.

$$\begin{aligned} P(1) &= (1)^4 - 2(1)^3 - 7(1)^2 + 8(1) + 12 \\ &= 12 \\ &\neq 0 \\ P(-1) &= (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12 \\ &= 0 \end{aligned}$$

$(x + 1)$ is a factor.

$$\begin{aligned} P(2) &= (2)^4 - 2(2)^3 - 7(2)^2 + 8(2) + 12 \\ &= 0 \end{aligned}$$

$(x - 2)$ is a factor.

4 Find the product of the two linear factors.

$(x + 1)(x - 2) = x^2 - x - 2$

5 Use long division to divide the quartic by the quadratic factor $x^2 - x - 2$.

$$\begin{array}{rl} & x^2 - x - 6 \\ x^2 - x - 2 & \overline{)\,x^4 - 2x^3 - 7x^2 + 8x + 12} \\ & \underline{x^4 - x^3 - 2x^2} \\ & \quad - x^3 - 5x^2 + 8x \\ & \quad \underline{- x^3 + x^2 + 2x} \\ & \qquad - 6x^2 + 6x + 12 \\ & \qquad \underline{- 6x^2 + 6x + 12} \\ & \qquad\qquad\qquad 0 \end{array}$$

6 Express the quartic in factorised form.

$$\begin{aligned} y &= (x + 1)(x - 2)(x^2 - x - 6) \\ &= (x + 1)(x - 2)(x - 3)(x + 2) \end{aligned}$$

7 To find the x-intercepts, solve $y = 0$.

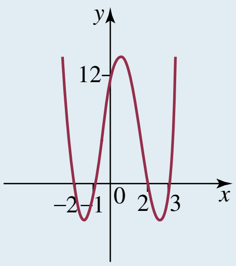
If $0 = (x + 1)(x - 2)(x - 3)(x + 2)$

$x = -1, 2, 3, -2.$

8 State the x-intercepts.

The x-intercepts are $-2, -1, 2, 3$.

9 Sketch the graph of the quartic.



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Complete this digital doc: WorkSHEET: Cubic and quartic functions (doc-14624)

Exercise 18.5 Quartic functions

assessON

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1a–d, 2a–d, 3–6, 9	Questions: 1c–f, 2c–f, 3, 4, 6, 7, 9	Questions: 1d–h, 2d–h, 3–10

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

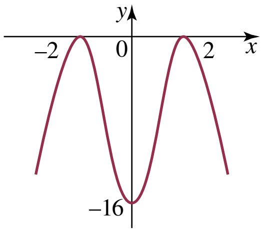
1. **WE13** Sketch the graph of each of the following showing all intercepts. You may like to verify the shape of the graph using a graphics calculator or another form of digital technology.
   a. $y = (x - 2)(x + 3)(x - 4)(x + 1)$
   b. $y = (x^2 - 1)(x + 2)(x - 5)$
   c. $y = 2x^4 + 6x^3 - 16x^2 - 24x + 32$
   d. $y = x^4 + 4x^3 - 11x^2 - 30x$
   e. $y = x^4 + 4x^3 - 12x - 9$
   f. $y = x^4 - 4x^2 + 4$
   g. $y = 30x - 37x^2 + 15x^3 - 2x^4$
   h. $y = 6x^4 + 11x^3 - 37x^2 - 36x + 36$
2. Sketch each of the following.
   a. $y = x^2(x - 1)^2$
   b. $y = -(x + 1)^2(x - 4)^2$
   c. $y = -x(x - 3)^3$
   d. $y = (2 - x)(x - 1)(x + 1)(x - 4)$
   e. $y = (x - a)(b - x)(x + c)(x + d), \quad a, b, c, d > 0$

Understanding

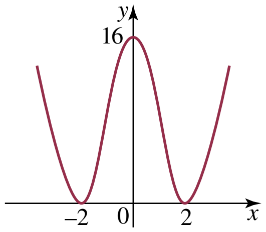
3. **MC** A quartic touches the x-axis at $x = -3$ and $x = 2$. It crosses the y-axis at $y = -9$. A possible equation is:
   A. $y = \frac{1}{4}(x + 3)^2(x - 2)^2$
   B. $y = -\frac{1}{6}(x + 3)^3(x - 2)$
   C. $y = -\frac{3}{8}(x + 3)(x - 2)^3$
   D. $y = -\frac{1}{4}(x + 3)^2(x - 2)^2$

4. **MC** Consider the function $f(x) = x^4 - 8x^2 - 16$.
   a. When factorised, $f(x)$ is equal to:
   A. $(x + 2)(x - 2)(x - 1)(x + 4)$
   B. $(x + 3)(x - 2)(x - 1)(x + 1)$
   C. $(x - 2)^3(x + 2)$
   D. $(x - 2)^2(x + 2)^2$
   b. The graph of $f(x)$ is best represented by:

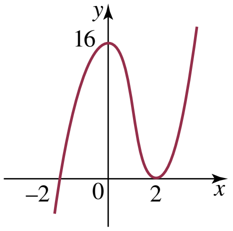
A.



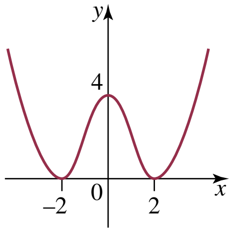
B.



C.



D.



Reasoning

5. Sketch the graph of each of the following functions.
   a. $y = x(x - 1)^3$
   b. $y = (2 - x)(x^2 - 4)(x + 3)$
   c. $y = x^4 - x^2$
   d. $y = 9x^4 - 30x^3 + 13x^2 + 20x + 4$
   e. $y = -(x - 2)^2(x + 1)^2$
   f. $y = x^4 - 6x^2 - 27$
   g. $y = (x + 2)^3(x - 3)$
   h. $y = 4x^2 - x^4$
   Verify your answers using a graphics calculator.
6. The function $f(x) = x^4 + ax^3 - 4x^2 + bx + 6$ has x-intercepts $(2, 0)$ and $(-3, 0)$. Find the values of a and b.

7. The functions $y = (a - 2b)x^4 - 3x - 2$ and $y = x^4 - x^3 + (a + 5b)x^2 - 5x + 7$ both have an x-intercept of 1. Find the value of a and b.

8. Patterns emerge when we graph polynomials with repeated factors, that is, polynomials of the form $P(x) = (x - a)^n,\ n > 1$. Describe what happens if:
   - **a.** n is even
   - **b.** n is odd.

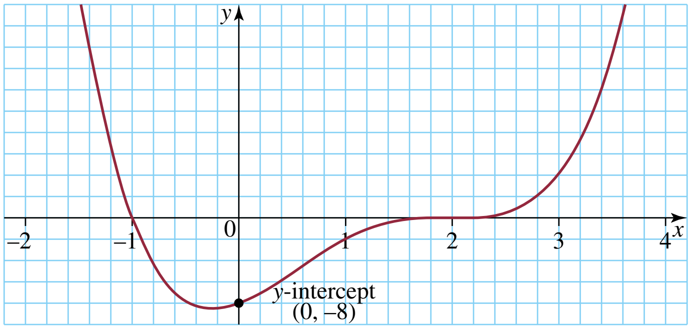
Problem solving

9. A carnival ride has a piece of the track modelled by the rule

$$h = -\frac{1}{300}x(x - 12)^2(x - 20) + 15,\quad 0 \le x \le 20$$

where x metres is the horizontal displacement from the origin and h metres is the vertical displacement of the track above the horizontal ground.
   - **a.** How high above the ground level is the track at the origin?
   - **b.** Use technology to sketch the function. Give the coordinates of any stationary points (that is, turning points or points of inflection).
   - **c.** How high above ground level is the track when $x = 3$?

10. Find the rule for the quartic function shown.



Reflection

What are the basic differences between cubic and quartic functions?

CHALLENGE 18.2

Two functions $f(x)$ and $g(x)$ are inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$. The domain of $f(g(x))$ must be the domain of $g(x)$, and the domain of $g(f(x))$ must be the domain of $f(x)$. Using this information, show that $f(x) = x^2,\ x \ge 0$ and $g(x) = \sqrt{x},\ x \ge 0$ are inverses of each other.

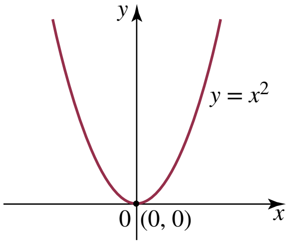
18.6 Transformations

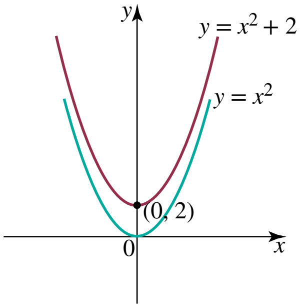
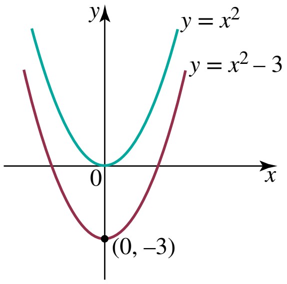
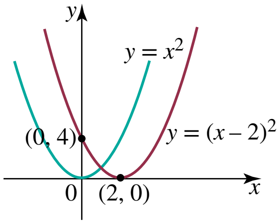
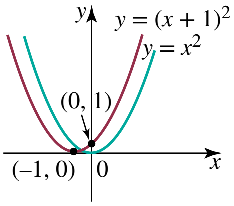
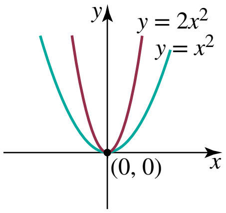
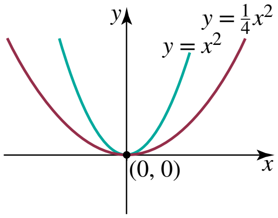
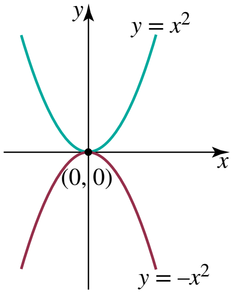
18.6.1 Transformations

- Once the basic shape of the graph of a particular function or relation is known, it is not difficult to predict the shape of a related function, which is a transformation of the basic function or relation. **Transformations** of parabolas have been dealt with previously, but for the sake of comparison with other functions, they are included in this chapter. Other functions and relations considered are circles, hyperbolas, exponential functions, cubic and quartic functions. Below is a summary of transformations of functions discussed previously.

18.6.2 Quadratic functions

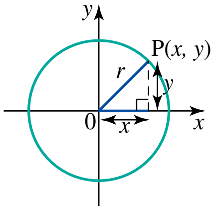
- The basic quadratic function is $y = x^2$. The shape of its graph is:

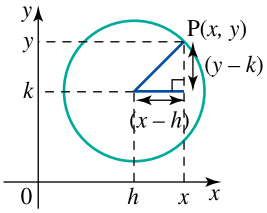


Transformation	Description	Graph
Vertical translation	Adding or subtracting a constant to $y = x^2$ moves the curve up or down the y-axis.	 
Horizontal translation	If the graph of $y = x^2$ is translated b units horizontally, the equation becomes $y = (x - b)^2$.	 
Dilation	If the graph of $y = x^2$ is dilated by a factor of a, the graph becomes narrower if $a > 1$ and wider if $0 < a < 1$.	 
Reflection	If the x^2 term is positive, the graph is concave up, while if there is a negative sign in front of the x^2 term, the graph is concave down.	

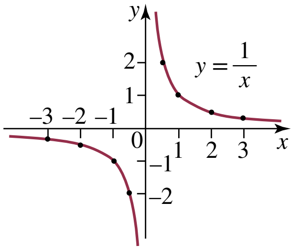
18.6.3 Circles

- The equation of a circle (relation) with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

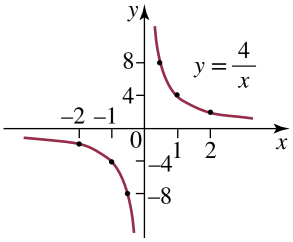
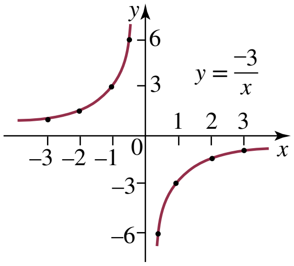


Transformation	Description	Graph
Translation	If the circle is translated h units to the right, parallel to the x-axis, and k units upwards, parallel to the y-axis, the equation of the circle, centre (h, k) becomes $(x - h)^2 + (y - k)^2 = r^2$.	

18.6.4 Hyperbolas

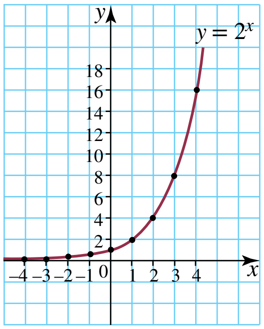


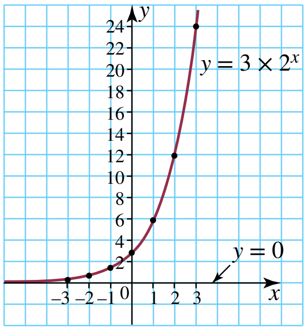
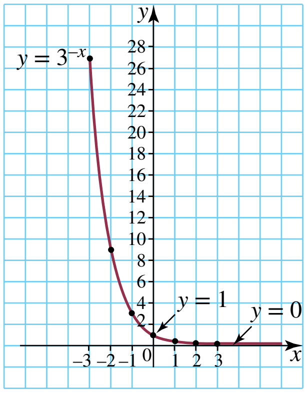
- The hyperbola is a function of the form $xy = k$ or $y = \frac{k}{x}$.
- The graph of $y = \frac{1}{x}$ has the shape

Transformation	Description	Graph
Dilation	Graphs of the form $y = \frac{k}{x}$ are the same basic shape as $y = \frac{1}{x}$, with y-values dilated by a factor of k.	
Negative values of k	Negative values of k cause the graph to be reflected across the y-axis.	

18.6.5 Exponential functions

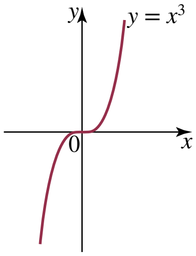
- These functions are of the form $y = a^x$, where $a \neq 1$. The basic shape has a y-intercept of 1.

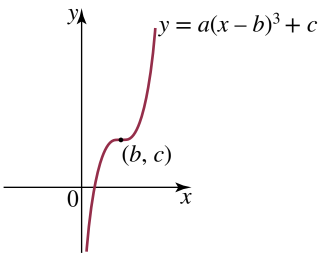


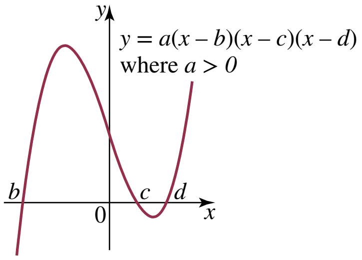
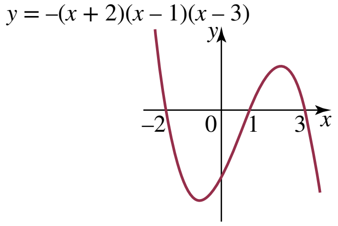
Transformation	Description	Graph
Functions for the form $y = k \times ax$	Multiplying by a factor of k causes the y-intercept to move to the point $(0, k)$.	
Functions with a negative exponent	This causes the graph to be reflected in the y-axis.	

18.6.6 Cubic functions

- The basic form of a cubic function is $y = x^3$. This can also be expressed in the form $y = a(x - b)^3 + c$, where $a = 1$, $b = 0$ and $c = 0$.

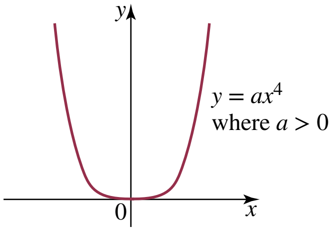


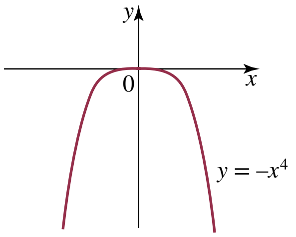
Transformation	Description	Graph
Translation	If $a \neq 1$, $b \neq 0$ and $c \neq 0$, the graph is translated $+b$ units in the x direction, $+c$ units in the y direction, and dilated by a factor of a in the y direction.	

Reflection	The cubic function can be expressed in factor form as $y = a(x - b)(x - c)(x - d)$, where b, c and d are the x-intercepts. If the value of a is negative, this causes the curve to be reflected in the x-axis.	 

18.6.7 Quartic functions

- The basic form of the quartic function $y = ax^4$, when a is positive, has the following shape.



Transformation	Description	Graph
Reflection	Negative values of a cause the graph to be reflected in the x-axis.	

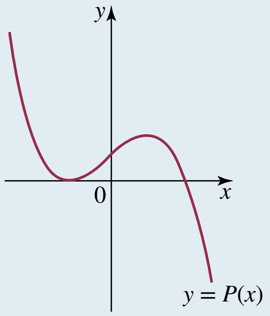
18.6.8 Transformation in general polynomials

- With knowledge of the transformations that occur in the functions just discussed, it is possible to generate many other graphs without knowing the equation of the original function. Consider a basic polynomial $y = P(x)$ and what happens to the shape of the curve as the function is changed.

WORKED EXAMPLE 14

Use the sketch of $y = P(x)$ shown at right to sketch:

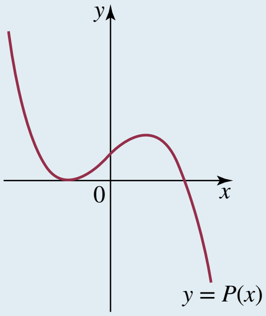
a $y = P(x) + 1$
b $y = P(x) - 1$
c $y = -P(x)$.



THINK **WRITE/DRAW**

a **1** Sketch the original $y = P(x)$.

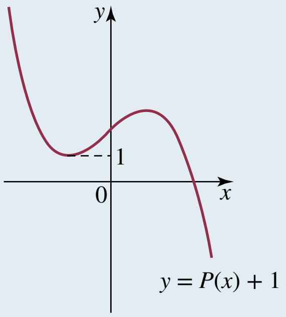
a



2 Consider the x-values. They remain unchanged — there is no horizontal translation.

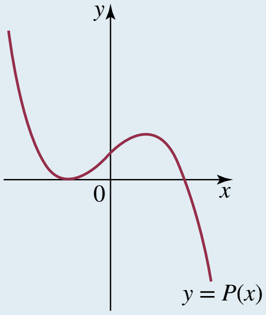
3 Consider the y-values. They are increased by 1 — the curve is shifted up 1 unit.

4 Sketch the graph of $y = P(x) + 1$ using a similar scale to the original.



b **1** Sketch the original $y = P(x)$.

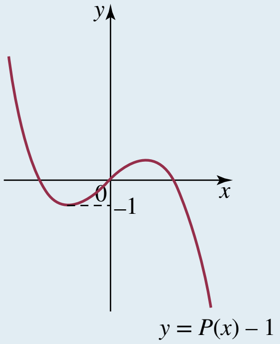
b



2 Consider the x-values. They remain unchanged — there is no horizontal translation.

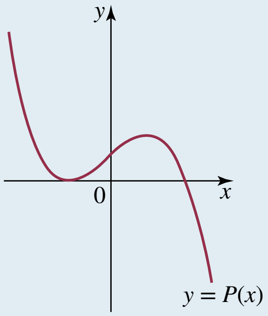
3 Consider the y-values. They are decreased by 1 — the curve is shifted down 1 unit.

4 Sketch the graph of $y = P(x) - 1$ using a similar scale to the original.



c **1** Sketch the original $y = P(x)$.

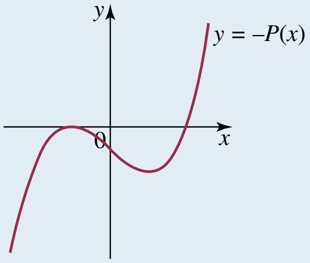
c



2 Consider the x-values. They remain unchanged — there is no horizontal translation.

3 Consider the y-values. They will all change sign — the curve will be reflected in the x-axis. That is, negative becomes positive and positive becomes negative.

4 Sketch the graph of $y = -P(x)$ using a similar scale to the original.



learnon RESOURCES — ONLINE ONLY

Try out this interactivity: Polynomial transformations (int-2794)

Exercise 18.6 Transformations

assesson

Individual pathways

■ PRACTISE	■ CONSOLIDATE	■ MASTER
Questions: 1–3, 5, 7	Questions: 1–4, 6, 7, 9	Questions: 1–10

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

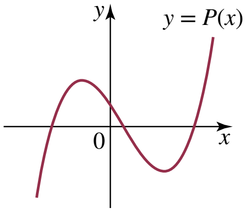
1. **WE14** Use the sketch of $y = P(x)$ shown at right to sketch:

a. $y = P(x) + 1$

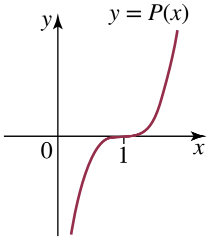
b. $y = P(x) - 2$

c. $y = -P(x)$

d. $y = 2P(x)$.

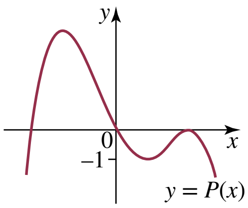


2. Consider the sketch of $y = P(x)$ shown at right. Sketch:
   a. $y = P(x) + 1$
   b. $y = -P(x)$
   c. $y = P(x + 2)$.



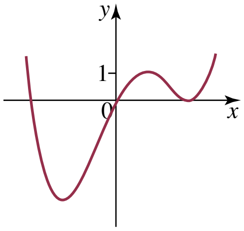
Understanding

3. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = -P(x)$.
4. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = 2P(x)$.
5. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = P(x) - 2$.
6. Consider the sketch of $y = P(x)$ shown below.

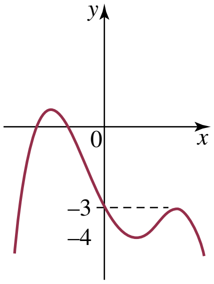


Give a possible equation for each of the following in terms of $P(x)$.

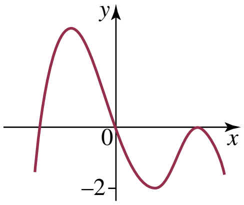
a.



b.



c.



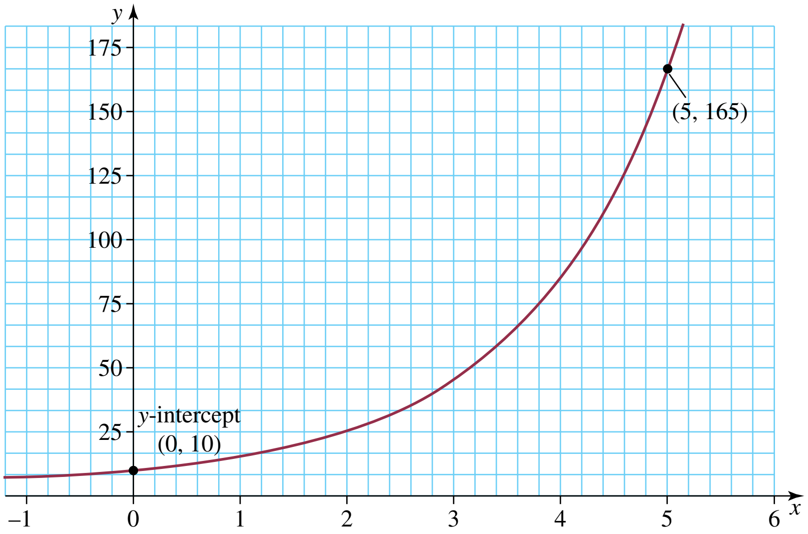
Reasoning

7. $y = x(x - 2)(x - 3)$ and $y = -2x(x - 2)(x - 3)$ are graphed on the same set of axes. Describe the relationship between the two graphs using the language of transformations.
8. If $y = -hr^{-q(x+p)} - r$, what translations take place from the original graph, $y = r^x$?

Problem solving

9. The graph of $y = \frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 2 parallel to the y-axis, translated 2 units to the left and up 1 unit. Find the equation of the resultant curve. Give the equations of any asymptotes.

10. The graph of an exponential function is shown.



Its general rule is given by $y = a(2^x) + b$.

a. Find the values of a and b.

b. Describe any transformations that had to be applied to the graph of $y = 2^x$ to achieve this graph.

Reflection

Why is it important to understand how transformations can affect the shape of a graph?

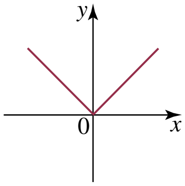
18.7 Review

18.7.1 Review questions

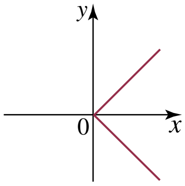
Fluency

1. Which of the following are functions?

a.



b.



2. Which of the following are functions? For each identified as functions, state the equation of the inverse function, if it exists.

a. $y = 2x - 7$ **b.** $x^2 + y^2 = \sqrt{30}$ **c.** $y = 2^x$ **d.** $y = \frac{1}{x + 1}$

3. If $f(x) = \sqrt{4 - x^2}$:

a. find:

i. $f(0)$ **ii.** $f(1)$ **iii.** $f(2)$

b. Does $f^{-1}(x)$ exist? If so, what is it equation?

4. Sketch each of the following curves, showing all intercepts.

a. $y = (x - 1)(x + 2)(x - 3)$ **b.** $y = (2x + 1)(x + 5)^2$

5. Give an example of the equation of a cubic that would just touch the x-axis and cross it at another point.

6. Match each equation with its type of curve.

a. $y = x^2 + 2$	**A.** circle
b. $x^2 + y^2 = 9$	**B.** cubic
c. $f(x) = \frac{2}{x + 2}$	**C.** exponential

d. $g(x) = 6^{-x}$ **D.** parabola

e. $h(x) = (x + 1)(x - 3)(x + 5)$ **E.** hyperbola

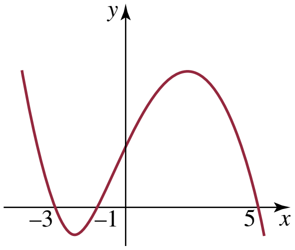
7. MC The equation for this graph could be:

A. $y = (x - 5)(x + 1)(x + 3)$

B. $y = (x - 3)(x - 1)(x + 5)$

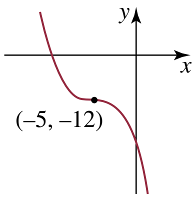
C. $y = (x - 3)(x + 1)(x + 5)$

D. $y = (5 - x)(1 + x)(3 + x)$

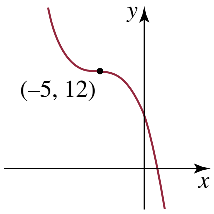


8. MC Which of the following shows the graph of $y = -2(x + 5)^3 - 12$?

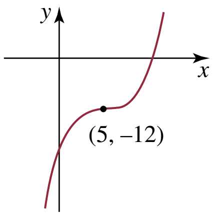
A.



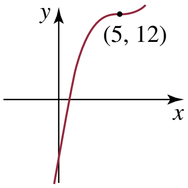
B.



C.



D.



9. Sketch:

a. $y = x(x - 2)(x + 11)$ **b.** $y = x^3 + 6x^2 - 15x + 8$ **c.** $y = -2x^3 + x^2$

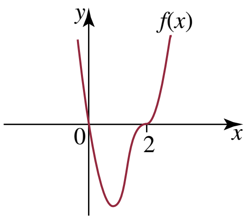
10. MC The rule for the graph shown could be:

A. $f(x) = x(x + 2)^3$

B. $f(x) = -x(x - 2)^2$

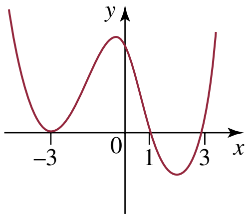
C. $f(x) = x^2(x - 2)^2$

D. $f(x) = x(x - 2)^3$

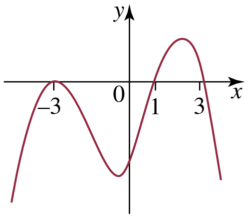


11. MC The graph of $y = (x + 3)^2(x - 1)(x - 3)$ is best represented by:

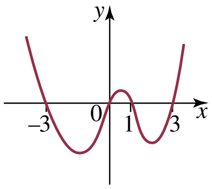
A.



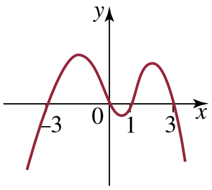
B.



C.

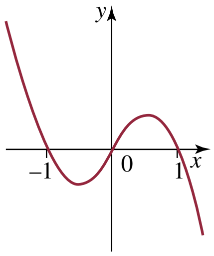


D.



12. Sketch the graph of $y = x^4 - 7x^3 + 12x^2 + 4x - 16$, showing all intercepts.

13. Consider the sketch of $y = P(x)$ shown at right. Sketch $y = -P(x)$.



14. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = P(x) + 3$.

Problem solving

15. Describe what happens to $f(x) = -2^x$ as $x \to \infty$ and $x \to -\infty$.

16. Find any points of intersection between $f(x) = x^2 - 4$ and $g(x) = x^3 + x^2 - 12$.

17. The concentration of alcohol (mg/L) in a bottle of champagne is modelled by $C = C_0 \times 0.33^{kt}$ where t represents the time in days after the bottle is opened. If the initial concentration is 80 mg/L and the concentration after 1 day is 70 mg/L, find the concentration remaining after:

a. 3 days **b.** 1 week **c.** 18 hours.

18. The number of hyenas, H, in the zoo is given by $H = 20(10^{0.1t})$, where t is the number of years since counting started. At the same time, the number of dingoes, D, is given by $D = 25(10^{0.05t})$.

a. Calculate the number of:

i. hyenas **ii.** dingoes when counting began.

b. Calculate the numbers of each after:

i. 1 year **ii.** 18 months.

c. Which of the animals is the first to reach a population of 40 and by how long?

d. After how many months are the populations equal and what is this population?

19. a. Consider the equation $f(x) = a(x - h)^2 + k$. By restricting the x-values, find the equation of the inverse function.

b. Show that the function $f(x) = \frac{a}{x} + b$ and its inverse function intersect on the line $y = x$.

20. A shend is a type of tropical pumpkin grown by the people of Outer Thrashia. The diameter (D m) of a shend increases over a number of months (m) according to the rule $D = 0.25 \times (10)^{0.01m}$.

a. Determine the diameter of the shend after 4 months.

b. If the shend is not harvested it will explode when it reaches a critical diameter of 0.5 metres. Show that it takes approximately 30 months for an unharvested shend to explode.

21. The surface area of a lake is evaporating at a rate of 5% per year due to climate change. To model this situation, the surface area of the lake (S km^2) over time is given by $S = 20\,000 \times 0.95^x$, where x is the time in years.

a. Explain whether this is an exponential relationship.

b. What is the surface area of the lake initially?

c. What will the surface area be in 10 years' time?

d. Plot a graph for this relationship.

e. What will the surface area be in 100 years' time?

f. Explain whether this is a realistic model.

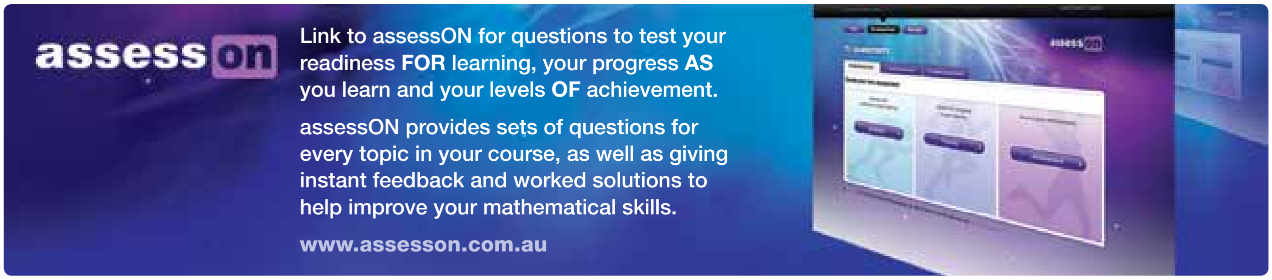
learnon RESOURCES — ONLINE ONLY

- **Try out this interactivity:** Word search: Topic 18 (int-2877)
- **Try out this interactivity:** Crossword: Topic 18 (int-2878)
- **Try out this interactivity:** Sudoku: Topic 18 (int-3893)
- **Complete this digital doc:** Concept map: Topic 18 (doc-14626)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

cubic functions
dilation
domain
exponential functions
function
function notation
horizontal line test
hyperbola
inflection point
inverse
many-to-many
many-to-one
one-to-many
one-to-one
quadratic functions
quartic functions
reflection
relation
transformations
translations
vertical line test



Investigation | Rich task

Shaping up!

Many beautiful patterns are created by starting with a single function or relation and transforming and repeating it over and over.

In this task you will apply what you have learned about functions, relations and transformations (dilations, reflections and translations) to explore mathematical patterns.

Exploring patterns using transformations

1. a. On the same set of axes, draw the graphs of:

- **i.** $y = x^2 - 4x + 1$
- **ii.** $y = x^2 - 3x + 1$
- **iii.** $y = x^2 - 2x + 1$
- **iv.** $y = x^2 + 2x + 1$
- **v.** $y = x^2 + 3x + 1$
- **vi.** $y = x^2 + 4x + 1$

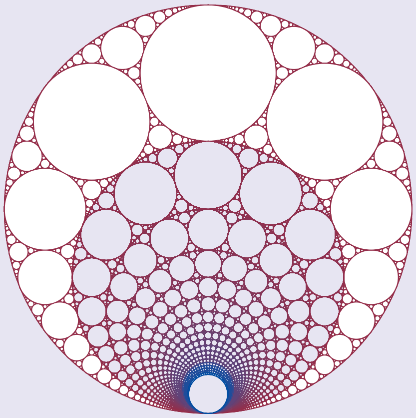
b. Describe the pattern formed by your graphs. Use mathematical terms such as intercepts, turning point, shape and transformations.

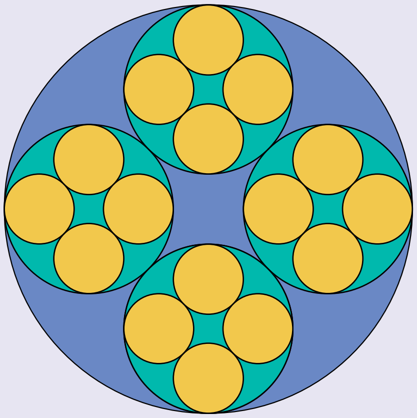
What you have drawn is referred to as a **family of curves** — curves in which the shape of the curve changes if the values of a, b and c in the general equation $y = ax^2 + bx + c$ change.

c. Explore the family of parabolas formed by changing the values of a and c. Comment on your findings.

d. Explore exponential functions belonging to the family of curves with equation $y = ka^x$, families of cubic functions with equations $y = ax^3$ or $y = ax^3 + bx^2 + cx + d$, and families of quartic functions with equations $y = ax^4$ or $y = ax^4 + bx^3 + cx^2 + dx + e$. Comment on your findings.

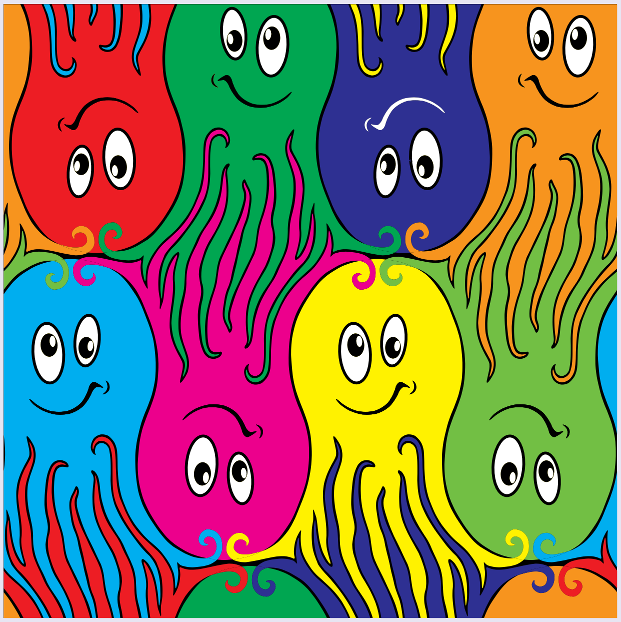
e. Choose one of the designs shown below and recreate it (or a simplified version of it). Record the mathematical equations used to complete the design.





Coming up with your design

2. Use what you know about transformations to functions and relations to create your own design from a basic graph. You could begin with a circle, add some line segments and then repeat the pattern with some change.
Record all the equations and restrictions you use.
It may be helpful to apply your knowledge of inverse functions too.
A digital technology will be very useful for this task.
Create a poster of your design to share with the class.



learnon RESOURCES — ONLINE ONLY

Complete this digital doc: Code puzzle: 2 important events of 1973 (doc-15951)

Answers

Topic 18 Functions and relations

Exercise 18.2 Functions and relations

1. a. One-to-many **b.** Many-to-one **c.** Many-to-one **d.** One-to-one **e.** One-to-one
f. Many-to-one **g.** Many-to-many **h.** Many-to-one **i.** One-to-one **j.** Many-to-one
k. One-to-one **l.** Many-to-one

2. a. b, c, d, e, f, h, i, j, k, l **b.** d, e, i

3. a. i. 1 **ii.** 7 **iii.** –5 **iv.** 16
b. i. 2 **ii.** 1 **iii.** 3 **iv.** 0
c. i. 3 **ii.** 2 **iii.** 6 **iv.** 9
d. i. 9 **ii.** 1 **iii.** 16 **iv.** $a^2 + 6a + 9$
e. i. 12 **ii.** 6 **iii.** –4 **iv.** 2

4. A, C, D

5. a, b, c, f

6. a. 3 **b.** 3 **c.** $\frac{5}{x} - 2x$ **d.** $\frac{10}{x^2} - x^2$ **e.** $\frac{10}{x+3} - x - 3$ **f.** $\frac{10}{x-1} - x + 1$

7. a. 3 **b.** −3 or 3 **c.** $\frac{1}{3}$ **d.** 2 or 3 **e.** −4 or 1 **f.** −1

8. a. $f(x) \to \infty$ **b.** $f(x) \to 0$ **c.** $f(x) \to 0$ **d.** $f(x) \to -\infty$ **e.** $f(x) \to 0$

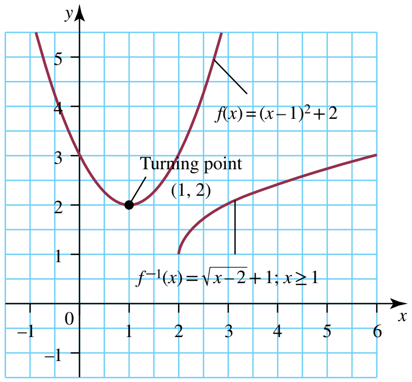
9. a. $(0, -4)$, $(2, 0)$ **b.** $(1, -2)$, $(-\frac{2}{3}, 3)$ **c.** $(2, 0)$, $(-2, 0)$ **d.** $(3, -4)$

10. a. $f^{-1}x = \dfrac{x+1}{2}$
b. $f^{-1}(x) = \sqrt{x+3}$ or $f^{-1}(x) = -\sqrt{x+3}$
c. $f(x)^{-1} = \sqrt{x-4} + 2$ or $f(x)^{-1} = -\sqrt{x-4} + 2$

11. a. i. The horizontal line test fails.
ii. An inverse function will exist for $f(x) = x(x-2)$, $x \le 1$ or $f(x) = x(x-2)$, $x \ge 1$.
b. i. The horizontal line test is upheld.
ii. $f^{-1}(x) = -\sqrt{4 - x}$, $x \le 4$.

12. a. $x = \pm 3$ **b.** $x = 2\frac{1}{3}$ **c.** $x = 28$

13. a. Ran = $[2, \infty)$ **b.** Many-to-one
c. and **f.**



d. Dom = $[1, \infty)$
e. $f^{-1}(x) = \sqrt{x-2} + 1$, Dom = $[2, \infty)$, Ran = $[1, \infty)$
g. (0, 3) and (3, 6)

Challenge 18.1

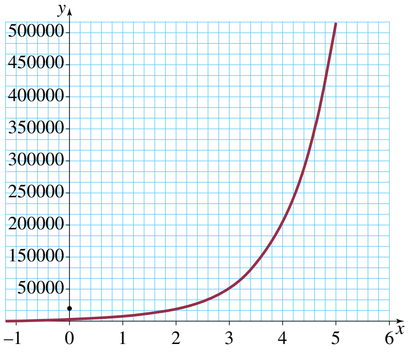
These graphs are inverse because they are the mirror images of each other through the line $y = x$.

Exercise 18.3 Exponential functions

1. a. 2000

b. 486 000

c.

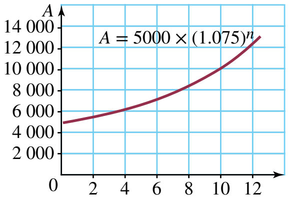


d. 1.26 h

2. a. \$5000

b. \$7717

c.



d. 10 years

3. a. C

b. D

4. A

5. B

6. a. \$883.50

b. \$821.66

c. $V = 950 \times (0.93)^n$

d. \$397.67

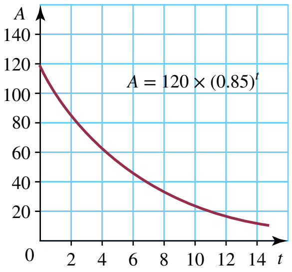
7. a. 102 mg

b. 86.7 mg

c. $A = 120 \times (0.85)^t$

d. 83.927 mg

e.



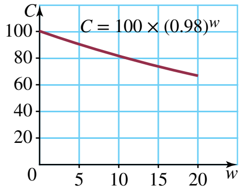
f. Approximately 210 years

8. a. i. 96.04%

ii. 90.39%

b. $C = 100(0.98)^w$

c.



d. 8 washings

9. a. 118 (million)

b. $a = 1.02$; $P = 118 \times (1.02)^n$

c.

Year	1990	1995	2000	2005	2010
Population	118	130	144	159	175

Calculated population is less accurate after 10 years.

d. 288 (million)

10. a. 32

b. 0.98

c. $T = 32 \times (0.98)^t$

d. 26.1, 21.4, 17.5, 14.3; values are close except for $t = 40$.

11. a. 3 dogs

b. 27 dogs

c. 3 years

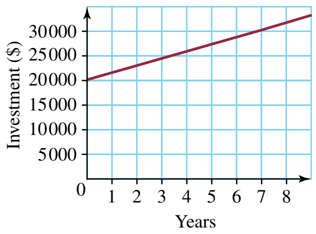
12. a. i. 39.85 mg

ii. 18.43 mg

b. More than 35.78 centuries

13. a. $A = 20000 \times 1.06^x$

b.



c. 7 years

d. 6 years — 1 year quicker

e. 9.05% p.a.

14. a. Approximately 20 200

b, c. Teacher to check.

15. a. $a = 100,\ b = 1.20,$ increase = 20%/min

b. $N = 146\,977 \times 0.70^m$

16. a. 500 °C

b. 125 °C

c. Between 5 and 6 hours once it has cooled to below 15 °C

17. a. 45 °C

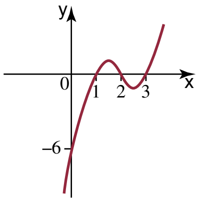
b. $T = 45 \times 0.95^t$

c. 10 °C

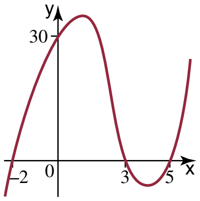
d. No. The line $T = 0$ is an asymptote.

Exercise 18.4 Cubic functions

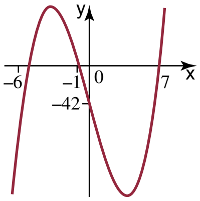
1. a.



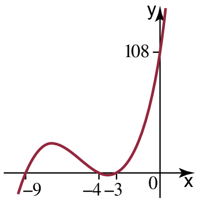
b.



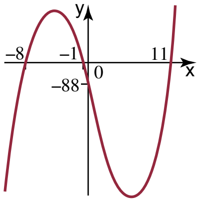
c.



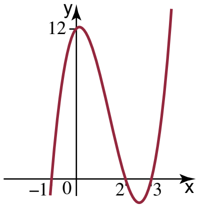
d.



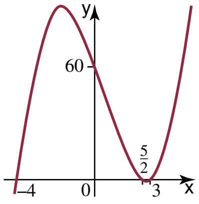
e.



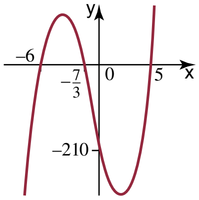
f.



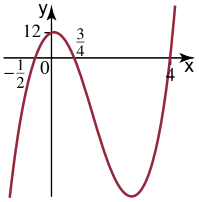
g.



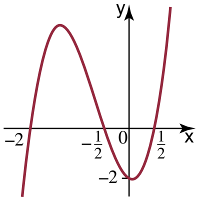
h.



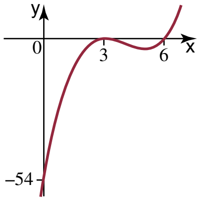
i.



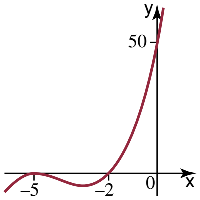
j.



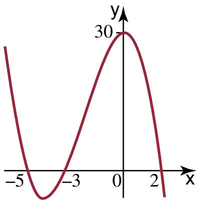
k.



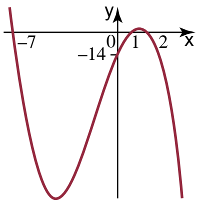
l.



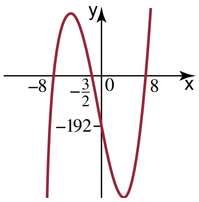
2. a.



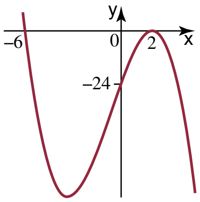
b.



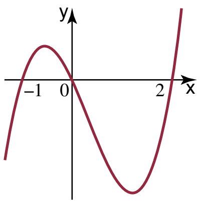
c.



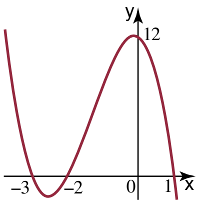
d.



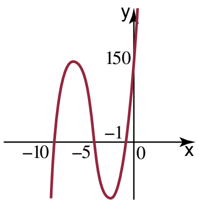
e.



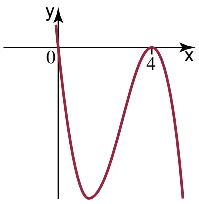
f.



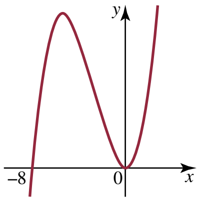
g.



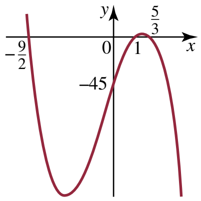
h.



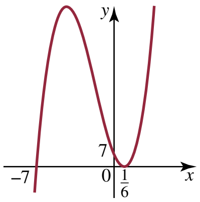
i.



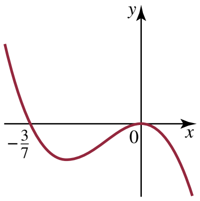
j.



k.



l.



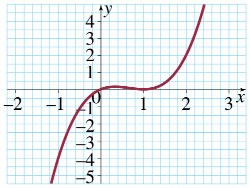
3. C

4. C

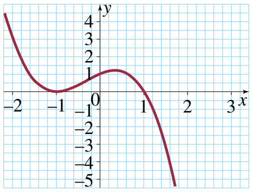
5. B

6. D

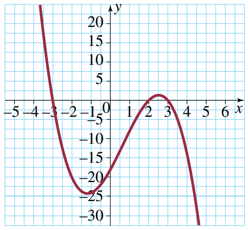
7. a.



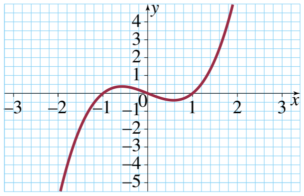
b.



c.



d.



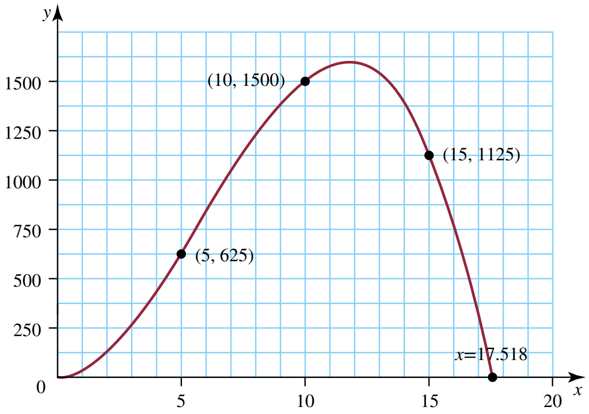
8. $a = 2,\ b = -7$

9. $a = \dfrac{-(27 + 11b)}{11}$

10. a. Check with your teacher.

b. $0 < x < 17.5$ **c.** 1377 cm^3

d.

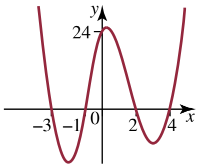


e. (11.6662, 1587.963); this is the value of x which creates the maximum volume.

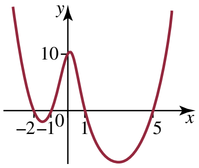
11. $y = 2(x + 2)(x - 2)(x - 5)$

Exercise 18.5 Quartic functions

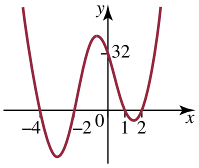
1. a.



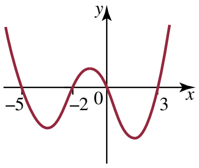
b.



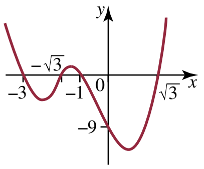
c.



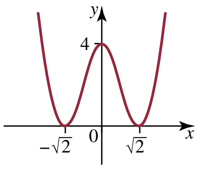
d.



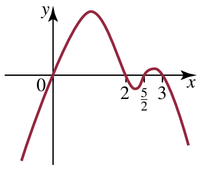
e.



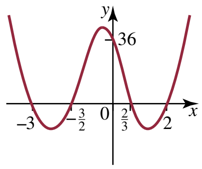
f.



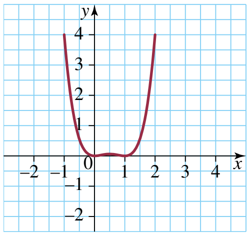
g.



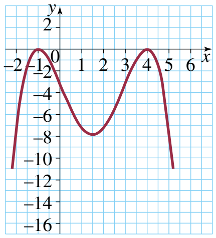
h.



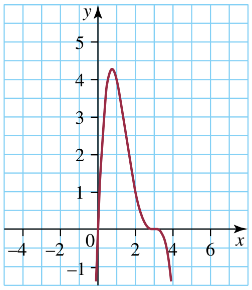
2. a.



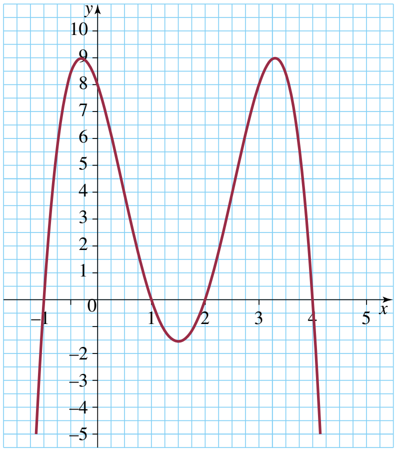
b.



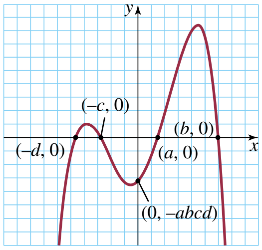
c.



d.



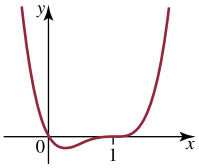
e.



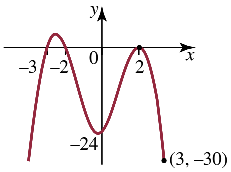
3. D

4. a. D **b.** B

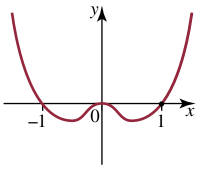
5. a.



b.



c.



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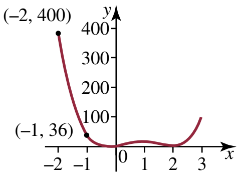
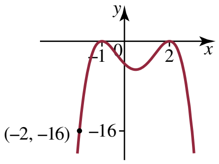
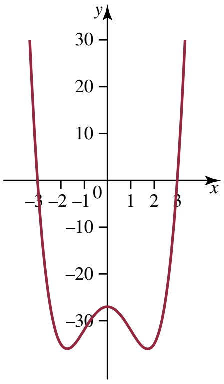


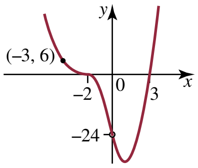
e.



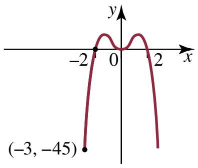
f.



g.



h.



6. $a = 4, \ b = -19$

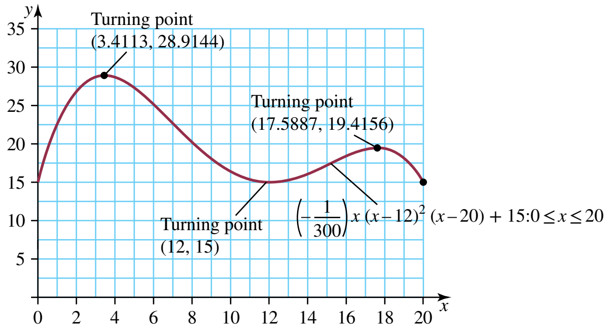
7. $a = 3, \ b = -1$

8. a. If n is even, the graph touches the x-axis.

b. If n is odd, the graph cuts the x-axis.

9. a. 15 m

b.



c. 28.77 m

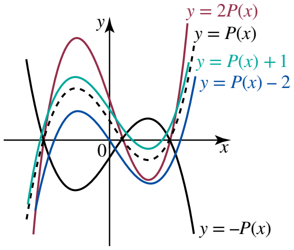
10. $y = (x + 1)(x - 2)^3$

Challenge 18.2

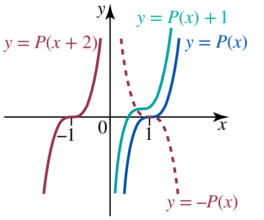
Check with your teacher.

Exercise 18.6 Transformations

1.



2.



3. They have the same x-intercepts, but $y = -P(x)$ is a reflection of $y = P(x)$ in the x-axis.

4. They have the same x-intercepts, but the y-values in $y = 2P(x)$ are all twice as large.

5. The entire graph is moved down 2 units. The shape is identical.

6. a. $y = -P(x)$ **b.** $y = P(x) - 3$ **c.** $y = 2P(x)$

7. The original graph has been reflected in the x-axis and dilated by a factor of 2 in the y direction. The location of the intercepts remains unchanged.

8. Dilation by a factor of h from the x-axis, reflection in the x-axis, dilation by a factor of $\frac{1}{q}$ from the y-axis, reflection in the y-axis, translation of p units left, translation of r units down.

9. $y = -\dfrac{2}{(x + 2)} + 1,\ x = -2, y = 1$

10. a. $y = 5(2^x) + 5$

b. Dilation by a factor of 5 parallel to the y-axis and translation of 5 units up. Graph asymptotes to $y = 5$.

18.7 Review

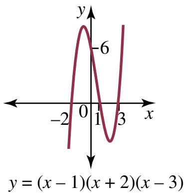
1. a

2. a, c, d, $\frac{x + 7}{2}$, $\log_2 x$, $\frac{1}{x} - 1$

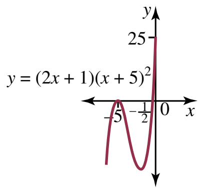
3. a. i. 2 **ii.** $\sqrt{3}$ **iii.** 0

b. No.

4. a.



b.



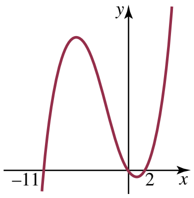
5. Check with your teacher. One possible answer is $y = (x - 1)(x - 2)^2$.

6. a. D **b.** A **c.** E **d.** C **e.** B

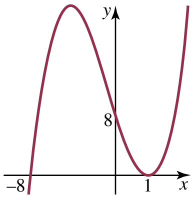
7. D

8. A

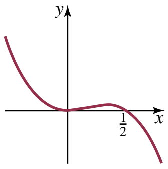
9. a.



b.



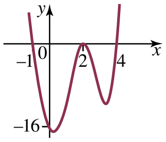
c.



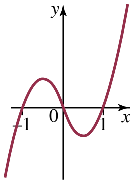
10. D

11. A

12.



13.



14. The entire graph is moved up 3 units. The shape is identical.

15. As $x \to \infty$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to 0$

16. (2, 0)

17. a. 53.59 mg/L **b.** 31.42 mg/L **c.** 72.38 mg/L

18. a. i. 20 **ii.** 25

b. i. $H = 25$; $D = 28$ **ii.** $H = 28$; $D = 30$

c. Hyenas after 3 years; dingoes after 4 years

d. After about 23 months; 31 animals

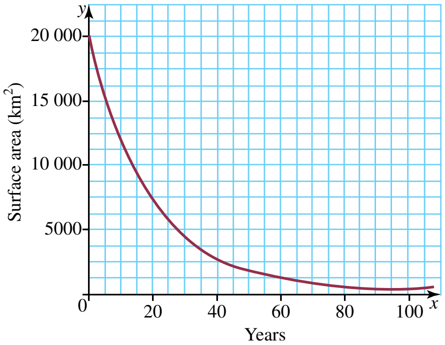
19. a. $f^{-1}(x) = \sqrt{\frac{x-k}{a}} + h, \quad x \geq \frac{k}{a}$ **b.** Check with your teacher.

20. a. 0.27 m **b.** Teacher to check.

21. a. Yes, because the relationship involves a variable as an exponent.

b. 20 000 km^2 **c.** 11 975 km^2

d.



e. 118 km^2

f. No this is not a realistic model as is it does not take into account changes to climate, rain, runoff from mountains, glaciers etc.

Investigation — Rich task

Check with your teacher.

TOPIC 19
Circle geometry

19.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

19.1.1 Why learn this?

For thousands of years humans have been fascinated by circles. Since they first looked upwards towards the sun and moon, which, from a distance at least, looked circular, humans have created circular monuments to nature. The most famous circular invention, one that has been credited as the most important invention of all, is the wheel. Scholars as early as Socrates and Plato have been fascinated with the sheer beauty of the properties of circles, and many scholars made a life's work out of studying them. Euclid was probably the most famous of these. It is in circle geometry that the concepts of congruence and similarity, studied earlier, have a powerful context.

19.1.2 What do you know?

assesson

1. **THINK** List what you know about circle geometry. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of circle geometry.

LEARNING SEQUENCE

learnon RESOURCES — ONLINE ONLY

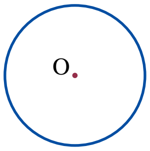
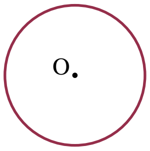
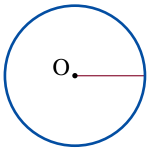
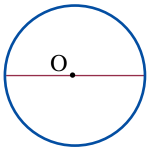
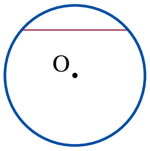
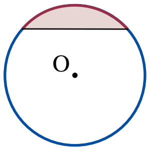
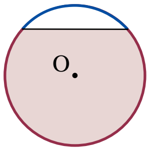
Watch this eLesson: The story of mathematics: Piscopia (eles-2022)

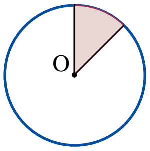
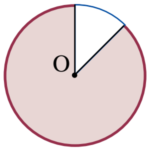
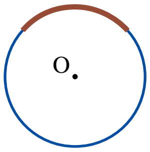
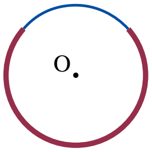
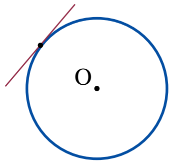
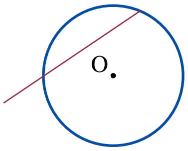
19.2 Angles in a circle

19.2.1 Circles

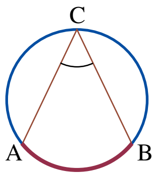
- A **circle** is a set of points that lie a fixed distance (the **radius**) from a fixed point (the **centre**).
- In circle geometry, there are many theorems that can be used to solve problems. It is important that we are also able to prove these theorems.
- To prove a theorem:
  1. state the aim of the proof
  2. use given information and previously established theorems to establish the result
  3. give a reason for each step of the proof
  4. state a clear conclusion.

19.2.2 Parts of a circle

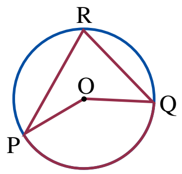
Part (name)	Description	Diagram
Centre	The middle point, equidistant from all points on the circumference. It is usually shown by a dot and labelled O.	
Circumference	The outside length or the boundary forming the circle. It is the circle's perimeter.	
Radius	A straight line from the centre to any point on the circumference	
Diameter	A straight line from one point on the circumference to another, passing through the centre	
Chord	A straight line from one point on the circumference to another	
Segment	The area of the circle between a chord and the circumference. The smaller segment is called the minor segment and the larger segment is the major segment.	 

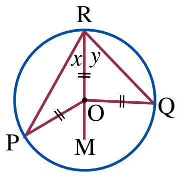
Part (name)	Description	Diagram
Sector	An area of a circle enclosed by 2 radii and the circumference	 
Arc	A portion of the circumference	 
Tangent	A straight line that touches the circumference at one point only	
Secant	A chord extended beyond the circumference on one side	

19.2.3 Angles in a circle



- In the diagram at right, chords AC and BC form the angle ACB. Arc AB has *subtended* angle ACB.
- **Theorem 1 Code**
- **The angle subtended at the centre of a circle is twice the angle subtended at the circumference, standing on the same arc.**





Proof:

Let $\angle PRO = x$ and $\angle QRO = y$
$RO = PO = QO$ (radii of the same circle are equal)
$\angle RPO = x$
and $\angle RQO = y$
$\angle POM = 2x$ (exterior angle of triangle)
and $\angle QOM = 2y$ (exterior angle of triangle)
$\angle POQ = 2x + 2y$
$= 2(x + y)$
which is twice the size of $\angle PRQ = x + y$.

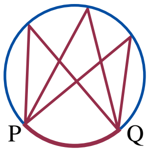
The angle subtended at the centre of a circle is twice the angle subtended at the circumference, standing on the same arc.

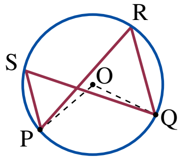
- **Theorem 2 Code** ⊠

  All angles that have their vertex on the circumference and are subtended by the same arc are equal.

  Proof:

  Join P and Q to O, the centre of the circle.





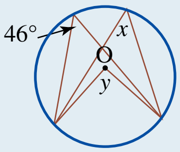
  Let∠PSQ $= x$
  ∠POQ $= 2x$ (angle at the centre is twice)
  the angle at the circumference)
  ∠PRQ $= x$ (angle at the circumference is half the angle of the centre)
  ∠PSQ = ∠PRQ.

  Angles at the circumference subtended by the same arc are equal.

  The application of the first two circle geometry theorems can be seen in the following worked example.

WORKED EXAMPLE 1

Find the values of the pronumerals in the diagram at right, giving reasons for your answers.

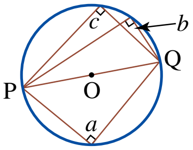


THINK	WRITE
1 Angles x and 46° are angles subtended by the same arc and both have their vertex on the circumference.	$x = 46°$ ⊠
2 Angles y and 46° stand on the same arc. The 46° angle has its vertex on the circumference and y has its vertex at the centre. The angle at the centre is twice the angle at the circumference.	$y = 2 \times 46°$ $= 92°$ ◬

- **Theorem 3 Code** ⌓

  Angles subtended by the diameter, that is, angles in a semicircle, are right angles.

  In the diagram at right, PQ is the diameter. Angles a, b and c are right angles. This theorem is in fact a special case of Theorem 1.



  Proof:

  ∠POQ = 180°(straight line)

  Let S refer to the angle at the circumference subtended by the diameter. In the figure, S could be at the points where a, b and c are represented on the diagram.

  ∠PSQ = 90° (angle at the circumference is half the angle at the centre)

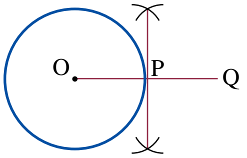
  Angles subtended by a diameter are right angles.

19.2.4 Constructing a tangent

There are a number of ways to construct a tangent to a circle, as explained using the following steps.

1. Draw a circle of radius 5 cm and centre O.
2. Draw a radius.
3. Call the point of intersection of the radius and the circumference, P.
4. Extend this radius through P to the point Q, 5 cm outside the circle.
5. Using O and Q as centres, draw intersecting arcs above and below the line OQ.
6. Draw a straight line joining the points of intersection. This line is the tangent.
7. What do you notice about the angle between OQ and the tangent?
8. Investigate another technique for constructing a tangent to a circle.

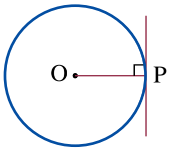
9. Write a set of instructions for this method of constructing a tangent.



- **Theorem 4 Code**

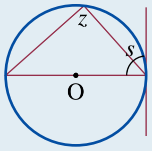
  If a radius is drawn to any point on the circumference and a tangent is drawn at the same point, then the radius will be perpendicular to the tangent.

  In the diagram at right, the radius is drawn to a point, P, on the circumference. The tangent to the circle is also drawn at P. The radius and the tangent meet at right angles, that is, the angle at P equals 90°.



WORKED EXAMPLE 2

Find the values of the pronumerals in the diagram at right, giving a reason for your answer.

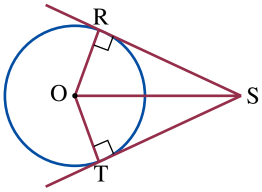


THINK	WRITE
1 Angle z is subtended by the diameter. Use an appropriate theorem to state the value of z.	$z = 90°$
2 Angle s is formed by a tangent and a radius, drawn to the point of contact. Apply the corresponding theorem to find the value of s.	$s = 90°$

- **Theorem 5 Code**

  The angle formed by two tangents meeting at an external point is bisected by a straight line joining the centre of the circle to that external point.

  Proof:



Consider ΔSOR and ΔSOT.

OR = OT (radii of the same circle are equal)

OS is common.

∠ORS = ∠OTS = 90° (angle between a tangent and radii is 90°)

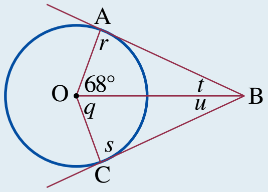
∴ ΔSOR ≅ ΔSOT (RHS)

So ∠ROS = ∠TOS and ∠OSR = ∠OST (corresponding angles in congruent triangles are equal).

The angle formed by two tangents meeting at an external point is bisected by a straight line joining the centre of the circle to the external point.

WORKED EXAMPLE 3

Given that BA and BC are tangents to the circle, find the values of the pronumerals in the diagram at right. Give reasons for your answers.



THINK	WRITE
1 Angles *r* and *s* are angles formed by the tangent and the radius, drawn to the same point on the circle. State their size.	$s = r = 90°$ ⊙
2 In the triangle ABO, two angles are already known and so angle *t* can be found using our knowledge of the sum of the angles in a triangle.	ΔABO: $t + 90° + 68° = 180°$ △ $t + 158° = 180°$ $t = 22°$
3 ∠ABC is formed by the two tangents, so the line BO, joining the vertex B with the centre of the circle, bisects this angle. This means that angles *t* and *u* are equal.	$\angle ABO = \angle CBO$ ⊙ $\angle ABO = t = 22°, \angle CBO = u$ $u = 22°$
4 ΔAOB and ΔCOB are similar triangles.	In ΔAOB andΔCOB $r + t + 68^\circ = 180^\circ$ △ $s + u + q = 180^\circ$ △ $r = s = 90^\circ$(proved previously) $t = u = 22^\circ$(proved previously $\therefore q = 68^\circ$

learnon RESOURCES — ONLINE ONLY

Try out this interactivity: Angles in a circle (int-2795)

Complete this digital doc: SkillSHEET: Using tests to prove congruent triangles (doc-5390)

Complete this digital doc: SkillSHEET: Corresponding sides and angles of congruent triangles (doc-5391)

Complete this digital doc: SkillSHEET: Using tests to prove similar triangles (doc-5392)

Complete this digital doc: SkillSHEET: Angles in a triangle (doc-5393)

Complete this digital doc: SkillSHEET: More angle relations (doc-5394)

Complete this digital doc: WorkSHEET: Circle geometry I (doc-14627)

Exercise 19.2 Angles in a circle

Individual pathways

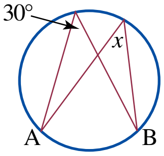
PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2a–c, 3a–d, 4–7	Questions: 1d–i, 2b–f, 3c–f, 4–10	Questions: 1–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

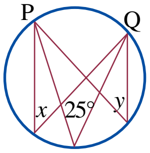
Fluency

1. **WE1** Find the values of the pronumerals in each of the following, giving reasons for your answers.

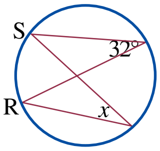
a.



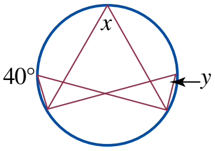
b.



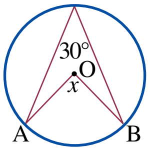
c.



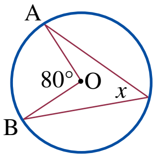
d.



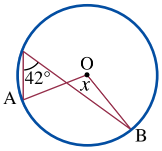
e.



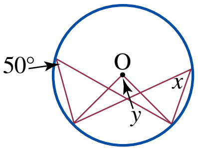
f.



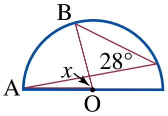
g.



h.

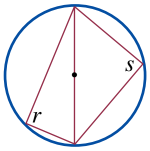


i.

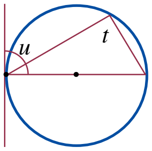


2. **WE2** Find the values of the pronumerals in each of the following figures, giving reasons for your answers.

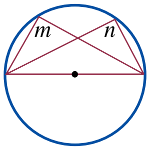
a.



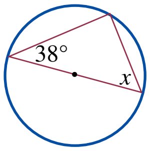
b.



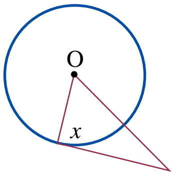
c.



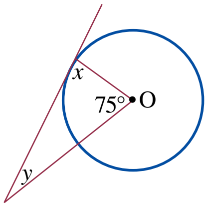
d.



e.



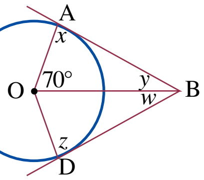
f.



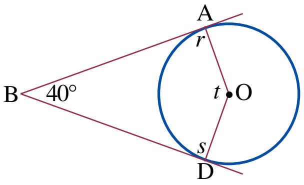
Understanding

3. **WE3** Given that AB and DB are tangents, find the value of the pronumerals in each of the following, giving reasons for your answers.

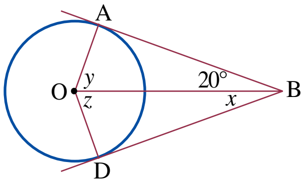
a.



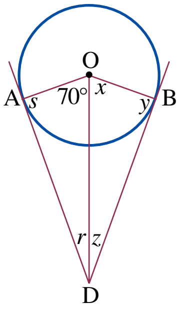
b.



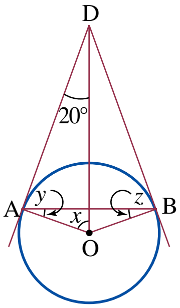
c.



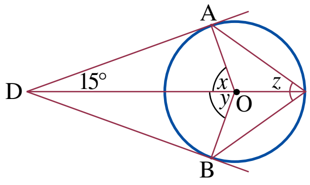
d.



e.

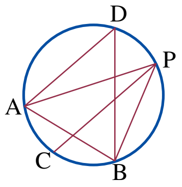


f.



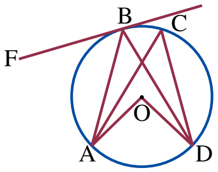
4. MC *Note:* There may be more than one correct answer.
In the diagram at right, which angle is subtended by the same arc as ∠APB?

A. ∠APC
B. ∠BPC
C. ∠ABP
D. ∠ADB



5. MC *Note:* There may be more than one correct answer.
Referring to the diagram at right, which of the statements is true?

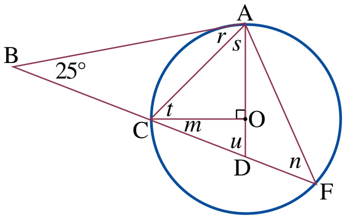
A. 2∠AOD = ∠ABD
B. ∠AOD = 2∠ACD
C. ∠ABF = ∠ABD
D. ∠ABD = ∠ACD



Reasoning

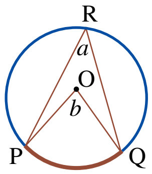
6. Values are suggested for the pronumerals in the diagram at right. AB is a tangent to a circle and O is the centre. In each case give reasons to justify suggested values.

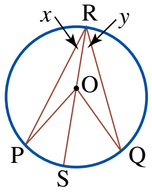
a. $s = t = 45°$
b. $r = 45°$
c. $u = 65°$
d. $m = 25°$
e. $n = 45°$



7. Set out below is the proof of this result: The angle at the centre of a circle is twice the angle at the circumference standing on the same arc.

Copy and complete the following to show that ∠POQ = 2 × ∠PRQ.
Construct a diameter through R. Let the opposite end of the diameter be S.

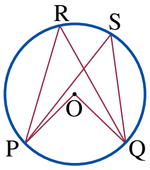




Let ∠ORP = x and ∠ORQ = y.
OR = OP (______________________________)
∠OPR = x (______________________________)
∠SOP = $2x$ (exterior angle equals ______________________________)
OR = OQ (______________________________)
∠OQR = ______________________________ (______________________________)
∠SOQ = ______________________________ (______________________________)
Now ∠PRQ = ____________ and ∠POQ = ____________.
Therefore ∠POQ = 2 × ∠PRQ.

8. Prove that the segments formed by drawing tangents from an external point to a circle are equal in length.

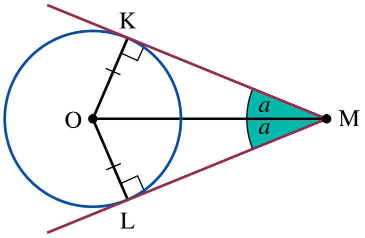
9. Use the figure drawn below to prove that angles subtended by the same arc are equal.



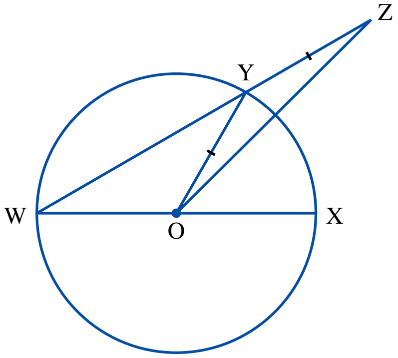
Problem solving

10. Use your knowledge of types of triangles, angles in triangles and the fact that the radius of a circle meets the tangent to the circle at right angles *to prove* the following theorem:

The angle formed between two tangents meeting at an external point is bisected by a line from the centre of the circle to the external point.



11. WX is the diameter of a circle with centre at O. Y is a point on the circle and WY is extended to Z so that OY = YZ. Prove that angle ZOX is three times angle YOZ.



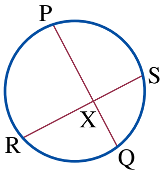
Reflection

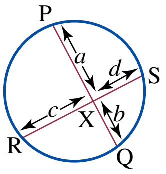
What are the common steps in proving a theorem?

19.3 Intersecting chords, secants and tangents

19.3.1 Intersecting chords

In the diagram below, chords PQ and RS intersect at X.





- **Theorem 6 Code**
  If the two chords intersect inside a circle, then the point of intersection divides each chord into two segments so that the product of the lengths of the segments for both chords is the same.

$$\mathbf{PX \times QX = RX \times SX \text{ or } a \times b = c \times d}$$

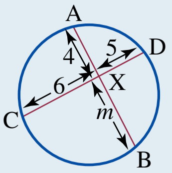
Proof:

Join PR and SQ.
Consider ΔPRX and ΔSQX.

$\angle PXR = \angle SXQ$ (vertically opposite angles are equal)
$\angle RSQ = \angle RPQ$ (angles at the circumference standing on the same arc are equal)
$\angle PRS = \angle PQS$ (angles at the circumference standing on the same arc are equal)
$\Delta PRX \sim \Delta SQX$ (equiangular)
$\frac{PX}{SX} = \frac{RX}{QX}$ (ratio of sides in similar triangles is equal)
or $PX \times QX = RX \times SX$

WORKED EXAMPLE 4

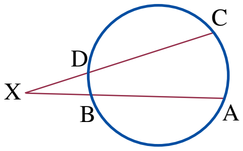
Find the value of the pronumeral.



THINK	WRITE
1 Chords AB and CD intersect at X. Point X divides each chord into two parts so that the products of the lengths of these parts are equal. Write this as a mathematical statement.	$AX \times BX = CX \times DX$
2 Identify the lengths of the line segments.	$AX = 4, BX = m, CX = 6, DX = 5$
3 Substitute the given lengths into the formula and solve for *m*.	$4m = 6 \times 5$ $m = \frac{30}{4}$ $= 7.5$

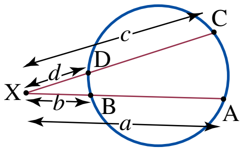
19.3.2 Intersecting secants

In the diagram below, chords CD and AB are extended to form secants CX and AX respectively. They intersect at X.



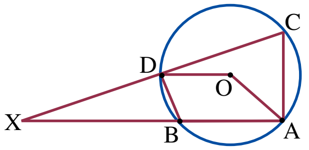
- **Theorem 7 Code**

If two secants intersect outside the circle as shown, then the following relationship is always true:



$$\mathbf{AX \times XB = XC \times DX \text{ or } a \times b = c \times d.}$$

Proof:
Join D and A to O, the centre of the circle.



Let $\angle DCA = x$.
$\angle DOA = 2x$ (angle at the centre is twice the angle at the circumference standing on the same arc)
Reflex $\angle DOA = 360° - 2x$ (angles in a revolution add to 360°)
$\angle DBA = 180° - x$ (angle at the centre is twice the angle at the circumference standing on the same arc)
$\angle DBX = x$ (angle sum of a straight line is 180°)
$\angle DCA = \angle DBX$

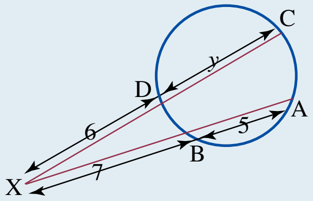
Consider ΔBXD and ΔCXA.
$\angle BXD$ is common.
$\angle DCA = \angle DBX$ (shown previously)
$\angle XAC = \angle XDB$ (angle sum of a triangle is 180°)
$\angle AXC \sim \Delta DXB$ (equiangular)

$$\frac{AX}{DX} = \frac{XC}{XB}$$

or $AX \times XB = XC \times DX$

WORKED EXAMPLE 5

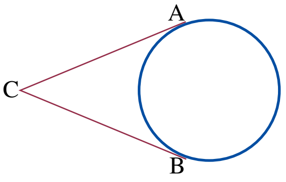
Find the value of the pronumeral.



THINK	WRITE
1 Secants XC and AX intersect outside the circle at X. Write the rule connecting the lengths of XC, DX, AX and XB.	$XC \times DX = AX \times XB$
2 State the length of the required line segments.	$XC = y + 6 \quad DX = 6$ $AX = 7 + 5 \quad XB = 7$ $= 12$
3 Substitute the length of the line segments and solve the equation for y.	$(y + 6) \times 6 = 12 \times 7$ $6y + 36 = 84$ $6y = 48$ $y = 8$

19.3.3 Intersecting tangents

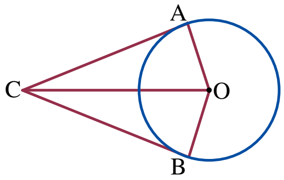
- In the following diagram, tangents AC and BC intersect at C and AC = BC.
- **Theorem 8 Code**
  If two tangents meet outside a circle, then the lengths from the external point to where they meet the circle are equal.



Proof:

Join A and B to O, the centre of the circle.

Consider ΔOCA and ΔOCB.



OC is common.

$OA = OB$ (radii of the same circle are equal)

$\angle OAC = \angle OBC$ (radius is perpendicular to tangent through the point of contact)

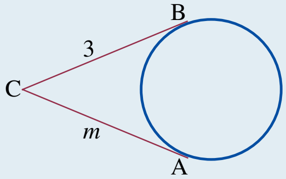
$\Delta OCA \cong \Delta OCB$ (RHS)

$AC = BC$ (corresponding sides of congruent triangles are equal).

If two tangents meet outside a circle, the lengths from the external point to the point of contact are equal.

WORKED EXAMPLE 6

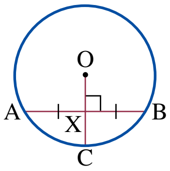
Find the value of the pronumeral.



THINK	WRITE
1 BC and AC are tangents intersecting at C. State the rule that connects the lengths BC and AC.	$AC = BC$
2 State the lengths of BC and AC.	$AC = m, \quad BC = 3$
3 Substitute the required lengths into the equation to find the value of *m*.	$m = 3$

19.3.4 Chords and radii

- In the diagram below, the chord AB and the radius OC intersect at X at 90°; that is, $\angle OXB = 90°$. OC bisects the chord AB; that is, $AX = XB$.

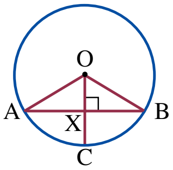


- **Theorem 9 Code**
- **If a radius and a chord intersect at right angles, then the radius bisects the chord.**

Proof:

Join OA and OB.

Consider ΔOAX and ΔOBX.



$OA = OB$ (radii of the same circle are equal)

$\angle OXB = \angle OXA$ (given)

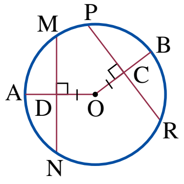
OX is common.

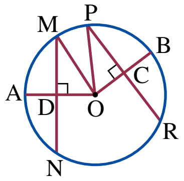
$\Delta OAX \cong \Delta OBX$ (RHS)

$AX = BX$ (corresponding sides in congruent triangles are equal)

If a radius and a chord intersect at right angles, then the radius bisects the chord.

- **The converse is also true:**
  If a radius bisects a chord, the radius and the chord meet at right angles.
- **Theorem 10**
  Chords equal in length are equidistant from the centre.
  This theorem states that if the chords MN and PR are of equal length, then OD = OC.
  Proof:
  Construct OA ⊥ MN and OB ⊥ PR.
  Then OA bisects MN and OB bisects PR (Theorem 9)
  Because MN = PR, MD = DN = PC = CR.
  Construct OM and OP, and consider ΔODM and ΔOCP.



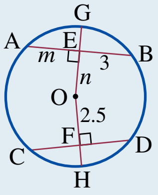


$$
\begin{aligned}
MD &= PC \text{ (shown above)}\\
OM &= OP \text{ (radii of the same circle are equal)}\\
\angle ODM &= \angle OCP = 90^\circ \text{ (by construction)}\\
\Delta ODM &\cong \Delta OCP \text{(RHS)}\\
\text{So } OD &= OC \text{ (corresponding sides in congruent triangles are equal)}
\end{aligned}
$$

Chords equal in length are equidistant from the centre.

WORKED EXAMPLE 7

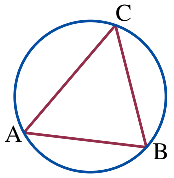
Find the values of the pronumerals, given that AB = CD.



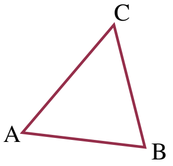
THINK	WRITE
1 Since the radius OG is perpendicular to the chord AB, the radius bisects the chord.	AE = EB
2 State the lengths of AE and EB.	$AE = m,\ EB = 3$
3 Substitute the lengths into the equation to find the value of *m*.	$m = 3$
4 AB and CD are chords of equal length and OE and OF are perpendicular to these chords. This implies that OE and OF are equal in length.	OE = OF
5 State the lengths of OE and OF.	$OE = n,\ OF = 2.5$
6 Substitute the lengths into the equation to find the value of *n*.	$n = 2.5$

19.3.5 The circumcentre of a triangle

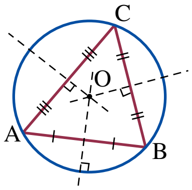
- In the diagram, a circle passes through the vertices of the triangle ABC.



- The circle is called the **circumcircle** of triangle ABC, and the centre of the circle is called the **circumcentre**.
- The circumcentre is located as follows.
  Draw any triangle ABC. Label the vertices.



Construct perpendicular bisectors of AB, AC and BC, and let the bisectors intersect at O. This means that OA = OB = OC, so a circle can be drawn through A, B and C with a centre at O.



Exercise 19.3 Intersecting chords, secants and tangents

Individual pathways

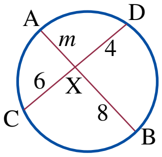
🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–7, 10	Questions: 1–8, 10	Questions: 1–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

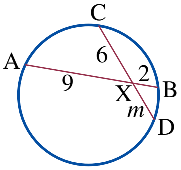
Fluency

1. WE4 Find the value of the pronumeral in each of the following.

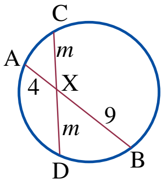
a.



b.

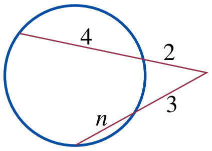


c.

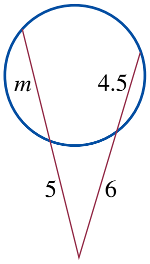


2. WE5 Find the value of the pronumeral in each of the following.

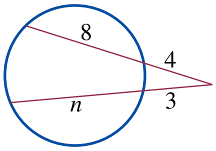
a.



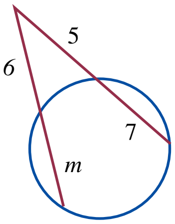
b.



c.

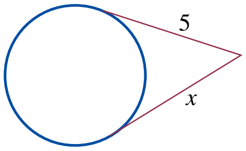


d.

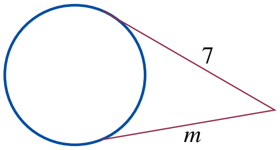


3. **WE6** Find the value of the pronumerals in each of the following.

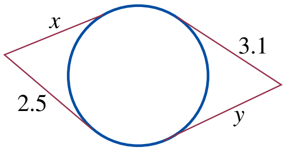
a.



b.

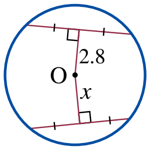


c.

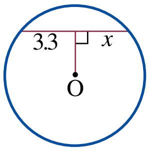


4. **WE7** Find the value of the pronumeral in each of the following.

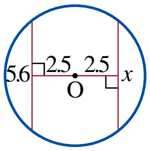
a.



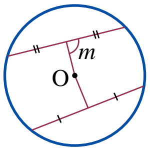
b.



c.



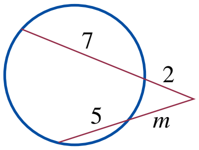
d.



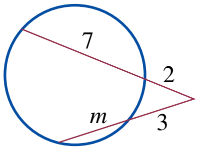
Understanding

5. **MC** *Note:* There may be more than one correct answer.
In which of the following figures is it possible to find the value of *m* through solving a linear equation?

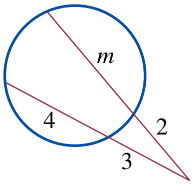
A.



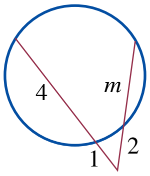
B.



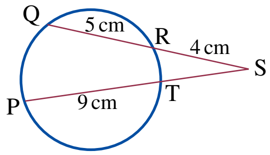
C.



D.



6. Find the length, ST, in the diagram below.



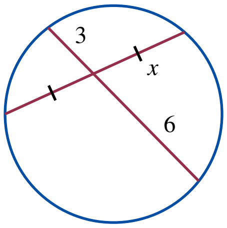
Reasoning

7. Prove the result: If a radius bisects a chord, then the radius meets the chord at right angles. Remember to provide reasons for your statements.
8. Prove the result: Chords that are an equal distance from the centre are equal in length. Provide reasons for your statements.
9. Prove that the line joining the centres of two intersecting circles bisects their common chord at right angles. Provide reasons for your statements.

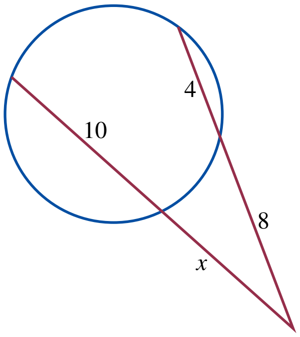
Problem solving

10. Calculate the pronumeral for each of the following diagrams.

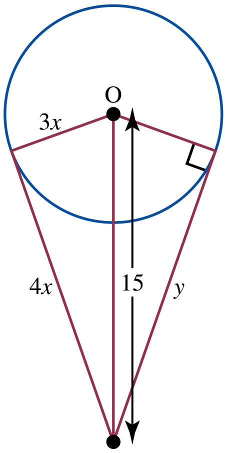
a.



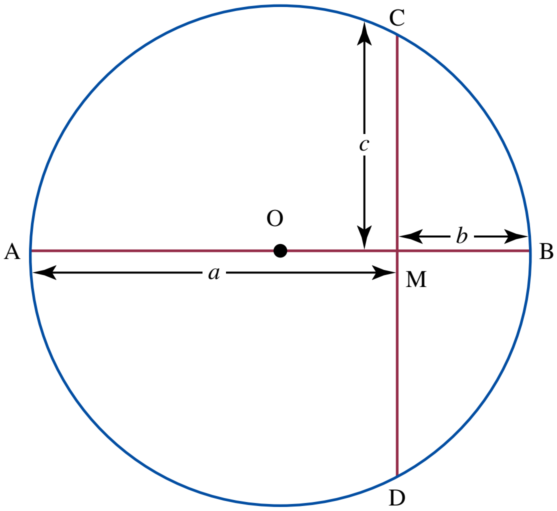
b.



c.



11. AOB is the diameter of the circle. CD is a chord perpendicular to AB and meeting AB at M.



a. Why is M the midpoint of CD?

b. If $CM = c, AM = a$ and $MB = b$, prove that $c^2 = ab$.

c. Explain why the radius of the circle is equal to $\dfrac{a+b}{2}$.

Reflection

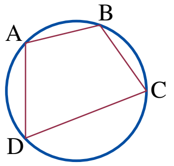
What techniques will you use to prove circle theorems?

CHALLENGE 19.1

An astroid is the curve traced by a point on the circumference of a small circle as it rolls around the inside circumference of a circle that is four times larger than it. Draw the shape of an astroid.

19.4 Cyclic quadrilaterals

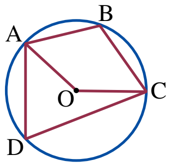
19.4.1 Quadrilaterals in circles



- A **cyclic quadrilateral** has all four vertices on the circumference of a circle; that is, the quadrilateral is inscribed in the circle.

  In the diagram at right, points A, B, C and D lie on the circumference; hence, ABCD is a cyclic quadrilateral.

  It can also be said that points A, B, C and D are **concyclic**; that is, the circle passes through all the points.



- **Theorem 11 Code**

  The opposite angles of a cyclic quadrilateral are supplementary (add to 180°).

  Proof:

  Join A and C to O, the centre of the circle.

  Let $\angle ABC = x$.

  Reflex $\angle AOC = 2x$ (angle at the centre is twice the angle at the circumference standing on the same arc)

  Reflex $\angle AOC = 360° - 2x$ (angles in a revolution add to 360°)

  $\angle ADC = 180° - x$ (angle at the centre is twice the angle at the circumference standing on the same arc)

  $\angle ABC + \angle ADC = 180°$

  Similarly, $\angle DAB + \angle DCB = 180°$.

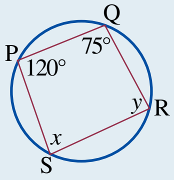
  Opposite angles in a cyclic quadrilateral are supplementary.

- **The converse is also true:**

  If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

WORKED EXAMPLE 8

Find the values of the pronumerals in the diagram below. Give reasons for your answers.

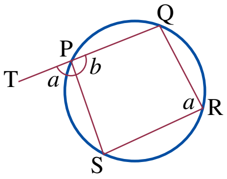


THINK	WRITE
1 PQRS is a cyclic quadrilateral, so its opposite angles are supplementary. First find the value of x by considering a pair of opposite angles $\angle PQR$ and $\angle RSP$ and forming an equation to solve.	$\angle PQR + \angle RSP = 180°$ (The opposite angles of a cyclic quadrilateral are supplementary.) $\angle PQR = 75°, \angle RSP = x$ $x + 75° = 180°$ $x = 105°$
2 Find the value of y by considering the other pair of opposite angles ($\angle SPQ$ and $\angle QRS$).	$\angle SPQ + \angle QRS = 180°$ $\angle SPQ = 120°, \angle QRS = y$ $y + 120° = 180°$ $y = 60°$

- **Theorem 12 Code**

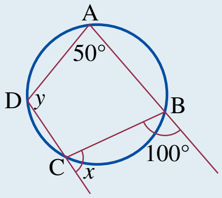
  The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

  Proof:
  $\angle QPS + \angle QRS = 180°$ (opposite angles of a cyclic quadrilateral)
  $\angle QPS + \angle SPT = 180°$ (adjacent angles on a straight line)
  Therefore $\angle SPT = \angle QRS$.
  The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



WORKED EXAMPLE 9

Find the value of the pronumerals in the diagram below.



THINK	WRITE
1 ABCD is a cyclic quadrilateral. The exterior angle, x, is equal to its interior opposite angle, $\angle DAB$.	$x = \angle DAB, \angle DAB = 50°$ So $x = 50°$.
2 The exterior angle, 100°, is equal to its interior opposite angle, $\angle ADC$.	$\angle ADC = 100°, \angle ADC = y$ So $y = 100°$.

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Complete this digital doc: SkillSHEET: Angles in a quadrilateral (doc-5396)

Complete this digital doc: WorkSHEET: Circle geometry II (doc-14628)

Exercise 19.4 Cyclic quadrilaterals

assesson

Individual Pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–6, 8	Questions: 1–8	Questions: 1–9

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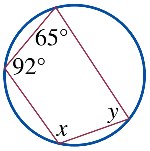
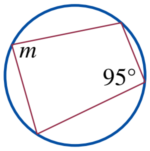
Fluency

1. **WE8** Find the values of the pronumerals in each of the following.

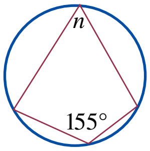
a.



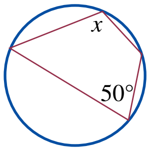
b.



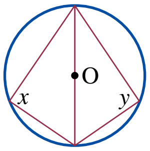
c.



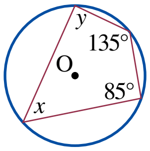
d.



e.

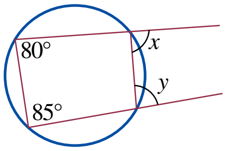


f.

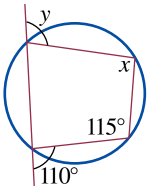


2. **WE9** Find the values of the pronumerals in each of the following.

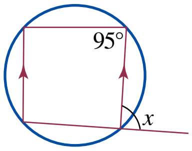
a.



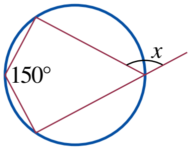
b.



c.



d.



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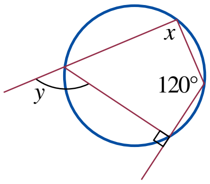

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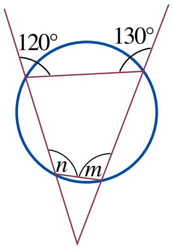
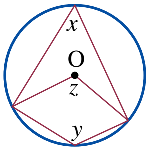


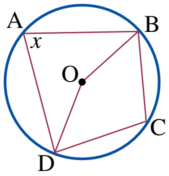
3. **MC** *Note:* There may be more than one correct answer.
Which of the following correctly states the relationship between x, y and z in the diagram shown?



A. $x = y$ and $x = 2z$
B. $x = 2y$ and $y + z = 180°$
C. $z = 2x$ and $y = 2z$
D. $x + y = 180°$ and $z = 2x$

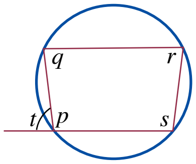
Understanding

4. The steps below show you how to set out the proof that the opposite angles of a cyclic quadrilateral are equal.



a. Find the size of ∠DOB.
b. Find the size of the reflex angle DOB.
c. Find the size of ∠BCD.
d. Find ∠DAB + ∠BCD.

5. **MC** *Note:* There may be more than one correct answer.



a. Which of the following statements is always true for the diagram shown?
A. $r = t$
B. $r = p$
C. $r = q$
D. $r = s$

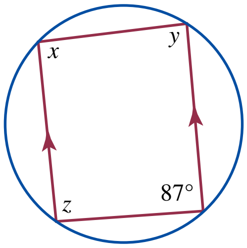
b. Which of the following statements is correct for the diagram shown?
A. $r + p = 180°$
B. $q + s = 180°$
C. $t + p = 180°$
D. $t = r$

Reasoning

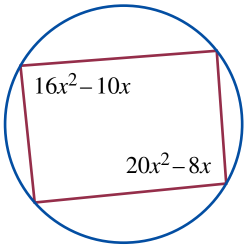
6. Prove that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

7. Calculate the values of the pronumerals in each of these diagrams.

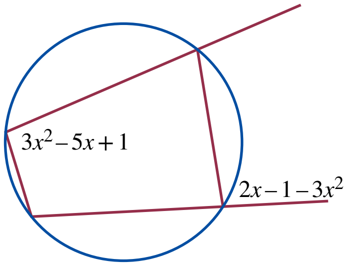
a.



b.

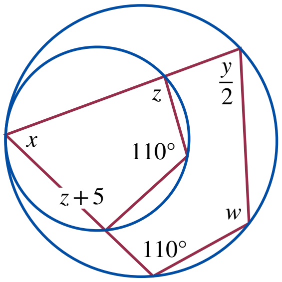


c.

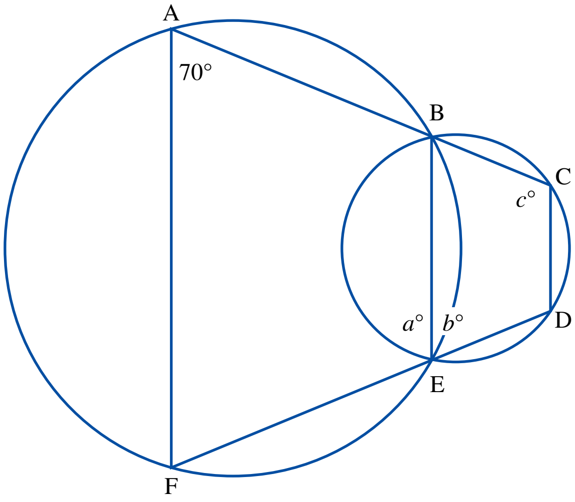


Problem solving

8. Calculate the value of each pronumeral in the diagram at right.



9. ∠FAB = 70°, ∠BEF = $a°$, ∠BED = $b°$ and ∠BCD = $c°$.



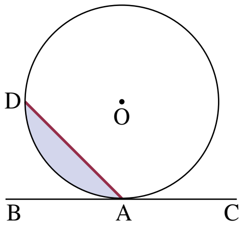
a. Find the values of a, b and c.
b. Prove that CD is parallel to AF.

Reflection

What is a cyclic quadrilateral?

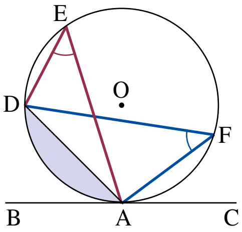
19.5 Tangents, secants and chords

19.5.1 The alternate segment theorem



- Consider the figure shown. Line BC is a tangent to the circle at the point A.
- A line is drawn from A to anywhere on the circumference, point D.
  The angle ∠BAD defines a segment (the shaded area).
  The unshaded part of the circle is called the alternate segment to ∠BAD.

Now consider angles subtended by the chord AD in the alternate segment, such as the angles marked in red and blue.



- The alternate segment theorem states that these are equal to the angle that made the segment, namely:

$$\angle BAD = \angle AED \text{ and } \angle BAD = \angle AFD$$

- **Theorem 13 Code**

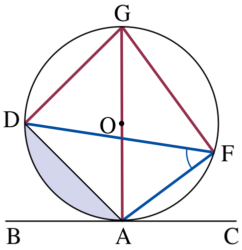
The angle between a tangent and a chord is equal to the angle in the alternate segment.

Proof:

We are required to prove that ∠BAD = ∠AFD.

Construct the diameter from A through O, meeting the circle at G.

Join G to the points D and F.



$$\begin{aligned} \angle BAG &= \angle CAG = 90^\circ && \text{(radii} \perp \text{tangent at point of contact)} \\ \angle GFA &= 90^\circ && \text{(angle in a semicircle is } 90^\circ) \\ \angle GDA &= 90^\circ && \text{(angle in a semicircle is } 90^\circ) \end{aligned}$$

Consider ΔGDA. We know that ∠GDA = 90°.

$$\begin{aligned} \angle GDA + \angle DAG + \angle AGD &= 180^\circ \\ 90^\circ + \angle DAG + \angle AGD &= 180^\circ \\ \angle DAG + \angle AGD &= 90^\circ \end{aligned}$$

∠BAG is also a right angle.

$$\angle BAG = \angle BAD + \angle DAG = 90^\circ$$

Equate the two results.

$$\angle DAG + \angle AGD = \angle BAD + \angle DAG$$

Cancel the equal angles on both sides.

$$\angle AGD = \angle BAD$$

Now consider the fact that both triangles DAG and DAF are subtended from the same chord (DA).

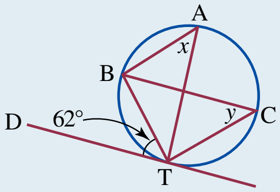
∠AGD = ∠AFD (Angles in the same segment standing on the same arc are equal).

Equate the two equations.

∠AFD = ∠BAD

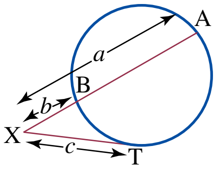
WORKED EXAMPLE 10

Find the value of *x* and *y*, giving reasons.



THINK	WRITE
1 Use the alternate segment theorem to find *x*.	$x = 62^\circ$ (angle between a tangent and a chord is equal to the angle in the alternate segment)
2 The value of *y* is the same as *x* because *x* and *y* are subtended by the same chord BT.	$y = 62^\circ$ (angles in the same segment standing on the same arc are equal)

19.5.2 Tangents and secants



- **Theorem 14 Code**

  If a tangent and a secant intersect as shown, the following relationship is always true:

  XA × XB = $(XT)^2$ or $a \times b = c^2$.

  Proof:

  Join BT and AT.

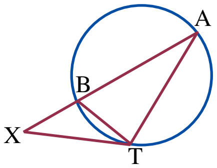
  Consider ΔTXB and ΔAXT.

  $$\begin{aligned} &\angle TXB \text{ is common.} \\ &\angle XTB = \angle XAT \text{ (angle between a tangent and a chord is equal to the angle in the alternate segment)} \\ &\angle XBT = \angle XTA \text{ (angle sum of a triangle is } 180^\circ) \end{aligned}$$

  ΔTXB~ΔAXT (equiangular)

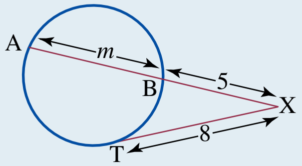
  So $\dfrac{XB}{XT} = \dfrac{XT}{XA}$

  or XA × XB = $(XT)^2$.



WORKED EXAMPLE 11

Find the value of the pronumeral.



THINK	WRITE
1 Secant XA and tangent XT intersect at X. Write the rule connecting the lengths of XA, XB and XT.	$XA \times XB = (XT)^2$
2 State the values of XA, XB and XT.	$XA = m + 5$, $XB = 5$, $XT = 8$
3 Substitute the values of XA, XB and XT into the equation and solve for *m*.	$(m + 5) \times 5 = 8^2$ $5m + 25 = 64$ $5m = 39$ $m = 7.8$

Exercise 19.5 Tangents, secants and chords

assessON

Individual pathways

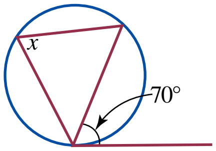
PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 4–6, 8, 10, 13–15, 17, 20, 22	Questions: 1–3, 5, 7, 9, 11, 13, 14, 16, 19, 21, 22	Questions: 1–23

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

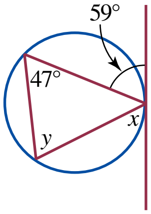
Fluency

1. **WE10** Find the value of the pronumerals in the following.

a.

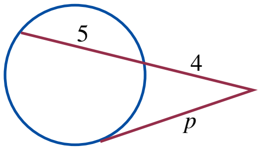


b.

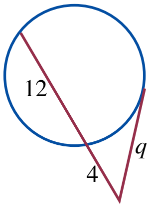


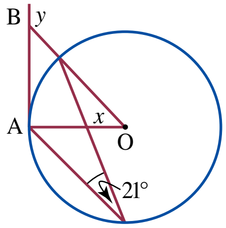
2. **WE11** Find the value of the pronumerals in the following.

a.



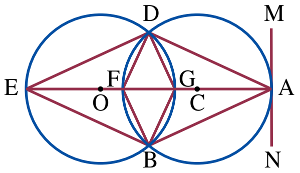
b.





3. Line AB is a tangent to the circle as shown in the figure on the right. Find the values of the angles labelled x and y.

Questions **4** to **6** refer to the figure on the right. The line MN is a tangent to the circle, and EA is a straight line. The circles have the same radius.



4. Find 6 different right angles.

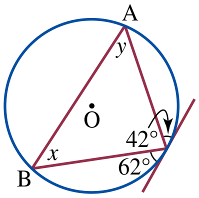
5. **MC** If ∠DAC = 20°, then ∠CFD and ∠FDG are respectively:

A. 70° and 50° **B.** 70° and 40° **C.** 40° and 70° **D.** 70° and 70°

6. **MC** A triangle similar to FDA is:

A. FDG **B.** FGB **C.** EDA **D.** GDE

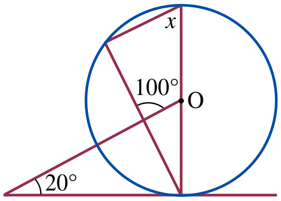
7. Find the values of the angles x and y in the figure at right.



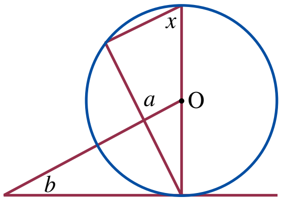
Understanding

8. Show that if the sum of the two given angles in question **7** is 90°, then the line AB must be a diameter.

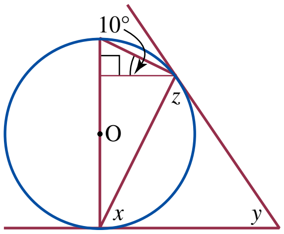
9. Find the value of x in the figure at right, given that the line underneath the circle is a tangent.



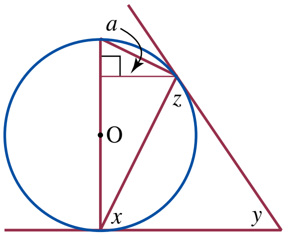
10. In the figure at right, express x in terms of a and b. This is the same drawing as in question **9**.



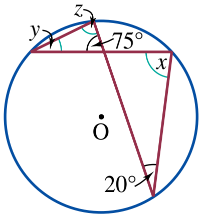
11. Two tangent lines to a circle meet at an angle y, as shown in the figure at right. Find the values of the angles x, y and z.



12. Solve question **11** in the general case (see the figure at right) and show that $y = 2a$. This result is important for space navigation (imagine the circle to be the Earth) in that an object at y can be seen by people at x and z at the same time.

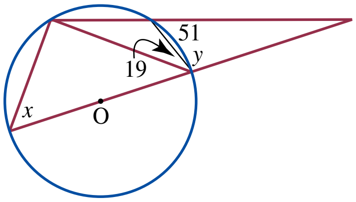


13. In the figure at right, find the values of the angles x, y and z.

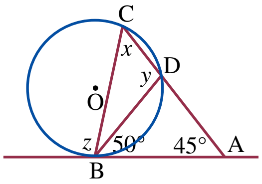


14. **MC** Examine the figure at right. The angles x and y (in degrees) are respectively:

A. 51 and 99 **B.** 51 and 129
C. 39 and 122 **D.** 51 and 122



Questions **15** to **17** refer to the figure at right. The line BA is a tangent to the circle at point B. Line AC is a chord that meets the tangent at A.



15. Find the values of the angles x and y.

16. **MC** The triangle which is similar to triangle BAD is:

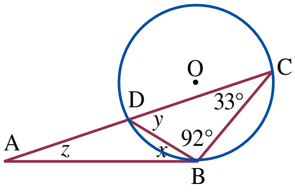
A. CAB **B.** BCD **C.** BDC **D.** AOB

17. **MC** The value of the angle z is:

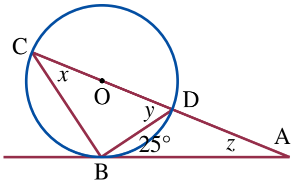
A. 50° **B.** 85° **C.** 95° **D.** 100°

Reasoning

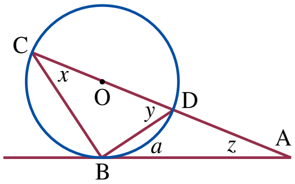
18. Find the values of the angles x, y and z in the figure at right. The line AB is tangent to the circle at B.



19. Find the values of the angles x, y and z in the figure at right. The line AB is tangent to the circle at B. The line CD is a diameter.



20. Solve question **19** in the general case; that is, express angles x, y and z in terms of a (see the figure at right).

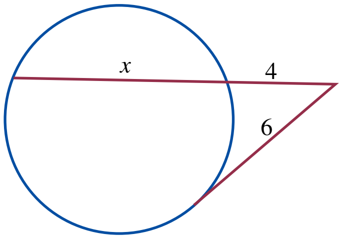


21. Prove that, when two circles touch, their centres and the point of contact are collinear.

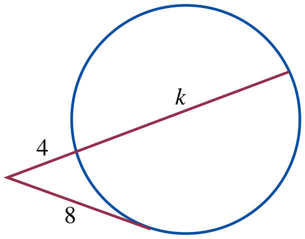
Problem solving

22. Find the value of the pronumerals in the following.

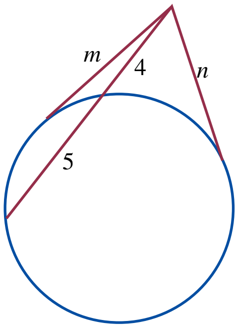
a.



b.



c.



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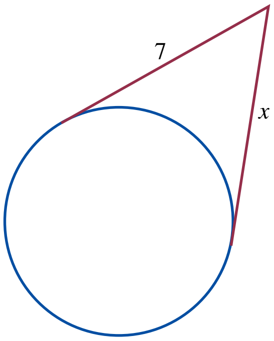

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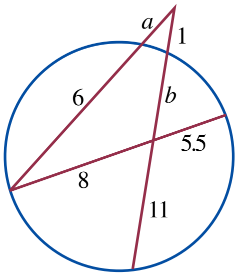

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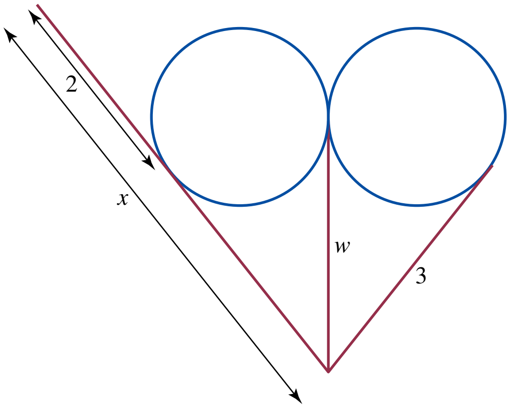
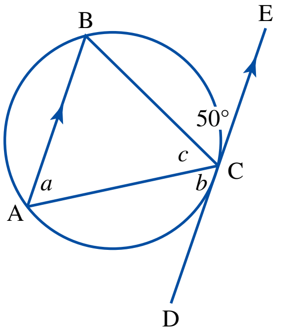


23. Find the values of a, b and c in each case.

a. $\angle BCE = 50°$ and $\angle ACE = c$



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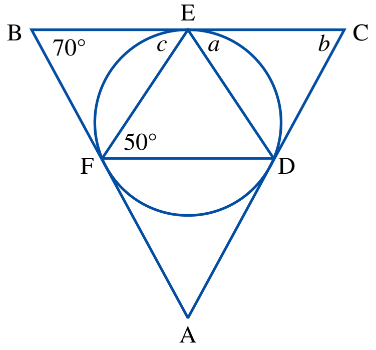


Reflection

Describe the alternate segment of a circle.

CHALLENGE 19.2

How can an annulus be cut into seven pieces with three straight lines?

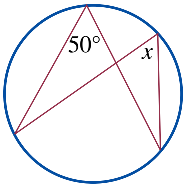
19.6 Review

19.6.1 Review questions

Fluency

1. Determine the values of the pronumerals in each of the following.

a.



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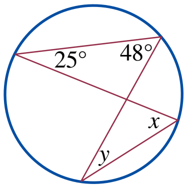
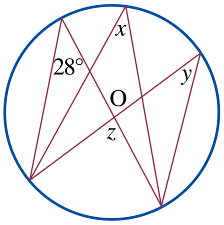


c.



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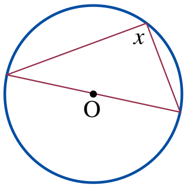
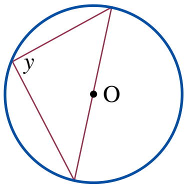


e.



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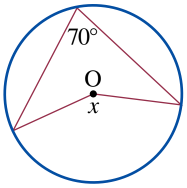
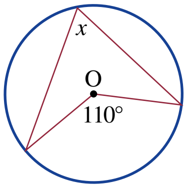
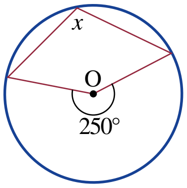


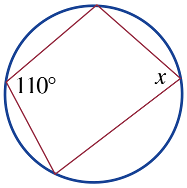
g.



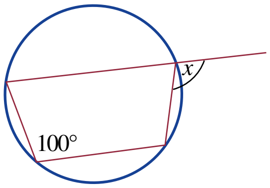
h.



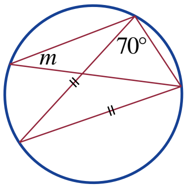
i.



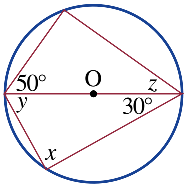
j.



k.

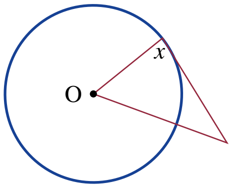


l.

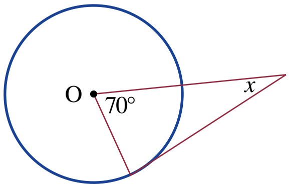


2. Find the value of the pronumeral in each case.

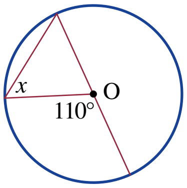
a.



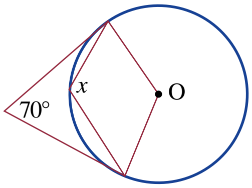
b.



c.

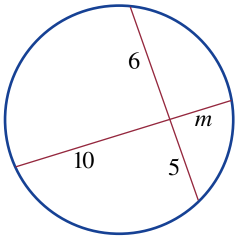


d.

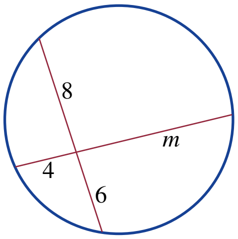


3. Find the value of m in each of the following.

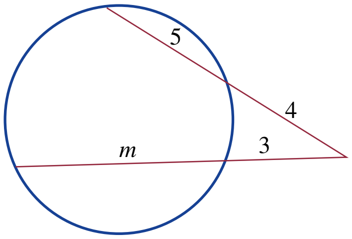
a.



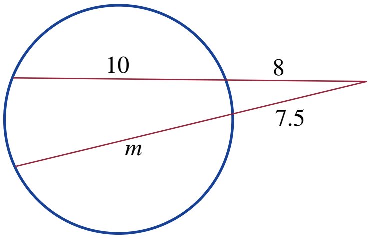
b.



c.

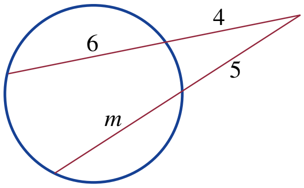


d.

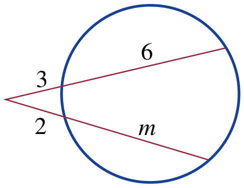


4. **MC** *Note:* There may be more than one correct answer.
In which of the following figures is it possible to get a reasonable value for the pronumeral?

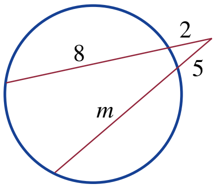
A.



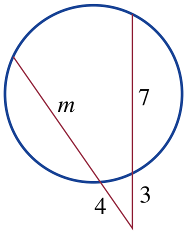
B.



C.

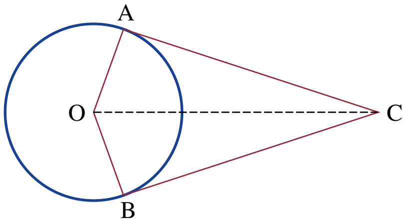


D.



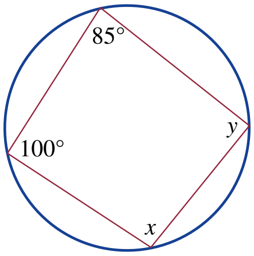
5. **MC** *Note:* There may be more than one correct answer.
Which of the following statements is true for the diagram shown?

A. AO = BO
B. AC = BC
C. ∠OAC = ∠OBC
D. ∠AOC = 90°

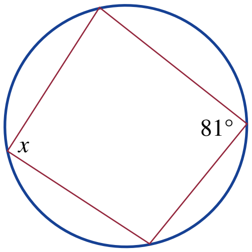


6. Find the values of the pronumerals in the following figures.

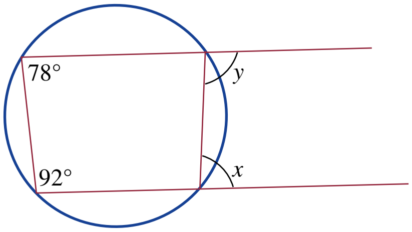
a.



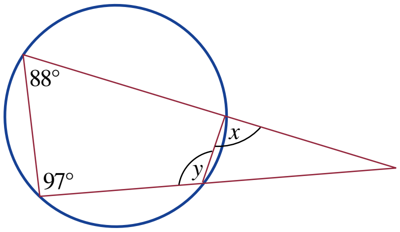
b.



c.

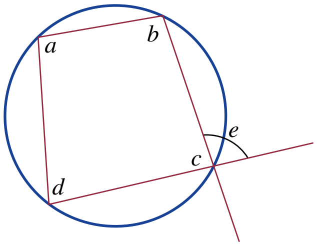


d.



7. **MC** Which of the following statements is *not* always true for the diagram at right?

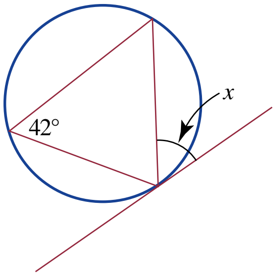
A. $\angle a + \angle c = 180°$
B. $\angle b + \angle d = 180°$
C. $\angle e + \angle c = 180°$
D. $\angle a + \angle e = 180°$



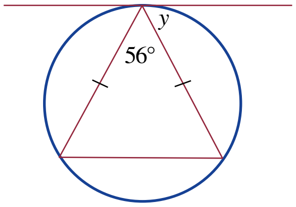
Problem solving

8. Find the values of the pronumerals in the following figures.

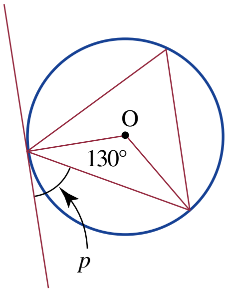
a.



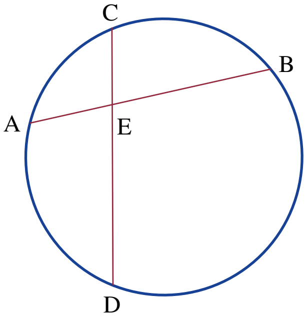
b.



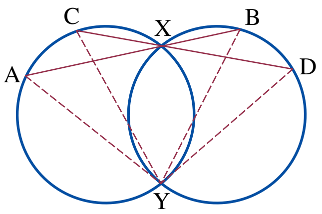
c.



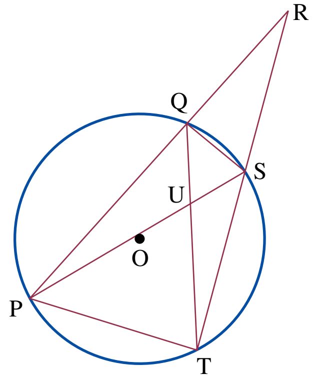
9. Two chords, AB and CD, intersect at E as shown. If AE = CE, prove that EB = ED.



10. Two circles intersect at X and Y. Two lines, AXB and CXD, intersect one circle at A and C, and the other at B and D, as shown.
Prove that ∠AYC = ∠BYD.



11. Name at least five pairs of equal angles in the following diagram.



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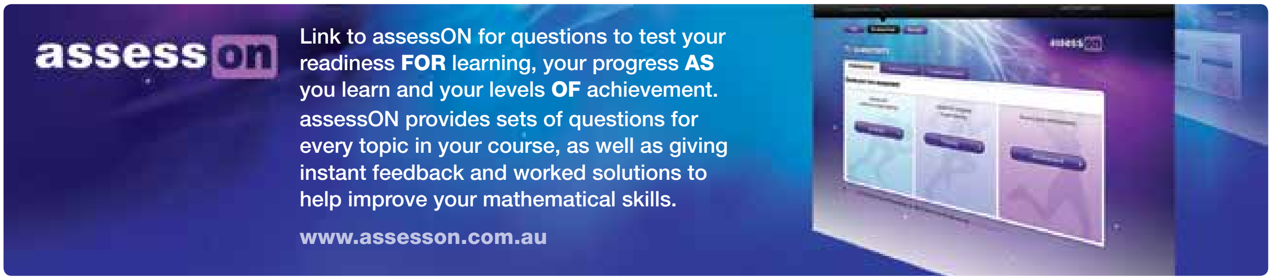
- **Try out this interactivity:** Word search: Topic 19 (int-2880)
- **Try out this interactivity:** Crossword: Topic 19 (int-2881)
- **Try out this interactivity:** Sudoku: Topic 19 (int-3894)
- **Complete this digital doc:** Concept map: Topic 19 (doc-14630)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

alternate segment theorem
angle
arc
chord
circle
circumcentre
circumcircle
circumference
concyclic
cyclic
cyclic quadrilateral
diameter
major segment
minor segment
radius
secant
sector
segment
subtend
tangent
theorem



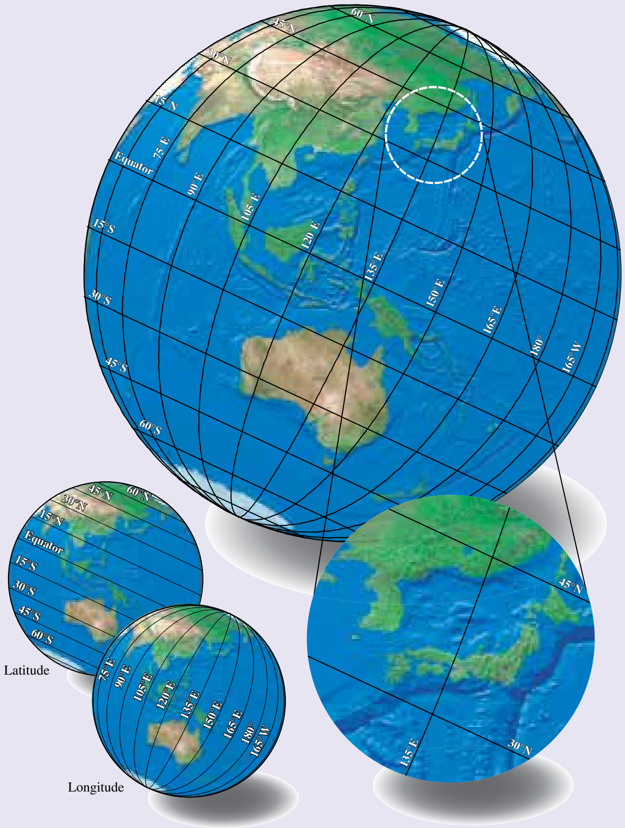
Investigation | Rich task

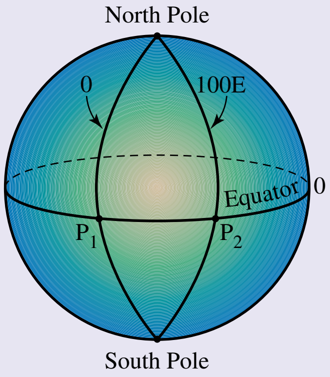
Variation of distance

The Earth approximates the shape of a sphere. Lines of longitude travel between the North and South poles, while lines of latitude travel east–west, parallel to the equator. While the lines of longitude are all approximately the same length, this is not the case with lines of latitude. The line of latitude at the equator is the maximum length and these lines decrease in length on approaching both the North and South poles.

This investigation looks at how the distance between points on two given lines of longitude and the same line of latitude changes as we move from the equator to the pole.

Consider two lines of longitude, 0° and 100°E. Take two points, P_1 and P_2, lying on the equator on lines of longitude 0° and 100°E respectively.





The distance (in km) between two points on the same line of latitude is given by the formula:

Distance = angle sector between the two points × 111 × cos (degree of latitude).

1. The size of the angle sector between P_1 and P_2 is 100° and these two points lie on 0° latitude. The distance between the points would be calculated as $100 \times 111 \times \cos 0°$. Determine this distance.
2. Move the two points to the 10° line of latitude. Calculate the distance between P_1 and P_2 in this position. Round your answer to the nearest kilometre.
3. Complete the following table showing the distance (rounded to the nearest kilometre) between the points P_1 and P_2 as they move from the equator towards the pole.

Latitude	Distance between P_1 and P_2 (km)
0°	
10°	
20°	
30°	
40°	
50°	
60°	
70°	
80°	
90°	

4. Describe what happens to the distance between P_1 and P_2 as we move from the equator to the pole. Is there a constant change? Explain.
5. You would perhaps assume that, at a latitude of 45°, the distance between P_1 and P_2 is half the distance between the points at the equator. This is not the case. At what latitude does this occur?
6. On the grid lines provided, sketch a graph displaying the change in distance between the points in moving from the equator to the pole.
7. Consider the points P_1 and P_2 on lines of longitude separated by 1°. On what line of latitude (to the nearest degree) would the points be 100 km apart?
8. Keeping the points P_1 and P_2 on the same line of latitude, and varying their lines of longitude, investigate the rate that the distance between them changes from the equator to the pole. Is it more or less rapid in comparison to what you found earlier?

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Complete this digital doc: Code puzzle: Where is 2 litres of hydrochloric acid produced each day? (doc-15953)

Answers

Topic 19 Circle geometry

Exercise 19.2 Angles in a circle

1. a. $x = 30°$ (theorem 2)
b. $x = 25°, \; y = 25°$ (theorem 2 for both angles)
c. $x = 32°$ (theorem 2)
d. $x = 40°, \; y = 40°$ (theorem 2 for both angles)
e. $x = 60°$ (theorem 1)
f. $x = 40°$ (theorem 1)
g. $x = 84°$ (theorem 1)
h. $x = 50°$ (theorem 2); $y = 100°$ (theorem 1)
i. $x = 56°$ (theorem 1)

2. a. $s = 90°, \; r = 90°$ (theorem 3 for both angles)
b. $u = 90°$ (theorem 4); $t = 90°$ (theorem 3)
c. $m = 90°, \; n = 90°$ (theorem 3 for both angles)
d. $x = 52°$ (theorem 3 and angle sum in a triangle = 180°)
e. $x = 90°$ (theorem 4)
f. $x = 90°$(theorem 4); $y = 15°$ (angle sum in a triangle = 180°)

3. a. $x = z = 90°$ (theorem 4); $y = w = 20°$ (theorem 5 and angle sum in a triangle = 180°)
b. $s = r = 90°$ (theorem 4); $t = 140°$ (angle sum in a quadrilateral = 360°)
c. $x = 20°$ (theorem 5); $y = z = 70°$ (theorem 4 and angle sum in a triangle = 180°)
d. $s = y = 90°$ (theorem 4); $x = 70°$ (theorem 5); $r = z = 20°$ (angle sum in a triangle = 180°)
e. $x = 70°$ (theorem 4 and angle sum in a triangle = 180°); $y = z = 20°$ (angle sum in a triangle = 180°)
f. $x = y = 75°$ (theorem 4 and angle sum in a triangle = 180°); $z = 75°$ (theorem 1)

4. D

5. B, D

6. a. Base angles of a right-angled isosceles triangle
b. $r + s = 90°, \; s = 45° \Rightarrow \; r = 45°$
c. u is the third angle in ΔABD, which is right-angled.
d. m is the third angle in ΔOCD, which is right-angled.
e. ∠AOC and ∠AFC stand on the same arc with ∠AOC at the centre and ∠AFC at the circumference.

7. OR = OP (radii of the circle)
∠OPR = x (equal angles lie opposite equal sides)
∠SOP = $2x$ (exterior angle equals the sum of the two interior opposite angles)
OR = OQ (radii of the circle)
∠OQR = y (equal angles lie opposite equal sides)
∠SOQ = $2y$ (exterior angle equals the sum of the two interior opposite angles)
Now ∠PRQ = $x + y$ and ∠POQ = $2x + 2y = 2(x + y)$.
Therefore ∠POQ = 2 × ∠PRQ.

8, 9. Check with your teacher.
10. Check with your teacher.
11. Check with your teacher.

Exercise 19.3 Intersecting chords, secants and tangents

1. a. $m = 3$ **b.** $m = 3$ **c.** $m = 6$

2. a. $n = 1$ **b.** $m = 7.6$ **c.** $n = 13$ **d.** $m = 4$

3. a. $x = 5$ **b.** $m = 7$ **c.** $x = 2.5, y = 3.1$

4. a. $x = 2.8$ **b.** $x = 3.3$ **c.** $x = 5.6$ **d.** $m = 90°$

5. B, C, D

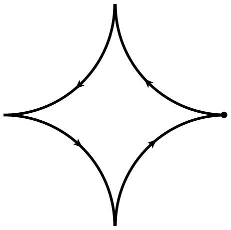
6. ST = 3 cm

7, **8**, **9**. Check with your teacher.

10. a. $x = 3\sqrt{2}$ **b.** $x = 6$ **c.** $x = 3,\ y = 12$

11. Check with your teacher.

Challenge 19.1



Exercise 19.4 Cyclic quadrilaterals

1. a. $x = 115°, y = 88°$ **b.** $m = 85°$ **c.** $n = 25°$
d. $x = 130°$ **e.** $x = y = 90°$ **f.** $x = 45°,\ y = 95°$

2. a. $x = 85°,\ y = 80°$ **b.** $x = 110°, y = 115°$ **c.** $x = 85°$
d. $x = 150°$ **e.** $x = 90°,\ y = 120°$ **f.** $m = 120°, n = 130°$

3. D

4. a. $2x$ **b.** $360° - 2x$
c. $180° - x$ **d.** $180°$

5. a. A
b. A, B, C, D

6. Check with your teacher.

7. a. $x = 93°, y = 87°, z = 93°$
b. $x = -2°$ or $\frac{5}{2}^{\circ}$
c. $x = \frac{2}{3}^{\circ}$ or $\frac{1}{2}^{\circ}$

8. $w = 110°, x = 70°, y = 140°, z = 87.5°$

9. a. $a = 110°, b = 70°$and $c = 110°$
b. Check with your teacher.

Exercise 19.5 Tangents, secants and chords

1. a. $x = 70°$
b. $x = 47°,\ \ y = 59°$

2. a. $p = 6$
b. $q = 8$

3. $x = 42°,\ \ y = 132°$

4. MAC, NAC, FDA, FBA, EDG, EBG

5. B

6. D

7. $x = 42°,\ \ y = 62°$

8. Answers will vary.

9. 60°

10. $x = 180° - a - b$

11. $x = 80°,\ y = 20°,\ z = 80°$

12. Answers will vary.

13. $x = 85°,\ y = 20°,\ z = 85°$

14. D

15. $x = 50°,\ y = 95°$

16. A

17. C

18. $x = 33°,\ y = 55°,\ z = 22°$

19. $x = 25°,\ y = 65°,\ z = 40°$

20. $x = a,\ y = 90° - a,\ z = 90° - 2a$

21. Check with your teacher.

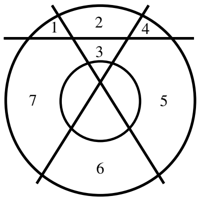
22. a. $x = 5$ **b.** $k = 12$ **c.** $m = 6,\ n = 6$

d. $x = 7$ **e.** $b = 4,\ a = 2$ **f.** $w = 3,\ x = 5$

23. a. $a = 50°,\ b = 50°$ and $c = 80°$

b. $a = 50°,\ b = 70°$ and $c = 70°$

Challenge 19.2



19.6 Review

1. a. $x = 50°$ **b.** $x = 48°, y = 25°$ **c.** $x = y = 28°, z = 56°$

d. $x = 90°$ **e.** $y = 90°$ **f.** $y = 140°$

g. $y = 55°$ **h.** $y = 125°$ **i.** $y = 70°$

j. $y = 100°$ **k.** $m = 40°$ **l.** $x = 90°, y = 60°, z = 40°$

2. a. $x = 90°$ **b.** $x = 20°$

c. $x = 55°$ **d.** $x = 125°$

3. a. $m = 3$ **b.** $m = 12$

c. $m = 9$ **d.** $m = 11.7$

4. A, B, D

5. A, B, C

6. a. $x = 95°, y = 80°$ **b.** $x = 99°$

c. $x = 78°, y = 92°$ **d.** $x = 97°, y = 92°$

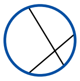
7. D

8. a. $x = 42°$

b. $y = 62°$

c. $p = 65°$

9. CE × ED = AE × EB



AE = CE (given)

∴ ED = EB

10. ∠AYC = ∠AXC

∠BXD = ∠BYD

But ∠AXC = ∠BXD

⇒ ∠AYC = ∠BYD

11. ∠PQT and ∠PST, ∠PTS and ∠RQS, ∠TPQ and ∠QSR, ∠QPS and ∠QTS, ∠TPS and ∠TQS, ∠PQS and ∠PTS, ∠PUT and ∠QUS, ∠PUQ and ∠TUS

Investigation — Rich task

1. 11 100 km

2. 10 931 km

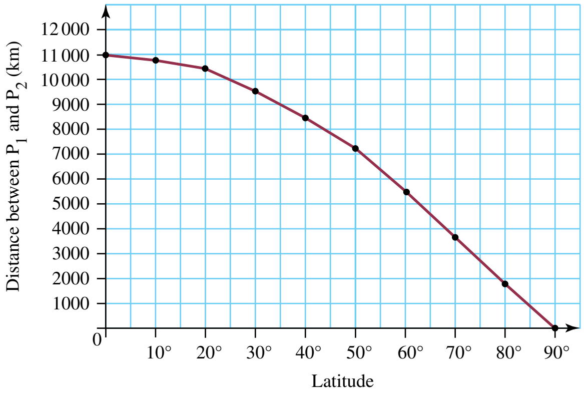
3.

Latitude	Distance between P_1 and P_2 (km)
0°	11 100
10°	10 931
20°	10 431
30°	9613
40°	8503
50°	7135
60°	5550
70°	3796
80°	1927
90°	0

4. The distance between P_1 and P_2 decreases from 11 100 km at the equator to 0 km at the pole. The change is not constant. The distance between the points decreases more rapidly on moving towards the pole.

5. Latitude 60°

6.



7. Latitude 26°

8. Answers will vary. Teacher to check.

TOPIC 20
Trigonometry II

20.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

20.1.1 Why learn this?

The ability to calculate distances using angles has long been critical. Early explorers, using rudimentary calculations, were able to navigate their way around the planet and were even able to map coastlines along the way. It is essential to be able to calculate distances that can't be physically measured. Our explorations have now turned towards the skies. By applying trigonometry we can approximate the distances to other planets and beyond.

20.1.2 What do you know?

assesson

1. **THINK** List what you know about trigonometry. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of trigonometry.

LEARNING SEQUENCE

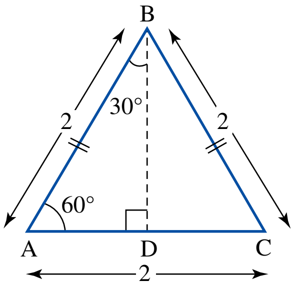
learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Agnesi (eles-2023)

20.2 The sine rule

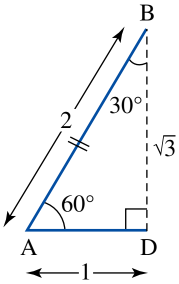
20.2.1 Exact values

- Most of the trigonometric values that we will deal with in this chapter are approximations.
- However, angles of 30°, 45° and 60° have exact values of sine, cosine and tangent.



- Consider an equilateral triangle, ABC, of side length 2 units.
  If the triangle is perpendicularly bisected, then two congruent triangles, ABD and CBD, are obtained. From triangle ABD it can be seen that altitude BD creates a right-angled triangle with angles of 60° and 30° and base length (AD) of 1 unit. The altitude BD is obtained using Pythagoras' theorem.

$$\begin{aligned}(AB)^2 &= (AD)^2 + (BD)^2\\ 2^2 &= 1^2 + (BD)^2\\ 4 &= 1 + (BD)^2\\ 4-1 &= (BD)^2\\ (BD)^2 &= 3\\ BD &= \sqrt{3}\end{aligned}$$



- Using triangle ABD and the three trigonometric ratios the following exact values are obtained:

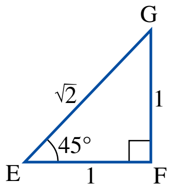
$$\sin B = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 30° = \frac{1}{2} \qquad \sin A = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 30° = \frac{\sqrt{3}}{2} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 60° = \frac{1}{2}$$

$$\tan B = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 30° = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \qquad \tan A = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 60° = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}$$

- Consider a right-angled isosceles triangle EFG whose equal sides are of 1 unit. The hypotenuse EG is obtained by using Pythagoras' theorem.

$$\begin{aligned}(EG)^2 &= (EF)^2 + (FG)^2\\ &= 1^2 + 1^2\\ &= 2\\ EG &= \sqrt{2}\end{aligned}$$



- Using triangle EFG and the three trigonometric ratios, the following exact values are obtained:

$$\sin E = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

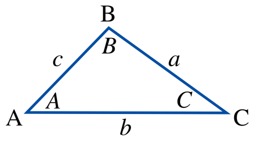
$$\cos E = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$\tan E = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 45° = \frac{1}{1} \text{ or } 1$$

Summary of exact values

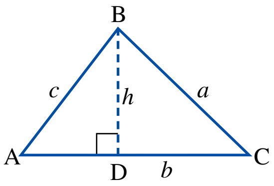
θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

20.2.2 The sine rule



- In non–right-angled triangles, it is usual to label the angles A, B and C, and the sides a, b and c, so that side a is the side opposite angle A, side b is the side opposite angle B and side c is the side opposite angle C.
- In a non–right-angled triangle, a perpendicular line, h, can be drawn from the angle B to side b.

  This divides the triangle into two right-angled triangles, ABD and CBD.
- Using triangle ABD and the sine trigonometric ratio for right-angled triangles, we obtain $\sin A = \frac{h}{c}$. Using triangle CBD and the sine trigonometric ratio for right-angled triangles, we obtain $\sin C = \frac{h}{a}$.



$\frac{h}{c} = \sin A$ and $\frac{h}{a} = \sin C$

- Transposing each equation to make h the subject, we obtain: $h = c\sin A$ and $h = a\sin C$. Since h is common to both triangles the two equations may be equated and we get $c\sin A = a\sin C$.

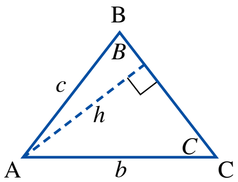
  Dividing both sides of the equation by $\sin A$ gives:

$$c = \frac{a\sin C}{\sin A}$$

  Dividing both sides of the equation by $\sin C$ gives:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

- In a similar way, if a perpendicular line is drawn from angle A to side a, the two right-angled triangles would give $h = c\sin B$ and $h = b\sin C$.



$h = c\sin B$ and
$h = b\sin C$

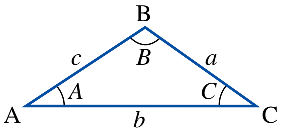
- This would give: $\frac{b}{\sin B} = \frac{c}{\sin C}$

  From this, the **sine rule** can be derived.
- In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

  Notes

  1. When using this rule, depending on the values given, any combination of the two equalities may be used to solve a particular triangle.
  2. To solve a triangle means to find all unknown side lengths and angles.

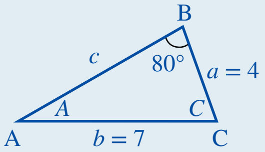


- The sine rule can be used to solve non–right-angled triangles if we are given:
  1. two angles and one side length
  2. two side lengths and an angle opposite one of these side lengths.

WORKED EXAMPLE 1

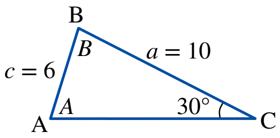
TI | CASIO

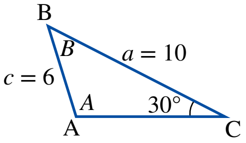
In the triangle ABC, $a = 4$ m, $b = 7$ m and $B = 80°$. Find A, C and c.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write down the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{b}{\sin B}$
4 Substitute the known values into the rule.	$\frac{4}{\sin A} = \frac{7}{\sin 80°}$
5 Transpose the equation to make $\sin A$ the subject.	$4 \sin 80° = 7 \sin A$ $\sin A = \frac{4 \sin 80°}{7}$
6 Evaluate.	$A = \sin^{-1}\left(\frac{4 \sin 80°}{7}\right)$ $\approx 34.246\,004\,71°$
7 Round off the answer to degrees and minutes.	$\approx 34°15'$
8 Determine the value of angle C using the fact that the angle sum of any triangle is 180°.	$C \approx 180° - (80° + 34°15')$ $= 65°45'$
9 Write down the sine rule to find c.	To find side length c: $\frac{c}{\sin C} = \frac{b}{\sin B}$
10 Substitute the known values into the rule.	$\frac{c}{\sin 65°45'} = \frac{b}{\sin 80°}$
11 Transpose the equation to make c the subject.	$c = \frac{7 \sin 65°45'}{\sin 80°}$
12 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 6.48 m

20.2.3 The ambiguous case

- If two side lengths and an angle opposite one of these side lengths are given, then two different triangles may be drawn. For example, if $a = 10$, $c = 6$ and $C = 30°$, two possible triangles could be created.





In the first case (above left), angle A is an acute angle, while in the second case (above right), angle A is an obtuse angle.

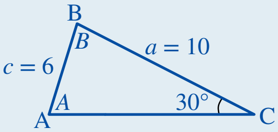
- When using the sine rule to find an angle, the inverse sine function is used. If we are finding an angle, given the sine value, it is important to rem|lember that an angle between 0° and 90° has the same sine value as its supplement. For example, sin 40° = 0.6427, and sin 140° = 0.6427.

WORKED EXAMPLE 2

TI | CASIO

In the triangle ABC, $a = 10$ m, $c = 6$ m and $C = 30°$. Find two possible values of A, and hence two possible values of B and b.

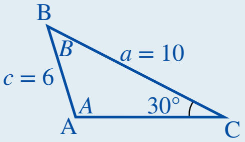
Case 1

THINK	**WRITE/DRAW**
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write down the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{c}{\sin C}$
4 Substitute the known values into the rule.	$\frac{10}{\sin A} = \frac{6}{\sin 30°}$ $10 \sin 30° = 6 \sin A$
5 Transpose the equation to make sin A the subject.	$\sin A = \frac{10 \sin 30°}{6}$
6 Evaluate angle A.	$A = \sin^{-1}\left(\frac{10 \sin 30°}{6}\right)$ $\approx 56.442\,690\,24°$
7 Round off the answer to degrees and minutes.	$A = 56°27'$
8 Determine the value of angle B, using the fact that the angle sum of any triangle is 180°.	$B \approx 180° - (30° + 56°27')$ $= 93°33'$
9 Write down the sine rule to find b.	To find side length b: $\frac{b}{\sin B} = \frac{c}{\sin C}$
10 Substitute the known values into the rule.	$\frac{b}{\sin 93°33'} = \frac{6}{\sin 30°}$
11 Transpose the equation to make b the subject.	$b = \frac{6 \sin 93°33'}{\sin 30°}$
12 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 11.98 m

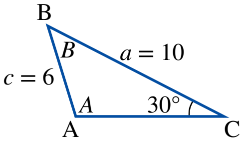
The values we have just obtained are only one set of possible answers for the given dimensions of the triangle ABC.

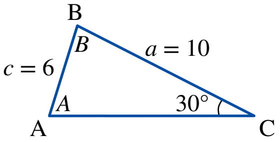
We are told that $a = 10$ m, $c = 6$ m and $C = 30°$. Since side a is larger than side c, it follows that angle A will be larger than angle C. Angle A must be larger than 30°; therefore it may be an acute angle or an obtuse angle.

Case 2

THINK	**WRITE/DRAW**
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Write down the alternative value for angle A. Simply subtract the value obtained for A in Case 1 from 180°.	To find the alternative angle A: If $\sin A = 0.8333$, then A could also be: $A \approx 180^\circ - 56^\circ 27'$ $= 123^\circ 33'$
3 Determine the alternative value of angle B, using the fact that the angle sum of any triangle is 180°.	$B \approx 180^\circ - (30^\circ + 123^\circ 33')$ $= 26^\circ 27'$
4 Write down the sine rule to find the alternative b.	To find side length b: $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
5 Substitute the known values into the rule.	$\dfrac{b}{\sin 26^\circ 27'} = \dfrac{6}{\sin 30^\circ}$
6 Transpose the equation to make b the subject.	$b = \dfrac{6 \sin 26^\circ 27'}{\sin 30^\circ}$
7 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 5.34 m

Hence, for this example there were two possible solutions as shown by the diagrams below.





- The ambiguous case does not apply to every question. Consider Worked example 1, in which we were required to solve the triangle ABC given $a = 4$ m, $b = 7$ m and $B = 80^\circ$. For angle A, we obtained $A = 34^\circ 15'$. However, angle A could also have been $A = 145^\circ 45'$ (since there are two possible values of A between 0° and 180° whose sine is the same; that is, $\sin 34^\circ 15' = 0.5628$ and $\sin 145^\circ 45' = 0.5628$).

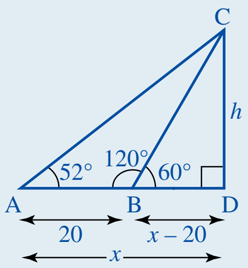
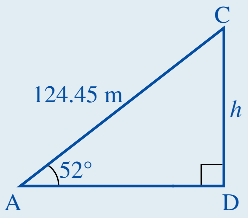
  We will now see whether or not $A = 145^\circ 45'$ is a possible solution.
  To obtain C subtract angles A and B from 180°.

$$\begin{aligned} C &= 180^\circ - (80^\circ + 145^\circ 45') \\ &= 180^\circ - 225^\circ 45' \\ &= -45^\circ 45' \text{ (not possible)} \end{aligned}$$

  Hence, for Worked example 1 only one possible solution exists.
- The ambiguous case exists if C is an acute angle and $a > c > a \sin C$, or any equivalent statement; for example, if B is an acute angle and $a > b > a \sin B$, and so.
  - In Worked example 2, where $a = 10$ m, $c = 6$ m and $C = 30^\circ$, there were two possible solutions because C was an acute angle and $a > c > a \sin C$, since $10 > 6 > 10 \times 0.5$.
  - In Worked example 1, where $a = 4$ m, $b = 7$ m and $B = 80^\circ$, there was only one possible solution because even though B was an acute angle, the condition $a > b > a \sin B$ could not be satisfied.

WORKED EXAMPLE 3

To calculate the height of a building, Kevin measures the angle of elevation to the top as 52°. He then walks 20 m closer to the building and measures the angle of elevation as 60°. How high is the building?

THINK	**WRITE/DRAW**
1 Draw a labelled diagram of the situation and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied for triangle ABC.	The sine rule can be used for triangle ABC since two angles and one side length have been given.
3 Determine the value of angle ACB, using the fact that the angle sum of any triangle is 180°.	$\angle ACB = 180° - (52° + 120°)$ $= 8°$
4 Write down the sine rule to find b (or AC).	To find side length b of triangle ABC: $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
5 Substitute the known values into the rule.	$\dfrac{b}{\sin 120°} = \dfrac{20}{\sin 8°}$
6 Transpose the equation to make b the subject.	$b = \dfrac{20 \times \sin 120°}{\sin 8°}$
7 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 124.45 m
8 Draw a diagram of the situation, that is, triangle ADC, labelling the required information. *Note:* There is no need to solve the rest of the triangle in this case as the values will not assist in finding the height of the building.	
9 Write down what is given for the triangle.	Have: angle and hypotenuse
10 Write down what is needed for the triangle.	Need: opposite side
11 Determine which of the trigonometric ratios is required (SOH–CAH–TOA).	$\sin\theta = \dfrac{O}{H}$
12 Substitute the given values into the appropriate ratio.	$\sin 52° = \dfrac{h}{124.45}$
13 Transpose the equation and solve for h.	$124.45 \sin 52° = h$ $h = 124.45 \sin 52°$
14 Round off the answer to 2 decimal places.	≈ 98.07
15 Answer the question.	The height of the building is 98.07 m.

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Exercise 20.2 The sine rule

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 2, 4, 5, 7, 10, 11, 14, 16, 18, 20, 22, 23, 24	Questions: 1, 3, 5, 6, 8, 10–12, 14, 16, 18, 20, 22, 24	Questions: 1, 5–17, 19, 21, 23, 24, 25

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Fluency

1. **WE1** In the triangle ABC, $a = 10$, $b = 12$ and $B = 58°$. Find A, C and c.

2. In the triangle ABC, $c = 17.35$, $a = 26.82$ and $A = 101°47'$. Find C, B and b.

3. In the triangle ABC, $a = 5$, $A = 30°$ and $B = 80°$. Find C, b and c.

4. In the triangle ABC, $c = 27$, $C = 42°$ and $A = 105°$. Find B, a and b.

5. In the triangle ABC, $a = 7$, $c = 5$ and $A = 68°$. Find the perimeter of the triangle.

6. Find all unknown sides and angles for the triangle ABC, given $A = 57°$, $B = 72°$ and $a = 48.2$.

7. Find all unknown sides and angles for the triangle ABC, given $a = 105$, $B = 105°$ and $C = 15°$.

8. Find all unknown sides and angles for the triangle ABC, given $a = 32$, $b = 51$ and $A = 28°$.

9. Find the perimeter of the triangle ABC if $a = 7.8$, $b = 6.2$ and $A = 50°$.

10. **MC** *Note:* There may be more than one correct answer.

In a triangle ABC, $B = 40°$, $b = 2.6$ and $c = 3$. The value of C is approximately:

A. 47° **B.** 48° **C.** 132° **D.** 133°

11. **WE2** In the triangle ABC, $a = 10$, $c = 8$ and $C = 50°$. Find two possible values of A, and hence two possible values of b.

12. In the triangle ABC, $a = 20$, $b = 12$ and $B = 35°$. Find two possible values for the perimeter of the triangle.

13. Find all unknown sides and angles for the triangle ABC, given $A = 27°$, $B = 43°$ and $c = 6.4$.

14. Find all unknown sides and angles for the triangle ABC, given $A = 100°$, $b = 2.1$ and $C = 42°$.

15. Find all unknown sides and angles for the triangle ABC, given $A = 25°$, $b = 17$ and $a = 13$.

Understanding

16. **WE3** To calculate the height of a building, Kevin measures the angle of elevation to the top as 48°. He then walks 18 m closer to the building and measures the angle of elevation as 64°. How high is the building?

Reasoning

17. A river has parallel banks that run directly east–west. From the south bank, Kylie takes a bearing to a tree on the north side. The bearing is 047°T. She then walks 10 m due east, and takes a second bearing to the tree. This is 305°T. Find:

a. her distance from the second measuring point to the tree

b. the width of the river, to the nearest metre.

18. A ship sails on a bearing of S20°W for 14 km; then it changes direction and sails for 20 km and drops anchor. Its bearing from the starting point is now N65°W.

a. How far is it from the starting point?

b. On what bearing did it sail the 20 km leg?

19. A cross-country runner runs at 8 km/h on a bearing of 150°T for 45 mins; then she changes direction to a bearing of 053°T and runs for 80 mins at a different speed until she is due east of the starting point.

a. How far was the second part of the run?

b. What was her speed for this section?

c. How far does she need to run to get back to the starting point?

20. From a fire tower, A, a fire is spotted on a bearing of N42°E. From a second tower, B, the fire is on a bearing of N12°W. The two fire towers are 23 km apart, and A is N63°W of B. How far is the fire from each tower?

21. **MC** A boat sails on a bearing of N15°E for 10 km and then on a bearing of S85°E until it is due east of the starting point. The distance from the starting point to the nearest kilometre is:

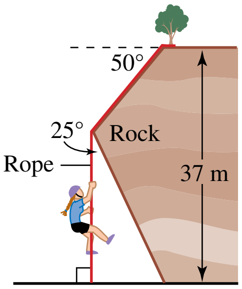
A. 10 km **B.** 38 km **C.** 113 km **D.** 114 km

22. **MC** A hill slopes at an angle of 30° to the horizontal. A tree that is 8 m tall and leaning downhill is growing at an angle of 10° m to the vertical and is part-way up the slope. The vertical height of the top of the tree above the slope is:

A. 7.37 m **B.** 8.68 m **C.** 10.84 m **D.** 15.04 m

23. A cliff is 37 m high. The rock slopes outward at an angle of 50° to the horizontal and then cuts back at an angle of 25° to the vertical, meeting the ground directly below the top of the cliff.

Carol wishes to abseil from the top of the cliff to the ground as shown in the diagram. Her climbing rope is 45 m long, and she needs 2 m to secure it to a tree at the top of the cliff. Will the rope be long enough to allow her to reach the ground?



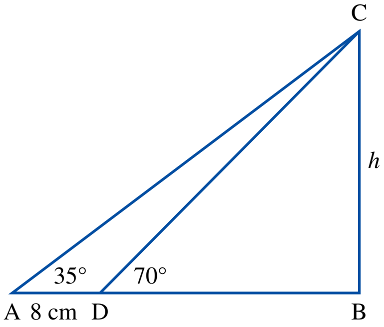
Problem solving

24. A yacht sets sail from a marina and sails on a bearing of 065°T for 3.5 km. It then turns and sails on a bearing of 127°T for another 5 km.

a. How far is the yacht from the marina?

b. On what bearing to the nearest minute should the yacht travel if it was to sail directly back to the marina?

25. Find the value of h, correct to 1 decimal place.

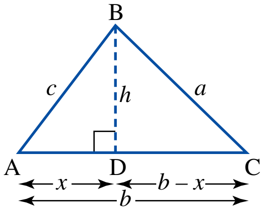


Reflection

In what situations can the sine rule be used?

20.3 The cosine rule

20.3.1 The cosine rule



- In any non–right-angled triangle ABC, a perpendicular line can be drawn from angle B to side b. Let D be the point where the perpendicular line meets side b, and let the length of the perpendicular line be h. Let the length AD = x units. The perpendicular line creates two right-angled triangles, ADB and CDB.
- Using triangle ADB and Pythagoras' theorem, we obtain:

$$c^2 = h^2 + x^2 \qquad [1]$$

Using triangle CDB and Pythagoras' theorem, we obtain:

$$a^2 = h^2 + (b - x)^2 \qquad [2]$$

Expanding the brackets in equation [2]:

$$a^2 = h^2 + b^2 - 2bx + x^2$$

Rearranging equation [2] and using $c^2 = h^2 + x^2$ from equation [1]:

$$\begin{aligned} a^2 &= h^2 + x^2 + b^2 - 2bx \\ &= c^2 + b^2 - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

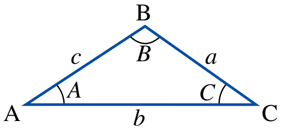
From triangle ABD, $x = c \cos A$; therefore $a^2 = b^2 + c^2 - 2bx$ becomes

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

- This is called the **cosine rule** and is a generalisation of Pythagoras' theorem.
- In a similar way, if the perpendicular line was drawn from angle A to side a or from angle C to side c, the two right-angled triangles would give $c^2 = a^2 + b^2 - 2ab \cos C$ and $b^2 = a^2 + c^2 - 2ac \cos B$ respectively. From this, the cosine rule can be stated:

  In any triangle ABC:

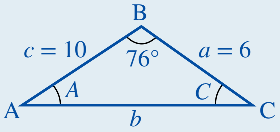
  $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$



- The cosine rule can be used to solve non–right-angled triangles if we are given:
  1. three sides of the triangle
  2. two sides of the triangle and the included angle (the angle between the given sides).

WORKED EXAMPLE 4

Find the third side of triangle ABC given $a = 6$, $c = 10$ and $B = 76°$.

THINK	**WRITE/DRAW**
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the cosine rule has been satisfied.	Yes, the cosine rule can be used since two side lengths and the included angle have been given.
3 Write down the appropriate cosine rule to find side b.	To find side b: $b^2 = a^2 + c^2 - 2ac \cos B$
4 Substitute the given values into the rule.	$= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 76°$
5 Evaluate.	$\approx 106.969\,372\,5$ $b \approx \sqrt{106.969\,372\,5}$
6 Round off the answer to 2 decimal places.	≈ 10.34

- *Note:* Once the third side has been found, the sine rule could be used to find other angles if necessary.
- If three sides of a triangle are known, an angle could be found by transposing the cosine rule to make cos A, cos B or cos C the subject.

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \Rightarrow \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

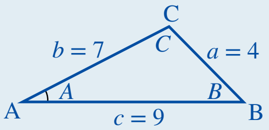
$$b^2 = a^2 + c^2 - 2ac \cos B \quad \Rightarrow \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \quad \Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

WORKED EXAMPLE 5

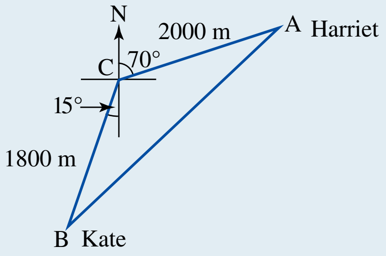
TI | CASIO

Find the smallest angle in the triangle with sides 4 cm, 7 cm and 9 cm.

THINK	**WRITE/DRAW**
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information. *Note:* The smallest angle will correspond to the smallest side.	 Let $a = 4$ $b = 7$ $c = 9$
2 Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since three side lengths have been given.
3 Write down the appropriate cosine rule to find angle A.	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
4 Substitute the given values into the rearranged rule.	$= \dfrac{7^2 + 9^2 - 4^2}{2 \times 7 \times 9}$
5 Evaluate.	$= \dfrac{114}{126}$
6 Transpose the equation to make A the subject by taking the inverse cos of both sides.	$A = \cos^{-1}\left(\dfrac{114}{126}\right)$ $\approx 25.208\,765\,3^\circ$
7 Round off the answer to degrees and minutes.	$\approx 25^\circ 13'$

WORKED EXAMPLE 6

Two rowers, Harriet and Kate, set out from the same point. Harriet rows N70°E for 2000 m and Kate rows S15°W for 1800 m. How far apart are the two rowers?

THINK	**WRITE/DRAW**
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	
2 Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since two side lengths and the included angle have been given.
3 Write down the appropriate cosine rule to find side c.	To find side c: $c^2 = a^2 + b^2 - 2ab \cos C$
4 Substitute the given values into the rule.	$= 2000^2 + 1800^2 - 2 \times 2000 \times 1800 \cos 125^\circ$
5 Evaluate.	$\approx 11\,369\,750.342$ $c \approx \sqrt{11\,369\,750.342}$
6 Round off the answer to 2 decimal places.	≈ 3371.91
7 Answer the question.	The rowers are 3371.91 m apart.

Exercise 20.3 The cosine rule

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3–5, 7, 9, 11, 13, 15	Questions: 1, 2, 4, 6–9, 11–13, 15, 16, 18	Questions: 1, 3, 4, 6, 7, 9–14, 16–19

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE4** Find the third side of triangle ABC given $a = 3.4$, $b = 7.8$ and $C = 80°$.

2. In triangle ABC, $b = 64.5$, $c = 38.1$ and $A = 58°34'$. Find a.

3. In triangle ABC, $a = 17$, $c = 10$ and $B = 115°$. Find b, and hence find A and C.

4. **WE5** Find the smallest angle in the triangle with sides 6 cm, 4 cm and 8 cm. (*Hint:* The smallest angle is opposite the smallest side.)

5. In triangle ABC, $a = 356$, $b = 207$ and $c = 296$. Find the largest angle.

6. In triangle ABC, $a = 23.6$, $b = 17.3$ and $c = 26.4$. Find the size of all the angles.

7. **WE6** Two rowers set out from the same point. One rows N30°E for 1500 m and the other rows S40°E for 1200 m. How far apart are the two rowers?

8. Maria cycles 12 km in a direction N68°W and then 7 km in a direction of N34°E.

  a. How far is she from her starting point?

  b. What is the bearing of the starting point from her finishing point?

Understanding

9. A garden bed is in the shape of a triangle, with sides of length 3 m, 4.5 m and 5.2 m.

  a. Calculate the smallest angle.

  b. Hence, find the area of the garden. (*Hint:* Draw a diagram, with the longest length as the base of the triangle.)

10. A hockey goal is 3 m wide. When Sophie is 7 m from one post and 5.2 m from the other, she shoots for goal. Within what angle, to the nearest degree, must the shot be made if it is to score a goal?

11. An advertising balloon is attached to two ropes 120 m and 100 m long. The ropes are anchored to level ground 35 m apart. How high can the balloon fly?

12. A plane flies in a direction of N70°E for 80 km and then on a bearing of S10°W for 150 km.

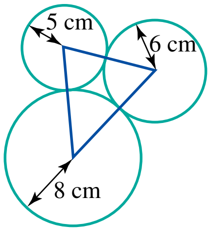
  a. How far is the plane from its starting point?

  b. What direction is the plane from its starting point?

13. Ship A is 16.2 km from port on a bearing of 053°T and ship B is 31.6 km from the same port on a bearing of 117°T. Calculate the distance between the two ships.

14. A plane takes off at 10.00 am from an airfield and flies at 120 km/h on a bearing of N35°W. A second plane takes off at 10.05 am from the same airfield and flies on a bearing of S80°E at a speed of 90 km/h. How far apart are the planes at 10.25 am?

Reasoning



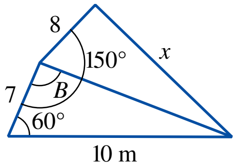
15. Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Find the largest angle in the triangle.

16. For the given shape at near right, determine:

a. the length of the diagonal

b. the magnitude (size) of angle B

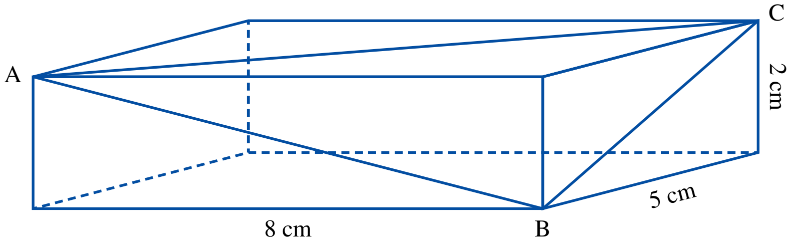
c. the length of x.



17. From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 minutes the angle of depression is 15°. How fast does the yacht sail?

Problem solving

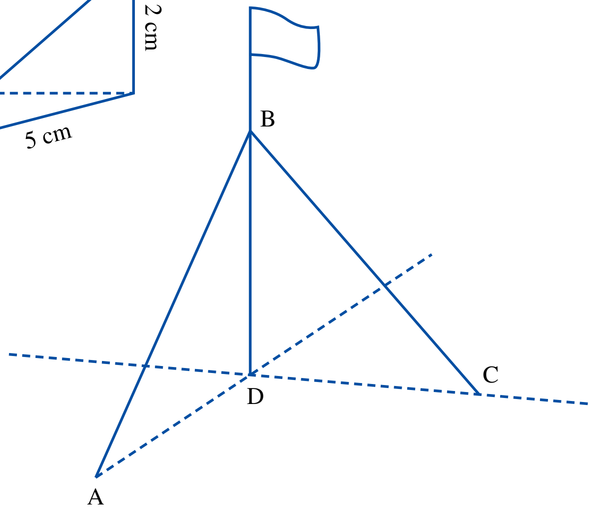
18. Find the measure of angles CAB, ABC and BCA. Give your answers correct to 2 decimal places.



19. A vertical flag pole DB is supported by two wires AB and BC. AB is 5.2 metres long, BC is 4.7 metres long and B is 3.7 metres above ground level. Angle ADC is a right angle.

a. Find the distance from A to C.

b. Calculate the angle between AB and BC.



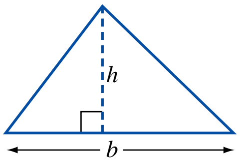
Reflection

In what situations would you use the sine rule rather than the cosine rule?

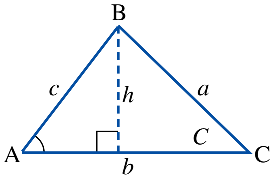
20.4 Area of triangles

20.4.1 Area of triangles

- The area of any triangle is given by the rule area $= \frac{1}{2}bh$, where b is the base and h is the perpendicular height of the triangle.



- However, often the perpendicular height is not given and needs to be calculated first. In the triangle ABC, b is the base and h is the perpendicular height of the triangle.
- Using the trigonometric ratio for sine:



$$\sin A = \frac{h}{c}$$

Transposing the equation to make h the subject, we obtain:

$$h = c \sin A$$

- Therefore, the area of triangle ABC becomes:

$$\boxed{\text{area} = \tfrac{1}{2} bc \sin A}$$

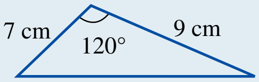
- Depending on how the triangle is labelled, the formula could read:

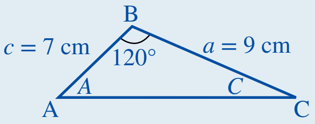
$$\text{area} = \tfrac{1}{2} ab \sin C \quad \text{area} = \tfrac{1}{2} ac \sin B \quad \text{area} = \tfrac{1}{2} bc \sin A$$

- The area formula may be used on any triangle provided that two sides of the triangle and the included angle (that is, the angle between the two given sides) are known.

WORKED EXAMPLE 7

Find the area of the triangle shown.

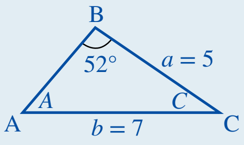


THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 9$ cm, $c = 7$ cm, $B = 120°$.
2 Check that the criterion for the area rule has been satisfied.	The area rule can be used since two side lengths and the included angle have been given.
3 Write down the appropriate rule for the area.	$\text{Area} = \frac{1}{2}ac \sin B$
4 Substitute the known values into the rule.	$= \frac{1}{2} \times 9 \times 7 \times \sin 120°$
5 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	$\approx 27.28 \text{ cm}^2$

WORKED EXAMPLE 8

TI | CASIO

A triangle has known dimensions of $a = 5$ cm, $b = 7$ cm and $B = 52°$. Find A and C and hence the area.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 5$, $b = 7$, B = 52°.
2 Check whether the criterion for the area rule has been satisfied.	The area rule cannot be used since the included angle has not been given.

3 Write down the sine rule to find A.

To find angle A:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

4 Substitute the known values into the rule.

$$\frac{5}{\sin A} = \frac{7}{\sin 52°}$$

5 Transpose the equation to make $\sin A$ the subject.

$$5 \sin 52° = 7 \sin A$$

$$\sin A = \frac{5 \sin 52°}{7}$$

6 Evaluate.

$$A = \sin^{-1}\left(\frac{5 \sin 52°}{7}\right)$$

$$\approx 34.254\,151\,87°$$

7 Round off the answer to degrees and minutes.

$$\approx 34°15'$$

8 Determine the value of the included angle, C, using the fact that the angle sum of any triangle is 180°.

$$C \approx 180° - (52° + 34°15')$$
$$= 93°45'$$

9 Write down the appropriate rule for the area.

$$\text{Area} = \tfrac{1}{2} ab \sin C$$

10 Substitute the known values into the rule.

$$\approx \tfrac{1}{2} \times 5 \times 7 \times \sin 93°45'$$

11 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.

$$\approx 17.46\text{ cm}^2$$

20.4.2 Heron's formula

- If the lengths of all the sides of the triangle are known but none of the angles are known, the cosine rule can be used to find an angle and then the area can be calculated. Alternatively, **Heron's formula** could be used to find the area.
- Heron's formula gives the area of a triangle as:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the *semi-perimeter* of the triangle; that is,

$$s = \tfrac{1}{2}(a + b + c)$$

Note: The proof of this formula is beyond the scope of this course.

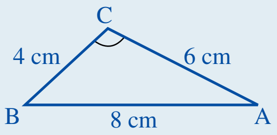
WORKED EXAMPLE 9

Find the area of the triangle with sides of 4 cm, 6 cm and 8 cm.

THINK

WRITE/DRAW

1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.



Let $a = 4, b = 6, c = 8$.

2 Determine which area rule will be used.

Since three side lengths have been given, use Heron's formula.

3 Write down the rule for Heron's formula.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

4 Write down the rule for s, the semi-perimeter of the triangle.	$s = \frac{1}{2}(a + b + c)$
5 Substitute the given values into the rule for the semi-perimeter.	$= \frac{1}{2}(4 + 6 + 8)$ $= 9$
6 Substitute all of the known values into Heron's formula.	$\text{Area} = \sqrt{9(9-4)(9-6)(9-8)}$
7 Evaluate.	$= \sqrt{9 \times 5 \times 3 \times 1}$ $= \sqrt{135}$ $\approx 11.618\,950\,04$
8 Round off the answer to 2 decimal places and include the appropriate unit.	$\approx 11.62\ \text{cm}^2$

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Complete this digital doc: WorkSHEET: Cosine rule and area of triangles (doc-14632)

Exercise 20.4 Area of triangles

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 12, 14, 16, 18, 20, 23	Questions: 1–6, 8, 9, 11, 13, 15, 17, 19, 21, 23	Questions: 1, 4, 5, 8–10, 12–14, 16–24

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE7** Find the area of the triangle ABC with $a = 7$, $b = 4$ and $C = 68°$.

2. Find the area of the triangle ABC with $a = 7.3$, $c = 10.8$ and $B = 104°40'$.

3. Find the area of the triangle ABC with $b = 23.1$, $c = 18.6$ and $A = 82°17'$.

4. A triangle has $a = 10$ cm, $c = 14$ cm and $C = 48°$. Find A and B and hence the area.

5. **WE8** A triangle has $a = 17$ m, $c = 22$ m and $C = 56°$. Find A and B and hence the area.

6. A triangle has $b = 32$ mm, $c = 15$ mm and $B = 38°$. Find A and C and hence the area.

7. **MC** In a triangle, $a = 15$ m, $b = 20$ m and $B = 50°$. The area of the triangle is:

A. 86.2 m^2 **B.** 114.9 m^2 **C.** 149.4 m^2 **D.** 172.4 m^2

8. **WE9** Find the area of the triangle with sides of 5 cm, 6 cm and 8 cm.

9. Find the area of the triangle with sides of 40 mm, 30 mm and 5.7 cm.

10. Find the area of the triangle with sides of 16 mm, 3 cm and 2.7 cm.

11. **MC** A triangle has sides of length 10 cm, 14 cm and 20 cm. The area of the triangle is:

A. 41 cm^2 **B.** 65 cm^2 **C.** 106 cm^2 **D.** 137 cm^2

12. A piece of metal is in the shape of a triangle with sides of length 114 mm, 72 mm and 87 mm. Find its area using Heron's formula.

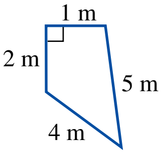
13. A triangle has the largest angle of 115°. The longest side is 62 cm and another side is 35 cm. Find the area of the triangle.

14. A triangle has two sides of 25 cm and 30 cm. The angle between the two sides is 30°. Find:

a. its area **b.** the length of its third side

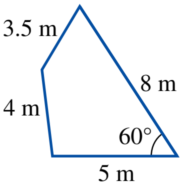
c. its area using Heron's formula.

Understanding



15. The surface of a fish pond has the shape shown in the diagram at right. How many goldfish can the pond support if each fish requires 0.3 m^2 surface area of water?

16. Find the area of this quadrilateral.

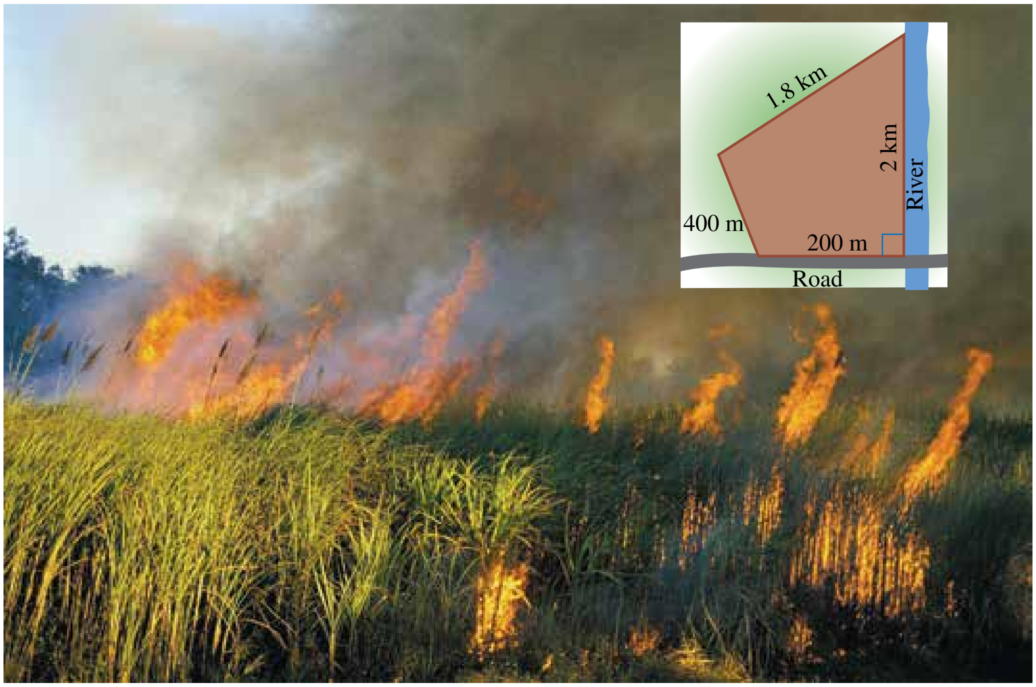


Reasoning

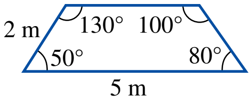
17. A parallelogram has diagonals of length 10 cm and 17 cm. An angle between them is 125°. Find:
- **a.** the area of the parallelogram
- **b.** the dimensions of the parallelogram.

18. A lawn is to be made in the shape of a triangle, with sides of length 11 m, 15 m and 17.2 m. How much grass seed, to the nearest kilogram, needs to be purchased if it is sown at the rate of 1 kg per 5 m^2?

19. A bushfire burns out an area of level grassland shown in the diagram. (*Note:* This is a sketch of the area and is not drawn to scale.) What is the area, in hectares, of the land that is burned?



20. An earth embankment is 27 m long and has a vertical cross-section shown in the diagram. Find the volume of earth needed to build the embankment.



21. MC A parallelogram has sides of 14 cm and 18 cm and an angle between them of 72°. The area of the parallelogram is:

A. 118.4 cm^2 **B.** 172.4 cm^2 **C.** 239.7 cm^2 **D.** 252 cm^2

22. MC An advertising hoarding is in the shape of an isosceles triangle, with sides of length 15 m, 15 m and 18 m. It is to be painted with two coats of purple paint. If the paint covers 12 m^2 per litre, the amount of paint needed, to the nearest litre, would be:

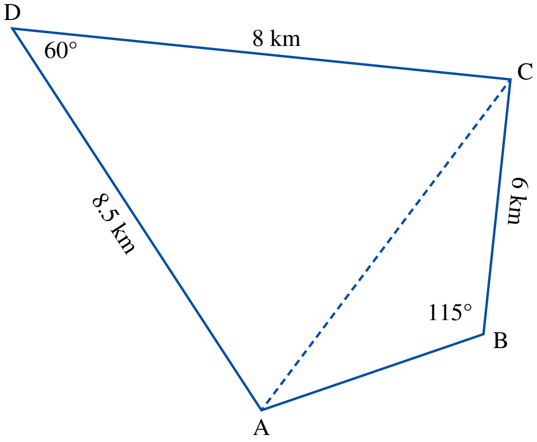
A. 9 L **B.** 18 L **C.** 24 L **D.** 36 L

Problem solving

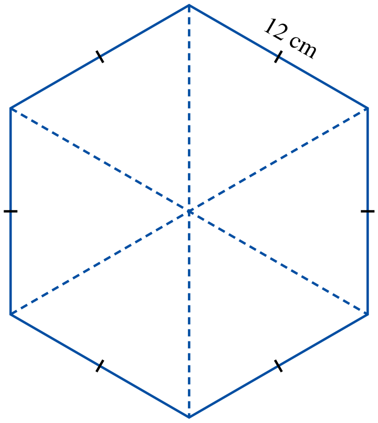
23. A surveyor measured the boundaries of a property as shown. The side AB could not be measured because it crossed through a marsh. The owner of the property wanted to know the total area and the length of the side AB.

a. Find the area of the triangle ACD.
b. Find the distance AC.
c. Find the angle CAB.
d. Find the angle ACB.
e. Find the length AB.
f. Find the area of the triangle ABC.
g. What is the area of the property?

Give all lengths correct to 2 decimal places and angles to the nearest degree.



24. A regular hexagon has sides of length 12 centimetres. It is divided into six smaller equilateral triangles. Find the area of the hexagon, giving your answer correct to 2 decimal places.



Reflection

List three formulas for finding the area of a triangle.

CHALLENGE 20.1

Cosine, sine and area rule applied to right-angled triangles

The cosine rule states that $c^2 = a^2 + b^2 - 2ab\cos C$. If $C° = 90°$ and $\cos 90° = 0$, then:

$c^2 = a^2 + b^2 - 2ab(0)$

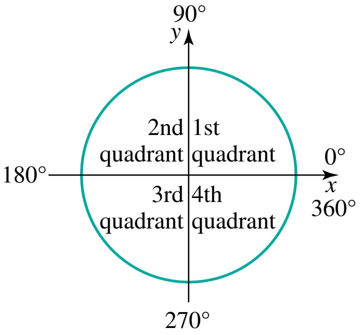
$c^2 = a^2 + b^2$.

Hence, when the cosine rule is applied to a right-angled triangle, you are left with Pythagoras' theorem for right-angled triangles.

Investigate what happens when the sine and area rules are applied to right-angled triangles.

20.5 The unit circle

20.5.1 The unit circle



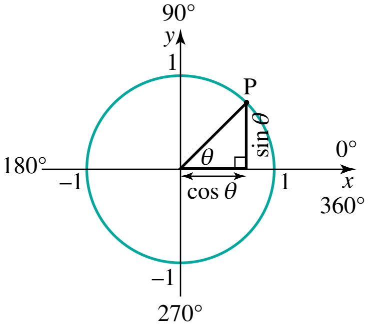
- A **unit circle** is a circle with a radius of 1 unit.
- The unit circle is divided into 4 quadrants.
- All angles in quadrant 1 are between 0° and 90°. All angles in quadrant 2 are between 90° and 180°, in quadrant 3 between 180° and 270°, and in quadrant 4 between 270° and 360°.
- The quadrants are numbered in the anticlockwise direction, and positive angles are measured anticlockwise from 0°. Negative angles are measured clockwise from 0°.

WORKED EXAMPLE 10

State the quadrant of the unit circle in which each of the following angles is found.

a 145° **b 282°**

THINK	WRITE
a The given angle is between 90° and 180°. State the appropriate quadrant.	**a** 145° is in quadrant 2.
b The given angle is between 270° and 360°. State the appropriate quadrant.	**b** 282° is in quadrant 4.



- If a right-angled triangle containing angle θ is constructed in quadrant 1 of the unit circle, then the value of sin θ can be found by measuring the length of the opposite side and the value of cos θ by measuring the length of the adjacent side.
- The point of intersection of the radius (which is one of the arms of angle θ) with the unit circle is P. From the diagram above, observe that cos θ represents the x-coordinate of point P and sin θ represents its y-coordinate. This observation provides us with the technique for finding sine and cosine of any angle in the unit circle, as shown in the diagram at the top of this page.
- To find the value of sine, cosine or tangent of any angle θ from the unit circle, follow these steps:
  1. Draw a unit circle.
  2. Construct the required angle so that its vertex is at the origin and the angle itself is measured from 0° (as marked on the x-axis) in an anticlockwise direction. Label the point of intersection of the radius and the unit circle, P.
  3. Use a ruler to find the coordinates of point P.
  4. Interpret the results: $x = \cos\theta$, $y = \sin\theta$ and $\dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$, where x and y are coordinates of P.

20.5.2 The 4 quadrants of the unit circle

For the following activity and exercise, we will need to be able to read values for sine and cosine from a unit circle.

Constructing your own unit circle

Step 1. Using graph paper, carefully draw a circle centred at the origin and with a radius of 5 cm. Label the x- and y-axes.

Step 2. On your graph, mark in 0°, 90°, 180°, 270° and 360°.

Step 3. Since we need a unit circle, 5 cm will represent 1 unit; that is, 5 cm = 1 unit. Carefully mark a scale on each axis, where each centimetre represents 0.2 units. (Draw as carefully as possible, since you will need to read values from your axes in Exercise 20.5.)

Use a unit circle to investigate the following.

1. If P has coordinates (x, y) and is located on the unit circle, what is the highest value that the x-coordinate can take? Hence, what is the largest value that the cosine of an angle can take?
2. What is the lowest value that the x-coordinate can take? Hence, what is the smallest value that the cosine of an angle can take?
3. What is the highest value that the y-coordinate can take? Hence, what is the largest value that the sine of an angle can take?
4. What is the lowest value that the y-coordinate can take? Hence, what is the smallest value that the sine of an angle can take?
5. Note that P could be in any of the four quadrants (depending on the size of the angle). Hence, its coordinates could take either positive or negative values, or zero.

   Copy and complete the table below to summarise whether sine, cosine and tangent are positive or negative for angles in each of the four quadrants.

	1st quadrant	**2nd quadrant**	**3rd quadrant**	**4th quadrant**
$\sin\theta$	positive (+)		negative (−)	
$\cos\theta$	positive (+)	negative (−)		
$\tan\theta$	positive (+)			negative (−)

6. Copy and complete the following sentences.

   Sine is positive in the ______ and ______ quadrants and is negative in the ______ and ______ quadrants.

   Cosine is positive in the ______ and ______ quadrants and is negative in the ______ and ______ quadrants.

7. Use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to work out whether the tangent of an angle in each quadrant is positive or negative. Copy and complete the following sentence: Tangent is positive in the ______ and ______ quadrants and is negative in the ______ and ______ quadrants.

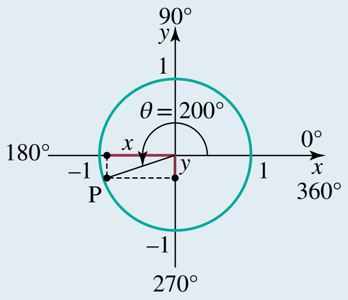
WORKED EXAMPLE 11 TI | CASIO

Find the approximate value of each of the following using the unit circle.

a **sin 200°**

b **cos 200°**

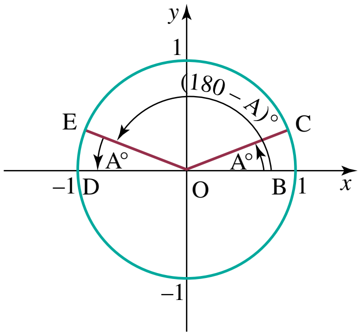
c **tan 200°**

THINK	WRITE/DRAW
Draw a unit circle and construct an angle of 200°. Label the point corresponding to the angle of 200° on the circle P. Highlight the lengths, representing the x- and y-coordinates of point P.	
a The sine of the angle is given by the y-coordinate of P. Find the y-coordinate of P by measuring the distance along the y-axis. State the value of sin 200°. (*Note:* The sine value will be negative as the y-coordinate is negative.)	**a** $\sin 200° = -0.3$
b The cosine of the angle is given by the x-coordinate of P. Find the x-coordinate of P by measuring the distance along the x-axis. State the value of cos 200°. (*Note:* Cosine is also negative in quadrant 3, as the x-coordinate is negative.)	**b** $\cos 200° = -0.9$
c $\tan 200° = \dfrac{\sin 200°}{\cos 200°}$	**c** $\dfrac{-0.3}{-0.9} = \dfrac{1}{3} = 0.3333$

- The approximate results obtained in Worked example 11 can be verified with the aid of a calculator:

  $$\sin 200° = -0.342\,020\,143,\ \cos 200° = -0.939\,692\,62 \text{ and } \tan 200° = 0.3640.$$

  Rounding these values to 1 decimal place would give -0.3, -0.9 and 0.4 respectively, which match the values obtained from the unit circle.
- Consider the special relationship between the sine, cosine and tangent of supplementary angles, say $A°$ and $(180 - A)°$.
- In the diagram, triangle OBC is reflected in the y-axis to form triangle ODE. Clearly, $\angle BOC = \angle DOE = A°$. It follows that the obtuse angle BOE equals $(180 - A)°$.
  From the diagram, $|BC| = |ED|$ and $|OB| = |OD|$.
  $ED = \sin(180 - A)$ and is positive.
  $OD = \cos(180 - A)$ and is negative.
  Thus, $\sin(180 - A) = \sin A$ and $\cos(180 - A) = -\cos A$
  It follows that $\tan(180 - A) = \dfrac{\sin(180 - A)}{\cos(180 - A)} = \dfrac{\sin A}{-\cos A} = -\tan A$.



Exercise 20.5 The unit circle

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1, 4–15, 18	Questions: Questions:1–6, 8, 10, 12, 15, 17, 18	Questions: 1–19

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE10** State which quadrant of the unit circle each of the following angles is in.

a. 60° **b.** 130° **c.** 310° **d.** 260°
e. 100° **f.** 185° **g.** 275° **h.** 295°

2. **MC** If $\theta = 43°$, the triangle drawn to show this would be in:

A. quadrant 1 **B.** quadrant 2 **C.** quadrant 3 **D.** quadrant 4

3. **MC** If $\theta = 295°$, the triangle drawn to show this would be in:

A. quadrant 1 **B.** quadrant 2 **C.** quadrant 3 **D.** quadrant 4

4. **WE11** Find the value of each of the following using the unit circle.

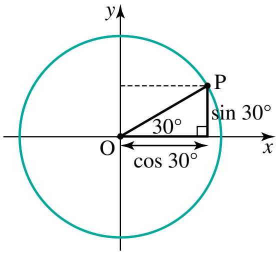
a. sin 20° **b.** cos 20° **c.** cos 100° **d.** sin 100°
e. sin 320° **f.** cos 320° **g.** sin 215° **h.** cos 215°

5. Use the unit circle to find each of the following.

a. sin 90° **b.** cos 90° **c.** sin 180° **d.** cos 180°
e. sin 270° **f.** cos 270° **g.** sin 360° **h.** cos 360°

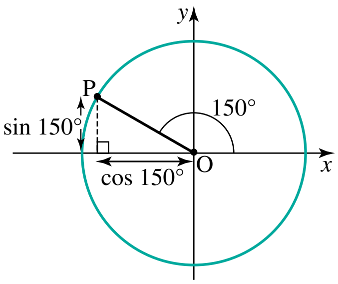
Understanding

6. On the unit circle, use a protractor to measure an angle of 30° from the positive x-axis. Mark the point P on the circle. Use this point to construct a triangle in quadrant 1 as shown.



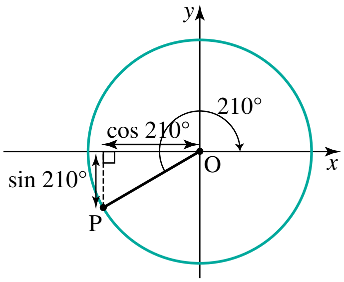
a. Find cos 30°. (Remember that the length of the adjacent side of the triangle is cos 30°.)
b. Find sin 30°. (This is the length of the opposite side of the triangle.)
c. Check your answers in **a** and **b** by finding these values with a calculator.

7. Using your graph of the unit circle, measure 150° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 2 as shown.



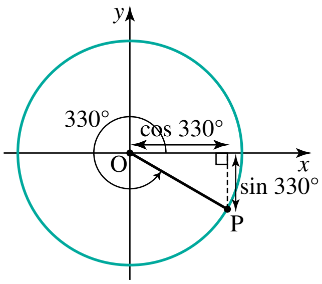
a. What angle does the radius OP make with the negative x-axis?
b. Remembering that $x = \cos\theta$, use your circle to find the value of cos 150°.
c. How does cos 150° compare to cos 30°?
d. Remembering that $y = \sin\theta$, use your circle to find the value of sin 150°.
e. How does sin 150° compare with sin 30°?

8. On the unit circle, measure 210° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 3 as shown.



a. What angle does the radius OP make with the negative x-axis?
b. Use your circle to find the value of cos 210°.
c. How does cos 210° compare to cos 30°?
d. Use your circle to find the value of sin 210°.
e. How does sin 210° compare with sin 30°?

9. On the unit circle, measure 330° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 4 as shown.



a. What angle does the radius OP make with the positive x-axis?
b. Use your circle to find the value of cos 330°.
c. How does cos 330° compare to cos 30°?
d. Use your circle to find the value of sin 330°.
e. How does sin 330° compare with sin 30°?

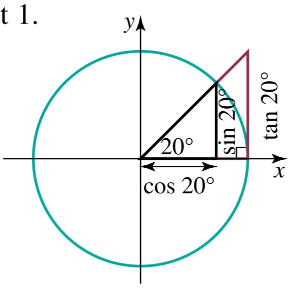
10. On the unit circle, draw an appropriate triangle for the angle of 20° in quadrant 1.

a. Find sin 20°. **b.** Find cos 20°.

c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
Accurately measure the length of the tangent between the x-axis and the point where it meets the hypotenuse and, hence, state the value of tan 20°.

d. What is the value of $\frac{\sin 20^\circ}{\cos 20^\circ}$?

e. How does tan 20° compare with $\frac{\sin 20^\circ}{\cos 20^\circ}$?



11. On the unit circle, draw an appropriate triangle for the angle of 135° in quadrant 2.

a. Find sin 135°, using sin 45°. **b.** Find cos 135°, using cos 45°.

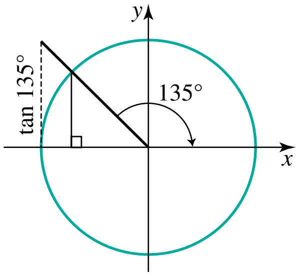
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.

Accurately measure the length of the tangent to where it meets the hypotenuse to find the value of tan 135°.

d. What is the value of $\frac{\sin 135^\circ}{\cos 135^\circ}$?

e. How does tan 135° compare with $\frac{\sin 135^\circ}{\cos 135^\circ}$?

f. How does tan 135° compare with tan 45°?



12. On the unit circle, draw an appropriate triangle for the angle of 220° in quadrant 3.

a. Find sin 220°. **b.** Find cos 220°.

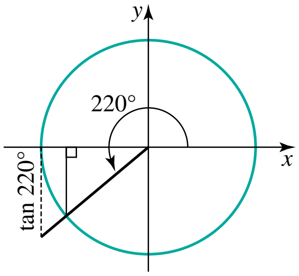
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.

Find tan 220° by accurately measuring the length of the tangent to where it meets the hypotenuse.

d. What is the value of $\frac{\sin 220^\circ}{\cos 220^\circ}$?

e. How does tan 220° compare with $\frac{\sin 220^\circ}{\cos 220^\circ}$?

f. How does tan 220° compare with tan 40°? (Use a calculator.)



13. On the unit circle, draw an appropriate triangle for the angle of 300° in quadrant 4.

a. Find sin 300°. **b.** Find cos 300°.

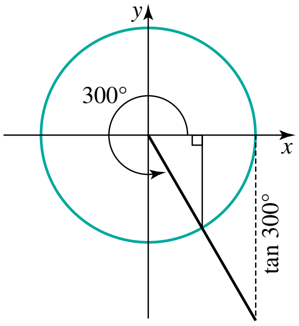
c. Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.

Find tan 300° by accurately measuring the length of the tangent to where it meets the hypotenuse.

d. What is the value of $\frac{\sin 300^\circ}{\cos 300^\circ}$?

e. How does tan 300° compare with $\frac{\sin 300^\circ}{\cos 300^\circ}$?

f. How does tan 300° compare with tan 60°? (Use a calculator.)



14. **MC** In a unit circle, the length of the radius is equal to:

A. sin θ **B.** cos θ **C.** tan θ **D.** 1

Reasoning

15. If $\sin x^\circ = p$, $0 \le x \le 90^\circ$, write each of the following in terms of p.

a. $\cos x^\circ$ **b.** $\sin(180 - x)^\circ$ **c.** $\cos(180 - x)^\circ$

16. Simplify $\sin(180-x)^\circ - \sin x^\circ$.

17. The point P $(\cos\theta, \sin\theta)$ lies in the first quadrant on the circumference of a unit circle. Find the new coordinates of P under the following transformations.

a. Dilated by a factor of 2 horizontally and then 3 vertically

b. Rotated 180° anticlockwise about the origin **c.** Reflected in both axes

Problem solving

18. Simplify:

a. $\dfrac{\sin(180^\circ - \theta)\cos(270^\circ + \theta)}{\tan(-\theta)}$, given that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

b. $\dfrac{\sin^2(90^\circ - \theta) - \cos^2(270^\circ - \theta)}{\cos(90^\circ - \theta) + \cos(-\theta)}$.

19. Simplify:

a. $\dfrac{\sin(90^\circ - x)\cos(90^\circ - x)}{\cos^2(x)}$

b. $\sin^2(180^\circ - x)\operatorname{cosec}(270^\circ + x) + \cos^2(360^\circ - x)\sec(180^\circ - x)$

where $\sec(x) = \dfrac{1}{\cos(x)}$ and $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$.

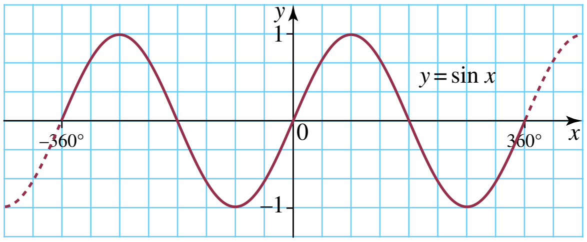
Reflection

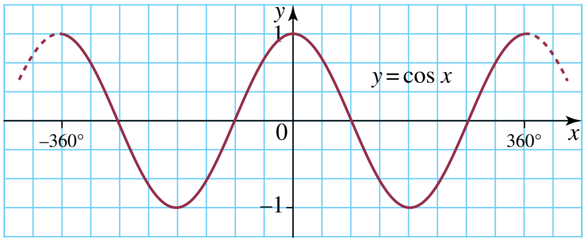
What is the length of the diameter of the unit circle?

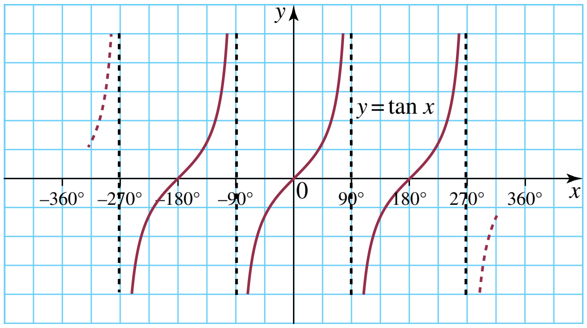
20.6 Trigonometric functions

20.6.1 Sine, cosine and tangent graphs

- The graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ are shown below.







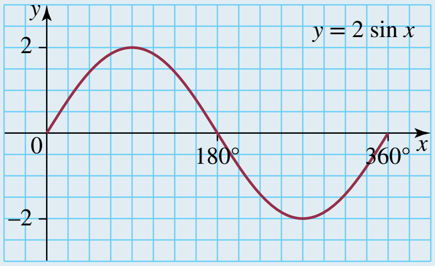
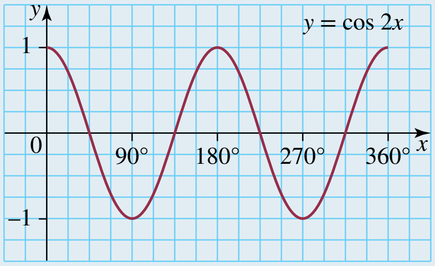
- Trigonometric graphs repeat themselves continuously in cycles, and hence they are called **periodic functions**.
- The period of the graph is the distance between repeating peaks or troughs. In the graphs shown above, the period between the repeating peaks for $y = \sin x$ and $y = \cos x$ is 360°. The graph of $y = \tan x$ shows a periodic function, but it is not of the same form as $y = \sin x$ and $y = \cos x$. The period of the basic tan graph is 180°, and asymptotes occur at $x = 90^\circ$ and intervals of 180°.

- The **amplitude** of a periodic graph is half the distance between the maximum and minimum values of the function. In the graphs of $y = \sin x$ and $y = \cos x$ shown on the previous page, the distance is half of two units (the distance between -1 and 1), hence the amplitude is 1 unit. Amplitude can also be described as the amount by which the graph goes above and below its mean value. In the above examples, the mean value lies along the x-axis. The amplitude of $y = \tan x$ is undefined.

WORKED EXAMPLE 12

TI | CASIO

Sketch the graphs of a $y = 2\sin x$ and b $y = \cos 2x$ for $0° \le x \le 360°$.

THINK	WRITE/DRAW
a **1** The graph must be drawn from 0° to 360°. **2** Compared to the graph of $y = \sin x$ each value of $\sin x$ has been multiplied by 2, therefore the amplitude of the graph must be 2. **3** Label the graph $y = 2\sin x$.	**a** 
b **1** The graph must be drawn from 0° to 360°. **2** Compared to the graph of $y = \cos x$, each value of x has been multiplied by 2, therefore the period of the graph must become 180°. **3** Label the graph $y = \cos 2x$.	**b** 

- For the graph of $y = a\sin nx$, or $y = a\cos nx$, the amplitude is a and the period becomes $\frac{360°}{n}$.
- If the graph has a negative value of a, the amplitude is the positive value of a, i.e. the amplitude is always $|a|$.
- For the graph of $y = a\tan nx$, the amplitude is undefined and the period is $\frac{180°}{n}$.

WORKED EXAMPLE 13

For each of the following graphs, state:

i the amplitude **ii the period.**

a $y = 2\sin 3x$ **b $y = \cos\frac{x}{3}$** **c $y = \tan 2x$**

THINK	WRITE
a The value of a is 2. The period is $\frac{360°}{n}$.	**a** **i** Amplitude = 2 **ii** Period $= \frac{360}{3}$ $= 120°$
b The value of a is 1. The period is $\frac{360°}{n}$.	**b** **i** Amplitude = 1 **ii** Period $= \frac{360}{\frac{1}{3}}$ $= 1080°$

c The tangent curve has infinite amplitude. The period is $\frac{180°}{2}$.	**c** **i** Amplitude infinite **ii** Period $= \frac{180°}{2}$ $= 90°$

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Try out this interactivity: Trigonometric functions (int-2796)

Exercise 20.6 Trigonometric functions

assessON

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–19, 22, 24, 27	Questions: 1–21, 23–27	Questions: 1–20, 22, 24, 26–28

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Fluency

1. Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin x													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
sin x													

2. On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from -1 to 1. Carefully plot the graph of $y = \sin x$ using the values from the table in question **1**.

3. How long does it take for the graph of $y = \sin x$ to complete one full cycle?

4. From your graph of $y = \sin x$, find the value of y for each of the following.

a. $x = 42°$
b. $x = 130°$
c. $x = 160°$
d. $x = 200°$
e. $x = 180°$
f. $x = 70°$
g. $x = 350°$
h. $x = 290°$

5. From your graph of $y = \sin x$, find the value of x for each of the following.

a. $y = 0.9$
b. $y = -0.9$
c. $y = 0.7$
d. $y = -0.5$
e. $y = -0.8$
f. $y = 0.4$

6. Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos x													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
cos x													

7. On graph paper, rule x- and y-axes and carefully mark a scale along each axis.
Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°.
Use 2 cm = 1 unit along the y-axis to show y-values from -1 to 1.
Carefully plot the graph of $y = \cos x$ using the values from the table in question **6**.

8. If you were to continue the graph of $y = \cos x$, what shape would you expect it to take?

9. Is the graph of $y = \cos x$ the same as the graph of $y = \sin x$? How does it differ? What features are the same?

10. Using the graph of $y = \cos x$, find a value of y for each of the following.

a. 48° **b.** 170° **c.** 180° **d.** 340°
e. 240° **f.** 140° **g.** 40° **h.** 165°

11. Using the graph of $y = \cos x$, find a value of x for each of the following.

a. $y = -0.5$ **b.** $y = 0.8$ **c.** $y = 0.7$
d. $y = -0.6$ **e.** $y = 0.9$ **f.** $y = -0.9$

12. Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
tan x													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
tan x													

13. On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from -2 to 2. Carefully plot the graph of $y = \tan x$ using the values from the table in question **12**.

14. If you were to continue the graph of $y = \tan x$, what shape would you expect it to take?

15. Is the graph of $y = \tan x$ the same as the graphs of $y = \sin x$ and $y = \cos x$? How does it differ? What features are the same?

16. Using the graph of $y = \tan x$, find a value of y for each of the following.

a. 60° **b.** 135° **c.** 310° **d.** 220°
e. 500° **f.** 590° **g.** 710° **h.** 585°

17. Using the graph of $y = \tan x$, find a value of x for each of the following.

a. $y = 1$ **b.** $y = 1.5$ **c.** $y = -0.4$ **d.** $y = -2$ **e.** $y = 0.2$ **f.** $y = -1$

18. **WE12** Sketch the following graphs

a. $y = \cos x,\ \text{for } x \in [-180°, 180°]$ **b.** $y = \sin x,\ \text{for } x \in [0°, 720°]$
c. $y = \sin 2x,\ \text{for } x \in [0°, 360°]$ **d.** $y = 2\cos x,\ \text{for } x \in [-360°, 0°]$

19. **WE13** For each of the graphs in question **18**, state

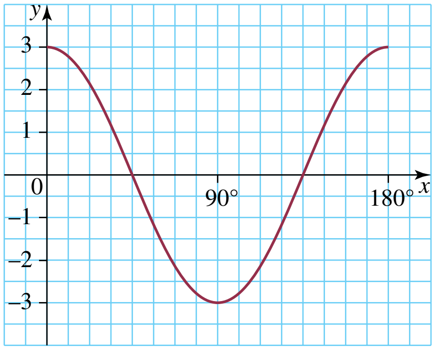
i. the period **ii.** the amplitude.

20. For each of the following, state:

i. the period **ii.** the amplitude.

a. $y = 3\cos 2x$ **b.** $y = 4\sin 3x$ **c.** $y = 2\cos \frac{x}{2}$
d. $y = \frac{1}{2}\sin \frac{x}{4}$ **e.** $y = -\sin x$ **f.** $y = -\cos 2x$

21. **MC** Parts **a** to **c** refer to the graph below.



a. The amplitude of the graph is:

A. 180° **B.** 90° **C.** 3 **D.** −3 **E.** 6

b. The period of the graph is:

A. 180° **B.** 360° **C.** 90° **D.** 3 **E.** −3

c. The equation of the graph could be:

A. $y = \cos x$ **B.** $y = \sin x$ **C.** $y = 3\cos\frac{x}{3}$

D. $y = 3\cos 2x$ **E.** $y = 3\sin 2x$

Understanding

22. Sketch each of the following graphs, stating the period and amplitude of each.

a. $y = 2\cos\frac{x}{3}$, for $x \in [0°, 360°]$ **b.** $y = -3\sin 2x$, for $x \in [0°, 360°]$

c. $y = 3\sin\frac{x}{2}$, for $x \in [-180°, 180°]$ **d.** $y = -\cos 3x$, for $x \in [0°, 360°]$

e. $y = 5\cos 2x$, for $x \in [0°, 180°]$ **f.** $y = -\sin 4x$, for $x \in [0°, 180°]$

23. Use technology to sketch the graphs of each of the following for $0° \le x \le 360°$.

a. $y = \cos x + 1$ **b.** $y = \sin 2x - 2$ **c.** $y = \cos(x - 60°)$ **d.** $y = 2\sin 4x + 3$

Reasoning

24. a. Sketch the graph of $y = \cos 2x$ for $x \in [0°, 360°]$.

i. What is the minimum value of y for this graph?

ii. What is the maximum value of y for this graph?

b. Using the answers obtained in part **a** write down the maximum and minimum values of $y = \cos 2x + 2$.

c. What would be the maximum and minimum values of the graph of $y = 2\sin x + 3$? Explain how you obtained these values.

25. a. Complete the table below by filling in the exact values of $y = \tan x$

x	0°	30°	60°	90°	120°	150°	180°
$y = \tan x$							

b. Sketch the graph of $y = \tan x$ for [0°, 180°].

c. What happens at $x = 90°$?

d. For the graph of $y = \tan x$, $x = 90°$ is called an asymptote. Write down when the next asymptote would occur.

e. State the period and amplitude of $y = \tan x$.

26. a. Sketch the graph of $y = \tan 2x$ for [0°, 180°].

b. When do the asymptotes occur?

c. State the period and amplitude of $y = \tan 2x$.

Problem solving

27. The height of the tide above the mean sea level on the first day of the month is given by the rule

$$h = 3\sin(30t°)$$

where t is the time in hours since midnight.

a. Sketch the graph of h versus t.

b. What was the height of the high tide?

c. Calculate the height of the tide at 8 am.

28. The temperature inside a house t hours after 3 am is given by the rule

$$T = 22 - 2\cos(15t°) \text{ for } 0 \le t \le 24$$

where t is the temperature in degrees Celsius.

a. What is the temperature inside the house at 9 am?

b. Sketch the graph of T versus t.

c. What is the warmest and coolest that it gets inside the house over the 24-hour period?

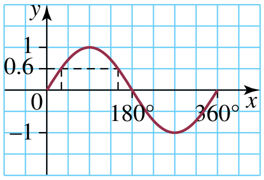
Reflection

For the graph of $y = a\tan nx$, what would be the period and amplitude?

20.7 Solving trigonometric equations

20.7.1 Solving trigonometric equations graphically

- Because of the periodic nature of circular functions, there are infinitely many solutions to trigonometric equations.
- Equations are usually solved within a particular domain (x-values), to restrict the number of solutions.
- The sine graph below shows the solutions between 0° and 360° for the equation $\sin x° = 0.6$.



In the example above, it can clearly be seen that there are two solutions to this equation, which are approximately $x = 37°$ and $x = 143°$.

- The smaller the period, the greater the number of solutions within a particular domain.
- It is difficult to obtain accurate answers from a graph. More accurate answers can be obtained using technology.

20.7.2 Solving trigonometric equations algebraically

Exact answers can be found for some trigonometric equations using the table at the top of page 846.

WORKED EXAMPLE 14

TI | CASIO

Solve the following equations.

a $\sin x = \dfrac{\sqrt{3}}{2},\ x \in [0°, 360°]$ **b** $\cos 2x = -\dfrac{1}{\sqrt{2}},\ x \in [0°, 360°]$

THINK	WRITE
a **1** The inverse operation of sine is $\sin^{-1}$.	**a** $x = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$
2 The first solution in the given domain from the table at the top of page 846 is $x = 60°$.	
3 Since sine is positive in the first and second quadrants, another solution must be $x = 180° - 60° = 120°$.	There are two solutions in the given domain, $x = 60°$ and $x = 120°$.
b **1** The inverse operation of cosine is $\cos^{-1}$.	**b** $2x = \cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$
2 From the table of values, $\cos^{-1}\dfrac{1}{\sqrt{2}} = 45°$.	
3 Cosine is negative in the second and third quadrants, which gives the first two solutions to the equation as: $180° - 45°$ and $180° + 45°$.	$2x = 135°, 225°$
4 Solve for x by dividing by 2.	$x = 67.5°, 112.5°$
5 Since the domain in this case is [0°, 360°] and the period has been halved, there must be 4 solutions altogether. The other 2 solutions can be found by adding the period onto each solution.	The period $= \dfrac{360°}{2} = 180°$ $x = 67.5° + 180°, 112.5° + 180°$ $x = 67.5°, 112.5°, 247.5°, 292.5°$

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Try out this interactivity: Trigonometry (int-2797)

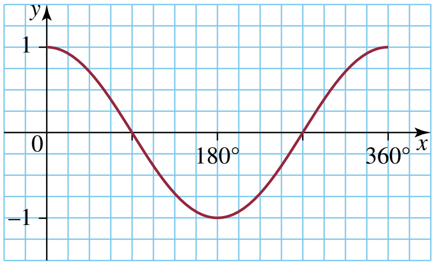
Exercise 20.7 Solving trigonometric equations

Individual pathways

🟩 PRACTISE	🟦 CONSOLIDATE	🟪 MASTER
Questions: 1–6	Questions: 1–7	Questions: 1–7

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency



1. Use the graph at right to find approximate answers to the following equations for the domain $0 \le x \le 360°$. Check your answers using a calculator.

- **i.** $\cos x = 0.9$
- **ii.** $\cos x = 0.3$
- **iii.** $\cos x = -0.2$
- **iv.** $\cos x = -0.6$

2. Solve the following equations for the domain $0° \le x \le 360°$.

a. $\sin x = \frac{1}{2}$ **b.** $\sin x = \frac{\sqrt{3}}{2}$ **c.** $\cos x = -\frac{1}{2}$ **d.** $\cos x = -\frac{1}{\sqrt{2}}$

e. $\sin x = 1$ **f.** $\cos x = -1$ **g.** $\sin x = -\frac{1}{2}$ **h.** $\sin x = -\frac{1}{\sqrt{2}}$

i. $\cos x = \frac{\sqrt{3}}{2}$ **j.** $\cos x = -\frac{\sqrt{3}}{2}$ **k.** $\sin x = 1$ **l.** $\cos x = 0$

Understanding

3. **WE14** Solve the following equations for the given values of x.

a. $\sin 2x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b.** $\cos 2x = -\frac{\sqrt{3}}{2}, x \in [0°, 360°]$

c. $\tan 2x = \frac{1}{\sqrt{3}}, x \in [0°, 360°]$ **d.** $\cos (4x) = \frac{-1}{2}, x \in [0°, 180°]$

e. $\sin 4x = -\frac{1}{2}, x \in [0°, 180°]$ **f.** $\sin 3x = -\frac{1}{\sqrt{2}}, x \in [-180°, 180°]$

g. $\tan 3x = -1, x \in [0°, 90°]$ **h.** $\cos 3x = 0, x \in [0°, 360°]$

Reasoning

4. Solve the following equations for $x \in [0°, 360°]$.

a. $2 \sin x - 1 = 0$ **b.** $2 \cos x = \sqrt{3}$ **c.** $\sqrt{2} \cos x - 1 = 0$ **d.** $\tan x + 1 = 0$

5. Sam measured the depth of water at the end of the Intergate jetty at various times on Friday 13 August 2010. The table below provides her results.

Time	6 am	7	8	9	10	11	12 pm	1	2	3	4	5	6	7	8	9
Depth	1.5	1.8	2.3	2.6	2.5	2.2	1.8	1.2	0.8	0.5	0.6	1.0	1.3	1.8	2.2	2.5

a. Plot the data.
b. Determine:
  i. the period **ii.** the amplitude.
c. Sam fishes from the jetty when the depth is a maximum. Specify these times for the next 3 days.
d. Sam's mother can moor her yacht when the depth is above 1.5 m. During what periods can she moor the yacht on Sunday 16 January?

Problem solving

6. Solve:
  a. $\sqrt{3}\sin(x°) = \cos(x°)$ for $0° \leq x \leq 360°$
  b. $2\sin(x°) + \cos(x°) = 0$ for $0° \leq x \leq 360°$.

7. Solve $2\sin^2(x°) + 3\sin(x°) - 2 = 0$ for $0° \leq x \leq 360°$.

Reflection

Explain why sine and cosine functions can be used to model situations that occur in nature such as tide heights and sound waves.

CHALLENGE 20.2

The grad (^{g}) is another measurement used when measuring the size of angles. A grad is equivalent to $\frac{1}{400}$ of a full circle. Write each of the following as grads (1 grad is written as 1^g).

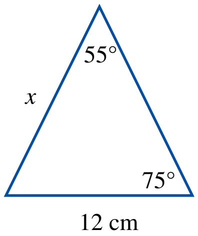
1 90° **2** 180° **3** 270° **4** 360°

20.8 Review

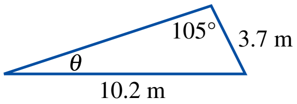
20.8.1 Review questions

Fluency

1. Find the value of x, correct to 1 decimal place.

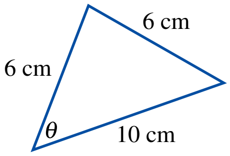


2. Find the value of θ, correct to the nearest minute.

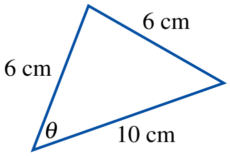


3. Find all unknown sides and angles of triangle ABC, given $a = 25$ m, $A = 120°$ and $B = 50°$.

4. Find the value of x, correct to 1 decimal place.



5. Find the value of θ, correct to the nearest degree.



6. A triangle has sides of length 12 m, 15 m and 20 m. Find the magnitude (size) of the largest angle.

7. A triangle has two sides of 18 cm and 25 cm. The angle between the two sides is 45°. Find:
  a. its area
  b. the length of its third side
  c. its area using Heron's formula.

8. If $\theta = 290°$, the triangle to show this would be drawn in which quadrant?

9. On the unit circle, draw an appropriate triangle for the angle 110° in quadrant 2.
  a. Find sin 110° and cos 110°, correct to 2 decimal places.
  b. Find tan 110°, correct to 2 decimal places.

10. The value of sin 53° is equal to:
  a. cos 53° **b.** cos 37° **c.** sin 37° **d.** tan 53°

11. Simplify $\frac{\sin 53°}{\sin 37°}$.

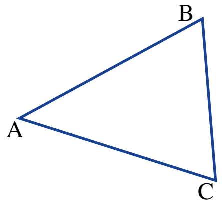
12. Draw a sketch of $y = \sin x$ from $0° \le x \le 360°$.

13. Draw a sketch of $y = \cos x$ from $0° \le x \le 360°$.

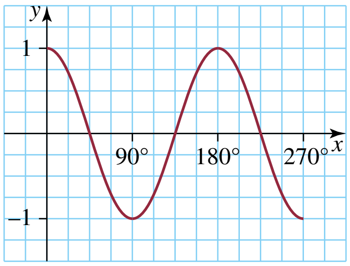
14. Draw a sketch of $y = \tan x$ from $0° \le x \le 360°$.

Non-calculator questions

15. Label this triangle so that $\frac{x}{\sin 46°} = \frac{y}{\sin 68°}$.



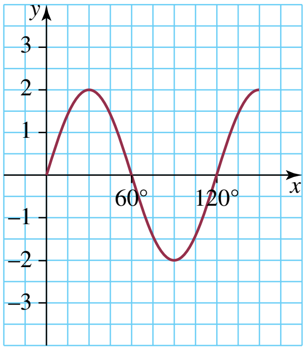
16. State the period and amplitude of each of the following graphs.
  a. $y = 2 \sin 3x$ **b.** $y = -3 \cos 2x$
  c.



17. Sketch the following graphs.
  a. $y = 2 \sin x, x \in [0°, 360°]$ **b.** $y = \cos 2x, x \in [-180°, 180°]$

18. Use technology to write down the solutions to the following equations for the domain $0° \le x \le 360°$ to 2 decimal places.
  a. $\sin x = -0.2$ **b.** $\cos 2x = 0.7$ **c.** $3 \cos x = 0.1$ **d.** $2 \tan 2x = 0.5$

19. Solve each of the following equations.
  a. $\sin x = -\frac{1}{2}, x \in [0°, 360°]$ **b.** $\cos x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$
  c. $\cos x = \frac{1}{\sqrt{2}}, x \in [0°, 360°]$ **d.** $\sin x = \frac{1}{\sqrt{2}}, x \in [0°, 360°]$



20. The equation that represents the graph on the right could be:

a. $y = 3 \sin 2x$ **b.** $y = 2 \cos 3x$
c. $y = 3 \cos 2x$ **d.** $y = 2 \sin 2x$
e. $y = 2 \sin 3x$

21. a. Use technology to help sketch the graph of $y = 2 \sin 2x - 3$.
b. Write down the period and the amplitude of the graph in part **a**.

22. Sketch the graphs of each of the following, stating
i. the period
ii. the amplitude.
a. $y = 2 \cos 2x, x \in [0°, 360°]$ **b.** $y = 3 \sin 4x, x \in [0°, 180°]$
c. $y = -2 \cos 3x, x \in [-60°, 60°]$ **d.** $y = 4 \sin 2x, x \in [-90°, 90°]$

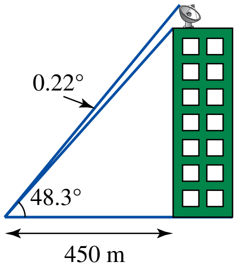
23. Solve each of the following equations for the given values of x.

a. $\cos 2x = \dfrac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b.** $\sin 3x = \dfrac{1}{2}, x \in [-90°, 90°]$
c. $\sin 2x = \dfrac{1}{\sqrt{2}}, x \in [0°, 360°]$ **d.** $\cos 3x = -\dfrac{1}{\sqrt{2}}, x \in [0°, 360°]$
e. $\sin 4x = 0, x \in [0°, 180°]$ **f.** $\tan 4x = 21, x \in [0°, 180°]$

24. Solve the following for $x \in [0°, 360°]$.
a. $2 \cos x - 1 = 0$ **b.** $2 \sin x = -\sqrt{3}$ **c.** $-\sqrt{2} \cos x + 1 = 0$ **d.** $\sqrt{2} \sin x + 1 = 0$

25. Sketch the graph of $y = \tan 2x, x \in [0°, 180°]$. Write down the period, amplitude and the equations of any asymptotes.

Problem solving



26. A satellite dish is placed on top of an apartment building as shown in the diagram. Find the height of the satellite dish.

27. Australian power points supply voltage, V, in volts, where $V = 240 \sin 18\,000t$ and t is measured in seconds.

a. Complete the table below and sketch the graph, showing the fluctuations in voltage over time.

t	v
0.000	
0.005	
0.010	
0.015	
0.020	
0.025	
0.030	
0.035	
0.040	

b. Find the times at which the maximum voltage output occurs.
c. How many seconds are there between times of maximum voltage output?
d. How many periods (or cycles) are there per second?

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- **Try out this interactivity:** Word search: Topic 20 (int-2883)
- **Try out this interactivity:** Crossword: Topic 20 (int-2884)
- **Try out this interactivity:** Sudoku: Topic 20 (int-3895)
- **Complete this digital doc:** Concept map: Topic 20 (doc-14634)

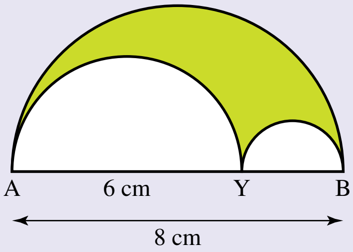
Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

ambiguous case
amplitude
area
asymptote
cosine
cosine graphs
cosine rule
exact value
Heron's formula
period
periodic function
quadrants
sine
sine graphs
sine rule
trigonometric equation
trigonometric function
unit circle

Investigation | Rich task

What's an arbelos?



As an introduction to this task, you are required to complete the following construction. The questions that follow require the application of measurement formulas, and an understanding of semicircles related to this construction.

1. Constructing an arbelos

- Rule a horizontal line AB 8 cm long.
- Find the midpoint of the line and construct a semicircle on top of the line with AB as the diameter.
- Mark Y as a point on AB such that AY = 6 cm.
- Find the midpoint of AY and draw a small semicircle inside the larger semicircle with AY as the diameter.
- Find the midpoint of YB and construct a semicircle (also inside the larger semicircle) with a diameter YB.

The shape enclosed by the three semicircles is known as an arbelos. The word, in Greek, means *shoemaker's knife* as it resembles the blade of a knife used by cobblers. The point Y is not fixed and can be located anywhere along the diameter of the larger semicircle, which can also vary in size.

2. Perimeter of an arbelos

The perimeter of an arbelos is the sum of the arc length of the three semicircles. Perform the following calculations, leaving each answer in terms of π.

a. Calculate the arc length of the semicircle with diameter AB.

b. Calculate the arc length of the semicircle with diameter AY.

c. Calculate the arc length of the semicircle on diameter YB.

d. Compare the largest arc length with the two smaller arc lengths. What do you conclude?

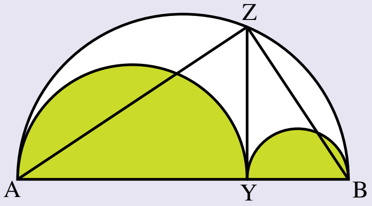
3. We can generalise the arc length of an arbelos. The point Y can be located anywhere on the line AB, which can also vary in length. Let the diameter AB be d cm, AY be d_1 cm and YB be d_2 cm. Prove that your conclusion from question **2d** holds true for any value of d, where $d_1 + d_2 = d$.

4. Area of an arbelos

The area of an arbelos may be treated as the area of a composite shape.

a. Using your original measurements, calculate the area of the arbelos you drew in question **1**. Leave your answer in terms of π.

The area of the arbelos can also be calculated using another method. We can draw the common tangent to the two smaller semicircles at their point of contact and extend this tangent to the larger semicircle. It is said that the area of the arbelos is the same as the area of the circle constructed on this common tangent as diameter.



YZ is the common tangent.

Triangles AYZ, BYZ and AZB are all right-angled triangles. We can use Pythagoras' theorem, together with a set of simultaneous equations, to determine the length of the tangent YZ.

b. **Complete the following.**

In ΔAYZ, $\mathrm{AZ}^2 = \mathrm{AY}^2 + \mathrm{YZ}^2$

$= 6^2 + \mathrm{YZ}^2$

In ΔBYZ, $\mathrm{BZ}^2 = \mathrm{BY}^2 + \mathrm{YZ}^2$

$=$ .... $+ \mathrm{YZ}^2$

Adding these two equations,

$\mathrm{AZ}^2 + \mathrm{BZ}^2 =$ .... + ...

But, in ΔAZB, $\mathrm{AZ}^2 + \mathrm{BZ}^2 = \mathrm{AB}^2$

.... + ... = ...

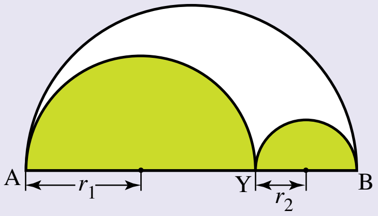
So,

YZ = ... (Leave your answer in surd form.)

c. Now calculate the area of the circle with diameter YZ. Is your answer the same as that calculated in question **4a**?

The area of an arbelos can be generalised.

Let the radii of the two smaller semicircles be r_1 and r_2.



5. Develop a formula for the area of the arbelos in terms of r_1 and r_2. Demonstrate the use of your formula by checking your answer to question **4a**.

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Complete this digital doc: Code puzzle: What is the periodic table? (doc-15955)

Answers

Topic 20 Trigonometry II

Exercise 20.2 The sine rule

1. 44°58′, 77°2′, 13.79
2. 39°18′, 38°55′, 17.21
3. 70°, 9.85, 9.4
4. 33°, 38.98, 21.98
5. 19.12
6. $C = 51^\circ$, $b = 54.66$, $c = 44.66$
7. $A = 60^\circ$, $b = 117.11$, $c = 31.38$
8. $B = 48^\circ 26'$, $C = 103^\circ 34'$, $c = 66.26$; or $B = 131^\circ 34'$, $C = 20^\circ 26'$, $c = 23.8$
9. 24.17
10. B, C
11. $A = 73^\circ 15'$, $b = 8.73$; or $A = 106^\circ 45'$, $b = 4.12$
12. 51.9 or 44.86
13. $C = 110^\circ$, $a = 3.09$, $b = 4.64$
14. $B = 38^\circ$, $a = 3.36$, $c = 2.28$
15. $B = 33^\circ 33'$, $C = 121^\circ 27'$, $c = 26.24$; or $B = 146^\circ 27'$, $C = 8^\circ 33'$, $c = 4.57$
16. 43.62 m
17. **a.** 6.97 m **b.** 4 m
18. **a.** 13.11 km **b.** N20°47′W
19. **a.** 8.63 km **b.** 6.48 km/h **c.** 9.90 km
20. 22.09 km from A and 27.46 km from B
21. C **22.** B
23. Yes, she needs 43 m altogether.
24. **a.** 7.3 km **b.** 282°3′
25. $h = 7.5$ cm

Exercise 20.3 The cosine rule

1. 7.95
2. 55.22
3. 23.08, 41°53′, 23°7′
4. 28°57′
5. 88°15′
6. $A = 61^\circ 15'$, $B = 40^\circ$, $C = 78^\circ 45'$
7. 2218 m
8. **a.** 12.57 km **b.** S35°1′E
9. **a.** 35°6′ **b.** 6.73 m^2
10. 23°
11. 89.12 m
12. **a.** 130 km **b.** S22°12′E
13. 28.5 km
14. 74.3 km
15. 70°49′
16. **a.** 8.89 m **b.** 77° **c.** $x = 10.07$ m
17. 1.14 km/h
18. $\angle CAB = 34.65^\circ$, $\angle ABC = 84.83^\circ$ and $\angle BCA = 60.52^\circ$
19. **a.** 4.6637 m **b.** 55.93°

Exercise 20.4 Area of triangles

1. 12.98
2. 38.14
3. 212.88

4. $A = 32^\circ 4'$, $B = 99^\circ 56'$, area = 68.95 cm^2

5. $A = 39^\circ 50'$, $B = 84^\circ 10'$, area = 186.03 m^2

6. $A = 125^\circ 14'$, $B = 16^\circ 46'$, area = 196.03 mm^2

7. C

8. 14.98 cm^2

9. 570.03 mm^2

10. 2.15 cm^2

11. B

12. 3131.41 mm^2

13. 610.38 cm^2

14. **a.** 187.5 cm^2 **b.** 15.03 cm^2 **c.** 187.47 cm^2

15. 17 goldfish

16. 22.02 m^2

17. **a.** Area = 69.63 cm^2 **b.** Dimensions are 12.08 cm and 6.96 cm.

18. 17 kg

19. 52.2 hectares

20. 175 m^3

21. C

22. B

23. **a.** 29.44 km^2 **b.** 8.26 km **c.** 41° **d.** 24°
e. 3.72 km **f.** 10.11 km^2 **g.** 39.55 km^2

24. 374.12 cm^2

Challenge 20.1

$a \sin B = b$
Area $= \frac{1}{2}ac$

Exercise 20.5 The unit circle

1. **a.** 1st **b.** 2nd **c.** 4th **d.** 3rd
e. 2nd **f.** 3rd **g.** 4th **h.** 4th

2. A

3. D

4. **a.** 0.35 **b.** 0.95 **c.** -0.17 **d.** 0.99
e. -0.64 **f.** 0.77 **g.** -0.57 **h.** -0.82

5. **a.** 1 **b.** 0 **c.** 0 **d.** -1
e. -1 **f.** 0 **g.** 0 **h.** 1

6. **a.** 0.87 **b.** 0.50

7. **a.** 30° **b.** -0.87 **c.** $\cos 150^\circ = -\cos 30^\circ$
d. 0.5 **e.** $\sin 150^\circ = \sin 30^\circ$

8. **a.** 30° **b.** -0.87 **c.** $\cos 210^\circ = -\cos 30^\circ$
d. -0.50 **e.** $\sin 210^\circ = -\sin 30^\circ$

9. **a.** 30° **b.** 0.87 **c.** $\cos 330^\circ = -\cos 30^\circ$
d. -0.50 **e.** $\sin 330^\circ = -\sin 30^\circ$

10. **a.** 0.34 **b.** 0.94 **c.** 0.36
d. 0.36 **e.** They are equal.

11. **a.** 0.71 **b.** -0.71 **c.** -1
d. -1 **e.** They are equal. **f.** $\tan 135^\circ = -\tan 45^\circ$

12. **a.** -0.64 **b.** -0.77 **c.** 0.84
d. 0.83 **e.** They are approx. equal. **f.** $\tan 220^\circ = \tan 40^\circ$

13. **a.** -0.87 **b.** 0.5 **c.** -1.73
d. -1.74 **e.** They are approx. equal. **f.** $\tan 300^\circ = -\tan 60^\circ$

14. D

15. **a.** $\sqrt{1 - p^2}$ **b.** p **c.** $-\sqrt{1 - p^2}$

16. 0

17. **a.** $(2\cos\theta, 3\sin\theta)$ **b.** $(-\cos\theta, -\sin\theta)$ **c.** $(-\cos\theta, -\sin\theta)$

18. **a.** $-\frac{1}{2}\sin(2\theta)$ **b.** $\cos(\theta) - \sin\theta)$

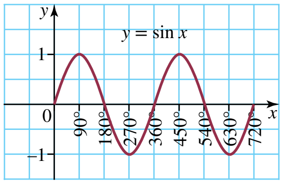
19. **a.** $\tan(\boldsymbol{x})$ **b.** $-\sec(\boldsymbol{x})$

Exercise 20.6 Trigonometric functions

1.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin *x*	0	0.5	0.87	1	0.87	0.5	0	−0.5°	−0.87	−1	−0.87	−0.5	0
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
sin *x*	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	

2.



3. 360°

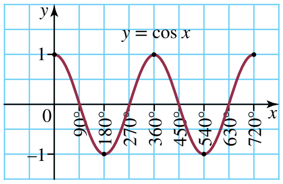
4. **a.** 0.7 **b.** 0.8 **c.** 0.35 **d.** −0.35
e. 0 **f.** 0.9 **g.** −0.2 **h.** −0.9

5. **a.** 64°, 116°, 424°, 476° **b.** 244°, 296°, 604°, 656° **c.** 44°, 136°, 404°, 496°
d. 210°, 330°, 570°, 690° **e.** 233°, 307°, 593°, 667° **f.** 24°, 156°, 384°, 516°

6.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos *x*	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
cos *x*	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1	

7.



8. The graph would continue with the cycle.

9. It is a very similar graph with the same shape; however, the sine graph starts at (0, 0), whereas the cosine graph starts at (0, 1).

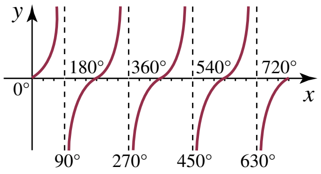
10. **a.** 0.7 **b.** −0.98 **c.** −1 **d.** 0.9
e. −0.5 **f.** −0.8 **g.** 0.8 **h.** −0.96

11. **a.** 120°, 240°, 480°, 600° **b.** 37°, 323°, 397°, 683° **c.** 46°, 314°, 406°, 674°
d. 127°, 233°, 487°, 593° **e.** 26°, 334°, 386°, 694° **f.** 154°, 206°, 514°, 566°

12.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
tan *x*	0	0.58	1.73	undef.	−1.73	−0.58	0	0.58	1.73	undef.	−1.73	−0.58	0
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
tan *x*	0.58	1.73	undef.	−1.73	−0.58	0	0.58	1.73	undef.	−1.73	−0.58	0	

13. $y = \tan x$



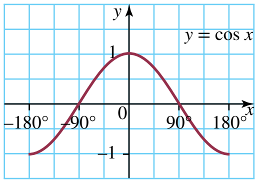
14. The graph would continue repeating every 180° as above.

15. Quite different. $y = \tan x$ has undefined values (asymptotes) and repeats every 180° rather than 360°. It also gives all y-values, rather than just values between −1 and 1.

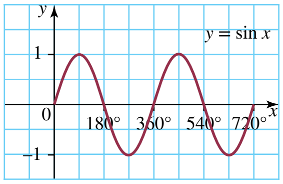
16. **a.** 1.7 **b.** −1 **c.** −1.2 **d.** 0.8
e. −0.8 **f.** 1.2 **g.** −0.2 **h.** 1

17. **a.** 45°, 225°, 405°, 585° **b.** 56°, 236°, 416°, 596° **c.** 158°, 338°, 518°, 698°
d. 117°, 297°, 477°, 657° **e.** 11°, 191°, 371°, 551° **f.** 135°, 315°, 495°, 675°

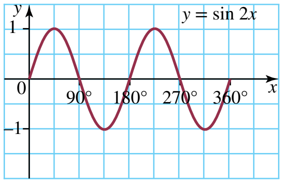
18. a.



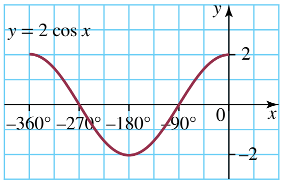
b.



c.



d.



19. a. i. 360° **ii.** 1

b. i. 360° **ii.** 1

c. i. 180° **ii.** 1

d. i. 360° **ii.** 2

20. a. i. 180° **ii.** 3

b. i. 120° **ii.** 4

c. i. 720° **ii.** 2

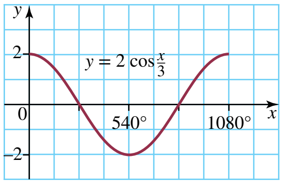
d. i. 1440° **ii.** $\frac{1}{2}$

e. i. 360° **ii.** 1

f. i. 180° **ii.** 1

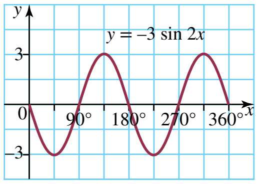
21. a. C **b.** A **c.** D

22. a.



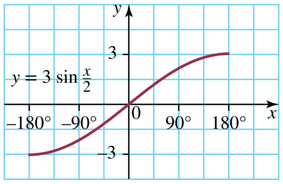
Period = 1080°
Amplitude = 2

b.



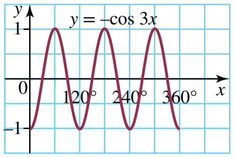
Period = 180°
Amplitude = 3

c.



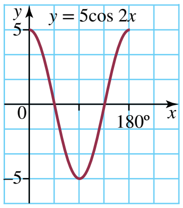
Period = 720°
Amplitude = 3

d.



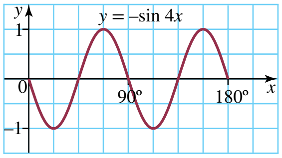
Period = 120°
Amplitude = 1

e.



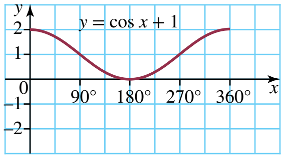
Period = 180°
Amplitude = 5

f.

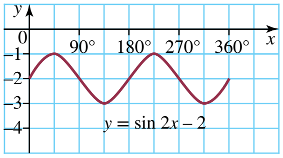


Period = 90°
Amplitude = 1

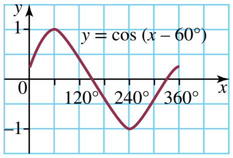
23. a.



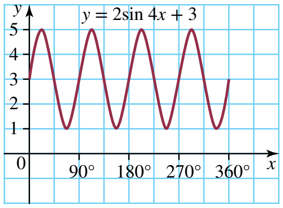
b.



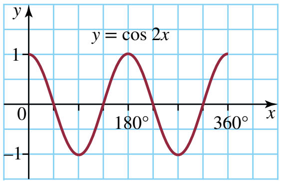
c.



d.



24. a.



i. -1 **ii.** 1

b. i. 3 **ii.** 1

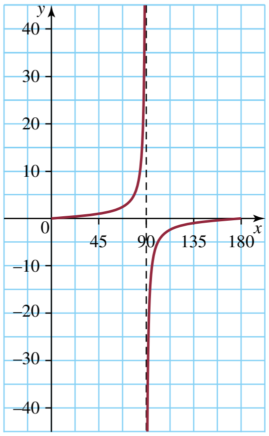
c. Max value of $\sin x = 1$, hence max value of $y = 2 \times 1 + 3 = 5$

d. Min value of $\sin x = -1$, hence min value of $y = 2 \times -1 + 3 = 1$

25. a.

x	0	30°	60°	90°	120°	150°	180°
y	0	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	undef	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0

b.

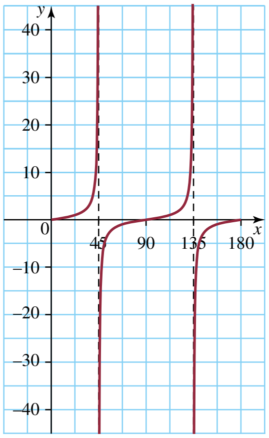


c. At $x = 90°$, y is undefined.

d. $x = 270°$

e. The period = 180°, amplitude is undefined.

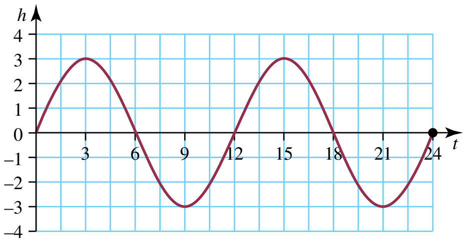
26. a.



b. $x = 45°$ and $x = 135°$

c. Period = 90° and amplitude is undefined.

27. a.

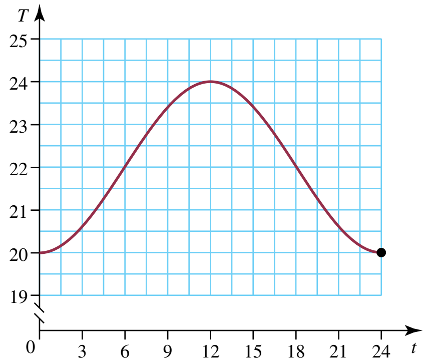


b. 3 metres

c. -2.6 metres

28. a. 22 °C

b.



c. Coolest 20 °C

d. Warmest 24 °C

Exercise 20.7 Solving trigonometric equations

1. Calculator answers

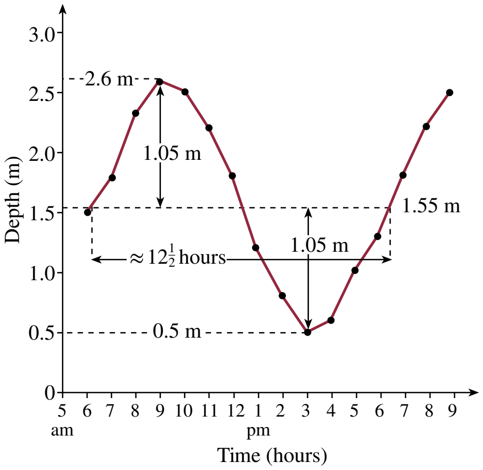
i. 25.84°, 334.16° **ii.** 72.54°, 287.46° **iii.** 101.54°, 258.46° **iv.** 126.87°, 233.13°

2. a. 30°, 150° **b.** 60°, 120° **c.** 120°, 240° **d.** 135°, 225°
e. 90° **f.** 180° **g.** 210°, 330° **h.** 225°, 315°
i. 30°, 330° **j.** 150°, 210° **k.** 90° **l.** 90°, 270°

3. a. 30°, 60°, 210°, 240° **b.** 75°, 105°, 255°, 285°
c. 15°, 105°, 195°, 285° **d.** 30°, 60°, 12°, 150°
e. 52.5°, 82.5°, 142.5°, 172.5° **f.** −165°, −135°, −45°, −15°, 75°, 105°
g. 45° **h.** 30°, 90°, 150°, 210°, 270°, 330°

4. a. 30°, 150° **b.** 30°, 330° **c.** 45°, 315° **d.** 135°, 315°

5. a.



b. i. $12\frac{1}{2}$ h **ii.** 1.05 m

c. 10.00 am, 10.30 pm, 11.00 am, 11.30 pm, noon
d. Until 1.45 am Sunday, 8 am to 2.15 pm and after 8.30 pm

6. a. $x =$ 30°, 210° **b.** $x =$ 153.43°, 333.43°

7. $x =$ 30°, 150°

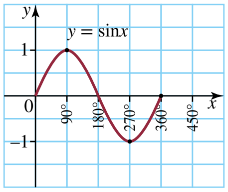
Challenge 20.2

1. 100^g
2. 200^g
3. 300^g
4. 400^g

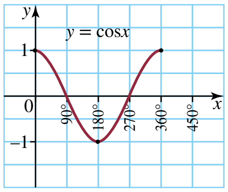
20.8 Review

1. 14.15 cm
2. 20°31′
3. $b = 22.11$ m, $c = 5.01$ m, $C = 10°$
4. 3.64 cm
5. 34°
6. 94°56′
7. a. 159.10 cm^2 **b.** 17.68 cm **c.** 159.10 cm^2
8. 4th quadrant
9. a. 0.94, −0.34 **b.** −2.75
10. b
11. tan 53°

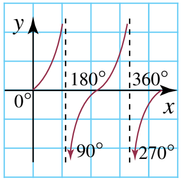
12.



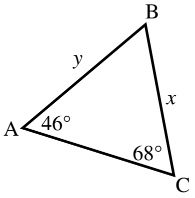
13.



14. $y = \tan x$

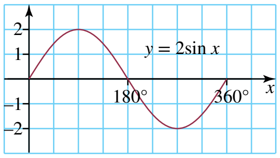


15.

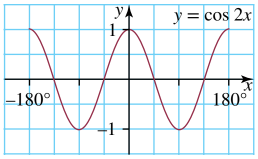


16. a. Period = 120°, amplitude = 2 **b.** Period = 180°, amplitude = 3 **c.** Period = 180°, amplitude = 0.5

17. a.



b.



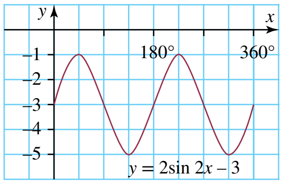
18. a. $x = 191.54, 348.46$ **b.** $x = 22.79, 157.21, 202.79, 337.21$

c. $x = 88.09, 271.91$ **d.** $x = 7.02, 97.02, 187.02, 277.02$

19. a. 210°, 330° **b.** 30°, 330° **c.** 45°, 315° **d.** 45°, 135°

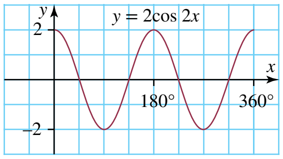
20. e

21. a.



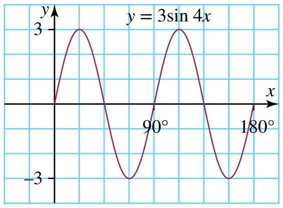
b. Period = 180°, amplitude = 2

22. a.



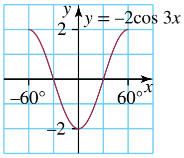
i. Period = 180°
ii. Amplitude = 2

b.



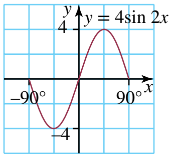
i. Period = 90°
ii. Amplitude = 3

c.



i. Period = 120°
ii. Amplitude = 2

d.



i. Period = 180°
ii. Amplitude = 4

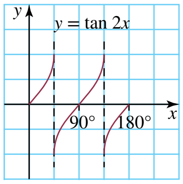
23. a. 15°, 165°, 195°, 345° **b.** –70°, 10°, 50°

c. 22.5°, 67.5, 202.5°, 247.5° **d.** 45°, 75°, 165°, 195°, 285°, 315°

e. 0, 45°, 90°, 135°, 180° **f.** 33.75°, 78.75°, 123.75°, 168.75°

24. a. 60°, 300° **b.** 240°, 300° **c.** 45, 315° **d.** 225, 315°

25.

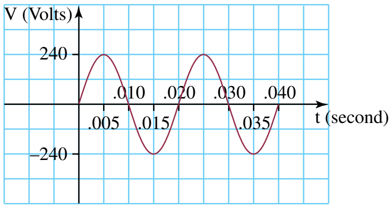


Period = 90, amplitude is undefined.
Asymptotes are at $x = 45°$ and $x = 135°$.

26. 3.92 m

27. a.

t	*V*
0.000	0
0.005	240
0.010	0
0.015	– 240
0.020	0
0.025	240
0.030	0
0.035	– 240
0.040	0



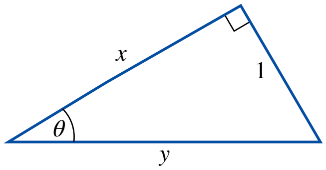
b. Maximum voltage occurs at $t = 0.005$ s, 0.025 s.

c. 0.02 s

d. 50 cycles per second

Investigation — Rich task

Take one triangle containing the angle θ and label its sides.



This gives

$$\tan(\theta) = \frac{1}{x}$$

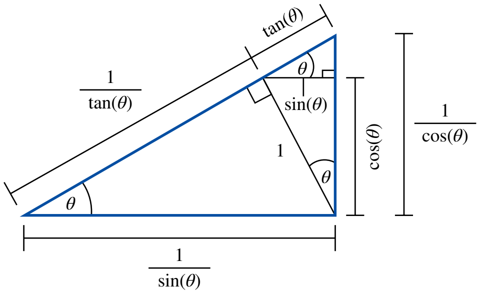
$$\Rightarrow x = \frac{1}{\tan(\theta)}$$

and

$$\sin(\theta) = \frac{1}{y}$$

$$\Rightarrow y = \frac{1}{\sin(\theta)}.$$

Continue until you get the following lengths.



$$\text{Area}_1 = \frac{1}{2}\sin(\theta)\cos(\theta)$$

$$\text{Area}_2 = \frac{90-\theta}{360}\pi$$

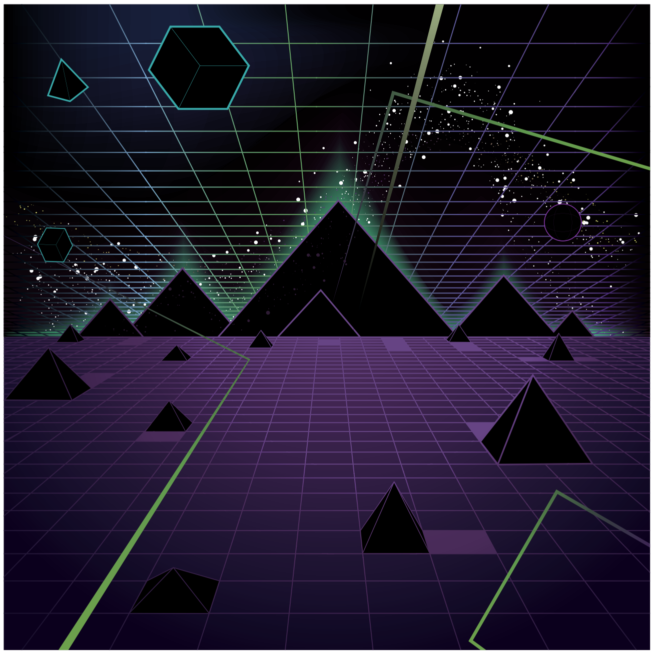
$$\text{Area}_3 = \frac{1}{2}\left(\frac{1}{\cos(\theta)} - \cos(\theta)\right)\sin(\theta)$$

$$\text{Area}_4 = \frac{1}{2} \times \frac{1}{\tan(\theta)} - \frac{90-\theta}{360}\pi$$

$$\text{Area}_5 = \frac{3\pi}{4}$$

TOPIC 21
Programming

21.1 Overview



Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the content and concepts covered in this topic.

21.1.1 Why learn this?

Computer programming is the process of creating instructions for a computer to follow to perform tasks.

In this topic the computer language JavaScript is used in most of the examples and questions. JavaScript has been chosen for its popularity and availability on nearly all web browsers.

This topic will introduce how a program can process and organise large amounts of data, and will explore generating many random numbers to run simulations. The topic will also explore the use of graphics as powerful tools to visualise simulations.

21.1.2 What do you know?

assesson

1. **THINK** Write what you know about computer programming languages. Use a thinking tool such as a concept map to display your ideas.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, combine all the ideas into one large concept map or poster.

LEARNING SEQUENCE

online only

21.2 Programs

21.2.1 Values

- Three of the most common types of values used in computer programming are:
  - **numbers** (numerical values): for example `1`, `-2`, `1.234`, `-14.345345`
  - **strings** (any text surrounded in quotation marks `" "`): for example `"In quotation marks"`,`"213.2"`, `"Subject: Year 10 Mathematics"`,`"!@#$%^&"`,`" "`
  - **Booleans** (used for logic, this type only has two possible values): either `false` or `true`.
- In JavaScript the expression `typeof value` returns either `"number"`, `"string"` or `"boolean"` if `value` is a number, string or Boolean respectively.

WORKED EXAMPLE 1

Answer these questions for each of the following values.

i Is the value a number, a string or Boolean?

ii Apply the expression `typeof value` to each value. What is the result of the expression `typeof value`?

a `"Brendan"` **b** `1.41421` **c** `false` **d** `"1.41421"`

THINK	WRITE
a i `"Brendan"` is in quotation marks, so this value is a string.	**a i** String
ii 1 Apply the expression `typeof value` to the value `"Brendan"`.	**ii** `typeof "Brendan"`
2 `typeof "Brendan"` is a `"string"`.	`"string"`
b i `1.41421` is a numerical value not in quotation marks, so this value is a number.	**b i** Number
ii 1 Apply the expression `typeof value` to the value `1.41421`.	**ii** `typeof 1.41421`
2 `typeof 1.41421` is a `"number"`.	`"number"`
c i `false` is a Boolean value.	**c i** Boolean
ii 1 Apply the expression `typeof value` to the value `false`.	**ii** `typeof false`
2 `typeof false` is a `"boolean"`.	`"boolean"`
d i `"1.41421"` is in quotation marks, so it is a string.	**d i** String
ii 1 Apply the expression `typeof value` to the value `"1.41421"`.	**ii** `typeof "1.41421"`
2 `typeof "1.41421"` is a `"string"`.	`"string"`

21.2.2 Variables

- Computer languages use memory locations to store values. These named containers are called **variables**. There are complex rules as to what is a valid variable name. For simplicity, this topic will restrict the variable names to three simple rules.

- Variable names:
  - must not start with a number
  - can only contain upper and lower case letters, numbers, and the underscore character (`_`), and cannot contain spaces
  - cannot be JavaScript keywords. The following are JavaScript keywords which should not be used as variable names.
    `abstract, arguments, Array, boolean, break, byte, case, catch, char, class, const, continue, Date, debugger, default, delete, do, double, else, enum, eval, export, extends, final, finally, float, for, function, goto, hasOwnProperty, if, implements, import, in, Infinity, instanceof, int, interface, isFinite, isNaN, isPrototypeOf, length, let, long, Math, name, NaN, native, new, null, Number, Object, package, private, protected, prototype, public, return, short, static, String, super, switch, synchronized, this, throw, throws, toString, transient, true, try, typeof, undefined, valueOf, var, void, volatile, while, with, yield, false`

WORKED EXAMPLE 2

Each of the following is a valid variable name. True or false?

a `someReallyLongName` **b** `3.14_nearly_pi`
c `number/divide` **d** `age_43`
e `while`

THINK	WRITE
a `someReallyLongName` is valid because it only uses letters.	**a** True
b `3.14_nearly_pi` is invalid because it uses a number as the first character.	**b** False
c `number/divide` is invalid because it uses a `/` character.	**c** False
d `age_43` is valid as it only uses letters, numbers and the `_` character. It also does not start with a number.	**d** True
e `while` is invalid because it is a JavaScript keyword.	**e** False

21.2.3 Numerical expressions

- In JavaScript, numerical expressions involving numbers, brackets, and plus and minus signs evaluate normally.
- Multiplication uses the character `*`.
- Division uses the character `/`.
- Fractions can be evaluated using the division character, but the numerator and denominator expressions must be put in brackets.

$$\frac{300+10+4}{1+9\times 11} = \texttt{(300 + 10 + 4)/(1 + 9 * 11)}$$

- The `%` symbol is used to find remainder after a division. For example, `32%10` evaluates to `2`, as 10 divides into 32 three times with a 2 remainder.
- JavaScript has a square root function: $\sqrt{x}$ = `Math.sqrt(x)`. This function returns only the positive square root.

WORKED EXAMPLE 3

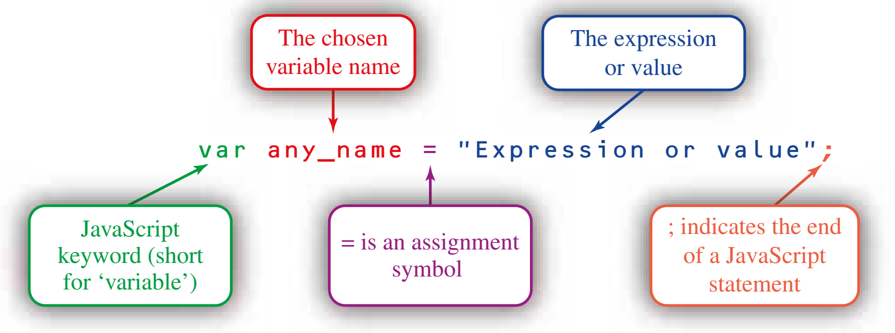
Write the following as JavaScript expressions.

a $(11 - 1 + 1.2) + (15 - 4)$	**b** $213 \times 32 \times 0.5$
c $720 \div (6 \times 5 \times 4 \times 3 \times 2 \times 1)$	**d** $\dfrac{17 + 13}{7 - 2}$
e The remainder from the division 19 ÷ 6	**f** $\sqrt{169}$

THINK	WRITE
a This is a simple numerical expression with no special characters required.	**a** `(11-1+1.2)+(15-4)`
b Substitute × with `*`.	**b** `213*32*0.5`
c Replace the × with a `*` and the ÷ with a `/`.	**c** `720/(6*5*4*3*2*1)`
d The fraction is a division. The numerator and denominator require brackets.	**d** `(17+13)/(7-2)`
e Use the modulus symbol, `%`, to find the remainder from the division 19 ÷ 6.	**e** `19%6`
f Use the `Math.sqrt` function to find the square root of 169.	**f** `Math.sqrt(169)`

21.2.4 Assigning

- When assigning to a variable for the first time, the statement should begin with the JavaScript keyword `var`. JavaScript uses the following assignment structure to assign a value or expression to a variable.



```
var any_name = "Expression or value";
```

WORKED EXAMPLE 4

Write JavaScript statements to make the following variable assignments.

a **Assign the value `1236` to the variable `population`.**

b **Assign `"Australia"` to the variable `country`.**

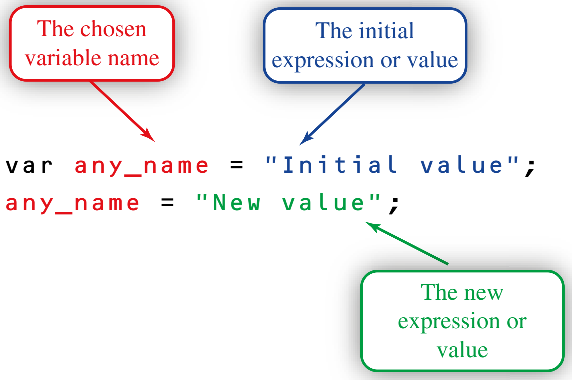
c **Assign the mathematical expression $5 \times 4 \times 3 \times 2 \times 1$ to the variable `factorial5`.**

THINK	WRITE
a **1** Identify the variable name, which in this case is `population`.	**a**
2 Identify the value, which in this case is `1236`.	

3 Apply the JavaScript assignment structure to write the statement.	`var population = 1236;`
b 1 Identify the variable name, which in this case is `country`.	**b**
2 Identify the value, which in this case is `"Australia"`.	
3 Apply the JavaScript assignment structure to write the statement.	`var country = "Australia";`
c 1 Identify the variable name, which in this case is `factorial5`.	**c**
2 Identify the mathematical expression, which in this case is $5 \times 4 \times 3 \times 2 \times 1$.	
3 Convert the mathematical expression into a JavaScript expression.	`5*4*3*2*1`
4 Apply the JavaScript assignment structure to write the statement.	`var factorial5 = 5*4*3*2*1;`

21.2.5 Reassigning

- A variable can change value and type as the program runs through a sequence of statements in order. When a variable is reassigned, the statement does not require the JavaScript keyword `var`. JavaScript uses the following sequence to assign and reassign a value or expression to a variable. The value of the variable changes with each assignment.



```
var any_name = "Initial value";
any_name = "New value";
```

WORKED EXAMPLE 5

Manually simulate running the following programs. For each program:

i what is the final type of the variable `value`

ii what is the final value stored in the variable `value`

iii what is the result of the expression `typeof value`?

a
```
var value = 9876;
value = true;
```

b
```
var value = 4+3+2+1;
value = (5*4*3*2*1)/value;
```

c
```
var value = false;
var another = "Victoria";
value = another;
another = true;
```

THINK			**WRITE**
a	**1**	Write the initial assignment.	**a** `value = 9876`
	2	Write the reassignment.	`value = true`
	i	`true` is one of the Boolean values, so this value is a Boolean.	**i** Boolean
	ii	The last value assigned to `value` was `true`.	**ii** `true`
	iii	Apply the expression `typeof value` to the value `true`.	**iii** `"boolean"`
b	**1**	Write the initial assignment.	**b** `value = 4+3+2+1`
	2	Evaluate the expression.	`value = 10`
	3	Write the reassignment.	`value = (5*4*3*2*1)/value`
	4	On the right-hand side of the assignment, substitute `10` for `value`.	`value = (5*4*3*2*1)/10`
	5	Substitute `120` for `(5*4*3*2*1)`.	`value = 120/10`
	6	Substitute `12` for `120/10`.	`value = 12`
	i	`12` is a numerical value not in quotation marks, so this value is a number.	**i** Number
	ii	The last value assigned to `value` was `12`.	**ii** `12`
	iii	Apply the expression `typeof value` to the value `12`.	**iii** `"number"`
c	**1**	Write the initial assignment to `value`.	**c** `value = false`
	2	Write the initial assignment to `another`.	`another = "Victoria"`
	3	Write the reassignment to `value`.	`value = another`
	4	Substitute `"Victoria"` for `another`.	`value = "Victoria"`
	5	Write the reassignment to `another`.	`another = true`
	i	`"Victoria"` is in quotation marks, so this value is a string.	**i** String
	ii	The last value assigned to `value` was `"Victoria"`.	**ii** `"Victoria"`
	iii	Apply the expression `typeof value` to the value `"Victoria"`.	**iii** `"string"`

21.2.6 If structure

- Decisions are based on Boolean values. In JavaScript, the `if` structure is used to make a decision to run a section of code if the decision value is true. For example, the following program will run `{statement 1, statement 2, ..., statement n}` if `decision` is `true`.

```
if (decision) {
 statement 1
 statement 2

 statement n
 }
```

WORKED EXAMPLE 6

Manually simulate running the following programs. For each program, what is the final value stored in the variable `data`?

a

```
var data = 22;
if (true) {
   data = data/2;
}
```

b

```
var data="Stays the same";
if (false) {
   data="Changed";
}
```

THINK	WRITE
a **1** Write the first assignment.	**a** `data = 22`
2 `decision` is `true`, so run the statement inside the `{}` block.	
3 Write the assignment.	`data = data/2`
4 On the right-hand side of the assignment, substitute `22` for `data`.	`data = 22/2`
5 Substitute `11` for `22/2`.	`data = 11`
6 Write the final value stored in `data`.	`11`
b **1** Write the first assignment.	**b** `data="Stays the same"`
2 `decision` is `false`, so ignore the statement inside the `{}` block.	
3 Write the final value stored in `data`.	`"Stays the same"`

21.2.7 If else structure

- In JavaScript, the `if else` structure is used to execute one of two different sections of code.
  1. Execute `{statement 1, statement 2, ..., statement n}` if `decision` is `true`.
  2. Execute `{statement A, statement B, ..., statement Z}` if `decision` is `false`.

```
if (decision) {
   statement 1
   statement 2

   statement n
} else {
   statement A
   statement B

   statement Z
}
```

WORKED EXAMPLE 7

Manually simulate running the following programs. For each program, what is the final value stored in the variable `results`?

a

```
var results = 1.23;
if (true) {
   results = 3.21;
} else {
   results = 2.13;
}
```

b

```
var results = 12;
var multiplyBy3 = false;
if (multiplyBy3) {
   results = 3*results;
} else {
   results = results/results;
}
```

THINK	WRITE
a **1** Write the first assignment.	**a** `results = 1.23`
2 `decision` is `true`, so run the statement inside the first `{}` block.	
3 Write the assignment.	`results = 3.21`
4 Write the final value stored in `results`.	`3.21`
b **1** Write the first assignment.	**b** `results = 12`
2 Write the second assignment.	`multiplyBy3 = false`
3 `decision multiplyBy3` is `false`, so run the statement inside the second `{}` block.	
4 Write the equation.	`results = results/results`
5 On the right-hand side of the assignment, substitute `12` for `results`.	`results = 12/12`
6 Substitute `1` for `12/12`.	`results = 1`
7 Write the final value stored in `results`.	`1`

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Exercise 21.2 Programs

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE1** Answer these questions for each of the following values.

i. Is the value a number, a string or Boolean?

ii. Apply the expression `typeof value` to each value. What is the result of the expression `typeof value`?

a. `"A string"`

b. `"-2.344"`

c. `false`

d. `"The first sentence."`

e. `-2.344`

f. `true`

g. `"1234567"`

h. `1234567`

2. **WE2** Each of the following is a valid variable name. True or false?

a. `value2||value1`
b. `number*multiply`
c. `thisIsAValidName`
d. `23.34_a_decimal`
e. `element_1_2`
f. `else`
g. `length_12`
h. `age_16`
i. `for`
j. `isThisOne_Valid`
k. `2`
l. `false`
m. `______`
n. `a name`
o. `and&&this`

3. **WE5** Manually simulate running the following programs. For each program:

i. what is the final type of the variable `result`
ii. what is the final result stored in the variable `result`
iii. what is the result of the expression `typeof result`?

a.
```
var result = 23.234;
result = false;
```
b.
```
var result = 123+2+32+2+222+2+1;
result = result / (61+1+16+1+111+1+1);
```
c.
```
var result = "Was this";
result = "Now this, but the same type";
```
d.
```
var result = true;
var another = "Another type and value";
result = another;
another = false;
```

4. **WE6** Manually simulate running the following programs. For each program, what is the final value stored in the variable `information`?

a.
```
var information = true;
if (false) {
  information = false;
}
```
b.
```
var information = 3;
if (true) {
  information = information*information*information;
}
```

5. **WE7** Manually simulate running the following programs. For each program, what is the final value stored in the variable `calculations`?

a.
```
var calculations = 3;
var tooSmall = false;
if (tooSmall) {
  calculations = calculations/2;
} else {
  calculations = calculations*calculations;
}
```
b.
```
var calculations = (1+2+3)*5;
if (true) {
  calculations = calculations/2;
} else {
  calculations = calculations/3;
}
```

Understanding

6. **WE3** Write the following as JavaScript expressions.

a. $2048 \div (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)$

b. $\dfrac{2.2 + 0.7 + 0.1}{3.2 - 0.1 \times 2}$

c. $(11 + 2) + \dfrac{15 - 4}{11} + 16$

d. The remainder from the division $22 \div 16$

e. $22 \times 12 \div 2$

f. $\sqrt{625 \div (5 \times 5)}$

g. $18 \times 9 \times 27 \times 2 \times 0.5$

7. **WE4** Write JavaScript statements to make the following variable assignments.

a. Assign `"Earth"` to the variable `planet`.

b. Assign the mathematical expression $2 \times 4 + 1$ to the variable `threeSquared`.

c. Assign the value `10.2` to the variable `timeInSeconds`.

d. Assign the mathematical expression $\dfrac{2 + 3 + 3}{2}$ to the variable `division`.

e. Assign the value `2` to the variable `distanceOverThereInKilometers`.

8. Manually simulate running the following programs. For each program:

i. what is the final type of the variable `a`

ii. what is the final result stored in the variable `a`

iii. what is the result of the expression `typeof a`?

a.

```
var a = 34;
var b = 12;
var c = 22;
b = c;
b = b*b;
a = c+b;
c = 34;
```

b.

```
var a = 480;
var r = 12;
var y = 12;
a = a/r;
var t = 23;
a = a + t;
a = a * y;
```

9. Manually simulate running the following program. What is the final value stored in the variable `number`?

```
var test1 = true;
var test2 = false;
var test3 = false;
var test4 = true;
var number = 322;
if (test1) { number = number / 2; }
if (test2) { number = 3 * number + 1; }
if (test3) { number = number / 2; }
if (test4) { number = 3 * number + 1; }
```

10. Manually simulate running the following program. What is the final value stored in the variable `time`?

```
var time = 100;
if (true) {
  time = time + time - 2;
  time = 20 * time;
} else {
  time = 2 * time + time * 4 + 2;
}
```

Reasoning

11. Write the rest of the program below to calculate the `perimeter` and `area` for a rectangle, given the `width` and `height` of the rectangle.

```
var width = 10;
var height = 12;
```

12. Write the rest of the program below to calculate the total `surface` area and `volume` of a box, given the `width`, `height` and `depth` of the `box`.

```
var width = 8;
var height = 5;
var depth = 3;
```

Problem solving

13. Given the equation $(fx + g)(hx + i) = ax^2 + bx + c$, write the rest of the program below to solve for `a`, `b` and `c` given `f`, `g`, `h` and `i`.

```
var f = 7;
var g = 12;
var h = 2;
var i = -3;
```

14. Given the equation $ax^2 + bx + c = (fx + g)(hx + i)$, write the rest of the program below to solve for `g`, `h` and `i` given `a`, `b`, `c` and `f`.

```
var a = 12;
var b = 39;
var c = 30;
var f = 3;
var g = 6;
```

21.3 Data structures

21.3.1 Data structures and numbers

- In JavaScript, data structures can be constructed from basic building blocks. The building blocks include numbers, strings and Booleans. These building blocks can be combined to represent more complex values.
- A basic piece of data associated with other data is called a **property**. For example, a person's data may be made up of many properties.

Property	Example value	Type
`firstName`	`"Tom"`	String
`surname`	`"Jones"`	String
`yearOfBirth`	`1940`	Number
`isAdult`	`true`	Boolean

- Numbers are the most widely used type in computer programs. JavaScript uses the number type to represent both integers and real numbers. Most other programming languages use different types to represent integers and real numbers.
- It is important to understand that there are limitations on the size and accuracy of numbers stored in JavaScript.
  - JavaScript can accurately store all integer values between `-9007199254740992` and `9007199254740992` inclusive.
  - JavaScript can approximately store real values between $-1.797\,693\,134\,823\,157 \times 10^{308}$ and $1.797\,693\,134\,823\,157 \times 10^{308}$. However, if a value is very close to 0, JavaScript approximates the value as `0`. JavaScript cannot represent very small real values between -5×10^{-324} and 5×10^{-324}; all values in that range are approximated as `0`.

WORKED EXAMPLE 8

Can the following values be stored? If so, will they be stored accurately or approximately?

a `12`
b `11.00`
c `10.01`
d `9007199254740992`
e `9007199254740997`
f $\mathbf{1.79 \times 10^{308}}$
g $\mathbf{1.80 \times 10^{308}}$
h $\mathbf{2.3 \times 10^{-380}}$

THINK	**WRITE**
a The value `12` is an integer and within the limits of an accurate integer.	**a** Can be stored accurately
b The value `11.00` is still considered an integer and within the limits of an accurate integer.	**b** Can be stored accurately
c The value `10.01` is not an integer, but is within the limits of an approximate real number.	**c** Can be stored approximately
d The value `9007199254740992` is an integer and just within the limits of an accurate integer.	**d** Can be stored accurately
e The value `9007199254740997` is an integer and is outside limits of an accurate integer, but is within the limits of an approximate real number.	**e** Can be stored approximately
f The value 1.79×10^{308} is outside limits of an accurate integer, but is within the limits of an approximate real number.	**f** Can be stored approximately
g The value 1.80×10^{308} is outside the limits of an approximate real number.	**g** Cannot be stored
h The value 2.3×10^{-380} is too small to be represented.	**h** Cannot be stored

21.3.2 Strings

- Strings are used to represent text. A **character** is a string of length 1. A string is a list of characters. Each character position in a string has an incrementing **index** starting at 0.

The string `"Year 10"`	
Character	**Index**
`"Y"`	0
`"e"`	1
`"a"`	2
`"r"`	3
`" "`	4
`"1"`	5
`"0"`	6

- The first character in a `string` of length `N` is accessed with index `0`, the second character is accessed with index `1`, the third character is accessed with index `2`, and the last character is accessed with index `N-1`.

`var string=`	`"`	`a`	`b`	`c`	`...`	`#`	`"`
Location		1st	2nd	3rd	...	N	
JavaScript index		`0`	`1`	`2`		`N-1`	

WORKED EXAMPLE 9

What is the JavaScript index of each of the following characters in the string `"abcdefgh"`?

a `"a"` **b** `"f"` **c** `"b"` **d** `"h"`

THINK	WRITE
a `"a"` is the 1st value, so it has an index `0`.	**a** `0`
b `"f"` is the 6th value, so it has an index `5`.	**b** `5`
c `"b"` is the 2nd value, so it has an index `1`.	**c** `1`
d `"h"` is the 8th value, so it has an index `7`.	**d** `7`

- Characters inside a string are accessed using an `index` with the expression `string[index]`.

WORKED EXAMPLE 10

Given the string assignment below, evaluate the following JavaScript expressions manually.

```
var string = "The quick brown fox";
```

a `string[1]` **b** `string[0]` **c** `string[12]` **d** `string[4+3]`

THINK	WRITE
a `string[1]` has an index of `1`, which is the 2nd character in the string.	**a** `"h"`
b `string[0]` has an index of `0`, which is the 1st character in the string.	**b** `"T"`
c `string[12]` has an index of `12`, which is the 13th character in the string.	**c** `"o"`
d `string[4+3]` has an index of `7`, which is the 8th character in the string.	**d** `"c"`

21.3.3 Booleans

- You will recall that Booleans are the simplest value type, and are often used in logic. A Boolean has only two possible values: `true` or `false`. The `value` of a Boolean can be toggled using the expression `!value`.
- Two numerical values can be compared with each other by using different combinations of symbols as shown in the following table. The comparisons return Boolean results as shown.

Expression	Description
`!value`	Opposite of `value`, where `value` is a Boolean
`x1===x2`	`true` if `x1` equals `x2`, otherwise `false`
`x1!==x2`	`true` if `x1` does not equal `x2`, otherwise `false`
`x1>x2`	`true` if `x1` is greater than `x2`, otherwise `false`
`x1>=x2`	`true` if `x1` is greater than or equal to `x2`, otherwise `false`
`x1<x2`	`true` if `x1` is less than `x2`, otherwise `false`
`x1<=x2`	`true` if `x1` is less than or equal to `x2`, otherwise `false`

WORKED EXAMPLE 11

Evaluate the following JavaScript expressions manually.

a `!false`
b `!true`
c `5.43===5.34`
d `1233.4!==1323.4`
e `18.4>-12.2`
f `6.8>=6.7`
g `34.015<34.025`
h `101.99<=101.01`

THINK	WRITE
a The opposite of `false` is `true`.	**a** `true`
b The opposite of `true` is `false`.	**b** `false`
c The statement “`5.43` equals `5.34`” is `false`.	**c** `false`
d The statement “`1233.4` does not equal `1323.4`” is `true`.	**d** `true`
e The statement “`18.4` is greater than `-12.2`” is `true`.	**e** `true`
f The statement “`6.8` is greater than or equal to `6.7`” is `true`.	**f** `true`
g The statement “`34.015` is less than `34.025`” is `true`.	**g** `true`
h The statement “`101.99` is less than or equal to `101.01`” is `false`.	**h** `false`

- The statement `boolean1&&boolean2` returns `true` if `boolean1` and `boolean2` are both `true`; otherwise, the statement returns `false`.

WORKED EXAMPLE 12

Evaluate the following JavaScript expressions manually.

a `true&&true`
b `false&&false`
c `true&&false`
d `false&&true`

THINK	WRITE
a Both the Booleans are `true`.	**a** `true`
b At least one Boolean is `false`.	**b** `false`
c At least one Boolean is `false`.	**c** `false`
d At least one Boolean is `false`.	**d** `false`

- The statement `boolean1||boolean2` returns `true` if either `boolean1` or `boolean2` are `true`; otherwise, the statement returns `false`.

WORKED EXAMPLE 13

Evaluate the following JavaScript expressions manually.

a `false||false`
b `true||false`
c `false||true`
d `true||true`

THINK	WRITE
a Both the Booleans are `false`.	**a** `false`
b At least one Boolean is `true`.	**b** `true`
c At least one Boolean is `true`.	**c** `true`
d At least one Boolean is `true`.	**d** `true`

21.3.4 Arrays

- Arrays are one method of combining multiple types into a list data structure. A variable can contain an array of the basic building blocks: numbers, strings and Booleans.
- An array takes the form `[value0, value1, value2,...]` where `value0, value1, value2, ...` are the different values of the array. An array can have zero, one or more values, and the values can be any type (i.e. number, string or Boolean).
- An array with many values can be split over multiple lines. For example:

```
var array = [
value0,
value1,
value2,
...
];
```

WORKED EXAMPLE 14

Each of the following is an array. True or false?

a `"abc"`
b `123.12`
c `[2.3,6.7,53]`
d `[]`
e `["abc"]`
f `["a", "b", "c"]`
g `true`
h `[14,false, " ",[3,3]]`

THINK	WRITE
a `"abc"` is in quotation marks, so this value is a string, not an array.	**a** False
b `123.12` is a numerical value not in quotation marks, so this value is a number, not an array.	**b** False
c `[2.3,6.7,53]` is a list of 3 numbers, so this is an array.	**c** True
d `[]` is an empty list, so this is an array.	**d** True
e `["abc"]` is a list of 1 string, so this is an array.	**e** True
f `["a","b","c"]` is a list of 3 strings, so this is an array.	**f** True
g `true` is a Boolean value, not an array.	**g** False
h `[14,false," ",[3,3]]` is a list of 4 different types of values so this is an array.	**h** True

- Values in an array are accessed in a similar fashion to how characters are accessed in a string. The values inside an array are accessed using an integer index starting at `0`. The first value in an `array` of length `N` is accessed with index `0`, the second value is accessed with index `1`, the third value is accessed with index `2`, and the last value is accessed with index `N-1`.

`var array=`	`[`	`"first"`	`,`	`"second"`	`,`	`"third"`	`,...,`	`"last"`	`];`
Location		1st		2nd		3rd	...	Nth	
JavaScript index		`0`		`1`		`2`		`N-1`	

WORKED EXAMPLE 15

What are the JavaScript indexes of the following values in the array `["one","two","three","four","five","last"]`?

a `"one"` **b** `"three"` **c** `"four"` **d** `"last"`

THINK	WRITE
a `"one"` is the 1st value, so it has an index of `0`.	**a** `0`
b `"three"` is the 3rd value, so it has an index of `2`.	**b** `2`
c `"four"` is the 4th value, so it has an index of `3`.	**c** `3`
d `"last"` is the 6th value, so it has an index of `5`.	**d** `5`

- A value inside an `array` is accessed by its `index` using the expression `array[index].`

WORKED EXAMPLE 16

Given the array assignment

`var array = ["String",false,14,51,"true",3];`

evaluate the following JavaScript expressions manually.

a `array[1]` **b** `array[3]`

c `array[2]` **d** `array[array[5]]`

THINK	WRITE
a `array[1]` has an index of `1`, which is the 2nd value in the array.	**a** `false`
b `array[4]` has an index of `4`, which is the 5th value in the array.	**b** `"true"`
c `array[2]` has an index of `2`, which is the 3rd value in the array.	**c** `14`

d **1** `array[5]` has an index of `5`, which is the 6th value in the array.	**d**
2 The 6th value is 3.	`array[5]=3`
3 Write the original expression.	`array[array[5]]`
4 Substitute 3 for `array[5]`.	`= array[3]`
5 `array[3]` has an index of `3`, which is the 4th value in the array.	`= 51`

- The expression `array.length` will return the number of values in the `array`.

WORKED EXAMPLE 17

Evaluate the following JavaScript expressions manually.

a `[].length`

b `[13,2].length`

c `[12.2,false,[12,11,10,9]].length`

THINK	WRITE
a `[]` is an empty list, so it has a length of `0`.	**a** `0`
b `[13,2]` is a list of `2` numbers.	**b** `2`
c `[12.2,false,[12,11,10,9]].length` is a list of `3` different types of values. *Note:* Do not count the number of values in the `[12,11,10,9]` array, as this is counted as one value in the outer array.	**c** `3`

21.3.5 Objects

- Arrays combine a list of different values, where each value is accessed using an integer index. If an integer is not descriptive enough, then a JavaScript **object** can be used to build data structures. Arrays use integer indexes to access the values in the array, whereas JavaScript objects require property names to access the values.
- The following structure assigns the JavaScript object with the properties `property1, property2, ..., propertyN` and values `value1, value2, ... , valueN` to the `variable`.

```
var variable = {
  property1: value1,
  property2: value2,

  propertyN: valueN
}
```

WORKED EXAMPLE 18

Assign a JavaScript object using the following information to the variable `singer`.

Property	Value
`firstName`	`"Tom"`
`surname`	`"Jones"`
`yearOfBirth`	`1940`
`isAdult`	`true`

THINK	**WRITE**
1 Start the assignment to `singer`.	`var singer = {`
2 Define the first property and value.	`firstName: "Tom",`
3 Define the second property and value.	`surname: "Jones",`
4 Define the third property and value.	`yearOfBirth: 1940,`
5 Define the last property and value.	`isAdult: true`
6 Close the brackets. Note that the last property does not require a trailing comma.	`}`

21.3.6 Accessing properties

- A `property` of an object `variable` is extracted using the expression `variable.property`.

```
var variable = {
  property: "store this",
  ignoreProperty: "ignore this"
}
var stored = variable.property;
```

WORKED EXAMPLE 19

Given the object assignment below, evaluate the following JavaScript expressions manually.

```
var person = {
  firstName: "Tom",
  surname: "Jones",
  yearOfBirth: 1940,
  isAdult: true
}
```

a `person.surname` **b** `person.isAdult`

c `person.yearOfBirth`

THINK	**WRITE**
a Access the `surname` property of `person`.	**a** `"Jones"`
b Access the `isAdult` property of `person`.	**b** `true`
c Access the `yearOfBirth` property of `person`.	**c** `1940`

21.3.7 Pointers

- A **pointer** is a reference to an object. Variables become pointers when they are assigned an array or object. Variables are not pointers when they are assigned a number, string or Boolean. Instead, a new copy is created.

```
// Pointers
var arrayOriginal = ["Array","array"];
var arrayPointer = arrayOriginal;var
```

```
objectOriginal = {x:1,y:2};
var objectPointer = objectOriginal;


// New copies are created
var stringOriginal = "String";
var stringNewCopy = stringOriginal;


var numberOriginal = 1;
var numberNewCopy = numberOriginal;


var booleanOriginal = true;
var boolanNewCopy = booleanOriginal;
```

WORKED EXAMPLE 20

In each of the following programs, the `variable` is a pointer. True or false?

a
```
var variable = ["Value1","Value2"];
```
b
```
var location = {
  x: 10,
  y: 20
}
var variable = location;
```
c
```
var original = true;
var variable = original;
```
d
```
var Anthony = "Tony";
var variable = Anthony;
```
e
```
var calculation = 10*2;
var variable = calculation;
```

THINK	WRITE
a The assignment to `variable` is an array, so `variable` is a pointer.	**a** True
b The variable `location` is an object, so `variable` is a pointer.	**b** True
c The variable `original` is a Boolean, so `variable` is not a pointer.	**c** False
d The variable `Anthony` is a string, so `variable` is not a pointer.	**d** False
e The variable `calculation` is a number, so `variable` is not a pointer.	**e** False

- In JavaScript, it is possible for two variables, `originalPointer` and `newPointer`, to point to a single object. That is, either of the two pointers will reference the same object.

```
var originalPointer = {
  property1: value1,
  property2: value2,
  ...
  propertyN: valueN
}
var newPointer = originalPointer;
```

WORKED EXAMPLE 21

Consider the following program.

```
var pointer1 = {
  colour:"Blue",
  bold:true,
  x:10,
  y:1
}
var pointer2 = pointer1;
pointer1.colour = "Red";
pointer2.bold = false;
pointer1.x = pointer2.y;
```

Evaluate the following JavaScript expressions after the program has run.

a `pointer2.colour`
b `pointer1.bold`
c `pointer2.x`
d `pointer1.y`

THINK / **WRITE**

1 `pointer1` and `pointer2` both reference the same object.

```
{
  colour:"Blue",
  bold:true,
  x:10,
  y:1
}
```

2 `pointer1` and `pointer2` both reference the same object.
`pointer1.colour = "Red";`
The above statement changes the object's property `colour` to `"Red"`.

```
{
  colour:"Red",
  bold:true,
  x:10,
  y:1
}
```

3 `pointer1` and `pointer2` both reference the same object.
`pointer2.bold = false;`
The above statement changes the object's property `bold` to `false`.

```
{
  colour:"Red",
  bold:false,
  x:10,
  y:1
}
```

4 `pointer1` and `pointer2` both reference the same object.
`pointer1.x = pointer2.y;`
The above statement changes the object's property `x` to equal the property `y`.

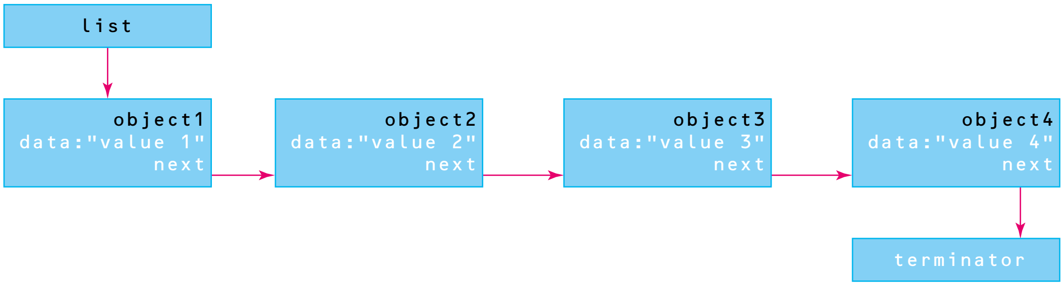
```
{
  colour:"Red",
  bold:false,
  x:1,
  y:1
}
```

THINK	WRITE
a `pointer2.colour` accesses the `colour` property.	**a** `"Red"`
b `pointer1.bold` accesses the `bold` property.	**b** `false`
c `pointer2.x` accesses the `x` property.	**c** `1`
d `pointer1.y` accesses the `y` property.	**d** `1`

21.3.8 Linked lists

- A **linked list** is a list of objects in which each object stores `data` and points to the `next` object in the list. The `list` points to the first object. The last object points to a `terminator` to indicate the end of the list.
- It is easier to build linked lists in reverse order because the pointers in each object point to the next item in the list. The last object in the list only points to the terminator, so it can be completely built. Once the last object has been built, the second last object can be completely built, and so on.

```
var terminator = {};
var object4 = { data:"value 4", next:terminator};
var object3 = { data:"value 3", next:object4};
var object2 = { data:"value 2", next:object3};
var object1 = { data:"value 1", next:object2};
var list = object1;
```



WORKED EXAMPLE 22

Represent the `list` of values `"a"`, `"b"` and `"c"` as a linked list.

THINK

1 Define a `terminator`.

2 Define the last object with the `data: "c"` and `next` pointing to the `terminator`.

3 Define the object with the `data: "b"` and `next` pointing to the previous object (`object3`).

4 Define the first object with the `data: "a"` and `next` pointing to the previous object (`object2`).

5 Define `list` pointing to the first object (`object1`).

WRITE

```
var terminator = {};
var object3 = {
  data:"c",
  next:terminator
}
var object2 = {
  data:"b",
  next:object3
}
var object1 = {
  data:"a",
  next:object2
}
var list = object1;
```

- Linked lists are useful data structures, as they can represent lists of data that have no set length and can change throughout the program.

learnon RESOURCES — ONLINE ONLY

Try out this interactivity: CodeBlocks Activity 2 (int-6574)

Exercise 21.3 Data structures

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE8** Can the following values be stored? If so, will they be stored accurately or approximately?

- **a.** 2.6×10^{-480}
- **b.** `244`
- **c.** `61.00`
- **d.** `-23.0001`
- **e.** `-9007199254740999`
- **f.** -1.79×10^{308}
- **g.** `9007199254740992`
- **h.** `26`
- **i.** `6.00`
- **j.** `-120.99`
- **k.** `-9007199254740992`
- **l.** $1.797\,693\,135 \times 10^{308}$
- **m.** 1.236×10^{-20}
- **n.** 2×10^{308}
- **o.** `-9007199254740994`
- **p.** $1.797\,693\,134 \times 10^{308}$

2. **WE11** Evaluate the following JavaScript expressions manually.

- **a.** `!true`
- **b.** `true===false`
- **c.** `true!==false`
- **d.** `"223"==="223"`
- **e.** `3.41!==3.42`
- **f.** `822.4>222.2`
- **g.** `61.18>=162.17`
- **h.** `32.15<224.25`
- **i.** `53===54`
- **j.** `3.4!==3.4`
- **k.** `82.4>-1222.2`
- **l.** `!!true`
- **m.** `1021.929<=10222.1`
- **n.** `!false`
- **o.** `!!!false`
- **p.** `"23"==="4"`
- **q.** `61.18>=162.17`
- **r.** `342.15<3224.25`
- **s.** `101.929<=122.1`

3. **WE12** Evaluate the following JavaScript expressions manually.

- **a.** `false&&false`
- **b.** `true&&false`

4. **WE13** Evaluate the following JavaScript expressions manually.

- **a.** `true||true`
- **b.** `false||false`

5. Evaluate the following JavaScript expressions manually.

- **a.** `true||false`
- **b.** `false&&true`
- **c.** `false||true`
- **d.** `true&&true`

6. **WE17** Evaluate the following JavaScript expressions manually.

- **a.** `[1,2,3,2].length`
- **b.** `["blah","blah","blah",true,["Hello"]].length`
- **c.** `[].length`
- **d.** `[111,12,24,42,41,4].length`
- **e.** `["A string",false,["Array","here"],12].length`

Understanding

7. **WE9** What is the JavaScript index of the following characters in the string
`"The quick brown fox jumped over the lazy dog"`?

- **a.** `"x"`
- **b.** `"c"`
- **c.** `"z"`
- **d.** `"p"`
- **e.** `"f"`
- **f.** `"g"`

8. **WE10** Given the string assignment below, evaluate the following JavaScript expressions manually.

`var someText = "Strings are a little bit like arrays.";`

- **a.** `someText[9]`
- **b.** `someText[19]`
- **c.** `someText[20+3]`

9. **WE14** Each of the following is an array. True or false?

- **a.** `[]`
- **b.** `["TIME"]`
- **c.** `false`
- **d.** `"TIME"`
- **e.** `1.1233`
- **f.** `["T","I","M","E"]`
- **g.** `[14,false,"Y","A",61.71,[1,2,3],161.7]`
- **h.** `[0.13,16,61.71,161.7,253]`

10. WE15 What are the JavaScript indexes of the following values in the array `["JAN","FEB","MAR","APR","MAY","JUN","JUL","AUG","SEP","OCT","NOV","DEC"]`?

a. `"DEC"` **b.** `"JUL"` **c.** `"MAR"` **d.** `"JAN"` **e.** `"JUN"` **f.** `"NOV"`

11. WE19 Given the object assignment below, evaluate the following JavaScript expressions manually.

```
var purchase = {
  item: "iPad",
  number: 1,
  cost: 765.00,
  paid: true
}
```

a. `purchase.cost` **b.** `purchase.item` **c.** `purchase.paid`

12. WE20 In each of the following programs, the `possiblePointer` is a pointer. True or false?

a.
```
var possiblePointer = ["A","B","C"].length;
```
b.
```
var string = "Some string";
var possiblePointer = string;
```
c.
```
var boolean = false;
var possiblePointer = boolean;
```
d.
```
var point = {
  x: 10,
  y: 20
}
var possiblePointer = point.x;
```
e.
```
var possiblePointer = ["An array"];
```
f.
```
var workThisOut = [(100+99+98-1)*0];
var possiblePointer = workThisOut;
```
g.
```
var objects = {d:"Data"};
var possiblePointer = objects;
```

13. WE21 Consider the following program.

```
var original = {
  firstName:"Wendy",
  surname:"Cook",
  age: 76
}
var newSurname = "Simons";
var pointer = original;
original.surname = newSurname;
pointer.firstName = "Clare";
original.age = 19;
```

Evaluate the following JavaScript expressions after the program has run.

a. `pointer.surname` **b.** `original.firstName`

c. `original.age`

Reasoning

14. WE22 Represent the list of objects `5`, `4` and `3` as a linked list called `list`.

15. Explain why a linked list is easier to build in reverse order.

Problem solving

16. Given the linked list below, write reassignment(s) required to remove the object with the `data"value 1"` from the list.

```
var terminator = {};
var growingList = terminator;
growingList = { data:"value 2", next:growingList};
growingList = { data:"value 1", next:growingList};
var list = growingList;
```

17. Given the linked list below, write reassignment(s) required to remove the object `with the data "value 3"` from the list.

```
var terminator = {};
var object4 = { data:"value 4", next:terminator};
var object3 = { data:"value 3", next:object4};
var object2 = { data:"value 2", next:object3};
var object1 = { data:"value 1", next:object2};
var list = object1;
```

21.4 Algorithms

- An **algorithm** is a step-by-step set of tasks to solve a particular problem. A program is the implementation of an algorithm.

21.4.1 Output

- The **console** is a special region in a web browser for monitoring the running of JavaScript programs. Most web browsers, including Chrome, Firefox, Safari, Internet Explorer, Microsoft Edge and Opera, allow you activate the console through the menus options.
- In order to see the result of an expression or value, the JavaScript `console.log` function can be used. This function outputs results to the console.

JavaScript statement	Output to console
`console.log("Output value");`	`Output value`

Note that the quotation marks in strings are not shown in the output.

WORKED EXAMPLE 23

What is the output to the console after each of the following statements runs?

a `console.log([true,"23",4.5][2]);`

b `console.log("Simple string");`

THINK	WRITE
a **1** Show the original statement.	**a** `console.log([true,"23",4.5][2]);`
2 `[true,"23",4.5][2]` is accessing the 3rd value in the array `[true,"23",4.5]`	`console.log(4.5);`
3 Write the output to console.	4.5

b **1** Show the original statement.

b `console.log("Simple string");`

2 Write the output to console. Note that the quotation marks in strings are not shown in the output.

```
Simple string
```

21.4.2 Comments

- Comments are added to code to give hints about function of the program. The comments are completely ignored by the computer, so they are only for our benefit.
- JavaScript comments start with `//`. All the text after `//` on a line is ignored by the computer. For example, consider the following line:

  `var sqrt2 = 1.4142; // An approximation to square root of 2`

  The first part of the line, `var sqrt2 = 1.4142;`, is the actual JavaScript code to be run.

  The second part (until the end of the line), `// An approximation to square root of 2`, is ignored by the computer and is only there for our own reference. The text could be changed and would make no difference to the running of the program.

WORKED EXAMPLE 24

Add a comment to each statement of the form `// variable=value` where `variable` is the variable name and `value` is the evaluated value.

a `var sqrt2 = 1+4/10;`

b `var pages = 12+11;`

c `var product = 1*2*3;`

THINK	WRITE
a **1** The variable name is `sqrt2`.	**a**
2 The calculated value is `1+4/10 = 1.4`.	
3 Append the comment.	`var sqrt2 = 1+4/10; // sqrt2=1.4`
b **1** The variable name is `pages`.	**b**
2 The calculated value is `12+11 = 23`.	`var pages = 12+11; // pages=23`
3 Append the comment.	
c **1** The variable name is `product`.	**c**
2 The calculated value is `1*2*3 = 6`.	
3 Append the comment.	`var product = 1*2*3;// product=6`

21.4.3 Defining a function

- In more complex programs, it is useful to wrap a section of code in a JavaScript function. This gives the section of code `{statement 1, statement 2, ..., statement n}` a descriptive `name`. A function with a `name` can be given zero, one or more inputs, `x1,x2,...`.

```
function name(x1,x2,...) {
  statement 1
  statement 2
```

```
  statement n
  return output;
}
var store = name(v1,v2,...);
```

- Once a function is defined, it can be called with the statement `name(v1,v2,...);`. The value `v1` is assigned to input `x1`, the value `v2` is assigned to input `x2`, and so on.
- In JavaScript a function `name` can `return` an internal value `output` to be `stored`.

WORKED EXAMPLE 25

What is the output to the console of each of the following programs?

a

```
function mean(a,b,c) {
   console.log((a+b+c)/3);
}
mean(11,100,12);
```

b

```
function probability(events,samples) {
  console.log(events/samples);
}
probability(2,200);
```

c

```
function willNotChange() {
  return "Same";
}
console.log(willNotChange());
```

THINK	WRITE
a **1** The input `a` equals `11`.	**a** `a = 11`
2 The input `b` equals `100`.	`b = 100`
3 The input `c` equals `12`.	`c = 12`
4 Write the output statement.	`console.log((a+b+c)/3)`
5 Substitute `11` for `a`, `100` for `b` and `12` for `c`.	`console.log((11+100+12)/3)`
6 Substitute `(11+100+12)/3` for `41`.	`console.log(41)`
7 Write the output to the console.	`41`
b **1** The input `events` equals `2`.	**b** `events = 2`
2 The input `samples` equals `200`.	`samples = 200`
3 Write the output statement.	`console.log(events/samples)`
4 Substitute `2` for `events` and `200` for samples.	`console.log(2/200)`
5 Substitute `2/200` for `0.01`.	`console.log(0.01)`
6 Write the output to the console.	`0.01`
c **1** The function has no inputs.	**c**
2 Write the return expression.	`willNotChange()="Same"`
3 Write the output statement.	`console.log(willNotChange());`

4 Substitute `"Same"` for `willNotChange()`.	`console.log("Same");`
5 Write the output to the console.	`Same`

21.4.4 Return at any point

- A function can have many return points within itself. Once the return point is reached, the function does not execute any further. This is useful when the result is known before running through the rest of the function.

```
function name() {
  ...
  if (decision) {
  return early;
  }
  ...
  return output;
}
```

WORKED EXAMPLE 26

What is the output to the console of the following program?

```
function isFirst(index) {
  var decision = index===0;
  if (decision) {
    return "Yes";
  }
  return "No";
}
var test = isFirst(0);
console.log(test);
```

THINK	WRITE
1 Write the assignment.	`test = isFirst(0)`
2 The input `a` equals `0`.	`index = 0`
3 Write the first assignment inside the function `isFirst`.	`decision = index===0`
4 Substitute `0` for `index`.	`decision = 0===0`
5 Substitute `true` for `0===0`.	`decision = true`
6 `decision` is `true`, so run the statement inside the first `{}` block. The function is now finished.	`return "Yes";`
7 Write the return expression.	`isFirst(0) = "Yes"`
8 Rewrite the assignment outside the function.	`test = isFirst(0)`
9 Substitute `"Yes"` for `isFirst(0)`.	`test = "Yes"`
10 Write the output statement.	`console.log(test);`
11 Substitute `"Yes"` for `test`.	`console.log("Yes");`
12 Write the output to console.	`Yes`

21.4.5 Designing an algorithm

- Designing an algorithm for a problem involves:
  - determining the inputs
  - determining the function name
  - breaking the problem into simple steps.

WORKED EXAMPLE 27

Design an algorithm for each of the problems below. For each problem:

i determine the inputs
ii determine the function name
iii break the problem into simple steps.

a Round a `decimal` down to the nearest unit.
b Count the number of squares with a given `side` length that fit inside a rectangle with a given `width` and `height`.

THINK	**WRITE**
a **i** Write the input.	**a** **i** `decimal`
ii Give the function a short meaningful name.	**ii** `roundDown`
iii **1** Write step 1.	**iii** Find the decimal part.
2 Write step 2.	Find the whole number part.
b **i** Write the inputs.	**b** **i** `side,width,height`
ii Write the output (function name).	**ii** `count`
iii **1** Write step 1.	**iii** Count the number of squares that fit along the width.
2 Write step 2.	Round the width count down to the nearest integer.
3 Write step 3.	Count the number of squares that fit up the height.
4 Write step 4.	Round the height count down to the nearest integer.
5 Write step 5.	Return the width count multiplied by the height count.

21.4.6 Implement an algorithm

- Implementing an algorithm in JavaScript involves:
  - designing an algorithm
  - writing function inputs
  - writing a JavaScript statement for each step
  - returning the required result (output).

WORKED EXAMPLE 28

Implement an algorithm as a function in JavaScript for each of the problems below.

a Round a `decimal` down to the nearest unit.
b Count the number of squares with a given `side` length that fit inside a rectangle with a given `width` and `height`.

THINK **WRITE**

a **1** Design an algorithm. **a**

2 Write the function input.

```
function roundDown(decimal) {
```

3 Write step 1.

```
  // Find the decimal part
  var decimalPart = decimal%1;
```

4 Write step 2.

```
  // Find the whole number part.
  var whole = decimal-decimalPart;
```

5 Return the required result.

```
  return whole;
}
```

b **1** Design an algorithm. **b**

2 Write the function inputs.

```
function count(side,width,height) {
```

3 Write step 1.

```
  // Count number squares that
  // fit along the width.
  var widthCount = width/side;
```

4 Write step 2.

```
  // Round the width count down
  // to the nearest unit.
  widthCount=roundDown(widthCount);
```

5 Write step 3.

```
  // Count number squares that
  // fit up the height.
  var heightCount = height/side;
```

6 Write step 4.

```
  // Round the height count down
  // to the nearest unit.
  heightCount=roundDown(heightCount);
```

7 Write step 5.

```
  var squares=widthCount*heightCount;
```

8 Return the required result.

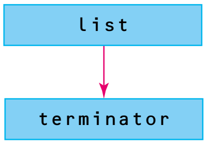
```
  return squares;
}
```

21.4.7 Linked list algorithms and empty lists

- The following algorithms are required to create and manipulate a list of data.
  - Create a terminator.
  - Create an empty list.
  - Add data to the start of the list.
  - Add data to the end of the list.
  - Add data at a particular location in the list.
  - Remove data from the start of the list.
  - Remove data from the end of the list.
  - Remove data at a particular location in the list.
  - Read the data at particular location in the list.
  - Change the data at particular location in the list.
  - Copy a list.
  - Sort a list.

Create an empty list

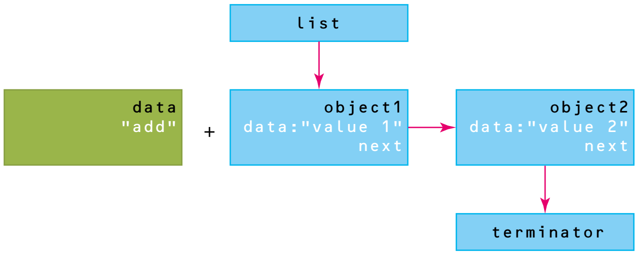
- In order to build on a list, the first thing required is an empty list. The variable `list` normally points to the start of the list, but in this case the start of an empty list is the `terminator`.

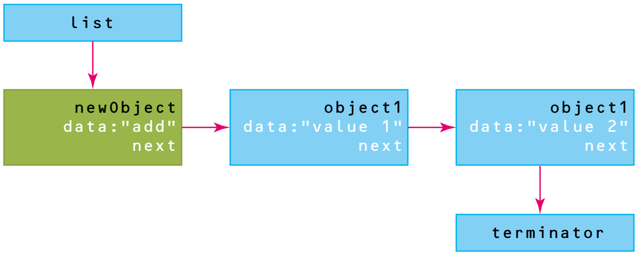


```
// Create a terminator.
var terminator = {};

// Define a function to create an empty list.
function empty() {
  return terminator;
}

// Store the empty list
var list = empty();
```

Add data

Once you have a list, new data can be added to the list.





WORKED EXAMPLE 29

Design and implement an algorithm to add `data` to the start of a linked `list` and return the updated list. Create an empty list called `blank`. Call the function defined above with the data `"Only"` and the `blank` list. Store the updated list in the variable `list`. Use the following steps.

a Determine the inputs.
b Determine the function name.
c Break the problem into simple steps.

d **Implement the algorithm as a function in JavaScript.**
  - **i** **Write function inputs.**
  - **ii** **Write JavaScript statements for each step.**
  - **iii** **Return the required result (output).**

e **Test the function.**

THINK	WRITE
a Write the inputs.	**a** `list, data`
b Give the function a short meaningful name.	**b** `addToStart`
c **1** Write step 1.	**c** Create a new object.
2 Write step 2.	Add the data to the new object.
3 Write step 3.	Point the new object to the first item in the original list.
4 Returning the required result.	Return the new object as the start of the new list.

d **i** Implement the algorithm. Start by writing the function inputs.

```
d function addToStart(list, data) {
```

ii **1** Comment on step 1.

2 Comment on step 2.

3 Comment on step 3.

```
    // Create a new object.
    // Add the data to the new object.
    // Point the new object to the
    // first item in the original list.
```

4 Implement steps 1, 2 and 3.

```
    var newObject = {
        data: data,
        next: list
    }
```

iii Return the required result.

```
    return newObject;
}
```

e **1** Test the new function. Start by defining the `terminator` and function `empty` as outlined previously in the chapter.

```
e var terminator = {};
function empty() {
    return terminator;
}
```

2 Create an empty list.

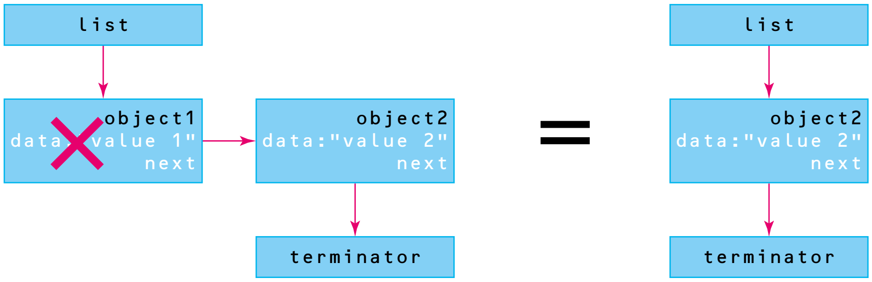
```
var blank = empty();
```

3 Call the function `addToStart` with the required inputs and store the result.

```
var data = "Only";
var list = addToStart(data, blank);
```

21.4.8 Remove data

- Once you have a populated list (a list that contains objects), objects can be removed from the list.



WORKED EXAMPLE 30

Design and implement an algorithm to remove the first object in a linked list, and return the updated list. Represent the `list3` of values `1,2` and `3` as the linked list. Create a function and call it with the `list3` list. Store the updated list in the variable `list2`.

a Determine the inputs.
b Determine the function name.
c Break the problem into simple steps.
d Implement the algorithm as a function in JavaScript.
  i Write the function inputs.
  ii Write a JavaScript statement for each step.
  iii Return the required result (output).
e Now test the function.

THINK	WRITE
a Write the inputs.	**a** `list`
b Give the function a short meaningful name.	**b** `removeFromStart`
c **1** Write step 1. **2** Return the required result.	**c** Create a pointer to the second object. Return the second object as the start of the new list.

d **i** **1** Implement the algorithm. Start by writing the function inputs.

d
```
function removeFromStart(list) {
```

ii **1** Comment on step 1.

2 Implement step 1.

ii
```
    // Create a pointer to
    // the second object.
    var secondObject = list.next;
```

iii Return the required result.

iii
```
    // Return the second object
    // as the start of the list.
    return secondObject;
  }
```

e **1** Test the new function. Start by defining a `terminator`.

e
```
var terminator = {};
```

2 Define the last object with the `data` value `3` and `next` pointing to the `terminator`.

```
var object3 = {
  data:3,
  next:terminator
}
```

3 Define the object with the `data` value `2` and `next` pointing to the previous object.

```
   var object2 = {
     data:2,
     next:object3
   }
```

4 Define the first object with the `data` value `1` and `next` pointing to the previous object.

```
var object1 = {
  data:1,
  next:object2
}
```

5 Define `list3` pointing to the first object.

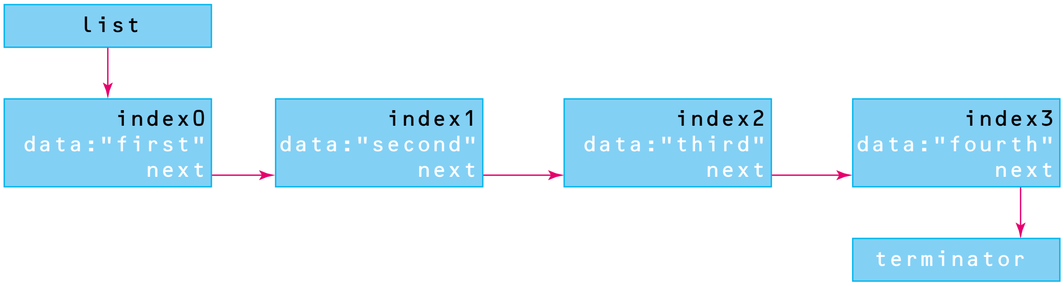
```
var list3 = object1;
```

6 Call the function `removeFromStart` with the required inputs and store the result.

```
var list2 = removeFromStart
  (list3);
```

21.4.9 Accessing data

- Once you have a populated list, objects can be accessed at any position in the list. Objects in a linked list are accessed in a similar way to objects in an array, using an integer index starting at `0`. The first object in a list of length `N` is accessed with index `0`, the second object is accessed with index `1`, the third object is accessed with index `2`, and the last object is accessed with index `N-1`.



WORKED EXAMPLE 31

Design and implement an algorithm to return the data at a particular `index` in a linked `list`. Represent the `list` of values `"Find"`, `"me"` and `"here."` as the linked list. Create a function, call it function with the `list` and access the second piece of data. Store the returned data in the variable `found`.

a Determine the inputs.

b Determine the function name.

c Break the problem into simple steps.

d Implement the algorithm as a function in JavaScript.

  i Write the function inputs.

  ii Write a JavaScript statement for each step.

e Now test the function.

THINK	WRITE
a Write the inputs.	**a** `list, index`
b Give the function a short meaningful name.	**b** `getData`
c 1 Write step 1.	**c** If the index is `0` or less, then return the data of the first object.
2 Write step 2.	Create a reduced list pointer to the second object.

3 Write step 3.

Create a new index one less than the input index.

4 Write step 4.

Return `getData` using the reduced list and index.

d **i** Implement the algorithm. Start by writing the function inputs.

i

```
function getData(list, index) {
```

ii **1** Comment on step 1.

ii

```
  // If index 0 or less then return
  // data of the first object.
```

2 Implement step 1.

```
  if (index <= 0) {
    return list.data;
  }
```

3 Comment on step 2.

```
  // Create a reduced list pointer
  // to the second object.
```

4 Implement step 2.

```
  var reduced = list.next;
```

5 Comment on step 3.

```
  // Create a new index one less
  // than the input index.
```

6 Implement step 3.

```
  var newIndex = index-1;
```

7 Comment on step 4.

```
  // Return getData using the
  // reduced list and index.
```

8 Implement step 4.

```
  return getData(reduced, newIndex);
}
```

e **1** Test the new function. Start by defining a `terminator`.

e

```
var terminator = {};
```

2 Define the last object with the `data`: `"here."` and `next` pointing to the `terminator`.

```
var object3 = {
  data:"here.",
  next:terminator
}
```

3 Define the object with the `data`: `"me"` and `next` pointing to the previous object.

```
var object2 = {
  data:"me",
  next:object3
}
```

4 Define the first object with the `data`: `"Find"` and `next` pointing to the previous object.

```
var object1 = {
  data:"Find",
  next:object2
}
```

5 Define a `list` pointing to the first object.

```
var list = object1;
```

6 Call the function `getData` with the `list` and index `1` (2nd value). Then store the result in found.

```
var found = getData(list,1);
```

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Exercise 21.4 Algorithms

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE23** What is the output to the console after each of the following statements runs?

a. `console.log("A string");`
b. `console.log(["A","B","C","D"][2]);`
c. `console.log(Math.sqrt(16));`
d. `console.log(true);`
e. `console.log(123*4/2);`
f. `console.log(true&&false);`

2. Match the following JavaScript comments with the appropriate program statements in the table below.

```
// set the depth to 1.4
// set the height to 2.3
// calculate the volume
// calculate the top area
// set the width to 3
```

Program	Comment
`var width = 3;`	
`var height = 2.3;`	
`var depth = 1.4;`	
`var area = width*height;`	
`var volume = area*depth;`	

3. **WE25** What is the output to the console of each of the following programs?

a.
```
function total(a,b,c,d,e) {
  console.log((a+b+c+d+e));
}
total(19,28,37,46,55);
```

b.
```
function boring() {
  return "Nothing to see
  here";
}
console.log(boring());
```

c.
```
function surfaceArea(width,height,depth){
  var faceTop = width*depth;
  var faceFront = width*height;
  var faceSide = height*depth;
  var area = 2*(faceTop + faceFront + faceSide);
  console.log(area);
}
surfaceArea(4,7,9);
```

4. **WE26** What is the output to the console of the following program?

```
function testLast(index, length) {
  var decision = index===length-1;
  if (decision) {
    return "Yes";
}
    return "No";
}
var isLast = testLast(100,100);
console.log(isLast);
```

5. Match the following algorithm design steps with their function implementations.

- Calculate the total of `a`, `b` and `c`.
- Calculate the average.
- Return the average.
- The function `average` requires three values: `a,b,c`.
- Store the number of values.

Function	**Design**

`function average(a,b,c) {`
`  var total = a+b+c;`
`  var number = 3;`
`  var output = total/number;`
`  return output;}`	

Understanding

6. **WE24** Add a comment to each statement of the form `// variable=value` where `variable` is the variable name and `value` is the evaluated value.

a. `var nearly_2 = 1+1/2+1/4+1/8+1/16+1/32;`
b. `var distanceKm = 2600/1000;`
c. `var seconds = 60*60*24*356.25;`

7. For each statement in the following program, find the `variable` name between var and `=`, then calculate the value assigned to this `variable`. Add a comment to the end of each statement indicating the progress: `// variable=value`.

```
var percentage=22+3;
var total=6+14;
var amount=(percentage/100)*total;
```

8. **WE27** Design an algorithm for each of the problems below. For each problem:

i. determine the inputs
ii. determine the function name
iii. break the problem into simple steps.

a. Test if a triangle with sides *a*, *b* and *c* is a right-angled triangle. Assume *c* is the largest value.
b. Test if a number is a positive integer.
c. Test if three numbers *a*, *b* and *c* are a Pythagorean triad. (A Pythagorean triad is 3 positive integers (whole numbers) that could form a right-angled triangle.) Assume *c* is the largest value.
d. Test if three numbers *x*, *y* and *z* are a Pythagorean triad. The variables *x*, *y* and *z* could be in any order.

9. **WE28** Implement an algorithm as a function in JavaScript for each of the problems given in question **8**.

Reasoning

10. **WE29** Design and implement an algorithm to add `data` to the end of a linked `list` and return the updated list. Use the following steps. Assume there is a terminator already defined.

```
var terminator = {};
```

Represent the `list3` of values `1`, `2` and `3` as a linked list. Call the function with the `list3` list and the value 4. Store the updated list in the variable `list4`.

a. Determine the inputs.
b. Determine the function name.
c. Break the problem into simple steps.

- **d.** Implement the algorithm as a function in JavaScript.
    - **i.** Write function inputs.
    - **ii.** Write JavaScript statements for each step.
    - **iii.** Return the required result (output).
- **e.** Test the function.

11. **WE30** Design and implement an algorithm to remove the last object in a linked `list` and return the updated list. Assume there is a terminator already defined.

```
var terminator = {};
```

Represent the `list3` of values `1`, `2` and `3` as the linked list. Create a function and call it with the `list3` list. Store the updated list in the variable `list2`.

- **a.** Determine the inputs.
- **b.** Determine the function name.
- **c.** Break the problem into simple steps.
- **d.** Implement the algorithm as a function in JavaScript.
    - **i.** Write the function inputs.
    - **ii.** Write a JavaScript statement for each step.
    - **iii.** Return the required result (output).
- **e.** Now test the function.

12. **WE31** Design and implement an algorithm to set the `data` at a particular `index` in a linked `list`. Represent the `list` of values `"Replace"`, `"me"` and `"here."` as the linked list. Create a function, call it with the list and set the third piece of data as `"there."`.

- **a.** Determine the inputs.
- **b.** Determine the function name.
- **c.** Break the problem into simple steps.
- **d.** Implement the algorithm as a function in JavaScript.
    - **i.** Write the function inputs.
    - **ii.** Write a JavaScript statement for each step.
- **e.** Now test the function.

Problem solving

13. Design and implement an algorithm to remove at an object at a particular `index` in a `list` and return an updated list.

Represent the `list3` of values `1`, `2` and `3` as the linked list. Create a function and call it with `list3` and index `1`. Store the updated list in the variable `list2`.

- **a.** Determine the inputs.
- **b.** Determine the function name.
- **c.** Break the problem into simple steps.
- **d.** Implement the algorithm as a function in JavaScript.
    - **i.** Write the function inputs.
    - **ii.** Write a JavaScript statement for each step.
    - **iii.** Return the required result (output).
- **e.** Now test the function.

14. Design and implement a function to return the maximum possible area with a given perimeter. (*Hint:* Of all the plane shapes, a circle has the maximum possible area for a given perimeter.)

21.5 Matrices

21.5.1 Two-dimensional arrays

- Section 21.3 introduced the concept of arrays. In a simple array, every item can be accessed with a single index. A simple array that requires only one index to access elements is called a **one-dimensional array**. An array of arrays is considered a **two-dimensional array**, as it requires two indexes to access a single element.

WORKED EXAMPLE 32

How many dimensions do the following arrays have?

a `[1,2,3]`
b `[[1,2],[3,4]]`
c `[[1],[2],[3]]`
d `[[1,2,3]]`
e `[[],[],[]]`

THINK	WRITE
a Each element in the array is a simple number.	**a** 1
b Each element in the array is another array.	**b** 2
c Each element in the array is still another array with of length 1.	**c** 2
d The only element in the array is another array.	**d** 2
e Each element in the array is a simple empty array.	**e** 2

21.5.2 Array indexes

- Values in a two-dimensional array are accessed with two indexes, both starting at `0`. The first index accesses the sub-array, and the second index accesses the value inside the sub-array.

`var array =`	`[[`	`1`	`,`	`2`	`,`	`3`	`],[`	`4`	`,`	`5`	`,`	`6`	`]];`
First index		0		0		0		1		1		1	
Second index		0		1		2		0		1		2	

WORKED EXAMPLE 33

Consider the following two-dimensional array.

`var array = [["a","b"],["c","d"],["e","f"]];`

i **What is the first JavaScript index required to access each of the following values in the array?**
ii **What is the second JavaScript index required to access each of the following values in the array?**

a `"a"`
b `"b"`
c `"f"`
d `"c"`

THINK	WRITE
a **i** `"a"` is in the 1st sub-array, so the first index is `0`.	**a** **i** `0`
ii `"a"` is the 1st in the sub-array, so the second index is `0`.	**ii** `0`

b **i** `"b"` is in the 1st sub-array, so the first index is `0`.	**b** **i** `0`
ii `"b"` is the 2nd in the sub-array, so the second index is `1`.	**ii** `1`
c **i** `"f"` is in the 3rd sub-array, so the first index is `2`.	**c** **i** `2`
ii `"f"` is the 2nd in the sub-array, so the second index is `1`.	**ii** `1`
d **i** `"c"` is in the 2nd sub-array, so the first index is `1`.	**d** **i** `1`
ii `"c"` is the 1st in the sub-array, so the second index is `0`.	**ii** `0`

- Values inside a two-dimensional `array` are accessed using two indexes `index1` and `index2` with the following expression `array[index1][index2]`.

WORKED EXAMPLE 34

Consider the two-dimensional array below.

`var array = [[3.4,2.1,3.9,8.3],[4.1,8.7,3.2,2.3]];`

Evaluate the following JavaScript expressions manually.

a `array[1][3]` **b** `array[0][1]` **c** `array[0][2]`

THINK	WRITE
a `array[1][3]` has a first index of `1` which is the 2nd sub-array. `array[1][3]` has a second index of `3` which is the 4th value in the 2nd sub-array.	**a** `2.3`
b `array[0][1]` has a first index of `0` which is the 1st sub-array. `array[0][1]` has a second index of `1` which is the 2nd value in the 1st sub-array.	**b** `2.1`
c `array[0][2]` has a first index of `0` which is the 1st sub-array. `array[0][2]` has a second index of `2` which is the 3rd value in the 1st sub-array.	**c** `3.9`

21.5.3 Matrices and arrays

- **Matrices** are a mathematical concept. A matrix is a rectangular array of numbers arranged in rows and columns.
- A JavaScript two-dimensional array can represent a matrix as long as all the sub-arrays have the same length (number of columns).

WORKED EXAMPLE 35

The arrays below can represent matrices. True or false?

a `[[1,2],[3,4],[5,6]]`

b `[[3],[3,3.1],[3,3.1,3.14]]`

c `[[[10,9],[8,7]],[[6,5],[4,3]]]`

THINK	**WRITE**
a `[[1,2],[3,4],[5,6]]` can represent a matrix because it is a two-dimensional array and all 3 sub-arrays have length 2.	**a** True
b `[[3],[3,3.1],[3,3.1,3.14]]` cannot represent a matrix because all 3 sub-arrays have different lengths.	**b** False
c `[[[10,9],[8,7]],[[6,5],[4,3]]]` cannot represent a matrix because it is a three-dimensional array.	**c** False

21.5.4 Matrix size

- The size of a matrix (also known as its order) is described with the number rows first then the number columns. The size of a matrix is described as $m \times n$, or 'm by n', where m is the number of rows and n is the number of columns.
- In JavaScript, there is a direct relationship between the size of a matrix and the array that represents it.
  – The number of rows in a matrix corresponds to the number of sub-arrays.
  – The number of columns in a matrix corresponds to the length of each sub-array.

WORKED EXAMPLE 36

Each of the arrays below represents a matrix. Answer the following for each array.

i Reformat the array so there is one sub-array per line.
ii How many rows does the corresponding matrix have?
iii How many columns does the corresponding matrix have?

a `[[1,2],[3,4],[5,6]]`
b `[[3,1,4]]`
c `[[3,1,9,8],[3,2,4,2]]`
d `[[4],[2],[2],[21]]`

THINK / **WRITE**

a **i** Reformatting the array helps to make the number of rows and columns clearer.

a i
```
[
    [1,2],
    [3,4],
    [5,6]
]
```

ii The array has 3 sub-arrays, so the matrix has 3 rows. — **ii** 3

iii Each sub-array has length 2, so the matrix has 2 columns. — **iii** 2

b **i** Reformatting the array helps to make the number of rows and columns clearer.

b i
```
[
    [3,1,4]
]
```

ii The array has 1 sub-array, so the matrix has 1 row. — **ii** 1

iii The sub-array has length 3, so the matrix has 3 columns. — **iii** 3

c **i** Reformatting the array helps to make the number of rows and columns clearer.

c i
```
[
    [3,1,9,8],
    [3,2,4,2]
]
```

ii The array has 2 sub-arrays, so the matrix has 2 rows. **ii** 2

iii Each sub-array has length 4, so the matrix has 4 columns. **iii** 4

d **i** Reformatting the array helps make the number of rows and columns clearer. **d** **i**

```
[
    [4],
    [2],
    [2],
    [21]
]
```

ii The array has 4 sub-arrays, so the matrix has 4 rows. **ii** 4

iii Each sub-array has length 1, so the matrix has 1 column. **iii** 1

21.5.5 Matrix representation

- A matrix can be converted into a two-dimensional array. For each row in the matrix, create a sub-array of the row. The final representation is an array of the sub-arrays in the same order they appear in the matrix.

Matrix	JavaScript two-dimensional array

$\begin{bmatrix} 11 & 12 & \dots & 19 \\ 21 & 22 & \dots & 29 \\ \vdots & \vdots & \ddots & \vdots \\ 71 & 72 & \dots & 79 \end{bmatrix}$	`[`
	`[11, 12, ..., 19],`
	`[21, 22, ..., 29],`
	`...`
	`[71, 72, ..., 79]`
	`]`

WORKED EXAMPLE 37

Represent each of the following matrices as a JavaScript two-dimensional array.

a $\begin{bmatrix} 22 & 41 & 15 \end{bmatrix}$

b $\begin{bmatrix} 2 & 4 & 5 & 62 \\ 1 & 3 & 1 & 31 \end{bmatrix}$

c $\begin{bmatrix} 2 \\ 9 \\ 7 \\ 1 \end{bmatrix}$

THINK	WRITE
a **1** Open the outer array.	**a** `[`
2 Convert the first/last row into a sub-array.	`[22, 41, 15]`
3 Close the outer array once all rows have been converted.	`]`
b **1** Open the outer array.	**b** `[`
2 Convert the first row into a sub-array.	`[2, 4, 5, 62],`
3 Convert the last row into a sub-array.	`[1, 3, 1, 31]`
4 Close the outer array once all rows have been converted.	`]`

c **1** Open the outer array.	**c** `[`
2 Convert the first row into a sub-array.	`[2],`
3 Convert the second row into a sub-array.	`[9],`
4 Convert the third row into a sub-array.	`[7],`
5 Convert the last row into a sub-array.	`[1]`
6 Close the outer array once all rows have been converted.	`]`

21.5.6 Matrix indexes

- Traditionally, elements in a matrix are referenced with two integer indexes starting at 1. This is different to the JavaScript method of referencing an array element using an index starting at `0`.

Matrix	**JavaScript two-dimensional array**
The matrix below has m rows and n columns. Elements of a matrix are accessed with two integer indexes starting at 1. The first index references the row and the second index references the column.	The two-dimensional array below has `m` rows and n columns. Elements of a two-dimensional array are accessed with two integer indexes starting at 0. The first index accesses the sub-array and the second index accesses value in the sub-array.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

```
[
  [a[0][0], a[0][1], ..., a[0][n-1]],
  [a[1][0], a[1][1], ..., a[1][n-1]],
  ...
  [a[m-1][0], a[m-1][1], ..., a[m-1][n-1]]
]
```

WORKED EXAMPLE 38

Consider the following matrix and two-dimensional JavaScript array.

$$A = \begin{bmatrix} 12 & 22 & 13 & 24 \\ 46 & 28 & 92 & 15 \end{bmatrix}$$

```
var A = [
  [12,22,13,24],
  [46,28,92,15]
];
```

Answer the following for each of the values below.

i **What is the first index required to access this value in the matrix *A*?**
ii **What is the second index required to access this value in the matrix *A*?**
iii **What is the first index required to access this value in the JavaScript array `A`?**
iv **What is the second index required to access this value in the JavaScript array `A`?**
v **Using the JavaScript indexes, write an expression to access this value in the JavaScript array `A`.**

a **13** **b** **15** **c** **12**

THINK	WRITE
a **i** 13 is in the 1st row of the matrix *A*.	**a** **i** 1
ii 13 is in the 3rd column of the matrix *A*.	**ii** 3
iii `13` is in the 1st sub-array, so the first index is `0`.	**iii** `0`
iv `13` is in the 3rd in the sub-array, so the second index is `2`.	**iv** `2`
v Use the JavaScript indexes `0` and `2`.	**v** `A[0][2]`
b **i** 15 is in the 2nd row of the matrix *A*.	**b** **i** 2
ii 15 is in the 4th column of the matrix *A*.	**ii** 4
iii `15` is in the 2nd sub-array, so the first index is `1`.	**iii** `1`
iv `15` is in the 4th in the sub-array, so the second index is `3`.	**iv** `3`
v Use the JavaScript indexes `1` and `3`.	**v** `A[1][3]`
c **i** 12 is in the 1st row of the matrix *A*.	**b** **i** 1
ii 12 is in the 1st column of the matrix *A*.	**ii** 1
iii `12` is in the 1st sub-array, so the first index is `0`.	**iii** `0`
iv `12` is in the 1st in the sub-array, so the second index is `0`.	**iv** `0`
v Use the JavaScript indexes `0` and `0`.	**v** `A[0][0]`

Exercise 21.5 Matrices

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE32** How many dimensions do the following arrays have?

a. `[[12],[5]]`
b. `[1]`
c. `[]`
d. `[[[2]]]`
e. `[[21,29,31,24,23]]`
f. `[[[]]]`
g. `[[[21,29],[31,24]]]`
h. `[[],[],[],[],[]]`
i. `[[101,22],[13,14],[36,44],[33,45]]`

2. **WE33** Consider the two-dimensional array below.

```
var matrix = [[1,2],[3,4],[5,6],[7,8]];
```

i. What is the first JavaScript index required to access each of the following values in the array?
ii. What is the second JavaScript index required to access each of the following values in the array?

a. `7` **b.** `1` **c.** `5` **d.** `6`

3. **WE34** Consider the two-dimensional array below.

```
var matrix = [[23,4,3],[4,6,7],[1,1,72]];
```

Evaluate the following JavaScript expressions manually.

a. `matrix[2][1]` **b.** `matrix[2][2]` **c.** `matrix[1][0]`

4. **WE35** The arrays below can represent matrices. True or false?

a. `[[1],[1,2],[1,2,3],[1,2,3,4]]`
b. `[[],[3]]`
c. `[[12,12],[223,14]]`
d. `[[[101,19],[84,47],[78,77]],[[26,5],[28,72],[28,79]]]`

Understanding

5. **WE36** Each of the arrays below represents a matrix. For each array:
   - **i.** how many rows does the corresponding matrix have?
   - **ii.** how many columns does the corresponding matrix have?
   - **a.** `[[1],[2],[3],[4],[5],[6]]`
   - **b.** `[[32.2,13.5,44.3,3.3,23.2,3.1]]`
   - **c.** `[[3,1,9],[31,12,14],[32,27,47],[3,24,44],[34,42,4]]`
   - **d.** `[[1,2,3,4,5],[6,7,8,9,10]]`

6. **WE37** Represent each of the following matrices as a JavaScript two-dimensional array.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 4 & 5 & 2 & 62 \\ 1 & 3 & 1 & 2 & 31 \end{bmatrix}$ **c.** $\begin{bmatrix} 21 & 11 \\ 19 & 12 \\ 71 & 82 \\ 12 & 81 \end{bmatrix}$

7. **WE38** Consider the following matrix and two-dimensional JavaScript array.

$$B = \begin{bmatrix} 6 & 8 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

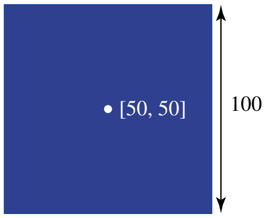
```
var B = [
   [6,8,2],
   [1,2,4]
];
```

Answer the following for each of the values below.
- **i.** What is the first index required to access this value in the matrix B?
- **ii.** What is the second index required to access this value in the matrix B?
- **iii.** What is the first index required to access this value in the JavaScript array `B`?
- **iv.** What is the second index required to access this value in the JavaScript array `B`?
- **v.** Using the JavaScript indexes, write an expression to access this value in the JavaScript array `B`.

a. 6 **b.** 4 **c.** 8

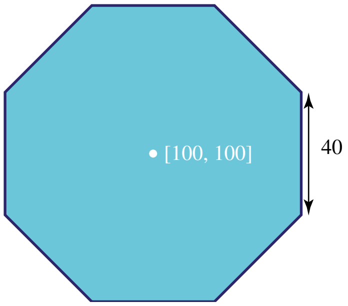
Reasoning

8. Create a JavaScript representation of a 4 by 4 matrix called `indexProduct` where the values are the product of the two corresponding JavaScript indexes.
9. Create a JavaScript representation of a 4 by 2 matrix called `square`. Each row of the matrix is the `[x,y]` coordinates of a corner of a unit square. The square is centred at the coordinates `[50,50]` and has side lengths of `100`.



Problem solving

10. Create a JavaScript representation of an 8 by 2 matrix called `octagon`. Each row of the matrix is the `[x,y]` corner coordinates of an octagon. The octagon is centred at coordinates `[100,100]` and has side lengths of `40`. Round all values to the closest integer.



21.6 Graphics

21.6.1 Web pages

- A simple way of drawing graphics on a computer screen is to create an HTML page with a **canvas**. The canvas is a defined area on your web page where the graphics can be drawn with JavaScript.

- The HTML page will be the same for all graphical examples in this section, but it can be modified for your own programs. The canvas provided has a set width of 600 and height of 500. These parameters can also be modified for your own programs. What will change for each example is the JavaScript used to draw on the canvas.
- Depending on what technology and internet access you have available, there are two options to build the HTML web page and the necessary JavaScript programs: you can build it either locally (stored on your own computer) or online.

Local web page

- The first option to build a web page is to create it locally on your computer.
- It is important to edit and create program files using a plain text editor. Do not use word processing programs such as Word, as they add extra information in the files for structure, formatting and so on. Many plain text editors are available, and some are designed for editing programs. Two simple plain text editors installed by default on most computers are Notepad on PCs and TextEdit (in plain text mode) on Macs.
- In order to use TextEdit in plain text mode on a Mac, you will need to change some preferences.
  1 Go to **TextEdit > Preferences > New Document** and set **Format** to **Plain text**. Also turn off all the Options.
  2 Go to **TextEdit > Preferences > Open and Save**. Under **When Opening a File**, check **Display HTML files as HTML code instead of formatted text.**

Format

Use the Format menu to choose settings for an open document.

○ Rich text ○ Wrap to page

● Plain text

Window Size

Width: 90 characters

Height: 30 lines

Font

Plain text font: Change... Courier 11

Rich text font: Change... Helvetica 12

Properties

Document properties are saved only with rich text files. Choose File > Show Properties to change the properties for an open document.

Author:

Organization:

Copyright:

Options

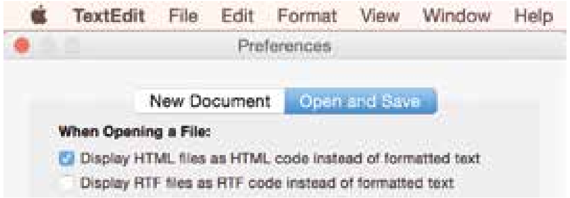
☐ Check spelling as you type ☐ Smart copy/paste

☐ Check grammar with spelling ☐ Smart quotes

☐ Correct spelling automatically ☐ Smart dashes

☐ Show ruler ☐ Smart links

☐ Data detectors ☐ Text replacement



When Opening a File:

☑ Display HTML files as HTML code instead of formatted text

☐ Display RTF files as RTF code instead of formatted text

- Once the preferences in TextEdit have been set correctly, you can create web pages locally on your computer using the following steps.
  1 On your computer, create a new directory for each web page.
  2 In the new directory, create a new standard HTML file called **index.html** with following content. (The content of this file is the same for all examples in this section.)

```
<!DOCTYPE html>
<html lang="en">
<head></head>
<body>
<canvas id="canvas" width="600" height="500"></canvas>
<script>
    var canvas = document.getElementById("canvas");
    var context = canvas.getContext("2d");
</script>
<script src="script.js"></script>
</body>
</html>
```

3 In the new directory, create a new JavaScript file called **script.js** with the following content. (The content of this file will change for different graphical examples.)

```
// Assume there exists an HTML page which
// provides a context to the canvas

var gradient=context.createLinearGradient(0,0,400,300);
gradient.addColorStop(0,"red");
gradient.addColorStop(1,"blue");
context.fillStyle=gradient;
context.fillRect(100,100,400,300);
```

4 Open the file **index.html** with any web browser on your computer and it should display a red and blue rectangle.

Online web page

- Your second option for building a web page is to do it online. There are a number of online JavaScript editors, such as JSFiddle and JS Bin, which allow you to edit HTML and JavaScript online without having to save files locally. The following instructions are for JS Bin, but similar steps will be required for other online JavaScript editors.

1 Open http://jsbin.com/?html,js,output in your web browser.

2 In the **HTML** section, add the following standard content (this will stay the same for all examples).

```
<!DOCTYPE html>
<html lang="en">
<head></head>
<body>
<canvas id="canvas" width="600" height="500"></canvas>
<script>
    var canvas = document.getElementById("canvas");
    var context = canvas.getContext("2d");
</script>
</body>
</html>
```

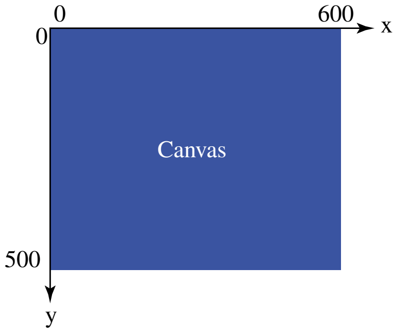
3 In the **JavaScript** section, add the following example content (this will change for different graphical examples).

```
// Assume there exists an HTML page which
// provides a context to the canvas

var gradient=context.createLinearGradient(0,0,400,300);
gradient.addColorStop(0,"red");
gradient.addColorStop(1,"blue");
context.fillStyle=gradient;
context.fillRect(100,100,400,300);
```

4 The output section should display a red and blue rectangle.

21.6.2 Canvas coordinates

- **Coordinates** `(x,y)` are used to reference every point on the canvas. The example canvas provided has a width of 600 and height of 500.
- The `x` value indicates how far right the point is. The left edge is at `x = 0` and the right edge is at `x = 600`.
- The `y` value indicates how far down the point is. The top edge is at `y = 0` and the bottom edge is at `y = 500`.



Lines

- The HTML page provides a `context` to the canvas. The `context` allows JavaScript to draw on the canvas.
- In order to draw a line, two coordinate pairs are required. The following code will draw a line from `(x1,y1)` to `(x2,y2)`.

```
// Assume there exists an HTML page which
// provides a context to the canvas

context.moveTo(x1,y1);
context.lineTo(x2,y2);
context.stroke();
```

- The `width` of line can be changed with `context.lineWidth = width;`.

WORKED EXAMPLE 39

Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a line:

a from `(5,15)` to `(500,500)`
b from `(300,10)` to `(50,400)` with width `15`
c from `(50,50)` to `(550,50)` to `(300,483)` back to `(50,50)`.

THINK	WRITE
a 1 Assume there exists a `context`.	**a**
2 Move to the first point, (`5,15`).	`context.moveTo(5,15);`
3 Create a line to the second point, (`500,500`).	`context.lineTo(500,500);`
4 Draw the line stroke.	`context.stroke();`
b 1 Assume there exists a `context`.	**b**
2 Set the line width.	`context.lineWidth = 15;`
3 Move to the first point, (`300,10`).	`context.moveTo(300,10);`
4 Create a line to the second point, (`50,400`).	`context.lineTo(50,400);`
5 Draw the line stroke.	`context.stroke();`
c 1 Assume there exists a `context`.	**c**
2 Move to the first point, (`50,50`).	`context.moveTo(50,50);`
3 Create a line to the second point, (`550,50`).	`context.lineTo(550,50);`
4 Create a line to the third point, (`300,483`).	`context.lineTo(300,483);`
5 Create a line to the fourth point, (`50,50`).	`context.lineTo(50,50);`
6 Draw the line stroke.	`context.stroke();`

21.6.3 Polygons

- In order to draw a polygon, a list of `m` coordinates (vertices) is required. The following code will draw a filled polygon from a list of coordinates `(x1,y1)`,`(x2,y2)`,`...`,`(xm,ym)`.

```
// Assume there exists an HTML page which
// provides a context to the canvas

context.beginPath();
context.moveTo(x1,y1);
context.lineTo(x2,y2);
...
context.lineTo(xm,xm);
context.closePath();
context.fill();
```

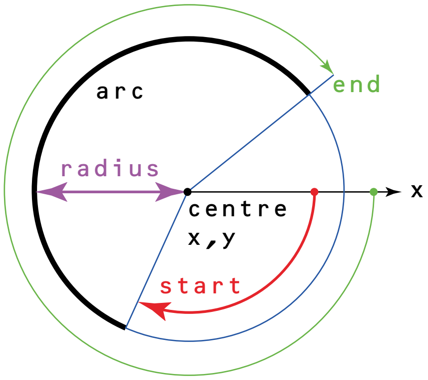
WORKED EXAMPLE 40

Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a filled rectangle with corners at `(100,50)` and `(500,250)`.

THINK	WRITE
1 Assume there exists a `context`.	
2 Store the 1st point, `(100,50)`.	`var x1 = 100;` `var y1 = 50;`
3 Infer and store the 2nd point, `(500,50)`, from the two given corners.	`var x2 = 500;` `var y2 = 50;`
4 Store the 3rd point, `(500,250)`.	`var x3 = 500;` `var y3 = 250;`
5 Infer and store the 4th point, `(500,50)`, from the two given corners.	`var x4 = 100;` `var y4 = 250;`
6 Restart the path.	`context.beginPath();`
7 Move to the 1st point, `(x1,y1)`.	`context.moveTo(x1,y1);`
8 Create a line to the 2nd point, `(x2,y2)`.	`context.lineTo(x2,y2);`
9 Create a line to the 3rd point, `(x3,y3)`.	`context.lineTo(x3,y3);`
10 Create a line to the 4th point, `(x4,y4)`.	`context.lineTo(x4,y4);`
11 Create a line to the beginning of the path.	`context.closePath();`
12 Fill in the closed path.	`context.fill();`

21.6.4 Circles

- To draw a circle we require the function
- `arc(x,y,radius,start,end,counterclockwise).`
  - – `x,y` define the centre.
  - – `radius` defines the radius of the `arc`.
  - – The two inputs `start` and `end` define the `start` and `end` points of the arc. They are measured in **radians** (another way to measure angles); 360° equals 2π radians.
  - – `counterclockwise` is an optional input and defines the direction the `arc` is drawn. The direction defaults to clockwise (`false`) if `counterclockwise` is omitted. The input `counterclockwise` can be set to `true` to reverse the direction to anticlockwise.
- To draw a circle, we require a full circuit, so the `start` is set to `0` and the `end` is set to 2π radians `(2*Math.PI)`. The following code will draw a filled circle with the centre `(x,y)` and a given `radius`.



```
// Assume there exists an HTML page which
// provides a context to the canvas

var end = 2*Math.PI;
context.beginPath();
context.arc(x, y, radius, 0, end);
context.closePath();
context.fill();
```

WORKED EXAMPLE 41

Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a filled circle:

a with centre `(100,100)` and radius `50`

b with centre `(50,200)` and diameter `80`.

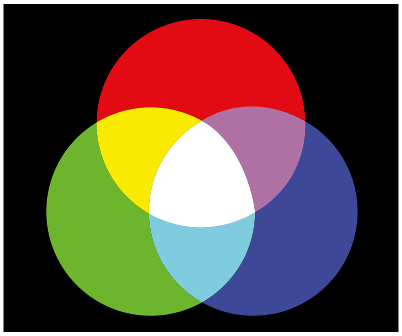
THINK	WRITE
a 1 Assume there exists a `context`.	**a**
2 Store the centre, `(100,100)`.	`var x = 100;` `var y = 100;`
3 Store the `radius` of `50`.	`var radius = 50;`
4 Store the `end` as a complete circuit.	`var end = 2*Math.PI;`
5 Restart the path.	`context.beginPath();`
6 Create a full arc at the `radius` around `x,y`.	`context.arc(x,y,radius,0,end);`
7 Close the path.	`context.closePath();`
8 Fill in the closed path.	`context.fill();`
b 1 Assume there exists a `context`.	**b**
2 Store the centre, `(50,200)`.	`var x = 50;` `var y = 200;`
3 Store the `diameter` of `80`.	`var diameter = 80;`
4 Calculate and store the `radius` of `50`.	`var radius = diameter/2;`
5 Store the `end` as a complete circuit.	`var end = 2*Math.PI;`
6 Restart the path.	`context.beginPath();`
7 Create a full arc at the `radius` around `x,y`.	`context.arc(x,y,radius,0,end);`
8 Close the path.	`context.closePath();`
9 Fill in the closed path.	`context.fill();`

21.6.5 Colour

- Graphical displays use light to display colour. This is additive colour. This is different from mixing paints, which is subtractive colour.
- The three primary colours used in computer graphics are red, green and **blue**. The following code generates a Venn diagram to show how red, green and **blue** add together.

```
context.fillStyle = "black";
context.fillRect(0,0,600,500);

// Forces the new shapes to add colours
context.globalCompositeOperation = "lighter";

context.fillStyle = "Red";
context.beginPath();
context.arc(300, 181, 160, 0, 2*Math.PI);
context.fill();

context.fillStyle = "Lime"; // Pure green
context.beginPath();
context.arc(220, 319, 160, 0, 2*Math.PI);
context.fill();

context.fillStyle = "Blue";
context.beginPath();
context.arc(380, 319, 160, 0, 2*Math.PI);
context.fill();
```

- There a number of predefined colours, but sometimes more control is required. Another method to define a colour is with the string `"rgb(red,green,blue)"` where `red`, `green` and `blue` define the intensity using integers between `0` and `255`.



WORKED EXAMPLE 42

Define a string of the form `"rgb(red,green,blue)"` for each of the following colours.

a **Green** **b** **Yellow** **c** **White** **d** **Dark red**

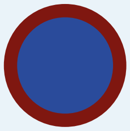
THINK	WRITE
a Set the green to maximum intensity, `255`, and set the other colours to `0`.	**a** `"rgb(0,255,0)"`
b It can be seen from the Venn diagram that in additive colour, yellow is a combination of red and green. Set the red and green to maximum intensity, `255`, and set blue to `0`.	**b** `"rgb(255,255,0)"`

c It can be seen from the Venn diagram that in additive colour, white is a combination of red, green and blue. Set all colours to maximum intensity, `255`.	**c** `"rgb(255,255,255)"`
d Reduce the red intensity to a value about midway between 0 and 255 and set the other colours to `0`.	**d** `"rgb(128,0,0)"`

- Colour can be added to lines with `context.strokeStyle = lineColour;`.
- Colour can be added to fill with `context.fillStyle = fillColour;`.

WORKED EXAMPLE 43

Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a blue circle with its centre at `(300,180)`, a radius of `70`, and an outline of width `15` and colour dark red.

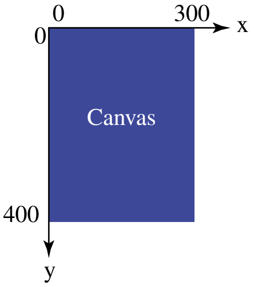


THINK	WRITE
1 Assume there exists a `context`.	
2 Store the centre, `(300,180)`.	`var x = 300;` `var y = 180;`
3 Store the `radius, 70`.	`var radius = 70;`
4 Store the `end` as a complete circuit.	`var end = 2*Math.PI;`
5 Store the blue fill colour.	`var fillColour = "rgb(0,0,255)";`
6 Store the dark red outline colour.	`var lineColour = "rgb(128,0,0)";`
7 Restart the path.	`context.beginPath();`
8 Set the fill colour of the circle.	`context.fillStyle = fillColour;`
9 Set the outline width.	`context.lineWidth = 15;`
10 Set the outline colour of the circle.	`context.strokeStyle = lineColour;`
11 Create a full arc at the `radius` around `x,y`.	`context.arc(x,y,radius,0,end);`
12 Close the path.	`context.closePath();`
13 Fill in the closed path.	`context.fill();`
14 Draw the outline.	`context.stroke();`

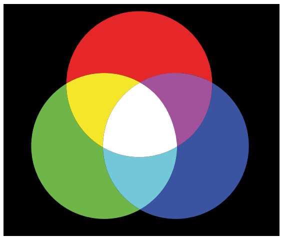
Exercise 21.6 Graphics

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency



1. A canvas has width 300 and height 400. On the canvas, what are the coordinates in the form `[x,y]` of:
   a. the centre?
   b. the middle of the right edge?
   c. the middle of the bottom edge?
   d. the right top corner?
2. **WE42** Define a string of the form `"rgb(red,green,blue)"` for each of the following colours.
   a. Blue
   b. Black
   c. Cyan (green and blue)
   d. Dark green
3. Match the colours below with the colour strings in the following table.
   Red, pink, yellow, blue, green, light green, black, light blue



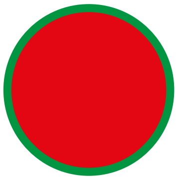
Colour string	Colour
`"rgb(128,255,128)"`	
`"rgb(0,0,0)"`	
`"rgb(255,255,0)"`	
`"rgb(255,128,128)"`	
`"rgb(255,0,0)"`	
`"rgb(0,255,0)"`	
`"rgb(128,128,255)"`	
`"rgb(0,0,255)"`	

4. Assume there exists an HTML page which provides a `context` to the canvas. Write a JavaScript statement to change the colour of a line to red. Use the string structure `"rgb(red,green,blue)"`.
5. Assume there exists an HTML page which provides a `context` to the canvas. Write a JavaScript statement to change the colour of a fill to yellow. Use the string structure `"rgb(red,green,blue)"`.

Understanding

6. **WE39** Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a line:
   a. from `(500,10)` to `(10,500)`
   b. from `(66,14)` to `(20,410)` with width `10`
   c. from `(86,3)` to `(50,150)` with width `3`
   d. from `(20,13)` to `(5,40)` with width `9`
   e. from `(50,50)` to `(550,50)` to `(550,450)` to `(50,450)` and back to `(50,50)`.
7. **WE40** Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a filled rectangle with corners at `(40,20)` and `(200,250)`.

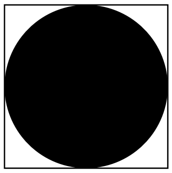
8. **WE41** Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a filled circle:
   a. with centre `(22,43)` and radius `20`.
   b. with centre `(100,100)` and diameter `160`.
9. **WE43** Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a red circle with centre `(200,280)`, diameter `200`, and an outline of width `8` and colour dark green.



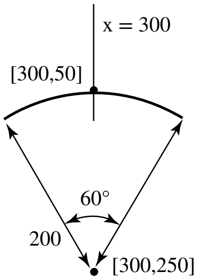
10. Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw lines to create a square with opposite corners at:
   - **a.** `(150,200)` and `(200,100)`
   - **b.** `(116,240)` and `(120,210)`
   - **c.** `(400,130)` and `(110,250)`.

Reasoning

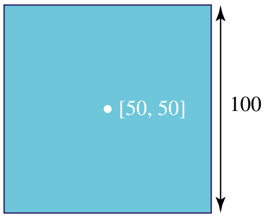
11. Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw two equal touching blue circles with centres `(200,300)` and `(400,300)`.
12. Assume there exists an HTML page which provides a `context` to the canvas. Write JavaScript code to draw a filled circle to fit inside a square with corners at `(200,150)` and `(400,300)`. Also draw the square with no fill.



13. Given that 180° equals π radians and equals `Math.PI` in JavaScript, write the JavaScript expression to represent the following angles in radians.
   - **a.** 90°
   - **b.** 60°
   - **c.** 30°
   - **d.** 45°
   - **e.** 161.2°
14. Assume there exists an HTML page which provides a `context` to the canvas. Draw a black arc with a radius of `200`. The arc has an angle of 60° and rotates about the point `[300,250]`. The arc is symmetrical about the line `x = 300`, and its highest point is `[300,50]`.

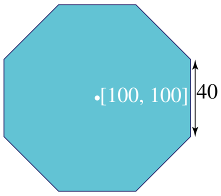


15. Assume there exists an HTML page which provides a `context` to the canvas. Draw a light blue square with a blue outline. The square is centred on the point `[50,50]` and has side lengths of `100`.

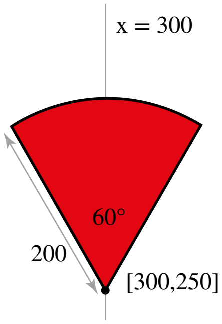


Problem solving

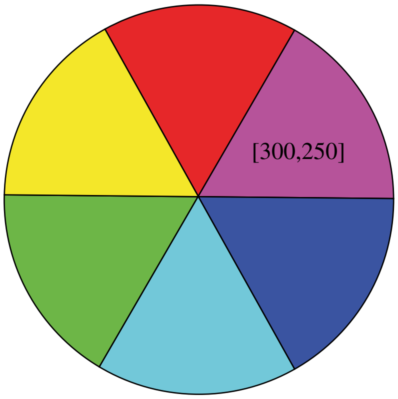
16. Assume there exists an HTML page which provides a `context` to the canvas. Draw a light blue octagon with a blue outline. The octagon is centred on the point `[100,100]` and has side lengths of `40`. Round all values to the closest integer.



17. Assume there exists an HTML page which provides a `context` to the canvas. Draw a red sector with a black outline. The sector has a radius of `200`. The sector point has an angle of 60° and coordinates `[300,250]`. The sector is symmetrical about the line `x = 300` and is pointing down.



18. Assume there exists an HTML page which provides a `context` to the canvas. Using JavaScript, draw a colour wheel as shown, centred at the point `[300,250]` with radius `200`. The fill colours have the extreme intensities of `0` and `255`. The outlines are black.



21.7 Simulations

- The more complex a problem is, the harder it is to analyse manually. Simulating a complex problem in computer programs can help our understanding of the problem, as we can change the variables and view the results.

21.7.1 Random numbers

- Some simulations require random events. JavaScript has a function named `Math.random` to generate numbers. Every time `Math.random()` is called, a different random number between `0` (inclusive) and `1` (exclusive, but can come very close) is returned. The following statement stores a random number in the variable `sample`.

```
var sample = Math.random();
```

- `Math.random()` returns a real number between `0` (inclusive) and `1` (exclusive). Often a larger or smaller range of random numbers is required. The value returned from `Math.random()` can be multiplied by a scaling factor to adjust the range. The following statement stores a random number between `0` (inclusive) and `scale` (exclusive) in the variable `sample`.

```
var sample = scale*Math.random();
```

WORKED EXAMPLE 44

Write an expression to return a random number between 0 and:

a **20** **b** **0.01** **c** **−3**

THINK	**WRITE**
a The `scale` factor is `20`.	**a** `20*Math.random()`
b The `scale` factor is `0.01`.	**b** `0.01*Math.random()`
c The `scale` factor is `-3`.	**c** `-3*Math.random()`

21.7.2 For loops

- Simulations often require that the same task be applied to every value in an array. **Loops** can execute a task many times on different data without needing multiple copies of the same code.
- The JavaScript keyword `for` provides a mechanism to loop over the same code many times. There are four sections where the `for` loop can be customised.
  - `initialise` provides a place to initialise a variable. This is usually involves setting an index variable to `0`.
  - If `repeat` is `true`, the code in the loop is repeated until `repeat` changes to `false`. The `repeat` is usually a Boolean expression to indicate if there is any more data to process.
  - `iterate` is executed each time, directly after the code block is executed. It is extremely important that `iterate` eventually changes the `repeat` expression to `false`. This step usually involves increasing the index variable.
  - `{statement 1, statement 2, ..., statement S}` is the section of code repeated every time `repeat` is `true`.

```
for (initialise; repeat; iterate) {
   statement 1
   statement 2

   statement S
}
```

Loop over an array

- The following structure is one way to repeat a section of code `{statement 1, statement 2, ..., statement S}` for each `item` in an `array`. The variable `i` is the array index and is also used to track the current loop iteration.

```
var array = [...];
var m = array.length;
for (var i = 0; i < m; i = i+1) {
   var item = array[i];
   statement 1
   statement 2

   statement S
}
```

WORKED EXAMPLE 45

How many times is the loop code executed in each of the following programs?

a

```
var employees = ["Ben","Tom","Tim"];
var payRates = [54.50,43.00,90.00];
var m = employees.length;
for (var i=0; i < m; i = i+1) {
   var item = employees[i];
   var payRate = payRates[i];
   console.log(item+ "gets payed "+payRate+" per hour");
}
```

b

```
var sum = 0;
var groups = [[2,3,5,7],[11,13,17,19]];
var m = groups.length;
for (var j=0; j < m; j = j+1) {
   var group = primes[j];
   sum = sum + group.length;
}
console.log(sum);
```

THINK	WRITE
a **1** The code follows the structure to loop over the array `employees=["Ben","Tom","Tim"]`.	**a**
2 The length of the `employees` array is 3.	3

b **1** The code follows the structure to loop over the array `groups=[[2,3,5,7],[11,13,17,19]]`. **b**

2 The array `groups` has `2` sub-arrays, so it has a length of 2. 2

21.7.3 Nested loops

- A single loop over a multi-dimensional array only gives you access to the sub-arrays. In order to access all the items in a multi-dimensional array, a **nested loop** is required.
- The following structure is one way to repeat a section of code `{statement 1, statement 2, ..., statement S}` for each `item` in a two-dimensional `array`. The variable `i` is the first array index and is also used to track the outer loop iteration. The variable `j` is the second array index and is also used to track the inner loop iteration.

```
var array = [[...],...,[...]];
var m = array.length;
for (var i = 0; i < m; i = i+1) {
   var n = array[i].length;
   for (var j = 0; j < n; j = j+1) {
      var item = array[i][j];
      statement 1
      statement 2

      statement S
   }
}
```

WORKED EXAMPLE 46

Given the following two-dimensional array assignment, write a nested loop to total all the values in the two-dimensional array. Store the total in the variable `sum`.

```
var array = [
   [1,2,3,4],
   [5,6,7,8]
];
```

THINK / **WRITE**

1 Write the initial array assignment.

```
var array = [
   [1,2,3,4],
   [5,6,7,8]
];
```

2 Initialise the `sum` with `0`.

```
var sum = 0;
```

3 Store the length of the array.

```
var m = array.length;
```

4 Start the outer `for` loop.

```
for (var i=0; im; i=i+1) {
```

5 Store the length of the sub-array.

```
   var n = array[i].length;
```

6 Start the inner `for` loop.

```
   for (var j=0; j<n; j=j+1) {
```

7 Store the current array `item`.

```
    var item = array[i][j];
```

8 Increase `sum` by the array `item`.

```
    sum = sum + item;
```

9 Close the inner `for` loop.

```
  }
```

10 Close the output `for` loop.

```
}
```

21.7.4 Initialising arrays

- It is not always practical to initialise arrays to the correct size and values straight away. Sometimes the values are generated by a program, or the initial size is unknown. The following statement stores an empty array in the variable `empty`.

```
var empty = [];
```

- Values can be added to the array using the push function. The following statement pushes the `item` to the end of existing `array`.

```
array.push(item);
```

WORKED EXAMPLE 47

Create a function called `randomNumbers` to return an array of `size` random numbers between `0` and `1`.

THINK / **WRITE**

1 Write the function inputs.

```
function randomNumbers(size) {
```

2 Create an empty array

```
  var array = [];
```

3 Start the `for` loop.

```
  for (var i=0; isize; i=i+1) {
```

4 Find a new random value.

```
    var random = Math.random();
```

5 Add the random number to the end of the array.

```
    array.push(random);
  }
```

6 Return the generated array.

```
  return array;
 }
```

WORKED EXAMPLE 48

Create a function called `randomBooleans` to return a matrix of Booleans with a given number of `rows` and `columns` where each value has a `probability` of being `true`.

THINK / **WRITE**

1 Write the function inputs.

```
function randomBooleans(
               rows,
               columns,
               probability) {
```

2 Create an empty matrix.

```
  var matrix = [];
```

3 Start the outer `for` loop.

```
  for (var i=0; irows; i=i+1) {
```

4 Create a new empty row.

```
    var newRow = [];
```

5 Start the inner `for` loop.

```
    for (var j=0; j < columns; j=j+1) {
```

6 Find a new random value.

```
      var random = Math.random();
```

7 Generate a random Boolean.

```
      var isTrue = randomprobability;
```

8 Add the random Boolean to the end of the array.

```
      newRow.push(isTrue);
    }
```

9 Store the new row in the new matrix.

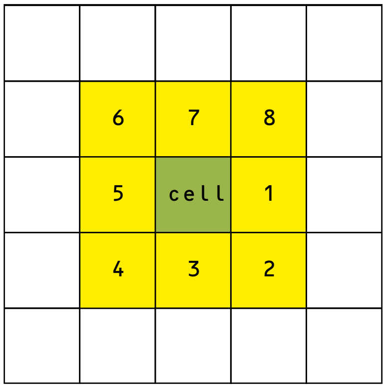
```
    matrix.push(newRow);
  }
```

10 Return the generated matrix.

```
  return matrix;
}
```

21.7.5 The Game of Life

- The Game of Life was invented by the mathematician John Conway. It has some simple rules but is very difficult to analyse without using a simulation. The next few subsections will work towards building a simulation of the Game of Life.
- The Game of Life consists of a matrix of cells. Each cell in the matrix has 8 adjacent cells. A cell can be either alive or empty.



	6	7	8	
	5	`cell`	1	
	4	3	2	

- If a given cell has a live cell adjacent to it, that cell is referred to as a neighbour of the first cell. Each cell lives or dies depending on how many neighbours it has. If a cell is alive and:
  – has only one or no neighbours, it will die of loneliness
  – has two or three neighbours, it will continue to live
  – has four or more neighbours, it will die from overcrowding.
- If a cell is empty and has three neighbours, the neighbours will reproduce and the cell will become alive.

WORKED EXAMPLE 49

Design and implement a function to return `true` if a cell lives according to John Conway's rules for the Game of Life. Return `false` if the cell stays empty or dies. The function `isAlive` has the Boolean input `alive` and number of `neighbours`.

a Break the problem into simple steps.

b Implement the algorithm as a function in JavaScript.

- **i Write the function inputs.**
- **ii Write a JavaScript statement for each step.**
- **iii Return the required result (output).**

THINK	WRITE
a **1** Write step 1.	**a** Initialise a variable to track the status of the new cell. A status of true indicates alive, a status of false indicates dead or empty.
2 Write step 2.	If the current cell is alive and has 2 or 3 neighbours set the new status to alive.
3 Write step 3.	If the current cell is alive and does not have 2 or 3 neighbours set the new status to dead.
4 Write step 4.	If the current cell is empty and has 3 neighbours set the new status to alive.
5 Write step 5.	If the current cell is empty and does not have 3 neighbours set the new status to empty.
6 Return the required result	Return the status.

b i Implement the algorithm. Start by writing the function inputs.

b i
```
functionisAlive(alive,neighbours) {
```

ii 1 Comment on step 1.

ii
```
// Initialise a variable to
// track the status of the new cell
// A status of true indicates alive,
// a status of false indicates
// dead or empty.
```

2 Implement step 1.

```
var newStatus = false;
```

3 Comment and implement step 2 and 3.

```
if (alive) {
 // If the current cell is alive
 // and has 2 or 3 neighbours set
 // the new status to alive.
 // If the current cell is alive
 // and does not have 2 or 3
 // neighbours set the new status
 // to dead.
 newStatus = neighbours===2 ||
             neighbours===3;
}
```

6 Comment and implement step 4 and 5.

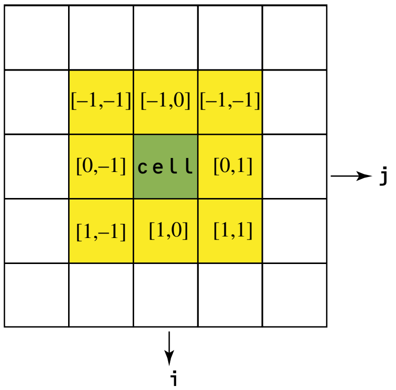
```
else {
  // If the current cell is empty
  // and has 3 neighbours set the
  // new status to alive.
  // If the current cell is empty
  // and does not have 3 neighbours
  // set the new status to empty.
  newStatus = neighbours === 3;
}
```

7 Return the required result.

```
  return newStatus;
}
```

21.7.6 Adjacent cells

- Computer programs often perform image manipulations such as blurring, sharpening and resizing. These operations involve treating an image as a large matrix and performing calculations based on adjacent cells. The locations of these adjacent cells can be considered as indexes `[i,j]` relative to the current cell. For example, the adjacent right cell is `0` units down and `1` unit right `[0,1]`.



WORKED EXAMPLE 50

Given the following list of adjacent cells (`adjacents`), create a function to `count` the number of adjacent `true` values at the cell location `[i,j]` of the input matrix.

```
var adjacents = [
  [0,1],[1,1],[1,0],[1,-1],[0,-1],[-1,-1],[-1,0],[-1,1]
];
```

THINK

1 Rewrite the list of adjacents.

WRITE

```
var adjacents = [
  [0,1],[1,1],[1,0],[1,-1],
  [0,-1],[-1,-1],[-1,0],[-1,1]
];
```

2 Write the function inputs.

```
function count(i,j,matrix) {
```

3 Initialise the `total` to `0`.

```
  var total = 0;
```

4 Store the number of adjacents.

```
  var tests = adjacents.length;
```

5 Start the `for` loop.

```
  for (var k=0; ktests; k=k+1) {
```

6 Offset the `i` index up or down.

```
      var ai = i + adjacents[k][0];
```

7 Offset the `j` index left or right.

```
      var aj = j + adjacents[k][1];
```

8 Check the adjacent row exists.

```
      if (matrix[ai]) {
```

9 Check the adjacent cell.

```
        if (matrix[ai][aj]) {
```

10 Increment the `total` if true.

```
          total = total+1;
```

11 Close the `if` statements.

```
      }
    }
```

12 Close the `for` loop.

```
  }
```

13 Return the generated `count` as the number of neighbours.

```
  return total;
}
```

21.7.7 Matrix transforms

- Some simulations require large amounts of information to be stored as a matrix. As part of the simulation, these matrices are transformed to produce new matrices. The following function `transforms` a `matrix` into a new `transformed` matrix.
  ```
  var transformed = transforms(matrix);
  ```

- For example, weather prediction involves extremely complex simulations. The following is a major oversimplification of predicting the weather. Fundamentally, weather simulations divide the surface of the Earth into a grid and convert relevant information into a matrix. A sequence of transformations is applied to the matrix to simulate weather propagation. For weather prediction to be useful, the simulated propagation has to be faster than real weather propagation. The following is an example of a transform function in a weather simulation.
  ```
  var weatherAtTime1 = timeTransform(weatherAtTime0);
  ```
- The following function structure can be used to iterate through all the elements in a matrix and generate a new transformed matrix of the same size.

```
function transforms(matrix) {
  var newMatrix = [];
  var m = matrix.length;
  for (var i = 0; i < m; i = i+1) {
    var newColumn = [];
    var n = matrix[i].length;
    for (var j = 0; j < n; j = j+1) {
      var item = matrix[i][j];
      // Processing to create a newItem
      newColumn.push(newItem);
    }
    newMatrix.push(newColumn);
  }
  return newMatrix;
}
```

WORKED EXAMPLE 51

Given the following adjacent cells and function definitions, create a function called `transforms` to accept a `matrix`. The `matrix` represents the current state of the cells in an instance of John Conway's Game of Life. Each element in the `matrix` is a Boolean cell. The cell is `true` if alive and `false` if empty.

```
function isAlive(alive, neighbours) {
var newStatus = false;
  if (alive) {
    newStatus = neighbours === 2||neighbours === 3;
  } else {
   newStatus = neighbours === 3;
  }
  return newStatus;
}
var adjacents = [
  [0,1],[1,1],[1,0],[1,-1],
  [0,-1],[-1,-1],[-1,0],[-1,1]
];
function count(i, j, matrix) {
  var total = 0;
  var tests = adjacents.length;
  for (var k=0; ktests; k = k+1) {
    var ai = i + adjacents[k][0];
    var aj = j + adjacents[k][1];
    if (matrix[ai]) {
      if (matrix[ai][aj]) {
        total = total+1;
    }
   }
 }
 return total;
}
```

THINK / **WRITE**

1 Write the function and input.

```
function transforms(matrix) {
```

2 Create an new empty matrix.

```
  var newMatrix = [];
```

3 Store the length of the matrix.

```
  var m = matrix.length;
```

4 Start the outer `for` loop.

```
  for (var i=0; im; i=i+1) {
```

5 Create a new empty row.

```
    var newRow = [];
```

6 Store the length of the row.

```
    var n = matrix[i].length;
```

7 Start the inner `for` loop.

```
    for (var j=0; jn; j=j+1) {
```

8 Store the current matrix `item`.

```
      var item = matrix[i][j];
```

9 Count the number of neighbours around the cell.

```
            var neighbours = count(i,j,matrix);
```

10 Determine if the cell is still alive in the new iteration.

```
            varnewItem =isAlive(item,
            neighbours);
```

11 Store the item in the new row.

```
            newRow.push(newItem);
```

12 Close the inner `for` loop.

```
        }
```

13 Store the new row in the new matrix.

```
        newMatrix.push(newRow);
```

14 Close the outer `for` loop.

```
    }
```

15 Return the new transformed matrix.

```
    return newMatrix;
}
```

21.7.8 Simulation animations

- Animating the results of a simulation can help visualise the process. The following program structure can be expanded and used to animate a simulation. The animation loop updates and redraws the simulation state a number of times a second, depending on the computer's speed. The details of the code are explained in the comments.

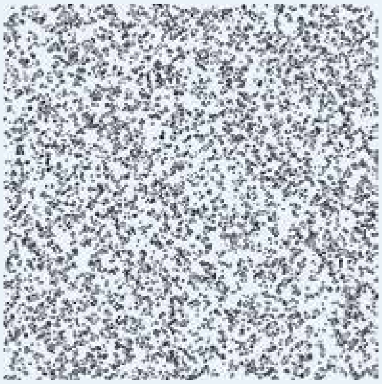
```
// Assume there exists an HTML page which
// provides a context to the canvas

// Create the first simulation state
var matrix = initialiseMatrix();

function draw() {
      // Clear the canvas
      context.clearRect(0,0,width,height);

      // Draw the current state
      // add your own code here

      // Update the state
      matrix = transforms(matrix);

      // Wait for the next animation time
      window.requestAnimationFrame(draw);
}

// Start the animation
window.requestAnimationFrame(draw);
```

WORKED EXAMPLE 52

Below are some functions created in previous worked examples. These will be used for the final simulation. Use them to create a loop to simulate John Conway's Game of Life. Create a `500` by `500` matrix of cells. Initially each cell has a `0.5` probability of being alive. Display the cells' progression as an animation.

```
function randomBooleans(rows, columns, probability) {
    var matrix = [];
    for (var i = 0; i < rows; i = i+1) {
      var newRow = [];
      for (var j = 0; j < columns; j = j+1) {
        var random = Math.random();
        var isTrue = random <= probability;
        newRow.push(isTrue);
     }
     matrix.push(newRow);
    }
    return matrix;
 }
 function isAlive(alive, neighbours) {
    var newStatus = false;
    if (alive) {
      newStatus = neighbours === 2||neighbours === 3;
    } else {
     newStatus = neighbours === 3;
    }
    return newStatus;
}
var adjacents = [
    [0,1],[1,1],[1,0],[1,-1],
    [0,-1],[-1,-1],[-1,0],[-1,1]
];
function count(i, j, matrix) {
   var total = 0;
   var tests = adjacents.length;
   for (var k=0; k<tests; k = k+1) {
     var ai = i + adjacents[k][0];
     var aj = j + adjacents[k][1];
     if (matrix[ai]) {
       if (matrix[ai][aj]) {
         total = total+1;
       }
     }
   }
   return total;
}
function transforms(matrix) {
   var newMatrix = [];
   var m = matrix.length;
```

```
    for (var i = 0; i < m; i = i+1) {
      var newRow = [];
      var n = matrix[i].length;
      for (var j = 0; j < n; j = j+1) {
          var item = matrix[i][j];
          var neighbours = count(i, j, matrix);
          var newItem = isAlive(item, neighbours);
          newRow.push(newItem);
     }
     newMatrix.push(newRow);
  }
  return newMatrix;
}
```



THINK | **WRITE**

1 Store the number of cell `rows` and `columns`.

```
var rows = 500;
var columns = 500;
```

2 Generate a `rows`-by-`columns` matrix of random Booleans. Each cell has a `0.5` probability of being alive.

```
var matrix = randomBooleans(
               rows,
               columns,
               0.5);
```

3 Define a `draw` function.

```
function draw() {
```

4 Clear the canvas each loop.

```
context.clearRect(0,0,600,500);
```

5 This part of the code is used to draw all the cells each time through the animation loop. The nested `for` loop iterates through the rows and columns. If the cell is alive, a 1 by 1 black rectangle is drawn at the cell location.

```
for (var i=0; i<rows; i=i+1) {
 for (var j=0; j<columns; j=j+1) {
   if (matrix[i][j]) {
     context.fillRect(j,i,1,1);
   }
   }
}
```

6 The matrix is updated according to John Conway's Game of Life rules. This is done by the predefined `transforms` function.

```
  matrix = transforms(matrix);
```

7 The animation loop is now complete. Request the next animation frame.

```
  window.requestAnimationFrame(draw);
}
```

8 This call starts the animation loop.

```
window.requestAnimationFrame(draw);
```

Exercise 21.7 Simulations

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. **WE44** Write an expression to return a random number between 0 and:

a. 1
b. 0.11
c. –0.22
d. 3
e. 2
f. –3
g. 5
h. 2.4
i. 1000

2. Write an expression to return a random number between 20 and 30.

3. **WE45** How many times is the following loop code executed?

```
var matrix = [[2,3],[5,7],[6,4],[1,8]];
var m = matrix.length;
for (var i=0; i < m; i = i+1) {
  var row = matrix[i];
  console.log(row[0]+row[1]);
}
```

4. How many times is the inner loop code executed?

```
var matrix = [[2,3,3],[5,7,2],[6,4,1],[1,8,0]];
var m = matrix.length;
for (var i=0; i < m; i = i+1) {
  var row = matrix[i];
  var n = row.length;
  for (var j=0; j < n; j = j+1) {
    console.log(row[j]);
   }
}
```

5. Consider the following program.

```
var sum = 0;
var speed = 12;
var times = [2,2,3,2,4,5];
var m = times.length;
for (var i=0; i = m; i = i+1) {
  var time = times[i];
  var distance = time*speed;
  sum = sum + distance;
}
console.log(sum);
```

a. How many times is the loop code executed?

b. What is the output to the console?

Understanding

6. **WE46** Given the following matrix assignment, write a nested loop to total all the values in the matrix. Store the average in the variable `mean`.

```
var matrix = [
  [1,2,3,4,5,6,7],
  [8,9,10,11,12,13,14],
  [15,16,17,18,19,20,21]
];
```

7. **WE47** Create a function called `randomBooleans` to return an array of size `required` random Boolean where `true` and `false` have an equal random chance.
8. Create a function called `randomGenders` to return an array of size `people`. Make the elements of the array random strings, either `"Male"` or `"Female"`, where `"Male"` has a 51% random chance.
9. **WE48** Create a function called `randomScaled` to return a matrix of random numbers between `0` and `scale` with a given number of `rows` and `columns`.
10. **a.** Create a function called `randomColours` to return a `height`-by-`width` matrix of colours where each colour string has a random probability according to the table below.

Colour	Probability
`"rgb(255,0,0)"`	`0.05`
`"rgb(0,255,0)"`	`0.10`
`"rgb(0,0,255)"`	`0.15`
`"rgb(0,0,0)"`	`0.70`

b. Assume there exists an HTML page which provides a `context` to the canvas. Write a JavaScript function to `draw` an `image` matrix where each element is a pixel with a colour string. (*Hint:* Use the function `context.fillRect(x,y,1,1)` to draw a filled 1-by-1 square with a corner at `x,y`.)

c. Call the two functions with the following code and describe the image generated on the canvas.

```
var image = randomColours(500,600);
draw(image);
```

Reasoning

11. **WE49** Design and implement a function to return `true` if a cell lives according to Nathan Thompson's rules for Highlife. Highlife has very similar rules to John Conway's Game of Life except for the rule for reproduction.

In Highlife, if a cell is alive and:

- has only one or no neighbours, it will die of loneliness
- has two or three neighbours, it will continue to live
- had four or more neighbours, it will die from overcrowding.

In Highlife, if a cell is empty and has three **or six** neighbours, the neighbours will reproduce and the cell will become alive.

Return `false` if the cell stays empty or dies. The function `isAlive` has the Boolean input `alive` and number of `neighbours`.

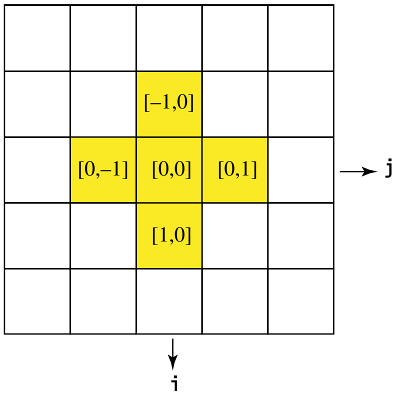
a. Break the problem into simple steps.

b. Implement the algorithm as a function in JavaScript.

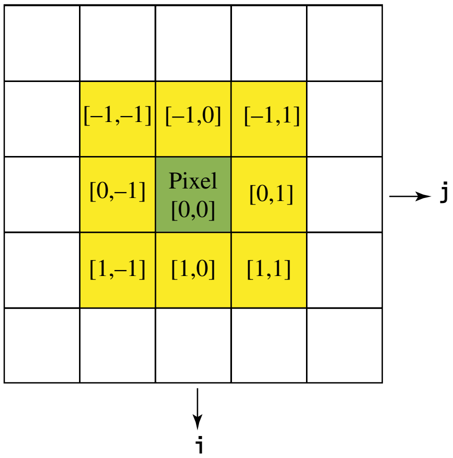
- **i.** Write the function inputs.
- **ii.** Write a JavaScript statement for each step.
- **iii.** Return the required result (output).

12. **WE50** Given the list of `closeCells`, create a function to `average` the value of all the close cells relative to the cell location `[i,j]` of the input `image` matrix.

```
var closeCells = [[0,0],[0,1],[1,0],[0,-1],[-1,0]];
```



13. **WE51** Given the following program definitions, create a function called `transforms` to accept a `matrix`. The `matrix` represents a greyscale image. Each element in the `matrix` represents a pixel's shade. The function `transforms` and generates a new matrix that is a blurred version of the original. The new matrix is generated by taking the average of a 3 by 3 grid centred around the pixel location.



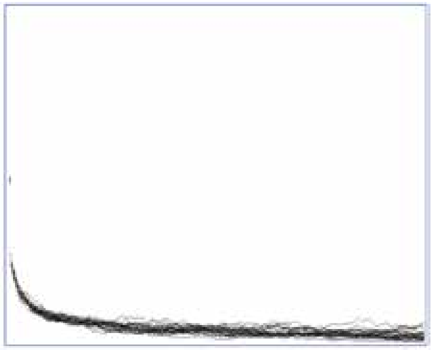
```
var grid = [
  [0,0],[0,1],[1,1],[1,0],[1,-1],[0,-1],[-1,-1],[-1,0],
[-1,1]
];
function smooth(i, j, matrix) {
  var count = 0;
  var total = 0;
  var tests = grid.length;
  for (var k=0; k<tests; k = k+1) {
    var ai = i + grid[k][0];
    var aj = j + grid[k][1];
    if (matrix[ai]) {
      if (typeof matrix[ai][aj] === "number") {
        count = count+1;
        total = total + matrix[ai][aj];
      }
    }
  }
```

```
    if (count === 0) {
        return 0;
    }
    var mean = total / count;
    return mean;
}
```

Problem solving

14. **WE52** Create a simulation loop to simulate Nathan Thompson's Highlife, using the rules given in question **11**. Create a `100` by `100` matrix of cells. Initially each cell has a `0.4` probability of being alive. Display the cells' progression as an animation. Each cell should be displayed as a `5` by `5` rectangle.

15. Create a simulation of John Conway's Game of Life. Use a `100` by `100` matrix of cells. Initially each cell has a `0.5` probability of being alive. The program should also draw a scatterplot showing the proportion of alive cells against the simulation iteration number. Run the simulation for 600 iterations and restart each time. The program should automatically restart the simulation 20 times and draw the 20 graphs on top of each other, with a result similar to the image on the right.



21.8 Review

21.8.1 Review questions

Fluency

1. Answer the questions below for each of the following values.
   - **i.** Is the value a number, string or Boolean?
   - **ii.** Apply the expression `typeof value` to the value. What is the result of the expression `typeof value`?
   - **a.** `false`
   - **b.** `"-99823.232"`
   - **c.** `-99823.232`
   - **d.** `true`
   - **e.** `0`
   - **f.** `"Some information as a string"`
2. Each of the following is a valid variable name. True or false?
   - **a.** `camelCase`
   - **b.** `snake-case`
   - **c.** `-negative`
   - **d.** `result&&other`
   - **e.** `to^power`
   - **f.** `some space`
   - **g.** `item_12_21`
   - **h.** `function`
   - **i.** `dimension_2`
   - **j.** `_2_`
3. Manually simulate running the following programs. For each program:
   - **i.** what is the final type of the variable `change`?
   - **ii.** what is the final result stored in the variable `change`?
   - **iii.** what is the result of the expression `typeof change`?
   - **a.**

```
var change = 3;
if (change < 5) {
  change = change*change;
}
```

b.
```
var change = 3;
var tooLarge = change >= 3;
if (tooLarge) {
  change = change - 3;
} else {
  change = change * change * change;
}
```

c.
```
var change = true||false;
change = true && change;
change = false && change;
```

d.
```
var change = "Start with this.";
change = "End with this.";
```

e.
```
var change = false;
var less = 1213;
change = less;
change = "A string";
less = 22;
less = true;
```

f.
```
var change = 123*2;
change = 12*22;
change = true;
```

g.
```
var change = true;
if (change) {
  change = false;
}
```

4. Can the following values be stored? If so, will they be stored accurately or approximately?

a. 1.6×10^{-200}

b. `9007199254740992`

c. $1.797\,693\,1 \times 10^{308}$

d. `−9007199254740995`

e. -1.79×10^{308}

f. `-9007199254740992`

g. `-9007199254740991`

h. `-9007199254740994`

i. `2412`

j. `14.00`

k. `-123.1`

l. `26`

m. `61.00`

n. `-866.99`

o. 1.2×10^{-800}

p. 3×10^{308}

q. `9007199254740991`

r. $1.797\,693\,2 \times 10^{308}$

5. Evaluate the following JavaScript expressions manually.

a. `!(true&&false)`

b. `false&&true`

c. `false||true`

d. `537===538`

e. `3.41!==3.41`

f. `["some","string","here",false,true,10,["Array"]].length`

g. `false!==true`

h. `"923"==="923"`

i. `false&&false`

j. `14401.9249<=1242.1`

k. `!!!(true||false)`

l. `"2"==="42"`

m. `[111,12,2932,32].length`

n. `[1,2,3,2,4,3,2].length`

o. `61363.1>=163242.17`

p. `34122.12315<1233224.25`

q. `[].length`

r. `33.33<74.275`

s. `21.122>=122.17`

t. `Math.sqrt(100)`

u. `true&&false` **v.** `truetrue`

w. `falsefalse` **x.** `!true`

y. `true===false` **z.** `32.4221!==32.4212`

6. What is the JavaScript index of each of the following characters in the string `"The review questions."`?

a. `"."` **b.** `"q"` **c.** `"v"`

d. `"w"` **e.** `"t"` **f.** `"T"`

7. Given the string assignment below, evaluate the following JavaScript expressions manually.

```
var heading = "The review questions.";
```

a. `heading[2]` **b.** `heading[5]` **c.** `heading[4+3]`

8. Each of the following is an array. True or false?

a. `-21.13`

b. `["Q","U","E","S","T","I","O","N"].length`

c. `["QUESTION"]` **d.** `true` **e.** `"DATA"`

f. `[true,false,"W","WEWE",120,[],[1,2]]`

g. `[0.13,16]` **h.** `[]`

9. What are the JavaScript indexes of the following values in the array shown below?

```
["Monday","Tuesday","Wednesday","Thursday","Friday",
"Saturday","Sunday"]
```

a. `"Wednesday"` **b.** `"Monday"` **c.** `"Sunday"`

d. `"Saturday"` **e.** `"Tuesday"`

10. Given the object assignment below, evaluate the following JavaScript expressions manually.

```
var day = {
   name: "Monday",
   day: 9,
   month: "May",
   year: 2016
}
```

a. `day.name` **b.** `day.month` **c.** `day.year`

11. In each of the following programs, the `aPointer` is a pointer. True or false?

a.
```
var aPointer = ["Some data","in","an","array"].length;
```

b.
```
var aPointer = { a:"complex",data:"structure"};
```

c.
```
var aPointer = "True";
```

d.
```
var someObject = {};
var aPointer = someObject;
```

e.
```
var person = {
   firstName: "Beth",
   class: "Mathematics"
};
```

f.
```
var aPointer = person;
```

g.
```
var aPointer = ["Some data","in","an","array"];
```

h.
```
var aPointer = 10;
```

i.
```
var aPointer = true;
var person = {
   firstName: "Beth",
   class: "Mathematics"
};
var aPointer = person.firstName;
```

12. Match the following JavaScript comments with the appropriate program statements in the table below.

```
// total surface area
// set the depth to 6
// set the height to 5
// calculate the top area
// set the width to 3
// calculate the front area
// calculate the side area
```

Program	Comment
`var width = 3;`	
`var height = 5;`	
`var depth = 6;`	
`var tArea = width*depth;`	
`var fArea = width*height;`	
`var sArea = depth*height;`	
`var area = 2*(tArea + fArea + sArea)`	

13. What is the output to the console of each of the following programs?

a.
```
function distance(x1,y1,x2,y2) {
    var diffX = x2 - x1;
    var diffY = y2 - y1;
    console.log(Math.sqrt(diffX*diffX+diffY*diffY));
}
distance(50,200,200,400);
```

b.
```
function doTheSame() {
    return "Nothing changes";
}
console.log(doTheSame());
```

c.
```
function edgeLength(width,height,depth) {
    var totalEdges = 4*(width+height+depth);
    console.log(totalEdges);
}
edgeLength(4,7,9);
```

d.
```
function median(sortedArray) {
    var arrayLength = sortedArray.length;
    var isOdd = arrayLength%2 === 1;
    if (isOdd) {
        var index = (arrayLength-1)/2;
        return sortedArray[index];
    }
    var index1 = arrayLength/2;
    var index2 = index1-1;
    return (sortedArray[index1]+sortedArray[index2])/2;
}
var theMedian = median([1,2,3,4,6,7,9,144]);
console.log(theMedian);
```

14. Match the following algorithm design steps with their function implementations.

- Return the square root of the sum of squares.
- The function `distance` requires two points, `p1` and `p2`.
- Calculate the difference in y coordinates.
- Calculate the difference in x coordinates.
- Sum the differences squared.

Function	Design
`function distance(p1,p2) {`	
`  var dx = p2.x-p1.x;`	
`  var dy = p2.y-p1.y;`	
`  var sum = dx*dx+dy*dy;`	
`  return Math.sqrt(sum);` `}`	

15. Consider the following arrays.

i. How many dimensions does each array have?

ii. The arrays below can represent matrices. True or false?

a. `[[13],[52],[2]]`
b. `[100]`
c. `[]`
d. `[[[[4]]]]`
e. `[[271,279,274,273],[2,9,4,3]]`
f. `[[23,4,7],[2,3],[2,3,3]]`
g. `[[271,27],[2,93],[1]]`
h. `[[1]]`

16. Consider the two-dimensional array below.

```
var matrix = [[19,28,37,46],[55,64,73,82]];
```

For each of the following values in the `matrix`:

i. what is the first JavaScript index required to access the value?

ii. what is the second JavaScript index required to access the value?

iii. what is the JavaScript expression required to access the value?

a. `64` **b.** `73` **c.** `46` **d.** `19` **e.** `28` **f.** `82`

17. Consider the two-dimensional array below.

```
var results = [[1,4,7,3],[7,6,7,3],[1,2,6,3],[4,6,7,31]];
```

Evaluate the following JavaScript expressions manually.

a. `results[2][0]` **b.** `results[3][1]` **c.** `results[3][3]`

18. Each of the arrays below represents a matrix. For each array:

i. how many rows does the corresponding matrix have?

ii. how many columns does the corresponding matrix have?

a. `[[1],[2]]`
b. `[[31,12,97,3,4,7],[321,122,124,4,3,2],[324,422,42,3,2,9]]`
c. `[[1,2,3,4,5,6,7,8,9,10],[10,9,8,7,6,5,4,3,2,1]]`
d. `[[12,3,23,9],[2,32,2,76],[3,2,6,42]]`

19. Represent each of the following matrices as a JavaScript two-dimensional array.

a. $\begin{bmatrix} 2 & 9 \\ 1 & 3 \\ 19 & 122 \\ 23 & 83 \\ 9 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 13 & 22 & 31 \\ 8 & 2 & 8 \\ 51 & 26 & 29 \end{bmatrix}$ **c.** $\begin{bmatrix} 72 & 24 & 56 \\ 14 & 31 & 61 \end{bmatrix}$

20. Consider the following matrix and two-dimensional JavaScript array.

$$C = \begin{bmatrix} 2 & 11 \\ 3 & 13 \\ 5 & 17 \\ 7 & 19 \end{bmatrix}$$

```
var C = [
   [2,11],
   [3,13],
   [5,17],
   [7,19]
];
```

Answer the following for each of the values below.

i. What is the first index required to access the value in the matrix C?
ii. What is the second index required to access the value in the matrix C?
iii. What is the first index required to access the value in the JavaScript array `C`?
iv. What is the second index required to access the value in the JavaScript array `C`?
v. Using the JavaScript indexes, write an expression to access the value in the JavaScript array `C`.

a. 17 **b.** 13 **c.** 2

21. A canvas has height 512 and width 1024. On the canvas, what is the coordinate in the form `[x,y]` of:

a. the centre? **b.** the middle of the left edge?
c. the middle of the top edge? **d.** the right bottom corner?

22. Match the colours below with the colour strings in the following table.

Light red, dark purple, dark green, dark yellow, light yellow, dark grey, light grey, blue

Colour string	Colour
`"rgb(128,0,128)"`	
`"rgb(0,0,255)"`	
`"rgb(0,128,0)"`	
`"rgb(255,255,128)"`	
`"rgb(200,200,200)"`	
`"rgb(128,128,0)"`	
`"rgb(255,128,128)"`	
`"rgb(50,50,50)"`	

23. Define a string of the form `"rgb(red,green,blue)"` for each of the following colours. for the following colours. Use the value `128` to represent a medium intensity of red, green or blue. For example, dark green is `"rgb(0,128,0)"` and light green is `"rgb(128,255,128)"`.

a. Red **b.** Blue **c.** Dark blue **d.** Green
e. Cyan **f.** Dark cyan **g.** Dark red **h.** Black
i. White **j.** Light blue **k.** Yellow

24. Assume there exists an HTML page which provides a `context` to the canvas. Write a JavaScript statement to change the colour of a line to blue. Use the string structure `"rgb(red,green,blue)"`.

25. Assume there exists an HTML page which provides a `context` to the canvas. Write a JavaScript statement to change the colour of a fill to cyan. Use the string structure `"rgb(red,green,blue)"`.

26. Write an expression to return a random number between 0 and:

a. `1` **b.** `-10` **c.** `255`

d. `3.14` **e.** `2*Math.PI` **f.** $\sqrt{2}$

g. 360

27. Write an expression to return a random number between 180 and 270.

28. How many times is the following loop code executed?

```
var array = [3,4,234,3,32];
for (var i=0; i < array.length; i = i+1) {
   var value = array[i];
   console.log(value);
}
```

29. Consider the following program.

```
var matrix = [[2,3,3,2,2],[5,7,2,4,5],[6,4,1,5,6]];
var m = matrix.length;
for (var i=0; i < m; i = i+1) {
   var row = matrix[i];
   var n = row.length;
   for (var j=0; j < n; j = j+1) {
      console.log(row[j]);
   }
}
```

a. How many times is the outer loop code executed?

b. How many times is the inner loop code executed?

30. Consider the following program.

```
var mass = 0;
var densities = [1000,500,100,10];
var volumes = [2,2,3,42];
var matrix = [densities,volumes];
var m = volumes.length;
for (var i=0; i < m; i = i+1) {
   var volume = matrix[0][i];
   var density = matrix[1][i];
   var weight = volume*density;
   mass = mass + weight;
}
console.log(mass);
```

a. How many times is the loop code executed?

b. What is the output to the console?

Problem solving

31. a. Design and implement an algorithm `sumMultiples(value1,value2,limit)` to sum all of the positive integers that are multiples of `value1` or `value2` below `limit`.

b. What is the output of the function from part **a** given `value1=7`, `value2=11` and `limit=1000`?

32. For each of the following problems:

i. write a program to solve the problem and output the solution to the console

ii. state the solution to the problem.

a. How many different ways are there to make 1 dollar using only 5-cent, 10-cent, 20-cent and 50-cent coins?

b. A prime number is an integer greater than 1 with only two whole number factors, itself and 1. What is the thousandth prime?

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Complete this digital doc: How to write your own program (doc-18769)

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

canvas
character
coordinates
linked list
loops
matrices
nested loop
object
one-dimensional array
pointer
property
radians
two-dimensional array

Answers

Topic 21 Programming

Exercise 21.2 Programs

1. a. i. String **ii.** `"string"`
b. i. String **ii.** `"string"`
c. i. Boolean **ii.** `"boolean"`
d. i. String **ii.** `"string"`
e. i. Number **ii.** `"number"`
f. i. Boolean **ii.** `"boolean"`
g. i. String **ii.** `"string"`
h. i. Number **ii.** `"number"`

2. a, b, d, f, I, k, l, n, o. False
c, e, g, h, j, m. True

3. a. i. Boolean **ii.** `false` **iii.** `"boolean"`
b. i. Number **ii.** `2` **iii.** `"number"`
c. i. String
ii. `"Now this, but the same type"`
iii. `"string"`
d. i. String
ii. `"Another type and value"`
iii. `"string"`

4. a. `true` **b.** `27`

5. a. `9` **b.** `15`

6. a. `2048/(2*2*2*2*2*2*2*2)`
b. `(2.2+0.7+0.1)/(3.2-0.1*2)`
c. `(11+2)+(15-4)/11+16`
d. `22%16`
e. `22*12/2`
f. `Math.sqrt(625/(5*5))`
g. `18*9*27*2*0.5`

7. a. `var planet = "Earth";`
b. `var threeSquared = 2*4+1;`
c. `var timeInSeconds = 10.2;`
d. `var division = (2+3+3)/2;`
e. `var distanceOverThereInKilometers = 2;`

8. a. i. Number **ii.** `506` **iii.** `"number"`
b. i. Number **ii.** `756` **iii.** `"number"`

9. `484`

10. `3960`

11. `var width = 10;`

12.
```
var height = 12;
var perimeter = 2*(width+height);
var area = width*height;
var width = 8;
var height = 5;
var depth = 3;
var topArea = width*depth;
var sideArea = height*depth;
```

```
var frontArea = width*height;
var surface = 2*(topArea+sideArea+frontArea);
var volume = width*height*depth;
```

13.

```
var f = 7;
var g = 12;
var h = 2;
var i = -3;
var a = f*h;
var b = f*i + g*h;
var c = g*i;
```

14.

```
var a = 12;
var b = 39;
var c = 30;
var f = 3;
var g = 6;
var h = a/f;
var i = c/g;
```

Exercise 21.3 Data structures

1. a, l, n. Cannot be stored
b, c, g, h, i, k. Can be stored accurately
d, e, f, j, m, o, p. Can be stored approximately

2. a, b, g, i, j, p, q. `false`
c–f, h, k–o, r, s. `true`

3. a. `false` **b.** `false`

4. a. `true` **b.** `false`

5. a. `true` **b.** `false` **c.** `true` **d.** `true`

6. a. `4` **b.** `5` **c.** `0`
d. `6` **e.** `4`

7. a. `18` **b.** `7` **c.** `38`
d. `23` **e.** `16` **f.** `43`

8. a. `"r"` **b.** `"e"` **c.** `"t"`

9. a, b, f, g, h. True **c, d, e.** True

10. a. `11` **b.** `6` **c.** `2`
d. `0` **e.** `6` **f.** `10`

11. a. `765` **b.** `"iPad"` **c.** `true`

12. a–d. False **e–g.** True

13. a. `"Simons"` **b.** `"Clare"` **c.** `19`

14.

```
var terminator = {};
var object3 = { data:3, next:terminator};
var object2 = { data:4, next:object3};
var object1 = { data:5, next:object2};
var list = object1;
```

15. The terminator is a blank object that references nothing, so it can be built first. The last object only points to the terminator so it can be built next.

If the first object is built first, the data can be added but the next has nothing to point to yet.

16. `list = list.next;`

17. `list.next.next = list.next.next.next;`

Exercise 21.4 Algorithms

1. a. `A string` **b.** `C` **c.** `4`

d. `true` **e.** `246` **f.** `false`

2.

Program	Comment
`var width = 3;`	`// set the width to 3`
`var height = 2.3;`	`// set the height to 2.3`
`var depth = 1.4;`	`// set the depth to 1.4`
`var area = width*height;`	`// calculate the top area`
`var volume = area*depth;`	`// calculate the volume`

3. a. `185`

b. `Nothing to see here`

c. `254`

4. `No`

5.

Function	Design
`function average(a,b,c) {`	The function `average` requires three values: `a,b,c.`
`  var total = a+b+c;`	Calculate the total of `a, b` and `c`.
`  var number = 3;`	Store the number of values.
`  var output = total/number;`	Calculate the average.
`  return output;` `}`	Return the average.

6. a.

```
var nearly_2 = 1+1/2+1/4+1/8+1/16+1/32;
 // nearly_2=1.96875
```

b.

```
var distanceKm = 2600/1000; //
    distanceKm=2.6
```

c.

```
var seconds = 60*60*24*356.25; //
    seconds= 30780000
```

7.

```
var percentage=22+3; // percentage=25
var total=6+14; // total=20
var amount=(percentage/100)*total; // amount=5
```

8. a. i. `a,b,c`

ii. Any reasonably descriptive name, e.g. `isRightAngled`

iii.

1. Use Pythagoras' theorem: $a^2 + b^2 = c^2$
2. Calculate the left-hand side: $a^2 + b^2$
3. Calculate the right-hand side: c^2
4. Return `true` if the left-hand side equals the right-hand side.

b. i. `number`

ii. Any reasonably descriptive name, e.g. `isNatural`

iii.

1. Return `false` if `number` is not positive.
2. Find the decimal part of `number.`
3. Return `true` if the decimal part is zero.
4. Return `false` if the decimal part does not equal zero.

c. i. `a,b,c`

ii. Any reasonably descriptive name, e.g. `isTriad`

iii.

1. If `a` is not a positive integer, return `false.`
2. If `b` is not a positive integer, return `false.`

3. If `c` is not a positive integer, return `false`.

4. Return `true` if `a, b` and `c` form a right-angled triangle.

d. i. `x,y,z`

ii. Any reasonably descriptive name, e.g. `isPythagoreanTriad`

iii. 1. If `x,y,z` is a Pythagorean triad, return `true`.

2. If `x,z,y` is a Pythagorean triad, return `true`.

3. If `y,z,x` is a Pythagorean triad, return `true`.

4. Otherwise, return `false`.

9. a.

```
function isRightAngled(a,b,c) {
  // Use the Pythagoras' theorem
  // a*a+b*b=c*c
  // Calculate left side a*a+b*b
  var left = a*a+b*b;
  // Calculate right side c*c
  var right = c*c;
  // Return true if left side equals
  // right side
  return left === right;
}
```

b.

```
function isNatural(number) {
  // Return false if number is not positive.
  if (number <= 0) { return false; }
  // Find the decimal part.
  var decimalPart = number%1;
  // Return true if the decimal part is zero
  // Return false if the decimal part
  // does not equal zero
  return decimalPart===0;
}
```

c.

```
function isTriad(a,b,c) {
  // If a is not a positive integer
  // return false.
  if (isNatural(a)===false) {
      return false;
      }
  // If b is not a positive integer
  // return false.
  if (isNatural(b)===false) {
      return false;
      }
  // If c is not a positive integer
  // return false.
  if (isNatural(c)===false) {
      return false;
      }

  // Return true if a, b and c form a

  // right angled triangle.

  return isRightAngled(a,b,c);

  }
```

d.

```
function isPythagoreanTriad(x,y,z) {
  // If x,y,z is a Pythagorean triad
  // return true.
  if (isTriad(x,y,z)) { return true; }
  // If x,z,y is a Pythagorean triad
  // return true.
  if (isTriad(x,z,y)) { return true; }
  // If y,z,x is a Pythagorean triad
```

```
    // return true.
    if (isTriad(y,z,x)) { return true; }
    // Otherwise return false.
  return false;
  }
```

10. There are many ways to implement the required algorithm. One method is shown below.

a. `list, data` **b.** `addToEnd`

c. **1.** If `list` is empty, return a new object with the `data` as the new list.

2. Create a reduced list pointer to point to the next item in the list.

3. Create an appended list by calling `addToEnd` with the reduced list and `data`.

4. Update the first object in the list to point to the appended list.

5. Return the updated list as the new list.

d.

```
var terminator = {};
function addToEnd(list, data) {
  // If list is empty return a new object
  // with the data as the new list.
  if (list === terminator) {
    return {
      data:data,
      next:terminator
    }
  }
  // Create a reduced list pointer to point
  // to the next item in the list.
  var reducedList = list.next;
  // Create an appended list by calling
  // addToEnd
  // with the reduced list and data.
  var appendedList = addToEnd(list.next, data);
  // Update the first object in the list
  // to point to the appended list.
  list.next = appendedList;
  // Return the updated list as the new
  // list.
  return list;
}
```

e.

```
var object3 = {
      data:3,
  next:terminator
}
var object2 = {
      data:2,
      next:object3
}
var object1 = {
      data:1,
      next:object2
}
var list3 = object1;
var list4 = addToEnd(list3, 4);
```

11. There are many ways to implement the required algorithm. One method is shown below.

a. `list` **b.** `removeFromEnd`

c. **1.** If the list has no objects, return the terminator.

2. If the list has only one object, return the terminator.

3. Create a sub-list pointer to point to the next item in the list.

4. Create a reduced list by calling `removeFromEnd` with the sub-list.

5. Update the first object in the list to point to the reduced list.

6. Return the updated list as the new list.

d.

```
var terminator = {};
function removeFromEnd(list) {
  // If list has no objects
  // return the terminator.
  if (list === terminator) {
    return terminator;
  }
  // If list has only one object
  // return the terminator.
  if (list.next === terminator) {
    return terminator;
  }
  // Create a sub list pointer to
  // point to the next item in the list.
  var subList = list.next;
  // Create a reduced list by calling
  // removeFromEnd with the sub list.
  var reducedList = removeFromEnd(subList)
  // Update the first object in the
  // list to point to the reduced list.
  list.next = reducedList;
  // Return the updated list as the new
  // list.
  return list;
}
```

e.

```
var object3 = {
    data:3,
    next:terminator
}
var object2 = {
    data:2,
    next:object3
}
var object1 = {
    data:1,
    next:object2
}
var list3 = object1;
var list2 = removeFromEnd(list3);
```

12. There are many ways to implement the required algorithm. One method is shown below.

a. `list, index, data` **b.** `setData`

c. **1.** If the index is 0 or less, then set the data of the first object and terminate the function.

2. Create a reduced list pointer to the second object.

3. Create a new index one less than the input index.

4. Call `setData` using the reduced list, reduced index and `data`.

d.

```
function setData(list, index, data) {
  // If index 0 or less then set
  // data of the first object and
  // terminate the function.
  if (index <= 0) {
    list.data = data;
    return;
  }
  // Create a reduced list pointer
  // to the second object.
  var reduced = list.next;
  // Create a new index one less
  // than the input index.
  var newIndex = index-1;
  // Call setData using the
```

```
  // reduced list, reduced index
  // and data.
  setData(reduced, newIndex, data);
}
```

e.

```
var terminator = {};
var object3 = {
  data:"here.",
  next:terminator
}
var object2 = {
  data:"me",
  next:object3
}
var object1 = {
  data:"Replace",
  next:object2
}
var list = object1;
setData(list,2,"there.");
```

13. There are many ways to implement the required algorithm. One method is shown below.

a. `list, index`

b. `removeAtIndex`

c. **1.** If the index is 0 or less, then return the next object in the list.

2. Create a reduced list pointer to the second object.

3. Create a new index one less than the input index.

4. Call `removeAtIndex` using the reduced list and index, then store the list.

5. Update the first object in the list to point to the list created above.

6. Return `removeAtIndex` using the reduced list and index.

d.

```
function removeAtIndex(list, index) {
  // If index 0 or less then return the
  // next object in the list.
  if (index <= 0) {
    return list.next;
  }
  // Create a reduced list pointer
  // to the second object.
var reduced = list.next;
  // Create a new index one less
  // than the input index.
var newIndex = index-1;
  // Call removeAtIndex using the
  // reduced list and index then
  // store the list.
var removedList = removeAtIndex(reduced, newIndex);
  // Update the first object in the
  // list to point to the list created above.
  list.next = removedList;
  // Return the updated list as the new list.
   return list;
}
```

e.

```
var terminator = {};
var object3 = {
    data:3,
    next:terminator
}
var object2 = {
    data:2,
    next:object3
}
```

```
var object1 = {
    data:1,
    next:object2
}
var list3 = object1;
var list2 = removeAtIndex(list3, 1);
```

14.
```
function maximumArea(perimeter) {
    // A circle contains the maximum possible
    // area with a perimeter.
    // Store an approximation to PI
    var PI = 3.141592653589793;
    // Calculate the radius of a circle with
    // a given perimeter
    var radius = perimeter / (2*PI);
    // Calculate the area of a circle with a
    // given radius
    var area = PI * radius * radius;
    // Return the area of a circle with
    // a given perimeter.
    return area;
}
```

Exercise 21.5 Matrices

1. a. 2 **b.** 1 **c.** 1 **d.** 3 **e.** 2
f. 3 **g.** 3 **h.** 2 **i.** 2

2. a. i. `3` **ii.** `0`
b. i. `0` **ii.** `0`
c. i. `2` **ii.** `0`
d. i. `2` **ii.** `1`

3. a. `1` **b.** `72` **c.** `4`

4. a. False **b.** False **c.** True **d.** False

5. a. i. 6 **ii.** 1
b. i. 1 **ii.** 6
c. i. 5 **ii.** 3
d. i. 2 **ii.** 5

6. a.
```
[
    [1,2,3],
    [5,6,2]
]
```
b.
```
[
    [2,4,5,2,62],
    [1,3,1,2,31]
]
```
c.
```
[
    [21,11],
    [19,12],
    [71,82],
    [12,81]
]
```

7. a. i. 1 **ii.** 1 **iii.** `0` **iv.** `0` **v.** `B[0][0]`
b. i. 2 **ii.** 3 **iii.** `1` **iv.** `2` **v.** `B[1][2]`
c. i. 1 **ii.** 2 **iii.** `0` **iv.** `1` **v.** `B[0][1]`

8.
```
var indexProduct = [
  [0,0,0,0],
  [0,1,2,3],
  [0,2,4,6],
  [0,3,6,9]
];
```

9. Any order of the vertices is correct.
```
var square = [
  [0,0],
  [100,0],
  [100,100],
  [0,100]
];
```

10. Any order of the vertices is correct.
```
var octagon = [
    [148,80],
    [148,120],
    [120,148],
    [80,148],
    [52,120],
    [52,80],
    [80,52],
    [120,52]
];
```

Exercise 21.6 Graphics

1. a. `[150,200]` **b.** `[300,200]` **c.** `[150,400]` **d.** `[300,0]`

2. a. `"rgb(0,0,255)"` **b.** `"rgb(0,0,0)"` **c.** `"rgb(0,255,255)"` **d.** `"rgb(0,128,0)"`

3.

Colour string	Colour
`"rgb(128,255,128)"`	Light green
`"rgb(0,0,0)"`	Black
`"rgb(255,255,0)"`	Yellow
`"rgb(255,128,128)"`	Pink
`"rgb(255,0,0)"`	Red
`"rgb(0,255,0)"`	Green
`"rgb(128,128,255)"`	Light blue
`"rgb(0,0,255)"`	Blue

4. `context.strokeStyle = "rgb(255,0,0)";`

5. `context.fillStyle = "rgb(255,255,0)";`

6. a.
```
context.moveTo(500,10);
context.lineTo(10,500);
context.stroke();
```

b.
```
context.lineWidth = 10;
context.moveTo(66,14);
context.lineTo(20,410);
context.stroke();
```

c.
```
context.lineWidth = 3;
context.moveTo(86,3);
context.lineTo(50,150);
context.stroke();
```

d.
```
context.lineWidth = 9;
context.moveTo(20,13);
context.lineTo(5,40);
context.stroke();
```

e.

```
context.moveTo(50,50);
context.lineTo(550,50);
context.lineTo(550,450);
context.lineTo(50,450);
context.lineTo(50,50);
context.stroke();
```

7. Many answers are possible. One method is shown below.

```
var x1 = 40;
var y1 = 20;
var x2 = 200;
var y2 = 20;
var x3 = 200;
var y3 = 250;
var x4 = 40;
var y4 = 250;
context.beginPath();
context.moveTo(x1,y1);
context.lineTo(x2,y2);
context.lineTo(x3,y3);
context.lineTo(x4,y4);
context.closePath();
context.fill();
```

8. a.

```
var x = 22;
var y = 43;
var radius = 20;
var end = 2*Math.PI;
context.beginPath();
context.arc(x,y,radius,0,end);
context.closePath();
context.fill();
```

b.

```
var x = 100;
var y = 100;
var diameter = 160;
var radius = diameter/2;
var end = 2*Math.PI;
context.beginPath();
context.arc(x,y,radius,0,end);
context.closePath();
context.fill();
```

9.

```
var x = 200;
var y = 280;
var diameter = 200;
var radius = diameter/2;
var end = 2*Math.PI;
var fillColour = "rgb(255,0,0)";
var lineColour = "rgb(0,128,0)";
context.beginPath();
context.fillStyle = fillColour;
context.lineWidth = 8;
context.strokeStyle = lineColour;
```

```
context.arc(x,y,radius,0,end);
context.closePath();
context.fill();
context.stroke();
```

10. a. Many answers are possible. One method is shown below.

```
var x1 = 150;
var y1 = 200;
var x2 = 200;
var y2 = 100;
// Find middle of the square
var mx = (x1+x2)/2;
var my = (y1+y2)/2;
// Find the difference of x and y from
// the centre.
var dx = x1-mx;
var dy = y1-my;
context.moveTo(mx-dx,my-dy);
context.lineTo(mx+dy,my-dx);
context.lineTo(mx+dx,my+dy);
context.lineTo(mx-dy,my+dx);
context.lineTo(mx-dx,my-dy);
context.stroke();
```

b.

```
var x1 = 116;
var y1 = 240;
var x2 = 120;
var y2 = 210;
// Find middle of the square
var mx = (x1+x2)/2;
var my = (y1+y2)/2;
// Find the difference of x and y from
// the centre.
var dx = x1-mx;
var dy = y1-my;
context.moveTo(mx-dx,my-dy);
context.lineTo(mx+dy,my-dx);
context.lineTo(mx+dx,my+dy);
context.lineTo(mx-dy,my+dx);
context.lineTo(mx-dx,my-dy);
context.stroke();
```

c.

```
var x1 = 400;
var y1 = 130;
var x2 = 110;
var y2 = 250;
// Find middle of the square
var mx = (x1+x2)/2;
var my = (y1+y2)/2;
// Find the difference of x and y from
// the centre.
var dx = x1-mx;
```

```
  var dy = y1-my;
  context.moveTo(mx-dx,my-dy);
  context.lineTo(mx+dy,my-dx);
  context.lineTo(mx+dx,my+dy);
  context.lineTo(mx-dy,my+dx);
  context.lineTo(mx-dx,my-dy);
  context.stroke();
```

11.
```
var x1 = 200;
var y1 = 300;
var x2 = 400;
var y2 = 300;
var diameter = x2-x1;
var radius = diameter/2;
var end = 2*Math.PI;
var fillColour = "rgb(0,0,255)";
context.fillStyle = fillColour;
context.beginPath();
context.arc(x1,y1,radius,0,end);
context.closePath();
context.fill();
context.beginPath();
context.arc(x2,y2,radius,0,end);
context.closePath();
context.fill();
```

12.
```
// Assume there exists a HTML page which
// provides a context to the canvas
var x1=200;
var y1=100;
var x2=400;
var y2=100;
var x3=400;
var y3=300;
var x4=200;
var y4=300;
context.beginPath();
context.moveTo(x1,y1);
context.lineTo(x2,y2);
context.lineTo(x3,y3);
context.lineTo(x4,y4);
context.closePath();
context.stroke();
var x = (x1+x3)/2;
var y = (y1+y3)/2;
var radius = (x3-x1)/2;
var end = 2*Math.PI;
context.beginPath();
context.arc(x, y, radius, 0, end);
context.closePath();
context.fill();
```

13. a. `Math.PI/2` **b.** `Math.PI/3` **c.** `Math.PI/6` **d.** `Math.PI/4`

e. `Math.PI*161.2/180`

14.
```
var x = 300;
var y = 250;
var radius = 200;
var startAngle = 240;
var endAngle = 300;
var startRatio = startAngle/360;
var endRatio = endAngle/360;
var start = 2*Math.PI * startRatio;
var end = 2*Math.PI * endRatio;
context.arc(x,y,radius,start,end);
context.stroke();
```

15. Many answers are possible. One method is shown below.
```
// Assume there exists a HTML page which
// provides a context to the canvas
context.beginPath();
context.fillStyle = "rgb(128,128,255)";
context.strokeStyle = "rgb(0,0,255)";
context.lineTo(0,0);
context.lineTo(100,0);
context.lineTo(100,100);
context.lineTo(0,100);
context.closePath();
context.fill();
context.stroke();
```

16. Many answers are possible. One method is shown below.
```
// Assume there exists a HTML page which
// provides a context to the canvas
context.beginPath();
context.fillStyle = "rgb(128,128,255)";
context.strokeStyle = "rgb(0,0,255)";
context.lineTo(148,80);
context.lineTo(148,120);
context.lineTo(120,148);
context.lineTo(80,148);
context.lineTo(52,120);
context.lineTo(52,80);
context.lineTo(80,52);
context.lineTo(120,52);
context.closePath();
context.fill();
context.stroke();
```

17. Many answers are possible. One method is shown below.
```
// Assume there exists a HTML page which
// provides a context to the canvas
var x = 300;
var y = 250;
var radius = 200;
```

```
var start = 2*Math.PI * 4 / 6;
var end = 2*Math.PI * 5 / 6;
context.beginPath();
context.fillStyle = "rgb(255,0,0)";
context.moveTo(x,y);
context.arc(x,y,radius,start,end);
context.lineTo(x,y);
context.fill();
context.stroke();
```

18. Many answers are possible. One method is shown below.

```
// Assume there exists a HTML page which
// provides a context to the canvas
function sector(colour, i) {
  var x = 300;
  var y = 250;
  var radius = 200;
  var start = 2*Math.PI * i / 6;
  var end = 2*Math.PI * (i+1) / 6;
  context.beginPath();
  context.fillStyle = colour;
  context.moveTo(x,y);
  context.arc(x,y,radius,start,end);
  context.lineTo(x,y);
  context.fill();
  context.stroke();
}
sector("rgb(0,0,255)",0);
sector("rgb(0,255,255)",1);
sector("rgb(0,255,0)",2);
sector("rgb(255,255,0)",3);
sector("rgb(255,0,0)",4);
sector("rgb(255,0,255)",5);
```

Exercise 21.7 Simulations

1. a. `Math.random()` **b.** `0.11*Math.random()` **c.** `-0.22*Math.random()`

d. `3*Math.random()` **e.** `2*Math.random()` **f.** `-3*Math.random()`

g. `5*Math.random()` **h.** `2.4*Math.random()` **i.** `1000*Math.random()`

2. `20+10*Math.random()`

3. 4

4. 12

5. a. 6 **b.** 216

6.

```
var matrix = [
  [1,2,3,4,5,6,7],
  [8,9,10,11,12,13,14],
  [15,16,17,18,19,20,21]
];
var sum = 0;
var count = 0;
var m = matrix.length;
for (var i=0; i<m; i=i+1) {
  var n = matrix[i].length;
  for (var j=0; j<n; j=j+1) {
    var item = matrix[i][j];
    sum = sum + item;
```

```
    count = count + 1;
  }
}
var mean = sum / count;
```

7.

```
function randomBooleans(required) {
  var array = [];
  for (var i=0; i<required; i=i+1) {
    var random = Math.random() <= 0.5;
    array.push(random);
  }
   return array;
}
```

8.

```
function randomGenders(people) {
  var array = [];
  for (var i=0; i<people; i=i+1) {
    var isMale = Math.random() <= 0.51;
    if (isMale) {
      array.push("Male");
    } else {
      array.push("Female");
     }
  }
  return array;
}
```

9.

```
function randomScaled(
            rows,
            columns,
            scale) {
  var matrix = [];
  for (var i=0; i<rows; i=i+1) {
    var newRow = [];
    for (var j=0; j<columns; j=j+1) {
      var random = scale*Math.random();
      newRow.push(random);
    }
    matrix.push(newRow);
  }
  return matrix;
}
```

10. a.

```
function randomColours(
            height,
            width) {
  var matrix = [];
  for (var i=0; i<height; i=i+1) {
    var newRow = [];
    for (var j=0; j<width; j=j+1) {
      var random = Math.random();
      if (random <= 0.05) {
        newRow.push("rgb(255,0,0)");
      } else if (random <= 0.15) {
        newRow.push("rgb(0,255,0)");
      } else if (random <= 0.30) {
        newRow.push("rgb(0,0,255)");
      } else {
        newRow.push("rgb(0,0,0)");
      }
    }
    matrix.push(newRow);
  }
   return matrix;
 }
```

b.

```
// Assume there exists a HTML page which
// provides a context to the canvas
function draw(image) {
   var rows = image.length;
   for (var i=0; i<rows; i=i+1) {
      var columns = image[i].length;
      for (var j=0; j<columns; j=j+1) {
         var colour = image[i][j];
         context.fillStyle = colour;
         context.fillRect(j,i,1,1);
      }
   }
}
```

c. The 600 by 500 canvas image is mainly black with random blue, green and red points.

11. a. 1. Initialise a variable to track the status of the new cell. A status of true indicates alive; a status of false indicates dead or empty.

a. If the current cell is alive and has 2 or 3 neighbours, set the new status to alive.

b. If the current cell is alive and does not have 2 or 3 neighbours, set the new status to dead.

c. If the current cell is empty and has 3 or 6 neighbours, set the new status to alive.

d. If the current cell is empty and does not have 3 or 6 neighbours, set the new status to empty.

5. Return the status.

b.

```
function isAlive(alive, neighbours) {
// Initialise a variable to track the
// status of the new cell. A status
// of true indicates alive, a status
// of false indicates dead or empty.
var newStatus = false;

if (alive) {

   // If the current cell is alive and
   // has 2 or 3 neighbours, set the new
   // status to alive.
   // If the current cell is alive and
   // does not have 2 or 3 neighbours,
   // set the new status to dead.
   newStatus =
   neighbours === 2 ||
   neighbours === 3;
} else {
   // If the current cell is empty and
   // has 3 or 6 neighbours, set the new
   // status to alive.
   // If the current cell is empty and
   // does not have 3 or 6 neighbours, set
   // the new status to empty.
   newStatus =
      neighbours === 3 ||
      neighbours === 6;
   }

   // Return the status.

   return newStatus;

}
```

12.

```
var closeCells = [[0,0],[0,1],[1,0],
    [0,-1],[-1,0]];
function average(i,j,image) {
   var total = 0;
   var count = 0;
   var tests = closeCells.length;
   for (var k=0; k<tests; k=k+1) {
      var ai = i + closeCells[k][0];
```

```
    var aj = j + closeCells[k][1];
    if (image[ai]) {
      if (typeof image[ai][aj] ===
          "number") {
          total = total + image[ai][aj];
          count = count + 1;
      }
    }
  }
  return total/count;
}
```

13.

```
function transforms(matrix) {
  var newMatrix = [];
  var rows = matrix.length;
  for (var i=0; i<rows; i=i+1) {
    var newRow = [];
    var columns = matrix[i].length;
    for (var j=0; j<columns; j=j+1) {
      newRow.push(smooth(i,j,matrix));
    }
    newMatrix.push(newRow);
  }
  return newMatrix;
}
```

14.

```
// Assume there exists a HTML page which
// provides a context to the canvas
function randomBooleans(rows, columns,
probability) {
  var matrix = [];
  for (var i = 0; i < rows; i = i+1) {
    var newRow = [];
    for (var j = 0; j < columns; j = j+1) {
      var random = Math.random();
      var isTrue = random <= probability;
      newRow.push(isTrue);
    }
    matrix.push(newRow);
  }
  return matrix;
}
function isAlive(alive, neighbours) {
  var newStatus = false;
  if (alive) {
    newStatus =
      neighbours === 2 ||
      neighbours === 3;
  } else {
    newStatus =
      neighbours === 3 ||
      neighbours === 6;
  }
  return newStatus;
}
var adjacents = [
  [0,1],[1,1],[1,0],[1,-1],
  [0,-1],[-1,-1],[-1,0],[-1,1]];
function count(i, j, matrix) {
  var total = 0;
  var tests = adjacents.length;
  for (var k=0; k<tests; k = k+1) {
    var ai = i + adjacents[k][0];
    var aj = j + adjacents[k][1];
```

```
      if (matrix[ai]) {
        if (matrix[ai][aj]) {
          total = total+1;
        }
      }
    }
  return total;
}
function transforms(matrix) {
  var newMatrix = [];
  var m = matrix.length;
  for (var i = 0; i < m; i = i+1) {
    var newRow = [];
    var n = matrix[i].length;
    for (var j = 0; j < n; j = j+1) {
      var item = matrix[i][j];
      var neighbours =
        count(i, j, matrix);
        var newItem =
          isAlive(item, neighbours);
        newRow.push(newItem);
    }
    newMatrix.push(newRow);
  }
  return newMatrix;
}
var rows = 100;
var columns = 100;
var matrix = randomBooleans(
  rows,
  columns,
  0.4);
function draw() {
  context.clearRect(0,0,600,500);
  for (var i=0; i<rows; i=i+1) {
    for (var j=0; j<columns; j=j+1) {
      if (matrix[i][j]) {
        context.fillRect(j*5,i*5,5,5);
      }
    }
  }
  matrix = transforms(matrix);
  window.requestAnimationFrame(draw);
}
window.requestAnimationFrame(draw);
```

15.

```
// Assume there exists a HTML page which
// provides a context to the canvas
function randomBooleans(rows, columns,
probability) {
  var matrix = [];
  for (var i = 0; i < rows; i = i+1) {
    var newRow = [];
    for (var j = 0; j < columns; j = j+1) {
      var random = Math.random();
      var isTrue = random <= probability;
      newRow.push(isTrue);
    }
    matrix.push(newRow);
  }
  return matrix;
}
function isAlive(alive, neighbours) {
```

```
  var newStatus = false;
  if (alive) {
    newStatus =
      neighbours === 2 ||
      neighbours === 3;
  } else {
    newStatus = neighbours === 3;
  }
  return newStatus;
}
var adjacents = [
  [0,1],[1,1],[1,0],[1,-1],
  [0,-1],[-1,-1],[-1,0],[-1,1]];
function count(i, j, matrix) {
  var total = 0;
  var tests = adjacents.length;
  for (var k=0; k<tests; k = k+1) {
    var ai = i + adjacents[k][0];
    var aj = j + adjacents[k][1];
    if (matrix[ai]) {
      if (matrix[ai][aj]) {
        total = total+1;
      }
    }
  }
  return total;
}
function transforms(matrix) {
  var newMatrix = [];
  var m = matrix.length;
  for (var i = 0; i < m; i = i+1) {
    var newRow = [];
    var n = matrix[i].length;
    for (var j = 0; j < n; j = j+1) {
      var item = matrix[i][j];
      var neighbours =
        count(i, j, matrix);
      var newItem =
        isAlive(item, neighbours);
      newRow.push(newItem);
    }
    newMatrix.push(newRow);
  }
  return newMatrix;
}
var rows = 100;
var columns = 100;
for (var loop = 0; loop < 20; loop++) {
  var matrix = randomBooleans(rows,
     columns, 0.5);
  for (var iteration = 0; iteration <= 600;
     iteration++) {
    var alive = 0;
    for (var i=0; i<rows; i=i+1) {
      for (var j=0; j<columns; j=j+1) {
        if (matrix[i][j]) {
          alive = alive + 1;
        }
      }
    }
    var ratio = alive / (rows * columns);
    context.fillRect(iteration,
```

```
            500*(1-ratio),1,1);
        matrix = transforms(matrix);
     }
  }
```

21.8 Review

1. a. i. Boolean **ii.** "boolean"
b. i. String **ii.** "string"
c. i. Number **ii.** "number"
d. i. Boolean **ii.** "boolean"
e. i. Number **ii.** "number"
f. i. String **ii.** "string"

2. a, g, i, j. True
b–f, h. False

3. a. i. Number **ii.** 9 **iii.** "number"
b. i. Number **ii.** 0 **iii.** "number"
c. i. Boolean **ii.** false **iii.** "boolean"
d. i. String" **ii.** End with this." **iii.** "string"
e. i. String **ii.** "A string" **iii.** "string"
f. i. Boolean **ii.** true **iii.** "boolean"
g. i. Boolean **ii.** false **iii.** "boolean"

4. a, c, d, e, h, k, n. Can be stored approximately
b, f, g, i, j, l, m, q. Can be stored accurately
o, p, r. Cannot be stored

5. a. true **b.** false **c.** true **d.** false **e.** false **f.** 7
g. true **h.** true **i.** false **j.** false **k.** false **l.** false
m. 4 **n.** 7 **o.** false **p.** true **q.** 0 **r.** true
s. false **t.** 10 **u.** false **v.** true **w.** false **x.** false
y. false **z.** true

6. a. 20 **b.** 11 **c.** 6 **d.** 9 **e.** 15 **f.** 0

7. a. "e" **b.** "e" **c.** "i"

8. a. False **b.** False **c.** True **d.** False **e.** False **f.** True
g. True **h.** True

9. a. 2 **b.** 0 **c.** 6 **d.** 5 **e.** 1

10. a. "Monday" **b.** "May" **c.** 2016

11. a. False **b.** True **c.** False **d.** True **e.** True **f.** True
g. False **h.** False **i.** False

12.

Program	Comment
`var width = 3;`	`// set the width to 3`
`var height = 5;`	`// set the height to 5`
`var depth = 6;`	`// set the depth to 6`
`var tArea = width*depth;`	`// calculate the top area`
`var fArea = width*height;`	`// calculate the front area`
`var sArea = depth*height;`	`// calculate the side area`
`var area = 2*(tArea + fArea + sArea)`	`// total surface area`

13. a. 250 **b.** Nothing changes **c.** 80 **d.** 5

14.

Function	Design
`function distance(p1,p2) {`	The function `distance` requires two points, `p1` and `p2`.
`var dx = p2.x-p1.x;`	Calculate the difference in x coordinates.
`var dy = p2.y-p1.y;`	Calculate the difference in y coordinates.
`var sum = dx*dx+dy*dy;`	Sum the differences squared.
`return Math.sqrt(sum);}`	Return the square root of the sum of squares.

15. a. i. 2 **ii.** True
b. i. 1 **ii.** False
c. i. 1 **ii.** False
d. i. 4 **ii.** False
e. i. 2 **ii.** True
f. i. 2 **ii.** False
g. i. 2 **ii.** False
h. i. 2 **ii.** True

16. a. i. `1` **ii.** `1` **iii.** `matrix[1][1]`
b. i. `1` **ii.** `2` **iii.** `matrix[1][2]`
c. i. `0` **ii.** `3` **iii.** `matrix[0][3]`
d. i. `0` **ii.** `0` **iii.** `matrix[0][0]`
e. i. `0` **ii.** `1` **iii.** `matrix[0][1]`
f. i. `1` **ii.** `3` **iii.** `matrix[1][3]`

17. a. `1` **b.** `6` **c.** `31`

18. a. i. `2` **ii.** `1`
b. i. `3` **ii.** `6`
c. i. `2` **ii.** `10`
d. i. `3` **ii.** `4`

19. a. `[[2,9],[1,3],[19,122],[23,83],[9,5]]`
b. `[[13,22,31],[8,2,8],[51,26,29]]`
c. `[[72,24,56],[14,31,61]]`

20. a. i. 3 **ii.** 2 **iii.** `2` **iv.** `1` **v.** `C[2][1]`
b. i. 2 **ii.** 2 **iii.** `1` **iv.** `1` **v.** `C[1][1]`
c. i. 1 **ii.** 1 **iii.** `0` **iv.** `0` **v.** `C[0][0]`

21. a. `[256,512]` **b.** `[0,512]` **c.** `[256,0]` **d.** `[512,1024]`

22.

Colour string	Colour
`"rgb(128,0,128)"`	Dark purple
`"rgb(0,0,255)"`	Blue
`"rgb(0,128,0)"`	Dark green
`"rgb(255,255,128)"`	Light yellow
`"rgb(200,200,200)"`	Light grey
`"rgb(128,128,0)"`	Dark yellow
`"rgb(255,128,128)"`	Light red
`"rgb(50,50,50)"`	Dark grey

23. a. `"rgb(255,0,0)"` **b.** `"rgb(0,0,255)"` **c.** `"rgb(0,0,128)"`
d. `"rgb(0,255,0)"` **e.** `"rgb(0,255,255)"` **f.** `"rgb(0,128,128)"`
g. `"rgb(128,0,0)"` **h.** `"rgb(0,0,0)"` **i.** `"rgb(255,255,255)"`
j. `"rgb(128,128,255)"` **k.** `"rgb(255,255,0)"`

24. `context.strokeStyle = "rgb(0,0,255)";`

25. `context.fillStyle = "rgb(0,255,255)";`

26. a. `Math.random()` **b.** `-10*Math.random()`

c. `255*Math.random()` **d.** `3.14*Math.random()`

e. `2*Math.PI*Math.random()` **f.** `Math.sqrt(2)*Math.random()`

g. `360*Math.random()`

27. `180 + 90*Math.random()`

28. 5

29. a. 3 **b.** 15

30. a. 5 **b.** `3720`

31. a.

```
function sumMultiples(value1,value2,limit) {
   var sum = 0;
   for (var i = 1; i < limit; i++) {
      if ((i%value1===0)||(i%value2===0)) {
         sum = sum + i;
      }
   }
   return sum;
}
110110
```

32. a. i. Many answers are possible. One solution is shown below.

```
var combinations = 0;
for (var fifties = 0; fifties <= 2; fifties++) {
   for (var twenties = 0; twenties <= 5; twenties++) {
      for (var tens = 0; tens <= 10; tens++) {
         for (var fives = 0; fives <= 20; fives++) {
            var amount = fifties*50+twenties*20+tens*10+fives*5;
            if (amount === 100) {
               combinations++;
            }
         }
      }
   }
}
console.log(combinations);
```

ii. `49`

b. i. Many answers are possible. One solution is shown below.

```
var primes = [];
for (var i = 2; primes.length < 1000; i++) {
   var isPrime = true;
   for (var j = 0; (j < primes.length) && isPrime; j++) {
      if (i%primes[j] === 0) {
         isPrime = false;
      }
   }
   if (isPrime) {
      primes.push(i);
   }
}
console.log(primes[1000-1]);
```

ii. `7919`

APPENDIX
CAS Calculator companion

The information below is a sample. Full CAS Calculator companions are located as interactives in the Reading panel of your learnON title and as PDFs in the Resources section of the *About this course* topic.

TOPIC 1

WORKED EXAMPLE 3

Simplify each of the following.

a $\left(2n^4\right)^3$ **b** $\left(3a^2b^7\right)^3$ **c** $\left(\dfrac{2x^3}{y^4}\right)^4$ **d** $\left(-4\right)^3$

TI | THINK

a–d On a Calculator page, use the brackets and complete the entry lines as:

$\left(2n^4\right)^3$

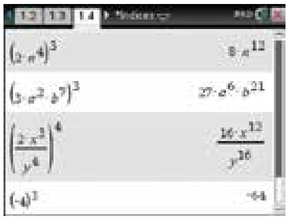
$\left(3a^2b^7\right)^3$

$\left(\dfrac{2x^3}{y^4}\right)^4$

$\left(-4\right)^3$

Press ENTER `enter` after each entry.

WRITE

a–d

$(2\cdot n^4)^3$ $8\cdot n^{12}$

$(3\cdot a^2\cdot b^7)^3$ $27\cdot a^6\cdot b^{21}$

$\left(\dfrac{2\cdot x^3}{y^4}\right)^4$ $\dfrac{16\cdot x^{12}}{y^{16}}$

$(-4)^3$ -64

$\left(2n^4\right)^3 = 8n^{12}$

$\left(3a^2b^7\right)^3 = 27a^6b^{21}$

$\left(\dfrac{2x^3}{y^4}\right)^4 = \dfrac{16x^{12}}{y^{16}}$

$\left(-4\right)^3 = -64$

CASIO | THINK

a–d On the Main screen, use the brackets and complete the entry lines as:

$\left(2n^4\right)^3$

$\left(3a^2b^7\right)^3$

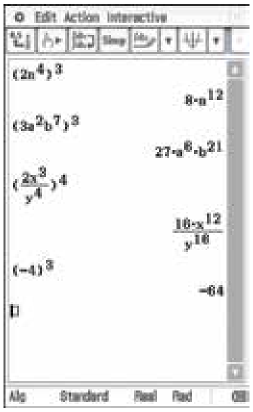
$\left(\dfrac{2x^3}{y^4}\right)^4$

$\left(-4\right)^3$

Press EXE after each entry.

WRITE

a–d



$(2n^4)^3$ $8\cdot n^{12}$

$(3a^2b^7)^3$ $27\cdot a^6\cdot b^{21}$

$\left(\dfrac{2x^3}{y^4}\right)^4$ $\dfrac{16\cdot x^{12}}{y^{16}}$

$(-4)^3$ -64

$\left(2n^4\right)^3 = 8n^{12}$

$\left(3a^2b^7\right)^3 = 27a^6b^{21}$

$\left(\dfrac{2x^3}{y^4}\right)^4 = \dfrac{16x^{12}}{y^{16}}$

$\left(-4\right)^3 = -64$

TOPIC 2

WORKED EXAMPLE 2

If $c = \sqrt{a^2 + b^2}$, calculate c if $a = 12$ and $b = -5$.

TI | THINK

In a new document, open a calculator page. To substitute values, use the symbol |. Press CTRL `ctrl` and then `=` to bring up the palette; use the Touchpad to select the | symbol. Then type 'and' or find it in the catalog 📖.

Complete the entry line as:

$c = \sqrt{a^2 + b^2} \mid a = 12$ and $b = -5$

Then press ENTER `enter`.

WRITE

$c=\sqrt{a^2+b^2} \mid a=12$ and $b=-5$ $c=13$

If $a = 12$ and $b = -5$, then $c = \sqrt{a^2 + b^2} = 13$.

CASIO | THINK

To type the equation, $\sqrt{\ }$ is on the Keyboard `Math1` screen. The vertical line | is on the Keyboard `Math3` screen.

Complete the entry line as:

$c = \sqrt{a^2 + b^2} \mid a = 12$ and $b = -5$

Then press EXE.

WRITE

$c=\sqrt{a^2+b^2} \mid a=12$ and $b=-5$

$c=13$

If $a = 12$ and $b = -5$ then $c = \sqrt{a^2 + b^2} = 13$.

TOPIC 2

WORKED EXAMPLE 13

Solve each of the following linear equations.

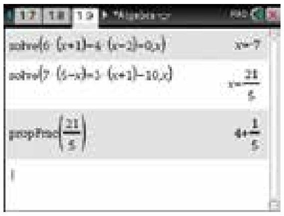
a $6(x + 1) - 4(x - 2) = 0$ **b** $7(5 - x) = 3(x + 1) - 10$

TI | THINK

a–b On a Calculator page, complete the entry lines as:
solve
$(6(x + 1) - 4(x - 2) = 0, x)$
solve
$(7(5 - x) = 3(x + 1) - 10, x)$
Press ENTER enter after each entry.

WRITE

a–b



solve(6·(x+1)−4·(x−2)=0,x) x=−7

solve(7·(5−x)=3·(x+1)−10,x) $x=\frac{21}{5}$

propFrac$\left(\frac{21}{5}\right)$ $4+\frac{1}{5}$

$6(x + 1) - 4(x - 2) = 0$
$\Rightarrow x = -7$
$7(5 - x) = 3(x + 1) - 10$
$\Rightarrow x = 4\frac{1}{5}$

CASIO | THINK

a–b On the Main screen, complete the entry lines as:
solve
$(6(x + 1) - 4(x - 2) = 0, x)$
solve
$(7(5 - x) = 3(x + 1) - 10, x)$
Press EXE after each entry.
Convert **b** to a proper fraction.

WRITE

solve(6(x+1)−4(x−2)=0)
{x=−7}
solve(7(5−x)=3(x+1)−10
$\left\{x=\frac{21}{5}\right\}$
propFrac$(\frac{21}{5})$
$4+\frac{1}{5}$

$6(x + 1) - 4(x - 2) = 0$
$\Rightarrow x = -7$
$7(5 - x) = 3(x + 1) - 10$
$\Rightarrow x = 4\frac{1}{5}$

TOPIC 3

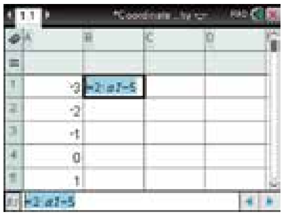
WORKED EXAMPLE 1

Plot the linear graph defined by the rule $y = 2x - 5$ for the x-values $-3, -2, -1, 0, 1, 2$ and 3.

TI | THINK

1 In a new document, on a Lists & Spreadsheet page, label column A as x and label column B as y.
Enter the x-values into column A.
Then in cell B1, complete the entry line as:
$= 2a1 - 5$
Then press ENTER enter.

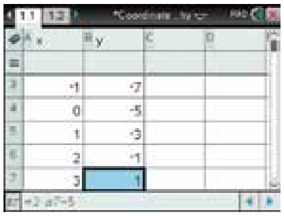
WRITE



	A	B	C	D
1	-3	=2·a1-5		
2	-2			
3	-1			
4	0			
5	1			

B1 =2·a1-5

2 Hold down the SHIFT key and the down arrow to fill down the y-values.



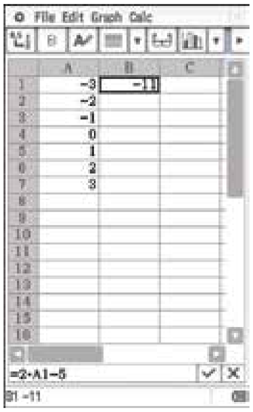
	A x	B y	C	D
3	-1	-7		
4	0	-5		
5	1	-3		
6	2	-1		
7	3	1		

B7 =2·a7-5

CASIO | THINK

1 It is possible to plot this graph using a calculator.
On the Spreadsheet screen, enter the x-values into column A.
Then in cell B1, complete the entry line as:
$= 2A1 - 5$
Then press EXE.

WRITE

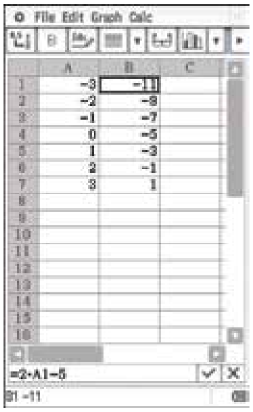


	A	B	C
1	−3	−11	
2	−2		
3	−1		
4	0		
5	1		
6	2		
7	3		

=2·A1−5
B1 −11

2 Highlight cell B1 to B7, then tap:

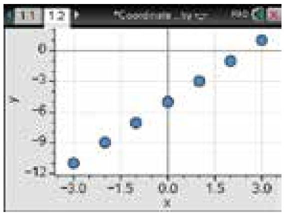
- Edit
- Fill Range
- OK.



	A	B	C
1	−3	−11	
2	−2	−9	
3	−1	−7	
4	0	−5	
5	1	−3	
6	2	−1	
7	3	1	

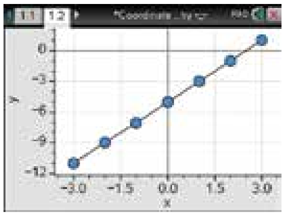
=2·A1−5
B1 −11

3 Open a Data & Statistics page. Press TAB `tab` to locate the label of the horizontal axis and select the variable *x*.
Press TAB `tab` again to locate the label of the vertical axis and select the variable *y*. The graph will be plotted as shown.



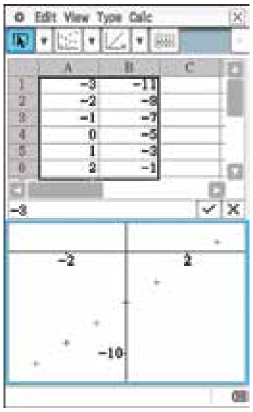
4 To join the dots with a line, press:

- MENU `menu`
- 2: Plot Properties `2`
- 1: Connect Data Points `1`.



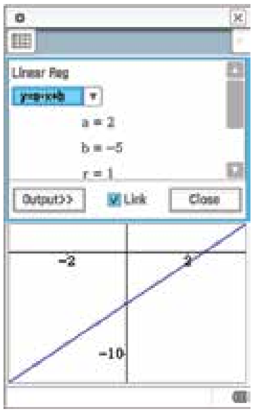
3 Highlight cells A1 to B7, then tap:

- Graph
- Scatter.



	A	B	C
1	−3	−11	
2	−2	−9	
3	−1	−7	
4	0	−5	
5	1	−3	
6	2	−1	

4 To join the dots with a line, tap the `Line` icon.
Note that the equation is given, if required.



Linear Reg
y=a·x+b
a = 2
b = −5
r = 1

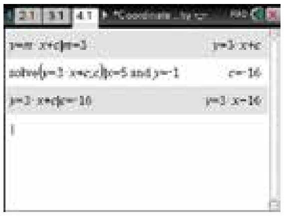
WORKED EXAMPLE 10

Find the equation of the straight line passing through the point (5, –1) with a gradient of 3.

TI | THINK

The equation can be found using a CAS calculator as follows.
In a new problem on a Calculator page, completed the entry lines as:
$y = m \times x + c | m = 3$
solve $(y = 3x + c, c) | x = 5$ and $y = -1$
$y = 3x + c | c = -16$
Press ENTER `enter` after each entry.

WRITE



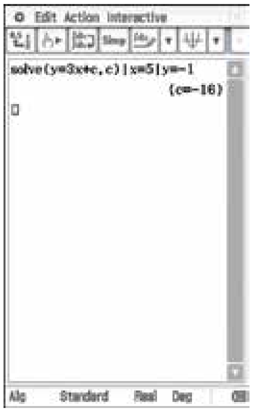
```
y=m·x+c|m=3                          y=3·x+c
solve(y=3·x+c,c)|x=5 and y=−1         c=−16
y=3·x+c|c=−16                        y=3·x−16
```

The equation is $y = 3x - 16$.

CASIO | THINK

On the Main screen, complete the entry lines as:
solve $(y = 3x + c, c) | x = 5 | y = -1$
Press EXE.

WRITE



```
solve(y=3x+c,c)|x=5|y=−1
                       {c=−16}
```

The equation is $y = 3x - 16$.

TOPIC 3

WORKED EXAMPLE 13

Find the distance between the points P(−1, 5) and Q(3, −2).

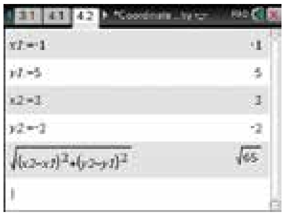
TI | THINK

On a Calculator page, complete the entry lines as:

$x1: = -1$
$y1: = 5$
$x2: = 3$
$y2: = -2$
$\sqrt{(x2 - x1)^2 + (y2 - y1)^2}$

Press ENTER [enter] after each entry.

WRITE


x1:=−1 −1
y1:=5 5
x2:=3 3
y2:=−2 −2
$\sqrt{(x2-x1)^2+(y2-y1)^2}$ $\sqrt{65}$

The distance between the two points is $\sqrt{65}$.

CASIO | THINK

On the Main screen, complete the entry lines as:

$\sqrt{(\boldsymbol{x} - \boldsymbol{a})^2 + (\boldsymbol{y} - \boldsymbol{b})^2} | \boldsymbol{x} = -1 | \ \boldsymbol{y} = 5 | \boldsymbol{a} = 3 | \boldsymbol{b} = -2$
$\boldsymbol{x}1 = \boldsymbol{x} = -1$
$\boldsymbol{y}1 = \boldsymbol{y} = 5$
$\boldsymbol{a} = \boldsymbol{x}2 = 3$
$\boldsymbol{b} = \boldsymbol{y}2 = -2$

WRITE

$\sqrt{(x-a)^2+(y-b)^2}|x=-1|y=5|$▶
$\sqrt{65}$

The distance between the two points is $\sqrt{65}$.

TOPIC 4

WORKED EXAMPLE 12

Find the point(s) of intersection between $y = x + 5$ and $y = \frac{6}{x}$

a algebraically **b graphically.**

TI | THINK

a 1 In a new problem, on a Calculator page, press:
- MENU [menu]
- 1: Actions [1]
- 1: Define [1].

Complete the entry line as:
Define $f1(x) = x + 5$

Repeat for the second function:

Define $f2(x) = \frac{6}{x}$

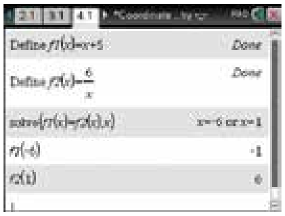
Press ENTER [enter] after each entry.

WRITE/DRAW


Define $f1(x)=x+5$ Done
Define $f2(x)=\frac{6}{x}$ Done

2 To find the intersection points algebraically, press:
- MENU [menu]
- 3: Algebra [3]
- 1: Solve [1].

Complete the entry line as:
solve $(f1(\boldsymbol{x}) = f2(\boldsymbol{x}), \boldsymbol{x})$


Define $f1(x)=x+5$ Done
Define $f2(x)=\frac{6}{x}$ Done
solve$(f1(x)=f2(x),x)$ x=−6 or x=1
$f1(-6)$ −1
$f1(1)$ 6

The points (−6, −1) and (1, 6) are the points of intersection.

CASIO | THINK

a Ensure the calculator is set to Standard mode.
On the Main screen, tap:
- Action
- Advanced
- solve.

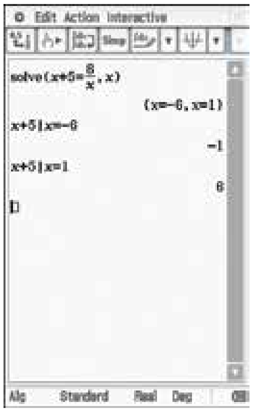
Complete the entry lines as:

$$\text{solve}\left(x + 5 = \frac{6}{x}, x\right)$$

Press EXE.

WRITE

a



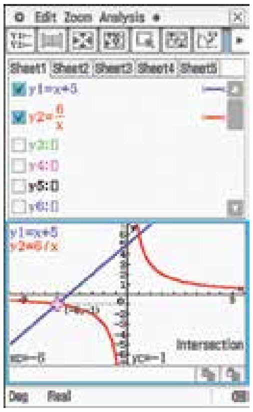
solve$(x+5=\frac{6}{x}, x)$
{x=−6, x=1}
x+5|x=−6
−1
x+5|x=1
6

The points of intersection are (−6, −1) and (1, 6).

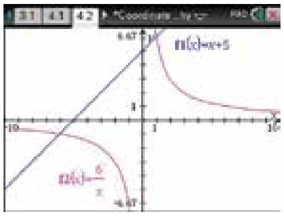
b In the Graph & Table page, complete the entry lines as:
$y1 = x + 5$
$y2 = \frac{6}{x}$
Then tap the graphing icon.
To find the points of intersection, tap:
- Analysis
- G-Solve
- Intersection.

To find the next point of inter-section, press the right arrow.

b



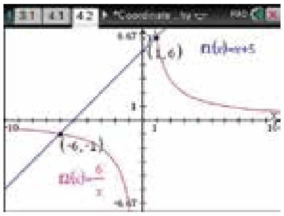
b **1** On a Graphs page, press the up arrow ▲ to select the function $f2(x)$, then press ENTER enter. The graph will be displayed. Now press TAB tab, select the function $f1(x)$ and press ENTER enter to draw the function. Apply colour if you would like to.



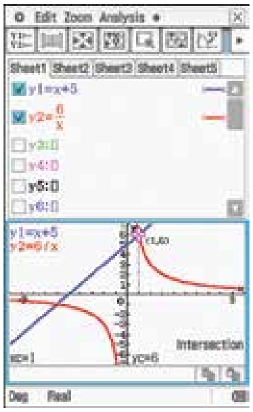
2 To find the points of intersection between the two graphs, press:

- MENU menu
- 6: Analyze Graph 6
- 4: Intersection 4.

Move the cursor to the left of one of the intersection points, press ENTER enter, then move the cursor to the right of this intersection point and press ENTER enter. The intersection point is displayed. Repeat for the other point of intersection.



The points $(-6, -1)$ and $(1, 6)$ are the points of intersection.



The points of intersection are $(-6, -1)$ and $(1, 6)$.

WORKED EXAMPLE 18

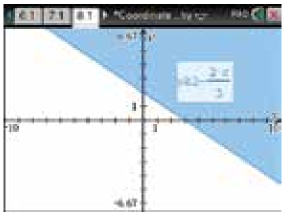
Identify the required region in the following pair of linear inequalities.

$2x + 3y \geq 6, y < 2x - 3$

TI | THINK

1 In a new problem, on a Graphs page at the function entry line, press the delete key del to delete the equals sign =. Complete the entry line as $y \geq 2 - \frac{2x}{3}$.
Then press ENTER enter.

WRITE/DRAW

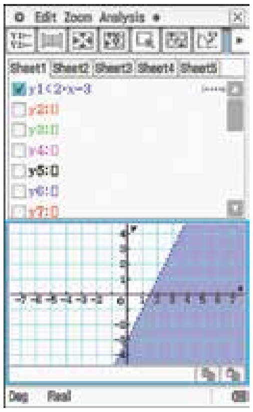


The graph region corresponding to $2x + 3y \geq 6$ is displayed.

CASIO | THINK

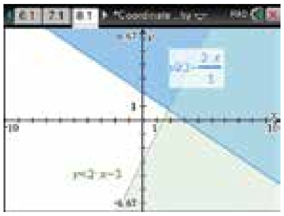
1 On the Graph & Table screen at the function entry line, type:
$y < 2x - 3$
To get the inequality tap ▼.
Tick the inequality.
Tap and the shaded region is displayed.

WRITE/DRAW



The graph region corresponding to $y < 2x - 3$ is displayed.

2 Press TAB tab. At the function entry line, press the delete key del to delete the equals sign =, then complete the entry line as $y < 2x - 3$. Then press ENTER enter. You may need to change the Line Colour and Fill Colour of this inequality to green to see the shaded region in dark green as shown.

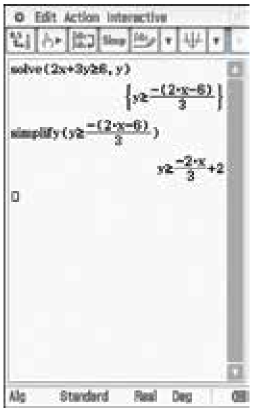


The shaded region indicated is the area corresponding to $2x + 3y \geq 6$ and $y < 2x - 3$.

2 On the Main screen, complete the entry line as: solve $(2x + 3y \geq 6, y)$ Highlight the previous answer and drag it to complete the entry line as:

$$\text{simplify}\left(y \geq \frac{-(2x - 6)}{3}\right)$$

Press EXE after each entry line.



solve$(2x+3y \geq 6, y)$

$$\left\{y \geq \frac{-(2 \cdot x - 6)}{3}\right\}$$

simplify$\left(y \geq \frac{-(2 \cdot x - 6)}{3}\right)$

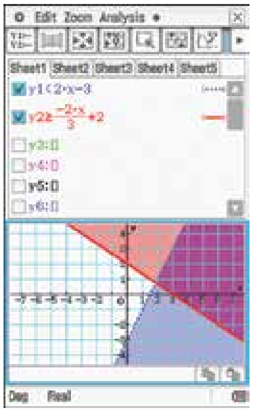
$$y \geq \frac{-2 \cdot x}{3} + 2$$

The inequality is given by $\left(y \geq \frac{-(2x - 6)}{3}\right)$

3 Go back to the Graph & Table screen and, at the function entry line, type:

$$y \geq -\frac{2x}{3} + 2$$

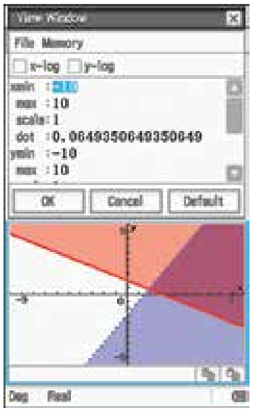
Tap and the shaded region is displayed.



☑ $y1 < 2 \cdot x - 3$
☑ $y2 \geq \frac{-2 \cdot x}{3} + 2$
☐ y3:□
☐ y4:□
☐ y5:□
☐ y6:□

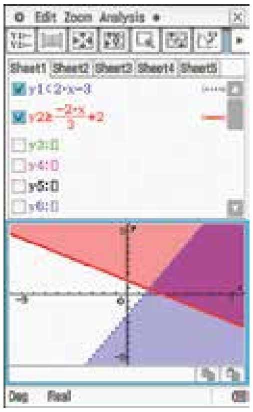
4 If the solution region is hard to see, fix this by setting an appropriate viewing window.

To do this, tap . Select the values as shown in the screenshot and tap OK.



xmin : -10
max : 10
scale : 1
dot : 0.0649350649350649
ymin : -10
max : 10

5 The darker shaded region to the top right is the area corresponding to $y < 2x - 3$ and $2x + 3y \geq 6$.

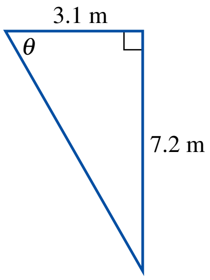


☑ $y1 < 2 \cdot x - 3$
☑ $y2 \geq \frac{-2 \cdot x}{3} + 2$
☐ y3:□
☐ y4:□
☐ y5:□
☐ y6:□

TOPIC 5

WORKED EXAMPLE 15

Find the size of angle θ:
a correct to the nearest second
b correct to the nearest minute.



TI | THINK

a On a Calculator page, in degree mode, complete the entry line as:

$\tan^{-1}\left(\frac{7.2}{3.1}\right)$

To convert the decimal degree into degrees, minutes and seconds, press:

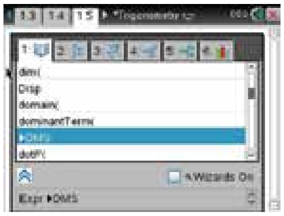
- CATALOG 🕮
- 1: 1
- d: D.

Scroll and select ▶ DMS. Then press ENTER enter.

b Using the same screen, round to the nearest minute.

WRITE/DRAW

a



```
tan⁻¹(7.2/3.1)                66.7054
(66.705436746106)▶DMS   66°42'19.5722"
```

$\theta = 66°42'20''$ correct to the nearest second.

b $\theta = 66°42'$ correct to the nearest minute.

CASIO | THINK

a On the Main screen, complete the entry line as:

$\tan^{-1}\left(\frac{7.2}{3.1}\right)$

To convert the decimal answer into degrees, minutes and seconds, tap:

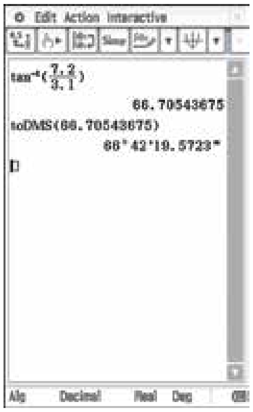
- Action
- Transformation
- DMS
- dms.

Highlight the decimal answer and drag across. Then press EXE.

b Using the same screen, round to the nearest minute.

WRITE/DRAW

a



```
tan⁻¹(7.2/3.1)
                66.70543675
toDMS(66.70543675)
            66°42'19.5723"
```

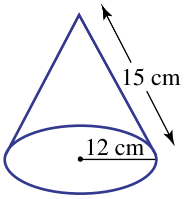
$\theta = 66°42'20''$ correct to the nearest second.

b $\theta = 66°42'$ correct to the nearest minute.

TOPIC 6

WORKED EXAMPLE 4

Find the total surface area of the cone shown.



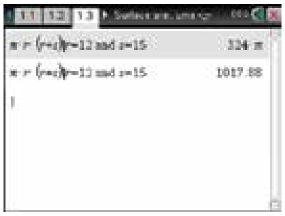
TI | THINK

On the Calculator page, complete the entry line as:

$\pi(r+s) \mid r = 12$ and $s = 15$

Press CTRL ctrl ENTER enter to get a decimal approximation.

WRITE



```
π·r·(r+s)|r=12 and s=15     324·π
π·r·(r+s)|r=12 and s=15    1017.88
```

The total surface area of the cone is $1017.9\,\text{cm}^2$ correct to 1 decimal place.

CASIO | THINK

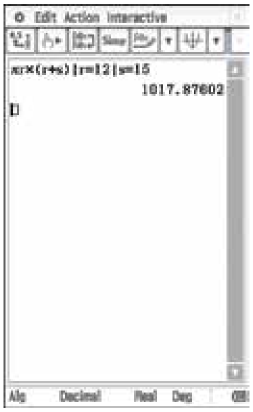
a-b On the Main screen, complete the entry lines as:

$\pi r(r+s) \mid r = 12 \mid s = 15$

From the bottom of the screen, tap the word 'standard' to change to decimal mode and an approximation.

WRITE

a-b



```
πr×(r+s)|r=12|s=15
                1017.87602
```

The total surface area of the cone is $1017.9\,\text{cm}^2$ correct to 1 decimal place.

WORKED EXAMPLE 3

Expand and simplify each of the following.

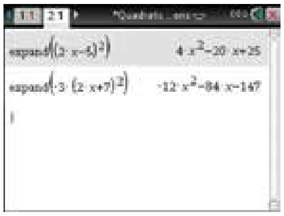
a $(2x-5)^2$ **b** $-3(2x+7)^2$

TI | THINK

a–b In a new problem, on a Calculator page, complete the entry lines as:
expand$(2x-5)^2$
expand$\left(-3(2x+7)^2\right)$
Press ENTER [enter] after each entry.

WRITE

a–b



expand$\left((2\cdot x-5)^2\right)$ $4\cdot x^2-20\cdot x+25$
expand$\left(-3\cdot(2\cdot x+7)^2\right)$ $-12\cdot x^2-84\cdot x-147$

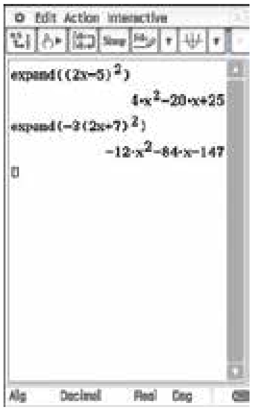
$$(2x-5)^2 = 4x^2-20x+25$$
$$-3(2x+7)^2 = -12x^2-84x-147$$

CASIO | THINK

a–b On the Main screen, complete the entry lines as:
expand$(2x-5)^2$
expand$\left(-3(2x+7)^2\right)$
Press EXE after each entry.

WRITE

a–b



expand$((2x-5)^2)$
$4\cdot x^2-20\cdot x+25$
expand$(-3(2x+7)^2)$
$-12\cdot x^2-84\cdot x-147$

$$(2x-5)^2 = 4x^2-20x+25$$
$$-3(2x+7)^2 = -12x^2-84x-147$$

WORKED EXAMPLE 11

Factorise the following by completing the square.

a x^2+4x+2 **b** x^2-9x+1

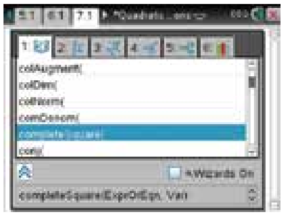
TI | THINK

a 1 In a new problem, on a Calculator page, to express a quadratic in the completing the square form, press:
- CATALOG [📖]
- 1: [1]
- C: [C]

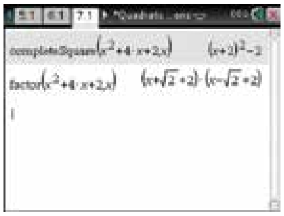
then scroll down, until completeSquare(and is highlighted, then press ENTER [enter]. Using the catalog is one method to show the syntax required for various commands.

WRITE

a



2 Complete the entry lines as:
completeSquare(x^2+4x+2, x)
factor(x^2+4x+2, x)
Press ENTER [enter] after each entry.



completeSquare$(x^2+4\cdot x+2,x)$ $(x+2)^2-2$
factor$(x^2+4\cdot x+2,x)$ $(x+\sqrt{2}+2)\cdot(x-\sqrt{2}+2)$

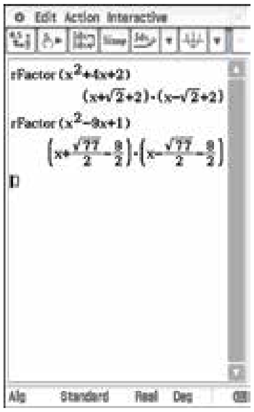
$$\begin{aligned} x^2+4x+2 &= (x+2)^2-2 \\ &= (x+\sqrt{2}+2)(x-\sqrt{2}+2) \end{aligned}$$

CASIO | THINK

a-b On the Main screen, complete the entry lines as:
rfactor (x^2+4x+2)
rfactor (x^2-9x+1)
Press EXE after each entry.

WRITE

a–b



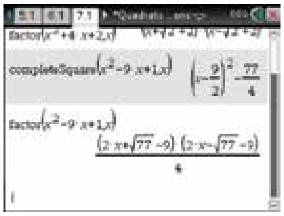
rFactor(x^2+4x+2)
$(x+\sqrt{2}+2)\cdot(x-\sqrt{2}+2)$
rFactor(x^2-9x+1)
$\left(x+\frac{\sqrt{77}}{2}-\frac{9}{2}\right)\cdot\left(x-\frac{\sqrt{77}}{2}-\frac{9}{2}\right)$

$$x^2+4x+2 = (x+\sqrt{2}+2)(x-\sqrt{2}+2)$$
$$x^2-9x+1 = \left(x+\frac{\sqrt{77}}{2}-\frac{9}{2}\right)\left(x-\frac{\sqrt{77}}{2}-\frac{9}{2}\right)$$

b Press:

- MENU menu
- 3: Algebra 3
- 5: Complete the Square 5.

Complete the entry lines as:
completeSquare($x^2 - 9x + 1, x$)
factor($x^2 - 9x + 1, x$)
Press ENTER enter after each entry.

b 

completeSquare$(x^2-9\cdot x+1,x)$ $\left(x-\frac{9}{2}\right)^2-\frac{77}{4}$

factor$(x^2-9\cdot x+1,x)$ $\frac{(2\cdot x+\sqrt{77}-9)\cdot(2\cdot x-\sqrt{77}-9)}{4}$

$$x^2 - 9x + 1 = \left(x - \frac{9}{2}\right)^2 - \frac{77}{4}$$
$$= \frac{(2x + \sqrt{77} - 9)(2x - \sqrt{77} - 9)}{4}$$

WORKED EXAMPLE 7

Determine the solution of each of the following quadratic equations by inspecting their corresponding graphs. Give answers to 1 decimal place where appropriate.

a $x^2 + x - 2 = 0$ **b** $2x^2 - 4x - 5 = 0$

TI | THINK

a In a new problem, on a Graphs page, complete the function entry line as:
$f1(x) = x^2 + x - 2$
Then press ENTER enter. The graph will be displayed.
To find the x-intercepts, press:

- MENU menu
- 6: Analyze Graph 1
- 1: Zero 1.

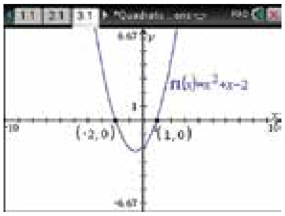
Move the cursor to the left of the zero, press ENTER enter, then move the cursor to the right of the zero and press ENTER enter. The coordinates of the x-intercept are displayed. Press ENTER enter to fix the coordinates on the graph. Repeat for the other x-intercept.

b On a Graphs page, complete the function entry line as:
$f1(x) = 2x^2 - 4x - 5$
Then press ENTER enter. The graph will be displayed.
To find the x-intercepts, press:

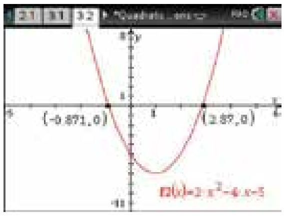
- MENU menu
- 6: Analyze Graph 1
- 1: Zero 1.

Move the cursor to the left of the zero, press ENTER enter, then move the cursor to the right of the zero and press ENTER enter. The coordinates of the x-intercept are displayed. Press ENTER enter to fix the coordinates on the graph. Repeat for the other x-intercept.

WRITE/DRAW

a 

$x^2 + x - 2 = 0$
$\Rightarrow x = 1$ or -2

b 

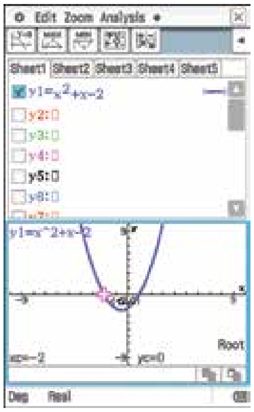
$2x^2 - 4x - 5 = 0$
$\Rightarrow x \approx -0.9$ or 2.9 correct to 1 decimal place.

CASIO | THINK

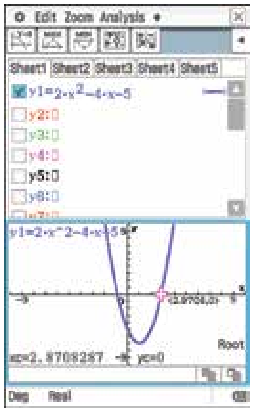
a On the Graph & Table screen, complete the function entry line as:
$y1 = x^2 + x - 2$
Press EXE.
Then tap the icon. The graph will be displayed. To find the x-intercepts, tap the icon.
To find the second root, tap the right arrow.
Note: Ensure that the graph arrows have been activated in the 'graph format' menu.

b On the Graph & Table screen, complete the function entry line as:
$y1 = 2x^2 - 4x - 5$
Press EXE.
Then tap the icon. The graph will be displayed. To find the x-intercepts, tap the icon.
To find the second root, tap the right arrow.

WRITE/DRAW

a 

$x^2 + x - 2 = 0$
$\Rightarrow x = 1$ or -2

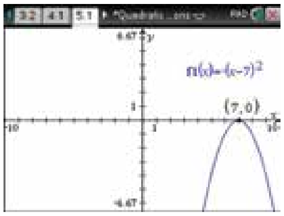
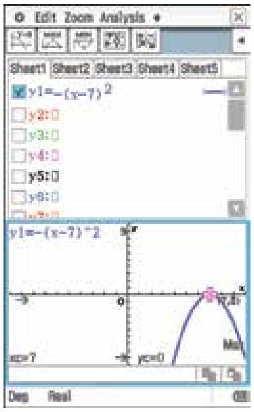
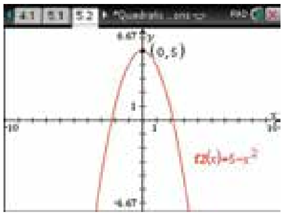
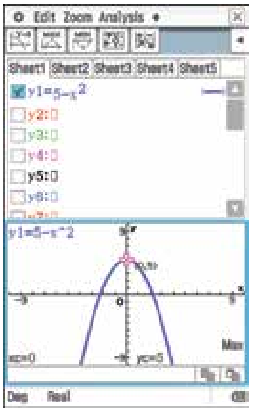
b 

$2x^2 - 4x - 5 = 0$
$\Rightarrow x \approx -0.9$ or 2.9 correct to 1 decimal place.

WORKED EXAMPLE 6

For each of the following graphs, give the coordinates of the turning point and state whether it is a maximum or a minimum.

a $y = -(x - 7)^2$ **b** $y = 5 - x^2$

TI | THINK	**WRITE**	**CASIO | THINK**	**WRITE**
a In a new problem, on a Graphs page, complete the function entry line as: $f1(x) = -(x - 7)^2$ Then press ENTER enter.	**a**  The turning point (7, 0) is a maximum.	**a** On the Graph & Table screen, complete the function entry line as: $y1 = -(x - 7)^2$ Tick the $y1$ box and tap the icon. To find the turning point, tap the MAX icon.	**a**  The turning point (7, 0) is a maximum.
b In a new problem, on a new Graphs page, complete the function entry line as: $f1(x) = 5 - x^2$ Then press ENTER enter.	**b**  The turning point (0, 5) is a maximum.	**b** On the Graph & Table screen, complete the function entry line as: $y1 = 5 - x^2$ Tick the $y1$ box and tap . To find the turning point, tap the MAX icon.	**b**  The turning point (0, 5) is a maximum.

TOPIC 12

WORKED EXAMPLE 9

The number of lollies in each of 8 packets is 11, 12, 13, 14, 16, 17, 18, 19.

Calculate the mean and standard deviation correct to 2 decimal places.

TI | THINK

In a new problem, on a Calculator page, complete the entry lines as shown. This stores the data values to the variable *'lollies'*.

lollies := {11, 12, 13, 14, 15, 16, 17, 18, 19}

Although we can find many summary statistics, to find the mean only, open a Calculator page and press:

- MENU menu
- 6: Statistics 6
- 3: List Math 3
- 3: Mean 3.

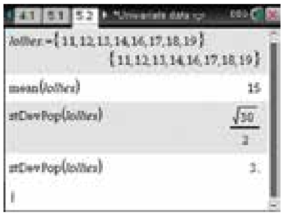
Press var var and select *'lollies'*, then press ENTER enter.

To find the population standard deviation only, press:

- MENU menu
- 6: Statistics 6
- 3: List Math 3
- 9: Population standard deviation 9.

Press var var and select *'lollies'*, then press ENTER enter. Press CTRL/ENTER ctrl enter to get a decimal approximation.

WRITE



```
lollies:={11,12,13,14,16,17,18,19}
                    {11,12,13,14,16,17,18,19}
mean(lollies)                              15
stDevPop(lollies)                        √30/2
stDevPop(lollies)                          3.
```

The mean number of lollies is 15 and the population standard deviation is $\sigma = 2.74$.

CASIO | THINK

On the Statistics screen, label list1 as 'x' and enter the values from the question.

Press EXE after entering each value.

To find the summary statistics, tap:

- Calc
- One-Variable.

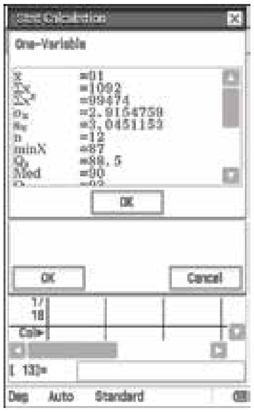
Set values as:

- XList: main\x
- Freq: 1.

Tap OK.

The standard deviation is shown as σ_x and the mean is shown as $\bar{x}$.

WRITE



```
x̄     =91
Σx    =1092
Σx²   =99474
σx    =2.9154759
sx    =3.0451153
n     =12
minX  =87
Q1    =88.5
Med   =90
```

The mean number of lollies is 15 with a standard deviation of 2.74.

TOPIC 12

WORKED EXAMPLE 14

Below are the scores for two students in eight Mathematics tests throughout the year:

John: 45, 62, 64, 55, 58, 51, 59, 62

Penny: 84, 37, 45, 80, 74, 44, 46, 50

a **Use the statistics function on a calculator to find the mean and standard deviation for each student.**

b **Which student had the better overall performance on the eight tests?**

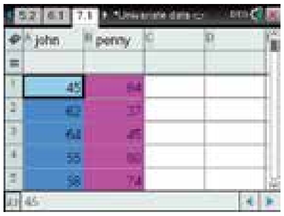
c **Which student was more consistent over the eight tests?**

TI | THINK

a **1** In a new problem, on a Lists & Spreadsheet page, label column A as 'john' and column B as 'penny'. Enter the data sets from the question. Press ENTER enter after each value.

WRITE

a



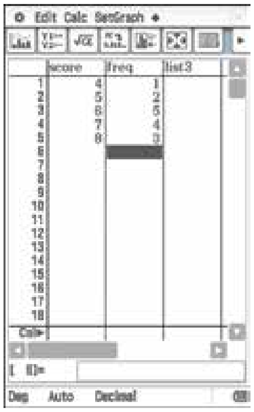
	john	penny	C	D
1	45	84		
2	62	37		
3	64	45		
4	55	80		
5	58	74		

CASIO | THINK

a **1** On the Statistics screen, label list1 as 'John' and list 2 as 'Penny'. Enter the data set as shown. Press EXE after each value.

WRITE

a



	score	freq	list3
1	4	1	
2	5	2	
3	6	5	
4	7	4	
5	8	3	

2 To find only the mean and standard deviation of each data set, open a Calculator page and complete the entry lines as:
mean(*john*)
stDevPop(*john*)
mean(*penny*)
stDevPop(*penny*)
Press CTRL ctrl ENTER enter after each entry to get a decimal approximation.

```
mean(john)                57
stDevPop(john)             6
mean(penny)             57.5
stDevPop(penny)  17.4212513902
```

John: $\overline{x} = 57, \sigma = 6$
Penny: $\overline{x} = 57.5, \sigma = 17.42$
correct to 2 decimal places.

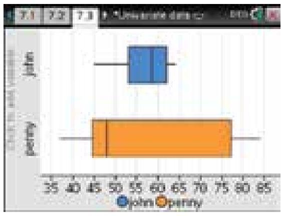
b–c To draw the two boxplots on the same Data & Statistics page, press TAB tab to locate the label of the horizontal axis and select the variable ‘john’. Then press:
- MENU menu
- 1: Plot Type 1
- 2: Box Plot 2.

Then press:
- MENU menu
- 2: Plot properties 2
- 5: Add X-variable 5

and select ‘penny’.
To change the colour of each boxplot, press CTRL ctrl MENU menu.
Then press:
- 6: Color 6
- 2: Fill Color 2.

Select whichever colour you like from the palette for each of the boxplots.

b–c

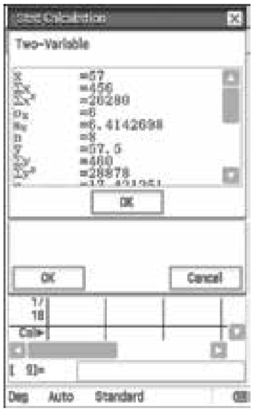


Penny performed slightly better overall as her mean mark was higher than John’s; however, John was more consistent as his standard deviation was lower than Penny’s.

2 To find only the mean and standard deviation of each data set, tap:
- Calc
- Two-Variable.

Set values as:
- XList: main\john
- YList: main\penny
- Freq: 1.

Tap OK.
The *x*-values relate to John and the *y*-values to Penny. Scroll down to see all the statistics.



```
Stat Calculation
Two-Variable
x̄     =57
Σx    =456
Σx²   =26280
σx    =6
sx    =6.4142698
n     =8
ȳ     =57.5
Σy    =460
Σy²   =28878
```

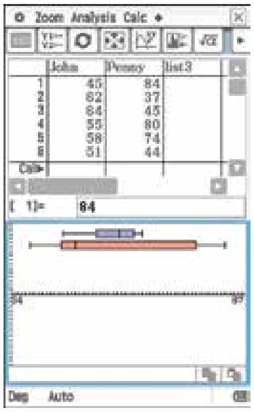
John: $\overline{x} = 57, \sigma = 6$
Penny: $\overline{x} = 57.5, \sigma = 17.42$
correct to 2 decimal places.

b–c To draw the two box-and-whisker plots on the same Statistics screen, tap:
-
- Settings
- 1.

Set values as:
- Type: MedBox
- XList: main\john
- Freq: 1.

Tap Set.
Repeat the process for Penny, but after tapping Settings, tap 2 and set XList: main\penny.
Tap Setgraph.
Tick StatGraph1 and StatGraph2.

b–c



	John	Penny	list3
1	45	84	
2	62	37	
3	64	45	
4	55	80	
5	58	74	
6	51	44	

Penny performed slightly better overall as her mean mark was higher than John’s; however, John was more consistent as his standard deviation was lower than Penny’s.

WORKED EXAMPLE 3

Mary sells business shirts in a department store. She always records the number of different styles of shirt sold during the day. The table below shows her sales over one week.

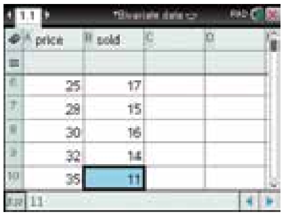
Price($)	14	18	20	21	24	25	28	30	32	35
Number of shirts sold	21	22	18	19	17	17	15	16	14	11

a **Construct a scatterplot of the data.**
b **State the type of correlation between the two variables and, hence, draw a corresponding conclusion.**

TI | THINK **WRITE/DRAW**

a–b **1** In a new document, on a Lists & Spreadsheet page, label column A as ‘price’ and label column B as ‘sold’. Enter the data as shown in the table.

a–b



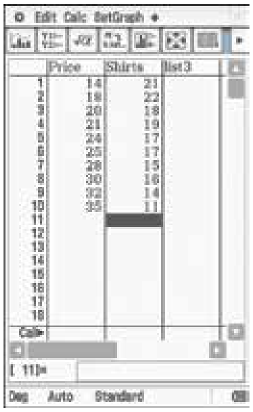
	A price	B sold	C	D
6	25	17		
7	28	15		
8	30	16		
9	32	14		
10	35	11		

B10 11

2 Open a Data & Statistics page. Press TAB `tab` to locate the label of the horizontal axis and select the variable ‘price’. Press TAB `tab` again to locate the label of the vertical axis and select the variable ‘sold’.

CASIO | THINK **WRITE/DRAW**

a–b **1** On the Statistic screen, label list1 as ‘Price’ and list 2 as ‘Shirts’, then enter the values from the question.
Press EXE after entering each value.



	Price	Shirts	list3
1	14	21	
2	18	22	
3	20	18	
4	21	19	
5	24	17	
6	25	17	
7	28	15	
8	30	16	
9	32	14	
10	35	11	

2 Tap:
- SetGraph
- Settings.

Set values as shown in the screenshot, then tap:
- Set
- y.

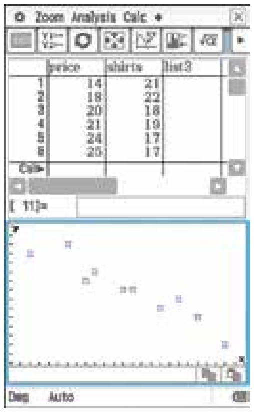
Tap OK.
The standard deviation is shown.

Draw: ● On ○ Off
Type: Scatter
XList: main\Price
YList: main\Shirts
Freq: 1
Mark: square

3 To change the colour of the scatterplot, place the pointer over one of the data points. Then press CTRL `ctrl` MENU `menu`.
Press:
3: Colour `3`
2: Fill Colour `2`.
Select a colour from the palette for the scatterplot. Press ENTER `enter`.

The scatterplot is shown, using a suitable scale for both axes. The points are close to forming a straight line. There is a strong negative, linear correlation between the two variables. The trend indicates that the price of a shirt appears to affect the number sold; that is, the more expensive the shirt, the fewer are sold.

3 The scatterplot will appear.



	price	shirts	list3
1	14	21	
2	18	22	
3	20	18	
4	21	19	
5	24	17	
6	25	17	

[11]=

The scatterplot is shown, using a suitable scale for both axes. The points are close to forming a straight line. There is a strong negative, linear correlation between the two variables. The trend indicates that the price of a shirt appears to affect the number sold; that is, the more expensive the shirt, the fewer are sold.

WORKED EXAMPLE 7

The percentages from two tests (English and Maths) for a group of 5 students are as shown.

Student	English (%)	Maths (%)
1	95	85
2	85	95
3	80	70
4	70	65
5	60	70

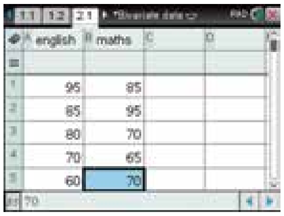
a Calculate the correlation coefficient between the two sets of results.
b Based on this value, describe the relationship between the English and Maths results for this group of students.

TI | THINK

a 1 In a new problem, on a Lists & Spreadsheet page, label column A as 'english' and label column B as 'maths'. Enter the data as in the table.

WRITE

a



	A english	B maths	C
1	95	85	
2	85	95	
3	80	70	
4	70	65	
5	60	70	

B5 70

CASIO | THINK

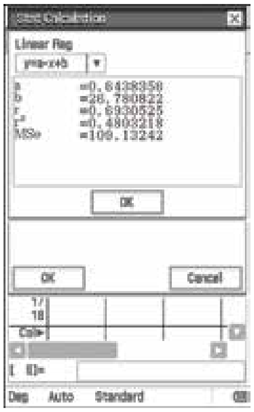
a On the Statistic screen, label list1 as 'English' and list 2 as 'Maths', then enter the values from the question. Press EXE after entering each value.
To find the correlation coefficient, tap:
- Calc
- Regression
- Linear Reg.

Set values as:
- XList: main\English
- YList: main\Maths
- Freq: 1.

Tap OK.
The correlation coefficient is represented by the 'r' value.

WRITE

a



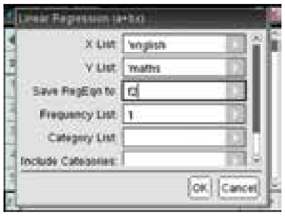
Linear Reg
y=a·x+b

a =0.6438356
b =26.780822
r =0.6930525
r^2 =0.4803218
MSe =109.13242

The correlation coefficient is 0.69.

2 Open a Calculator page.
Press:
- MENU menu
- 4: Statistics 4
- 1: Stat Calculations 1
- 4: Linear Regression (a + bx) 4.

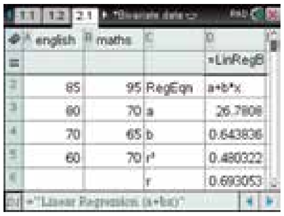
Select 'english' as the X List and 'maths' as the Y List, and leave the next fields as shown. TAB tab to OK and press ENTER enter.



X List: 'english
Y List: 'maths
Save RegEqn to: f2
Frequency List: 1
Category List:
Include Categories:

b The value of the correlation is shown as r, and its value is stored in the variable *stat*. *r*.

b



	A english	B maths	C	D
=				=LinRegB
2	85	95	RegEqn	a+b*x
3	80	70	a	26.7806
4	70	65	b	0.643836
5	60	70	r²	0.480322
6			r	0.693053

D1 ="Linear Regression (a+bx)"

A correlation coefficient of 0.69 indicates the relationship between English and Maths marks for this group of students is only moderate. This seems to indicate that students who are good at English are not necessarily good at Maths, and vice versa.

b

b The correlation coefficient of 0.69 is only a moderate correlation. Therefore, students who do well in English don't necessarily do well in Maths and vice versa.

WORKED EXAMPLE 11

The Australian women's national basketball team, the Opals, competed at the 2008 Olympic Games in Beijing, winning a silver medal. These are the heights (in metres) of the 12 team members:

1.73, 1.65, 1.8, 1.83, 1.96, 1.88, 1.63, 1.88, 1.83, 1.88, 1.8, 1.96

Provide calculations and explanations as evidence to verify or refute the following statements.

a The mean height of the team is greater than their median height.

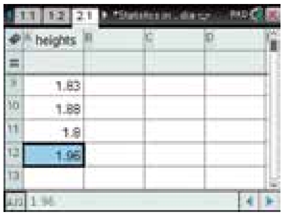
b The range of the heights of the 12 players is almost 3 times their interquartile range.

c Only 5 players are on the court at any one time. A team of 5 players can be chosen such that their mean, median and modal heights are all the same.

TI | THINK **WRITE**

a 1 In a new problem, on a Lists & Spreadsheet page, label column A as 'heights'. Enter the data from the question.

a



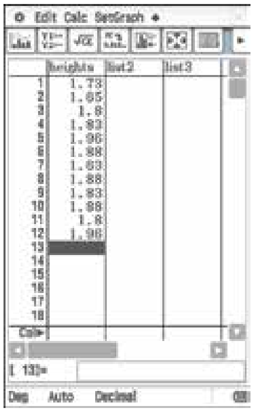
	A heights	B	C	D
=				
9	1.83			
10	1.88			
11	1.8			
12	1.96			
13				

A12 1.96

CASIO | THINK **WRITE**

a On the Statistics screen, label list1 as 'heights'. Enter the data in the table as shown.
Press EXE after each value.

a



	heights	list2	list3
1	1.73		
2	1.65		
3	1.8		
4	1.83		
5	1.96		
6	1.88		
7	1.63		
8	1.88		
9	1.83		
10	1.88		
11	1.8		
12	1.96		
13			
14			
15			
16			
17			
18			

[13]=

2 Open a Calculator page and complete the entry lines as:
mean(*heights*)
median(*heights*)
Press ENTER `enter` after each entry.

```
mean(heights)      1.81917
median(heights)    1.83
```

The mean heights are less than the median heights, so the statement is false.

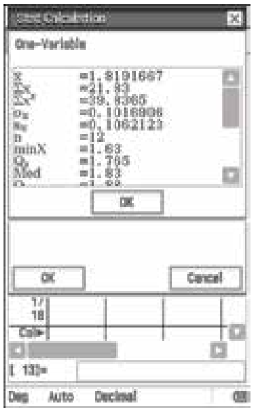
To find the statistics summary, tap:

- Calc
- One-Variable.

Set values as:

- XList: main\height
- Freq: 1.

Tap OK.
Scroll down to find the median.



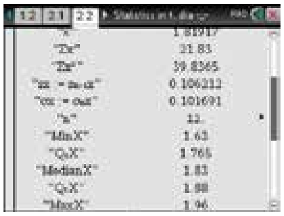
```
One-Variable
x̄     =1.8191667
Σx    =21.83
Σx²   =39.8365
σx    =0.1016806
sx    =0.1062123
n     =12
minX  =1.63
Q1    =1.765
Med   =1.83
```

The mean height (1.82 m) is less than the median height (1.83 m), so the statement is false.

b To find all the summary statistics, open the Calculator page and press:

- MENU `menu`
- 6: Statistics `6`
- 1: Stat Calculations `1`
- 1: One-Variable Statistics `1`.

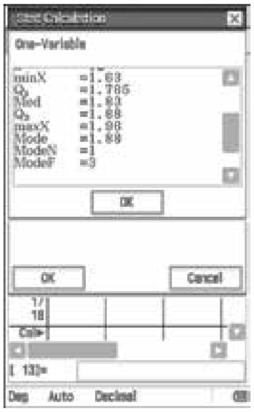
Select 1 as the number of lists. Then on the One-Variable Statistics page, select ‘heights’ as the X1 List and leave the frequency as 1. Leave the next two fields empty and TAB `tab` to OK and then press ENTER `enter`.

b 

```
"x̄"                 1.81917
"Σx"                21.83
"Σx²"               39.8365
"sx := sn-1x"       0.106212
"σx := σnx"         0.101691
"n"                 12.
"MinX"              1.63
"Q1X"               1.765
"MedianX"           1.83
"Q3X"               1.88
"MaxX"              1.96
```

The range is max − min $= 1.96 - 1.63 = 0.33Q1$ $= 1.765$ and $Q3 = 1.88$
$IQR = Q3 - Q1 = 1.88 - 1.765$ $= 0.115$.
Now $2.\,91QR \approx$ range so the statement is true.

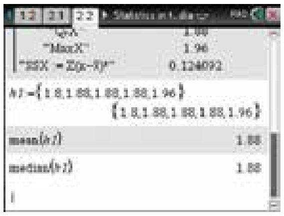
b More statistics can be found from the statistics summary.

b 

```
One-Variable
minX  =1.63
Q1    =1.765
Med   =1.83
Q3    =1.88
maxX  =1.96
Mode  =1.88
ModeN =1
ModeF =3
```

The range is max − min $= 1.96 - 1.63 = 0.33$
$Q1 = 1.765$ and $Q3 = 1.88$
$IQR = Q3 - Q1$
$= 1.88 - 1.765 = 0.115$.
Now $2.\,91QR \approx$ range so the statement is true.

c Note that the mode is 1.88 as three players have this height. If you chose one player shorter and one player taller than the mode by the same amount, then the 5 heights chosen from this sample will have their mode, median and mean all equal.
To verify this, open a Calculator page and complete the entry lines as:
h1: = {1.8, 1.88, 1.88, 1.88, 1.96}
mean(*h*1)
median(*h*1)
Press ENTER `enter` after each entry.

c 

```
"Q3X"                  1.88
"MaxX"                 1.96
"SSX := Σ(x-x̄)²"       0.124092
h1:={1.8,1.88,1.88,1.88,1.96}
                 {1.8,1.88,1.88,1.88,1.96}
mean(h1)               1.88
median(h1)             1.88
```

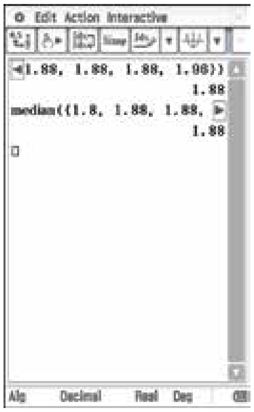
The mode, median and mean of the sample chosen are all equal.

c Note that the mode is 1.88 as three players have this height.
If we chose one player shorter and one player larger than the mode by the same amount, then the 5 chosen from this sample will have their mode, median and mean all equal.
To verify this complete the entry lines as:

$$\text{mean}\left(\{1.8, 1.88, 1.88, 1.88, 1.96\}\right)$$

$$\text{median}\left(\{1.8, 1.88, 1.88, 1.88, 1.96\}\right)$$

Press EXE after each entry.

c 

```
◄1.88, 1.88, 1.88, 1.96})
                        1.88
median({1.8, 1.88, 1.88, ►
                        1.88
```

The mode, median and mean of the sample chosen are all equal.

TOPIC 15

WORKED EXAMPLE 9

Calculate the future value of an investment of $4000 at 6% p.a. for 2 years with interest compounded quarterly.

TI | THINK

Use the finance functions available on the calculator for this question.
On a Calculator page, press:

- MENU `menu`
- 8: Finance `8`
- 2: TVM Functions `2`
- 5: Future Value `5`.

Complete the entry line as:
tvmFV (8, 1.5, 4000, 0)
Press ENTER `enter`.
Note that the number of compounding periods is 8, that is 4 times a year for 2 years, and the interest is $\frac{6}{4} = 1.5\%$ quarterly.

WRITE

tvmFV(8,1.5,4000,0)   -4505.97

The future value is $4505.97.

CASIO | THINK

On the Financial screen, select:

- Compound Interest.

Enter the values as shown in the screenshot.
The FV is left blank. Tap it and it will calculate the value.
Note that the number of compounding periods is 8, that is 4 times a year for 2 years, and the interest is $\frac{6}{4} = 1.5\%$ quarterly.

WRITE

Compound Interest
N 8
I% 1.5
PV 4000
PMT 0
FV -4505.970348
P/Y 1
C/Y 1

The future value is $4505.97.

TOPIC 16

WORKED EXAMPLE 23

Evaluate $\log_5 35 + \log_5 15 - \log_5 21$.

TI | THINK

On a Calculator page, press CTRL `ctrl` log `10x` and complete the entry line as:
$\log_5 35 + \log_5 15 - \log_5 21$
Then press ENTER `enter`.

WRITE

$\log_5(35)+\log_5(15)-\log_5(21)$ 2

$\log_5 35 + \log_5 15 - \log_5 21 = 2$

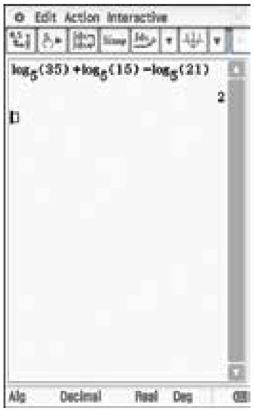
CASIO | THINK

On the `Math1` keyboard screen, tap:

- `log■□`.

Complete the entry line as:
$\log_5(35) + \log_5(15) - \log_5(21)$
Then press EXE.

WRITE



$\log_5(35)+\log_5(15)-\log_5(21)$
2

$\log_5 35 + \log_5 15 - \log_5 21 = 2$

TOPIC 17

WORKED EXAMPLE 6

If $P(x) = 2x^3 + x^2 - 3x - 4$, find:

a $P(1)$ **b** $P(-2)$ **c** $P(a)$ **d** $P(2b)$ **e** $P(x + 1)$

TI | THINK

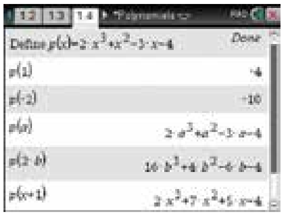
a–e On a Calculator page, press:

- MENU $\boxed{\text{menu}}$
- 1: Actions $\boxed{1}$
- 1: Define $\boxed{1}$.

Complete the entry lines as:
Define
$p(x) = 2x^3 + x^2 - 3x - 4$
$p(1)$
$p(-2)$
$p(a)$
$p(2b)$
$p(x + 1)$
Press ENTER $\boxed{\text{enter}}$ after each entry.

WRITE

a–e



Define $p(x)=2\cdot x^3+x^2-3\cdot x-4$ Done
$p(1)$ -4
$p(-2)$ -10
$p(a)$ $2\cdot a^3+a^2-3\cdot a-4$
$p(2\cdot b)$ $16\cdot b^3+4\cdot b^2-6\cdot b-4$
$p(x+1)$ $2\cdot x^3+7\cdot x^2+5\cdot x-4$

$P(1) = -4$
$P(-2) = -10$
$P(a) = 2a^3 + a^2 - 3a - 4$
$P(2b) = 16b^3 + 4b^2 - 6b - 4$
$P(x + 1) = 2x^3 + 7x^2 + 5x - 4$

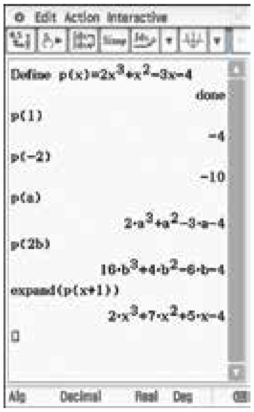
CASIO | THINK

a–e On the Main screen, complete the entry lines as:
Define
$p(x) = 2x^3 + x^2 - 3x - 4$
$p(1)$
$p(-2)$
$p(a)$
$p(2b)$
$p(x + 1)$
expand$(p(x + 1))$

Press EXE after each entry.

WRITE

a–e



Define $p(x)=2x^3+x^2-3x-4$
done
$p(1)$
-4
$p(-2)$
-10
$p(a)$
$2\cdot a^3+a^2-3\cdot a-4$
$p(2b)$
$16\cdot b^3+4\cdot b^2-6\cdot b-4$
expand$(p(x+1))$
$2\cdot x^3+7\cdot x^2+5\cdot x-4$

$P(1) = -4$
$P(-2) = -10$
$P(a) = 2a^3 + a^2 - 3a - 4$
$P(2b) = 16b^3 + 4b^2 - 6b - 4$
$P(x + 1) = 2x^3 + 7x^2 + 5x - 4$

TOPIC 18

WORKED EXAMPLE 5

Find any points of intersection between $= f(x) = 2x + 1$ *and* $g(x)\frac{1}{x}$.

TI | THINK

1 In a new problem, on a Calculator page, press:

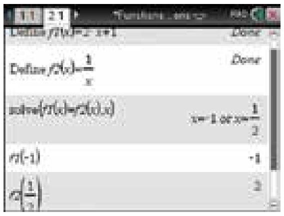
- MENU $\boxed{\text{menu}}$
- 1: Actions $\boxed{1}$
- 1: Define $\boxed{1}$.

Complete the entry lines as:
Define $f1(x) = 2x + 1$
Define $f2(x) = \frac{1}{x}$
Press ENTER $\boxed{\text{enter}}$ after each entry.
Then press:

- MENU $\boxed{\text{menu}}$
- 3: Algebra $\boxed{3}$
- 1: Solve $\boxed{1}$.

Complete the entry lines as:
solve $(f1(x) = f2(x), x)$
$f1(-1)$
$f2\left(\frac{1}{2}\right)$
Press ENTER $\boxed{\text{enter}}$ after each entry.

WRITE



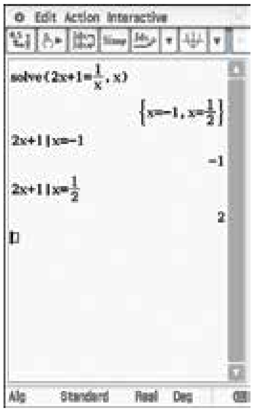
Define $f1(x)=2\cdot x+1$ Done
Define $f2(x)=\frac{1}{x}$ Done
solve$(f1(x)=f2(x),x)$ $x=-1$ or $x=\frac{1}{2}$
$f1(-1)$ -1
$f2\left(\frac{1}{2}\right)$ 2

The points of intersection are $(-1, -1)$ and $\left(\frac{1}{2}, 2\right)$.

CASIO | THINK

On the Main screen, complete the entry lines as:
$2x + 1|x = -1$
$2x + 1|x = \frac{1}{2}$
solve $\left(2x + 1 = \frac{1}{x}, x\right)$
Press EXE.

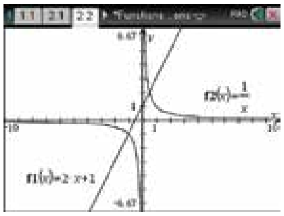
WRITE



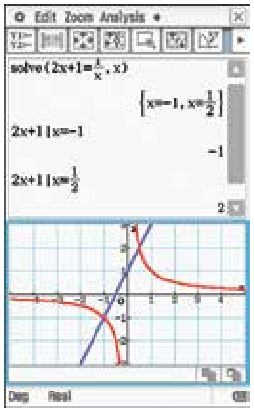
solve$\left(2x+1=\frac{1}{x},x\right)$
$\left\{x=-1, x=\frac{1}{2}\right\}$
$2x+1|x=-1$
-1
$2x+1|x=\frac{1}{2}$
2

The points of intersection are $(-1, -1)$ and $\left(\frac{1}{2}, 2\right)$.

2 Alternatively, open a Graphs page in the current document. Since the functions have already been entered, just select the functions and press ENTER enter. The graphs will be displayed.



Alternatively, tap . Highlight each equation and drag it down to the axis below. The window may need to be adjusted by tapping .



solve$(2x+1=\frac{1}{x},x)$

$\{x=-1, x=\frac{1}{2}\}$

$2x+1|x=-1$

-1

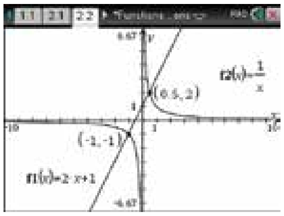
$2x+1|x=\frac{1}{2}$

2

3 The viewing window may need to be altered to see the graphs more clearly. To find the points of intersection between the two functions, press:

- MENU menu
- 6: Analyze Graph 6
- 4: Intersection 4.

Move the cursor to the left of the intersection point, and press ENTER enter. Then move the cursor to the right of the intersection point and press ENTER enter. The intersection point is displayed.
Repeat for the other intersection point.

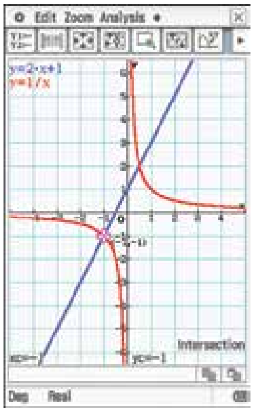


The points of intersection are (−1, −1) and (0.5, 2).

To find the points of intersection, tap:

- Resize
- Analysis
- G-Solve
- Intersection.

To find the other intersection point, tap the right arrow.



The points of intersection are (−1, −1) and (0.5, 2).

WORKED EXAMPLE 12

Sketch the graphs of a $y = 2\sin x$ and b $y = \cos 2x$ for $0° \le x \le 360°$.

TI | THINK

a 1 In a new problem, on a Graphs page, ensure the Graphs & Geometry Settings are set to the degrees mode, as shown in the screenshot. To do this, press:

- MENU menu
- 9: Settings 9.

TAB tab to Graphing Angle and select ‘Degree’. TAB tab to OK and press ENTER enter.

WRITE

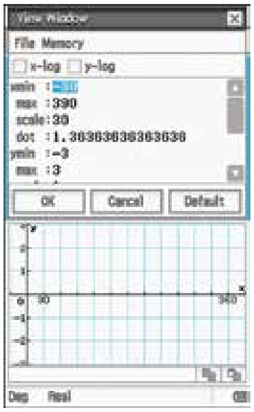
a

Display Digits: Fix 2
Graphing Angle: Degree
Geometry Angle: Degree
☐ Automatically hide plot labels
☑ Show axis and values

CASIO | THINK

On a Graph & Table screen, set an appropriate viewing window as shown.

WRITE

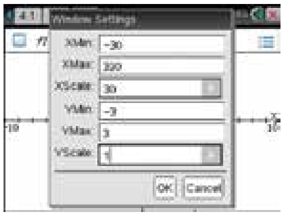


☐ x-log ☐ y-log
xmin :
max :390
scale :30
dot :1.30303030303030
ymin :−3
max :3

2 To set an appropriate viewing window, press:

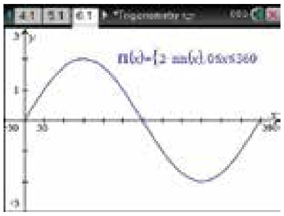
- MENU `menu`
- 4: Window/Zoom `4`
- 1: Window Settings `1`.

Select the values as shown in the screenshot.
TAB `tab` to OK and press ENTER `enter`.



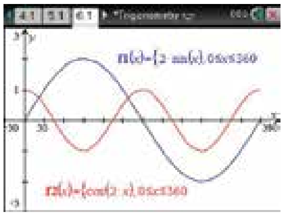
Window Settings
XMin: −30
XMax: 390
XScale: 30
YMin: −3
YMax: 3
YScale: 1

3 Complete the function entry line as:
$f1 = 2\sin(x)\,|\,0 \le x \le 360$
Press ENTER `enter`. The graph is displayed as required for $0° \le x \le 360°$.



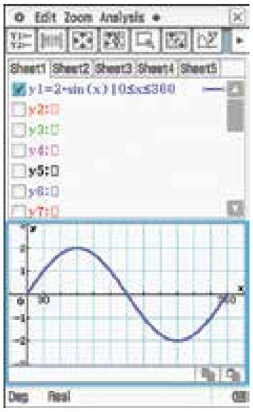
b Complete the function entry line as:
$f1 = \cos(2x)\,|\,0 \le x \le 360$
Press ENTER `enter`, and the graph is displayed, as required only for $0° \le x \le 360°$.

b



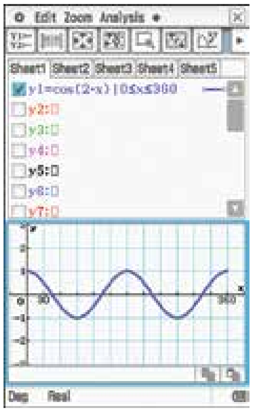
a Complete the function entry line as:
$y1 = 2\sin(x)\,|\,0 \le x \le 360$
Press EXE.
Tap the graph tab and the graph is displayed as required only for $0° \le x \le 360°$.

a



b Complete the function entry line as:
$y1 = \cos(2x)\,|\,0 \le x \le 360$
Press EXE.
Tap the graph tab and the graph is displayed as required only for $0° \le x \le 360°$.

b



GLOSSARY

3-dimensional: a shape that occupies space (a solid); that is, one that has dimensions in three directions — length, width and height

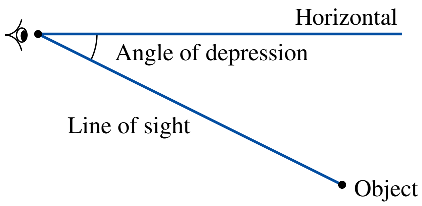
Addition Law of probability: if A and B are mutually exclusive events, then $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B)$ or $\mathrm{P}(A \cup B) = \mathrm{P}(A) + P(B)$.

algebraic fractions: fractions that contain pronumerals (letters)

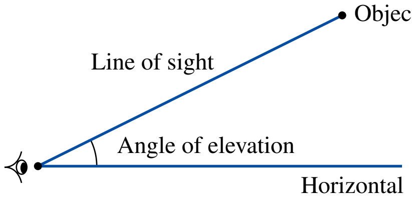
algorithm: a step-by-step set of tasks to solve a particular problem. A program is an implementation of an algorithm.

amplitude: half the distance between the maximum and minimum values of a function

angle of depression: the angle measured down from the horizontal line (through the observation point) to the line of vision



angle of elevation: the angle measured up from the horizontal line (through the observation point) to the line of vision



area: the amount of flat surface enclosed by the shape. It is measured in square units, such as square metres, m^2, or square kilometres, km^2.

Associative Law: a method of combining two numbers or algebraic expressions is associative if the result of the combination of these objects does not depend on the way in which the objects are grouped. Addition and multiplication obey the Associative Law, but subtraction and division are not associative.

asymptote: a line that a graph approaches but never meets

back-to-back stem-and-leaf plot: a method for comparing two data distributions by attaching two sets of 'leaves' to the same 'stem' in a stem-and-leaf plot; for example, comparing the pulse rate before and after exercise

Pulse rate

Before		After
9 8 8 8	6	
8 6 6 4 1 1 0	7	
8 8 6 2	8	6 7 8 8
6 0	9	0 2 2 4 5 8 9 9
4	10	0 4 4
0	11	8
	12	4 4
	13	
	14	6

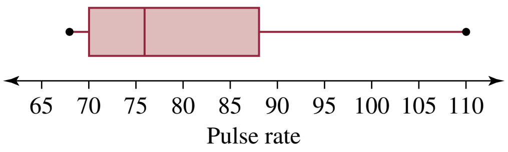
base: the digit at the bottom of a number written in index form. For example, in 6^4, the base is 6. This tells us that 6 is multiplied by itself four times.

bivariate data: sets of data where each piece is represented by two variables

Boolean: a JavaScript data type with two possible values: `true` or `false`. JavaScript Booleans are used to make logical decisions.

boundary line: indicates whether the points on a line satisfy the inequality

boxplots (box-and-whisker plots): a graphical representation of the 5-number summary; that is, the lowest score, lower quartile, median, upper quartile and highest score, for a particular set of data



by eye (line of best fit): carefully looking at the data and drawing a line so that there is an equal number of points above and below the line

canvas: a defined area on your web page where graphics can be drawn with JavaScript

capacity: the maximum amount of fluid that can be contained in an object. It is usually applied to the measurement of liquids and is measured in units such as millilitres (mL), litres (L) and kilolitres (kL).

Cartesian plane: the area formed by a horizontal line with a scale (x-axis) joined to a vertical line with a scale (y-axis). The point of intersection of the lines is called the origin.

census: collection of data from a population (e.g. all Year 10 students) rather than a sample

centre (of circle): middle point of a circle, equidistant (equal in distance) from all points on its circumference

centre of enlargement: the point from which the enlargement of an image is measured

character: in programming, a string of length `1`. A JavaScript character is used to represent a letter, digit or symbol.

circle (equation): the general equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

circumcentre: the centre of a circle drawn so that it touches all three vertices of a triangle

circumcircle: a circle drawn so that it touches all three vertices of a triangle

Closure Law: when an operation is performed on an element (or elements) of a set, the result produced must also be an element of that set.

coincident: lines that lie on top of each other

collinear points: points that all lie on the same straight line

Commutative Law: a method of combining two numbers or algebraic expressions is commutative if the result of the combination does not depend on the order in which the objects are given. For example, the addition of 2 and 3 is commutative, since $2 + 3 = 3 + 2$. However, subtraction is not commutative, since $2 - 3 \neq 3 - 2$.

compass bearings: directions measured in degrees from the north–south line in either a clockwise or anticlockwise direction. To write the compass bearing we need to state whether the angle is measured from the north or south, the size of the angle and whether the angle is measured in the direction of east or west; for example, N27°W, S32°E.

complement (of a set): the complement of a set, A, written A', is the set of elements that are in ξ but not in A.

complementary events: events that have no common elements and together make up the sample space. If A and A' are complementary events, then $P(A) + P(A') = 1$.

completing the square: a procedure used to transform an algebraic expression into a perfect square

composite figure: a figure made up of more than one basic shape

compound interest: the interest earned by investing a sum of money (the principal) when each successive interest payment is added to the principal for the purpose of calculating the next interest payment. The formula used for compound interest is: $A = P(1 + R)^n$, where A is the amount to which the investment grows, P is the principal or initial amount invested, R is the interest rate per compounding period (as a decimal) and n is the number of compounding periods. The compound interest is calculated by subtracting the principal from the amount: $CI = A - P$.

compounded value: the value of the investment with accrued interest included

compounding period: the period of time over which interest is calculated

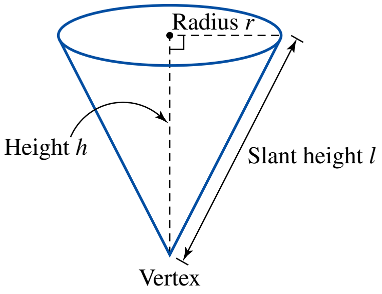
concave polygon: a polygon with at least one reflex interior angle

concyclic (points): points that lie on the circumference of a circle

conditional probability: where the probability of an event is conditional (depends) on another event occurring first. For two events A and B, the conditional probability of event B, given that event A occurs, is denoted by $\mathrm{P}(B|A)$ and can be calculated using the formula:

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}, \mathrm{P}(A) \neq 0.$$

cone: a solid formed by taking a circular base and a point not in the plane of the circle, called the vertex, which lies above or below the circle, and joining the vertex to each point on the circumference of the circle.



congruent triangles: there are five standard congruence tests for triangles: SSS (side, side, side), SAS (side, included angle, side), ASA (two angles and one side), AAS (two angles and a non-included side) and RHS (right angle, hypotenuse, side).

conjugate surds: surds that, when multiplied together, result in a rational number. For example, $(\sqrt{a} + \sqrt{b})$ and $(\sqrt{a} - \sqrt{b})$ are conjugate surds, because $(\sqrt{a} + \sqrt{b}) \times (\sqrt{a} - \sqrt{b}) = a - b$.

console: a special region in a web browser for monitoring the running of JavaScript programs

converse: the reverse of a statement

convex polygon: a polygon with no interior reflex angles

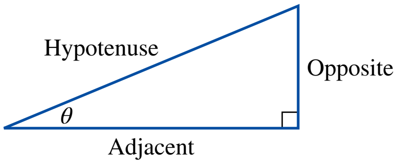
coordinates (programming): a pair of values (typically `x` and `y`) that represent a point on the screen

correlation: a measure of the relationship between two variables. Correlation can be classified as linear, non-linear, positive, negative, weak, moderate or strong.

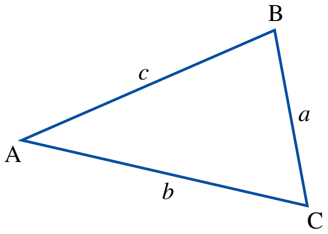
correlation coefficient r: the value of r indicates the strength of the relationship between two variables. Its range is $-1 \leq r \leq +1$, -1 being a strong negative relationship and $+1$ being a strong positive relationship. The closer the value of r is to 0, the less strong the relationship between the variables.

cosine (cos) ratio: the ratio of the adjacent side to the hypotenuse in a right-angled triangle.

So, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.



cosine rule: in any triangle ABC, $c^2 = a^2 + b^2 - 2ab\cos \mathrm{C}$.



cube: a polyhedron with 6 faces. All faces are squares of the same size.

cubic functions: the basic form of a cubic function is $y = ax^3$. These functions can have 1, 2 or 3 roots.

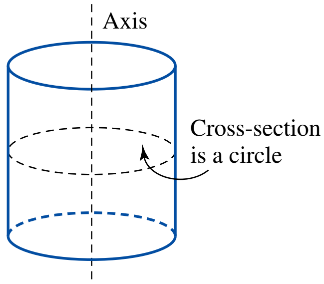
cumulative frequency: the total of all frequencies up to and including the frequency for a particular score in a frequency distribution

cumulative frequency curve: a line graph that is formed when the cumulative frequencies of a set of data are plotted against the end points of their respective class intervals and then joined up by straight-line segments. It is also called an ogive.

cyclic quadrilateral: a quadrilateral that has all four vertices on the circumference of a circle. That is, the quadrilateral is inscribed in the circle.

cyclical pattern: a pattern that displays fluctuations that repeat but will usually take longer than a year to repeat

cylinder: a solid that has parallel circular discs of equal radius at the ends. The centres of the discs form the axis of the cylinder.



Cylinder

data: various forms of information

degree (angle): a unit used to measure the size of an angle

degree (of a polynomial): the degree of a polynomial in x is the highest power of x in the expression.

denominator: the lower number of a fraction that represents the number of equal fractional parts a whole has been divided into

dependent events: successive events in which one event affects the occurrence of the next

dependent variable: this variable is graphed on the y-axis.

depreciation: the reduction in the value of an item as it ages over a period of time. The formula used is $A = P(1 - R)^n$, where A is the depreciated value of the item, P is its initial value, R is the percentage the item depreciates each year (expressed as a decimal) and n is the number of years the item has depreciated.

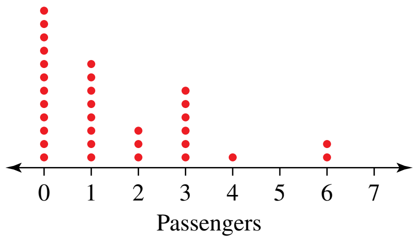
deviation: the difference between a data value and the mean

direction (correlation): if the points in a scatterplot show an upward pattern, the direction is said to be positive; if the pattern is downward, the direction is negative.

discriminant: referring to the quadratic equation $ax^2 + bx + c = 0$, the discriminant is given by $\Delta = b^2 - 4ac$. It is the expression under the square-root sign in the quadratic formula and can be used to determine the number and type of solutions of a quadratic equation.

domain (of a function or relation): the set of all allowable values of x

dot plot: this graphical representation uses one dot to represent a single observation. Dots are placed in columns or rows, so that each column or row corresponds to a single category or observation.



Eighth Index Law: terms with fractional indices can be written as surds. For example, $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

elimination method: a method used to solve simultaneous equations. This method combines the two equations into a third equation involving only one of the variables.

enlargement (dilation): a scaled-up (or down) version of a figure in which the transformed figure is in proportion to the original figure; that is, the two figures are similar

equation: a statement that asserts that two expressions are equal in value. An equation must have an equal sign. For example, $x + 4 = 12$.

equilateral triangle: a triangle with all sides equal in length, and all angles equal to 60°

event space: a list of all the possible outcomes obtained from a probability experiment. It is written as ξ or S, and the list is enclosed in a pair of curled brackets { }. It is also called the sample space.

experimental probability: the probability of an event based on the outcomes of experiments, simulations or surveys

exponential decay: a quantity that decreases by a constant percentage in each fixed period of time. This growth can be modelled by exponential functions of the type $y = ka^x$, where $0 < a < 1$.

exponential functions: relationships of the form $y = a^x$, where $a \neq 1$, are called exponential functions with base a.

exponential growth: a quantity that grows by a constant percentage in each fixed period of time. This growth can be modelled by exponential functions of the type $y = ka^x$, where $a > 1$.

extrapolation: the process of predicting a value of a variable outside the range of the data

factor theorem: if $P(x)$ is a polynomial, and $P(a) = 0$ for some number a, then $P(x)$ is divisible by $(x - a)$.

Fifth Index Law: to remove brackets containing a product, raise every part of the product to the index outside the brackets. So, $(ab)^m = a^m b^m$.

First Index Law: when terms with the same base are multiplied, the indices are added. So, $a^m \times a^n = a^{m+n}$.

flat-rate loan: a loan in which the interest charged is simple interest

FOIL: a diagrammatic method of expanding a pair of brackets. The letters in FOIL represent the order of the expansion: First, Outer, Inner and Last.

form (correlation): the general shape of a pattern. For example, if the points in a scatterplot show an approximate linear pattern, it is described as having linear form.

Fourth Index Law: to remove brackets, multiply the indices inside the brackets by the index outside the brackets. Where no index is shown, assume that it is 1. So, $(a^m)^n = a^{mn}$.

frequency: the number of times a particular score appears

function: a process that takes a set of x-values and produces a related set of y-values. For each distinct x-value, there is only one related y-value. They are usually defined by a formula for $f(x)$ in terms of x; for example, $f(x) = x^2$.

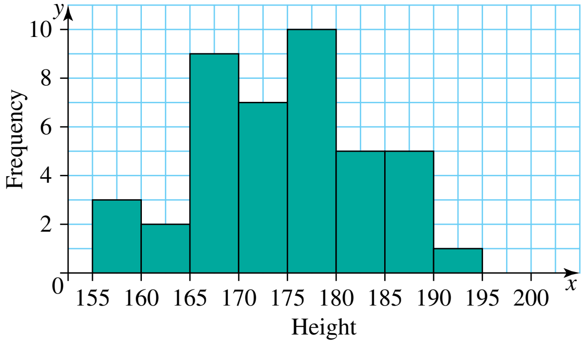
future value: the future value of a loan or investment

general upward or downward trend: a graph that overall goes up or down over time

half plane: the solution that is the region on one side of the line

Heron's formula: this formula is used to find the area of a triangle when all three sides are known. The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the sides and s is the semi-perimeter or $s = \frac{a+b+c}{2}$.

histogram: a special type of column graph, in which no gaps are left between columns and each column straddles an x-axis score. The x-axis scale is continuous and usually a half-interval is left before the first column and after the last column.



hyperbola: the graph of $y = \frac{1}{x}$ is a rectangular hyperbola with asymptotes on the x- and y-axes.

hypotenuse: the longest side of a right-angled triangle. It is the side opposite the right angle.

Identity Law: when 0 is added to an expression or the expression is multiplied by 1, the value of the variable does not change. For example, $x + 0 = x$ and $x \times 1 = x$.

image (similar figures): the enlarged (or reduced) figure produced

independent events: successive events that have no effect on each other

independent variable: this is the x-axis (or horizontal) variable

index (power or exponent): the number expressing the power to which a number or pronumeral is raised. For example, in the expression 3^2, the index is 2. Plural: *indices*

index (programming): a integer that points to a particular item in an array

inequality: when one algebraic expression or one number is greater than or less than another

inequations: similar to equations, but contain an inequality sign instead of an equal sign. For example, $x = 3$ is an equation, but $x < 3$ is an inequation.

integers (Z): These include the positive and negative whole numbers, as well as zero; that is, … , −3, −2, −1, 0, 1, 2, …

interpolation: the process of predicting a value of a variable from within the range of the data

interquartile range: the difference between the upper (or third) quartile, Q_U (or Q_3), and the lower (or first) quartile, Q_L (or Q_1); that is, $IQR = Q_U - Q_L = Q_3 - Q_1$. It is the range of approximately the middle half of the data.

Inverse Law: when the additive inverse of a number or pronumeral is added to itself, the sum is equal to 0. When the multiplicative inverse of a number or pronumeral is multiplied by itself, the product is equal to 1. So, $x + (-x) = 0$ and $x \times \frac{1}{x} = 1$.

irrational numbers (I): numbers that cannot be written as fractions. Examples of irrational numbers include surds, π and non-terminating, non-recurring decimals.

isosceles triangle: a triangle with two sides equal in length

lay-by: a method used to purchase an item whereby the purchaser makes regular payments to the retailer, who retains the item until the complete price is paid

line of best fit: a straight line that best fits the data points of a scatterplot that appear to follow a linear trend. It is positioned on the scatterplot so that there is approximately an equal number of data points on either side of the line, and so that all the points are as close to the line as possible.

line segment: a line segment or interval is a part of a line with end points.

linear graphs: consist of an infinite number of points that can be joined to form a straight line

linked list: a list of objects. Each object stores data and points to the next object in the list. The last object points to a terminator to indicate the end of the list.

literal equation: an equation that includes two or more pronumerals or variables

logarithm: the power to which a given positive number b, called the base, must be raised in order to produce the number x. The logarithm of x, to the base b, is denoted by $\log_b x$. Algebraically: $\log_b x = y \leftrightarrow b^y = x$; for example, $\log_{10}100 = 2$ because $10^2 = 100$.

logarithmic equation: an equation that requires the application of the laws of indices and logarithms to solve

loop: in JavaScript, a process that executes the same code many times with different data each time

many-to-many relation: a relation in which one range value may yield more than one domain value and vice versa

many-to-one relation: a function or mapping that takes the same value for at least two different elements of its domain

matrix: a rectangular array of numbers arranged in rows and columns

maximal domain: the limit of the x-values that a function can have

mean: one measure of the centre of a set of data. It is given by mean $= \dfrac{\text{sum of all scores}}{\text{number of scores}}$ or $\bar{x} = \dfrac{\Sigma x}{n}$.

When data are presented in a frequency distribution table, $\bar{x} = \dfrac{\sum (f \times x)}{n}$.

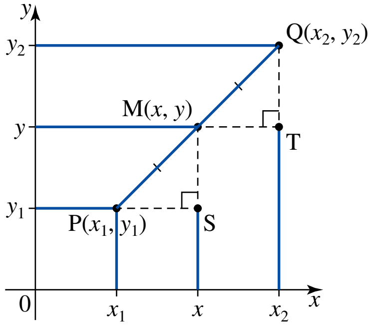
measures of central tendency: mean, median and mode

measures of spread: range, interquartile range, standard deviation

median: one measure of the centre of a set of data. It is the middle score for an odd number of scores arranged in numerical order. If there is an even number of scores, the median is the mean of the two middle scores when they are ordered. Its location is determined by the rule $\dfrac{n+1}{2}$.

For example, the median value of the set 1 3 3 4 5 6 8 9 9 is 5, while the median value for the set 1 3 3 4 5 6 8 9 9 10 is the mean of 5 and 6 (5.5).

midpoint: the midpoint of a line segment is the point that divides the segment into two equal parts. The coordinates of the midpoint M between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.



mode: one measure of the centre of a set of data. It is the score that occurs most often. There may be no mode (all scores occur once), one mode or more than one mode (two or more scores occur equally frequently).

moderate (correlation): describes a relationship in which the points on the scatterplot are reasonably close together and approximate a linear pattern

Multiplication Law of probability: if events A and B are independent, then: $\text{P}(A \text{ and } B) = \text{P}(A) \times \text{P}(B)$ or $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$.

mutually exclusive events: events that cannot occur together. On a Venn diagram, two mutually exclusive events will appear as disjoint sets.

natural numbers: the set of positive integers, or counting numbers; that is, the set 1, 2, 3, …

negatively skewed: showing larger amounts of data as the values of the data increase

nested loop: a loop within a loop. The outer loop contains an inner loop. The first iteration of the outer loop triggers a full cycle of the inner loop until the inner loop completes. This triggers the second iteration of the outer loop, which triggers a full cycle of the inner loop again. This process continues until the outer loop finishes a full cycle.

Null Factor Law: if $a \times b = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$; used when solving quadratic equations

number (programming): a JavaScript data type that represents a numerical value

object: a general JavaScript data type that can have many properties. Each property is a name–value pair so that the property has a name to reference a value.

odds: relates to probabilities in gambling. They are given as ratios, such as $5-1$, $\frac{5}{1}$ or $5:1$.

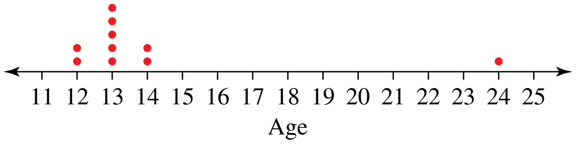
ogive (cumulative frequency polygon): a graph formed by joining the top right-hand corners of the columns of a cumulative frequency histogram

one-dimensional array: a simple array of values in which the values can be of any type except for another array

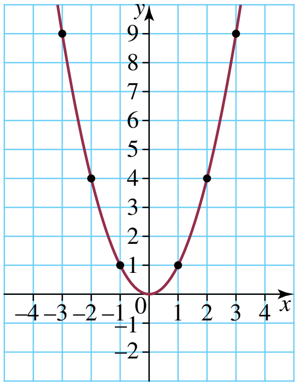
one-to-many relation: a relation in which there may be more than one range value for one domain value but only one domain value for each range value

one-to-one relation: refers to the relationship between two sets such that every element of the first set corresponds to one and only one element of the second set

outlier: a piece of data that is considerably different from the rest of the values in a set of data; for example, 24 is the outlier in the set of ages {12, 12, 13, 13, 13, 13, 13, 14, 14, 24}.

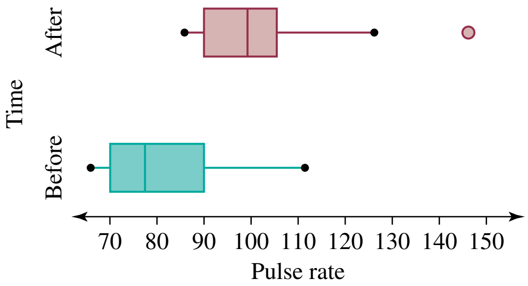


parabola: the graph of a quadratic function has the shape of a parabola. For example, the typical shape is that of the graph of $y = x^2$.

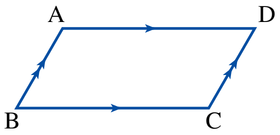


parallel: parallel lines in a plane never meet, no matter how far they are extended. Parallel lines have the same gradient.

parallel boxplots: two or more boxplots drawn on the same scale to visually compare the five-number summaries of the data sets. These boxplots compare the pulse rates of the same group of people before and after exercise.



parallelogram: a quadrilateral with both pairs of opposite sides parallel



percentile: the value below which a given percentage of all scores lie. For example, the 20th percentile is the value below which 20% of the scores in the set of data lie.

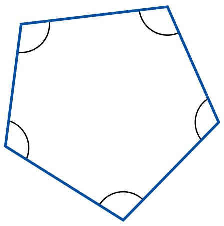
periodic functions: functions that have graphs that repeat themselves continuously in cycles, for example, graphs of $y = \sin x$ and $y = \cos x$. The period of the graph is the distance between repeating peaks or troughs.

perpendicular: perpendicular lines are at right angles to each other. The product of the gradients of two perpendicular lines is -1.

pi (π): the Greek letter π represents the ratio of the circumference of any circle to its diameter. The number π is irrational, with an approximate value of $\frac{22}{7}$, and a decimal value of $\pi = 3.141\,59\ldots$.

pointer: in JavaScript, a variable that points to a JavaScript object or array. Multiple pointers can point to the same object or array.

polygon: a plane figure bounded by line segments



polynomial: an expression containing only non-negative integer powers of a variable

population: the whole group from which a sample is drawn

positively skewed: showing larger amounts of data as the values of the data decrease

primary data: data collected by the user

principal: an amount of money borrowed or invested

probability: the likelihood or chance of a particular event (result) occurring.

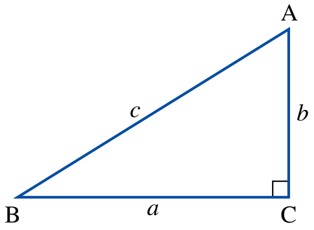
$$\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}.$$

The probability of an event occurring ranges from 0 (impossible — will not occur) to 1 (certainty — will definitely occur).

proof: an argument that shows why a statement is true

property (programming): references a value on an object. A complex object may have many properties. Each property has a unique name on the object.

Pythagoras' theorem: in any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. This is often expressed as $c^2 = a^2 + b^2$.



quadratic formula: gives the roots of the quadratic equation $ax^2 + bx + c = 0$. It is expressed as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

quantiles: percentiles expressed as decimals. For example, the 95th percentile is the same as the 0.95 quantile.

quartic functions: the basic form of a quartic function is $y = ax^4$. If the value of a is positive, the curve is upright, whereas a negative value of a results in an inverted graph. A maximum of 4 roots can result.

quartiles: values that divide an ordered set into four (approximately) equal parts. There are three quartiles — the first (or lower) quartile Q_1, the second quartile (or median) Q_2 and the third (or upper) quartile Q_3.

radian: a unit used to measure angles; 360° equals 2π radians

radius: the straight line from a circle's centre to any point on its circumference

random pattern: a pattern that does not show any regular fluctuation

range: the difference between the highest and lowest scores in a set of data; that is, range = highest score − lowest score

range (of a function or relation): the set of all allowable values of y

rational numbers (Q): numbers that can be written as fractions, where the denominator is not zero

real numbers (R): the set of all rational and irrational numbers

reciprocal: a number by which a given number is multiplied to result in 1

rectangular prism (cuboid): a solid that has six rectangular faces and a uniform cross-section

recurring decimals: These decimals have one or more digits repeated continuously; for example, 0.999 …. They can be expressed exactly by placing a dot or horizontal line over the repeating digits; for example, $8.343\,434 = 8.\dot{3}\dot{4}$ or $8.\overline{34}$.

reducible-interest-rate loan: a loan in which the interest charged is compound interest, but the amount of the loan and the interest are repaid with regular repayments

regular polygon: a polygon with sides of the same length and interior angles of the same size

relation: a set of ordered pairs

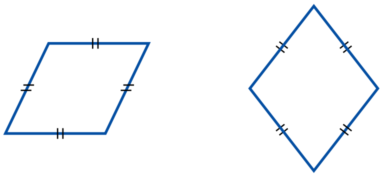
relative frequency: represents the frequency of a particular score divided by the total sum of the frequencies. It is given by the rule:

$$\text{relative frequency of a score} = \frac{\text{frequency of the score}}{\text{total sum of frequencies}}.$$

remainder theorem: if a polynomial $P(x)$ is divided by $x - a$, where a is any real number, the remainder is $P(a)$.

required region: the region that contains the points that satisfy an inequality

rhombus: a parallelogram with all sides equal



sample: part of a population chosen so as to give information about the population as a whole

sample space: *see* **Event space**.

scale factor: the ratio of the corresponding sides in similar figures, where the enlarged (or reduced) figure is referred to as the image and the original figure is called the object.

$$\text{scale factor} = \frac{\text{image length}}{\text{object length}}$$

scatterplot: a graphical representation of bivariate data that displays the degree of correlation between two variables. Each piece of data on a scatterplot is shown by a point. The x-coordinate of this point is the value of the independent variable and the y-coordinate is the corresponding value of the dependent variable.

seasonal pattern: a pattern that displays fluctuations that repeat at the same time over a particular time interval (such as a week, month or quarter) and usually last less than a year

Second Index Law: when terms with the same base are divided, the indices are subtracted. So, $a^m \div a^n = a^{m-n}$.

secondary data: data collected by others

Seventh Index Law: a term with a negative index can be expressed with a positive index using this law.

So, $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$.

similar figures: figures that have identical shape but different size. The corresponding angles in similar figures are equal in size, and the corresponding sides are in the same ratio, called a scale factor.

similar triangles: triangles that have similar shape but different size. There are four standard tests to determine whether two triangles are similar: AAA (angle, angle, angle), SAS (side, angle, side), SSS (side, side, side) and RHS (right angle, hypotenuse side).

simple interest: the interest accumulated when the interest payment in each period is a fixed fraction of the principal. The formula used is $I = \dfrac{P \times r \times T}{100}$, where I is the interest earned (in \$) when a principal of \$$P$ is invested at an interest rate of r% p.a. for a period of T years.

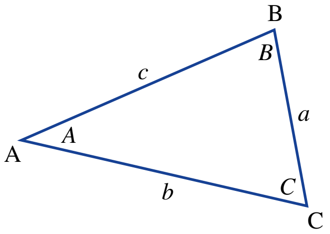
simultaneous: occurring at the same time

simultaneous equations (linear): two (or more) linear graphs that have the same solution

sine (sin) ratio: the ratio of the opposite side to the hypotenuse in a right-angled triangle.

So, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$.

sine rule: in any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



Sixth Index Law: to remove brackets containing a fraction, multiply the indices of both the numerator and denominator by the index outside the brackets. So, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

sphere: a solid for which all points on the surface are equidistant from the centre

standard deviation: a measure of the variability of spread of a data set. It gives an indication of the degree to which the individual data values are spread around the mean.

strength (correlation): an indication of how closely the points on a scatterplot fit a straight line

string (programming): a JavaScript data type that represents text

strong (correlation): describes a relationship in which the points on a scatterplot are close together and show a definite linear pattern

subjective probability: probability that is based on one or more of the following: judgements, opinions, assessments, estimations and conjectures by individuals. It may also involve beliefs, emotions and bias.

substitution method: a method used to solve simultaneous equations. It is useful when one (or both) of the equations has one of the variables as the subject.

successive discount: a discount applied after an initial discount has been applied

supplementary (angles): angles that add to 180°

surds: roots of numbers that do not have an exact answer, so they are irrational numbers. Surds themselves are exact numbers; for example, $\sqrt{6}$ or $\sqrt[3]{5}$.

symmetrical: the identical size, shape and arrangement of parts of an object on opposite sides of a line or plane

system of equations: a set of two or more equations with the same variables

tangent (tan) ratio: the ratio of the opposite side to the adjacent side in a right-angled triangle. So, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

terminating decimals: decimals that have a fixed number of places; for example, 0.6 and 2.54

theorems: rules or laws

theoretical probability: given by the rule $\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$ or $\text{P}(E) = \frac{n(E)}{n(S)}$, where $n(E)$ = number of times or ways an event, E, can occur and $n(S)$ = number of elements in the sample space or number of ways all outcomes can occur, given all the outcomes are equally likely

Third Index Law: any term (excluding 0) with an index of 0 is equal to 1. So, $a^0 = 1$.

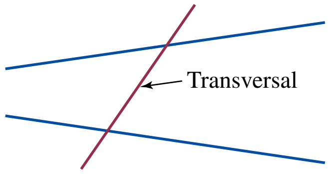
time series: a sequence of measurements taken at regular intervals (daily, weekly, monthly and so on) over a certain period of time. They are used for analysing general trends and making predictions for the future.

total surface area (TSA): the area of the outside surface of a 3-dimensional figure

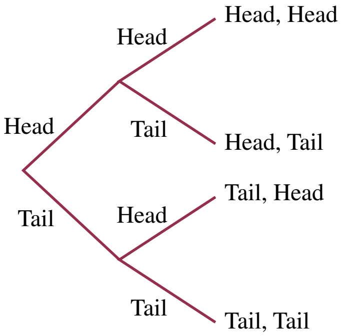
transcendental number: a non-recurring decimal that is not the root of any polynomial equation with rational coefficients. A transcendental number is an irrational number but not an algebraic number. Pi (π) is an example of a transcendental number.

transformations (quadratics): changes that occur to the basic parabola $y = x^2$ in order to obtain another graph. Examples of transformations are translations, reflections or dilations. Transformations can also be applied to non-quadratic functions.

transversal: a line that meets two or more other lines in a plane



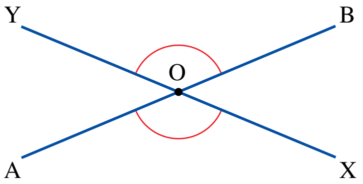
tree diagrams: branching diagrams that list all the possible outcomes of a probability experiment. This diagram shows the outcomes when a coin is tossed twice.



trend: a general tendency in a set of data
trend line: the line of best fit that is drawn on a time series graph, which is used to forecast future values
trial: the number of times a probability experiment is conducted
trigonometric ratios: three different ratios of one side of a triangle to another. The three ratios are the sine, cosine and tangent ratios.
true bearings: directions that are written as the number of degrees (3 digits) from north in a clockwise direction, followed by the word *true* or T; for example, due east would be 090° true or 090°T
two-dimensional array: an array of one-dimensional arrays
two-step chance experiment: a probability experiment that involves two trials
two-way tables: tables that list all the possible outcomes of a probability experiment in a logical manner

Hair colour	Hair type		Total
Red	1	1	2
Brown	8	4	12
Blonde	1	3	4
Black	7	2	9
Total	17	10	27

unit circle: a circle with its centre at the origin and having a radius of 1 unit
univariate data: data relating to a single variable
variable (programming): a named container or memory location that holds a value
vertically opposite angles: when two lines intersect, four angles are formed at the point of intersection, and two pairs of vertically opposite angles result. Vertically opposite angles are equal.



volume: the amount of space a 3-dimensional object occupies. The units used are cubic units, such as cubic centimetres (cm^3) and cubic metres (m^3).
weak (correlation): describes a relationship in which the points on a scatterplot are far apart

INDEX