Instructor's Solutions Manual

PARTIAL DIFFERENTIAL EQUATIONS

with FOURIER SERIES and BOUNDARY VALUE PROBLEMS

Second Edition

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Contents

Preface

This manual contains solutions with notes and comments to problems from the textbook

Partial Differential Equations
with Fourier Series and Boundary Value Problems
Second Edition

Most solutions are supplied with complete details and can be used to supplement examples from the text. There are also many figures and numerical computions on Mathematica that can be very useful for a classroom presentation. Certain problems are followed by discussions that aim to generalize the problem under consideration.

I hope that these notes will sevre their intended purpose:

- To check the level of difficulty of assigned homework problems;
- To verify an answer or a nontrivial computation; and
- to provide worked solutions to students or graders.

As of now, only problems from Chapters 1–7 are included. Solutions to problems from the remaining chapters will be posted on my website

www.math.missouri.edu/ nakhle

as I complete them and will be included in future versions of this manual.

I would like to thank users of the first edition of my book for their valuable comments. Any comments, corrections, or suggestions from Instructors would be greatly appreciated. My e-mail address is

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Errata

The following mistakes appeared in the first printing of the second edition.

Corrections in the text and figures

p. 224, Exercise #13 is better done after Section 4.4.
p. 268, Exercise #8(b), n should be even.
p.387, Exercise#12, use $y_2 = I_0(x)$ not $y_2 = J_1(x)$.
p. 425 Figures 5 and 6: Relabel the ticks on the x-axis as $-\pi$, $-\pi/2$, $\pi/2$, π, instead of -2π, $-\pi$, π, 2π.
p. 467, l. (-3): Change reference (22) to (20).
p. 477 l. 10: $(xt) \leftrightarrow (x, t)$.
p. 477 l. 19: Change "interval" to "triangle"
p. 487, l.1: Change "is the equal" to "is equal"
p. 655, l.13: Change $\ln|\ln(x^2+y^2)|$ to $\ln(x^2+y^2)$.

Corrections to Answers of Odd Exercises

Section 7.2, # 7: Change i to $-i$.
Section 7.8, # 13: $f(x) = 3$ for $1 < x < 3$ not $1 < x < 2$.
Section 7.8, # 35: $\sqrt{\frac{2}{\pi}}\frac{(e^{-iw}-1)}{w}\sum_{j=1}^{3} j\sin(jw)$
Section 7.8, # 37: $i\sqrt{\frac{2}{\pi}}\frac{1}{w^3}$
Section 7.8, # 51: $\frac{3}{\sqrt{2\pi}}[\delta_1 - \delta_0]$.
Section 7.8, # 57: The given answer is the derivative of the real answer, which should be

$$\frac{1}{\sqrt{2\pi}}\Big((x+2)\big(\mathcal{U}_{-2}-\mathcal{U}_0\big)+(-x+2)\big(\mathcal{U}_0-\mathcal{U}_1\big)+\big(\mathcal{U}_1-\mathcal{U}_3\big)+(-x+4)\big(\mathcal{U}_3-\mathcal{U}_4\big)\Big)$$

Section 7.8, # 59: The given answer is the derivative of the real answer, which should be

$$\frac{1}{2}\frac{1}{\sqrt{2\pi}}\Big((x+3)\big(\mathcal{U}_{-3}-\mathcal{U}_{-2}\big)+(2x+5)\big(\mathcal{U}_{-2}-\mathcal{U}_{-1}\big)+(x+4)\big(\mathcal{U}_{-1}-\mathcal{U}_0\big)$$
$$+(-x+4)\big(\mathcal{U}_0-\mathcal{U}_1\big)+(-2x+5)\big(\mathcal{U}_1-\mathcal{U}_2\big)+(-x+3)\big(\mathcal{U}_2-\mathcal{U}_3\big)\Big)$$

Section 7.10, # 9: $\frac{1}{2}\big[t\sin(x+t)+\frac{1}{2}\cos(x+t)-\frac{1}{2}\cos(x-t)\big]$.

Any suggestion or correction would be greatly appreciated. Please send them to my e-mail address

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Solutions to Exercises 1.1

1. If u_1 and u_2 are solutions of (1), then

$$\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}=0 \quad \text{and} \quad \frac{\partial u_2}{\partial t}+\frac{\partial u_2}{\partial x}=0.$$

Since taking derivatives is a linear operation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(c_1u_1+c_2u_2)+\frac{\partial}{\partial x}(c_1u_1+c_2u_2) &= c_1\frac{\partial u_1}{\partial t}+c_2\frac{\partial u_2}{\partial t}+c_1\frac{\partial u_1}{\partial x}+c_2\frac{\partial u_2}{\partial x}\\ &= c_1\overbrace{\left(\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}\right)}^{=0}+c_2\overbrace{\left(\frac{\partial u_2}{\partial t}++\frac{\partial u_2}{\partial x}\right)}^{=0}=0,\end{aligned}$$

showing that $c_1u_1+c_2u_2$ is a solution of (1).
2. (a) General solution of (1): $u(x,\,t)=f(x-t)$. On the x-axis ($t=0$): $u(x,\,0)=xe^{-x^2}=f(x-0)=f(x)$. So $u(x,\,t)=f(x-t)=(x-t)e^{-(x-t)^2}$.
3. (a) General solution of (1): $u(x,\,t)=f(x-t)$. On the t-axis ($x=0$): $u(0,\,t)=t=f(0-t)=f(-t)$. Hence $f(t)=-t$ and so $u(x,\,t)=f(x-t)=-(x-t)=t-x$.
4. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

So (1) becomes

$$(a+b)\frac{\partial u}{\partial \alpha}+(c+d)\frac{\partial u}{\partial \beta}=0.$$

Let $a=1,\ b=1,\ c=1,\ d=-1$. Then

$$2\frac{\partial u}{\partial \alpha}=0 \quad\Rightarrow\quad \frac{\partial u}{\partial \alpha}=0,$$

which implies that u is a function of β only. Hence $u=f(\beta)=f(x-t)$.
5. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

Recalling the equation, we obtain

$$\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}=0 \quad\Rightarrow\quad (b-a)\frac{\partial u}{\partial \alpha}+(d-c)\frac{\partial u}{\partial \beta}=0.$$

Let $a=1,\ b=2,\ c=1,\ d=1$. Then

$$\frac{\partial u}{\partial \alpha}=0 \quad\Rightarrow\quad u=f(\beta) \quad\Rightarrow\quad u(x,\,t)=f(x+t),$$

where f is an arbitrary differentiable function (of one variable).

6. The solution is very similar to Exercise 5. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

Recalling the equation, we obtain

$$2\frac{\partial u}{\partial t}+3\frac{\partial u}{\partial x}=0 \quad\Rightarrow\quad (2b+3a)\frac{\partial u}{\partial \alpha}+(2d+3c)\frac{\partial u}{\partial \beta}=0.$$

Let $a=0,\ b=1,\ c=2,\ d=-3$. Then the equation becomes

$$2\frac{\partial u}{\partial \alpha}=0 \quad\Rightarrow\quad u=f(\beta) \quad\Rightarrow\quad u(x,\,t)=f(cx+dt)=f(2x-3t),$$

where f is an arbitrary differentiable function (of one variable).

7. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

The equation becomes

$$(b-2a)\frac{\partial u}{\partial \alpha}+(d-2c)\frac{\partial u}{\partial \beta}=2.$$

Let $a=1,\ b=2,\ c=-1,\ d=0$. Then

$$2\frac{\partial u}{\partial \beta}=2 \quad\Rightarrow\quad \frac{\partial u}{\partial \beta}=1.$$

Solving this ordinary differential equation in β, we get $u=\beta+f(\alpha)$ or $u(x,\,t)=-x+f(x+2t)$.

8. Let $\alpha=c_1x+d_1t$, $\beta=c_2x+d_2t$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= c_1\frac{\partial u}{\partial \alpha}+c_2\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= d_1\frac{\partial u}{\partial \alpha}+d_2\frac{\partial u}{\partial \beta}.\end{aligned}$$

The equation becomes

$$(ad_1+bc_1)\frac{\partial u}{\partial \alpha}+(ad_2+bc_2)\frac{\partial u}{\partial \beta}=u.$$

Let $c_1=a,\ d_1=-b,\ c_2=0,\ d_2=\frac{1}{a}\ (a\neq 0)$. Then

$$\frac{\partial u}{\partial \beta}=u \quad\Rightarrow\quad u=f(\alpha)e^{\beta}.$$

Hence $u(x,\,t)=f(ax-bt)e^{\frac{t}{a}}$.

9. (a) The general solution in Exercise 5 is $u(x,\,t)=f(x+t)$. When $t=0$, we get $u(x,\,0)=f(x)=1/(x^2+1)$. Thus

$$u(x,\,t)=f(x+t)=\frac{1}{(x+t)^2+1}.$$

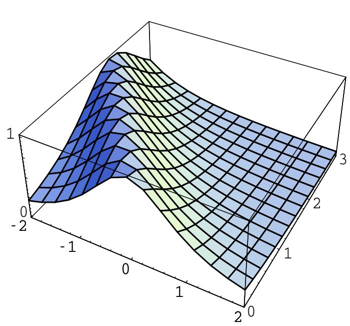


Figure for Exercise 9(b).

(c) As t increases, the wave $f(x)=\frac{1}{1+x^2}$ moves to the left.

10. (a) The directional derivative is zero along the vector $(a,\,b)$.
(b) Put the equation in the form $\frac{\partial u}{\partial x}+\frac{b}{a}\frac{\partial u}{\partial y}=0\ (a\neq 0)$. The characteristic curves are obtained by solving

$$\frac{dy}{dx}=\frac{b}{a} \quad\Rightarrow\quad y=\frac{b}{a}x+C \quad\Rightarrow\quad y-\frac{b}{a}x=C.$$

Let $\phi(x, y) = y - \frac{b}{a}x$. The characteristic curves are the level curves of ϕ.
(c) The solution of $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$ is of the form $u(x, y) = f(\phi(x, y)) = f(y - \frac{b}{a}x)$, where f is a differentiable function of one variable.
11. The characteristic curves are obtained by solving

$$\frac{dy}{dx} = x^2 \quad \Rightarrow \quad y = \frac{1}{3}x^3 + C \quad \Rightarrow \quad y - \frac{1}{3}x^3 = C.$$

Let $\phi(x, y) = y - \frac{1}{3}x^3$. The characteristic curves are the level curves of ϕ. The solution of is of the form $u(x, y) = f(\phi(x, y)) = f(y - \frac{1}{3}x^3)$, where f is a differentiable function of one variable.

12. We follow the method of characteristic curves. Let's find the characteristic curves. For $x \neq 0$,

$$\frac{\partial u}{\partial x} + \frac{y}{x}\frac{\partial u}{\partial y} = 0;$$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow y = Cx,$$

where we have used Theorem 1, Appendix A.1, to solve the last differential equation. Hence the characteristic curves are $\frac{y}{x} = C$ and the solution of the partial differential equation is $u(x, y) = f\left(\frac{y}{x}\right)$. To verify the solution, we use the chain rule and get $u_x = -\frac{y}{x^2}f'\left(\frac{y}{x}\right)$ and $u_y = \frac{1}{x}f'\left(\frac{y}{x}\right)$. Thus $xu_x + yu_y = 0$, as desired.

13. To find the characteristic curves, solve $\frac{dy}{dx} = \sin x$. Hence $y = -\cos x + C$ or $y + \cos x = C$. Thus the solution of the partial differential equation is $u(x, y) = f(y + \cos x)$. To verify the solution, we use the chain rule and get $u_x = -\sin x f'(y + \cos x)$ and $u_y = f'(y + \cos x)$. Thus $u_x + \sin x u_y = 0$, as desired.
14. Put the equation in the form $\frac{\partial u}{\partial x} + xe^{-x^2}\frac{\partial u}{\partial y} = 0$. The characteristic curves are obtained by solving

$$\frac{dy}{dx} = xe^{-x^2} \quad \Rightarrow \quad y = -\frac{1}{2}e^{-x^2} + C \quad \Rightarrow \quad y + \frac{1}{2}e^{-x^2} = C.$$

Let $\phi(x, y) = y + \frac{1}{2}e^{-x^2}$. The characteristic curves are the level curves of ϕ. The solution is of the form $u(x, y) = f(\phi(x, y)) = f(y + \frac{1}{2}e^{-x^2})$, where f is a differentiable function of one variable.

Exercises 1.2

1. We have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

So

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 v}{\partial t \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial t} = -\frac{\partial^2 u}{\partial x^2}.$$

Assuming that $\frac{\partial^2 v}{\partial t \partial x} = \frac{\partial^2 v}{\partial x \partial t}$, it follows that $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, which is the one dimensional wave equation with $c = 1$. A similar argument shows that v is a solution of the one dimensional wave equation.

2. (a) For the wave equation in u, the appropriate initial conditions are $u(x, 0) = f(x)$, as given, and $u_t(x, 0) = -v_x(x, 0) = h'(x)$. (b) For the wave equation in v, the appropriate initial conditions are $v(x, 0) = h(x)$, as given, and $v_t(x, 0) = -u_x(x, 0) = f'(x)$.

3. $u_{xx} = F''(x + ct) + G''(x + ct)$, $u_{tt} = c^2 F''(x + ct) + c^2 G(x - ct)$. So $u_{tt} = c^u_{xx}$, which is the wave equation.

4. (a) Using the chain rule in two dimensions:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\
&= \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta \partial \alpha} + \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2} \\
&= \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + \frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) \\
&= c^2\frac{\partial^2 u}{\partial \alpha^2} - c^2\frac{\partial^2 u}{\partial \beta \partial \alpha} - c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2} \\
&= c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Substituting into the wave equation, it follows that

$$c^2\frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2} = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

(b) The last equation says that $\frac{\partial u}{\partial \beta}$ is constant in α. So

$$\frac{\partial u}{\partial \beta} = g(\beta)$$

where g is an arbitrary differentiable function.

(c) Integrating the equation in (b) with respect to β, we find that $u = G(\beta) + F(\alpha)$, where G is an antiderivative of g and F is a function of α only.

(d) Thus $u(x, t) = F(x + ct) + G(x - ct)$, which is the solution in Exercise 3.

5. (a) We have $u(x, t) = F(x + ct) + G(x - ct)$. To determine F and G, we use the initial data:

$$u(x, 0) = \frac{1}{1 + x^2} \quad \Rightarrow \quad F(x) + G(x) = \frac{1}{1 + x^2}; \quad (1)$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \Rightarrow \quad cF'(x) - cG'(x) = 0$$
$$\Rightarrow \quad F'(x) = G'(x) \quad \Rightarrow \quad F(x) = G(x) + C, \qquad (2)$$

where C is an arbitrary constant. Plugging this into (1), we find

$$2G(x) + C = \frac{1}{1+x^2} \quad \Rightarrow \quad G(x) = \frac{1}{2}\left[\frac{1}{1+x^2} - C\right];$$

and from (2)

$$F(x) = \frac{1}{2}\left[\frac{1}{1+x^2} + C\right].$$

Hence

$$u(x,\, t) = F(x+ct) + G(x-ct) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right].$$

6. We have $u(x,\, t) = F(x+ct) + G(x-ct)$. To determine F and G, we use the initial data:

$$u(x,\, 0) = e^{-x^2} \quad \Rightarrow \quad F(x) + G(x) = e^{-x^2}; \quad (1)$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \Rightarrow \quad cF'(x) - cG'(x) = 0$$
$$\Rightarrow \quad F'(x) = G'(x) \quad \Rightarrow \quad F(x) = G(x) + C, \qquad (2)$$

where C is an arbitrary constant. Plugging this into (1), we find

$$2G(x) + C = e^{-x^2} \quad \Rightarrow \quad G(x) = \frac{1}{2}\left[e^{-x^2} - C\right];$$

and from (2)

$$F(x) = \frac{1}{2}\left[e^{-x^2} + C\right].$$

Hence

$$u(x,\, t) = F(x+ct) + G(x-ct) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right].$$

7. We have $u(x,\, t) = F(x+ct) + G(x-ct)$. To determine F and G, we use the initial data:

$$u(x,\, 0) = 0 \quad \Rightarrow \quad F(x) + G(x) = 0; \quad (1)$$

$$\frac{\partial u}{\partial t}(x,0) = -2xe^{-x^2} \quad \Rightarrow \quad cF'(x) - cG'(x) = -2xe^{-x^2}$$
$$\Rightarrow \quad cF(x) - cG(x) = \int -2xe^{-x^2}\, dx = e^{-x^2} + C$$
$$\Rightarrow \quad F(x) - G(x) = \frac{e^{-x^2}}{c} + C, \qquad (2)$$

where we rewrote C/c as C to denote the arbitrary constant. Adding (2) and (1), we find

$$2F(x) = \frac{e^{-x^2}}{c} + C \quad \Rightarrow \quad F(x) = \frac{1}{2c}\left[e^{-x^2} + C\right];$$

and from (1)

$$G(x) = -\frac{1}{2c}\left[e^{-x^2} + C\right].$$

Hence

$$u(x,\, t) = F(x+ct) + G(x-ct) = \frac{1}{2c}\left[e^{-(x+ct)^2} - e^{-(x-ct)^2}\right].$$

8. We have $u(x,t) = F(x+ct) + G(x-ct)$. To determine F and G, we use the initial data:

$$u(x,0) = 0 \quad \Rightarrow \quad F(x)+G(x) = 0 \quad \Rightarrow \quad F(x) = -G(x) \quad \Rightarrow \quad F'(x) = -G'(x); \quad (1)$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{x}{(1+x^2)^2} \quad \Rightarrow \quad cF'(x) - cG'(x) = \frac{x}{(1+x^2)^2}$$

$$\Rightarrow \quad 2cF'(x) = \frac{x}{(1+x^2)^2} \quad \Rightarrow \quad F'(x) = \frac{x}{2c(1+x^2)^2}, \text{(from (1))}$$

$$\Rightarrow \quad F(x) = \int \frac{x}{2c(1+x^2)^2}\,dx = \frac{-1}{4c(1+x^2)} + C,$$

where C is an arbitrary constant. From (1),

$$G(x) = -F(x) = \frac{1}{4c(1+x^2)} - C;$$

where the C here is the same as the C in the definition of $F(x)$. So

$$u(x,t) = F(x+ct) + G(x-ct) = \frac{1}{4c}\left[\frac{-1}{(1+(x+ct))^2} + \frac{1}{(1+(x-ct))^2}\right].$$

9. As the hint suggests, we consider two separate problems: The problem in Exercise 5 and the one in Exercise 7. Let $u_1(x,t)$ denote the solution in Exercise 5 and $u_2(x,t)$ the solution in Exercise 7. It is straightforward to verify that $u = u_1 + u_2$ is the desired solution. Indeed, because of the linearity of derivatives, we have $u_{tt} = (u_1)_{tt} + (u_2)_{tt} = c^2(u_1)_{xx} + c^2(u_2)_{xx}$, because u_1 and u_2 are solutions of the wave equation. But $c^2(u_1)_{xx} + c^2(u_2)_{xx} = c^2(u_1+u_2)_{xx} = u_{xx}$ and so $u_{tt} = c^2 u_{xx}$, showing that u is a solution of the wave equation. Now $u(x,0) = u_1(x,0) + u_2(x,0) = 1/(1+x^2) + 0$, because $u_1(x,0) = 1/(1+x^2)$ and $u_2(x,0) = 0$. Similarly, $u_t(x,0) = -2xe^{-x^2}$; thus u is the desired solution. The explicit formula for u is

$$u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right] + \frac{1}{2c}\left[e^{-(x+ct)^2} - e^{-(x-ct)^2}\right].$$

10. Reasoning as in the previous exercise, we find the solution to be $u = u_1 + u_2$, wehre u_1 is the solution in Exercise 6 and u_2 is the solution in Exercise 8. Thus,

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] + \frac{1}{4c}\left[\frac{-1}{(1+(x+ct))^2} + \frac{1}{(1+(x-ct))^2}\right].$$

11. We have

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(-L\frac{\partial I}{\partial t} - RI\right) = -L\frac{\partial^2 I}{\partial x \partial t} - R\frac{\partial I}{\partial x};$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV \quad \Rightarrow \quad \frac{\partial V}{\partial t} = \frac{-1}{C}\left[\frac{\partial I}{\partial x} + GV\right];$$

so

$$\frac{\partial^2 V}{\partial t^2} = \frac{-1}{C}\left[\frac{\partial^2 I}{\partial t \partial x} + G\frac{\partial V}{\partial t}\right].$$

To check that V verifies (1), we start with the right side

$$\begin{aligned}
&LC\frac{\partial^2 V}{\partial t^2} + (RC+LG)\frac{\partial V}{\partial t} + RGV \\
&= LC\frac{-1}{C}\left[\frac{\partial^2 I}{\partial t\partial x} + G\frac{\partial V}{\partial t}\right] + (RC+LG)\frac{-1}{C}\left[\frac{\partial I}{\partial x} + GV\right] + RGV \\
&= -L\frac{\partial^2 I}{\partial t\partial x} - R\frac{\partial I}{\partial x} - \frac{LG}{C}\overbrace{\left[C\frac{\partial V}{\partial t} + \frac{\partial I}{\partial x} + GV\right]}^{=)} \\
&= -L\frac{\partial^2 I}{\partial t\partial x} - R\frac{\partial I}{\partial x} = \frac{\partial^2 V}{\partial x^2},
\end{aligned}$$

which shows that V satisfies (1). To show that I satisfies (1), you can proceed as we did for V or you can note that the equations that relate I and V are interchanged if we interchange L and C, and R and G. However, (1) remains unchanged if we interchange L and C, and R and G. So I satisfies (1) if and only if V satisfies (1).

12. The function being graphed is the solution (2) with $c = L = 1$:

$$u(x,\, t) = \sin \pi x \cos \pi t.$$

In the second frame, $t = 1/4$, and so $u(x,\, t) = \sin \pi x \cos \pi/4 = \frac{\sqrt{2}}{2} \sin \pi x$. The maximum of this function (for $0 < x < \pi$ is attained at $x = 1/2$ and is equal to $\frac{\sqrt{2}}{2}$, which is a value greater than 1/2.

13. The function being graphed is

$$u(x,\, t) = \sin \pi x \cos \pi t - \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

In frames 2, 4, 6, and 8, $t = \frac{m}{4}$, where $m = 1$, 3, 5, and 7. Plugging this into $u(x,\, t)$, we find

$$u(x,\, t) = \sin \pi x \cos \frac{m\pi}{4} - \frac{1}{2} \sin 2\pi x \cos \frac{m\pi}{2} + \frac{1}{3} \sin 3\pi x \cos \frac{3m\pi}{4}.$$

For $m = 1$, 3, 5, and 7, the second term is 0, because $\cos \frac{m\pi}{2} = 0$. Hence at these times, we have, for, $m = 1$, 3, 5, and 7,

$$u(x,\, \frac{m}{4}) = \sin \pi x \cos \pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

To say that the graph of this function is symmetric about $x = 1/2$ is equivalent to the assertion that, for $0 < x < 1/2$, $u(1/2 + x,\, \frac{m}{4}) = u(1/2 - x,\, \frac{m}{4})$. Does this equality hold? Let's check:

$$\begin{aligned} u(1/2 + x,\, \frac{m}{4}) &= \sin \pi(x + 1/2) \cos \frac{m\pi}{4} + \frac{1}{3} \sin 3\pi(x + 1/2) \cos \frac{3m\pi}{4} \\ &= \cos \pi x \cos \frac{m\pi}{4} - \frac{1}{3} \cos 3\pi x \cos \frac{3m\pi}{4}, \end{aligned}$$

where we have used the identities $\sin \pi(x + 1/2) = \cos \pi x$ and $\sin 2\pi(x + 1/2) = -\cos 3\pi x$. Similalry,

$$\begin{aligned} u(1/2 - x,\, \frac{m}{4}) &= \sin \pi(1/2 - x) \cos \frac{m\pi}{4} + \frac{1}{3} \sin 3\pi(1/2 - x) \cos \frac{3m\pi}{4} \\ &= \cos \pi x \cos \frac{m\pi}{4} - \frac{1}{3} \cos 3\pi x \cos \frac{3m\pi}{4}. \end{aligned}$$

So $u(1/2 + x,\, \frac{m}{4}) = u(1/2 - x,\, \frac{m}{4})$, as expected.

14. Note that the condition $u(0,\, t) = u(1,\, t)$ holds in all frames. It states that the ends of the string are held fixed at all time. It is the other condition on the first derivative that is specific to the frames in question.

The function being graphed is

$$u(x,\, t) = \sin \pi x \cos \pi t - \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

As we just stated, the equality $u(0,\, t) = u(1,\, t)$ holds for all t. Now

$$\frac{\partial}{\partial x} u(x,\, t) = \pi \cos \pi x \cos \pi t - \pi \cos 2\pi x \cos 2\pi t + \pi \cos 3\pi x \cos 3\pi t.$$

To say that $\frac{\partial}{\partial x} u(0,\, t) = 0$ and $\frac{\partial}{\partial x} u(1,\, t)$ means that the slope of the graph (as a function of x) is zero at $x = 0$ and $x = 1$. This is a little difficult to see in the frames 2, 4, 6, and 8 in Figure 8, but follows by plugging $t = 1/4$, $3/4$, $5/4$ and $7/4$ and $x = 0$ or 1 in the derivative. For example, when $x = 0$ and $t = 1/4$, we obtain:

$$\frac{\partial}{\partial x}u(0,\,1/4) = \pi\cos\pi/4 - \pi\cos\pi/2 + \pi\cos 3\pi/4 = \pi\cos\pi/4 + \pi\cos 3\pi/4 = 0.$$

15. Since the initial velocity is 0, from (10), we have

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = f(x) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}$ implies that

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sin\frac{2\pi x}{L}.$$

The equation is satisfied with the choice $b_1 = 0$, $b_2 = 1$, and all other b_n's are zero. This yields the solution

$$u(x,\,t) = \sin\frac{2\pi x}{L}\cos\frac{2c\pi t}{L}.$$

Note that the condition $u_t(x,\,0) = 0$ is also satisfied.

16. Since the initial velocity is 0, from (10), we have

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = f(x) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}$ implies that

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}.$$

Clearly, this equation is satisfied with the choice $b_1 = \frac{1}{2}$, $b_3 = \frac{1}{4}$, and all other b_n's are zero. This yields the solution

$$u(x,\,t) = \frac{1}{2}\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}\cos\frac{3c\pi t}{L}.$$

Note that the condition $u_t(x,\,0) = 0$ is also satisfied.

17. Same reasoning as in the previous exercise, we find the solution

$$u(x,\,t) = \frac{1}{2}\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}\cos\frac{3c\pi t}{L} + \frac{2}{5}\sin\frac{7\pi x}{L}\cos\frac{7c\pi t}{L}.$$

18. Since the initial displacement is 0, we use the functions following (1):

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = 0$ is clearly satisfied. To satisfy the second intial condition, we proceed as follows:

$$\begin{aligned} g(x) &= \sin\frac{\pi x}{L} = \frac{\partial}{\partial t}u(x,\,0) \\ &= \frac{\partial}{\partial t}\sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L} \\ &= \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\frac{\partial}{\partial t}\left(\sin\frac{cn\pi t}{L}\right)\Big|_{t=0} \\ &= \left(\sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right)\Big|_{t=0} \\ &= \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\frac{cn\pi}{L}. \end{aligned}$$

Thus

$$\sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n^* \frac{cn\pi}{L}\sin\frac{n\pi x}{L}\frac{cn\pi}{L}.$$

This equality holds if we take $b_1^* \frac{c\pi}{L} = 1$ or $b_1^* = \frac{L}{c\pi}$, and all other $b_n^* = 0$. Thus

$$u(x,\,t) = \frac{L}{c\pi}\sin\frac{\pi x}{L}\sin\frac{c\pi t}{L}.$$

This solution satisfies both initial conditions.

19. Reasoning as in the previous exercise, we satrt with the solution

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = 0$ is clearly satisfied. To satisfy the second intial condition, we must have

$$\begin{aligned} \frac{1}{4}\sin\frac{3\pi x}{L} - \frac{1}{10}\sin\frac{6\pi x}{L} &= \left[\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}\right)\right]_{t=0} \\ &= \sum_{n=1}^{\infty} \frac{cn\pi}{L} b_n^* \sin\frac{n\pi x}{L}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{4} &= \frac{3c\pi}{L}b_3^* \quad\Rightarrow\quad b_3^* = \frac{L}{12c\pi}; \\ -\frac{1}{10} &= \frac{6c\pi}{L}b_6^* \quad\Rightarrow\quad b_6^* = -\frac{L}{60c\pi}; \end{aligned}$$

and all other b_n^8 are 0. Thus

$$u(x,\,t) = \frac{L}{12c\pi}\sin\frac{3\pi x}{L}\sin\frac{3c\pi t}{L} - \frac{L}{60c\pi}\sin\frac{6\pi x}{L}\sin\frac{6c\pi t}{L}.$$

20. Write the initial condition as $u(x,\,0) = \frac{1}{4}\sin\frac{4\pi x}{L}$, then proceed as in Exercises 15 or 16 and you will get

$$u(x,\,t) = \frac{1}{4}\sin\frac{4\pi x}{L}\cos\frac{4c\pi t}{L}.$$

21. (a) We have to show that $u(\frac{1}{2},\,t)$ is a constant for all $t > 0$. With $c = L = 1$, we have

$$u(x,\,t) = \sin 2\pi x\cos 2\pi t \Rightarrow u(1/2,\,t) = \sin\pi\cos 2\pi t = 0 \quad \text{for all } t > 0.$$

(b) One way for $x = 1/3$ not to move is to have $u(x, t) = \sin 3\pi x \cos 3\pi t$. This is the solution that corresponds to the initial condition $u(x, 0) = \sin 3\pi x$ and $\frac{\partial u}{\partial t}(x, 0) = 0$. For this solution, we also have that $x = 2/3$ does not move for all t.

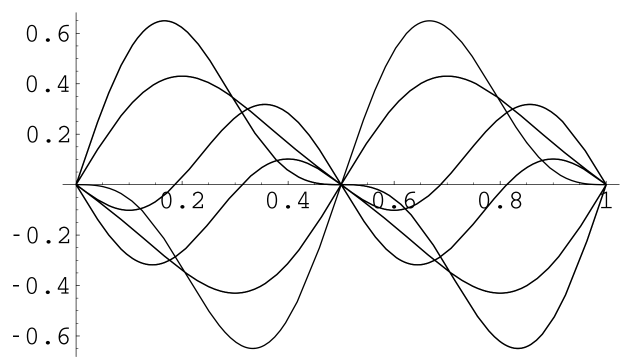
22. (a) Reasoning as in Exercise 17, we find the solution to be

$$u(x, t) = \frac{1}{2}\sin 2\pi x \cos 2\pi t + \frac{1}{4}\sin 4\pi x \cos 4\pi t.$$

(b) We used Mathematica to plot the shape of the string at times $t = 0$ to $t = 1$ by increments of .1. The string returns to some of its previous shapes. For example, when $t = .1$ and when $t = .9$, the string has the same shape.

```
u[x_, t_] = 1 / 2 Sin[2 Pi x] Cos[2 Pi t] + 1 / 4 Sin[4 Pi x] Cos[4 Pi t]
tt = Table[u[x, t], {t, 0, 1, .1}];
Plot[Evaluate[tt], {x, 0, 1}]
```

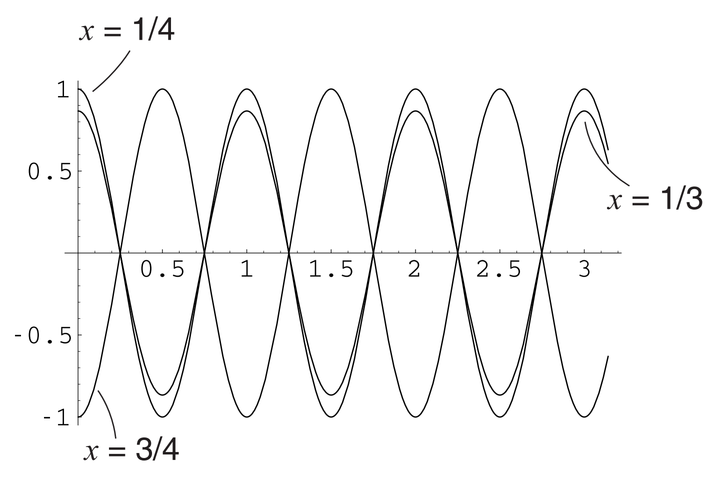
$\frac{1}{2}$ Cos[2 π t] Sin[2 π x] + $\frac{1}{4}$ Cos[4 π t] Sin[4 π x]



The point $x = 1/2$ does not move. This is clear: If we put $x = 1/2$ in the solution, we obtain $u(1/2, t) = 0$ for all t.

23. The solution is $u(x, t) = \sin 2\pi x \cos 2\pi t$. The motions of the points $x = 1/4, 1/3$, and $3/4$ are illustrated by the following graphs. Note that the point $x = 1/2$ does not move, so the graph that describes its motion is identically 0.

```
u[x_, t_] = Sin[2 Pi x] Cos[2 Pi t]
Plot[{u[1 / 4, t], u[1 / 3, t], u[3 / 4, t]}, {t, 0, Pi}]
```



In each case, we have a cosine wave, namely $u(x_0, t) = \sin 2\pi x_0 \cos 2\pi t$, scaled by a factor $\sin 2\pi x_0$.

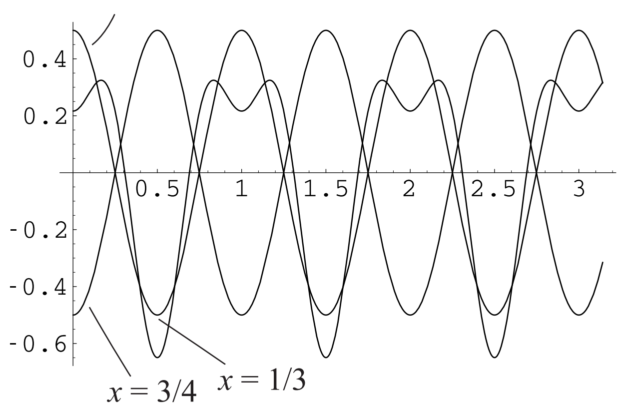
24. The solution in Exercise 22 is

$$u(x, t) = \frac{1}{2}\sin 2\pi x \cos 2\pi t + \frac{1}{4}\sin 4\pi x \cos 4\pi t.$$

The motions of the points $x = 1/4$, $1/3$, $1/2$, and $3/4$ are illustrated by the following graphs. As in the previous exercise, the point $x = 1/2$ does not move, so the graph that describes its motion is identically 0.

```
u[x_, t_] = 1 / 2 Sin[2 Pi x] Cos[2 Pi t] + 1 / 4 Sin[4 Pi x] Cos[4 Pi t]
Plot[{u[1 / 4, t], u[1 / 3, t], u[3 / 4, t]}, {t, 0, Pi}]
```

$$\frac{1}{2}\text{Cos}[2\pi t]\ \text{Sin}[2\pi x] + \frac{1}{4}\text{Cos}[4\pi t]\ \text{Sin}[4\pi x]$$



Unlike the previous exercises, here the motion of a point is not always a cosine wave. In fact, it is the sum of two scaled cosine waves: $u(x_0, t) = \frac{1}{2}\sin 2\pi x_0 \cos 2\pi t + \frac{1}{4}\sin 4\pi x_0 \cos 4\pi t$.

25. The solution (2) is

$$u(x, t) = \sin\frac{\pi x}{L}\cos\frac{\pi c t}{L}.$$

Its initial conditions at time $t_0 = \frac{3L}{2c}$ are

$$u(x, \frac{3L}{2c}) = \sin\frac{\pi x}{L}\cos\left(\frac{\pi c}{L}\cdot\frac{3L}{2c}\right) = \sin\frac{\pi x}{L}\cos\frac{3\pi}{2} = 0;$$

and

$$\frac{\partial u}{\partial t}(x, \frac{3L}{2c}) = -\frac{\pi c}{L}\sin\frac{\pi x}{L}\sin\left(\frac{\pi c}{L}\cdot\frac{3L}{2c}\right) = -\frac{\pi c}{L}\sin\frac{\pi x}{L}\sin\frac{3\pi}{2} = \frac{\pi c}{L}\sin\frac{\pi x}{L}.$$

26. We have

$$\begin{aligned} u(x, t + (3L)/(2c)) &= \sin\frac{\pi x}{L}\cos\frac{\pi c}{L}\cdot(t + \frac{3L}{2c}) \\ &= \sin\frac{\pi x}{L}\cos\left(\frac{\pi c t}{L} + \frac{3\pi}{2}\right) = \sin\frac{\pi x}{L}\sin\frac{\pi c t}{L}, \end{aligned}$$

where we used the indentity $\cos(a + \frac{3\pi}{2}) = \sin a$. Thus $u(x, t + (3L)/(2c))$ is equal to the solution given by (7). Call the latter solution $v(x, t)$. We know that $v(x, t)$ represents the motion of a string that starts with initial shape $f(x) = 0$ and intial velocity $g(x) = \frac{\pi c}{L}\sin\frac{\pi x}{L}$. Now, appealing to Exercise 25, we have that at time $t_0 = (3L)/(2c)$, the shape of the solution u is $u(x, (3L)/(2c)) = 0$ and its velocity is

$\frac{\partial u}{\partial t}(x, \frac{3L}{2c}) = \frac{\pi c}{L} \sin \frac{\pi x}{L}$. Thus the subsequent motion of the string, u, at time $t + t_0$ is identical to the motion of a string, v, starting at time $t = 0$, whenever v has initial shape $u(x, t_0)$ and the initial velocity $\frac{\partial u}{\partial t}(x, t_0)$.

27. (a) The equation is equivalent to

$$-\frac{1}{r}\frac{\partial u}{\partial t} - \frac{\kappa}{r}\frac{\partial u}{\partial x} = u.$$

The solution of this equation follows from Exercise 8, Section 1.1, by taking $a = -\frac{1}{r}$ and $b = -\frac{\kappa}{r}$. Thus

$$u(x, t) = f(-\frac{1}{r}x + \frac{\kappa}{r}t)e^{-rt}.$$

Note that this equivalent to

$$u(x, t) = f(x - \kappa t)e^{-rt},$$

by replacing the function $x \mapsto f(x)$ in the first formula by $x \mapsto f(-rx)$. This is acceptable because f is arbitrary.
(b) The number of particles at time $t \geq 0$ is given by $\int_{-\infty}^{\infty} u(x, t)\, dx$. We have $M = \int_{-\infty}^{\infty} u(x, 0)\, dx$. But $u(x, 0) = f(x)$, so $M = \int_{-\infty}^{\infty} f(x)\, dx$. For $t > 0$, the number of particles is

$$\begin{aligned}\int_{-\infty}^{\infty} u(x, t)\, dx &= \int_{-\infty}^{\infty} f(x - \kappa t)e^{-rt}\, dx \\ &= e^{-rt}\int_{-\infty}^{\infty} f(x - \kappa t)\, dx = e^{-rt}\int_{-\infty}^{\infty} f(x)\, dx = Me^{-rt},\end{aligned}$$

where, in evaluating the integral $\int_{-\infty}^{\infty} f(x - \kappa t)\, dx$, we used the change of variables $x \leftrightarrow x - \kappa t$, and then used $M = \int_{-\infty}^{\infty} f(x)\, dx$.

Solutions to Exercises 2.1

1. (a) $\cos x$ has period 2π. (b) $\cos \pi x$ has period $T = \frac{2\pi}{\pi} = 2$. (c) $\cos \frac{2}{3}x$ has period $T = \frac{2\pi}{2/3} = 3\pi$. (d) $\cos x$ has period 2π, $\cos 2x$ has period π, 2π, 3π,:. A common period of $\cos x$ and $\cos 2x$ is 2π. So $\cos x + \cos 2x$ has period 2π.

2. (a) $\sin 7\pi x$ has period $T = \frac{2\pi}{7\pi} = 2/7$. (b) $\sin n\pi x$ has period $T = \frac{2\pi}{n\pi} = \frac{2}{n}$. Since any integer multiple of T is also a period, we see that 2 is also a period of $\sin n\pi x$. (c) $\cos mx$ has period $T = \frac{2\pi}{m}$. Since any integer multiple of T is also a period, we see that 2π is also a period of $\cos mx$. (d) $\sin x$ has period 2π, $\cos x$ has period 2π; $\cos x + \sin x$ so has period 2π. (e) Write $\sin^2 2x = \frac{1}{2} - \frac{\cos 4x}{2}$. The function $\cos 4x$ has period $T = \frac{2\pi}{4} = \frac{\pi}{2}$. So $\sin^2 2x$ has period $\frac{\pi}{2}$.

3. (a) The period is $T = 1$, so it suffices to describe f on an interval of length 1. From the graph, we have

$$f(x) = \begin{cases} 0 & \text{if } -\frac{1}{2} \le x < 0, \\ 1 & \text{if } 0 \le x < \frac{1}{2}. \end{cases}$$

For all other x, we have $f(x+1) = f(x)$.
(b) f is continuous for all $x \neq \frac{k}{2}$, where k is an integer. At the half-integers, $x = \frac{2k+1}{2}$, using the graph, we see that $\lim_{h\to x^+} f(h) = 0$ and $\lim_{h\to x^-} f(h) = 1$. At the integers, $x = k$, from the graph, we see that $\lim_{h\to x^+} f(h) = 1$ and $\lim_{h\to x^-} f(h) = 0$. The function is piecewise continuous.
(c) Since the function is piecewise constant, we have that $f'(x) = 0$ at all $x \neq \frac{k}{2}$, where k is an integer. It follows that $f'(x+) = 0$ and $f'(x-) = 0$ (Despite the fact that the derivative does not exist at these points; the left and right limits exist and are equal.)

4. The period is $T = 4$, so it suffices to describe f on an interval of length 4. From the graph, we have

$$f(x) = \begin{cases} x+1 & \text{if } -2 \le x \le 0, \\ -x+1 & \text{if } 0 < x < 2. \end{cases}$$

For all other x, we have $f(x+4) = f(x)$. (b) The function is continuous at all x. (c) (c) The function is differentiable for all $x \neq 2k$, where k is an integer. Note that f' is also 4-periodic. We have

$$f'(x) = \begin{cases} 1 & \text{if } -2 < x \le 0, \\ -1 & \text{if } 0 < x < 2. \end{cases}$$

For all other $x \neq 2k$, we have $f(x+4) = f(x)$. If $x = 0, \pm 4, \pm 8, \ldots$, we have $f'(x+) = 1$ and $f'(x-) = -1$. If $x = \pm 2, \pm 6, \pm 10, \ldots$, we have $f'(x+) = -1$ and $f'(x-) = 1$.

5. This is the special case $p = \pi$ of Exercise 6(b).

6. (a) A common period is $2p$. (b) The orthogonality relations are

$$\int_{-p}^{p} \cos\frac{m\pi x}{p} \cos\frac{n\pi x}{p}\,dx = 0 \quad \text{if } m \neq n,\ m,\, n = 0, 1, 2, \ldots;$$

$$\int_{-p}^{p} \sin\frac{m\pi x}{p} \sin\frac{n\pi x}{p}\,dx = 0 \quad \text{if } m \neq n,\ m,\, n = 1, 2, \ldots;$$

$$\int_{-p}^{p} \cos\frac{m\pi x}{p} \sin\frac{n\pi x}{p}\,dx = 0 \quad \text{for all } m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots.$$

These formulas are established by using various addition formulas for the cosine and sine. For example, to prove the first one, if $m \neq n$, then

$$\begin{aligned} &\int_{-p}^{p} \cos\frac{m\pi x}{p} \cos\frac{n\pi x}{p}\,dx \\ &= \frac{1}{2}\int_{-p}^{p} \left[\cos\frac{(m+n)\pi x}{p} + \cos\frac{(m-n)\pi x}{p}\right] dx \\ &= \frac{1}{2}\left[\frac{p}{m+n)\pi}\sin\frac{(m+n)\pi x}{p} + \frac{p}{m-n)\pi}\sin\frac{(m-n)\pi x}{p}\right]\Big|_{-p}^{p} = 0. \end{aligned}$$

7. Suppose that Show that $f_1, f_2, \ldots, f_n, \ldots$ are T-periodic functions. This means that $f_j(x+T) = f(x)$ for all x and $j = 1, 2, \ldots, n$. Let $s_n(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)$. Then

$$\begin{aligned} s_n(x+T) &= = a_1 f_1(x+T) + a_2 f_2(x+T) + \cdots + a_n f_n(x+T) \\ &= a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x) = s_n(x); \end{aligned}$$

which means that s_n is T-periodic. In general, if $s(x) = \sum_{j=1}^{\infty} a_j f_j(x)$ is a series that converges for all x, where each f_j is T-periodic, then

$$s(x+T) = \sum_{j=1}^{\infty} a_j f_j(x+T) = \sum_{j=1}^{\infty} a_j f_j(x) = s(x);$$

and so $s(x)$ is T-periodic.

8. (a) Since $|\cos x| \leq 1$ and $|\cos \pi x| \leq 1$, the equation $\cos x + \cos \pi x = 2$ holds if and only if $\cos x = 1$ and $\cos \pi x = 1$. Now $\cos x = 1$ implies that $x = 2k\pi$, k and integer, and $\cos \pi x = 1$ implies that $x = 2m$, m and integer. So $\cos x + \cos \pi x = 2$ implies that $2m = 2k\pi$, which implies that $k = m = 0$ (because π is irrational). So the only solution is $x = 0$. (b) Since $f(x) = \cos x + \cos \pi x = 2$ takes on the value 2 only at $x = 0$, it is not periodic.

9. (a) Suppose that f and g are T-periodic. Then $f(x+T) \cdot g(x+T) = f(x) \cdot g(x)$, and so $f \cdot g$ is T periodic. Similarly,

$$\frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)},$$

and so f/g is T periodic.
(b) Suppose that f is T-periodic and let $h(x) = f(x/a)$. Then

$$\begin{aligned} h(x+aT) &= f\left(\frac{x+aT}{a}\right) = f\left(\frac{x}{a} + T\right) \\ &= f\left(\frac{x}{a}\right) \quad \text{(because } f \text{ is } T\text{-periodic)} \\ &= h(x). \end{aligned}$$

Thus h has period aT. Replacing a by $1/a$, we find that the function $f(ax)$ has period T/a.
(c) Suppose that f is T-periodic. Then $g(f(x+T)) = g(f(x))$, and so $g(f(x))$ is also T-periodic.

10. (a) $\sin x$ has period 2π, so $\sin 2x$ has period $2\pi/2 = \pi$ (by Exercise 9(b)).
(b) $\cos \frac{1}{2}x$ has period 4π and $\sin 2x$ has period π (or any integer multiple of it). So a common period is 4π. Thus $\cos \frac{1}{2}x + 3 \sin 2x$ had period 4π (by Exercise 7)
(c) We can write $\frac{1}{2+\sin x} = g(f(x))$, where $g(x) = 1/x$ and $f(x) = 2 + \sin x$. Since f is 2π-periodic, it follows that $\frac{1}{2+\sin x}$ is 2π-periodic, by Exercise 9(c).
(d) Same as part (c). Here we let $f(x) = \cos x$ and $g(x) = e^x$. Then $e^{\cos x}$ is 2π-periodic.

11. Using Theorem 1,

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = \int_0^{\pi} f(x)\, dx = \int_0^{\pi} \sin x\, dx = 2.$$

12. Using Theorem 1,

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = \int_0^{\pi} f(x)\, dx = \int_0^{\pi} \cos x\, dx = 0.$$

13.

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi/2} 1\,dx = \pi/2.$$

14. Using Theorem 1,

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} x^2\,dx = \frac{\pi^3}{3}.$$

15. Let $F(x) = \int_a^x f(t)\,dt$. If F is 2π-periodic, then $F(x) = F(x+2\pi)$. But

$$F(x+2\pi) = \int_a^{x+2\pi} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+2\pi} f(t)\,dt = F(x) + \int_x^{x+2\pi} f(t)\,dt.$$

Since $F(x) = F(x+2\pi)$, we conclude that

$$\int_x^{x+2\pi} f(t)\,dt = 0.$$

Applying Theorem 1, we find that

$$\int_x^{x+2\pi} f(t)\,dt = \int_0^{2\pi} f(t)\,dt = 0.$$

The above steps are reversible. That is,

$$\begin{aligned}\int_0^{2\pi} f(t)\,dt = 0 \quad &\Rightarrow \quad \int_x^{x+2\pi} f(t)\,dt = 0\\ &\Rightarrow \quad \int_a^x f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+2\pi} f(t)\,dt = \int_a^{x+2\pi} f(t)\,dt\\ &\Rightarrow \quad F(x) = F(x+2\pi);\end{aligned}$$

and so F is 2π-periodic.

16. We have

$$F(x+T) = \int_a^{x+T} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt.$$

So F is T periodic if and only if

$$\begin{aligned}F(x+T) = F(x) \quad &\Leftrightarrow \quad \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt = \int_a^x f(t)\,dt\\ &\Leftrightarrow \quad \int_x^{x+T} f(t)\,dt = 0\\ &\Leftrightarrow \quad \int_a^{a+T} f(t)\,dt = 0,\end{aligned}$$

where the last assertion follows from Theorem 1.

17. By Exercise 16, F is 2 periodic, because $\int_0^2 f(t)\,dt = 0$ (this is clear from the graph of f). So it is enough to describe F on any interval of length 2. For $0 < x < 2$, we have

$$F(x) = \int_0^x (1-t)\,dt = t - \frac{t^2}{2}\Big|_0^x = x - \frac{x^2}{2}.$$

For all other x, $F(x+2) = F(x)$. (b) The graph of F over the interval $[0, 2]$ consists of the arch of a parabola looking down, with zeros at 0 and 2. Since F is 2-periodic, the graph is repeated over and over.

18. (a) We have

$$\begin{aligned}\int_{nT}^{(n+1)T} f(x)\,dx &= \int_0^T f(s+nT)\,ds \quad (\text{let } x = s+nT,\ dx = ds)\\ &= \int_0^T f(s)\,ds \quad (\text{because } f \text{ is } T\text{-periodic})\\ &= \int_0^T f(x)\,dx.\end{aligned}$$

(b)

$$\begin{aligned}\int_{(n+1)T}^{a+T} f(x)\,dx &= \int_{nT}^{a} f(s+T)\,ds \quad (\text{let } x = s+T,\ dx = ds)\\ &= \int_{nT}^{a} f(s)\,ds \quad (\text{because } f \text{ is } T\text{-periodic})\\ &= \int_{nT}^{a} f(x)\,dx.\end{aligned}$$

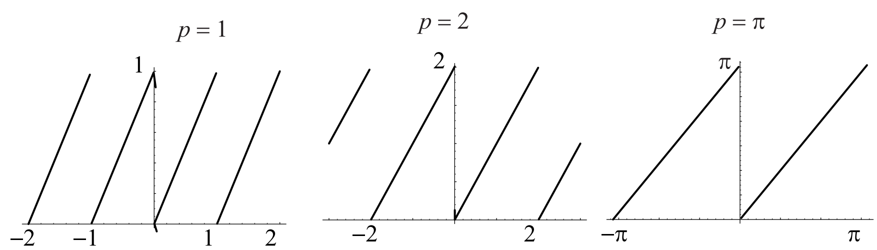
(c)

$$\begin{aligned}\int_a^{a+T} f(x)\,dx &= \int_a^{(n+1)T} f(x)\,dx + \int_{(n+1)T}^{a+T} f(x)\,dx\\ &= \int_a^{(n+1)T} f(x)\,dx + \int_{nT}^{a} f(x)\,dx \quad (\text{by (b)})\\ &= \int_{nT}^{(n+1)T} f(x)\,dx = \int_0^T f(x)\,dx \quad (\text{by (a)}).\end{aligned}$$

19. (a) The plots are shown in the following figures.

The Mathematica command to plot the graphs is the following:

```
p=Pi
Plot[x - p Floor[x/p], {x, -Pi, Pi}]
```

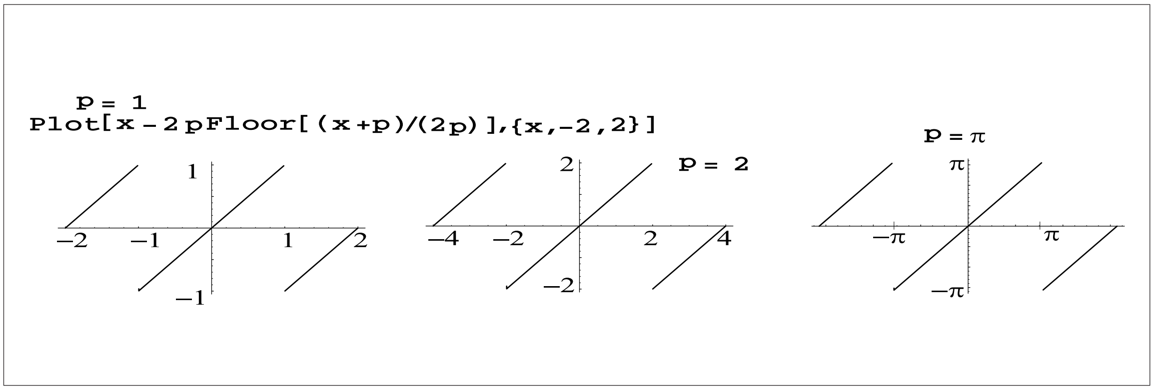


(b) Let us show that $f(x) = x - p\left[\frac{x}{p}\right]$ is p-periodic.

$$\begin{aligned}f(x+p) &= x+p-p\left[\frac{x+p}{p}\right] = x+p-p\left[\frac{x}{p}+1\right] = x+p-p\left(\left[\frac{x}{p}\right]+1\right)\\ &= x-p\left[\frac{x}{p}\right] = f(x).\end{aligned}$$

From the graphs it is clear that $f(x) = x$ for all $0 < x < p$. To see this from the formula, use the fact that $[t] = 0$ if $0 \le t < 1$. So, if $0 \le x < p$, we have $0 \le \frac{x}{p} < 1$, so $\left[\frac{x}{p}\right] = 0$, and hence $f(x) = x$.

20. (a) Plot of the function $f(x) = x - 2p\left[\frac{x+p}{2p}\right]$ for $p = 1,\ 2,$ and π.

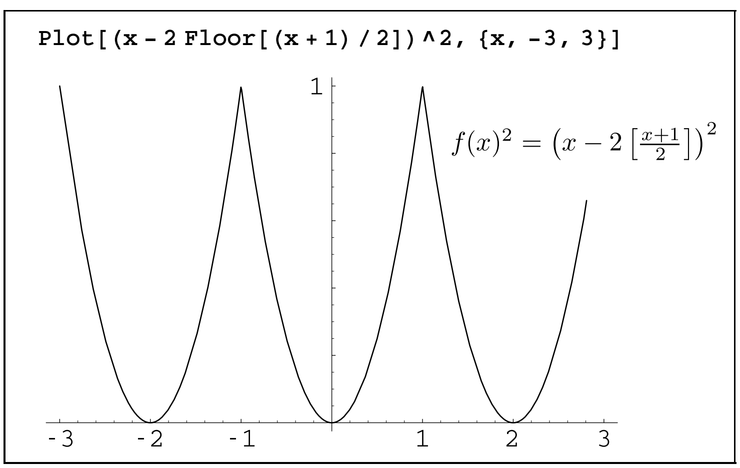


(b)

$$\begin{aligned} f(x+2p) &= (x+2p) - 2p\left[\frac{(x+2p)+p}{2p}\right] = (x+2p) - 2p\left[\frac{x+p}{2p}+1\right] \\ &= (x+2p) - 2p\left(\left[\frac{x+p}{2p}\right]+1\right) = x - 2p\left[\frac{x+p}{2p}\right] = f(x). \end{aligned}$$

So f is $2p$-periodic. For $-p < x < p$, we have $0 < \frac{x+p}{2p} < 1$, hence $\left[\frac{x+p}{2p}\right] = 0$, and so $f(x) = x - 2p\left[\frac{x+p}{2p}\right] = x$.

21. (a) With $p = 1$, the function f becomes $f(x) = x - 2\left[\frac{x+1}{2}\right]$, and its graph is the first one in the group shown in Exercise 20. The function is 2-periodic and is equal to x on the interval $-1 < x < 1$. By Exercise 9(c), the function $g(x) = h(f(x)$ is 2-periodic for any function h; in particular, taking $h(x) = x^2$, we see that $g(x) = f(x)^2$ is 2-periodic. (b) $g(x) = x^2$ on the interval $-1 < x < 1$, because $f(x) = x$ on that interval. (c) Here is a graph of $g(x) = f(x)^2 = \left(x - 2\left[\frac{x+1}{2}\right]\right)^2$, for all x.

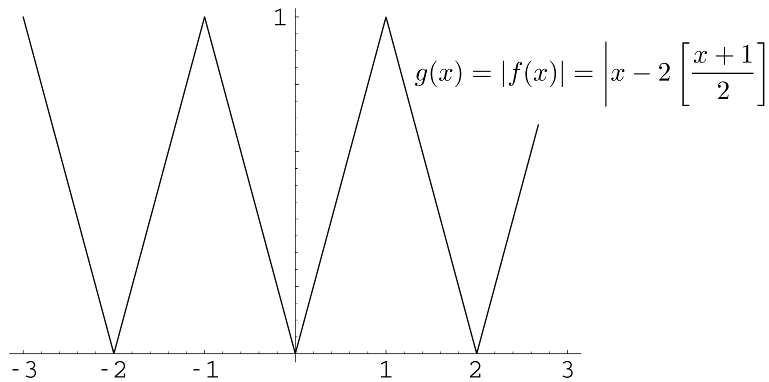


22. (a) As in Exercise 21, the function $f(x) = x - 2\left[\frac{x+1}{2}\right]$ is 2-periodic and is equal to x on the interval $-1 < x < 1$. So, by Exercise 9(c), the function

$$g(x) = |f(x)| = \left|x - 2\left[\frac{x+1}{2}\right]\right|$$

is 2-periodic and is clearly equal to $|x|$ for all $-1 < x < 1$. Its graph is a triangular wave as shown in (b).

```
Plot[Abs[x - 2 Floor[(x + 1) / 2]], {x, -3, 3}]
```



(c) To obtain a triangular wave of arbitrary period $2p$, we use the $2p$-periodic function

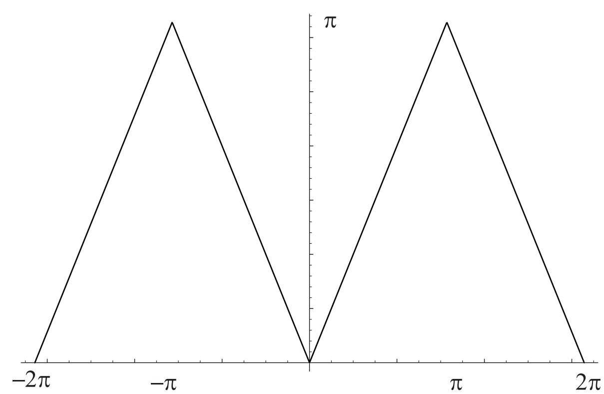
$$f(x) = x - 2p\left[\frac{x+p}{2p}\right],$$

which is equal to x on the interval $-p < x < p$. Thus,

$$g(x) = \left| x - 2p\left[\frac{x+p}{2p}\right] \right|$$

is a $2p$-periodic triangular wave, which equal to $|x|$ in the interval $-p < x < p$. The following graph illustrates this function with $p = \pi$.

```
p = Pi
Plot[Abs[x - 2 p Floor[(x + p) / (2 p)]], {x, -2 Pi, 2 Pi}]
```



23. (a) Since $f(x + 2p) = f(x)$, it follows that $g(f(x + 2p)) = g(f(x))$ and so $g(f(x))$ is $2p$-periodic. For $-p < x < p$, $f(x) = x$ and so $g(f(x)) = g(x)$.
(b) The function $e^{g(x)}$, with $p = 1$, is the 2-periodic extension of the function which equals e^x on the interval $-1 < x < 1$. Its graph is shown in Figure 1, Section 2.6 (with $a = 1$).

24. Let $f_1, f_2, \ldots, f_n$ be the continuous components of f on the interval $[0, T]$, as described prior to the exercise. Since each f_j is continuous on a closed and bounded interval, it is bounded: That is, there exists $M > 0$ such that $|f_j(x)| \le M$ for all x in the domain of f_j. Let M denote the maximum value of M_j for $j = 1, 2, \ldots, n$. Then $|f(x)| \le M$ for all x in $[0, T]$ and so f is bounded.

25. We have

$$\begin{aligned} |F(a+h) - F(a)| &= \left| \int_0^a f(x)\,dx - \int_0^{a+h} f(x)\,dx \right| \\ &= \left| \int_a^{a+h} f(x)\,dx \right| \leq M \cdot h, \end{aligned}$$

where M is a bound for $|f(x)|$, which exists by the previous exercise. (In deriving the last inequality, we used the following property of integrals:

$$\left| \int_a^b f(x)\,dx \right| \leq (b-a) \cdot M,$$

which is clear if you interpret the integral as an area.) As $h \to 0$, $M \cdot h \to 0$ and so $|F(a+h) - F(a)| \to 0$, showing that $F(a+h) \to F(a)$, showing that F is continuous at a.
(b) If f is continuous and $F(a) = \int_0^a f(x)\,dx$, the fundamental theorem of calculus implies that $F'(a) = f(a)$. If f is only piecewise continuous and a_0 is a point of continuity of f, let (x_{j-1}, x_j) denote the subinterval on which f is continuous and a_0 is in (x_{j-1}, x_j). Recall that $f = f_j$ on that subinterval, where f_j is a continuous component of f. For a in (x_{j-1}, x_j), consider the functions $F(a) = \int_0^a f(x)\,dx$ and $G(a) = \int_{x_{j-1}}^a f_j(x)\,dx$. Note that $F(a) = G(a) + \int_0^{x_{j-1}} f(x)\,dx = G(a) + c$. Since f_j is continuous on (x_{j-1}, x_j), the fundamental theorem of calculus implies that $G'(a) = f_j(a) = f(a)$. Hence $F'(a) = f(a)$, since F differs from G by a constant.

26. We have

$$F(a) = \int_0^{a+T} f(x)\,dx - \int_0^a f(x)\,dx.$$

By the previous exercise, F is a sum of two continuous and piecewise smooth functions. (The first term is a translate of $\int_0^a f(x)\,dx$ by T, and so it is continuous and piecewise smooth.) Thus F is continuous and piecewise smooth. Since each term is differentiable the points of continuity of f, we conclude that F is also differentiable at the points of continuity of f.
(b) By Exercise 25, we have, at the points where f is continuous, $F'(a) = f(a + T) - f(a) = 0$, because f is periodic with period T. Thus F is piecewise constant.
(c) A piecewise constant function that is continuous is constant (just take left and right limits at points of discontinuity.) So F is constant.

27. (a) The function $\sin \frac{1}{x}$ does not have a right or left limit as $x \to 0$, and so it is not piecewise continuous. (To be piecewise continuous, the left and right limits must exist.) The reason is that $1/x$ tends to $+\infty$ as $x \to 0^+$ and so $\sin 1/x$ oscillates between $+1$ and -1. Similarly, as $x \to 0^-$, $\sin 1/x$ oscillates between $+1$ and -1. See the graph.
(b) The function $f(x) = x \sin \frac{1}{x}$ and $f(0) = 0$ is continuous at 0. The reason for this is that $\sin 1/x$ is bounded by 1, so, as $x \to 0$, $x \sin 1/x \to 0$, by the squeeze theorem. The function, however, is not piecewise smooth. To see this, let us compute its derivative. For $x \neq 0$,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

As $x \to 0^+$, $1/x \to +\infty$, and so $\sin 1/x$ oscillates between $+1$ and -1, while $\frac{1}{x} \cos \frac{1}{x}$ oscillates between $+\infty$ and $-\infty$. Consequently, $f'(x)$ has no right limit at 0. Similarly, it fails to have a left limit at 0. Hence f is not piecewise smooth. (Recall that to be piecewise smooth the left and right limits of the derivative have to exist.)
(c) The function $f(x) = x^2 \sin \frac{1}{x}$ and $f(0) = 0$ is continuous at 0, as in case (b).

Also, as in (b), the function is not piecewise smooth. To see this, let us compute its derivative. For $x \neq 0$,

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

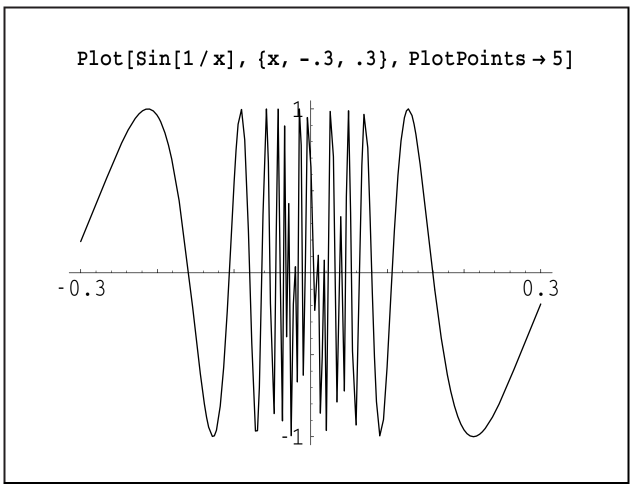
As $x \to 0^+$, $1/x \to +\infty$, and so $2x \sin 1/x \to 0$, while $\cos \frac{1}{x}$ oscillates between $+1$ and -1. Hence, as $x \to 0^+$, $2x \sin \frac{1}{x} - \cos \frac{1}{x}$ oscillates between $+1$ and -1, and so $f'(x)$ has no right limit at 0. Similarly, it fails to have a left limit at 0. Hence f is not piecewise smooth. (d) The function $f(x) = x^3 \sin \frac{1}{x}$ and $f(0) = 0$ is continuous at 0, as in case (b). It is also smooth. We only need to check the derivative at $x = 0$. We have

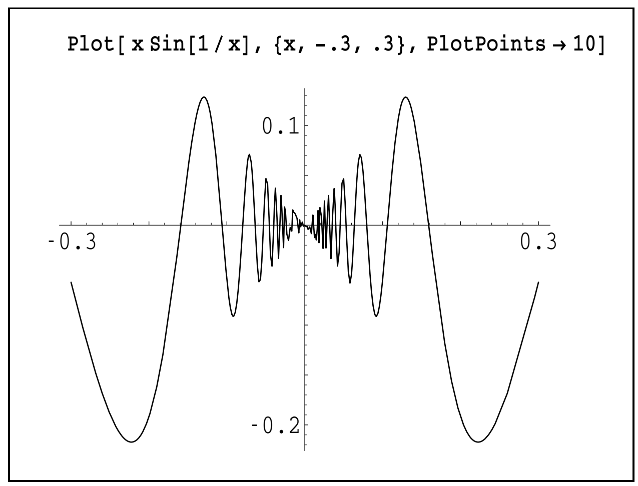
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^3 \sin \frac{1}{h}}{h} \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0.$$

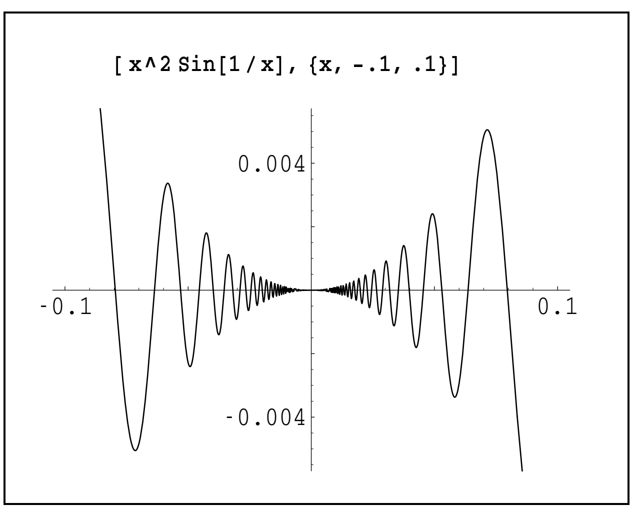
For $x \neq 0$, we have

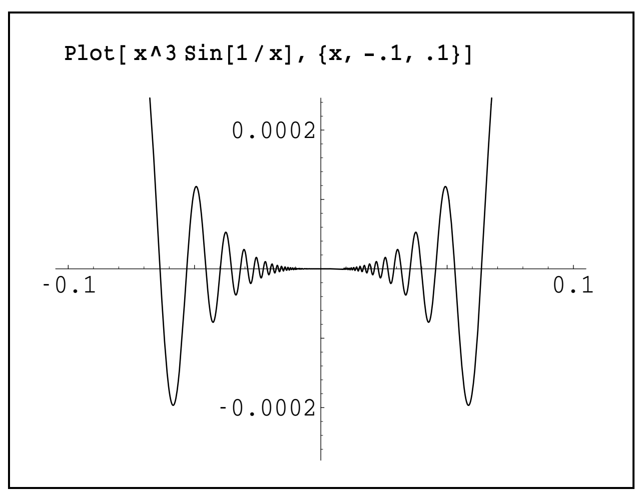
$$f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}.$$

Since $f'(x) \to 0 = f'(0)$ as $x \to 0$, we conclude that f' exists and is continuous for all x.



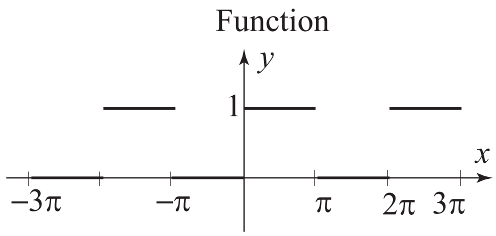


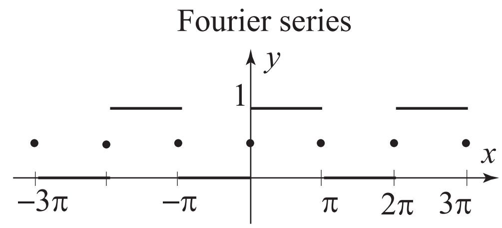




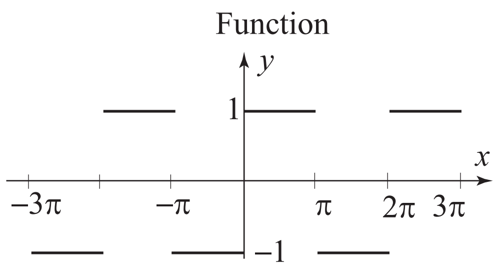
Solutions to Exercises 2.2

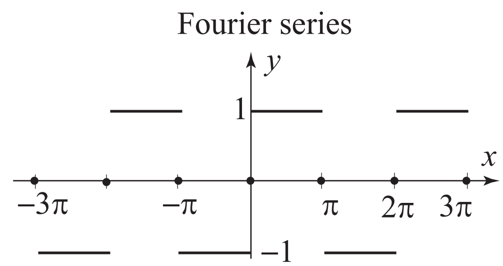
1. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is $\frac{1}{2}$.



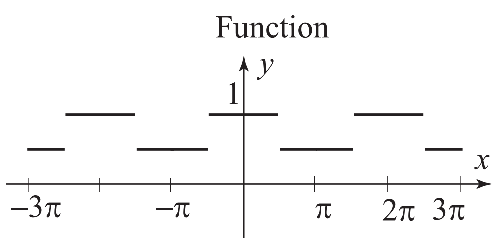


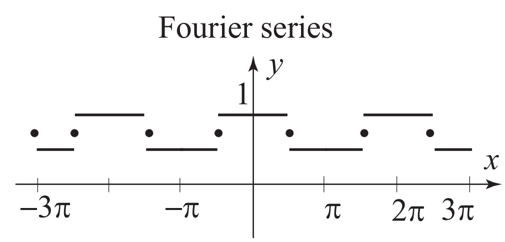
2. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is 0 in this case.



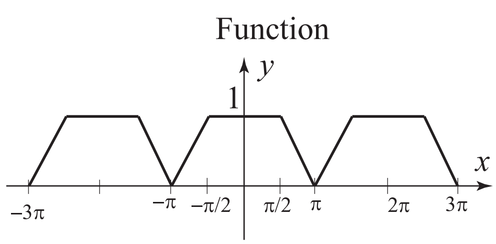


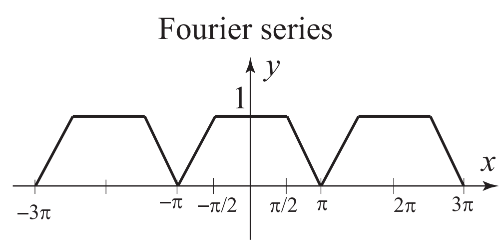
3. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is 3/4 in this case.





4. Since the function is continuous and piecewise smooth, it is equal to its Fourier series.





5. We compute the Fourier coefficients using he Euler formulas. Let us first note that since $f(x) = |x|$ is an even function on the interval $-\pi < x < \pi$, the product $f(x)\sin nx$ is an odd function. So

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{|x|\sin nx}^{\text{odd function}}\, dx = 0,$$

because the integral of an odd function over a symmetric interval is 0. For the other coefficients, we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= = \frac{1}{2\pi}\int_{-\pi}^{0} (-x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{2\pi}x^2\Big|_0^{\pi} = \frac{\pi}{2}. \end{aligned}$$

In computing a_n $(n \geq 1)$, we will need the formula

$$\int x \cos ax\,dx = \frac{\cos(a\,x)}{a^2} + \frac{x\,\sin(a\,x)}{a} + C \quad (a \neq 0),$$

which can be derived using integration by parts. We have, for $n \geq 1$,

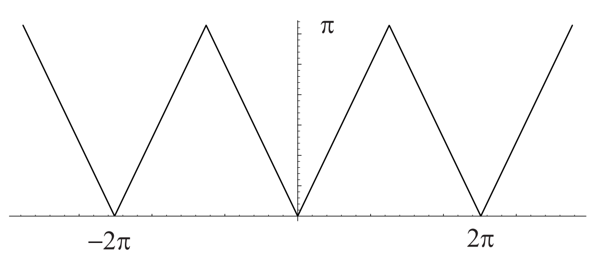
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\big[(-1)^n - 1\big] \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier series is

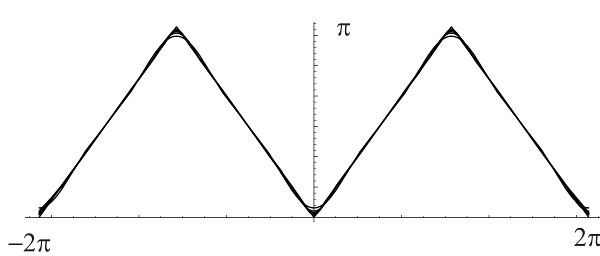
$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos(2k+1)x\,.$$

In[2]:=
```
s[n_, x_] := Pi / 2 - 4 / Pi Sum[1 / (2 k + 1) ^2 Cos[(2 k + 1) x], {k, 0, n}]
```

In[25]:=
```
partialsums = Table[s[n, x], {n, 1, 7}];
f[x_] = x - 2 Pi Floor[(x + Pi) / (2 Pi)]
g[x_] = Abs[f[x]]
Plot[g[x], {x, -3 Pi, 3 Pi}]
Plot[Evaluate[{g[x], partialsums}], {x, -2 Pi, 2 Pi}]
```



The function g(x) = | x |
and its periodic extension



Partial sums of
the Fourier series. Since we are summing over the odd integers, when n = 7, we are actually summing the 15th partial sum.

6. We compute the Fourier coefficients using he Euler formulas. Let us first note that $f(x)$ is an odd function on the interval $-\pi < x < \pi$, so

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

because the integral of an odd function over a symmetric interval is 0. A similar argument shows that $a_n = 0$ for all n. This leaves b_n. We have, for $n \geq 1$,

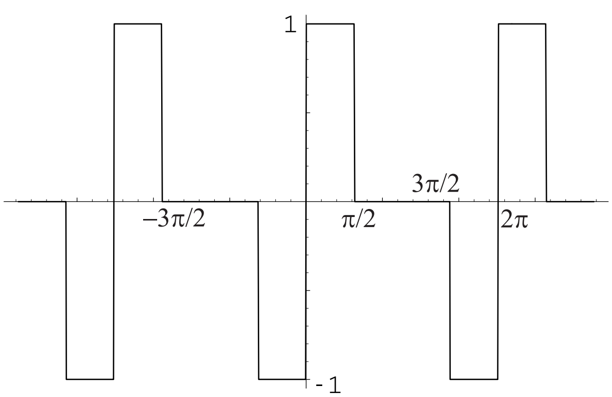
$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi/2}\sin nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi/2}\sin nx\,dx \\ &= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]\Big|_{0}^{\pi/2} \\ &= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n}\left[1 - \cos\frac{n\pi}{2}\right] \end{aligned}$$

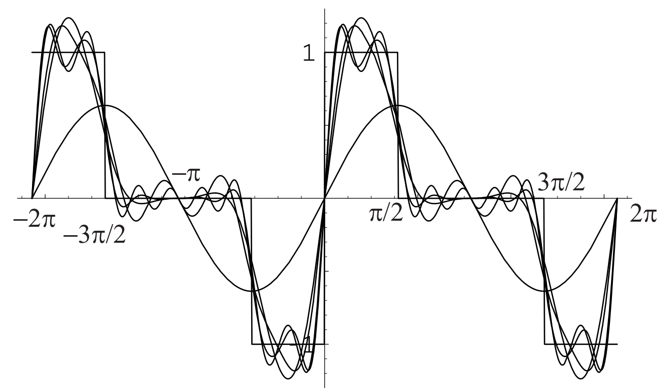
Thus the Fourier series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin nx.$$

(b) The new part here is defining the periodic function. We willuse the same construction as in the previous exercise. We will use the Which command to define a function by several formulas over the interval - 2 Pi to 2 Pi. We then use the construction of Exrcises 20 and 22 of Section 2.1 to define the periodic extension.

```
s[n_, x_] := 2 / Pi Sum[(1 - Cos[ k Pi / 2]) / k Sin[k x], {k, 1, n}]

partialsums = Table[s[n, x], {n, 1, 10, 2}];
f[x_] = Which[x < -Pi / 2, 0, -Pi / 2 < x < 0, -1, 0 < x < Pi / 2, 1, x > Pi / 2, 0]
g[x_] = x - 2 Pi Floor[(x + Pi) / ( 2 Pi)]
h[x_] = f[g[x]]
Plot[h[x], {x, -3 Pi, 3 Pi}]
Plot[Evaluate[{h[x], partialsums}], {x, -2 Pi, 2 Pi}]
```





The partial sums of the Fourier series converge to the function at all points of continuity. At the points of discontinuity, we observe a Gibbs phenomenon. At the points of discontinuity, the Fourier series is converging to 0 or $\pm 1/2$. depending on the location of the point of discontinuity: For all $x = 2k\pi$, the Fourier series converges to 0; and for all $x = (2k+1)\pi/2$, the Fourier series converges to $(-1)^k/2$.

7. f is even, so all the b_n's are zero. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = \frac{2}{\pi}.$$

We will need the trigonometric identity

$$\sin a \cos b = \frac{1}{2}\big(\sin(a-b) + \sin(a+b)\big).$$

For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\big(\sin(1-n)x + \sin(1+n)x\big)\,dx \\
&= \frac{1}{\pi}\left[\frac{-1}{1-n}\cos(1-n)x - \frac{1}{(1+n)}\cos(1+n)x\right]\Big|_0^{\pi} \quad (\text{if } n \neq 1) \\
&= \frac{1}{\pi}\left[\frac{-1}{1-n}(-1)^{1-n} - \frac{1}{(1+n)}(-1)^{1+n} + \frac{1}{1-n} + \frac{1}{1+n}\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{4}{\pi(1-n^2)} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

If $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\,dx = 0.$$

Thus, the Fourier series is : $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1}\cos 2kx\,.$

8. You can compute the Fourier series directly as we did in Exercise 7, or you can use Exercise 7 and note that $|\cos x| = |\sin(x + \frac{\pi}{2})|$. This identity can be verified by comparing the graphs of the functions or as follows:

$$\sin(x + \frac{\pi}{2}) = \sin x \overbrace{\cos\frac{\pi}{2}}^{=0} + \cos x \overbrace{\sin\frac{\pi}{2}}^{=1} = \cos x.$$

So

$$\begin{aligned}
|\cos x| &= |\sin(x + \frac{\pi}{2})| \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1}\cos\Big(2k(x + \frac{\pi}{2})\Big) \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2 - 1}\cos 2kx,
\end{aligned}$$

where we have used

$$\cos\Big(2k(x + \frac{\pi}{2})\Big) = \cos 2kx\,\cos k\pi - \sin 2kx\,\sin k\pi = (-1)^k \cos 2kx.$$

9. Just some hints:
(1) f is even, so all the b_n's are zero.
(2)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

(3) Establish the identity

$$\int x^2 \cos(ax)\, dx = \frac{2\,x\,\cos(a\,x)}{a^2} + \frac{\left(-2+a^2\,x^2\right)\,\sin(a\,x)}{a^3} + C \qquad (a \neq 0),$$

using integration by parts.

10. The function $f(x) = 1 - \sin x + 3\cos 2x$ is already given by its own Fourier series. If you try to compute the Fourier coefficients by using the Euler formulas, you will find that all a_ns and b_n's are 0 except $a_0 = 1$, $a_2 = 3$, and $b_1 = -1$. This is because of the orthogonality of the trigonometric system. Let us illustrate this by computing the b_n's ($n \geq 1$):

$$\begin{aligned}
b_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - \sin x + 3\cos 2x)\sin nx\, dx \\
&= \frac{1}{2\pi}\overbrace{\int_{-\pi}^{\pi} \sin nx\, dx}^{=0 \text{ for all } n} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \sin nx\, dx + \frac{3}{2\pi}\overbrace{\int_{-\pi}^{\pi} \cos 2x \sin nx\, dx}^{=0 \text{ for all } n} \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \sin nx\, dx \\
&= \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

which shows that $b_1 = -1$, while all other b_ns are 0.

11. We have $f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $g(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$. Both functions are given by their Fourier series.

12. The function is clearly odd and so, as in Exercise 6, all the a_ns are 0. Also,

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 x - x^3)\sin nx\, dx.$$

To compute this integral, we use integration by parts, as follows:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \overbrace{(\pi^2 x - x^3)}^{u}\overbrace{\sin nx}^{v'}\, dx \\
&= \frac{2}{\pi}(\pi^2 x - x^3)\frac{-\cos nx}{n}\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} (\pi^2 - 3x^2)\frac{-\cos nx}{n}\, dx \\
&= \frac{2}{\pi}\int_0^{\pi} \overbrace{(\pi^2 - 3x^2)}^{u}\frac{\overbrace{\cos nx}^{v'}}{n}\, dx \\
&= \frac{2}{\pi}(\pi^2 - 3x^2)\frac{\sin nx}{n^2}\Big|_0^{\pi} - \frac{12}{\pi}\int_0^{\pi} x\frac{\sin nx}{n^2}\, dx \\
&= \frac{12}{\pi}x\frac{(-\cos nx)}{n^3}\Big|_0^{\pi} + \frac{12}{\pi}\overbrace{\int_0^{\pi} \frac{\cos nx}{n^3}\, dx}^{=0} \\
&= 12\frac{(-\cos n\pi)}{n^3} = 12\frac{(-1)^{n+1}}{n^3}
\end{aligned}$$

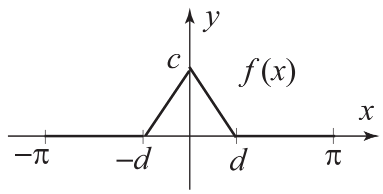
Thus the Fourier series is

$$12\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}\sin nx.$$

13. You can compute directly as we did in Example 1, or you can use the result of Example 1 as follows. Rename the function in Example 1 $g(x)$. By comparing graphs, note that $f(x) = -2g(x+\pi)$. Now using the Fourier series of $g(x)$ from Example, we get

$$f(x) = -2\sum_{n=1}^{\infty} \frac{\sin n(\pi + x)}{n} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

14. The function is even. Its graph is as follows:



Since $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, by interpreting the integral as an area, we see that a_0 is $1/(2\pi)$ times the area under the graph of $f(x)$, above the x-axis, from $x=-d$ to $x=d$. Thus $a_0 = \frac{1}{2\pi}cd$. To compute a_n, we use integration by parts:

$$\begin{aligned} a_n &= \frac{2c}{d\pi}\int_0^d \overbrace{(x-d)}^{u}\overbrace{\cos nx}^{v'}\,dx \\ &= \frac{2c}{d\pi}(x-d)\frac{\sin nx}{n}\Big|_0^d - \frac{2}{\pi}\int_0^d \frac{\sin nx}{n}\,dx \\ &= \frac{2c}{d\pi}\frac{-\cos nx}{n^2}\Big|_0^d = \frac{2c}{d\pi n^2}\left(1-\cos(nd)\right). \end{aligned}$$

Using the identity $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$, we find that

$$b_n = \frac{4c}{d\pi n^2}\sin^2\frac{nd}{2}.$$

Thus the Fourier series is

$$\frac{cd}{2\pi} + \frac{4c}{d\pi}\sum_{n=1}^{\infty}\frac{\sin^2\frac{nd}{2}}{n^2}\cos nx.$$

15. f is even, so all the b_n's are zero. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} e^{-x}\,dx = -\frac{e^{-x}}{\pi}\Big|_0^{\pi} = \frac{1-e^{-\pi}}{\pi}.$$

We will need the integral identity

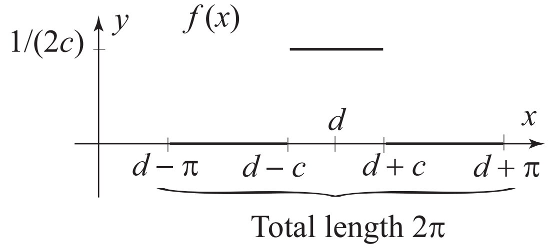
$$\int e^{ax}\cos(bx)\,dx = \frac{a\,e^{a\,x}\,\cos(b\,x)}{a^2+b^2} + \frac{b\,e^{a\,x}\,\sin(b\,x)}{a^2+b^2} + C \quad (a^2+b^2 \neq 0),$$

which can be established by using integration by parts; alternatively, see Exercise 17, Section 2.6. We have, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{n}{n^2+1}e^{-x}\sin nx - \frac{1}{n^2+1}e^{-x}\cos nx\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi(n^2+1)}\left[-e^{-\pi}(-1)^n+1\right] = \frac{2(1-(-1)^n e^{-\pi})}{\pi(n^2+1)}. \end{aligned}$$

Thus the Fourier series is $\displaystyle \frac{e^{\pi}-1}{\pi e^{\pi}} + \frac{2}{\pi e^{\pi}}\sum_{n=1}^{\infty}\frac{1}{n^2+1}\left(e^{\pi}-(-1)^n\right)\cos nx\,.$

16. In general, the function is neither even nor odd. Its graph is the following:



As in Exercise 14, we interpret a_0 as a scaled area and obtain $a_0 = \frac{1}{2\pi}(2c)/(2c) = 1/(2\pi)$. For the other coefficients, we will compute the integrals over the interval $(d-\pi, d+\pi)$, whose length is 2π. This is possible by Theorem 1, Section 2.1, because the integrands are 2π-periodic. We have

$$\begin{aligned} a_n &= \frac{1}{2c\pi}\int_{d-c}^{d+c} \cos nx\, dx \\ &= \frac{1}{2cn\pi}\sin nx\Big|_{d-c}^{d+c} = \frac{1}{2cn\pi}\left(\sin n(d+c) - \sin n(d-c)\right) \\ &= \frac{1}{cn\pi}\cos(nd)\sin(nc). \end{aligned}$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{2c\pi}\int_{d-c}^{d+c} \sin nx\, dx \\ &= -\frac{1}{2cn\pi}\cos nx\Big|_{d-c}^{d+c} = \frac{1}{2cn\pi}\left(\cos n(d-c) - \cos n(d+c)\right) \\ &= \frac{1}{cn\pi}\sin(nd)\sin(nc). \end{aligned}$$

Thus the Fourier series is

$$\frac{1}{2\pi} + \frac{1}{c\pi}\sum_{n=1}^{\infty}\left(\frac{\cos(nd)\sin(nc)}{n}\cos nx + \frac{\sin(nd)\sin(nc)}{n}\sin nx\right).$$

Another way of writing the Fourier series is as follows:

$$\begin{aligned} &\frac{1}{2\pi} + \frac{1}{c\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(nc)}{n}\left(\cos(nd)\cos nx + \sin(nd)\sin nx\right)\right\} \\ &= \frac{1}{2\pi} + \frac{1}{c\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(nc)}{n}\cos n(x-d)\right\}. \end{aligned}$$

17. Setting $x = \pi$ in the Fourier series expansion in Exercise 9 and using the fact that the Fourier series converges for all x to $f(x)$, we obtain

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

where we have used $\cos n\pi = (-1)^n$. Simplifying, we find

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

18. Take $x = \frac{\pi}{2}$ in the Fourier series of Exercise 13, note that $\sin(n\frac{\pi}{2}) = 0$ if $n = 2k$ and $\sin(n\frac{\pi}{2}) = (-1)^k$ if $n = 2k+1$, and get

$$\frac{\pi}{2} = 2\sum_{k=0}^{\infty}\frac{(-1)^{2k+2}(-1)^k}{2k+1} = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1},$$

and the desired series follows upon dividing by 2.

19. (a) Let $f(x)$ denote the function in Exercise 1 and $w(x)$ the function in Example 5. Comparing these functions, we find that $f(x) = \frac{1}{\pi} w(x)$. Now using the Fourier series of w, we find

$$f(x) = \frac{1}{\pi}\left[\frac{\pi}{2} + 2\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}\right] = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}.$$

(b) Let $g(x)$ denote the function in Exercise 2 and $f(x)$ the function in (a). Comparing these functions, we find that $g(x) = 2f(x) - 1$. Now using the Fourier series of f, we find

$$g(x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}.$$

(c) Let $k(x)$ denote the function in Figure 13, and let $f(x)$ be as in (a). Comparing these functions, we find that $k(x) = f\left(x + \frac{\pi}{2}\right)$. Now using the Fourier series of f, we get

$$\begin{aligned} k(x) &= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)(x+\frac{\pi}{2})}{2k+1} \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\Big(\sin[(2k+1)x]\overbrace{\cos[(2k+1)\frac{\pi}{2}]}^{=0} + \cos[(2k+1)x]\overbrace{\sin[(2k+1)\frac{\pi}{2}]}^{=(-1)^k}\Big) \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos[(2k+1)x]. \end{aligned}$$

(d) Let $v(x)$ denote the function in Exercise 3, and let $k(x)$ be as in (c). Comparing these functions, we find that $v(x) = \frac{1}{2}\big(k(x) + 1\big)$. Now using the Fourier series of k, we get

$$v(x) = \frac{3}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos[(2k+1)x].$$

20. (a) Let $f(x)$ denote the function in Figure 14 and let $|\sin x|$. By comparing graphs, we see that $|\sin x| + \sin x = 2f(x)$. So

$$f(x) = \frac{1}{2}\left(|\sin x| + \sin x\right).$$

Now the Fourier series of $\sin x$ is $\sin x$ and the Fourier series of $|\sin x|$ is computed in Exercise 7. Combining these two series, we obtain

$$f(x) = \frac{1}{2}\sin x + \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1}\cos 2kx.$$

In particular, $f(x)$ has only one nonzero b_n; namely, $b_1 = \frac{1}{2}$. All other b_ns are 0.
(b) Let $g(x)$ denote the function in Figure 15. Then $g(x) = f(x + \frac{\pi}{2})$. Hence, using

$\sin(a + \frac{\pi}{2}) = \cos a$, we obtain

$$\begin{aligned} g(x) &= f\big(x + \frac{\pi}{2}\big) \\ &= \frac{1}{2}\sin\big(x + \frac{\pi}{2}\big) + \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k)^2-1}\cos\Big[2k\big(x+\frac{\pi}{2}\big)\Big] \\ &= \frac{1}{\pi} + \frac{1}{2}\cos x \\ &\quad -\frac{2}{\pi}\sum_{k=1}^{\infty}\Big[\frac{1}{(2k)^2-1}\cos(2kx)\overbrace{\cos(2k\frac{\pi}{2})}^{(-1)^k} - \sin(2kx)\overbrace{\sin(2k\frac{\pi}{2})}^{=0}\Big] \\ &= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{k=1}^{\infty}\Big[\frac{(-1)^k}{(2k)^2-1}\cos(2kx)\Big]. \end{aligned}$$

21. (a) Interpreting the integral as an area (see Exercise 16), we have

$$a_0 = \frac{1}{2\pi}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{8}.$$

To compute a_n, we first determine the equation of the function for $\frac{\pi}{2} < x < \pi$. From Figure 16, we see that $f(x) = \frac{2}{\pi}(\pi - x)$ if $\frac{\pi}{2} < x < \pi$. Hence, for $n \geq 1$,

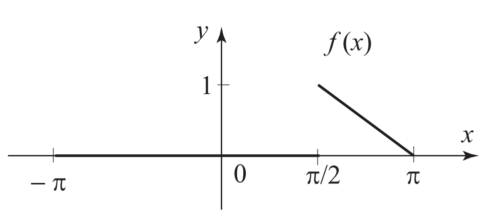
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{\pi/2}^{\pi}\frac{2}{\pi}\overbrace{(\pi - x)}^{u}\overbrace{\cos nx}^{v'}\,dx \\ &= \frac{2}{\pi^2}(\pi - x)\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi} + \frac{2}{\pi^2}\int_{\pi/2}^{\pi}\frac{\sin nx}{n}\,dx \\ &= \frac{2}{\pi^2}\left[\frac{-\pi}{2n}\sin\frac{n\pi}{2}\right] - \frac{2}{\pi^2 n^2}\cos nx\Big|_{\pi/2}^{\pi} \\ &= -\frac{2}{\pi^2}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]. \end{aligned}$$

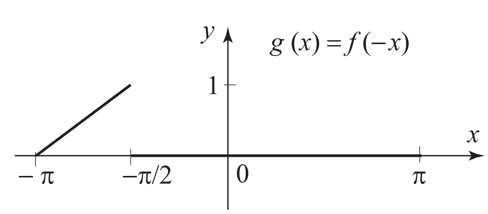
Also,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{\pi/2}^{\pi}\frac{2}{\pi}\overbrace{(\pi - x)}^{u}\overbrace{\sin nx}^{v'}\,dx \\ &= -\frac{2}{\pi^2}(\pi - x)\frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} - \frac{2}{\pi^2}\int_{\pi/2}^{\pi}\frac{\cos nx}{n}\,dx \\ &= \frac{2}{\pi^2}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]. \end{aligned}$$

Thus the Fourier series representation of f is

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Big\{-\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]\cos nx \\ &\qquad + \left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]\sin nx\Big\}. \end{aligned}$$





(b) Let $g(x) = f(-x)$. By performing a change of variables $x \leftrightarrow -x$ in the Fourier series of f, we obtain (see also Exercise 24 for related details) Thus the Fourier series representation of f is

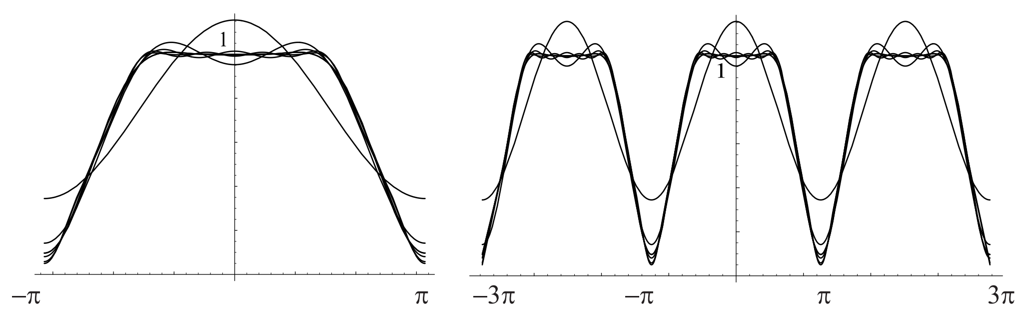
$$\begin{aligned} g(x) &= \frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Big\{-\Big[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\Big]\cos nx \\ &\qquad - \Big[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big]\sin nx\Big\}. \end{aligned}$$

22. By comparing graphs, we see that the function in Exercise 4 (call it $k(x)$) is the sum of the three functions in Exercises 19(c) and 21(a) and (b) (call them h, f, and g, respectively). Thus, adding and simplifying, we obtain

$$\begin{aligned} k(x) &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos nx \\ &\quad +\frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Big\{-\Big[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\Big]\cos nx \\ &\qquad + \Big[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big]\sin nx\Big\} \\ &\quad +\frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Big\{-\Big[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\Big]\cos nx \\ &\qquad - \Big[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big]\sin nx\Big\} \\ &= \frac{3}{4} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\Big\{\Big[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\Big]\cos nx\Big\}. \end{aligned}$$

We illustrate the convergence of the Fourier series in the following figure.

```
s[n_, x_] := 3 / 4 - 4 / Pi^2 Sum[((-1)^k - Cos[k Pi / 2]) Cos[k x] / k^2, {k, 1, n}]
partialsums = Table[s[n, x], {n, 1, 7}];
Plot[Evaluate[partialsums], {x, - Pi, Pi}]
Plot[Evaluate[partialsums], {x, - 3 Pi, 3 Pi}]
```



23. This exercise is straightforward and follows from the fact that the integral is linear.

24. (a) This part follows by making appropriate changes of variables. We have

$$\begin{aligned} a(g,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)\,dx \\ &= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x)\,(-1)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = a(f,\,0). \end{aligned}$$

Similarly,

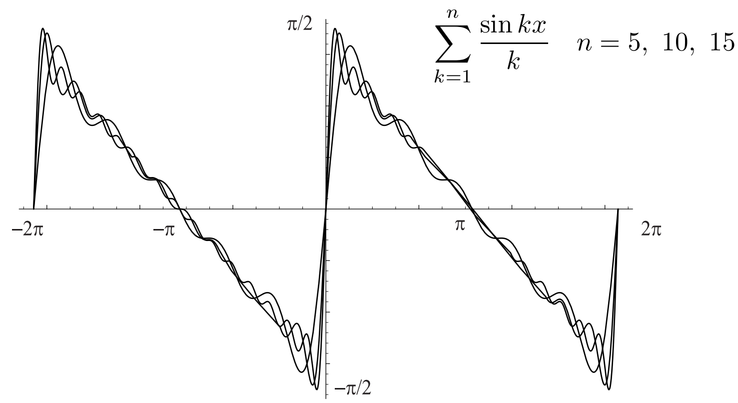
$$\begin{aligned}
a(g,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos(-nx)\,(-1)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a(f,\,n); \\
b(g,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\sin(-nx)\,(-1)dx = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -b(f,\,n).
\end{aligned}$$

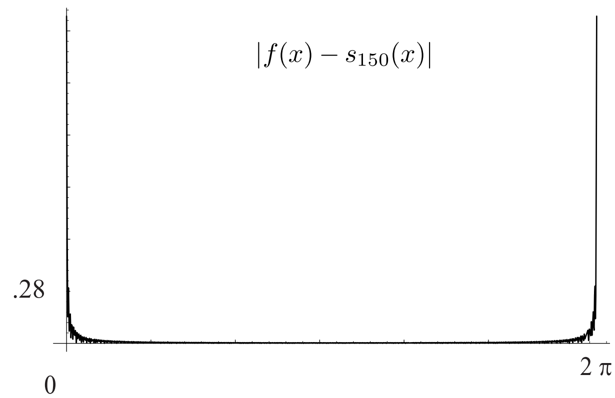
(b) As in part (a),

$$\begin{aligned}
a(h,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\alpha)\,dx \\
&= \frac{1}{2\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = a(f,\,0);\ \text{(by Theorem 1, Section 2.1)} \\
a(h,\,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\alpha)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\cos(n(x+\alpha))\,dx \\
&= \frac{\cos n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\cos nx\,dx - \frac{\sin n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\sin nx\,dx \\
&= \frac{\cos n\alpha}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx - \frac{\sin n\alpha}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&\qquad \text{(by Theorem 1, Section 2.1)} \\
&= a(f,\,n)\cos n\alpha - b(f,\,n)\sin n\alpha; \\
b(h,\,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\alpha)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\sin(n(x+\alpha))\,dx \\
&= \frac{\cos n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\sin nx\,dx + \frac{\sin n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\cos nx\,dx \\
&= b(f,\,n)\cos n\alpha + a(f,\,n)\sin n\alpha.
\end{aligned}$$

25. For (a) and (b), see plots.

(c) We have $s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$. So $s_n(0) = 0$ and $s_n(2\pi) = 0$ for all n. Also, $\lim_{x\to 0^+} f(x) = \frac{\pi}{2}$, so the difference between $s_n(x)$ and $f(x)$ is equal to $\pi/24$ at $x = 0$. But even we look near $x = 0$, where the Fourier series converges to $f(x)$, the difference $|s_n(x)-f(x)|$ remains larger than a positive number, that is about .28 and does not get smaller no matter how large n. In the figure, we plot $|f(x) - s_{150}(x)|$. As you can see, this difference is 0 everywhere on the interval $(0,\,2\pi)$, except near the points 0 and 2π, where this difference is approximately .28. The precise analysis of this phenomenon is done in the following exercise.





26. (a) We have

$$s_N\left(\frac{\pi}{N}\right) = \sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{n} = \sum_{n=1}^{N} \frac{\pi}{N} \cdot \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}}.$$

(b) Consider the Riemann integral

$$I = \int_0^\pi \frac{\sin x}{x}\, dx,$$

where the integrand $f(x) = \frac{\sin x}{x}$ may be considered as a continuous function on the interval $[0, \pi]$ if we define $f(0) = 0$. This is because $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Since the definite integral of a continuous function on a closed interval is the limit of Riemann sums, it follows that the integral I is the limit of Riemann sums corresponding to the function $f(x)$ on the interval $[0, \pi]$. We now describe these Riemann sums. Let $\Delta x = \frac{\pi}{N}$ and partition the interval $[0, \pi]$ into N subintervals of equal length Δx. Form the Nth Riemann sum by evaluating the function $f(x)$ at the right endpoint of each subinterval. We have N of these endpoints and they are $\frac{n\pi}{N}$, for $n = 1, 2, \ldots, N$. Thus,

$$\begin{aligned} I &= \lim_{N\to\infty} \sum_{n=1}^{N} f\left(\frac{n\pi}{N}\right) \Delta x \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\pi}{N} \cdot \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{n} \\ &= \lim_{N\to\infty} s_N\left(\frac{\pi}{N}\right). \end{aligned}$$

Hence

$$\lim_{N\to\infty} s_N\left(\frac{\pi}{N}\right) = \int_0^\pi \frac{\sin x}{x}\, dx,$$

as desired.
(c) From the Taylor series

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty < x < \infty),$$

we obtain, for all $x \neq 0$,

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}, \quad (-\infty < x < \infty,\ x \neq 0).$$

The right side is a power series that converges for all $x \neq 0$. It is also convergent for $x = 0$ and its value for $x = 0$ is 1. Since the left side is also equal to 1 (in the limit), we conclude that

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}, \quad (-\infty < x < \infty).$$

(d) A power series can be integrated term-by-term within its radius of convergence. Thus

$$\begin{aligned}
\int_0^\pi \frac{\sin x}{x}\,dx &= \int_0^\pi \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\right) dx \\
&= \sum_{n=0}^{\infty}\int_0^\pi \left((-1)^n \frac{x^{2n}}{(2n+1)!}\right) dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{x^{2n+1}}{2n+1}\Big|_0^\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2n+1}.
\end{aligned}$$

We have thus expressed the integral I as an alternating series. Using a property of an alternating series with decreasing coefficients, we know that I is greater than any partial sum that is obtained by leaving out terms starting with a negative term; and I is smaller than any partial sum that is obtained by leaving out terms starting with a positive term. So if $I = \sum_{n=0}^{\infty}(-1)^n a_n$, where a_n are positive and decreasing to 0, then $I > \sum_{n=0}^{N}(-1)^n a_n$ if N is even and $I < \sum_{n=0}^{N}(-1)^n a_n$ if N is odd. So

$$\sum_{n=0}^{5} \frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2n+1} < I < \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2n+1}.$$

More explicitely,

$$\pi - \frac{1}{3!}\frac{\pi^3}{3} + \frac{1}{5!}\frac{\pi^5}{5} - \frac{1}{7!}\frac{\pi^7}{7} + + \frac{1}{9!}\frac{\pi^9}{9} - - \frac{1}{11!}\frac{\pi^{11}}{11} < I < \pi - \frac{1}{3!}\frac{\pi^3}{3} + \frac{1}{5!}\frac{\pi^5}{5} - \frac{1}{7!}\frac{\pi^7}{7} + + \frac{1}{9!}\frac{\pi^9}{9}.$$

With the help of a calculator, we find that

$$1.8519 < I < 1.85257,$$

which is slightly better than what is in the text.
(e) We have

$$\left|f\left(\frac{\pi}{N}\right) - s_N\left(\frac{\pi}{N}\right)\right| = \left|\frac{1}{2}(\pi - \frac{\pi}{N}) - s_N\left(\frac{\pi}{N}\right)\right|.$$

As $N \to \infty$, we have $\frac{1}{2}(\pi - \frac{\pi}{N}) \to \frac{\pi}{2}$ and $s_N\left(\frac{\pi}{N}\right) \to I$ so

$$\lim_{N\to\infty}\left|f\left(\frac{\pi}{N}\right) - s_N\left(\frac{\pi}{N}\right)\right| = |\frac{\pi}{2} - I|.$$

Using part (f), we find that this limit is between $1.86 - \frac{\pi}{2} \approx .2892$ and $1.85 - \frac{\pi}{2} \approx .2792$.

(f) The fact that there is a hump on the graph of $s_n(x)$ does not contradict the convergence theorem. This hump is moving toward the endpoints of the interval. So if you fix $0 < x < 2\pi$, the hump will eventually move away from x (toward the endpoints) and the partial sums will converge at x.

27. The graph of the sawtooth function is symmetric with respect to the point $\pi/2$ on the interval $0, \pi)$; that is, we have $f(x) = -f(2\pi - x)$. The same is true for the partial sums of the Fourier series. So we expect an overshoots of the partial sums near π of the same magnitude as the overshoots near 0. More precisely, since $s_N(x) = \sum_{n=1}^{N} \frac{\sin nx}{n}$, it follows that

$$s_N\left(2\pi - \frac{\pi}{N}\right) = \sum_{n=1}^{N} \frac{\sin\left(n(2\pi - \frac{\pi}{N})\right)}{n} = \sum_{n=1}^{N} \frac{\sin\left(n(-\frac{\pi}{N})\right)}{n} = -\sum_{n=1}^{N} \frac{\sin(n\frac{\pi}{N})}{n}.$$

So, by Exercise 26(b), we have

$$\lim_{N\to\infty} s_N\left(2\pi - \frac{\pi}{N}\right) = \lim_{N\to\infty} -s_N\left(\frac{\pi}{N}\right) = -\int_0^{\pi} \frac{\sin x}{x}\, dx.$$

Similarly,

$$\left| f\left(2\pi - \frac{\pi}{N}\right) - s_N\left(2\pi - \frac{\pi}{N}\right)\right| = \left|\frac{1}{2}(-\pi + \frac{\pi}{N}) + s_N\left(\frac{\pi}{N}\right)\right|.$$

As $N \to \infty$, we have $\frac{1}{2}(-\pi + \frac{\pi}{N}) \to -\frac{\pi}{2}$ and $s_N\left(\frac{\pi}{N}\right) \to I$ so

$$\lim_{N\to\infty}\left| f\left(2\pi - \frac{\pi}{N}\right) - s_N\left(2\pi - \frac{\pi}{N}\right)\right| = |-\frac{\pi}{2} + I| \approx .27.$$

The overshoot occurs at $2\pi - \frac{\pi}{N+1} = \frac{(2N+1)\pi}{N+1}$ (using the result from Exercise 26(f)).

28. This exercise is very much like Exercise 26. We outline the details.
(a) We have

$$s_N\left(\pi - \frac{\pi}{N}\right) = 2\sum_{n=1}^{N}(-1)^{n+1}\frac{\sin\left(n\pi - \frac{n\pi}{N}\right)}{n} = 2\sum_{n=1}^{N} \frac{\pi}{N}\cdot\frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}},$$

where we have used the identity $\sin(n\pi - \alpha) = -\cos n\pi \sin\alpha = (-1)^{n+1}\sin\alpha$.
(b) As in Exercise 26(b), we have Thus,

$$\lim_{N\to\infty} s_N\left(\pi - \frac{\pi}{N}\right) = 2\lim_{N\to\infty}\sum_{n=1}^{N} \frac{\pi}{N}\cdot\frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}} \quad = \quad 2\int_0^{\pi} \frac{\sin x}{x}\, dx,$$

as desired.
(c) We have

$$\left| f\left(\pi - \frac{\pi}{N}\right) - s_N\left(\pi - \frac{\pi}{N}\right)\right| = \left|\pi - \frac{\pi}{N} - s_N\left(\pi - \frac{\pi}{N}\right)\right|.$$

As $N \to \infty$, we have $\pi - \frac{\pi}{N} \to \pi$ and $s_N\left(\pi - \frac{\pi}{N}\right) \to 2I$ so

$$\lim_{N\to\infty}\left| f\left(\pi - \frac{\pi}{N}\right) - s_N\left(\frac{\pi}{N}\right)\right| = |\pi - 2I|.$$

Using Exercise 26(f), we find that this limit is between $2(1.86) - \pi \approx .578$ and $2(1.85) - \pi \approx .559$.

Solutions to Exercises 2.3

1. (a) and (b) Since f is odd, all the a_n's are zero and

$$\begin{aligned} b_n &= \frac{2}{p}\int_0^p \sin\frac{n\pi}{p}\,dx \\ &= \frac{-2}{n\pi}\cos\frac{n\pi}{p}\Big|_0^\pi = \frac{-2}{n\pi}\big[(-1)^n - 1\big] \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus the Fourier series is $\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin\frac{(2k+1)\pi}{p}x$. At the points of discontinuity, the Fourier series converges to the average value of the function. In this case, the average value is 0 (as can be seen from the graph.

2. (a) and (b). The function is odd. Using the Fourier series from Exercise 13, Section 2.2, we have

$$t = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt \qquad \text{for } -\pi < t < \pi.$$

Let $t = \frac{\pi x}{p}$, then

$$\frac{\pi x}{p} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{p} \qquad \text{for } -\pi < \frac{\pi x}{p} < \pi.$$

Hence

$$x = \frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi}{p}x) \quad \text{for } -p < x < p;$$

which yields the desired Fourier series. At the points of discontinuity, the Fourier series converges to the average value of the function. In this case, the average value is 0 (as can be seen from the graph.

3. (a) and (b) The function is even so all the b_n's are zero,

$$a_0 = \frac{1}{p}\int_0^p a\big[(1 - (\frac{x}{p})^2\big]\,dx = \frac{a}{p}(x - \frac{1}{3p^2}x^3)\Big|_0^p = \frac{2}{3}a;$$

and with the help of the integral formula from Exercise 9, Section 2.2, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2a}{p}\int_0^p (1 - \frac{x^2}{p^2})\cos\frac{n\pi x}{p}\,dx = -\frac{2a}{p^3}\int_0^p x^2\cos\frac{n\pi x}{p}\,dx \\ &= -\frac{2a}{p^3}\left[2x\frac{p^2}{(n\pi)^2}\cos\frac{n\pi x}{p} + \frac{p^3}{(n\pi)^3}(-2 + \frac{(n\pi)^2}{p^2})x^2\sin\frac{n\pi x}{p}\right]\Big|_0^p \\ &= -\frac{4a(-1)^n}{n^2\pi^2}. \end{aligned}$$

Thus the Fourier series is $\frac{2}{3}a + 4a\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n\pi)^2}\cos(\frac{n\pi}{p}x)$. Note that the function is continuous for all x.

4. (a) and (b) The function is even. It is also continuous for all x. Using the Fourier series from Exercise 9, Section 2.2, we have

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nt \qquad \text{for } -\pi < t < \pi.$$

Let $t = \frac{\pi x}{p}$, then

$$(\frac{\pi x}{p})^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\frac{\pi x}{p} \qquad \text{for } -\pi < \frac{\pi x}{p} < \pi.$$

Hence

$$x^2 = \frac{p^2}{3} + \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\frac{\pi x}{p} \qquad \text{for } -p < x < p,$$

which yields the desired Fourier series.

5. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x = 0$ to $x = p$, we see that $a_0 = 0$. Now, using integration by parts, we obtain

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^p -\left(\frac{2c}{p}\right)(x - p/2)\cos\frac{n\pi}{p}x\,dx = -\frac{4c}{p^2}\int_0^p \overbrace{(x-p/2)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\
&= -\frac{4c}{p^2}\left[\overbrace{\frac{p}{n\pi}(x-p/2)\sin\frac{n\pi}{p}x\Big|_{x=0}^{p}}^{=0} - \frac{p}{n\pi}\int_0^p \sin\frac{n\pi}{p}x\,dx\right] \\
&= -\frac{4c}{p^2}\frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_{x=0}^{p} = \frac{4c}{n^2\pi^2}(1-\cos n\pi) \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{8c}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{8c}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\frac{\pi}{p}x\right]}{(2k+1)^2}.$$

6. (a) and (b) The function is even. It is not continuous at $\pm d + 2kp$. At these points, the Fourier series converges to $c/2$. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x = 0$ to $x = p$, we see that $a_0 = cd/p$. We have

$$\begin{aligned}
a_n &= \frac{2c}{p}\int_0^d \cos\frac{n\pi}{p}x\,dx \\
&= \frac{2c}{n\pi}\sin\frac{n\pi}{p}x\Big|_{x=0}^{d} = \frac{2c}{n\pi}\sin\frac{n\pi d}{p}.
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{cd}{p} + \frac{2c}{\pi}\sum_{k=0}^{\infty} \frac{\sin\frac{n\pi d}{p}}{n}\cos(\frac{n\pi}{p}x).$$

7. The function in this exercise is similar to the one in Example 3. Start with the Fourier series in Example 3, multiply it by $1/c$, then change $2p \leftrightarrow p$ (this is not a change of variables, we are merely changing the notation for the period from $2p$ to p) and you will get the desired Fourier series

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{2n\pi x}{p}\right)}{n}.$$

The function is odd and has discontinuities at $x = \pm p + 2kp$. At these points, the Fourier series converges to 0.

8. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x=0$ to $x=d$, we see that $a_0=(cd)/(2p)$. Now, using integration by parts, we obtain

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^d -\left(\frac{c}{d}\right)(x-d)\cos\frac{n\pi}{p}x\,dx = -\frac{2c}{dp}\int_0^d \overbrace{(x-d)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\
&= -\frac{2c}{dp}\left[\overbrace{\frac{p}{n\pi}(x-d)\sin\frac{n\pi}{p}x\Big|_{x=0}^{d}}^{=0} - \frac{p}{n\pi}\int_0^d \sin\frac{n\pi}{p}x\,dx\right] \\
&= -\frac{2c}{dp}\frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_{x=0}^{d} = \frac{2cp}{dn^2\pi^2}\left(1-\cos\frac{n\pi d}{p}\right).
\end{aligned}$$

Thus the Fourier series is

$$f(x)=\frac{cd}{2p}+\frac{2cp}{d\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi d}{p}}{n^2}\cos\left(n\frac{\pi}{p}x\right).$$

9. The function is even; so all the b_n's are 0,

$$a_0=\frac{1}{p}\int_0^p e^{-cx}\,dx=-\frac{1}{cp}e^{-cx}\Big|_0^p=\frac{1-e^{-cp}}{cp};$$

and with the help of the integral formula from Exercise 15, Section 2.2, for $n\geq 1$,

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^p e^{-cx}\cos\frac{n\pi x}{p}\,dx \\
&= \frac{2}{p}\frac{1}{n^2\pi^2+p^2c^2}\left[n\pi p e^{-cx}\sin\frac{n\pi x}{p}-p^2ce^{-cx}\cos\frac{n\pi x}{p}\right]\Big|_0^p \\
&= \frac{2pc}{n^2\pi^2+p^2c^2}\left[1-(-1)^n e^{-cp}\right].
\end{aligned}$$

Thus the Fourier series is

$$\frac{1}{pc}(1-e^{-cp})+2cp\sum_{n=1}^{\infty}\frac{1}{c^2p^2+(n\pi)^2}(1-e^{-cp}(-1)^n)\cos(\frac{n\pi}{p}x).$$

10. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x=0$ to $x=p$, we see that $a_0=(1/p)(c+(p-c)/2)=(p+c)/(2p)$. Now, using integration by parts, we obtain

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^c \cos\frac{n\pi}{p}x\,dx+\frac{2}{p}\int_c^p -\left(\frac{1}{p-c}\right)(x-p)\cos\frac{n\pi}{p}x\,dx \\
&= \frac{2}{p}\frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^c-\frac{2}{p(p-c)}\int_c^p \overbrace{(x-p)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\
&= \frac{2}{n\pi}\sin\frac{n\pi c}{p}-\frac{2}{p(p-c)}\left[\frac{p}{n\pi}(x-p)\sin\frac{n\pi}{p}x\Big|_c^p-\frac{p}{n\pi}\int_c^p\sin\frac{n\pi}{p}x\,dx\right] \\
&= \frac{2}{n\pi}\sin\frac{n\pi c}{p}-\frac{2}{p(p-c)}\left[-\frac{p}{n\pi}(c-p)\sin(\frac{n\pi}{p}c)+\frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_c^p\right] \\
&= -\frac{2}{p(p-c)}\frac{p^2}{n^2\pi^2}\left[(-1)^n-\cos\frac{n\pi c}{p}\right] \\
&= \frac{2p}{\pi^2(c-p)n^2}\left[(-1)^n-\cos\frac{n\pi c}{p}\right].
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{p+c}{2p} + \frac{2p}{\pi^2(c-p)} \sum_{n=1}^{\infty} \frac{(-1)^n - \cos\frac{n\pi c}{p}}{n^2} \cos\left(\frac{n\pi}{p}x\right).$$

11. We note that the function $f(x) = x\sin x$ $(-\pi < x < \pi)$ is the product of $\sin x$ with a familiar function, namely, the 2π-periodic extension of x $(-\pi < x < \pi)$. We can compute the Fourier coefficients of $f(x)$ directly or we can try to relate them to the Fourier coefficients of $g(x) = x$. In fact, we have the following useful fact.

Suppose that $g(x)$ is an odd function and write its Fourier series representation as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where b_n is the nth Fourier coefficient of g. Let $f(x) = g(x)\sin x$. Then f is even and its nth cosine Fourier coefficients, a_n, are given by

$$a_0 = \frac{b_1}{2}, \qquad a_1 = \frac{b_2}{2}, \qquad a_n = \frac{1}{2}\left[b_{n+1} - b_{n-1}\right] \quad (n \geq 2).$$

To prove this result, proceed as follows:

$$\begin{aligned} f(x) &= \sin x \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} b_n \sin x \sin nx \\ &= \sum_{n=1}^{\infty} \frac{b_n}{2}\big[-\cos[(n+1)x] + \cos[(n-1)x]\big]. \end{aligned}$$

To write this series in a standard Fourier series form, we reindex the terms, as follows:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty}\left(-\frac{b_n}{2}\cos[(n+1)x]\right) + \sum_{n=1}^{\infty}\left(\frac{b_n}{2}\cos[(n-1)x]\right) \\ &= \sum_{n=2}^{\infty}\left(-\frac{b_{n-1}}{2}\cos nx\right) + \sum_{n=0}^{\infty}\left(\frac{b_{n+1}}{2}\cos nx\right) \\ &= \sum_{n=2}^{\infty}\left(-\frac{b_{n-1}}{2}\cos nx\right)\frac{b_1}{2} + \frac{b_2}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{b_{n+1}}{2}\cos nx\right) \\ &= \frac{b_1}{2} + \frac{b_2}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{b_{n+1}}{2} - \frac{b_{n-1}}{2}\right)\cos nx. \end{aligned}$$

This proves the desired result. To use this result, we recall the Fourier series from Exercise 2: For $-\pi < x < \pi$, $g(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$; so $b_1 = 2$, $b_2 = -1/2$ and $b_n = \frac{2(-1)^{n+1}}{n}$ for $n \geq 2$. So, $f(x) = x\sin x = \sum_{n=1}^{\infty} a_n \cos nx$, where $a_0 = 1$, $a_1 = -1/2$, and, for $n \geq 2$,

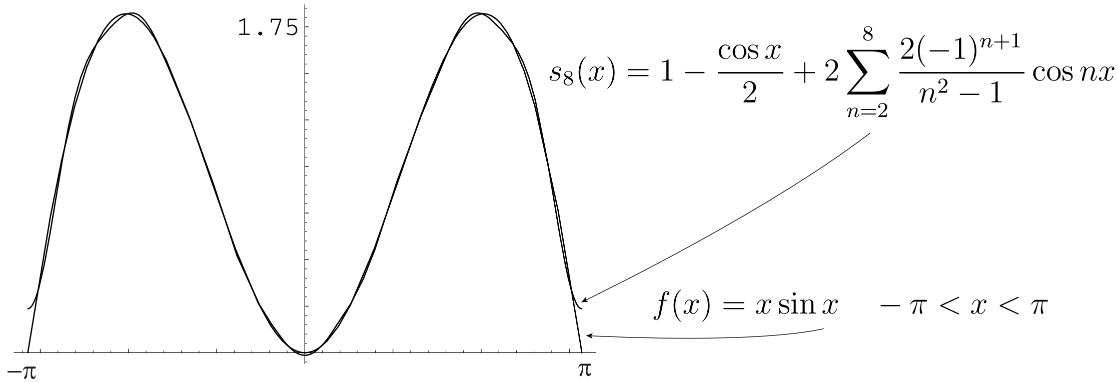
$$a_n = \frac{1}{2}\left[\frac{2(-1)^{n+2}}{n+1} - \frac{2(-1)^n}{n-1}\right] = \frac{2(-1)^{n+1}}{n^2-1}.$$

Thus, for $-\pi < x < \pi$,

$$x\sin x = 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx.$$

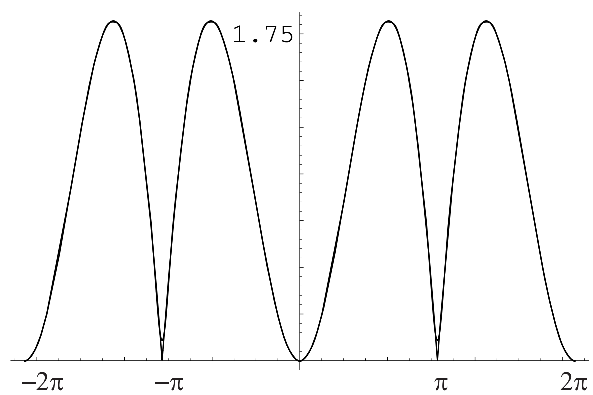
The convergence of the Fourier series is illustrated in the figure. Note that the partial sums converge uniformly on the entire real line. This is a consequence of the fact that the function is piecewise smooth and continuous for all x. The following is the 8th partial sum.

```
s[n_, x_] = 1 - Cos[x] / 2 + 2 Sum[(-1) ^ (k + 1) / (k^2 - 1) Cos[k x], {k, 2, n}];

Plot[Evaluate[{x Sin[x], s[8, x]}], {x, - Pi, Pi}]
```



To plot the function over more than one period, we can use the Floor function to extend it outside the interval $(-\pi, \pi)$. In what follows, we plot the function and the 18th partial sum of its Fourier series. The two graphs are hard to distinguish from one another.

```
f[x_] = x Sin[x];
g[x_] = x - 2 Pi Floor[(x + Pi) / ( 2 Pi)];
h[x_] = f[g[x]];
Plot[Evaluate[{h[x], s[18, x]}], {x, -2 Pi, 2 Pi}]
```



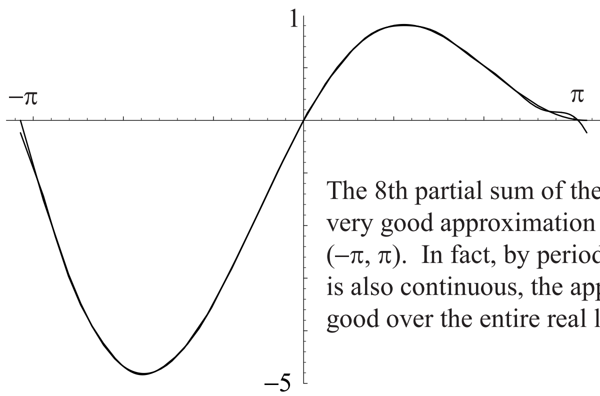
12. The Fourier series of the function $f(x) = (\pi - x)\sin x$ $(-\pi < x < \pi)$ can be obtained from that of the function in Exercise 11, as follows. Call the function in Exercise 11 $g(x)$. Then, on the interval $-\pi < x < \pi$, we have

$$\begin{aligned} f(x) &= (\pi - x)\sin x = \pi \sin -x \sin x \\ &= \pi \sin x - g(x). \end{aligned}$$

Since $\pi \sin x$ is its own Fourier series, using the Fourier series from Exercise 11, we obtain

$$
\begin{aligned}
f(x) &= \pi \sin x - \left(1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx\right) \\
&= \pi \sin x - 1 + \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{2(-1)^{n}}{n^2-1} \cos nx.
\end{aligned}
$$

```
s[n_, x_] =
  Pi Sin[x] - 1 + Cos[x] / 2 - 2 Sum[(-1) ^ (k + 1) / (k^2 - 1) Cos[k x], {k, 2, n}];
Plot[Evaluate[{(Pi - x) Sin[x], s[8, x]}], {x, - Pi, Pi}]
```



The 8th partial sum of the Fourier series provides a very good approximation of the function on the interval (−π, π). In fact, by periodicity, because the function is also continuous, the approximation is uniformly good over the entire real line.

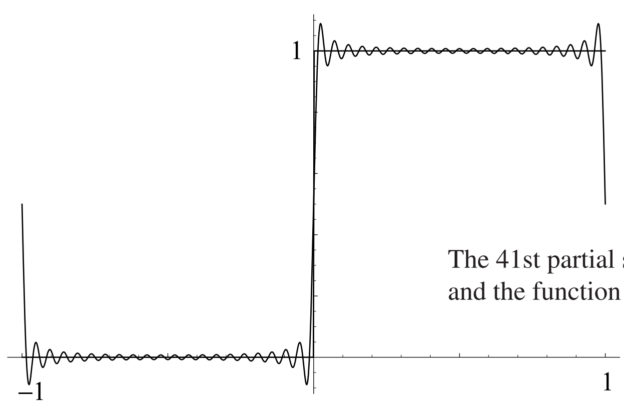
13. Take $p = 1$ in Exercise 1, call the function in Exercise 1 $f(x)$ and the function in this exercise $g(x)$. By comparing graphs, we see that

$$
g(x) = \frac{1}{2}\left(1 + f(x)\right).
$$

Thus the Fourier series of g is

$$
\frac{1}{2}\left(1 + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin(2k+1)\pi x\right) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin(2k+1)\pi x.
$$

```
f[x_] = Which[x < 0, 0, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 1 / 2 + 2 / Pi Sum[1 / (2 k + 1) Sin[(2 k + 1 ) Pi x], {k, 0, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]

Which[x < 0, 0, 0 < x < 1, 1, x > 1, 0]
```



The 41st partial sum of the Fourier series and the function on the interval (-1, 1).

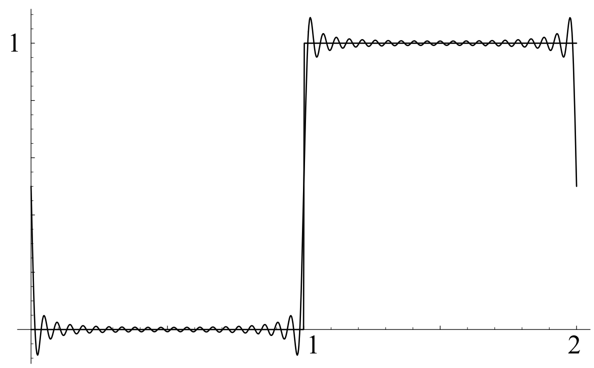
14. Call the function in Exercise 13 $g(x)$ and the function in this exercise $h(x)$. By comparing graphs of the 2-periodic extensions, we see that

$$h(x) = g(-x).$$

Thus the Fourier series of h is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin[(2k+1)\pi(-x)] = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin[(2k+1)\pi x].$$

```
h[x_] = Which[x < 1, 0, 1 < x < 2, 1, x > 2, 0]
s[n_, x_] = 1 / 2 - 2 / Pi Sum[1 / (2 k + 1) Sin[(2 k + 1) Pi x], {k, 0, n}];
Plot[Evaluate[{h[x], s[20, x]}], {x, 0, 2}]

Which[x < 1, 0, 1 < x < 2, 1, x > 2, 0]
```



15. To match the function in Example 2, Section 2.2, take $p = a = \pi$ in Example 2 of this section. Then the Fourier series becomes

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x,$$

which is the Fourier series of Example 2, Section 2.2.

16. (a) As $d \to p$, the function becomes the constant function $f(x) = c$. Its Fourier series is itself; in particular, all the Fourier coefficients are 0, except $a_0 = c$. This is clear if we let $d \to p$ in the formulas for the Fourier coefficients, because

$$\lim_{d\to p}\frac{cd}{p} = 0 \quad \text{and} \quad \lim_{d\to p}\frac{2c}{\pi}\frac{\sin\frac{dn\pi}{p}}{n} = 0.$$

(b) If $c = p/d$ and $d \to 0$, interesting things happen. The function tends to 0 pointwise, except at $x = 0$, where it tends to ∞. Also, for any fixed d (no matter how small), the area under the graph of f and above the x-axis, from $x = -d$ to $x = d$, is $2p$. Taking $c = p/d$, we find that $a_0 = 1$ for all d and, hence, $a_0 \to 1$, as $d \to 0$. For $n \geq 1$, we have

$$\lim_{d\to 0} a_n = \lim_{d\to 0}\frac{2p}{d\pi}\frac{\sin\frac{dn\pi}{p}}{n} = \lim_{d\to 0}\frac{2p}{\pi}\frac{n\pi}{p}\frac{\cos\frac{dn\pi}{p}}{n} = 2,$$

by using l'Hospital's rule. Thus, even though the function tends to 0, its Fourier coefficients are not tending to 0. In fact, the Fourier coefficients are tending to 1 for a_0 and to 2 for all other a_n. These limiting Fourier coefficients do not correspond to any function! That is, there is no function with Fourier coefficients given by $a_0 = 1$, $a_n = 2$ and $b_n = 0$ for all $n \geq 1$. There is, however, a generalized function (a constant multiple of the Dirac delta function) that has precisely those coefficients. You may have encountered the Dirac delta function previously. You can read more about it in Sections 7.8-7.10.

17. (a) Take $x = 0$ in the Fourier series of Exercise 4 and get

$$0 = \frac{p^2}{3} - \frac{4p^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

(b) Take $x = p$ in the Fourier series of Exercise 4 and get

$$p^2 = \frac{p^2}{3} - \frac{4p^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Summing over the even and odd integers separately, we get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2}.$$

But $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4} \frac{\pi^2}{6}$. So

$$\frac{\pi^2}{6} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{\pi^2}{24} \quad \Rightarrow \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

18. To derive (4), repeat the proof of Theorem 1 until you get to the equation (7). Then continue as follows:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right) \cos nx \, dx \quad (\text{let } t = \frac{px}{\pi}) \\ &= \frac{1}{p} \int_{-p}^{p} f(t) \cos\left(\frac{n\pi}{p} t\right) \, dt, \end{aligned}$$

which is formula (4). Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right) \sin nx \, dx \quad (\text{let } t = \frac{px}{\pi}) \\ &= \frac{1}{p} \int_{-p}^{p} f(t) \sin\left(\frac{n\pi}{p} t\right) \, dt, \end{aligned}$$

which is formula (5).

19. This is very similar to the proof of Theorem 2(i). If $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$, then, for all x,

$$f(-x) = \sum_{n=1}^{\infty} b_n \sin(-\frac{n\pi}{p} x) = -\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x = -f(x),$$

and so f is odd. Conversely, suppose that f is odd. Then $f(x) \cos \frac{n\pi}{p} x$ is odd and, from (10), we have $a_n = 0$ for all n. Use (5), (9), and the fact that $f(x) \sin \frac{n\pi}{p} x$ is even to get the formulas for the coefficients in (ii).

20. (a) f_e is even because

$$f_e(-x) = \frac{f(-x) + f(- - x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x).$$

f_o is odd because

$$f_o(-x) = \frac{f(-x) - f(--x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x).$$

(b) We have

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x).$$

To show that this decomposition is unique, suppose that $f(x) = g(x) + h(x)$, where g is even and h is odd. Then $f(-x) = g(-x) + h(-x) = g(x) - h(x)$, and so

$$f(x) + f(-x) = g(x) + h(x) + g(x) - h(x) = 2g(x);$$

equivalently,

$$g(x) = \frac{f(x) + f(-x)}{2} = f_e(x).$$

By considering $f(x) - f(-x)$, we obtain that $h(x) = f_o(x)$; and hence the decomposition is unique.

(c) If $f(x)$ is $2p$-periodic, then clearly $f(-x)$ is also $2p$-periodic. So f_e and f_o are both $2p$-periodic, being linear combinations of $2p$-periodic functions.

(d) If $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p})$, then

$$f(-x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi(-x)}{p} + b_n \sin \frac{n\pi(-x)}{p}) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{p} - b_n \sin \frac{n\pi x}{p});$$

and so

$$f_e(x) = \frac{f(x) + f(-x)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p},$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p}.$$

21. From the graph, we have

$$f(x) = \begin{cases} -1 - x & \text{if } -1 < x < 0, \\ 1 + x & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 - x & \text{if } -1 < x < 0, \\ -1 + x & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} -x & \text{if } -1 < x < 0, \\ x & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Note that, $f_e(x) = |x|$ for $-1 < x < 1$. The Fourier series of f is the sum of the Fourier series of f_e and f_o. From Example 1 with $p = 1$,

$$f_e(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x].$$

From Exercise 1 with $p = 1$,

$$f_o(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin[(2k+1)\pi x].$$

Hence

$$f(x) = \frac{1}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\left[-\frac{\cos[(2k+1)\pi x]}{\pi(2k+1)^2} + \frac{\sin[(2k+1)\pi x]}{2k+1}\right].$$

22. From the graph, we have

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 & \text{if } -1 < x < 0, \\ 1-x & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \begin{cases} \frac{x}{2}+1 & \text{if } -1 < x < 0, \\ 1-\frac{x}{2} & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{x}{2} \quad (-1 < x < 1).$$

As expected, $f(x) = f_e(x) + f_o(x)$. Let $g(x)$ be the function in Example 2 with $p = 1$ and $a = 1/2$. Then $f_e(x) = g(x) + 1/2$. So from Example 2 with $p = 1$ and $a = 1/2$, we obtain

$$\begin{aligned} f_e(x) &= \frac{1}{2} + \frac{1}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x] \\ &= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x]. \end{aligned}$$

From Exercise 2 with $p = 1$,

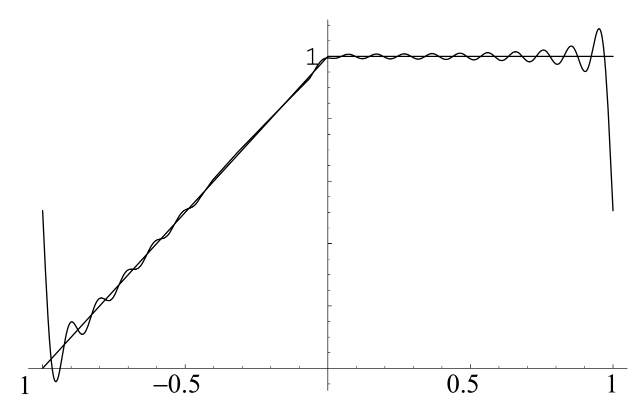
$$f_o(x) = \frac{1}{2}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

Hence

$$\begin{aligned} f(x) &= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x] + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) \\ &= \frac{3}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n^2}\cos(n\pi x) + \frac{(-1)^{n+1}}{n}\sin(n\pi x). \end{aligned}$$

Let's illustrate the convergence of the Fourier series. (This is one way to check that our answer is correct.)

```
f[x_] = Which[x < -1, 0, -1 < x < 0, x + 1, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 3 / 4 + 1 / Pi Sum[(1 - (-1) ^k) / (Pi k^2) Cos[ k Pi x] +
      (-1) ^ (k + 1) / k Sin[k Pi x], {k, 1, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]
```



23. From the graph, we have

$$f(x) = \begin{cases} -2x - 1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 & \text{if } -1 < x < 0, \\ -1 + 2x & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} -x & \text{if } -1 < x < 0, \\ x & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} -x - 1 & \text{if } -1 < x < 0, \\ 1 - x & \text{if } 0 < x < 1, \end{cases}$$

As expected, $f(x) = f_e(x) + f_o(x)$. Note that, $f_e(x) = |x|$ for $-1 < x < 1$. Let $g(x)$ be the function in Example 2 with $p = 1$. Then $f_e(x) = g(x)$. So from Example 1 with $p = 1$, we obtain

$$f_e(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x].$$

Note that $f_o(x) = 1 - x$ for $0 < x < 2$. The Fourier series of f_o follows from Exercise 7 with $p = 2$. Thus

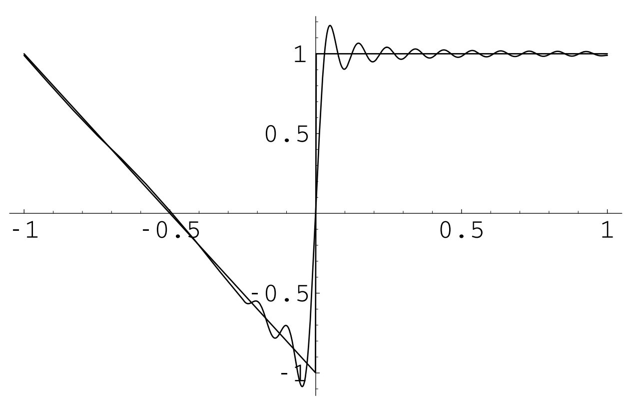
$$f_o(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}.$$

Hence

$$\begin{aligned} f(x) &= \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x] + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(n\pi x) + \frac{\sin(n\pi x)}{n} \right]. \end{aligned}$$

Let's illustrate the convergence of the Fourier series. (This is one way to check that our answer is correct.)

```
Clear[s]
f[x_] = Which[x < -1, 0, -1 < x < 0, -2 x - 1, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 1 / 2 + 2 / Pi Sum[- (1 - (-1) ^k) / (Pi k^2) Cos[ k Pi x] +
        Sin[ k Pi x] / ( k), {k, 1, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]
```



24. From the graph, we have

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0, \\ 1 - x & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 + x & \text{if } -1 < x < 0, \\ 0 & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} \frac{1+x}{2} & \text{if } -1 < x < 0, \\ \frac{1-x}{2} & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} \frac{-1-x}{2} & \text{if } -1 < x < 0, \\ \frac{1-x}{2} & \text{if } 0 < x < 1. \end{cases}$$

Note that, as expected, $f(x) = f_e(x) + f_o(x)$. We have $f_e(x) = g(x)$, where $g(x)$ is the function in Example 2 with $p = 1$ and $c = 1/2$. So

$$f_e(x) = \frac{1}{4} + \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x].$$

Also, $f_o(x) = \frac{x-1}{2}$ for $0 < x < 2$. So, from Example 3, with $p = 1$ and $c = 1/2$, we obtain

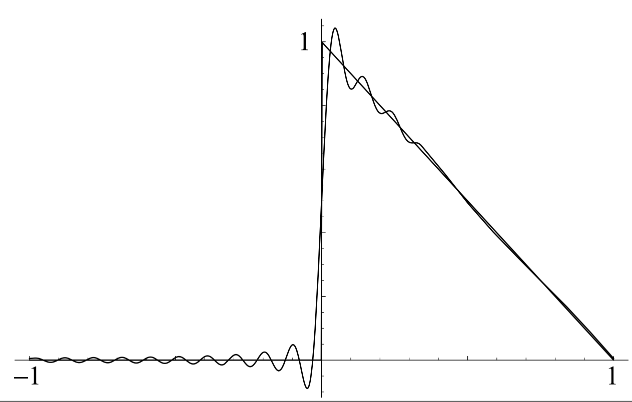
$$f_o(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n}.$$

Hence

$$\begin{aligned} f(x) &= \frac{1}{4} + \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x] + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n} \\ &= \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(n\pi x) + \frac{\sin(n\pi x)}{n} \right]. \end{aligned}$$

Let's illustrate the convergence of the Fourier series. (This is one way to check that our answer is correct.)

```
Clear[s]
f[x_] = Which[x < 0, 0, 0 < x < 1, 1 - x, x > 1, 0]
s[n_, x_] = 1 / 4 + 1 / Pi Sum[(1 - (-1) ^k) / ( Pi k^2) Cos[ k Pi x] +
        Sin[ k Pi x] / ( k), {k, 1, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]
```



25. Since f is $2p$-periodic and continuous, we have $f(-p) = f(-p + 2p) = f(p)$. Now

$$a_0' = \frac{1}{2p} \int_{-p}^{p} f'(x)\, dx = \frac{1}{2p} f(x)\Big|_{-p}^{p} = \frac{1}{2p}(f(p) - f(-p)) = 0.$$

Integrating by parts, we get

$$\begin{aligned} a_n' &= \frac{1}{p}\int_{-p}^{p} f'(x)\cos\frac{n\pi x}{p}\,dx \\ &= \frac{1}{p}\overbrace{f(x)\cos\frac{n\pi x}{p}\Big|_{-p}^{p}}^{=0} + \frac{n\pi}{p}\overbrace{\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx}^{b_n} \\ &= \frac{n\pi}{p}b_n. \end{aligned}$$

Similarly,

$$\begin{aligned} b_n' &= \frac{1}{p}\int_{-p}^{p} f'(x)\sin\frac{n\pi x}{p}\,dx \\ &= \frac{1}{p}\overbrace{f(x)\sin\frac{n\pi x}{p}\Big|_{-p}^{p}}^{=0} - \frac{n\pi}{p}\overbrace{\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx}^{a_n} \\ &= -\frac{n\pi}{p}a_n. \end{aligned}$$

26. From Exercise 25, the Fourier series representation of f' is

$$f'(x) = a_0' + \sum_{n=1}^{\infty}(a_n'\cos\frac{n\pi x}{p} + b_n'\sin\frac{n\pi x}{p}) = \sum_{n=1}^{\infty}\frac{n\pi}{p}b_n\cos\frac{n\pi x}{p} - \frac{n\pi x}{p}a_n\sin\frac{n\pi x}{p}.$$

This series is precisely the series that we obtain by differentiating term by term the Fourier series of $f(x)$. That is,

$$\begin{aligned} f'(x) &= \left(a_0 + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p})\right)' \\ &= \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right)' \\ &= \sum_{n=1}^{\infty}\frac{n\pi}{p}b_n\cos\frac{n\pi x}{p} - \frac{n\pi x}{p}a_n\sin\frac{n\pi x}{p}. \end{aligned}$$

27. The function in Exercise 5 is piecewise smooth and continuous, with a piecewise smooth derivative. We have

$$f'(x) = \begin{cases} -\frac{2c}{p} & \text{if } 0 < x < p, \\ \frac{2c}{p} & \text{if } -p < x < 0. \end{cases}$$

The Fourier series of f' is obtained by differentiating term by term the Fourier series of f (by Exercise 26). So

$$f'(x) = \frac{8c}{\pi^2}\sum_{k=0}^{\infty}\frac{-1}{(2k+1)^2}\frac{(2k+1)\pi}{p}\sin\frac{(2k+1)\pi}{p}x = -\frac{8c}{p\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{(2k+1)\pi}{p}x.$$

Now the function in Exercise 1 is obtained by multiplying $f'(x)$ by $-\frac{p}{2c}$. So to obtain the Fourier series in Exercise 1, we multiply the Fourier series of f' by $-\frac{p}{2c}$ and get

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{(2k+1)\pi}{p}x.$$

28. The function in Exercise 4 is piecewise smooth and continuous, with a piecewise smooth derivative. We have $f'(x) = 2x$ for $-p < x < p$. The Fourier series of f' is obtained by differentiating term by term the Fourier series of f (by Exercise 26). So

$$f'(x) = \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{-1}{n^2}\left(\frac{n\pi}{p}\right)\sin\frac{(2k+1)\pi}{p}x = \frac{4p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{p}x.$$

Now the function in Exercise 2 is obtained by multiplying $f'(x)$ by $\frac{1}{2}$. So to obtain the Fourier series in Exercise 2, we multiply the Fourier series of f' by $\frac{1}{2}$ and get

$$\frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{p}x.$$

29. The function in Exercise 8 is piecewise smooth and continuous, with a piecewise smooth derivative. We have

$$f'(x) = \begin{cases} -\frac{c}{d} & \text{if } 0 < x < d, \\ 0 & \text{if } d < |x| < p, \\ \frac{c}{d} & \text{if } -d < x < 0. \end{cases}$$

The Fourier series of f' is obtained by differentiating term by term the Fourier series of f (by Exercise 26). Now the function in this exercise is obtained by multiplying $f'(x)$ by $-\frac{d}{c}$. So the desired Fourier series is

$$-\frac{d}{c}f'(x) = -\frac{d}{c}\frac{2cp}{d\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos\frac{dn\pi}{p}}{n^2}\left(-\frac{n\pi}{p}\right)\sin\frac{n\pi}{p}x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{dn\pi}{p}}{n}\sin\frac{n\pi}{p}x.$$

30. The function in Exercise 10 is piecewise smooth and continuous, with a piecewise smooth derivative. So we can differentiate its Fourier series term by term to obtain the Fourier series of $f'(x)$. We have

$$f'(x) = \begin{cases} -\frac{1}{p-c} & \text{if } c < x < p, \\ 0 & \text{if } |x| < c, \\ \frac{1}{p-c} & \text{if } -p < x < -c. \end{cases}$$

Hence

$$(c-p)f'(x) = \begin{cases} 1 & \text{if } c < x < p, \\ 0 & \text{if } |x| < c, \\ -1 & \text{if } -p < x < -c, \end{cases}$$

which is the desired function. Differentiating term by term the Fourier series of f (by Exercise 26), we obtain

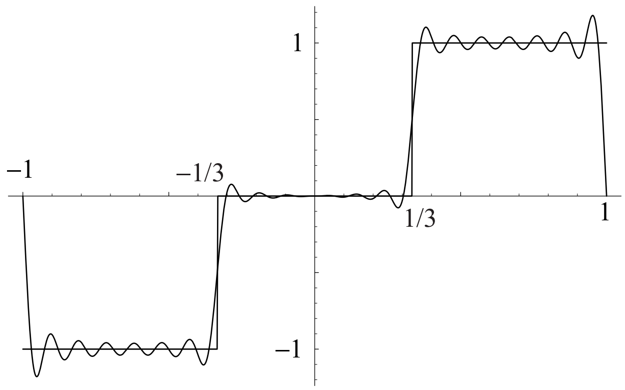
$$f'(x) = \frac{2p}{\pi^2(c-p)}\sum_{n=1}^{\infty}-\frac{(-1)^n-\cos\frac{n\pi c}{p}}{n^2}\left(\frac{n\pi}{p}\right)\sin\left(\frac{n\pi}{p}x\right).$$

Simplifying, we get the desired Fourier series

$$\begin{aligned}(c-p)f'(x) &= (c-p)\frac{2p}{\pi^2(c-p)}\sum_{n=1}^{\infty}-\frac{(-1)^n-\cos\frac{n\pi c}{p}}{n^2}\left(\frac{n\pi}{p}\right)\sin\left(\frac{n\pi}{p}x\right) \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi c}{p}-(-1)^n}{n}\sin\left(\frac{n\pi}{p}x\right).\end{aligned}$$

The convergence of the Fourier series is illustrated in the figure for the case $p = 1$ and $c = 1/3$.

```
Clear[s]
f[x_] =
 Which[x < -1, 0, -1 < x < -1 / 3, -1, -1 / 3 < x < 1 / 3, 0, 1 / 3 < x < 1, 1, x > 1, 0]
s[n_, x_] = 2 / Pi Sum[(Cos[k Pi / 3] - (-1) ^k) / (k) Sin[ k Pi x], {k, 1, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]
```



31. (a) If $\lim_{n\to\infty}\cos nx = 0$ for some x, then any subsequence of $(\cos nx)$ also converges to 0, in particular, $\lim_{n\to\infty}\cos(2nx) = 0$. Furthermore, $\lim_{n\to\infty}\cos^2 nx = 0$. But $\cos^2 nx = \frac{1+\cos(2nx)}{2}$, and taking the limit as $n \to \infty$ on both sides we get $0 = \frac{1+0}{2}$ or $0 = 1/2$, which is obviously a contradiction. Hence $\lim_{n\to\infty}\cos nx = 0$ holds for no x.
(b) If $\sum_{n=0}\infty \cos nx$ converges for some x, then by the nth term test, we must have $\lim_{n\to\infty}\cos nx = 0$. But this limit does not hold for any x; so the series does not converge for any x.

32. (a) The function 2π-periodic function given by $f(x) = \frac{1}{2}(\pi - x)$ if $0 < x < 2\pi$ is piecewise continuous with derivative equal to $-1/2$ for all $x \neq 2k\pi$. At the points where the derivative fails to exist, the left and right limits of the derivative exist and equal to -1/2. So the function is piecewise smooth.
(b) If we differentiate term by term the Fourier series $\sum_{n=0}^{\infty}\frac{\sin nx}{n}$, we obtain the Fourier series $\sum_{n=0}^{\infty}\cos nx$, which is not convergent for any x, by Exercise 30. Thus the differentiated series does not converge for any x, and, in particular, it does not converge to $f'(x)$. Hence the series cannot be differentiated term by term.

33. The function $F(x)$ is continuous and piecewise smooth with $F'(x) = f(x)$ at all the points where f is continuous (see Exercise 25, Section 2.1). So, by Exercise 26, if we differentiate the Fourier series of F, we get the Fourier series of f. Write

$$F(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{p}x + B_n \sin\frac{n\pi}{p}x\right)$$

and

$$f(x) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right).$$

Note that the a_0 term of the Fourier series of f is 0 because by assumption $\int_0^{2p} f(x)\,dx = 0$. Differentiate the series for F and equate it to the series for f and get

$$\sum_{n=1}^{\infty}\left(-A_n\frac{n\pi}{p}\sin\frac{n\pi}{p}x + \frac{n\pi}{p}B_n\cos\frac{n\pi}{p}x\right) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right).$$

Equate the nth Fourier coefficients and get

$$\begin{aligned} -A_n\frac{n\pi}{p} = b_n \quad &\Rightarrow \quad A_n = -\frac{p}{n\pi}b_n; \\ B_n\frac{n\pi}{p} = a_n \quad &\Rightarrow \quad B_n = \frac{p}{n\pi}a_n. \end{aligned}$$

This derives the nth Fourier coefficients of F for $n \geq 1$. To get A_0, note that $F(0) = 0$ because of the definition of $F(x) = \int_0^x f(t)\,dt$. So

$$0 = F(0) = A_0 + \sum_{n=1}^{\infty} A_n = A_0 + \sum_{n=1}^{\infty} -\frac{p}{n\pi}b_n;$$

and so $A_0 = \sum_{n=1}^{\infty} \frac{p}{n\pi}b_n$. We thus obtained the Fourier series of F in terms of the Fourier coefficients of f; more precisely,

$$F(x) = \frac{p}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n} + \sum_{n=1}^{\infty}\left(-\frac{p}{n\pi}b_n\cos\frac{n\pi}{p}x + \frac{p}{n\pi}a_n\sin\frac{n\pi}{p}x\right).$$

The point of this result is to tell you that, in order to derive the Fourier series of F, you can integrate the Fourier series of f term by term. Furthermore, the only assumption on f is that it is piecewise smooth and integrates to 0 over one period (to guarantee the periodicity of F.) Indeed, if you start with the Fourier series of f,

$$f(t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}t + b_n\sin\frac{n\pi}{p}t\right),$$

and integrate term by term, you get

$$\begin{aligned} F(x) &= \int_0^x f(t)\,dt = \sum_{n=1}^{\infty}\left(a_n\int_0^x \cos\frac{n\pi}{p}t\,dt + b_n\int_0^x \sin\frac{n\pi}{p}t\,dt\right) \\ &= \sum_{n=1}^{\infty}\left(a_n\left(\frac{p}{n\pi}\right)\sin\frac{n\pi}{p}t\Big|_0^x\,dt + b_n\left(-\frac{p}{n\pi}\right)\cos\frac{n\pi}{p}t\Big|_0^x\right) \\ &= \frac{p}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n} + \sum_{n=1}^{\infty}\left(-\frac{p}{n\pi}b_n\cos\frac{n\pi}{p}x + \frac{p}{n\pi}a_n\sin\frac{n\pi}{p}x\right), \end{aligned}$$

as derived earlier. See the following exercise for an illustration.

34. The function in Exercise 2 meets all the requirements of the previous exercise that allow term-by-term integration of its Fourier series. Note that if you integrate the function in Exercise 2 (called $f(x)$), you get the function in Exercise 4 (called $F(x)$) Thus, using the Fourier series in Exercise 2, we obtain

$$\begin{aligned} F(x) &= \int_0^x f(t)\,dt\frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^x \sin(\frac{n\pi}{p}t)\,dt \\ &= \frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{p}{n\pi}\right)\cos(\frac{n\pi}{p}t)\Big|_0^x \\ &= \frac{2p^2}{\pi^2}\overbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}}^{=\pi^2/(12)} - \frac{2p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(\frac{n\pi}{p}x) \\ &= \frac{p^2}{6} - \frac{2p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(\frac{n\pi}{p}x), \end{aligned}$$

where we used Exercise 17(a) to evaluate the infinite series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \pi^2/(12)$. This gives us the Fourier series of $F(x)$, which is the function in Exercise 4.

Solutions to Exercises 2.4

1. The even extension is the function that is identically 1. So the cosine Fourier series is just the constant 1. The odd extension yields the function in Exercise 1, Section 2.3, with $p = 1$. So the sine series is

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}.$$

This is also obtained by evaluating the integral in (4), which gives

$$b_n = 2\int_0^1 \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos n\pi x\Big|_0^1 = \frac{2}{n\pi}(1-(-1)^n).$$

2. (a) The odd extension of the function is the 2π-periodic function given by $f_2(x) = (\pi - x)$ for $0 < x < 2\pi$. This is 2 times the sawtooth function (Example 1, Section 2.1). Thus the sine series in this case is

$$2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

The even extension is the triangular 2π-periodic function given by

$$f_1(x) = \begin{cases} \pi - x & \text{if } 0 \le x < \pi, \\ \pi + x & \text{if } -\pi < x < 0, \end{cases}$$

The cosine series follows from Example 2, Section 2.3 (with $a = p = \pi$). We have

$$f_1(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}.$$

3. The even extension is the function in Exercise 4, Section 2.3, with $p = 1$. So the cosine Fourier series is

$$\frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}\cos n\pi x.$$

In evaluating the sine Fourier coefficients, we will use the formula

$$\int x^2 \sin ax\,dx = -\left(\frac{\left(-2 + a^2 x^2\right)\cos(a\,x)}{a^3}\right) + \frac{2\,x\,\sin(a\,x)}{a^2} + C \quad (a \neq 0),$$

which is obtained using integration by parts. For $n \ge 1$, we have

$$\begin{aligned} b_n &= 2\int_0^1 x^2\sin(n\pi x)\,dx \\ &= -2\left[\frac{(-2 + (n\pi)^2\,x^2)\cos(n\pi\,x)}{(n\pi)^3} - \frac{2\,x\,\sin(n\pi\,x)}{(n\pi)^2}\right]\Big|_0^1 \\ &= -2\left[\frac{(-2 + (n\pi)^2)(-1)^n}{(n\pi)^3} + \frac{2}{(n\pi)^3}\right] \\ &= 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^3}((-1)^n - 1)\right]. \end{aligned}$$

Thus the sine series representation

$$2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^3}((-1)^n - 1)\right]\sin(n\pi x).$$

4. (a) Let us use the formulas for the coefficients from Theorem 1. We have

$$\begin{aligned} b_n &= \int_1^2 (x-1)\sin\frac{n\pi}{2}x\,dx \\ &= \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2}x - \frac{2}{n\pi}x\cos\frac{n\pi}{2}x + \frac{2}{n\pi}\cos\frac{n\pi}{2}x\Big|_1^2 \\ &= -\frac{2}{n\pi}(-1)^n - \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2}. \end{aligned}$$

Thus the sine Fourier series is

$$\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}(-1)^n - \left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x.$$

For the cosine coefficients, we have $a_0 = 1/4$ (compute an area under the graph of f) and, for $n \geq 1$,

$$\begin{aligned} a_n &= \int_1^2 (x-1)\cos\frac{n\pi}{2}x\,dx \\ &= \left(\frac{2}{n\pi}\right)^2 \cos\frac{n\pi}{2}x + \frac{2}{n\pi}x\sin\frac{n\pi}{2}x - \frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big|_1^2 \\ &= \left(\frac{2}{n\pi}\right)^2\left[(-1)^n - \cos\frac{n\pi}{2}\right]. \end{aligned}$$

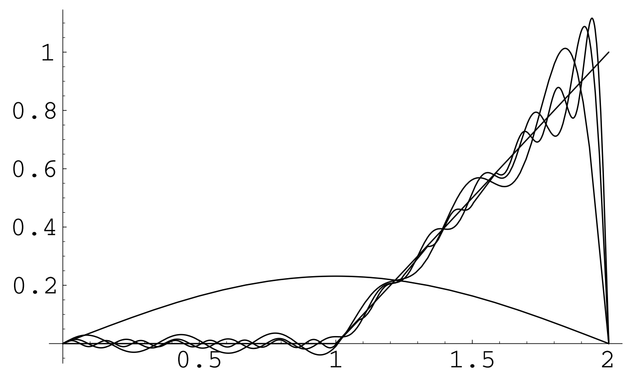
Thus, we have the cosine series

$$f_1(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)^2\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi}{2}x.$$

(b) The graphs of the partial sums of the since and cosine series both converge to $f(x)$ on the interval $[0, 2]$, as expected. The partial sums differ outside this interval. In fact, the sine series is odd while the cosine series is even. There is also a major difference in the way the series converge to the function. The odd extension, which gives us the sine series, is discontinuous at $x = \pm 2$ and the discontinuities there are similar to those of the sawtooth function. As a consequence, the partial sums of the Fourier series display a Gibbs phenomenon at these points. Moreover, since the Fourier coefficients are of the order $1/n$, the partial sums converge slowly to the function. By contrast, the even extension, which yields the cosine series, is piecewise smooth and continuous every where. Consequently, its Fourier series converges uniformly on the entire real line. The cosine coefficients are of the order $1/n^2$, and this makes the series converge quite fast, faster than the Fourier sine series.

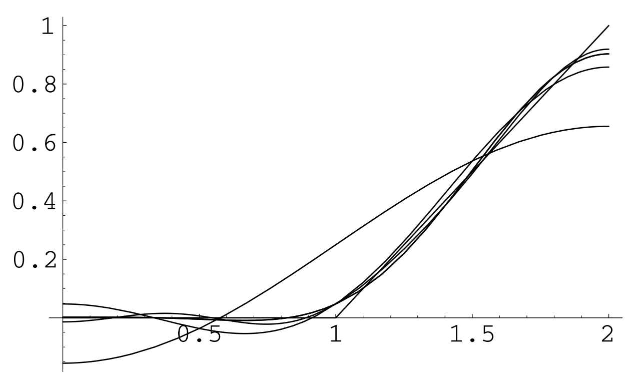
We first plot the sine series then the cosine series on the interval [0 , p]

```
b[k_] = -2 / (k Pi) (-1) ^k - (2 / (k Pi)) ^2 Sin[k Pi / 2];
ss[n_, x_] :=  Sum[b[k] Sin[k Pi / 2 x ], {k, 1, n}];
partialsineseries = Table[ss[n, x], {n, 1, 31, 10}];
f[x_] = If[0 < x < 1, 0, x - 1]
Plot[Evaluate[{partialsineseries, f[x]}], {x, 0, 2}]
```



Here is the cosine series:

```
a[k_] = (2 / (k Pi)) ^2 ( (-1) ^k - Cos[k Pi / 2]);
cs[n_, x_] :=  1 / 4 + Sum[a[k] Cos[k Pi / 2 x ], {k, 1, n}];
partialcosineseries = Table[cs[n, x], {n, 1, 5}];
f[x_] = If[0 < x < 1, 0, x - 1]
Plot[Evaluate[{partialcosineseries, f[x]}], {x, 0, 2}, PlotRange → All]
```



5. (a) Cosine series:

$$a_0 = \frac{1}{p}\int_a^b dx = \frac{b-a}{p};$$

$$a_n = \frac{2}{p}\int_a^b \cos\frac{n\pi x}{p}\,dx = \frac{2}{p}\frac{p}{n\pi}\left(\sin\frac{n\pi b}{p} - \sin\frac{n\pi a}{p}\right);$$

thus the even extension has the cosine series

$$f_e(x) = \frac{b-a}{p} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi b}{p} - \sin\frac{n\pi a}{p}\right)\cos\frac{n\pi x}{p}.$$

Sine series:

$$b_n = \frac{2}{p}\int_a^b \sin\frac{n\pi x}{p}\,dx = \frac{2}{p}\frac{p}{n\pi}\left(\cos\frac{n\pi a}{p} - \cos\frac{n\pi b}{p}\right);$$

thus the odd extension has the sine series

$$f_o(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi a}{p} - \cos\frac{n\pi b}{p}\right)\sin\frac{n\pi x}{p}.$$

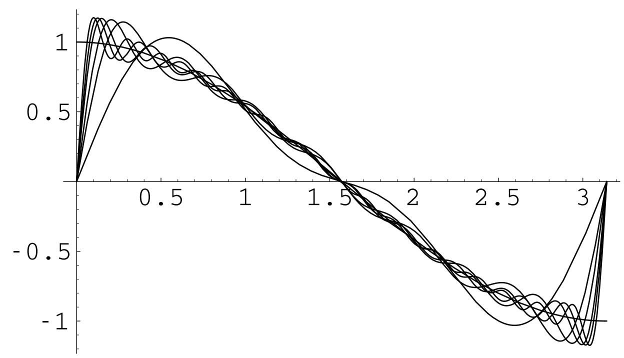
6. The even extension is the function $f_1(x) = \cos x$ for all x. Hence the Fourier series expansion is just $\cos x$. For the odd extension, we have, for $n > 1$,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}\cos x \sin nx\,dx \\ &= \frac{2}{\pi}\left[\frac{\cos(1-n)x}{2(1-n)} - \frac{\cos(1+n)x}{2(1+n)}\right]\Big|_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{(-1)^{n-1}}{(1-n)} - \frac{(-1)^{n+1}}{(1+n)} - \frac{1}{(1-n)} + \frac{1}{(1+n)}\right] \\ &= \frac{2n}{\pi}\frac{1+(-1)^n}{n^2-1}. \end{aligned}$$

For $n = 1$, you can easily show that $b_1 = 0$. Thus the sine Fourier series is

$$\frac{2}{\pi}\sum_{n=2}^{\infty} n\frac{1+(-1)^n}{n^2-1}\sin nx = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{k}{(2k)^2-1}\sin(2kx).$$

```
b[k_] = 2 k / Pi (1 + (-1) ^k) / (k^2 - 1);
ss[n_, x_] := Sum[b[k] Sin[k x ], {k, 2, n}];
partialsineseries = Table[ss[n, x], {n, 5, 30, 5}];
f[x_] = Cos[x]
Plot[Evaluate[{partialsineseries, f[x]}], {x, 0, Pi}]
```



7. The even extension is the function $|\cos x|$. This is easily seen by plotting the graph. The cosine series is (Exercise 8, Section 2.2):

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2-1}\cos(2nx).$$

Sine series:

$$
\begin{aligned}
b_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x \sin 2nx\, dx \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} [\sin(1+2n)x - \sin(1-2\,n)x)]\, dx \\
&= \frac{2}{\pi}\Big[\frac{1}{-1+2\,n} + \frac{1}{1+2\,n} - \frac{\overbrace{\cos(\frac{(-1+2\,n)\,\pi}{2})}^{=0}}{-1+2\,n} - \frac{\overbrace{\cos(\frac{(1+2\,n)\,\pi}{2})}^{=0}}{1+2\,n}\Big] \\
&= \frac{8}{\pi}\frac{n}{4n^2-1};
\end{aligned}
$$

thus the odd extension has the sine series

$$f_o(x) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nx.$$

8. The even 2π-periodic extension of the function $f(x) = x\sin x$ for $0 < x < \pi$ is given on the interval $(-\pi,\, \pi)$ by $f_1(x) = x\sin x$. Its Fourier series is computed in Exercise 11, Section 2.3. We have

$$f_1(x) = 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx.$$

The odd extension is given by $f_2(x) = |x|\sin x$ for $-\pi < x < \pi$. For the sine series, we have, for $n \neq 1$, Sine series:

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x\sin x \sin nx\, dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\big[-\cos(n+1)x + \cos(n-1)x\big]\, dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\big[\cos(n+1)x + \cos(n-1)x\big]\, dx \\
&= \frac{1}{\pi}\Bigg[\frac{-1}{(n+1)^2}\cos(n+1)x - \frac{x}{n+1}\sin(n+1)x \\
&\quad + \cos(n-1)x + \frac{1}{(n-1)^2}\cos(n-1)x + \frac{x}{n-1}\sin(n-1)x + \cos(n-1)x\Bigg]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-1}{(n+1)^2}(-1)^{n+1} + \frac{(-1)^{n-1}}{(n-1)^2} + \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right] \\
&= \frac{1}{\pi}\left[1+(-1)^n\right]\frac{-4n}{(n^2-1)^2},
\end{aligned}
$$

and

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\sin^2 x\, dx = \frac{1}{\pi}\int_0^{\pi} x(1-\cos 2x)\, dx = \frac{\pi}{2}.$$

Thus

$$f_2(x) = \frac{\pi}{2}\sin x - \frac{4}{\pi}\sum_{n=2}^{\infty}\left[1+(-1)^n\right]\frac{n}{(n^2-1)^2}\sin nx.$$

9. We have

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\, dx.$$

To evaluate this integral, we will use integration by parts to derive the following two formulas: for $a \neq 0$,

$$\int x\sin(ax)\, dx = -\frac{x\,\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} + C,$$

and

$$\int x^2 \sin(ax)\, dx = \frac{2\cos(a\,x)}{a^3} - \frac{x^2\cos(a\,x)}{a} + \frac{2\,x\sin(a\,x)}{a^2} + C.$$

So

$$\int x(1-x)\sin(ax)\, dx$$
$$= \frac{-2\cos(a\,x)}{a^3} - \frac{x\cos(a\,x)}{a} + \frac{x^2\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} - \frac{2\,x\sin(a\,x)}{a^2} + C.$$

Applying the formula with $a = n\pi$, we get

$$\int_0^1 x(1-x)\sin(n\pi x)\, dx$$
$$= \frac{-2\cos(n\pi\,x)}{(n\pi)^3} - \frac{x\cos(n\pi\,x)}{n\pi} + \frac{x^2\cos(n\pi\,x)}{n\pi} + \frac{\sin(n\pi\,x)}{(n\pi)^2} - \frac{2\,x\sin(n\pi\,x)}{(n\pi)^2}\Big|_0^1$$
$$= \frac{-2\left((-1)^n-1\right)}{(n\pi)^3} - \frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} = \frac{-2\left((-1)^n-1\right)}{(n\pi)^3}$$
$$= \begin{cases} \frac{4}{(n\pi)^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

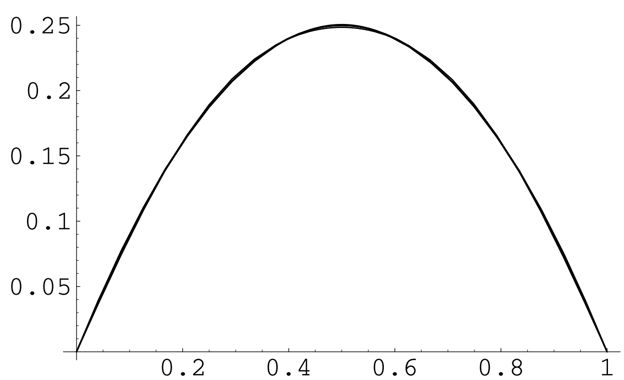
Thus

$$b_n = \begin{cases} \frac{8}{(n\pi)^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

Hence the sine series in

$$\frac{8}{\pi^3}\sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^3}.$$

```
b[k_] = 8 / Pi^3 1 / (2 k + 1) ^3;
ss[n_, x_] := Sum[b[k] Sin[(2 k + 1) Pi x ], {k, 0, n}];
partialsineseries = Table[ss[n, x], {n, 1, 5}];
f[x_] = x (1 - x)
Plot[Evaluate[{partialsineseries, f[x]}], {x, 0, 1}]
```



Perfect!

10. We use the fact that the function is the sum of three familiar functions to derive its sine series without excessive computations. Write

$$1 - x^2 = 1 - x + x(1-x) = g_1(x) + g_2(x) + g_3(x),$$

where $g_1(x) = 1$, $g_2(x) = -x$, and $g_3(x) = x(1-x)$. For g_1, use Exercise 1; for g_2, use Example 1; and and for g_3, use Exercise 9. Putting all this together, we find the sine coefficient

$$\begin{aligned} b_n &= \frac{2}{n\pi}(1-(-1)^n) - \frac{2(-1)^{n+1}}{n\pi} + \frac{-4\left((-1)^n - 1\right)}{(n\pi)^3} \\ &= \frac{2}{n\pi} - 4\frac{\left((-1)^n - 1\right)}{(n\pi)^3}. \end{aligned}$$

Thus, we have the sine series

$$2\sum_{n=1}^{\infty}\left(\frac{1}{n\pi} - 2\frac{\left((-1)^n - 1\right)}{(n\pi)^3}\right)\sin n\pi x.$$

11. The function is its own sine series.

12. Sine series expansion:

$$\begin{aligned} b_n &= 2\int_0^1 \sin(\frac{\pi}{2}x)\sin(n\pi x)\,dx \\ &= \left[\frac{\sin(n\pi - \pi/2)x}{n\pi - \pi/2} - \frac{\sin(n\pi + \pi/2)x}{n\pi + \pi/2}\right]\Big|_0^1 \\ &= \frac{\sin(n\pi - \pi/2)}{n\pi - \pi/2} - \frac{\sin(n\pi + \pi/2)}{n\pi + \pi/2} \\ &= -\frac{\cos(n\pi)}{n\pi - \pi/2} - \frac{\cos(n\pi)}{n\pi + \pi/2} = -(-1)^n\left[\frac{1}{n\pi + \pi/2} + \frac{1}{n\pi - \pi/2}\right] \\ &= (-1)^{n+1}\frac{2n\pi}{(n\pi)^2 - (\pi/2)^2} = \frac{(-1)^{n+1}}{\pi}\frac{8n}{4n^2 - 1}. \end{aligned}$$

Thus the sine series is

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{4n^2 - 1}\sin n\pi x.$$

13. We have

$$\sin\pi x\cos\pi x = \frac{1}{2}\sin 2\pi x.$$

This yields the desired 2-periodic sine series expansion.

14. We have

$$(1+\cos\pi x)\sin\pi x = \sin\pi x + \frac{1}{2}\sin 2\pi x.$$

This yields the desired 2-periodic sine series expansion.

15. We have

$$\begin{aligned} b_n = 2\int_0^1 e^x \sin n\pi x\,dx &= = \frac{2e^x}{1+(n\pi)^2}\left(\sin n\pi x - n\pi\cos n\pi x\right)\Big|_0^1 \\ &= \frac{2e}{1+(n\pi)^2}\left(n\pi(-1)^{n+1}\right) + \frac{2n\pi}{1+(n\pi)^2} \\ &= \frac{2n\pi}{1+(n\pi)^2}\left(1 + e(-1)^{n+1}\right). \end{aligned}$$

Hence the sine series is

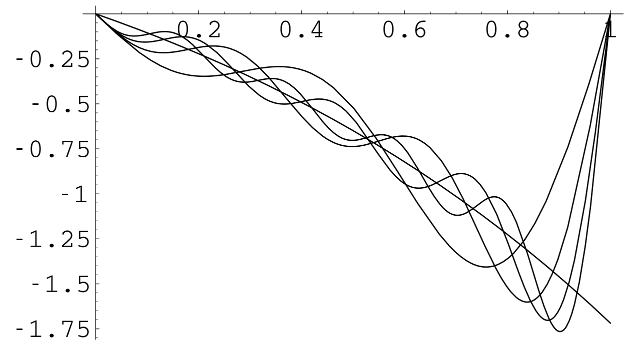
$$\sum_{n=1}^{\infty}\frac{2n\pi}{1+(n\pi)^2}\left(1 + e(-1)^{n+1}\right)\sin n\pi x.$$

16. Just add the sine series of 1 and the sine series of $-e^x$ for $0 < x < 1$ and you will get the sine series

$$\sum_{n=1}^{\infty}\left[\frac{2}{\pi n}(1-(-1)^n) - \frac{2n\pi}{1+(n\pi)^2}\left(1 + e(-1)^{n+1}\right)\right]\sin n\pi x.$$

Here is an illustration of the since series convergence.

```
b[k_] = 2 / (k Pi) (1 - (-1) ^k) - (2 k Pi) / (1 + (k Pi) ^2 ) (1 + E (-1) ^ (k + 1));
ss[n_, x_] :=  Sum[b[k] Sin[k Pi x ], {k, 1, n}];
partialsineseries = Table[ss[n, x], {n, 3, 10, 2}];
f[x_] = 1 - E^x
Plot[Evaluate[{partialsineseries, f[x] }], {x, 0, 1}, PlotRange → All]
```



17. (b) Sine series expansion:

$$\begin{aligned}
b_n &= \frac{2}{p}\int_0^a \frac{h}{a}x\sin\frac{n\pi x}{p}\,dx + \frac{2}{p}\int_a^p \frac{h}{a-p}(x-p)\sin\frac{n\pi x}{p}\,dx\\
&= \frac{2h}{ap}\Big[-x\frac{p}{n\pi}\cos\frac{n\pi x}{p}\Big|_0^a + \frac{p}{n\pi}\int_0^a \cos\frac{n\pi x}{p}\,dx\Big]\\
&\quad+\frac{2h}{(a-p)p}\Big[(x-p)\frac{(-p)}{n\pi}\cos(\frac{n\,\pi\,x}{p})\Big|_a^p + \int_a^p \frac{p}{n\pi}\cos\frac{n\,\pi\,x}{p}\,dx\Big]\\
&= \frac{2h}{pa}\Big[\frac{-ap}{n\pi}\cos\frac{n\pi a}{p} + \frac{p^2}{(n\pi)^2}\sin\frac{n\pi a}{p}\Big]\\
&\quad+\frac{2h}{(a-p)p}\Big[\frac{p}{n\pi}(a-p)\cos\frac{n\pi a}{p} - \frac{p^2}{(n\pi)^2}\sin\frac{n\pi a}{p}\Big]\\
&= \frac{2hp}{(n\pi)^2}\sin\frac{n\pi a}{p}\Big[\frac{1}{a} - \frac{1}{a-p}\Big]\\
&= \frac{2hp^2}{(n\pi)^2(p-a)a}\sin\frac{n\pi a}{p}.
\end{aligned}$$

Hence, we obtain the given Fourier series.

Solutions to Exercises 2.5

1. We have

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1,\\ -1 & \text{if } -1 < x < 0;\end{cases}$$

The Fourier series representation is

$$f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin(2k+1)\pi x.$$

The mean square error (from (5)) is

$$E_N = \frac{1}{2}\int_{-1}^{1} f^2(x)\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2).$$

In this case, $a_n = 0$ for all n, $b_{2k} = 0$, $b_{2k+1} = \frac{4}{\pi(2k+1)}$, and

$$\frac{1}{2}\int_{-1}^{1} f^2(x)\,dx = \frac{1}{2}\int_{-1}^{1} dx = 1.$$

So

$$E_1 = 1 - \frac{1}{2}(b_1^2) = 1 - \frac{8}{\pi^2} \approx 0.189.$$

Since $b_2 = 0$, it follows that $E_2 = E_1$. Finally,

$$E_1 = 1 - \frac{1}{2}(b_1^2 + b_3^2) = 1 - \frac{8}{\pi^2} - \frac{8}{9\,\pi^2} \approx 0.099.$$

2. We have

$$f(x) = \begin{cases} x & \text{if } -\pi < x < \pi, \\ f(x+2\pi) & \text{otherwise.} \end{cases}$$

Its Fourier series representation is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

The mean square error (from (5)) is

$$\begin{aligned} E_N &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \\ &= \frac{1}{6\pi}x^3\Big|_{-\pi}^{\pi} - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \\ &= \frac{\pi^2}{3} - 2\sum_{n=1}^{N}\frac{1}{n^2}, \end{aligned}$$

where, in the last equality, we used $a_n = 0$ and $b_n = 2\frac{(-1)^{n+1}}{n}$ for all n. So

$$\begin{aligned} E_1 &= \frac{\pi^2}{3} - 2 \approx 1.29; \\ E_2 &= E_1 - 2/(2^2) \approx .79; \\ E_3 &= E_2 - 2/(3^2) \approx .57. \end{aligned}$$

3. We have $f(x) = 1 - (x/\pi)^2$ for $-\pi < x < \pi$. Its Fourier series representation is

$$f(x) = \frac{2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n\pi)^2}\cos nx.$$

Thus $a_0 = \frac{2}{3}$, $a_n = 4\frac{(-1)^{n+1}}{(n\pi)^2}$, and $b_n = 0$ for all $n \geq 1$. The mean square error (from (5)) is

$$\begin{aligned} E_N &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 - (x/\pi)^2\right)^2\,dx - \frac{4}{9} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 - 2\frac{x^2}{\pi^2} + \frac{x^4}{\pi^4}\right)\,dx - \frac{4}{9} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4} \\ &= \frac{1}{\pi}\left[x - \frac{2\,x^3}{3\pi^2} + \frac{x^5}{5\pi^4}\right]\Big|_0^{\pi} - \frac{4}{9} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4} \\ &= 1 - \frac{2}{3} + \frac{1}{5} - \frac{4}{9} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4} \\ &= \frac{4}{45} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4}. \end{aligned}$$

So

$$\begin{aligned} E_1 &= \frac{4}{45} - \frac{8}{\pi^4} \approx .0068; \\ E_2 &= E_1 - \frac{8}{\pi^4}\frac{1}{2^4} \approx .0016; \\ E_3 &= E_2 - \frac{8}{\pi^4}\frac{1}{3^4} \approx .0006. \end{aligned}$$

4. We have $f(x) = x^2$ for $-1 < x < 1$. Its Fourier series representation is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$$

Thus $a_0 = \frac{1}{3}$, $a_n = 4\frac{(-1)^n}{(n\pi)^2}$, and $b_n = 0$ for all $n \geq 1$. The mean square error (from (5)) is

$$\begin{aligned} E_N &= \frac{1}{2}\int_{-1}^{1} f^2(x)\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \\ &= \int_0^1 x^4\,dx - \frac{1}{9} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4} \\ &= \frac{1}{5} - \frac{1}{9} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4} \\ &= \frac{4}{45} - \frac{8}{\pi^4}\sum_{n=1}^{N}\frac{1}{n^4}. \end{aligned}$$

So

$$\begin{aligned} E_1 &= \frac{4}{45} - \frac{8}{\pi^4} \approx .0068; \\ E_2 &= E_1 - \frac{8}{\pi^4}\frac{1}{2^4} \approx .0016; \\ E_3 &= E_2 - \frac{8}{\pi^4}\frac{1}{3^4} \approx .0006. \end{aligned}$$

5. We have

$$\begin{aligned} E_N &= \frac{1}{2}\int_{-1}^{1} f^2(x)\,dx - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2) \\ &= 1 - \frac{1}{2}\sum_{n=1}^{N} b_n^2 = 1 - \frac{8}{\pi^2}\sum_{1 \leq n \text{ odd} \leq N}\frac{1}{n^2}. \end{aligned}$$

With the help of a calculator, we find that $E_{39} = .01013$ and $E_{41} = .0096$. So take $N = 41$.

6. This exercise is a well-known application of the definite integral and can be found in any calculus textbook.

7. (a) Parseval's identity with $p = \pi$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Applying this to the given Fourier series expansion, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \quad\Rightarrow\quad \frac{\pi^2}{12} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \quad\Rightarrow\quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

For (b) and (c) see the solution to Exercise 17, Section 2.3.

8. Applying Parseval's identity, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} \quad\Rightarrow\quad \frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \quad\Rightarrow\quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

9. We have $f(x) = \pi^2 x - x^3$ for $-\pi < x < \pi$ and, for $n \geq 1$, $b_n = \frac{12}{n^3}(-1)^{n+1}$. By Parseval's identity

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{12}{n^3}\right)^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\pi^2 x - x^3\right)^2 \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\pi^4 x^2 - 2\pi^2 x^4 + x^6\right)dx \\
&= \frac{1}{\pi}\left(\frac{\pi^4}{3}x^3 - \frac{2\pi^2}{5}x^5 + \frac{x^7}{7}\right)\Big|_0^{\pi} \\
&= \pi^6\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{8}{105}\pi^6.
\end{aligned}$$

Simplifying, we find that

$$\zeta(6) = \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{(8)(2)}{(105)(144)}\pi^6 = \frac{\pi^6}{945}.$$

13. For the given function, we have $b_n = 0$ and $a_n = \frac{1}{n^2}$. By Parseval's identity, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} \quad\Rightarrow\quad \int_{-\pi}^{\pi} f^2(x)\,dx = \pi\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi\zeta(4) = \frac{\pi^5}{90},$$

where we have used the table preceding Exercise 7 to compute $\zeta(4)$.

15. Let us write the terms of the function explicitly. We have

$$f(x) = \sum_{n=0}^{\infty}\frac{\cos nx}{2^n} = 1 + \frac{\cos x}{2} + \frac{\cos 2x}{2^2} + \cdots.$$

Thus for the given function, we have

$$b_n = 0 \text{ for all } n, a_0 = 1,\ a_n = \frac{1}{2^n} \text{ for } n \geq 1.$$

By Parseval's identity, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx &= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}a_n^2 \\
&= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2^n)^2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{4^n} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{4^n}.
\end{aligned}$$

To sum the last series, we use a geometric series: if $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3},$$

and so

$$\int_{-\pi}^{\pi} f^2(x)\,dx = 2\pi(\frac{1}{2} + \frac{1}{2}\frac{4}{3}) = \frac{7\pi}{3}.$$

17. For the given function, we have

$$a_0 = 1,\ a_n = \frac{1}{3^n},\ b_n = \frac{1}{n} \text{ for } n \geq 1.$$

By Parseval's identity, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx &= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\
&= 1 + \frac{1}{2}\sum_{n=1}^{\infty}(\frac{1}{(3^n)^2} + \frac{1}{n^2}) \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{9^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{9^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}$$

Using a geometric series, we find

$$\sum_{n=0}^{\infty} \frac{1}{9^n} = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}.$$

By Exercise 7(a),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

So

$$\int_{-\pi}^{\pi} f^2(x)\,dx = 2\pi(\frac{1}{2} + \frac{1}{2}\frac{9}{8} + \frac{1}{2}\frac{\pi^2}{6}) = \frac{17\pi}{8} + \frac{\pi^3}{6}.$$

Solutions to Exercises 2.6

1. From Example 1, for $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$,

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a - in} e^{inx} \quad (-\pi < x < \pi);$$

consequently,

$$e^{-ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a + in} e^{inx} \quad (-\pi < x < \pi),$$

and so, for $-\pi < x < \pi$,

$$\begin{aligned} \cosh ax &= \frac{e^{ax} + e^{-ax}}{2} \\ &= \frac{\sinh \pi a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1}{a + in} + \frac{1}{a - in} \right) e^{inx} \\ &= \frac{a \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} e^{inx}. \end{aligned}$$

2. From Example 1, for $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$,

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a - in} e^{inx} \quad (-\pi < x < \pi);$$

consequently,

$$e^{-ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a + in} e^{inx} \quad (-\pi < x < \pi),$$

and so, for $-\pi < x < \pi$,

$$\begin{aligned} \sinh ax &= \frac{e^{ax} - e^{-ax}}{2} \\ &= \frac{\sinh \pi a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1}{a - in} - \frac{1}{a + in} \right) e^{inx} \\ &= \frac{i \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{n}{n^2 + a^2} e^{inx}. \end{aligned}$$

3. In this exercise, we will use the formulas $\cosh(iax) = \cos ax$ and $\sinh(iax) = i \sin ax$, for all real a and x. (An alternative method is used in Exercise 4.) To prove these formulas, write

$$\cosh(iax) = \frac{e^{iax} + e^{-iax}}{2} = \cos ax,$$

by Euler's identity. Similarly,

$$\sinh(iax) = \frac{e^{iax} - e^{-iax}}{2} = i \sin ax.$$

If a is not an integer, then $ia \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$, and we may apply the result of Exercise 1 to expand e^{iax} in a Fourier series:

$$\begin{aligned} \cos(ax) &= \cosh(iax) = \frac{(ia)\sinh(i\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + (ia)^2} e^{inx} \\ &= \frac{-a \sin(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 - a^2} e^{inx}. \end{aligned}$$

4. We have

$$\sin ax = \frac{e^{iax} - e^{-iax}}{2}.$$

Since $a \neq 0, \pm 1, \pm 2, \ldots$, we can use Example 1 to expand e^{iax} and e^{-iax}. We have, for $-\pi < x < \pi$,

$$\begin{aligned} e^{iax} &= \frac{\sinh(i\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(ia) - in} e^{inx} \\ &= \frac{e^{i\pi a} - e^{-i\pi a}}{2i\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a - n} e^{inx} \\ &= \frac{\sin(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a - n} e^{inx}. \end{aligned}$$

Similarly,

$$e^{-iax} = \frac{\sin(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a + n} e^{inx}.$$

Hence

$$\begin{aligned} \sin ax &= \frac{e^{iax} - e^{-iax}}{2i} \\ &= \frac{\sin \pi a}{2i\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1}{a - n} - \frac{1}{a + n} \right) e^{inx} \\ &= -\frac{i \sin \pi a}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{n}{a^2 - n^2} e^{inx}. \end{aligned}$$

5. Use identities (1); then

$$\begin{aligned} \cos 2x + 2 \sin 3x &= \frac{e^{2ix} + e^{-2ix}}{2} + 2 \frac{e^{3ix} - e^{-3ix}}{2i} \\ &= ie^{-3ix} + \frac{e^{-2ix}}{2} + \frac{e^{2ix}}{2} - ie^{3ix}. \end{aligned}$$

6. Use identities (1); then

$$\begin{aligned} \sin 3x &= \frac{e^{3ix} - e^{-3ix}}{2i} \\ &= i \frac{e^{-3ix}}{2} - i \frac{e^{3ix}}{2}. \end{aligned}$$

7. You can use formulas (5)–(8) to do this problem, or you can start with the Fourier series in Exercise 3 and rewrite it as follows:

$$\begin{aligned} \cos ax &= \frac{-a \sin(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{n^2 - a^2} \\ &= \frac{-a \sin(\pi a)}{\pi} \sum_{n=-\infty}^{-1} \frac{(-1)^n e^{inx}}{n^2 - a^2} - \frac{a \sin(\pi a)}{\pi} \frac{1}{-a^2} - \frac{a \sin(\pi a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{inx}}{n^2 - a^2} \\ &= \frac{\sin(\pi a)}{\pi a} - \frac{a \sin(\pi a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{-n} e^{-inx}}{(-n)^2 - a^2} - \frac{a \sin(\pi a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{inx}}{n^2 - a^2} \\ &= \frac{\sin(\pi a)}{\pi a} - \frac{a \sin(\pi a)}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\overbrace{e^{inx} + e^{-inx}}^{=2\cos nx}}{n^2 - a^2} \\ &= \frac{\sin(\pi a)}{\pi a} - 2 \frac{a \sin(\pi a)}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2 - a^2}. \end{aligned}$$

8. You can use formulas (5)–(8) to do this problem, or you can start with the Fourier series in Exercise 4 and rewrite it as follows:

$$\begin{aligned}
\sin ax &= -\frac{i\sin\pi a}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{n}{a^2-n^2}e^{inx}\\
&= \frac{i\sin(\pi a)}{\pi}\sum_{n=-\infty}^{-1}\frac{(-1)^n n e^{inx}}{n^2-a^2}+\frac{i\sin(\pi a)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n e^{inx}}{n^2-a^2}\\
&= \frac{i\sin(\pi a)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(-n)e^{-inx}}{(-n)^2-a^2}+\frac{i\sin(\pi a)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n e^{inx}}{n^2-a^2}\\
&= \frac{i\sin(\pi a)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-a^2}\overbrace{(e^{inx}-e^{-inx})}^{=2i\sin nx}\\
&= -\frac{2\sin(\pi a)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-a^2}\sin nx.
\end{aligned}$$

9. If $m=n$ then

$$\frac{1}{2p}\int_{-p}^{p}e^{i\frac{m\pi}{p}x}e^{-i\frac{n\pi}{p}x}\,dx=\frac{1}{2p}\int_{-p}^{p}e^{i\frac{m\pi}{p}x}e^{-i\frac{m\pi}{p}x}\,dx=\frac{1}{2p}\int_{-p}^{p}dx=1.$$

If $m\neq n$, then

$$\begin{aligned}
\frac{1}{2p}\int_{-p}^{p}e^{i\frac{m\pi}{p}x}e^{-i\frac{n\pi}{p}x}\,dx &= \frac{1}{2p}\int_{-p}^{p}e^{i\frac{(m-n)\pi}{p}x}\,dx\\
&= \frac{-i}{2(m-n)\pi}e^{i\frac{(m-n)\pi}{p}x}\Big|_{-p}^{p}\\
&= \frac{-i}{2(m-n)\pi}\left(e^{i(m-n)\pi}-e^{-i(m-n)\pi}\right)\\
&= \frac{-i}{2(m-n)\pi}\left(\cos[(m-n)\pi]-\cos[-(m-n)\pi]\right)=0.
\end{aligned}$$

10. From (6), we have

$$c_n+c_{-n}=a_n,$$

and

$$c_n-c_{-n}=-ib_n\Rightarrow b_n=i(c_n-c_{-n}).$$

11. The function in Example 1 is piecewise smooth on the entire real line and continuous at $x=0$. By the Fourier series representation theorem, its Fourier series converges to the value of the function at $x=0$. Putting 4x=04 in the Fourier series we thus get

$$f(0)=1=\frac{\sinh\pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in).$$

The doubly infinite sum is to be computed by taking symmetric partial sums, as follows:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in) &= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(-1)^n}{a^2+n^2}(a+in)\\
&= a\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(-1)^n}{a^2+n^2}+i\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(-1)^n n}{a^2+n^2}.
\end{aligned}$$

But

$$\sum_{n=-N}^{N}\frac{(-1)^n n}{a^2+n^2}=0,$$

because the summand is an odd function of n. So

$$1 = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a+in) = \frac{a \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2},$$

which is equivalent to the desired identity.

12. Starting with the Fourier series of Example 1, we have, for $-\pi < x\pi$ and $a \neq 0$,

$$\begin{aligned} e^{ax} &= \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a+in)e^{inx} \\ &= \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)}{a^2+n^2}\big(\cos nx + i \sin nx\big) \\ &= \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \big[(a \cos nx - n \sin nx) + i(n \cos nx + a \sin nx)\big]}{a^2+n^2} \\ &= \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a \cos nx - n \sin nx)}{a^2+n^2}, \end{aligned}$$

where in the last step we have used the fact that the symmetric partial sums of an odd integrand in n adds-up to 0 (see the previous exercise).

13. (a) At points of discontinuity, the Fourier series in Example 1 converges to the average of the function. Consequently, at $x = \pi$ the Fourier series converges to $\frac{e^{a\pi}+e^{-a\pi}}{2} = \cosh(a\pi)$. Thus, plugging $x = \pi$ into the Fourier series, we get

$$\cosh(a\pi) = \frac{\sinh(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}(a+in) \overbrace{e^{in\pi}}^{=(-1)^n} = \frac{\sinh(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(a+in)}{a^2+n^2}.$$

The sum $\sum_{n=-\infty}^{\infty} \frac{in}{a^2+n^2}$ is the limit of the symmetric partial sums

$$i \sum_{n=-N}^{N} \frac{n}{a^2+n^2} = 0.$$

Hence $\sum_{n=-\infty}^{\infty} \frac{in}{a^2+n^2} = 0$ and so

$$\cosh(a\pi) = \frac{\sinh(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} \quad \Rightarrow \quad \coth(a\pi) = \frac{a}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2},$$

upon dividing both sides by $\sinh(a\pi)$. Setting $t = a\pi$, we get

$$\coth t = \frac{t}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(\frac{t}{\pi})^2+n^2} = \sum_{n=-\infty}^{\infty} \frac{t}{t^2+(\pi n)^2},$$

which is (b). Note that since a is not an integer, it follows that t is not of the form $k\pi i$, where $k$is an integer.

14. Start with the expansion from Exercise 13(b): For $t \neq 0, \pm i\pi \pm 2i\pi, \ldots$,

$$\coth t = \sum_{n=-\infty}^{\infty} \frac{t}{t^2+(\pi n)^2}.$$

Take $t = ix$ with $x \neq k\pi$; then

$$\coth(ix) = \sum_{n=-\infty}^{\infty} \frac{ix}{(ix)^2+(\pi n)^2} = i \sum_{n=-\infty}^{\infty} \frac{x}{(\pi n)^2-x^2}.$$

But

$$\coth(ix) = \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = \frac{2\cos x}{2i \sin x} = -i \cot x.$$

So

$$-i \cot x = \sum_{n=-\infty}^{\infty} \frac{ix}{(ix)^2 + (\pi n)^2} = i \sum_{n=-\infty}^{\infty} \frac{x}{(\pi n)^2 - x^2};$$

equivalently, for $x \neq k\pi$,

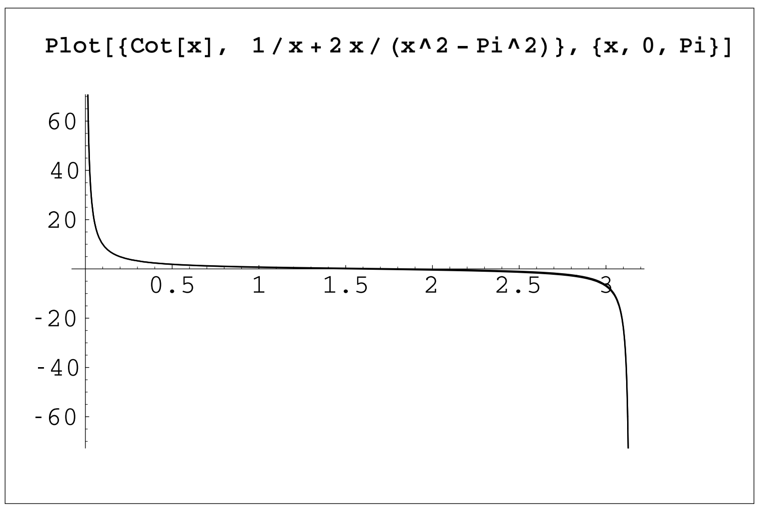
$$\cot x = \sum_{n=-\infty}^{\infty} \frac{x}{x^2 - (\pi n)^2}.$$

Additional remarks: For a fixed x in the domain of convergence of the series, the series converges absolutely. By writing the $n = 0$ term separately and taking symmetric partial sums, we get, for $x \neq k\pi$,

$$\cot x = \frac{1}{x} + 2 \sum_{n=1}^{\infty} \frac{x}{x^2 - (\pi n)^2}.$$

This is a pretty good approximation of the cotangent function by a sum of simple rational functions. See the figure for the approximation obtained by using only one term from the series on the interval $(0, \pi)$. The approximation of $\cot x$ that we derived tells you how badly the cotangent behaves around $k\pi$. For example, around $x = 0$, $\cot x$ is as bad (or as good) as $1/x$.

The approximation of $\cot x$ by rational functions is also useful in solving equations like $\cot x = x$, which arise frequently in applications (see Section 3.6).



15. (a) This is straightforward. Start with the Fourier series in Exercise 1: For $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$, and $-\pi < x < \pi$, we have

$$\cosh ax = \frac{a \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} e^{inx}.$$

On the left side, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh^2(ax)\, dx &= \frac{1}{\pi} \int_0^{\pi} \frac{\cosh(2ax) + 1}{2}\, dx \\ &= \frac{1}{2\pi} \Big[x + \frac{1}{2a} \sinh(2ax)\Big]\Big|_0^{\pi} = \frac{1}{2\pi} \Big[\pi + \frac{1}{2a} \sinh(2a\pi)\Big]. \end{aligned}$$

On the right side of Parseval's identity, we have

$$\frac{(a \sinh \pi a)^2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + a^2)^2}.$$

Hence

$$\frac{1}{2\pi}\big[\pi + \frac{1}{2a}\sinh(2a\pi)\big] = \frac{(a\sinh\pi a)^2}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+a^2)^2}.$$

Simplifying, we get

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+a^2)^2} = \frac{\pi}{2(a\sinh\pi a)^2}\big[\pi + \frac{1}{2a}\sinh(2a\pi)\big].$$

(b) This part is similar to part (a). Start with the Fourier series of Exercise 2: For $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$, and $-\pi < x < \pi$, we have

$$\sinh ax = \frac{i\sinh\pi a}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{n}{n^2+a^2}e^{inx}.$$

On the left side, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh^2(ax)\,dx &= \frac{1}{\pi}\int_0^{\pi}\frac{\cosh(2ax)-1}{2}\,dx\\ &= \frac{1}{2\pi}\big[-x + \frac{1}{2a}\sinh(2ax)\big]\Big|_0^{\pi} = \frac{1}{2\pi}\big[-\pi + \frac{1}{2a}\sinh(2a\pi)\big].\end{aligned}$$

On the right side of Parseval's identity, we have

$$\frac{\sinh^2\pi a}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{n^2}{(n^2+a^2)^2}.$$

Hence

$$\frac{1}{2\pi}\big[-\pi + \frac{1}{2a}\sinh(2a\pi)\big] = \frac{\sinh^2(\pi a)}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{n^2}{(n^2+a^2)^2}.$$

Simplifying, we get

$$\sum_{n=-\infty}^{\infty}\frac{n^2}{(n^2+a^2)^2} = \frac{\pi}{2\sinh^2(\pi a)}\big[-\pi + \frac{1}{2a}\sinh(2a\pi)\big].$$

16. (a) Following the hint, for a given w, we have

$$\frac{d}{dz}\frac{e^{z+w}}{e^z} = \frac{e^z e^{z+w} - e^z e^{z+w}}{(e^z)^2} = 0.$$

(This calculation uses the fact that $\frac{d}{dz}e^u = e^u\frac{du}{dz}$.) Since $\frac{d}{dz}\frac{e^{z+w}}{e^z} = 0$, we conclude that $\frac{e^{z+w}}{e^z} = C$ or $e^{z+w} = Ce^z$ for all z, where C is a constant (that depends on w). Take $z = 0$. It follows that $C = e^w$ and hence $e^{z+w} = e^z e^w$.
(b) We have

$$\begin{aligned}e^{i\theta} &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} = 1 + i\theta + i^2\frac{\theta^2}{2!} + +i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \cdots\\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\\ &= \cos\theta + i\sin\theta.\end{aligned}$$

(c) Replace θ by $-\theta$ in (b) and you get

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta,$$

which is the complex conjugate of $\cos\theta + i\sin\theta$. Thus

$$e^{-i\theta} = \overline{e^{i\theta}}.$$

(d) For any complex number, $|z|^2 = z \cdot \overline{z}$. Thus

$$\left|e^{i\theta}\right|^2 = e^{i\theta} \cdot \overline{e^{i\theta}} = e^{i\theta} \cdot e^{-i\theta} = 1.$$

So $\left|e^{i\theta}\right| = 1$.
(e) $e^{i\theta}+e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$. Thus $\cos\theta = (e^{i\theta}+e^{-i\theta})/2$. Similarly, $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$ (f)

$$\begin{aligned}
e^{i\pi/2} &= \cos(\pi/2) + i\sin(\pi/2) = i;\\
e^{3i\pi/4} &= \cos(3\pi/4) + i\sin(3\pi/4) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2};\\
e^{i\pi} &= \cos\pi + i\sin\pi = -1;\\
e^{2i\pi} &= \cos(2\pi) + i\sin(2\pi) = 1;\\
e^{16i\pi} &= \cos(16\pi) + i\sin(16\pi) = 1.
\end{aligned}$$

(g) We have

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1 = e^z.$$

17. (a) In this exercise, we let a and b denote real numbers such that $a^2 + b^2 \neq 0$. Using the linearity of the integral of complex-valued functions, we have

$$\begin{aligned}
I_1 + iI_2 &= \int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx\\
&= \int \left(e^{ax}\cos bx + ie^{ax}\sin bx\right)dx\\
&= \int e^{ax}\overbrace{(\cos bx + i\sin bx)}^{e^{ibx}}\,dx\\
&= \int e^{ax}e^{ibx}\,dx = \int e^{x(a+ib)}\,dx\\
&= \frac{1}{a+ib}e^{x(a+ib)} + C,
\end{aligned}$$

where in the last step we used the formula $\int e^{\alpha x}\,dx = \frac{1}{\alpha}e^{\alpha x} + C$ (with $\alpha = a + ib$), which is valid for all complex numbers $\alpha \neq 0$ (see Exercise 19 for a proof).
(b) Using properties of the complex exponential function (Euler's identity and the fact that $e^{z+w} = e^z e^w$), we obtain

$$\begin{aligned}
I_1 + iI_2 &= \frac{1}{a+ib}e^{x(a+ib)} + C\\
&= \frac{\overline{(a+ib)}}{(a+ib)\cdot\overline{(a+ib)}}e^{ax}e^{ibx} + C\\
&= \frac{a-ib}{a^2+b^2}e^{ax}\big(\cos bx + i\sin bx\big) + C\\
&= \frac{e^{ax}}{a^2+b^2}\left[\big(a\cos bx + b\sin bx\big) + i\big(-b\cos bx + a\sin bx\big)\right] + C.
\end{aligned}$$

(c) Equating real and imaginary parts in (b), we obtain

$$I_1 = \frac{e^{ax}}{a^2+b^2}\big(a\cos bx + b\sin bx\big)$$

and

$$I_2 = \frac{e^{ax}}{a^2+b^2}\big(-b\cos bx + a\sin bx\big).$$

18. (a) Use Euler's identity and properties of the exponential as follows:

$$\begin{aligned}
\cos n\theta + i\sin n\theta &= e^{in\theta} = \left(e^{i\theta}\right)^n\\
&= (\cos\theta + i\sin\theta)^n.
\end{aligned}$$

(b) With $n = 2$, the identity in (a) becomes

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= (\cos\theta)^2 + (i\sin\theta)^2 + 2i\cos\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.\end{aligned}$$

Equating real and imaginary parts, we obtain

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

(c) Proceeding as in (b) with $n = 3$, we obtain

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos\theta)^3 + 3\cos\theta(i\sin\theta)^2 + 3(\cos\theta)^2(i\sin\theta) + (i\sin\theta)^3 \\ &= (\cos\theta)^3 - 3\cos\theta(\sin\theta)^2 + i\big(3(\cos\theta)^2\sin\theta - (\sin\theta)^3\big).\end{aligned}$$

Equating real and imaginary parts, we obtain

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

But $\sin^2\theta = 1 - \cos^2\theta$, so

$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta.$$

Similarly, using $\cos^2\theta = 1 - \sin^2\theta$, we obtain

$$\sin 3\theta = -4\sin^3\theta + 3\sin\theta.$$

19. The purpose of this exercise is to show you that the familiar formula from calculus for the integral of the exponential function,

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C,$$

holds for all nonzero complex numbers a. Note that this formula is equivalent to

$$\frac{d}{dx}e^{ax} = ae^{ax},$$

where the derivative of $\frac{d}{dx}e^{ax}$ means

$$\frac{d}{dx}e^{ax} = \frac{d}{dx}\big(\operatorname{Re}(e^{ax}) + i\operatorname{Im}(e^{ax})\big) = \frac{d}{dx}\operatorname{Re}(e^{ax}) + i\frac{d}{dx}\operatorname{Im}(e^{ax}).$$

Write $a = \alpha + i\beta$, where α and β are real numbers. Then

$$\begin{aligned}e^{ax} &= e^{x(\alpha+i\beta)} = e^{\alpha x}e^{i\beta x} \\ &= e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x.\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dx}e^{ax} &= \frac{d}{dx}e^{\alpha x}\cos\beta x + i\frac{d}{dx}e^{\alpha x}\sin\beta x \\ &= \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x + i\big(\alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x\big) \\ &= e^{\alpha x}\big(\alpha\cos\beta x - \beta\sin\beta x + i\big(\alpha\sin\beta x + \beta\cos\beta x\big)\big) \\ &= e^{\alpha x}(\cos\beta x + i\sin\beta x)(\alpha + i\beta) \\ &= ae^{\alpha x}e^{i\beta x} = ae^{\alpha x + i\beta x}ae^{iax},\end{aligned}$$

as claimed.

20. By definition of the integral of a complex-valued function, we have

$$\int_a^b (h(t)+g(t))\,dt = \int_a^b \operatorname{Re}(h(t)+g(t))\,dt + i\int_a^b \operatorname{Im}(h(t)+g(t))\,dt.$$

But $\operatorname{Re}(h(t)+g(t)) = \operatorname{Re}(h(t)) + \operatorname{Re}(g(t))$ and $\operatorname{Im}(h(t)+g(t)) = \operatorname{Im}(h(t)) + \operatorname{Im}(g(t))$. So

$$\begin{aligned}\int_a^b (h(t)+g(t))\,dt &= \int_a^b \operatorname{Re}(h(t))\,dt + i\int_a^b \operatorname{Im}(h(t))\,dt \\ &\quad + \int_a^b \operatorname{Re}(g(t))\,dt + i\int_a^b \operatorname{Im}(g(t))\,dt \\ &= \int_a^b h(t)\,dt + \int_a^b g(t)\,dt.\end{aligned}$$

In a similar way, we can show that

$$\int_a^b \alpha h(t)\,dt = \alpha\int_a^b h(t)\,dt$$

for any complex number α. Combining the two properties, we obtain that

$$\int_a^b (\alpha h(t)+\beta g(t))\,dt = \alpha\int_a^b h(t)\,dt + \beta\int_a^b g(t)\,dt.$$

21. By Exercise 19,

$$\begin{aligned}\int_0^{2\pi} \left(e^{it}+2e^{-2it}\right)dt &= \frac{1}{i}e^{it} + \frac{2}{-2i}e^{-2it}\Big|_0^{2\pi} \\ &= -i\overbrace{e^{2\pi i}}^{=1} + i\overbrace{e^{-2\pi i}}^{=1} - (-i+i) = 0.\end{aligned}$$

Of course, this result follows from the orthogonality relations of the complex exponential system (formula (11), with $p=\pi$).

22. Taking a hint from the real-valued case, and using Exercise 19, we integrate by parts and obtain

$$\begin{aligned}\int_0^{\pi} \overbrace{t}^{=u}\,\overbrace{e^{2it}\,dt}^{=dv} &= t\frac{1}{2i}e^{2it}\Big|_0^{\pi} - \frac{1}{2i}\int_0^{\pi} e^{2it}\,dt \\ &= -i\frac{\pi}{2}e^{2i\pi} + \frac{1}{4}e^{2it}\Big|_0^{\pi} \\ &= -i\frac{\pi}{2} + \frac{1}{4}(1-1) = -i\frac{\pi}{2}\end{aligned}$$

23. First note that

$$\frac{1}{\cos t - i\sin t} = \frac{1}{e^{-it}} = e^{it}.$$

Hence

$$\int \frac{1}{\cos t - i\sin t}dt = \int e^{it}dt = -ie^{it} + C = \sin t - i\cos t + C.$$

24. Using linearity and integrating term by term, we have

$$\int_0^{2\pi} (3t - 2\cos t + 2i\sin t)\,dt = 6\pi.$$

Hence

$$\overline{\int_0^{2\pi} (3t - 2\cos t + 2i\sin t)\,dt} = \overline{6\pi} = 6\pi.$$

25. First note that

$$\frac{1+it}{1-it} = \frac{(1+it)^2}{(1-it)(1+it)} = \frac{1-t^2+2it}{1+t^2} = \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}.$$

Hence

$$\begin{aligned}\int \frac{1+it}{1-it}dt &= \int \frac{1-t^2}{1+t^2}\,dt + i\int \frac{2t}{1+t^2}\,dt \\ &= \int (-1 + \frac{2}{1+t^2}\,dt + i\int \frac{2t}{1+t^2}\,dt \\ &= -t + 2\tan^{-1} t + i\ln(1+t^2) + C.\end{aligned}$$

22. For $n \neq 0$, using the orthogonality relations, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \overbrace{(1-t)}^{=u}\,\overbrace{e^{-int}\,dt}^{=dv} &= (1-t)\frac{1}{-in}e^{-int}\Big|_{-\pi}^{\pi} + \frac{1}{-in}\overbrace{\int_{-\pi}^{\pi} e^{-int}\,dt}^{=0} \\ &= \frac{i}{n}\big[(1-\pi)e^{-in\pi} - (1+\pi)e^{in\pi}\big] \\ &= (-1)^n\frac{i}{n} - 2\pi = (-1)^{n+1}\frac{2i\pi}{n}.\end{aligned}$$

If $n = 0$, the integral becomes

$$\int_{-\pi}^{\pi} (1-t)\,dt = 2\pi.$$

Solutions to Exercises 2.7

1. (a) General solution of $y'' + 2y' + y = 0$. The characteristic equation is $\lambda^2+2\lambda+1 = 0$ or $(\lambda+1)^2 = 0$. It has one double characteristic root $\lambda = -1$. Thus the general solution of the homogeneous equation $y'' + 2y' + y = 0$ is

$$y = c_1 e^{-t} + c_2 t e^{-t}.$$

To find a particular solution of $y'' + 2y' + y = 25\cos 2t$, we apply Theorem 1 with $\mu = 1$, $c = 2$, and $k = 1$. The driving force is already given by its Fourier series: We have $b_n = a_n = 0$ for all n, except $a_2 = 25$. So $\alpha_n = \beta_n = 0$ for all n, except $\alpha_2 = \frac{A_2 a_2}{A_2^2+B_2^2}$ and $\beta_2 = \frac{B_2 a_2}{A_2^2+B_2^2}$, where $A_2 = 1 - 2^2 = -3$ and $B_2 = 4$. Thus $\alpha_2 = \frac{-75}{25} = -3$ and $\beta_2 = \frac{100}{25} = 4$, and hence a particular solution is $y_p = -3\cos 2t + 4\sin 2t$. Adding the general solution of the homogeneous equation to the particular solution, we obtain the general solution of the differential equation $y'' + 2y' + y = 25\cos 2t$

$$y = c_1 e^{-t} + c_2 t e^{-t} - 3\cos 2t + 4\sin 2t.$$

(b) Since $\lim_{t\to\infty} c_1 e^{-t} + c_2 t e^{-t} = 0$, it follows that the steady-state solution is

$$y_s = -3\cos 2t + 4\sin 2t.$$

3. (a) General solution of $4y'' + 4y' + 17y = 0$. The characteristic equation is $4\lambda^2 + 4\lambda + 17 = 0$. Its characteristic roots are

$$\lambda = \frac{-2 \pm \sqrt{4-(4)(17)}}{4} = \frac{-2 \pm \sqrt{-64}}{4} = -\frac{1}{2} \pm 2i.$$

Thus the general solution of the homogeneous equation is

$$y = c_1 e^{-t/2}\cos 2t + c_2 e^{-t/2}\sin 2t.$$

It is easy to see that $y = 1/17$ is a particular solution of $4y'' + 4y' + 17y = 1$. (This also follows from Theorem 1.) Hence the general solution is

$$y = c_1 e^{-t/2}\cos 2t + c_2 e^{-t/2}\sin 2t + \frac{1}{17}.$$

(b) Since $\lim_{t\to\infty} c_1 e^{-t/2}\cos 2t + c_2 e^{-t/2}\sin 2t = 0$, it follows that the steady-state solution is

$$y_s = \frac{1}{17}.$$

5. (a) To find a particular solution (which is also the steady-state solution) of $y'' + 4y' + 5y = \sin t - \frac{1}{2}\sin 2t$, we apply Theorem 1 with $\mu = 1$, $c = 4$, and $k = 5$. The driving force is already given by its Fourier series: We have $b_n = a_n = 0$ for all n, except $b_1 = 1$ and $b_2 = -1/2$. So $\alpha_n = \beta_n = 0$ for all n, except, possibly, α_1, α_2, β_1, and $beta_2$. We have $A_1 = 4$, $A_2 = 1$, $B_1 = 4$, and $B_2 = 8$. So

$$\begin{aligned}
\alpha_1 &= \frac{-B_1 b_1}{A_1^2 + B_1^2} = \frac{-4}{32} = -\frac{1}{8},\\
\alpha_2 &= \frac{-B_2 b_2}{A_2^2 + B_2^2} = \frac{4}{65} = \frac{4}{65},\\
\beta_1 &= \frac{A_1 b_1}{A_1^2 + B_1^2} = \frac{4}{32} = \frac{1}{8},\\
\beta_2 &= \frac{A_2 b_2}{A_2^2 + B_2^2} = \frac{-1/2}{65} = -\frac{1}{130}.
\end{aligned}$$

Hence the steady-state solution is

$$y_p = -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{4}{65}\cos 2t - \frac{1}{130}\sin 2t.$$

(b) We have

$$\begin{aligned}
y_p &= -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{4}{65}\cos 2t - \frac{1}{130}\sin 2t,\\
(y_p)' &= \frac{1}{8}\sin t + \frac{1}{8}\cos t - \frac{8}{65}\sin 2t - \frac{1}{65}\cos 2t,\\
(y_p)'' &= \frac{1}{8}\cos t - \frac{1}{8}\sin t - \frac{16}{65}\cos 2t + \frac{2}{65}\sin 2t,\\
(y_p)'' + 4(y_p)' + 5y_p &= \left(\frac{1}{8} + \frac{4}{8} - \frac{5}{8}\right)\cos t + \left(-\frac{1}{8} + \frac{4}{8} + \frac{5}{8}\right)\sin t\\
&\quad + \left(\frac{2}{65} - \frac{32}{65} - \frac{5}{130}\right)\sin 2t + \left(-\frac{16}{65} - \frac{4}{65} + \frac{20}{65}\right)\cos 2t\\
&= \sin t + \left(\frac{2}{75} - \frac{32}{65} - \frac{5}{130}\right)\sin 2t - \frac{1}{2}\sin 2t,
\end{aligned}$$

which shows that y_p is a solution of the nonhomogeneous differential equation.

9. (a) Natural frequency of the spring is

$$\omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{10.1} \approx 3.164.$$

(b) The normal modes have the same frequency as the corresponding components of driving force, in the following sense. Write the driving force as a Fourier series $F(t) = a_0 + \sum_{n=1}^{\infty} f_n(t)$ (see (5). The normal mode, $y_n(t)$, is the steady-state response of the system to $f_n(t)$. The normal mode y_n has the same frequency as f_n. In our case, F is 2π-periodic, and the frequencies of the normal modes are computed in Example 2. We have $\omega_{2m+1} = 2m+1$ (the n even, the normal mode is 0). Hence the frequencies of the first six nonzero normal modes are 1, 3, 5, 7, 9, and 11. The closest one to the natural frequency of the spring is $\omega_3 = 3$. Hence, it is expected that y_3 will dominate the steady-state motion of the spring.

13. According to the result of Exercise 11, we have to compute $y_3(t)$ and for this purpose, we apply Theorem 1. Recall that y_3 is the response to $f_3 = \frac{4}{3\pi}\sin 3t$, the component of the Fourier series of $F(t)$ that corresponds to $n = 3$. We have $a_3 = 0$, $b_3 = \frac{4}{3\pi}$, $\mu = 1$, $c = .05$, $k = 10.01$, $A_3 = 10.01 - 9 = 1.01$, $B_3 = 3(.05) = .15$,

$$\alpha_3 = \frac{-B_3 b_3}{A_3^2 + B_3^2} = \frac{-(.15)(4)/(3\pi)}{(1.01)^2 + (.15)^2} \approx -.0611 \quad \text{and} \quad \beta_3 = \frac{A_3 b_3}{A_3 + B_3^2} \approx .4111.$$

So

$$y_3 = -.0611\cos 3t + .4111\sin 3t.$$

The amplitude of y_3 is $\sqrt{.0611^2 + .4111^2} \approx .4156$.

17. (a) In order to eliminate the 3rd normal mode, y_3, from the steady-state solution, we should cancel out the component of F that is causing it. That is, we must remove $f_3(t) = \frac{4\sin 3t}{3\pi}$. Thus subtract $\frac{4\sin 3t}{3\pi}$ from the input function. The modified input function is

$$F(t) - \frac{4\sin 3t}{3\pi}.$$

Its Fourier series is he same as the one of F, without the 3rd component, $f_3(t)$. So the Fourier series of the modified input function is

$$\frac{4}{\pi}\sin t + \frac{4}{\pi}\sum_{m=2}^{\infty}\frac{\sin(2m+1)t}{2m+1}.$$

(b) The modified steady-state solution does not have the y_3-component that we found in Exercise 13. We compute its normal modes by appealing to Theorem 1 and using as an input function $F(t) - f_3(t)$. The first nonzero mode is y_1; the second nonzero normal mode is y_5. We compute them with the help of Mathematica. Let us first enter the parameters of the problem and compute α_n and β_n, using the definitions from Theorem 1. The input/output from Mathematica is the following

```
Clear[a, mu, p, k, alph, bet, capa, capb, b, y]
mu = 1;
c = 5 / 100;
k = 1001 / 100;
p = Pi;
a0 = 0;
a[n_] = 0;
b[n_] = 2 / (Pi n) (1 - Cos[ n Pi]);
alph0 = a0 / k;
capa[n_] = k - mu (n Pi / p) ^2
capb[n_] = c n Pi / p
alph[n_] = (capa[n] a[n] - capb[n] b[n]) / (capa[n] ^2 + capb[n] ^2)
bet[n_] = (capa[n] b[n] + capb[n] a[n]) / (capa[n] ^2 + capb[n] ^2)
```

$$\frac{1001}{100} - n^2$$

$$\frac{n}{20}$$

$$-\frac{1 - \mathrm{Cos}[n\,\pi]}{10\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi}$$

$$\frac{2\left(\frac{1001}{100} - n^2\right)(1 - \mathrm{Cos}[n\,\pi])}{n\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi}$$

It appears that

$$\alpha_n = \frac{-(1-\cos(n\,\pi))}{10\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi} \quad \text{and} \quad \beta_n = \frac{2\left(\frac{1001}{100} - n^2\right)(1-\cos(n\,\pi))}{n\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi}$$

Note how these formulas yield 0 when n is even. The first two nonzero modes of the modified solution are

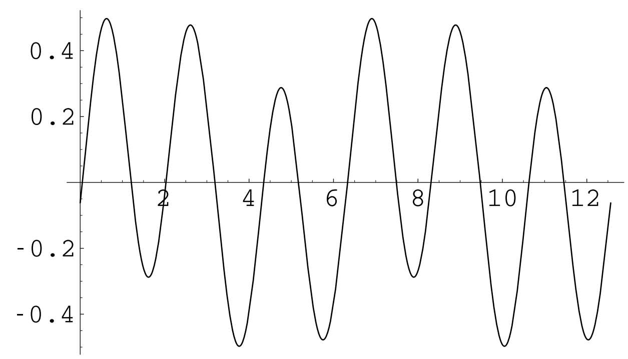
$$y_1(t) = \alpha_1 \cos t + \beta_1 \sin t = -.0007842 \cos t + .14131 \sin t$$

and

$$y_5(t) = \alpha_5 \cos 5t + \beta_5 \sin 5t - .00028 \cos 5t - .01698 \sin 5t.$$

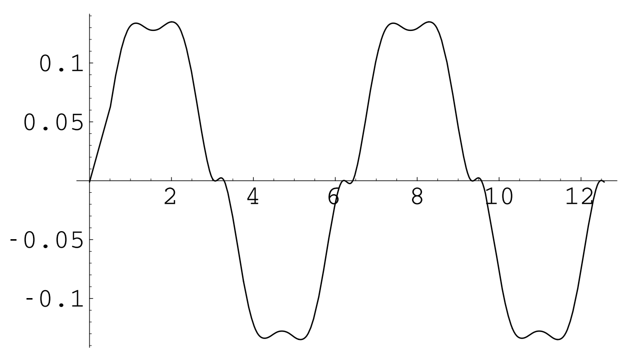
(c) In what follows, we use 10 nonzero terms of the original steady-state solution and compare it with 10 nonzero terms of the modified steady-state solution. The graph of the original steady-state solution looks like this:

```
steadystate[t_] = Sum[alph[n] Cos[n t] + bet[n] Sin[n t], {n, 1, 20}];
Plot[Evaluate[steadystate[t]], {t, 0, 4 Pi}]
```



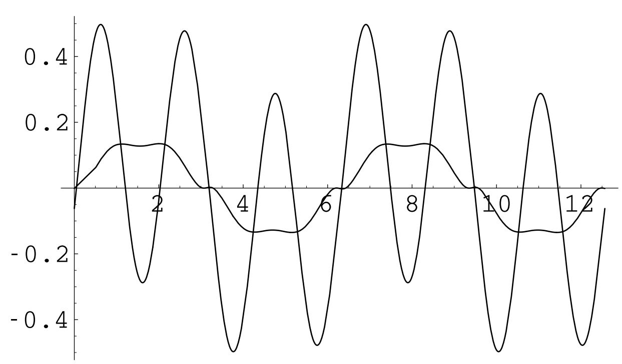
The modified steady-state is obtained by subtracting y_3 from the steady-state. Here is its graph.

```
modifiedsteadystate[t_] = steadystate[t] - (alph[3] Cos[3 t] + bet[3] Sin[3 t]);
Plot[Evaluate[modifiedsteadystate[t]], {t, 0, 4 Pi}]
```



In order to compare, we plot both functions on the same graph.

```
Plot[Evaluate[{steadystate[t], modifiedsteadystate[t]}], {t, 0, 4 Pi}]
```

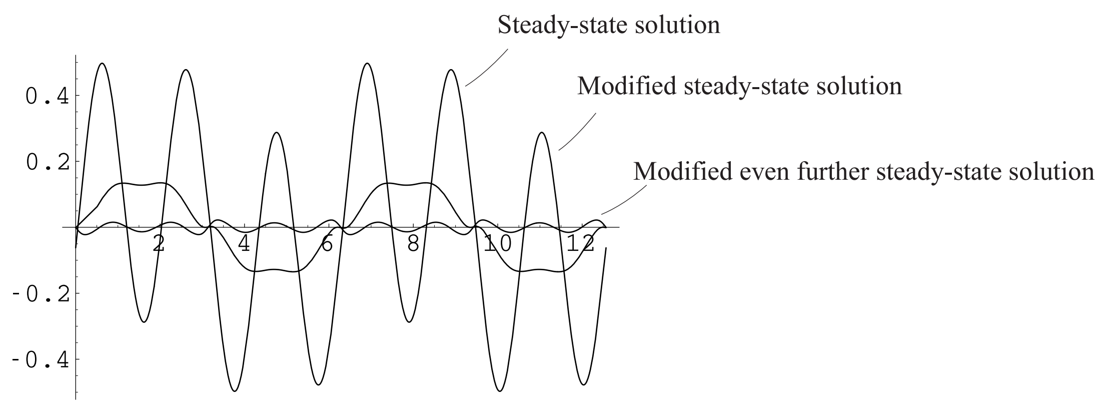


It seems like we were able to reduce the amplitude of the steady-state solution by a factor of 2 or 3 by removing the third normal mode. Can we do better? Let us analyze the amplitudes of the normal modes. These are equal to $\sqrt{\alpha_n^2 + \beta_n^2}$. We have the following numerical values:

```
amplitudes = N[Table[Sqrt[alph[n]^2 + bet[n]^2], {n, 1, 20}]]

{0.141312, 0., 0.415652, 0., 0.0169855, 0., 0.00466489, 0., 0.00199279, 0.,
 0.00104287, 0., 0.000616018, 0., 0.000394819, 0., 0.000268454, 0., 0.000190924, 0.}
```

It is clear from these values that y_3 has the largest amplitude (which is what we expect) but y_1 also has a relatively large amplitude. So, by removing the first component of F, we remove y_1, and this may reduce the oscillations even further. Let's see the results. We will plot the steady-state solution y_s, $y_s - y_3$, and $y_s - y_1 - y_3$.

```
modifiedfurther[t_] = modifiedsteadystate[t] - (alph[1] Cos[ t] + bet[1] Sin[ t]);
Plot[
 Evaluate[{modifiedfurther[t], steadystate[t], modifiedsteadystate[t]}], {t, 0, 4 Pi}]
```



21. (a) The input function $F(t)$ is already given by its Fourier series: $F(t) = 2\cos 2t + \sin 3t$. Since the frequency of the component $\sin 3t$ of the input function is 3 and is equal to the natural frequency of the spring, resonance will occur (because there is no damping in the system). The general solution of $y'' + 9y = 2\cos 2t + \sin 3t$ is $y = y_h + y_p$, where y_h is the general solution of $y'' + 9y = 0$ and y_p is a particular solution of the nonhomogeneous equation. We have $y_h = c_1 \sin 3t + c_2 \cos 3t$ and, to find y_p, we apply Exercise 20 and get

$$y_p = \left(\frac{a_2}{A_2}\cos 2t + \frac{b_2}{A_2}\sin 2t\right) + R(t),$$

where $a_2 = 2$, $b_2 = 0$, $A_2 = 9 - 2^2 = 5$, $a_{n_0} = 0$, $b_{n_0} = 1$, and

$$R(t) = -\frac{t}{6}\cos 3t.$$

Hence

$$y_p = \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t$$

and so the general solution is

$$y = c_1 \sin 3t + c_2 \cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t.$$

(b) To eliminate the resonance from the system we must remove the component of F that is causing resonance. Thus add to $F(t)$ the function $-\sin 3t$. The modified input function becomes $F_{\text{modified}}(t) = 2\cos 2t$.

25. The general solution is $y = c_1 \sin 3t + c_2 \cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t$. Applying the initial condition $y(0) = 0$ we get $c_2 + \frac{2}{5} = 0$ or $c_2 = -\frac{2}{5}$. Thus

$$y = c_1 \sin 3t - \frac{2}{5}\cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t.$$

Applying the initial condition $y'(0) = 0$, we obtain

$$\begin{aligned} y' &= 3c_1 \cos 3t + \frac{6}{5}\sin 3t - \frac{6}{5}\sin 2t - \frac{1}{6}\cos 3t + \frac{t}{2}\sin 3t, \\ y'(0) &= 3c_1 - \frac{1}{6}, \\ y'(0) = 0 &\Rightarrow c_1 = \frac{1}{18}. \end{aligned}$$

Thus

$$y = \frac{1}{18}\sin 3t - \frac{2}{5}\cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t.$$

Solutions to Exercises 2.9

1.
$$|f_n(x)| = \left|\frac{\sin nx}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

The sequence converges uniformly to 0 for all real x, because $\frac{1}{\sqrt{n}}$ controls its size independently of x.

5. If $x = 0$ then $f_n(0) = 0$ for all n. If $x \neq 0$, then applying l'Hospital's rule, we find
$$\lim_{n\to\infty} |f_n(x)| = |x| \lim_{n\to\infty} \frac{n}{e^{-nx}} = |x| \lim_{n\to\infty} \frac{1}{|x|e^{-n}} = 0.$$

The sequence does not converge uniformly on any interval that contains 0 because $f_n(\frac{1}{n}) = e^{-1}$, which does not tend to 0.

6. As in Exercise 5, the sequence converges to pointwise to 0 for all x, but not uniformly on any interval that contains 0 because $f_n(\frac{1}{n}) = e^{-1} - e^{-2}$, which does not tend to 0.

7. The sequence converges to 0 for all x because the degree of n in the denominator is greater than its degree in the numerator. For each n, the extreme points of $f_n(x)$ occur when $f_n'(x) = 0$ or
$$\frac{n - n^3 x^2}{(1 + n^2 x^2)^2} = 0 \quad \Rightarrow \quad n - n^3 x = 0 \quad \Rightarrow \quad x = \frac{\pm 1}{n}.$$

Since $f_n(\frac{1}{n}) = \frac{1}{2}$ the sequence does not converge to 0 uniformly on the interval $[0, \infty)$ (or any interval that contains 0).

9. $\left|\frac{\cos kx}{k^2}\right| \leq \frac{1}{k^2} = M_k$ for all x. Since $\sum M_k < \infty$ (p-series with $p > 1$), the series converges uniformly for all x.

14. $\left|\frac{x}{10}\right| \leq (\frac{9}{10})^k = M_k$ for all $|x| \leq 9$. Since $\sum (\frac{9}{10})^k < \infty$ (geometric series with ratio $9/10 < 1$), the series converges uniformly for all $|x| \leq 9$.

17. $\left|\frac{(-1)^k}{|x|+k^2}\right| \leq \frac{1}{k^2} = M_k$ for all x. Since $\sum M_k < \infty$ (p-series with $p > 1$), the series converges uniformly for all x.

24. (a) If $x = 0$, the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k} e^{-kx}$ is obviously convergent. If $x > 0$, say $x \geq \delta > 0$, then
$$\left|\frac{\sin kx}{k} e^{-kx}\right| \leq \frac{e^{-k\delta}}{k} = M_k.$$

The series $\sum M_k = \sum \frac{e^{-k\delta}}{k}$ is convergent by comparison to the geometric series $\sum \left(e^{-\delta}\right)^k < \infty$. By the Weierstrass M-test, the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k} e^{-kx}$ converges uniformly for all $x \geq \delta > 0$. Since $x > 0$ is arbitrary, it follows that the series converges for all $x > 0$.
(b) The differentiated series is $\sum_{k=1}^{\infty} (\cos kx - \sin kx) e^{-kx}$. For $x \geq \delta > 0$, we have
$$\left|(\cos kx - \sin kx)e^{-kx}\right| \leq 2e^{-k\delta} = M_k.$$

The series $\sum_k M_k = 2\sum_k e^{-k\delta}$ is convergent (geometric series $\sum_k \left(e^{-\delta}\right)^k < \infty$. By the Weierstrass M-test, the series $\sum_{k=1}^{\infty} (\cos kx - \sin kx)e^{-kx}$ converges uniformly for all $x \geq \delta > 0$. We proved in (a) that the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k} e^{-kx}$ converges uniformly for all $x \geq \delta > 0$. Hence by Theorem 4, since the series and the differentiated series are uniformly convergent for $x \geq \delta > 0$, it follows that we can differentiate the series term by term and get
$$\left(\sum_{k=1}^{\infty} \frac{\sin kx}{k} e^{-kx}\right)' = \sum_{k=1}^{\infty} \left(\frac{\sin kx}{k} e^{-kx}\right)' = \sum_{k=1}^{\infty} (\cos kx - \sin kx) e^{-kx}.$$

Since $x > 0$ as arbitrary, we see that term-by-term differentiation is justified for all $x > 0$. Note that this process can be repeated as often as we wish, because, as we differentiate a term in the series, we obtain polynomials in k times e^{-kx}. Since e^{-kx} dominates the polynomial, just like it did with the first derivative, the differentiated series will converge uniformly.

Solutions to Exercises 2.10

3. By Theorem 2(c), the series converges for all $x \neq 2n\pi$. For $x = 2n\pi$, the series $\sum_{k=1}^{\infty} \frac{\cos 2kn\pi}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent.

4. This series converges uniformly for all x by the Weierstrass M-test, because

$$\left| \frac{\sin 3kx}{k^2} \right| \leq \frac{1}{k^2} \quad \text{(for all } x\text{)}$$

and $\sum_k \frac{1}{k^2}$ is convergent.

5. The cosine part converges uniformly for all x, by the Weierstrass M-test. The sine part converges for all x by Theorem 2(b). Hence the given series converges for all x.

9. (a) If $\lim_{k\to\infty} \sin kx = 0$, then

$$\lim_{k\to\infty} \sin^2 kx = 0 \quad \Rightarrow \quad \lim_{k\to\infty} (1 - \cos^2 kx) = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \cos^2 kx = 1 \quad (*).$$

Also, if $\lim_{k\to\infty} \sin kx = 0$, then $\lim_{k\to\infty} \sin(k+1)x) = 0$. But $\sin(k+1)x = \sin kx \cos x + \cos kx \sin x$, so

$$\begin{aligned} 0 = \lim_{k\to\infty} (\overbrace{\sin kx \cos x}^{\to 0} + \cos kx \sin x) \quad &\Rightarrow \quad \lim_{k\to\infty} \cos kx \sin x = 0 \\ &\Rightarrow \quad \lim_{k\to\infty} \cos kx = 0 \text{ or } \sin x = 0. \end{aligned}$$

By (*), $\cos kx$ does not tend to 0, so $\sin x = 0$, implying that $x = m\pi$. Consequently, if $x \neq m\pi$, then $\lim k \to \infty \sin kx$ is not 0 and the series $\sum_{k=1}^{\infty} \sin kx$ does not converge by the nth term test, which proves (b).

10. (a) The series $\sum_{k=1}^{\infty} \frac{\cos kx}{k}$ converges uniformly on $[0.2, \frac{\pi}{2}]$ by Theorem 2(a). If we differentiate the series term by term, we obtain the series $-\sum_{k=1}^{\infty} \sin kx$, which does not converge for any x in the interval $[0.2, \frac{\pi}{2}]$ by Exercise 9. So the given series is uniformly convergent on $[0.2, \frac{\pi}{2}]$, but cannot be differentiated term by term.
(b) A similar situation occurs with the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ on any interval $[a, b]$ where $0 < a < b < 2\pi$. We omit the proof.

Solutions to Exercises 3.1

1. $u_{xx} + u_{xy} = 2u$ is a second order, linear, and homogeneous partial differential equation. $u_x(0, y) = 0$ is linear and homogeneous.

2. $u_{xx} + xu_{xy} = 2$, is a second order, linear, and nonhomogeneous (because of the right side) partial differential equation. $u(x, 0) = 0$, $u(x, 1) = 0$ are linear and homogeneous.

3. $u_{xx} - u_t = f(x, t)$ is a second order, linear, and nonhomogeneous (because of the right side) partial differential equation. $u_t(x, 0) = 2$ is linear and nonhomogeneous.

4. $u_{xx} = u_t$ is a second order, linear, and homogeneous partial differential equation. $u(x, 0) = 1$ and $u(x, 1) = 0$ are linear and nonhomogeneous (because of the first condition.

5. $u_t u_x + u_{xt} = 2u$ is second order and nonlinear because of the term $u_t u_x$. $u(0, t) + u_x(0, t) = 0$ is linear and homogeneous.

6. $u_{xx} + e^t u_{tt} = u \cos x$ is a second order, linear, and homogeneous partial differential equation. $u(x, 0) + u(x, 1) = 0$ is linear and homogeneous.

7. (a) $u_{xx} = u_{yy} = 0$, so $u_{xx} + u_{yy} = 0$.
(b) $u_{xx} = 2$, $u_{yy} = -2$, so $u_{xx} + u_{yy} = 0$. (c) We have $u = \frac{x}{x^2+y^2}$. So

$$u_x = \frac{-x^2 + y^2}{(x^2 + y^2)^2}, \quad u_y = \frac{-2\,x\,y}{(x^2 + y^2)^2},$$

and

$$u_{xx} = \frac{2\,(x^3 - 3\,x\,y^2)}{(x^2 + y^2)^3}, \quad u_{yy} = \frac{-2\,(x^3 - 3\,xy^2)}{(x^2 + y^2)^3} \quad \Rightarrow \quad \Delta u = u_{xx} + u_{yy} = 0.$$

(d) We have $u = \frac{y}{x^2+y^2}$. Switching x and y in (c), it follows immediately that

$$u_y = \frac{-y^2 + x^2}{(x^2 + y^2)^2}, \quad u_x = \frac{-2\,x\,y}{(x^2 + y^2)^2},$$

and

$$u_{yy} = \frac{2\,(y^3 - 3\,y\,x^2)}{(x^2 + y^2)^3}, \quad u_{xx} = \frac{-2\,(y^3 - 3\,x^2 y)}{(x^2 + y^2)^3} \quad \Rightarrow \quad \Delta u = u_{xx} + u_{yy} = 0.$$

(e) We have $u = \ln(x^2 + y^2)$, so

$$u_x = \frac{2\,x}{x^2 + y^2}, \quad u_y = \frac{2\,y}{x^2 + y^2},$$

and

$$u_{xx} = \frac{-2\,(x^2 - y^2)}{(x^2 + y^2)^2}, \quad u_{yy} = \frac{2\,(x^2 - y^2)}{(x^2 + y^2)^2} \quad \Rightarrow \quad \Delta u = u_{xx} + u_{yy} = 0.$$

(f) We have $u = e^y \cos x$, so

$$u_x = -e^y \sin(x), \quad u_y = e^y \cos(x),$$

and

$$u_{xx} = -e^y \cos(x), \quad u_{yy} = e^y \cos(x) \quad \Rightarrow \quad \Delta u = u_{xx} + u_{yy} = 0.$$

(g) We have $u = \ln(x^2 + y^2) + e^y \cos x$. Since u is the sum of two solutions of Laplace's equation (by (e) and (f)), it is itself a solution of Laplace's equation.

8. We note that the function is symmetric with respect to x, y, and z, so we only need to compute the derivatives with respect to x (or any other one of the variables) and then replace x by the other variables to get the other derivatives. We have

$$u_x = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad u_y = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad u_z = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

and

$$u_{xx} = \frac{2\,x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \quad u_{yy} = \frac{-x^2 + 2\,y^2 - z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \quad u_{zz} = \frac{-x^2 - y^2 + 2\,z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}};$$

so

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0.$$

9. (a) Let $u(x, y) = e^{ax}e^{by}$. Then

$$\begin{aligned} u_x &= ae^{ax}e^{by} \\ u_y &= be^{ax}e^{by} \\ u_{xx} &= a^2e^{ax}e^{by} \\ u_{yy} &= b^2e^{ax}e^{by} \\ u_{xy} &= abe^{ax}e^{by}. \end{aligned}$$

So

$$\begin{aligned} & Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \\ \Leftrightarrow\ & Aa^2e^{ax}e^{by} + 2Babe^{ax}e^{by} + Cb^2e^{ax}e^{by} \\ & +Dae^{ax}e^{by} + Ebe^{ax}e^{by} + Fe^{ax}e^{by} = 0 \\ \Leftrightarrow\ & e^{ax}e^{by}\big(Aa^2 + 2Bab + Cb^2 + Da + Eb + F\big) = 0 \\ \Leftrightarrow\ & Aa^2 + 2Bab + Cb^2 + Da + Eb + F = 0, \end{aligned}$$

because $e^{ax}e^{by} \neq 0$ for all x and y.
(b) By (a), in order to solve

$$u_{xx} + 2u_{xy} + u_{yy} + 2\,u_x + 2\,u_y + u = 0,$$

we can try $u(x, y) = e^{ax}e^{by}$, where a and b are solutions of

$$a^2 + 2ab + b^2 + 2a + 2b + 1 = 0.$$

But

$$a^2 + 2ab + b^2 + 2a + 2b + 1 = (a + b + 1)^2.$$

So $a + b + 1 = 0$. Clearly, this equation admits infinitely many pairs of solutions (a, b). Here are four possible solutions of the partial differential equation:

$$\begin{aligned} a = 1,\ b = -2 \quad &\Rightarrow \quad u(x, y) = e^x e^{-2y} \\ a = 0,\ b = -1 \quad &\Rightarrow \quad u(x, y) = e^{-y} \\ a = -1/2,\ b = -1/2 \quad &\Rightarrow \quad u(x, y) = e^{-x/2}e^{-y/2} \\ a = -3/2,\ b = 1/2 \quad &\Rightarrow \quad u(x, y) = e^{-3x/2}e^{y/2} \end{aligned}$$

10. According to Exercise 9(a), to find solutions of

$$u_{tt} + 2Bu_t - c^2u_{xx} + Au = 0 \quad (c > 0,\ A > 0)$$

we can we can try $u(x, y) = e^{ax}e^{by}$, where a and b are solutions of

$$a^2 - c^2b^2 + 2Ba + A = 0 \quad \text{or} \quad a(a + 2B) - c^2b^2 + A = 0.$$

This equation admits infinitely many pairs of solutions $(a,\, b)$. Here are two possible solutions of the partial differential equation:

$$\begin{aligned} a=0,\ b=\sqrt{A}/c \quad &\Rightarrow \quad u(t,\,x)=e^{\sqrt{A}x/c} \\ a=-2B,\ b=c\sqrt{A}/c \quad &\Rightarrow \quad u(t,\,x)=e^{-2Bt}e^{\sqrt{A}x/c}. \end{aligned}$$

12. Consider the nonlinear equation

$$(1) \qquad u_t + A(u)u_x = 0, \text{with initial condition } u(x,0)=\phi(x), \quad (*)$$

where $A(u)$ is a function of u.

Let $x(t)$ denote a solution of the first order differential equation

$$(2) \qquad \frac{dx}{dt} = A(u(x(t),t)). \quad (**)$$

This is a characteristic curve of (*). Restrict a solution u of (*) to a characteristic curve and consider the function of t given by $t \mapsto u((x(t),\, t))$. The derivative with respect to t of this function is

$$\begin{aligned} \frac{d}{dt}u((x(t),\, t)) &= u_x((x(t),\, t))\frac{d}{dt}x(t) + u_t((x(t),\, t)) \\ &= u_x((x(t),\, t))A(u(x(t),t)) + u_t((x(t),\, t)) = 0, \end{aligned}$$

since u is a solution of (*). Thus, $t \mapsto u((x(t),\, t))$ is constant on the characteristic curves. But this implies that $t \mapsto A(u((x(t),\, t)))$ is constant on the characteristic curves, and hence from (**) it follows that the characteristic curves have constant derivatives and so *the characteristic curves are straight lines with slopes* $A(u(x(t),\, t)))$. Setting $t=0$. We conclude that the slope of a characteristic line is $A(u(x(0),\, 0))) = A(\phi(x(0)))$. We write these lines in the form

$$x = tA(\phi(x(0))) + x(0)\,.$$

We complete the solution of (*) by solving for $x(0)$ in the preceding equation to get an implicit relation for the characteristic lines, of the form $L(x,t)=x(0)$. The final solution of (*) will be of the form $u(x,t)=f(L(x,t))$, where f is a function chosen so as to satisfy the initial condition in (*). That is $f(L(x,0))=\phi(x)$.

13. We follow the outlined solution in Exercise 12. We have

$$A(u)=\ln(u),\ \phi(x)=e^x, \quad \Rightarrow \quad A(u(x(t)),\, t)) = A(\phi(x(0))) = \ln(e^{x(0)}) = x(0).$$

So the characteristic lines are

$$x = tx(0)+x(0) \quad \Rightarrow \quad x(0)=L(x\, t)=\frac{x}{t+1}.$$

So $u(x,\, t)=f(L(x,\, t))=f\left(\frac{x}{t+1}\right)$. The condition $u(x,\, 0)=e^x$ implies that $f(x)=e^x$ and so

$$u(x,\, t)=e^{\frac{x}{t+1}}.$$

Check: $u_t=-e^{\frac{x}{t+1}}\frac{x}{(t+1)^2}$, $u_x=e^{\frac{x}{t+1}}\frac{1}{t+1}$,

$$u_t+\ln(u)u_x = -e^{\frac{x}{t+1}}\frac{x}{(t+1)^2}+\frac{x}{t+1}e^{\frac{x}{t+1}}\frac{1}{t+1}=0.$$

14. We follow the outlined solution in Exercise 12. We have

$$A(u)=u+1,\ \phi(x)=x^2, \quad \Rightarrow \quad A(u(x(t)),\, t)) = A(\phi(x(0))) = x(0)^2+1.$$

So the characteristic lines are

$$x = t(x(0)^2 + 1) + x(0) \quad \Rightarrow \quad tx(0)^2 + x(0) + t - x = 0.$$

Solving this quadratic equation in $x(0)$, we find two solutions

$$x(0) = \frac{-1 \pm \sqrt{1 - 4t(t - x)}}{2t},$$

and so

$$u(x, t) = f\left(\frac{-1 \pm \sqrt{1 - 4t(t - x)}}{2t}\right).$$

We now use the initial condition $u(x, 0) = x^2$ to determine the sign in front of the radical. Since we cannot simply set $t = 0$ in the formula, because of the denominator, we will take the limit as $t \to 0$ and write:

$$\lim_{t\to 0} u(x, t) = x^2 \quad \Rightarrow \quad \lim_{t\to 0} f\left(\frac{-1 \pm \sqrt{1 - 4t(t - x)}}{2t}\right) = x^2.$$

Now

$$\lim_{t\to 0} \frac{-1 - \sqrt{1 - 4t(t - x)}}{2t}$$

does not exist, but using l'Hospital's rule, we find that

$$\lim_{t\to 0} \frac{-1 + \sqrt{1 - 4t(t - x)}}{2t} = \lim_{t\to 0} \frac{(1 - 4t(t - x))^{-1/2}(-8t + 4x)}{4} = x.$$

So assuming that f is continuous, we get

$$x^2 = f\left(\lim_{t\to 0} \frac{-1 + \sqrt{1 - 4t(t - x)}}{2t}\right) = f(x),$$

and hence

$$u(x, t) = \left(\frac{-1 + \sqrt{1 - 4t(t - x)}}{2t}\right)^2.$$

15. We will just modify the solution from the previous exercise. We have

$$A(u) = u + 2, \ \phi(x) = x^2, \quad \Rightarrow \quad A(u(x(t)), t)) = A(\phi(x(0))) = x(0)^2 + 2.$$

So the characteristic lines are

$$x = t(x(0)^2 + 2) + x(0) \quad \Rightarrow \quad tx(0)^2 + x(0) + 2t - x = 0.$$

Solving for $x(0)$, we find

$$x(0) = \frac{-1 \pm \sqrt{1 - 4t(2t - x)}}{2t},$$

and so

$$u(x, t) = f\left(\frac{-1 \pm \sqrt{1 - 4t(2t - x)}}{2t}\right).$$

Now

$$\lim_{t\to 0} \frac{-1 - \sqrt{1 - 4t(2t - x)}}{2t}$$

does not exist, but using l'Hospital's rule, we find that

$$\lim_{t\to 0} \frac{-1 + \sqrt{1 - 4t(2t - x)}}{2t} = \lim_{t\to 0} \frac{(1 - 4t(2t - x))^{-1/2}(-16t + 4x)}{4} = x.$$

So

$$u(x,\,t) = \left(\frac{-1 + \sqrt{1 - 4t(2t - x)}}{2t}\right)^2.$$

16. We have

$$A(u) = u^2,\ \phi(x) = x, \quad \Rightarrow \quad A(u(x(t)),\,t)) = A(\phi(x(0))) = x(0)^2.$$

So the characteristic lines are

$$x = tx(0)^2 + x(0) \quad \Rightarrow \quad tx(0)^2 + x(0) - x = 0.$$

Solving for $x(0)$, we find

$$x(0) = \frac{-1 \pm \sqrt{1 + 4tx}}{2t},$$

and so

$$u(x,\,t) = f\left(\frac{-1 \pm \sqrt{1 + 4tx}}{2t}\right).$$

Now

$$\lim_{t \to 0} \frac{-1 - \sqrt{1 + 4tx}}{2t}$$

does not exist, but using l'Hospital's rule, we find that

$$\lim_{t \to 0} \frac{-1 + \sqrt{1 + 4tx}}{2t} = \lim_{t \to 0} \frac{(1 + 4tx)^{-1/2}(4x)}{4} = x.$$

So

$$u(x,\,t) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

17. We have

$$A(u) = u^2,\ \phi(x) = \sqrt{x}, \quad \Rightarrow \quad A(u(x(t)),\,t)) = A(\phi(x(0))) = x(0).$$

So the characteristic lines are

$$x = tx(0) + x(0) \quad \Rightarrow \quad x(0)(t + 1) - x = 0.$$

Solving for $x(0)$, we find

$$x(0) = \frac{x}{t + 1},$$

and so

$$u(x,\,t) = f\left(\frac{x}{t + 1}\right).$$

Now

$$u(x,\,0) = f(x) = \sqrt{x}.$$

So

$$u(x,\,t) = \sqrt{\frac{x}{t + 1}}.$$

Solutions to Exercises 3.3

1. The solution is

$$u(x,\,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(b_n\cos c\frac{n\pi t}{L}+b_n^*\sin c\frac{n\pi t}{L}\right),$$

where b_n are the Fourier sine coefficients of f and b_n^* are $\frac{L}{cn\pi}$ times the Fourier coefficients of g. In this exercise, $b_n^*=0$, since $g=0$, $b_1=0.05$; and $b_n=0$ for all $n>1$, because f is already given by its Fourier sine series (period 2). So $u(x,\,t)=0.05\sin\pi x\cos t$.

2. The solution is

$$u(x,\,t)=\sum_{n=1}^{\infty}\sin n\pi x\left(b_n\cos nt+b_n^*\sin nt\right),$$

where b_n are the Fourier sine coefficients of f and b_n^* are $\frac{1}{n}$ times the Fourier coefficients of g. In this exercise, $b_n^*=0$ since $g=0$. To get the Fourier coefficients of f, we note that $f(x)=\sin\pi x\cos\pi x=\frac{1}{2}\sin(2\pi x)$. So $b_2=\frac{1}{2}$, and all other $b_n=0$. So $u(x,\,t)=0.5\sin(2\pi x)\cos(2t)$.

3. The solution is

$$u(x,\,t)=\sum_{n=1}^{\infty}\sin(n\pi x)\left(b_n\cos(n\pi t)+b_n^*\sin(n\pi t)\right),$$

where $b_n^*=0$ since $g=0$. The Fourier coefficients of f are $b_1=1$, $b_2=3$, $b_5=-1$ and all other $b_n=0$. So

$$u(x,\,t)=\sin\pi x\cos(\pi t)+3\sin(2\pi x)\cos(2\pi t)-\sin(5\pi x)\cos(5\pi t).$$

4. The solution is

$$u(x,\,t)=\sum_{n=1}^{\infty}\sin(n\pi x)\left(b_n\cos(n\pi t)+b_n^*\sin(n\pi t)\right),$$

where $b_2^*=\frac{1}{2\pi}$, all other $b_n^*=0$, $b_1=1$, $b_3=\frac{1}{2}$, $b_7=3$ and all other $b_n=0$. So

$$\begin{aligned}u(x,\,t)&=\sin\pi x\cos(\pi t)+\frac{1}{2\pi}\sin(2\pi x)\sin(2\pi t)\\&\quad+\frac{1}{2}\sin(3\pi x)\cos(3\pi t)+3\sin(7\pi x)\cos(7\pi t).\end{aligned}$$

5. (a) The solution is

$$u(x,\,t)=\sum_{n=1}^{\infty}\sin(n\pi x)\left(b_n\cos(4n\pi t)+b_n^*\sin(4n\pi t)\right),$$

where b_n is the nth sine Fourier coefficient of f and b_n^* is $L/(cn)$ times the Fourier coefficient of g, where $L=1$ and $c=4$. Since $g=0$, we have $b_n^*=0$

for all n. As for the Fourier coefficients of f, we can get them by using Exercise 17, Section 2.4, with $p = 1$, $h = 1$, and $a = 1/2$. We get

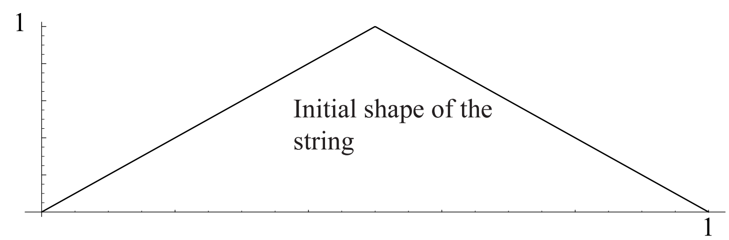
$$b_n = \frac{8}{\pi^2} \sin \frac{\frac{n\pi}{2}}{n^2}.$$

Thus

$$\begin{aligned} u(x,\,t) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} \sin(n\pi x) \cos(4n\pi t) \\ &= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x) \cos(4(2k+1)\pi t). \end{aligned}$$

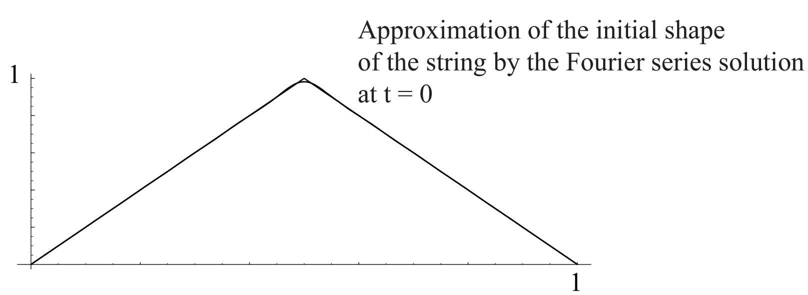
(b) Here is the initial shape of the string. Note the new Mathematica command that we used to define piecewise a function. (Previously, we used the If command.)

```
Clear[f]
f[x_] := 2 x   /; 0 < x < 1 / 2
f[x_] := 2 ( 1 - x)   /; 1 / 2 < x < 1
Plot[f[x], {x, 0, 1}]
```



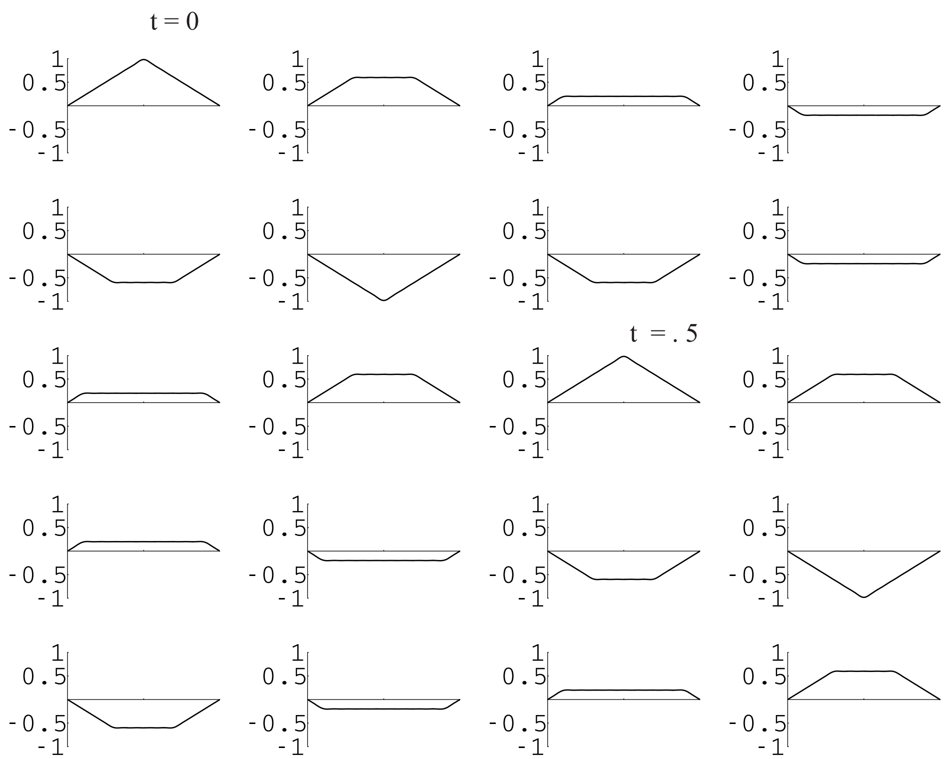
Because the period of $\cos(4(2k+1)\pi t)$ is $1/2$, the motion is periodic in t with period $1/2$. This is illustrated by the following graphs. We use two different ways to plot the graphs: The first uses simple Mathematica commands; the second one is more involved and is intended to display the graphs in a convenient array.

```
Clear[partsum]
partsum[x_, t_] :=
 8 / Pi ^ 2 Sum[Sin[(-1) ^ k (2 k + 1) Pi x] Cos[4 (2 k + 1) Pi t] / (2 k + 1) ^ 2, {k, 0, 10}]
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}]
```



Here is the motion in an array.

```
tt = Table[
   Plot[Evaluate[ partsum[x, t] ], {x, 0, 1}, PlotRange → {{0, 1}, {-1, 1}},
    Ticks → {{.5}, {-1, -.5, .5, 1}}, DisplayFunction → Identity], {t, 0, 1, 1 / 20}];
Show[GraphicsArray[Partition[tt, 4]]]
```



The first frame is the initial shape at t = 0. Subsequent frames occur in icrements of time of size 1/20.

6. (a) Using the formula from the text with $c = 1/\pi$ and $L = 1$, we find the solution

$$u(x,\,t) = \sum_{n=1}^{\infty} \sin n\pi x \left(b_n \cos nt + b_n^* \sin nt\right),$$

where b_n is the nth sine Fourier coefficient of f and

$$\begin{aligned} b_n^* &= \frac{2}{n}\int_0^1 g(x)\sin n\pi x\,dx \\ &= -\frac{4}{\pi n^2}\cos n\pi x\Big|_0^1 = \frac{4}{\pi n^2}\big(1-(-1)^n\big). \end{aligned}$$

We now compute the Fourier coefficients of f. We have

$$b_n = \frac{2}{30}\int_{1/3}^{2/3}(x-1/3)\sin n\pi x\,dx + \frac{2}{30}\int_{2/3}^{1}(1-x)\sin n\pi x\,dx.$$

We evaluate the integral with the help of the following identity, that can be derived using integration by parts:

$$\int (a+bx)\sin cx\,dx = -\frac{a+bx}{c}\cos cx + \frac{b}{c^2}\sin cx + C \quad (c \neq 0).$$

Thus

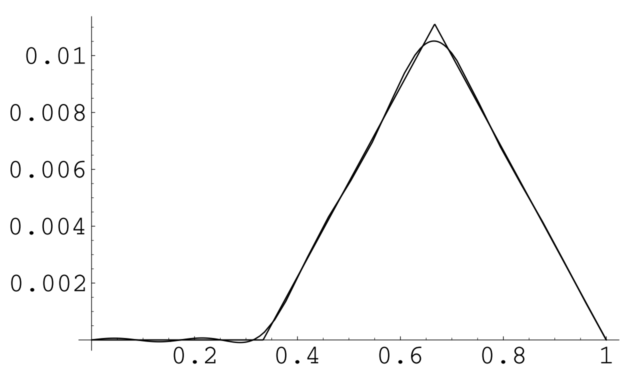
$$\begin{aligned}
b_n &= \frac{1}{15}\left[-\frac{x-1/3}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_{1/3}^{2/3} \\
&\quad + \frac{1}{15}\left[\frac{x-1}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_{2/3}^{1} \\
&= \frac{1}{15}\left[-\frac{1/3}{n\pi}\cos\frac{2n\pi}{3} + \frac{1}{n^2\pi^2}\sin\frac{2n\pi}{3} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{3}\right] \\
&\quad + \frac{1}{15}\left[\frac{1/3}{n\pi}\cos\frac{2n\pi}{3} + \frac{1}{n^2\pi^2}\sin\frac{2n\pi}{3}\right] \\
&= \frac{1}{15\,n^2\pi^2}\left[2\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right] \\
&= \frac{1}{15\,n^2\pi^2}\sin\frac{n\pi}{3}\left[4\cos\frac{n\pi}{3} - 1\right]
\end{aligned}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty}\frac{\sin n\pi x}{\pi n^2}\left(\frac{1}{15\,\pi}\sin\frac{n\pi}{3}\left[4\cos\frac{n\pi}{3}-1\right]\cos nt + 4\big(1-(-1)^n\big)\sin nt\right).$$

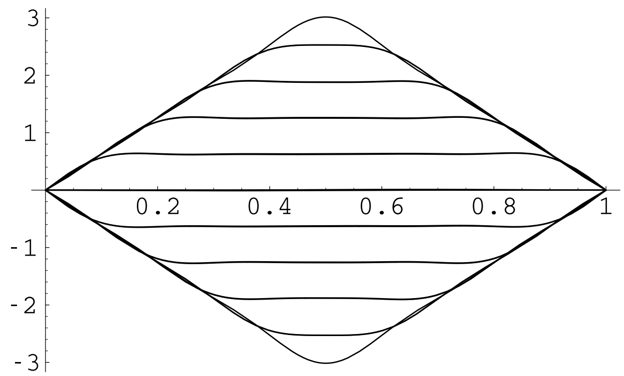
(b) Here is the initial shape of the string approximated using a partial sum of the series solution at time $t = 0$.

```
Clear[partsum, n, t, f]
Clear[f]
f[x_] := 0   /; 0 < x < 1 / 3
f[x_] := 1 / 30 (x - 1 / 3)  /; 1 / 3 < x < 2 / 3
f[x_] := 1 / 30 (1 - x)  /;  2 / 3 < x < 1
partsum[x_, t_] =
  Sum[ Sin[n Pi x] (1 / n^2 Sin[n Pi / 3] / (15 Pi^2) (4 Cos[ n Pi / 3] - 1) Cos[n t]
      + 4 / (n^2 Pi) (1 - (-1) ^n) Sin[ n t])
   , {n, 1, 10}];
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}]
```



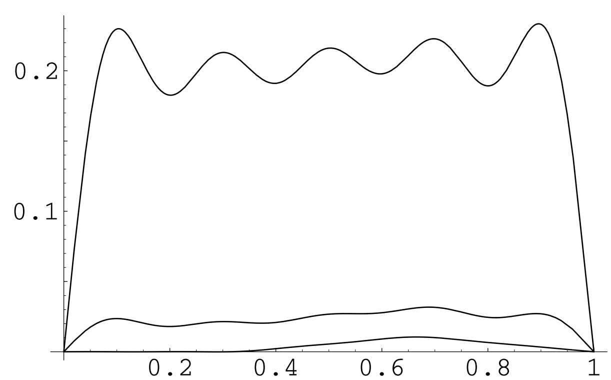
Because the period of $\cos nt$ and $\sin nt$ is 2π, the motion is 2π-periodic in t. This is illustrated by the following graphs.

```
Plot[Evaluate[ Table[partsum[x, t] , {t, 0, 2 Pi, 2 Pi / 20}]], {x, 0, 1}]
```



It is interesting to note that soon after the string is released, the initial velocity affects the motion in such a way that the initial shape does not appear to have a significant impact on the subsequent motion. Here are snapshots of the string very soon after it is released.

```
Plot[Evaluate[ {partsum[x, 0], partsum[x, .01], partsum[x, .1]}], {x, 0, 1}]
```



7. (a) Using the formula from the text with $c = 4$ and $L = 1$, we find the solution

$$u(x, t) = \sum_{n=1}^{\infty} \sin n\pi x \left(b_n \cos 4n\pi t + b_n^* \sin 4n\pi t\right),$$

where b_n is the nth sine Fourier coefficient of f and

$$\begin{aligned} b_n^* &= \frac{2}{4n\pi} \int_0^1 g(x) \sin n\pi x \, dx \\ &= -\frac{1}{2\pi^2 n^2} \cos n\pi x \Big|_0^1 = \frac{1}{2\pi^2 n^2} \big(1 - (-1)^n\big). \end{aligned}$$

The Fourier coefficients of f. We have

$$b_n = 8 \int_0^{1/4} x \sin n\pi x \, dx + 2 \int_{1/4}^{3/4} \sin n\pi x \, dx + 8 \int_{3/4}^{1} (1 - x) \sin n\pi x \, dx.$$

We evaluate the integral with the help of the identity

$$\int (a+bx)\sin cx\,dx = -\frac{a+bx}{c}\cos cx + \frac{b}{c^2}\sin cx + C \quad (c \neq 0).$$

We obtain

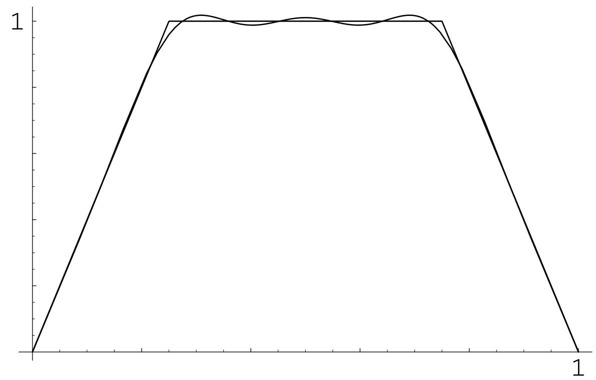
$$\begin{aligned}
b_n &= 8\left[-\frac{x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_0^{1/4} \\
&\quad -\frac{2}{n\pi}\cos n\pi x\Big|_{1/4}^{3/4} \\
&\quad +8\left[\frac{x-1}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_{3/4}^{1} \\
&= 8\left[-\frac{1/4}{n\pi}\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right] \\
&\quad -\frac{2}{n\pi}\cos\frac{3n\pi}{4} + \frac{2}{n\pi}\cos\frac{n\pi}{4} \\
&\quad +8\left[\frac{1/4}{n\pi}\cos\frac{3n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{3n\pi}{4}\right] \\
&= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{4} + \frac{8}{n^2\pi^2}\sin\frac{3n\pi}{4} \\
&= \frac{8}{n^2\pi^2}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \sin n\pi x \left(\frac{8}{n^2\pi^2}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right)\cos 4n\pi t\right. \\
&\quad \left. + \frac{1}{2\pi^2 n^2}\left(1-(-1)^n\right)\sin 4n\pi t\right).
\end{aligned}$$

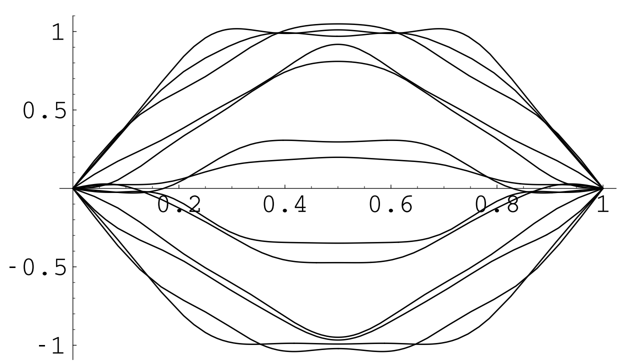
(b) Here is the initial shape of the string approximated using a partial sum of the series solution at time $t = 0$.

```
Clear[partsum, n, t, f]
Clear[f]
f[x_] := 4 x   /; 0 < x < 1 / 4
f[x_] := 1   /; 1 / 4 < x < 3 / 4
f[x_] := 4 (1 - x)  /; 3 / 4 < x < 1
partsum[x_, t_] = Sum[ Sin[n Pi x] (
     8 / (Pi ^ 2 n ^ 2) ( Sin[n Pi / 4] + Sin[3 n Pi / 4]) Cos[4 n Pi t]
      + 1 / (2 n ^ 2 Pi ^ 2) (1 - (-1) ^ n) Sin[4 n Pi t]
    )
   , {n, 1, 10}];
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}, PlotRange → All]
```



Because the period of $\cos 4n\pi t$ and $\sin 4n\pi t$ is $1/2$, the motion is $1/2$-periodic in t. This is illustrated by the following graphs.

```
Clear[tt, t]
Plot[Evaluate[ Table[partsum[x, t] , {t, 0, 1 / 2 , .04}]], {x, 0, 1}]
```



8. (a) The solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(nt),$$

where b_n is the nth sine Fourier coefficient of f. We will get b_n from the series that we found in Exercise 8, Section 2.4. We have, for $0 < x < \pi$,

$$x \sin x = \frac{\pi}{2} \sin x - \frac{4}{\pi} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2 - 1)^2} \sin nx.$$

Let $x = \pi t$. Then, for $0 < t < 1$,

$$\pi t \sin \pi t = \frac{\pi}{2} \sin \pi t - \frac{4}{\pi} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2 - 1)^2} \sin n\pi t.$$

Equivalently, for $0 < x < 1$,

$$x \sin \pi x = \frac{1}{2} \sin \pi x - \frac{4}{\pi^2} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2 - 1)^2} \sin n\pi x.$$

So

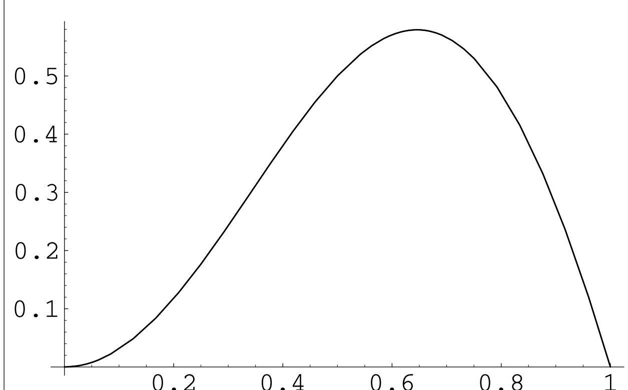
$$b_1 = \frac{1}{2} \quad \text{and} \quad b_n = [1 + (-1)^n] \frac{-4n}{\pi^2 (n^2 - 1)^2} \ (n \geq 2).$$

Thus

$$\begin{aligned} u(x, t) &= \frac{\sin \pi x \cos t}{2} - \frac{4}{\pi^2} \sum_{n=2}^{\infty} [1 + (-1)^n] \frac{n}{(n^2 - 1)^2} \sin(n\pi x) \cos(nt) \\ &= \frac{\sin \pi x \cos t}{2} - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{n}{(4n^2 - 1)^2} \sin(2n\pi x) \cos(2nt). \end{aligned}$$

(b) Here is the initial shape of the string.

```
Clear[partsum, n, t, f]
Clear[f]
f[x_] = x Sin[Pi x]
partsum[x_, t_] =
  Sin[Pi x] Cos[t] / 2 - 16 / Pi^2 Sum[ Sin[2 n Pi x] Cos[2 n t] n / (4 n^2 - 1) ^2
     , {n, 1, 10}];
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}, PlotRange → All]

x Sin[π x]
```



9. The solution is

$$u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x) \left(b_n \cos(n\pi t) + b_n^* \sin(n\pi t)\right),$$

where $b_1^* = \frac{1}{\pi}$ and all other $b_n^* = 0$. The Fourier coefficients of f are

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx.$$

To evaluate this integral, we will use integration by parts to derive first the formula: for $a \neq 0$,

$$\int x\sin(ax)\,dx = -\frac{x\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} + C,$$

and

$$\int x^2\sin(ax)\,dx = \frac{2\cos(a\,x)}{a^3} - \frac{x^2\cos(a\,x)}{a} + \frac{2\,x\sin(a\,x)}{a^2} + C;$$

thus

$$\begin{aligned}
&\int x(1-x)\sin(ax)\,dx\\
&= \frac{-2\cos(a\,x)}{a^3} - \frac{x\cos(a\,x)}{a} + \frac{x^2\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} - \frac{2\,x\sin(a\,x)}{a^2} + C.
\end{aligned}$$

Applying the formula with $a = n\pi$, we get

$$\begin{aligned}
&\int_0^1 x(1-x)\sin(n\pi x)\,dx\\
&= \frac{-2\cos(n\pi\,x)}{(n\pi)^3} - \frac{x\cos(n\pi\,x)}{n\pi} + \frac{x^2\cos(n\pi\,x)}{n\pi} + \frac{\sin(n\pi\,x)}{(n\pi)^2} - \frac{2\,x\sin(n\pi\,x)}{(n\pi)^2}\Big|_0^1\\
&= \frac{-2\left((-1)^n-1\right)}{(n\pi)^3} - \frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} = \frac{-2\left((-1)^n-1\right)}{(n\pi)^3}\\
&= \begin{cases} \frac{4}{(n\pi)^3} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}
\end{aligned}$$

Thus

$$b_n = \begin{cases} \frac{8}{(n\pi)^3} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even,}\end{cases}$$

and so

$$u(x,\,t) = \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)\cos((2k+1)\pi t)}{(2k+1)^3} + \frac{1}{\pi}\sin(\pi x)\sin(\pi t).$$

10. (a) Using the formula from the text with $c = 1$ and $L = 1$, we find the solution

$$u(x,\,t) = \sum_{n=1}^{\infty}\sin n\pi x b_n\cos n\pi t,$$

where b_n is the nth sine Fourier coefficient of f,

$$b_n = 8\int_0^{1/4} x\sin n\pi x\,dx - 8\int_{1/4}^{3/4}(x-1/2)\sin n\pi x\,dx + 8\int_{3/4}^{1}(x-1)\sin n\pi x\,dx.$$

We evaluate the integral as we did in Exercise 7:

$$
\begin{aligned}
b_n &= 8\left[-\frac{x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_0^{1/4} \\
&\quad -8\left[-\frac{(x-1/2)}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_{1/4}^{3/4} \\
&\quad +8\left[\frac{1-x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]\Big|_{3/4}^{1} \\
&= 8\left[-\frac{1/4}{n\pi}\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right] \\
&\quad -8\left[-\frac{1/4}{n\pi}\cos\frac{3n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{3n\pi}{4} - \frac{1/4}{n\pi}\cos\frac{n\pi}{4} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right] \\
&\quad +8\left[\frac{-1/4}{n\pi}\cos\frac{3n\pi}{4} - \frac{1}{n^2\pi^2}\sin\frac{3n\pi}{4}\right] \\
&= \frac{16}{n^2\pi^2}\left(\sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right).
\end{aligned}
$$

Thus

$$
u(x,\,t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin n\pi x\cos n\pi t}{n^2}\left(\sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right).
$$

11. (a) From (11),

$$
u_n(x,\,t) = \sin\frac{n\pi x}{L}\left(b_n\cos\frac{cn\pi t}{L} + b_n^*\sin\frac{cn\pi t}{L}\right),
$$

hence

$$
\begin{aligned}
u_n(x,\,t+\frac{2L}{nc}) &= \sin\frac{n\pi x}{L}\left(b_n\cos\frac{cn\pi(t+\frac{2L}{nc})}{L} + b_n^*\sin\frac{cn\pi(t+\frac{2L}{nc})}{L}\right) \\
&= \sin\frac{n\pi x}{L}\left(b_n\cos(\frac{cn\pi t}{L}+2\pi) + b_n^*\sin(\frac{cn\pi t}{L}+2\pi)\right) \\
&= \sin\frac{n\pi x}{L}\left(b_n\cos\frac{cn\pi t}{L} + b_n^*\sin\frac{cn\pi t}{L}\right) = u_n(x,\,t),
\end{aligned}
$$

so $u_n(x,\,t)$ has time period $\frac{2L}{nc}$. Since any positive integer multiple of a period is also a period, we conclude that $u_n(x,\,t)$ has time period $\frac{2L}{c}$, and since this period is independent of n, it follows that any linear combination of u_n's has period $\frac{2L}{c}$. This applies to an infinite sum also. Hence the series solution (8) is time periodic with period $\frac{2L}{c}$, which proves (b). To prove (c), note that

$$\begin{aligned}
u_n(x,\, t+\frac{L}{c}) &= \sin\frac{n\pi x}{L}\left(b_n\cos\frac{cn\pi(t+\frac{L}{c})}{L} + b_n^*\sin\frac{cn\pi(t+\frac{L}{c})}{L}\right)\\
&= \sin\frac{n\pi x}{L}\left(b_n\cos(\frac{cn\pi t}{L}+n\pi) + b_n^*\sin(\frac{cn\pi t}{L}+n\pi)\right)\\
&= (-1)^n\sin\frac{n\pi x}{L}\left(b_n\cos\frac{cn\pi t}{L} + b_n^*\sin\frac{cn\pi t}{L}\right)\\
&= \sin(\frac{n\pi x}{L}-n\pi)\left(b_n\cos\frac{cn\pi t}{L} + b_n^*\sin\frac{cn\pi t}{L}\right)\\
&= -\sin(\frac{n\pi(L-x)}{L})\left(b_n\cos\frac{cn\pi t}{L} + b_n^*\sin\frac{cn\pi t}{L}\right)\\
&= -u_n(L-x,\, t).
\end{aligned}$$

Since this relation holds for every n, summing over n, we obtain

$$u(x,\, t+\frac{L}{c}) = \sum_{n=1}^{\infty} u(x,\, t+\frac{L}{c}) = -\sum_{n=1}^{\infty} u(L-x,\, t) = -u(L-x,\, t).$$

Think of the string at time t as part of an odd $2L$-periodic graph, so that we can talk about $u(x,\, t)$ for x outside the interval $[0,\, L]$. Moreover, we have $u(-x,\, t) = -u(x,\, t)$ and $u(x+2L, t) = u(x,\, t)$. This graph moves as t varies and returns to its original shape at $t+\frac{2L}{c}$. The shape of the string at time $t+\frac{L}{c}$ (half a period) is obtained by translating the portion over the interval $[-L,\, 0]$ to the interval $[0,\, L]$. Thus, $u(x,\, t+\frac{L}{c}) = u(x-L,\, t) = -u(L-x,\, t)$, since u is odd.

12. In this exercise, we solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2},$$

where $0 < x < L$, $t > 0$, subject to the conditions

$$u(0,\, t) = 0, \quad u(L,\, t) = 0,$$

and

$$u(x,\, 0) = f(x), \quad \frac{\partial u}{\partial t}(x,\, 0) = g(x)\ 0 < x < L.$$

(a) To find the product solutions, plug $u(x,\, t) = X(x)T(t)$ into the equation and get

$$\begin{aligned}
XT'' + 2kXT' &= c^2X''T;\\
\frac{T''}{c^2T} + 2k\frac{T'}{c^2T} &= \frac{X''}{X} \quad \text{(Divide both sides by } c^2XT).
\end{aligned}$$

Since the left side is a function of t only and the right side is a function of x only, the variables are separated and the equality holds only if

$$\frac{T''}{T} + 2k\frac{T'}{T} = C \quad \text{and} \quad \frac{X''}{X} = C,$$

where C is a separation constant. Equivalently, we have the following equations in X and T:

$$T'' + 2kT' = CT \Rightarrow T'' + 2kT' - CT = 0$$

and

$$X'' - CX = 0.$$

Plugging $u = XT$ into the boundary conditions, we arrive at $X(0) = 0$ and $X(L) = 0$, as we did in the text when solving equation (1), subject to (2) and (3). Also, as in the text, the nontrivial solutions of the initial value problem $X'' - CX = 0$, $X(0) = 0$ and $X(L) = 0$, occur when $C = -\mu^2 < 0$. Otherwise, if $C \geq 0$, then the only solution of the initial value problem is $X = 0$. Thus, we must solve the equations

$$X'' + \mu^2 X = 0, \quad X(0) = 0, \; X(L) = 0,$$

$$T'' + 2kT' + (\mu c)^2 T = 0,$$

where now μ is the separation constant.
(b) As in the text, for the solution of equation (1), the nontrivial solutions in X are

$$X = X_n = \sin \frac{n\pi}{L} x, \; n = 1, 2, \ldots,$$

and these correspond to the separation constant

$$\mu = \mu_n = \frac{n\pi}{L}.$$

(c) To determine the solutions in T we have to solve $T''+2k\,T'+(\frac{n\pi}{L}c)^2 T = 0$. This is a second order linear differential equation with constant coefficients (Appendix A.2). Its general solution has three different cases depending on the roots of the characteristic equation

$$\lambda^2 + 2k\lambda + \left(\frac{n\pi}{L}c\right)^2 = 0.$$

By the quadratic formula, the roots of this equation are

$$\lambda = -k \pm \sqrt{k^2 - \left(\frac{n\pi}{L}c\right)^2}.$$

Case I: $k^2 - \left(\frac{n\pi}{L}c\right)^2 > 0$; equivalently, $k > \frac{n\pi}{L}c > 0$ or $\frac{Lk}{\pi c} > n > 0$. For these positive integers n, we have two characteristic roots

$$\lambda_1 = -k + \sqrt{k^2 - \left(\frac{n\pi}{L}c\right)^2} \quad \text{and} \quad \lambda_2 = -k - \sqrt{k^2 - \left(\frac{n\pi}{L}c\right)^2}.$$

To these roots correspond the following solutions:

$$f_1(t) = e^{(-k+\sqrt{\Delta})t} \quad \text{and} \quad f_2(t) = e^{(-k-\sqrt{\Delta})t},$$

where $\Delta = k^2 - \left(\frac{n\pi}{L}c\right)^2$. By taking linear combinations of these solutions, we obtain new solutions of the equation. Consider the following linear combinations:

$$\frac{f_1 + f_2}{2} \quad \text{and} \quad \frac{f_1 - f_2}{2}.$$

We have

$$\begin{aligned} \frac{f_1 + f_2}{2} &= \frac{1}{2}\left(e^{(-k+\sqrt{\Delta})t} + e^{(-k-\sqrt{\Delta})t}\right) \\ &= e^{-kt}\frac{e^{\sqrt{\Delta}t} + e^{-\sqrt{\Delta}t}}{2} = e^{-kt}\cosh(\sqrt{\Delta}t). \end{aligned}$$

Similarly, we have

$$\frac{f_1 + f_2}{2} = e^{-kt} \sinh(\sqrt{\Delta} t).$$

It is not difficult to show that these solutions are linearly independent. Thus the general solution in this case is

$$T_n = e^{-kt}(a_n \cosh \lambda_n t + b_n \sinh \lambda_n t),$$

where

$$\lambda_n = \sqrt{k^2 - \left(\frac{n\pi}{L}c\right)^2}.$$

Case II: $k^2 - \left(\frac{n\pi}{L}c\right)^2 = 0$; equivalently, $k = \frac{n\pi}{L}c > 0$ or $n = \frac{Lk}{\pi c}$. In this case, we have a double characteristic root, $\lambda = -k$, and the corresponding solutions are

$$f_1(t) = e^{-kt} \quad \text{and} \quad f_2(t) = t\, e^{-kt}.$$

The general solution in this case is

$$T_n = e^{-kt}(a_n + b_n t) \qquad n = \frac{Lk}{\pi c}.$$

Case III: $k^2 - \left(\frac{n\pi}{L}c\right)^2 < 0$; equivalently, $k < \frac{n\pi}{L}c > 0$ or $\frac{Lk}{\pi c} < n$. For these positive integers n, we have two complex conjugate characteristic roots

$$\lambda_1 = -k + i\sqrt{k^2 - \left(\frac{n\pi}{L}c\right)^2} \quad \text{and} \quad \lambda_2 = -k - i\sqrt{k^2 - \left(\frac{n\pi}{L}c\right)^2}.$$

Thus the general solution in this case is

$$T_n = e^{-kt}(a_n \cos \lambda_n t + b_n \sin \lambda_n t),$$

where

$$\lambda_n = \sqrt{-k^2 + \left(\frac{n\pi}{L}c\right)^2}.$$

(d) When $\frac{kL}{\pi c}$ is not a positive integer, Case II does not occur. So only Cases I and II are to be considered. These yield the product solutions

$$\sin \frac{n\pi x}{L} T_n(t),$$

where T_n is as described in (c). Adding all the product solutions, we get solution is

$$\begin{aligned} u(x,\, t) \quad &= \quad e^{-kt} \sum_{1 \le n < \frac{kL}{\pi c}} \sin \frac{n\pi}{L} x(a_n \cosh \lambda_n t + b_n \sinh \lambda_n t) \\ &\quad + e^{-kt} \sum_{\frac{kL}{\pi c} < n < \infty} \sin \frac{n\pi}{L} x(a_n \cos \lambda_n t + b_n \sin \lambda_n t), \end{aligned}$$

where these sums run over integers only. Setting $t = 0$ and using the initial condition, we obtain

$$u(x,\, 0) = f(x) = \sum_{1 \le n < \frac{kL}{\pi c}} a_n \sin \frac{n\pi}{L} x + \sum_{\frac{kL}{\pi c} < n < \infty} a_n \sin \frac{n\pi}{L} x.$$

The two sums can be combined into one sum, and we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x.$$

Thus a_n is the nth Fourier sine series coefficient of f. Hence

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \ldots .$$

To determine b_n, we differentiate with respect to time, set $t = 0$, and use the initial conditions. This yields

$$g(x) = \sum_{n=1}^{\infty} (-k a_n + \lambda_n b_n) \sin \frac{n\pi}{L} x.$$

Thus

$$-k a_n + \lambda_n b_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \ldots .$$

Having determined a_n previously, this last equation can be used to determine b_n. More precisely,

$$b_n = \frac{k a_n}{\lambda_n} + \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \ldots .$$

(e) If $\frac{kL}{\pi c} = n$ for some positive integer n, the solution $u(x, t)$ is as in (d) with the following additional term that corresponds to T_n from Case II in (c) (since $n = \frac{kL}{\pi c}$, we have $\frac{n\pi}{L} = \frac{k}{c}$) :

$$\sin(\frac{k}{c} x)(a_{\frac{kL}{\pi c}} e^{-kt} + b_{\frac{kL}{\pi c}} t e^{-kt}).$$

To determine a_n and b_n, we use the initial conditions, as we did in part (d). From the initial displacement, we find that all the a_n, including the case $n = \frac{kL}{\pi c}$, are determined by the Fourier coefficients of f:

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx, \quad n = 1, 2, \ldots .$$

When we differentiate (with respect to t) the series solution, the derivative of the term corresponding to $n = \frac{kL}{\pi c}$ is

$$\sin(\frac{k}{c} x) \left[b_{\frac{kL}{\pi c}} e^{-kt} - k e^{-kt} \left(a_{\frac{kL}{\pi c}} + b_{\frac{kL}{\pi c}} t \right) \right].$$

Setting $t = 0$, we get

$$\sin(\frac{k}{c} x) \left[b_{\frac{kL}{\pi c}} - k a_{\frac{kL}{\pi c}} \right].$$

Thus, all the b_ns are determined as in part (d), except for $n = \frac{kL}{\pi c}$, we have

$$-k a_{\frac{kL}{\pi c}} + b_{\frac{kL}{\pi c}} = \frac{2}{L} \int_0^L g(x) \sin \frac{k}{c} x \, dx$$

or

$$b_{\frac{kL}{\pi c}} = k a_{\frac{kL}{\pi c}} + \frac{2}{L} \int_0^L g(x) \sin \frac{k}{c} x \, dx.$$

13. To solve

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = \sin x, \quad \frac{\partial u}{\partial t}(x,0) = 0,$$

we follow the method of the previous exercise. We have $c = 1$, $k = .5$, $L = \pi$, $f(x) = \sin x$, and $g(x) = 0$. Thus the real number $\frac{Lk}{c\pi} = .5$ is not an integer and we have $n > \frac{kL}{\pi}$ for all n. So only Case III from the solution of Exercise 12 needs to be considered. Thus

$$u(x,\, t) = \sum_{n=1}^{\infty} e^{-.5t} \sin nx \big(a_n \cos \lambda_n t + b_n \sin \lambda_n t,$$

where

$$\lambda_n = \sqrt{(.5n)^2 - 1}.$$

Setting $t = 0$, we obtain

$$\sin x = \sum_{n=1}^{\infty} a_n \sin nx.$$

Hence $a_1 = 1$ and $a_n = 0$ for all $n > 1$. Now since

$$b_n = \frac{k a_n}{\lambda_n} + \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx, \quad n = 1, 2, \ldots,$$

it follows that $b_n = 0$ for all $n > 1$ and and the solution takes the form

$$u(x,\, t) = e^{-.5t} \sin x \big(\cos \lambda_1 t + b_1 \sin \lambda_1 t \big),$$

where $\lambda_1 = \sqrt{(.5)^2 - 1} = \sqrt{.75} = \frac{\sqrt{3}}{2}$ and

$$b_1 = \frac{k a_1}{\lambda_1} = \frac{1}{\sqrt{3}}.$$

So

$$u(x,\, t) = e^{-.5t} \sin x \big(\cos(\frac{\sqrt{3}}{2} t) + \frac{1}{\sqrt{3}} \sin(\frac{\sqrt{3}}{2} t) \big).$$

14. We follow the solution outlined in Exercise 12. We have $k = \frac{1}{2}$, $c = 1$, $L = \pi$, $f(x) = x \sin x$, $g(x) = 0$. Since $\frac{kL}{\pi c} = \frac{1}{2}$ is not an integer and is less than 1, the solution is given by

$$u(x,\, t) = e^{-\frac{1}{2}t} \sum_{n=1}^{\infty} \sin nx \left(a_n \cos \lambda_n t + b_n \sin \lambda_n t\right),$$

where, for $n \geq 1$,

$$\lambda_n = \frac{1}{2}\sqrt{4n^2 - 1},$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \sin nx\, dx$$

and

$$-\frac{1}{2}a_n + \lambda_n b_n = 0 \quad \Rightarrow \quad b_n = \frac{1}{\sqrt{4n^2-1}}a_n.$$

To compute a_n, we treat the case $n = 1$ separately. Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$, we have

$$\int x \sin^2 x\,dx = \int x\frac{(1-\cos 2x)}{2}\,dx = \frac{x^2}{4} - \frac{\cos 2\,x}{8} - \frac{x\,\sin 2\,x}{4} + C;$$

so

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin^2 x\,dx = \frac{2}{\pi}\left(\frac{x^2}{4} - \frac{\cos 2\,x}{8} - \frac{x\,\sin 2\,x}{4}\right)\Big|_0^\pi = \frac{\pi}{2},$$

and hence

$$b_1 = \frac{1}{2\lambda_1}a_1 = \frac{\pi}{2\sqrt{3}}.$$

For $n > 1$, using the formula $\sin x \sin nx = \frac{1}{2}(\cos((n-1)x) - \cos(n+1)x)$ and integration by parts, we first derive the formula

$$\begin{aligned}
&\int x\sin x \sin nx\,dx \\
&= \frac{1}{2}\int x(\cos((n-1)x) - \cos(n+1)x)\,dx \\
&= \frac{\cos((-1+n)\,x)}{2(-1+n)^2} + \frac{x\,\sin((-1+n)\,x)}{2(-1+n)} - \frac{\cos((1+n)\,x)}{2(1+n)^2} - \frac{x\,\sin((1+n)\,x)}{2(1+n)} + C
\end{aligned}$$

Thus, for $n \geq 2$,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\frac{(-1)^{n-1}-1}{2(-1+n)^2} - \frac{2}{\pi}\frac{(-1)^{n+1}-1}{2(1+n)^2} \\
&= \begin{cases} \frac{-2}{\pi(-1+n)^2} + \frac{2}{\pi(1+n)^2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \\
&= \begin{cases} \frac{-8n}{\pi(n^2-1)^2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Consequently,

$$a_{2k} = \frac{-16k}{\pi(4k^2-1)^2},\ a_{2k+1} = 0,\ b_{2k} = \frac{-16k}{\pi(4k^2-1)^2}\frac{1}{\sqrt{16k^2-1}},\ b_{2k+1} = 0,$$

and so

$$\begin{aligned}
u(x,\,t) &= \frac{\pi}{2}e^{-.5t}\sin x\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right) \\
&\quad -\frac{16}{\pi}e^{-.5t}\sum_{k=1}^\infty \frac{k\sin(2kx)}{(4k^2-1)^2}\left[\cos\left(\sqrt{4k^2-\frac{1}{4}}\,t\right) + \frac{1}{\sqrt{16k^2-1}}\sin\left(\sqrt{4k^2-\frac{1}{4}}\,t\right)\right].
\end{aligned}$$

15. (a) We use Exercise 12 to solve

$$\frac{\partial^2 u}{\partial t^2} + 3\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$u(0,t) = u(\pi,t) = 0,$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 10.$$

We have $c = 1$, $k = 3/2$, $L = \pi$, $f(x) = 0$, and $g(x) = 10$. So $\frac{kL}{c\pi} = 3/2$, which is not an integer. We distinguish two cases: $n < 3/2$ (that is, $n = 1$) and $n > 3/2$ (that is, $n > 1$). For $n = 1$, we have $\lambda_1 = \sqrt{5}/2$, $a_1 = 0$ and

$$b_1 = \frac{2}{\lambda_1 \pi} \int_0^{\pi} 10 \sin x \, dx = \frac{80}{\pi\sqrt{5}}.$$

The corresponding solution in T is

$$\frac{80}{\pi\sqrt{5}} e^{-1.5t} \sinh \frac{\sqrt{5}}{2} t.$$

For $n > 1$, we have

$$\lambda_n = \sqrt{|(3/2)^2 - n^2|} = \frac{1}{2}\sqrt{4n^2 - 9}$$

and

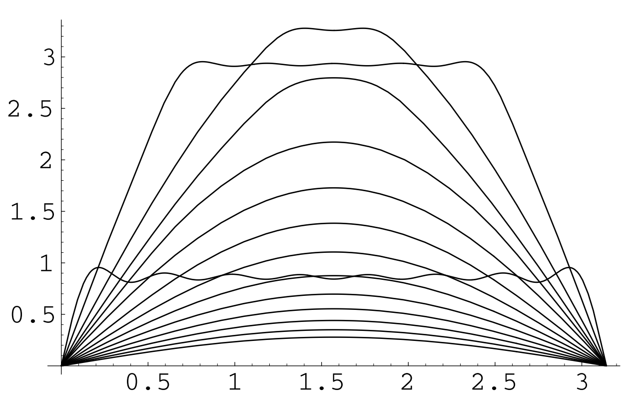
$$\begin{aligned} b_n &= \frac{2}{\pi \lambda_n} \int_0^{\pi} \sin nx \, dx \\ &= \frac{40}{n\pi\sqrt{4n^2 - 9}} (1 - (-1)^n) \\ &= \begin{cases} \frac{80}{(2k+1)\pi\sqrt{4(2k+1)^2 - 9}} & \text{if } n = 2k+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} u(x,\,t) &= \frac{80}{\pi\sqrt{5}} e^{-1.5t} \sin x \sinh \frac{\sqrt{5}}{2} t \\ &\quad + \frac{80}{\pi} e^{-1.5t} \sum_{k=1}^{\infty} \frac{\sin[(2k+1)x] \sin[\sqrt{(2k+1)^2 - 9/4}\, t]}{(2k+1)\pi\sqrt{4(2k+1)^2 - 9}}. \end{aligned}$$

(b) The following figure shows that the string rises and reaches a maximum height of approximately 3, and then starts to fall back down. As it falls down, it never rises up again. That is the string does not oscillate, but simply falls back to its initial rest position.

```
u[x_, t_] = 80 / (Pi Sqrt[5]) E^(-3 / 2 t) Sin[x] Sinh[Sqrt[5] / 2 t] +
   40 / Pi E^(-3 / 2 t) Sum[Sin[(2 j + 1) x]
      Sin[Sqrt[(2 j  + 1)^2  - 9 / 4] t] / ((2 j + 1) Sqrt[(2 j + 1)^2 - 9 / 4]), {j, 1, 7}];

tt = Table[u[x, t], {t, .1, 8, .6}];

Plot[Evaluate[tt], {x, 0, Pi}, PlotRange → All]
```



Solutions to Exercises 3.4

1. We will use (5), since $g^* = 0$. The odd extension of period 2 of $f(x) = \sin \pi x$ is $f^*(x) = \sin \pi x$. So

$$u(x, t) = \frac{1}{2}\big[\sin(\pi(x + \frac{t}{\pi})) + \sin(\pi(x - \frac{t}{\pi}))\big] = \frac{1}{2}\big[\sin(\pi x + t) + \sin(\pi x - t)\big].$$

2. Reasoning as in Exercise 1, we find

$$u(x, t) = \frac{1}{2}\big[\sin(\pi x + t)\cos(\pi x + t) + \sin(\pi x - t)\cos(\pi x - t)\big].$$

3. We have $f^*(x) = \sin \pi x + 3 \sin 2\pi x$, $g^*(x) = \sin \pi x$. Both f^* and g^* are periodic with period 2 and they are odd functions. So, the integral of g^* over one period, $\int_{-1}^{1} g^*(x)\, dx = 0$. This implies that an antiderivative of g^* is also 2-periodic. Indeed, let

$$G(x) = \int_0^x \sin \pi s\, ds = -\frac{1}{\pi} \cos \pi s\Big|_0^x = \frac{1}{\pi}(1 - \cos \pi x).$$

This is clearly 2-periodic. By (6) (or (4)), we have

$$\begin{aligned} u(x, t) &= \frac{1}{2}\big[\sin(\pi(x+t)) + 3\sin(2\pi(x+t)) + \sin(\pi(x-t)) + 3\sin(2\pi(x-t))\big] \\ &\quad + \frac{1}{2\pi}\Big[\big[1 - \cos\pi(x+t)\big] - \big[1 - \cos\pi(x-t)\big]\Big] \\ &= \frac{1}{2}\Big[\sin(\pi(x+t)) + 3\sin(2\pi(x+t)) + \sin(\pi(x-t)) + 3\sin(2\pi(x-t))\Big] \\ &\quad + \frac{1}{2\pi}\big[\cos\pi(x-t) - \cos\pi(x+t)\big]. \end{aligned}$$

4. For $-1 < x < 1$, we have

$$g^*(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ -1 & \text{if } -1 < x < 0. \end{cases}$$

An antiderivative of this function on the interval $[-1, 1]$ is given by

$$G(x) = \begin{cases} = & x \quad \text{if } 0 < x < 1, \\ & -x \quad \text{if } -1 < x < 0. \end{cases}$$

We define G outside the interval $[-1, 1]$ by extending it periodically. Thus $G(x + 2) = G(x)$ for all x. By (6),

$$u(x, t) = \frac{1}{2}\big[G(x + t) - G(x - t)\big].$$

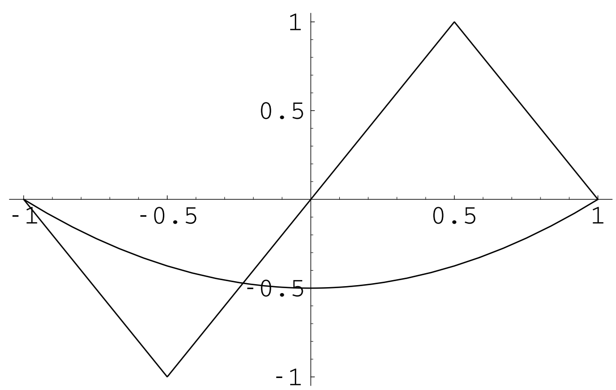
5. The solution is of the form

$$\begin{aligned} u(x, t) &= \frac{1}{2}\big[f^*(x-t) + f(x+t)\big] + \frac{1}{2}\big[G(x+t) - G(x-t)\big] \\ &= \frac{1}{2}\big[\big(f^*(x-t) - G(x-t)\big) + \big(f^*(x+t) + G(x+t)\big)\big], \end{aligned}$$

where f^* is the odd extension of f and G is as in Example 3. In the second equality, we expressed u as the average of two traveling waves: one wave traveling to the right and one to the left. Note that the waves are not the same, because of the G term. We enter the formulas in Mathematica and illustrate the motion of the string.

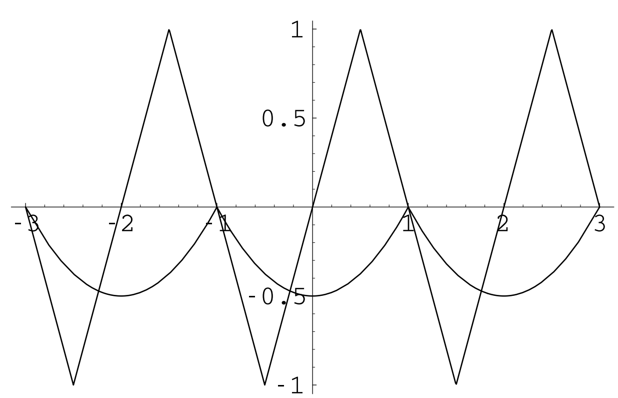
The difficult part in illustrating this example is to define periodic functions with *Mathematica.* This can be done by appealing to results from Section 2.1. We start by defining the odd extensions of f and G (called big g) on the interval [-1, 1].

```
Clear[f, bigg]
f[x_] := 2 x   /; -1 / 2 < x < 1 / 2
f[x_] := 2 ( 1 - x)  /; 1 / 2 < x < 1
f[x_] := -2 ( 1 + x)  /; -1 < x < -1 / 2
bigg[x_] = 1 / 2 x^2 - 1 / 2

Plot[{f[x], bigg[x]}, {x, -1, 1}]
```



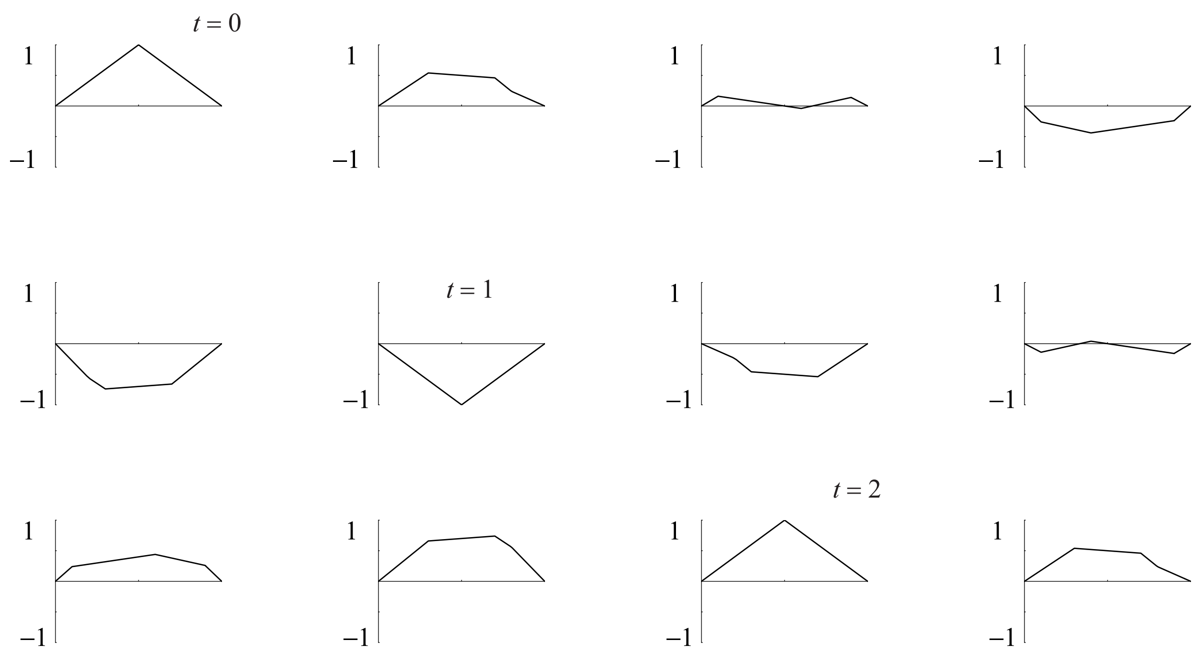
Here is a tricky Mathematica construction. (Review Section 2.1.)

```
extend[x_] := x - 2 Floor[(x + 1) / 2]
periodicf[x_] := f[extend[x]]
periodicbigg[x_] := bigg[extend[x]]
Plot[{periodicf[x], periodicbigg[x]}, {x, -3 , 3}]
```



Because f^* and G are 2-periodic, it follows immediately that $f^*(x \pm ct)$ and $G(x \pm ct)$ are $2/c$-periodic in t. Since $c = 1$, u is 2-periodic in t. The following is an array of snapshots of u. You can also illustrate the motion of the string using Mathematica (see the Mathematica notebooks). Note that in this array we have graphed the exact solution and not just an approximation using a Fourier series. This is a big advantage of the d'Alembert's solution over the Fourier series solution.

```
u[x_, t_] := 1 / 2 (periodicf[x - t] + periodicf[x + t]) +
  1 / 2 (periodicbigg[x - t] - periodicbigg[x + t])
tt = Table[
   Plot[Evaluate[ u[x, t] ], {x, 0, 1}, PlotRange → {{0, 1}, {-1, 1}},
    Ticks → {{.5}, {-1, -.5, .5, 1}}, DisplayFunction → Identity], {t, 0, 2.3, 1 / 5}];
Show[GraphicsArray[Partition[tt, 4]]]
```



6. The solution is of the form

$$u(x,\, t) = \frac{1}{2}\big[G(x+t) - G(x-t)\big],$$

where $G(x)$ is a continuous antiderivative of the odd extension of g (denoted g also). Since $g(-x) = -g(x)$, we have

$$G(x) = \begin{cases} \int_{-1}^{x} -\cos \pi t\, dt & \text{if } -1 < x < 0 \\ \int_{-1}^{0} -\cos \pi t\, dt + \int_0^x \cos \pi t\, dt & \text{if } 0 < x < 1. \end{cases}$$

But

$$\int_{-1}^{0} -\cos \pi t\, dt = 0,$$

so

$$G(x) = \begin{cases} -\frac{1}{\pi} \sin \pi x & \text{if } -1 < x < 0 \\ \frac{1}{\pi} \sin \pi x & \text{if } 0 < x < 1. \end{cases}$$

From this we conclude that, for all x,

$$G(x) = \frac{1}{\pi}\big|\sin \pi x\big|.$$

7. Very much like Exercise 4.

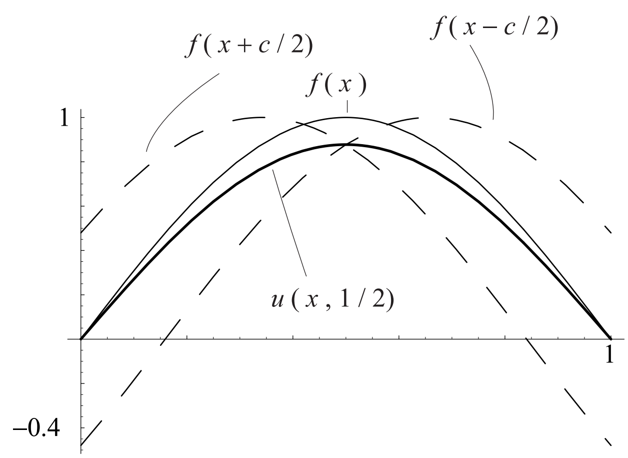
8. Very much like Exercise 8 but even easier. We have, for all x,

$$G(x) = \frac{1}{\pi} \cos \pi x.$$

9. You can use Exercise 11, Section 3.3, which tells us that the time period of motion is $T = \frac{2L}{c}$. So, in the case of Exercise 1, $T = 2\pi$, and in the case of Exercise 5, $T = 2$. You can also obtain these results directly by considering the formula for $u(x, t)$. In the case of Exercise 1, $u(x, t) = \frac{1}{2}\big[\sin(\pi x + t) + \sin(\pi x - t)\big]$ so $u(x, t + 2\pi) = \frac{1}{2}\big[\sin(\pi x + t2\pi) + \sin(\pi x - t2\pi)\big] = u(x, t)$. In the case of Exercise 5, use the fact that f^* and G are both 2-periodic.

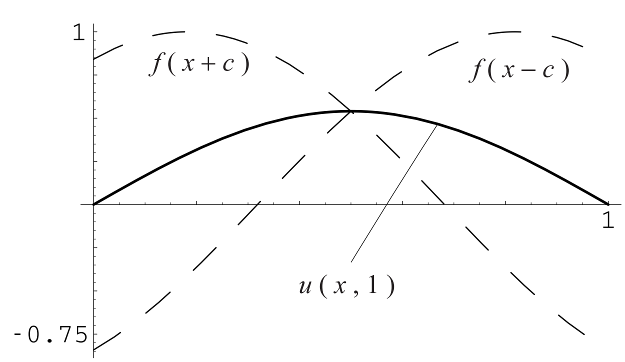
10. Here is a plot with Mathematica.

```
Clear[f, u, c]
c = 1 / Pi
f[x_] = Sin[Pi x]
u[x_, t_] := 1 / 2 (f[x - c t] + f[x + c t])
Plot[{f[x], f[x - c / 2], f[x + c / 2], u[x, 1 / 2]}, {x, 0, 1}]
```



The string at time $t = 1$:

```
Plot[{f[x - 1 / Pi ], f[x + 1 / Pi ], u[x, 1 ]}, {x, 0, 1}]
```



12. Let us recall some facts about periodic functions. (a) If f is T-periodic, then any translate of f, say $f(x+\tau)$ is also T-periodic. (b) If f is T-periodic, then any dilate of f, say $f(\pm\alpha x)$ is $T/|\alpha|$-periodic ($\alpha \neq 0$). (c) Linear combinations of T-periodic functions are also T-periodic. Properties (a)–(c) are straightforward to prove. (See Exercise 9, Section 2.1.) In d'Alembert's solution, f^* and G are $2L$-periodic. So, the functions $f^*(ct)$ and $G(ct)$ are $2L/c$-periodic. Thus, for fixed x, the translated functions $f^*(x \pm ct)$ and $G(\pm cx)$ are $2L/c$-periodic in the t-variable. Hence $u(x, t)$ is $2L/c$-periodic in t, being a linear combination of $2L/c$-periodic functions.

13. We have

$$u(x, t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds,$$

where f^* and g^* are odd and $2L$-periodic. So

$$u(x, t+\frac{L}{c}) = \frac{1}{2}\left[f^*(x+ct+L) + f^*(x-ct-L)\right] + \frac{1}{2c}\int_{x-ct-L}^{x+ct+L} g^*(s)\,ds.$$

Using the fact that f^* is odd, $2L$-period, and satisfies $f^*(L-x) = f^*(x)$ (this property is given for f but it extends to f^*), we obtain

$$\begin{aligned} f^*(x+ct+L) &= f^*(x+ct+L-2L) = f^*(x+ct-L) \\ &= -f^*(L-x-ct) = -f^*(L-(x+ct)) = -f^*(x+ct). \end{aligned}$$

Similarly

$$\begin{aligned} f^*(x-ct-L) &= -f^*(L-x+ct) \\ &= -f^*(L-x+ct) = -f^*(L-(x-ct)) = -f^*(x-ct). \end{aligned}$$

Also $g^*(s+L) = -g^*(-s-L) = -g^*(-s-L+2L) = -g^*(L-s) = -g^*(s)$, by the given symmetry property of g. So, using a change of variables, we have

$$\frac{1}{2c}\int_{x-ct-L}^{x+ct+L} g^*(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s+L)\,ds = -\frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds.$$

Putting these identities together, it follows that $u(x, t+\frac{L}{c}) = -u(x, t)$.

14. We have

$$u(x, t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right] + \frac{1}{2c}[G(x+ct) - G(x-ct)],$$

where f^* is odd and $2L$-periodic and G is $2L$-periodic. In fact, G is even (see Exercise 17 or see below). For any $2L$-periodic even function ϕ, we have

$$\phi(L+a) = \phi(L+a-2L) = \phi(a-L) = \phi(L-a).$$

And for any $2L$-periodic odd function ψ, we have

$$\psi(L+a) = \psi(L+a-2L) = \psi(a-L) = -\psi(L-a).$$

Using these two facts and that f^* is odd and G is even, we get Then

$$\begin{aligned} u(x, L) &= \frac{1}{2}\left[f^*(L+ct) + f^*(L-ct)\right] + \frac{1}{2c}[G(L+ct) - G(L-ct)] \\ &= \frac{1}{2}\left[-f^*(L-ct) + f^*(L-ct)\right] + \frac{1}{2c}[G(L-ct) - G(L-ct)] = 0. \end{aligned}$$

To have a complete proof, let us show that G is even. We know that g^* is $2L$-periodic and odd. For any x, we have

$$\begin{aligned} G(-x) &= \int_a^{-x} g^*(t)\,dt = \int_{-a}^{x} g^*(-t)(-1)dt = \int_{-a}^{x} g^*(t)\,dt \\ &= \overbrace{\int_{-a}^{a} g^*(t)\,dt}^{=0} + \overbrace{\int_a^x g^*(t)\,dt}^{G(x)} \\ &= G(x), \end{aligned}$$

where we have used that g^* is odd to infer that its integral over a symmetric interval is 0.
(b) Checking the initial condition is easier. We have

$$u(x,\,0) = \frac{1}{2}\,[f^*(x) + f^*(x)] + \frac{1}{2c}[G(x) - G(x)] = f^*(x),$$

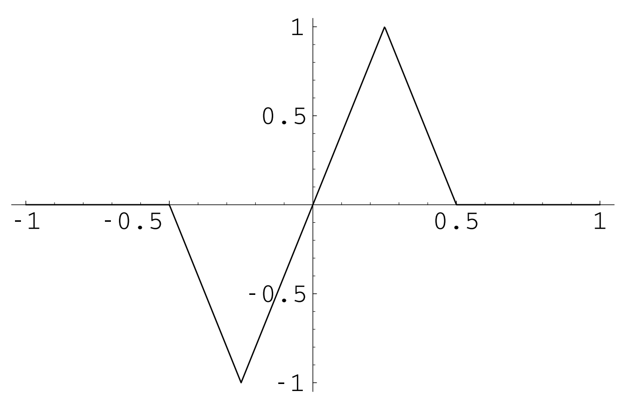
and

$$u_t(x,\,0) = \frac{1}{2}\,[cf^*(x) - cf^*(x)] + \frac{1}{2c}[cg(x) + cg(x)] = g(x).$$

Here we have used the fact that $G' = g$. You can check that this equality is true at the points of continuity of g.

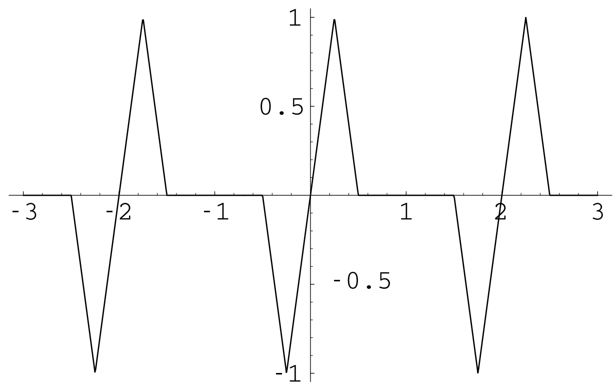
15. Let us define the function $f(x)$ on $(0,\,1)$ and then plot it.

```
Clear[f]
f[x_] := 4 x   /; -1 / 4 < x < 1 / 4
f[x_] := 4 ( 1 / 2 - x)   /; 1 / 4 < x < 1 / 2
f[x_] := 0   /; 1 / 2 < x < 1
f[x_] := 4 ( -1 / 2 - x)   /; -1 / 2 < x < -1 / 4
f[x_] := 0   /; -1  < x < -1 / 2
Plot[ f[x], {x, -1, 1}]
```



Let us now plot the periodic extension of f

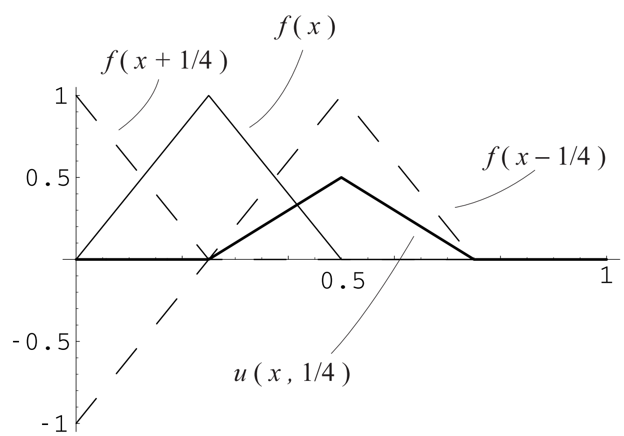
```
extend[x_] := x - 2 Floor[(x + 1) / 2]
periodicf[x_] := f[extend[x]]
Plot[periodicf[x], {x, -3 , 3}]
```



Periodic extension of f

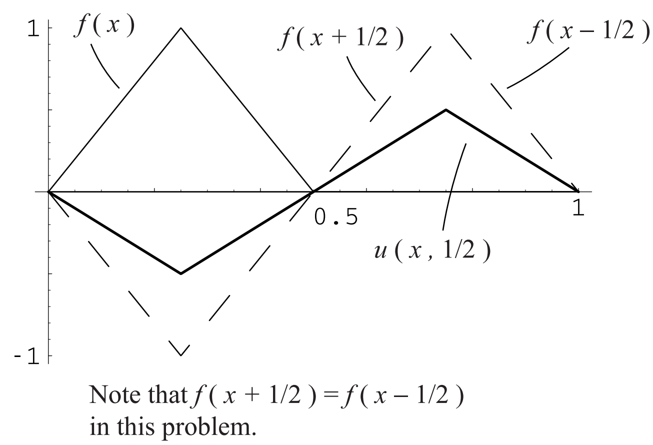
We are now ready to define u and plot it for any value of t. We plot $u(x, 1/4)$ along with $f(x-1/4)$ and $f(x+1/4)$ to illustrate the fact that u is the average of these two traveling waves.

```
u[x_, t_] := 1 / 2 (periodicf[x - t] + periodicf[x + t])
Plot[
 Evaluate[{periodicf[x], periodicf[x + 1 / 4], periodicf[x - 1 / 4], u[x, 1 / 4]}],
 {x, 0, 1}, PlotRange → All]
```



Next, we show the string at $t = 1/2$.

```
u[x_, t_] := 1 / 2 (periodicf[x - t] + periodicf[x + t])
Plot[
 Evaluate[{periodicf[x], periodicf[x + 1 / 2], periodicf[x - 1 / 2], u[x, 1 / 2]}],
 {x, 0, 1}, PlotRange → All]
```



(b) It is clear from the graph of $u(x, 1/4)$ that the points $.75 < x < 1$ are still at rest when $t = 1/4$.
(c) Because $f^*(x) = 0$ for $.5 < x < 1.5$ and $u(x, t) = \frac{1}{2}(f(x+t) + f(x-t))$, a point x_0 in the interval $(1/2, 1)$ will feel a vibration as soon as the right traveling wave reaches it. Since the wave is traveling at speed $c = 1$ and is supported on the interval $(0, 1/2)$, the vibration will reach the point x_0 after traveling a distance $x_0 - 1/2$. Since the wave is traveling at speed $c = 1$, it will take $x_0 - 1/2$ time for the point x_0 to start move from rest. For all values of $t < x_0 - 1/2$, the point x_0 will remain at rest. For example, if $3/4 < x_0 < 1$, then this point will remain at rest for all time $t < 1/4$, as we saw in the previously.
(d) For an arbitrary value of $c > 0$, the wave will travel at the speed of c. Thus it will take $(x_0 - 1/2)/c$ time to reach the point x_0 located in the interval $1/2 < x < 1$.

16. (a) From (8), Section 3.3, with $b_n^* = 0$ (since $g = 0$), we have

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x b_n \cos\left(c\frac{n\pi}{L} t\right) \\ &= \frac{1}{2} \sum_{n=1}^{\infty} b_n \Big(\sin \frac{n\pi}{L}(x - ct) + \sin \frac{n\pi}{L}(x + ct) \Big), \end{aligned}$$

where we have used the formula $\sin a \cos b = \frac{1}{2}\big(\sin(a-b) + \sin(a+b)\big)$.
(b) Since, for all x,

$$f^*(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x,$$

putting $x+ct$ and $x-ct$ in place of x, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin[\frac{n\pi}{L}(x-ct)] + \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin[\frac{n\pi}{L}(x+ct)] \\ &= \frac{1}{2}\big(f^*(x-ct)+f^*(x+ct)\big). \end{aligned}$$

17. (a) To prove that G is even, see Exercise 14(a). That G is $2L$-periodic follows from the fact that g is $2L$-periodic and its integral over one period is 0, because it is odd (see Section 2.1, Exercise 15).

Since G is an antiderivative of g^*, to obtain its Fourier series, we apply Exercise 33, Section 3.3, and get

$$G(x) = A_0 - \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n}\cos\frac{n\pi}{L}x,$$

where $b_n(g)$ is the nth Fourier sine coefficient of g^*,

$$b_n(g) = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

and

$$A_0 = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n}.$$

In terms of b_n^*, we have

$$\frac{L}{\pi}\frac{b_n(g)}{n} = \frac{2}{n\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx = cb_n^*,$$

and so

$$\begin{aligned} G(x) &= \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n} - \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n}\cos\frac{n\pi}{L}x \\ &= \sum_{n=1}^{\infty} cb_n^*\Big(1-\cos(\frac{n\pi}{L}x)\Big). \end{aligned}$$

(b) From (a), it follows that

$$\begin{aligned} G(x+ct)-G(x-ct) &= \sum_{n=1}^{\infty} cb_n^*\Big[\Big(1-\cos(\frac{n\pi}{L}(x+ct))\Big)-\Big(1-\cos(\frac{n\pi}{L}(x-ct))\Big)\Big] \\ &= \sum_{n=1}^{\infty} -cb_n^*\Big[\cos(\frac{n\pi}{L}(x+ct))-\cos(\frac{n\pi}{L}(x-ct))\Big] \end{aligned}$$

(c) Continuing from (b) and using the notation in the text, we obtain

$$\begin{aligned} \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds &= \frac{1}{2c}\left[G(x+ct)-G(x-ct)\right] \\ &= \sum_{n=1}^{\infty} -b_n^*\frac{1}{2}\Big[\cos(\frac{n\pi}{L}(x+ct))-\cos(\frac{n\pi}{L}(x-ct))\Big] \\ &= \sum_{n=1}^{\infty} b_n^*\sin(\frac{n\pi}{L}x)\sin(\frac{n\pi}{L}ct) \\ &= \sum_{n=1}^{\infty} b_n^*\sin(\frac{n\pi}{L}x)\sin(\lambda_n t). \end{aligned}$$

(d) To derive d'Alembert's solution from (8), Section 3.3, proceed as follows:

$$\begin{aligned} u(x,\,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)\cos(\lambda_n t) + \sum_{n=1}^{\infty} b_n^* \sin(\frac{n\pi}{L}x)\sin(\lambda_n t) \\ &= \frac{1}{2}\big(f^*(x-ct)+f^*(x+ct)\big) + \frac{1}{2c}\left[G(x+ct)-G(x-ct)\right], \end{aligned}$$

where in the last equality we used Exercise 16 and part (c).

18. (a) Recall that u and its partial derivatives are functions of x and t. By differentiating under the integral sign, we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2}\int_0^L \frac{d}{dt}\left[u_t^2 + c^2 u_x^2\right] dx \\ &= \frac{1}{2}\int_0^L \left[2u_t u_{tt} + 2c^2 u_x u_{xt}\right] dx. \end{aligned}$$

(b) But $u_{tt} = c^2 u_{xx}$, so

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^L \left[c^2 u_t u_{xx} + c^2 u_x u_{xt}\right] dx \\ &= c^2 \int_0^L \left[u_t u_{xx} + u_x u_{xt}\right] dx \\ &= c^2 \int_0^L \left(u_t\, u_x\right)_x dx \\ &= c^2 u_t\, u_x\Big|_0^L = c^2[u_t(L,\,t)\,u_x(L,\,t) - u_t(0,\,t)\,u_x(0,\,t)]. \end{aligned}$$

(c) Since u is a solution of the wave equation representing a string with fixed ends, we have that the velocity at the endpoints is 0. That is, $u_t(0,\,t) = u_t(L,\,t) = 0$.
(d) From (c) and (d), we conclude that

$$\frac{d}{dt}E(t) = 0.$$

Thus $E(t)$ is constant for all t. Setting $t = 0$ in the formula for $E(t)$ and using the initial conditions, we obtain

$$E(0) = \frac{1}{2}\int_0^L \left(u_t^2(x,\,0) + c^2 u_x^2(x,\,0)\right) dx = \frac{1}{2}\int_0^L \left(g^2(x) + c^2[f'(x)]^2\right) dx.$$

Since E is constant in time, this gives the energy of the string for all $t > 0$.

19. (a) The characteristic lines are lines with slope $\pm 1/c = \pm 1$ in the xt-plane.
(b) The interval of dependence of a point $(x_0,\,t_0)$ is the interval on the x-axis, centered at x_0, with radius ct_0. For the point $(.5,\,.2)$ this interval is $.3 \le x \le .7$. For the point $(.3,\,2)$ this interval is $-1.8 \le x \le 2.3$.
(c) The horizontal, semi-infinite, strip S is bounded by the line $x = 0$ and $x = 1$ and lies in the upper half-plane. The region I is the triangular region with base on the interval $(0,\,1)$, bounded by the lines through $(0, 0)$ at slope 1, and through $(1,\,0)$ at slope -1. The three vertices of I are $(0,\,0)$, $(1/2,\,1/2)$, and $(0,\,1)$. The point $(.5,\,.2)$ is in I but the point $(.3,\,2)$ is not

in I.
(d) For the points $(x,\,t)$ in I, $u(x,\,t)$ depends only on f and g and not their periodic extension. We have

$$\begin{aligned} u(x,\,t) &= = \frac{1}{2}\big[f(x-t)+f(x+t)\big]+\frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ &= \frac{1}{2}\int_{x-t}^{x+t} s\,ds \\ &= \frac{1}{4}s^2\Big|_{x-t}^{x+t} = \frac{1}{4}\big[(x+t)^2-(x-t)^2\big] = xt. \end{aligned}$$

20. Let $P_1=(x_0,\,t_0)$ be an arbitrary point in the region II. Form a characteristic parallelogram with vertices P_1, P_2, Q_1, Q_2, as shown in Figure 7 in Section 3.4. The vertices P_2 and Q_2 are on the characteristic line $x-t=0$ and the vertex Q_1 is on the boundary line $x=0$. From Proposition 1, we have $u(P_1)=u(Q_1)+u(Q_2)-u(P_2)$. We will find $u(P_2)$ and $u(Q_2)$ by using the formula $u(x,\,t)=xt$ from Exercise 19(d), because P_2 and Q_2 are in the region I. Also, $u(Q_1)=0$, because of the boundary condition $u(0,\,t)=0$ for all $t>0$.

The point Q_2 is the intersection point of the characteristic lines $x+t=x_0+t_0$ and $x-t=0$. Adding the equations, we get $2x=x_0+t_0$ or $x=\frac{x_0+t_0}{2}$. From $x=t$, we obtain $t=\frac{x_0+t_0}{2}$, and so $Q_2=(\frac{x_0+t_0}{2},\,\frac{x_0+t_0}{2})$. The coordinates of Q_1 and P_2 are computed similarly. The point Q_1 is the intersection point of the characteristic line $x-t=x_0-t_0$ and the boundary $x=0$. Thus $t=t_0-x_0$ and so $Q_1=(0,\,t_0-x_0)$. Similarly, the point P_2 is the intersection point of the characteristic lines $x+t=t_0-x_0$ and $x-t=0$. Adding the equations, we get $2x=t_0-x_0$ or $x=\frac{t_0-x_0}{2}$. Thus $P_2=(\frac{-x_0+t_0}{2},\,\frac{-x_0+t_0}{2})$. Let us simplify the notation and write $P_1=(x,\,t)$ instead of $(x_0,\,t_0)$. Then, for $P_1=(x,\,t)$ in the region II,

$$\begin{aligned} u(x,\,t) &= u(Q_2)-u(P_2) \\ &= u(\frac{x+t}{2},\,\frac{x+t}{2})-u(\frac{-x+t}{2},\,\frac{-x+t}{2}) \\ &= \frac{x+t}{2}\cdot\frac{x+t}{2}-\frac{-x+t}{2}\cdot\frac{-x+t}{2} \\ &= \frac{1}{4}\big[(x+t)^2-(-x+t)^2\big] \\ &= tx. \end{aligned}$$

Interestingly, the formula for $u(x,\,t)$ for $(x,\,t)$ in region II is the same as the one for $(x,\,t)$ in the region I. In particular, this formula for $u(x,\,t)$ satisfies the wave equation and the boundary condition at $x=0$. The other conditions in the wave problem do not concern the points in the region II and thus should not be checked.

21. Follow the labeling of Figure 8 in Section 3.4. Let $P_1=(x_0,\,t_0)$ be an arbitrary point in the region II. Form a characteristic parallelogram with vertices P_1, P_2, Q_1, Q_2, as shown in Figure 8 in Section 3.4. The vertices P_2 and Q_1 are on the characteristic line $x+2t=1$ and the vertex Q_2 is on the boundary line $x=1$. From Proposition 1, we have

$$u(P_1)=u(Q_1)+u(Q_2)-u(P_2)=u(Q_1)-u(P_2),$$

because $u(Q_2) = 0$. We will find $u(P_2)$ and $u(Q_1)$ by using the formula $u(x, t) = -4t^2 + x - x^2 + 8tx$ from Example 4, because P_2 and Q_1 are in the region I.

The point Q_1 is the intersection point of the characteristic lines $x - 2t = x_0 - 2t_0$ and $x + 2t = 1$. Adding the equations and then solving for x, we get

$$x = \frac{x_0 + 1 - 2t_0}{2}.$$

The second coordinate of Q_1 is then

$$t = \frac{1 - x_0 + 2t_0}{4}.$$

The point Q_2 is the intersection point of the characteristic line $x + 2t = x_0 + 2t_0$ and $x = 1$. Thus

$$t = \frac{x_0 + 2t_0 - 1}{2}.$$

The point P_2 is the intersection point of the characteristic lines $x + 2t = 1$ and $x - 2t = 1 - (x_0 + 2t_0 - 1)$. Solving for x and t, we find the coordinates of P_2 to be

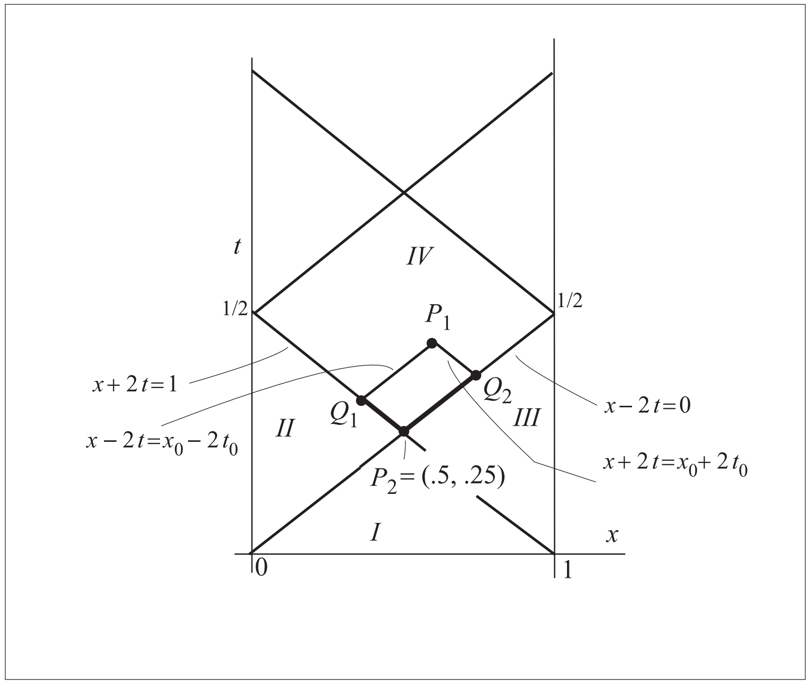
$$x = \frac{3 - x_0 - 2t_0}{2} \quad \text{and} \quad t = \frac{-1 + x_0 + 2t_0}{4}.$$

To simplify the notation, replace x_0 and t_0 by x and y in the coordinates of the points Q_1 and P_2 and let $\phi(x, t) = -4t^2 + x - x^2 + 8tx$. We have

$$\begin{aligned} u(x, t) &= u(Q_1) - u(P_2) \\ &= \phi\left(\frac{x + 1 - 2t}{2}, \frac{1 - x + 2t}{4}\right) - \phi\left(\frac{3 - x - 2t}{2}, \frac{-1 + x + 2t}{4}\right) \\ &= 5 - 12t - 5x + 12tx, \end{aligned}$$

where the last expression was derived after a few simplifications that we omit. It is interesting to note that the formula satisfies the wave equation and the boundary condition $u(1, t) = 0$ for all $t > 0$. Its restriction to the line $x + 2t = 1$ (part of the boundary of region I) reduces to the formula for $u(x, t)$ for (x, t) in region I. This is to be expected since u is continuous in (x, t).

22. Label the problem as in the following figure:



Let $P_1 = (x_0,\, t_0)$ be an arbitrary point in the region IV. Form a characteristic parallelogram with vertices P_1, P_2, Q_1, Q_2, as shown in the figure. The point P_2 is on the intersection of $x - 2t = 0$ and $x + 2t = 1$. Its coordinates are found to be (.5, .25). Since P_2 belongs to three regions, we can compute $u(P_2)$ from one of three formulas that we have thus far. Let's use the formula for the points in region I. We have (from Example 4)

$$u(P_2) = u(.5,\, .25) = -4(.25)^2 + .5 - (.5)^2 + 8(.25)(.5) = 1.$$

The point Q_1 is on the intersection of the lines $x - 2t = x_0 - 2t_0$ and $x + 2t = 1$. Its coordinates are

$$\left(\frac{x_0 + 1 - 2t_0}{2},\, t_0 - \frac{x_0}{2}\right).$$

Since Q_1 is in region II, we compute $u(Q_1)$ using the formula from Example 5. We find

$$\begin{aligned} u\left(\frac{x_0 + 1 - 2t_0}{2},\, t_0 - \frac{x_0}{2}\right) &= \frac{x_0 + 1 - 2t_0}{2} \\ &\quad +4\frac{x_0 + 1 - 2t_0}{2} \cdot \left(t_0 - \frac{x_0}{2}\right) \\ &= \frac{1}{2} + t - 4\,t^2 - \frac{x}{2} + 4\,t\,x - x^2. \end{aligned}$$

We now consider the point Q_2 on the intersection of $x - 2t = 0$ and $x + 2t = x_0 + 2t_0$. We have

$$Q_2 = \left(\frac{x_0 + 2t_0}{2},\, \frac{x_0 + 2t_0}{4}\right).$$

Since Q_2 is in region III, we compute $u(Q_2)$ using the formula from Exercise 21. We find

$$\begin{aligned} u\left(\frac{x_0+2t_0}{2}, \frac{x_0+2t_0}{4}\right) &= 5-12\frac{x_0+2t_0}{4}-5\frac{x_0+2t_0}{2} \\ &\quad +12\frac{x_0+2t_0}{4}\frac{x_0+2t_0}{2} \\ &= 5-11\,t+8\,t^2-\frac{11\,x}{2}+8\,t\,x+2\,x^2. \end{aligned}$$

From Proposition 1, we have

$$\begin{aligned} u(P_1) &= u(Q_1)+u(Q_2)-u(P_2) \\ &= \frac{1}{2}+t-4\,t^2-\frac{x}{2}+4\,t\,x-x^2 \\ &\quad +5-11\,t+8\,t^2-\frac{11\,x}{2}+8\,t\,x+2\,x^2-1 \\ &= \frac{9}{2}-10\,t+4\,t^2-6\,x+12\,t\,x+x^2 \end{aligned}$$

It is not difficult to show that this function satisfies the wave equation. As further verification, you can check hat this formula yields 1 when evaluated at (.5, .25), which are the coordinates of the points P_2.

Solutions to Exercises 3.5

1. Multiply the solution in Example 1 by $\frac{78}{100}$ to obtain

$$u(x,\,t)=\frac{312}{\pi}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)^2t}}{2k+1}\sin(2k+1)x.$$

2. The function f is given by its Fourier series. So

$$u(x,\,t)=30\,\sin x\,e^{-t}.$$

3. We have

$$u(x,\,t)=\sum_{n=1}^{\infty}b_ne^{-n^2t}\sin(nx),$$

where

$$\begin{aligned}
b_n &= \frac{66}{\pi}\int_0^{\pi/2}x\sin nx\,dx+\frac{66}{\pi}\int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\\
&= \frac{66}{\pi}\left(-\frac{x\,\cos(n\,x)}{n}+\frac{\sin(n\,x)}{n^2}\right)\Big|_0^{\pi/2}\\
&\quad+\frac{66}{\pi}\left(\frac{-(\pi-x)\,\cos(n\,x)}{n}-\frac{\sin(n\,x)}{n^2}\right)\Big|_{\pi/2}^{\pi}\\
&= \frac{66}{\pi}\left(-\frac{\frac{\pi}{2}\,\cos(n\frac{\pi}{2})}{n}+\frac{\sin(n\frac{\pi}{2})}{n^2}+\frac{\frac{\pi}{2}\cos(n\frac{\pi}{2})}{n}+\frac{\sin(n\frac{\pi}{2})}{n^2}\right)\\
&= \frac{132}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2}\\
&= \begin{cases}132\frac{(-1)^k}{\pi(2k+1)^2} & \text{if } n=2k+1,\\ 0 & \text{if } n=2k.\end{cases}
\end{aligned}$$

Thus

$$u(x,\,t)=\frac{132}{\pi}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)^2t}\sin((2k+1)x)}{(2k+1)^2}.$$

4. We have

$$u(x,\,t)=\sum_{n=1}^{\infty}b_ne^{-n^2t}\sin nx,$$

where b_n is the nth sine Fourier coefficient of the function $f(x)=100$ if $0<x<\pi/2$ and 0 if $\pi/2<x<\pi$. Instead of computing b_n, we will appeal to Exercise 29, Section 3.3, with $d=\pi/2$ and $p=\pi$. We get

$$b_n=\frac{200}{\pi n}\left(1-\cos\frac{n\pi}{2}\right).$$

Thus

$$u(x,\,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\left(1-\cos\frac{n\pi}{2}\right)e^{-n^2t}\frac{\sin nx}{n}.$$

5. We have

$$u(x,\,t)=\sum_{n=1}^{\infty}b_ne^{-(n\pi)^2t}\sin(n\pi x),$$

where

$$\begin{aligned} b_n &= 2\int_0^1 x\sin(n\pi x)\,dx = 2\left[-\frac{x\,\cos(n\,\pi\,x)}{n\,\pi} + \frac{\sin(n\,\pi\,x)}{n^2\,\pi^2}\right]\Big|_0^1 \\ &= -2\frac{\cos n\,\pi}{n\,\pi} = 2\frac{(-1)^{n+1}}{n\pi}. \end{aligned}$$

So

$$u(x,\,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-(n\pi)^2 t}\sin(n\pi x)}{n}.$$

6. We have

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t}\sin(n\pi x),$$

where

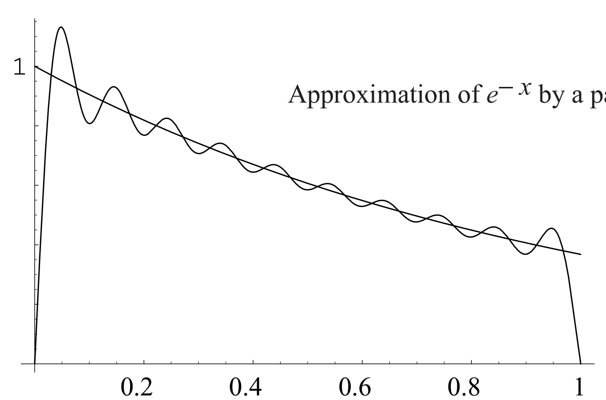
$$\begin{aligned} b_n &= 2\int_0^1 e^{-x}\sin(n\pi x)\,dx \\ &= \frac{2e^{-x}}{1+(n\pi)^2}\big(-\sin n\pi x - n\pi\cos n\pi x\big)\Big|_0^1 \\ &= \frac{2e^{-1}}{1+(n\pi)^2}\big(-n\pi(-1)^n\big) - \frac{2}{1+(n\pi)^2}\big(-n\pi\big) \\ &= \frac{2n\pi}{1+(n\pi)^2}\big(1+(-1)^{n+1}e^{-1}\big). \end{aligned}$$

So

$$u(x,\,t) = 2\pi\sum_{n=1}^{\infty}\frac{n}{1+(n\pi)^2}\big(1+(-1)^{n+1}e^{-1}\big)e^{-(n\pi)^2 t}\sin(n\pi x).$$

Putting $t = 0$, we obtain the sine series expansion of e^{-x} for $0 < x < 1$. Let's check this result with a graph.

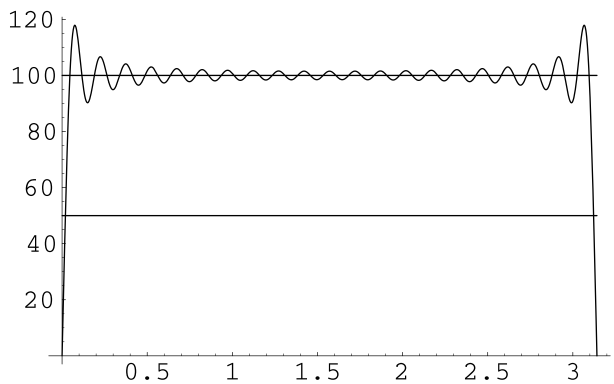
```
Clear[f, partsum]
f[x_] := E^(-x)
partsum[x_, t_, n_] :=
  2 Pi Sum[k / (1 + k^2 Pi^2)
     (1 + (-1)^(k + 1) E^(-1)) E^(- k^2 Pi^2 t) Sin[k Pi x], {k, 0, n}];
Plot[Evaluate[{f[x], partsum[x, 0, 20]}], {x, 0, 1}]
```



Approximation of e^{-x} by a partial sum of its sine series.

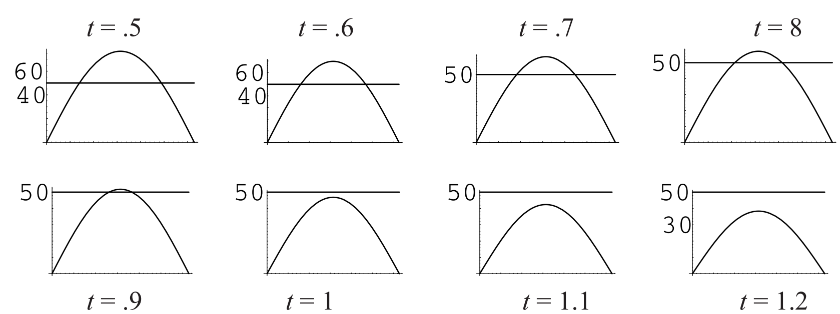
7. (a) From the figures in Example 1, we can see that when t is equal to 1 the temperature is already below 50 degrees. We will give a more accurate approximation. Let us first check the solution by plotting an approximation of the initial temperature distribution.

```
Clear[f, partsum]
f[x_] := 100
partsum[x_, t_, n_] :=
  400 / Pi Sum[ E^(-(2 k + 1)^2 t) Sin[(2 k + 1) x] / (2 k + 1), {k, 0, n}];
Plot[Evaluate[{f[x], 50, partsum[x, 0, 20]}], {x, 0, Pi}, PlotRange → All]
```



Let us now examine the temperature distribution, starting with $t = .5$ to $t = 1.2$ with increments of $t = .1$.

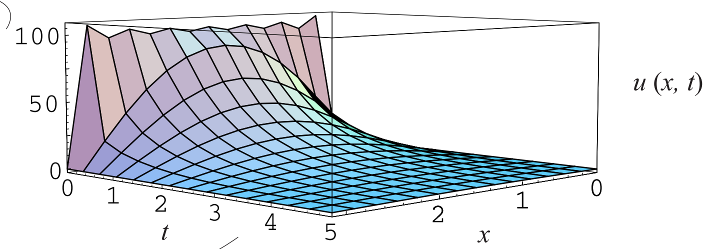
```
tt = Table[
   Plot[Evaluate[{ partsum[x, t, 10] , 50}], {x, 0, Pi},
    PlotRange → All, DisplayFunction → Identity], {t, 0.5, 1.2, 1/10}];
Show[GraphicsArray[Partition[tt, 4]]]
```



It appears that the maximum temperature dips under 50 degree between $t = .9$ and $t = 1$. You can repeat the plot for $t = .9$ to $t = 1$, using increments of .01 and get a more accurate value of t.
(b)

```
Plot3D[partsum[x, t, 20], {x, 0, Pi},
 {t, 0, 5}, PlotRange -> All, ViewPoint → {3, 3, .4}]
```

At time t = 0, the initial temperature is 100, except at the endpoints, which are kept at 0 degree.



As t increases, the temperature across the bar tends to 0 uniformly.

This 3 dimensional plot shows the temperature across the whole bar as a function of time.

To prove that the temperature tends uniformly to 0 in the bar, you can proceed as follows (this is more than what the question is asking for): for $t > 0$ and all x,

$$\begin{aligned} |u(x,\,t)| &= \left| \frac{400}{\pi} \sum_{k=0}^{\infty} e^{-(2k+1)^2 t} \frac{\sin(2k+1)x}{2k+1} \right| \\ &= \frac{400}{\pi} \sum_{k=1}^{\infty} e^{-kt} = \frac{400}{\pi} \sum_{k=1}^{\infty} \left(e^{-t}\right)^k \\ &= \frac{400}{\pi} \left(1 - \frac{1}{1-e^{-t}}\right). \end{aligned}$$

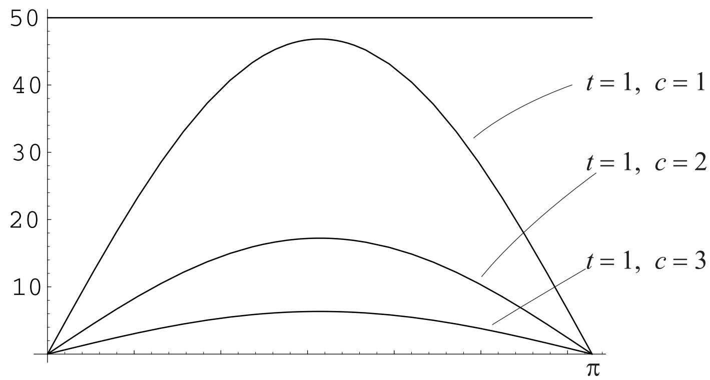
As $t \to \infty$, the last displayed expression tends to 0, which shows that $u(x,\,t)$ tends to 0 uniformly in x.

8. (a) Applying (4), we see that, for an arbitrary $c > 0$, the solution becomes

$$u(x,\,t) = \frac{400}{\pi} \sum_{k=0}^{\infty} e^{-c^2(2k+1)^2 t} \frac{\sin(2k+1)x}{2k+1}.$$

(b) As c increases, the rate of heat transfer increases as illustrated by the following figures. This is also

```
Clear[f, partsum]
f[x_] := 100
partsum[x_, t_, n_, c_] :=
  400 / Pi Sum[ E^(-(2 k + 1) ^2 t c) Sin[(2 k + 1) x] / (2 k + 1), {k, 0, n}];
Plot[
 Evaluate[{50, partsum[x, 1, 20, 1], partsum[x, 1, 20, 2], partsum[x, 1, 20, 3]}],
 {x, 0, Pi}, PlotRange → All]
```



9. (a) The steady-state solution is a linear function that passes through the points (0, 0) and (1, 100). Thus, $u(x) = 100x$.
(b) The steady-state solution is a linear function that passes through the points (0, 100) and (1, 100). Thus, $u(x) = 100$. This is also obvious: If you keep both ends of the insulated bar at 100 degrees, the steady-state temperature will be 100 degrees.

10. (a) Recall that u_1 satisfies $(u_1)_t = c^2(u_1)_{xx}$, and u_2 satisfies $(u_2)_t = 0$ and $(u_2)_{xx} = 0$. So

$$\begin{aligned} u_t &= (u_1+u_2)_t = (u_1)_t + (u_2)_t = (u_1)_t = c^2(u_1)_{xx} \\ &= c^2(u_1)_{xx} + c^2(u_2)_{xx} = c^2\left[(u_1)_{xx} + (u_2)_{xx}\right] = c^2 u_{xx}. \end{aligned}$$

Thus u satisfies (6). Now recall that $u_1(0) = T_1$, $u_1(L) = T_2$, and $u_2(0, t) = 0 = u_2(L, t)$. Thus

$$u(0, t) = u_1(0) + u_2(0, t) = T_1 \text{ and } u(L, t) = u_1(L) + u_2(L, t) = T_2.$$

Thus u satisfies (7). Now $u_1(x) = \frac{T_2-T_1}{L}x + T_1$ and $u_2(x, 0) = f(x) - u_1(t)$. So

$$u(x, 0) = u_1(x) + u_2(x, 0) = f(x),$$

showing that u satisfies (8).
(b) According to (14), we have (see the integral formula in Exercise 7, Sec-

tion 3.3)

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L \left[f(x) - \left(\frac{T_2 - T_1}{L}x + T_1\right)\right] \sin\frac{n\pi}{L}x\,dx \\
&= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx - \frac{2}{L}\int_0^L \left(\frac{T_2 - T_1}{L}x + T_1\right)\sin\frac{n\pi}{L}x\,dx \\
&= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx - \frac{2}{L}\left[-\frac{L}{n\pi}\left(\frac{T_2 - T_1}{L}x + T_1\right)\cos\frac{n\pi}{L}x \right. \\
&\quad \left. + \frac{T_2 - T_1}{L}\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi}{L}x\right]\Big|_0^L \\
&= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx - \frac{2}{n\pi}\big(T_1 - (-1)^n T_2\big).
\end{aligned}$$

11. We use the solution outlined in the text. We have the steady-state solution

$$u_1(x) = \frac{0 - 100}{1}\,x + 100 = 100(1 - x).$$

Then

$$u_2(x,\,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t}\sin(n\pi x),$$

where

$$\begin{aligned}
b_n &= 2\int_0^1 \big(30\,\sin(\pi x) - 100(1 - x)\big)\,\sin(n\pi x)\,dx \\
&= \overbrace{60\int_0^1 \sin(\pi x)\,\sin(n\pi x)\,dx}^{=30 \text{ if } n=1,\ 0 \text{ if } n\neq 1} - \overbrace{200\int_0^1 (1 - x)\,\sin(n\pi x)\,dx}^{=\frac{200}{n\pi}} \\
&= \begin{cases} 30 - \frac{200}{\pi} & \text{if } n = 1, \\ -\frac{200}{n\pi} & \text{if } n > 1. \end{cases}
\end{aligned}$$

Thus So

$$u(x,\,t) = u_1(x) + u_2(x,\,t) = 100(1-x) + 30\,e^{-\pi^2 t}\sin\pi x - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(n\pi)^2 t}\sin(n\pi x)}{n}.$$

12. Let us use (13) and the formula from Exercise 10. We have

$$u(x,\,t) = u_1(x) + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin n\pi x,$$

where $u_1(x) = 100$ and

$$\begin{aligned}
b_n &= 100\int_0^1 x(1 - x)\sin n\pi x\,dx - 200\frac{1 - (-1)^n}{n\pi} \\
&= 200\frac{(1 - (-1)^n)}{(n\pi)^3} - 200\frac{1 - (-1)^n}{n\pi} \\
&= 200\frac{(1 - (-1)^n)}{(n\pi)^3} - 200\frac{1 - (-1)^n}{n\pi}.
\end{aligned}$$

Thus

$$\begin{aligned} u(x,\,t) &= 100 + 200\sum_{n=1}^{\infty}\left[\frac{(1-(-1)^n)}{(n\pi)^3} - \frac{1-(-1)^n}{n\pi}\right] e^{-n^2\pi^2 t}\sin n\pi x \\ &= 100 + 400\sum_{k=0}^{\infty}\left[\frac{1}{((2k+1)\pi)^3} - \frac{1}{(2k+1)\pi}\right] e^{-(2k+1)^2\pi^2 t}\sin[(2k+1)\pi x]. \end{aligned}$$

13. We have $u_1(x) = -\frac{50}{\pi}x + 100$. We use (13) and the formula from Exercise 10, and get (recall the Fourier coefficients of f from Exercise 3)

$$\begin{aligned} u(x,\,t) &= -\frac{50}{\pi}x + 100 \\ &\quad + \sum_{n=1}^{\infty}\left[\frac{132}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2} - 100\left(\frac{2-(-1)^n}{n\pi}\right)\right] e^{-n^2 t}\sin nx. \end{aligned}$$

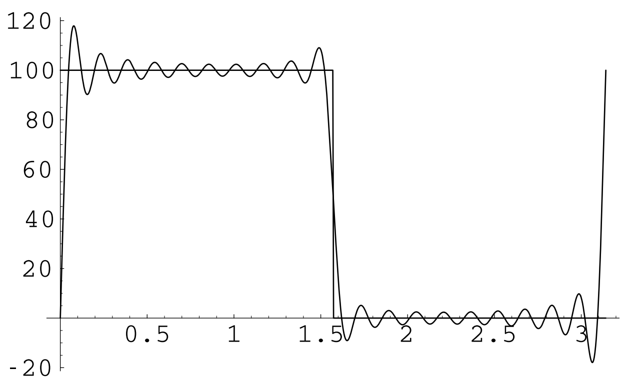
14. We have $u_1(x) = \frac{100}{\pi}x$. We use (13) and the formula from Exercise 10, and get (recall the Fourier coefficients of f from Exercise 4)

$$\begin{aligned} u(x,\,t) &= \frac{100}{\pi}x \\ &\quad + \sum_{n=1}^{\infty}\left[\frac{200}{\pi n}\left(1-\cos\frac{n\pi}{2}\right) + 200\frac{(-1)^n}{n\pi}\right] e^{-n^2 t}\sin nx \\ &= \frac{100}{\pi}x + \frac{200}{\pi}\sum_{n=1}^{\infty}\left[1+(-1)^n - \cos\frac{n\pi}{2}\right]\frac{e^{-n^2 t}\sin nx}{n} \\ &= \frac{100}{\pi}x + \frac{100}{\pi}\sum_{n=1}^{\infty}[2-(-1)^n]\,\frac{e^{-(2n)^2 t}\sin(2nx)}{n}. \end{aligned}$$

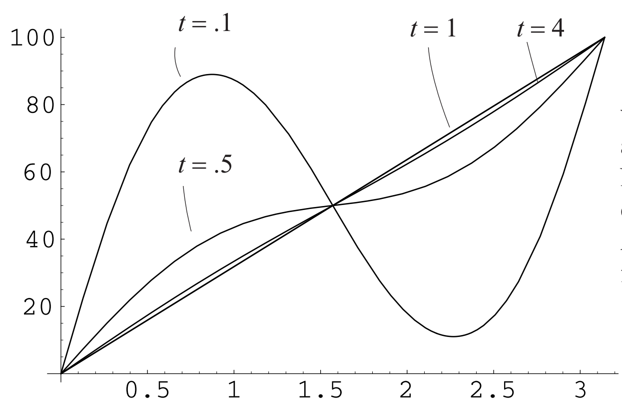
It is good at this point to confirm our results with some graphs. We will plot a partial sum of the series solution at $t = 0$ (this should be close to the initial temperature distribution). We also plot the partial sum of the series solution for large t (the graphs should be close to the steady-state solution $u_1(x)$).

The graphs suggest that $u(1/2,\,t) = 50$ for all t. Can you explain this result using the formula for u and on physical grounds?

```
Clear[f, partsum, n , k, sstate]
f[x_] := 100   /; 0 < x < Pi / 2
f[x_] := 0  /; Pi / 2 < x < Pi
sstate[x_] = 100 / Pi x
partsum[x_, t_, n_] := sstate[x] +
   100 / Pi Sum[ (2 - (-1) ^k ) E^ (- (2 k) ^2 t) Sin[2 k  x] / k, {k, 1, n}];
Plot[Evaluate[{f[x], partsum[x, 0, 20]}], {x, 0, Pi}]
Plot[Evaluate[{sstate[x], partsum[x, .1, 20],
   partsum[x, .5, 20], partsum[x, 1, 20], partsum[x, 4, 20]}], {x, 0, Pi}]
```



Initial temperature distribution and its approximation using a partial sum of the series solution at time $t = 0$.



Steady-state temperature distribution along with the temperature distribution at various values of t.
Note how quickly the temperature converges to the steady-state distribution:
At $t = 4$, we can't distinguish $u(x, 4)$ from the steady-state temperature distribution.

15. From (14), we have $b_n = 0$, since $f(x) - u_1(x) = 0$. Hence $u_2(x, t) = 0$ and so $u(x, t) = u_1(x) =$ steady-state solution. This is obvious on physical grounds: If the initial temperature distribution is already in equilibrium, then it will not change.

16. This problem is very much like Exercise 10(a). We sketch the details. Let $u_1(x, t) = u_1(x) = \frac{B-A}{L}x + A$. Then u_1 satisfies the wave equation

$$u_{tt} = c^2 u_{xx},$$

the boundary conditions

$$u_1(0, t) = A \quad \text{and} \quad u_1(L, t) = B,$$

and the initial conditions

$$u_1(x,\,0) = u_1(x) = \frac{B-A}{L}x + A \quad \text{and} \quad (u_1)_t(x,\,0) = 0.$$

Let $u_2(x,\,t)$ denote the solution of the wave equation with 0 boundary data (that is $u_2(0,\,t) = 0$ and $u_2(L,\,t) = 0$) and satisfying the initial conditions

$$u(x,\,0) = f(x) - u_1(x) \quad \text{and} \quad u_t(x,\,0) = g(x).$$

It is straightforward to show that

$$u(x,\,t) = u_1(x) + u_2(x,\,t)$$

is a solution of the nonhomogeneous wave problem

$$u_{tt} = c^2 u_{xx};$$

$$u(0,\,t) = A, \quad u(L,\,t) = B;$$

$$u(x,\,0) = f(x), \quad u_t(x,\,0) = g(x).$$

We can appeal to the solution of the wave equation from Sections 3.3 or 3.4 to express the solution u_2 in terms of the initial conditions. For example, using the solution from Section 3.3, we obtain

$$u_2(x,\,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x\,(b_n \cos\lambda_n t + b_n^* \sin\lambda_n t)\,,$$

where $\lambda_n = \frac{cn\pi}{L}$,

$$b_n = \frac{2}{L}\int_0^L (f(x) - u_1(x))\sin\frac{n\pi}{L}x\,dx,$$

and

$$b_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx.$$

17. Fix $t_0 > 0$ and consider the solution at time $t = t_0$:

$$u(x,\,t_0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x e^{-\lambda_n^2 t_0}.$$

We will show that this series converges uniformly for all x (not just $0 \le x \le L$) by appealing to the Weierstrass M-test. For this purpose, it suffices to establish the following two inequalities:
(a) $\left|b_n \sin\frac{n\pi}{L}xe^{-\lambda_n^2 t_0}\right| \le M_n$ for all x; and
(b) $\sum_{n=1}^{\infty} M_n < \infty$.
To establish (a), note that

$$\begin{aligned}
|b_n| &= \left|\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right| \le \frac{2}{L}\int_0^L \left|f(x)\sin\frac{n\pi}{L}x\right|\,dx \\
&\quad \text{(The absolute value of the integral is} \\
&\quad \le \text{ the integral of the absolute value.)} \\
&\le \frac{2}{L}\int_0^L |f(x)|\,dx = A \text{ (because } |\sin u| \le 1 \text{ for all } u).
\end{aligned}$$

Note that A is a finite number because f is bounded, so its absolute value is bounded and hence its integral is finite on $[0, L]$. We have

$$\begin{aligned}\left|b_n \sin\frac{n\pi}{L}x e^{-\lambda_n^2 t_0}\right| &\leq A e^{-\lambda_n^2 t_0} = A e^{-\frac{c^2\pi^2 t_0}{L^2}n^2} \\ &\leq A\left(e^{-\frac{c^2\pi^2 t_0}{L^2}}\right)^n = A r^n,\end{aligned}$$

where $r = e^{-\frac{c^2\pi^2 t_0}{L^2}} < 1$. Take $M_n = A r^n$. Then a holds and $\sum M_n$ is convergent because it is a geometric series with ratio $r < 1$.

Solutions to Exercises 3.6

1. Since the bar is insulated and the temperature inside is constant, there is no exchange of heat, and so the temperature remains constant for all $t > 0$. Thus $u(x, t) = 100$ for all $t > 0$. This is also a consequence of (2), since in this case all the a_n's are 0 except $a_0 = 100$.

2. The initial temperature distribution is already given by its Fourier cosine series. We have $a_1 = 1$ and $a_n = 0$ for all $n \neq 1$. Thus

$$u(x, t) = e^{-(n\pi)^2} \cos \pi x.$$

3. We apply (4):

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx,$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi/2} 100\, x\, dx + \frac{1}{\pi}\int_0^{\pi/2} 100\,(\pi - x)\, dx \\ &= \frac{50}{\pi} x^2 \Big|_0^{\pi/2} - \frac{50}{\pi}(\pi - x)^2 \Big|_{\pi/2}^{\pi} = \frac{25}{2}\pi + \frac{25}{2}\pi = 25\pi, \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi/2} 100\, x \cos nx\, dx + \frac{2}{\pi}\int_0^{\pi/2} 100\,(\pi - x)\cos nx\, dx \\ &= \frac{200}{\pi}\left(\frac{\cos n\,x}{n^2} + \frac{x\,\sin n\,x}{n}\right)\Big|_0^{\pi/2} \\ &\quad + \frac{200}{\pi}\left(-\frac{\cos n\,x}{n^2} + \frac{(\pi - x)\,\sin n\,x}{n}\right)\Big|_{\pi/2}^{\pi} \\ &= \frac{200}{\pi}\left(\frac{\cos n\frac{\pi}{2}}{n^2} + \frac{\frac{\pi}{2}\sin n\frac{\pi}{2}}{n} + \frac{\cos n\frac{\pi}{2}}{n^2} - \frac{\frac{\pi}{2}\sin n\frac{\pi}{2}}{n} - \frac{1}{n^2} - \frac{\cos n\,\pi}{n^2}\right) \\ &= \frac{200}{\pi n^2}\left(2\cos n\frac{\pi}{2} - (1 + (-1)^n)\right) \\ &= \begin{cases} 0 & \text{if } n = 4k+1 \text{ or } n = 4k+3, \\ 0 & \text{if } n = 4k \text{ or } n = 4k+3, \\ -\frac{800}{\pi n^2} & \text{if } n = 4k+2 = 2(2k+1). \end{cases} \end{aligned}$$

So

$$u(x, t) = 25\pi - \frac{200}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-4(2k+1)^2 t} \cos(2(2k+1)x),$$

4. Refer to Example 1. We need to expand $f(x)$ in a cosine series. The series can be derived from Exercise 6, Section 2.3, with $d = 1/2$, $p = 1$, and $c = 100$. We have, for $0 < x < 1$,

$$f(x) = \frac{100}{2} + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} \cos n\pi x.$$

But

$$\frac{\sin\frac{n\pi}{2}}{=} \begin{cases} 0 & \text{if } n = 2k, \\ (-1)^k & \text{if } n = 2k+1, \end{cases}$$

so

$$f(x) = 50 + \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[(2k+1)\pi x],$$

and the solution of the heat problem becomes

$$u(x, t) = 50 + \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[(2k+1)\pi x] e^{-(2k+1)^2 \pi^2 t} \quad (t > 0,\ 0 < x < 1).$$

5. Apply the separation of variables method as in Example 1; you will arrive at the following equations in X and T:

$$X'' - kX = 0, \quad X(0) = 0,\ X'(L) = 0$$
$$T' - kc^2 T = 0 \qquad .$$

We now show that the separation constant k has to be negative by ruling out the possibilities $k = 0$ and $k > 0$.

If $k = 0$ then $X'' = 0 \Rightarrow X = ax + b$. Use the initial conditions $X(0) = 0$ implies that $b = 0$, $X'(L) = 0$ implies that $a = 0$. So $X = 0$ if $k = 0$.

If $k > 0$, say $k = \mu^2$, where $\mu > 0$, then

$$\begin{aligned} X'' - \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cosh \mu x + c_2 \sinh \mu x; \\ X(0) = 0 \quad &\Rightarrow \quad 0 = c_1;\ X = c_2 \sinh \mu x; \\ X'(L) = 0 \quad &\Rightarrow \quad 0 = c_2 \mu \cosh(\mu L) \\ &\Rightarrow \quad c_2 = 0, \end{aligned}$$

because $\mu \neq 0$ and $\cosh(\mu L) \neq 0$. So $X = 0$ if $k > 0$. This leaves the case $k = -\mu^2$, where $\mu > 0$. In this case

$$\begin{aligned} X'' + \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cos \mu x + c_2 \sin \mu x; \\ X(0) = 0 \quad &\Rightarrow \quad 0 = c_1;\ X = c_2 \sin \mu x; \\ X'(L) = 0 \quad &\Rightarrow \quad 0 = c_2 \mu \cos(\mu L) \\ &\Rightarrow \quad c_2 = 0 \text{ or } \cos(\mu L) = 0. \end{aligned}$$

To avoid the trivial solution, we set $\cos(\mu L) = 0$, which implies that

$$\mu = (2k+1)\frac{\pi}{2L}, \quad k = 0, 1, \ldots .$$

Plugging this value of k in the equation for T, we find

$$T' + \mu^2 c^2 T = 0 \quad \Rightarrow \quad T(t) = B_k e^{-\mu^2 c^2 t} = B_k e^{-\left((2k+1)\frac{\pi}{2L}\right)^2 c^2 t}.$$

Forming the product solutions and superposing them, we find that

$$u(x, t) = \sum_{k=0}^{\infty} B_k e^{-\mu^2 c^2 t} = \sum_{k=0}^{\infty} B_k e^{-\left((2k+1)\frac{\pi}{2L}\right)^2 c^2 t} \sin\left[(2k+1)\frac{\pi}{2L} x\right].$$

To determine the coefficients B_k, we use the initial condition and proceed as in Example 1:

$$u(x,\,0) = f(x) \quad\Rightarrow\quad f(x) = \sum_{k=0}^{\infty} B_k \sin\left[(2k+1)\frac{\pi}{2L}x\right];$$

$$\Rightarrow\quad f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right] = \sum_{k=0}^{\infty} B_k \sin\left[(2k+1)\frac{\pi}{2L}x\right]\sin\left[(2n+1)\frac{\pi}{2L}x\right]$$

$$\Rightarrow\quad \int_0^L f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx = \sum_{k=0}^{\infty} B_k \int_0^L \sin\left[(2k+1)\frac{\pi}{2L}x\right]\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx$$

$$\Rightarrow\quad \int_0^L f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx = B_n \int_0^L \sin^2\left[(2n+1)\frac{\pi}{2L}x\right]\,dx,$$

where we have integrated the series term by term and used the orthogonality of the functions $\sin\left[(2k+1)\frac{\pi}{2L}x\right]$ on the interval $[0,\,L]$. The orthogonality can be checked directly by verifying that

$$\int_0^L \sin\left[(2k+1)\frac{\pi}{2L}x\right]\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx = 0$$

if $n \neq k$. Solving for B_n and using that

$$\int_0^L \sin^2\left[(2n+1)\frac{\pi}{2L}x\right]\,dx = \frac{L}{2}$$

(check this using a half-angle formula), we find that

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx.$$

6. In order to apply the result of Exercise 5, we must compute the Fourier coefficients of the series expansion of $f(x) = \sin x$ in terms of the functions $\sin\left[\frac{(2n+1)}{2}x\right]$ $(n = 0,\,1,\,2,\,\ldots)$ on the interval $[0,\,\pi]$. That is, we must find B_n in the series expansion

$$\sin x = \sum_{n=0}^{\infty} B_n \sin\left[\frac{(2n+1)}{2}x\right].$$

According to Exercise 5, we have

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \sin\left[\frac{(2n+1)}{2}x\right] dx\\
&= \frac{1}{\pi}\int_0^{\pi} \left(\cos[(n-\frac{1}{2})x] - \cos[(n+\frac{3}{2})x]\right) dx\\
&= \frac{1}{\pi}\left[\frac{1}{n-\frac{1}{2}}\sin[(n-\frac{1}{2})x] - \frac{1}{n+\frac{3}{2}}\sin[(n+\frac{3}{2})x]\right]\Big|_0^{\pi}\\
&= \frac{2}{\pi}\left[\frac{1}{2n-1}\sin[(n\pi-\frac{\pi}{2})] - \frac{1}{2n+3}\sin[(n\pi+\frac{3\pi}{2})]\right]\\
&= \frac{2}{\pi}\left[\frac{-(-1)^n}{2n-1} + \frac{(-1)^n}{2n+3}\right]\\
&= \frac{8}{\pi}\frac{(-1)^{n+1}}{(2n-1)(2n+3)}.
\end{aligned}$$

We can now apply Exercise 5 and get the solution

$$u(x,\,t) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+3)}e^{-(2n+1)^2t/4}\sin\left[\frac{(2n+1)}{2}x\right].$$

7. In the first part of this solution, we will justify term-by-term integration of the series solution. Recall that the solution is given by

$$u(x,\,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos\frac{n\pi x}{L},$$

where a_n is the cosine Fourier coefficient of f. Since f is a temperature distribution, we can suppose that it is bounded; hence $|f(x)| \le M$ for all x, where M is a constant. Hence

$$\begin{aligned}
|a_n| &= \frac{2}{L}\left|\int_0^L f(x)\cos\frac{n\pi}{L}\,dx\right| \le \frac{2}{L}\int_0^L \overbrace{|f(x)|}^{\le M}\,\overbrace{\left|\cos\frac{n\pi x}{L}\right|}^{\le 1}\,dx\\
&\le \frac{2}{L}\int_0^L M\,dx = \frac{2}{L}\,L\,M = 2M,
\end{aligned}$$

and so, for $t > 0$,

$$|u(x,\,t)| \le |a_0| + \sum_{n=1}^{\infty}\left|a_n e^{-\lambda_n^2 t}\cos\frac{n\pi x}{L}\right| \le |a_0| + 2M\sum_{n=1}^{\infty} e^{-\lambda_n^2 t} < \infty.$$

To see that the series $\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}$ is convergent, write

$$e^{-\lambda_n^2 t} = \left(e^{-\frac{tc^2\pi^2}{L^2}}\right)^{n^2} < \left(e^{-\frac{tc^2\pi^2}{L^2}}\right)^{n} = r^n,$$

where $r = e^{-\frac{tc^2\pi^2}{L^2}} < 1$. Thus the series converges by comparison to a geometric series with ratio $0 < r < 1$. From this it follows that, for fixed $t > 0$, the series $a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos\frac{n\pi x}{L}$ converges uniformly for all x, by the Weierstrass M-test. Thus we can integrate the series term by term (Theorem 3, Section 2.7) and get

$$\begin{aligned}\frac{1}{L}\int_0^L u(x,t)\,dx &= \overbrace{\frac{1}{L}\int_0^L a_0\,dx}^{=a_0}+\sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\,\overbrace{\frac{1}{L}\int_0^L \cos\frac{n\pi x}{L}\,dx}^{=0}\\ &= a_0=\frac{1}{L}\int_0^L f(x)\,dx,\end{aligned}$$

where the last equality follows from the definition of a_0. Thus at any time $t>0$, the average of the temperature distribution inside the bar is constant and equals to the average of the initial temperature distribution $f(x)$. This makes sens on physical grounds since the bar is insulated and there is no exchange of heat with the surrounding, the temperature may vary but the total heat inside the bar remains the same, and so the average temperature remains the same at any time $t>0$.

8. By Exercise 7, the average temperature in Example 1 is a_0. We now show that the steady-state temperature is also a_0. Take the limit as $t\to\infty$, assuming that we can interchange the limit and the summation sign:

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left(a_0+\sum_{n=1}^{\infty} a_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x\right)\\ &= a_0+\sum_{n=1}^{\infty}\Big[a_n\cos\frac{n\pi}{L}x\,\overbrace{\lim_{t\to\infty} e^{-\left(\frac{cn\pi}{L}\right)^2 t}}^{=0}\Big]\\ &= a_0.\end{aligned}$$

If you really want to justify the interchange of the limit and the summation sign, you can use the estimate that we found in Exercise 7, as follows. Let $r=e^{-\frac{tc^2\pi^2}{L^2}}$. Then

$$\left|\sum_{n=1}^{\infty} a_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x\right| \le \sum_{n=1}^{\infty} r^n=\frac{r}{1-r}.$$

As $t\to\infty$, $r\to 0$, showing that

$$\lim_{t\to\infty}\sum_{n=1}^{\infty} a_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x=0.$$

9. This is a straightforward application of Exercise 7. For Exercise 1 the average is 100. For Exercise 2 the average is $a_0=0$.

10. Using the relation $\sin ax\sin bx=\frac{1}{2}\big[\cos(a-b)x-\cos(a+b)x\big]$, we obtain, for $m \neq n$,

$$\begin{aligned}
&\int_0^L \sin \mu_m x \sin \mu_n x \, dx \\
&= \frac{1}{2} \int_0^L \left[\cos(\mu_m - \mu_n)x - \cos(\mu_m + \mu_n)x \right] dx \\
&= \frac{1}{2(\mu_m - \mu_n)} \sin(\mu_m - \mu_n)L - \frac{1}{2(\mu_m + \mu_n)} \sin(\mu_m + \mu_n)L \\
&= \frac{1}{2(\mu_m - \mu_n)} \left[\sin \mu_m L \cos \mu_n L - \cos \mu_m L \sin \mu_n L \right] \\
&\quad - \frac{1}{2(\mu_m + \mu_n)} \left[\sin \mu_m L \cos \mu_n L + \cos \mu_m L \sin \mu_n L \right] \\
&= \left[\mu_m \sin \mu_n L \cos \mu_m L - \mu_n \cos \mu_n L \sin \mu_m L \right] / \left(\mu_n^2 - \mu_m^2 \right) .
\end{aligned}$$

We now show that $\mu_m \sin \mu_n L \cos \mu_m L - \mu_n \cos \mu_n L \sin \mu_m L = 0$, by using (7), which states that $\tan(\mu L) = -\frac{\mu}{\kappa}$ for all $\mu = \mu_j$. In particular, $\cos \mu L \neq 0$. Dividing through by $\cos \mu_m L \cos \mu_n L \neq 0$, we find

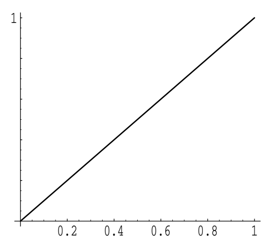
$$\begin{aligned}
&\mu_m \sin \mu_n L \cos \mu_m L - \mu_n \cos \mu_n L \sin \mu_m L = 0 \\
\Leftrightarrow \quad & \mu_m \tan \mu_n L - \mu_n \tan \mu_m L = 0 \\
\Leftrightarrow \quad & -\mu_m \frac{\mu_n}{\kappa} + \mu_n \frac{\mu_m}{\kappa} = 0,
\end{aligned}$$

which establishes the desired orthogonality relation.

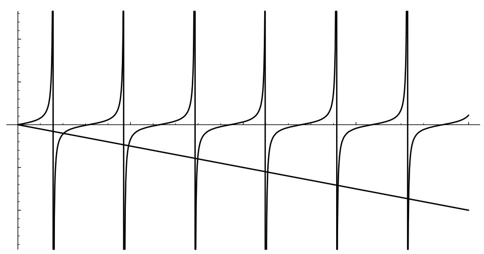
11. To do this problem, as well as Exercises 12-14, we will need the numerical values from Example 3. This will be done entirely on Mathematica.

In the first window, we plot $f(x)$, $\tan x$, and $-x$, and find numerical values for the roots of $\tan x = -x$.

```
f[x_]:=x
Plot[f[x],{x,0,1}]
```




```
Plot[{Tan[x], -x}, {x,0,20}]
```



```
approxsol={3,5,8,11,14.3,17.5};
sol=Table[
        FindRoot[Tan[x]==-x,{x,approxsol[[j]]}],
        {j,1,6}]
```

{{x → 2.02876}, {x → 4.91318}, {x → 7.97867}, {x → 11.0855}, {x → 14.2074}, {x → 17.3364}}

In the second window, we form a partial sum of the series solution. This requires computing numerically the generalized Fourier coefficients of $f(x)$. We then evaluate the partial sum solution at time $t = 0$ (this should approximate $f(x)$), and plot it against $f(x)$.

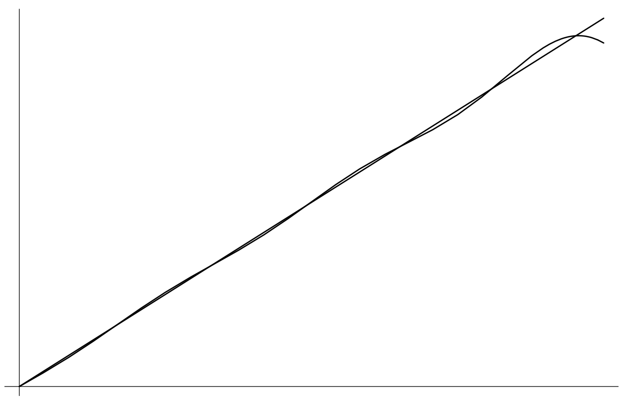
```
u6[x_,t_]= Sum[c[j] Exp[- sol[[j,1,2]]^2 t] Sin[sol[[j,1,2]] x],
{j,1,6}]
```

$e^{-4.11586\, t}$ c[1] Sin[2.02876 x] + $e^{-24.1393\, t}$ c[2] Sin[4.91318 x] + $e^{-63.6591\, t}$ c[3] Sin[7.97867 x] + $e^{-122.889\, t}$ c[4] Sin[11.0855 x] + $e^{-201.851\, t}$ c[5] Sin[14.2074 x] + $e^{-300.55\, t}$ c[6] Sin[17.3364 x]

```
c[j_]:=1/Integrate[Sin[sol[[j,1,2]] x]^2,{x,0,1}] *\
Integrate[f[x] Sin[sol[[j,1,2]] x] ,{x,0,1}]

tableofcoeff=Table[Chop[c[j]],{j,1,6}]
```

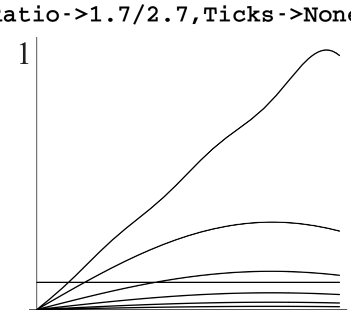
{0.729175, -0.156164, 0.0613973, -0.0321584, 0.0196707, -0.0132429}

```
Plot[Evaluate[{f[x],u6[x,0]}],{x,0,1},
AspectRatio->1.7/2.7,Ticks->None]
```



In the third window, we plot the series solution at various values of t.

```
tabplot=Table[
u6[x,t],
{t,0,1,.2}];
Plot[Evaluate[{tabplot,.1}],{x,0,1},PlotRange->{0,1},
AspectRatio->1.7/2.7,Ticks->None]
```



12., **13.**, and **14.** follow by repeating the same commands as in Exercise 11 for the appropriate function $f(x)$.

15. Following the solution in Example 2, we arrive at the following equations

$$T' - kc^2 T = 0,$$
$$X'' - kX = 0, \quad X'(0) = 0,\ X'(L) = -\kappa X(L),$$

where k is a separation constant. If $k = 0$ then

$$\begin{aligned} X'' = 0 &\;\Rightarrow\; X = ax + b, \\ X'(0) = 0 &\;\Rightarrow\; a = 0 \\ X'(L) = -\kappa X(L) &\;\Rightarrow\; b = 0; \end{aligned}$$

so $k = 0$ leads to trivial solutions. If $k = \alpha^2 > 0$, then, as in the text, we conclude that $X = 0$. This leaves the case $k = -\mu^2 < 0$. In this case $X(x) = c_1 \cos \mu x + c_2 \sin \mu x$. From $X'(0) = 0$ it follows that $c_2 = 0$ or $X = c_2 \cos \mu x$. Set $c_2 = 1$ to simplify the presentation. From $X'(L) = -\kappa X(L)$, we find that $-\mu \sin \mu L = -\kappa \cos \mu L$. Hence μ is a solution of the equation

$$\frac{\mu}{\kappa} = \frac{\cos \mu L}{\sin \mu L} \quad \text{or} \quad \cot \mu L = \frac{1}{\kappa}\,\mu.$$

Let μ_n denote the nth positive solution of this equation, and let $X_n(x) = \cos \mu_n x$. Solving the equation for T for each μ_n, we find

$$T = T_n = c_n e^{-c^2 \mu_n^2 t}$$

and hence, by superposing the product solutions, we arrive at

$$u(x,\, t) = \sum_{n=1}^{\infty} c_n \cos \mu_n x e^{-c^2 \mu_n^2 t}.$$

To prove that the X_n's are orthogonal, see Exercise 19. The proof involves ideas that will be studied in greater detail in Chapter 6, in the context of Sturm-Liouville theory.

Using the initial condition, it follows that

$$u(x,\, 0) = f(x) = \sum_{n=1}^{\infty} c_n X_n(x).$$

Multiplying both sides by X_m and integrating term by term, it follows from the orthogonality of the X_n that

$$\int_0^L f(x) X_m(x)\, dx = \sum_{n=1}^{\infty} c_n \overbrace{\int_0^L X_n(x) X_m(x)\, dx}^{=0 \text{ if } m \neq n}.$$

Hence

$$\int_0^L f(x) X_m(x)\, dx = c_n \int_0^L X_n^2(x)\, dx$$

or

$$c_n = \frac{1}{\kappa_n} \int_0^L f(x) X_n(x)\, dx$$

where

$$\kappa_n = \int_0^L X_n^2(x)\, dx\,.$$

Let us evaluate κ_n. Recall that μ_n is a solution of $\cot L\mu = \frac{1}{\kappa}\mu$. We have

$$\begin{aligned}\kappa_n &= \int_0^L \cos^2 \mu_n x\,dx = \frac{1}{2}\int_0^L (1+\cos 2\mu_n x)\,dx\\ &= \frac{L}{2}\frac{1}{4\mu_n}\sin 2\mu_n x\Big|_0^L = \frac{L}{2}\frac{1}{4\mu_n}\sin 2\mu_n L.\end{aligned}$$

In order to use what we know about μ_n, namely, that it is a solution of $\cot L\mu = \frac{1}{\kappa}\mu$, we express the sine in terms of the cotangent, as follows. We have

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha = \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha+\cos^2\alpha}\\ &= 2\frac{\cot\alpha}{\cot^2\alpha+1},\end{aligned}$$

where the last identity follows upon dividing by $\cos^2\alpha$. Using this in the expression for κ_n, we find

$$\begin{aligned}\kappa_n &= \frac{L}{2}+\frac{1}{4\mu_n}\cdot 2\frac{\cot\mu_n L}{1+\cot^2\mu_n L}\\ &= \frac{L}{2}+\frac{1}{2\mu_n}\frac{(\mu_n/\kappa)}{1+(\mu_n/\kappa)^2}\\ &= \frac{L}{2}+\frac{\kappa}{2(\kappa^2+\mu_n^2)}.\end{aligned}$$

16. (a) Take a product solution, $u(x,t) = X(x)T(t)$, plug it into the heat equation $u_t = c^2 u_{xx}$, proceed formally by using the typical argument of the separation of variables method, and you will get

$$\begin{gathered}T'X = c^2X''T\\ \tfrac{T'}{c^2T} = \tfrac{X''}{X}\\ \tfrac{T'}{c^2T} = k \quad\text{and}\quad \tfrac{X''}{X} = k\\ T' - kc^2T = 0 \quad\text{and}\quad X'' - kX = 0,\end{gathered}$$

where k is a separation constant. Doing the same with the periodic boundary conditions, you will find

$$\begin{aligned}X(-\pi)T(t) = X(\pi)T(t) \quad&\text{and}\quad X'(-\pi)T(t) = X'(\pi)T(t)\\ X(-\pi) = X(\pi) \quad&\text{and}\quad X'(-\pi) = X'(\pi).\end{aligned}$$

(b) If $k = \alpha^2 > 0$, then the general solution of the $X'' - \alpha^2 X = 0$ is $X = c_1\cosh\alpha x + c_2\sinh\alpha x$ (see Appendix A.2). But

$$\begin{aligned}X(\pi) = X(-\pi) \quad&\Rightarrow\quad c_1\cosh\alpha\pi + c_2\sinh\alpha\pi = c_1\cosh\alpha\pi - c_2\sinh\alpha\pi\\ &\Rightarrow\quad c_2\sinh\alpha\pi = 0\\ &\Rightarrow\quad c_2 = 0,\end{aligned}$$

because $\sinh\alpha\pi \neq 0$. Using the second boundary condition, we obtain

$$\alpha c_1\sinh\alpha\pi = -\alpha c_1\sinh\alpha\pi \quad\Rightarrow\quad c_1 = 0.$$

So the case $k = \alpha^2 > 0$ leads only to trivial solutions.
(c) If $k = 0$, then the solution of $X'' = 0$ is $X = ax + b$. The boundary condition $X'(-\pi) = -X'(\pi)$ implies that $a = -a$ or $a = 0$. Hence X is constant, and we denote it by a_0. Note that this is an acceptable solution of the differential equation and the periodic boundary conditions.
(d) If $k = -\mu^2 < 0$, the solution in X is $X = c_1 \cos \mu x + c_2 \sin \mu x$. Using the boundary conditions, we have

$$\begin{aligned} X(-\pi) = X(\pi) \quad &\Rightarrow \quad c_1 \cos \mu\pi - c_2 \sin \mu\pi = c_1 \cos \mu\pi + c_2 \sin \mu\pi \\ &\Rightarrow \quad c_2 \sin \mu\pi = 0 \\ &\Rightarrow \quad c_2 = 0 \quad \text{or} \quad \sin \mu\pi = 0. \end{aligned}$$

If $c_2 = 0$, then the condition $X'(-\pi) = X'(\pi)$ implies that $-c_2\mu \sin \mu\pi = c_2\mu \sin \mu\pi$, which in turn implies that $\sin \mu\pi = 0$. So, either way, we have the condition $\sin \mu\pi = 0$, which implies that $\mu = \pm m$, where $m = 1, 2, 3, \ldots$. In order not to duplicate solutions, we take $\mu = m$, where $m = 1, 2, 3, \ldots$, and obtain the solutions

$$X_m = a_m \cos mx + b_m \sin mx.$$

The corresponding solutions in T are

$$T' - m^2c^2T = 0 \quad \Rightarrow \quad T = e^{-m^2c^2t}.$$

Therefore, The product solutions are

$$X_m(x)T_m(t) = \big(a_m \cos mx + b_m \sin mx\big)e^{-m^2c^2t} \quad (m \geq 1).$$

Superposing the product solutions, we obtain

$$u(x,\, t) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)e^{-m^2c^2t}.$$

Setting $t = 0$ and using the boundary conditions, it follows that

$$u(x,\, 0) = f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

Thinking of this as the Fourier series of f, it follows that the a_n and b_n are the Fourier coefficients of f and are given by the Euler formula in Section 2.2. Here we are considering f as a 2π-periodic function of the angle x.

17. Part (a) is straightforward as in Example 2. We omit the details that lead to the separated equations:

$$T' - kT = 0,$$
$$X'' - kX = 0, \quad X'(0) = -X(0),\ X'(1) = -X(1),$$

where k is a separation constant.
(b) If $k = 0$ then

$$\begin{aligned} X'' = 0 \quad &\Rightarrow \quad X = ax + b, \\ X'(0) = -X(0) \quad &\Rightarrow \quad a = -b \\ X'(1) = -X(1) \quad &\Rightarrow \quad a = -(a + b) \ \Rightarrow\ 2a = -b; \\ &\Rightarrow \quad a = b = 0. \end{aligned}$$

So $k = 0$ leads to trivial solutions.
(c) If $k = \alpha^2 > 0$, then

$$\begin{aligned}
X'' - \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cosh \mu x + c_2 \sinh \mu x;\\
X'(0) = -X(0) \quad &\Rightarrow \quad \mu c_2 = -c_1\\
X'(1) = -X(1) \quad &\Rightarrow \quad \mu c_1 \sinh \mu + \mu c_2 \cosh \mu = -c_1 \cosh \mu - c_2 \sinh \mu\\
&\Rightarrow \quad \mu c_1 \sinh \mu - c_1 \cosh \mu = -c_1 \cosh \mu - c_2 \sinh \mu\\
&\Rightarrow \quad \mu c_1 \sinh \mu = -c_2 \sinh \mu\\
&\Rightarrow \quad \mu c_1 \sinh \mu = \frac{c_1}{\mu} \sinh \mu.
\end{aligned}$$

Since $\mu \neq 0$, $\sinh \mu \neq 0$. Take $c_1 \neq 0$ and divide by $\sinh \mu$ and get

$$\mu c_1 = \frac{c_1}{\mu} \;\Rightarrow\; \mu^2 = 1 \;\Rightarrow\; k = 1.$$

So $X = c_1 \cosh x + c_2 \sinh x$. But $c_1 = -c_2$, so

$$X = c_1 \cosh x + c_2 \sinh x = c_1 \cosh x - c_1 \sinh x = c_1 e^{-x}.$$

Solving the equation for T, we find $T(t) = e^t$; thus we have the product solution

$$c_0 e^{-x} e^t,$$

where, for convenience, we have used c_0 as an arbitrary constant.
(d) If $k = -\alpha^2 < 0$, then

$$\begin{aligned}
X'' + \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cos \mu x + c_2 \sin \mu x;\\
X'(0) = -X(0) \quad &\Rightarrow \quad \mu c_2 = -c_1\\
X'(1) = -X(1) \quad &\Rightarrow \quad -\mu c_1 \sin \mu + \mu c_2 \cos \mu = -c_1 \cos \mu - c_2 \sin \mu\\
&\Rightarrow \quad -\mu c_1 \sin \mu - c_1 \cos \mu = -c_1 \cos \mu - c_2 \sin \mu\\
&\Rightarrow \quad -\mu c_1 \sin \mu = -c_2 \sin \mu\\
&\Rightarrow \quad -\mu c_1 \sin \mu = \frac{c_1}{\mu} \sin \mu.
\end{aligned}$$

Since $\mu \neq 0$, take $c_1 \neq 0$ (otherwise you will get a trivial solution) and divide by c_1 and get

$$\mu^2 \sin \mu = -\sin \mu \;\Rightarrow\; \sin \mu = 0 \;\Rightarrow\; \mu = n\pi,$$

where n is an integer. So $X = c_1 \cos n\pi x + c_2 \sin n\pi x$. But $c_1 = -c_2 \mu$, so

$$X = -c_1 \big(n\pi \cos n\pi x - \sin n\pi x \big).$$

Call $X_n = n\pi \cos n\pi x - \sin n\pi x$.
(e) To establish the orthogonality of the X_n's, treat the case $k = 1$ separately. For $k = -\mu^2$, we refer to the boundary value problem

$$X'' + \mu_n^2 X = 0, \; X(0) = -X'(0), \; X(1) = -X'(1),$$

that is satisfied by the X_n's, where $\mu_n = n\pi$. We establish orthogonality using a trick from Sturm-Liouville theory (Chapter 6, Section 6.2). Since

$$X_m'' = \mu_m^2 X_m \text{ and } X_n'' = \mu_n^2 X_n,$$

multiplying the first equation by X_n and the second by X_m and then subtracting the resulting equations, we obtain

$$\begin{gathered} X_n X_m'' = \mu_m^2 X_m X_n \text{ and } X_m X_n'' = \mu_n^2 X_n X_m \\ X_n X_m'' - X_m X_n'' = (\mu_n^2 - \mu_m^2) X_m X_n \\ \big(X_n X_m' - X_m X_n'\big)' = (\mu_n^2 - \mu_m^2) X_m X_n \end{gathered}$$

where the last equation follows by simply checking the validity of the identity $X_n X_m'' - X_m X_n'' = \big(X_n X_m' - X_m X_n'\big)'$. So

$$\begin{aligned} (\mu_n^2 - \mu_m^2) \int_0^1 X_m(x) X_n(x)\, dx &= \int_0^1 \big(X_n(x) X_m'(x) - X_m(x) X_n'(x)\big)'\, dx \\ &= X_n(x) X_m'(x) - X_m(x) X_n'(x)\Big|_0^1, \end{aligned}$$

because the integral of the derivative of a function is the function itself. Now we use the boundary conditions to conclude that

$$\begin{aligned} & X_n(x) X_m'(x) - X_m(x) X_n'(x)\Big|_0^1 \\ &= X_n(1) X_m'(1) - X_m(1) X_n'(1) - X_n(0) X_m'(0) + X_m(0) X_n'(0) \\ &= -X_n(1) X_m(1) + X_m(1) X_n(1) + X_n(0) X_m(0) - X_m(0) X_n(0) \\ &= 0. \end{aligned}$$

Thus the functions are orthogonal. We still have to verify the orthogonality when one of the X_n's is equal to e^{-x}. This can be done by modifying the argument that we just gave.
(f) Superposing the product solutions, we find that

$$u(x,\, t) = c_0 e^{-x} e^t + \sum_{n=1}^\infty c_n T_n(t) X_n(x).$$

Using the initial condition, it follows that

$$u(x,\, 0) = f(x) = c_0 e^{-x} + \sum_{n=1}^\infty c_n X_n(x).$$

The coefficients in this series expansion are determined by using the orthogonality of the X_n's in the usual way. Let us determine c_0. Multiplying both sides by e^{-x} and integrating term by term, it follows from the orthogonality of the X_n that

$$\int_0^1 f(x) e^{-x}\, dx = c_0 \int_0^1 e^{-2x} dx + \sum_{n=1}^\infty c_n \overbrace{\int_0^1 X_n(x) e^{-x}\, dx}^{=0}.$$

Hence

$$\int_0^1 f(x) e^{-x}\, dx = c_0 \int_0^1 e^{-2x} dx = c_0 \frac{1 - e^{-2}}{2}.$$

Thus

$$c_0 = \frac{2e^2}{e^2 - 1} \int_0^1 f(x) e^{-x}\, dx.$$

In a similar way, we prove that

$$c_n = \frac{1}{\kappa_n} \int_0^1 f(x) X_n(x)\, dx$$

where

$$\kappa_n = \int_0^1 X_n^2(x)\, dx\,.$$

This integral can be evaluated as we did in Exercise 15 or by straightforward computations, using the explicit formula for the X_n's, as follows:

$$\begin{aligned}
\int_0^1 X_n^2(x)\, dx &= \int_0^1 \big(n\pi \cos n\pi x - \sin n\pi x\big)^2\, dx \\
&= \int_0^1 \big(n^2\pi^2 \cos^2 n\pi x + \sin^2 n\pi x - 2n\pi \cos(n\pi x)\sin(n\pi x)\big)\, dx \\
&= \overbrace{\int_0^1 n^2\pi^2 \cos^2 n\pi x\, dx}^{=(n^2\pi^2)/2} + \overbrace{\int_0^1 \sin^2 n\pi x\, dx}^{=1/2} \\
&\quad -2n\pi \overbrace{\int_0^1 \cos(n\pi x)\sin(n\pi x)\, dx}^{=0} \\
&= \frac{n^2\pi^2 + 1}{2}.
\end{aligned}$$

19. (a) Following the solution in Exercise 16, we arrive at the following equations

$$T' - kT = 0,$$
$$X'' - kX = 0, \quad X'(0) = X(0),\; X'(1) = -X(1),$$

where k is a separation constant. We will not repeat the details.
(b) If $k = 0$ then

$$\begin{aligned}
X'' = 0 \quad &\Rightarrow \quad X = ax + b, \\
X'(1) = -X(1) \quad &\Rightarrow \quad a = -a \quad \Rightarrow \quad a = 0 \\
X'(0) = X(0) \quad &\Rightarrow \quad a = b = 0;
\end{aligned}$$

so $k = 0$ leads to trivial solutions. If $k = \alpha^2 > 0$, then

$$X'' - \alpha^2 X = 0 \quad \Rightarrow \quad X = c_1 \cosh \alpha x + c_2 \sinh \alpha x,$$
$$X(0) = c_1,\; X'(0) = \alpha c_2,$$
$$X(1) = c_1 \cosh \alpha + c_2 \sinh \alpha,\; X'(1) = \alpha c_1 \sinh \alpha + \alpha c_2 \cosh \alpha,$$
$$X'(0) = X(0) \quad \Rightarrow \quad c_1 = \alpha c_2$$
$$X'(1) = -X(1) \quad \Rightarrow \quad -(c_1 \cosh \alpha + c_2 \sinh \alpha) = \alpha c_1 \sinh \alpha + \alpha c_2 \cosh \alpha.$$

Using $\alpha c_2 = c_1$, we obtain from the last equation

$$-(c_1 \cosh \alpha + \frac{c_1}{\alpha} \sinh \alpha) = \alpha c_1 \sinh \alpha + c_1 \cosh \alpha.$$

Either $c_1 = 0$, in which case we have a trivial solution, or

$$-(\alpha \cosh \alpha + \sinh \alpha) = \alpha(\alpha \sinh \alpha + \cosh \alpha)$$
$$-2\alpha \cosh \alpha = (\alpha^2 + 1) \sinh \alpha.$$

If $\alpha > 0$, then the left side is < 0 while the right side is > 0. So there are no positive solutions. Also, if $\alpha < 0$, then the left side is > 0 while the right side is < 0. So there are no negative solutions. Hence the only solution that satisfies the equation and the boundary conditions is the trivial solution in the case $k = \alpha^2 > 0$.

(c) The separation constant must be negative: $k = -\mu^2 < 0$. In this case $X(x) = c_1 \cos \mu x + c_2 \sin \mu x$. From $X'(0) = X(0)$ it follows that $c_1 = \mu c_2$. Hence $X(x) = c_2 \mu \cos \mu x + c_2 \sin \mu x$, where $c_2 \neq 0$. From $X'(1) = -X(1)$, we find that $-c_2\mu^2 \sin \mu + c_2 \mu \cos \mu = -c_2 \mu \cos \mu - c_2 \sin \mu$. Hence μ is a solution of the equation

$$-\mu^2 \sin \mu + \mu \cos \mu = -\mu \cos \mu - \sin \mu \quad \Rightarrow \quad 2\mu \cos \mu = (\mu^2 - 1) \sin \mu$$

Note that $\mu = \pm 1$ s not a solution and $\cos \mu = 0$ is not a possibility, since this would imply $\sin \mu = 0$ and the two equations have no common solutions. So we can divide by $\cos \mu(\mu^2 - 1)$ and conclude that μ satisfies the equation $\tan \mu = \frac{2\mu}{\mu^2 - 1}$. This equation has infinitely many solutions $\mu = \pm \mu_n$, that can be seen on a graph. We take only the positive μ_n since we do not gain any new solutions in X for $-\mu_n$. Write the corresponding solutions

$$X = X_n = c_n(\mu_n \cos \mu_n x + \sin \mu_n), \quad (n = 1, 2, , \ldots)$$

The corresponding solutions in T are

$$T_n(t) = e^{-\mu_n^2 t}.$$

(e) We can prove that the X_n are orthogonal on the interval $[0, 1]$ as follows. (The ideas in the proof will be studied in greater detail in Chapter 6, in the context of Sturm-Liouville theory. They will also appear when we establish the orthogonality of Bessel functions (Chapter 4) and Legendre polynomials (Chapter 5).) We will not need the explicit formulas for X_n and X_m. We will work with the fact hat these are both solutions of the same boundary value problem

$$X'' + \mu^2 X = 0, \; X(0) = X'(0), \; X(1) = -X'(1),$$

corresponding to different values of $\mu = \mu_m$ and $\mu = \mu_n$, with $\mu_m \neq \mu_n$. So

$$X_m'' = \mu_m^2 X_m \text{ and } X_n'' = \mu_n^2 X_n.$$

Multiplying the first equation by X_n and the second by X_m and then subtracting the resulting equations, we obtain

$$X_n X_m'' = \mu_m^2 X_m X_n \text{ and } X_m X_n'' = \mu_n^2 X_n X_m$$
$$X_n X_m'' - X_m X_n'' = (\mu_n^2 - \mu_m^2) X_m X_n$$
$$\big(X_n X_m' - X_m X_n'\big)' = (\mu_n^2 - \mu_m^2) X_m X_n$$

where the last equation follows by simply checking the validity of the identity $X_n X_m'' - X_m X_n'' = \big(X_n X_m' - X_m X_n'\big)'$. So

$$\begin{aligned}(\mu_n^2 - \mu_m^2)\int_0^1 X_m(x)X_n(x)\,dx &= \int_0^1 \big(X_n(x)X_m'(x) - X_m(x)X_n'(x)\big)'\,dx\\ &= X_n(x)X_m'(x) - X_m(x)X_n'(x)\Big|_0^1,\end{aligned}$$

because the integral of the derivative of a function is the function itself. Now we use the boundary conditions to conclude that

$$\begin{aligned}&X_n(x)X_m'(x) - X_m(x)X_n'(x)\Big|_0^1\\ &= X_n(1)X_m'(1) - X_m(1)X_n'(1) - X_n(0)X_m'(0) + X_m(0)X_n'(0)\\ &= -X_n(1)X_m(1) + X_m(1)X_n(1) - X_n(0)X_m(0) + X_m(0)X_n(0)\\ &= 0.\end{aligned}$$

Thus the functions are orthogonal.
(f) Superposing the product solutions, we find that

$$u(x, t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x).$$

Using the initial condition, it follows that

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} c_n X_n(x).$$

Multiplying both sides by X_m and integrating term by term, it follows from the orthogonality of the X_n that

$$\int_0^1 f(x)X_m(x)\,dx = \sum_{n=1}^{\infty} c_n \overbrace{\int_0^1 X_n(x)X_m(x)\,dx}^{=0 \text{ if } m\neq n}.$$

Hence

$$\int_0^1 f(x)X_m(x)\,dx = c_n \int_0^1 X_n^2(x)\,dx$$

or

$$c_n = \frac{1}{\kappa_n}\int_0^1 f(x)X_n(x)\,dx$$

where

$$\kappa_n = \int_0^1 X_n^2(x)\,dx\,.$$

Solutions to Exercises 3.7

3. We use the double sine series solution that is outlined in the text.

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where a and b are the dimensions of the membrane,

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}},$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy,$$

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy.$$

Applying these formulas in our case, we find

$$\lambda_{mn} = \sqrt{m^2+n^2},$$

$$B^*_{mn} = \begin{cases} \frac{2}{\lambda_{1,2}} = \frac{2}{\sqrt{5}} & \text{if } (m, , n) = (1, 2), \\ 0 & \text{otherwise,} \end{cases}$$

because of the orthogonality of the functions $\sin m\pi x\,\sin n\pi y$ and the fact that g contains only the function $\sin\pi x\,\sin 2\imath y$. We also have

$$\begin{aligned} B_{mn} &= 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin m\pi x\,\sin n\pi y\,dx\,dy \\ &= 4\int_0^1 x(1-x)\sin m\pi x\,dx\int_0^1 y(1-y)\,\sin n\pi y\,dy \\ &= \begin{cases} \frac{64}{(mn\pi^2)^3} & \text{if } m \text{ and } n \text{ are both odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Refer to Example 1 of this section or Exercise 9, Section 3.3, for the computation of the definite integrals. Thus

$$u(x, y, t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{64\,\sin((2k+1)\pi x)\sin((2l+1)\pi y)}{((2k+1)(2l+1)\pi^2)^3}\cos\sqrt{(2k+1)^2+(2l+1)^2}t + \frac{2}{\sqrt{5}}\sin\pi x\sin(2\pi y)\sin(\sqrt{5}\,t).$$

5. We proceed as in Exercise 3. We have

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin m\pi x\sin n\pi y,$$

where $\lambda_{mn} = \sqrt{m^2+n^2}$, $B_{mn} = 0$, and

$$\begin{aligned} B^*_{mn} &= \frac{4}{\sqrt{m^2+n^2}}\int_0^1\int_0^1 \sin m\pi x\,\sin n\pi y\,dx\,dy \\ &= \frac{4}{\sqrt{m^2+n^2}}\int_0^1 \sin m\pi x\,dx\int_0^1 \sin n\pi y\,dy \\ &= \begin{cases} \frac{16}{\sqrt{m^2+n^2}(mn)\pi^2} & \text{if } m \text{ and } n \text{ are both odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$u(x, y, t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{16\sin((2k+1)\pi x)\sin((2l+1)\pi y)}{\sqrt{(2k+1)^2+(2l+1)^2}\,(2k+1)(2l+1)\pi^2}\sin\sqrt{(2k+1)^2+(2l+1)^2}t$$

12. We use the double sine series solution that is outlined in the text.

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\lambda_{mn}^2 t},$$

where a and b are the dimensions of the membrane,

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}},$$

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy.$$

Applying these formulas in our case, we find

$$\lambda_{mn} = \pi\sqrt{m^2+n^2},$$

$$\begin{aligned} A_{mn} &= 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin m\pi x\,\sin n\pi y\,dx\,dy \\ &= \begin{cases} \frac{64}{(mn\pi^2)^3} & \text{if } m \text{ and } n \text{ are both odd,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

We have used the result of Exercise 3. Thus

$$u(x, y, t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{64\sin((2k+1)\pi x)\sin((2l+1)\pi y)}{((2k+1)(2l+1)\pi^2)^3}e^{-\left((2k+1)^2+(2l+1)^2\right)\pi^2 t}.$$

Solutions to Exercises 3.8

1. The solution is given by

$$u(x,\,y)=\sum_{n=1}^{\infty}B_n\sin(n\pi x)\sinh(n\pi y),$$

where

$$\begin{aligned}B_n &= \frac{2}{\sinh(2n\pi)}\int_0^1 x\sin(n\pi x)\,dx\\ &= \frac{2}{\sinh(2n\pi)}\left[-\frac{x\,\cos(n\,\pi\,x)}{n\,\pi}+\frac{\sin(n\,\pi\,x)}{n^2\,\pi^2}\right]\Big|_0^1\\ &= \frac{2}{\sinh(2n\pi)}\frac{-(-1)^n}{n\,\pi}=\frac{2}{\sinh(2n\pi)}\frac{(-1)^{n+1}}{n\,\pi}.\end{aligned}$$

Thus,

$$u(x,\,y)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sinh(2n\pi)}\sin(n\pi x)\sinh(n\pi y).$$

3. The solution is given by

$$u(x,\,y)=\sum_{n=1}^{\infty}A_n\sin(\frac{n\pi x}{2})\sinh(\frac{n\pi}{2}(1-y))+\sum_{n=1}^{\infty}D_n\sin(n\pi y)\sinh(n\pi x),$$

where

$$\begin{aligned}A_n &= \frac{2}{\sinh(\frac{n\pi}{2})}\int_0^2 100\sin(\frac{n\pi x}{2})\,dx\\ &= \frac{2}{\sinh(\frac{n\pi}{2})}\left[-\frac{200}{n\pi}\cos\frac{n\pi x}{2}\right]\Big|_0^2\\ &= \frac{200}{n\pi}(1-\cos(-1)^n)\\ &= \begin{cases}\frac{400}{n\pi} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even;}\end{cases}\end{aligned}$$

and

$$\begin{aligned}D_n &= \frac{2}{\sinh(2n\pi)}\int_0^1 100(1-y)\sin(n\pi y)\,dy\\ &= \frac{200}{\sinh(2n\pi)}\left[\frac{(-1+y)\,\cos(n\,\pi\,y)}{n\,\pi}-\frac{\sin(n\,\pi\,y)}{n^2\,\pi^2}\right]\Big|_0^1\\ &= \frac{200}{n\pi\sinh(2n\pi)}\end{aligned}$$

Thus,

$$u(x,\,y)=\frac{400}{\pi}\sum_{k=0}^{\infty}\frac{\sin(\frac{(2k+1)\pi x}{2})\sinh(\frac{(2k+1)\pi}{2}(1-y))}{(2k+1)\sinh(\frac{(2k+1)\pi}{2})}+\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi y)\sinh(n\pi x)}{n\pi\sinh(2n\pi)}.$$

7. (a) From Example 1, the solution is

$$u(x,\,y)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a},$$

where

$$\begin{aligned} B_n &= \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f_2(x)\sin\frac{n\pi x}{a}\,dx \\ &= \frac{1}{\sinh(\frac{n\pi b}{a})}\overbrace{\frac{2}{a}\int_0^a f_2(x)\sin\frac{n\pi x}{a}\,dx}^{b_n} \\ &= \frac{1}{\sinh(\frac{n\pi b}{a})}b_n, \end{aligned}$$

where b_n is the sine Fourier coefficient of $f_2(x)$, as given by (3), Section 2.4. Replacing B_n by its value in terms of b_n in the series solution, we obtain (a). In the general case, we refer the series solution (9). It is easy to see, as we just did, that

$$\begin{aligned} A_n &= \frac{1}{\sinh(\frac{n\pi b}{a})}\overbrace{\frac{2}{a}\int_0^a f_1(x)\sin\frac{n\pi x}{a}\,dx}^{b_n(f_1)} = \frac{1}{\sinh(\frac{n\pi b}{a})}b_n(f_1), \\ C_n &= \frac{1}{\sinh(\frac{n\pi a}{b})}\overbrace{\frac{2}{b}\int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy}^{b_n(g_1)} = \frac{1}{\sinh(\frac{n\pi a}{b})}b_n(g_1), \\ D_n &= \frac{1}{\sinh(\frac{n\pi a}{b})}\overbrace{\frac{2}{b}\int_0^b g_2(y)\sin\frac{n\pi y}{b}\,dy}^{b_n(g_2)} = \frac{1}{\sinh(\frac{n\pi a}{b})}b_n(g_2), \end{aligned}$$

where, $b_n(f)$ denotes the nth sine Fourier coefficient of f. So, in terms of the Fourier coefficients of the boundary functions, the general solution (9) becomes

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} b_n(f_1)\sin\frac{n\pi}{a}x\frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}} + \sum_{n=1}^{\infty} b_n(f_2)\sin\frac{n\pi}{a}x\frac{\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi b}{a}} \\ &\quad \sum_{n=1}^{\infty} b_n(g_1)\sin\frac{n\pi}{b}y\frac{\sinh\frac{n\pi}{b}(a-x)}{\sinh\frac{n\pi a}{b}} + \sum_{n=1}^{\infty} b_n(g_2)\sin\frac{n\pi}{b}y\frac{\sinh\frac{n\pi}{b}x}{\sinh\frac{n\pi a}{b}}. \end{aligned}$$

Solutions to Exercises 3.9

1. We apply (2), with $a = b = 1$:

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y,$$

where

$$\begin{aligned}
E_{mn} &= \frac{-4}{\pi^2(m^2+n^2)} \int_0^1 \int_0^1 x \sin m\pi x \sin n\pi y \, dx \, dy \\
&= \frac{-4}{\pi^2(m^2+n^2)} \int_0^1 x \sin m\pi x \, dx \overbrace{\int_0^1 \sin n\pi y \, dy}^{=\frac{1-(-1)^n}{n\pi}} \\
&= \frac{-4}{\pi^4(m^2+n^2)} \frac{1-(-1)^n}{n} \left(-\frac{x \cos(m\,\pi\,x)}{m} + \frac{\sin(m\,x)}{m^2 \pi} \right) \Big|_0^1 \\
&= \frac{4}{\pi^4(m^2+n^2)} \frac{1-(-1)^n}{n} \frac{(-1)^m}{m}.
\end{aligned}$$

Thus

$$u(x, y) = \frac{8}{\pi^4} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m^2+(2k+1)^2)m(2k+1)} \sin m\pi x \sin((2k+1)\pi y).$$

3. We use the solution in the text:

$$u(x, y) = u_1(x, y) + u_2(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y + u_2(x, y),$$

where u_1 is the solution of an associated Poisson problem with zero boundary data, and u_2 is the solution of the Dirichlet problem with the given boundary data. We have

$$\begin{aligned}
E_{mn} &= \frac{-4}{\pi^2(m^2+n^2)} \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin n\pi y \, dx \, dy \\
&= \frac{-4}{\pi^2(m^2+n^2)} \overbrace{\int_0^1 \sin \pi x \sin m\pi x \, dx}^{0 \text{ if } m \neq 1,\ 1/2 \text{ if } m=1} \overbrace{\int_0^1 \sin n\pi y \, dy}^{=\frac{1-(-1)^n}{n\pi}} \\
&= \begin{cases} 0 & \text{if } m \neq 1, \\ \frac{-2}{\pi^3(1+n^2)} \frac{(1-(-1)^n)}{n} & \text{if } m = 1 \end{cases}.
\end{aligned}$$

Thus

$$u_1(x, y) = \frac{-4}{\pi^3} \sin(\pi x) \sum_{k=0}^{\infty} \frac{1}{(1+(2k+1)^2)(2k+1)} \sin((2k+1)\pi y).$$

To find u_2, we apply the results of Section 3.8. We have

$$\sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y,$$

where

$$\begin{aligned} B_n &= \frac{2}{\sinh(n\pi)} \int_0^1 x \sin n\pi x \, dx \\ &= \frac{2}{\sinh(n\pi)} \frac{(-1)^{n+1}}{n\pi}. \end{aligned}$$

Thus

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(n\pi)} \sin n\pi x \sinh n\pi y,$$

and so

$$\begin{aligned} u(x, y) &= \frac{-4}{\pi^3} \sin(\pi x) \sum_{k=0}^{\infty} \frac{1}{(1 + (2k+1)^2)(2k+1)} \sin((2k+1)\pi y) \\ &\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(n\pi)} \sin n\pi x \sinh n\pi y. \end{aligned}$$

5. We will use an eigenfunction expansion based on the eigenfunctions $\phi(x, y) = \sin m\pi x \sin n\pi y$, where $\Delta\pi(x, y) = -\pi^2 (m^2 + n^2) \sin m\pi x \sin n\pi y$. So plug

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y$$

into the equation $\Delta u = 3u - 1$, proceed formally, and get

$$\begin{gathered} \Delta \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y \right) = 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1 \\ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \Delta \left(\sin m\pi x \sin n\pi y \right) = 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1 \\ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} -E_{mn} \pi^2 (m^2 + n^2) \sin m\pi x \sin n\pi y \\ = 3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1 \\ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(3 + \pi^2 (m^2 + n^2)\right) E_{mn} \sin m\pi x \sin n\pi y = 1. \end{gathered}$$

Thinking of this as the double sine series expansion of the function identically 1, it follows that $\left(3 + \pi^2 (m^2 + n^2)\right) E_{mn}$ are double Fourier sine coefficients, given by (see (8), Section 3.7)

$$\begin{aligned} \left(3 + \pi^2 (m^2 + n^2)\right) E_{mn} &= 4 \int_0^1 \int_0^1 \sin m\pi x \sin n\pi y \, dx \, dy \\ &= 4 \frac{1 - (-1)^m}{m\pi} \frac{1 - (-1)^n}{n\pi} \\ &= \begin{cases} 0 & \text{if either } m \text{ or } n \text{ is even,} \\ \frac{16}{\pi^2 m n} & \text{if both } m \text{ and } n \text{ are even.} \end{cases} \end{aligned}$$

Thus

$$E_{mn} = \begin{cases} 0 & \text{if either } m \text{ or } n \text{ is even,} \\ \frac{16}{\pi^2 m n \left(3 + \pi^2 (m^2 + n^2)\right)} & \text{if both } m \text{ and } n \text{ are even,} \end{cases}$$

and so

$$u(x, y) = \frac{16}{\pi^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\sin((2k+1)\pi x) \sin((2l+1)\pi y)}{(2k+1)\,(2l+1)\left(3 + \pi^2 \left((2k+1)^2 + (2l+1)^2\right)\right)}.$$

8. Following the outline in the text, leading to the solution (21), we first compute

$$b_m(y) = 2\int_0^1 x \sin m\pi x\, dx = \frac{2(-1)^{m+1}}{m\pi};$$

then

$$u(x,\, y) = \sum_{m=1}^{\infty} E_m(y) \sin m\pi x,$$

where

$$\begin{aligned}
E_m(y) &= \frac{-1}{m\pi \sinh(m\pi)}\Big[\sinh(m\pi(1-y)) \int_0^y \sinh(m\pi s) \frac{2(-1)^{m+1}}{m\pi}\, ds + \\
&\quad \sinh(m\pi y) \int_y^1 \sinh(m\pi(1-s)) \frac{2(-1)^{m+1}}{m\pi}\, ds\Big] \\
&= \frac{2(-1)^m}{m^2\pi^2 \sinh(m\pi)}\Big[\sinh(m\pi(1-y)) \overbrace{\int_0^y \sinh(m\pi s)\, ds}^{\frac{1}{m\pi}(\cosh(m\pi y)-1)} \\
&\quad + \sinh(m\pi y) \overbrace{\int_y^1 \sinh(m\pi(1-s))\, ds}^{\frac{1}{m\pi}(\cosh(m\pi(1-y))-1)}\Big] \\
&= \frac{2(-1)^m}{m^3\pi^3 \sinh(m\pi)}\Big[\sinh(m\pi(1-y))(\cosh(m\pi y) - 1) \\
&\quad + \sinh(m\pi y)(\cosh(m\pi(1-y)) - 1)\Big] \\
&= \frac{2(-1)^m}{m^3\pi^3 \sinh(m\pi)}\Big[\sinh(m\pi) - \sinh(m\pi(1-y)) - \sinh(m\pi y)\Big],
\end{aligned}$$

where in the last step we used the identity $\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b$. So

$$u(x,\, y) = \frac{2}{\pi^3} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^3 \sinh(m\pi)} \big(\sinh(m\pi) - \sinh(m\pi(1-y)) - \sinh(m\pi y)\big) \sin m\pi x.$$

Solutions to Exercises 3.10

1. We use a combination of solutions from (2) and (3) and try a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} \sin mx \big[A_m \cosh m(1-y) + B_m \sinh my\big].$$

(If you have tried a different form of the solution, you can still do the problem, but your answer may look different from the one derived here. The reason for our choice is to simplify the computations that follow.) The boundary conditions on the vertical sides are clearly satisfied. We now determine A_m and B_m so as to satisfy the conditions on the other sides. Starting with $u(1, 0) = 100$, we find that

$$100 = \sum_{m=1}^{\infty} A_m \cosh m \sin mx.$$

Thus $A_m \cosh m$ is the sine Fourier coefficient of the function $f(x) = 100$. Hence

$$A_m \cosh m = \frac{2}{\pi}\int_0^{\pi} 100 \sin mx\, dx \quad \Rightarrow \quad A_m = \frac{200}{\pi m \cosh m}\left[1-(-1)^m\right].$$

Using the boundary condition $u_y(x, 1) = 0$, we find

$$0 = \sum_{m=1}^{\infty} \sin mx \big[A_m(-m)\sinh[m(1-y)] + mB_m \cosh my\big]\Big|_{y=1}.$$

Thus

$$0 = \sum_{m=1}^{\infty} mB_m \sin mx \cosh m.$$

By the uniqueness of Fourier series, we conclude that $mB_m \cosh m = 0$ for all m. Since $m \cosh m \neq 0$, we conclude that $B_m = 0$ and hence

$$\begin{aligned} u(x, y) &= \frac{200}{\pi}\sum_{m=1}^{\infty} \frac{[1-(-1)^m]}{m \cosh m} \sin mx \cosh[m(1-y)] \\ &= \frac{400}{\pi}\sum_{k=0}^{\infty} \frac{\sin[(2k+1)x]}{(2k+1)\cosh(2k+1)} \cosh[(2k+1)(1-y)]. \end{aligned}$$

2. Try a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} \sin mx \big[A_m \cosh m(1-y) + B_m \sinh my\big].$$

Using the boundary conditions on the horizontal sides, starting with $u(x, 0) = 100$, we find that

$$100 = \sum_{n=1}^{\infty} A_m \cosh m \sin mx.$$

Thus $A_m \cosh m$ is the sine Fourier coefficient of the function $f(x) = 100$ (as in the previous exercise). Hence

$$A_m = \frac{200}{\pi m \cosh m}\left[1 - (-1)^m\right].$$

Using the boundary condition $u_y(x, 1) = \sin 2x$, we find

$$\sin 2x = \sum_{n=1}^{\infty} m B_m \sin mx \cosh m.$$

By the uniqueness of Fourier series, we conclude that $mB_m \cosh m = 0$ for all $m \neq 2$ and $2B_2 \cosh 2 = 1$ or $B_2 = \frac{1}{2\cosh 2}$. Thus

$$\begin{aligned} u(x, y) &= \frac{1}{2\cosh 2} \sin(2x) \sinh(2y) \\ &\quad + \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)x]}{(2k+1)\cosh(2k+1)} \cosh[(2k+1)(1-y)]. \end{aligned}$$

3. Try a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} \sin mx \big[A_m \cosh m(1-y) + B_m \sinh my\big].$$

Using the boundary conditions on the horizontal sides, starting with $u_y(x, 1) = 0$, we find

$$0 = \sum_{n=1}^{\infty} m B_m \sin mx \cosh m.$$

Thus $B_m = 0$ for all m and so

$$u(x, y) = \sum_{n=1}^{\infty} \sin mx A_m \cosh m(1-y).$$

We now work with the Robin condition $u(x, 0) + 2u_y(x, 0) = g(x)$ and get

$$\begin{aligned} &\sum_{n=1}^{\infty} \sin mx A_m \cosh m - 2\sum_{n=1}^{\infty} \sin mx A_m m \sinh m = g(x) \\ &\Rightarrow \quad g(x) = \sum_{n=1}^{\infty} A_m \sin mx \big[\cosh m - 2m \sinh m\big] \\ &\Rightarrow \quad A_m \big[\cosh m - 2m \sinh m\big] = \frac{2}{\pi} \int_0^{\pi} g(x) \sin mx \, dx \\ &\Rightarrow \quad A_m = \frac{2}{\pi[\cosh m - 2m \sinh m]} \int_0^{\pi} g(x) \sin mx \, dx, \end{aligned}$$

which determines the solution.

5. We combine solutions of different types from Exercise 4 and try a solution of the form

$$u(x, y) = A_0 + B_0 y + \sum_{m=1}^{\infty} \cos \frac{m\pi}{a} x \big[A_m \cosh[\frac{m\pi}{a}(b-y)] + B_m \sinh[\frac{m\pi}{a} y]\big].$$

Using the boundary conditions on the horizontal sides, starting with $u_y(x, b) = 0$, we find that

$$0 = B_0 + \sum_{m=1}^{\infty} \frac{m\pi}{a} B_m \cos \frac{m\pi}{a} x \cosh \left[\frac{m\pi}{a} b\right].$$

Thus $B_0 = 0$ and $B_m = 0$ for all $m \geq 1$ and so

$$A_0 + \sum_{m=1}^{\infty} A_m \cos \frac{m\pi}{a} x \cosh[\frac{m\pi}{a}(b - y)].$$

Now, using $u(x, 0) = g(x)$, we find

$$g(x) = A_0 + \sum_{m=1}^{\infty} A_m \cosh[\frac{m\pi}{a} b] \cos \frac{m\pi}{a} x.$$

Recognizing this as a cosine series, we conclude that

$$A_0 = \frac{1}{a} \int_0^a g(x)\, dx$$

and

$$A_m \cosh[\frac{m\pi}{a} b] = \frac{2}{a} \int_0^a g(x) \cos \frac{m\pi}{a} x\, dx;$$

equivalently, for $m \geq 1$,

$$A_m = \frac{2}{a \cosh[\frac{m\pi}{a} b]} \int_0^a g(x) \cos \frac{m\pi}{a} x\, dx.$$

7. The solution is immediate form (5) and (6). We have

$$u(x, y) = \sum_{m=1}^{\infty} B_m \sin mx \sinh my,$$

where B_m are given by (6) with $f(x) = \sin 2x$. By the orthogonality of the trigonometric system, we have $B_m = 0$ for all $m \neq 2$ and $B_2 = \frac{1}{2\cosh(2\pi)}$. Thus

$$u(x, y) = \frac{1}{2\cosh(2\pi)} \sin 2x \sinh 2y.$$

9. We follow the solution in Example 3. We have

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where

$$u_1(x, y) = \sum_{m=1}^{\infty} B_m \sin mx \sinh my,$$

with

$$B_m = \frac{2}{\pi m \cosh(m\pi)} \int_0^{\pi} \sin mx\, dx = \frac{2}{\pi m^2 \cosh(m\pi)} (1 - (-1)^m);$$

and

$$u_2(x, y) = \sum_{m=1}^{\infty} A_m \sin mx \cosh[m(\pi - y)],$$

with

$$A_m = \frac{2}{\pi \cosh(m\pi)} \int_0^\pi \sin mx \, dx = \frac{2}{\pi m \cosh(m\pi)}(1 - (-1)^m).$$

Hence

$$u(x, y) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(1 - (-1)^m)}{m \cosh(m\pi)} \sin mx \left[\frac{\sinh my}{m} + \cosh[m(\pi - y)] \right]$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{(2k+1)\cosh[(2k+1)\pi]} \left[\frac{\sinh[(2k+1)y]}{(2k+1)} + \cosh[(2k+1)(\pi - y)] \right].$$

Solutions to Exercises 4.1

1. We could use Cartesian coordinates and compute u_x, u_y, u_{xx}, and u_{yy} directly from the definition of u. Instead, we will use polar coordinates, because the expression $x^2 + y^2 = r^2$, simplifies the denominator, and thus it is easier to take derivatives. In polar coordinates,

$$u(x, y) = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} = r^{-1}\cos\theta\,.$$

So

$$u_r = -r^{-2}\cos\theta, \quad u_{rr} = 2r^{-3}\cos\theta, \quad u_\theta = -r^{-1}\sin\theta, \quad u_{\theta\theta} = -r^{-1}\cos\theta.$$

Plugging into (1), we find

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{2\cos\theta}{r^3} - \frac{\cos\theta}{r^3} - \frac{\cos\theta}{r^3} = 0 \quad (\text{if } r \neq 0).$$

If you used Cartesian coordinates, you should get

$$u_{xx} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^2} \quad \text{and} \quad u_{yy} = -\frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^2}.$$

2. Reasoning as in Exercises 1, we will use polar coordinates:

$$u(x, y) = \frac{y}{x^2 + y^2} = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r} = r^{-1}\sin\theta\,.$$

So

$$u_r = -r^{-2}\sin\theta, \quad u_{rr} = 2r^{-3}\sin\theta, \quad u_\theta = r^{-1}\cos\theta, \quad u_{\theta\theta} = -r^{-1}\sin\theta.$$

Plugging into (1), we find

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{2\sin\theta}{r^3} - \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} = 0 \quad (\text{if } r \neq 0).$$

3. In polar coordinates:

$$u(r, \theta) = \frac{1}{r} \Rightarrow u_r = -\frac{1}{r^2},\; u_{rr} = \frac{2}{r^3},\; u_\theta = 0,\; u_{\theta\theta} = 0.$$

Plugging into (1), we find

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{2}{r^3} - \frac{1}{r^3} = \frac{1}{r^3} \quad (\text{for } r \neq 0).$$

Thus u does not satisfy Laplace's equation.

4. In cylindrical coordinates:

$$u(\rho, \phi, z) = \frac{z}{\rho} \Rightarrow u_{\rho\rho} = \frac{2z}{r^3},\; u_{\phi\phi} = 0,\; u_{zz} = 0.$$

Plugging into (2), we find

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2z}{r^3} - \frac{z}{r^3} + 0 + 0 = \frac{z}{r^3} \quad (\text{for } \rho \neq 0).$$

5. In spherical coordinates:

$$u(r,\,\theta,\,\phi) = r^3 \Rightarrow u_{rr} = 6r,\; u_\theta = 0, u_{\theta\theta} = 0,\; u_{\phi\phi} = 0.$$

Plugging into (3), we find

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\,\frac{\partial u}{\partial \theta} + \csc^2\theta\,\frac{\partial^2 u}{\partial \phi^2}\Big) = 6r + 6r = 12r.$$

6. In polar coordinates:

$$\begin{aligned} u(x,\,y) &= \ln(x^2+y^2) = 2\ln r;\\ \nabla^2 u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\\ &= -\frac{2}{r^2} + \frac{2}{r^2} + 0 = 0 \quad (\text{for } r \neq 0). \end{aligned}$$

7. In spherical coordinates:

$$u(r,\,\theta,\,\phi) = r^{-1} \Rightarrow u_r = -\frac{1}{r^2},\; u_{rr} = \frac{2}{r^3},\; u_\theta = 0, u_{\theta\theta} = 0,\; u_{\phi\phi} = 0.$$

Plugging into (3), we find

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\,\frac{\partial u}{\partial \theta} + \csc^2\theta\,\frac{\partial^2 u}{\partial \phi^2}\Big) = \frac{2}{r^3} - \frac{2}{r^3} = 0 \quad (\text{for } r \neq 0).$$

8. In polar coordinates:

$$\begin{aligned} u(r,\,\theta) = \frac{\theta\sin\theta}{r} \quad \Rightarrow \quad & u_r = -\frac{\theta\sin\theta}{r^2},\; u_{rr} = \frac{2\theta\sin\theta}{r^3},\\ & u_\theta = \frac{\sin\theta + \theta\cos\theta}{r},\; u_{\theta\theta} = \frac{2\cos\theta - \theta\sin\theta}{r}. \end{aligned}$$

Plugging into (1), we find

$$\nabla^2 u = \frac{2\theta\sin\theta}{r^3} - \frac{\theta\sin\theta}{r^3} + \frac{2\cos\theta - \theta\sin\theta}{r^3} = \frac{2\cos\theta}{r^3} \quad (\text{for } r \neq 0).$$

Thus u does not satisfy Laplace's equation.

9. (a) If $u(r,\theta,\phi)$ depends only on r, then all partial derivatives of u with respect to θ and ϕ are 0. So (3) becomes

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\,\frac{\partial u}{\partial \theta} + \csc^2\theta\,\frac{\partial^2 u}{\partial \phi^2}\Big) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}.$$

(b) If $u(r,\theta,\phi)$ depends only on r and θ, then all partial derivatives of u with respect to ϕ are 0. So (3) becomes

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\,\frac{\partial u}{\partial \theta}\Big).$$

10. From (7), we have

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi}.$$

Now $\rho = r\sin\theta$, $r = (\rho^2 + z^2)^{\frac{1}{2}}$, $z = r\cos\theta$, $\theta = \tan^{-1}\frac{\rho}{z}$. Hence, by the chain rule in two dimensions,

$$u_\rho = u_r r_\rho + u_\theta \theta_\rho = \frac{\rho}{r} u_r + \frac{z}{\rho^2 + z^2} u_\theta = \sin\theta u_r + \frac{\cos\theta}{r} u_\theta.$$

So

$$\begin{aligned} \nabla^2 u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r\sin\theta}\left[\sin\theta u_r + \frac{\cos\theta}{r} u_\theta\right] + \frac{1}{r^2\sin^2\theta} u_{\phi\phi} \\ &= u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left[u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\phi\phi}\right], \end{aligned}$$

which is (8).

11. (a) This part follows from the fact that differentiation is a linear operation. Hence

$$\nabla^2(\alpha u + \beta v) = \alpha\nabla^2(u) + \beta\nabla^2(v).$$

So if $\nabla^2 u = 0$ and $\nabla^2 v = 0$, then $\nabla^2(\alpha u + \beta v) = 0$, implying that $\alpha u + \beta v$ is harmonic.
(b) Take $u(x, y) = x$ and $v(x, y) = x$. Clearly, $u_{xx} = u_{yy} = 0$ so u and hence v are harmonic. But $uv = x^2$; so $(uv)_{xx} = 2$ and $(uv)_{yy} = 0$. Hence $\nabla^2(uv) = 2$, implying that uv is not harmonic.
(c) If u is harmonic then $u_{xx} + u_{yy} = 0$. We have

$$(u^2)_{xx} = (2uu_x)_x = 2(u_x)^2 + 2uu_{xx} \quad \text{and} \quad (u^2)_{yy} = (2uu_y)_y = 2(u_y)^2 + 2uu_{yy}.$$

If u^2 is also harmonic, then

$$\begin{aligned} 0 = (u^2)_{xx} + (u^2)_{yy} &= 2(u_x)^2 + 2uu_{xx} + 2(u_y)^2 + 2uu_{yy} \\ &= 2((u_x)^2 + (u_y)^2) + 2u\overbrace{(u_{xx} + u_{yy})}^{=0} \\ &= 2((u_x)^2 + (u_y)^2), \end{aligned}$$

which implies that both u_x and u_y are identically 0. This latter condition implies that u is constant on its domain of definition, if the domain is connected. This result is not true if the domain of the function u is not connected. For example, suppose that $u(x, y) = 1$ if $x > 1$ and $u(x, y) = -1$ if $x < -1$. Clearly, u is not constant and both u and u^2 are harmonic. (For the definition of connected set, see my book "Applied Complex Analysis and Partial Differential Equations.") Examples of connected sets are any disk, the plane, the upper half-plane, any triangular region any annulus.
(d) This part is similar to (c). We suppose that the domain of definition of the functions is connected. (Here again, the result is not true if the domain is not connected.) Suppose that u, v, and $u^2 + v^2$ are harmonic. Then

$$u_{xx} + u_{yy} = 0; \qquad v_{xx} + v_{yy} = 0;$$

$$2((u_x)^2 + (u_y)^2) + 2u\overbrace{(u_{xx} + u_{yy})}^{=0} + 2((v_x)^2 + (v_y)^2) + 2v\overbrace{(v_{xx} + v_{yy})}^{=0}.$$

The last equation implies that $(u_x)^2 + (u_y)^2 + (v_x)^2 + (v_y)^2 = 0$, which, in turn, implies that $u_x = u_y = v_x = v_y = 0$. Hence u and v are constant.

Solutions to Exercises 4.2

1. We appeal to the solution (5) with the coefficients (6). Since $f(r) = 0$, then $A_n = 0$ for all n. We have

$$\begin{aligned} B_n &= \frac{1}{\alpha_n J_1(\alpha_n)^2} \int_0^2 J_0(\frac{\alpha_n r}{2}) r\, dr \\ &= \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s) s\, ds \quad (\text{let } s = \frac{\alpha_n}{2} r) \\ &= \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} [s J_1(s)] \Big|_0^{\alpha_n} \\ &= \frac{4}{\alpha_n^2 J_1(\alpha_n)} \quad \text{for all } n \geq 1. \end{aligned}$$

Thus

$$u(r, t) = 4 \sum_{n=1}^{\infty} \frac{J_0(\frac{\alpha_n r}{2})}{\alpha_n^2 J_1(\alpha_n)} \sin(\frac{\alpha_n t}{2}).$$

2. We appeal to the solution (5) with the coefficients (6). Since $f(r) = 1 - r^2$, we can use the result of Example 2 and get

$$A_n = \frac{8}{\alpha_n^3 J_1(\alpha_n)}.$$

For the B_n's, we can compute as in Exercise 1 and get

$$\begin{aligned} B_n &= \frac{1}{5\alpha_n J_1(\alpha_n)^2} \int_0^1 J_0(\alpha_n r) r\, dr \\ &= \frac{1}{5\alpha_n^3 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s) s\, ds \quad (\text{let } s = \alpha_n r) \\ &= \frac{1}{5\alpha_n^3 J_1(\alpha_n)^2} [s J_1(s)] \Big|_0^{\alpha_n} \\ &= \frac{1}{5\alpha_n^2 J_1(\alpha_n)} \quad \text{for all } n \geq 1. \end{aligned}$$

Thus

$$u(r, t) = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{5\alpha_n^3 J_1(\alpha_n)} \left[40 \cos(10\alpha_n t) + \alpha_n \sin(10\alpha_n t)\right].$$

3. As in Exercise 1, $f(r) = 0 \;\Rightarrow\; A_n = 0$ for all n. We have

$$\begin{aligned} B_n &= \frac{2}{\alpha_n J_1(\alpha_n)^2} \int_0^{\frac{1}{2}} J_0(\alpha_n r) r\, dr \\ &= \frac{2}{\alpha_n^3 J_1(\alpha_n)^2} \int_0^{\alpha_n/2} J_0(s) s\, ds \\ &= \frac{2}{\alpha_n^3 J_1(\alpha_n)^2} [s J_1(s)] \Big|_0^{\alpha_n/2} \\ &= \frac{J_1(\frac{\alpha_n}{2})}{\alpha_n^2 J_1(\alpha_n)^2} \quad \text{for all } n \geq 1. \end{aligned}$$

Thus

$$u(r, t) = \sum_{n=1}^{\infty} \frac{J_1(\frac{\alpha_n}{2})}{\alpha_n^2 J_1(\alpha_n)^2} J_0(\alpha_n r) \sin(\alpha_n t).$$

4. As in Exercise 1, $f(r) = 0 \;\Rightarrow\; A_n = 0$ for all n. We have

$$B_n = \frac{2}{\alpha_n J_1(\alpha_n)^2} \int_0^1 J_0(\alpha_3 r) J_0(\alpha_n r) r\, dr = 0 \quad \text{for } n \neq 3 \text{ by orthogonality.}$$

For $n = 3$,

$$B_3 = \frac{2}{\alpha_3 J_1(\alpha_3)^2} \int_0^1 J_0(\alpha_3 r)^2 r\,dr = \frac{2}{\alpha_3 J_1(\alpha_3)^2} \frac{1}{2} J_1(\alpha_3)^2 = \frac{1}{\alpha_3},$$

where we have used the orthogonality relation (12), Section 4.8, with $p = 0$. Thus

$$u(r, t) = \frac{1}{\alpha_3} J_0(\alpha_3 r) \sin(\alpha_3 t).$$

5. Since $g(r) = 0$, we have $B_n = 0$ for all n. We have

$$A_n = \frac{2}{J_1(\alpha_n)^2} \int_0^1 J_0(\alpha_1 r) J_0(\alpha_n r) r\,dr = 0 \quad \text{for } n \neq 1 \text{ by orthogonality.}$$

For $n = 1$,

$$A_1 = \frac{2}{J_1(\alpha_1)^2} \int_0^1 J_0(\alpha_1 r)^2 r\,dr = 1,$$

where we have used the orthogonality relation (12), Section 4.8, with $p = 0$. Thus

$$u(r, t) = J_0(\alpha_1 r) \cos(\alpha_1 t).$$

6. $a = 2$, $c = 1$, $f(r) = 1 - r$, and $g(r) = 0$. We have

$$g(r) = 0 \quad \Rightarrow \quad B_n = 0 \text{ for all } n;$$

Also,

$$\begin{aligned}
A_n &= \frac{1}{2J_1(\alpha_n)^2} \int_0^2 (1 - r) J_0(\alpha_n r/2) r\,dr \\
&= \frac{2}{\alpha_n^2 J_1(\alpha_n)^2} \int_0^{\alpha_n} (1 - \frac{2s}{\alpha_n}) J_0(s) s\,ds \quad (\text{let } s = \alpha_n r/2) \\
&= \frac{2}{\alpha_n^2 J_1(\alpha_n)^2} \left[sJ_1(s)\Big|_0^{\alpha_n} - \frac{2}{\alpha_n} \int_0^{\alpha_n} s^2 J_1(s)\,ds \right] \\
&= \frac{2}{\alpha_n J_1(\alpha_n)} - \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} \left[s^2 J_1(s)\Big|_0^{\alpha_n} - \int_0^{\alpha_n} sJ_1(s)\,ds \right] \\
&= \frac{-2}{\alpha_n J_1(\alpha_n)} + \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} \int_0^{\alpha_n} sJ_1(s)\,ds \\
&= \frac{-2}{\alpha_n J_1(\alpha_n)} - \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} \int_0^{\alpha_n} sJ_0'(s)\,ds \\
&= \frac{-2}{\alpha_n J_1(\alpha_n)} - \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} \left[sJ_0(s)\Big|_0^{\alpha_n} - \int_0^{\alpha_n} J_0(s)\,ds \right] \\
&= \frac{-2}{\alpha_n J_1(\alpha_n)} + \frac{4}{\alpha_n^3 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s)\,ds
\end{aligned}$$

Thus

$$u(r, t) = 2 \sum_{n=1}^{\infty} \frac{2 \int_0^{\alpha_n} J_0(s)\,ds - \alpha_n^2 J_1(\alpha_n)}{\alpha_n^3 J_1(\alpha_n)^2} J_0\left(\frac{\alpha_n r}{2}\right) \cos\left(\frac{\alpha_n t}{2}\right).$$

7. Using orthogonality as in Exercise 5, since $f(r) = J_0(\alpha_3\, r)$, we find that $A_n = 0$ for all $n \neq 3$ and $A_3 = 1$. For the B_n's, we have

$$\begin{aligned}
B_n &= \frac{2}{\alpha_n J_1(\alpha_n)^2} \int_0^1 (1 - r^2) J_0(\alpha_n r) r\,dr \\
&= \frac{2}{\alpha_n^5 J_1(\alpha_n)^2} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s) s\,ds \quad (\text{let } s = \alpha_n r).
\end{aligned}$$

The last integral can be evaluated by parts: $u=\alpha_n^2-s^2,\ du=-2s,\ J_0(s)s\,ds=dv,\ v=sJ_1(s)$. So

$$\begin{aligned}
B_n &= \frac{2}{\alpha_n^5 J_1(\alpha_n)^2}\left[(\alpha_n^2-s^2)sJ_1(s)\Big|_0^{\alpha_n}+2\int_0^{\alpha_n} s^2 J_1(s)\,ds\right]\\
&= \frac{4}{\alpha_n^5 J_1(\alpha_n)^2}\int_0^{\alpha_n} s^2 J_1(s)\,ds\\
&= \frac{4}{\alpha_n^5 J_1(\alpha_n)^2}s^2 J_2(s)\Big|_0^{\alpha_n} \quad \text{(by (7), Section 4.8)}\\
&= \frac{4J_2(\alpha_n)}{\alpha_n^3 J_1(\alpha_n)^2}\\
&= \frac{8}{\alpha_n^4 J_1(\alpha_n)} \quad \left(\text{using } J_2(\alpha_n)=\frac{2}{\alpha_n}J_1(\alpha_n)\right).
\end{aligned}$$

Thus

$$u(r,\,t)=J_0(\alpha_3 r)\cos(\alpha_3 t)+8\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)}{\alpha_n^4 J_1(\alpha_n)}\sin(\alpha_n t).$$

8. Since $g(r)=0$, we have $B_n=0$ for all n. We have

$$\begin{aligned}
A_n &= \frac{2}{J_1(\alpha_n)^2}\int_0^1 \frac{1}{128}(3-4r^2+r^4)J_0(\alpha_n r)r\,dr\\
&= \frac{1}{64\alpha_n^6 J_1(\alpha_n)^2}\int_0^{\alpha_n}(3\alpha_n^4-4\alpha_n^2 s^2+s^4)J_0(s)s\,ds \quad (\text{let } s=\alpha_n r).
\end{aligned}$$

We now integrate by parts: $u=3\alpha_n^4-4\alpha_n^2 s^2+s^4,\ du=4(s^3-2\alpha_n^2 s)\,ds,\ J_0(s)s\,ds=dv,\ v=sJ_1(s)$. Then

$$\begin{aligned}
A_n &= \frac{1}{64\alpha_n^6 J_1(\alpha_n)^2}\left[(3\alpha_n^4-4\alpha_n^2 s^2+s^4)sJ_1(s)\Big|_0^{\alpha_n}-4\int_0^{\alpha_n}(s^2-2\alpha_n^2)s^2 J_1(s)\,ds\right]\\
&= -\frac{1}{16\alpha_n^6 J_1(\alpha_n)^2}\int_0^{\alpha_n}(s^2-2\alpha_n^2)s^2 J_1(s)\,ds
\end{aligned}$$

Integrate by parts again: $u=s^2-2\alpha_n^2,\ du=2s\,ds,\ s^2J_1(s)\,ds=dv,\ v=s^2J_2(s)$. Then

$$\begin{aligned}
A_n &= -\frac{1}{16\alpha_n^6 J_1(\alpha_n)^2}\left[(s^2-2\alpha_n^2)s^2J_2(s)\Big|_0^{\alpha_n}-2\int_0^{\alpha_n}s^3J_2(s)\,ds\right]\\
&= -\frac{1}{16\alpha_n^6 J_1(\alpha_n)^2}\left[-\alpha_n^4 J_2(\alpha_n)-2\ s^3 J_3(s)\,ds\Big|_0^{\alpha_n}\right]\\
&= \frac{1}{16\alpha_n^3 J_1(\alpha_n)^2}\left[\alpha_n J_2(\alpha_n)+2J_3(\alpha_n)\right].
\end{aligned}$$

Now recall that

$$J_{p-1}(x)+J_{p+1}(x)=\frac{2p}{x}J_p(x).$$

Take $p=1$ and $x=\alpha_n$, and get

$$J_3(\alpha_n)=\frac{4}{\alpha_n}J_2(\alpha_n)-J_1(\alpha_n).$$

Hence $\alpha_n J_2(\alpha_n)+2J_3(\alpha_n)=\frac{8}{\alpha_n}J_2(\alpha_n)=\frac{16}{\alpha_n^2}J_1(\alpha_n)$. Therefore,

$$A_n=\frac{1}{\alpha_n^5 J_1(\alpha_n)}$$

and the solution becomes

$$u(r,\,t)=\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)}{\alpha_n^5 J_1(\alpha_n)}\cos(\alpha_n t).$$

9. (a) Modifying the solution of Exercise 3, we obtain

$$u(r,\, t) = \sum_{n=1}^{\infty} \frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2} J_0(\alpha_n r)\sin(\alpha_n ct).$$

(b) Under suitable conditions that allow us to interchange the limit and the summation sign (for example, if the series is absolutely convergent), we have, for a given $(r,\, t)$,

$$\begin{aligned}
\lim_{c\to\infty} u(r,\, t) &= \lim_{c\to\infty} \sum_{n=1}^{\infty} \frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2} J_0(\alpha_n r)\sin(\alpha_n ct) \\
&= \sum_{n=1}^{\infty} \lim_{c\to\infty} \frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2} J_0(\alpha_n r)\sin(\alpha_n ct) \\
&= 0,
\end{aligned}$$

because $\lim_{c\to\infty} \frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2} = 0$ and $\sin(\alpha_n ct)$ is bounded. If we let $u_1(r,\, t)$ denote the solution corresponding to $c = 1$ and $u_c(r,\, t)$ denote the solution for arbitrary $c > 0$. Then, it is easy t check that

$$u_c(r,\, t) = \frac{1}{c} u_1(r,\, ct).$$

This shows that if c increases, the time scale speeds proportionally to c, while the displacement decreases by a factor of $\frac{1}{c}$.

10. Separating variables, we look for solutions of the form $u(r,\, t) = R(r)T(t)$. Plugging this into the heat equation, we find

$$RT' = c^2\left[R''T + \frac{1}{r}R'T\right] \quad \Rightarrow \quad \frac{T'}{c^2T} = \frac{R''}{R} + \frac{1}{R}\frac{R'}{R} = -\lambda^2,$$

where λ^2 is a separation constant. (The choice of a nonnegative sign for the separation constant will be justified momentarily.) Hence we arrive at two ordinary differential equations:

$$\begin{gathered}
T' + \lambda^2 c^2 T = 0; \\
r^2 R'' + rR' + \lambda^2 r^2 R = 0.
\end{gathered}$$

Note that the equation in T is first order. Its solution is an exponential function

$$T(t) = Ce^{-c^2\lambda^2 t}.$$

If we had chosen a negative separation constant $-\lambda^2 < 0$, the solution in T becomes

$$T(t) = Ce^{c^2\lambda^2 t},$$

which blows up exponentially with $t > 0$. This does not make sense on physical grounds and so we discard this solution and the corresponding separation constant, and take the separation to be nonnegative. The boundary conditions on R are

$$R(0) \text{ is finite and } R(a) = 0.$$

The solution of the parametric form of Bessel's equation

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0$$

is

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r).$$

In order for $R(0)$ to be finite, we take $c_2 = 0$, because Y_0 is not bounded near zero and $J_0(0) = 1$. The boundary condition $R(a) = c_1 J_0(\lambda a) = 0$ implies that

$$\lambda a = \alpha_n \quad \Rightarrow \lambda = \lambda_n = \frac{\alpha_n}{a}, \; n = 1, 2, \ldots,$$

where α_n is the nth positive zero of J_0. Solving the corresponding equation in T, we find

$$T_n(t) = A_n e^{c^2 \lambda_n^2 t}.$$

Thus the product solutions are

$$A_n J_0(\lambda_n r) e^{c^2 \lambda_n^2 t}.$$

These satisfy the heat equation and the boundary conditions. In order to satisfy the given initial condition, we form a series solution by superposing the product solutions and take

$$u(r, t) = \sum_{n=1}^{\infty} A_n e^{-c^2 \lambda_n^2 t} J_0(\lambda_n r).$$

Setting $t = 0$, we find that

$$f(r) = u(r, 0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r).$$

Hence the A_n's are the Bessel Fourier coefficients of f and are given by

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r) r\, dr.$$

11. (a) With $f(r) = 100$, we have from Exercise 10,

$$\begin{aligned} A_n &= \frac{200}{a^2 J_1(\alpha_n)} \int_0^a J_0(\frac{\alpha_n r}{a} r) r\, dr \\ &= \frac{200}{\alpha_n^2 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s) s\, ds \quad (\text{let } \frac{\alpha_n r}{a} = s \\ &= \frac{200}{\alpha_n^2 J_1(\alpha_n)^2} J_1(s) s \Big|_0^{\alpha_n} = \frac{200}{\alpha_n J_1(\alpha_n)}. \end{aligned}$$

Hence

$$u(r, t) = 200 \sum_{n=1}^{\infty} \frac{J_0(\frac{\alpha_n r}{a})}{\alpha_n J_1(\alpha_n)} e^{-\frac{c^2 \alpha_n^2 t}{a^2}}.$$

The solution represents the time evolution of the temperature of the disk, starting with an initial uniform temperature distribution of 100° and with its boundary held at 0°. The top and bottom faces of the disk are assumed to be completely insulated so that no heat flows through them.

(b) The maximum value of the temperature should occur at the center, since heat is being lost only along the circumference of the disk. Of all the points in the disk, the center is least affected by this loss of heat. If we take $a = 1$, $c = 1$, and $t = 3$ in the solution to part (a), we obtain

$$u(r, 3) = 200 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)} e^{-3\alpha_n^2}.$$

To plot the solution, we need the zeros of $J_0(x)$. You can find these zeros by using the FindRoot command to solve $J_0(x)=0$ or, better yet, you can use the built-in values, as we now show. First, load the following package

```
<< NumericalMath‘BesselZeros‘
```

Here are the first 7 zeros of $J_0(x)$.

```
BesselJZeros[0, 7]
```

```
{2.40483, 5.52008, 8.65373, 11.7915, 14.9309, 18.0711, 21.2116}
```

To get the 3rd zero, you can use the following

```
BesselJZeros[0, 7][[3]]
```

```
8.65373
```

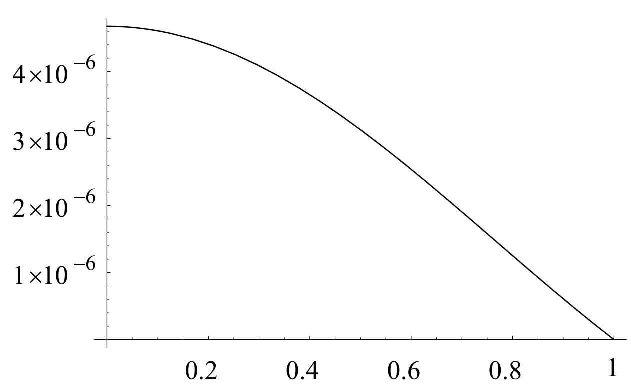
An approximation to our solution, using 2 terms of the series solution is

```
Clear[u]
u[r_] = 200 Sum[BesselJ[0, r BesselJZeros[0, 7][[j]]] /
      (BesselJZeros[0, 7][[j]] BesselJ[1, BesselJZeros[0, 7][[j]]])
    E^(-3 BesselJZeros[0, 7][[j]]^2), {j, 1, 2}]
```

200 (2.33781 × 10^{-8} BesselJ[0, 2.40483 r] − 1.06105 × 10^{-40} BesselJ[0, 5.52008 r])

Note that the coefficients are very small. We will not get a better approximation by adding more terms.

```
Plot[u[r], {r, 0, 1}]
```



12. (a) Take (8), Section 4.8, with $p = 1$. Then

$$\int J_1(x)\,dx = -J_0(x) + C.$$

Take (7), Section 4.8, with $p = 0$. Then

$$\int xJ_0(x)\,dx = x\,J_1(x) + C.$$

(b) From (5), Section 4.8, we obtain

$$J_{p+1}(x) = J_{p-1}(x) - 2J_p'(x).$$

Integrate both sides, then

$$\int J_{p+1}(x)\,dx = \int J_{p-1}(x)\,dx - 2J_p(x) + C.$$

(c) The formula is true for $n = 0$, since it reduces to

$$\int J_1(x)\,dx = -J_0(x) + C,$$

which is true by (a). Assume the formula is true for $n \geq 1$ and let us prove it for $n+1$. We have from (b), with $p = 2(n+1)$ and the induction hypothesis,

$$\begin{aligned}\int J_{2(n+1)+1}(x)\,dx &= \int J_{2(n+1)-1}(x)\,dx - 2J_{2n+2}(x) + C\\ &= -J_0(x) - 2\sum_{k=1}^{n} J_{2k}(x) - 2J_{2n+2}(x) + C\\ &= -J_0(x) - 2\sum_{k=1}^{n+1} J_{2k}(x) + C,\end{aligned}$$

which shows that formula holds for $n+1$ and thus it holds for all n by induction.

Take $n = 1$ and get

$$\begin{aligned}\int J_3(x)\,dx &= -J_0(x) - 2\sum_{k=1}^{1} J_{2k}(x) + C\\ &= -J_0(x) - 2J_2(x) + C.\end{aligned}$$

Take $n = 2$ and get

$$\begin{aligned}\int J_5(x)\,dx &= -J_0(x) - 2\sum_{k=1}^{2} J_{2k}(x) + C\\ &= -J_0(x) - 2J_2(x) - 2J_4(x)C.\end{aligned}$$

(d) Starting with (3), Section 4.8, we have

$$\begin{aligned}xJ_p'(x) + pJ_p(x) &= xJ_{p-1}(x)\\ \int xJ_p'(x)\,dx + p\int J_p(x)\,dx &= \int xJ_{p-1}(x)\,dx.\end{aligned}$$

Integrate by parts:

$$\int xJ_p'(x)\,dx = xJ_p(x) - \int J_p(x)\,dx.$$

So

$$\begin{aligned} xJ_p(x) - \int J_p(x)\,dx + p\int J_p(x)\,dx &= \int xJ_{p-1}(x)\,dx \\ xJ_p(x) + (p-1)\int J_p(x)\,dx &= \int xJ_{p-1}(x)\,dx \\ xJ_{p+1}(x) + p\int J_{p+1}(x)\,dx &= \int xJ_p(x)\,dx, \end{aligned}$$

where the last equality follows fromt eh previousone by changing p to $p+1$.
(e) From (d) and (c), we have

$$\begin{aligned} \int xJ_{2n}(x)\,dx &= xJ_{2n+1}(x) + 2n\int J_{2n+1}(x)\,dx + C \\ &= xJ_{2n+1}(x) + 2n\big[-J_0(x) - 2\sum_{k=1}^{n} J_{2k}(x)\big] + C \\ &= xJ_{2n+1}(x) - 2n\,J_0(x) - 4n\sum_{k=1}^{n} J_{2k}(x) + C. \end{aligned}$$

Take $n=1$, then

$$\int xJ_2(x)\,dx = xJ_3(x) - 2J_0(x) - 4J_2(x) + C.$$

Take $n=2$, then

$$\int xJ_4(x)\,dx = xJ_5(x) - 4J_0(x) - 8J_2(x) - 8J_4(x) + C.$$

Solutions to Exercises 4.3

1. The condition $g(r,\theta) = 0$ implies that $a^*_{mn} = 0 = b^*_{mn}$. Since $f(r,\theta)$ is proportional to $\sin 2\theta$, only $b_{2,n}$ will be nonzero, among all the a_{mn} and b_{mn}. This is similar to the situation in Example 2. For $n = 1, 2, \ldots$, we have

$$\begin{aligned} b_{2,n} &= \frac{2}{\pi J_3(\alpha_{2,n})^2} \int_0^1 \int_0^{2\pi} (1-r^2)r^2 \sin 2\theta J_2(\alpha_{2,n} r) \sin 2\theta r \, d\theta \, dr \\ &= \frac{2}{\pi J_3(\alpha_{2,n})^2} \int_0^1 \overbrace{\int_0^{2\pi} \sin^2 2\theta \, d\theta}^{=\pi} (1-r^2) r^3 J_2(\alpha_{2,n} r)\, dr \\ &= \frac{2}{J_3(\alpha_{2,n})^2} \int_0^1 (1-r^2) r^3 J_2(\alpha_{2,n} r)\, dr \\ &= \frac{2}{J_3(\alpha_{2,n})^2} \frac{2}{\alpha_{2,n}^2} J_4(\alpha_{2,n}) = \frac{4 J_4(\alpha_{2,n})}{\alpha_{2,n}^2 J_3(\alpha_{2,n})^2}, \end{aligned}$$

where the last integral is evaluated with the help of formula (15), Section 4.3. We can get rid of the expression involving J_4 by using the identity

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$

With $p = 3$ and $x = \alpha_{2,n}$, we get

$$\overbrace{J_2(\alpha_{2,n})}^{=0} + J_4(\alpha_{2,n}) = \frac{6}{\alpha_{2,n}} J_3(\alpha_{2,n}) \quad \Rightarrow \quad J_4(\alpha_{2,n}) = \frac{6}{\alpha_{2,n}} J_3(\alpha_{2,n}).$$

So

$$b_{2,n} = \frac{24}{\alpha_{2,n}^3 J_3(\alpha_{2,n})}.$$

Thus

$$u(r,\,\theta,\,t) = 24 \sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n}^3 J_3(\alpha_{2,n})} \cos(\alpha_{2,n} t).$$

2. The condition $g(r,\theta) = 0$ implies that $a^*_{mn} = 0 = b^*_{mn}$. Since $f(r,\theta)$ is proportional to $\cos\theta$, only $a_{1,n}$ will be nonzero, among all the a_{mn} and b_{mn}. This is similar to the situation in Example 2. For $n = 1, 2, \ldots$, we have

$$\begin{aligned} a_{1,n} &= \frac{2}{9\pi J_2(\alpha_{1,n})^2} \int_0^3 \int_0^{2\pi} (9-r^2) r \cos\theta J_1\!\left(\frac{\alpha_{1,n} r}{3}\right) \cos\theta r \, d\theta \, dr \\ &= \frac{2}{9\pi J_2(\alpha_{1,n})^2} \int_0^3 \overbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}^{=\pi} (9-r^2) r^2 J_1\!\left(\frac{\alpha_{1,n} r}{3}\right) dr \\ &= \frac{2}{9 J_2(\alpha_{1,n})^2} \int_0^3 (9-r^2) r^2 J_1\!\left(\frac{\alpha_{1,n} r}{3}\right) dr \\ &= \frac{2}{9 J_2(\alpha_{1,n})^2} \frac{2 \cdot 3^5}{\alpha_{1,n}^2} J_3(\alpha_{1,n}) = \frac{108 J_3(\alpha_{1,n})}{\alpha_{1,n}^2 J_2(\alpha_{1,n})^2}, \end{aligned}$$

where the last integral is evaluated with the help of formula (15), Section 4.3. We can get rid of the expression involving J_3 by using the identity

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$

With $p = 2$ and $x = \alpha_{1,n}$, we get

$$\overbrace{J_1(\alpha_{1,n})}^{=0} + J_3(\alpha_{1,n}) = \frac{4}{\alpha_{1,n}} J_2(\alpha_{1,n}) \quad \Rightarrow \quad J_3(\alpha_{1,n}) = \frac{4}{\alpha_{1,n}} J_2(\alpha_{1,n}).$$

So

$$a_{1,n} = \frac{432}{\alpha_{1,n}^3 J_2(\alpha_{1,n})}.$$

Thus

$$u(r,\,\theta,\,t) = 432\cos\theta \sum_{n=1}^{\infty} \frac{J_1(\frac{\alpha_{1,n}r}{3})}{\alpha_{1,n}^3 J_2(\alpha_{1,n})}\ \cos(\frac{\alpha_{1,n}}{3}t).$$

3. Since $f(r,\theta)$ is proportional to $\sin\theta$, only $b_{1,n}$ will be nonzero, among all the a_{mn} and b_{mn}. This is similar to the situation in Example 2. For $n = 1, 2, \ldots$, we have

$$\begin{aligned} b_{1,n} &= \frac{1}{2\pi J_2(\alpha_{1,n})^2} \int_0^2 \int_0^{2\pi} (4-r^2) r \sin\theta J_1(\frac{\alpha_{1,n}r}{2}) \sin\theta r\, d\theta\, dr \\ &= \frac{1}{2J_2(\alpha_{1,n})^2} \int_0^2 (4-r^2) r^2 J_1(\frac{\alpha_{1,n}r}{2})\, dr \\ &= \frac{32 J_3(\alpha_{1,n})}{\alpha_{1,n}^2 J_2(\alpha_{1,n})^2} \\ &= \frac{128}{\alpha_{1,n}^3 J_2(\alpha_{1,n})} \end{aligned}$$

where we evaluated the last integral with the help of formula (15), Section 4.3, and then simplified with the help of the identity

$$J_3(\alpha_{1,n}) = \frac{4}{\alpha_{1,n}} J_2(\alpha_{1,n})$$

(see Exercise 2).

For the part of the solution coming from the condition $g(r,\theta) = 1$, see the solution to Exercise 1, Section 4.2. Thus

$$u(r,\,\theta,\,t) = 128\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\frac{\alpha_{1,n}r}{2})}{\alpha_{1,n}^3 J_2(\alpha_{1,n})}\ \cos(\frac{\alpha_{1,n}}{2}t) + 4\sum_{n=1}^{\infty} \frac{J_0(\frac{\alpha_{0,n}r}{2})}{\alpha_{0,n}^2 J_1(\alpha_{0,n})}\ \sin(\frac{\alpha_{0,n}}{2}t).$$

4. The condition $g(r,\theta) = 0$ implies that $a_{mn}^* = 0 = b_{mn}^*$. Also, by orthogonality, or simply by considering the special form of $f(r,\theta)$, we see that all a_{mn} and b_{mn} are zero except for $b_{3,2}$ which should be 1. Thus

$$u(r,\,\theta,\,t) = J_3(\alpha_{32}\, r)\sin 3\theta \cos \alpha_{3,2} t.$$

5. We have $a_{mn} = b_{mn} = 0$. Also, all a_{mn}^* and b_{mn}^* are zero except $b_{2,n}^*$. We have

$$b_{2,n}^* = \frac{2}{\pi\alpha_{2,n} J_3(\alpha_{2,n})^2} \int_0^1 \int_0^{2\pi} (1-r^2) r^2 \sin 2\theta J_2(\alpha_{2,n} r) \sin 2\theta r\, d\theta\, dr.$$

The integral was computed in Exercise 1. Using the computations of Exercise 1, we find

$$b_{2,n}^* = \frac{24}{\alpha_{2,n}^4 J_3(\alpha_{2,n})}.$$

hus

$$u(r,\,\theta,\,t) = 24\sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n}^4 J_3(\alpha_{2,n})}\ \sin(\alpha_{2,n} t).$$

6. Let us write the solution as

$$u(r,\,\theta,\,t) = u_1(r,\,\theta,\,t) + u_2(r,\,\theta,\,t),$$

where u_1 is the part that comes from $f(r,\theta) = 1 - r^2$ and u_2 is the part that comes form $g(r,\theta) = J_0(r)$. For u_1, we can use Example 2, Section 4.2, and get

$$u_1(r,\,\theta,\,t) = 8\sum_{n=1}^{\infty} \frac{J_0(\alpha_{0,n}r)}{\alpha_{0,n}^3 J_1(\alpha_{0,n})}\cos(\alpha_{0,n}t).$$

To compute u_2, we need to do some work. We first note that only $a_{0,n}^*$ is nonzero because $g(r,\,\theta) = J_0(r)$ depends only on r. We have

$$a_{0,n}^* = \frac{2}{\alpha_{0,n}J_1(\alpha_{0,n})^2}\int_0^1 J_0(r)J_0(\alpha_{0,n}r)r\,dr.$$

We evaluate the integral by using the Bessel differential equation, as we did when we established the orthogonality of Bessel functions (see the proof of Theorem 3, Section 4.8). The function $y = J_0(r)$ is a solution of

$$(ry')' + ry = 0.$$

The function $R_n(r) = J_0(\alpha_{0,n}r)$ is a solution of

$$(rR_n')' + \alpha_{0,n}^2 rR_n = 0 \qquad R_n(1) = 0.$$

Multiplying the first differential equation by R_n and the second by y and subtracting, we obtain

$$R_n(ry')' - y(rR_n')' = (\alpha_{0,n}^2 - 1)ryR_n \quad\Rightarrow\quad (\alpha_{0,n}^2 - 1)ryR_n = [r(R_ny' - R_n'y)]'.$$

Integrating from 0 to 1 gives

$$\begin{aligned}(\alpha_{0,n}^2 - 1)\int_0^1 rJ_0(r)J_0(\alpha_{0,n}r)\,dr &= [r(R_ny' - R_n'y)]\Big|_0^1 \\ &= -R_n'(1)y(1) = -\alpha_{0,n}J_0'(\alpha_{0,n})J_0(1) \\ &= \alpha_{0,n}J_1(\alpha_{0,n})J_0(1),\end{aligned}$$

because $J_0'(x) = -J_1(x)$. Since $\alpha_{0,n} > 1$, we can divide by $\alpha_{0,n}^2 - 1$ and get

$$\int_0^1 J_0(r)J_0(\alpha_{0,n}r)r\,dr = \frac{\alpha_{0,n}J_1(\alpha_{0,n})J_0(1)}{\alpha_{0,n}^2 - 1}.$$

So

$$a_{0,n}^* = \frac{2J_0(1)}{(\alpha_{0,n}^2 - 1)J_1(\alpha_{0,n})}$$

and the solution is

$$u(r,\,\theta,\,t) = 8\sum_{n=1}^{\infty}\frac{J_0(\alpha_{0,n}r)}{\alpha_{0,n}^3 J_1(\alpha_{0,n})}\cos(\alpha_{0,n}t) + \sum_{n=1}^{\infty}\frac{2J_0(1)\,J_0(\alpha_{0,n}r)}{(\alpha_{0,n}^2 - 1)J_1(\alpha_{0,n})}\sin(\alpha_{0,n}t).$$

7. The partial differential equation is

$$u_{tt} = c^2\big(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\big), \quad \text{for } 0 < r < a,\ 0 \le \theta < 2\pi,\ t > 0.$$

The boundary condition is

$$u(a,\,\theta,\,t) = 0, \quad \text{for } 0 \le \theta < 2\pi,\ t > 0.$$

The initial conditions are

$$u(r,\,\theta,\,0) = 0 \text{ and } u_t(r,\,\theta,\,0) = g(r,\,\theta), \quad \text{for } 0 < r < a,\ 0 \le \theta < 2\pi.$$

(b) Assume that $u(r,\theta,t)=R(r)\Theta(\theta)T(t)$. Since $u(r,\theta,0)=0$ for all $0<r<a$ and $0\le\theta<2\pi$, we set $T(t)=0$ to avoid having to take $R(r)=0$ or $\Theta(\theta)=0$, which would lead to trivial solutions. Separating variables, as in the text, we arrive at

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{\alpha_{mn}r}{a}\right)\left[a_{mn}^*\cos m\theta+b_{mn}^*\sin m\theta\right]\sin\left(\frac{\alpha_{mn}ct}{a}\right).$$

(c) From (b),

$$u_t(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_{mn}c}{a}J_m\left(\frac{\alpha_{mn}r}{a}\right)\left[a_{mn}^*\cos m\theta+b_{mn}^*\sin m\theta\right]\cos\left(\frac{\alpha_{mn}ct}{a}\right)$$

and hence

$$\begin{aligned}g(r,\theta)&=u_t(r,\theta,0)\\&=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_{mn}c}{a}J_m\left(\frac{\alpha_{mn}r}{a}\right)\left[a_{mn}^*\cos m\theta+b_{mn}^*\sin m\theta\right]\\&=\overbrace{\sum_{n=1}^{\infty}\frac{\alpha_{0n}c}{a}a_{0n}^*J_0\left(\frac{\alpha_{0n}r}{a}\right)}^{=a_0^*(r)}\\&\quad+\sum_{m=1}^{\infty}\left\{\overbrace{\sum_{n=1}^{\infty}\frac{\alpha_{mn}c}{a}a_{mn}^*J_m\left(\frac{\alpha_{mn}r}{a}\right)}^{=a_m^*(r)}\cos m\theta+\overbrace{\sum_{n=1}^{\infty}\frac{\alpha_{mn}c}{a}b_{mn}^*J_m\left(\frac{\alpha_{mn}r}{a}\right)}^{=b_m^*(r)}\sin m\theta\right\}.\end{aligned}$$

(d) Thinking of the result of (c) as a Fourier series (for fixed r and $0\le\theta<2\pi$)

$$g(r,\theta)=a_0^*(r)+\sum_{m=1}^{\infty}\left(a_m^*(r)\cos m\theta+b_m^*(r)\sin m\theta\right),$$

we see that we must have

$$a_0^*(r)=\frac{1}{2\pi}\int_0^{2\pi}g(r,\theta)\,d\theta=\sum_{n=1}^{\infty}\frac{\alpha_{0n}c}{a}a_{0n}^*J_0\left(\frac{\alpha_{0n}r}{a}\right);$$

$$a_m^*(r)=\frac{1}{2\pi}\int_0^{\pi}g(r,\theta)\cos m\theta\,d\theta=\sum_{n=1}^{\infty}\frac{\alpha_{mn}c}{a}a_{mn}^*J_m\left(\frac{\alpha_{mn}r}{a}\right);$$

and

$$b_m^*(r)=\frac{1}{2\pi}\int_0^{\pi}g(r,\theta)\sin m\theta\,d\theta=\sum_{n=1}^{\infty}\frac{\alpha_{mn}c}{a}b_{mn}^*J_m\left(\frac{\alpha_{mn}r}{a}\right),$$

for $m=1,2,\ldots$.
(e) We now let r vary in $[0,a]$ and view the expansions in (d) as Bessel series. Then, by Theorem 2, Section 4.8, we must have

$$\begin{aligned}a_{0n}^*&=\frac{a}{\alpha_{0n}c}\frac{2}{a^2J_1(\alpha_{0n})^2}\int_0^a a_0^*(r)J_0\left(\frac{\alpha_{0n}r}{a}\right)r\,dr\\&=\frac{1}{\pi\alpha_{0n}caJ_1(\alpha_{0n})^2}\int_0^a\int_0^{2\pi}g(r,\theta)J_0\left(\frac{\alpha_{0n}r}{a}\right)r\,d\theta\,dr,\end{aligned}$$

which implies (17). Also, for $m=1,2,\ldots$,

$$\begin{aligned}a_{mn}^*&=\frac{a}{\alpha_{mn}c}\frac{2}{a^2J_{m+1}(\alpha_{mn})^2}\int_0^a a_m^*(r)J_m\left(\frac{\alpha_{mn}r}{a}\right)r\,dr\\&=\frac{2}{\pi\alpha_{mn}caJ_{m+1}(\alpha_{mn})^2}\int_0^a\int_0^{2\pi}g(r,\theta)J_m\left(\frac{\alpha_{mn}r}{a}\right)\cos(m\theta)\,r\,d\theta\,dr,\end{aligned}$$

which implies (18), and

$$\begin{aligned} b^*_{mn} &= \frac{a}{\alpha_{mn}c}\,\frac{2}{a^2 J_{m+1}(\alpha_{mn})^2}\int_0^a b^*_m(r) J_m\left(\frac{\alpha_{mn} r}{a}\right) r\,dr \\ &= \frac{2}{\pi \alpha_{mn} c a J_{m+1}(\alpha_{mn})^2}\int_0^a \int_0^{2\pi} g(r,\,\theta) J_m\left(\frac{\alpha_{mn} r}{a}\right)\sin(m\theta)\, r\,d\theta\,dr, \end{aligned}$$

which implies (19).

8. (a) Subproblem #1

$$\begin{aligned} (u_1)_{tt} &= c^2\nabla^2 u_1, && 0<r<a,\ 0\le\theta<2\pi,\ t>0,\\ u_1(a,\,\theta,\,t) &= 0, && 0\le\theta<2\pi,\ t>0,\\ u_1(r,\,\theta,\,0) &= f(r,\,\theta), && 0<r<a,\ 0\le\theta<2\pi,\\ (u_1)_t(r,\,\theta,\,0) &= 0, && 0<r<a,\ 0\le\theta<2\pi. \end{aligned}$$

Subproblem #2

$$\begin{aligned} (u_2)_{tt} &= c^2\nabla^2 u_2, && 0<r<a,\ 0\le\theta<2\pi,\ t>0,\\ u_2(a,\,\theta,\,t) &= 0, && 0\le\theta<2\pi,\ t>0,\\ u_2(r,\,\theta,\,0) &= 0, && 0<r<a,\ 0\le\theta<2\pi,\\ (u_2)_t(r,\,\theta,\,0) &= g(r,\,\theta), && 0<r<a,\ 0\le\theta<2\pi. \end{aligned}$$

By linearity, or superposition, it follows that

$$u(r,\,\theta,\,t) = u_1(r,\,\theta,\,t) + u_2(r,\,\theta,\,t)$$

is a solution of

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u, && 0<r<a,\ 0\le\theta<2\pi,\ t>0,\\ u(a,\,\theta,\,t) &= 0, && 0\le\theta<2\pi,\ t>0,\\ u(r,\,\theta,\,0) &= f(r,\,\theta), && 0<r<a,\ 0\le\theta<2\pi,\\ u_t(r,\,\theta,\,0) &= g(r,\,\theta), && 0<r<a,\ 0\le\theta<2\pi. \end{aligned}$$

(b) By Example 1,

$$u_1(r,\,\theta,\,t) = \sum_{m=0}^\infty \sum_{n=1}^\infty J_m\left(\frac{\alpha_{mn} r}{a}\right)\left[a_{mn}\cos m\theta + b_{mn}\sin m\theta\right]\cos\left(\frac{\alpha_{mn} ct}{a}\right),$$

with the a_{mn} and b_{mn} as given by (12)–(14). By Exercise 7,

$$u_2(r,\,\theta,\,t) = \sum_{m=0}^\infty \sum_{n=1}^\infty J_m\left(\frac{\alpha_{mn} r}{a}\right)\left[a^*_{mn}\cos m\theta + b^*_{mn}\sin m\theta\right]\sin\left(\frac{\alpha_{mn} ct}{a}\right),$$

with the a^*_{mn} and b^*_{mn} as given by (17)–(19). This implies the solution (16).

9. (a) For $l=0$ and all $k\ge 0$, the formula follows from (7), Section 4.8, with $p=k$:

$$\int r^{k+1} J_k(r)\,dr = r^{k+1} J_{k+1}(r) + C.$$

(b) Assume that the formula is true for l (and all $k\ge 0$). Integrate by parts, using

$u = r^{2l}$, $dv = r^{k+1}J_{k+1}(r)\,dr$, and hence $du = 2lr^{2l-1}dr$ and $v = r^{k+1}J_{k+1}(r)$:

$$\begin{aligned}\int r^{k+1+2l}J_k(r)\,dr &= \int r^{2l}[r^{k+1}J_k(r)]\,dr\\ &= r^{2l}\,r^{k+1}J_{k+1}(r) - 2l\int r^{2l-1}r^{k+1}J_{k+1}(r)\,dr\\ &= r^{k+1+2l}J_{k+1}(r) - 2l\int r^{k+2l}J_{k+1}(r)\,dr\\ &= r^{k+1+2l}J_{k+1}(r) - 2l\int r^{(k+1)+1+2(l-1)}J_{k+1}(r)\,dr\end{aligned}$$

and so, by the induction hypothesis, we get

$$\begin{aligned}\int r^{k+1+2l}J_k(r)\,dr &= r^{k+1+2l}J_{k+1}(r) - 2l\sum_{n=0}^{l-1}\frac{(-1)^n 2^n (l-1)!}{(l-1-n)!}r^{k+2l-n}J_{k+n+2}(r) + C\\ &= r^{k+1+2l}J_{k+1}(r)\\ &\quad+\sum_{n=0}^{l-1}\frac{(-1)^{n+1}2^{n+1}l!}{(l-(n+1))!}r^{k+1+2l-(n+1)}J_{k+(n+1)+1}(r) + C\\ &= r^{k+1+2l}J_{k+1}(r) + \sum_{m=1}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}r^{k+1+2l-m}J_{k+m+1}(r) + C\\ &= \sum_{m=0}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}r^{k+1+2l-m}J_{k+m+1}(r) + C,\end{aligned}$$

which completes the proof by induction for all integers $k \geq 0$ and all $l \geq 0$.

10. $a = c = 1$, $f(r,\theta) = (1-r^2)(\frac{1}{4} - r^2)r^3\sin\theta$, $g(r,\theta) = 0$. Since $g(r,\theta) = 0$, then a^*_{mn} and b^*_{mn} are 0. Since $f(r,\theta)$ is proportional to $\sin 3\theta$, by orthogonality, only b_{3n} can possibly be nonzero. All other coefficients a_{mn} and b_{mn} must be 0. We have

$$\begin{aligned}b_{3n} &= \frac{2}{J_4(\alpha_{3n})^2}\int_0^1 (1-r^2)(\frac{1}{4} - r^2)r^3 J_3(\alpha_{3n}r)r\,dr\\ &= \frac{2}{\alpha_{3n}^9 J_4(\alpha_{3n})^2}\int_0^{\alpha_{3n}} (\alpha_{3n}^2 - s^2)(\frac{1}{4}\alpha_{3n}^2 - s^2)J_3(s)s^4\,ds \quad (\text{where } s = \alpha_{3n}r)\\ &= \frac{1}{2\alpha_{3n}^9 J_4(\alpha_{3n})^2}\int_0^{\alpha_{3n}} \left(4s^2 - 5\alpha_{3n}^2 s^2 + \alpha_{3n}^4\right)s^4 J_3(s)\,ds\\ &= \frac{1}{2\alpha_{3n}^9 J_4(\alpha_{3n})^2}\left(4I_2 - 5\alpha_{3n}^2 I_1 + \alpha_{3n}^4 I_0\right),\end{aligned}$$

where, using the result of the previous exercise,

$$\begin{aligned}I_l &\equiv \int_0^{\alpha_{3n}} s^{4+2l}J_3(s)\,ds\\ &= \sum_{m=0}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}s^{4+2l-m}J_{4+m}(s)\Big|_0^{\alpha_{3n}}\\ &= l!\sum_{m=0}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}\alpha_{3n}^{4+2l-m}J_{4+m}(\alpha_{3n}).\end{aligned}$$

Thus

$$\begin{aligned} I_0 &= \alpha_{3n}^4 J_4(\alpha_{3n}) \\ I_1 &= \alpha_{3n}^6 J_4(\alpha_{3n}) - 2\alpha_{3n}^5 J_5(\alpha_{3n}), \\ I_2 &= \alpha_{3n}^8 J_4(\alpha_{3n}) - 4\alpha_{3n}^7 J_5(\alpha_{3n}) + 8\alpha_{3n}^6 J_6(\alpha_{3n}). \end{aligned}$$

Hence

$$\begin{aligned} 4I_2 - 5\alpha_{3n}^2 I_1 + \alpha_{3n}^4 I_0 &= 4\alpha_{3n}^8 J_4(\alpha_{3n}) - 16\alpha_{3n}^7 J_5(\alpha_{3n}) + 32\alpha_{3n}^6 J_6(\alpha_{3n}) \\ &\quad -5\alpha_{3n}^8 J_4(\alpha_{3n}) + 10\alpha_{3n}^7 J_5(\alpha_{3n}) + \alpha_{3n}^8 J_4(\alpha_{3n}) \\ &= -6\alpha_{3n}^7 J_5(\alpha_{3n}) + 32\alpha_{3n}^6 J_6(\alpha_{3n}). \end{aligned}$$

Thus

$$b_{3n} = \frac{-6\alpha_{3n}^7 J_5(\alpha_{3n}) + 32\alpha_{3n}^6 J_6(\alpha_{3n})}{2\alpha_{3n}^9 J_4(\alpha_{3n})^2} = \frac{-3\alpha_{3n} J_5(\alpha_{3n}) + 16 J_6(\alpha_{3n})}{\alpha_{3n}^3 J_4(\alpha_{3n})^2},$$

and the solution is

$$u(r, \theta, t) = \sum_{n=1}^{\infty} \frac{-3\alpha_{3n} J_5(\alpha_{3n}) + 16 J_6(\alpha_{3n})}{\alpha_{3n}^3 J_4(\alpha_{3n})^2} J_3(\alpha_{3n} r) \sin 3\theta \cos(\alpha_{3n} t).$$

Note: Using the identity

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x),$$

we can simplify the b_{3n} into

$$b_{3n} = \frac{-40(\alpha_{3n}^2 - 32)}{\alpha_{3n}^5 J_4(\alpha_{3n})}.$$

11. The solution is similar to the solution of the wave equation. The only difference is the equation in T, which is in this case

$$T' = -c^2 \lambda T$$

with solution

$$T(t) = Ce^{-c^2 \lambda t} = Ce^{-c^2 \alpha_{mn}^2 t/a}.$$

The other parts of the product solutions are as in the wave equation

$$J_m\left(\frac{\alpha_{mn} r}{a}\right)\left[a_{mn} \cos m\theta + b_{mn} \sin m\theta\right].$$

Thus the solution is

$$u(r, \theta, t) == \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{\alpha_{mn} r}{a}\right)\left[a_{mn} \cos m\theta + b_{mn} \sin m\theta\right] e^{-c^2 \alpha_{mn}^2 t/a^2}.$$

Using the initial condition $u(r, \theta, 0) = f(r, \theta)$, we see that the coefficients a_{mn} and b_{mn} are determined by (12)–(14), as in the case of the wave equation.

12. $a = c = 1$, $f(r, \theta) = (1 - r^2) r \sin\theta$. Using Example 2, we see that the solution is given by

$$u(r, \theta, t) == 16 \sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1n} r)}{\alpha_{1n}^3 J_2(\alpha_{1n})} e^{-\alpha_{1n}^2 t}.$$

(b) We have $u(r, \theta, 0) = (1 - r^2) r \sin\theta$. Since $(1 - r^2) r \geq 0$, the maximum occurs when $\theta = \pi/2$ ($\sin\theta = 1$) at the value of r in $[0, 1]$ that maximizes the function

$h(r) = r(1-r^2) = r - r^3$. We have $h'(r) = 1 - 3r^2$. So $h'(r) = 0 \Rightarrow r = 1/\sqrt{3}$. So the maximum occurs at $\theta = \pi/2$ and $r = 1/\sqrt{3}$. The maximum value is $f\big(1/\sqrt{3},\, \pi/2\big) = \frac{2}{3\sqrt{3}}$.
(c) We have

$$u(r,\,\theta,\,1) == 16\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1n}r)}{\alpha_{1n}^3 J_2(\alpha_{1n})} e^{-\alpha_{1n}^2}.$$

The maximum will occur when $\theta = \pi/2$. Also

$$u(r,\,\theta,\,2) == 16\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1n}r)}{\alpha_{1n}^3 J_2(\alpha_{1n})} e^{-2\alpha_{1n}^2}.$$

The maximum will occur when $\theta = \pi/2$.

13. The proper place for this problem is in the next section, since its solution invovles solving a Dirichlet problem on the unit disk. The initial steps are similar to the solution of the heat problem on a rectangle with nonzero boundary data (Exercise 11, Section 3.8). In order to solve the problem, we consider the following two subproblems: Subproblem #1 (Dirichlet problem)

$$\begin{aligned} (u_1)_{rr} + \frac{1}{r}(u_1)_r + \frac{1}{r^2}(u_1)_{\theta\theta} &= 0, && 0 < r < 1,\ 0 \le \theta < 2\pi, \\ u_1(1,\,\theta) &= \sin 3\theta, && 0 \le \theta < 2\pi. \end{aligned}$$

Subproblem #2 (to be solved after finding $u_1(r,\,\theta)$ from Subproblem #1)

$$\begin{aligned} (u_2)_t &= (u_2)_{rr} + \tfrac{1}{r}(u_2)_r + \tfrac{1}{r^2}(u_2)_{\theta\theta}, && 0 < r < 1,\ 0 \le \theta < 2\pi,\ t > 0, \\ u_2(1,\,\theta,\,t) &= 0, && 0 \le \theta < 2\pi,\ t > 0, \\ u_2(r,\,\theta,\,0) &= -u_1(r,\,\theta), && 0 < r < 1,\ 0 \le \theta < 2\pi. \end{aligned}$$

You can check, using linearity (or superposition), that

$$u(r,\,\theta,\,t) = u_1(r,\,\theta) + u_2(r,\,\theta,\,t)$$

is a solution of the given problem.

The solution of subproblem #1 follows immediately from the method of Section 4.5. We have

$$u_2(r,\,\theta) = r^3 \sin 3\theta.$$

We now solve subproblem #2, which is a heat problem with 0 boundary data and initial temperature distribution given by $-u_2(r,\,\theta) = -r^3\sin 3\theta$. reasoning as in Exercise 10, we find that the solution is

$$u_2(r,\,\theta,\,t) = \sum_{n=1}^{\infty} b_{3n} J_3(\alpha_{3n}r)\sin(3\theta) e^{-\alpha_{3n}^2 t},$$

where

$$\begin{aligned} b_{3n} &= \frac{-2}{\pi J_4(\alpha_{3n})^2} \int_0^1 \int_0^{2\pi} r^3 \sin^2 3\theta J_3(\alpha_{3n}r) r\, d\theta\, dr \\ &= \frac{-2}{J_4(\alpha_{3n})^2} \int_0^1 r^4 J_3(\alpha_{3n}r)\, dr \\ &= \frac{-2}{J_4(\alpha_{3n})^2} \frac{1}{\alpha_{3n}^5} \int_0^{\alpha_{3n}} s^4 J_3(s)\, ds \quad (\text{where } \alpha_{3n}r = s) \\ &= \frac{-2}{J_4(\alpha_{3n})^2} \frac{1}{\alpha_{3n}^5} s^4 J_4(s)\Big|_0^{\alpha_{3n}} \\ &= \frac{-2}{\alpha_{3n} J_4(\alpha_{3n})}. \end{aligned}$$

Hence

$$u(r,\,\theta,\,t) = r^3 \sin 3\theta - 2\sin(3\theta) \sum_{n=1}^{\infty} \frac{J_3(\alpha_{3n} r)}{\alpha_{3n} J_4(\alpha_{3n})} e^{-\alpha_{3n}^2 t}.$$

Exercises 4.4

1. Since f is already given by its Fourier series, we have from (4)

$$u(r,\,\theta) = r\cos\theta = x.$$

2. Since f is already given by its Fourier series, we have from (4)

$$u(r,\,\theta) = r^2\sin 2\theta = 2r^2\sin\theta\cos\theta = 2xy.$$

3. Recall the Fourier series of the sawtooth function

$$f(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$$

(see Example 1, Section 2.2). From (4)

$$u(r,\,\theta) = \sum_{n=1}^{\infty} r^n\frac{\sin n\theta}{n}.$$

4. Let us compute the Fourier coefficients of f (or use Example 4, Section 2.2). We have

$$a_0 = \frac{1}{2\pi}\int_0^{\pi}(\pi-\theta)\,d\theta = -\frac{1}{4\pi}(\pi-\theta)^2\Big|_0^{\pi} = \frac{\pi}{4};$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi}(\pi-\theta)\cos n\theta\,d\theta \\ &= \frac{1}{\pi}\left[\frac{1}{n}(\pi-\theta)\sin n\theta\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin n\theta\,d\theta\right] \\ &= \frac{-1}{\pi n^2}\cos n\theta\Big|_0^{\pi} = \frac{1-(-1)^n}{\pi n^2}; \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi}(\pi-\theta)\sin n\theta\,d\theta \\ &= \frac{1}{\pi}\left[\frac{-1}{n}(\pi-\theta)\cos n\theta\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos n\theta\,d\theta\right] \\ &= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{1}{n^2}\sin n\theta\Big|_0^{\pi}\right] = \frac{1}{n}. \end{aligned}$$

Hence

$$f(\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{\pi n^2}\cos n\theta + \frac{1}{n}\sin n\theta\right);$$

and

$$u(r,\,\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{\pi n^2}\cos n\theta + \frac{1}{n}\sin n\theta\right)r^n.$$

5. Let us compute the Fourier coefficients of f. We have

$$a_0 = \frac{50}{\pi}\int_0^{\pi/4}d\theta = \frac{25}{2};$$

$$a_n = \frac{100}{\pi}\int_0^{\pi/4}\cos n\theta\,d\theta = \frac{100}{n\pi}\sin n\theta\Big|_0^{\pi} = \frac{100}{n\pi}\sin\frac{n\pi}{4};$$

$$b_n = \frac{100}{\pi}\int_0^{\pi/4} \sin n\theta\, d\theta = -\frac{100}{n\pi}\cos n\theta\Big|_0^{\pi} = \frac{100}{n\pi}(1-\cos\frac{n\pi}{4}).$$

Hence

$$f(\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\cos n\theta + (1-\cos\frac{n\pi}{4})\sin n\theta\right);$$

and

$$u(r,\,\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\cos n\theta + (1-\cos\frac{n\pi}{4})\sin n\theta\right)r^n.$$

6. In Cartesian coordinates,

$$u(x,\,y) = \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) = \tan^{-1}\left(\frac{y}{1-x}\right).$$

Let $U(x,\,y) = u(-x+1,\,y)$. Note that U is a solution of Laplace's equation if and only if u is a solution of Laplace's equation. But

$$U(x,\,y) = u(-x+1,\,y) = \tan^{-1}\left(\frac{y}{1-(-x+1)}\right) = \tan^{-1}\left(\frac{y}{x}\right) = \theta + 2k\pi.$$

Back to polar coordinates, it is clear that U is a solution of Laplace's equation.

7. Using (6), we have

$$u(r,\,\theta) = \sum_{n=1}^{\infty} r^n\frac{\sin n\theta}{n} = \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) = \tan^{-1}\left(\frac{y}{1-x}\right).$$

Let $-\frac{\pi}{2} < T < \frac{\pi}{2}$, and set

$$T = u(x,\,y) = \tan^{-1}\left(\frac{y}{1-x}\right) \quad\Rightarrow\quad \tan T = \frac{y}{1-x}$$
$$\Rightarrow\quad y = \tan(T)\,(1-x).$$

Thus the isotherms are straight lines through the point $(1,\,0)$, with slope $-\tan T$.

8. $u(r\,\theta) = r\cos\theta = x$. So $u(x\,y) = T$ if and only if $x = T$, which shows that the isotherms lie on vertical lines.

9. $u(r\,\theta) = 2r^2\sin\theta\cos\theta = 2xy$. So $u(x\,y) = T$ if and only if $2xy = T$ if and only if $y = \frac{T}{2x}$, which shows that the isotherms lie on hyperbolas centered at the origin.

10. (a) Since f is already given by its Fourier series, we have from (4)

$$u(r,\,\theta) = 1 + r^2\sin 2\theta = 1 + 2r^2\cos\theta\sin\theta = 1 + 2xy.$$

(b) $u(x,\,y) = T$ if and only if $1+2xy = T$ if and only if $y = \frac{T-1}{2x}$, which shows that the isotherms lie on hyperbolas centered at the origin.

11. (a) If $u(r,\,\theta) = u(\theta)$ is independent of r and satisfies Laplace's equation, then, because $u_r = 0$ and $u_{rr} = 0$, Laplace's equation implies that

$$\frac{1}{r^2}u_{\theta\theta} = 0 \quad\text{or}\quad u_{\theta\theta} = 0.$$

The solution of this equation is

$$u(\theta) = a\theta + b,$$

where a and b are constants.
(b) Using the boundary conditions

$$\begin{aligned} u(0) = T_1 &\quad\Rightarrow\quad b = T_1 \\ u(\alpha) = T_2 &\quad\Rightarrow\quad a\alpha + T_1 = T_2 \\ &\quad\Rightarrow\quad a = \frac{T_2 - T_1}{\alpha}. \end{aligned}$$

Thus

$$u(\theta) = \frac{T_2 - T_1}{\alpha}\theta + T_1$$

is a solution that satifies the boundary conditions $u(0) = T_1$ and $u(\alpha) = T_2$. On the circular boundary, u satsifies $u(\theta) = \frac{T_2 - T_1}{\alpha}\theta + T_1$.

12. (a) If $u(r, \theta) = u(r)$ is independent of θ and satisfies Laplace's equation, then, because $u_{\theta\theta} = 0$, Laplace's equation implies that

$$u_{rr} + \frac{1}{r}u_r = 0 \quad \text{or} \quad r^2 u_{rr} + r u_r = 0.$$

This is an Euler equation (appendix A.3), with indicial equation

$$\rho(\rho - 1) + \rho = 0 \quad \text{or} \quad \rho^2 = 0.$$

We have one double indicial root $\rho = 0$ and thus the solution is

$$u(r) = a \ln r + b$$

(see the solution of Euler's equation in Appendix A.3).
(b) Using the boundary conditions

$$\begin{aligned} u(1/2) = 1 &\quad\Rightarrow\quad -a \ln 2 + b = 1 \\ u(1) = 2 &\quad\Rightarrow\quad b = 2 \\ &\quad\Rightarrow\quad a = \frac{1}{\ln 2}. \end{aligned}$$

Thus

$$u(r) = \frac{\ln r}{\ln 2} + 2$$

is a solution of the Dirichlet problem in the annulus that satifies the given boundary conditions. (It is a good idea to verify the solution by plugging into the equation and checking the boundary conditions.)

13. We follow the steps in Example 4 (with $\alpha = \frac{\pi}{4}$) and arrive at the same equation in Θ and R. The solution in Θ is

$$\Theta_n(\theta) = \sin(4n\theta), \quad n = 1, 2, \ldots,$$

and the equation in R is

$$r^2 R'' + r R' - (4n)^2 R = 0.$$

The indicial equation for this Euler equation is

$$\rho^2 - (4n)^2 = 0 \quad\Rightarrow\quad \rho = \pm 4n.$$

Taking the bounded solutions only, we get

$$R_n(r) = r^{4n}.$$

Thus the product solutions are $r^{4n}\sin 4\theta$ and the series solution of the problem is of the form

$$u(r,\,\theta) = \sum_{n=1}^{\infty} b_n r^{4n} \sin 4n\theta.$$

To determine b_n, we use the boundary condition:

$$\begin{aligned} u_r(r,\,\theta)\big|_{r=1} = \sin\theta \quad &\Rightarrow \quad \sum_{n=1}^{\infty} b_n 4n r^{4n-1} \sin 4n\theta\big|_{r=1} = \sin\theta \\ &\Rightarrow \quad \sum_{n=1}^{\infty} b_n 4n \sin 4n\theta = \sin\theta \\ &\Rightarrow \quad 4nb_n = \frac{2}{\pi/4}\int_0^{\pi/4} \sin\theta\sin 4n\theta\,d\theta. \end{aligned}$$

Thus

$$\begin{aligned} b_n &= \frac{2}{\pi n}\int_0^{\pi/4} \sin\theta\sin 4n\theta\,d\theta \\ &= \frac{1}{\pi n}\int_0^{\pi/4} [-\cos[(4n+1)\theta] + \cos[(4n-1)\theta]]\,d\theta \\ &= \frac{1}{\pi n}\left[-\frac{\sin[(4n+1)\theta]}{4n+1} + \frac{\sin[(4n-1)\theta]}{4n-1}\right]\Big|_0^{\pi/4} \\ &= \frac{1}{\pi n}\left[-\frac{\sin[(4n+1)\frac{\pi}{4}}{4n+1} + \frac{\sin[(4n-1)\frac{\pi}{4}}{4n-1}\right] \\ &= \frac{1}{\pi n}\left[-\frac{\cos(n\pi)\sin\frac{\pi}{4}]}{4n+1} - \frac{\cos(n\pi)\sin\frac{\pi}{4}}{4n-1}\right] \\ &= \frac{(-1)^n}{\pi n}\frac{\sqrt{2}}{2}\left[\frac{-1}{4n+1} - \frac{1}{4n-1}\right] \\ &= \frac{(-1)^{n+1}\sqrt{2}}{\pi}\,\frac{4}{16n^2-1}. \end{aligned}$$

Hence

$$u(r,\,\theta) = \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{16n^2-1} r^{4n}\sin 4n\theta.$$

14. Modify the solution of Exercise 13 as follows. The solution in Θ is

$$\Theta_n(\theta) = \sin(2n\theta), \quad n = 1,\,2,\,\ldots,$$

and the equation in R is

$$r^2R'' + rR' - (2n)^2R = 0$$

with indicial equation

$$\rho^2 - (2n)^2 = 0 \quad \Rightarrow \quad \rho = \pm 2n.$$

Taking the bounded solutions only, we get

$$R_n(r) = r^{2n}.$$

Thus the product solutions are $r^{2n} \sin 2\theta$ and the series solution of the problem is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^{2n} \sin 2n\theta.$$

To determine b_n, we use the boundary condition:

$$u_r(r, \theta)\big|_{r=1} = \theta \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n 2n r^{2n-1} \sin 2n\theta\big|_{r=1} = \theta$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} b_n 2n \sin 2n\theta = \theta$$

$$\Rightarrow \quad 2n b_n = \frac{2}{\pi/2} \int_0^{\pi/2} \theta \sin 2n\theta \, d\theta.$$

Thus

$$\begin{aligned} b_n &= \frac{2}{\pi n} \int_0^{\pi/2} \theta \sin 2n\theta \, d\theta \\ &= \frac{2}{\pi n} \Big[\frac{1}{(2n)^2} \sin 2n\theta - \frac{\theta}{2n} \cos 2n\theta \Big] \Big|_0^{\pi/2} \\ &= \frac{(-1)^{n+1}}{2n^2}. \end{aligned}$$

Hence

$$u(r, \theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^2} r^{2n} \sin 2n\theta.$$

15. This is immediate by superposition and linearity of the equation. Indeed,

$$\nabla^2 u = \nabla^2 u_1 + \nabla^2 u_2 = 0 + 0 = 0.$$

We now check the boundary conditions. For $\theta = 0$, we have

$$u(r, 0) = u_1(r, 0) + u_2(r, 0) = T_1 + 0 = T_1.$$

For $\theta = \alpha$, we have

$$u(r, \alpha) = u_1(r, \alpha) + u_2(r, \alpha) = T_2 + 0 = T_2.$$

For $r = a$, we have

$$u(a, \theta) = u_1(a, \theta) + u_2(a, \theta) = g(\theta) + f(\theta) - g(\theta) = f(\theta).$$

16. We flow the method of Exercise 15. Accordingly, we consider two Dirichlet problems. The first one can be described by Figure 7(b) by taking $\alpha = \pi/4$, $T_1 = 0$ and $T_2 = 1$. The solution in this case depends only on θ and is $u_1(\theta) = \frac{4}{\pi}\theta$. The second problem is described by Figure 7(c) with $\alpha = \pi/4$, $f(\theta) = 3 \sin 4\theta$, and $g(\theta) = \frac{4}{\pi}\theta$ (the values of u_1 on the circular boundary). Thus the boundary values in Figue 7(c) are given by $u_1(\theta) = 3 \sin 4\theta - \frac{4}{\pi}\theta$. The solution of this problem is follows by applying the method of Example 4. We find that

$$u_2(r, \theta) = \sum_{n=1}^{\infty} b_n r^{4n} \sin(4n\theta),$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi/4}\int_0^{\pi/4}\big(3\sin 4\theta - \frac{4}{\pi}\theta\big)\sin(4n\theta)\,d\theta \\ &= \overbrace{\frac{8}{\pi}\int_0^{\pi/4} 3\sin(4\theta)\sin(4n\theta)\,d\theta}^{3 \text{ if } n=1,\ 0 \text{ otherwise}} - \frac{32}{\pi^2}\int_0^{\pi/4}\theta\sin(4n\theta)\,d\theta. \end{aligned}$$

Also

$$\begin{aligned} \frac{32}{\pi^2}\int_0^{\pi/4}\theta\sin(4n\theta)\,d\theta &= \frac{32}{\pi^2}\left[\frac{1}{16n^2}\sin(4n\theta) - \frac{\theta}{4n}\cos(4n\theta)\right]\Big|_0^{\pi/4} \\ &= \frac{32}{\pi^2}\left[-\frac{\pi}{16n}\cos(n\pi)\right] = 2\frac{(-1)^{n+1}}{\pi n}. \end{aligned}$$

So

$$b_1 = 3 + \frac{2}{\pi}$$

and, for $n \geq 2$,

$$b_n = 2\frac{(-1)^{n+1}}{\pi n}.$$

Thus

$$\begin{aligned} u(r,\,\theta) &= u_1(\theta) + u_2(r,\,\theta) \\ &= \frac{4}{\pi}\theta + \big(3 + \frac{2}{\pi}\big)\sin 4\theta + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\pi n} r^{4n}\sin(4n\theta). \end{aligned}$$

17. Since u satsifies Laplace's equation in the disk, the separation of variables method and the fact that u is 2π-periodic in θ imply that u is given by the series (4), where the coefficients are to be determined from the Neumann boundary condition. From

$$u(r,\,\theta) = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\big[a_n\cos n\theta + b_n\sin n\theta\big],$$

it follows that

$$u_r(r,\,\theta) = \sum_{n=1}^{\infty}\left(n\frac{r^{n-1}}{a^n}\right)\big[a_n\cos n\theta + b_n\sin n\theta\big].$$

Using the boundary condition $u_r(a,\,\theta) = f(\theta)$, we obtain

$$f(\theta) = \sum_{n=1}^{\infty}\frac{n}{a}\big[a_n\cos n\theta + b_n\sin n\theta\big].$$

In this Fourier series expansion, the $n = 0$ term must be 0. But the $n = 0$ term is given by

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$

thus the compatibility condition

$$\int_0^{2\pi} f(\theta)\,d\theta = 0$$

must hold. Once this condition is satsified, we determine the coefficients a_n and b_n by using the Euler formulas, as follows:

$$\frac{n}{a}a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

and

$$\frac{n}{a}b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

Hence

$$a_n = \frac{a}{n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \quad \text{and} \quad b_n = \frac{a}{n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

Note that a_0 is still arbitray. Indeed, the solution of a Neumann problem is not unique. It can be determined only up to an additive constant (which does not affect the value of the normal derivative at the boundary).

18. Since u satsifies Laplace's equation in the unit disk, it is of the form

$$u(r,\,\theta) = a_0 + \sum_{n=1}^{\infty} r^n\big[a_n\cos n\theta + b_n\sin n\theta\big].$$

Hence

$$\begin{aligned} &u_r(r,\,\theta) + 2u(r,\,\theta)\\ &\quad = \sum_{n=1}^{\infty} nr^{n-1}\big[a_n\cos n\theta + b_n\sin n\theta\big] + 2a_0 + \sum_{n=1}^{\infty} 2r^n\big[a_n\cos n\theta + b_n\sin n\theta\big]\\ &\quad = 2a_0 + \sum_{n=1}^{\infty}(2r^n + nr^{n-1})\big[a_n\cos n\theta + b_n\sin n\theta\big]. \end{aligned}$$

Using the boundary condition $u_r(1,\,\theta) + 2u(1,\,\theta) = 100 - 2\cos 2\theta$, we obtain

$$2a_0 + \sum_{n=1}^{\infty}(2+n)\big[a_n\cos n\theta + b_n\sin n\theta\big] = 100 - 2\cos 2\theta.$$

By orthogonality (or uniqueness of Fourier coefficients), we conclude that

$$\begin{aligned} 2a_0 = 100 \quad &\Rightarrow \quad a_0 = 50\\ 3a_1 = -2 \quad &\Rightarrow \quad a_1 = -\frac{2}{3}\\ (2+n)a_n = 0\ (n \geq 2) \quad &\Rightarrow \quad a_n = 0\ (n \geq 2);\\ (2+n)b_n = 0\ (n \geq 1) \quad &\Rightarrow \quad b_n = 0\ (n \geq 1). \end{aligned}$$

Thus

$$u(r,\,\theta) = 50 - \frac{2}{3}r\cos\theta,$$

as can be verified directly by using the equation and the boundary conditions.

19. For $|z| < 1$,

$$\log(1-z)^{-1} = -\log(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}.$$

Taking imaginary parts and using $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, we get

$$\begin{aligned}
\log(1-z)^{-1} &= \sum_{n=1}^{\infty} r^n \frac{e^{in\theta}}{n} \\
&= \sum_{n=1}^{\infty} r^n \frac{\cos n\theta + i\sin n\theta}{n} \\
&= \sum_{n=1}^{\infty} r^n \frac{\cos n\theta}{n} + i\sum_{n=1}^{\infty} r^n \frac{\sin n\theta}{n}; \\
\text{Im}\left(\log(1-z)^{-1}\right) &= \text{Im}\left(\sum_{n=1}^{\infty} r^n \frac{\cos n\theta}{n} + i\sum_{n=1}^{\infty} r^n \frac{\sin n\theta}{n}\right) \\
&= \sum_{n=1}^{\infty} r^n \frac{\sin n\theta}{n}.
\end{aligned}$$

But

$$\begin{aligned}
\text{Im}\left(\log(1-z)^{-1}\right) &= -\text{Im}\left(\log(1-z)\right) \\
&= -\text{Arg}\,(1-z) = -\text{Arg}\,[(1-x) - iy] = \text{Arg}\,[(1-x)+iy] \\
&= \tan^{-1}\left(\frac{y}{1-x}\right) \\
&= \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right).
\end{aligned}$$

This is valid for all $0 \le r < 1$ and all θ, since in that case $x^2 + y^2 < 1$ and therefore the point $(1-x,\, y)$ always lies in the right half-plane, where $\text{Arg}\,(x+iy)$ can be expressed as $\tan^{-1}(y/x)$. Here $\text{Arg}\,z$ denotes the rincipal branch of the argument (see Section 12.5 for further discussion).

20. For $|z| < 1$,

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}.$$

Taking imaginary parts and using $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, we get

$$\begin{aligned}
\log(1+z) &= \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{e^{in\theta}}{n} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\cos n\theta + i\sin n\theta}{n} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\cos n\theta}{n} + i\sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\sin n\theta}{n}; \\
\text{Im}\left(\log(1+z)\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\sin n\theta}{n}
\end{aligned}$$

.

But

$$\begin{aligned}\text{Im } (log(1+z)) &= \text{Im } (\log(1+z)) \\ &= \text{Arg } (1+z) = \text{Arg } [(1+x)+iy] \\ &= \tan^{-1}\left(\frac{y}{1+x}\right) \\ &= \tan^{-1}\left(\frac{r\sin\theta}{1+r\cos\theta}\right).\end{aligned}$$

This is valid for all $0 \le r < 1$ and all θ.
(b) Taking real parts yields

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}r^n\frac{\cos n\theta}{n} &= \text{Re } (\log(1+z)) \\ &= \ln|(1+z)| = \ln\left((1+x)^2+y^2\right)^{1/2} \\ &= \frac{1}{2}\ln\left((1+x)^2+y^2\right) \\ &= \frac{1}{2}\ln\left(1+2r\cos\theta+r^2\right),\end{aligned}$$

where $x = r\cos\theta$ and $y = \sin\theta$.

21. Using the fact that the solutions must be bounded as $r \to \infty$, we see that $c_1 = 0$ in the first of the two equations in (3), and $c_2 = 0$ in the second of the two equations in (3). Thus

$$R(r) = R_n(r) = c_n r^{-n} = \left(\frac{r}{a}\right)^n \quad \text{for } n = 0,\, 1,\, 2,\, \ldots.$$

The general solution becomes

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\left(a_n\cos n\theta + b_n\sin n\theta\right), \quad r > a.$$

Setting $r = a$ and using the boundary condition, we obtain

$$f(\theta) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\theta + b_n\sin n\theta\right),$$

which implies that the a_n and b_n are the Fourier coefficients of f and hence are given by (5).

22. The Fourier coefficients of the boundary function are computed in Example 1. Applying the result of the previous exercise, we find

$$u(r,\,\theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\frac{\sin n\theta}{r^n}, \quad r > 1.$$

This is precisely the solution in Example 1, except that r is replaced by $\frac{1}{r}$. So to find the isotherms, we can proceed exactly as in Example 2, change r to $\frac{1}{r}$, and get: for $0 < T < 100$,

$$u(x,\,y) = T \quad \Leftrightarrow \quad x^2+y^2-1 = -2\left(\tan\frac{\pi T}{100}\right)y.$$

If $T = 50$, we find that $y = 0$ is the corresponding isotherm. If $T \neq 50$, obtain

$$x^2 + \left(y + \tan\frac{\pi T}{100}\right)^2 = \sec^2\frac{\pi T}{100}.$$

Thus the isotherms lie on circles with centers at $(0, -\tan\frac{\pi T}{100})$ and radii $\left|\sec\frac{\pi T}{100}\right|$.

Note that the isotherms here are the complementary portions of the circles found in Example 3, and these correspond to the complementary temperature $T - 100$.

23. Using Exercise 5 and formula (6), we find

$$\begin{aligned}
u(r, \theta) &= \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sin\left(\frac{n\pi}{4}\right)\cos n\theta + \left(1-\cos\left(\frac{n\pi}{4}\right)\right)\sin n\theta}{r^n}\right)\\
&= \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin n\theta}{nr^n} - \frac{n\left(\theta-\frac{\pi}{4}\right)}{nr^n}\right)\\
&= \frac{25}{2} + \frac{100}{\pi}\left[\tan^{-1}\left(\frac{r^{-1}\sin\theta}{1-r^{-1}\cos\theta}\right) - \tan^{-1}\left(\frac{r^{-1}\sin\left(\theta-\frac{\pi}{4}\right)}{1-r^{-1}\cos\left(\theta-\frac{\pi}{4}\right)}\right)\right]\\
&= \frac{25}{2} + \frac{100}{\pi}\left[\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) - \tan^{-1}\left(\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{r-\cos\left(\theta-\frac{\pi}{4}\right)}\right)\right].
\end{aligned}$$

To find the isotherms, let $0 < T < 100$, then

$$\begin{aligned}
u(r, \theta) = T \quad &\Leftrightarrow \quad T = \frac{25}{2} + \frac{100}{\pi}\left[\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) - \tan^{-1}\left(\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{r-\cos\left(\theta-\frac{\pi}{4}\right)}\right)\right]\\
&\Leftrightarrow \quad \frac{\pi}{100}\left(T-\frac{25}{2}\right) = \tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) - \tan^{-1}\left(\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{r-\cos\left(\theta-\frac{\pi}{4}\right)}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\tan\left[\frac{\pi}{100}\left(T-\frac{25}{2}\right)\right]\\
&\quad = \tan\left[\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) - \tan^{-1}\left(\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{r-\cos\left(\theta-\frac{\pi}{4}\right)}\right)\right]\\
&\quad = \frac{\frac{\sin\theta}{r-\cos\theta} - \frac{\sin\left(\theta-\frac{\pi}{4}\right)}{r-\cos\left(\theta-\frac{\pi}{4}\right)}}{1+\frac{\sin\theta}{r-\cos\theta}\cdot\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{r-\cos\left(\theta-\frac{\pi}{4}\right)}}\\
&\quad = \frac{r\left(\sin\theta-\sin\left(\theta-\frac{\pi}{4}\right)\right) - \sin\theta\cos\left(\theta-\frac{\pi}{4}\right) - \cos\theta\sin\left(\theta-\frac{\pi}{4}\right)}{r^2 - r\left(\cos\theta+\cos\left(\theta-\frac{\pi}{4}\right)\right) + \cos\theta\cos\left(\theta-\frac{\pi}{4}\right) + \sin\theta\sin\left(\theta-\frac{\pi}{4}\right)}\\
&\quad = \frac{r\left(\sin\theta-\sin\left(\theta-\frac{\pi}{4}\right)\right) - \sin\frac{\pi}{4}}{r^2 - r\left(\cos\theta+\cos\left(\theta-\frac{\pi}{4}\right)\right) + \cos\frac{\pi}{4}}\\
&\quad = \frac{r\left(\sqrt{2}\sin\theta - \sin\theta + \cos\theta\right) - 1}{\sqrt{2}\,r^2 - r\left(\sqrt{2}\cos\theta + \cos\theta + \sin\theta\right) + 1}\\
&\quad = \frac{x + \left(\sqrt{2}-1\right)y - 1}{\sqrt{2}\left(x^2+y^2\right) - \left(\sqrt{2}+1\right)x - y + 1}.
\end{aligned}$$

Let

$$K = \cot\left[\frac{\pi}{100}\left(T-\frac{25}{2}\right)\right].$$

The equation becomes of the isotherms becomes

$$\sqrt{2}\,(x^2+y^2)-\left(\sqrt{2}+1\right)x-y+1=Kx+K(\sqrt{2}-1)y-K$$

or

$$x^2-\left(1+\frac{1}{\sqrt{2}}+\frac{K}{\sqrt{2}}\right)x+y^2-\left(\frac{1}{\sqrt{2}}+\left(1-\frac{1}{\sqrt{2}}\right)K\right)y=-\frac{1}{\sqrt{2}}\,(1+K)\,.$$

Hence

$$\begin{aligned}
&\left(x-\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}+\frac{K}{\sqrt{2}}\right)\right)^2+\left(y-\frac{1}{2}\left(\frac{1}{\sqrt{2}}+\left(1-\frac{1}{\sqrt{2}}\right)K\right)\right)^2\\
&= -\frac{1}{\sqrt{2}}\,(1+K)\\
&\quad +\frac{1}{4}\left(1+\frac{1}{\sqrt{2}}+\frac{K}{\sqrt{2}}\right)^2+\frac{1}{4}\left(\frac{1}{\sqrt{2}}+\left(1-\frac{1}{\sqrt{2}}\right)K\right)^2\\
&= \frac{1}{4}\left(1+\sqrt{2}+\frac{1}{2}\right)+\frac{1}{8}-\frac{1}{\sqrt{2}}+\frac{K^2}{8}+\frac{1}{4}\left(1-\sqrt{2}+\frac{1}{2}\right)K^2\\
&= \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)(1-K^2)\,.
\end{aligned}$$

Thus the isotherms are circles with centers at

$$\left(\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}+\frac{K}{\sqrt{2}}\right),\,\frac{1}{2}\left(\frac{1}{\sqrt{2}}+\left(1-\frac{1}{\sqrt{2}}\right)K\right)\right)$$

and radii

$$\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)(1-K^2)}.$$

It is now straightforward to check that all these circles pass through the points $(1,\,0)$ and $\left(\frac{1}{\sqrt{2}},\,\frac{1}{\sqrt{2}}\right)$.

24. (a) As in the text, we have

$$R_n(r)=c_1(\frac{r}{R_2})^n+c_2(\frac{r}{R_2})^{-n}\quad\text{for } n=1,\,2,\,\ldots,$$

and

$$R_0(r)=c_1+c_2\ln(\frac{r}{R_2}).$$

The condition $R(R_2)=0$ implies that

$$0=R_n(R_2)=c_1(\frac{R_2}{R_2})^n+c_2(\frac{R_2}{R_2})^{-n}\quad\Rightarrow\quad c_1=-c_2;$$

and

$$0=R_0(R_2)=c_1+c_2\ln(\frac{R_2}{R_2})\quad\Rightarrow\quad c_1=0.$$

Hence

$$R_n(r)=c_2\left[(\frac{r}{R_2})^{-n}-(\frac{r}{R_2})^n\right]\quad\text{for } n=1,\,2,\,\ldots,$$

and

$$R_0(r)=c_2\ln(\frac{r}{R_2}).$$

Hence the general solution is

$$u(r,\theta)=A_0\ln(\frac{r}{R_2})+\sum_{n=1}^{\infty}[A_n\cos n\theta+B_n\sin n\theta]\left[(\frac{r}{R_2})^{-n}-(\frac{r}{R_2})^n\right].$$

Using the boundary condition, we obtain

$$f_1(\theta) = u(R_1,\,\theta) = A_0 \ln(\frac{R_1}{R_2}) + \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta] \left[(\frac{R_1}{R_2})^{-n} - (\frac{R_1}{R_2})^{n} \right],$$

which is the Fourier series of $f_1(\theta)$. Thus

$$A_0 \ln(\frac{R_1}{R_2}) = a_0 = \frac{1}{2\pi} \int_0^{2\pi} f_1(\theta)\, d\theta;$$

$$A_n \left[(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n} \right] = a_n = \frac{1}{\pi} \int_0^{2\pi} f_1(\theta) \cos n\theta\, d\theta;$$

and

$$B_n \left[(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n} \right] = b_n = \frac{1}{\pi} \int_0^{2\pi} f_1(\theta) \sin n\theta\, d\theta.$$

Solving for A_n, B_n in terms of a_n and b_n and plugging back into the solution u, we obtain

$$\begin{aligned} u(r,\theta) &= a_0 \frac{\ln(\frac{r}{R_2})}{\ln(\frac{R_1}{R_2})} + \sum_{n=1}^{\infty} [a_n \cos n\theta + b_n \sin n\theta] \frac{(\frac{r}{R_2})^{-n} - (\frac{r}{R_2})^{n}}{(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n}} \\ &= a_0 \frac{\ln r - \ln R_2}{\ln R_1 - \ln R_2} + \sum_{n=1}^{\infty} [a_n \cos n\theta + b_n \sin n\theta] \frac{R_1^n}{r^n} \frac{R_2^{2n} - r^{2n}}{R_2^{2n} - R_1^{2n}}. \end{aligned}$$

(b) Using the condition $R(R-1) = 0$ and proceeding as in (a), we find

$$R_n(r) = c_2 \left[(\frac{r}{R_1})^{n} - (\frac{r}{R_1})^{-n} \right] \quad \text{for } n = 1,\, 2,\, \ldots,$$

and

$$R_0(r) = c_2 \ln(\frac{r}{R_1}).$$

Hence the general solution is

$$u^*(r,\theta) = A_0^* \ln(\frac{r}{R_1}) + \sum_{n=1}^{\infty} [A_n^* \cos n\theta + B_n^* \sin n\theta] \left[(\frac{r}{R_1})^{n} - (\frac{r}{R_1})^{-n} \right].$$

Using the boundary condition, we obtain

$$f_2(\theta) = u^*(R_2,\,\theta) = A_0^* \ln(\frac{R_2}{R_1}) + \sum_{n=1}^{\infty} [A_n^* \cos n\theta + B_n^* \sin n\theta] \left[(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n} \right],$$

which is the Fourier series of $f_2(\theta)$. Thus

$$A_0^* \ln(\frac{R_2}{R_1}) = a_0^* = \frac{1}{2\pi} \int_0^{2\pi} f_2(\theta)\, d\theta;$$

$$A_n^* \left[(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n} \right] = a_n^* = \frac{1}{\pi} \int_0^{2\pi} f_2(\theta) \cos n\theta\, d\theta;$$

and

$$B_n^* \left[(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n} \right] = b_n^* = \frac{1}{\pi} \int_0^{2\pi} f_2(\theta) \sin n\theta\, d\theta.$$

Solving for A_n^*, B_n^* in terms of a_n^* and b_n^* and plugging back into the solution u^*, we obtain

$$\begin{aligned} u^*(r,\theta) &= a_0^* \frac{\ln(\frac{r}{R_1})}{\ln(\frac{R_2}{R_1})} + \sum_{n=1}^{\infty} [a_n^* \cos n\theta + b_n^* \sin n\theta] \frac{(\frac{r}{R_1})^{n} - (\frac{r}{R_1})^{-n}}{(\frac{R_2}{R_1})^{n} - (\frac{R_1}{R_2})^{n}} \\ &= a_0^* \frac{\ln r - \ln R_1}{\ln R_2 - \ln R_1} + \sum_{n=1}^{\infty} [a_n^* \cos n\theta + b_n^* \sin n\theta] \frac{R_2^n}{r^n} \frac{r^{2n} - R_1^{2n}}{R_2^{2n} - R_1^{2n}}, \end{aligned}$$

where a_0^*, a_n^*, and b_n^* are the Fourier coefficients of $f_2(\theta)$.
(c) Just add the solutions in (a) and (b).

25. The hint does it.

26. (a) The function is already given by its Fourier series. We have

$$f(\theta) = 50 - 50\cos\theta.$$

(b) The steady-state solution follows by usng the Fourier series of f. We have

$$u(r, \theta) = 50 - 50\,r\cos\theta.$$

(d) To find the isotherms, we must solve $u(r, \theta) = T$, where T is any value between 0 and 100. Using $x = r\cos\theta$, we find

$$50 - 50\,r\cos\theta = T \Rightarrow 50 - 50\,x = T.$$

Thus the isotherms are portions of the vertical lines

$$x = \frac{50 - T}{50}$$

inside the unit disk.
(e) We follow the steps outlined in Exercise 25. First subproblem: $\nabla^2 u_1 = 0$, $u_1(1, \theta) = 50 - 50\cos\theta$. Its solution is

$$u_1(r, \theta) = 50 - 50\,r\cos\theta.$$

Second subproblem: $v_t = \nabla^2 v$, $v(1, \theta, t) = 0$, $v(r, \theta, 0) = -50 + 50\,r\cos\theta$. We next find the Bessel series expanson of $v(r, \theta, 0)$. It is clear that only a_{0n} and a_{1n} are nonzero and that all the remaining coefficients a_{mn} and b_{mn} are 0. From Example 1, Section 4.8, we find that, for $0 < r < 1$,

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0(\alpha_{0,n} r)}{\alpha_{0,n} J_1(\alpha_{0,n})}.$$

Thus, for $0 < r < 1$,

$$-50 = -100\sum_{n=1}^{\infty} \frac{J_0(\alpha_{0,n} r)}{\alpha_{0,n} J_1(\alpha_{0,n})}.$$

Also, from Execrise 20, Section 4.8, we have, for $0 < r < 1$,

$$r = 2\sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n} J_2(\alpha_{1,n})}.$$

So, for $0 < r < 1$,

$$50\,r = 100\sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n} J_2(\alpha_{1,n})}.$$

Thus the solution is

$$\begin{aligned} u(r, \theta, t) &= 50 - 50\,r\cos\theta - 100\sum_{n=1}^{\infty} \frac{J_0(\alpha_{0,n} r)}{\alpha_{0,n} J_1(\alpha_{0,n})} e^{-\alpha_{0,n}^2 t} \\ &\quad +100\cos\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n} J_2(\alpha_{1,n})} e^{-\alpha_{1,n}^2 t}. \\ &= 100\sum_{n=1}^{\infty} \frac{J_0(\alpha_{0,n} r)}{\alpha_{0,n} J_1(\alpha_{0,n})}\left(1 - e^{-\alpha_{0,n}^2 t}\right) \\ &\quad -100\cos\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n} J_2(\alpha_{1,n})}\left(1 - e^{-\alpha_{1,n}^2 t}\right). \end{aligned}$$

27. (a) $a = c = 1$, $F(r,\theta) = (1-r^2)r\sin\theta$, $G(\theta) = \sin 2\theta$. Using the notation of Exercise 25, we have

$$u_1(r,\theta) = r^2 \sin 2\theta.$$

Also by Example 2, Section 4.3, we have

$$F(r,\theta) = (1-r^2)r\sin\theta = 16\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n}^3 J_2(\alpha_{1,n})}.$$

We now find the Bessel series expansion of u_1. Since u_1 is proportional to $\sin 2\theta$, we only only need b_{2n} (all other a_{mn} and b_{mn} are 0). We have

$$\begin{aligned} b_{2,n} &= \frac{2}{J_3(\alpha_{2,n})^2} \int_0^1 r^2 J_2(\alpha_{2,n} r) r\, dr \\ &= \frac{2}{\alpha_{2,n}^4 J_3(\alpha_{2,n})^2} \int_0^{\alpha_{2,n}} s^3 J_2(s)\, ds \quad (s = \alpha_{2,n}\, r) \\ &= \frac{2}{\alpha_{2,n}^4 J_3(\alpha_{2,n})^2} \left[s^3 J_3(s) \right] \Big|_0^{\alpha_{2,n}} \\ &= \frac{2}{\alpha_{2,n} J_3(\alpha_{2,n})}. \end{aligned}$$

Thus

$$u_1(r,\theta) = r^2 \sin 2\theta = 2\sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n} J_3(\alpha_{2,n})}$$

and

$$v(r,\theta,t) = 16\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n}^3 J_2(\alpha_{1,n})} e^{-\alpha_{1,n}^2 t} - 2\sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n} J_3(\alpha_{2,n})} e^{-\alpha_{2,n}^2 t};$$

and so

$$\begin{aligned} u(r,\theta,t) &= r^2 \sin 2\theta + 16\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n}^3 J_2(\alpha_{1,n})} e^{-\alpha_{1,n}^2 t} \\ &\quad -2\sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n} J_3(\alpha_{2,n})} e^{-\alpha_{2,n}^2 t} \\ &= 16\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\alpha_{1,n} r)}{\alpha_{1,n}^3 J_2(\alpha_{1,n})} e^{-\alpha_{1,n}^2 t} + 2\sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n} J_3(\alpha_{2,n})} \left(1 - e^{-\alpha_{2,n}^2 t}\right) \end{aligned}$$

(b) The steady-state solution is

$$u_1(r,\theta) = r^2 \sin 2\theta = 2\sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n} J_3(\alpha_{2,n})}.$$

28. (a) For $z = re^{i\theta}$ with $0 \le r = |z| < 1$, the geometric series is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} r^n e^{in\theta}.$$

Also $\overline{z} = re^{-i\theta}$, $\left|\overline{z}\right| = r$. So if $|z| < 1$ then $\left|\overline{z}\right| < 1$ and we have

$$\frac{1}{1-\overline{z}} = \sum_{n=0}^{\infty} \left(\overline{z}\right)^n = \sum_{n=0}^{\infty} r^n e^{-in\theta}.$$

For $|z| < 1$, we have

$$\begin{aligned}
\frac{1}{1-z} + \frac{1}{1-\overline{z}} - 1 &= \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=0}^{\infty} r^n e^{-in\theta} - 1 \\
&= 1 + \sum_{n=0}^{\infty} r^n \left(e^{in\theta} + e^{-in\theta}\right) \\
&= 1 + 2\sum_{n=0}^{\infty} r^n \cos n\theta,
\end{aligned}$$

where we have used $e^{in\theta} + e^{-in\theta} = 2\cos n\theta$. Using properties of the complex conjugate, we see that

$$\frac{1}{1-\overline{z}} = \frac{1}{\overline{1-z}} = \overline{\left(\frac{1}{1-z}\right)}.$$

So, because for any complex number w, $w + \overline{w} = 2\operatorname{Re} w$, we infer that

$$\frac{1}{1-z} + \frac{1}{1-\overline{z}} = 2\operatorname{Re}\left(\frac{1}{1-z}\right).$$

Comparing with the previous equations, we find that

$$2\operatorname{Re}\left(\frac{1}{1-z}\right) - 1 = 1 + 2\sum_{n=0}^{\infty} r^n \cos n\theta;$$

equivalently,

$$1 + 2\operatorname{Re}\left(\frac{1}{1-z} - 1\right) = 1 + 2\sum_{n=0}^{\infty} r^n \cos n\theta,$$

for all $0 \le r < 1$ and all θ.
(b) From (a) we have

$$\begin{aligned}
1 + 2\sum_{n=0}^{\infty} r^n \cos n\theta &= \frac{1}{1-z} + \frac{1}{1-\overline{z}} - 1 \\
&= \frac{2 - z - \overline{z}}{(1-z)(1-\overline{z})} - 1 \\
&= \frac{2 - (z+\overline{z}) - (1 - (z+\overline{z}) + |z|^2)}{1 - (z + \overline{z} + |z|^2} \\
&= \frac{1-r^2}{1 - 2r\cos\theta + r^2},
\end{aligned}$$

where we hae used $z \cdot \overline{z} = |z|^2 = r^2$ and $z + \overline{z} = 2\operatorname{Re} z = 2x = 2r\cos\theta$.

29. (a) Recalling the Euler formulas for the Fourier coefficients, we have

$$
\begin{aligned}
u(r,\,\theta) = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n [a_n \cos n\theta + b_n \sin n\theta] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\,d\phi \cos n\theta + \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\,d\phi \sin n\theta\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{1}{\pi}\int_0^{2\pi} f(\phi)\big[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\big]\,d\phi\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n(\theta-\phi)\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right]\,d\phi.
\end{aligned}
$$

(b) Continuing from (a) and using Exercise 28, we obtain

$$
\begin{aligned}
u(r,\,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{1-\left(\frac{r}{a}\right)^2}{1-2\left(\frac{r}{a}\right)\cos\theta + \left(\frac{r}{a}\right)^2} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P\big(r/a,\,\theta-\phi\big)\,d\phi.
\end{aligned}
$$

Solutions to Exercises 4.5

1. Using (2) and (3), we have that

$$u(\rho,\, z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho) \sinh(\lambda_n z), \quad \lambda_n = \frac{\alpha_n}{a},$$

where $\alpha_n = \alpha_{0,n}$ is the nth positive zero of J_0, and

$$\begin{aligned}
A_n &= \frac{2}{\sinh(\lambda_n h) a^2 J_1(\alpha_n)^2} \int_0^a f(\rho) J_0(\lambda_n \rho)\rho\, d\rho \\
&= \frac{200}{\sinh(2\alpha_n) J_1(\alpha_n)^2} \int_0^1 J_0(\alpha_n \rho)\rho\, d\rho \\
&= \frac{200}{\sinh(2\alpha_n)\alpha_n^2 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s) s\, ds \quad (\text{let } s = \alpha_n \rho) \\
&= \frac{200}{\sinh(2\alpha_n)\alpha_n^2 J_1(\alpha_n)^2} [J_1(s)s]\Big|_0^{\alpha_n} \\
&= \frac{200}{\sinh(2\alpha_n)\alpha_n J_1(\alpha_n)}.
\end{aligned}$$

So

$$u(\rho,\, z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\sinh(\alpha_n z)}{\sinh(2\alpha_n)\alpha_n J_1(\alpha_n)}.$$

2. Using (2) and (3), we have that

$$u(\rho,\, z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho) \sinh(\lambda_n z), \quad \lambda_n = \frac{\alpha_n}{a},$$

where $\alpha_n = \alpha_{0,n}$ is the nth positive zero of J_0, and

$$\begin{aligned}
A_n &= \frac{2}{\sinh(\lambda_n h) a^2 J_1(\alpha_n)^2} \int_0^a f(\rho) J_0(\lambda_n \rho)\rho\, d\rho \\
&= \frac{2}{\sinh(2\alpha_n) J_1(\alpha_n)^2} \int_0^1 (100-\rho^2) J_0(\alpha_n \rho)\rho\, d\rho \\
&= \frac{2}{\sinh(2\alpha_n)\alpha_n^4 J_1(\alpha_n)^2} \int_0^{\alpha_n} (100\alpha_n^2 - s^2) s J_0(s)\, ds \quad (\text{let } s = \alpha_n \rho) \\
&= \frac{2}{\sinh(2\alpha_n)\alpha_n^4 J_1(\alpha_n)^2} \left[(100\alpha_n^2 - s^2) s J_1(s)\Big|_0^{\alpha_n} + 2\int_0^{\alpha_n} s^2 J_1(s)\, ds \right] \\
&= \frac{2}{\sinh(2\alpha_n)\alpha_n^4 J_1(\alpha_n)^2} \left[(99\alpha_n^3 J_1(\alpha_n) + 2[s^2 J_2(s)]\Big|_0^{\alpha_n} \right] \\
&= \frac{2}{\sinh(2\alpha_n)\alpha_n^4 J_1(\alpha_n)^2} \left[99\alpha_n^3 J_1(\alpha_n) + 2\alpha_n^2 J_2(\alpha_n) \right] \\
&= \frac{2}{\sinh(2\alpha_n)\alpha_n^3 J_1(\alpha_n)} (99\alpha_n^2 + 4) \quad \left(\text{because } J_2(\alpha_n) = \frac{2}{\alpha_n} J_1(\alpha_n)\right)
\end{aligned}$$

So

$$u(\rho,\, z) = 2 \sum_{n=1}^{\infty} \frac{99\alpha_n^2 + 4}{\sinh(2\alpha_n)\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n \rho)\sinh(\alpha_n z).$$

3. We have

$$\begin{aligned} A_n &= \frac{200}{\sinh(2\alpha_n)J_1(\alpha_n)^2}\int_0^{\frac{1}{2}} J_0(\alpha_n\rho)\rho\,d\rho \\ &= \frac{200}{\sinh(2\alpha_n)\alpha_n^2 J_1(\alpha_n)^2}\int_0^{\frac{\alpha_n}{2}} J_0(s)s\,ds \quad (\text{let } s=\alpha_n\rho) \\ &= \frac{200}{\sinh(2\alpha_n)\alpha_n^2 J_1(\alpha_n)^2}[J_1(s)s]\Big|_0^{\frac{\alpha_n}{2}} \\ &= \frac{100 J_1(\frac{\alpha_n}{2})}{\sinh(2\alpha_n)\alpha_n J_1(\alpha_n)^2}. \end{aligned}$$

So

$$u(\rho,\,z) = 100\sum_{n=1}^{\infty}\frac{J_1(\frac{\alpha_n}{2})J_0(\alpha_n\rho)\sinh(\alpha_n z)}{\sinh(2\alpha_n)\alpha_n J_1(\alpha_n)^2}.$$

4. We have

$$\begin{aligned} A_n &= \frac{2}{\sinh(\lambda_n h)a^2 J_1(\alpha_n)^2}\int_0^a f(\rho)J_0(\lambda_n\rho)\rho\,d\rho \\ &= \frac{140}{\sinh(2\alpha_n)J_1(\alpha_n)^2}\int_0^1 J_0(\rho)J_0(\alpha_n\rho)\rho\,d\rho. \end{aligned}$$

In evaluating the last integral, we can proceed as we did in Exercise 6, Section 4.3. The details are very similar and so will be omitted. The result is

$$\int_0^1 J_0(\rho)J_0(\alpha_n\rho)\rho\,d\rho. = \frac{\alpha_n J_0(1)J_1(\alpha_n)}{\alpha_n^2-1}.$$

So

$$u(\rho,\,z) = 140J_0(1)\sum_{n=1}^{\infty}\frac{\alpha_n J_0(\alpha_n\rho)\sinh(\alpha_n z)}{\sinh(2\alpha_n)(\alpha_n^2-1)J_1(\alpha_n)}.$$

5. (a) We proceed exactly as in the text and arrive at the condition $Z(h)=0$ which leads us to the solutions

$$Z(z)=Z_n(z)=\sinh(\lambda_n(h-z)), \quad \text{where } \lambda_n=\frac{\alpha_n}{a}.$$

So the solution of the problem is

$$u(\rho,\,z)=\sum_{n=1}^{\infty} C_n J_0(\lambda_n\rho)\sinh(\lambda_n(h-z)),$$

where

$$C_n = \frac{2}{a^2 J_1(\alpha_n)^2\sinh(\lambda_n h)}\int_0^a f(\rho)J_0(\lambda_n\rho)\rho\,d\rho.$$

(b) The problem can be decomposed into the sum of two subproblems, one treated in the text and one treated in part (a). The solution of the problem is the sum of the solutions of the subproblems:

$$u(\rho,\,z)=\sum_{n=1}^{\infty}\Big(A_n J_0(\lambda_n\rho)\sinh(\lambda_n z)+C_n J_0(\lambda_n\rho)\sinh(\lambda_n(h-z))\Big),$$

where

$$A_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f_2(\rho) J_0(\lambda_n \rho)\rho \, d\rho,$$

and

$$C_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f_1(\rho) J_0(\lambda_n \rho)\rho \, d\rho.$$

6. Apply the result of Exercise 5. We have

$$\begin{aligned} A_n = C_n &= \frac{200}{J_1(\alpha_n)^2 \sinh(2\alpha_n)} \int_0^1 J_0(\alpha_n \rho)\rho \, d\rho \\ &= \frac{200}{\alpha_n^2 J_1(\alpha_n)^2 \sinh(2\alpha_n)} \int_0^{\alpha_n} J_0(s) s \, ds \\ &= \frac{200}{\alpha_n^2 J_1(\alpha_n)^2 \sinh(2\alpha_n)} J_1(s) s \Big|_0^{\alpha_n} \\ &= \frac{200}{\alpha_n J_1(\alpha_n) \sinh(2\alpha_n)}. \end{aligned}$$

Hence the solution

$$u(\rho, z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)}{\alpha_n J_1(\alpha_n) \sinh(2\alpha_n)} \left[\sinh(\alpha_n z) + \sinh(\alpha_n(2 - z))\right].$$

Using identities for the hyperbolic functions, you can check that

$$\sinh(\alpha_n z) + \sinh(\alpha_n(2 - z)) = 2 \sinh \alpha_n \cosh(\alpha_n(z - 1))$$

and

$$\sinh(2\alpha_n) = 2 \sinh \alpha_n \cosh \alpha_n.$$

So

$$\begin{aligned} u(\rho, z) &= 400 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)}{\alpha_n J_1(\alpha_n) \sinh(2\alpha_n)} \sinh \alpha_n \cosh(\alpha_n(z - 1)) \\ &= 200 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)}{\alpha_n J_1(\alpha_n) \cosh(\alpha_n)} \cosh(\alpha_n(z - 1)). \end{aligned}$$

7. Let $x = \lambda\rho$, then

$$\frac{d}{d\rho} = \lambda \frac{d}{dx}$$

and the equation

$$\rho^2 R'' + \rho R' - \lambda^2 \rho^2 R = 0,$$

which is the same as

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} - \lambda^2 \rho^2 R = 0,$$

becomes

$$\lambda^2 \rho^2 \frac{d^2 R}{dx^2} + \lambda\rho \frac{dR}{dx} - \lambda^2 \rho^2 R = 0$$

or

$$x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} - x^2 R = 0,$$

which is the modified Bessel equation of order 0. Its general solution is

$$R(x) = c_1 I_0(x) + c_2 K_0(x)$$

or

$$R(\rho) = c_1 I_0(\lambda\rho) + c_2 K_0(\lambda\rho).$$

8. (a) To separate variables in the problem

$$\begin{aligned} u_{\rho\rho} + \tfrac{1}{\rho} u_\rho + u_{zz} &= 0 \quad \text{for } 0 < \rho < a,\ 0 < z < h, \\ u(\rho, 0) = u(\rho, h) &= 0 \quad \text{for } 0 < \rho < a, \\ u(a, z) &= f(z) \quad \text{for } 0 < z < h, \end{aligned}$$

let $u(\rho, z) = R(\rho)Z(z)$. We obtain

$$R''Z + \frac{1}{\rho}R'Z + RZ'' = 0$$

or

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} = -\frac{Z''}{Z} = \lambda.$$

Hence

$$\begin{aligned} Z'' + \lambda Z &= 0, \\ \rho R'' + R' - \lambda\rho R &= = 0. \end{aligned}$$

Separating variables in the boundary conditions, we have

$$u(\rho, 0) = 0 = u(\rho, h) \quad \Rightarrow Z(0) = 0 = Z(h).$$

(b) The boundary value problem

$$Z'' + \lambda Z = 0 \quad Z(0) = 0 = Z(h)$$

has nontrivial solutions given by

$$Z_n(z) = \sin\frac{n\pi z}{h} \ (n = 1, 2, \ldots).$$

These correspond to the separation constant

$$\lambda = \lambda_n = \left(\frac{n\pi}{h}\right)^2.$$

(c) The equation in R becomes

$$\rho R'' + R' - \left(\frac{n\pi}{h}\right)^2 \rho R = 0$$

and, by Exercise 7, this has nontrivial solutions

$$R(\rho) = c_1 I_0\left(\frac{n\pi}{h}\rho\right) + c_2 K_0\left(\frac{n\pi}{h}\rho\right).$$

For $R(\rho)$ to remain bounded as $\rho \to 0^+$, we must take $c_2 = 0$ and hence we conclude that

$$R_n(\rho) = B_n I_0\left(\frac{n\pi}{h}\rho\right),$$

where B_n is a constant. We can now write the series solution of the problem as

$$u(\rho,\, z) = \sum_{n=1}^{\infty} B_n I_0\left(\frac{n\pi}{h}\rho\right) \sin\frac{n\pi z}{h}.$$

To determine B_n, we use the boundary condition

$$f(z) = u(a,\, z) = \sum_{n=1}^{\infty} B_n I_0\left(\frac{n\pi}{h}a\right) \sin\frac{n\pi z}{h}.$$

Thinking of this as a sine series of f, we conclude that

$$B_n I_0\left(\frac{n\pi}{h}a\right) = \frac{2}{h}\int_0^h f(z)\sin\frac{n\pi z}{h}\,dz$$

or

$$B_n = \frac{2}{h\, I_0\left(\frac{n\pi}{h}a\right)}\int_0^h f(z)\sin\frac{n\pi z}{h}\,dz.$$

9. We use the solution in Exercise 8 with $a = 1,\ h = 2,\ f(z) = 10z$. Then

$$\begin{aligned} B_n &= \frac{1}{I_0\left(\frac{n\pi}{2}\right)}\int_0^2 0z\sin\frac{n\pi z}{2}\,dz \\ &= \frac{40}{n\pi\, I_0\left(\frac{n\pi}{2}\right)}(-1)^{n+1}. \end{aligned}$$

Thus

$$u(\rho,\, z) = \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\, I_0\left(\frac{n\pi}{2}\right)} I_0\left(\frac{n\pi}{2}\rho\right)\sin\frac{n\pi z}{2}.$$

10. Adding the solutions to Exercises 6 and 9, we obtain, for $0 < \rho < 1$ and $0 < z < 2$,

$$\begin{aligned} u(\rho,\, z) &= 200\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\rho)}{\alpha_n J_1(\alpha_n)\cosh(\alpha_n)}\cosh(\alpha_n(z-1)) \\ &\quad + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\, I_0\left(\frac{n\pi}{2}\right)} I_0\left(\frac{n\pi}{2}\rho\right)\sin\frac{n\pi z}{2}. \end{aligned}$$

11. We have $a = 10,\ h = 6,$

$$\begin{aligned} u(\rho,\, 0) &= 56,\ \text{for } 0 < \rho < 10, \\ u(\rho,\, 6) &= 78,\ \text{for } 0 < \rho < 10, \\ u(10,\, z) &= \begin{cases} 56 & \text{for } 0 < z < 4, \\ 78 & \text{for } 4 < z < 6. \end{cases} \end{aligned}$$

Hence the solution is

$$\begin{aligned} u(\rho,\, z) &= 156\sum_{n=1}^{\infty}\frac{J_0(\frac{\alpha_n\rho}{10})}{\alpha_n J_1(\alpha_n)}\,\frac{\sinh(\frac{\alpha_n z}{10})}{\sinh(\frac{3}{5}\alpha_n)} \\ &\quad +112\sum_{n=1}^{\infty}\frac{J_0(\frac{\alpha_n\rho}{10})}{\alpha_n J_1(\alpha_n)}\,\frac{\sinh(\frac{\alpha_n(6-z)}{10})}{\sinh(\frac{3}{5}\alpha_n)} + \sum_{n=1}^{\infty} B_n I_0(\frac{n\pi\rho}{6})\sin\frac{n\pi z}{6}, \end{aligned}$$

where

$$\begin{aligned} B_n &= \frac{1}{3I_0(\frac{10n\pi}{6})} \int_0^6 u(10,\, z) \sin \frac{n\pi z}{6}\, dz \\ &= \frac{1}{3I_0(\frac{5n\pi}{3})} \left[56 \int_0^4 \sin \frac{n\pi z}{6}\, dz + 78 \int_4^6 \sin \frac{n\pi z}{6}\, dz \right] \\ &= \frac{2}{n\pi I_0(\frac{5n\pi}{3})} \left[56 + 22 \cos \frac{2n\pi}{3} - 78(-1)^n \right]. \end{aligned}$$

Solutions to Exercises 4.6

1. Write (1) in polar coordinates:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -k\phi \qquad \phi(a,\,\theta) = 0.$$

Consider a product solution $\phi(r,\,\theta) = R(r)\Theta(\theta)$. Since θ is a polar angle, it follows that

$$\Theta(\theta + 2\pi) = \Theta(\theta);$$

in other words, Θ is 2π-periodic. Plugging the product solution into the equation and simplifying, we find

$$\begin{aligned} R''\Theta + \tfrac{1}{r}R'\Theta + \tfrac{1}{r^2}R\Theta'' &= -kR\Theta; \\ \left(R'' + \tfrac{1}{r}R' + kR\right)\Theta &= -\tfrac{1}{r^2}R\Theta''; \\ r^2\tfrac{R''}{R} + r\tfrac{R'}{R} + kr^2 &= -\tfrac{\Theta''}{\Theta}; \end{aligned}$$

hence

$$r^2\frac{R''}{R} + r\frac{R'}{R} + kr^2 = \lambda,$$

and

$$-\frac{\Theta''}{\Theta} = \lambda \quad \Rightarrow \quad \Theta'' + \lambda\Theta = 0,$$

where λ is a separation constant. Our knowledge of solutions of second order linear ode's tells us that the last equation has 2π-periodic solutions if and only if

$$\lambda = m^2, \quad m = 0,\, \pm 1,\, \pm 2,\, \ldots.$$

This leads to the equations

$$\Theta'' + m^2\Theta = 0,$$

and

$$r^2\frac{R''}{R} + r\frac{R'}{R} + kr^2 = m^2 \quad \Rightarrow \quad r^2R'' + rR' + (kr^2 - m^2)R = 0.$$

These are equations (3) and (4). Note that the condition $R(a) = 0$ follows from $\phi(a,\,\theta) = 0 \Rightarrow R(a)\Theta(\theta) = 0 \Rightarrow R(a) = 0$ in order to avoid the constant 0 solution.

2. For $n = 1,\,2,\,\ldots$, let $\phi_{0,n}(r,\,\theta) = J_0(\lambda_{0,n}r)$ and for $m = 1,\,2,\,\ldots$ and $n = 1,\,2,\,\ldots$, let $\phi_{m,n}(r,\,\theta) = J_m(\lambda_{mn}r)\cos m\theta$ and $\psi_{m,n}(r,\,\theta) = J_m(\lambda_{mn}r)\sin m\theta$. The orthogonality relations for these functions state that if $(m\,n) \neq (m',\,n')$ then

$$\int_0^{2\pi}\int_0^a \phi_{mn}(r,\,\theta)\phi_{m'n'}(r,\,\theta)r\,dr\,d\theta = 0,$$

$$\int_0^{2\pi}\int_0^a \psi_{mn}(r,\,\theta)\psi_{m'n'}(r,\,\theta)r\,dr\,d\theta = 0,$$

and for all $(m\,n)$ and $(m',\,n')$

$$\int_0^{2\pi}\int_0^a \phi_{mn}(r,\,\theta)\psi_{m'n'}(r,\,\theta)r\,dr\,d\theta = 0.$$

The proofs follow from the orthogonality of the trigonometric functions and the Bessel functions. For example, to prove the first identity, write

$$\begin{aligned} \int_0^{2\pi}\int_0^a \phi_{mn}(r,\,\theta)\phi_{m'n'}(r,\,\theta)r\,dr\,d\theta &= \int_0^{2\pi}\cos m\theta\cos m'\theta d\theta \int_0^a J_m(\lambda_n r)J'_m(\lambda_{n'}r)r\,dr \\ &= I_1 I_2. \end{aligned}$$

If $m \neq m'$, then $I_1 = 0$ by the orthogonality of the 2π-periodic cosine functions. If $m = m'$, then necessarily $n \neq n'$ (otherwise $(m, n) = (m', n')$) and then

$$I_2 = \int_0^a J_m(\lambda_n r) J_m(\lambda_{n'} r) r\, dr = 0,$$

by the orthogonality relations for Bessel functions ((11), Section 4.8). Either way, we have $I = I_1 I_2 = 0$. The other relations are established similarly. We omit the proof.

3. Let ϕ_{mn} and ψ_{mn} be as in Exercise 2. We will evaluate

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^a \phi_{mn}^2(r, \theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} \cos^2 m\theta d\theta \int_0^a J_m(\lambda_n r)^2 r\, dr \\ &= I_1 I_2, \end{aligned}$$

where $m \neq 0$. We have

$$I_1 = \int_0^{2\pi} \cos^2 m\theta = \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} = \pi,$$

and

$$I_2 = \int_0^a J_m(\lambda_n r)^2 r\, dr = \frac{a^2}{2} J_{m+1}^2(\alpha_{mn}),$$

by (12), Section 4.8. Thus

$$I = \frac{\pi a^2}{2} J_{m+1}^2(\alpha_{mn}).$$

If $m = 0$, then

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^a \phi_{0n}^2(r, \theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^a J_0(\lambda_0 r)^2 r\, dr \\ &= \pi a^2 J_{m+1}^2(\alpha_{mn}). \end{aligned}$$

Finally,

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^a \psi_{mn}^2(r, \theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} \sin^2 m\theta d\theta \int_0^a J_m(\lambda_n r)^2 r\, dr \\ &= \frac{\pi a^2}{2} J_{m+1}^2(\alpha_{mn}). \end{aligned}$$

5. We proceed as in Example 1 and try

$$u(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn} \cos m\theta + B_{mn} \sin m\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(r, \theta),$$

where $\phi_{mn}(r, \theta) = J_m(\lambda_{mn} r)(A_{mn} \cos m\theta + B_{mn} \sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2 \phi_{mn}$, and get

$$\begin{aligned}
\nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\right) &= 1-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\,\theta)\right) = 1-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}-\lambda_{mn}^2\phi_{mn}(r,\,\theta) = 1-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(1-\alpha_{mn}^2)\phi_{mn}(r,\,\theta) = 1.
\end{aligned}$$

We recognize this expansion as the expansion of the function 1 in terms of the functions ϕ_{mn}. Because the right side is independent of θ, it follows that all A_{mn} and B_{mn} are zero, except $A_{0,n}$. So

$$\sum_{n=1}^{\infty}(1-\alpha_{mn}^2)A_{0,n}J_0(\alpha_{0,n})r) = 1,$$

which shows that $(1-\alpha_{mn}^2)A_{0,n} = a_{0,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 0 of the function 1. This series is computed in Example 1, Section 4.8. We have

$$1 = \sum_{n=1}^{\infty}\frac{2}{\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r) \quad 0 < r < 1.$$

Hence

$$(1-\alpha_{mn}^2)A_{0,n} = \frac{2}{\alpha_{0,n}J_1(\alpha_{0,n})} \quad \Rightarrow \quad A_{0,n} = \frac{2}{(1-\alpha_{mn}^2)\alpha_{0,n}J_1(\alpha_{0,n})};$$

and so

$$u(r,\,\theta) = \sum_{n=1}^{\infty}\frac{2}{(1-\alpha_{mn}^2)\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r).$$

6. We proceed as in the previous exercise and try

$$u(r,\,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta),$$

where $\phi_{mn}(r,\,\theta) = J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2\phi_{mn} = -\alpha_{mn}^2\phi_{mn}$, and get

$$\begin{aligned}
\nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\right) &= 3\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta) + r\sin\theta\\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\,\theta)\right) = 3\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta) + r\sin\theta\\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}-\alpha_{mn}^2\phi_{mn}(r,\,\theta) = 3\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta) + r\sin\theta\\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-3-\alpha_{mn}^2)\phi_{mn}(r,\,\theta) = r\sin\theta.
\end{aligned}$$

We recognize this expansion as the expansion of the function $r\sin\theta$ in terms of the functions ϕ_{mn}. Because the right side is proportional to $\sin\theta$, it follows that all A_{mn} and B_{mn} are zero, except $B_{1,n}$. So

$$\sin\theta\sum_{n=1}^{\infty}(-3-\alpha_{1n}^2)B_{1,n}J_1(\alpha_{1n}r) = r\sin\theta,$$

which shows that $(-3-\alpha_{1n}^2)B_{1,n} = b_{1,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 1 of the function r. This series is computed in Exercise 20, Section 4.8. We have

$$r = \sum_{n=1}^{\infty} \frac{2}{\alpha_{1,n} J_2(\alpha_{1,n})} J_1(\alpha_{1,n} r) \quad 0 < r < 1.$$

Hence

$$(-3-\alpha_{1n}^2)B_{1,n} = \frac{2}{\alpha_{1,n} J_2(\alpha_{1,n})} \quad \Rightarrow \quad B_{1,n} = \frac{-2}{(3+\alpha_{1,n}^2)\alpha_{1,n} J_2(\alpha_{1,n})};$$

and so

$$u(r,\theta) = \sin\theta \sum_{n=1}^{\infty} \frac{-2}{(3+\alpha_{1,n}^2)\alpha_{1,n} J_2(\alpha_{1,n})} J_1(\alpha_{1,n} r).$$

7. We proceed as in the previous exercise and try

$$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \phi_{mn}(r,\theta),$$

where $\phi_{mn}(r,\theta) = J_m(\lambda_{mn} r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2\phi_{mn} = -\alpha_{mn}^2\phi_{mn}$, and get

$$\begin{aligned}
\nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\theta)\right) &= 2 + r^3\cos 3\theta \\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\theta)\right) = 2 + r^3\cos 3\theta \\
&\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\alpha_{mn}^2\phi_{mn}(r,\theta) = 2 + r^3\cos 3\theta.
\end{aligned}$$

We recognize this expansion as the expansion of the function $2 + r^3\cos 3\theta$ in terms of the functions ϕ_{mn}. Because of the special form of the right side, it follows that all A_{mn} and B_{mn} are zero, except $A_{0,n}$ and $A_{3,n}$. So

$$\sum_{n=1}^{\infty} -\alpha_{0n}^2 A_{0,n} J_0(\alpha_{0n} r) = 2$$

and

$$\cos 3\theta \sum_{n=1}^{\infty} -\alpha_{3n}^2 A_{3,n} J_3(\alpha_{3n} r) = r^3\cos 3\theta.$$

Using the series in Example 1, Section 4.8:

$$1 = \sum_{n=1}^{\infty} \frac{2}{\alpha_{0,n} J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r) \quad 0 < r < 1,$$

we find that

$$2 = \sum_{n=1}^{\infty} \frac{4}{\alpha_{0,n} J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r) \quad 0 < r < 1,$$

and so

$$\sum_{n=1}^{\infty} -\alpha_{0n}^2 A_{0,n} J_0(\alpha_{0n} r) = \sum_{n=1}^{\infty} \frac{4}{\alpha_{0,n} J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r),$$

from which we conclude that

$$-\alpha_{0n}^2 A_{0,n} = \frac{4}{\alpha_{0,n} J_1(\alpha_{0,n})}$$

or

$$A_{0,n} = \frac{-4}{\alpha_{0,n}^3 J_1(\alpha_{0,n})}.$$

From the series in Exercise 20, Section 4.8 (with $m = 3$): We have

$$r^3 = \sum_{n=1}^{\infty} \frac{2}{\alpha_{3,n} J_4(\alpha_{3,n})} J_3(\alpha_{3,n} r) \quad 0 < r < 1,$$

we conclude that

$$\sum_{n=1}^{\infty} \frac{2}{\alpha_{3,n} J_4(\alpha_{3,n})} J_3(\alpha_{3,n} r) = \sum_{n=1}^{\infty} -\alpha_{3n}^2 A_{3,n} J_3(\alpha_{3n} r)$$

and so

$$A_{3,n} = \frac{-2}{\alpha_{3,n}^3 J_4(\alpha_{3,n})}.$$

Hence

$$u(r, \theta) = \sum_{n=1}^{\infty} \frac{-4}{\alpha_{0,n}^3 J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r) + \cos 3\theta \sum_{n=1}^{\infty} \frac{-2}{\alpha_{3,n}^3 J_4(\alpha_{3,n})} J_3(\alpha_{3,n} r).$$

8. We proceed as in Exercise 5. Let

$$u(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(r, \theta),$$

where $\phi_{mn}(r, \theta) = J_m(\lambda_{mn} r)(A_{mn} \cos m\theta + B_{mn} \sin m\theta)$. Plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2 \phi_{mn}$, and get

$$\begin{aligned} \nabla^2 \left(\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(r, \theta) \right) &= r^2 \\ &\Rightarrow \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} -\alpha_{mn}^2 \phi_{mn}(r, \theta) = r^2. \end{aligned}$$

We recognize this expansion as the expansion of the function 1 in terms of the functions ϕ_{mn}. Because the right side is independent of θ, it follows that all A_{mn} and B_{mn} are zero, except $A_{0,n}$. So

$$\sum_{n=1}^{\infty} -\alpha_{mn}^2 A_{0,n} J_0(r) = r^2,$$

which shows that $-\alpha_{mn}^2 A_{0,n} = a_{0,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 0 of the function r^2. By Theorem 2, Section 4.8, we have

$$\begin{aligned} a_{0,n} &= \frac{2}{J_1(\alpha_{0,n})^2} \int_0^1 r^2 J_0(\alpha_{0,n}) r \, dr \\ &= \frac{2}{\alpha_{0,n}^4 J_1(\alpha_{0,n})^2} \int_0^{\alpha_{0,n}} s^3 J_0(s) \, ds \quad (\text{let } \alpha_{0,n})r = s) \\ &\quad u = s^2,\ du = 2s\, ds,\ dv = s J_0(s)\, ds,\ v = s J_1(s) \\ &= \frac{2}{\alpha_{0,n}^4 J_1(\alpha_{0,n})^2} \left[s^3 J_1(s) \Big|_0^{\alpha_{0,n}} - 2 \int_0^{\alpha_{0,n}} s^2 J_1(s) \, ds \right] \\ &= \frac{2}{\alpha_{0,n}^4 J_1(\alpha_{0,n})^2} \left[\alpha_{0,n}^3 J_1(\alpha_{0,n}) - 2s^2 J_2(s) \Big|_0^{\alpha_{0,n}} \right] \\ &= \frac{2}{\alpha_{0,n}^4 J_1(\alpha_{0,n})^2} \left[\alpha_{0,n}^3 J_1(\alpha_{0,n}) - 2\alpha_{0,n}^2 J_2(\alpha_{0,n}) \right]. \end{aligned}$$

Hence

$$A_{0,n} = \frac{-2}{\alpha_{0,n}^4 J_1(\alpha_{0,n})^2} \left[\alpha_{0,n} J_1(\alpha_{0,n}) - 2J_2(\alpha_{0,n})\right],$$

and so

$$u(r,\,\theta) = \sum_{n=1}^{\infty} \frac{-2}{\alpha_{0,n}^4 J_1(\alpha_{0,n})^2} \left[\alpha_{0,n} J_1(\alpha_{0,n}) - 2J_2(\alpha_{0,n})\right] J_0(\alpha_{0,n} r).$$

9. Let

$$h(r) = \begin{cases} r & \text{if } 0 < r < 1/2, \\ 0 & \text{if } 1/2 < r < 1. \end{cases}$$

Then the equation becomes $\nabla^2 u = f(r,\,\theta)$, where $f(r,\,\theta) = h(r)\sin\theta$. We proceed as in the previous exercise and try

$$u(r,\,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \phi_{mn}(r,\,\theta),$$

where $\phi_{mn}(r,\,\theta) = J_m(\lambda_{mn} r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2\phi_{mn} = -\alpha_{mn}^2\phi_{mn}$, and get

$$\begin{aligned} \nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\right) &= h(r)\sin\theta \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\,\theta)\right) = h(r)\sin\theta \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\alpha_{mn}^2\phi_{mn}(r,\,\theta) = h(r)\sin\theta. \end{aligned}$$

We recognize this expansion as the expansion of the function $h(r)\sin\theta$ in terms of the functions ϕ_{mn}. Because the right side is proportional to $\sin\theta$, it follows that all A_{mn} and B_{mn} are zero, except $B_{1,n}$. So

$$\sin\theta\sum_{n=1}^{\infty} -\alpha_{1n}^2 B_{1,n} J_1(\alpha_{1n} r) = h(r)\sin\theta,$$

which shows that $-\alpha_{1n}^2 B_{1,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 1 of the function $h(r)$:

$$\begin{aligned} -\alpha_{1n}^2 B_{1,n} &= \frac{2}{J_2(\alpha_{1,n})^2}\int_0^{1/2} r^2 J_1(\alpha_{1,n} r)\,dr \\ &= \frac{2}{\alpha_{1,n}^3 J_2(\alpha_{1,n})^2}\int_0^{\alpha_{1,n}/2} s^2 J_1(s)\,ds \\ &= \frac{2}{\alpha_{1,n}^3 J_2(\alpha_{1,n})^2} s^2 J_2(s)\Big|_0^{\alpha_{1,n}/2} \\ &= \frac{J_2(\alpha_{1,n}/2)}{2\alpha_{1,n} J_2(\alpha_{1,n})^2}. \end{aligned}$$

Thus

$$u(r,\,\theta) = \sin\theta\sum_{n=1}^{\infty} -\frac{J_2(\alpha_{1,n}/2)}{2\alpha_{1,n}^3 J_2(\alpha_{1,n})^2} J_1(\alpha_{1,n} r).$$

11. We have $u(r,\theta) = u_1(r,\theta)+u_2(r,\theta)$, where $u_1(r,\theta)$ is the solution of $\nabla^2 u = 1$ and $u(1,\theta) = 0$, and $u_2(r,\theta)$ is the solution of $\nabla^2 u = 0$ and $u(1,\theta) = \sin 2\theta$. By Example 1, we have

$$u_1(r,\theta) = \sum_{n=1}^{\infty} \frac{-2}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n r);$$

while u_2 is easily found by the methods of Section 4.4:

$$u_2(r\,\theta) = r^2 \sin 2\theta.$$

Thus,

$$u(r,\theta) = r^2 \sin 2\theta + \sum_{n=1}^{\infty} \frac{-2}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n r).$$

Solutions to Exercises 4.7

1. Bessel equation of order 3. Using (7), the first series solution is

$$J_3(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+3)!}\left(\frac{x}{2}\right)^{2k+3}=\frac{1}{1\cdot 6}\frac{x^3}{8}-\frac{1}{1\cdot 24}\frac{x^5}{32}+\frac{1}{2\cdot 120}\frac{x^7}{128}+\cdots.$$

5. Bessel equation of order $\frac{3}{2}$. The general solution is

$$\begin{aligned} y(x) &= c_1 J_{\frac{3}{2}}+c_2 J_{-\frac{3}{2}} \\ &= c_1\left(\frac{1}{1\cdot\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^{\frac{3}{2}}-\frac{1}{1\cdot\Gamma(\frac{7}{2})}\left(\frac{x}{2}\right)^{\frac{7}{2}}+\cdots\right) \\ &\quad +c_2\left(\frac{1}{1\cdot\Gamma(\frac{-1}{2})}\left(\frac{x}{2}\right)^{-\frac{3}{2}}-\frac{1}{1\cdot\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}}+\cdots\right). \end{aligned}$$

Using the basic property of the gamma function and (15), we have

$$\begin{gathered} \Gamma(\tfrac{5}{2})=\tfrac{3}{2}\Gamma(\tfrac{3}{2})=\tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2})=\tfrac{3}{4}\sqrt{\pi} \\ \Gamma(\tfrac{7}{2})=\tfrac{5}{2}\Gamma(\tfrac{5}{2})=\tfrac{15}{8}\sqrt{\pi} \\ -\tfrac{1}{2}\Gamma(-\tfrac{1}{2})=\Gamma(\tfrac{1}{2})=\sqrt{\pi}\ \Rightarrow\ \Gamma(-\tfrac{1}{2})=-2\sqrt{\pi}. \end{gathered}$$

So

$$\begin{aligned} y(x) &= c_1\sqrt{\frac{2}{\pi x}}\left(\frac{4}{3}\frac{x^2}{4}-\frac{8}{15}\frac{x^4}{16}+\cdots\right) \\ &\quad c_2\sqrt{\frac{2}{\pi x}}(-1)\left(-\frac{1}{2}\frac{2}{x}-\frac{x}{2}-\cdots\right) \\ &= c_1\sqrt{\frac{2}{\pi x}}\left(\frac{x^2}{3}-\frac{x^4}{30}+\cdots\right)+c_2\sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}+\frac{x}{2}-\cdots\right) \end{aligned}$$

9. Divide the equation through by x^2 and put it in the form

$$y''+\frac{1}{x}y'+\frac{x^2-9}{x^2}y=0 \quad \text{for } x>0.$$

Now refer to Appendix A.6 for terminology and for the method of Frobenius that we are about to use in this exercise. Let

$$p(x)=\frac{1}{x} \quad \text{for} \quad q(x)=\frac{x^2-9}{x^2}.$$

The point $x=0$ is a singular point of the equation. But since $x\,p(x)=1$ and $x^2\,q(x)=x^2-9$ have power series expansions about 0 (in fact, they are already given by their power series expansions), it follows that $x=0$ is a regular singular point. Hence we may apply the Frobenius method. We have already found one series solution in Exercise 1. To determine the second series solution, we consider the indicial equation

$$r(r-1)+p_0 r+q_0=0,$$

where $p_0=1$ and $q_0=-9$ (respectively, these are the constant terms in the series expansions of $xp(x)$ and $x^2q(x)$). Thus the indicial equation is

$$r-9=0 \quad \Rightarrow \quad r_1=3,\ r_2=-3.$$

The indicial roots differ by an integer. So, according to Theorem 2, Appendix A.6, the second solution y_2 may or may not contain a logarithmic term. We have, for $x>0$,

$$y_2=ky_1\ln x+x^{-3}\sum_{m=0}^{\infty}b_m x^m=ky_1\ln x+\sum_{m=0}^{\infty}b_m x^{m-3},$$

where $a_0 \neq 0$ and $b_0 \neq 0$, and k may or may not be 0. Plugging this into the differential equation

$$x^2y'' + xy' + (x^2 - 9)y = 0$$

and using the fact that y_1 is a solution, we have

$$\begin{aligned}
y_2 &= ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3} \\
y_2' &= ky_1' \ln x + k\frac{y_1}{x} + \sum_{m=0}^{\infty} (m-3)b_m x^{m-4}; \\
y_2'' &= ky_1'' \ln x + k\frac{y_1'}{x} + k\frac{xy_1' - y_1}{x^2} + \sum_{m=0}^{\infty} (m-3)(m-4)b_m x^{m-5} \\
&= ky_1'' \ln x + 2k\frac{y_1'}{x} - k\frac{y_1}{x^2} + \sum_{m=0}^{\infty} (m-3)(m-4)b_m x^{m-5};
\end{aligned}$$

$$\begin{aligned}
&x^2y_2'' + xy_2' + (x^2-9)y_2 \\
&= kx^2y_1'' \ln x + 2kxy_1' - ky_1 + \sum_{m=0}^{\infty} (m-3)(m-4)b_m x^{m-3} \\
&\quad + kxy_1' \ln x + ky_1 + \sum_{m=0}^{\infty} (m-3)b_m x^{m-3} \\
&\quad + (x^2-9)\left[ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3}\right] \\
&= k \ln x \big[\overbrace{x^2y_1'' + xy_1' + (x^2-9)y_1}^{=0}\big] \\
&\quad + 2kxy_1' + \sum_{m=0}^{\infty} \big[(m-3)(m-4)b_m + (m-3)b_m - 9b_m\big]x^{m-3} \\
&\quad + x^2 \sum_{m=0}^{\infty} b_m x^{m-3} \\
&= 2kxy_1' + \sum_{m=0}^{\infty} (m-6)mb_m x^{m-3} + \sum_{m=0}^{\infty} b_m x^{m-1}.
\end{aligned}$$

To combine the last two series, we use reindexing as follows

$$\begin{aligned}
&\sum_{m=0}^{\infty} (m-6)mb_m x^{m-3} + \sum_{m=0}^{\infty} b_m x^{m-1} \\
&\quad = -5b_1x^{-2} + \sum_{m=2}^{\infty} (m-6)mb_m x^{m-3} + \sum_{m=2}^{\infty} b_{m-2} x^{m-3} \\
&\quad = -5b_1x^{-2} + \sum_{m=2}^{\infty} \big[(m-6)mb_m + b_{m-2}\big]x^{m-3}.
\end{aligned}$$

Thus the equation

$$x^2y_2'' + xy_2' + (x^2-9)y_2 = 0$$

implies that

$$2kxy_1' - 5b_1x^{-2} + \sum_{m=2}^{\infty} \big[(m-6)mb_m + b_{m-2}\big]x^{m-3} = 0.$$

This equation determines the coefficients b_m ($m \geq 1$) in terms of the coefficients of y_1. Furthermore, it will become apparent that k cannot be 0. Also, b_0 is arbitrary but by assumption $b_0 \neq 0$. Let's take $b_0 = 1$ and determine the the first five b_m's.

Recall from Exercise 1

$$y_1 = \frac{1}{1\cdot 6}\frac{x^3}{8} - \frac{1}{1\cdot 24}\frac{x^5}{32} + \frac{1}{2\cdot 120}\frac{x^7}{128} + \cdots .$$

So

$$y_1' = \frac{3}{1\cdot 6}\frac{x^2}{8} - \frac{5}{1\cdot 24}\frac{x^4}{32} + \frac{7}{2\cdot 120}\frac{x^6}{128} + \cdots$$

and hence (taking $k = 1$)

$$2kxy_1' = \frac{6k}{1\cdot 6}\frac{x^3}{8} - \frac{10k}{1\cdot 24}\frac{x^5}{32} + \frac{14k}{2\cdot 120}\frac{x^7}{128} + \cdots .$$

The lowest exponent of x in

$$2kxy_1' - 5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}$$

is x^{-2}. Since its coefficient is $-5b_1$, we get $b_1 = 0$ and the equation becomes

$$2xy_1' + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}.$$

Next, we consider the coefficient of x^{-1}. It is $(-4)2b_2 + b_0$. Setting it equal to 0, we find

$$b_2 = \frac{b_0}{8} = \frac{1}{8}.$$

Next, we consider the constant term, which is the $m = 3$ term in the series. Setting its coefficient equal to 0, we obtain

$$(-3)3b_3 + b_1 = 0 \quad \Rightarrow \quad b_3 = 0$$

because $b_1 = 0$. Next, we consider the term in x, which is the $m = 4$ term in the series. Setting its coefficient equal to 0, we obtain

$$(-2)4b_4 + b_2 = 0 \quad \Rightarrow \quad b_4 = \frac{1}{8}b_2 = \frac{1}{64}.$$

Next, we consider the term in x^2, which is the $m = 5$ term in the series. Setting its coefficient equal to 0, we obtain $b_5 = 0$. Next, we consider the term in x^3, which is the $m = 6$ term in the series plus the first term in $2kxy_1'$. Setting its coefficient equal to 0, we obtain

$$0 + b_4 + \frac{k}{8} = 0 \quad \Rightarrow \quad k = -8b_4 = -\frac{1}{8}.$$

Next, we consider the term in x^4, which is the $m = 7$ term in the series. Setting its coefficient equal to 0, we find that $b_7 = 0$. It is clear that $b_{2m+1} = 0$ and that

$$y_2 \approx -\frac{1}{8}y_1 \ln x + \frac{1}{x^3} + \frac{1}{8\,x} + \frac{1}{64}x + \cdots$$

Any nonzero constant multiple of y_2 is also a second linearly independent solution of y_1. In particular, $384\,y_2$ is an alternative answer (which is the answer given in the text).

10. Linear independence for two solutions is equivalent to one not being a constant multiple of the other. Since J_p is not a constant multiple of Y_p, it follows that

$x^p J_p$ are not a constant multiple of $x^p Y_p$ and hence $x^p J_p$ and $x^p Y_p$ are linearly independent. We now verify that they are solutions of

$$xy'' + (1-2p)y' + xy = 0.$$

We have

$$\begin{aligned}
y &= x^p J_p(x),\\
y' &= px^{p-1}J_p(x) + x^p J_p',\\
y'' &= p(p-1)x^{p-2}J_p(x) + 2px^{p-1}J_p'(x) + x^p J_p''(x),\\
xy'' + (1-2p)y' + xy &= x\big[p(p-1)x^{p-2}J_p(x) + 2px^{p-1}J_p'(x) + x^p J_p''(x)\big]\\
&\quad +(1-2p)\big[px^{p-1}J_p(x) + x^p J_p'(x)\big] + x\,x^p J_p(x)\\
&= x^{p-1}\big[x^2 J_p''(x) + [2px + (1-2p)x]J_p'(x)\\
&\quad +[p(p-1) + (1-2p)p + x^2]J_p(x)\big]\\
&= x^{p-1}\big[x^2 J_p''(x) + xJ_p'(x) + (x^2 - p^2)J_p(x)\big]\\
&= 0
\end{aligned}$$

because J_p satisfies $x^2 J_p''(x) + xJ_p'(x) + (x^2-p^2)J_p(x) = 0$. A similar proof shows that $x^p Y_p$ is also a solution of $xy'' + (1-2p)y' + xy = 0$.

11. The equation is of the form given in Exercise 10 with $p = 0$. Thus its general solution is

$$y = c_1 J_0(x) + c_2 Y_0(x).$$

This is, of course, the general solution of Bessel's equation of order 0. Indeed, the given equation is equivalent to Bessel's equation of order 0.

12. The equation is of the form given in Exercise 10 with $p = 1/2$. Thus its general solution is

$$y = c_1 x^{1/2} J_{1/2}(x) + c_2 x^{1/2} Y_{1/2}(x).$$

But it is also clear that

$$y = a\cos x + b\sin x$$

is the general solution of $y'' + y = 0$. In fact, the two solutions are equivalent since $x^{1/2}\,J_{1/2}(x)$ is a constant multiple of $\sin x$ (see Example 1) and $x^{1/2}\,Y_{1/2}(x)$ is a constant multiple of $\cos x$. (From Example 1, it follows that $x^{1/2}\,J_{-1/2}(x)$ is a constant multiple of $\cos x$. But, by (10), $Y_{1/2}(x) = -J_{-1/2}(x)$. So $x^{1/2}\,Y_{1/2}(x)$ is a constant multiple of $\cos x$ as claimed.)

13. The equation is of the form given in Exercise 10 with $p = 3/2$. Thus its general solution is

$$y = c_1 x^{3/2} J_{3/2}(x) + c_2 x^{3/2} Y_{3/2}(x).$$

Using Exercise 22 and (1), you can also write this general solution in the form

$$\begin{aligned}
y &= c_1 x\left[\frac{\sin x}{x} - \cos x\right] + c_2 x\left[-\frac{\cos x}{x} - \sin x\right]\\
&= c_1\left[\sin x - x\cos x\right] + c_2\left[-\cos x - x\sin x\right].
\end{aligned}$$

In particular, two linearly independent solution are

$$y_1 = \sin x - x\cos x \quad \text{and} \quad y_2 = \cos x + x\sin x.$$

This can be verified directly by using the differential equation (try it!).

14. The equation is of the form given in Exercise 10 with $p = 2$. Thus its general solution is

$$y = c_1 x^2 J_2(x) + c_2 x^2 Y_2(x).$$

15. All three parts, (a)–(c), follow directly from the series form of $J_p(x)$ (see (7)).

17. We have

$$\begin{aligned}
y &= x^{-p}u,\\
y' &= -px^{-p-1}u + x^{-p}u',\\
y'' &= p(p+1)x^{-p-2}u + 2(-p)x^{-p-1}u' + x^{-p}u'',\\
xy'' + (1+2p)y' + xy &= x\big[p(p+1)x^{-p-2}u - 2px^{-p-1}u' + x^{-p}u''\big]\\
&\quad +(1+2p)\big[-px^{-p-1}u + x^{-p}u'\big] + x\,x^{-p}u\\
&= x^{-p-1}\big[x^2u'' + [-2px + (1+2p)x]u'\\
&\quad +[p(p+1) - (1+2p)p + x^2]u\big]\\
&= x^{-p-1}\big[x^2u'' + xu' + (x^2-p^2)u\big].
\end{aligned}$$

Thus, by letting $y = x^{-p}u$, we transform the equation

$$xy'' + (1+2p)y' + xy = 0$$

into the equation

$$x^{-p-1}\big[x^2u'' + xu' + (x^2-p^2)u\big] = 0,$$

which, for $x > 0$, is equivalent to

$$x^2u'' + xu' + (x^2-p^2)u = 0,$$

a Bessel equation of ordr $p > 0$ in u. The general solution of the last equation is

$$u = c_1 J_p(x) + c_2 Y_p(x).$$

Thus the general solution of the original equation is

$$Y = c_1 x^{-p} J_p(x) + c_2 x^{-p} Y_p(x).$$

18. For $x > 0$, let $z = \sqrt{x}$ and $\phi(z) = y(x)$. Then $\frac{dz}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2z}$ and, by the chain rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\phi(z) = \frac{d\phi}{dz}\frac{dz}{dx} = \frac{d\phi}{dz}\frac{1}{2\sqrt{x}} = \frac{d\phi}{dz}\frac{1}{2z},\\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left[\frac{d\phi}{dz}\frac{1}{2\sqrt{x}}\right]\\
&= \frac{d}{dx}\left[\frac{d\phi}{dz}\right]\frac{1}{2\sqrt{x}} + \frac{d}{dx}\left[\frac{1}{2\sqrt{x}}\right]\frac{d\phi}{dz}\\
&= \frac{d^2\phi}{dz^2}\frac{dz}{dx}\frac{1}{2z} - \frac{1}{4}x^{-3/2}\frac{d\phi}{dz}\\
&= \frac{1}{4}\left[\frac{d^2\phi}{dz^2}z^{-2} - z^{-3}\frac{d\phi}{dz}\right].
\end{aligned}$$

So

$$xy'' + y' + \frac{1}{4}y = 0 \quad\Rightarrow\quad z^2\frac{1}{4}\left[\frac{d^2\phi}{dz^2}z^{-2} - z^{-3}\frac{d\phi}{dz}\right] + \frac{d\phi}{dz}\frac{1}{2\,z} + \frac{1}{4}\phi = 0$$

$$\Rightarrow\quad \frac{d^2\phi}{dz^2} + z^{-1}\frac{d\phi}{dz} + \phi = 0$$

$$\Rightarrow\quad z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0.$$

This is a Bessel equation of order 0 in ϕ. Its general solution is

$$\phi(z) = c_1 J_0(z) + c_2 Y_0(z).$$

Thus the solution of the original equation is

$$y(x) = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x}).$$

21. Using (7),

$$\begin{aligned}
J_{-\frac{1}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2k-\frac{1}{2}} \\
&= \sqrt{\frac{2}{x}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\frac{1}{2})}\frac{x^{2k}}{2^{2k}} \\
&= \sqrt{\frac{2}{x}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{2^{2k}k!}{(2k)!\sqrt{\pi}}\frac{x^{2k}}{2^{2k}} \quad \text{(by Exercise 44(a))} \\
&= \sqrt{\frac{2}{\pi\,x}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{x^{2k}}{(2k)!} = \sqrt{\frac{2}{\pi\,x}}\cos x.
\end{aligned}$$

22. (a) Using (7),

$$\begin{aligned}
J_{\frac{3}{2}}(x) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\frac{3}{2}+1)}\left(\frac{x}{2}\right)^{2k+\frac{3}{2}} \\
&= \sqrt{\frac{2}{x}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+2+\frac{1}{2})}\frac{x^{2k+2}}{2^{2k+2}} \\
&= \sqrt{\frac{2}{\pi x}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{2\,2^{2k+1}k!}{(2k+3)(2k+1)!}\frac{x^{2k+2}}{2^{2k+2}} \\
&\quad (\Gamma(k+2+\frac{1}{2}) = \Gamma(k+1+\frac{1}{2})\Gamma(k+1+\frac{1}{2}) \text{ then use Exercise 44(b))} \\
&= \sqrt{\frac{2}{\pi\,x}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)}{(2k+3)!}x^{2k+2} \quad \text{(multiply and divide by } (2k+2)) \\
&= \sqrt{\frac{2}{\pi\,x}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k)}{(2k+1)!}x^{2k} \quad \text{(change } k \text{ to } k-1) \\
&= \sqrt{\frac{2}{\pi\,x}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}[(2k+1)-1]}{(2k+1)!}x^{2k} \\
&= \sqrt{\frac{2}{\pi\,x}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k)!}x^{2k} - \sqrt{\frac{2}{\pi\,x}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k+1)!}x^{2k} \\
&= \sqrt{\frac{2}{\pi\,x}}\left(-\cos x + \frac{\sin x}{x}\right).
\end{aligned}$$

25. (a) Let $u = \frac{2}{a}\, e^{-\frac{1}{2}(at-b)}$, $Y(u) = y(t)$, $e^{-at+b} = \frac{a^2}{4}\, u^2$; then

$$\frac{dy}{dt} = \frac{dY}{du}\frac{du}{dt} = Y'\,(-e^{-\frac{1}{2}(at-b)});\ \frac{d^2y}{dt^2} = \frac{d}{du}\left(Y'\,(-e^{-\frac{1}{2}(at-b)})\right) = Y''e^{-at+b} + Y'\frac{a}{2}e^{-\frac{1}{2}(at-b)}.$$

So

$$Y''e^{-at+b} + Y'\frac{a}{2}e^{-\frac{1}{2}(at-b)} + Ye^{-at+b} = 0 \quad \Rightarrow \quad Y'' + \frac{a}{2}Y'e^{-\frac{1}{2}(at-b)} + Y = 0,$$

upon multiplying by e^{at-b}. Using $u = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$, we get

$$Y'' + \frac{1}{u}Y' + Y = 0 \quad \Rightarrow \quad u^2Y'' + uY' + u^2Y = 0,$$

which is Bessel's equation of order 0.
(b) The general solution of $u^2Y'' + uY' + u^2Y = 0$ is $Y(u) = c_1 J_0(u) + c_2 Y_0(u)$. But $Y(u) = y(t)$ and $u = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$, so

$$y(t) = c_1 J_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}) + c_2 Y_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).$$

(c) (i) If $c_1 = 0$ and $c_2 \neq 0$, then

$$y(t) = c_2 Y_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).$$

As $t \to \infty$, $u \to 0$, and $Y_0(u) \to -\infty$. In this case, $y(t)$ could approach either $+\infty$ or $-\infty$ depending on the sign of c_2. $y(t)$ would approach infinity linearly as near 0, $Y_0(x) \approx \ln x$ so $y(t) \approx \ln\left(\frac{2}{a}e^{-\frac{1}{2}(at-b)}\right) \approx At$.
(ii) If $c_1 \neq 0$ and $c_2 = 0$, then

$$y(t) = c_1 J_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).$$

As $t \to \infty$, $u(t) \to 0$, $J_0(u) \to 1$, and $y(t) \to c_1$. In this case the solution is bounded.
(ii) If $c_1 \neq 0$ and $c_2 \neq 0$, as $t \to \infty$, $u(t) \to 0$, $J_0(u) \to 1$, $Y_0(u) \to -\infty$. Since Y_0 will dominate, the solution will behave like case (i).

It makes sense to have unbounded solutions because eventually the spring wears out and does not affect the motion. Newton's laws tell us the mass will continue with unperturbed momentum, i.e., as $t \to \infty$, $y'' = 0$ and so $y(t) = c_1 t + c_2$, a linear function, which is unbounded if $c_1 \neq 0$.

31. (a) From $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt$, we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1;$$

$$\Gamma(2) = \int_0^\infty te^{-t}\, dt = \overbrace{-te^{-t}\Big|_0^\infty}^{=0} + \int_0^\infty e^{-t}\, dt = 1.$$

(b) Using (15) and the basic property of the gamma function

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{3}{4}\sqrt{\pi}$$
$$-\tfrac{3}{2}\Gamma(-\tfrac{3}{2}) = \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi} \text{ (see Exercise 5)} \ \Rightarrow\ \Gamma(-\tfrac{3}{2}) = \tfrac{4}{3}\sqrt{\pi}.$$

33. (a) In (13), let $u^2 = t$, $2u\, du = dt$, then

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt = \int_0^\infty u^{2(x-1)}e^{-u^2}(2u)du = 2\int_0^\infty u^{2x-1}e^{-u^2}\, du.$$

(b) Using (a)

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 2\int_0^\infty u^{2x-1}e^{-u^2}du2\int_0^\infty v^{2y-1}e^{-v^2}dv\\ &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}u^{2x-1}v^{2y-1}du\,dv.\end{aligned}$$

(c) Switching to polar coordinates: $u = r\cos\theta$, $v = r\sin\theta$, $u^2+v^2 = r^2$, $dudv = rdrd\theta$; for (u, v) varying in the first quadrant ($0 \le u < \infty$ and $0 \le v < \infty$), we have $0 \le \theta \le \frac{\pi}{2}$, and $0 \le r < \infty$, and the double integral in (b) becomes

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}(r\cos\theta)^{2x-1}(r\sin\theta)^{2y-1}rdrd\theta\\ &= 2\int_0^{\frac{\pi}{2}}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta\,\overbrace{2\int_0^\infty r^{2(x+y)-1}e^{-r^2}dr}^{=\Gamma(x+y)}\\ &\qquad\text{(use (a) with } x+y \text{ in place of } x)\\ &= 2\Gamma(x+y)\int_0^{\frac{\pi}{2}}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta,\end{aligned}$$

implying (c).

34. Using Exercise 33 with $x = y = \frac{1}{2}$,

$$[\Gamma(\frac{1}{2})]^2 = 2\Gamma(1)\int_0^{\frac{\pi}{2}}\cos^{1-1}\theta\sin^{1-1}\theta d\theta = 2\int_0^{\frac{\pi}{2}}d\theta = 2\frac{\pi}{2} = \pi,$$

so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

35. From Exercise 33(a) and Exercise 34,

$$\sqrt{\pi} = \Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}du,$$

so

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-u^2}du \quad\Rightarrow\quad \sqrt{\pi} = \int_{-\infty}^\infty e^{-u^2}du,$$

because e^{-u^2} is an even function.

38. Let $I = \int_0^{\pi/2}\cos^5\theta\sin^6\theta\,d\theta$. Applying Exercise 33, we take $2x-1 = 5$ and $2y-1 = 6$, so $x = 3$ and $y = \frac{7}{2}$. Then

$$\frac{\Gamma(3)\,\Gamma(\frac{7}{2})}{\Gamma(3+\frac{7}{2})} = 2I.$$

Now, $\Gamma(3) = 2! = 2$, and

$$\Gamma(3+\frac{7}{2}) = (2+\frac{7}{2})\Gamma(2+\frac{7}{2}) = (\frac{4+7}{2})(\frac{2+7}{2})\Gamma(1+\frac{7}{2}) = \frac{11}{2}\frac{9}{2}\frac{7}{2}\Gamma(\frac{7}{2}) = \frac{693}{8}\Gamma(\frac{7}{2}).$$

So

$$I = \frac{1}{2}\frac{\Gamma(3)\,\Gamma(\frac{7}{2})}{\Gamma(3+\frac{7}{2})} = \frac{\Gamma(\frac{7}{2})}{\frac{693}{8}\Gamma(\frac{7}{2})} = \frac{8}{693}.$$

40. Let $I = \int_0^{\pi/2}\sin^{2k}\theta\,d\theta$. Applying Exercise 33, we take $2x-1 = 0$ and $2y-1 = 2k$, so $x = \frac{1}{2}$ and $y = k+\frac{1}{2}$. Then

$$2I = \frac{\Gamma(\frac{1}{2})\,\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2}+k+\frac{1}{2})} = \frac{\Gamma(\frac{1}{2})\,\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}.$$

Now, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(k+1) = k!$, and $\Gamma(k+\frac{1}{2}) = \frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$ (see Exercise 44). So

$$I = \frac{\sqrt{\pi}}{2(k!)}\frac{(2k)!}{2^{2k}k!}\sqrt{\pi} = \frac{\pi}{2}\frac{(2k)!}{2^{2k}(k!)^2}.$$

41. Let $I = \int_0^{\pi/2} \sin^{2k+1}\theta\, d\theta$. Applying Exercise 33, we take $2x - 1 = 0$ and $2y - 1 = 2k + 1$, so $x = \frac{1}{2}$ and $y = k + 1$. Then

$$2I = \frac{\Gamma(\frac{1}{2})\,\Gamma(k+1)}{\Gamma(k+1+\frac{1}{2})} = \frac{\sqrt{\pi}\,k!}{(k+\frac{1}{2})\Gamma(k+\frac{1}{2})} = \frac{2\sqrt{\pi}\,k!}{(2k+1)\Gamma(k+\frac{1}{2})}.$$

As in (a), we now use $\Gamma(k+\frac{1}{2}) = \frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$, simplify, and get

$$I = \frac{2^{2k}(k!)^2}{(2k+1)!}.$$

44. (a) Proof by induction. If $n = 0$, the formula clearly holds since by (15) $\Gamma(1/2) = \sqrt{\pi}$. Now suppose that he formula holds for n and let us establish it for $n + 1$. That is, suppose that

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi}$$

(the induction hypothesis) and let us prove that

$$\Gamma(n+1+\frac{1}{2}) = \frac{(2(n+1))!}{2^{2(n+1)}\,(n+1)!}\sqrt{\pi}.$$

Starting with the right side, we have

$$\begin{aligned}
\frac{(2(n+1))!}{2^{2(n+1)}\,(n+1)!}\sqrt{\pi} &= \frac{(2n+2)(2n+1)((2n)!)}{2^2 2^{2n}\,(n+1)(n!)}\sqrt{\pi} \\
&= (n+\frac{1}{2})\frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi} \\
&= (n+\frac{1}{2})\Gamma(n+\frac{1}{2}) \quad \text{(by the induction hypothesis)} \\
&= \Gamma(n+\frac{1}{2}+1),
\end{aligned}$$

by the basic property of the gamma function. Thus the formula is true for $n + 1$ and hence for all $n \geq 0$.

(b) We use (a) and the basic property of the gamma function:

$$\begin{aligned}
\Gamma(n+\frac{1}{2}+1) &= (n+\frac{1}{2})\Gamma(n+\frac{1}{2}) \\
&= (n+\frac{1}{2})\frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi} = \frac{(2n+1)}{2}\,\frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi} \\
&= \frac{(2n+1)!}{2^{2n+1}\,n!}\sqrt{\pi}.
\end{aligned}$$

Solutions to Exercises 4.8

1. (a) Using the series definition of the Bessel function, (7), Section 4.7, we have

$$\begin{aligned}
\frac{d}{dx}[x^{-p}J_p(x)] &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^p k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^p k!\Gamma(k+p+1)}\frac{d}{dx}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k 2k}{2^p k!\Gamma(k+p+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{2k-1}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^p (k-1)!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k-1}\\
&= -\sum_{m=0}^{\infty}\frac{(-1)^m}{2^p m!\Gamma(m+p+2)}\left(\frac{x}{2}\right)^{2m+1} \qquad (\text{set } m=k-1)\\
&= -x^{-p}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+p+2)}\left(\frac{x}{2}\right)^{2m+p+1} = -x^{-p}J_{p+1}(x).
\end{aligned}$$

To prove (7), use (1):

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \quad \Rightarrow \quad \int x^p J_{p-1}(x)\,dx = x^p J_p(x) + C.$$

Now replace p by $p+1$ and get

$$\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C,$$

which is (7). Similarly, starting with (2),

$$\begin{aligned}
\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x) \quad &\Rightarrow \quad -\int x^{-p}J_{p+1}(x)\,dx = x^{-p}J_p(x) + C\\
&\Rightarrow \quad \int x^{-p}J_{p+1}(x)\,dx = -x^{-p}J_p(x) + C.
\end{aligned}$$

Now replace p by $p-1$ and get

$$\int x^{-p+1} J_p(x)\,dx = -x^{-p+1} J_{p-1}(x) + C,$$

which is (8).
(b) To prove (4), carry out the differentiation in (2) to obtain

$$x^{-p}J_p'(x) - px^{-p-1}J_p(x) = -x^{-p}J_{p+1}(x) \quad \Rightarrow \quad xJ_p'(x) - pJ_p(x) = -xJ_{p+1}(x),$$

upon multiplying through by x^{p+1}. To prove (5), add (3) and (4) and then divide by x to obtain

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x).$$

To prove (6), subtract (4) from (3) then divide by x.

2. (a) Since $\frac{d}{dx}J_0(x) = -J_1(x)$, the critical points (maxima or minima) of J_0 occur where its derivative vanishes, hence at the zeros of $J_1(x)$.
(b) At the maxima or minima of J_p, we have $J_p'(x) = 0$. From (5), this implies that, at a critical point, $J_{p-1}(x) = J_{p+1}(x)$. Also, from (3) and (4) it follows that

$$x = p\frac{J_p(x)}{J_{p-1}(x)} = p\frac{J_p(x)}{J_{p+1}(x)}.$$

3. $\displaystyle\int xJ_0(x)\,dx = xJ_1(x) + C$, by (7) with $p=0$.

4. $\int x^4 J_3(x)\,dx = x^4 J_4(x) + C$, by (7) with $p = 3$.

5. $\int J_1(x)\,dx = -J_0(x) + C$, by (8) with $p = 1$.

6. $\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) + C$, by (8) with $p = 3$.

7. $\int x^3 J_2(x)\,dx = x^3 J_3(x) + C$, by (7) with $p = 2$.

8.

$$\begin{aligned}\int x^3 J_0(x)\,dx &= \int x^2[xJ_0(x)]\,dx \\ &\quad x^2 = u,\, xJ_0(x)\,dx = dv,\;\; 2x\,dx = du,\, v = xJ_1(x) \\ &= x^3 J_1(x) - 2\int x^2 J_1(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C \\ &= x^3 J_1(x) - 2x^2\left(\frac{2}{x}J_1(x) - J_0(x)\right) + C \text{ (use (6) with } p = 1) \\ &= (x^3 - 4x)J_1(x) + 2x^2 J_0(x) + C.\end{aligned}$$

9.

$$\begin{aligned}\int J_3(x)\,dx &= \int x^2[x^{-2}J_3(x)]\,dx \\ &\quad x^2 = u,\, x^{-2}J_3(x)\,dx = dv,\;\; 2x\,dx = du,\, v = -x^{-2}J_2(x) \\ &= -J_2(x) + 2\int x^{-1}J_2(x)\,dx = -J_2(x) - 2x^{-1}J_1(x) + C \\ &= J_0(x) - \frac{2}{x}J_1(x) - \frac{2}{x}J_1(x) + C\text{(use (6) with } p = 1) \\ &= J_0(x) - \frac{4}{x}J_1(x) + C.\end{aligned}$$

10. We assume throughout this exercise that n is a positive integer. Let $u = J_0(x)$, then $du = -J_1(x)$. If $n \neq -1$, then

$$\int_0^\infty J_1(x)[J_0(x)]^n\,dx - \int_{x=0}^\infty u^n\,du = -\frac{1}{n+1}[J_0(x)]^{n+1}\Big|_0^\infty = \frac{-1}{n+1}(0-1) = \frac{1}{n+1},$$

where we have used $\lim_{x\to\infty} J_0(x) = 0$ and $J_0(0) = 1$.

11. By (2) with $p = \frac{1}{2}$,

$$\begin{aligned}J_{3/2}(x) &= -x^{\frac{1}{2}}\frac{d}{dx}\left[x^{-\frac{1}{2}}J_{\frac{1}{2}}(x)\right] = -x^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{d}{dx}\left(\frac{\sin x}{x}\right) \\ &= -\sqrt{\frac{2}{\pi}}x^{\frac{1}{2}}\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).\end{aligned}$$

12. By (2) with $p = \frac{3}{2}$,

$$\begin{aligned}J_{5/2}(x) &= -x^{\frac{3}{2}}\frac{d}{dx}\left[x^{-\frac{3}{2}}J_{\frac{3}{2}}(x)\right] = -x^{\frac{3}{2}}\sqrt{\frac{2}{\pi}}\frac{d}{dx}\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right) \\ &= -\sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\left(\frac{\cos x}{x^3} - 3\frac{\sin x}{x^4} + \frac{\sin x}{x^2} + 2\frac{\cos x}{x^3}\right) \\ &= -\sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\left(3\frac{\cos x}{x^3} - 3\frac{\sin x}{x^4} + \frac{\sin x}{x^2}\right) \\ &= \sqrt{\frac{2}{\pi x}}\left((\frac{3}{x^2} - 1)\sin x - \frac{3}{x}\cos x\right).\end{aligned}$$

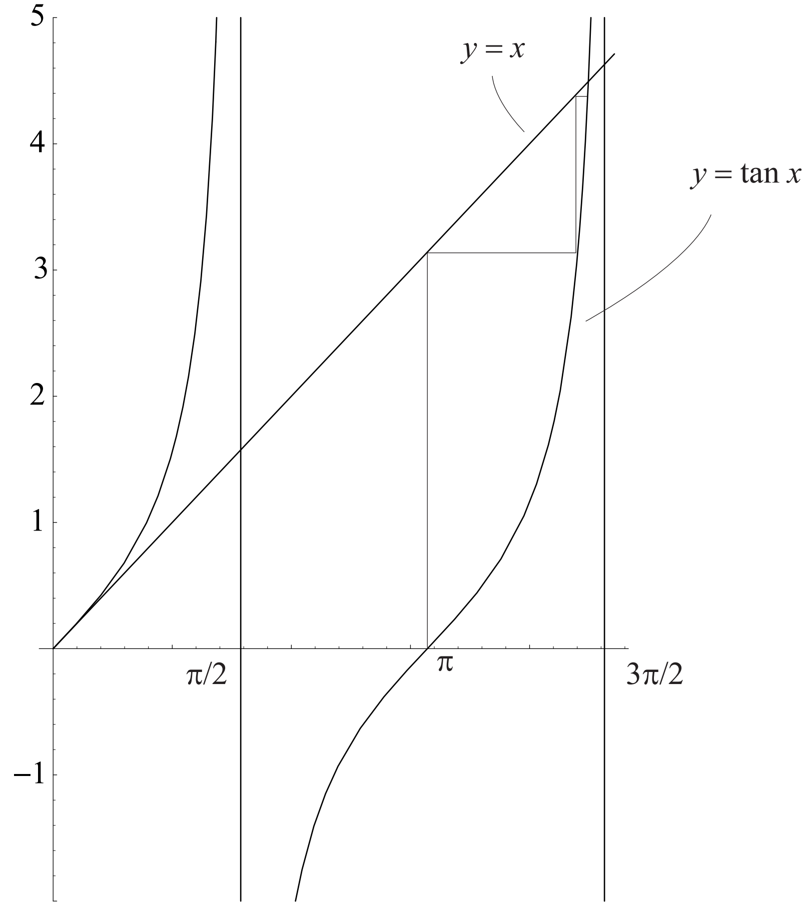
13. Use (6) with $p = 4$. Then

$$\begin{aligned}
J_5(x) &= \frac{8}{x}J_4(x) - J_3(x)\\
&= \frac{8}{x}\left[\frac{6}{x}J_3(x) - J_2(x)\right] - J_3(x) \quad \text{(by (6) with } p = 3)\\
&= \left(\frac{48}{x^2} - 1\right)J_3(x) - \frac{8}{x}J_2(x)\\
&= \left(\frac{48}{x^2} - 1\right)\left(\frac{4}{x}J_2(x) - J_1(x)\right) - \frac{8}{x}J_2(x) \quad \text{(by (6) with } p = 2)\\
&= \left(\frac{192}{x^3} - \frac{12}{x}\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x)\\
&= \frac{12}{x}\left(\frac{16}{x^2} - 1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \left(\frac{48}{x^2} - 1\right)J_1(x)\\
&\quad \text{(by (6) with } p = 1)\\
&= -\frac{12}{x}\left(\frac{16}{x^2} - 1\right)J_0(x) + \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right)J_1(x).
\end{aligned}$$

14. From the formula $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, it follows that the zeros of $J_{1/2}(x)$ are located precisely at the zeros of $\sin x$. The latter are located at $n\pi$, $n = 0, 1, 2, \ldots$.

15. (a) and (b) We will do this problem on Mathematica. Let us make some comments about the equation $\frac{\sin x}{x} = \cos x$. This equation is equivalent to the equation $\tan x = x$, which we have encountered in Section 3.6. It is clear from the graph that this equation has one root x_k in each interval $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k = 0, 1, 2, \ldots$. We have $x_0 = 0$ and the remaininng roots can be obtained by iteration as follows. Take the case $k = 1$. Start at $z_1 = \pi$ and go up to the line $y = x$ then over to the right to the graph of $\tan x$. You will intersect this graph at the point $z_2 = \pi + \tan^{-1}\pi$. Repeat hese steps by going up to the line $y = x$ and then over to the right to the graph of $y = \tan x$. You will intersect the graph at $z_2 = \pi + \tan^{-1} z_1$. As you continue this process, you will approach the value of x_1. To find x_k, start at $z_1 = k\pi$ and construct the sequence $z_n = k\pi + \tan^{-1} z_{n-1}$. Let us compute some numerical values and then compare them with the roots that we find by using built-in commands from Mathematica.

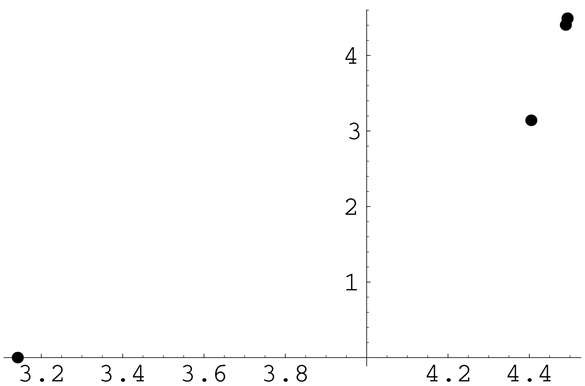
```
grph1 = Plot[{x, Tan[x]}, {x, 0, 3 Pi / 2}, AspectRatio → 1.5,
  PlotRange → {-2, 5}]
```



```
z[1] = Pi
z[k_] := N[ArcTan[z[k - 1]] + Pi]
Table[z[k], {k, 1, 10}]
```

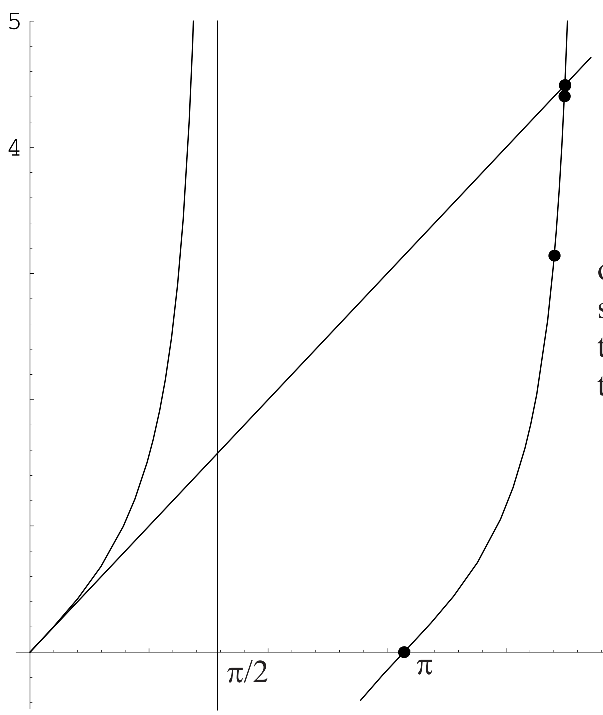
Iterative process to approximate the first root of $\tan x = x$

```
π, 4.40422, 4.48912, 4.49321, 4.4934, 4.49341, 4.49341, 4.49341, 4.49341, 4.49341

pts = Table[{z[k], Tan[z[k]]}, {k, 1, 5}]

{{π, 0}, {4.40422, 3.14159},
 {4.48912, 4.40422}, {4.49321, 4.48912}, {4.4934, 4.49321}}

grph2 = ListPlot[pts, PlotStyle → PointSize[0.02]]
```



Plot of points generated by iterative process.

```
Show[{grph1, grph2}, AspectRatio → 1.5,
  PlotRange → {-2, 5}]
```



The points $(\pi + \tan^{-1} z_k,\ z_k)$ on the graph of $\tan x$, showing a fast convergence to to the root of the equation $\tan x = x$.

If you want to use a built-in command to find the root, you can do the following (compare the result with the iterative process above).

```
FindRoot[Tan[x] == x, {x, 4}]

{x → 4.49341}
```

16. (a) There are $j-1$ roots in $(0, a)$. They are

$$x_k = \frac{\alpha_{pk}}{\alpha_{pj}}\, a \quad \text{for } k = 1, 2, \ldots, j-1,$$

which are just the $j-1$ positive zeros of $J_p(x)$, rescaled by the factor $\frac{a}{\alpha_{pj}}$ (the reciprocal of the scale factor occuring in the argument of $J_p\left(\frac{\alpha_{pj}}{a}x\right)$.

17. (a) From (17),

$$\begin{aligned} A_j &= \frac{2}{J_1(\alpha_j)^2}\int_0^1 f(x)J_0(\alpha_j x)x\,dx = \frac{2}{J_1(\alpha_j)^2}\int_0^c J_0(\alpha_j x)x\,dx \\ &= \frac{2}{\alpha_j^2 J_1(\alpha_j)^2}\int_0^{c\alpha_j} J_0(s)s\,ds \quad (\text{let } \alpha_j x = s) \\ &= \frac{2}{\alpha_j^2 J_1(\alpha_j)^2} J_1(s)s\Big|_0^{c\alpha_j} = \frac{2cJ_1(\alpha_j)}{\alpha_j J_1(\alpha_j)^2}. \end{aligned}$$

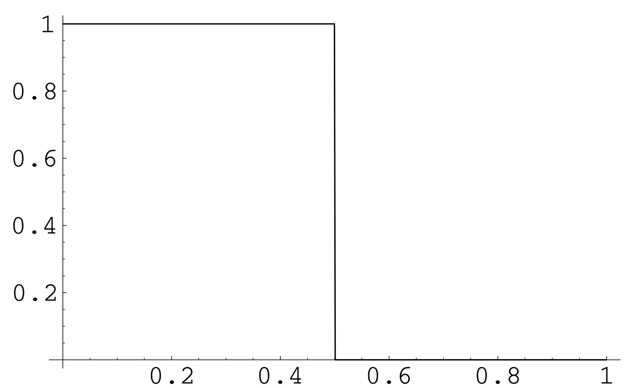
Thus, for $0 < x < 1$,

$$f(x) = \sum_{j=1}^{\infty} \frac{2cJ_1(\alpha_j)}{\alpha_j J_1(\alpha_j)^2} J_0(\alpha_j x).$$

(b) The function f is piecewise smooth, so by Theorem 2 the series in (a) converges to $f(x)$ for all $0 < x < 1$, except at $x = c$, where the series converges to the average value $\frac{f(c+)+f(c-)}{2} = \frac{1}{2}$.

18. Done on Mathematica

```
f[x_] := 1  /; 0 < x < 1 / 2
f[x_] := 0  /; 1 / 2 < x < 1
Plot[f[x], {x, 0, 1}]
```



To describe the Bessel series, we need the zeros of $J_0(x)$. These are built-in the program. To recall them, we need a package.

```
<< NumericalMath`BesselZeros`
```

Here are the first 20 zeros of $J_0(x)$.

```
zerolist=BesselJZeros[0, 20]
```

```
{2.40483, 5.52008, 8.65373, 11.7915, 14.9309, 18.0711,
 21.2116, 24.3525, 27.4935, 30.6346, 33.7758, 36.9171, 40.0584,
 43.1998, 46.3412, 49.4826, 52.6241, 55.7655, 58.907, 62.0485}
```

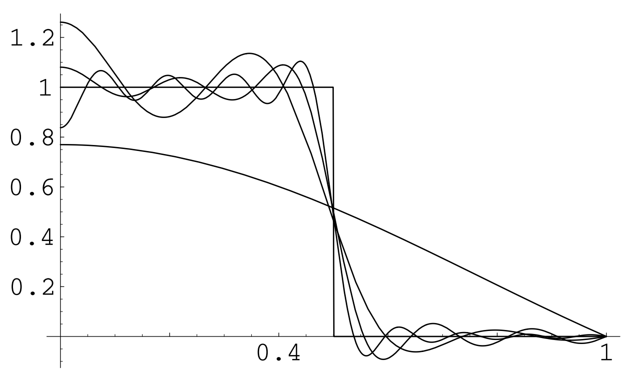
Here are the first and second partial sums

```
partsum[x, 1]
partsum[x, 2]
```

```
0.769756 BesselJ[0, 2.40483 x]

0.769756 BesselJ[0, 2.40483 x] + 0.661472 BesselJ[0, 5.52008 x]
```

Here are graphs of some partial sums , compared with the graph of the function

```
tt := Table[partsum[x, n], {n, 1, 20, 5}]
Plot[Evaluate[{tt, f[x]}], {x, 0, 1}]
```



19. (a) This is a Bessel series of order 4. By Theorem 2, we have

$$\begin{aligned} A_j &= \frac{2}{J_5(\alpha_{4,j})^2}\int_0^1 x^4 J_4(\alpha_{4,j}x)x\,dx \\ &= \frac{2}{\alpha_{4,j}^6 J_1(\alpha_{4,j})^2}\int_0^{\alpha_{4,j}} J_4(s)s^5\,ds \quad (\text{let } \alpha_{4,j}x = s) \\ &= \frac{2}{\alpha_{4,j}^6 J_5(\alpha_{4,j})^2} J_5(s)s^5\Big|_0^{\alpha_{4,j}} = \frac{2}{\alpha_{4,j}J_5(\alpha_{4,j})}. \end{aligned}$$

Thus, for $0 < x < 1$,

$$x^4 = 2\sum_{j=1}^{\infty}\frac{J_4(\alpha_{4,j}x)}{\alpha_{4,j}J_5(\alpha_{4,j})}.$$

To describe the Bessel series, we need the zeros of $J_0(x)$. These are built-in the program. To recall them, load the package as previously.

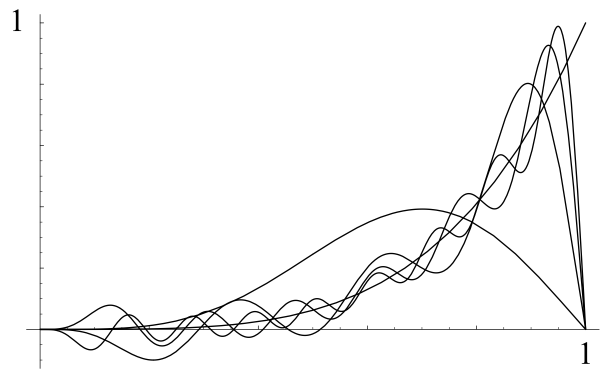
```
zerolist=BesselJZeros[4, 20]

{7.58834, 11.0647, 14.3725, 17.616, 20.8269, 24.019,
 27.1991, 30.371, 33.5371, 36.699, 39.8576, 43.0137, 46.1679,
 49.3204, 52.4716, 55.6217, 58.7708, 61.9192, 65.067, 68.2142}
```

Now the partial sums of the Bessel series can be defined as follows:

```
partsum[x_, n_] := Sum[2 / (zerolist[[j]] BesselJ[5, zerolist[[j]]])
   BesselJ[4, zerolist[[j]] x], {j, 1, n}]
```

Here are graphs of some partial sums , compared with the graph of the function

```
tt := Table[partsum[x, n], {n, 1, 20, 5}]
Plot[Evaluate[{tt, x^4}], {x, 0, 1}]
```



20. This is a Bessel series of order m. By Theorem 2, we have

$$\begin{aligned} A_j &= \frac{2}{J_{m+1}(\alpha_{m,j})^2}\int_0^1 x^m J_m(\alpha_{m,j}x)x\,dx \\ &= \frac{2}{\alpha_{m,j}^{m+2}J_{m+1}(\alpha_{m,j})^2}\int_0^{\alpha_{m,j}} J_m(s)s^{m+1}\,ds \quad (\text{let } \alpha_{m,j}x = s) \\ &= \frac{2}{\alpha_{m,j}^{m+2}J_{m+1}(\alpha_{m,j})^2} J_{m+1}(s)s^{m+1}\Big|_0^{\alpha_{m,j}} = \frac{2}{\alpha_{m,j}J_{m+1}(\alpha_{m,j})}. \end{aligned}$$

Thus, for $0 < x < 1$,

$$x^m = 2\sum_{j=1}^{\infty} \frac{J_m(\alpha_{m,j}x)}{\alpha_{m,j}J_{m+1}(\alpha_{m,j})}.$$

21. (a) Take $m = 1/2$ in the series expansion of Exercise 20 and you'll get

$$\sqrt{x} = 2\sum_{j=1}^{\infty} \frac{J_{1/2}(\alpha_j x)}{\alpha_j\, J_{3/2}(\alpha_j)} \quad \text{for } 0 < x < 1,$$

where α_j is the jth positive zero of $J_{1/2}(x)$. By Example 1, Section 4.7, we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi\, x}}\sin x.$$

So

$$\alpha_j = j\pi \quad \text{for } j = 1,\, 2,\, \ldots .$$

(b) We recall from Exercise 11 that

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi\, x}} \left(\frac{\sin x}{x} - \cos x \right).$$

So the coefficients are

$$\begin{aligned} A_j &= \frac{2}{\alpha_j\, J_{3/2}(\alpha_j)} = \frac{2}{j\pi\, J_{3/2}(j\pi)} \\ &= \frac{2}{j\pi \sqrt{\frac{2}{\pi\, j\pi}} \left(\frac{\sin j\pi}{j\pi} - \cos j\pi \right)} \\ &= (-1)^{j-1} \sqrt{\frac{2}{j}} \end{aligned}$$

and the Bessel series expansion becomes, or $0 < x < 1$,

$$\sqrt{x} = \sum_{j=1}^{\infty} (-1)^{j-1} \sqrt{\frac{2}{j}} J_{1/2}(\alpha_j x).$$

(c) Writing $J_{1/2}(x)$ in terms of $\sin x$ and simplifying, this expansion becomes

$$\begin{aligned} \sqrt{x} &= \sum_{j=1}^{\infty} (-1)^{j-1} \sqrt{\frac{2}{j}} J_{1/2}(\alpha_j x) \\ &= \sum_{j=1}^{\infty} (-1)^{j-1} \sqrt{\frac{2}{j}} \sqrt{\frac{2}{\pi\, \alpha_j}} \sin \alpha_j \\ &= \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \frac{\sin(j\pi x)}{\sqrt{x}}. \end{aligned}$$

Upon multiplying both sides by $\sqrt{x}$, we obtain

$$x = \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sin(j\pi x) \quad \text{for } 0 < x < 1,$$

which is the familiar Fourier sine series (half-range expansion) of the function $f(x) = x$.

23. Just repeat the computation in Exercises 20 with $m = 2$ and you will get, for $0 < x < 1$,

$$x^2 = 2 \sum_{j=1}^{\infty} \frac{1}{\alpha_{2,j} J_3(\alpha_{2,j})} J_2(\alpha_{2,j} x).$$

24. Repeat the computation in Exercises 20 with $m = 3$ and you will get, for $0 < x < 1$,

$$x^3 = 2 \sum_{j=1}^{\infty} \frac{1}{\alpha_{3,j} J_4(\alpha_{3,j})} J_3(\alpha_{3,j} x).$$

25. By Theorem 2 with $p = 1$, we have

$$\begin{aligned}
A_j &= \frac{2}{J_2(\alpha_{1,j})^2}\int_{\frac{1}{2}}^{1} J_1(\alpha_{1,j}x)\,dx \\
&= \frac{2}{\alpha_{1,j}J_2(\alpha_{1,j})^2}\int_{\frac{\alpha_{1,j}}{2}}^{\alpha_{1,j}} J_1(s)\,ds \quad (\text{let } \alpha_{1,j}x = s) \\
&= \frac{2}{\alpha_{1,j}J_2(\alpha_{1,j})^2}[-J_0(s)\Big|_{\frac{\alpha_{1,j}}{2}}^{\alpha_{1,j}} \quad (\text{by (8) with } p = 1) \\
&= \frac{-2\big[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\big]}{\alpha_{1,j}J_2(\alpha_{1,j})^2} \\
&= \frac{-2\big[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\big]}{\alpha_{1,j}J_0(\alpha_{1,j})^2},
\end{aligned}$$

where in the last equality we used (6) with $p = 1$ at $x = \alpha_{1,j}$ (so $J_0(\alpha_{1,j}) + J_2(\alpha_{1,j}) = 0$ or $J_0(\alpha_{1,j}) = -J_2(\alpha_{1,j})$). Thus, for $0 < x < 1$,

$$f(x) = -2\sum_{j=1}^{\infty} \frac{-2\big[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\big]}{\alpha_{1,j}J_0(\alpha_{1,j})^2} J_1(\alpha_{1,j}x).$$

26. By Theorem 2 with $p = 0$ and $a = 3$, we have

$$\begin{aligned}
A_j &= \frac{2}{9J_1(\alpha_{0,j})^2}\int_0^3 x^2 J_0(\frac{\alpha_{0,j}x}{3})x\,dx \\
&= \frac{18}{\alpha_{0,j}^4 J_1(\alpha_{0,j})^2}\int_0^{\alpha_{0,j}} J_0(s)s^3\,ds \quad (\text{let } \frac{\alpha_{1,j}x}{3} = s) \\
&\quad (\text{let } u = s^2,\ dv = sJ_0(s),\ du = 2s,\ v = sJ_1(s).) \\
&= \frac{18}{\alpha_{0,j}^4 J_1(\alpha_{0,j})^2}\Big[s^3 J_1(s)\Big|_0^{\alpha 0,j} - 2\int_0^{\alpha 0,j} s^2 J_1(s)\,ds\Big] \\
&= \frac{18}{\alpha_{0,j}^4 J_1(\alpha_{0,j})^2}\Big[\alpha_{0,j}^3 J_1(\alpha_{0,j}) - 2s^2 J_2(s)\Big|_0^{\alpha_{0,j}}\Big] \\
&= \frac{18}{\alpha_{0,j}^4 J_1(\alpha_{0,j})^2}\Big[\alpha_{0,j}^3 J_1(\alpha_{0,j}) - 2\alpha 0, j^2 J_2(\alpha_{0,j})\Big] \\
&= \frac{18}{\alpha_{0,j}^2 J_1(\alpha_{0,j})^2}\Big[\alpha_{0,j}J_1(\alpha_{0,j}) - 2\Big(\frac{2}{\alpha_{0,j}}J_1(\alpha_{0,j}) - \overbrace{J_0(\alpha_{0,j})}^{=0}\Big)\Big] \\
&\qquad\qquad (\text{by (6) with } p = 1) \\
&= \frac{18(\alpha_{0,j}^2 - 4)}{\alpha_{0,j}^3 J_1(\alpha_{0,j})}
\end{aligned}$$

Thus, for $0 < x < 3$,

$$x^2 = 18\sum_{j=1}^{\infty} \frac{\alpha_{0,j}^2 - 4}{\alpha_{0,j}^3 J_1(\alpha_{0,j})} J_0(\frac{\alpha_{0,j}x}{3}).$$

27. By Theorem 2 with $p = 2$, we have

$$\begin{aligned}
A_j &= \frac{2}{J_3(\alpha_{2,j})^2}\int_{\frac{1}{2}}^{1} x^{-1}J_2(\alpha_{2,j}x)\,dx \\
&= \frac{2}{J_3(\alpha_{2,j})^2}\int_{\frac{\alpha_{2,j}}{2}}^{\alpha_{2,j}} J_2(s)s^{-1}\,ds \quad (\text{let } \alpha_{2,j}x = s) \\
&= \frac{2}{J_3(\alpha_{2,j})^2}[-s^{-1}J_1(s)\Big|_{\frac{\alpha_{2,j}}{2}}^{\alpha_{2,j}} \\
&= \frac{2}{J_1(\alpha_{2,j})^2}\left[-\frac{J_1(\alpha_{2,j})}{\alpha_{2,j}} + 2\frac{J_1\left(\frac{\alpha_{2,j}}{2}\right)}{\alpha_{2,j}}\right] \\
&\quad (\text{because } J_3 = -J_1 \text{ at a zero of } J_2 \text{ by (6)}) \\
&= \frac{-2\big[J_1(\alpha_{2,j}) - 2J_1(\frac{\alpha_{2,j}}{2})\big]}{\alpha_{2,j}J_1(\alpha_{2,j})^2}.
\end{aligned}$$

Thus, for $0 < x < 1$,

$$f(x) = -2\sum_{j=1}^{\infty}\frac{J_1(\alpha_{2,j}) - 2J_1(\frac{\alpha_{2,j}}{2})}{\alpha_{2,j}J_1(\alpha_{2,j})^2}J_2(\alpha_{2,j}x).$$

28. By Theorem 2 with $p = 3$, we have

$$\begin{aligned}
A_j &= \frac{2}{J_4(\alpha_{3,j})^2}\int_{\frac{1}{30}}^{1} x^{-3}J_3(\alpha_{3,j}x)\,x\,dx \\
&= \frac{2\alpha_{3,j}}{J_4(\alpha_{3,j})^2}\int_{\frac{\alpha_{3,j}}{30}}^{\alpha_{3,j}} J_3(s)s^{-2}\,ds \quad (\text{let } \alpha_{3,j}x = s) \\
&= \frac{2\alpha_{3,j}}{J_4(\alpha_{3,j})^2}[-s^{-2}J_2(s)\Big|_{\frac{\alpha_{3,j}}{30}}^{\alpha_{3,j}} \\
&= \frac{-2\alpha_{3,j}}{J_4(\alpha_{3,j})^2}\left[[-\frac{900\,J_2\left(\frac{\alpha_{3,j}}{30}\right)}{\alpha_{3,j}} + \frac{J_2\left(\alpha_{3,j}\right)}{\alpha_{3,j}^2}\right] \\
&= \frac{-2\big[J_2(\alpha_{3,j}) - 900\,J_2(\frac{\alpha_{3,j}}{30})\big]}{\alpha_{3,j}J_2(\alpha_{3,j})^2},
\end{aligned}$$

because $J_4 = -J_2$ at a zero of J_3 by (6). Thus, for $0 < x < 1$,

$$f(x) = -2\sum_{j=1}^{\infty}\frac{J_2(\alpha_{3,j}) - 900\,J_2(\frac{\alpha_{3,j}}{30})}{\alpha_{3,j}J_2(\alpha_{3,j})^2}J_3(\alpha_{3,j}x).$$

29. By Theorem 2 with $p = 1$, we have

$$\begin{aligned}
A_j &= \frac{1}{2\,J_2(\alpha_{1,j})^2}\int_0^2 J_1(\alpha_{2,j}x/2)x\,dx \\
&= \frac{2}{\alpha_{1,j}^2 J_2(\alpha_{1,j})^2}\int_0^{\alpha_{1,j}} J_1(s)s\,ds \quad (\text{let } \alpha_{1,j}x/2 = s).
\end{aligned}$$

Since we cannot evaluate the definite integral in a simpler form, just leave it as it is and write the Bessel series expansion as

$$1 = \sum_{j=1}^{\infty}\frac{2}{\alpha_{1,j}^2 J_2(\alpha_{1,j})^2}\left[\int_0^{\alpha_{1,j}} J_1(s)s\,ds\right]J_1(\alpha_{1,j}x/2) \quad \text{for } 0 < x < 2.$$

30. This problem has a short answer. We want to expand the function $f(x) = J_0(x)$ for $0 < x < \alpha_{0,1}$ in a series expansion in terms of the functions

$$J_0\left(\frac{\alpha_{0j}}{\alpha_{01}}x\right), \quad j = 1, 2, \ldots .$$

Note that the first function in this set is

$$J_0\left(\frac{\alpha_{01}}{\alpha_{01}}x\right) = J_0(x) = f(x).$$

So f is its own Bessel series expansion in this system. We (must) have $A_1 = 1$ and $A_j = 0$ for all $j > 0$. This will come out of the formula for the Bessel coefficients, using orthogonality.

31. $p = 1$, $y = c_1 J_1(\lambda x) + c_2 Y_1(\lambda x)$. For y to be bounded near 0, we must take $c_2 = 0$. For $y(1) = 0$, we must take $\lambda = \lambda_j = \alpha_{1,j}$, $j = 1, 2, \ldots$; and so $y = y_i = c_{1,j} J_1(\alpha_{1,j} x)$.

33. $p = \frac{1}{2}$, $y = c_1 J_{\frac{1}{2}}(\lambda x) + c_2 Y_{\frac{1}{2}}(\lambda x)$. For y to be bounded near 0, we must take $c_2 = 0$. For $y(\pi) = 0$, we must take $\lambda = \lambda_j = \frac{\alpha_{\frac{1}{2},j}}{\pi} = j$, $j = 1, 2, \ldots$ (see Exercises 21); and so

$$y = y_i = c_{1,j} J_1(\frac{\alpha_{\frac{1}{2},j}}{\pi}x) = c_{1,j}\sqrt{\frac{2}{\pi x}}\sin(jx)$$

(see Example 1, Section 4.7).

35. (b) Bessel's equation of order 0 is $xy'' + y' + xy = 0$. Take $y = x^{-\frac{1}{2}}u$. Then $y' = x^{-\frac{1}{2}}[u' - \frac{1}{2x}u]$ and

$$\begin{aligned} y'' &= x^{-\frac{1}{2}}[u'' - \frac{1}{2x}u' + \frac{1}{2x^2}u - \frac{1}{2x}(u' - \frac{1}{2x}u)] \\ &= x^{-\frac{1}{2}}[u'' - \frac{1}{x}u' + \frac{3}{4x^2}u]. \end{aligned}$$

Substituting into Bessel's equation and simplifying, we find

$$\left(xu'' - u' + \frac{3}{4x}u\right) + \left(u' - \frac{1}{2x}u\right) + xu = 0 \quad \Rightarrow \quad u'' + \left(1 + \frac{1}{4x^2}\right)u = 0 \quad (*).$$

Since $J_0(x)$ satisfies Bessel's equation, it follows that $u = \sqrt{x}J_0(x)$ is a solution of the last displayed equation $(*)$.
(c) Let $v = \sin x$. Then $v'' + v = 0$. Multiplying this equation by u and $(*)$ by v, and then subtracting yields

$$uv'' - vu'' = \frac{1}{4x^2}uv \quad \Rightarrow \quad \left(uv' - u'v\right)' = \frac{1}{4x^2}uv,$$

which implies (c), since $\frac{1}{4x^2}uv = -(u'' + u)v$ from (b) (multiply the displayed equation in (b) by v).
(d) is clear from (b).
(e) Integrating both sides of the equation

$$\left(uv' - u'v\right)' = \frac{1}{4x^2}uv$$

from (c), we obtain

$$\int \frac{1}{4x^2}uv\,dx = uv' - u'v + C.$$

So for any positive integer k,

$$\begin{aligned} \int_{2k\pi}^{(2k+1)\pi} \frac{1}{4x^2}u(x)\sin x\,dx &= uv' - u'v\Big|_{2k\pi}^{(2k+1)\pi} = u\cos x - u'\sin x\Big|_{2k\pi}^{(2k+1)\pi} \\ &= -\big(u(2k\pi) + u((2k+1)\pi)\big). \end{aligned}$$

(f) Consider the integrand in (e): $\frac{u(x)\sin x}{4x^2}$. This is a continuous function on $[2k\pi,\,(2k+1)\pi]$. The function $\frac{\sin x}{4x^2}$ does not change signs in the interval $(2k\pi,\,(2k+1)\pi)$ (it is either always positive or always negative depending on the sign of $\sin x$). Let us consider the case when k is even. Then $\frac{\sin x}{4x^2} \geq 0$ for all x in $(2k\pi,\,(2k+1)\pi)$. Assume that $u(x)$ does not vanish in $(2k\pi,\,(2k+1)\pi)$. So $u(x) > 0$ or $u(x) < 0$ for all x in $(2k\pi,\,(2k+1)\pi)$. Take the case $u(x) > 0$ for all x in $(2k\pi,\,(2k+1)\pi)$. Then $\frac{u(x)\sin x}{4x^2} > 0$ for all x in $(2k\pi,\,(2k+1)\pi)$, which implies that

$$\int_{2k\pi}^{(2k+1)\pi} \frac{u(x)\sin x}{4x^2}\,dx > 0.$$

By continuity of u, it follows that $u(2k\pi) \geq 0$ and $u((2k+1)\pi) \geq 0$, and so (e) implies that

$$\int_{2k\pi}^{(2k+1)\pi} \frac{u(x)\sin x}{4x^2}\,dx \leq 0,$$

which is a contradiction. Thus $u(x)$ is not positive for all x in $(2k\pi,\,(2k+1)\pi)$ and so it must vanish at some point inside this interval. A similar argument shows that $u(x)$ cannot be negative for all x in $(2k\pi,\,(2k+1)\pi)$. A similar argument works when k is odd and shows that in all situations, u must vanish at some point in $(2k\pi,\,(2k+1)\pi)$.

(g) From (f) it follows that u has infinitely many zeros on the positive real line; at least one in each interval of the form $(2k\pi,\,(2k+1)\pi)$. Since these intervals are disjoint, these zeros are distinct. Now J_0 and u have the same zeros, and so the same applies to the zeros of J_0.

Note: You can repeat the above proof with $\sin x$ replaced by $\sin(x-a)$ and show that u (and hence J_0) has at least one zero in every interval of the form $(a,\,a+\pi)$ for any $a > 0$.

36. (a) We know that the function $J_p\left(\frac{\alpha_{pk}}{a}x\right)$ is a solution of the parametric form of Bessel's equation of order $p > 0$,

$$x^2y'' + xy' + (\lambda_k^2x^2 - p^2)y = 0,$$

where $\lambda_k = \frac{\alpha_{pk}}{a}$ and α_k is the kth positive zero of J_p. Rewrite the left side of the equation as follows: for $x > 0$,

$$\begin{aligned}
x^2y'' + xy' + (\lambda_k^2x^2 - p^2)y &= xy'' + y' + \frac{\lambda_k^2x^2 - p^2}{x}y && \text{(Divide by } x.\text{)}\\
&= \big(xy'\big)' + \lambda_k^2 xy - \frac{p^2}{x}y && \text{(Use product rule.)}\\
&= (xy')\big(xy'\big)' + \lambda_k^2x^2yy' - p^2yy' && \text{(Multiply by } xy'.\text{)}\\
&= \big[\tfrac{1}{2}(xy')^2\big]' + (\lambda_k^2x^2 - p^2)\big[\tfrac{1}{2}y^2\big]' && \text{(Chain rule.)}
\end{aligned}$$

Hence $J_p\left(\frac{\alpha_{pk}}{a}x\right)$ is a solution of

$$\big[\frac{1}{2}(xy')^2\big]' + (\lambda_k^2x^2 - p^2)\big[\frac{1}{2}y^2\big]' = 0$$

or

$$\big[(xy')^2\big]' + (\lambda_k^2x^2 - p^2)\big[y^2\big]' = 0.$$

(b) and (c) Integrate from 0 to a and use that $y(a) = 0$ (because $y(a) = J_p(\alpha_{pk}) = 0$)

$$\begin{aligned}
(xy')^2\Big|_0^a + \int_0^a (\lambda_k^2x^2 - p^2)\big[y^2\big]'dx &= 0 && \text{(The integral cancels the first derivative.)}\\
\Big[(xy')^2 + (\lambda_k^2x^2 - p^2)y^2\Big]\Big|_0^a - 2\lambda_k^2\int_0^a y^2x\,dx &= 0 && \text{(Integrate by parts.)}\\
(ay'(a))^2 - 2\lambda_k^2\int_0^a y^2x\,dx &= 0 && y(a) = 0,\ y(0) = 0 \text{ if } p > 0.)
\end{aligned}$$

Thus, with $y = J_p\left(\frac{\alpha_{pk}}{a}x\right)$, we obtain

$$[y'(a)]^2 = \frac{2\lambda_k^2}{a^2}\int_0^a J_p\left(\frac{\alpha_{pk}}{a}x\right)^2 x\,dx.$$

(d) For $y = J_p(\lambda_k x)$, by (4), we have

$$\lambda_k x J_p'(\lambda_k x) - pJ_p(\lambda_k x) = -\lambda_k x J_{p+1}(\lambda_k x)$$

or

$$xy'(x) - py(x) = -\lambda_k x J_{p+1}(\lambda_k x).$$

Substituting $x = a$ and using $y(a) = 0$, we get $ay'(a) = -\lambda_k a J_{p+1}(\alpha_k)$. Thus from (c)

$$\int_0^a J_p\left(\frac{\alpha_{pk}}{a}x\right)^2 x\,dx = \frac{a^2}{2\lambda_k^2}\lambda_k^2 J_{p+1}(\alpha_k)^2 = \frac{a^2}{2}[J_{p+1}(\alpha_k)]^2.$$

One more formula. To complement the integral formulas from this section, consider the following interesting formula. Let a, b, c, and p be positive real numbers with $a \neq b$. Then

$$\int_0^c J_p(ax)\,J_p(bx)x\,dx = \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_{p-1}(ac) - bJ_p(ac)J_{p-1}(bc)\big].$$

To prove this formula, we note that $y_1 = J_p(ax)$ satisfies

$$x^2y_1'' + xy_1' + (a^2x^2 - p^2)y_1 = 0$$

and $y_2 = J_p(bx)$ satisfies

$$x^2y_2'' + xy_2' + (b^2x^2 - p^2)y_2 = 0.$$

Write these equations in the form

$$(xy_1')' + y_1' + \frac{a^2x^2 - p^2}{x}y_1 = 0$$

and

$$(xy_2')' + y_2' + \frac{b^2x^2 - p^2}{x}y_1 = 0.$$

Multiply the first by y_2 and the second by $y - 1$, subtract, simplify, and get

$$y_2(xy_1')' - y_1(xy_2')' = y_1y_2(b^2 - a^2)x.$$

Note that

$$y_2(xy_1')' - y_1(xy_2')' = \frac{d}{dx}\big[y_2(xy_1') - y_1(xy_2')\big].$$

So

$$(b^2 - a^2)y_1y_2x = \frac{d}{dx}\big[y_2(xy_1') - y_1(xy_2')\big],$$

and, after integrating,

$$(b^2 - a^2)\int_0^c y_1(x)y_2(x)x\,dx = \big[y_2(xy_1') - y_1(xy_2')\big]\Big|_0^c = x\big[y_2y_1' - y_1y_2'\big]\Big|_0^c.$$

On the left, we have the desired integral times $(b^2 - a^2)$ and, on the right, we have

$$c\big[J_p(bc)aJ_p'(ac) - bJ_p(ac)J_p'(bc)\big] - c\big[aJ_p(0)J_p'(0) - bJ_p(0)J_p'(0)\big].$$

Since $J_p(0) = 0$ if $p > 0$ and $J_0'(x) = -J_1(x)$, it follows that $J_p(0)J_p'(0) - J_p(0)J_p'(0) = 0$ for all $p > 0$. Hence the integral is equal to

$$I = \int_0^c J_p(ax)\,J_p(bx)x\,dx = \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_p'(ac) - bJ_p(ac)J_p'(bc)\big].$$

Now using the formula

$$J_p'(x) = \frac{1}{2}\big[J_{p-1}(x) - J_{p+1}(x)\big],$$

we obtain

$$I = \frac{c}{2(b^2 - a^2)}\big[aJ_p(bc)\big(J_{p-1}(ac) - J_{p+1}(ac)\big) - bJ_p(ac)\big(J_{p-1}(bc) - J_{p+1}(bc)\big)\big].$$

Simplify with the help of the formula

$$J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x)$$

and you get

$$\begin{aligned} I &= \frac{c}{2(b^2 - a^2)}\Big[aJ_p(bc)\big(J_{p-1}(ac) - (\frac{2p}{ac}J_p(ac) - J_{p-1}(ac))\big) \\ &\qquad -bJ_p(ac)\big(J_{p-1}(bc) - (\frac{2p}{bc}J_p(bc) - J_{p-1}(bc))\big)\Big] \\ &= \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_{p-1}(ac) - bJ_p(ac)J_{p-1}(bc)\big], \end{aligned}$$

as claimed.

Note that this formula implies the orthogonality of Bessel functions. In fact its proof mirrors the proof of orthogonality from Section 4.8.

Solutions to Exercises 4.9

1. We have

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos\big(-x\sin\theta\big)\,d\theta = \frac{1}{\pi}\int_0^\pi \cos\big(x\sin\theta\big)\,d\theta.$$

So

$$J_0(0) = \frac{1}{\pi}\int_0^\pi d\theta = 1.$$

For $n \neq 0$,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos\big(n\theta - x\sin\theta\big)\,d\theta;$$

so

$$J_n(0) = \frac{1}{\pi}\int_0^\pi \cos n\theta\,d\theta = 0.$$

2. We have

$$\begin{aligned}
|J_n(x)| &= \left|\frac{1}{\pi}\int_0^\pi \cos\big(n\theta - x\sin\theta\big)\,d\theta\right| \\
&\leq \frac{1}{\pi}\int_0^\pi \overbrace{\big|\cos\big(n\theta - x\sin\theta\big)\big|}^{\leq 1}\,d\theta \\
&\leq \frac{1}{\pi}\int_0^\pi d\theta = 1.
\end{aligned}$$

3. taking imaginary parts from teh series in Exampe 2, we obtain

$$\begin{aligned}
\sin x &= \sum_{n=-\infty}^{\infty} J_n(x)\sin\frac{n\pi}{2} \\
&= \sum_{n=1}^{\infty} J_n(x)\sin\frac{n\pi}{2} + \sum_{-1}^{n=-\infty} J_n(x)\sin\frac{n\pi}{2} \\
&= \sum_{n=1}^{\infty} J_n(x)\sin\frac{n\pi}{2} - \sum_{n=1}^{\infty} J_{-n}(x)\sin\Big(\frac{n\pi}{2}\Big) \\
&= \sum_{k=0}^{\infty} J_{2k+1}(x)(-1)^k - \sum_{k=0}^{\infty} J_{-(2k+1)}(x) \\
&= 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(x).
\end{aligned}$$

4. (a) For any real number w, we have

$$\left|e^{iw}\right| = |\cos w + i\sin w| = \sqrt{\cos^2 + \sin^2 w} = 1.$$

If x and θ are real numbers then so is $x\sin\theta$, and hence

$$\left|e^{ix\sin\theta}\right| = 1.$$

(b) For fixed x, apply Parseval's identity to the Fourier series

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta},$$

whose Fourier coefficients are $J_n(x)$, and you get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|e^{ix\sin\theta}\right|^2\,d\theta=\sum_{n=-\infty}^{\infty}|J_n(x)|^2$$

or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}1\,d\theta=J_0(x)^2+2\sum_{n=1}^{\infty}J_n(x)^2,$$

hence

$$1=J_0(x)^2+2\sum_{n=1}^{\infty}J_n(x)^2.$$

(c) Since the series converges (for fxed x), its nth term goes to 0. Hence $\lim_{n\to\infty}J_n(x)=0$.

5. All the terms in the series

$$1=J_0(x)^2+2\sum_{n=1}^{\infty}J_n(x)^2$$

are nonnegative. Since they all add-up to 1, each must be less than or equal to 1. Hence

$$J_0(x)^2\leq 1\Rightarrow|J_0(x)|\leq 1$$

and, for $n\geq 2$,

$$2J_n(x)^2\leq 1\Rightarrow|J_n(x)|\leq\frac{1}{\sqrt{2}}.$$

6. (4) implies that

$$\cos\big(x\sin\theta\big)+i\sin\big(x\sin\theta\big)=\sum_{n=-\infty}^{\infty}J_n(x)\big(\cos n\theta+i\sin n\theta\big).$$

taking real parts, we get

$$\begin{aligned}\cos\big(x\sin\theta\big)&=\sum_{n=-\infty}^{\infty}J_n(x)\cos n\theta\\&=J_0(x)+\sum_{n=-1}^{-\infty}J_n(x)\cos n\theta+\sum_{n=1}^{\infty}J_n(x)\cos n\theta\\&=J_0(x)+\sum_{n=1}^{\infty}J_{-n}(x)\cos n\theta+\sum_{n=1}^{\infty}J_n(x)\cos n\theta\\&=J_0(x)+\sum_{n=1}^{\infty}(-1)^nJ_n(x)\cos n\theta+\sum_{n=1}^{\infty}J_n(x)\cos n\theta\\&=J_0(x)+\sum_{n=1}^{\infty}\Big[\big(1+(-1)^n\big)J_n(x)\cos n\theta\Big]\\&=J_0(x)+2\sum_{n=1}^{\infty}J_{2n}(x)\cos\big(2n\theta\big).\end{aligned}$$

The second identity follows similarly by taking imaginary parts.

7. We have $f(t) = f(\pi - t)$ and $g(t) = -g(\pi - t)$. So

$$\begin{aligned}
\int_0^{\pi} f(t)g(t)\,dt &= \int_0^{\pi/2} f(t)g(t)\,dt + \int_{\pi/2}^{\pi} f(t)g(t)\,dt \\
&= \int_0^{\pi/2} f(t)g(t)\,dt - \int_{\pi/2}^{0} f(\pi - x)g(\pi - x)\,dx \quad (\text{let } t = \pi - x) \\
&= \int_0^{\pi/2} f(t)g(t)\,dt - \int_0^{\pi/2} f(x)g(x)\,dx = 0.
\end{aligned}$$

8. (a) Apply Exercise 7 with $f(t) = \cos(x \sin t)$ and $g(t) = \cos t$. Part (b) is true for even n and not true if n is odd. If n is even, apply Exercise 7 with $f(t) = \sin(x \sin t)$ and $g(t) = \sin(nt)$.

Solutions to Exercises 5.1

1. Start with Laplace's equation in spherical coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2}+\cot\theta\frac{\partial u}{\partial \theta}+\csc^2\theta\,\frac{\partial^2 u}{\partial \phi^2}\Big)=0, \tag{1}$$

where $0<r<a$, $0<\phi<2\pi$, and $0<\theta<\pi$. To separate variable, take a product solution of the form

$$u(r,\,\theta,\,\phi)=R(r)\Theta(\theta)\Phi(\phi)=R\Theta\Phi,$$

and plug it into (1). We get

$$R''\Theta\Phi+\frac{2}{r}R'\Theta\Phi+\frac{1}{r^2}\Big(R\Theta''\Phi+\cot\theta R\Theta'\Phi+\csc^2\theta\,R\Theta\Phi''\Big)=0.$$

Divide by $R\Theta\Phi$ and multiply by r^2:

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\csc^2\theta\,\frac{\Phi''}{\Phi}=0.$$

Now proceed to separate the variables:

$$r^2\frac{R''}{R}+2r\frac{R'}{R}=-\left(\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\csc^2\theta\,\frac{\Phi''}{\Phi}\right).$$

Since the left side is a function of r and the right side is a function of ϕ and θ, each side must be constant and the constants must be equal. So

$$r^2\frac{R''}{R}+2r\frac{R'}{R}=\mu$$

and

$$\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\csc^2\theta\,\frac{\Phi''}{\Phi}=-\mu.$$

The equation in R is equivalent to (3). Write the second equation in the form

$$\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\mu=-\csc^2\theta\,\frac{\Phi''}{\Phi};$$

$$\sin^2\theta\left(\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\mu\right)=-\frac{\Phi''}{\Phi}.$$

This separates the variables θ and ϕ, so each side must be constant and the constant must be equal. Hence

$$\sin^2\theta\left(\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\mu\right)=\nu$$

and

$$\nu=-\frac{\Phi''}{\Phi}\quad\Rightarrow\quad\Phi''+\nu\Phi=0.$$

We expect 2π-periodic solutions in Φ, because ϕ is and azimuthal angle. The only way for the last equation to have 2π-periodic solutions that are essentially different is to set $\nu=m^2$, where $m=0,\,1,\,2,\,\ldots$. This gives the two equations

$$\Phi''+m^2\Phi=0$$

(equation (5)) and

$$\sin^2\theta\left(\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\mu\right)=m^2,$$

which is equivalent to (6).

3. From Section 5.5, we have

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{2}(3x^2-1).$$

So

$$P_0(\cos\theta)=1,\quad P_1(\cos\theta)=\cos\theta,\quad P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1).$$

Next, we verify that these functions are solutions of (7) with $\mu=n(n+1)$, with $n=0,1,2$, respectively, or $\mu=0,2,6$. (Recall the values of μ from (14).) For $\mu=0$, (7) becomes

$$\Theta''+\cot\theta\Theta'=0,$$

and clearly the constant function $P_0(\cos\theta)=1$ is a solution. For $\mu=2$, (7) becomes

$$\Theta''+\cot\theta\Theta'+2\Theta=0.$$

Taking $\Theta=P_1(\cos\theta)=\cos\theta$, we have $\Theta'=-\sin\theta$, and $\Theta''=-\cos\theta$. Plugging into the equation, we find

$$\Theta''+\cot\theta\Theta'+2\Theta=-\cos\theta+\frac{\cos\theta}{\sin\theta}(-\sin\theta)+2\cos\theta=0.$$

Finally, for $\mu=6$, (7) becomes

$$\Theta''+\cot\theta\Theta'+6\Theta=0.$$

Taking $\Theta=P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1)$, we have

$$\Theta'=-3\sin\theta\cos\theta,$$

$$\Theta''=-3\left(\cos^2\theta-\sin^2\theta\right),$$

$$\begin{aligned}\Theta''+\cot\theta\Theta'+6\Theta &= -3\left(\cos^2\theta-\sin^2\theta\right)+\frac{\cos\theta}{\sin\theta}(-3\sin\theta\cos\theta)\\ &\quad +3(3\cos^2\theta-1)\\ &= 3(\cos^2\theta+\sin^2\theta)-3=0\end{aligned}$$

4. When $m=0$, $P_1^0(\cos\theta)=P_1(\cos\theta)$ was treated in Exercise 3. When $m=1$, $P_1^1(\cos\theta)=-\sqrt{1-\cos^2\theta}=\sqrt{\sin^2\theta}=\sin\theta$. You should note that $0\le\theta\le\pi$, so $\sin\theta\ge 0$, thus the positive sign in front of the square root. With $m=1$ and $\mu=1$, equation (6) becomes

$$\Theta''+\cot\theta\Theta'+(2-\csc^2\theta)\Theta=0.$$

Taking $\Theta=P_1^1(\cos\theta)=\sin\theta$, we have $\Theta'=\cos\theta$, $\Theta''=-\sin\theta$. So

$$\begin{aligned}\Theta''+\cot\theta\Theta'+(2-\csc^2\theta)\Theta &= -\sin\theta+\frac{\cos\theta}{\sin\theta}\cos\theta+\left(2-\frac{1}{\sin^2\theta}\right)\sin\theta\\ &= \sin\theta+\frac{1}{\sin\theta}\overbrace{\left(\cos^2\theta-1\right)}^{=-\sin^2\theta}=0.\end{aligned}$$

Solutions to Exercises 5.2

1. This problem is similar to Example 2. Note that f is its own Legendre series:

$$f(\theta) = 20\,(P_1(\cos\theta) + P_0(\cos\theta)).$$

So really there is no need to compute the Legendre coefficients using integrals. We simply have $A_0 = 20$ and $A_1 = 20$, and the solution is

$$u(r,\,\theta) = 20 + 20\,r\,\cos\theta.$$

3. We have

$$u(r,\,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta),$$

with

$$\begin{aligned} A_n &= \frac{2n+1}{2}\int_0^{\pi} f(\theta) P_n(\cos\theta)\,\sin\theta\,d\theta \\ &= \frac{2n+1}{2}\int_0^{\frac{\pi}{2}} 100\,P_n(\cos\theta)\,\sin\theta\,d\theta + \frac{2n+1}{2}\int_{\frac{\pi}{2}}^{\pi} 20\,P_n(\cos\theta)\,\sin\theta\,d\theta. \end{aligned}$$

Let $x = \cos\theta$, $dx = -\sin\theta\,d\theta$. Then

$$A_n = 50(2n+1)\int_0^1 P_n(x)\,dx + 10\,(2n+1)\int_{-1}^0 P_n(x)\,dx.$$

The case $n = 0$ is immediate by using $P_0(x) = 1$,

$$A_0 = 50\int_0^1 dx + 10\int_{-1}^0 dx = 60.$$

For $n > 0$, the integrals are not straightforward and you need to refer to Exercise 10, Section 5.6, where they are evaluated. Quoting from this exercise, we have

$$\int_0^1 P_{2n}(x)dx = 0, \quad n = 1, 2, \ldots,$$

and

$$\int_0^1 P_{2n+1}(x)\,dx = \frac{(-1)^n (2n)!}{2^{2n+1}(n!)^2(n+1)}, \quad n = 0, 1, 2, \ldots.$$

Since $P_{2n}(x)$ is an even function, then, for $n > 0$,

$$\int_{-1}^0 P_{2n}(x)\,dx = \int_0^1 P_{2n}(x)\,dx = 0.$$

Hence for $n > 0$,

$$A_{2n} = 0.$$

Now $P_{2n+1}(x)$ is an odd function, so

$$\int_{-1}^0 P_{2n+1}(x)\,dx = -\int_0^1 P_{2n+1}(x)\,dx = -\frac{(-1)^n (2n)!}{2^{2n+1}(n!)^2(n+1)}.$$

Hence for $n = 0,\,1,\,2,\,\ldots$,

$$\begin{aligned} A_{2n+1} &= 50(4n+3)\int_0^1 P_{2n+1}(x)\,dx + 10\,(4n+3)\int_{-1}^0 P_{2n+1}(x)\,dx \\ &= 50(4n+3)\frac{(-1)^n (2n)!}{2^{2n+1}(n!)^2(n+1)} - 10\,(4n+3)\frac{(-1)^n (2n)!}{2^{2n+1}(n!)^2(n+1)} \\ &= 40\,(4n+3)\frac{(-1)^n (2n)!}{2^{2n+1}(n!)^2(n+1)} \\ &= 20\,(4n+3)\frac{(-1)^n (2n)!}{2^{2n}(n!)^2(n+1)}. \end{aligned}$$

So

$$u(r,\,\theta)=60+20\sum_{n=0}^{\infty}(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}r^{2n+1}P_{2n+1}(\cos\theta).$$

5. Solution We have

$$u(r,\,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta),$$

with

$$\begin{aligned}A_n &= \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\,\sin\theta\,d\theta\\ &= \frac{2n+1}{2}\int_0^{\frac{\pi}{2}}\cos\theta\,P_n(\cos\theta)\,\sin\theta\,d\theta\\ &= \frac{2n+1}{2}\int_0^{1} x\,P_n(x)\,dx,\end{aligned}$$

where, as in Exercise 3, we made the change of variables $x=\cos\theta$. At this point, we have to appeal to Exercise 11, Section 5.6, for the evaluation of this integral. (The cases $n=0$ and 1 can be done by referring to the explicit formulas for the P_n, but we may as well at this point use the full result of Exercise 11, Section 5.6.) We have

$$\int_0^1 xP_0(x)\,dx=\frac{1}{2};\int_0^1 xP_1(x)\,dx=\frac{1}{3};$$

$$\int_0^1 x\,P_{2n}(x)\,dx=\frac{(-1)^{n+1}(2n-2)!}{2^{2n}((n-1)!)^2 n(n+1)};\quad n=1,2,\ldots;$$

and

$$\int_0^1 x\,P_{2n+1}(x)\,dx=0;\quad n=1,2,\ldots.$$

Thus,

$$A_0=\frac{1}{2}\frac{1}{2}=\frac{1}{4};\quad A_1=\frac{3}{2}\frac{1}{3}=\frac{1}{2};\quad A_{2n+1}=0,\quad n=1,\,2,\,3,\ldots;$$

and for $n=1,\,2,\,\ldots$,

$$A_{2n}=\frac{2(2n)+1}{2}\frac{(-1)^{n+1}(2n-2)!}{2^{2n}((n-1)!)^2 n(n+1)}=(4n+1)\frac{(-1)^{n+1}(2n-2)!}{2^{2n+1}((n-1)!)^2 n(n+1)}.$$

So

$$u(r,\,\theta)=\frac{1}{4}+\frac{1}{2}r\cos\theta+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(4n+1)(2n-2)!}{2^{2n+1}((n-1)!)^2 n(n+1)}r^{2n}P_{2n}(\cos\theta).$$

7. (a) From (8)

$$u(r,\theta)=50+25\sum_{n=0}^{\infty}(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}r^{2n+1}P_{2n+1}(\cos\theta)\,.$$

Setting $\theta=\frac{\pi}{2}$, we get

$$\begin{aligned}u(r,\,\frac{\pi}{2}) &= 50+25\sum_{n=0}^{\infty}(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}r^{2n+1}P_{2n+1}(\cos\frac{\pi}{2})\\ &= 50+25\sum_{n=0}^{\infty}(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}r^{2n+1}\overbrace{P_{2n+1}(0)}^{=0}\\ &= 50\end{aligned}$$

This is expected, since the points with $\theta = \frac{\pi}{2}$ are located halfway between the boundary with The temperature at these points is the average value $\frac{100+0}{2}$ or 50.
(d) For $\theta = 0$, we have

$$\begin{aligned}
&\frac{u(r,\,0) + u(r,\,\pi)}{2}\\
&= 50 + 25\sum_{n=0}^{\infty}(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}r^{2n+1}\overbrace{(P_{2n+1}(\cos 0) + P_{2n+1}(\cos\pi))}^{=0}\\
&= 50,
\end{aligned}$$

because $P_{2n+1}(\cos 0) = P_{2n+1}(1) = 1$ and $P_{2n+1}(\cos\pi) = P_{2n+1}(-1) = -1$. This makes sense because the average of temperatures equidistant from $u = 50$ in a $\pm z$ symmetrical environment ought to be 50 itself.

9. We first describe the boundary function. From the figure, we see that there is an angle θ_0, with $\cos\theta_0 = -\frac{1}{3}$, and such that

$$f(\theta) = \begin{cases} 70 & \text{if } 0 \le \theta \le \theta_0,\\ 55 & \text{if } \theta_0 \le \theta \le \pi. \end{cases}$$

Appealing to (5), we have

$$\begin{aligned}
A_n &= \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\,\sin\theta\,d\theta\\
&= \frac{2n+1}{2}\int_0^{\theta_0} 70\cos\theta\,P_n(\cos\theta)\,\sin\theta\,d\theta\\
&\quad+\frac{2n+1}{2}\int_{\theta_0}^{\pi} 55\cos\theta\,P_n(\cos\theta)\,\sin\theta\,d\theta\\
&= \frac{2n+1}{2}\int_{-1}^{-\frac{1}{3}} 55\,P_n(t)\,dt + \frac{2n+1}{2}\int_{-\frac{1}{3}}^{1} 70\,P_n(t)\,dt,
\end{aligned}$$

where we have used the substitution $\cos\theta = t$, $dt = -\sin\theta d\theta$, and $\cos\theta_0 = -\frac{1}{3}$. For $n = 0$, we have $P_0(x) = 1$ and so

$$A_0 = \frac{1}{2}\left[(55)(\frac{2}{3} + (70)(\frac{4}{3})\right] = 65.$$

For $n \ge 1$, we appeal to Exercise 9(a) and (c), Section 5.6. The results that we need state that for $n = 1,\,2,\,\ldots,$

$$\int_x^1 P_n(t)\,dt = \frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)],$$

and

$$\int_{-1}^x P_n(t)\,dt = \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)]\,.$$

Thus, for $n \ge 1$,

$$\begin{aligned}
A_n &= \frac{2n+1}{2}\left[\frac{55}{2n+1}\left(P_{n+1}(-\frac{1}{3}) - P_{n-1}(-\frac{1}{3})\right)\right.\\
&\quad\left.+\frac{70}{2n+1}\left(P_{n-1}(-\frac{1}{3}) - P_{n+1}(-\frac{1}{3})\right)\right]\\
&= \frac{15}{2}\left(P_{n-1}(-\frac{1}{3}) - P_{n+1}(-\frac{1}{3})\right).
\end{aligned}$$

So

$$u(r,\,\theta) = 65 + \frac{15}{2}\sum_{n=1}^{\infty}\left(P_{n-1}(-\frac{1}{3}) - P_{n+1}(-\frac{1}{3})\right)\left(\frac{r}{3}\right)^n P_n(\cos\theta).$$

Solutions to Exercises 5.3

1. (c) Starting with (4) with $n = 2$, we have

$$Y_{2,m}(\theta, \phi) = \sqrt{\frac{5}{4\pi}\frac{(2-m)!}{(2+m)!}}\, P_2^m(\cos\theta)e^{im\phi},$$

where $m = -2, -1, 0, 1, 2$. To compute the spherical harmonics explicitely, we will need the explicit formula for the associated Legendre functions from Example 1, Section 5.7. We have

$$P_2^{-2}(x) = \tfrac{1}{8}(1-x^2); \quad P_2^{-1}(x) = \tfrac{1}{2}x\sqrt{1-x^2}; \quad P_2^0(x) = P_2(x) = \tfrac{3x^2-1}{2};$$
$$P_2^1(x) = -3x\sqrt{1-x^2}; \quad P_2^2(x) = 3(1-x^2).$$

So

$$\begin{aligned} Y_{2,-2}(\theta, \phi) &= \sqrt{\frac{5}{4\pi}\frac{(2+2)!}{(2-2)!}}\, P_2^{-2}(\cos\theta)e^{-2i\phi} \\ &= \sqrt{\frac{5}{4\pi}\frac{4!}{1}}\,\frac{1}{8}\overbrace{(1-\cos^2\theta)}^{=\sin^2\theta}e^{-2i\phi} \\ &= \sqrt{\frac{30}{\pi}}\,\frac{1}{8}\sin^2\theta e^{-2i\phi} = \frac{3}{4}\sqrt{\frac{5}{6\,\pi}}\,\sin^2\theta e^{-2i\phi}; \end{aligned}$$

$$\begin{aligned} Y_{2,-1}(\theta, \phi) &= \sqrt{\frac{5}{4\pi}\frac{(2+1)!}{(2-1)!}}\, P_2^{-1}(\cos\theta)e^{-i\phi} \\ &= \sqrt{\frac{5}{4\pi}\frac{3!}{1!}}\,\frac{1}{2}\cos\theta\,\overbrace{\sqrt{1-\cos^2\theta}}^{=\sin\theta}\,e^{-i\phi} \\ &= \sqrt{\frac{15}{2\pi}}\,\frac{1}{2}\cos\theta\sin\theta e^{-i\phi} = \frac{3}{2}\sqrt{\frac{5}{6\,\pi}}\,\cos\theta\sin\theta e^{-i\phi}. \end{aligned}$$

Note that since $0 \le \theta \le \pi$, we have $\sin\theta \ge 0$, and so the equality $\sqrt{1-\cos^2\theta} = \sin\theta$ that we used above does hold. Continuing the list of spherical harmonics, we have

$$\begin{aligned} Y_{2,0}(\theta, \phi) &= \sqrt{\frac{5}{4\pi}\frac{(2+0)!}{(2-0)!}}\, P_2(\cos\theta)e^{-i\phi} \\ &= \sqrt{\frac{5}{4\pi}}\,\frac{3\cos^2\theta - 1}{2} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1). \end{aligned}$$

The other spherical harmonics are computed similarly; or you can use the identity in Exercise 4. We have

$$\begin{aligned} Y_{2,2} &= (-1)^2\,\overline{Y_{2,-2}} = \overline{Y_{2,-2}} = \frac{3}{4}\sqrt{\frac{5}{6\,\pi}}\,\sin^2\theta\overline{e^{-2i\phi}} \\ &= \frac{3}{4}\sqrt{\frac{5}{6\,\pi}}\,\sin^2\theta e^{2i\phi}. \end{aligned}$$

In the preceding computation, we used two basic properties of the operation of complex conjugation:

$$\overline{az} = a\overline{z} \quad \text{if } a \text{ is a real number;}$$

and

$$\overline{e^{ia}} = e^{-ia} \quad \text{if } a \text{ is a real number.}$$

Finally,

$$\begin{aligned} Y_{2,1} &= (-1)^1\,\overline{Y_{2,-1}} = -\overline{Y_{2,-1}} = -\frac{3}{2}\sqrt{\frac{5}{6\,\pi}}\,\cos\theta\sin\overline{e^{-i\phi}} \\ &= -\frac{3}{2}\sqrt{\frac{5}{6\,\pi}}\,\cos\theta\sin e^{i\phi}. \end{aligned}$$

■

3. To prove (5), we use the definition of the spherical harmonics, (4), and write

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\pi} Y_{n,m}(\theta,\phi)\overline{Y}_{n',m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi \\ &= C\overbrace{\int_0^{2\pi} e^{im\phi}\,e^{im'\phi}d\phi}^{=I}\overbrace{\int_0^{\pi} P_n^m(\cos\theta)\,P_{n'}^{m'}(\cos\theta)\,\sin\theta d\theta}^{=II}, \end{aligned}$$

where C is a constant that depends on m, n, m', and n'. If $m \neq m'$, then $I = 0$, by the orthogonality of the complex exponential functions (see (11), Section 2.6, with $p = \pi$). If $m = m'$ but $n \neq n'$, then $II = 0$ by the orthogonality of the associate Legendre functions. (Make the change of variable $\cos\theta = x$ and then use (6), Section 5.7.) Thus if $(m, n) \neq (m', n')$, then the above double integral is 0. Next we prove (6), but first note that the left side of (6) is what you would get if you take $(m, n) = (m', n')$ in (5). This is because

$$Y_{m,n}(\theta,\,\phi)\overline{Y_{m,n}}(\theta,\,\phi) = |Y_{m,n}(\theta,\,\phi)|^2\,.$$

Here again we are using a property of complex numbers, which states that

$$z\,\overline{z} = |z|^2 \quad \text{for any complex number } z.$$

Using (4), we have

$$\begin{aligned} &\int_0^{\pi} Y_{n,m}(\theta,\phi)\overline{Y}_{n,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi \\ &= \overbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{im\phi}\,e^{-im\phi}d\phi}^{=I}\,\overbrace{\frac{(2n+1)(n-m)!}{2(n+m)!}\int_0^{\pi}\left[P_n^m(\cos\theta)\right]^2\,\sin\theta d\theta}^{=II}. \end{aligned}$$

We have

$$I = \frac{1}{2\pi}\int_0^{2\pi} e^{im\phi}\,e^{-im\phi}d\phi = \frac{1}{2\pi}\int_0^{2\pi} d\phi = 1,$$

because $e^{im\phi}\,e^{-im\phi} = 1$. Also, $II = 1$ by (7), Section 5.7. (You need to make a change of variables in the integral in II, $x = \cos\theta,\ dx = -\sin\theta\,d\theta$, before applying the result of Section 5.7). ■

5. (a) If $m = 0$, the integral becomes

$$\int_0^{2\pi} \phi\,d\phi = \frac{1}{2}\phi^2\Big|_0^{2\pi} = 2\pi^2.$$

Now suppose that $m \neq 0$. Using integration by parts, with $u = \phi,\ du = d\phi,\ dv = e^{-im\phi},\ v = \frac{1}{-im}e^{-im\phi}$, we obtain:

$$\int_0^{2\pi} \overbrace{\phi}^{=u}\,\overbrace{e^{-im\phi}}^{=dv}\,d\phi \;=\; \left[\phi\,\frac{1}{-im}e^{-im\phi}\right|_0^{2\pi} + \frac{1}{im}\int_0^{2\pi} e^{-im\phi}\,d\phi$$

We have

$$e^{-im\phi}\big|_{\phi=2\pi} = \left[\cos(m\phi) - i\sin(m\phi)\right|_{\phi=2\pi} = 1,$$

and

$$\begin{aligned}\int_0^{2\pi} e^{-im\phi}\,d\phi &= \int_0^{2\pi} \big(\cos m\phi - i\sin m\phi\big)\,d\phi \\ &= \begin{cases} 0 & \text{if } m \neq 0, \\ 2\pi & \text{if } m = 0 \end{cases}\end{aligned}$$

So if $m \neq 0$,

$$\int_0^{2\pi} \phi\, e^{-im\phi}\,d\phi = \frac{2\pi}{-im} = \frac{2\pi}{m} i.$$

Putting both results together, we obtain

$$\int_0^{2\pi} \phi\, e^{-im\phi}\,d\phi = \begin{cases} \frac{2\pi}{m} i & \text{if } m \neq 0, \\ 2\pi^2 & \text{if } m = 0. \end{cases}$$

(b) Using $n = 0$ and $m = 0$ in (9), we get

$$\begin{aligned}A_{0,0} &= \frac{1}{2\pi}\sqrt{\frac{1}{4\pi}\frac{0!}{0!}}\overbrace{\int_0^{2\pi}\phi\,d\phi}^{=2\pi^2}\int_0^{\pi} P_0(\cos\theta)\sin\theta\,d\theta \\ &= \pi\frac{1}{2\sqrt{\pi}}\int_0^{\pi}\frac{1}{\sqrt{2}}\sin\theta\,d\theta \\ &= \pi\frac{1}{2\sqrt{\pi}}\overbrace{\int_0^{\pi}\sin\theta\,d\theta}^{=2} = \sqrt{\pi}\end{aligned}$$

Using $n = 1$ and $m = 0$ in (9), we get

$$\begin{aligned}A_{1,0} &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}\frac{1}{1}}\int_0^{2\pi}\phi\;d\phi\int_0^{\pi} P_1(\cos\theta)\sin\theta\,d\theta \\ &= \frac{1}{4\pi}\sqrt{\frac{3}{\pi}}(2\pi^2)\overbrace{\int_{-1}^{1} P_1(x)\,dx}^{=0} \\ &= 0,\end{aligned}$$

where we used $\int_{-1}^{1} P_1(x)\,dx = 0$, because $P_1(x) = x$ is odd. Using $n = 1$ and $m = -1$ in (9), and appealing to the formulas for the associated Legendre functions from Section 5.7, we get

$$\begin{aligned}A_{1,-1} &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}\frac{2!}{0!}}\overbrace{\int_0^{2\pi}\phi\, e^{i\phi}\,d\phi}^{=\frac{2\pi}{(-1)}i}\int_0^{\pi} P_1^{-1}(\cos\theta)\sin\theta\,d\theta \\ &= -i\sqrt{\frac{3}{2\pi}}\frac{1}{2}\overbrace{\int_0^{\pi}\sin^2\theta\,d\theta}^{=\frac{\pi}{2}} \quad \Big(P_1^{-1}(\cos\theta) = \frac{1}{2}\sin\theta\Big) \\ &= -\frac{i}{4}\sqrt{\frac{3\pi}{2}}.\end{aligned}$$

Using $n = 1$ and $m = 1$ in (9), and appealing to the formulas for the associated

Legendre functions from Section 5.7, we get

$$\begin{aligned} A_{1,1} &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}\frac{0!}{2!}}\overbrace{\int_0^{2\pi}\phi\, e^{-i\phi}\,d\phi}^{=\frac{2\pi}{1}i}\int_0^{\pi} P_1^1(\cos\theta)\sin\theta\,d\theta \\ &= i\sqrt{\frac{3}{8\pi}}\overbrace{\int_0^{\pi} -\sin^2\theta\,d\theta}^{=-\frac{\pi}{2}} \quad (P_1^1(\cos\theta) = -\sin\theta) \\ &= -\frac{i}{4}\sqrt{\frac{3\pi}{2}}. \end{aligned}$$

(c) The formula for $A_{n,0}$ contains the integral $\int_0^{\pi} P_n^0(\cos\theta)\sin\theta\,d\theta$. But $P_n^0 = P_n$, the nth Legendre polynomial; so

$$\begin{aligned} \int_0^{\pi} P_n^0(\cos\theta)\sin\theta\,d\theta &= \int_0^{\pi} P_n(\cos\theta)\sin\theta\,d\theta \\ &= \int_{-1}^{1} P_n(x)\,dx \\ &= 0 \quad (n = 1,\, 2,\, \ldots), \end{aligned}$$

where the last equality follows from the orthogonality of Legendre polynomials (take $m = 0$ in Theorem 1, Section 5.6, and note that $P_0(x) = 1$, so $\int_{-1}^{1}(1)P_n(x)\,dx = 0$, as desired.)

7. (a) The boundary function is given by

$$f(\theta,\phi) = \begin{cases} 100 & \text{if } -\frac{\pi}{4}\phi \le \frac{\pi}{4}; \\ 0 & \text{otherwise.} \end{cases}$$

The solution $u(r,\theta,\phi)$ is given by (11), where the coefficients A_{nm} are given by (8). Since the integrand in (8) is 2π-periodic in ϕ, the outer limits in the integral in (8) can be changed to $-\pi$ to π without affecting the value of the integral. So we have

$$\begin{aligned} A_{nm} &= \int_{-\pi}^{\pi}\int_0^{\pi} f(\theta,\,\phi)\overline{Y}_{n,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi \\ &= 100\int_{-\pi/4}^{\pi/4}\int_0^{\pi}\overline{Y}_{n,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi\,. \end{aligned}$$

(b) Using the explicit formulas for the spherical harmonics from Exercise 1, obtain the coefficients given in the table.

If $m = n = 0$, then

$$A_{0,0} = \frac{100}{2\sqrt{\pi}}\int_{-\pi/4}^{\pi/4}\overbrace{\int_0^{\pi}\sin\theta\,d\theta}^{=2}\,d\phi = 50\sqrt{\pi}.$$

If $m = 0$ but $n \ne 0$, then A_{nm} contains the integral $\int_{-1}^{1} P_n(x)\,dx$ $(n = 1,\, 2,\, \ldots)$, which is equal to 0 by the orthogonality of the Legendre polynomials; and so $A_{mn} = 0$ if $m = 0$ and $n \ne 0$. For the other coefficients, let us compute using the formula for the spherical harmonics:

$$\begin{aligned} A_{nm} &= 100\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\int_{-\pi/4}^{\pi/4} e^{-im\phi}\,d\phi \int_0^{\pi} P_n^m(\cos\theta)\sin\theta\,d\theta \\ &= 100\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\overbrace{\int_{-\pi/4}^{\pi/4}(\cos m\phi - i\sin m\phi)\,d\phi}^{=\frac{2}{m}\sin\frac{m\pi}{4}} \int_{-1}^{1} P_n^m(s)\,ds \\ &= \frac{200}{m}\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\sin\frac{m\pi}{4}\int_{-1}^{1} P_n^m(s)\,ds. \end{aligned}$$

The integral $I_{nm} = \int_{-1}^{1} P_n^m(s)\,ds$ can be evaluated for particular values of m and n (or you can use the general formula from Exercises 6). For example, if $m = n = 1$, then

$$\int_{-1}^{1} P_1^1(s)\,ds = -\overbrace{\int_{-1}^{1}\sqrt{1-s^2}\,ds}^{=\text{half area of a disk of radius 1}} = -\frac{\pi}{2}.$$

So

$$A_{1,1} = \frac{200}{1}\sqrt{\frac{3}{4\pi}\frac{0!}{2!}}\sin\frac{\pi}{4}(-\frac{\pi}{2}) = -25\sqrt{3\pi}.$$

The other coefficients are computed similarly by appealing to the explicit formulas for the associated Legendre functions from Section 5.7 and using the preceding formulas.

(c) Using an approximation of the solution with n running from 0 to 2 and m running from $-n$ to n, we get

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}r^n Y_{n,m}(\theta,\,\phi) \approx \sum_{n=0}^{2}\sum_{m=-n}^{n} A_{nm}r^n Y_{n,m}(\theta,\,\phi).$$

Now substituting the explicit formulas and values, we obtain

$$\begin{aligned} &u(r,\theta,\phi) \\ &\approx r^0Y_{0,0}A_{0,0} + r^1Y_{1,-1}A_{1,-1} + r^1Y_{1,1}A_{1,1} + r^2Y_{2,-2}A_{2,-2} + r^2Y_{2,2}A_{2,2} \\ &= \frac{1}{2\sqrt{\pi}}50\sqrt{\pi} + r\left[\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\phi}\,25\sqrt{3\pi} + \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\phi}\,25\sqrt{3\pi}\right] \\ &\quad + r^2\left[\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\theta e^{-2i\phi}100\sqrt{\frac{5}{6\pi}} + \frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\theta e^{2i\phi}100\sqrt{\frac{5}{6\pi}}\right] \\ &= 25 + 75\,\frac{\sqrt{2}}{2}r\sin\theta\overbrace{\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)}^{=\cos\phi} + 125\,r^2\frac{1}{\pi}\sin^2\theta\overbrace{\left(\frac{e^{2i\phi}+e^{-2i\phi}}{2}\right)}^{=\cos 2\phi} \\ &= 25 + 75\,\frac{\sqrt{2}}{2}\,r\sin\theta\cos\phi + 125\,r^2\frac{1}{\pi}\sin^2\theta\cos 2\phi\,. \end{aligned}$$

■

9. We apply (11). Since f is its own spherical harmonics series, we have

$$u(r,\,\theta,\,\phi) = Y_{0,0}(\theta,\,\phi) = \frac{1}{2\sqrt{\pi}}.$$ ■

11. The solution is very much like the solution of Exercise 7. In our case, we have

$$\begin{aligned}
A_{nm} &= 50\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\int_{-\pi/3}^{\pi/3} e^{-im\phi}\,d\phi \int_0^{\pi} P_n^m(\cos\theta)\sin\theta\,d\theta \\
&= 50\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\overbrace{\int_{-\pi/3}^{\pi/3}(\cos m\phi - i\sin m\phi)\,d\phi}^{=\frac{2}{m}\sin\frac{m\pi}{3}}\int_{-1}^{1} P_n^m(s)\,ds \\
&= \frac{100}{m}\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\sin\frac{m\pi}{3}\int_{-1}^{1} P_n^m(s)\,ds \\
&= \frac{100}{m}\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\sin\frac{m\pi}{3} I_{nm},
\end{aligned}$$

where $I_{nm} = \int_{-1}^{1} P_n^m(s)\,ds$. Several values of I_{nm} were found previously. You can also use the result of Exercise 6 to evaluate it. ■

Solutions to Exercises 5.4

5. We apply Theorem 3 and note that since f depends only on r and not on θ or ϕ, the series expansion should also not depend on θ or ϕ. So all the coefficients in the series are 0 except for the coefficients $A_{j,0,0}$, which we will write as A_j for simplicity. Using (16) with $m = n = 0$, $a = 1$, $f(r, \theta, \phi) = 1$, and $Y_{0,0}(\theta, \phi) = \frac{1}{2\sqrt{\pi}}$, we get

$$\begin{aligned}
A_j &= \frac{2}{j_1^2(\alpha_{\frac{1}{2},j})}\int_0^1\int_0^{2\pi}\int_0^{\pi} j_0(\lambda_{0,j}\,r)\frac{1}{2\sqrt{\pi}}r^2\sin\theta\,d\theta\,d\phi\,dr \\
&= \frac{1}{\sqrt{\pi}\,j_1^2(\alpha_{\frac{1}{2},j})}\overbrace{\int_0^{2\pi}d\phi}^{=2\pi}\overbrace{\int_0^{\pi}\sin\theta\,d\theta}^{=2}\int_0^1 j_0(\lambda_{0,j}\,r)r^2\,dr \\
&= \frac{4\sqrt{\pi}}{j_1^2(\alpha_{\frac{1}{2},j})}\int_0^1 j_0(\lambda_{0,j}\,r)r^2\,dr,
\end{aligned}$$

where $\lambda_{0,j} = \alpha_{\frac{1}{2},j}$, the jth zero of the Bessel function of order $\frac{1}{2}$. Now

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

(see Example 1, Section 4.7), so the zeros of $J_{1/2}$ are precisely the zeros of $\sin x$, which are $j\pi$. Hence

$$\lambda_{0,j} = \alpha_{\frac{1}{2},j} = j\pi.$$

Also, recall that

$$j_0(x) = \frac{\sin x}{x}$$

(Exercises 38, Section 4.8), so

$$\begin{aligned}
\int_0^1 j_0(\lambda_{0,j}\,r)r^2\,dr &= \int_0^1 j_0(j\pi\,r)r^2\,dr = \int_0^1\frac{\sin(j\pi r)}{j\pi\,r}r^2\,dr \\
&= \frac{1}{j\pi}\overbrace{\int_0^1\sin(j\pi\,r)r\,dr}^{=\frac{(-1)^{j+1}}{j\pi}} \\
&= \frac{(-1)^{j+1}}{(j\pi)^2},
\end{aligned}$$

where the last integral follows by integration by parts. So,

$$A_j = \frac{4\sqrt{\pi}}{j_1^2(j\pi)}\frac{(-1)^{j+1}}{(j\pi)^2}.$$

This can be simplified by using a formula for j_1. Recall from Exercise 38, Section 4.7,

$$j_1(x) = \frac{\sin x - x\cos x}{x^2}.$$

Hence

$$j_1^2(j\pi) = \left[\frac{\sin(j\pi) - j\pi\cos(j\pi)}{(j\pi)^2}\right]^2 = \left[\frac{-\cos(j\pi)}{j\pi}\right]^2 = \left[\frac{(-1)^{j+1}}{j\pi}\right]^2 = \frac{1}{(j\pi)^2},$$

and

$$A_j = 4(-1)^{j+1}\sqrt{\pi},$$

and so the series expansion becomes: for $0 < r < 1$,

$$\begin{aligned} 1 &= \sum_{j=1}^{\infty} 4(-1)^{j+1}\sqrt{\pi}\frac{\sin j\pi r}{j\pi r} Y_{0,0}(\theta,\,\phi) \\ &= \sum_{j=1}^{\infty} 4(-1)^{j+1}\sqrt{\pi}\frac{\sin j\pi r}{j\pi r}\frac{1}{2\sqrt{\pi}} \\ &= \sum_{j=1}^{\infty} 2(-1)^{j+1}\frac{\sin j\pi r}{j\pi r}. \end{aligned}$$

It is interesting to note that this series is in fact a half range sine series expansion. Indeed, multiplying both sides by r, we get

$$r = \frac{2}{\pi}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\sin j\pi r}{j} \qquad (0 < r < 1),$$

which is a familiar sines series expansion (compare with Example 1, Section 2.4).■

7. Reasoning as we did in Exercise 5, we infer that the series representation is of the form

$$\sum_{j=1}^{\infty} A_j Y_{0,0}(\theta,\,\phi) j_0(\alpha_{\frac{1}{2},\,j} r),$$

where

$$\begin{aligned} A_j &= \frac{2}{j_1^2(\alpha_{\frac{1}{2},\,j})}\int_0^1\int_0^{2\pi}\int_0^{\pi} j_0(\lambda_{0,\,j}\,r)\frac{1}{2\sqrt{\pi}} r^4 \sin\theta\, d\theta\, d\phi\, dr \\ &= \frac{1}{\sqrt{\pi}\, j_1^2(\alpha_{\frac{1}{2},\,j})} \overbrace{\int_0^{2\pi} d\phi}^{=2\pi} \overbrace{\int_0^{\pi}\sin\theta\, d\theta}^{=2} \int_0^1 j_0(\lambda_{0,\,j}\,r) r^4\, dr \\ &= \frac{4\sqrt{\pi}}{j_1^2(\alpha_{\frac{1}{2},\,j})}\int_0^1 j_0(\lambda_{0,\,j}\,r) r^4\, dr, \end{aligned}$$

where $\lambda_{0,\,j} = \alpha_{\frac{1}{2},\,j}$, the jth zero of the Bessel function of order $\frac{1}{2}$. Again, from Exercise 5:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$\lambda_{0,\,j} = \alpha_{\frac{1}{2},\,j} = j\pi,$$

and

$$j_0(x) = \frac{\sin x}{x}.$$

So

$$\begin{aligned} \int_0^1 j_0(\lambda_{0,\,j}\,r) r^4\, dr &= \int_0^1 j_0(j\pi\, r) r^4\, dr = \int_0^1 \frac{\sin(j\pi r)}{j\pi\, r} r^4\, dr \\ &= \frac{1}{j\pi}\overbrace{\int_0^1 \sin(j\pi\, r) r^3\, dr}^{=(-6+j^2\pi^2)\frac{(-1)^{j+1}}{(j\pi)^3}} \\ &= \frac{(-6+j^2\pi^2)(-1)^{j+1}}{(j\pi)^4}, \end{aligned}$$

where the last integral follows by three integrations by parts. So,

$$A_j = \frac{4\sqrt{\pi}}{j_1^2(j\pi)}\frac{(-1)^{j+1}(-6+j^2\pi^2)}{(j\pi)^4}.$$

From Exercise 5,

$$j_1^4(j\pi) = \left[\frac{\sin(j\pi) - j\pi\cos(j\pi)}{(j\pi)^2}\right]^4 = \left[\frac{-\cos(j\pi)}{j\pi}\right]^4 = \left[\frac{(-1)^{j+1}}{j\pi}\right]^4 = \frac{1}{(j\pi)^4},$$

so

$$A_j = 4(-1)^{j+1}\sqrt{\pi}\frac{(-6 + j^2\pi^2)}{(j\pi)^3},$$

and so the series expansion becomes: for $0 < r < 1$,

$$\begin{aligned} r^2 &= \sum_{j=1}^{\infty} 4(-1)^{j+1}\sqrt{\pi}\frac{\sin j\pi r}{(j\pi)^3 r}(-6 + j^2\pi^2)\frac{1}{2\sqrt{\pi}} \\ &= \sum_{j=1}^{\infty} 2(-1)^{j+1}(-6 + j^2\pi^2)\frac{\sin j\pi r}{(j\pi)^3 r}. \end{aligned}$$

Here again, it is interesting to note that this series is a half range sine series expansion. Indeed, multiplying both sides by r, we get

$$r^3 = \sum_{j=1}^{\infty} 2(-1)^{j+1}(-6 + j^2\pi^2)\frac{\sin j\pi r}{(j\pi)^3} \qquad (0 < r < 1),$$

which is a sines series expansion of the function $f(r) = r^3$, $0 < r < 1$.

Solutions to Exercises 5.5

1. Putting $n=0$ in (9), we obtain

$$P_0(x) = \frac{1}{2^0}\sum_{m=0}^{0}(-1)^m \frac{(0-2m)!}{m!(0-m)!(0-2m)!}x^{0-2m}.$$

The sum contains only one term corresponding to $m=0$. Thus

$$P_0(x) = (-1)^0 \frac{0!}{0!0!0!}x^0 = 1,$$

because $0! = 1$. For $n=1$, formula (9) becomes

$$P_1(x) = \frac{1}{2^1}\sum_{m=0}^{M}(-1)^m \frac{(2-2m)!}{m!(1-m)!(1-2m)!}x^{1-2m},$$

where $M = \frac{1-1}{2} = 0$. Thus the sum contains only one term corresponding to $m=0$ and so

$$P_1(x) = \frac{1}{2^1}(-1)^0 \frac{2!}{0!1!1!}x^1 = x.$$

For $n=2$, we have $M = \frac{2}{2} = 1$ and (9) becomes

$$\begin{aligned}
P_2(x) &= \frac{1}{2^2}\sum_{m=0}^{1}(-1)^m \frac{(4-2m)!}{m!(2-m)!(2-2m)!}x^{2-2m} \\
&= \frac{1}{2^2}\overbrace{(-1)^0\frac{4!}{0!2!2!}x^2}^{m=0} + \frac{1}{2^2}\overbrace{(-1)^1\frac{(4-2)!}{1!1!0!}x^0}^{m=1} \\
&= \frac{1}{4}6x^2 + \frac{1}{4}(-1)2 = \frac{3}{2}x^2 - \frac{1}{2}.
\end{aligned}$$

For $n=3$, we have $M = \frac{3-1}{2} = 1$ and (9) becomes

$$\begin{aligned}
P_3(x) &= \frac{1}{2^3}\sum_{m=0}^{1}(-1)^m \frac{(6-2m)!}{m!(3-m)!(3-2m)!}x^{3-2m} \\
&= \frac{1}{2^3}(-1)^0\frac{6!}{0!3!3!}x^3 + \frac{1}{2^3}(-1)^1\frac{4!}{1!2!1!}x^1 \\
&= \frac{5}{2}x^3 - \frac{3}{2}x.
\end{aligned}$$

For $n=4$, we have $M = \frac{4}{2} = 2$ and (9) becomes

$$\begin{aligned}
P_4(x) &= \frac{1}{2^4}\sum_{m=0}^{2}(-1)^m \frac{(8-2m)!}{m!(4-m)!(4-2m)!}x^{4-2m} \\
&= \frac{1}{2^4}\frac{8!}{0!4!4!}x^4 - \frac{1}{2^4}\frac{6!}{1!3!2!}x^2 + \frac{1}{2^4}\frac{4!}{2!2!0!}x^0 \\
&= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}$$

3. By the first formula in Exercise 2(b) (applied with n and $n+1$), we have

$$\begin{aligned}
P_{2n}(0) - P_{2n+2}(0) = &= P_{2n}(0) - P_{2(n+1)}(0)\\
&= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} - (-1)^{n+1}\frac{(2(n+1))!}{2^{2(n+1)}((n+1)!)^2}\\
&= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} + (-1)^n \frac{(2n+2)(2n+1)((2n)!)}{4\,2^{2n}(n+1)^2(n!)^2}\\
&= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\left[1 + \frac{(2n+2)(2n+1)}{4\,(n+1)^2}\right]\\
&= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\left[1 + \frac{2(n+1)(2n+1)}{4\,(n+1)^2}\right]\\
&= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\left[1 + \frac{(2n+1)}{2\,(n+1)}\right]\\
&= (-1)^n \frac{(2n)!(4n+3)}{2^{2n+1}(n!)^2(n+1)}
\end{aligned}$$

5. Using the explicit formulas for the Legendre polynomials, we find

$$\begin{aligned}
\int_{-1}^{1} P_3(x)\,dx &= \int_{-1}^{1} (\frac{5}{2}x^3 - \frac{3}{2}x)\,dx\\
&= (\frac{5}{8}x^4 - \frac{3}{4}x^2)\Big|_{-1}^{1} = 0
\end{aligned}$$

Another faster way to see the answer is to simply note that P_3 is an odd function, so its integral over any symmetric interval is 0. There is yet another more important reason for this integral to equal 0. In fact,

$$\int_{-1}^{1} P_n(x)\,dx = 0 \qquad \text{for all } n \neq 0.$$

This is a consequence of orthogonality that you will study in Section 5.6.

7. Using the explicit formulas for the Legendre polynomials, we find

$$\begin{aligned}
\int_{0}^{1} P_2(x)\,dx &= \int_{0}^{1} (\frac{3}{2}x^2 - \frac{1}{2})\,dx\\
&= (\frac{1}{2}x^3 - \frac{1}{2}x)\Big|_{0}^{1} = 0
\end{aligned}$$

This integral is a special case of a more general formula presented in Exercise 10, Section 5.6.

9. This is Legendre's equation with $n(n+1) = 30$ so $n = 5$. Its general solution is of the form

$$\begin{aligned}
y &= c_1 P_5(x) + c_2 Q_5(x)\\
&= c_1 \frac{1}{8}(63x^5 - 70x^3 + 15x) + c_2\left(1 - 15x^2 + 30x^4 + \cdots\right)\\
&= c_1(63x^5 - 70x^3 + 15x) + c_2\left(1 - 15x^2 + 30x^4 + \cdots\right)
\end{aligned}$$

In finding $P_5(x)$, we used the given formulas in the text. In finding the first few terms of $Q_5(x)$, we used (3) with $n = 5$. (If you are comparing with the answers in your textbook, just remember that c_1 and c_2 are arbitrary constants.)

11. This is Legendre's equation with $n(n+1) = 0$ so $n = 0$. Its general solution is of the form

$$\begin{aligned} y &= c_1 P_0(x) + c_2 Q_0(x) \\ &= c_1 + c_2 \left(x - \frac{-2}{6} x^3 + \frac{24}{5!} x^5 + \cdots \right) \\ &= c_1 + c_2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) \end{aligned}$$

where we used $P_0(x) = 1$ and (4) with $n = 0$.

13. This is Legendre's equation with $n(n+1) = 6$ or $n = 2$. Its general solution is $y = c_1 P_2(x) + c_2 Q_2(x)$. The solution will be bounded on $[-1, 1]$ if and only if $c_2 = 0$; that's because P_2 is bounded in $[-1, 1]$ but Q_2 is not. Now, using $P_2(x) = \frac{1}{2}(3x^2 - 1)$, we find

$$y(0) = c_1 P_2(0) + c_2 Q_2(0) = -\frac{c_1}{2} + c_2 Q_2(0)$$

If $c_2 = 0$, then $c_1 = 0$ and we obtain the zero solution, which is not possible (since we are given $y'(0) = 1$, the solution is not identically 0). Hence $c_2 \neq 0$ and the solutions is not bounded.

15. This is Legendre's equation with $n(n+1) = 56$ or $n = 7$. Its general solution is $y = c_1 P_7(x) + c_2 Q_2(x)$. The solution will be bounded on $[-1, 1]$ if and only if $c_2 = 0$. Since $P_7(x)$ is odd, we have $P_7(0) = 0$ and so

$$1 = y(0) = c_1 P_7(0) + c_2 Q_7(0) = c_2 Q_2(0).$$

This shows that $c_2 \neq 0$ and the solutions is not bounded.

17. (To do this problem we can use the recurrence relation for the coefficients, as we have done below in the solution of Exercise 19. Instead, we offer a different solution based on an interesting observation.) This is Legendre's equation with $n(n+1) = \frac{3}{4}$ or $n = \frac{1}{2}$. Its general solution is still given by (3) and (4), with $n = \frac{1}{2}$:

$$y = c_1 y_1 + c_2 y_2,$$

where

$$\begin{aligned} y_1(x) &= 1 - \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!} x^2 + \frac{(\frac{1}{2}-2)\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+3)}{4!} x^4 + \cdots \\ &= 1 - \frac{3}{8} x^2 - \frac{21}{128} x^4 + \cdots \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= x - \frac{(\frac{1}{2}-1)(\frac{1}{2}+2)}{3!} x^3 + \frac{(\frac{1}{2}-3)(\frac{1}{2}-1)(\frac{1}{2}+2)(\frac{1}{2}+4)}{5!} x^5 + \cdots \\ &= x + \frac{5}{24} x^3 + \frac{15}{128} x^5 + \cdots \end{aligned}$$

Since $y_1(0) = 1$ and $y_2(0) = 0$, $y_1'(0) = 0$ and $y_2'(0) = 1$ (differentiate the series term by term, then evaluate at $x = 0$), it follows that the solution is $y = y_1(x) + y_2(x)$, where y_1 and y_2 are as describe above.

19. This is Legendre's equation with $\mu = 3$. Its solutions have series expansions $y = \sum_{m=0}^{\infty} a_m x^m$ for $-1 < x < 1$, where

$$a_{m+2} = \frac{m(m+1) - \mu}{(m+2)(m+1)} a_m \,.$$

Since $y(0) = 0$ and $y'(0) = 1$, we find that $a_0 = 0$ and $a_1 = 1$. Now because the recurrence relation is a two-step recurrence, we obtain

$$0 = a_0 = a_2 = a_4 = \cdots$$

The odd-indexed coefficients are determined from a_1. Taking $\mu = 3$ and $m = 1$ in the recurrence relation, we find:

$$a_3 = \frac{2-3}{(3)(2)}a_1 = -\frac{1}{6}.$$

Now using $m = 3$, we find

$$a_5 = \frac{3(4)-3}{(5)(4)}a_3 = \frac{9}{20}\,(-\frac{1}{6}) = -\frac{3}{40},$$

and so forth. Thus the solution is

$$y = x - \frac{x^3}{6} - \frac{3}{40}x^5 + \cdots.$$

23. Using the reduction of order formula in Exercise 22 (with $n = 0$) and the explicit formula for $P_0(x) = 1$, we find for $-1 < x < 1$,

$$\begin{aligned}
Q_0(x) &= P_0(x)\int \frac{1}{[P_0(x)]^2(1-x^2)}\,dx \\
&= \int \frac{1}{1-x^2}\,dx \\
&= \frac{1}{2}\int \left(\frac{1}{1+x} - \frac{1}{-1+x}\right)dx \\
&= \frac{1}{2}\,(\ln|1+x| - \ln|x-1|) = \frac{1}{2}\ln\Big(\frac{|1+x|}{|x-1|}\Big) \\
&= \frac{1}{2}\ln\Big(\frac{1+x}{1-x}\Big).
\end{aligned}$$

In evaluating the integral, we use the partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{2}\left(\frac{1}{1+x} - \frac{1}{-1+x}\right).$$

You should verify this identity or derive it using the standard methods from calculus for partial fraction decomposition. The second comment concerns the absolute value inside the logarithm. Because $-1 < x < 1$, we have $|1+x| = 1+x$ and $|-1+x| = 1-x$. So there is no need to keep the absolute values.

29. (a) Since

$$\left|(x + i\sqrt{1-x^2}\cos\theta)^n\right| = \left|x + i\sqrt{1-x^2}\cos\theta\right|^n,$$

it suffices to prove the inequality

$$\left|x + i\sqrt{1-x^2}\cos\theta\right| \leq 1,$$

which in turn will follow from

$$\left|x + i\sqrt{1-x^2}\cos\theta\right|^2 \leq 1.$$

For any complex number $\alpha + i\beta$, we have $|\alpha + i\beta|^2 = \alpha^2 + \beta^2$. So

$$\begin{aligned}
\left|x + i\sqrt{1-x^2}\cos\theta\right|^2 &= x^2 + (\sqrt{1-x^2}\cos\theta)^2 \\
&= x^2 + (1-x^2)\cos^2\theta \\
&\leq x^2 + (1-x^2) = 1,
\end{aligned}$$

which proves the desired inequality.
(b) Using Laplace's formula, we have, for $-1 \le x \le 1$,

$$\begin{aligned} |P_n(x)| &= \frac{1}{\pi}\left|\int_0^\pi (x + i\sqrt{1-x^2}\cos\theta)^n \, d\theta\right| \\ &\le \frac{1}{\pi}\int_0^\pi \left|(x + i\sqrt{1-x^2}\cos\theta)^n\right| d\theta \\ &\le \frac{1}{\pi}\int_0^\pi d\theta \qquad \text{(by (a))} \\ &= 1 \end{aligned}$$

31. Recall that if α is any real number and k is a nonnegative integer, the kth binomial coefficient is

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!} \quad \text{for } k \ge 1$$

and $\binom{\alpha}{0} = 1$. With this notation, the binomial theorem asserts that

$$(1+x)^\alpha = \sum_{k=0}^\infty \binom{\alpha}{k} x^k, \quad -1 < x < 1.$$

(a) Set $\alpha = -\frac{1}{2}$, and obtain that

$$\frac{1}{\sqrt{1+v}} = \sum_{k=0}^\infty (-1)^k \frac{(2k)!}{2^{2k}(k!)^2} v^k, \qquad |v| < 1.$$

Solution We have

$$\begin{aligned} \binom{-1/2}{k} &= \frac{1}{k!} \overbrace{\frac{-1}{2}\left(\frac{-1}{2} - 1\right)\cdots\left(\frac{-1}{2} - (k-1)\right)}^{k \text{ factors}} \\ &= \frac{1}{k!}\frac{-1}{2}\left(\frac{-3}{2}\right)\cdots\left(\frac{-(2k-1)}{2}\right) \\ &= \frac{1}{k!}\frac{(-1)^k}{2^k}[1\cdot 3\cdots(2k-1)] \\ &= \frac{1}{k!}\frac{(-1)^k}{2^k}\frac{[1\cdot 2\cdot 3\cdot 4\cdots(2k-1)\cdot(2k)]}{2\cdot 4\cdots(2k)} \\ &= \frac{1}{k!}\frac{(-1)^k}{2^k}\frac{(2k)!}{(2\cdot 1)\cdot(2\cdot 2)\cdots(2\cdot k)} \\ &= \frac{1}{k!}\frac{(-1)^k}{2^k}\frac{(2k)!}{2^k(k!)} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} \end{aligned}$$

So, for $|v| < 1$, the binomial series gives

$$\frac{1}{\sqrt{1+v}} = (1+v)^{-1/2} = \sum_{k=0}^\infty \binom{-1/2}{k} v^k = \sum_{k=0}^\infty (-1)^k \frac{(2k)!}{2^{2k}(k!)^2} v^k$$

(b) We have

$$\begin{aligned}
\frac{1}{\sqrt{1+(-2xu+u^2)}} &= \sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{2^{2k}(k!)^2}(-2xu+u^2)^k \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{2^{2k}(k!)^2}\sum_{j=0}^{k}\binom{k}{j}(-2xu)^{k-j}u^{2j} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^k \frac{(2k)!}{2^{2k}(k!)^2}(-1)^{k-j}\frac{k!}{j!(k-j)!}2^{k-j}x^{k-j}u^{k-j}u^{2j} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^j \frac{(2k)!}{2^{k+j}j!k!(k-j)!}x^{k-j}u^{k+j}
\end{aligned}$$

Let $n = k + j$, so $k = n - j$ and j cannot exceed $n/2$, because $0 \leq j \leq k$. Hence

$$\begin{aligned}
\frac{1}{\sqrt{1+(-2xu+u^2)}} &= \sum_{n=0}^{\infty}u^n\sum_{j=0}^{n/2}(-1)^j \frac{(2(n-j))!}{2^n(n-j)!(n-2j)!}x^{n-2j} \\
&= \sum_{n=0}^{\infty}u^n P_n(x)
\end{aligned}$$

Solutions to Exercises 5.6

1. Bonnet's relation says: For $n = 1, 2, \ldots,$

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x).$$

We have $P_0(x) = 1$ and $P_1(x) = x$. Take $n = 1$, then

$$\begin{aligned} 2P_2(x) + P_0(x) &= 3xP_1(x), \\ 2P_2(x) &= 3x \cdot x - 1, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1). \end{aligned}$$

Take $n = 2$ in Bonnet's relation, then

$$\begin{aligned} 3P_3(x) + 2P_1(x) &= 5xP_2(x), \\ 3P_3(x) &= 5x\big(\frac{1}{2}(3x^2 - 1)\big) - 2x, \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x. \end{aligned}$$

Take $n = 3$ in Bonnet's relation, then

$$\begin{aligned} 4P_4(x) + 3P_2(x) &= 7xP_3(x), \\ 4P_4(x) &= 7x\big(\frac{5}{2}x^3 - \frac{3}{2}x\big) - \frac{3}{2}\big(x^2 - 1\big), \\ P_4(x) &= \frac{1}{4}\left[\frac{35}{2}x^4 - 15x^2 + \frac{3}{2}\right]. \end{aligned}$$

3.

$$\int_{-1}^{1} x\, P_7(x)\, dx = \int_{-1}^{1} P_1(x)\, P_7(x)\, dx = 0,$$

by Theorem 1(i).

5. By Bonnet's relation with $n = 3$,

$$\begin{aligned} 7xP_3(x) &= 4P_4(x) + 3P_2(x), \\ xP_3(x) &= \frac{4}{7}P_4(x) + \frac{3}{7}P_2(x). \end{aligned}$$

So

$$\begin{aligned} \int_{-1}^{1} x\, P_2(x)P_3(x)\, dx &= \int_{-1}^{1} \left(\frac{4}{7}P_4(x) + \frac{3}{7}P_2(x)\right) P_2(x)\, dx \\ &= \frac{4}{7}\int_{-1}^{1} P_4(x)\, P_2(x)\, dx + \frac{3}{7}\int_{-1}^{1} [P_2(x)]^2\, dx \\ &= 0 + \frac{3}{7}\frac{2}{5} = \frac{6}{35}, \end{aligned}$$

where we have used Theorem 1(i) and (ii) to evaluate the last two integrals.

7. $\displaystyle\int_{-1}^{1} x^2\, P_7(x)\, dx = 0$, because x^2 is even and $P_7(x)$ is odd. So their product is odd, and the integral of any odd function over a symmetric interval is 0.

9. (a) Write (4) in the form

$$(2n+1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t).$$

Integrate from x to 1,

$$\begin{aligned}(2n+1)\int_x^1 P_n(t)\,dt &= \int_x^1 P'_{n+1}(t)\,dt - \int_x^1 P'_{n-1}(t)\,dt\\ &= P_{n+1}(t)\Big|_x^1 - P_{n-1}(t)\Big|_x^1\\ &= \big(P_{n-1}(x) - P_{n+1}(x)\big) + \big(P_{n+1}(1) - P_{n-1}(1)\big).\end{aligned}$$

By Example 1, we have $P_{n+1}(1) - P_{n-1}(1) = 0$. So for $n = 1, 2, \ldots$

$$\int_x^1 P_n(t)\,dt = \frac{1}{2n+1}\left[P_{n-1}(x) - P_{n+1}(x)\right].$$

(b) First let us note that because P_n is even when is even and odd when is odd, it follows that $P_n(-1) = (-1)^n P_n(1) = (-1)^n$. Taking $x = -1$ in (a), we get

$$\int_{-1}^1 P_n(t)\,dt = \frac{1}{2n+1}[P_{n-1}(-1) - P_{n+1}(-1)] = 0,$$

because $n-1$ and $n+1$ are either both even or both odd, so $P_{n-1}(-1) = P_{n+1}(-1)$.
(c) We have

$$0 = \int_{-1}^1 P_n(t)\,dt = \int_{-1}^x P_n(t)\,dt + \int_x^1 P_n(t)\,dt.$$

So

$$\begin{aligned}\int_{-1}^x P_n(t)\,dt &= -\int_x^1 P_n(t)\,dt\\ &= -\frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)]\\ &= \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)]\end{aligned}$$

10. (a) Replace n by $2n$ in the formula in Exercise 9(a), then take $x = 0$, and get

$$\int_0^1 P_{2n}(t)\,dt = \frac{1}{4n+1}[P_{2n-1}(0) - P_{2n+1}(0)] = 0,$$

because $P_{2n-1}(x)$ and $P_{2n+1}(x)$ are odd functions, so $P_{2n-1}(0) = 0 = P_{2n+1}(0)$.

(b) Applying the result of Exercise 9(a) with $x = 0$, we find

$$\begin{aligned}\int_0^1 P_{2n+1}(t)\,dt &= \frac{1}{2(2n+1)+1}[P_{2n}(0) - P_{2n+2}(0)] = \frac{1}{4n+3}[P_{2n}(0) - P_{2n+2}(0)]\\ &= \frac{1}{4n+3}(-1)^n\frac{(2n)!(4n+3)}{2^{2n+1}(n!)^2(n+1)} = (-1)^n\frac{(2n)!}{2^{2n+1}(n!)^2(n+1)},\end{aligned}$$

by the result of Exercise 3, Section 5.5.

11. (a)

$$\int_0^1 xP_0(x)\,dx = \int_0^1 x\,dx = \frac{1}{2};$$

$$\int_0^1 xP_1(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$$

(b) Using Bonnet's relation (with $2n$ in place of n):

$$xP_{2n} = \frac{1}{4n+1}\left[(2n+1)P_{2n+1}(x) + 2nP_{2n-1}(x)\right].$$

So, using Exercise 10 (with n and $n-1$), we find

$$\begin{aligned}
\int_0^1 x\,P_{2n}(x)\,dx &= \frac{1}{4n+1}\left[(2n+1)\int_0^1 P_{2n+1}(x)\,dx + 2n\int_0^1 P_{2n-1}(x)\,dx\right] \\
&= \frac{1}{4n+1}\left[(2n+1)\frac{(-1)^n(2n)!}{2^{2n+1}(n!)^2(n+1)} + 2n\frac{(-1)^{n-1}(2(n-1))!}{2^{2(n-1)+1}((n-1)!)^2 n}\right] \\
&= \frac{(-1)^n}{4n+1}\frac{(2n-2)!}{2^{2n}(n!)^2(n+1)}\left[(2n+1)\frac{(2n)(2n-1)}{2} - 4(n+1)n^2\right] \\
&= \frac{(-1)^n}{4n+1}\frac{(2n-2)!}{2^{2n}(n!)^2(n+1)}\left[-n(4n+1)\right] \\
&= \frac{(-1)^{n+1}(2n-2)!}{2^{2n}((n-1)!)^2 n(n+1)}
\end{aligned}$$

Using Bonnet's relation (with $2n+1$ in place of n):

$$xP_{2n+1} = \frac{1}{4n+3}\left[(2n+2)P_{2n+2}(x) + (2n+1)P_{2n}(x)\right].$$

Now using Exercise 10, it follows that $\int_0^1 x\,P_{2n+1}(x)\,dx = 0$.

13. We will use $D^n f$ to denote the nth derivative of f. Using Exercise 12,

$$\int_{-1}^1 (1-x^2)\,P_{13}(x)\,dx = \frac{(-1)^{13}}{2^{13}(13)!}\int_{-1}^1 D^{13}[(1-x^2)]\,(x^2-1)^{13}\,dx = 0$$

because $D^{13}[(1-x^2)] = 0$.

15. Using Exercise 12,

$$\begin{aligned}
\int_{-1}^1 x^n\,P_n(x)\,dx &= \frac{(-1)^n}{2^n n!}\int_{-1}^1 D^n[x^n]\,(x^2-1)^n\,dx \\
&= \frac{(-1)^n}{2^n}\int_{-1}^1 (x^2-1)^n\,dx \\
&= \frac{(-1)^n}{2^n}\int_0^\pi (\cos^2\theta-1)^n\,\sin\theta d\theta = \frac{1}{2^n}\int_0^\pi \sin^{2n+1}\theta d\theta \\
&\qquad \text{(Let } x=\cos\theta,\ dx=-\sin\theta d\theta.) \\
&= \frac{1}{2^n}\int_0^{\pi/2}\sin^{2n+1}\theta d\theta + \frac{1}{2^n}\int_{\pi/2}^{\pi}\sin^{2n+1}\theta d\theta \\
&= \frac{1}{2^n}\int_0^{\pi/2}\sin^{2n+1}\theta d\theta + \frac{1}{2^n}\int_0^{\pi/2}\sin^{2n+1}(\theta+\frac{\pi}{2})d\theta \\
&= \frac{1}{2^n}\int_0^{\pi/2}\sin^{2n+1}\theta d\theta + \frac{1}{2^n}\int_0^{\pi/2}\cos^{2n+1}\theta d\theta \qquad (\sin(\theta+\frac{\pi}{2}) = \cos\theta) \\
&= \frac{2}{2^n}\int_0^{\pi/2}\sin^{2n+1}\theta d\theta = \frac{2}{2^n}\frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{2^{n+1}(n!)^2}{(2n+1)!},
\end{aligned}$$

where we have used the results of Exercises 41 and 42, Section 4.7.

17. Using Exercise 12,

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)\,P_2(x)\,dx &= \frac{(-1)^2}{2^2 2!}\int_{-1}^{1} D^2[\ln(1-x)]\,(x^2-1)^2\,dx\\ &= \frac{1}{8}\int_{-1}^{1}\frac{-1}{(1-x)^2}\,(x-1)^2(x+1)^2\,dx\\ &= \frac{-1}{8}\int_{-1}^{1}(x+1)^2\,dx = \frac{-1}{24}(x+1)^3\Big|_{-1}^{1} = \frac{-1}{3}.\end{aligned}$$

21. For $n > 0$, we have

$$D^n[\ln(1-x)] = \frac{-(n-1)!}{(1-x)^n}$$

. So

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)\,P_n(x)\,dx &= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} D^n[\ln(1-x)]\,(x^2-1)^n\,dx\\ &= \frac{(-1)^{n+1}(n-1)!}{2^n n!}\int_{-1}^{1}\frac{(x+1)^n(x-1)^n}{(1-x)^n}\,dx\\ &= \frac{-1}{2^n n}\int_{-1}^{1}(x+1)^n\,dx = \frac{-1}{2^n n(n+1)}(x+1)^{n+1}\Big|_{-1}^{1}\\ &= \frac{-2}{n(n+1)}\end{aligned}$$

For $n = 0$, we use integration by parts. The integral is a convergent improper integral (the integrand has a problem at 1)

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)\,P_0(x)\,dx &= \int_{-1}^{1}\ln(1-x)\,dx\\ &= -(1-x)\ln(1-x) - x\Big|_{-1}^{1} = -2+2\ln 2.\end{aligned}$$

To evaluate the integral at $x = 1$, we used $\lim_{x\to 1}(1-x)\ln(1-x) = 0$.

22. The integral can be reduced to the one in Exercise 21, as follows: Let $t = -x$, then

$$\begin{aligned}\int_{-1}^{1} \ln(1+x)P_n(x)\,dx &= \int_{-1}^{1}\ln(1-t)P_n(-t)\,dt\\ &= (-1)^n\int_{-1}^{1}\ln(1-t)P_n t\,dt\\ &= (-1)^{n+1}\frac{2}{n(n+1)}\end{aligned}$$

if $n > 0$, and $-2+2\ln 2$ if $n = 0$.

23. Using Bonnet's relation and then the result of Exercise 21: for $n > 1$,

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)xP_n(x)\,dx &= \frac{1}{2n+1}\int_{-1}^{1}\ln(1-x)\big[(n+1)P_{n+1}(x)+nP_{n-1}\big]\,dx\\ &= \frac{n+1}{2n+1}\int_{-1}^{1}\ln(1-x)P_{n+1}(x)\,dx\\ &\quad+\frac{n}{2n+1}\int_{-1}^{1}\ln(1-x)P_{n-1}\,dx\\ &= \frac{n+1}{2n+1}\frac{-2}{(n+1)(n+2)}+\frac{n}{2n+1}\frac{-2}{(n-1)n}\\ &= \frac{-2}{2n+1}\Big[\frac{1}{n+2}+\frac{1}{n-1}\Big] = \frac{-2}{2n+1}\frac{2n+1}{(n+2)(n-1)}\\ &= \frac{-2}{(n+2)(n-1)}\end{aligned}$$

The cases $n = 0$ and $n = 1$ have to be computed separately. For $n = 0$, we have

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)x\,dx &= \frac{1}{2}(x^2-1)\ln(1-x)\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}(1+x)\,dx \\ &= -1.\end{aligned}$$

For the integration by parts, we used $u = \ln(1-x)$, $dv = x\,dx$, $du = \frac{-1}{1-x}\,dx$, $v = \frac{1}{2}(x^2-1)$. For $n = 1$, applying Bonnet's relation as described previously, we obtain

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)xP_1(x)\,dx &= \frac{1}{3}\int_{-1}^{1}\ln(1-x)\big[2P_2(x)+1P_0\big]\,dx \\ &= \frac{2}{3}\int_{-1}^{1}\ln(1-x)P_2(x)\,dx + \frac{1}{3}\int_{-1}^{1}\ln(1-x)P_0\,dx\end{aligned}$$

Now use the results of Exercise 21:

$$\begin{aligned}\int_{-1}^{1} \ln(1-x)xP_1(x)\,dx &= \frac{2}{3}\frac{-2}{2(2+1)} + \frac{1}{3}(-2+2\ln 2) \\ &= \frac{-8}{9} + \frac{2}{3}\ln 2\end{aligned}$$

26. (a) Using the result of Exercise 10, we have

$$\begin{aligned}A_k &= \frac{2k+1}{2}\int_0^1 f(x)P_k(x)\,dx \\ &= \frac{2k+1}{2}\int_0^1 P_k(x)\,dx \\ &= \frac{2k+1}{2}\begin{cases} 0 & \text{if } k = 2n,\ n = 1, 2, \ldots, \\ (-1)^n \frac{(2n)!}{2^{2n+1}(n!)^2(n+1)} & \text{if } k = 2n+1,\ n = 0, 1, \ldots, \end{cases} \\ &= \begin{cases} 0 & \text{if } k = 2n,\ n = 1, 2, \ldots, \\ (-1)^n \frac{(4n+3)(2n)!}{2^{2n+1}(n!)^2(n+1)} & \text{if } k = 2n+1,\ n = 0, 1, \ldots. \end{cases}\end{aligned}$$

Also,

$$A_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

Hence

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{4n+3}{4n+4}\right)\frac{(2n)!}{2^{2n}(n!)^2}\,P_{2n+1}(x)\,.$$

27. Let $f(x) = |x|$, $-1 \le x \le 1$. (a) We have

$$\begin{aligned}A_0 &= \frac{1}{2}\int_{-1}^{1}|x|P_0(x)\,dx \\ &= \int_0^1 x\,dx = \frac{1}{2} \\ A_1 &= \frac{2+1}{2}\int_{-1}^{1}|x|\,x\,dx = 0\end{aligned}$$

(Because the integrand is odd.)

$$\begin{aligned}A_2 &= \frac{5}{2}\int_{-1}^{1}|x|\,\frac{1}{2}(3x^2-1)\,dx \\ &= \frac{5}{2}\int_0^1 x\,(3x^2-1)\,dx = \frac{5}{4}.\end{aligned}$$

(b) In general $A_{2k+1} = 0$, because f is even, and

$$\begin{aligned} A_{2n} &= \frac{4n+1}{2}\int_{-1}^{1} |x| P_{2n}(x)\,dx \\ &= (4n+1)\int_0^1 x\,P_{2n}(x)\,dx \\ &= (4n+1)\frac{(-1)^{n+1}(2n-2)!}{2^{2n}((n-1)!)^2 n(n+1)}. \end{aligned}$$

by Exercise 11

29. Call the function in Exercise 28 $g(x)$. Then

$$g(x) = \frac{1}{2}\left(|x| + x\right) = \frac{1}{2}\left(|x| + P_1(x)\right).$$

Let B_k denote the Legendre coefficient of g and A_k denote the Legendre coefficient of $f(x) = |x|$, for $-1 < x < 1$. Then, because $P_1(x)$ is its own Legendre series, we have

$$B_k = \begin{cases} \frac{1}{2}A_k & \text{if } k \neq 1 \\ \frac{1}{2}\left(A_k + 1\right) & \text{if } k = 1 \end{cases}$$

Using Exercise 27 to compute A_k, we find

$$B_0 = \frac{1}{2}A_0 = \frac{1}{4},\ B_1 = \frac{1}{2} + \frac{1}{2}A_1 = \frac{1}{2} + 0 = \frac{1}{2}, B_{2n+1} = 0,\ n = 1, 2, \ldots,$$

and

$$B_{2n} = \frac{1}{2}A_{2n} = \frac{(-1)^{n+1}(2n-2)!}{2^{2n+1}((n-1)!)^2 n}\left(\frac{4n+1}{n+1}\right).$$

30. (a) With the help of Exercise 21, we find

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-1}^{1} \ln(1-x)P_0(x)\,dx = \ln 2 - 1; \\ A_n &= \frac{2n+1}{2}\int_{-1}^{1} \ln(1-x)P_n(x)\,dx \quad (n = 1, 2, \ldots) \\ &= \frac{2n+1}{2}\frac{(-2)}{n(n+1)} = \frac{-(2n+1)}{n(n+1)}. \end{aligned}$$

31. (a) For $-1 < x < 1$, we use the result of Exercise 30 and get and the Legendre series

$$\begin{aligned} \ln(1+x) &= \ln(1-(-x)) \\ &= \ln 2 - 1 - \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\,P_n(-x) \\ &= \ln 2 - 1 - \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)}\,P_n(x), \end{aligned}$$

because $P_n(-x) = (-1)^n P_n(x)$.

Solutions to Exercises 6.1

1. Let $f_j(x)=\cos(j\pi x)$, $j=0,1,2,3$, and $g_j(x)=\sin(j\pi x)$, $j=1,2,3$. We have to show that $\int_0^2 f_j(x)g_k(x)\,dx=0$ for all possible choices of j and k. If $j=0$, then

$$\int_0^2 f_j(x)g_k(x)\,dx=\int_0^2 \sin k\pi x\,dx=\frac{-1}{k\pi}\cos(k\pi x)\Big|_0^2=0.$$

If $j\neq 0$, and $j=k$, then using the identity $\sin\alpha\cos\alpha=\frac{1}{2}\sin 2\alpha$,

$$\begin{aligned}\int_0^2 f_j(x)g_j(x)\,dx &= \int_0^2 \cos(j\pi x)\sin(j\pi x)\,dx\\ &= \frac{1}{2}\int_0^2 \sin(2j\pi x)\,dx\\ &= \frac{-1}{4j\pi}\cos(2j\pi x)\Big|_0^2=0.\end{aligned}$$

If $j\neq 0$, and $j\neq k$, then using the identity

$$\sin\alpha\cos\beta=\frac{1}{2}\big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\big),$$

we obtain

$$\begin{aligned}\int_0^2 f_j(x)g_k(x)\,dx &= \int_0^2 \sin(k\pi x)\cos(j\pi x)\,dx\\ &= \frac{1}{2}\int_0^2 \big(\sin(k+j)\pi x+\sin(k-j)\pi x\big)\,dx\\ &= \frac{-1}{2\pi}\Big(\frac{1}{k+j}\cos(k+j)\pi x+\frac{1}{k-j}\cos(k-j)\pi x\Big)\Big|_0^2=0.\end{aligned}$$

5. Let $f(x)=1$, $g(x)=2x$, and $h(x)=-1+4x$. We have to show that

$$\int_{-1}^1 f(x)g(x)w(x)\,dx=0,\quad \int_{-1}^1 f(x)h(x)w(x)\,dx=0,\quad \int_{-1}^1 g(x)h(x)w(x)\,dx=0.$$

Let's compute:

$$\int_{-1}^1 f(x)g(x)w(x)\,dx=\int_{-1}^1 2x\sqrt{1-x^2}\,dx=0,$$

because we are integrating an odd function over a symmetric interval. For the second integral, we have

$$\begin{aligned}\int_{-1}^1 f(x)h(x)w(x)\,dx &= \int_{-1}^1 (-1+4x^2)\sqrt{1-x^2}\,dx\\ &= \int_0^\pi (-1+4\cos^2\theta)\sin^2\theta\,d\theta\\ &\qquad (x=\cos\theta,\ dx=-\sin\theta\,d\theta, \sin\theta\geq 0 \text{ for } 0\leq\theta\leq\pi.)\\ &= -\int_0^\pi \sin^2\theta\,d\theta+4\int_0^\pi(\cos\theta\sin\theta)^2\,d\theta\\ &= -\overbrace{\int_0^\pi \frac{1-\cos 2\theta}{2}\,d\theta}^{=\frac{\pi}{2}}+4\int_0^\pi \Big(\frac{1}{2}\sin(2\theta)\Big)^2\,d\theta\\ &= -\frac{\pi}{2}+\overbrace{\int_0^\pi \frac{1-\cos(4\theta)}{2}\,d\theta}^{=\frac{\pi}{2}}=0\end{aligned}$$

For the third integral, we have

$$\int_{-1}^{1} g(x)h(x)w(x)\,dx = \int_{-1}^{1} 2x(-1+4x^2)\sqrt{1-x^2}\,dx = 0,$$

because we are integrating an odd function over a symmetric interval.

9. In order for the functions 1 and $a+bx+x^2$ to be orthogonal, we must have

$$\int_{-1}^{1} 1\cdot(a+bx+x^2)\,dx = 0$$

Evaluating the integral, we find

$$\begin{aligned} ax+\frac{b}{2}x^2+\frac{1}{3}x^3\Big|_{-1}^{1} &= 2a+\frac{2}{3}=0 \\ a &= -\frac{1}{3}. \end{aligned}$$

In order for the functions x and $\frac{1}{3}+bx+x^2$ to be orthogonal, we must have

$$\int_{-1}^{1} 1\cdot(\frac{1}{3}+bx+x^2)x\,dx = 0$$

Evaluating the integral, we find

$$\begin{aligned} \frac{1}{6}x^2+\frac{b}{3}x^3+\frac{1}{4}x^4\Big|_{-1}^{1} &= \frac{b}{3}=0 \\ b &= 0. \end{aligned}$$

13. Using Theorem 1, Section 5.6, we find the norm of $P_n(x)$ to be

$$\|P_n\| = \left(\int_{-1}^{1} P_n(x)^2\,dx\right)^{\frac{1}{2}} = \left(\frac{2}{2n+1}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{2n+1}}.$$

Thus the orthonormal set of functions obtained from the Legendre polynomials is

$$\frac{\sqrt{2}}{\sqrt{2n+1}}P_n(x),\ n=0,\,2,\,\ldots.$$

17. For Legendre series expansions, the inner product is defined in terms of integration against the Legendre polynomials. That is,

$$(f,\,P_j) = \int_{-1}^{1} f(x)P_j(x)\,dx = \frac{2}{2j+1}A_j$$

where A_j is the Legendre coefficient of f (see (7), Section 5.6). According to the generalized Parseval's identity, we have

$$\begin{aligned} \int_{-1}^{1} f^2(x)\,dx &= \sum_{j=0}^{\infty}\frac{|(f,\,P_j)|^2}{\|P_j\|^2} \\ &= \sum_{j=0}^{\infty}\left(\frac{2}{2j+1}A_j\right)^2\frac{2}{2j+1} \\ &= \sum_{j=0}^{\infty}\frac{2}{2j+1}A_j^2. \end{aligned}$$

(The norm $\|P_j\|$ is computed in Exercise 13.)

Solutions to Exercises 6.2

1. Sturm-Liouville form: $\big(xy'\big)' + \lambda y = 0$, $p(x) = x$, $q(x) = 0$, $r(x) = 1$. Singular problem because $p(x) = 0$ at $x = 0$.

5. Divide the equation through by x^2 and get $\frac{y''}{x} - \frac{y'}{x^2} + \lambda\frac{y}{x} = 0$. Sturm-Liouville form: $\big(\frac{1}{x}y'\big)' + \lambda\frac{y}{x} = 0$, $p(x) = \frac{1}{x}$, $q(x) = 0$, $r(x) = \frac{1}{x}$. Singular problem because $p(x)$ and $r(x)$ are not continuous at $x = 0$.

9. Sturm-Liouville form: $\big((1-x^2)\,y'\big)' + \lambda y = 0$, $p(x) = 1-x^2$, $q(x) = 0$, $r(x) = 1$. Singular problem because $p(\pm 1) = 0$.

13. Before we proceed with the solution, we can use our knowledge of Fourier series to guess a family of orthogonal functions that satisfy the Sturm-Liouville problem: $y_k(x) = \sin\frac{2k+1}{2}x$, $k = 0, 1, 2, \ldots$. It is straightforward to check the validity of our guess. Let us instead proceed to derive these solutions. We organize our solution after Example 2. The differential equation fits the form of (1) with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. In the boundary conditions, $a = 0$ and $b = \pi$, with $c_1 = d_2 = 1$ and $c_2 = d_1 = 0$, so this is a regular Sturm-Liouville problem.

We consider three cases.

CASE 1: $\lambda < 0$. Let us write $\lambda = -\alpha^2$, where $\alpha > 0$. Then the equation becomes $y'' - \alpha^2 y = 0$, and its general solution is $y = c_1 \sinh\alpha x + c_2\cosh\alpha x$. We need $y(0) = 0$, so substituting into the general solution gives $c_2 = 0$. Now using the condition $y'(\pi) = 0$, we get $0 = c_1\alpha\cosh\alpha\pi$, and since $\cosh x \neq 0$ for all x, we infer that $c_1 = 0$. Thus there are no nonzero solutions in this case.
CASE 2: $\lambda = 0$. Here the general solution of the differential equation is $y = c_1 x + c_2$, and as in Case 1 the boundary conditions force c_1 and c_2 to be 0. Thus again there is no nonzero solution.

CASE 3: $\lambda > 0$. In this case we can write $\lambda = \alpha^2$ with $\alpha > 0$, and so the equation becomes $y'' + \alpha^2 y = 0$. The general solution is $y = c_1\cos\alpha x + c_2 \sin\alpha x$. From $y(0) = 0$ we get $0 = c_1\cos 0 + c_2\sin 0$ or $0 = c_1$. Thus $y = c_2\sin\alpha x$. Now we substitute the other boundary condition to get $0 = c_2\alpha\cos\alpha\pi$. Since we are seeking nonzero solutions, we take $c_2 \neq 0$. Thus we must have $\cos\alpha\pi = 0$, and hence $\alpha = \frac{2k+1}{2}$. Since $\lambda = \alpha^2$, the problem has eigenvalues

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2,$$

and corresponding eigenfunctions

$$y_k = \sin\frac{2k+1}{2}x,\ \ k = 0, 1, 2, \ldots.$$

17. Case I If $\lambda = 0$, the general solution of the differential equation is $X = ax + b$. As in Exercise 13, check that the only way to satisfy the boundary conditions is to take $a = b = 0$. Thus $\lambda = 0$ is not an eigenvalue since no nontrivial solutions exist.

Case II If $\lambda = -\alpha^2 < 0$, then the general solution of the differential equation is $X = c_1\cosh\alpha x + c_2\sinh\alpha x$. We have $X' = c_1\alpha\sinh x + c_2\alpha\cosh\alpha x$. In order to have nonzero solutions, we suppose throughout the solution that c_1 or c_2 is nonzero. The first boundary condition implies

$$c_1 + \alpha c_2 = 0 \quad c_1 = -\alpha c_2.$$

Hence both c_1 and c_2 are nonzero. The second boundary condition implies that

$$c_1(\cosh\alpha + \alpha\sinh\alpha) + c_2(\sinh\alpha + \alpha\cosh\alpha) = 0.$$

Using $c_1 = -\alpha c_2$, we obtain

$$\begin{aligned} -\alpha c_2(\cosh\alpha + \alpha\sinh\alpha) + c_2(\sinh\alpha + \alpha\cosh\alpha) &= 0 \quad (\text{divide by} c_2 \neq 0)\\ \sinh\alpha(1-\alpha^2) &= 0\\ \sinh\alpha = 0 \text{ or } 1-\alpha^2 &= 0 \end{aligned}$$

Since $\alpha \neq 0$, it follows that $\sinh\alpha \neq 0$ and this implies that $1-\alpha^2 = 0$ or $\alpha = \pm 1$. We take $\alpha = 1$, because the value -1 does not yield any new eigenfunctions. For $\alpha = 1$, the corresponding solution is

$$X = c_1 \cosh x + c_2 \sinh x = -c_2 \cosh x + c_2 \sinh x,$$

because $c_1 = -\alpha c_2 = -c_2$. So in this case we have one negative eigenvalue $\lambda = -\alpha^2 = -1$ with corresponding eigenfunction $X = \cosh x - \sinh x$.

Case III If $\lambda = \alpha^2 > 0$, then the general solution of the differential equation is

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x\,.$$

We have $X' = -c_1\alpha \sin \alpha x + c_2 \alpha \cos \alpha x$. In order to have nonzero solutions, one of the coefficients c_1 or c_2 must be $\neq 0$. Using the boundary conditions, we obtain

$$\begin{aligned} c_1 + \alpha c_2 &= 0 \\ c_1(\cos\alpha - \alpha\sin\alpha) + c_2(\sin\alpha + \alpha\cos\alpha) &= 0 \end{aligned}$$

The first equation implies that $c_1 = -\alpha c_2$ and so both c_1 and c_2 are $neq 0$. From the second equation, we obtain

$$\begin{aligned} -\alpha c_2(\cos\alpha - \alpha\sin\alpha) + c_2(\sin\alpha + \alpha\cos\alpha) &= 0 \\ -\alpha(\cos\alpha - \alpha\sin\alpha) + (\sin\alpha + \alpha\cos\alpha) &= 0 \\ \sin\alpha(\alpha^2+1) &= 0 \end{aligned}$$

Since $\alpha^2 + 1 \neq 0$, then $\sin\alpha = 0$, and so $\alpha = n\pi$, where $n = 1, 2, \ldots$. Thus the eigenvalues are

$$\lambda_n = (n\pi)^2$$

with corresponding eigenfunctions

$$y_n = -n\pi \cos n\pi x + \sin n\pi x, \qquad n = 1, 2, \ldots.$$

21. If $\lambda = \alpha^2$, then the solutions are of the form $c_1 \cosh \alpha x + c_2 \sinh \alpha x$. Using the boundary conditions, we find

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\ y(1) = 0 \quad &\Rightarrow \quad c_2 \sinh \alpha = 0. \end{aligned}$$

But $\alpha \neq 0$, hence $\sinh\alpha \neq 0$, and so $c_2 = 0$. There are no nonzero solutions if $\lambda > 0$ and so the problem as no positive eigenvalues. This does not contradict Theorem 1 because if we consider the equation $y'' - \lambda y = 0$ as being in the form (1), then $r(x) = -1 < 0$ and so the problem is a singular Sturm-Liouville problem to which Theorem 1 does not apply.

25. The eigenfunctions in Example 2 are $y_j(x) = \sin jx$, $j = 1, 2, \ldots$. Since f is one of these eigenfunctions, it is equal to its own eigenfunction expansion.

29. You can verify the orthogonality directly by checking that

$$\int_0^{2\pi} \sin\frac{nx}{2}\sin\frac{mx}{2}\,dx = 0 \quad \text{if } m \neq n \ (m,\, n \text{ integers}).$$

You can also quote Theorem 2(a) because the problem is a regular Sturm-Liouville problem.

33. (a) From Exercise 36(b), Section 4.8, with $y = J_0(\lambda r)$ and $p = 0$, we have

$$2\lambda^2 \int_0^a [y(r)]^2 r\,dr = [a y'(a)]^2 + \lambda^2 a^2 [y(a)]^2.$$

But y satisfies the boundary condition $y'(a) = -\kappa y(a)$, so

$$\begin{aligned} 2\lambda^2 \int_0^a [y(r)]^2 r\,dr &= a^2\kappa^2[y(a)]^2 + \lambda^2 a^2 [y(a)]^2; \\ \int_0^a [y(r)]^2 r\,dr &= \frac{a^2}{2}\left[\kappa^2 \frac{[J_0(\lambda_k a)]^2}{\lambda_k^2} + [J_0(\lambda_k a)]^2\right] \\ &= \frac{a^2}{2}\left[[J_0(\lambda_k a)]^2 + [J_1(\lambda_k a)]^2\right], \end{aligned}$$

because, by (7), $[J_0(\lambda_k a)]^2 = \left[\frac{\lambda_k}{\kappa} J_1(\lambda_k a)\right]^2$.

(b) Reproduce the sketch of proof of Theorem 1. The given formula for the coefficients is precisely formula (5) in this case.

34. (a) We apply the result of Exercise 33. For $0 < r < 1$, we have

$$f(r) = 100 = \sum_{k=1}^{\infty} A_k J_0(\lambda_k r),$$

where λ_k ($k = 1, 2, \ldots$) are the positive roots of (7), with $a = \kappa = 1$, and

$$\begin{aligned} A_k &= \frac{2}{[J_0(\lambda_k)]^2 + [J_1(\lambda_k)]^2} \int_0^1 100 J_0(\lambda_k r) r\,dr \\ &= \frac{200}{[J_0(\lambda_k)]^2 + [\lambda_k^2 J_1(\lambda_k)]^2} \int_0^{\lambda_k} J_0(s) s\,ds \quad (\text{Let } \lambda_k r = s.) \\ &= \frac{200}{\lambda_k^2\left([J_0(\lambda_k)]^2 + [J_1(\lambda_k)]^2\right)} s J_1(s)\Big|_0^{\lambda_k} \\ &= \frac{200\, J_1(\lambda_k)}{\lambda_k\left([J_0(\lambda_k)]^2 + [J_1(\lambda_k)]^2\right)}. \end{aligned}$$

(b) By the definition of λ_k, we know that it is a positive root of (7), which, in this case, becomes $\lambda J_1(\lambda) = J_0(\lambda)$. Thus, for all k, $\lambda_k J_1(\lambda_k) = J_0(\lambda_k)$ or $J_1(\lambda_k) = \frac{J_0(\lambda_k)}{\lambda_k}$, and so, for $0 < r < 1$,

$$\begin{aligned} 100 &= 200 \sum_{k=1}^{\infty} \frac{J_0(\lambda_k)}{\lambda_k^2\left([J_0(\lambda_k)]^2 + [J_1(\lambda_k)]^2\right)} J_0(\lambda_k r) \\ &= 200 \sum_{k=1}^{\infty} \frac{J_1(\lambda_k)}{\lambda_k^2 [J_0(\lambda_k)]^2 + [J_0(\lambda_k)]^2} J_0(\lambda_k r) \\ &= 200 \sum_{k=1}^{\infty} \frac{J_1(\lambda_k)}{[J_0(\lambda_k)]^2\left(1 + \lambda_k^2\right)} J_0(\lambda_k r). \end{aligned}$$

35. Because the initial and boundary values do not depend on θ, it follows that the problem has a radially symmetric solution that does not depend on θ. Following the steps in the solution of Exercise 10, Section 4.2, we find that

$$u(r, t) = \sum_{k=1}^{\infty} A_k e^{-\lambda_k^2 t} J_0(\lambda_k r) \quad (t > 0, 0 < r < 1),$$

where, due to the Robin boundary condition, the radial part R in the product solutions must satisfy the boundary condition $R'(1) = -R(1)$. (If you separate variables, you will arrive at the equation in R that is treated in Example 6.) Considering that $R(r) = J_0(\lambda r)$, this condition gives $\lambda J_0'(\lambda) = -J_0(\lambda)$. Using $J_0'(r) = -J_1(r)$, we see that λ is a solution of equation (7): $\lambda J_1(\lambda) = J_0(\lambda)$. Setting $t = 0$ and using $u(r, 0) = 100$, we find that

$$100 = \sum_{k=1}^{\infty} A_k J_0(\lambda_k r),$$

which is the series expansion in Exercise 34. Thus

$$u(r, t) = 200 \sum_{k=1}^{\infty} \frac{J_0(\lambda_k)}{\lambda_k^2\big([J_0(\lambda_k)]^2 + [J_1(\lambda_k)]^2\big)} e^{-\lambda_k^2 t} J_0(\lambda_k r) \quad (t > 0, 0 < r < 1).$$

Solutions to Exercises 6.3

1. (a) The initial shape of the chain is given by the function

$$f(x) = -.01(x - .5), \quad 0 < x < .5,$$

and the initial velocity of the chain is zero. So the solution is given by (10), with $L = .5$ and $B_j = 0$ for all j. Thus

$$u(x, t) = \sum_{j=1}^{\infty} A_j J_0\left(\alpha_j \sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\, t\right).$$

To compute A_j, we use (11), and get

$$\begin{aligned} A_j &= \frac{2}{J_1^2(\alpha_j)} \int_0^{.5} (-.01)(x - .5) J_0\left(\alpha_j\sqrt{2x}\right) dx \\ &= \frac{-.02}{J_1^2(\alpha_j)} \int_0^{.5} (x - .5) J_0\left(\alpha_j\sqrt{2x}\right) dx \end{aligned}$$

Make the change of variables $s = \alpha_j \sqrt{2x}$, or $s^2 = 2\alpha_j^2 x$, so $2s\,ds = 2\alpha_j^2\,dx$ or $dx = \frac{s}{\alpha_j^2}\,ds$. Thus

$$\begin{aligned} A_j &= \frac{-.02}{J_1^2(\alpha_j)} \int_0^{\alpha_j} (\frac{.5}{\alpha_j^2} s^2 - .5) J_0(s) \frac{s}{\alpha_j^2}\, ds \\ &= \frac{-.01}{\alpha_j^4 J_1^2(\alpha_j)} \int_0^{\alpha_j} (s^2 - \alpha_j^2) J_0(s) s\, ds \\ &= \frac{.01}{\alpha_j^4 J_1^2(\alpha_j)} \left[2\frac{\alpha_j^4}{\alpha_j^2} J_2(\alpha_j) \right] \\ &= \frac{.02}{\alpha_j^2 J_1^2(\alpha_j)} J_2(\alpha_j), \end{aligned}$$

where we have used the integral formula (15), Section 4.3, with $a = \alpha = \alpha_j$. We can give our answer in terms of J_1 by using formula (6), Section 4.8, with $p = 1$, and $x = \alpha_j$. Since α_j is a zero of J_0, we obtain

$$\frac{2}{\alpha_j} J_1(\alpha_j) = J_0(\alpha_j) + J_2(\alpha_j) = J_2(\alpha_j).$$

So

$$A_j = \frac{.02}{\alpha_j^2 J_1^2(\alpha_j)} \frac{2}{\alpha_j} J_1(\alpha_j) = \frac{.04}{\alpha_j^3 J_1(\alpha_j)}.$$

Thus the solution is

$$u(x, t) = \sum_{j=1}^{\infty} \frac{.04}{\alpha_j^3 J_1(\alpha_j)} J_0\left(\alpha_j\sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\, t\right),$$

where $g \approx 9.8\ m/sec^2$.

Going back to the questions, to answer (a), we have the normal modes

$$u_j(x, t) = \frac{.04}{\alpha_j^3 J_1(\alpha_j)} J_0\left(\alpha_j\sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\, t\right).$$

The frequency of the jth normal mode is

$$\nu_j = \sqrt{2g}\,\frac{\alpha_j}{4\pi}.$$

A six-term approximation of the solution is

$$u(x,\,t) \approx \sum_{j=1}^{6} \frac{.04}{\alpha_j^3 J_1(\alpha_j)} J_0\left(\alpha_j \sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\,t\right).$$

At this point, we use Mathematica (or your favorite computer system) to approximate the numerical values of the coefficients. Here is a table of relevant numerical data.

j	1	2	3	4	5	6
α_j	2.40483	5.52008	8.65373	11.7915	14.9309	18.0711
ν_j	.847231	1.94475	3.04875	4.15421	5.26023	6.36652
A_j	.005540	−.000699	.000227	−.000105	.000058	−.000036

Table 1 Numerical data for Exercise 1.

Exercises 6.4

1. This is a special case of Example 1 with $L = 2$ and $\lambda = \alpha^4$. The values of α are the positive roots of the equation

$$\cos 2\alpha = \frac{1}{\cosh 2\alpha}.$$

There are infinitely many roots, α_n ($n = 1, 2, \ldots$), that can be approximated with the help of a computer (see Figure 1). To each α_n corresponds one eigenfunction

$$X_n(x) = \cosh \alpha_n x - \cos \alpha_n x - \frac{\cosh 2\alpha_n - \cos 2\alpha_n}{\sinh 2\alpha_n - \sin 2\alpha_n} (\sinh \alpha_n x - \sin \alpha_n x).$$

5. There are infinitely many eigenvalues $\lambda = \alpha^4$, where α is a positive root of the equation

$$\cos \alpha = \frac{1}{\cosh \alpha}.$$

As in Example 1, the roots of this equation, α_n ($n = 1, 2, \ldots$), can be approximated with the help of a computer (see Figure 1). To each α_n corresponds one eigenfunction

$$X_n(x) = \cosh \alpha_n x - \cos \alpha_n x - \frac{\cosh \alpha_n - \cos \alpha_n}{\sinh \alpha_n - \sin \alpha_n} (\sinh \alpha_n x - \sin \alpha_n x).$$

The eigenfunction expansion of $f(x) = x(1-x)$, $0 < x < 1$, is

$$f(x) = \sum_{n=1}^{\infty} A_n X_n(x),$$

where

$$A_n == \frac{\int_0^1 x(1-x) X_n(x)\, dx}{\int_0^1 X_n^2(x)\, dx}.$$

After computing several of these coefficients, it was observed that:

$$\int_0^1 X_n^2(x)\, dx = 1 \quad \text{for all } n = 1, 2, \ldots,$$

$$A_{2n} = 0 \quad \text{for all } n = 1, 2, \ldots.$$

The first three nonzero coefficients are

$$A_1 = .1788, \quad A_3 = .0331, \quad A_5 = .0134.$$

So

$$f(x) \approx .1788\, X_1(x) + .0331\, X_3(x) + .0134\, X_5(x),$$

where X_n described explicitely in Example 1. We have

$$\begin{aligned}
X_1(x) &= \cosh(4.7300\, x) - \cos(4.7300\, x) + .9825\,(\sin(4.7300\, x) - \sinh(4.7300\, x)),\\
X_2(x) &= \cosh(1.0008\, x) - \cos(1.0008\, x) + 1.0008\,(\sin(1.0008\, x) - \sinh(1.0008\, x)),\\
X_3(x) &= \cosh(10.9956\, x) - \cos(10.9956\, x) + \sin(10.9956\, x) - \sinh(10.9956\, x),\\
X_4(x) &= \cosh(14.1372\, x) - \cos(14.1372\, x) + \sin(14.1372\, x) - \sinh(14.1372\, x),\\
X_5(x) &= \cosh(17.2788\, x) - \cos(17.2788\, x) + \sin(17.2788\, x) - \sinh(17.2788\, x).
\end{aligned}$$

9. Assume that μ and X are an eigenvalue and a corresponding eigenfunction of the Sturm-Liouville problem

$$X'' + \mu X = 0, \quad X(0) = 0,\ X(L) = 0.$$

Differentiate twice to see that X also satisfies the fourth order Sturm-Liouville problem

$$X^{(4)} - \lambda X = 0,$$
$$X(0) = 0,\ X''(0) = 0,\ X(L) = 0,\ X''(L) = 0\,.$$

If α and X are an eigenvalue and a corresponding eigenfunction of

$$X'' + \mu X = 0, \quad X(0) = 0,\ X(L) = 0,$$

then differentiating twice the equation, we find

$$X^{(4)} + \mu X'' = 0, \quad X(0) = 0,\ X(L) = 0.$$

But $X'' = -\mu X$, so $X^{(4)} - \mu^2 X = 0$ and hence X satisfies the equation $X^{(4)} - \lambda X = 0$ with $\lambda = \mu^2$. Also, from $X(0) = 0,\ X(L) = 0$ and the fact that $X'' = -\mu X$, it follows that $X''(0) = 0$ and $X''(L) = 0$.

Exercises 6.6

1. $u_{xxyy} = 0$, $u_{xxxx} = 4!$, $u_{yyyy} = -4!$, $\nabla^4 u = 0$.

3. Let $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = x \cdot u(x, y)$. We know that u is harmonic (Exercise 1, Section 4.1), and so v is biharmonic by Example 1, with $A = 0$, $B = 1$, $C = D = 0$.

5. Express v in Cartesian coordinates as follows:

$$\begin{aligned} v &= r^2 \cos(2\theta)(1 - r^2) \\ &= r^2[\cos^2\theta - \sin^2\theta](1 - r^2) \\ &= (x^2 - y^2)\big(1 - (x^2 + y^2)\big). \end{aligned}$$

Let $u = x^2 - y^2$. Then u is harmonic and so v is biharmonic by Example 1, with $A = 1$, $D = 1$, $B = C = 0$.

7. Write $v = r^2 \cdot r^n \cos n\theta$ and let $u = r^n \cos n\theta$. Then u is harmonic (use the Laplacian in polar coordinates to check this last assertion) and so v is biharmonic, by Example 1 with $A = 1$ and $B = C = D = 0$.

9. Write $v = ar^2 \ln r + br^2 + c \ln r + d = \phi + \psi$, where $\phi = [ar^2 + c] \ln r$ and $\psi = br^2 + d$. From Example 1, it follows that ψ is biharmonic. Also, $\ln r$ is harmonic (check the Laplacian in polar coordinates) and so, by Example 1, ϕ is biharmonic. Consequently, v is biharmonic, being the sum of two biharmonic functions.

11. Since the boundary values are independent of θ, try for a solution a function independent of θ and of the form $u = ar^2 \ln r + br^2 + c \ln r + d$. We have

$$u_r = 2ar \ln r + ar + 2br + \frac{c}{r} = 2ar \ln r + r(a + 2b) + \frac{c}{r}.$$

Now use the boundary conditions to solve for the unknown coefficients as follows:

$$\begin{aligned} u(1, \theta) = 0 &\quad\Rightarrow\quad b + d = 0, \\ u(2, \theta) = 0 &\quad\Rightarrow\quad 4a \ln 2 + 4b + c \ln 2 + d = 0, \\ u_r(1, \theta) = 1 &\quad\Rightarrow\quad a + 2b + c = 1, \\ u_r(2, \theta) = 1 &\quad\Rightarrow\quad 4a \ln 2 + 2a + 4b + \frac{c}{2} = 1. \end{aligned}$$

Solving this system of four equations in four unknowns, we find

$$a = \frac{1}{3 - 4\ln 2}, \quad b = \frac{-2\ln 2}{3 - 4\ln 2}, \quad c = \frac{2}{3 - 4\ln 2}, \quad d = \frac{2\ln 2}{3 - 4\ln 2},$$

and hence the solution

$$u(r, \theta) = \frac{1}{3 - 4\ln 2}\big(r^2 \ln r - 2\ln(2) r^2 + 2\ln r + 2\ln 2\big).$$

13. We follow the method of Theorem 1, as illustrated by Example 2. First, solve the Dirichlet problem $\nabla^2 w = 0$, $w(1, \theta) = \cos 2\theta$, for $0 \le r < 1$, $0 \le \theta \le 2\pi$. The solution in this case is $w(r, \theta) = r^2 \cos 2\theta$. (This is a simple application of the method of Section 4.4, since the boundary function is already given by its Fourier series.) We now consider a second Dirichlet problem on the unit disk with boundary values $v(1, \theta) = \frac{1}{2}\big(w_r(1, \theta) - g(\theta)\big)$. Since $g(\theta) = 0$ and $w_r(r, \theta) = 2r \cos 2\theta$, it follows that $v(1, \theta) = \cos 2\theta$. The solution of th Dirichlet problem in v is $v(r, \theta) = r^2 \cos 2\theta$. Thus the solution of biharmonic problem is

$$u(r, \theta) = (1 - r^2) r^2 \cos 2\theta + r^2 \cos 2\theta = 2r^2 \cos 2\theta - r^4 \cos 2\theta.$$

This can be verified directly by plugging into the biharmonic equation and the boundary conditions.

15. We just give the answers details: $w(r,\,\theta)=1$, $\nabla^2 v=0$, $v(1,\,\theta)=\frac{1}{2}(1-\cos\theta)$, $v(r,\,\theta)=-\frac{1}{2}r\cos\theta$; $u(r,\,\theta)=-\frac{1}{2}r\cos\theta(1-r^2)+1$.

17. $u(1,\,0)=0$ implies that $w=0$ and so $v(1,\,\theta)=-\frac{g(\theta)}{2}$. So

$$v(r,\,\theta)=-\frac{1}{2}\big[a_0+\sum_{n=1}^{\infty} r^n(\cos n\theta+b_n\sin n\theta)\big],$$

where a_n and b_n are the Fourier coefficients of g. Finally,

$$u(r,\theta)=(1-r^2)v(r,\theta)=-\frac{1}{2}(1-r^2)\Big[a_0+\sum_{n=1}^{\infty} r^n(\cos n\theta+b_n\sin n\theta)\Big].$$

Exercises 6.7

12. Correction to the suggested proof: $y_2 = I_0$ and not J_1.

15. Repeat Steps 1–3 of Example 2 without any change to arrive at the solution

$$u(r, t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(k_n^2 ct) + \beta_n \sin(k_n^2 ct)\right)\phi_n(r),$$

where the k_ns are the positive roots of (16). Putting $t = 0$ we get $0 = \sum_{n=1}^{\infty} \alpha_n \phi_n(r)$, which implies that $\alpha_n = 0$ for all n. Differentiating with respect to t then setting $t = 0$, we obtain

$$g(r) = \sum_{n=1}^{\infty} k_n^2 c\beta_n \phi_n(r),$$

which implies that

$$\int_0^1 g(r)\phi_n(r) r\, dr = k_n^2 c\beta_n \int_0^1 \phi_n^2(r) r\, dr$$

or

$$\beta_n = \frac{\int_0^1 g(r)\phi_n(r) r\, dr}{k_n^2 c\beta_n \int_0^1 \phi_n^2(r) r\, dr}.$$

The integral $\int_0^1 \phi_n^2(r) r\, dr$ is given by (23).

17. Let $u_1(r, t)$ denote the solution of the problem in Example 3 and let $u_2(r, t)$ denote the solution in Example 3. Then, by linearity or superposition, $u(r, t) = u_1(r, t) + u_2(r, t)$ is the desired solution.

19. Sketch of the solution:

$$\begin{aligned}
u(r, t) &= \sum_{n=1}^{\infty} A_n(t)\phi_n(r), \\
u_t(r, t) &= \sum_{n=1}^{\infty} A_n'(t)\phi_n(r), \\
\nabla^4 u(r, t) &= \sum_{n=1}^{\infty} k_n^4 A_n(t)\phi_n(r) \\
u_t = -c^2\nabla^4 u &\Rightarrow A_n'(t) = -c^2 k_n^4 \\
&\Rightarrow A_n(t) = A_n e^{-c^2 k_n^4 t},
\end{aligned}$$

where A_n is a constant. So

$$u(r, t) = \sum_{n=1}^{\infty} A_n e^{-c^2 k_n^4 t}\phi_n(r).$$

Setting $t = 0$, we get, for $0 \le r < 1$,

$$100 = \sum_{n=1}^{\infty} A_n \phi_n(r).$$

Thus A_n is as in Example 1.

20. Sketch of the solution:

$$\begin{aligned} u(r,\,t) &= \sum_{n=1}^{\infty} \alpha_n \phi_n(r), \\ \nabla^4 u(r,\,t) &= \sum_{n=1}^{\infty} k_n^4 \alpha_n \phi_n(r) \\ 100 = \nabla^4 u &\to 100 \sum_{n=1}^{\infty} k_n^4 \alpha_n \phi_n(r) \\ &\to k_n^4 \alpha_n = A_n \text{ or } \alpha_n = \frac{A_n}{k_n}, \end{aligned}$$

where A_n is as in Example 1.

Solutions to Exercises 7.1

1. We have
$$f(x) = \begin{cases} 1 & \text{if } -a < x < a, \quad (a > 0) \\ 0 & \text{otherwise,} \end{cases}$$
This problem is very similar to Example 1. From (3), if $\omega \neq 0$, then
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = \frac{1}{\pi}\int_{-a}^{a}\cos\omega t\,dt = \left[\frac{\sin\omega t}{\pi\omega}\right]_{-a}^{a} = \frac{2\sin a\omega}{\pi\omega}.$$
If $\omega = 0$, then
$$A(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt = \frac{1}{\pi}\int_{-a}^{a} dt = \frac{2a}{\pi}.$$
Since $f(x)$ is even, $B(\omega) = 0$. For $|x| \neq a$ the function is continuous and Theorem 1 gives
$$f(x) = \frac{2a}{\pi}\int_0^{\infty}\frac{\sin a\omega\cos\omega x}{\omega}\,d\omega.$$
For $x = \pm a$, points of discontinuity of f, Theorem 1 yields the value $1/2$ for the last integral. Thus we have the Fourier integral representation of f
$$\frac{2a}{\pi}\int_0^{\infty}\frac{\sin a\omega\cos\omega x}{\omega}\,d\omega = \begin{cases} 1 & \text{if } |x| < a, \\ 1/2 & \text{if } |x| = a, \\ 0 & \text{if } |x| > a. \end{cases}$$
■

3. We have
$$f(x) = \begin{cases} 1 - \cos x & \text{if } -\pi/2 < x < \pi/2, \\ 0 & \text{otherwise.} \end{cases}$$
The solution follows by combining Examples 1 and 2 and using the linearity of the Fourier integral, or we can compute directly and basically repeat the computation in these examples. Since f is even, $B(w) = 0$ for all w, and
$$\begin{aligned} A(w) &= \frac{2}{\pi}\int_0^{\pi/2}\big(1 - \cos t\big)\cos wt\,dt \\ &= \frac{2\sin(\pi w/2)}{\pi w} - \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{2}\left[\cos(w+1)t + \cos(w-1)t\right]\,dt \\ &= \frac{2\sin(\pi w/2)}{\pi w} - \frac{1}{\pi}\left[\frac{\sin(w+1)t}{w+1} + \frac{\sin(w-1)t}{w-1}\right]\Bigg|_0^{\pi/2} \qquad (w \neq 1) \\ &= \frac{2\sin(\pi w/2)}{\pi w} - \frac{1}{\pi}\left[\frac{\sin(w+1)\frac{\pi}{2}}{w+1} + \frac{\sin(w-1)\frac{\pi}{2}}{w-1}\right] \qquad (w \neq \pm 1) \\ &= \frac{2\sin(\pi w/2)}{\pi w} - \frac{1}{\pi}\left[\frac{\cos\frac{w\pi}{2}}{w+1} - \frac{\cos\frac{w\pi}{2}}{w-1}\right] \\ &= \frac{2\sin(\pi w/2)}{\pi w} - \frac{2}{\pi}\frac{1}{1-w^2}\cos\frac{w\pi}{2}, \end{aligned}$$
If $w = \pm 1$,
$$\begin{aligned} A(w) &= \frac{2\sin(\pm\pi/2)}{\pm\pi} - \frac{2}{\pi}\int_0^{\pi/2}\cos^2 t\,dt \\ &= \frac{2}{\pi} - \frac{2}{\pi}\int_0^{\pi/2}\frac{1+\cos 2t}{2}dt = \frac{2}{\pi} - \frac{1}{2}. \end{aligned}$$
(Another way to get this answer is to use the fact that $A(w)$ is continuous (whenever f is integrable) and take the limit of $A(w)$ as w tends to ± 1. You will get $\frac{1}{2}$. Applying the Fourier integral representation, we obtain:
$$\frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin(\pi w/2)}{w} - \frac{\cos\frac{\pi w}{2}}{1-w^2}\right)\cos wx dw = f(x) = \begin{cases} 1 - \cos x & \text{if } |x| \leq \frac{\pi}{2}, \\ 0 & \text{if } |x| > \frac{\pi}{2}. \end{cases}$$

5. since $f(x) = e^{-|x|}$ is even, $B(w) = 0$ for all w, and

$$\begin{aligned} A(w) &= \frac{2}{\pi}\int_0^\infty e^{-t}\cos wt\,dt \\ &= \frac{2}{\pi}\frac{e^{-t}}{1+w^2}\left[-\cos wt + w\sin wt\right]\Big|_0^\infty \\ &= \frac{2}{\pi}\frac{1}{1+w^2}, \end{aligned}$$

where we have used the result of Exercise 17, Sec. 2.6, to evaluate the integral. Applying the Fourier integral representation, we obtain:

$$e^{-|x|} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+w^2}\cos wx\,dw.$$

7. The function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

is neither even nor odd. We will need to compute both $A(w)$ and $B(w)$. For $w \neq 0$,

$$A(w) = \frac{1}{\pi}\int_0^1 \cos wt\,dt = \frac{1}{\pi}\frac{\sin w}{w};$$

$$B(w) = \frac{1}{\pi}\int_0^1 \sin wt\,dt = \frac{1}{\pi}\frac{1-\cos w}{w}.$$

If $w = 0$, just take limits of $A(w)$ and $B(w)$ as w tends to 0. You will find: $A(0) = \frac{1}{\pi}$ and $B(0) = 0$. Thus we have the Fourier integral representation of f

$$\frac{1}{\pi}\int_0^\infty \big(\sin w\cos wx + (1-\cos w)\sin wx\big),dw = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1, \\ 1/2 & \text{if } x = 0 \text{ or } x = 1, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

9. The function

$$f(x) = \begin{cases} x & \text{if } -1 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \\ -2-x & \text{if } -2 < x < -1, \\ 0 & \text{otherwise,} \end{cases}$$

is odd, as can be seen from its graph. Hence $A(w) = 0$ and

$$\begin{aligned} B(w) &= \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt \\ &= \frac{2}{\pi}\int_0^1 t\sin wt\,dt + \frac{2}{\pi}\int_1^2 (2-t)\sin wt\,dt \\ &= \frac{2}{\pi}\left[\frac{-t}{w}\cos wt\Big|_0^1 + \int_0^1 \frac{\cos wt}{w}\,dt\right] \\ &\quad + \frac{2}{\pi}\left[-\frac{2-t}{w}\cos wt\Big|_1^2 + \int_1^2 \frac{\cos wt}{w}\,dt\right] \\ &= \frac{2}{\pi w^2}\left[2\sin w - \sin 2w\right]. \end{aligned}$$

Since f is continuous, we obtain the Fourier integral representation of

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{w^2}\left[2\sin w - \sin 2w\right]\sin wx,dw.$$

11. For the function

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}
A(w) &= \frac{1}{\pi}\int_0^{\pi} \sin t \cos wt\, dt \\
&= \frac{1}{2\pi}\int_0^{\pi} \left[\sin(1+w)t + \sin(1-w)t\right]\, dt \\
&= -\frac{1}{2\pi}\left[\frac{\cos(1+w)t}{1+w} + \frac{\cos(1-w)t}{1-w}\right]\Bigg|_0^{\pi} \qquad (w \neq 1) \\
&= -\frac{1}{2\pi}\left[\frac{\cos(w+1)\pi}{1+w} + \frac{\cos(1-w)\pi}{1-w} - \frac{1}{1+w} - \frac{1}{1-w}\right] \\
&= \frac{1}{2\pi}\left[\frac{\cos w\pi}{1+w} + \frac{\cos w\pi}{1-w} + \frac{2}{1-w^2}\right] \\
&= \frac{1}{\pi(1-w^2)}\left[\cos w\pi + 1\right].
\end{aligned}$$

$$\begin{aligned}
B(w) &= \frac{1}{\pi}\int_0^{\pi} \sin t \sin wt\, dt \\
&= -\frac{1}{2\pi}\int_0^{\pi} \left[\cos(1+w)t - \cos(w-1)t\right]\, dt \\
&= -\frac{1}{2\pi}\left[\frac{\sin(1+w)t}{1+w} - \frac{\sin(1-w)t}{1-w}\right]\Bigg|_0^{\pi} \qquad (w \neq 1) \\
&= -\frac{1}{2\pi}\left[\frac{\sin(w+1)\pi}{1+w} - \frac{\sin(1-w)\pi}{1-w}\right] \\
&= \frac{\sin w\pi}{\pi(1-w^2)}.
\end{aligned}$$

Thus the Fourier integral representation of f is

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \left[\frac{1+\cos w\pi}{1-w^2}\cos wx + \frac{\sin w\pi}{1-w^2}\sin wx\right] dx.$$

13. (a) Take $x = 1$ in the Fourier integral representation of Example 1:

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin w \cos w}{w}\, dw = \frac{1}{2} \quad \Rightarrow \quad \int_0^{\infty} \frac{\sin w \cos w}{w}\, dw = \frac{\pi}{4}.$$

(b) Integrate by parts: $u = \sin^2 w$, $du = 2\sin w \cos w\, dw$, $dv = \frac{1}{w^2}dw$, $v = -\frac{1}{w}$:

$$\int_0^{\infty} \frac{\sin^2 \omega}{\omega^2}\, d\omega = \overbrace{\frac{\sin^2 w}{w}\Bigg|_0^{\infty}}^{=0} + 2\int_0^{\infty} \frac{\sin w \cos w}{w^2} dw = \frac{\pi}{2},$$

by (a).

15. $\displaystyle\int_0^{\infty} \frac{\sin \omega \cos 2\omega}{\omega}\, d\omega = 0.$ **Solution** Just take $x = 2$ in Example 1.

17.

$$\int_0^{\infty} \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \begin{cases} 0 & \text{if } x < 0, \\ \pi/2 & \text{if } x = 0, \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

Solution. Define

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \pi/2 & \text{if } x = 0, \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

Let us find the Fourier integral representation of f:

$$A(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-x}\cos wx\,dx = \frac{1}{1+w^2}$$

(see Exercise 5);

$$B(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-x}\sin wx\,dx = \frac{w}{1+w^2},$$

(see Exercise 17, Sec. 2.6). So

$$f(x) = \int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw,$$

which yields the desired formula.

19. Let $f(x) = e^{-a|x|}$, where $a > 0$. Then f, satisfies the conditions of Theorem 1, f is continuous and even. So it has a Fourier integral representation of the form

$$e^{-a\,|x|} = \int_0^\infty A(w)\cos wx\,dx,$$

where

$$\begin{aligned} A(w) &= \frac{2}{\pi}\int_0^\infty e^{-ax}\cos wx\,dx \\ &= \frac{2a}{\pi(a^2+w^2)}, \end{aligned}$$

where the integral is evaluated with the help of Exercise 17, Sec. 2.6 (see Exercise 5 of this section). For $x \geq 0$, the Fourier integral representation becomes

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos wx}{a^2+w^2}\,dw \quad (x \geq 0).$$

23. (a)

$$\begin{aligned} \text{Si}(-x) &= \int_0^{-x} \frac{\sin t}{t}\,dt \\ &= \int_0^{x} \frac{\sin(-u)}{(-u)}(-1)\,du \qquad \text{(Let } u=-t.\text{)} \\ &= -\int_0^{x} \frac{\sin u}{u}\,du = -\text{Si}(x). \end{aligned}$$

Thus $\text{Si}(x)$ is an odd function.
(b) $\lim_{x\to-\infty}\text{Si}(x) = \lim_{x\to\infty} -\text{Si}(x) - \frac{\pi}{2}$, by (8).
(c) We have

$$\begin{aligned} \sin t &= \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!} \qquad -\infty < t < \infty \\ \frac{\sin t}{t} &= \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!} \qquad -\infty < t < \infty \quad \text{(Divide by } t.\text{)} \end{aligned}$$

By a well-known property of power series, a power series may be integrated term by term within its radius of convergence. Here we are dealing with power series that converge for all t; thus we may integrate them term by term on any interval.

In particular, we have

$$\begin{aligned}
\operatorname{Si}(x) &= \int_0^x \frac{\sin t}{t}\,dt \\
&= \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n+1)!}\right) dt \\
&= \sum_{n=0}^{\infty}\left\{(-1)^n \frac{1}{(2n+1)!}\int_0^x t^{2n}dt\right\} \\
&= \sum_{n=0}^{\infty}\left\{(-1)^n \frac{1}{(2n+1)!}\frac{1}{2n+1}t^{2n+1}\Big|_0^x\right\} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!(2n+1)}x^{2n+1}.
\end{aligned}$$

25. We have

$$\operatorname{Si}(a) = \int_0^a \frac{\sin t}{t}\,dt, \qquad \operatorname{Si}(b) = \int_0^b \frac{\sin t}{t}\,dt.$$

So

$$\operatorname{Si}(b) - \operatorname{Si}(a) = \int_0^b \frac{\sin t}{t}\,dt - \int_0^a \frac{\sin t}{t}\,dt = \int_a^b \frac{\sin t}{t}\,dt.$$

27. Let $u = 1 - \cos x$, $du = \sin x$, $dv = \frac{1}{x^2}\,dx$, $v = -\frac{1}{x}$. Then

$$\begin{aligned}
\int_a^b \frac{1-\cos x}{x^2}\,dx &= -\frac{1-\cos x}{x}\Big|_a^b + \int_a^b \frac{\sin x}{x}\,dx \\
&= \frac{\cos b - 1}{b} - \frac{\cos a - 1}{a} + \operatorname{Si}(b) - \operatorname{Si}(a),
\end{aligned}$$

where we have used the result of Exercise 25.

Solutions to Exercises 7.2

1. In computing $\widehat{f}$, the integral depends on the values of f on the interval $(-1, 1)$. Since on this interval f is odd, it follows that $f(x)\cos wx$ is odd and $f(x)\sin wx$ is even on the interval $(-1, 1)$. Thus

$$\begin{aligned}
\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\overbrace{\int_{-1}^{1} f(x)\cos wx\,dx}^{=0} - \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} f(x)\sin wx\,dx \\
&= -\frac{2i}{\sqrt{2\pi}}\int_{0}^{1} \sin wx\,dx \\
&= \frac{2i}{\sqrt{2\pi}}\frac{\cos wx}{w}\Big|_0^1 \\
&= i\sqrt{\frac{2}{\pi}}\frac{\cos w - 1}{w}.
\end{aligned}$$

3. We evaluate the Fourier transform with the help of Euler's identity: $e^{-iwx} = \cos wx - i\sin wx$. We also use the fact that the integral of on odd function over a symmetric interval is 0, while the integral of an even function over a symmetric interval $[-a, a]$ is twice the integral over $[0, a]$.

$$\begin{aligned}
\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \sin x(\cos wx - i\sin wx)\,dx \\
&= \frac{1}{\sqrt{2\pi}}\overbrace{\int_{-\pi}^{\pi} \sin x\cos wx\,dx}^{=0} - \frac{i}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \sin x\sin wx\,dx \\
&= -\frac{2i}{\sqrt{2\pi}}\int_{0}^{\pi} \frac{1}{2}\left[(-\cos(w+1)x + \cos(w-1)x\right]\,dx.
\end{aligned}$$

If $w = 1$,

$$\widehat{f}(1) = -\frac{i}{\sqrt{2\pi}}\int_0^{\pi}\left[(-\cos 2x + 1\right]\,dx = -\frac{i\sqrt{\pi}}{\sqrt{2}}.$$

If $w = 1$, we find similarly that

$$\widehat{f}(-1) = \frac{i\sqrt{\pi}}{\sqrt{2}}.$$

(You may also note that $\widehat{f}$ is odd in this case.) For $w \neq \pm 1$, we have

$$\begin{aligned}
\widehat{f}(w) &= \frac{i}{\sqrt{2\pi}}\left(\frac{1}{w+1}\sin(w+1)x - \frac{1}{w-1}\sin(w-1)x)\right|_0^{\pi} \\
&= \frac{i}{\sqrt{2\pi}}\Big(\frac{1}{w+1}\sin(w+1)\pi - \frac{1}{w-1}\sin(w-1)\pi\Big) \\
&= \frac{i}{\sqrt{2\pi}}\Big(\frac{1}{w+1}(-1)\sin w\pi - \frac{1}{w-1}(-1)\sin w\pi\Big) \\
&= \frac{i}{\sqrt{2\pi}}\sin w\pi\Big(-\frac{1}{w+1} + \frac{1}{w-1}\Big) \\
&= i\sqrt{\frac{2}{\pi}}\frac{\sin w\pi}{w^2-1}.
\end{aligned}$$

Check that as $w \to \pm 1$, $\widehat{f}(w) \to \widehat{f}(\pm 1)$.

5. Use integration by parts to evaluate the integrals:

$$\begin{aligned}
\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos wx - i \sin wx)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos wx\, dx - \frac{i}{\sqrt{2\pi}} \overbrace{\int_{-1}^{1} (1-|x|) \sin wx\, dx}^{=0} \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \overbrace{(1-x)}^{u} \overbrace{\cos wx\, dx}^{dv} \\
&= \frac{2}{\sqrt{2\pi}} \Big((1-x) \frac{\sin wx}{w} \Big)\Big|_0^1 + \frac{2}{\sqrt{2\pi} w} \int_0^1 \sin wx\, dx \\
&= -\sqrt{\frac{2}{\pi}} \frac{\cos wx}{w^2}\Big|_0^1 \\
&= \sqrt{\frac{2}{\pi}} \frac{1-\cos w}{w^2}.
\end{aligned}$$

7. Before we compute the transform, let us evaluate the integral

$$I = \int x e^{-iwx}\, dx \quad (w \neq 0)$$

by parts. Let $u = x$, $dv = e^{-iwx}\, dx$; then $du = dx$ and $v = \frac{1}{-iw} e^{-iwx} = \frac{i}{w} e^{-iwx}$, because $\frac{1}{i} = -i$. So

$$\int x e^{-iwx}\, dx = i \frac{x}{w} e^{-iwx} - \frac{i}{w} \int e^{-iwx} dx = i \frac{x}{w} e^{-iwx} + \frac{1}{w^2} e^{-iwx} + C.$$

We are now ready to compute the transform. For $w \neq 0$, we have

$$\begin{aligned}
\widehat{f}(w) &= \frac{2}{\sqrt{2\pi}} \int_0^{10} x e^{-iwx}\, dx \\
&= \frac{2}{\sqrt{2\pi}} \left[i \frac{x}{w} e^{-iwx} + \frac{1}{w^2} e^{-iwx} \right]\Big|_0^{10} \\
&= \sqrt{\frac{2}{\pi}} \left[i \frac{10}{w} e^{-10\, iw} + \frac{e^{-10\, iw} - 1}{w^2} \right].
\end{aligned}$$

In terms of sines and cosine, we have

$$\begin{aligned}
\widehat{f}(w) &= \sqrt{\frac{2}{\pi}} \left[i \frac{10}{w} \big(\cos(10\, w) - i \sin(10\, w) \big) + \frac{\cos(10w) - i \sin(10w) - 1}{w^2} \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{10\, \sin(10\, w)}{w} + \frac{\cos(10w) - 1}{w^2} + i \left(\frac{10\, \cos(10\, w)}{w} - \frac{\sin(10w)}{w^2} \right) \right].
\end{aligned}$$

8. Consider the function in Example 1, which depends on the real number $a > 0$, and denote this function by $f_a(x)$ instead of $f(x)$. Now check that $g_n(x) = n\sqrt{\frac{\pi}{2}} f_{1/n}(x)$. We have from Example 1 (with $a = 1/n$)

$$\widehat{f_{1/n}}(w) = \sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{w}{n}\right)}{w}.$$

So

$$\widehat{g_n}(w) = n\sqrt{\frac{\pi}{2}}\widehat{f_{1/n}}(w) = n\sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{\pi}}\frac{\sin\left(\frac{w}{n}\right)}{w} = n\frac{\sin\left(\frac{w}{n}\right)}{w}.$$

For fixed $w \neq 0$, we have

$$\lim_{n\to\infty}\widehat{g_n}(w) = \lim_{n\to\infty}\frac{\sin\left(\frac{w}{n}\right)}{\frac{w}{n}} = 1.$$

If $w = 0$, we have

$$\widehat{g_n}(0) = n\sqrt{\frac{\pi}{2}}\widehat{f_{1/n}}(0) = n\sqrt{\frac{\pi}{2}}\cdot\frac{1}{n}\sqrt{\frac{2}{\pi}} = 1$$

(see the graph of $\widehat{f_{1/n}}$ in Figure 2).

9. In Exercise 1,

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \times (\text{area between graph of } f(x) \text{ and the } x\text{axis}) = 0.$$

In Exercise 7,

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \times (\text{area between graph of } f(x) \text{ and the } x\text{axis}) = \frac{100}{\sqrt{2\pi}}.$$

11. We use the identity $\mathcal{F}\mathcal{F}(f)(x) = f(-x)$. In case f is even, this identity becomes $\mathcal{F}\mathcal{F}(f)(x) = f(-x)$. Consider $g(x) = \frac{1}{1+x^2}$ and, for convenience, let us write it as $g(w) = \frac{1}{1+w^2}$. Recall from Exercise 5 that

$$\mathcal{F}(e^{-|x|})(w) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2} = \sqrt{\frac{2}{\pi}}g(w).$$

Multiplying both sides by $\sqrt{\frac{\pi}{2}}$ and using the linearity of the Fourier transform, it follows that

$$\mathcal{F}(\sqrt{\frac{\pi}{2}}e^{-|x|})(w) = g(w).$$

So

$$\mathcal{F}g = \mathcal{F}\mathcal{F}\left(\sqrt{\frac{\pi}{2}}e^{-|x|}\right) = \sqrt{\frac{\pi}{2}}e^{-|x|},$$

by the reciprocity relation. If we use the symbol w as a variable in the Fourier transform, we get

$$\mathcal{F}\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}}e^{-|w|},$$

13. We argue as in Exercise 11. Consider $g(x) = \frac{\sin ax}{x}$ where we assume $a > 0$. For the case $a < 0$, use $\sin(-ax) = -\sin ax$ and linearity of the Fourier transform. Let $f(x) = 1$ if $|x| < a$ and 0 otherwise. Recall from Example 1 that

$$\mathcal{F}(f(x))(w) = \sqrt{\frac{2}{\pi}}\frac{\sin aw}{w} = \sqrt{\frac{2}{\pi}}g(w).$$

Multiplying both sides by $\sqrt{\frac{\pi}{2}}$ and using the linearity of the Fourier transform, it follows that

$$\mathcal{F}(\sqrt{\frac{\pi}{2}}f(x))(w) = g(w).$$

So

$$\mathcal{F}g = \mathcal{F}\mathcal{F}\left(\sqrt{\frac{\pi}{2}}f(x)\right) = \sqrt{\frac{\pi}{2}}f(x),$$

by the reciprocity relation. Using the symbol w as a variable, we get

$$\mathcal{F}\left(\frac{\sin ax}{x}\right) = \sqrt{\frac{\pi}{2}}f(w) = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |w| < a, \\ 0 & \text{otherwise.} \end{cases}$$

15. Consider the function in Example 2,

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

If $a > 0$, then (see Exercise 17)

$$\mathcal{F}(f(ax))(w) = \frac{1}{a}\frac{1 - iw/a}{\sqrt{2\pi}(1 + (w/a)^2)} = \frac{a - iw}{\sqrt{2\pi}(a^2 + w^2)}.$$

But

$$f(ax) = \begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Hence

$$g(w) = \frac{a - iw}{a^2 + w^2} = \sqrt{2\pi}\mathcal{F}(f(ax))(w);$$

and so

$$\mathcal{F}(g) = \sqrt{2\pi}f(-ax) = \begin{cases} \sqrt{2\pi}e^{ax} & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Using the variable w,

$$\mathcal{F}(g)(w) = \begin{cases} \sqrt{2\pi}e^{aw} & \text{if } w < 0, \\ 0 & \text{if } w > 0. \end{cases}$$

If $a < 0$, use the identity

$$\mathcal{F}(f(-x))(w) = \mathcal{F}(f(x))(-w)$$

(see Exercise 17 with $a = -1$) to conclude that

$$\mathcal{F}(g)(w) = \begin{cases} \sqrt{2\pi}e^{aw} & \text{if } w > 0, \\ 0 & \text{if } w < 0. \end{cases}$$

17. (a) Consider first the case $a > 0$. Using the definition of the Fourier transform and a change of variables

$$\begin{aligned} \mathcal{F}(f(ax))(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}\,dx \\ &= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\frac{\omega}{a}x}\,dx \qquad (ax = X,\ dx = \frac{1}{a}dX) \\ &= \frac{1}{a}\mathcal{F}(f)(\frac{w}{a}). \end{aligned}$$

If $a < 0$, then

$$\begin{aligned} \mathcal{F}(f(ax))(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}\,dx \\ &= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)e^{-i\frac{\omega}{a}x}\,dx \\ &= -\frac{1}{a}\mathcal{F}(f)(\frac{w}{a}). \end{aligned}$$

Hence for all $a \neq 0$, we can write

$$= \frac{1}{|a|}\mathcal{F}(f)\left(\frac{\omega}{a}\right).$$

(b) We have

$$\mathcal{F}(e^{-|x|})(w) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}.$$

By (a),

$$\mathcal{F}(e^{-2|x|})(w) = \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{1}{1+(w/2)^2} = \sqrt{\frac{2}{\pi}}\frac{2}{4+w^2}.$$

(c) Let $f(x)$ denote the function in Example 2. Then $g(x) = f(-x)$. So

$$\widehat{g}(w) = \widehat{f}(-w) = \frac{1+iw}{\sqrt{2\pi}(1+w^2)}.$$

Let $h(x) = e^{-|x|}$. Then $h(x) = f(x) + g(x)$. So

$$\widehat{h}(w) = \widehat{f}(w) + \widehat{g}(w) = \frac{1-iw}{\sqrt{2\pi}(1+w^2)} + \frac{1+iw}{\sqrt{2\pi}(1+w^2)} = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}.$$

19. Using the definition of the Fourier transform

$$\begin{aligned}
\mathcal{F}(f(x-a))(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega(x+a)}\,dx \qquad (x-a=X,\ dx=dX) \\
&= e^{-i\omega a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = e^{-ia\omega}\mathcal{F}(f)(\omega).
\end{aligned}$$

21. We have $\mathcal{F}(e^{-x^2} = \frac{1}{\sqrt{2}}e^{-w^2/4}$, by Theorem 5. Using Exercise 20, we have

$$\begin{aligned}
\mathcal{F}(\frac{\cos x}{e^{x^2}}) &= \mathcal{F}(\cos x e^{-x^2}) \\
&= \frac{1}{2\sqrt{2}}\Big(e^{-\frac{(w-a)^2}{4}} + e^{-\frac{(w+a)^2}{4}}\Big)
\end{aligned}$$

23. We have $\mathcal{F}(\frac{1}{1+x^2}) = \sqrt{\frac{\pi}{2}}e^{-|w|}$. Using Exercise 20, we have

$$\begin{aligned}
\mathcal{F}(\frac{\cos x + \cos 2x}{1+x^2}) &= \mathcal{F}(\frac{\cos x}{1+x^2}) + \mathcal{F}(\frac{\cos 2x}{1+x^2}) \\
&= \frac{1}{2}\sqrt{\frac{\pi}{2}}\Big(e^{-|w-1|} + e^{-|w+1|} + e^{-|w-2|} + e^{-|w+2|}\Big).
\end{aligned}$$

25 Let

$$g(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and note that $f(x) = \cos(x)\,g(x)$. Now $\mathcal{F}(g(x)) = \sqrt{\frac{2}{\pi}\frac{\sin w}{w}}$. Using Exercise 20, we have

$$\begin{aligned}
\mathcal{F}(f(x)) &= \mathcal{F}(\cos x g(x)) \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\Big(\frac{\sin(w-1)}{w-1} + \frac{\sin(w+1)}{w+1}\Big).
\end{aligned}$$

27. Call $g(x)$ the function in the figure. The support of g is the interval (a, b), that has length $b-a$. Translate the function by $\frac{b+a}{2}$ in order to center its support at the origin, over the interval with endpoints at $\left(-\frac{b-a}{2}, \frac{b-a}{2}\right)$. Denote the translated function by $f(x)$. Thus, $g(x) = f\left(x - \frac{b+a}{2}\right)$, where $f(x) = h$ if $|x| < \frac{b-a}{2}$ and 0 otherwise. By Example 1 and Exercise 19,

$$\begin{aligned}
\widehat{g}(w) &= e^{-i\frac{b+a}{2}w}\widehat{f}(w) \\
&= e^{-i\frac{b+a}{2}w}h \cdot \sqrt{\frac{2}{\pi}}\frac{\sin\left(\frac{b-a}{2}w\right)}{w} \\
&= \sqrt{\frac{2}{\pi}}he^{-i\frac{b+a}{2}w}\frac{\sin\left(\frac{b-a}{2}w\right)}{w}.
\end{aligned}$$

29. Take $a = 0$ and relabel $b = a$ in Exercise 27, you will get the function $f(x) = h$ if $0 < x < a$. Its Fourier transform is

$$\widehat{f}(w) == \sqrt{\frac{2}{\pi}}he^{-i\frac{a}{2}w}\frac{\sin\left(\frac{a\,w}{2}\right)}{w}.$$

Let $g(x)$ denote the function in the figure. Then $g(x) = \frac{1}{a}x\, f(x)$ and so, by Theorem 3,

$$\begin{aligned}
\widehat{g}(w) &= \frac{1}{a}i\frac{d}{dw}\widehat{f}(w) \\
&= i\sqrt{\frac{2}{\pi}}\frac{h}{a}\frac{d}{dw}\left[e^{-i\frac{a}{2}w}\frac{\sin\left(\frac{a\,w}{2}\right)}{w}\right] \\
&= i\sqrt{\frac{2}{\pi}}\frac{h}{a}\frac{e^{-i\frac{a}{2}w}}{w}\left[a\frac{\cos\left(\frac{a\,w}{2}\right) - i\sin\left(\frac{a\,w}{2}\right)}{2} - \frac{\sin\left(\frac{a\,w}{2}\right)}{w}\right] \\
&= i\sqrt{\frac{2}{\pi}}\frac{h}{a}e^{-i\frac{a}{2}w}\left[\frac{-2\sin\left(\frac{a\,w}{2}\right) + a\,w\,e^{-i\frac{a}{2}w}}{2w^2}\right].
\end{aligned}$$

31. Let $f(x) = 1$ if $-a < x < a$ and 0 otherwise (Example 1), and let $g(x)$ denote the function in the figure. Then $g(x) = x^2 f(x)$ and, by Theorem 3,

$$\begin{aligned}
\widehat{g}(w) &= -\frac{d^2}{dw^2}\widehat{f}(w) \\
&= -\sqrt{\frac{2}{\pi}}\frac{d^2}{dw^2}\frac{\sin aw}{w} \\
&= -\sqrt{\frac{2}{\pi}}\frac{2aw\cos aw + a^2w^2\sin aw - 2\sin aw}{w^3}.
\end{aligned}$$

33. Let $g(x)$ denote the function in this exercise. By the reciprocity relation, since the function is even, we have $\mathcal{F}(\mathcal{F}(g)) = g(-x) = g(x)$. Taking inverse Fourier transforms, we obtain $\mathcal{F}^{-1}(g) = \mathcal{F}(g)$. Hence it is enough to compute the Fourier transform. We use the notation and the result of Exercise 34. We have

$$g(x) = 2f_{2a}(x) - f_a(x).$$

Verify this identity by drawing the graphs of f_{2a} and f_a and then drawing the graph of $f_{2a}(x) - f_a(x)$. With the help of this identity and the result of Exercise 34, we have

$$\begin{aligned}\widehat{g}(w) &= 2\widehat{f_{2a}}(w) - \widehat{f_a}(w)\\ &= 2\frac{8a}{\sqrt{2\pi}}\frac{\sin^2(aw)}{4(aw)^2} - \frac{4a}{\sqrt{2\pi}}\frac{\sin^2\left(\frac{aw}{2}\right)}{(aw)^2}\\ &= \frac{4}{a\sqrt{2\pi}}\left[\frac{\sin^2(aw)}{w^2} - \frac{\sin^2\left(\frac{aw}{2}\right)}{w^2}\right].\end{aligned}$$

34. Let $f(x)$ denote the function in Exercise 5 and $f_a(x)$ the function in the figure of this exercise. Then $f_a(x) = f\left(\frac{x}{a}\right)$ and so

$$\begin{aligned}\widehat{f_a}(w) &= \widehat{f\left(\frac{x}{a}\right)}(w) = a\widehat{f}(aw)\\ &= a\sqrt{\frac{2}{\pi}}\frac{1-\cos(aw)}{(aw)^2} \quad \text{(Exercise 5)}\\ &= a\sqrt{\frac{2}{\pi}}\frac{2\sin^2\left(\frac{aw}{2}\right)}{(aw)^2}\\ &= \frac{4a}{\sqrt{2\pi}}\frac{\sin^2\left(\frac{aw}{2}\right)}{(aw)^2}.\end{aligned}$$

35. (a) Apply Theorem 5 with $a = 2$, then

$$\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}}e^{-w^2/4}.$$

36. (a) Theorem 5 with $a = 1$,

$$\mathcal{F}(e^{-\frac{1}{2}x^2+2}) = e^2\mathcal{F}(e^{-\frac{1}{2}x^2}) = e^2e^{-\frac{1}{2}w^2}.$$

37. By Exercise 27,

$$\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}}e^{-w^2/4}.$$

By Theorem 3(i)

$$\begin{aligned}\mathcal{F}(xe^{-x^2}) &= i\frac{d}{dw}\left(\frac{1}{\sqrt{2}}e^{-w^2/4}\right)\\ &= \frac{i}{2\sqrt{2}}e^{-w^2/4}.\end{aligned}$$

39. Let

$$g(x) = \begin{cases} 1 & \text{if } |x| < 1,\\ 0 & \text{otherwise,}\end{cases}$$

and note that $f(x) = xg(x)$. Now $\mathcal{F}(g(x)) = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}$. By Theorem 3(i)

$$\begin{aligned}\mathcal{F}(f(x)) &= \mathcal{F}(xg(x))\\ &= i\frac{d}{dw}\left(\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}\right)\\ &= i\sqrt{\frac{2}{\pi}}\frac{w\cos w - \sin w}{w^2}.\end{aligned}$$

41. We have $\mathcal{F}(\frac{1}{1+x^2}) = \sqrt{\frac{\pi}{2}}e^{-|w|}$. So if $w > 0$

$$\mathcal{F}(\frac{x}{1+x^2})(w) = i\sqrt{\frac{\pi}{2}}\frac{d}{dw}e^{-w} = -i\sqrt{\frac{\pi}{2}}e^{-w}.$$

If $w < 0$

$$\mathcal{F}(\frac{x}{1+x^2})(w) = i\sqrt{\frac{\pi}{2}}\frac{d}{dw}e^{w} = i\sqrt{\frac{\pi}{2}}e^{w}.$$

If $w = 0$,

$$\mathcal{F}(\frac{x}{1+x^2})(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx = 0$$

(odd integrand). We can combine these answers into one formula

$$\mathcal{F}(\frac{x}{1+x^2})(w) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,(w)e^{-|w|}.$$

43. Using linearity,

$$\mathcal{F}\Big((1-x^2)e^{-x^2}\Big) = \mathcal{F}\Big(e^{-x^2}\Big) - \mathcal{F}\Big(x^2e^{-x^2}\Big).$$

By Theorem 3(ii) and Exercise 27,

$$\begin{aligned}\mathcal{F}\Big((1-x^2)e^{-x^2}\Big) &= \frac{1}{\sqrt{2}}e^{-w^2/4} + \frac{1}{\sqrt{2}}\frac{d^2}{dw^2}e^{-w^2/4} \\ &= \frac{1}{4\sqrt{2}}e^{-w^2/4}\left(2+w^2\right).\end{aligned}$$

45. Theorem 3 (i) and Exercise 19:

$$\begin{aligned}\mathcal{F}(xe^{-\frac{1}{2}(x-1)^2}) &= i\frac{d}{dw}\Big(\mathcal{F}(e^{-\frac{1}{2}(x-1)^2})\Big) \\ &= i\frac{d}{dw}\Big(e^{-iw}\mathcal{F}(e^{-\frac{1}{2}x^2})\Big) \\ &= i\frac{d}{dw}\Big(e^{-iw}e^{-\frac{1}{2}w^2})\Big) = i\frac{d}{dw}\Big(e^{-\frac{1}{2}w^2-iw})\Big) \\ &= i(-w-i)e^{-\frac{1}{2}w^2-iw} \\ &= (1-iw)e^{-\frac{1}{2}w^2-iw}.\end{aligned}$$

49.

$$\widehat{h}(\omega) = e^{-\omega^2}\cdot\frac{1}{1+\omega^2} = \mathcal{F}(\frac{1}{\sqrt{2}}e^{-x^2/4})\cdot\mathcal{F}(\sqrt{\frac{\pi}{2}}e^{-|x|}).$$

Hence

$$\begin{aligned}h(x) &= f*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2}}e^{-\frac{(x-t)^2}{4}}\sqrt{\frac{\pi}{2}}e^{-|t|}dt \\ &= \frac{1}{2\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{(x-t)^2}{4}}e^{-|t|}dt.\end{aligned}$$

51. Let

$$\widehat{f}(\omega) = \begin{cases}1 & \text{if } |\omega| < 1, \\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\widehat{h}(\omega) = \widehat{f}(\omega)e^{-\frac{1}{2}\omega^2}.$$

We have $\mathcal{F}^{-1}(e^{-\frac{1}{2}\omega^2}) = e^{-\frac{1}{2}x^2}$ and $f(x) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{x}$. Thus

$$\begin{aligned} h(x) &= f * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\sqrt{\frac{2}{\pi}}\frac{\sin t}{t}dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\frac{\sin t}{t}dt. \end{aligned}$$

53. Let $f(x) = xe^{-x^2/2}$ and $g(x) = e^{-x^2}$.
(a) $\mathcal{F}(f)(w) = -iwe^{-\frac{w^2}{2}}$, and $\mathcal{F}(g)(w) = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}}$.
(b)

$$\begin{aligned} \{\,*\,\} &= \{\,\cdot\,\} \\ &= -i\frac{w}{\sqrt{2}}e^{-\frac{w^2}{4}}e^{-\frac{w^2}{2}} \\ &= -i\frac{w}{\sqrt{2}}e^{-3\frac{w^2}{4}}. \end{aligned}$$

(c) With the help of Theorem 3

$$\begin{aligned} f * g &= \mathcal{F}^{-1}\Big(-i\frac{w}{\sqrt{2}}e^{-3\frac{w^2}{4}}\Big) \\ &= \mathcal{F}^{-1}\Big(i\frac{1}{\sqrt{2}}\frac{4}{6}\frac{d}{dw}e^{-3\frac{w^2}{4}}\Big) \\ &= \frac{1}{\sqrt{2}}\frac{2}{3}\mathcal{F}^{-1}\Big(i\frac{d}{dw}e^{-3\frac{w^2}{4}}\Big) = \frac{1}{\sqrt{2}}\frac{2}{3}x\mathcal{F}^{-1}\Big(e^{-3\frac{w^2}{4}}\Big) \\ &= \frac{2}{3\sqrt{3}}xe^{-\frac{1}{3}x^2}. \end{aligned}$$

In computing $\mathcal{F}^{-1}\Big(e^{-3\frac{w^2}{4}}\Big)$, use Exercise 10(a) and (5) to obtain

$$\mathcal{F}^{-1}\Big(e^{-aw^2}\Big) = \frac{1}{\sqrt{2a}}e^{-\frac{(-x)^2}{4a}} = \frac{1}{\sqrt{2a}}e^{-\frac{x^2}{4a}}.$$

55. We use the uniqueness of the Fourier transform, which states that if $\widehat{h} = \widehat{k}$ then $h = k$.
(a) Let $h = f * g$ and $k = g * f$ and use Theorem 4. Then

$$\mathcal{F}(h) = \mathcal{F}(f)\cdot\mathcal{F}(g) = \mathcal{F}(g)\cdot\mathcal{F}(f) = \mathcal{F}(g * f).$$

hence $f * g = g * f$.
(b) Let $u = f * (g * h)$ and $v = (f * g) * h$. Then

$$\begin{aligned} \mathcal{F}(u) &= \mathcal{F}(f)\cdot\mathcal{F}(g * h) = \mathcal{F}(f)\cdot(\mathcal{F}(g)\cdot\mathcal{F}(h)) \\ &= (\mathcal{F}(f)\cdot\mathcal{F}(g))\cdot\mathcal{F}(h) = \mathcal{F}(f * g)\cdot\mathcal{F}(h) \\ &= \mathcal{F}((f * g) * h) = \mathcal{F}(v). \end{aligned}$$

Hence $u = v$ or $f * (g * h) = (f * g) * h$.
(c) Use Exercise19:

$$\mathcal{F}(f_a) = \mathcal{F}(f(x-a)) = e^{-iaw}\mathcal{F}(f).$$

So

$$\begin{aligned} \mathcal{F}((f * g)_a) &= \mathcal{F}(f * g(x-a)) = e^{-iaw}\mathcal{F}(f * g) \\ &= \overbrace{e^{-iaw}\mathcal{F}(f)}^{\mathcal{F}(f_a)}\mathcal{F}(g) = \mathcal{F}(f_a)\mathcal{F}(g) = \mathcal{F}(f_a * g). \end{aligned}$$

Thus $(f * g)_a = f_a * g$. Similarly, $(f * g)_a = f * (g_a)$.

57. Recall that f is integrable means that

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

If f and g are integrable, then

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g(t)\, dt.$$

So, using properties of the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f * g(x)| dx &= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g(t)\, dt \right| dx \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-t) g(t)|\, dx\, dt \\
&\quad \text{(Interchange order of integration.} \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overbrace{\int_{-\infty}^{\infty} |f(x-t)|\, dx}^{=\int_{-\infty}^{\infty} |f(x)| dx} |g(t)|\, dt \\
&\quad \text{(Change variables in the inner integral } X = x - t.) \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx \int_{-\infty}^{\infty} |g(t)|\, dt < \infty;
\end{aligned}$$

thus $f * g$ is integrable.

60. Suppose that $\widehat{f}(w) = 0$ for all w not in $[a, b]$. This fact is expressed by saying that the support of $\widehat{f}$ is contained in $[a, b]$. Suppose also that support of $\widehat{g}$ is contained in $[c, d]$, where $[a, b]$ is disjoint from $[c, d]$. Consider the function $h = f * g$. Then for all w, $\widehat{h}(w) = \widehat{f}(w) \cdot \widehat{g}(w)$. In order for the product $\widehat{f}(w) \cdot \widehat{g}(w)$ to be nonzero, w has to be simultaneously in $[a, b]$ and $[c, d]$. But this is impossible, since $[a, b]$ and $[c, d]$ are disjoint. So

$$\widehat{h}(w) = \widehat{f}(w) \cdot \widehat{g}(w) = 0$$

for w, and hence $f * g = 0$. To complete the solution of this exercise, we must construct nonzero functions f and g with the stipulated Fourier transform properties. Let

$$f(x) = \sqrt{\frac{\pi}{2}} \frac{\sin^2 x}{x^2}.$$

From Example 1 (and Exercise 10(b)), we have

$$\widehat{f}(w) = \begin{cases} 1 - \frac{|w|}{2} & \text{if } |w| < 2, \\ 0 & \text{othewise.} \end{cases}$$

The function $\widehat{f}$ has support contained in $[-2, 2]$, because it is zero outside $|w| < 2$. To construct a function whose Fourier transform is supported outside the support of $[-2, 2]$, it suffices to take f and translate its Fourier transform by, say, 5. This is done by multiplying f by e^{5ix} (see Exercise 20(a). Let $g(x) = e^{5ix} f(x)$. Then

$$\widehat{g}(w) = \begin{cases} 1 - \frac{|w-5|}{2} & \text{if } |w-5| < 2, \\ 0 & \text{othewise.} \end{cases}$$

This Fourier transform is just the translate of $\widehat{f}$. It is supported in the interval $[3, 7]$. By the first part of the solution, $f * g = 0$.

Solutions to Exercises 7.3

1.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = \frac{1}{1+x^2}, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

Follow the solution of Example 1. Fix t and Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w,\,t) = -w^2\widehat{u}(w,\,t),$$
$$\widehat{u}(w,\,0) = \mathcal{F}\big(\frac{1}{1+x^2}\big) = \sqrt{\frac{\pi}{2}}e^{-|w|}, \quad \frac{d}{dt}\widehat{u}(w,\,0) = 0.$$

Solve the second order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)\cos wt + B(w)\sin wt.$$

Using $\frac{d}{dt}\widehat{u}(w,\,0) = 0$, we get

$$-A(w)w\sin wt + B(w)w\cos wt\Big|_{t=0} = 0 \;\Rightarrow\; B(w)w = 0 \;\Rightarrow\; B(w) = 0.$$

Hence

$$\widehat{u}(w,\,t) = A(w)\cos wt.$$

Using $\widehat{u}(w,\,0) = \sqrt{\frac{\pi}{2}}e^{-|w|}$, we see that $A(w) = \sqrt{\frac{\pi}{2}}e^{-|w|}$ and so

$$\widehat{u}(w,\,t) = \sqrt{\frac{\pi}{2}}e^{-|w|}\cos wt.$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \int_{-\infty}^{\infty} e^{-|w|}\cos wt\, e^{ixw}\,dw.$$

3.

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = e^{-x^2}.$$

Fix t and Fourier transform the problem with respect to the variable x:

$$\frac{d}{dt}\widehat{u}(w,\,t) = -\frac{w^2}{4}\widehat{u}(w,\,t),$$
$$\widehat{u}(w,\,0) = \mathcal{F}\big(e^{-x^2}\big) = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}}.$$

Solve the first order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)e^{-\frac{w^2}{4}}.$$

Using $\widehat{u}(w,\,0) = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}}$, we get $A(w) = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}}$. Hence

$$\widehat{u}(w,\,t) = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}}e^{-\frac{w^2}{4}t} = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}(1+t)}.$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{4}(1+t)}\, e^{ixw}\, dw.$$

5.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Fix t and Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w,\,t) = -c^2 w^2 \widehat{u}(w,\,t),$$
$$\widehat{u}(w,\,0) = \mathcal{F}\Big(\sqrt{\frac{\sin x}{x}}\Big)(w) = \widehat{f}(w) = \begin{cases} 1 & \text{if } |w| < 1 \\ 0 & \text{if } |w| > 1, \end{cases}$$
$$\frac{d}{dt}\widehat{u}(w,\,0) = 0.$$

Solve the second order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)\cos cwt + B(w)\sin cwt.$$

Using $\frac{d}{dt}\widehat{u}(w,\,0) = 0$, we get

$$\widehat{u}(w,\,t) = A(w)\cos cwt.$$

Using $\widehat{u}(w,\,0) = \widehat{f}(w)$, we see that

$$\widehat{u}(w,\,t) = \widehat{f}(w)\cos wt.$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)\cos cwt\, e^{ixw}\, dw = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos cwt\, e^{ixw}\, dw.$$

7.

$$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial t} = 0,$$
$$u(x,0) = f(x)\,.$$

Fourier transform the problem with respect to the variable x:

$$\frac{d}{dt}\widehat{u}(w,\,t) = -\frac{iw}{3}\widehat{u}(w,\,t),$$
$$\widehat{u}(w,\,0) = \mathcal{F}\big(u(x,\,0)\big)(w) = \widehat{f}(w).$$

Solve the first order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)e^{-i\frac{w}{3}t}.$$

Using the transformed initial condition, we get

$$\widehat{u}(w,\,t) = \widehat{f}(w)e^{-i\frac{w}{3}t}.$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)e^{-i\frac{w}{3}t}\, e^{ixw}\, dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)e^{iw(x-\frac{t}{3})}\, dw = f(x - \frac{t}{3}).$$

9.

$$t^2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0,$$
$$u(x,0) = 3\cos x\,.$$

The solution of this problem is very much like the solution of Exercise 7. However, there is a difficulty in computing the Fourier transform of $\cos x$, because $\cos x$ is not integrable on the real line. One can make sense of the Fourier transform by treating $\cos x$ as a generalized function, but there is no need for this in this solution, since we do not need the exact formula of the Fourier transform, as you will see shortly.

Let $f(x) = 3\cos x$ and Fourier transform the problem with respect to the variable x:

$$t^2 iw\widehat{u}(w,\,t) - \frac{d}{dt}\widehat{u}(w,\,t) = 0,$$
$$\widehat{u}(w,\,0) = \mathcal{F}\big(3\cos x)\big)(w) = \widehat{f}(w).$$

Solve the first order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)e^{-i\frac{w}{3}t^3}.$$

Using the transformed initial condition, we get

$$\widehat{u}(w,\,t) = \widehat{f}(w)e^{-i\frac{w}{3}t^3}.$$

Taking inverse Fourier transforms, we get

$$\begin{aligned} u(x,\,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)e^{-i\frac{w}{3}t^3}\,e^{ixw}\,dw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)e^{iw(x+\frac{t^3}{3})}\,dw \\ &= f(x+\frac{t^3}{3}) = 3\cos(x+\frac{t^3}{3}). \end{aligned}$$

11.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t},$$
$$u(x,0) = f(x)\,.$$

Fourier transform the problem with respect to the variable x:

$$\frac{d}{dt}\widehat{u}(w,\,t) - iw\widehat{u}(w,\,t) = 0,$$
$$\widehat{u}(w,\,0) = \widehat{f}(w).$$

Solve the first order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)e^{iwt}.$$

Using the transformed initial condition, we get

$$\widehat{u}(w,\,t) = \widehat{f}(w)e^{iwt}.$$

Taking inverse Fourier transforms, we get

$$\begin{aligned} u(x,\,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)e^{-iwt}\,e^{ixw}\,dw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)e^{iw(x+t)}\,dw \\ &= f(x+t). \end{aligned}$$

13.

$$\frac{\partial u}{\partial t} = t\,\frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = f(x)\,.$$

Fix t and Fourier transform the problem with respect to the variable x:

$$\frac{d}{dt}\widehat{u}(w,\,t) + tw^2\widehat{u}(w,\,t) = 0,$$
$$\widehat{u}(w,\,0) = \widehat{f}(w).$$

Solve the first order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)e^{-\frac{t^2w^2}{2}}.$$

Use the initial condition: $A(w) = \widehat{f}(w)$. Hence

$$\widehat{u}(w,\,t) = \widehat{f}(w)e^{-\frac{t^2w^2}{2}}.$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)e^{-\frac{t^2w^2}{2}}\,e^{ixw}\,dw.$$

15.

$$\frac{\partial^2 u}{\partial t^2} + 2\,\frac{\partial u}{\partial t} = -u,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x)\,.$$

Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w,\,t) + 2\frac{d}{dt}\widehat{u}(w,\,t) + \widehat{u}(w,\,t) = 0,$$
$$\widehat{u}(w,\,0) = \widehat{f}(w), \quad .\frac{d}{dt}\widehat{u}(w,\,t) = \widehat{g}(w).$$

Solve the second order differential equation in $\widehat{u}(w,\,t)$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$. Since we have a repeated root $\lambda = -1$, the general solution is of the form:

$$\widehat{u}(w,\,t) = A(w)e^{-t} + B(w)te^{-t}.$$

The initial condition $\widehat{u}(w,\,0) = \widehat{f}(w)$ implies that $\widehat{f}(w) = A(w)$. So $\widehat{u}(w,\,t) = \widehat{f}(w)e^{-t} + B(w)te^{-t}$. The condition $\frac{d}{dt}\widehat{u}(w,\,t) = \widehat{g}(w)$ implies that $-\widehat{f}(w) + B(w) = \widehat{g}(w)$ or $B(w) = \widehat{f}(w) + \widehat{g}(w)$. So

$$\widehat{u}(w,\,t) = \widehat{f}(w)e^{-t} + (\widehat{f}(w) + \widehat{g}(w))te^{-t}.$$

Hence Taking inverse Fourier transforms, we get

$$\begin{aligned}
u(x,\,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\widehat{f}(w)e^{-t} + (\widehat{f}(w) + \widehat{g}(w))te^{-t}\right)e^{ixw}\,dw \\
&= e^{-t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)\,e^{ixw}\,dw + te^{-t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\widehat{f}(w) + \widehat{g}(w)\right)e^{ixw}\,dw \\
&= e^{-t}f(x) + te^{-t}(f(x) + g(x).
\end{aligned}$$

17.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4}$$
$$u(x,0) = \begin{cases} 100 & \text{if } |x| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w, t) = w^4\widehat{u}(w, t),$$

$$\widehat{u}(w, 0) = \widehat{f}(w) = 100\sqrt{\frac{2}{\pi}}\frac{\sin 2w}{w}.$$

Solve the second order differential equation in $\widehat{u}(w, t)$:

$$\widehat{u}(w, t) = A(w)e^{-w^2 t} + B(w)e^{w^2 t}.$$

Because a Fourier transform is expected to tend to 0 as $w \to \pm\infty$, if we fix $t > 0$ and let $w \to \infty$ or $w \to -\infty$, we see that one way to make $\widehat{u}(w, t) \to 0$ is to take $B(w) = 0$. Then $\widehat{u}(w, t) = A(w)e^{-w^2 t}$, and from the initial condition we obtain $B(w) = \widehat{f}(w)$. So

$$\widehat{u}(w, t) = \widehat{f}(w)e^{-w^2 t} = 100\sqrt{\frac{2}{\pi}}\frac{\sin 2w}{w}e^{-w^2 t}.$$

Taking inverse Fourier transforms, we get

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 100\sqrt{\frac{2}{\pi}}\frac{\sin 2w}{w}e^{-w^2 t}\, e^{ixw}\, dw \\ &= \frac{100}{\pi}\int_{-\infty}^{\infty} \frac{\sin 2w}{w}e^{-w^2 t}\, e^{ixw}\, dw. \end{aligned}$$

19.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial t \partial x^2},$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)\,.$$

Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w, t) = -w^2\frac{d}{dt}\widehat{u}(w, t),$$

$$\widehat{u}(w, 0) = \widehat{f}(w), \quad \frac{d}{dt}\widehat{u}(w, 0) = \widehat{g}(w).$$

Solve the second order differential equation in $\widehat{u}(w, t)$. The characteristic equation is $\lambda^2 + w^2\lambda = 0$, with roots $\lambda = 0$ or $\lambda = -w^2$. The general solution is of the form:

$$\widehat{u}(w, t) = A(w) + B(w)e^{-w^2 t}.$$

The initial condition $\widehat{u}(w, 0) = \widehat{f}(w)$ implies that $\widehat{f}(w) = A(w) + B(w)$ or $A(w) = \widehat{f}(w) - B(w)$. So

$$\widehat{u}(w, t) = \widehat{f}(w) + B(w)(e^{-w^2 t} - 1).$$

The condition $\frac{d}{dt}\widehat{u}(w, 0) = \widehat{g}(w)$

$$\widehat{g}(w) = -B(w)w^2 \;\Rightarrow\; B(w) = -\frac{\widehat{g}(w)}{w^2}.$$

So

$$\widehat{u}(w, t) = \widehat{u}(w, t) = \widehat{f}(w) - \frac{\widehat{g}(w)}{w^2}(e^{-w^2 t} - 1) = \widehat{f}(w) + \frac{\widehat{g}(w)}{w^2}(1 - e^{-w^2 t}).$$

Taking inverse Fourier transforms, we get

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Big(\widehat{f}(w) + \frac{\widehat{g}(w)}{w^2}(1 - e^{-w^2 t})\Big)\, e^{ixw}\, dw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(w)\, e^{ixw}\, dw + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\widehat{g}(w)}{w^2}(1 - e^{-w^2 t})\big)\, e^{ixw}\, dw \\ &= f(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\widehat{g}(w)}{w^2}(1 - e^{-w^2 t})\big)\, e^{ixw}\, dw. \end{aligned}$$

21. (a) To verify that

$$u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

is a solution of the boundary value problem of Example 1 is straightforward. You just have to plug the solution into the equation and the initial and boundary conditions and see that the equations are verified. The details are sketched in Section 3.4, following Example 1 of that section.
(b) In Example 1, we derived the solution as an inverse Fourier transform:

$$u(x,\,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\big[\widehat{f}(w)\,\cos cwt + \frac{1}{cw}\widehat{g}(w)\,\sin cwt\big]e^{iwx}\,dx.$$

Using properties of the Fourier transform, we will show that

$$(1)\qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)\,\cos cwte^{iwx}\,dw = \frac{1}{2}[f(x-ct)+f(x+ct)];$$

$$(2)\qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{w}\widehat{g}(w)\,\sin cwt\,e^{iwx}\,dw = \frac{1}{2}\int_{x-ct}^{x+ct} g(s)\,ds.$$

To prove (1), note that

$$\cos cwt = \frac{e^{icwt}+e^{-icwt}}{2},$$

so

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)\,\cos cwte^{iwx}\,dw\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)\,(\frac{e^{icwt}+e^{-icwt}}{2})e^{iwx}\,dw\\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)\,e^{iw(x+ct)}\,dw\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)\,e^{iw(x-ct)}\,dw\right]\\
&= \frac{1}{2}[f(x+ct)+f(x-ct)];
\end{aligned}$$

because the first integral is simply the inverse Fourier transform of $\widehat{f}$ evaluated at $x+ct$, and the second integral is the inverse Fourier transform of $\widehat{f}$ evaluated at $x-ct$. This proves (1). To prove (2), we note that the left side of (2) is an inverse Fourier transform. So (2) will follow if we can show that

$$(3)\qquad \mathcal{F}\left\{\int_{x-ct}^{x+ct} g(s)\,ds\right\} = \frac{2}{w}\widehat{g}(w)\,\sin cwt.$$

Let G denote an antiderivative of g. Then (3) is equivalent to

$$\mathcal{F}\left(G(x+ct)-G(x-ct)\right)(w) = \frac{2}{w}\widehat{G'}(w)\,\sin cwt.$$

Since $\widehat{G'} = iw\widehat{G}$, the last equation is equivalent to

$$(4)\qquad \mathcal{F}\left(G(x+ct)\right)(w) - \mathcal{F}\left(G(x-ct)\right)(w) = 2i\widehat{G}(w)\,\sin cwt.$$

Using Exercise 19, Sec. 7.2, we have

$$\begin{aligned}
\mathcal{F}\left(G(x+ct)\right)(w) - \mathcal{F}\left(G(x-ct)\right)(w) &= e^{ictw}\mathcal{F}(G)(w) - e^{-ictw}\mathcal{F}(G)(w)\\
&= \mathcal{F}(G)(w)\left(e^{ictw}-e^{-ictw}\right)\\
&= 2i\widehat{G}(w)\,\sin cwt,
\end{aligned}$$

where we have applied the formula

$$\sin ctw = \frac{e^{ictw} - e^{-ictw}}{2i}.$$

This proves (4) and completes the solution.

23. Fourier transform the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} + k\,\frac{\partial u}{\partial x}, \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x)\,, \end{aligned}$$

and get

$$\begin{aligned} \frac{d}{dt}\widehat{u}(w,\,t) &= -c^2 w^2\,\widehat{u}(w,\,t) + ikw\widehat{u}(w,\,t) = \widehat{u}(w,\,t)(-c^2w^2 + ikw), \\ \widehat{u}(w,\,0) &= \widehat{f}(w). \end{aligned}$$

Solve the first order differential equation in $\widehat{u}(w,\,t)$ and get

$$\widehat{u}(w,\,t) = A(w)e^{(-c^2w^2+ikw)t}.$$

The initial condition $\widehat{u}(w,\,0) = \widehat{f}(w)$ implies

$$\widehat{u}(w,\,t) = \widehat{f}(w)e^{(-c^2w^2+ikw)t}.$$

Taking inverse Fourier transforms, we get

$$\begin{aligned} u(x,\,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(w)e^{(-c^2w^2+ikw)t}\,e^{ixw}\,dw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(w)e^{-c^2w^2t}\,e^{iw(x+kt)}\,dw. \end{aligned}$$

25. Using the Fourier transform, we obtain

$$\begin{aligned} \frac{d^2}{dt^2}\widehat{u}(w,\,t) &= c^2\,w^4\widehat{u}(w,\,t), \\ \widehat{u}(w,\,0) &= \widehat{f}(w), \qquad \frac{d}{dt}\widehat{u}(w,\,0) = \widehat{g}(w)\,. \end{aligned}$$

Thus

$$\widehat{u}(w,\,t) = A(w)e^{-cw^2t} + B(w)e^{cw^2t}.$$

Using the initial conditions:

$$\widehat{u}(w,\,0) = \widehat{f}(w) \;\Rightarrow\; A(w) + B(w) = \widehat{f}(w) \;\Rightarrow\; A(w) = \widehat{f}(w) - B(w);$$

and

$$\begin{aligned} \frac{d}{dt}\widehat{u}(w,\,0) = \widehat{g}(w) \quad &\Rightarrow \quad -cw^2A(w) + cw^2B(w) = \widehat{g}(w) \\ &\Rightarrow \quad -A(w) + B(w) = \frac{\widehat{g}(w)}{cw^2} \\ &\Rightarrow \quad -\widehat{f}(w) + 2B(w) = \frac{\widehat{g}(w)}{cw^2} \\ &\Rightarrow \quad B(w) = \frac{1}{2}\left(\widehat{f}(w) + \frac{\widehat{g}(w)}{cw^2}\right); \\ &\Rightarrow \quad A(w) = \frac{1}{2}\left(\widehat{f}(w) - \frac{\widehat{g}(w)}{cw^2}\right). \end{aligned}$$

Hence

$$\begin{aligned}
\widehat{u}(w,t) &= \frac{1}{2}\left(\widehat{f}(w) - \frac{\widehat{g}(w)}{cw^2}\right)e^{-cw^2 t} + \frac{1}{2}\left(\widehat{f}(w) + \frac{\widehat{g}(w)}{cw^2}\right)e^{cw^2 t}. \\
&= \widehat{f}(w)\frac{(e^{cw^2 t} + e^{-cw^2 t})}{2} + \frac{\widehat{g}(w)}{cw^2}\frac{(e^{cw^2 t} - e^{-cw^2 t})}{2} \\
&= \widehat{f}(w)\cosh(cw^2 t) + \frac{\widehat{g}(w)}{cw^2}\sinh(cw^2 t)
\end{aligned}$$

Taking inverse Fourier transforms, we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\widehat{f}(w)\cosh(cw^2 t) + \frac{\widehat{g}(w)}{cw^2}\sinh(cw^2 t)\right)e^{ixw}\,dw.$$

27. Fourier transform the boundary value problem:

$$\begin{aligned}
\frac{d}{dt}\widehat{u}(w,t) &= -ic^2 w^3\,\widehat{u}(w,t), \\
\widehat{u}(w,0) &= \widehat{f}(w).
\end{aligned}$$

Solve the equation in $\widehat{u}(w,t)$:

$$\widehat{u}(w,t) = A(w)e^{-ic^2 w^3 t}.$$

Apply the initial condition:

$$\widehat{u}(w,t) = \widehat{f}(w)e^{-ic^2 w^3 t}.$$

Inverse Fourier transform:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(w)e^{-ic^2 w^3 t}\,e^{ixw}\,dw.$$

Solutions to Exercises 7.4

1. Repeat the solution of Example 1 making some adjustments: $c = \frac{1}{2}$, $g_t(x) = \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{x^2}{t}}$,

$$\begin{aligned}
u(x,t) &= f * g_t(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{(x-s)^2}{t}}\, ds \\
&= \frac{20}{\sqrt{t\pi}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{t}}\, ds \quad (v = \frac{x-s}{\sqrt{t}},\ dv = -\frac{1}{\sqrt{t}}\, ds) \\
&= \frac{20}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{t}}}^{\frac{x+1}{\sqrt{t}}} e^{-v^2}\, ds \\
&= 10 \left(\operatorname{erf}(\frac{x+1}{\sqrt{t}}) - \operatorname{erf}(\frac{x-1}{\sqrt{t}}) \right).
\end{aligned}$$

3. Let us use an approach similar to Example 2. Fourier transform the boundary value problem and get:

$$\begin{aligned}
\frac{d}{dt}\widehat{u}(w,t) &= -w^2 \widehat{u}(w,t) \\
\widehat{u}(w,0) &= \mathcal{F}(70e^{-\frac{x^2}{2}}) = 70e^{-\frac{w^2}{2}}.
\end{aligned}$$

Solve the equation in $\widehat{u}$:

$$\widehat{u}(w,t) = A(w)e^{-w^2 t}.$$

Apply the boundary condition:

$$\widehat{u}(w,t) = 70e^{-\frac{w^2}{2}})e^{-w^2 t} = 70e^{-w^2(t+\frac{1}{2})}.$$

Inverse Fourier transform:

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}\left(70e^{-w^2(t+\frac{1}{2})}\right) \qquad (\frac{1}{2a} = t + \frac{1}{2}) \\
&= \frac{70}{\sqrt{2t+1}} \mathcal{F}^{-1}\left(\sqrt{2t+1}\, e^{-\frac{w^2}{2a}}\right) \qquad (a = \frac{1}{2t+1}) \\
&= \frac{70}{\sqrt{2t+1}} e^{-\frac{x^2}{2(2t+1)}},
\end{aligned}$$

where we have used Theorem 5, Sec. 7.2.

5. Apply (4) with $f(s) = s^2$:

$$\begin{aligned}
u(x,t) &= f * g_t(x) \\
&= \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^2 e^{-\frac{(x-s)^2}{t}}\, ds.
\end{aligned}$$

You can evaluate this integral by using integration by parts twice and then appealing to Theorem 5, Section 7.2. However, we will use a different technique based on the operational properties of the Fourier transform that enables us to evaluate a much more general integral. Let n be a nonnegative integer and suppose that f and $s^n f(s)$ are integrable and tend to 0 at $\pm\infty$. Then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^n f(s)\, ds = (i)^n \left[\frac{d^n}{dw^n} \mathcal{F}(f)(w) \right]_{w=0}.$$

This formula is immediate if we recall Theorem 3(ii), Section 7.2, and that

$$\widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s)\, ds.$$

We will apply this formula with

$$f(s) = \frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}.$$

We have

$$\begin{aligned}
\mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}\right)(w) &= e^{-iwx}\mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{s^2}{t}}\right)(w) \quad \text{(by Exercise 19, Sec. 7.2)}\\
&= e^{-iwx}e^{-w^2 t} = e^{-(iwx+w^2t)} \quad \text{(See the proof of Th. 1.)}
\end{aligned}$$

So

$$\begin{aligned}
u(x, t) &= \frac{1}{\sqrt{2t}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s^2 e^{-\frac{(x-s)^2}{t}}\, ds\\
&= -\left[\frac{d^2}{dw^2}\mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}\right)(w)\right]_{w=0}\\
&= -\left[\frac{d^2}{dw^2} e^{-(iwx+w^2t)}\right]_{w=0}\\
&= -\left[\frac{d}{dw} - e^{-(iwx+w^2t)}(ix+2wt)\right]_{w=0}\\
&= \left[-e^{-(iwx+w^2t)}(ix+2wt)^2 + 2te^{-(iwx+w^2t)}\right]_{w=0}\\
&= x^2 + 2t.
\end{aligned}$$

You can check the validity of this answer by plugging it back into the heat equation. The initial condition is also obviously met: $u(x, 0) = x^2$.

The approach that we took can be used to solve the boundary value problem with $f(x)x^n$ as initial temperature distribution. See the end of this section for interesting applications.

7. Proceed as in Example 2. Fourier transform the problem:

$$\frac{du}{dt}\widehat{u}(w, t) = -t^2 w^2 \widehat{u}(w, t), \quad \widehat{u}(w, 0) = \widehat{f}(w).$$

Solve for $\widehat{u}(w, t)$:

$$\widehat{u}(w, t) = \widehat{f}(w) e^{-w^2 \frac{t^3}{3}}.$$

Inverse Fourier transform and note that

$$u(x, t) = f * \mathcal{F}^{-1}\left(e^{-w^2 \frac{t^3}{3}}\right).$$

With the help of Theorem 5, Sec. 7.2 (take $a = \frac{t^3}{3}$), we find

$$\mathcal{F}^{-1}\left(e^{-w^2 \frac{t^3}{3}}\right) = \frac{\sqrt{3}}{\sqrt{2t^3}} e^{-x^2 \frac{3}{4t^3}}.$$

Thus

$$\begin{aligned}
u(x, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)\frac{\sqrt{3}}{\sqrt{2t^3}} e^{-\frac{3(x-s)^2}{4t^3}}\, ds\\
&= \frac{\sqrt{3}}{2\sqrt{\pi t^3}}\int_{-\infty}^{\infty} f(s) e^{-\frac{3(x-s)^2}{4t^3}}\, ds.
\end{aligned}$$

9. Fourier transform the problem:

$$\frac{du}{dt}\widehat{u}(w, t) = -e^{-t} w^2 \widehat{u}(w, t), \quad \widehat{u}(w, 0) = \widehat{f}(w).$$

Solve for $\widehat{u}(w, t)$:

$$\widehat{u}(w, t) = \widehat{f}(w)e^{-w^2(1-e^{-t})}.$$

Inverse Fourier transform and note that

$$u(x, t) = f * \mathcal{F}^{-1}\left(e^{-w^2(1-e^{-t})}\right).$$

With the help of Theorem 5, Sec. 7.2 (take $a = 1 - e^{-t}$), we find

$$\mathcal{F}^{-1}\left(e^{-w^2(1-e^{-t})}\right) = \frac{1}{\sqrt{2}\sqrt{1-e^{-t}}}e^{-\frac{x^2}{4(1-e^{-t})}}.$$

Thus

$$u(x, t) = \frac{1}{2\sqrt{\pi}\sqrt{1-e^{-t}}}\int_{-\infty}^{\infty} f(s)e^{-\frac{(x-s)^2}{4(1-e^{-t})}}\, ds.$$

11. This is a generalization of Exercise 9 that can be solved by similar methods. Fourier transform the problem:

$$\frac{du}{dt}\widehat{u}(w, t) = -a(t)w^2\widehat{u}(w, t), \quad \widehat{u}(w, 0) = \widehat{f}(w).$$

Let $B(t)$ denote an antiderivative of $a(t)$, hence $B(t) = \int_0^t a(\tau)\, d\tau$. Then $\widehat{u}(w, t)$:

$$\widehat{u}(w, t) = \widehat{f}(w)e^{-w^2 B(t)}.$$

Inverse Fourier transform and note that

$$u(x, t) = f * \mathcal{F}^{-1}\left(e^{-w^2 B(t)}\right).$$

With the help of Theorem 5, Sec. 7.2 (take $a = B(t)$), we find

$$\mathcal{F}^{-1}\left(e^{-w^2 B(t)}\right) = \frac{1}{\sqrt{2\, B(t)}}e^{-\frac{x^2}{4B(t)}}.$$

Thus

$$u(x, t) = \frac{1}{2\sqrt{\pi B(t)}}\int_{-\infty}^{\infty} f(s)e^{-\frac{(x-s)^2}{4B(t)}}\, ds.$$

13. If in Exercise 9 we take

$$f(x) = \begin{cases} 100 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

then the solution becomes

$$u(x, t) = \frac{50}{\sqrt{\pi}\sqrt{1-e^{-t}}}\int_{-1}^{1} e^{-\frac{(x-s)^2}{4(1-e^{-t})}}\, ds.$$

Let $z = \frac{x-s}{2\sqrt{1-e^{-t}}}$, $dz = \frac{-ds}{2\sqrt{1-e^{-t}}}$. Then

$$\begin{aligned} u(x, t) &= \frac{50}{\sqrt{\pi}\sqrt{1-e^{-t}}}2\sqrt{1-e^{-t}}\int_{\frac{x-1}{2\sqrt{1-e^{-t}}}}^{\frac{x+1}{2\sqrt{1-e^{-t}}}} e^{-z^2}\, dz \\ &= \frac{100}{\sqrt{\pi}}\int_{\frac{x-1}{2\sqrt{1-e^{-t}}}}^{\frac{x+1}{2\sqrt{1-e^{-t}}}} e^{-z^2}\, dz \\ &= 50\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{1-e^{-t}}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{1-e^{-t}}}\right)\right]. \end{aligned}$$

As t increases, the expression $\operatorname{erf}\left(\frac{x+1}{2\sqrt{1-e^{-t}}}\right)-\operatorname{erf}\left(\frac{x-1}{2\sqrt{1-e^{-t}}}\right)$ approaches very quickly $\operatorname{erf}\left(\frac{x+1}{2}\right)-\operatorname{erf}\left(\frac{x-1}{2}\right)$, which tells us that the temperature approaches the limiting distribution

$$50\left[\operatorname{erf}\left(\frac{x+1}{2}\right)-\operatorname{erf}\left(\frac{x-1}{2}\right)\right].$$

You can verify this assertion using graphs.

15. (a) From the definition

$$\begin{aligned}\operatorname{erf}(-x) &= \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-z^2}\,dz\\ &= -\frac{2}{\sqrt{\pi}}\int_0^{x} e^{-(-z)^2}\,dz \qquad (z=-Z,\ dz=-dZ)\\ &= -\operatorname{erf}(x)\end{aligned}$$

Thus erf is an odd function.
(b) From the definition

$$\begin{aligned}\operatorname{erf}(0) &= \frac{2}{\sqrt{\pi}}\int_0^{0} e^{-z^2}\,dz=0,\\ \operatorname{erf}(\infty) &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-z^2}\,dz=1,\end{aligned}$$

by (4)m Sec. 4.2.
(c) By the fundamental theorem of calculus,

$$\begin{aligned}\frac{d}{dx}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{x} e^{-z^2}\,dz\\ &= \frac{2}{\sqrt{\pi}}e^{-x^2}>0\end{aligned}$$

for all x. Thus erf is strictly increasing.
(d) We have

$$\begin{aligned}\frac{d}{dx}(x\operatorname{erf}(x)+\frac{1}{\sqrt{\pi}}e^{-x^2} &= \operatorname{erf}(x)+x\frac{d}{dx}\operatorname{erf}(x)-\frac{2x}{\sqrt{\pi}}e^{-x^2}\\ &= \operatorname{erf}(x)+\frac{2x}{\sqrt{\pi}}e^{-x^2}-\frac{2x}{\sqrt{\pi}}e^{-x^2} \qquad \text{(by (c))}\\ &= \operatorname{erf}(x).\end{aligned}$$

Thus

$$\int \operatorname{erf}(x)\,dx=(x\operatorname{erf}(x)+\frac{1}{\sqrt{\pi}}e^{-x^2}+C.$$

(e) From the power series expansion of

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!} \qquad (-\infty<z<\infty),$$

we obtain the power series

$$e^{-z^2}=\sum_{n=0}^{\infty}\frac{(-z^2)^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{n!} \qquad (-\infty<z<\infty).$$

Since the power series expansion for e^{-z^2} converges for all z, we can integrate it term by term over any interval. So we have

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz \\
&= \frac{2}{\sqrt{\pi}}\int_0^x \left(\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{n!}\right)dz \\
&= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x z^{2n}\,dz = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}z^{2n+1}\Big|_0^x \\
&= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}.
\end{aligned}$$

17. (a) If

$$f(x) = \begin{cases} T_0 & \text{if } a < x < b, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$u(x,\,t) = \frac{T_0}{2c\sqrt{\pi t}}\int_a^b e^{-\frac{(x-s)^2}{4c^2t}}\,ds.$$

(b) Let $z = \frac{x-s}{2c\sqrt{t}}$, $dz = \frac{-ds}{2c\sqrt{t}}$. Then

$$\begin{aligned}
u(x,\,t) &= \frac{T_0}{2c\sqrt{\pi t}}2c\sqrt{t}\int_{\frac{x-b}{2c\sqrt{t}}}^{\frac{x-a}{2c\sqrt{t}}} e^{-z^2}\,dz \\
&= \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{x-a}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-b}{2c\sqrt{t}}\right)\right].
\end{aligned}$$

19. Using (4) with $f(x) = T_0$, then

$$u(x,\,t) = \frac{T_0}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4c^2t}}\,ds.$$

Let $z = \frac{x-s}{2c\sqrt{t}}$, $dz = \frac{-ds}{2c\sqrt{t}}$. Then

$$u(x,\,t) = \frac{T_0}{2c\sqrt{\pi t}}2c\sqrt{t}\int_{-\infty}^{\infty} e^{-z^2}\,dz = T_0,$$

by (4), Sec. 7.2.

21.

23. The solution of the heat problem with initial data $f(x)$ is given by (4), which we recall as follows

$$u(x,\,t) = g_t * f(x),$$

where $g_t(x)$ is Gauss's kernel,

$$g_t(x) = \frac{1}{c\sqrt{2t}}e^{-x^2/(4c^2t)}.$$

Let $u_2(x,\,t)$ denote the solution of the heat problem with initial data $f(x-a)$. Applying (4) with f replaced by $(f)_a(x) = f(x-a)$, we obtain

$$u_2(x,\,t) = g_t * (f)_a(x).$$

By Exercise 47(c), Sec. 7.2, we know that the convolution operation commutes with translation. So

$$u_2(x,\,t) = g_t * (f)_a(x) = (g_t * f)_a(x) = u(x-a,\,t).$$

25. Let $u_2(x,\, t)$ denote the solution of the heat problem with initial temperature distribution $f(x) = e^{-(x-1)^2}$. Let $u(x,\, t)$ denote the solution of the problem with initial distribution e^{-x^2}. Then, by Exercise 23, $u_2(x,\, t) = u(x-1,\, t)$

By (4), we have

$$u(x,\, t) = \frac{1}{c\sqrt{2t}} e^{-x^2/(4c^2t)} * e^{-x^2}.$$

We will apply Exercise 24 with $a = \frac{1}{4c^2t}$ and $b = 1$. We have

$$\begin{aligned} \frac{ab}{a+b} &= \frac{1}{4c^2t} \times \frac{1}{\frac{1}{4c^2t}+1} \\ &= \frac{1}{1+4c^2t} \\ \frac{1}{\sqrt{2(a+b)}} &= \frac{1}{\sqrt{2(\frac{1}{4c^2t}+1)}} \\ &= \frac{c\sqrt{2t}}{\sqrt{4c^2t+1}}. \end{aligned}$$

So

$$\begin{aligned} u(x,\, t) &= \frac{1}{c\sqrt{2t}} e^{-x^2/(4c^2t)} * e^{-x^2} \\ &= \frac{1}{c\sqrt{2t}} \cdot \frac{c\sqrt{2t}}{\sqrt{4c^2t+1}} e^{-\frac{x^2}{1+4c^2t}} \\ &= \frac{1}{\sqrt{4c^2t+1}} e^{-\frac{x^2}{1+4c^2t}}, \end{aligned}$$

and hence

$$u_2(x,\, t) = \frac{1}{\sqrt{4c^2t+1}} e^{-\frac{(x-1)^2}{1+4c^2t}}.$$

27. We proceed as in Exercise 25. Let $u_2(x,\, t)$ denote the solution of the heat problem with initial temperature distribution $f(x) = e^{-(x-2)^2/2}$. Let $u(x,\, t)$ denote the solution of the problem with initial distribution $e^{-x^2/2}$. Then, by Exercise 23, $u_2(x,\, t) = u(x-2,\, t)$

By (4), we have

$$u(x,\, t) = \frac{1}{c\sqrt{2t}} e^{-x^2/(4c^2t)} * e^{-x^2/2}.$$

We will apply Exercise 24 with $a = \frac{1}{4c^2t}$ and $b = 1/2$. We have

$$\begin{aligned} \frac{ab}{a+b} &= \frac{1}{8c^2t} \times \frac{1}{\frac{1}{4c^2t}+\frac{1}{2}} \\ &= \frac{1}{2+4c^2t} \\ \frac{1}{\sqrt{2(a+b)}} &= \frac{1}{\sqrt{2(\frac{1}{4c^2t}+\frac{1}{2})}} \\ &= \frac{2c\sqrt{t}}{\sqrt{4c^2t+2}}. \end{aligned}$$

So

$$\begin{aligned} u(x,\, t) &= \frac{1}{c\sqrt{2t}} e^{-x^2/(4c^2t)} * e^{-x^2/2} \\ &= \frac{1}{c\sqrt{2t}} \cdot \frac{2c\sqrt{t}}{\sqrt{4c^2t+2}} e^{-\frac{x^2}{2+4c^2t}} \\ &= \frac{1}{\sqrt{2c^2t+1}} e^{-\frac{x^2}{2+4c^2t}}, \end{aligned}$$

and hence

$$u_2(x,\,t) = \frac{1}{\sqrt{2c^2t+1}} e^{-\frac{(x-2)^2}{2+4c^2t}}.$$

29. Parts (a)-(c) are obvious from the definition of $g_t(x)$.
(d) The total area under the graph of $g_t(x)$ and above the x-axis is

$$\begin{aligned}\int_{-\infty}^{\infty} g_t(x)\,dx &= \frac{1}{c\sqrt{2t}} \int_{-\infty}^{\infty} e^{-x^2/(4c^2t)}\,dx \\ &= \frac{2c\sqrt{t}}{c\sqrt{2t}} \int_{-\infty}^{\infty} e^{-z^2}\,dz \quad (z = \frac{x}{2c\sqrt{t}},\ dx = 2c\sqrt{t}\,dz)\end{aligned}$$

$$\sqrt{2}\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{2\pi},$$

by (4), Sec. 7.2.
(e) To find the Fourier transform of $g_t(x)$, apply (5), Sec. 7.2, with

$$a = \frac{1}{4c^2t}, \quad \frac{1}{\sqrt{2a}} = 2c\sqrt{2t}, \quad \frac{1}{4a} = c^2t.$$

We get

$$\begin{aligned}\widehat{g}_t(w) &= \frac{1}{c\sqrt{2t}} \mathcal{F}\left(e^{-x^2/(4c^2t)}\right)\,dx \\ &= \frac{1}{c\sqrt{2t}} \times 2c\sqrt{2t}e^{-c^2t\omega^2} \\ &= e^{-c^2t\omega^2}.\end{aligned}$$

(f) If f is an integrable and piecewise smooth function, then at its points of continuity, we have

$$\lim_{t\to 0} g_t * f(x) = f(x).$$

This is a true fact that can be proved by using properties of Gauss's kernel. If we interpret $f(x)$ as an initial temperature distribution in a heat problem, then the solution of this heat problem is given by

$$u(x,\,t) = g_t * f(x).$$

If $t \to 0$, the temperature $u(x,\,t)$ should approach the initial temperature distribution $f(x)$. Thus $\lim_{t\to 0} g_t * f(x) = f(x)$.

Alternatively, we can use part (e) and argue as follows. Since

$$\lim_{t\to 0} \mathcal{F}\left(g_t\right)(\omega) = \lim_{t\to 0} e^{-c^2t\omega^2} = 1,$$

So

$$\lim_{t\to 0} \mathcal{F}\left(g_t * f\right) = \lim_{t\to 0} \mathcal{F}\left(g_t\right)\mathcal{F}\left(f\right) = \mathcal{F}\left(f\right).$$

You would expect that the limit of the Fourier transform be the transform of the limit function. So taking inverse Fourier transforms, we get $\lim_{t\to 0} g_t * f(x) = f(x)$. (Neither one of the arguments that we gave is rigorous.)

A generalization of Exercise 5 Suppose that you want to solve the heat equation $u_t = u_{xx}$ subject to the initial condition $u(x,\,0) = x^n$ where n is a nonnegative integer. We have already done the case $n = 0$ (in Exercise 19) and $n = 2$ (in Exercise 5). For the general case, proceed as in Exercise 5 and apply (4) with $f(s) = s^n$:

$$\begin{aligned}u(x,\,t) &= f * g_t(x) \\ &= \frac{1}{\sqrt{2t}}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^n e^{-\frac{(x-s)^2}{t}}\,ds.\end{aligned}$$

Use the formula

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s^n f(s)\,ds = (i)^n \left[\frac{d^n}{dw^n}\mathcal{F}(f)(w)\right]_{w=0},$$

with

$$f(s) = \frac{1}{\sqrt{2t}}e^{-\frac{(x-s)^2}{t}}$$

and

$$\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(x-s)^2}{t}}\right)(w) = e^{-(iwx+w^2t)}$$

(see the solution of Exercise 5). So

$$u(x,\,t) = (i)^n \left[\frac{d^n}{dw^n}e^{-(iwx+w^2t)}\right]_{w=0}.$$

To compute this last derivative, recall the Taylor series formula

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

So knowledge of the Taylor series gives immediately the values of the derivatives at a. Since

$$e^{aw} = \sum_{n=0}^{\infty}\frac{(aw)^n}{n!},$$

we get

$$\left[\frac{d^j}{dw^j}e^{aw}\right]_{w=0} = a^j.$$

Similarly,

$$\left[\frac{d^k}{dw^k}e^{bw^2}\right]_{w=0} = \begin{cases} 0 & \text{if } k \text{ is odd,}\\ \frac{b^j(2j)!}{j!} & \text{if } k = 2j.\end{cases}$$

Returning to $u(x,\,t)$, we compute the nth derivative of $e^{-(iwx+w^2t)}$ using the Leibniz rule and use the what we just found and get

$$\begin{aligned} u(x,\,t) &= (i)^n \left[\frac{d^n}{dw^n}e^{-w^2t}e^{-iwx}\right]_{w=0}\\ &= (i)^n \sum_{j=0}^{n}\binom{n}{j}\frac{d^j}{dw^j}e^{-w^2t}\cdot\frac{d^{n-j}}{dw^{n-j}}e^{-iwx}\Big|_{w=0}\\ &= (i)^n \sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}\frac{d^{2j}}{dw^{2j}}e^{-w^2t}\cdot\frac{d^{n-2j}}{dw^{n-2j}}e^{-iwx}\Big|_{w=0}\\ &= (i)^n \sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}\frac{(-t)^j(2j)!}{j!}\cdot(-ix)^{n-2j}\\ &= \sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}\frac{t^j(2j)!}{j!}\cdot x^{n-2j}. \end{aligned}$$

For example, if $n = 2$,

$$u(x,\,t) = \sum_{j=0}^{1}\binom{2}{2j}\frac{t^j(2j)!}{j!}\cdot x^{2-2j} = x^2 + 2t,$$

which agrees with the result of Exercise 5. If $n = 3$,

$$u(x, t) = \sum_{j=0}^{1} \binom{3}{2j} \frac{t^j (2j)!}{j!} \cdot x^{3-2j} = x^3 + 6tx.$$

You can easily check that this solution verifies the heat equation and $u(x, 0) = x^3$. If $n = 4$,

$$u(x, t) = \sum_{j=0}^{2} \binom{4}{2j} \frac{t^j (2j)!}{j!} \cdot x^{4-2j} = x^4 + 12tx^2 + 12t^2.$$

Here too, you can check that this solution verifies the heat equation and $u(x, 0) = x^4$.

We now derive a recurrence relation that relates the solutions corresponding to $n-1$, n, and $n+1$. Let $u_n = u_n(x, t)$ denote the solution with initial temperature distribution $u_n(x, 0) = x^n$. We have the following recurrence relation

$$u_{n+1} = xu_n + 2ntu_{n-1}.$$

The proof of this formula is very much like the proof of Bonnet's recurrence formula for the Legendre polynomials (Section 5.6). Before we give the proof, let us verify the formula with $n = 3$. The formula states that $u_4 = 4u_3 + 6tu_2$. Sine $u_4 = x^4 + 12tx^2 + 12t^2$, $u_3 = x^3 + 6tx$, and $u_2 = x^2 + 2t$, we see that the formula is true for $n = 3$. We now prove the formula using Leibniz rule of differentiation. As in Section 5.6, let us use the symbol D^n to denote the nth derivative, We have

$$\begin{aligned}
u_{n+1}(x, t) &= (i)^{n+1} \left[\frac{d^{n+1}}{dw^{n+1}} e^{-(iwx+w^2t)} \right]_{w=0} \\
&= (i)^{n+1} \left[D^{n+1} e^{-(iwx+w^2t)} \right]_{w=0} \\
&= (i)^{n+1} \left[D^n \left(D e^{-(iwx+w^2t)} \right) \right]_{w=0} \\
&= (i)^{n+1} \left[D^n \left(-(ix+2wt) e^{-(iwx+w^2t)} \right) \right]_{w=0} \\
&= (i)^{n+1} \left[D^n \left(-(ix+2wt) e^{-(iwx+w^2t)} \right) \right]_{w=0} \\
&= (i)^{n+1} \left[-D^n \left(e^{-(iwx+w^2t)} \right) (ix+2wt) - 2ntD^{n-1} \left(e^{-(iwx+w^2t)} \right) \right]_{w=0} \\
&= x(i)^n \, D^n \left(e^{-(iwx+w^2t)} \right) \Big|_{w=0} + 2nt \, (i)^{n-1} D^{n-1} \left(e^{-(iwx+w^2t)} \right) \Big|_{w=0} \\
&= xu_n + 2ntu_{n-1}.
\end{aligned}$$

Solutions to Exercises 7.5

1. To solve the Dirichlet problem in the upper half-plane with the given boundary function, we use formula (5). The solution is given by

$$\begin{aligned} u(x, y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\,ds \\ &= \frac{50y}{\pi}\int_{-1}^{1} \frac{ds}{(x-s)^2+y^2} \\ &= \frac{50}{\pi}\left\{\tan^{-1}\left(\frac{1+x}{y}\right)+\tan^{-1}\left(\frac{1-x}{y}\right)\right\}, \end{aligned}$$

where we have used Example 1 to evaluate the definite integral.

3. We appeal to the Poisson integral formula (5). In evaluating the integral, we will come across the improper integral

$$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{s^2+y^2}\,ds,$$

which is a nontrivial integral. In order to evaluate it, we will appeal to results about Fourier transforms. Fix $y > 0$ and consider

$$h(s) = \frac{y\sqrt{2}}{\sqrt{\pi}(s^2+y^2)} \qquad (-\infty < s < \infty).$$

The Fourier transform of $h(s)$ is

$$\begin{aligned} \widehat{h}(w) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{e^{-isw}}{s^2+y^2}\,ds \\ &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos sw - i\sin sw}{s^2+y^2}\,ds \\ &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos sw}{s^2+y^2}\,ds, \end{aligned}$$

because the sine integral has an odd integrand, so its value is 0. Let us now use a table of Fourier transforms to evaluate the Fourier transform of h (see also Exercise 12, Section 7.2). From entry 7, form the table of Fourier transforms, Appendix B.1, we have

$$\widehat{h}(w) = e^{-y|w|} \qquad (y > 0,\ -\infty < w < \infty).$$

So

$$\text{(1)} \qquad \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos sw}{s^2+y^2}\,ds = e^{-y|w|} \qquad (y > 0,\ -\infty < w < \infty).$$

We are now ready to evaluate the Poisson integral for the given boundary function

$f(x)$:

$$\begin{aligned}
u(x, y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\,ds \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\,ds \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(x-s)}{s^2+y^2}\,ds \quad \text{(Change variables } s = x - S.\text{)} \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos(x-s)}{s^2+y^2}\,ds \\
&= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos x \cos s + \sin x \sin s}{s^2+y^2}\,ds \\
&= \frac{y\,\cos x}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{s^2+y^2}\,ds + \frac{y\,\sin x}{\pi}\overbrace{\int_{-\infty}^{\infty} \frac{\sin s}{s^2+y^2}\,ds}^{=0} \\
&\quad \text{(The integrand is odd in the second integral.)} \\
&= \cos x\,\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos s}{s^2+y^2}\,ds \\
&= \cos x\,e^{-y},
\end{aligned}$$

where we have used (1) with $w = 1$. It is instructive t check that we have indeed found the (bounded) solution to our Dirichlet problem: $u(x, y) = \cos x\,e^{-y}$. We have $u_{xx} = -cosx\,e^{-y}$ and $u_{yy} = cosxe^{-y}$. So $u_{xx} + u_{yy} = 0$. This shows that u is a solution of Laplace's equation. Does it satisfy the boundary condition? Let's see: $u(x\,0) = \cos x = f(x)$. Yes, it does. Can you guess the solution of the Dirichlet problem with boundary condition $f(x) = \sin x$?

5. Appealing to (4) in Section 7.5, with $y = y_1,\ y_2, y_1 + y_2$, we find

$$\mathcal{F}(P_{y_1})(w) = e^{-y_1|w|}, \quad \mathcal{F}(P_{y_2})(w) = e^{-y_2|w|}, \quad \mathcal{F}(P_{y_1+y_2})(w) = e^{-(y_1+y_2)|w|}.$$

Hence

$$\mathcal{F}(P_{y_1})(w)\cdot\mathcal{F}(P_{y_2})(w) = e^{-y_1|w|}e^{-y_2|w|} = e^{-(y_1+y_2)|w|} = \mathcal{F}(P_{y_1+y_2})(w).$$

But

$$\mathcal{F}(P_{y_1})(w)\cdot\mathcal{F}(P_{y_2})(w) = \mathcal{F}(P_{y_1} * P_{y_2})(w),$$

Hence

$$\mathcal{F}(P_{y_1+y_2})(w) = \mathcal{F}(P_{y_1} * P_{y_2})(w);$$

and so $P_{y_1+y_2} = P_{y_1} * P_{y_2}$.

7. (a) If $f(x) = \frac{1}{4+x^2}$, then in terms of the Poisson kernel, we have from (3), Sec. 7.5,

$$f(x) = \frac{1}{2}\sqrt{\frac{\pi}{2}}P_2(x).$$

The solution of the Dirichlet problem with boundary value $f(x)$ is then

$$u(x, y) = P_y * f(x) = \frac{1}{2}\sqrt{\frac{\pi}{2}}P_y * P_2(x) = \frac{1}{2}\sqrt{\frac{\pi}{2}}P_{2+y}(x),$$

by Exercise 5. More explicitely, using (3), Sec. 7.5, with $y + 2$ in place of y, we obtain

$$u(x, y) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{\pi}}\frac{y+2}{x^2+(y+2)^2} = \frac{1}{2}\frac{y+2}{x^2+(y+2)^2}$$

(b) The boundary function, $\frac{1}{4+x^2}$, takes values between 0 and 1/4. For $0 < T \leq \frac{1}{4}$, the points in the upper half-plane with $u(x, y) = T$ satisfy the equation

$$x^2 + (y+2)^2 - \tfrac{1}{2T}(y+2) = 0$$

$$x^2 + (y + 2 - \tfrac{1}{4T})^2 = \tfrac{1}{16T^2}.$$

These points lie on the circle with radius $\frac{1}{4|T|}$ and center at $(0, \frac{1}{4T} - 2)$.

9. Modify the solution of Example 1(a) to obtain that, in the present case, the solution is

$$u(x, y) = \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right].$$

To find the isotherms, we must determine the points (x, y) such that $u(x, y) = T$. As in the solution of Example 1(b), these points satisfy

$$x^2 + \left(y - a\cot(\frac{\pi T}{T_0})\right)^2 = \left(a\csc(\frac{\pi T}{T_0})\right)^2.$$

Hence the points belong to the arc in the upper half-plane of the circle with center $(0, a\cot(\frac{\pi T}{T_0}))$ and radius $a\csc(\frac{\pi T}{T_0})$. The isotherm corresponding to $T = \frac{T_0}{2}$ is the arc of the circle

$$x^2 + \left(y - a\cot(\frac{\pi}{2})\right)^2 = \left(a\csc(\frac{\pi}{2})\right)^2,$$

or

$$x^2 + y^2 = a^2.$$

Thus the isotherm in this case is the upper semi-circle of radius a and center at the origin.

13. Parts (a)-(c) are clear. Part (e) fellows from a table. For (d), you can use (e) and the fact that the total area under the graph of $P_y(x)$ and above the x-axis is $\sqrt{2\pi}\widehat{P_y}(0) = \sqrt{2\pi}e^{-0} = \sqrt{2\pi}$.
(f) If f is an integrable function and piecewise smooth, consider the Dirichlet problem with boundary values $f(x)$. Then we know that the solution is $u(x, y) = P_y * f(x)$. In particular, the solution tends to the boundary function as $y \to 0$. But this means that $\lim_{y\to 0} P_y * f(x) = f(x)$.

The proof of this fact is beyond the level of the text. Another way to justify the convergence is to take Fourier transforms. We have

$$\mathcal{F}(P_y * f)(w) = \mathcal{F}(P_y)(w)\cdot\mathcal{F}(f)(w) = e^{-y|w|}\mathcal{F}(f)(w).$$

Since $\lim_{y\to 0} e^{-y|w|} = 1$, it follows that

$$\lim_{y\to 0}\mathcal{F}(P_y * f)(w) = \lim_{y\to 0} e^{-y|w|}\mathcal{F}(f)(w) = \mathcal{F}(f)(w).$$

Taking inverse Fourier transforms, we see that $\lim_{y\to 0} P_y * f(x) = f(x)$.

The argument that we gave is not rigorous, since we did not justify that the inverse Fourier transform of a limit of functions is the limit of the inverse Fourier transforms.

Solutions to Exercises 7.6

1. The even extension of $f(x)$ is

$$f_e(x) = \begin{cases} 1 & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier transform of $f_e(x)$ is computed in Example 1, Sec. 7.2 (with $a = 1$). We have, for $w \geq 0$,

$$\mathcal{F}_c(f)(w) = \mathcal{F}(f_e)(w) = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}.$$

To write f as an inverse Fourier cosine transform, we appeal to (6). We have, for $x > 0$,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)(w)\cos wx\,dw,$$

or

$$\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wx\,dw = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1, \\ \frac{1}{2} & \text{if } x = 1. \end{cases}$$

Note that at the point $x = 1$, a point of discontinuity of f, the inverse Fourier transform is equal to $(f(x+) + f(x-))/2$.

3. The even extension of $f(x)$ is $f_e(x) = 3\,e^{-2|x|}$. We compute the Fourier transform of $f_e(x)$ by using Exercises 5 and 17(a), Sec. 7.2 (with $a = 2$). We have

$$\mathcal{F}(f_e)(w) = 3\sqrt{\frac{2}{\pi}}\frac{2}{4+w^2}.$$

So, for $w \geq 0$,

$$\mathcal{F}_c(f)(w) = \mathcal{F}(f_e)(w) = \sqrt{\frac{2}{\pi}}\frac{6}{4+w^2}.$$

To write f as an inverse Fourier cosine transform, we appeal to (6). We have, for $x > 0$,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)(w)\cos wx\,dw,$$

or

$$3e^{-2x} = \frac{12}{\pi}\int_0^\infty \frac{\cos wx}{4+w^2}\,dw.$$

5. The even extension of $f(x)$ is

$$f_e(x) = \begin{cases} \cos x & \text{if } -2\pi < x < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Let's compute the Fourier cosine transform using definition (5), Sec. 7.6:

$$\begin{aligned}
\mathcal{F}_c(f)(w) &= = \sqrt{\frac{2}{\pi}}\int_0^{2\pi}\cos x\cos wx\,dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^{2\pi}\frac{1}{2}\left[\cos(w+1)x + \cos(w-1)x\right]dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(w+1)x}{w+1} + \frac{\sin(w-1)x}{w-1}\right]\Big|_0^{2\pi} \qquad (w \neq 1) \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin 2(w+1)\pi}{w+1} + \frac{\sin 2(w-1)\pi}{w-1}\right] \qquad (w \neq 1) \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin 2\pi w}{w+1} + \frac{\sin 2\pi w}{w-1}\right] \qquad (w \neq 1) \\
&= \sqrt{\frac{2}{\pi}}\sin 2\pi w\frac{w}{w^2-1} \qquad (w \neq 1).
\end{aligned}$$

Also, by l'Hospital's rule, we have

$$\lim_{w\to 0}\sqrt{\frac{2}{\pi}}\sin 2\pi w\frac{w}{w^2-1}=\sqrt{2\pi},$$

which is the value of the cosine transform at $w=1$.

To write f as an inverse Fourier cosine transform, we appeal to (6). We have, for $x>0$,

$$\frac{2}{\pi}\int_0^\infty \frac{w}{w^2-1}\sin 2\pi w\,\cos wx\,dw=\begin{cases}\cos x & \text{if } 0<x<2\pi,\\ 0 & \text{if } x>2\pi.\end{cases}$$

For $x=2\pi$, the integral converges to $1/2$. So

$$\frac{2}{\pi}\int_0^\infty \frac{w}{w^2-1}\sin 2\pi w\,\cos 2\pi w\,dw=\frac{1}{2}.$$

7. The odd extension of f is

$$f_o(x)=\begin{cases}1 & \text{if } 0<x<1,\\ -1 & \text{if } -1<x<0\\ 0 & \text{otherwise.}\end{cases}$$

To avoid computing the Fourier transform of f_o from scratch, let us introduce the function

$$g(x)=\begin{cases}1 & \text{if } -\frac{1}{2}<x<\frac{1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

From Example 1, Sec. 7.2, we have

$$\mathcal{F}(g)(w)=\sqrt{\frac{2}{\pi}}\frac{\sin\frac{w}{2}}{w}.$$

Note that $f_o(x)=g(x-\frac{1}{2})+g(x-\frac{1}{2})$. So

$$\begin{aligned}\mathcal{F}(f_o)(w) &= \mathcal{F}(g(x-\frac{1}{2}))(w)+\mathcal{F}(g(x-\frac{1}{2}))(w)\\ &= e^{-\frac{1}{2}iw}\mathcal{F}(g)(w)+e^{\frac{1}{2}iw}\mathcal{F}(g)(w) \qquad \text{(Exercise 20, Sec. 7.2)}\\ &= \mathcal{F}(g)(w)\overbrace{\left(e^{-\frac{1}{2}iw}++e^{\frac{1}{2}iw}\right)}^{-2i\sin\frac{w}{2}}\\ &= -2i\sqrt{\frac{2}{\pi}}\frac{\sin^2 w2}{w}.\end{aligned}$$

Applying (10), Sec. 7.6, we find

$$\mathcal{F}_s(f)(w)=i\mathcal{F}(f_o)(w)=2\sqrt{\frac{2}{\pi}}\frac{\sin^2 w2}{w}=\sqrt{\frac{2}{\pi}}\frac{1-cosw}{w}.$$

The inverse sine transform becomes

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{1-cosw}{w}\sin wx\,dw.$$

9. Applying the definition of the transform and using Exercise 17, Sec. 2.6 to evaluate the integral,

$$\begin{aligned}\mathcal{F}_s(e^{-2x})(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\sin wx\,dx\\ &= \sqrt{\frac{2}{\pi}}\frac{e^{-2x}}{4+w^2}\left[-w\cos wx-2\sin wx\right]\Big|_{x=0}^{\infty}\\ &= \sqrt{\frac{2}{\pi}}\frac{w}{4+w^2}.\end{aligned}$$

The inverse sine transform becomes

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{w}{4+w^2} \sin wx \, dw.$$

11. The odd extension is

$$f_o(x) = \begin{cases} \sin 2x & \text{if } -\pi < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

To evaluate the Fourier transform of f_o, write $f_o(x) = \sin 2x \cdot g(x)$, where

$$g(x) = \begin{cases} 1 & \text{if } -\pi < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Exercise 20(b), Sec. 7.2, we have

$$\mathcal{F}(f_o) = \mathcal{F}(\sin 2x g(x)) = \frac{1}{2i} \left(\mathcal{F}(g)(w-2) - \mathcal{F}(g)(w+2) \right).$$

By Example 1, Sec. 7.2,

$$\mathcal{F}(g)(w) = \sqrt{\frac{2}{\pi}} \frac{\sin \pi w}{w}.$$

So

$$\begin{aligned} \mathcal{F}(f_o) &= \sqrt{\frac{2}{\pi}} \frac{1}{2i} \left(\frac{\sin \pi(w-2)}{w-2} - \frac{\sin \pi(w+2)}{w+2} \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{2i} \left(\frac{\sin \pi w}{w-2} - \frac{\sin \pi w}{w+2} \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{2}{i} \frac{\sin \pi w}{w^2-4}. \end{aligned}$$

Applying (10), Sec. 7.6, we find

$$\mathcal{F}_s(f)(w) = i\mathcal{F}(f_o)(w) = 2\sqrt{\frac{2}{\pi}} \frac{\sin \pi w}{w^2-4}.$$

The inverse sine transform becomes

$$f(x) = \frac{4}{\pi} \int_0^\infty \frac{\sin \pi w}{w^2-4} \sin wx \, dw.$$

13. We have $f_e(x) = \frac{1}{1+x^2}$. So

$$\mathcal{F}_c\left(\frac{1}{1+x^2} \right) = \mathcal{F}\left(\frac{1}{1+x^2} \right) = \sqrt{\frac{\pi}{2}} e^{-w} \quad (w > 0),$$

by Exercise 11, Sec. 7.2.

15. We have $f_o(x) = \frac{x}{1+x^2}$. So

$$\mathcal{F}_s\left(\frac{1}{1+x^2} \right) = i\mathcal{F}\left(\frac{x}{1+x^2} \right) = \sqrt{\frac{\pi}{2}} e^{-w} \quad (w > 0),$$

by Exercise 35, Sec. 7.2.

17. We have $f_e(x) = \frac{\cos x}{1+x^2}$. So

$$\mathcal{F}_c\left(\frac{\cos x}{1+x^2} \right) = \mathcal{F}\left(\frac{\cos x}{1+x^2} \right) = \sqrt{\frac{\pi}{2}} \left(e^{-|w-1|} + e^{-(w+1)} \right) \quad (w > 0),$$

by Exercises 11 and 20(b), Sec. 7.2.

19. Let f_e denote the even extension of f so that $f_e(x) = f(x)$ for all $x > 0$. Then, for $w > 0$,

$$\begin{aligned}
\mathcal{F}_c(f)(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty f_e(x) \cos wx \, dx \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty f_e(x) \cos wx \, dx \\
&\quad \text{(The integrand is even.)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f_e(x)(\cos wx - i \sin wx) \, dx \\
&\quad \text{(The integral of } f_e(x) \sin wx \text{ is 0 because the integrand is odd.)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f_e(x) e^{-iwx} \, dx \\
&= \mathcal{F}(f_e)(w).
\end{aligned}$$

We proceed in a similar way when dealing with the sine transform. Let f_o denote the odd extension of f. The $f(x) = f_o(x)$ for $x > 0$ and

$$\begin{aligned}
\mathcal{F}_s(f)(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty f_o(x) \sin wx \, dx \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^\infty f_o(x) \sin wx \, dx \\
&\quad \text{(The integrand is even.)} \\
&= i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f_o(x)(\cos wx - i \sin wx) \, dx \\
&\quad \text{(The integral of } f_o(x) \cos wx \text{ is 0 because the integrand is odd.)} \\
&= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^\infty f_o(x) e^{-iwx} \, dx \\
&= i\mathcal{F}(f_o)(w).
\end{aligned}$$

21. From the definition of the inverse transform, we have $\mathcal{F}_c f = \mathcal{F}_c^{-1} f$. So $\mathcal{F}_c \mathcal{F}_c f = \mathcal{F}_c \mathcal{F}_c^{-1} f = f$. Similarly, $\mathcal{F}_s \mathcal{F}_s f = \mathcal{F}_s \mathcal{F}_s^{-1} f = f$.

Solutions to Exercises 7.7

1. Fourier sine transform with respect to x:

$$\frac{d}{dt}\widehat{u}_s(w,\,t) = -w^2\widehat{u}_s(w,\,t) + \sqrt{\tfrac{2}{\pi}}w\,\overbrace{u(0,\,t)}^{=0}$$

$$\frac{d}{dt}\widehat{u}_s(w,\,t) = -w^2\widehat{u}_s(w,\,t).$$

Solve the first-order differential equation in $\widehat{u}_s(w,\,t)$ and get

$$\widehat{u}_s(w,\,t) = A(w)e^{-w^2t}.$$

Fourier sine transform the initial condition

$$\widehat{u}_s(w,\,0) = A(w) = \mathcal{F}_s(f(x))(w) = T_0\sqrt{\frac{2}{\pi}}\frac{1-\cos bw}{w}.$$

Hence

$$\widehat{u}_s(w,\,t) = \sqrt{\frac{2}{\pi}}\frac{1-\cos bw}{w}e^{-w^2t}.$$

Taking inverse Fourier sine transform:

$$u(x,\,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos bw}{w}e^{-w^2t}\sin wx\,dw.$$

3. As in Exercise 1, Fourier sine transform with respect to x:

$$\frac{d}{dt}\widehat{u}_s(w,\,t) = -w^2\widehat{u}_s(w,\,t) + \sqrt{\tfrac{2}{\pi}}w\,\overbrace{u(0,\,t)}^{=0}$$

$$\frac{d}{dt}\widehat{u}_s(w,\,t) = -w^2\widehat{u}_s(w,\,t).$$

Solve the first-order differential equation in $\widehat{u}_s(w,\,t)$ and get

$$\widehat{u}_s(w,\,t) = A(w)e^{-w^2t}.$$

Fourier sine transform the initial condition

$$\widehat{u}_s(w,\,0) = A(w) = \mathcal{F}_s(f(x))(w) = \sqrt{\frac{2}{\pi}}e^{-w}.$$

Hence

$$\widehat{u}_s(w,\,t) = \sqrt{\frac{2}{\pi}}e^{-w}e^{-w^2t} = \sqrt{\frac{2}{\pi}}e^{-(w^2t+w)}.$$

Taking inverse Fourier sine transform:

$$u(x,\,t) = \frac{2}{\pi}\int_0^\infty e^{-(w^2t+w)}\sin wx\,dw.$$

5. If you Fourier cosine the equations (1) and (2), using the Neumann type condition

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

you will get

$$\frac{d}{dt}\widehat{u}_c(w,\,t) = c^2\Big[-w^2\widehat{u}_c(w,\,t) - \sqrt{\tfrac{2}{\pi}}\,\overbrace{\frac{d}{dx}u(0,\,t)}^{=0}\Big]$$

$$\frac{d}{dt}\widehat{u}_c(w,\,t) = -c^2w^2\widehat{u}_c(w,\,t).$$

Solve the first-order differential equation in $\widehat{u}_c(w,\,t)$ and get

$$\widehat{u}_c(w,\,t) = A(w)e^{-c^2w^2t}.$$

Fourier cosine transform the initial condition

$$\widehat{u}_c(w,\,0) = A(w) = \mathcal{F}_c(f)(w).$$

Hence

$$\widehat{u}_s(w,\,t) = \mathcal{F}_c(f)(w)e^{-c^2w^2t}.$$

Taking inverse Fourier cosine transform:

$$u(x,\,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)(w)e^{-c^2w^2t}\cos wx\,dw.$$

7. Solution (4) of Example 1 reads

$$u(x,\,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \widehat{f}_s(w)e^{-c^2w^2t}\sin wt\,dw,$$

where

$$\widehat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_{s=0}^\infty f(s)\sin ws\,ds.$$

So

$$\begin{aligned}
u(x,\,t) &= \frac{2}{\pi}\int_{w=0}^\infty\int_{s=0}^\infty f(s)\sin ws\,ds e^{-c^2w^2t}\sin wt\,dw\\
&= \frac{2}{\pi}\int_{s=0}^\infty f(s)\int_{w=0}^\infty \sin ws\,\sin wt\,e^{-c^2w^2t}dwds\\
&\qquad \text{Use } \sin A\sin B = \frac{1}{2}\big[\cos(A-B)-\cos(A+B)\big].)\\
&= \frac{1}{\sqrt{2\pi}}\int_{s=0}^\infty f(s)\sqrt{\frac{2}{\pi}}\int_{w=0}^\infty \big[\cos(w(x-s)) - \cos w(x+s)\big]e^{-c^2w^2t}dwds.
\end{aligned}$$

Evaluate the inner integral by appealing to a result of Example 4, Sec. 7.2, which we recall in the following convenient notation:

$$\sqrt{\frac{2}{\pi}}\int_{w=0}^\infty e^{-aw^2}\cos xw\,dw = \frac{1}{\sqrt{2a}}e^{-\frac{x^2}{4a}}.$$

Hence

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_{w=0}^\infty \big[\cos(w(x-s)) - \cos w(x+s)\big]e^{-c^2w^2t}dw\\
&= \sqrt{\frac{2}{\pi}}\int_{w=0}^\infty \cos(w(x-s))e^{-c^2w^2t}dw - \sqrt{\frac{2}{\pi}}\int_{w=0}^\infty \cos w(x+s)e^{-c^2w^2t}dw\\
&= \frac{1}{\sqrt{2}\,c\sqrt{t}}\Big[e^{-\frac{(x-s)^2}{4c^2t}} - e^{-\frac{(x+s)^2}{4c^2t}}\Big];
\end{aligned}$$

and so

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_0^\infty f(s)\left[e^{-\frac{(x-s)^2}{4c^2t}} - e^{-\frac{(x+s)^2}{4c^2t}}\right]ds\,.$$

9. (a) Taking the sine transform of the heat equation (1) and using $u(0,t) = T_0$ for $t > 0$, we get

$$\frac{d}{dt}\,\widehat{u}_s(w,t) = c^2\Big[-w^2\widehat{u}_s(w,t) + \sqrt{\frac{2}{\pi}}wu(0,\,t)\Big];$$

or

$$\frac{d}{dt}\,\widehat{u}_s(w,t)+c^2\omega^2\widehat{u}_s(w,t)=c^2\sqrt{\frac{2}{\pi}}wT_0.$$

Taking the Fourier sine transform of the boundary condition $u(x,0)=0$ for $x>0$, we get $\widehat{u}_s(w,0)=0$.
(b) A particular solution of the differential equation can be guessed easily: $\widehat{u}_s(w,\,t)=\sqrt{\frac{2}{\pi}}\frac{T_0}{w}$. The general solution of the homogeneous differential equation:

$$\frac{d}{dt}\,\widehat{u}_s(w,t)+c^2\omega^2\widehat{u}_s(w,t)=0$$

is $\widehat{u}_s(w,\,t)=A(w)e^{-c^2w^2t}$. So the general solution of the nonhomogeneous differential equation is

$$\widehat{u}_s(w,\,t)=A(w)e^{-c^2w^2t}\sqrt{\frac{2}{\pi}}\frac{T_0}{w}.$$

Using $\widehat{u}_s(w,\,0)=A(w)\sqrt{\frac{2}{\pi}}\frac{T_0}{w}=0$, we find $A(w)=-\sqrt{\frac{2}{\pi}}\frac{T_0}{w}$. So

$$\widehat{u}_s(w\,t)=\sqrt{\frac{2}{\pi}}\frac{T_0}{w}-\sqrt{\frac{2}{\pi}}\frac{T_0}{w}e^{-c^2w^2t}.$$

Taking inverse sine transforms, we find

$$\begin{aligned}
u(x,t) &= \frac{2}{\pi}\int_0^\infty\left(\frac{T_0}{w}-\frac{T_0}{w}e^{-c^2w^2t}\right)\sin wx\,dw\\
&= T_0\overbrace{\frac{2}{\pi}\int_0^\infty\frac{\sin wx}{w}\,dw}^{=\operatorname{sgn}(x)=1}-\frac{2T_0}{\pi}\int_0^\infty\frac{\sin wx}{w}e^{-c^2w^2t}\,dw\\
&= T_0-\frac{2T_0}{\pi}\int_0^\infty\frac{\sin wx}{w}e^{-c^2w^2t}\,dw
\end{aligned}$$

11. We will solve the Dirichlet problem

$$u_{xx}+u_{yy}=0\qquad(0<x<1,\ 0<y)$$

$$u(0,\,y)=0,\ \tfrac{\partial u}{\partial y}(x,\,0)=0,\ u(1,\,y)=e^{-y}.$$

Cosine transform with respect to y:

$$\tfrac{d^2}{dx^2}\widehat{u}_c(x,\,w)-w^2\widehat{u}_c(x,\,w)-\sqrt{\tfrac{2}{\pi}}\overbrace{\frac{\partial u}{\partial y}(x,\,0)}^{=0}=0$$

$$\tfrac{d^2}{dx^2}\widehat{u}_c(x,\,w)=w^2\widehat{u}_c(x,\,w).$$

The general solution is

$$\widehat{u}_c(x,\,w)=A(w)\cosh wx+B(w)\sinh wx.$$

Using

$$\widehat{u}_c(0,\,w)=0\quad\text{and}\quad\widehat{u}_c(1,\,w)=\mathcal{F}_c(e^{-y})=\sqrt{\frac{2}{\pi}}\frac{1}{1+w^2},$$

we get

$$A(w)=0\quad\text{and}\quad B(w)=\sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}\cdot\frac{1}{\sinh w}.$$

Hence

$$\widehat{u}_c(x,\,w)=\sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}\frac{\sinh wx}{\sinh w}.$$

Taking inverse cosine transforms:

$$u(x,\, y) = \frac{2}{\pi} \int_0^\infty \frac{1}{1+w^2} \frac{\sinh wx}{\sinh w} \cos wy \, dw.$$

13. Proceed as in Exercise 11 using the Fourier sine transform instead of the cosine transform and the condition $u(x\,0) = 0$ instead of $u_y(x,\, 0) = 0$. This yields

$$\tfrac{d^2}{dx^2}\widehat{u}_s(x,\, w) - w^2 \widehat{u}_s(x,\, w) + \sqrt{\tfrac{2}{\pi}}\, \overbrace{u(x,\, 0)}^{=0} = 0$$

$$\tfrac{d^2}{dx^2}\widehat{u}_s(x,\, w) = w^2 \widehat{u}_s(x,\, w).$$

The general solution is

$$\widehat{u}_s(x,\, w) = A(w) \cosh wx + B(w) \sinh wx.$$

Using

$$\widehat{u}_s(0,\, w) = 0 \quad \text{and} \quad \widehat{u}_s(1,\, w) = \mathcal{F}_s(e^{-y}) = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2},$$

we get

$$A(w) = 0 \quad \text{and} \quad B(w) = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2} \cdot \frac{1}{\sinh w}.$$

Hence

$$\widehat{u}_s(x,\, w) = \sqrt{\frac{2}{\pi}} \frac{w}{1+w^2} \frac{\sinh wx}{\sinh w}.$$

Taking inverse sine transforms:

$$u(x,\, y) = \frac{2}{\pi} \int_0^\infty \frac{w}{1+w^2} \frac{\sinh wx}{\sinh w} \sin wy \, dw.$$

15. We will solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \qquad (0 < x,\; 0 < y)$$

$$u(x,\, 0) = 0, \quad u(0,\, y) = f(y),$$

where

$$f(y) = \begin{cases} 1 & \text{if } 0 < y < 1, \\ 0 & \text{if } 1 < y. \end{cases}$$

Sine transform with respect to y:

$$\tfrac{d^2}{dx^2}\widehat{u}_s(x,\, w) - w^2 \widehat{u}_s(x,\, w) + \sqrt{\tfrac{2}{\pi}}\, w\, \overbrace{u(x,\, 0)}^{=0} = 0$$

$$\tfrac{d^2}{dx^2}\widehat{u}_s(x,\, w) = w^2 \widehat{u}_s(x,\, w).$$

The general solution is

$$\widehat{u}_s(x,\, w) = A(w) e^{-wx} + B(w) e^{wx}.$$

Since we expect a Fourier transform to be bounded and since w and x are > 0, we discard the term in e^{wx} and take

$$\widehat{u}_s(x,\, w) = A(w) e^{-wx}.$$

Transforming the boundary condition:

$$\widehat{u}_s(0,\, w) = A(w) = \mathcal{F}_s(f(y))(w) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos w}{w}.$$

Hence

$$\widehat{u}_s(x,\, w) = \sqrt{\frac{2}{\pi}}\frac{1-\cos w}{w}e^{-wx}.$$

Taking inverse sine transforms:

$$u(x,\, y) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos w}{w}e^{-wx}\sin wy\, dw.$$

Solutions to Exercises 7.8

1. As we move from left to right at a point x_0, if the graph jumps by c units, then we must add the scaled Dirac delta function by $c\delta_{x_0}(x)$. If the jump is upward, c is positive; and if the jump is downward, c is negative. With this in mind, by looking at the graph, we see that

$$f'(x) = \frac{1}{2}\,\delta_{-2}(x) + \frac{1}{2}\,\delta_{-1}(x) - \frac{1}{2}\,\delta_1(x) - \frac{1}{2}\,\delta_2(x) = \frac{1}{2}\left(\delta_{-2}(x) + \delta_{-1}(x) - \delta_1(x) - \delta_2(x)\right).$$

3. We reason as in Exercise 1; furthermore, here we have to add the nonzero part of the derivative. We have

$$f'(x) = \delta_{-1}(x) - 2\,\delta_2(x) + \begin{cases} 0 & \text{if } x < -2, \\ -1 & \text{if } -2 < x < -1, \\ 1 & \text{if } -1 < x < 1, \\ 0 & \text{if } x > 1, \end{cases}$$

Using unit step functions, we can write

$$f'(x) = -\big(\mathcal{U}_{-2}(x) - \mathcal{U}_{-1}(x)\big) + \big(\mathcal{U}_{-1}(x) - \mathcal{U}_1(x)\big) + \delta_{-1}(x) - 2\,\delta_2(x).$$

Exercises **5**, **7**, **9**, and **11** can be done by reasoning as in Exercises 1 and 3. See the Answers in the back of the text.

13. We do this problem by reversing the steps in the solutions of the previous exercises. Since $f(x)$ has zero derivative for $x < -2$ or $x > 3$, it is therefore constant on these intervals. But since $f(x)$ tends to zero as $x \to \pm\infty$, we conclude that $f(x) = 0$ for $x < -2$ or $x > 3$. At $x = -2$, we have a jump upward by one unit, then the function stays constant for $-2 < x < -1$. At $x = -1$, we have another jump upward by one unit, then the function stays constant for $-1 < x < -1$. At $x = 1$, we have another jump upward by one unit, then the function stays constant for $1 < x < 3$. At $x = 3$, we have a jump downward by three units, then the function stays constant for $x > 3$. Summing up, we have

$$f(x) = \begin{cases} 0 & \text{if } x < -2, \\ 1 & \text{if } -2 < x < -1, \\ 2 & \text{if } -1 < x < 1, \\ 3 & \text{if } 1 < x < 3, \\ 0 & \text{if } 3 < x. \end{cases}$$

15. We reason as in the previous exercise and find that $f(x) = 0$ for $x < -1$. At $x = -1$, we have a jump downward by one unit, then the function has derivative $f'(x) = 2x$ for $-1 < x < 1$, implying that the function is $f(x) = x^2 + c$ on this interval. Since the graph falls by 1 unit at $x = -1$, we see that $f(x) = x^2 - 2$ for $-1 < x < 1$. At $x = 1$, we have a jump upward by one unit, then the function stays constant and equals 0 for $x > 1$. Summing up, we have

$$f(x) = \begin{cases} 0 & \text{if } x < -1, \\ x^2 - 2 & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x. \end{cases}$$

17. We use the definition (7) of the derivative of a generalized function and the fact that the integral against a delta function δ_a picks up the value of the function at a. Thus

$$\begin{aligned}\langle \phi'(x),\, f(x)\rangle &= \langle \phi(x),\, -f'(x)\rangle = -\langle \phi(x),\, f'(x)\rangle \\ &= -\langle \delta_0(x) - \delta_1(x),\, f'(x)\rangle = -f'(0) + f'(1).\end{aligned}$$

19. From Exercise 7, we have $\phi'(x) = \frac{1}{a}\big(\mathcal{U}_{-2a}(x) - \mathcal{U}_{-a}(x)\big) - \frac{1}{a}\big(\mathcal{U}_{a}(x) - \mathcal{U}_{2a}(x)\big)$. Using the definition of the unit step function, we find

$$\begin{aligned}\langle \phi'(x),\, f(x)\rangle &= \langle \frac{1}{a}\big(\mathcal{U}_{-2a}(x) - \mathcal{U}_{-a}(x)\big) - \frac{1}{a}\big(\mathcal{U}_{a}(x) - \mathcal{U}_{2a}(x)\big),\, f(x)\rangle \\ &= \int_{-\infty}^{\infty}\Big[\frac{1}{a}\big(\mathcal{U}_{-2a}(x) - \mathcal{U}_{-a}(x)\big) - \frac{1}{a}\big(\mathcal{U}_{a}(x) - \mathcal{U}_{2a}(x)\big)\Big] f(x)\, dx \\ &= \frac{1}{a}\int_{-2a}^{-a} f(x)\, dx - \frac{1}{a}\int_{a}^{2a} f(x)\, dx.\end{aligned}$$

21. From Exercise 7, we have $\phi'(x) = \frac{1}{a}\big(\mathcal{U}_{-2a}(x) - \mathcal{U}_{-a}(x)\big) - \frac{1}{a}\big(\mathcal{U}_{a}(x) - \mathcal{U}_{2a}(x)\big)$. Using (9) (or arguing using jumps on the graph), we find

$$\phi''(x) = \frac{1}{a}\big(\delta_{-2a}(x) - \delta_{-a}(x)\big) - \frac{1}{a}\big(\delta_{a}(x) - \delta_{2a}(x)\big) = \frac{1}{a}\big(\delta_{-2a}(x) - \delta_{-a}(x) - \delta_{a}(x) + \delta_{2a}(x)\big).$$

23. From Figure 19 we see that ϕ has no jumps on the graph. So, as a generalized function, the derivative has no deltas in it and

$$\phi'(x) = \begin{cases} 0 & \text{if } x < -2, \\ -1 & \text{if } -2 < x < -1, \\ 1 & \text{if } -1 < x < 1, \\ -1 & \text{if } 1 < x < 2, \\ 0 & \text{if } x > 2. \end{cases}$$

In terms of unit step functions, we have

$$\phi'(x) = -\big(\mathcal{U}_{-2}(x) - \mathcal{U}_{-1}(x)\big) + \big(\mathcal{U}_{-1}(x) - \mathcal{U}_{1}(x)\big) - \big(\mathcal{U}_{1}(x) - \mathcal{U}_{2}(x)\big).$$

Thus, using (9),

$$\phi''(x) = -\big(\delta_{-2}(x) - \delta_{-1}(x)\big) + \big(\delta_{-1}(x) - \delta_{1}(x)\big) - \big(\delta_{1}(x) - \delta_{2}(x)\big) = -\delta_{-2} + 2\delta_{-1} - 2\delta_{1} + \delta_{2}.$$

25. Using the definition of ϕ and the definition of a derivative of a generalized function, and integrating by parts, we find

$$\begin{aligned}
\langle \phi'(x),\, f(x)\rangle &= -\langle \phi(x),\, f'(x)\rangle = -\int_{-\infty}^{\infty} \phi(x) f'(x)\,dx \\
&= -\int_{-1}^{0} 2(x+1)f'(x)\,dx - \int_0^1 -2(x-1)f'(x)\,dx \\
&= -2(x+1)f(x)\Big|_{-1}^{0} + 2\int_{-1}^{0} f(x)\,dx + 2(x-1)f(x)\Big|_0^1 - 2\int_0^1 f(x)\,dx \\
&= -2f(0) + 2\int_{-1}^{0} f(x)\,dx + 2f(0) - 2\int_0^1 f(x)\,dx \\
&= \langle 2\big(\mathcal{U}_{-1}(x) - \mathcal{U}_0(x)\big),\, f(x)\rangle - \langle 2\big(\mathcal{U}_0(x) - \mathcal{U}_1(x)\big),\, f(x)\rangle. \\
&= \langle 2\big(\mathcal{U}_{-1}(x) - \mathcal{U}_0(x)\big) - 2\big(\mathcal{U}_0(x) - \mathcal{U}_1(x)\big),\, f(x)\rangle.
\end{aligned}$$

Thus

$$\phi'(x) = 2\big(\mathcal{U}_{-1}(x) - \mathcal{U}_0(x)\big) - 2\big(\mathcal{U}_0(x) - \mathcal{U}_1(x)\big).$$

Reasoning similarly, we find

$$\begin{aligned}
\langle \phi''(x),\, f(x)\rangle &= -\langle \phi'(x),\, f'(x)\rangle = -\int_{-\infty}^{\infty} \phi'(x) f'(x)\,dx \\
&= -2\int_{-1}^{0} f'(x)\,dx + 2\int_0^1 f'(x)\,dx \\
&= -2\big(f(0) - f(-1)\big) + 2\big(f(1) - f(0)\big) = 2f(-1) - 4f(0) + 2f(1) \\
&= \langle 2\delta_{-1} - 4\delta_0 + 2\delta_1,\, f(x)\rangle.
\end{aligned}$$

Thus

$$\phi''(x) = 2\delta_{-1} - 4\delta_0 + 2\delta_1.$$

27. We use definition (7) of the derivative of a generalized function and the fact that the integral against a delta function δ_a picks up the value of the function at a:

$$\langle \delta_0'(x),\, f(x)\rangle = -\langle \delta_0(x),\, f'(x)\rangle = -f'(0);$$

similarly,

$$\langle \delta_\alpha'(x),\, f(x)\rangle = -\langle \delta_\alpha(x),\, f'(x)\rangle = -f'(\alpha)$$

and

$$\langle \delta_\alpha^{(n)}(x),\, f(x)\rangle = (-1)^n \langle \phi(x),\, f^{(n)}(x)\rangle = -f^{(n)}(\alpha).$$

29. We use (13) and the linearity of the Fourier transform:

$$\mathcal{F}\big(3\delta_0 - 2\delta_{-2}\big) = \frac{1}{\sqrt{2\pi}}\big(3 - 2e^{2iw}\big).$$

31. We have $(\operatorname{sgn} x)' = 2\delta_0(x)$. This is clear if you draw the graph of $\operatorname{sgn} x$, you will see a jump of 2 units at $x = 0$, otherwise it is constant. Hence

$$\mathcal{F}\big((\operatorname{sgn} x)'\big) = \mathcal{F}\big(2\delta_0(x)\big) = \frac{2}{\sqrt{2\pi}}.$$

But

$$\mathcal{F}\big((\operatorname{sgn} x)'\big) = i\,w\,\mathcal{F}\big(\operatorname{sgn} x\big).$$

So

$$\mathcal{F}\big(\operatorname{sgn} x\big) = -\frac{i}{w}\mathcal{F}\big((\operatorname{sgn} x)'\big) = -\frac{i}{w}\cdot\frac{2}{\sqrt{2\pi}} = -i\sqrt{\frac{2}{\pi}}\,\frac{1}{w}.$$

33. Using the operational property in Theorem 3(i), Section 7.2, we find

$$\begin{aligned}
\mathcal{F}\big(x\,(\mathcal{U}_{-1}-\mathcal{U}_1)\big) &= i\frac{d}{dw}\mathcal{F}\big(\mathcal{U}_{-1}-\mathcal{U}_1\big)\\
&= i\frac{d}{dw}\left[-\frac{i}{\sqrt{2\pi}\,w}e^{iw}+\frac{i}{\sqrt{2\pi}\,w}e^{-iw}\right]\\
&= \frac{-i(i)}{\sqrt{2\pi}}\frac{d}{dw}\left[\frac{e^{iw}-e^{-iw}}{w}\right]\\
&= \frac{1}{\sqrt{2\pi}}\frac{d}{dw}\left[\frac{2i\sin w}{w}\right] \qquad (\text{Recall } e^{iu}-e^{-iu}=2i\sin u)\\
&= \frac{2i}{\sqrt{2\pi}}\left[\frac{w\,\cos w-\sin w}{w^2}\right]\\
&= i\sqrt{\frac{2}{\pi}}\left[\frac{w\,\cos w-\sin w}{w^2}\right].
\end{aligned}$$

The formula is good at $w=0$ if we take the limit as $w\to 0$. You will get

$$\mathcal{F}\big(x\,(\mathcal{U}_{-1}-\mathcal{U}_1)\big) = \lim_{w\to 0} i\sqrt{\frac{2}{\pi}}\left[\frac{w\,\cos w-\sin w}{w^2}\right] = i\sqrt{\frac{2}{\pi}}\lim_{w\to 0}\frac{-w\,\sin w}{2w} = 0.$$

(Use l'Hospital's rue.) Unlike the Fourier transform in Exercise 31, the transform here is a nice continuous function. There is a major difference between the transforms of the two exercises. In Exercise 31, the function is not integrable and its Fourier transform exists only as a generalized function. In Exercise 33, the function is integrable and its Fourier transform exists in the usual sense of Section 7.2. In fact, look at the transform in Exercise 31, it is not even defined at $w=0$.

An alternative way to do this problem is to realize that

$$\phi'(x) = -\delta_{-1}-\delta_1+\mathcal{U}_{-1}-\mathcal{U}_1.$$

So

$$\begin{aligned}
\mathcal{F}\big(\phi'(x)\big) &= \mathcal{F}\big(-\delta_{-1}-\delta_1+\mathcal{U}_{-1}-\mathcal{U}_1\big)\\
&= \frac{1}{\sqrt{2\pi}}\left(-e^{iw}-e^{-iw}-i\frac{e^{iw}}{w}+i\frac{e^{-iw}}{w}\right).
\end{aligned}$$

But

$$\mathcal{F}\big(\phi'(x)\big) = i\,w\,\mathcal{F}\big(\phi(x)\big).$$

So

$$\begin{aligned}
\mathcal{F}\big(\phi(x)\big) &= \frac{i}{w}\mathcal{F}\big(e^{iw}+e^{-iw}+i\frac{e^{iw}}{w}-i\frac{e^{-iw}}{w}\big)\\
&= \frac{i}{\sqrt{2\pi}\,w}\left[2\cos w+\frac{i}{w}(2i\sin w)\right]\\
&= i\sqrt{\frac{2}{\pi}}\left[\frac{w\cos w-\sin w}{w^2}\right].
\end{aligned}$$

35. Using linearity and (15), we find

$$\begin{aligned}
\mathcal{F}\left(\sum_{j=-3}^{3} j\left(\mathcal{U}_j-\mathcal{U}_{j+1}\right)\right) &= -\frac{i}{\sqrt{2\pi}\,w}\sum_{j=-3}^{3} j\left(e^{-ijw}-e^{-i(j+1)w}\right)\\
&= -\frac{i}{\sqrt{2\pi}\,w}\sum_{j=-3}^{3} je^{-ijw}\left(1-e^{-iw}\right)\\
&= -\frac{i}{\sqrt{2\pi}\,w}\left(1-e^{-iw}\right)\sum_{j=-3}^{3} je^{-ijw}\\
&= -\frac{i}{\sqrt{2\pi}\,w}\left(1-e^{-iw}\right)\sum_{j=1}^{3} j\left(e^{-ijw}-e^{ijw}\right)\\
&= -\frac{i}{\sqrt{2\pi}\,w}\left(1-e^{-iw}\right)\sum_{j=1}^{3} j(-2i)\sin(jw)\\
&= \sqrt{\frac{2}{\pi}}\frac{\left(e^{-iw}-1\right)}{w}\sum_{j=1}^{3} j\sin(jw).
\end{aligned}$$

37. Write $\tau^2(x)=x^2\mathcal{U}_0(x)$, then use the operational properties

$$\begin{aligned}
\mathcal{F}\left(\tau^2(x)\right) &= \mathcal{F}\left(x^2\mathcal{U}_0(x)\right)\\
&= -\frac{d^2}{dw^2}\mathcal{F}\left(\mathcal{U}_0(x)\right)\\
&= -\frac{-i}{\sqrt{2\pi}}\frac{d^2}{dw^2}\left[\frac{1}{w}\right]\\
&= \frac{2i}{\sqrt{2\pi}}\frac{1}{w^3}=i\sqrt{\frac{2}{\pi}}\frac{1}{w^3}.
\end{aligned}$$

39. We have

$$f'(x)=\frac{1}{2}\left(\delta_{-2}(x)+\delta_{-1}(x)-\delta_1(x)-\delta_2(x)\right).$$

So

$$\begin{aligned}
\mathcal{F}\left(f'(x)\right) &= \mathcal{F}\left(\frac{1}{2}\left(\delta_{-2}(x)+\delta_{-1}(x)-\delta_1(x)-\delta_2(x)\right)\right)\\
&= \frac{1}{2\sqrt{2\pi}}\left[e^{2iw}+e^{iw}-e^{-iw}-e^{-2iw}\right]\\
&= \frac{i}{\sqrt{2\pi}}\left[\sin(2w)+\sin w\right].
\end{aligned}$$

Hence

$$\mathcal{F}\left(f(x)\right)=\frac{-i}{w}\mathcal{F}\left(f'(x)\right)=\frac{-i}{w}\frac{i}{\sqrt{2\pi}}\left[\sin(2w)+\sin w\right]=\frac{1}{\sqrt{2\pi}}\frac{\sin(2w)+\sin w}{w}.$$

41. We have

$$f'(x)=-\mathcal{U}_{-2}(x)+2\mathcal{U}_{-1}(x)-\mathcal{U}_1(x)+\delta_{-1}(x)-2\,\delta_2(x).$$

So

$$\begin{aligned}\mathcal{F}(f'(x)) &= \mathcal{F}(-\mathcal{U}_{-2}(x) + 2\mathcal{U}_{-1}(x) - \mathcal{U}_1(x) + \delta_{-1}(x) - 2\,\delta_2(x))\\ &= -\frac{-i}{\sqrt{2\pi}}\frac{e^{2iw}}{w} + 2\frac{-i}{\sqrt{2\pi}}\frac{e^{iw}}{w} - \frac{-i}{\sqrt{2\pi}}\frac{e^{-iw}}{w} + \frac{1}{\sqrt{2\pi}}e^{iw} - \frac{2}{\sqrt{2\pi}}e^{-2iw}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{F}(f(x)) &= \frac{-i}{w}\mathcal{F}(f'(x))\\ &= \frac{1}{\sqrt{2\pi}\,w}\left[\frac{e^{2iw}}{w} - 2\frac{e^{iw}}{w} + \frac{e^{-iw}}{w} - ie^{iw} + 2ie^{-2iw}\right].\end{aligned}$$

43. Given $f(x) = e^{-x}$ if $x > 0$ and 0 otherwise, we see that $f'(x) = \delta_0(x) - e^{-x}$ if $x \geq 0$ and 0 otherwise. Hence $f'(x) = \delta_0(x) - f(x)$ and so

$$\begin{aligned}\mathcal{F}(f'(x)) &= \mathcal{F}(\delta_0(x) - f(x))\,;\\ iw\mathcal{F}(f(x)) &= \frac{1}{\sqrt{2\pi}} - \mathcal{F}(f(x))\,;\\ &\Rightarrow\ \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+iw}\\ &\Rightarrow\ \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\frac{1-iw}{1+w^2}.\end{aligned}$$

45. You may want to draw a graph to help you visualize the derivatives. For $f(x) = \sin x$ if $|x| < \pi$ and 0 otherwise, we have $f"(x) = \cos x$ if $|x| < \pi$ and 0 otherwise. Note that since f is continuous, we do not add delta functions at the endpoints $x = \pm\pi$ when computing f'. For f'', the graph is discontinuous at $x = \pm\pi$ and we have

$$f''(x) = -\delta_{-\pi} + \delta_{\pi} - \sin x$$

if $|x| \leq \pi$ and 0 otherwise. Thus

$$f''(x) = -\delta_{-\pi} + \delta_{\pi} - f(x) \quad \text{for all } x.$$

Taking the Fourier transform, we obtain

$$\begin{aligned}\mathcal{F}(f''(x)) &= \mathcal{F}(-\delta_{-\pi} + \delta_{\pi} - f(x))\,;\\ -w^2\mathcal{F}(f(x)) &= -\frac{1}{\sqrt{2\pi}}e^{i\pi w} + \frac{1}{\sqrt{2\pi}}e^{-i\pi w} - \mathcal{F}(f(x))\,;\\ &\Rightarrow\ (1+w^2)\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\overbrace{\left(e^{i\pi w} + e^{-i\pi w}\right)}^{=2\cos(\pi w)}\\ &\Rightarrow\ \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\frac{\cos(\pi w)}{1+w^2}.\end{aligned}$$

47. In computing the Fourier transform $\mathcal{F}(f * f)$, it is definitely better to use the formula $\mathcal{F}(f * f) = \mathcal{F}(f) \cdot \mathcal{F}(f)$, since we already have $\mathcal{F}(f) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}$. So

$$\mathcal{F}(f * f) = \frac{2}{\pi}\frac{a^2}{(a^2+w^2)^2}.$$

The second method is used here only to practice this method, which is a lot more useful in other situations. From Example 8, we have

$$f'' = a^2 f - 2a\delta_0.$$

So from

$$\frac{d^2}{dx^2}\big(f*f\big) = \frac{d^2 f}{dx^2} * f$$

we have

$$\begin{aligned}
\mathcal{F}\left(\frac{d^2}{dx^2}\big(f*f\big)\right) &= \mathcal{F}\left(\frac{d^2 f}{dx^2} * f\right) = \mathcal{F}\left(\big(a^2 f - 2a\delta_0\big) * f\right);\\
-w^2\mathcal{F}(f*f) &= \mathcal{F}\left(\big(a^2 f * f - 2a\delta_0 * f\big)\right)\\
&= a^2\mathcal{F}(f*f) - 2a\mathcal{F}\overbrace{(\delta_0 * f)}^{f/\sqrt{2\pi}}\\
&= a^2\mathcal{F}(f*f) - \frac{2a}{\sqrt{2\pi}}\mathcal{F}(f).
\end{aligned}$$

Thus

$$\mathcal{F}(f*f) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}\mathcal{F}(f).$$

Writing $\mathcal{F}(f*f) = \mathcal{F}(f)\cdot\mathcal{F}(f)$, we conclude that

$$\mathcal{F}(f) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}$$

and that

$$\mathcal{F}(f*f) = \frac{2}{\pi}\frac{a^2}{(a^2+w^2)^2}.$$

49. From Example 9, we have

$$f' = \delta_{-1} - \delta_1.$$

So from

$$\frac{d}{dx}\big(f*f\big) = \frac{df}{dx} * f$$

we have

$$\begin{aligned}
\frac{d}{dx}\big(f*f\big) &= \frac{df}{dx} * f = \big(\delta_{-1} - \delta_1\big) * f;\\
&= \delta_{-1} * f - \delta_1 * f; = \frac{1}{\sqrt{2\pi}}\big(f(x+1) - f(x-1)\big).
\end{aligned}$$

Using the explicit formula for f, we find

$$\frac{d}{dx}\big(f*f\big) = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } -2 < x < 0,\\ -\frac{1}{\sqrt{2\pi}} & \text{if } 0 < x < 2,\\ 0 & \text{otherwise,}\end{cases}$$

as can be verified directly from the graph of $f*f$ in Figure 18.

51. Using (20), we have

$$\begin{aligned}
\phi * \psi &= (3\delta_{-1}) * (\delta_2 - \delta_1)\\
&= 3\delta_{-1} * \delta_2 - 3\delta_{-1} * \delta_1\\
&= \frac{1}{\sqrt{2\pi}}\left[3\delta_{-1+2} - 3\delta_{-1+1}\right] = \frac{3}{\sqrt{2\pi}}\left[\delta_1 - \delta_0\right]
\end{aligned}$$

53. Using (20), we have

$$\begin{aligned}\phi*\psi &= (\delta_{-1}+2\delta_2)*(\delta_{-1}+2\delta_2)\\ &= \delta_{-1}*\delta_{-1}+2\delta_{-1}*\delta_2+2\delta_2*\delta_{-1}+4\delta_2*\delta_2\\ &= \frac{1}{\sqrt{2\pi}}\left[\delta_{-2}+4\delta_1+4\delta_4\right].\end{aligned}$$

55. Following the method of Example 9, we have

$$\begin{aligned}\frac{d}{dx}(\phi*\psi) &= \frac{d\phi}{dx}*\psi\\ &= (\delta_{-1}-\delta_1)*((\mathcal{U}_{-1}-\mathcal{U}_1)x).\end{aligned}$$

Using (19) and the explicit formula for ψ, it follows that

$$\begin{aligned}\frac{d}{dx}(\phi*\psi) &= \frac{1}{\sqrt{2\pi}}\big[(\text{left translate by 1 unit of }\psi)-(\text{right translate by 1 unit of }\psi)\big)\\ &= \begin{cases}\frac{1}{\sqrt{2\pi}}(x+1) & \text{if } -2<x<0\\ \frac{1}{\sqrt{2\pi}}(-x+1) & \text{if } 0<x<2\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Integrating $\frac{d}{dx}(\phi*\psi)$ and using the fact that $\phi*\psi$ equal 0 for large $|x|$, we find

$$\begin{aligned}\phi*\psi(x) &= \begin{cases}\frac{1}{\sqrt{2\pi}}(\frac{x^2}{2}+x) & \text{if } -2<x<0\\ \frac{1}{\sqrt{2\pi}}(-\frac{x^2}{2}+x) & \text{if } 0<x<2\\ 0 & \text{otherwise}\end{cases}\\ &= \frac{1}{\sqrt{2\pi}}(\frac{x^2}{2}+x)\big(\mathcal{U}_{-2}-\mathcal{U}_0\big)+\frac{1}{\sqrt{2\pi}}(-\frac{x^2}{2}+x)\big(\mathcal{U}_0-\mathcal{U}_2\big).\end{aligned}$$

57. Following the method of Example 9, we have

$$\begin{aligned}\frac{d}{dx}(\phi*\psi) &= \frac{d\psi}{dx}*\phi\\ &= (\delta_{-1}-\delta_1)*(\mathcal{U}_{-1}-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)\\ &= \delta_{-1}*\mathcal{U}_{-1}-\delta_{-1}*\mathcal{U}_1+\delta_{-1}*\mathcal{U}_2-\delta_{-1}*\mathcal{U}_3-\delta_1*\mathcal{U}_{-1}\\ &\quad +\delta_1*\mathcal{U}_1-\delta_1*\mathcal{U}_2+\delta_1*\mathcal{U}_3\\ &= \frac{1}{\sqrt{2\pi}}(\mathcal{U}_{-2}-\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_0+\mathcal{U}_2-\mathcal{U}_3+\mathcal{U}_4)\\ &= \frac{1}{\sqrt{2\pi}}((\mathcal{U}_{-2}-\mathcal{U}_0)-(\mathcal{U}_0-\mathcal{U}_1)-(\mathcal{U}_3-\mathcal{U}_4)).\end{aligned}$$

Integrating $\frac{d}{dx}(\phi*\psi)$ and using the fact that $\phi*\psi$ equal 0 for large $|x|$ and that there are no discontinuities on the graph, we find

$$\begin{aligned}
\phi * \psi(x) &= \begin{cases} \frac{1}{\sqrt{2\pi}}(x+2) & \text{if } -2<x<0 \\ \frac{1}{\sqrt{2\pi}}(-x+2) & \text{if } 0<x<1 \\ \frac{1}{\sqrt{2\pi}} & \text{if } 1<x<3 \\ \frac{1}{\sqrt{2\pi}}(-x+4) & \text{if } 3<x<4 \\ 0 & \text{otherwise} \end{cases} \\
&= \frac{1}{\sqrt{2\pi}}\Big((x+2)\big(\mathcal{U}_{-2}-\mathcal{U}_0\big)+(-x+2)\big(\mathcal{U}_0-\mathcal{U}_1\big) \\
&\quad +\big(\mathcal{U}_1-\mathcal{U}_3\big)+(-x+4)\big(\mathcal{U}_3-\mathcal{U}_4\big)\Big).
\end{aligned}$$

59. We have

$$\phi = \frac{1}{2}(\mathcal{U}_{-2}-\mathcal{U}_{-1})+(\mathcal{U}_{-1}-\mathcal{U}_1)+\frac{1}{2}(\mathcal{U}_1-\mathcal{U}_2)$$

and

$$\psi = \mathcal{U}_{-1}-\mathcal{U}_1.$$

So

$$\begin{aligned}
\frac{d}{dx}(\phi*\psi) &= \frac{d\psi}{dx}*\phi \\
&= (\delta_{-1}-\delta_1)*\left(\frac{1}{2}(\mathcal{U}_{-2}-\mathcal{U}_{-1})+(\mathcal{U}_{-1}-\mathcal{U}_1)+\frac{1}{2}(\mathcal{U}_1-\mathcal{U}_2)\right) \\
&= \frac{1}{2}\delta_{-1}*\mathcal{U}_{-2}-\frac{1}{2}\delta_{-1}*\mathcal{U}_{-1}+\delta_{-1}*\mathcal{U}_{-1}-\delta_{-1}*\mathcal{U}_1+\frac{1}{2}\delta_{-1}*\mathcal{U}_1-\frac{1}{2}\delta_{-1}*\mathcal{U}_2 \\
&\quad -\frac{1}{2}\delta_1*\mathcal{U}_{-2}+\frac{1}{2}\delta_1*\mathcal{U}_{-1}-\delta_1*\mathcal{U}_{-1}+\delta_1*\mathcal{U}_1)-\frac{1}{2}\delta_1*\mathcal{U}_1+\frac{1}{2}\delta_1*\mathcal{U}_2 \\
&= \frac{1}{\sqrt{2\pi}}\big(\frac{1}{2}\mathcal{U}_{-3}-\frac{1}{2}\mathcal{U}_{-2}+\mathcal{U}_{-2}-\mathcal{U}_0)+\frac{1}{2}\mathcal{U}_0-\frac{1}{2}\mathcal{U}_1 \\
&\quad -\frac{1}{2}\mathcal{U}_{-1}+\frac{1}{2}\mathcal{U}_0-\mathcal{U}_0+\mathcal{U}_2)-\frac{1}{2}\mathcal{U}_2+\frac{1}{2}\mathcal{U}_3\big) \\
&= \frac{1}{\sqrt{2\pi}}\big(\frac{1}{2}\mathcal{U}_{-3}+\frac{1}{2}\mathcal{U}_{-2}-\frac{1}{2}\mathcal{U}_{-1}-\mathcal{U}_0-\frac{1}{2}\mathcal{U}_1+\frac{1}{2}\mathcal{U}_2+\frac{1}{2}\mathcal{U}_3\big) \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\big(\mathcal{U}_{-3}+\mathcal{U}_{-2}-\mathcal{U}_{-1}-2\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3\big) \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\times\begin{cases} 1 & \text{if } -3<x<-2 \\ 2 & \text{if } -2<x<-1 \\ 1 & \text{if } -1<x<0 \\ -1 & \text{if } 0<x<1 \\ -2 & \text{if } 1<x<2 \\ -1 & \text{if } 2<x<3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Integrating $\frac{d}{dx}(\phi*\psi)$ and using the fact that $\phi*\psi$ equal 0 for large $|x|$ and that there are no discontinuities on the graph, we find

$$\begin{aligned}
\phi * \psi(x) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \times \begin{cases} (x+3) & \text{if } -3 < x < -2 \\ (2x+5) & \text{if } -2 < x < -1 \\ (x+4) & \text{if } -1 < x < 0 \\ (-x+4) & \text{if } 0 < x < 1 \\ (-2x+5) & \text{if } 1 < x < 2 \\ (-x+3) & \text{if } 2 < x < 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\Big((x+3)\big(\mathcal{U}_{-3} - \mathcal{U}_{-2}\big) + (2x+5)\big(\mathcal{U}_{-2} - \mathcal{U}_{-1}\big) + (x+4)\big(\mathcal{U}_{-1} - \mathcal{U}_0\big) \\
&\quad + (-x+4)\big(\mathcal{U}_0 - \mathcal{U}_1\big) + (-2x+5)\big(\mathcal{U}_1 - \mathcal{U}_2\big) + (-x+3)\big(\mathcal{U}_2 - \mathcal{U}_3\big)\Big).
\end{aligned}$$

Solutions to Exercises 7.9

1. Proceed as in Example 1 with $c = 1/2$. Equation (3) becomes in this case

$$\begin{aligned} u(x, t) &= \frac{2}{\sqrt{2t}} e^{-x^2/t} * \delta_1(x) \\ &= \frac{1}{\sqrt{\pi t}} e^{-(x-1)^2/t}, \end{aligned}$$

since the effect of convolution by δ_1 is to shift the function by 1 unit to the right and multiply by $\frac{1}{\sqrt{2\pi}}$.

3. We use the superposition principle (see the discussion preceeding Example 4). If ϕ is the solution of $u_t = \frac{1}{4}u_{xx} + \delta_0$, $u(x, 0) = 0$ and ψ is the solution of $u_t = \frac{1}{4}u_{xx} + \delta_1$, $u(x, 0) = 0$, then $\phi + \psi$ is the solution of $u_t = \frac{1}{4}u_{xx} + \delta_0 + \delta_1$, $u(x, 0) = 0$. Applying Examples 2 and 4, we find

$$\phi(x, t) = \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-x^2/t} - \frac{2|x|}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \frac{x^2}{t}\right)$$

and

$$\psi = \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-(x-1)^2/t} - \frac{2|x-1|}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \frac{(x-1)^2}{t}\right).$$

5. We use the superposition principle (see the discussion preceeding Example 4). If ϕ is the solution of $u_t = \frac{1}{4}u_{xx} + \delta_0$, $u(x, 0) = 0$ and ψ is the solution of $u_t = \frac{1}{4}u_{xx}$, $u(x, 0) = \mathcal{U}_0(x)$, then you can check that $\phi + \psi$ is the solution of $u_t = \frac{1}{4}u_{xx} + \delta_0$, $u(x, 0) = \mathcal{U}_0(x)$. By Examples 1,

$$\phi(x, t) = \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-x^2/t} - \frac{2|x|}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \frac{x^2}{t}\right)$$

and by Exercise 20, Section 7.4,

$$\psi(x, t) = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{t}}\right).$$

7. See the end of Section 7.4 for related topics.

9. Apply Theorem 2 with $c = 1$ and $f(x, t) = \cos ax$; then

$$\begin{aligned} u(x, t) &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-s)}} e^{-(x-y)^2/(4(t-s))} \cos(ay)\, dyds \\ &= \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2(t-s)}} e^{-y^2/(4(t-s))} \cos(a(x-y))\, dyds \\ &\qquad \text{(Change variables } x - y \leftrightarrow y) \\ &= \cos(ax) \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2(t-s)}} e^{-y^2/(4(t-s))} \cos(ay)\, dyds \\ &\qquad \text{(Integral of odd function is 0.} \\ &= \cos(ax) \int_0^t e^{-a^2(t-s)} ds \quad \text{(Fourier transform of a Guaussian.)} \\ &= \frac{1}{a^2} \cos(ax)\big(1 - e^{-a^2 t}\big). \end{aligned}$$

10. Very similar to Exercise 9. The answer is

$$u(x, t) = \frac{1}{a^2} \sin(ax)\big(1 - e^{-a^2 t}\big),$$

as you can check by using the equations.

11. Write the Fourier series of $f(x)$ in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big).$$

It is not hard to derive the solution of

$$u_t = u_{xx} + a_0, \quad u(x,\,0) = 0,$$

as $u_0(x,\,t) = a_0 t$. Now use superposition and the results of Exercises 9 and 10 to conclude that the solution of

$$u_t = u_{xx} + a_0 + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big)$$

is

$$u(x,\,t) = a_0 t + \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 t}}{n^2} \left(a_n \cos nx + b_n \sin nx\right).$$

Solutions to Exercises 7.10

1. Apply Proposition 1 with $f(x, s) = e^{-(x+s)^2}$, then

$$\begin{aligned}
\frac{dU}{dx} &= f(x, x) + \int_0^x \frac{\partial}{\partial x} f(x, s)\, ds \\
&= e^{-4x^2} - \int_0^x 2(x+s)e^{-(x+s)^2}\, ds \\
&= e^{-4x^2} - \int_x^{2x} 2ve^{-v^2}\, dv \quad ((v = x+s) \\
&= e^{-4x^2} + e^{-v^2}\Big|_x^{2x} = 2e^{-4x^2} - e^{-x^2}.
\end{aligned}$$

3. Use the product rule for differentiation and Proposition 1 and get

$$\begin{aligned}
\frac{d}{dx}\left(x^2 \int_0^x f(x, s)\, ds\right) &= 2x \int_0^x f(x, s)\, ds + x^2 \frac{d}{dx} \int_0^x f(x, s)\, ds \\
&= 2x \int_0^x f(x, s)\, ds + x^2 \left(f(x, x) + \int_0^x \frac{\partial}{\partial x} f(x, s)\, ds\right).
\end{aligned}$$

5. Following Theorem 1, we first solve $\phi_t = \phi_{xx}$, $\phi(x, 0, s) = e^{-s}x^2$, where $s > 0$ is fixed. The solution is $\phi(x, t, s) = e^{-s}(2t + x^2)$ (see the solution of Exercise 5, Section 7.4). The the desired solution is given by

$$\begin{aligned}
u(x, t) &= \int_0^t \phi(x, t-s, s) ds \\
&= \int_0^t e^{-s}(2(t-s) + x^2) ds \\
&= -2te^{-s} + 2se^{-s} + 2e^{-s} - x^2e^{-s}\Big|_0^t \\
&= -2 + 2t + x^2 + e^{-t}(2 - x^2).
\end{aligned}$$

7. Following Theorem 2, we first solve $\phi_{tt} = \phi_{xx}$, $\phi(x, 0, s) = 0$, $\phi_t(x, 0, s) = \cos x$ where $s > 0$ is fixed. By d'Alembert's method, the solution is

$$\phi(x, t, s) = \frac{1}{2} \int_{x-t}^{x+t} \cos y\, dy = \frac{1}{2}\big[\sin(x+t) - \sin(x-t)\big].$$

The the desired solution is given by

$$\begin{aligned}
u(x, t) &= \int_0^t \phi(x, t-s, s) ds \\
&= \frac{1}{2} \int_0^t \big[\sin(x+t-s) - \sin(x-t+s)\big] ds \\
&= \frac{1}{2}\big[\cos(x+t-s) + \cos(x-t+s)\big]\Big|_0^t \\
&= \frac{1}{2}\big[2\cos x - \cos(x+t) - \cos(x-t)\big].
\end{aligned}$$

9. Following Theorem 2, we first solve $\phi_{tt} = \phi_{xx}$, $\phi(x, 0, s) = 0$, $\phi_t(x, 0, s) = \cos(s + x)$ where $s > 0$ is fixed. By d'Alembert's method, the solution is

$$\phi(x, t, s) = \frac{1}{2}\int_{x-t}^{x+t} \cos(s + y)\, dy = \frac{1}{2}\big[\sin(s + x + t) - \sin(s + x - t)\big].$$

The the desired solution is given by

$$\begin{aligned} u(x, t) &= \int_0^t \phi(x, t - s, s) ds \\ &= \frac{1}{2}\int_0^t \big[\sin(x + t) - \sin(x - t + 2s)\big] ds \\ &= \frac{1}{2}\Big[s \sin(x + t) + \frac{1}{2}\cos(x - t + 2s)\Big]\Big|_0^t \\ &= \frac{1}{2}\Big[t \sin(x + t) + \frac{1}{2}\cos(x + t) - \frac{1}{2}\cos(x - t)\Big]. \end{aligned}$$

11. We use superposition and start by solving $\phi_{tt} = \phi_{xx}$, $\phi(x, 0, s) = 0$, $\phi_t(x, 0, s) = x \cos s$ where $s > 0$ is fixed. By d'Alembert's method, the solution is

$$\phi(x, t, s) = \frac{1}{2}\int_{x-t}^{x+t} y \cos s\, dy = \frac{\cos s}{4}\big[(x + t)^2 - (x - t)^2\big] = xt \cos s.$$

By Theorem 2, the solution of $\phi_{tt} = \phi_{xx} + x \cos t$, $\phi(x, 0) = 0$, $\phi_t(x, 0) = 0$ is given by

$$\begin{aligned} \phi(x, t) &= \int_0^t \phi(x, t - s, s) ds \\ &= \int_0^t x \overbrace{(t - s)}^{u} \overbrace{\cos s\, ds}^{dv} \quad \text{(Integrate by parts.)} \\ &= x \left[(t - s) \sin s\Big|_0^t + \int_0^t \sin s\, ds\right] \\ &= x(1 - \cos t). \end{aligned}$$

Next, we solve $\psi_{tt} = \psi_{xx}$, $\psi(x, 0) = 0$, $\psi_t(x, 0) = x$. D'Alembert's method implies that

$$\psi(x, t) = \frac{1}{2}\int_{x-t}^{x+t} y\, dy = \frac{1}{4}y^2\Big|_{x-t}^{x+t} = xt.$$

Thus the solution of the desired problem is

$$u(x, t) = \phi(x, t) + \psi(x, t) = x(1 - \cos t) + xt = x(1 + t - \cos t).$$

The validity of this solution can be checked by plugging it back into the equation and the initial conditions.

13. start by solving $\phi_{tt} = \phi_{xx}$, $\phi(x, 0, s) = 0$, $\phi_t(x, 0, s) = \delta_0(x)$ where $s > 0$ is fixed. By d'Alembert's method, the solution is

$$\phi(x, t, s) = \frac{1}{2}\int_{x-t}^{x+t} \delta_0(y)\, dy = \frac{1}{2}\big[\mathcal{U}_0(x + t) - \mathcal{U}_0(x - t)\big].$$

By Theorem 2, the solution of $u_{tt} = u_{xx} + \delta_0(x)$, $u(x, 0) = 0$, $\phi_t(x, 0) = 0$ is given by

$$\begin{aligned}\phi(x, t) &= \int_0^t \phi(x, t-s, s)ds \\ &= \frac{1}{2}\int_0^t \big[\mathcal{U}_0(x+t-s) - \mathcal{U}_0(x-t+s)\big]ds \\ &= \frac{1}{2}\big[-\tau(x+t-s) - \tau(x-t+s)\big]\Big|_0^t \\ &= -\tau(x) + \frac{1}{2}\big[\tau(x+t) + \tau(x-t)\big],\end{aligned}$$

where $\tau = \tau_0$ is the antiderivative of $\mathcal{U}_0$ described in Example 2, Section 7.8,

Solutions to Exercises A.1

1. We solve the equation $y'+y=1$ in two different ways. The first method basically rederives formula (2) instead of just appealing to it.
Using an integrating factor. In the notation of Theorem 1, we have $p(x)=1$ and $q(x)=1$. An antiderivative of $p(x)$ is thus $\int 1\cdot dx = x$. The integrating factor is

$$\mu(x) = e^{\int p(x)\,dx} = e^x.$$

Multiplying both sides of the equation by the integrating factor, we obtain the equivalent equation

$$e^x[y'+y] = e^x;$$

$$\frac{d}{dx}[e^x y] = e^x,$$

where we have used the product rule for differentiation to set $\frac{d}{dx}[e^x y] = e^x[y'+y]$. Integrating both sides of the equation gets rid of the derivative on the left side, and on the right side we obtain $\int e^x\,dx = e^x + C$. Thus,

$$e^x y = e^x + C \quad \Rightarrow \quad y = 1 + Ce^{-x},$$

where the last equality follows by multiplying by e^{-x} the previous equality. This gives the solution $y = 1+Ce^{-x}$ up to one arbitrary constant, as expected from the solution of a first order differential equation.
Using formula (2). We have, with $p(x)=1$, $\int p(x)\,dx = x$ (note how we took the constant of integration equal 0):

$$y = e^{-x}\left[C + \int 1\cdot e^x\,dx\right] = e^{-x}[C+e^x] = 1 + Ce^{-x}.$$

2. We apply (2) directly. We have $p(x)=2x$, an integrating factor is

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2},$$

and the solution becomes

$$\begin{aligned} y &= e^{-x^2}\left[C + \int xe^{x^2}\right] \\ &= e^{-x^2}\left[C + \frac{1}{2}e^{x^2}\right] = Ce^{-x^2} + \frac{1}{2}. \end{aligned}$$

3. This is immediate:

$$y' = -.5\,y \quad \Rightarrow \quad y = Ce^{-.5x}.$$

4. Put the equation in the form $y' - 2y = x$. Then from (2)

$$\begin{aligned} y &= e^{-\int -2dx}\left[C + \int xe^{\int -2\,dx}dx\right] \\ &= e^{2x}\left[C + \int xe^{-2x}dx\right] \\ &= e^{2x}\left[C - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right] = Ce^{2x} - \frac{1}{2}x - \frac{1}{4}, \end{aligned}$$

where we used integration by parts to evaluate the last integral: $u=x$, $dv = e^{-2x}\,dx$, $du = dx$, $v = -\frac{1}{2}e^{-2x}$.

5. According to (2),

$$y = e^x \left[C + \int \sin x e^{-x} dx \right].$$

To evaluate the integral, use integration by parts twice

$$\begin{aligned}
\int \sin x e^{-x} dx &= -\sin x e^{-x} + \int e^{-x} \cos x \, dx \\
&= -\sin x e^{-x} + \cos x(-e^{-x}) - \int e^{-x} \sin x \, dx; \\
2 \int \sin x e^{-x} dx &= -e^{-x}\big(\sin x + \cos x\big) \\
\int \sin x e^{-x} dx &= -\frac{1}{2} e^{-x}\big(\sin x + \cos x\big).
\end{aligned}$$

So

$$y = e^x \left[C - \frac{1}{2} e^{-x}\big(\sin x + \cos x\big) \right] = Ce^x - \frac{1}{2}\big(\sin x + \cos x\big).$$

6. We have $\int p(x)\, dx = -x^2$, so according to (2),

$$\begin{aligned}
y &= e^{x^2} \left[C + \int x^3 e^{-x^2} dx \right] \\
&= e^{x^2} \left[C - \frac{1}{2} x^2 e^{-x^2} + \frac{1}{2} \int e^{-x^2} 2x \, dx \right] \\
&= e^{x^2} \left[C - \frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} \right] \\
&= Ce^{x^2} - \frac{1}{2} x^2 - \frac{1}{2}.
\end{aligned}$$

7. Put the equation in standard form:

$$y' + \frac{1}{x} y = \frac{\cos x}{x}.$$

An integrating factor is

$$e^{\int p(x)\, dx} = e^{\int \frac{1}{x}\, dx} e^{\ln x} = x.$$

Multiply by the integrating factor (you will see that we just took an unnecessary detour):

$$\begin{aligned}
xy' + y &= \cos x; \\
\frac{d}{dx}\big[xy\big] &= \cos x; \\
xy &= \int \cos x \, dx = \sin x + C; \\
y &= \frac{\sin x}{x} + \frac{C}{x}.
\end{aligned}$$

8. We use an integrating factor

$$e^{\int p(x)\, dx} = e^{\int -2/x \, dx} = e^{-2 \ln x} = \frac{1}{x^2}.$$

Then

$$\begin{aligned}
\frac{1}{x^2}y' - \frac{2}{x^3}\,y &= 1;\\
\frac{d}{dx}\left[\frac{1}{x^2}y\right] &= 1\\
\frac{1}{x^2}y &= x + C\\
y &= x^3 + Cx^2.
\end{aligned}$$

9. We use an integrating factor

$$e^{\int p(x)\,dx} = e^{\int \tan x\,dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x} = \sec x.$$

Then

$$\begin{aligned}
\sec x y' - \sec x \tan x\, y &= \sec x \cos x;\\
\frac{d}{dx}\,[y\ \sec x] &= 1\\
y \sec x &= x + C =\\
y &= x\cos x + C\cos x.
\end{aligned}$$

10. Same integrating factor as in Exercise 9. The equation becomes

$$\begin{aligned}
\sec x y' - \sec x \tan x\, y &= \sec^3 x;\\
\frac{d}{dx}\,[y\ \sec x] &= \sec^3 x\\
y \sec x &= \int \sec^3 x\,dx.
\end{aligned}$$

To evaluate the last integral, you can use integration by parts:

$$\begin{aligned}
\int \sec^3 x\,dx &= \int \overbrace{\sec x}^{v}\,\overbrace{\sec^2 x\,dx}^{du}\\
&\quad (u = \tan x,\ dv = \tan x \sec x)\\
&= \tan x \sec x - \int \tan^2 x \sec x\,dx\\
&\quad (\tan^2 = \sec^2 - 1)\\
&= \tan x \sec x - \int \sec^3 x\,dx + \int \sec x\,dx;\\
2\int \sec^3 x\,dx &= \tan x \sec x + \int \sec x\,dx.
\end{aligned}$$

Now we use a known trick to evaluate the last integral:

$$\begin{aligned}
\int \sec x\,dx &= \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x}\,dx\\
&= \ln|\tan x + \sec x| + C,
\end{aligned}$$

where the last integral follows by setting $u = \tan x + \sec x$, $du = \sec x(\tan x + \sec x)$. Hence

$$\begin{aligned} y \sec x &= \frac{1}{2} \tan x \sec x + \frac{1}{2} \ln|\tan x + \sec x| + C + C; \\ y &= \frac{1}{2} \tan x + \frac{\cos x}{2} \ln|\tan x + \sec x| + C \cos x. \end{aligned}$$

11.

$$\begin{aligned} y' = y &\Rightarrow y = Ce^x; \\ y(0) = 1 &\Rightarrow 1 = C \\ &\Rightarrow y = e^x. \end{aligned}$$

12. An integrating factor is e^{2x}, so

$$\begin{aligned} y' + 2y = 1 &\Rightarrow e^{2x}[y' + 2y] = e^{2x}; \\ &\Rightarrow \frac{d}{dx}\left[e^{2x} y\right] = e^{2x} \\ &\Rightarrow ye^{2x} = \int e^{2x}\, dx = \frac{1}{2}e^{2x} + C \\ &\Rightarrow y = \frac{1}{2} + Ce^{-2x}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(0) = 2 &\Rightarrow 2 = \frac{1}{2} + C \\ &\Rightarrow C = \frac{3}{2} \\ &\Rightarrow y = \frac{1}{2} + \frac{3}{2}e^{-2x}. \end{aligned}$$

13. An integrating factor is $e^{\frac{x^2}{2}}$, so

$$\begin{aligned} e^{\frac{x^2}{2}} y' + xe^{\frac{x^2}{2}} y = xe^{\frac{x^2}{2}} &\Rightarrow \frac{d}{dx}\left[e^{\frac{x^2}{2}} y\right] = e^{\frac{x^2}{2}} x \\ &\Rightarrow ye^{\frac{x^2}{2}} = \int xe^{\frac{x^2}{2}}\, dx = e^{\frac{x^2}{2}} + C \\ &\Rightarrow y = 1 + Ce^{-\frac{x^2}{2}}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(0) = 0 &\Rightarrow 0 = 1 + C \\ &\Rightarrow C = -1 \\ &\Rightarrow y = 1 - e^{-\frac{x^2}{2}}. \end{aligned}$$

14. An integrating factor is x, so

$$\begin{aligned} xy' + y = \sin x &\Rightarrow \frac{d}{dx}[xy] = \sin x \\ &\Rightarrow xy = \int \sin x\, dx = -\cos x + C \\ &\Rightarrow y = -\frac{\cos x}{x} + \frac{C}{x}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(\pi)=1 \quad &\Rightarrow \quad 1=\frac{1}{\pi}+\frac{C}{\pi} \\ &\Rightarrow \quad C=\pi-1 \\ &\Rightarrow \quad y=-\frac{\cos x}{x}+\frac{\pi-1}{x}=\frac{\pi-1-\cos x}{x}. \end{aligned}$$

15. An integrating factor is x, so

$$\begin{aligned} x^2y'+2xy=x \quad &\Rightarrow \quad \frac{d}{dx}\left[x^2y\right]=x \\ &\Rightarrow \quad x^2y=\int x\,dx=\frac{1}{2}x^2+C \\ &\Rightarrow \quad y=\frac{1}{2}+\frac{C}{x^2}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(-1)=-2 \quad &\Rightarrow \quad -2=\frac{1}{2}+C \\ &\Rightarrow \quad C=-\frac{5}{2} \\ &\Rightarrow \quad y=\frac{1}{2}-\frac{5}{2\,x^2}. \end{aligned}$$

16. An integrating factor is $1/x^3$, so

$$\begin{aligned} \frac{y'}{x^2}-\frac{2}{x^3}y=\frac{1}{x^4} \quad &\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{x^2}y\right]=\frac{1}{x^4} \\ &\Rightarrow \quad \frac{1}{x^2}y=\int\frac{1}{x^4}\,dx=-\frac{1}{3}x^{-3}+C \\ &\Rightarrow \quad y=-\frac{1}{3}x^{-1}+Cx^2. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(1)=0 \quad &\Rightarrow \quad 0=-\frac{1}{3}+C \\ &\Rightarrow \quad C=\frac{1}{3} \\ &\Rightarrow \quad y=-\frac{1}{3\,x}+\frac{x^2}{3}. \end{aligned}$$

17. An integrating factor is $\sec x$ (see Exercise 9), so

$$\begin{aligned} \sec x y'+y\tan x\sec x=\tan x\sec x \quad &\Rightarrow \quad \frac{d}{dx}\left[y\sec x\right]=\sec x\tan x \\ &\Rightarrow \quad y\sec x=\int\tan x\sec x\,dx=\sec x+C \\ &\Rightarrow \quad y=1+C\cos x. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(0)=1 \quad &\Rightarrow \quad 1=1+C \\ &\Rightarrow \quad C=0 \\ &\Rightarrow \quad y=1. \end{aligned}$$

18. Use the solution of the previous exercise, $y = 1 + C\cos x$. The initial condition implies:

$$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad 2 = 1 + C \\ &\Rightarrow \quad C = 1 \\ &\Rightarrow \quad y = 1 + \cos x. \end{aligned}$$

19. An integrating factor is

$$e^{\int e^x\,dx} = e^{e^x}.$$

The equation becomes

$$\begin{aligned} e^{e^x} y' + e^{e^x} e^x y = e^x e^{e^x} \quad &\Rightarrow \quad \frac{d}{dx}\left[e^{e^x} y\right] = e^x e^{e^x} \\ &\Rightarrow \quad e^{e^x} y = \int e^{e^x} e^x\,dx = e^{e^x} + C \\ &\Rightarrow \quad y = 1 + Ce^{-e^x}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad 2 = 1 + Ce^{-1} \quad \Rightarrow \quad 1 = Ce^{-1} \\ &\Rightarrow \quad C = e \\ &\Rightarrow \quad y = 1 + ee^{-e^x} = 1 + e^{1-e^x}. \end{aligned}$$

20. An integrating factor is e^x. The equation becomes

$$\begin{aligned} e^x y' + e^x y = e^x e^x \quad &\Rightarrow \quad \frac{d}{dx}\left[e^x y\right] = e^{2x} \\ &\Rightarrow \quad e^x y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C \\ &\Rightarrow \quad y = \frac{1}{2}e^x + Ce^{-x}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(3) = 0 \quad &\Rightarrow \quad 0 = \frac{1}{2}e^3 + Ce^{-3} \quad \Rightarrow \quad -\frac{1}{2}e^3 = Ce^{-3} \\ &\Rightarrow \quad C = -\frac{1}{2}e^6 \\ &\Rightarrow \quad y = \frac{1}{2}e^x - \frac{1}{2}e^{6-x} = \frac{1}{2}\left(e^x - e^{6-x}\right). \end{aligned}$$

21. (a) Clear.
(b) e^x as a linear combination of the functions $\cosh x$, $\sinh x$: $e^x = \cosh x + \sinh x$.
(c) Let a, b, $c\,d$ be any real numbers such that $ad - bc \neq 0$. Let $y_1 = ae^x + be^{-x}$ and $y_2 = ce^x + de^{-x}$. Then y_1 and y_2 are solutions, since they are linear combinations of two solutions. We now check that y_1 and y_2 are linearly independent:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} ae^x + be^{-x} & ce^x + de^{-x} \\ ae^x - be^{-x} & ce^x - de^{-x} \end{vmatrix} \\ &= -ad + bc - (ad - bc) = -2(ad - bc) \neq 0. \end{aligned}$$

Hence y_1 and y_2 are linearly independent by Theorem 7.

22. (a) Clear.
(b) In computing the determinant, we use an expansion along the first row:

$$\begin{aligned}
W(e^x, e^{-x}, \cosh x) &= \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} \\
&= e^x \begin{vmatrix} -e^{-x} & \sinh x \\ e^{-x} & \cosh x \end{vmatrix} - e^{-x} \begin{vmatrix} -e^x & \sinh x \\ e^x & \cosh x \end{vmatrix} \\
&\quad + \cosh x \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} \\
&= e^x \left(-e^{-x} \cosh x - e^{-x} \sinh x\right) - e^{-x} \left(e^x \cosh x - e^x \sinh x\right) \\
&\quad \cosh x \left(e^x e^{-x} + e^x e^{-x}\right) \\
\equiv 0.
\end{aligned}$$

(c) Clearly the function 1 is a solution of $y''' - y' = 0$. The set of solutions $\{1, e^x, e^{-x}\}$ is a fundamental set if and only if it is linearly independent (Theorem 7). Let us compute the Wronskian. Expand along the 3rd column:

$$\begin{aligned}
W(e^x, e^{-x}, 1) &= \begin{vmatrix} e^x & e^{-x} & 1 \\ e^x & -e^{-x} & 0 \\ e^x & e^{-x} & 0 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} - 0 \cdot \begin{vmatrix} e^x & e^{-x} \\ e^x & e^{-x} \end{vmatrix} \\
&\quad +0 \cdot \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\
&= 2.
\end{aligned}$$

23. (a) Clear (b)

$$\begin{aligned}
W(x, x^2) &= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} \\
&= x^2
\end{aligned}$$

(c) Note that the Wronskian vanishes at $x = 0$. This does not contradict Theorem 2, because if you put the equation in standard form it becomes

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0.$$

The coefficients functions are not continuous at $x = 0$. So we cannot apply Theorem 2 on any interval that that contains 0. Note that $W(x, x^2) \neq 0$ if $x \neq 0$.

24. (a) and first part of (b) are clear. In standard form, the equation becomes

$$y'' - \frac{1+x}{x} y' + \frac{1}{x} y = 0.$$

Since the coefficient functions are not continuous at 0, Theorem 6 will not apply if we use an initial condition at $x = 0$. So there is no contradiction with Theorem 6.
(c)

$$\begin{aligned} W(e^x, 1+x) &= \begin{vmatrix} e^x & 1+x \\ e^x & 1 \end{vmatrix} \\ &= e^x - e^x - xe^x = -xe^x. \end{aligned}$$

The Wronskian is clearly nonzero on $(0, \infty)$ and hence the solutions are linearly independent on $(0, \infty)$.

25. The general solution is

$$y = c_1 e^x + c_2 e^{2x} + 2x + 3.$$

Let's use the initial conditions:

$$\begin{array}{lll} y(0) = 0 & \Rightarrow \; c_1 + c_2 + 3 = 0 & (*) \\ y'(0) = 0 & \Rightarrow \; c_1 + 2c_2 + 2 = 0 & (**) \\ \text{Subtract (*) from (**)} & \Rightarrow \; c_2 - 1 = 0; \; c_2 = 1 & \\ \text{Substitute into (*)} & \Rightarrow \; c_1 + 4 = 0; \; c_1 = -4. & \end{array}$$

Thus, $y = -4e^x + e^{2x} + 2x + 3$.

26. The general solution is

$$y = c_1 e^x + c_2 e^{2x} + 2x + 3.$$

From the initial conditions,

$$\begin{array}{lll} y(0) = 1 & \Rightarrow \; c_1 + c_2 + 3 = 1 & (*) \\ y'(0) = -1 & \Rightarrow \; c_1 + 2c_2 + 2 = -1 & (**) \\ \text{Subtract (*) from (**)} & \Rightarrow \; c_2 - 1 = -2; \; c_2 = -1 & \\ \text{Substitute into (*)} & \Rightarrow \; c_1 + 2 = 1; \; c_1 = -1. & \end{array}$$

Thus, $y = -e^x - e^{2x} + 2x + 3$.

27. The function $e^{x-1} (= e^{-1}e^x)$ is a constant multiple of the e^x. Hence it is itself a solution of the homogeneous equation. Similarly, $e^{2(x-1)}$ is a solution of the homogeneous equation. Using the same particular solution as in Example 3, we obtain the general solution

$$y = c_1 e^{x-1} + c_2 e^{2(x-1)} + 2x + 3.$$

From the initial conditions,

$$\begin{array}{lll} y(1) = 0 & \Rightarrow \; c_1 + c_2 + 5 = 0 & (*) \\ y'(1) = 2 & \Rightarrow \; c_1 + 2c_2 + 2 = 2 & (**) \\ \text{Subtract (*) from (**)} & \Rightarrow \; c_2 - 3 = 2; \; c_2 = 5 & \\ \text{Substitute into (*)} & \Rightarrow \; c_1 + 10 = 0; \; c_1 = -10. & \end{array}$$

Thus, $y = -10e^{x-1} + 5e^{2(x-1)} + 2x + 3$.

28. As in the previous exercise, the general solution

$$y = c_1 e^{x-1} + c_2 e^{2(x-1)} + 2x + 3.$$

From the initial conditions,

$$\begin{aligned}
&y(1)=1 &&\Rightarrow\ c_1+c_2+5=1 && (*)\\
&y'(1)=-1 &&\Rightarrow\ c_1+2c_2+2=-1 && (**)\\
&\text{Subtract (*) from (**)} &&\Rightarrow\ c_2-3=-2;\ c_2=1\\
&\text{Substitute into (*)} &&\Rightarrow\ c_1+6=1;\ c_1=-5.
\end{aligned}$$

Thus, $y=-5e^{x-1}+e^{2(x-1)}+2x+3$.

29. As in the previous exercise, here it is easier to start with the general solution

$$y=c_1e^{x-2}+c_2e^{2(x-2)}+2x+3.$$

From the initial conditions,

$$\begin{aligned}
&y(2)=0 &&\Rightarrow\ c_1+c_2+7=0 && (*)\\
&y'(1)=1 &&\Rightarrow\ c_1+2c_2+2=1 && (**)\\
&\text{Subtract (*) from (**)} &&\Rightarrow\ c_2-5=1;\ c_2=6\\
&\text{Substitute into (*)} &&\Rightarrow\ c_1+13=0;\ c_1=-13.
\end{aligned}$$

Thus, $y=-13e^{x-2}+6e^{2(x-2)}+2x+3$.

30. Start with the general solution

$$y=c_1e^{x-3}+c_2e^{2(x-3)}+2x+3.$$

From the initial conditions,

$$\begin{aligned}
&y(3)=3 &&\Rightarrow\ c_1+c_2+9=3 && (*)\\
&y'(1)=1 &&\Rightarrow\ c_1+2c_2+2=3 && (**)\\
&\text{Subtract (*) from (**)} &&\Rightarrow\ c_2-7=0;\ c_2=7\\
&\text{Substitute into (*)} &&\Rightarrow\ c_1+16=3;\ c_1=-13.
\end{aligned}$$

Thus, $y=-13e^{x-3}+7e^{2(x-3)}+2x+3$.

31. Project Problem: Abel's formula for $n=3$.

(a) Let y_1, y_2, y_3 be any three solutions of the third order equation

$$y'''+p_2(x)y''+p_1(x)y'+p_0(x)y=0.$$

Then the Wronskian is (expand along the 3rd row):

$$\begin{aligned}
W(y_1,\,y_2,\,y_3) &= \begin{vmatrix} y_1 & y_2 & y_3\\ y_1' & y_2' & y_3'\\ y_1'' & y_1'' & y_3'' \end{vmatrix}\\
&= y_1''\begin{vmatrix} y_2 & y_3\\ y_2' & y_3'\end{vmatrix} - y_2''\begin{vmatrix} y_1 & y_3\\ y_1' & y_3'\end{vmatrix}\\
&\quad +y_3''\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}\\
&= (y_2y_3'-y_2'y_3)y_1''-(y_1y_3'-y_1'y_3)y_2''\\
&\quad +(y_1y_2'-y_1'y_2)y_3''.
\end{aligned}$$

Using the product rule to differentiate W, we find

$$\begin{aligned} W'(y_1\,,\,y_2,\,y_3) &= (y_2y_3' - y_2'y_3)y_1''' + (y_2'y_3' + y_2y_3'' - y_2''y_3 - y_2'y_3')y_1'' \\ &\quad -(y_1y_3' - y_1'y_3)y_2''' - (y_1'y_3' + y_1y_3'' - y_1''y_3 - y_1'y_3')y_2'' \\ &\quad +(y_1y_2' - y_1'y_2)y_3''' + (y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2')y_3'' \\ &= (y_2y_3' - y_2'y_3)y_1''' - (y_1y_3' - y_1'y_3)y_2''' + (y_1y_2' - y_1'y_2)y_3'''. \end{aligned}$$

(b) For each $j = 1, 2, 3$, we have, from the differential equation,

$$y_j''' = -p_2(x)y_j'' - p_1(x)y_j' - p_0(x)y_j.$$

Substituting this into the formula for W', we obtain

$$\begin{aligned} W' &= -(y_2y_3' - y_2'y_3)\big(p_2(x)y_1'' + p_1(x)y_1' + p_0(x)y_1\big) \\ &\quad +(y_1y_3' - y_1'y_3)\big(p_2(x)y_2'' + p_1(x)y_2' + p_0(x)y_2\big) \\ &\quad -(y_1y_2' - y_1'y_2)\big(p_2(x)y_3'' + p_1(x)y_3' + p_0(x)y_3\big) \\ &= -p_2(x)(y_2y_3' - y_2'y_3)y_1'' + p_2(x)(y_1y_3' - y_1'y_3)y_2'' \\ &\quad -p_2(x)(y_1y_2' - y_1'y_2)y_3'' \\ &= -p_2(x)W. \end{aligned}$$

(c) From (b) W satisfies a the first order differential equation $W' = -p_2W$, whose solution is

$$W = Ce^{-\int p_2(x)\,dx}.$$

32. (a) Let $f(x) = |x^3|$,

$$f(x) = \begin{cases} x^3 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

Clearly $f'(x)$ exists if $x \neq$. For $x = 0$, we have

$$\begin{aligned} f'(0) &= \lim_{h\to} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to} \frac{|h|^3}{h} = \lim_{h\to} \frac{h^2|h|}{h} \\ &= \lim_{h\to} h|h| = 0. \end{aligned}$$

Hence the derivative exists for all x and is given by

$$f'(x) = \begin{cases} 3x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$$

Another way of writing the derivative is $f'(x) = 3x^2 \operatorname{sgn} x$, where $\operatorname{sgn} x = -1, 0$, or 1, according as $x < 0$, $x = 0$, or $x > 0$. We have

$$\begin{aligned} W(x^3, |x^3|) &= \begin{vmatrix} x^3 & |x^3| \\ 3x^2 & 3x^2 \operatorname{sgn} x \end{vmatrix} \\ &= 3x^5 \operatorname{sgn} x - 3x^2|x^3|. \end{aligned}$$

You can easily verify that the last expression is identically 0 by considering the cases $x > 0$, $x = 0$, and $x < 0$ separately.
(b) The functions x^3 and $|x^3|$ are linearly independent on the real line because one is not a constant multiple of the other.
(c) This does not contradict Theorem 7 because x^3 and $|x^3|$ are not solution of a differential equation of the form (3).

Using Mathematica **to solve ODE**

Let us start with the simplest command that you can use to solve an ode. It is the DSolve command. We illustrate by examples the different applications of this command. The simplest case is to solve y' = y

```
DSolve[y'[x] == y[x], y[x], x]
```

$\{\{y[x] \to e^{x} C[1]\}\}$

The answer is y= C e^x as you expect. Note how *Mathematica* denoted the constant by C[1]. The enxt example is a 2nd order ode

```
DSolve[y''[x] == y[x], y[x], x]
```

$\{\{y[x] \to e^{x} C[1] + e^{-x} C[2]\}\}$

Here we need two arbitrary constants C[1] and C[2]. Let's do an intial value problem.

Solving an Initial Value Problem

Here is how you would solve y'' = y, y(0)=0, y'(0)= 1

```
DSolve[{ y''[x] == y[x], y[0] == 0, y'[0] == 1}, y[x], x]
```

$\left\{\left\{y[x] \to \frac{1}{2} e^{-x} \left(-1 + e^{2x}\right)\right\}\right\}$

As you see, the initial value problem has a unique solution (there are no arbitrary constants in the answer.

Plotting the Solution

First we need to learn to extract the solution from the output. Here is how it is done. First, solve the problem and call the output solution:

```
solution = DSolve[{ y''[x] == y[x], y[0] == 0, y'[0] == 1}, y[x], x]
```

$$\left\{\left\{y[x] \to \frac{1}{2} e^{-x} (-1 + e^{2x})\right\}\right\}$$

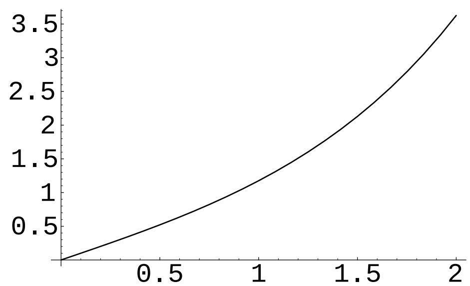
Extract the solution y(x) as follows:

```
y[x_] = y[x] /. solution[[1]]
```

$$\frac{1}{2} e^{-x} (-1 + e^{2x})$$

Now you can plot the solution:

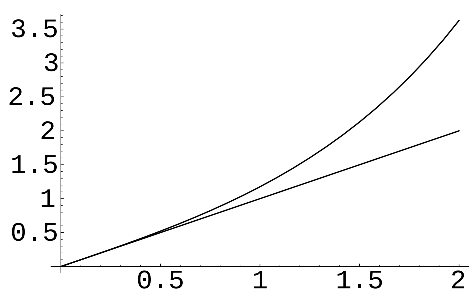
```
Plot[y[x], {x, 0, 2}]
```



The solution y(x) goes through the point (0, 0). This confirms the initial condition y(0) = 0.

Note the intial conditions on the graph: y(0)=0 and y'(0)=1. To confirm that y'(0)=1 (the slope of the graph at x=0 is 1), plot the tangent line (line with slope 1)

```
Plot[{y[x], x}, {x, 0, 2}]
```



The tangent to the solution y(x) at x = 0 is the line y = x, whose slope is 1. Thus y'(0) = 1, because the derivative is equal to the slope of the tangent line.

The Wronskian

The Wronskian is a determinat, so we can compute it using the Det command. Here is an illustration.

```
Clear[y]
sol1 = DSolve[y''[x] + y[x] == 0, y[x], x]
```

```
{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}
```

Two solutions of the differential equation are obtained different values to the constants c1 and c2. For

```
Clear[ y1, y2 ]
y1[x_] = c1 Cos[x];
y2[x_] = c2 Sin[x];
```

Their Wronskian is

```
w[x_] = Det[{{y1[x], y2[x]}, {y1'[x], y2'[x]}}]
```

$c1\,c2\,\mathrm{Cos}[x]^2 + c1\,c2\,\mathrm{Sin}[x]^2$

Let's simplify using the trig identity cos^2 x + sin^2 x =1

```
Simplify[w[x]]
```

```
c1 c2
```

The Wronskian is nonzero if c1\= 0 and c2 \= 0. Let us try a different problem with a nonhomogeneous

```
Clear[y]
sol2 = DSolve[y''[x] + y[x] == 1, y[x], x]
```

```
{{y[x] → 1 + C[1] Cos[x] + C[2] Sin[x]}}
```

Two solutions of the differential equation are obtained different values to the constants c1 and c2. For

```
Clear[ y1, y2 ]
y1[x_] = 1 +  Cos[x];
y2[x_] = 1 + Sin[x];
```

These solutions are clearly linearly independent (one is not a multiple of the other). Their Wronskian is

```
Clear[w]
w[x_] = Det[{{y1[x], y2[x]}, {y1'[x], y2'[x]}}]
```

$\mathrm{Cos}[x] + \mathrm{Cos}[x]^2 + \mathrm{Sin}[x] + \mathrm{Sin}[x]^2$

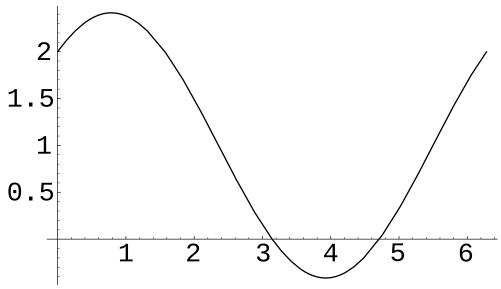
Let's simplify using the trig identity cos^2 x + sin^2 x =1

```
Simplify[w[x]]
```

```
1 + Cos[x] + Sin[x]
```

Let's plot w(x):

```
Plot[w[x], {x, 0, 2 Pi}]
```



The Wronskian does vanish at some values of x without being identically 0. Does this contradict Theorem 7 of Appendix A.1?

Solutions to Exercises A.2

1.

Equation: $y'' - 4\,y' + 3\,y = 0;$

Characteristic equation: $\lambda^2 - 4\,\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0;$

Characteristic roots: $\lambda_1 = 1; \quad \lambda_2 = 3$

General solution: $y = c_1 e^{x} + c_2 e^{3x}$

2.

Equation: $y'' - y' - 6\,y = 0;$

Characteristic equation: $\lambda^2 - \lambda - 6 = 0$

$(\lambda + 2)(\lambda - 3) = 0;$

Characteristic roots: $\lambda_1 = -2; \quad \lambda_2 = 3$

General solution: $y = c_1 e^{-2x} + c_2 e^{3x}$

3.

Equation: $y'' - 5\,y' + 6\,y = 0;$

Characteristic equation: $\lambda^2 - 5\,\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0;$

Characteristic roots: $\lambda_1 = 2; \quad \lambda_2 = 3$

General solution: $y = c_1 e^{2x} + c_2 e^{3x}$

4.

Equation: $2\,y'' - 3\,y' + y = 0;$

Characteristic equation: $2\lambda^2 - 3\,\lambda + 1 = 0$

$(\lambda - 1)(2\lambda - 1) = 0;$

Characteristic roots: $\lambda_1 = 1; \quad \lambda_2 = \dfrac{1}{2}$

General solution: $y = c_1 e^{x} + c_2 e^{x/2}$

5

Equation: $y'' + 2\,y' + y = 0;$

Characteristic equation: $\lambda^2 + 2\,\lambda + 1 = 0$

$(\lambda + 1)^2 = 0;$

Characteristic roots: $\lambda_1 = -1$ (double root)

General solution: $y = c_1 e^{-x} + c_2 x\, e^{-x}$

6.

Equation: $4\,y'' - 13\,y' + 9\,y = 0;$

Characteristic equation: $4\lambda^2 - 13\,\lambda + 9 = 0$

$(\lambda - 1)(4\lambda - 9) = 0;$

Characteristic roots: $\lambda_1 = 1; \quad \lambda_2 = \dfrac{9}{4}$

General solution: $y = c_1\, e^{x} + c_2\, e^{-9\,x/4}$

7.

$$\begin{aligned} &\text{Equation:} & 4\,y'' - 4\,y' + y &= 0;\\ &\text{Characteristic equation:} & 4\lambda^2 - 4\,\lambda + 1 &= 0\\ & & (2\lambda - 1)^2 &= 0;\\ &\text{Characteristic roots:} & \lambda_1 = 1/2 \text{ (double root)};& \\ &\text{General solution:} & y = c_1\,e^{x/2} + c_2\,x\,e^{x/2}& \end{aligned}$$

8.

$$\begin{aligned} &\text{Equation:} & 4\,y'' - 12\,y' + 9\,y &= 0;\\ &\text{Characteristic equation:} & 4\lambda^2 - 12\,\lambda + 9 &= 0\\ & & (2\lambda - 3)^2 &= 0;\\ &\text{Characteristic roots:} & \lambda_1 = 3/2 \text{ (double root)};& \\ &\text{General solution:} & y = c_1\,e^{3\,x/2} + c_2\,x\,e^{3\,x/2}& \end{aligned}$$

9.

$$\begin{aligned} &\text{Equation:} & y'' + y &= 0;\\ &\text{Characteristic equation:} & \lambda^2 + 1 &= 0\\ &\text{Characteristic roots:} & \lambda_1 = i \quad \lambda_2 = -i;& \\ &\text{Case III:} & \alpha = 0, \quad \beta = 1;& \\ &\text{General solution:} & y = c_1\cos x + c_2\sin x& \end{aligned}$$

10.

$$\begin{aligned} &\text{Equation:} & 9\,y'' + 4\,y &= 0;\\ &\text{Characteristic equation:} & 9\lambda^2 + 4 &= 0\\ & & \lambda^2 &= -\tfrac{4}{9}\\ &\text{Characteristic roots:} & \lambda_1 = \frac{4}{9}i \quad \lambda_2 = -\frac{4}{9}i;& \\ &\text{Case III:} & \alpha = 0, \quad \beta = \frac{4}{9};& \\ &\text{General solution:} & y = c_1\cos\left(\tfrac{4}{9}x\right) + c_2\sin\left(\tfrac{4}{9}x\right)& \end{aligned}$$

11.

$$\begin{aligned} &\text{Equation:} & y'' - 4\,y &= 0;\\ &\text{Characteristic equation:} & \lambda^2 - 4 &= 0\\ & & (\lambda - 2)(\lambda + 2) &= 0\\ &\text{Characteristic roots:} & \lambda_1 = 2 \quad \lambda_2 = -2;& \\ &\text{General solution:} & y = c_1 e^{2x} + c_2 e^{-2x}.& \end{aligned}$$

12. $y'' + 3\,y' + 3\,y = 0.$

$$\begin{aligned} &\text{Equation:} & y'' + 3\,y' + 3\,y &= 0;\\ &\text{Characteristic equation:} & \lambda^2 + 3\,\lambda + 3 &= 0;\\ &\text{Characteristic roots:} & \lambda = \frac{-3 \pm \sqrt{9 - 12}}{2} = \frac{-3 \pm i\sqrt{3}}{2};& \\ & & \lambda_1 = \frac{-3 + i\sqrt{3}}{2}, \quad \lambda_2 = \frac{-3 - i\sqrt{3}}{2};& \\ &\text{Case III:} & \alpha = -\frac{3}{2}, \quad \beta = \frac{\sqrt{3}}{2};& \\ &\text{General solution:} & y = c_1 e^{-\frac{3}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{3}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right)& \end{aligned}$$

13.

Equation:	$y'' + 4\,y' + 5\,y = 0;$
Characteristic equation:	$\lambda^2 + 4\,\lambda + 5 = 0;$
Characteristic roots:	$\lambda = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i;$
	$\lambda_1 = -2 + i, \quad \lambda_2 = -2 - i;$
Case III:	$\alpha = -2, \quad \beta = 1;$
General solution:	$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

14.

Equation:	$y'' - 2\,y' + 5\,y = 0;$
Characteristic equation:	$\lambda^2 - 2\,\lambda + 5 = 0;$
Characteristic roots:	$\lambda = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i;$
	$\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i;$
Case III:	$\alpha = 1, \quad \beta = 2;$
General solution:	$y = c_1 e^{x} \cos\big(2x\big) + c_2 e^{x} \sin\big(2x\big).$

15.

Equation:	$y'' + 6\,y' + 13\,y = 0;$
Characteristic equation:	$\lambda^2 + 6\,\lambda + 13 = 0;$
Characteristic roots:	$\lambda = -3 \pm \sqrt{-4} = -3 \pm 2i;$
	$\lambda_1 = -3 + 2i, \quad \lambda_2 = -3 - 2i;$
Case III:	$\alpha = -3, \quad \beta = 2;$
General solution:	$y = c_1 e^{-3x} \cos(2x) + c_2 e^{-3x} \sin(2x)$

16.

Equation:	$2\,y'' - 6\,y' + 5\,y = 0;$
Characteristic equation:	$2\lambda^2 - 6\,\lambda + 5 = 0;$
Characteristic roots:	$\lambda = \dfrac{3 \pm \sqrt{-1}}{2} = \dfrac{3 \pm i}{2};$
	$\lambda_1 = \dfrac{3}{2} + \dfrac{i}{2}, \quad \lambda_2 = \dfrac{3}{2} - \dfrac{i}{2};$
Case III:	$\alpha = \dfrac{3}{2}, \quad \beta = \dfrac{1}{2};$
General solution:	$y = c_1 e^{3x/2} \cos(x/2) + c_2 e^{3x/2} \sin(x/2)$

17.

Equation:	$y''' - 2\,y'' + y' = 0;$
Characteristic equation:	$\lambda^3 - 2\lambda^2 + \lambda = 0$
	$\lambda(\lambda - 1)^2 = 0;$
Characteristic roots:	$\lambda_1 = 0; \quad \lambda_2 = 1$ (double root)
General solution:	$y = c_1 + c_2\, e^{x} + c_3\, x\, e^{x}$

18.

Equation: $y''' - 3\,y'' + 2\,y = 0;$

Characteristic equation:
$$\begin{aligned} \lambda^3 - 3\lambda^2 + 2 &= 0 \\ (\lambda - 1)(\lambda^2 - 2\lambda - 2) &= 0; \end{aligned}$$

Characteristic roots: $\lambda_1 = 1,\ \lambda = 1 \pm \sqrt{-1}$

$\lambda_1 = 1,\ \lambda_2 = 1 + i,\ \lambda_3 = 1 - i;$

General solution: $y = c_1\, e^x + c_2\, e^x \cos x + c_3\, e^x \sin x$

19.

Equation: $y^{(4)} - 2\,y'' + y = 0;$

Characteristic equation:
$$\begin{aligned} \lambda^4 - 2\lambda^2 + 1 &= 0 \\ \left(\lambda^2 - 1\right)^2 &= 0; \end{aligned}$$

Characteristic roots: two double roots, $\lambda_1 = 1,\ \lambda_2 = -1;$

General solution: $y = c_1\, e^x + c_2\, e^{-x} + c_3\, x\, e^x + c_4\, x\, e^{-x}$

20.

Equation: $y^{(4)} - y = 0;$

Characteristic equation:
$$\begin{aligned} \lambda^4 - 1 &= 0; \\ (\lambda^2 - 1)(\lambda^2 + 1) &= 0 \end{aligned}$$

Characteristic roots: $\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = i, \lambda_4 = -i;$

General solution: $y = c_1\, e^x + c_2\, e^{-x} + c_3\, \cos x + c_4 \sin x$

21.

Equation: $y''' - 3\,y'' + 3y' - y = 0;$

Characteristic equation:
$$\begin{aligned} \lambda^3 - 3\lambda^2 + 3\lambda - 1 &= 0 \\ (\lambda - 1)^3 &= 0; \end{aligned}$$

Characteristic roots: $\lambda_1 = 1$ (multiplicity 3);

General solution: $y = c_1\, e^x + c_2\, x\, e^x + c_3\, x^2 e^x$

22.

Equation: $y''' + y = 0;$

Characteristic equation:
$$\begin{aligned} \lambda^3 + 1 &= 0 \\ (\lambda - 1)(\lambda^2 - \lambda + 1) &= 0; \end{aligned}$$

Characteristic roots: $\lambda_1 = 1,\ \lambda = \dfrac{1 \pm \sqrt{-3}}{2}$

$\lambda_1 = 1,\ \lambda_2 = \dfrac{1 + i\sqrt{3}}{2}, \lambda_3 = \dfrac{1 - i\sqrt{3}}{2};$

General solution: $y = c_1\, e^x + c_2 e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_3\, e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)$

23.

$$\begin{array}{lrcl} \text{Equation:} & y^{(4)} - 6\,y'' + 8y' - 3y & = & 0; \\ \text{Characteristic equation:} & \lambda^4 - 6\lambda^2 + 8\lambda - 3 & = & 0 \\ & (\lambda-1)^3(\lambda+3) & = & 0; \\ \text{Characteristic roots:} & \lambda_1 = 1 \text{ (multiplicity 3)}, \lambda_4 = -3; & & \\ \text{General solution:} & y = c_1\, e^x + c_2\, x\, e^x + c_3\; x^2\, e^x + c_4 e^{-3x}. & & \end{array}$$

24.

$$\begin{array}{lrcl} \text{Equation:} & y^{(4)} + 4\,y''' + 6\,y'' + 4y' + y & = & 0; \\ \text{Characteristic equation:} & \lambda^4 + 4\lambda^2 + 6\lambda^2 + 4\lambda + 1 & = & 0 \\ & (\lambda+1)^4 & = & 0; \\ \text{Characteristic roots:} & \lambda_1 = -1 \text{ (multiplicity 4)}; & & \\ \text{General solution:} & y = c_1\, e^{-x} + c_2\, x\, e^{-x} + c_3\; x^2\, e^{-x} + c_4\, x^3\, e^{-x}. & & \end{array}$$

25.

$$\begin{array}{lrcl} \text{Equation:} & y'' - 4\,y' + 3\,y & = & e^{2x}; \\ \text{Homogeneous equation:} & y'' - 4\,y' + 3\,y & = & 0; \\ \text{Characteristic equation:} & \lambda^2 - 4\lambda + 3 & = & 0 \\ & (\lambda-1)(\lambda-3) & = & 0; \\ \text{Characteristic roots:} & \lambda_1 = 1,\ \lambda_2 = 3; & & \\ \text{Solution of homogeneous equation:} & y_h = c_1\, e^x + c_2\, e^{3x}. & & \end{array}$$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned} y_p &= Ae^{2x}; \\ y_p' &= 2Ae^{2x}; \\ y_p'' &= 4Ae^{2x}. \end{aligned}$$

Plug into the equation $y'' - 4\,y' + 3\,y = e^{2x}$:

$$\begin{aligned} 4Ae^{2x} - 4(2Ae^{2x}) + 3Ae^{2x} &= e^{2x} \\ -Ae^{2x} &= e^{2x}; \\ A &= -1. \end{aligned}$$

Hence $y_p = -e^{2x}$ and so the general solution

$$y_g = c_1\, e^x + c_2\, e^{3x} - e^{2x}.$$

26.

$$\begin{array}{lrcl} \text{Equation:} & y'' -\ y' - 6\,y & = & e^x; \\ \text{Homogeneous equation:} & y'' -\ y' - 6\,y & = & 0; \\ \text{Characteristic equation:} & \lambda^2 - \lambda - 6 & = & 0 \\ & (\lambda+2)(\lambda-3) & = & 0; \\ \text{Characteristic roots:} & \lambda_1 = -2,\ \lambda_2 = 3; & & \\ \text{Solution of homogeneous equation:} & y_h = c_1\, e^{-2x} + c_2\, e^{3x}. & & \end{array}$$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned} y_p &= Ae^x; \\ y_p' &= Ae^x; \\ y_p'' &= Ae^x. \end{aligned}$$

Plug into the equation $y'' - y' - 6\,y = e^x$:

$$\begin{aligned} Ae^x - Ae^x - 6Ae^x &= e^x \\ -6Ae^x &= e^x; \\ A &= -1/6. \end{aligned}$$

Hence $y_p = -e^x/6$ and so the general solution

$$y_g = c_1\, e^{-2x} + c_2\, e^{3x} - \frac{e^x}{6}.$$

27.

Equation:	$y'' - 5\,y' + 6\,y$	$=$	$e^x + x;$
Homogeneous equation:	$y'' - 5\,y' + 6\,y$	$=$	$0;$
Characteristic equation:	$\lambda^2 - 5\,\lambda + 6$	$=$	0
	$(\lambda - 2)(\lambda - 3)$	$=$	$0;$
Characteristic roots:	$\lambda_1 = 2,\ \lambda_2 = 3;$		
Solution of homogeneous equation:	$y_h = c_1\, e^{2x} + c_2\, e^{3x}.$		

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned} y_p &= Ae^x + Bx + C; \\ y_p' &= Ae^x + B; \\ y_p'' &= Ae^x. \end{aligned}$$

Plug into the equation $y'' - 5\,y' + 6\,y = e^x + x$:

$$\begin{aligned} Ae^x - 5\,(Ae^x + B) + 6\,(Ae^x + Bx + C) &= e^x + x \\ 2Ae^x &= e^x; \\ 6B &= 1 \\ 6C - 5B &= 0. \end{aligned}$$

Hence

$$A = 1/2, \quad B = 1/6, \quad C = 5/36; \quad y_p = \frac{e^x}{2} + \frac{x}{6} + \frac{5}{36};$$

and so the general solution

$$y_g = c_1\, e^{2x} + c_2\, e^{3x} + \frac{e^x}{2} + \frac{x}{6} + \frac{5}{36}.$$

28.

$$\begin{array}{lrcl}
\text{Equation:} & 2\,y'' - 3\,y' + y & = & e^{2x} + \sin x; \\
\text{Homogeneous equation:} & 2\,y'' - 3\,y' + y & = & 0; \\
\text{Characteristic equation:} & 2\,\lambda^2 - 3\,\lambda + 1 & = & 0 \\
 & (2\,\lambda - 1)(\lambda - 1) & = & 0; \\
\text{Characteristic roots:} & \lambda_1 = 1/2,\ \lambda_2 = 1; & & \\
\text{Solution of homogeneous equation:} & y_h = c_1\, e^{x/2} + c_2\, e^{x}. & &
\end{array}$$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned}
y_p &= A\cos x + B\sin x + Ce^{2x}; \\
y_p' &= -A\sin x + B\cos x + 2Ce^{2x}; \\
y_p'' &= -A\cos x - B\sin x + 4Ce^{2x}.
\end{aligned}$$

Plug into the equation $2\,y'' - 3\,y' + y = e^{2x} + \sin x$:

$$\begin{aligned}
-2A\cos x - 2B\sin x + 8Ce^{2x} + 3A\sin x - 3B\cos x - 6Ce^{2x} & \\
+A\cos x + B\sin x + Ce^{2x} &= e^{2x} + \sin x \\
-A - 3B &= 0; \\
3A - B &= 1 \\
3C &= 1.
\end{aligned}$$

Hence

$$A = 3/10, \quad B = -1/10, \quad C = 1/3; \quad y_p = \frac{3}{10}\cos x - \frac{1}{10}\sin x + \frac{1}{3}e^{2x};$$

and so the general solution

$$y_g = c_1\, e^{x/2} + c_2\, e^{x} + \frac{3}{10}\cos x - \frac{1}{10}\sin x + \frac{1}{3}e^{2x}.$$

29.

$$\begin{array}{lrcl}
\text{Equation:} & y'' - 4\,y' + 3\,y & = & x\,e^{-x}; \\
\text{Homogeneous equation:} & y'' - 4\,y' + 3\,y & = & 0; \\
\text{Characteristic equation:} & \lambda^2 - 4\,\lambda + 3 & = & 0 \\
 & (\lambda - 1)(\lambda - 3) & = & 0; \\
\text{Characteristic roots:} & \lambda_1 = 1,\ \lambda_2 = 3; & & \\
\text{Solution of homogeneous equation:} & y_h = c_1\, e^{x} + c_2\, e^{3x}. & &
\end{array}$$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned}
y_p &= (Ax + B)e^{-x}; \\
y_p' &= e^{-x}(Ax - B + A); \\
y_p'' &= e^{-x}(Ax - B - 2A).
\end{aligned}$$

Plug into the equation $y'' - 4\,y' + 3\,y = e^{-x}$:

$$\begin{aligned} e^{-x}(Ax - B - 2A) - 4e^{-x}(Ax - B + A) + 3(Ax + B)e^{-x} &= e^{-x} \\ 8A &= 1; \\ -6A + 8B &= 0. \end{aligned}$$

Hence

$$A = 1/8, \quad B = 3/32; \quad y_p = (\frac{x}{8} + \frac{3}{32})e^{-x};$$

and so the general solution

$$y_g = c_1\, e^{x} + c_2\, e^{3x} + (\frac{x}{8} + \frac{3}{32})e^{-x}.$$

30.

Equation:	$y'' - 4\,y = \cosh x;$
Homogeneous equation:	$y'' - 4\,y = 0;$
Characteristic equation:	$\lambda^2 - 4 = 0$
Characteristic roots:	$\lambda_1 = -2,\ \lambda_2 = 2;$
Solution of homogeneous equation:	$y_h = c_1\, \cosh 2x + c_2\, \sinh 2x.$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned} y_p &= A\cosh x + B\sinh x; \\ y_p' &= A\sinh x + B\cosh x; \\ y_p'' &= A\cosh x + B\sinh x. \end{aligned}$$

Plug into the equation $y'' - 4\,y = \cosh x$:

$$\begin{aligned} A\cosh x + B\sinh x - 4A\cosh x - 4B\sinh x; &= \cosh x \\ -3A &= 1; \\ -3B &= 0. \end{aligned}$$

Hence

$$A = -1/3, \quad B = 0; \quad y_p = -\frac{1}{3}\cosh x;$$

and so the general solution

$$y_g = c_1\, \cosh 2x + c_2\, \sinh 2x - \frac{1}{3}\cosh x.$$

31. Using the half-angle formula,

Equation:	$y'' + 4\,y = \frac{1}{2} - \frac{1}{2}\cos 2x;$
Homogeneous equation:	$y'' + 4\,y = 0;$
Characteristic equation:	$\lambda^2 + 4 = 0$
Characteristic roots:	$\lambda_1 = -2i,\ \lambda_2 = 2i;$
Solution of homogeneous equation:	$y_h = c_1\, \cos 2x + c_2\, \sin 2x.$

To find a particular solution, we apply the method of undetermined coefficients. We also notice that 1/8 is a solution of $y'' + 4y = 1/8$, so we really do not need to look for the solution corresponding to the term 1/8 of the nonhomogeneous part. We try

$$\begin{aligned} y_p &= \frac{1}{8} + Ax\,\cos 2x + Bx\,\sin 2x; \\ y_p' &= A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x; \\ y_p'' &= -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x. \end{aligned}$$

Plug into the equation $y'' + 4\,y = \frac{1}{2} - \frac{1}{2}\cos 2x$:

$$\begin{aligned} -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x \\ +4\left(\frac{1}{8} + Ax\,\cos 2x + Bx\,\sin 2x\right) &= \frac{1}{2} - \frac{1}{2}\cos 2x; \\ -4A\sin 2x + 4B\cos 2x &= -\frac{1}{2}\cos 2x; \\ 4A = 0;\ 4B = \frac{1}{2}. \end{aligned}$$

Hence

$$A = 0, \quad B = -\frac{1}{8}; \quad y_p = \frac{1}{8} - \frac{x}{8}\sin 2x;$$

and so the general solution

$$y_g = c_1\,\cos 2x + c_2\,\sin 2x + \frac{1}{8} - \frac{x}{8}\sin 2x.$$

32.

Equation: $y'' + 4\,y = x\,\sin 2x;$

Homogeneous equation: $y'' + 4\,y = 0;$

Characteristic equation: $\lambda^2 + 4 = 0$

Characteristic roots: $\lambda_1 = -2i,\ \lambda_2 = 2i;$

Solution of homogeneous equation: $y_h = c_1\,\cos 2x + c_2\,\sin 2x.$

To find a particular solution, we apply the method of undetermined coefficients:

$$\begin{aligned} y_p &= x(Ax + B)\,\cos 2x + x(Cx + D)\,\sin 2x; \\ y_p' &= (2Ax + B)\cos 2 - 2(Ax^2 + Bx)\sin 2x + (2Cx + D)\sin 2x + 2(Cx^2 + Dx)\cos 2x; \\ y_p'' &= 2Ax\cos 2x - 4(2Ax + B)\sin 2x - 4(Ax^2 + Bx)\cos 2x \\ &\quad +2Cx\sin 2x + 4(2Cx + D)\cos 2x - 4(Cx^2 + Dx)\sin 2x. \end{aligned}$$

Plug into the equation $y'' + 4\,y = x\,\sin 2x$:

$$\begin{aligned} 2Ax\cos 2x - 4(2Ax + B)\sin 2x + 2Cx\sin 2x + 4(2Cx + D)\cos 2x &= x\,\sin 2x; \\ (2A + 4(2Cx + D)))\cos 2x + (2C - 4(2Ax + B)))\sin 2x &= x\,\sin 2x; \\ -8A = 1;\ 8C = 0; \\ -4B + 2C = 0 \\ 2A + 4D = 0. \end{aligned}$$

Hence

$$A=-\frac{1}{8},\ C=0,\ B=0,\ D=\frac{1}{16};\quad y_p=-\frac{1}{8}x^2\cos 2x+\frac{1}{16}x\sin 2x;$$

and so the general solution

$$y_g=c_1\,\cos 2x+c_2\,\sin 2x-\frac{1}{8}x^2\cos 2x+\frac{1}{16}x\sin 2x.$$

33.

Equation:	$y''+y$	$=$	$\frac{1}{2}+\frac{1}{2}\cos 2x;$
Homogeneous equation:	$y''+y$	$=$	$0;$
Characteristic equation:	λ^2+1	$=$	0
Characteristic roots:	$\lambda_1=-i,\ \lambda_2=i;$		
Solution of homogeneous equation:	$y_h=c_1\,\cos x+c_2\,\sin x.$		

To find a particular solution, we apply the method of undetermined coefficients. We also use our experience and simplify the solution by trying

$$\begin{aligned} y_p &= \frac{1}{2}+A\cos 2x; \\ y_p' &= -2A\sin 2x; \\ y_p'' &= -4A\cos 2x. \end{aligned}$$

Plug into the equation $y''+y=\frac{1}{2}+\frac{1}{2}\cos 2x$:

$$\begin{aligned} -3A\cos 2x+\frac{1}{2} &= \frac{1}{2}+\frac{1}{2}\cos 2x; \\ -3A &= \frac{1}{2}; \\ A &= -\frac{1}{6}. \end{aligned}$$

Hence

$$y_p=-\frac{1}{6}\cos 2x+\frac{1}{2};$$

and so the general solution

$$y_g=c_1\,\cos 2x+c_2\,\sin 2x-\frac{1}{6}\cos 2x+\frac{1}{2}.$$

34.

Equation:	$y''+2\,y'+2\,y$	$=$	$e^{-x}\,\cos 2x;$
Homogeneous equation:	$y''+2\,y'+2y$	$=$	$0;$
Characteristic equation:	$\lambda^2+2\lambda+2$	$=$	0
Characteristic roots:	$\lambda_1=-1-i,\ \lambda_2=-1+i;$		
Solution of homogeneous equation:	$y_h=c_1\,e^{-x}\cos x+c_2\,e^{-x}\sin x.$		

To find a particular solution, we apply the method of undetermined coefficients:

$$\begin{aligned}
y_p &= e^{-x}x\big(A\cos x + B\sin x\big);\\
y_p' &= -e^{-x}x\big(A\cos x + B\sin x\big)\\
&\quad +e^{-x}\big[\big(A\cos x + B\sin x\big) + x\big(-A\sin x + B\cos x\big)\big];\\
y_p'' &= e^{-x}x\big(A\cos x + B\sin x\big)\\
&\quad -e^{-x}\big[\big(A\cos x + B\sin x\big) + x\big(-A\sin x + B\cos x\big)\big]\\
&\quad -e^{-x}\big[\big(A\cos x + B\sin x\big) + x\big(-A\sin x + B\cos x\big)\big]\\
&\quad e^{-x}\big[\big(-A\sin x + B\cos x\big) + \big(-A\sin x + B\cos x\big) + x\big(-A\cos x - B\sin x\big)\big]\\
&= -2e^{-x}\big[\big(A\cos x + B\sin x\big) + x\big(-A\sin x + B\cos x\big)\big]\\
&\quad +2e^{-x}\big(-A\sin x + B\cos x\big).
\end{aligned}$$

Plug into the equation $y'' + 2\,y' + 2\,y = e^{-x}\,\cos 2x$:

$$\begin{aligned}
-2e^{-x}\big[\big(A\cos x + B\sin x\big) + x\big(-A\sin x + B\cos x\big)\big]&\\
+2e^{-x}\big(-A\sin x + B\cos x\big)&\\
-2e^{-x}x\big(A\cos x + B\sin x\big)&\\
+2e^{-x}\big[\big(A\cos x + B\sin x\big) + x\big(-A\sin x + B\cos x\big)\big]&\\
2e^{-x}x\big(A\cos x + B\sin x\big) &= e^{-x}\,\cos 2x;\\
2e^{-x}\big(-A\sin x + B\cos x\big) &= e^{-x}\,\cos 2x.
\end{aligned}$$

Hence

$$A = 0,\ 2B = 1 \ \Rightarrow\ B = \frac{1}{2};\quad y_p = \frac{1}{2}x\,e^{-x}\sin x;$$

and so the general solution

$$y_g = c_1\,e^{-x}\cos x + c_2\,e^{-x}\sin x + \frac{1}{2}xe^{-x}\sin x.$$

35.

Equation:	$y'' + 2\,y' + y = e^{-x};$
Homogeneous equation:	$y'' + 2\,y' + y = 0;$
Characteristic equation:	$\lambda^2 + 2\,\lambda + 1 = 0$
Characteristic roots:	$\lambda_1 = -1$ (double root);
Solution of homogeneous equation:	$y_h = c_1\,e^{-x} + c_2\,x\,e^{-x}.$

To find a particular solution, we apply the method of undetermined coefficients:

$$\begin{aligned}
y_p &= A\,x^2\,e^{-x};\\
y_p' &= -A\,x^2\,e^{-x} + 2A\,x\,e^{-x};\\
y_p'' &= A\,x^2\,e^{-x} - 2A\,x\,e^{-x} + 2A\,e^{-x} - 2A\,x\,e^{-x}.
\end{aligned}$$

Plug into the equation $y'' + 2\,y' + y = e^{-x}$:

$$\begin{aligned}
A\,x^2\,e^{-x} - 2A\,x\,e^{-x} + 2A\,e^{-x} - 2A\,x\,e^{-x}&\\
+2\big(-A\,x^2\,e^{-x} + 2A\,x\,e^{-x}\big) + A\,x^2\,e^{-x} &= e^{-x};\\
2Ae^{-x} &= e^{-x};\\
A = \frac{1}{2}.&
\end{aligned}$$

Hence

$$y_p = \frac{1}{2}x^2 e^{-x};$$

and so the general solution

$$y_g = c_1\, e^{-x} + c_2\, xe^{-x} + \frac{1}{2}x^2 e^{-x}.$$

36.

$$\begin{array}{lrcl}
\text{Equation:} & y'' + 2y' + y & = & x\,e^{-x} + 6; \\
\text{Homogeneous equation:} & y'' + 2\,y' + y & = & 0; \\
\text{Characteristic equation:} & \lambda^2 + 2\,\lambda + 1 & = & 0 \\
\text{Characteristic roots:} & \lambda_1 = -1 \text{ (double root)}; & & \\
\text{Solution of homogeneous equation:} & y_h = c_1\, e^{-x} + c_2\, x\, e^{-x}. & &
\end{array}$$

For a particular solution, try

$$\begin{aligned}
y_p &= 6 + A\, x^3\, e^{-x}; \\
y_p' &= -A\, x^3\, e^{-x} + 3A\, x^2\, e^{-x}; \\
y_p'' &= A\, x^3\, e^{-x} - 3A\, x^2\, e^{-x} + 6A\, x\, e^{-x} - 3A\, x^2\, e^{-x}.
\end{aligned}$$

Plug into the equation $y'' + 2\,y' + y = 6 + x\,e^{-x}$:

$$\begin{aligned}
A\, x^3\, e^{-x} - 3A\, x^2\, e^{-x} + 6A\, x\, e^{-x} - 3A\, x^2\, e^{-x} & \\
-A\, x^3\, e^{-x} + 3A\, x^2\, e^{-x} + 6 + A\, x^3\, e^{-x} &= 6 + x\, e^{-x}; \\
6A\, x\, e^{-x} &= x\, e^{-x}; \\
A = \frac{1}{6}. &
\end{aligned}$$

Hence

$$y_p = 6 + \frac{1}{6}x^3 e^{-x};$$

and so the general solution

$$y_g = c_1\, e^{-x} + c_2\, xe^{-x} + \frac{1}{6}x^3 e^{-x} + 6.$$

37.

$$\begin{array}{lrcl}
\text{Equation:} & y'' - y' - 2\,y & = & x^2 - 4; \\
\text{Homogeneous equation:} & y'' - y' - 2\,y & = & 0; \\
\text{Characteristic equation:} & \lambda^2 - \lambda - 2 & = & 0 \\
\text{Characteristic roots:} & \lambda_1 = -1, \quad \lambda_2 = 2; & & \\
\text{Solution of homogeneous equation:} & y_h = c_1\, e^{-x} + c_2\, e^{2x}. & &
\end{array}$$

For a particular solution, try

$$\begin{aligned}
y_p &= A\, x^2 + B\, x + C; \\
y_p' &= 2A\, x + B; \\
y_p'' &= 2A.
\end{aligned}$$

Plug into the equation $y'' - y' - 2\,y = x^2 - 4$:

$$\begin{aligned} 2A - 2A\,x - B - 2A\,x^2 - 2B\,x - 2C &= x^2 - 4; \\ -2A &= 1; \\ A = -\frac{1}{2}; & \\ -2A - 2B = 0 &\Rightarrow 1 - 2B = 0; \\ & B = \frac{1}{2}; \\ 2A - 2C - B = 4 &\Rightarrow -\frac{1}{2} - \frac{1}{2} - 2C = 4 \\ & C = \frac{5}{4}. \end{aligned}$$

Hence

$$y_p = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{5}{4};$$

and so the general solution

$$y_g = c_1\,e^{-x} + c_2\,e^{2x} - \frac{1}{2}x^2 + \frac{1}{2}x + \frac{5}{4}.$$

38.

Equation: $y'' - y' - 2\,y = 2\,x^2 + x\,e^x$;

Homogeneous equation: $y'' - y' - 2\,y = 0$;

Characteristic equation: $\lambda^2 - \lambda - 2 = 0$

Characteristic roots: $\lambda_1 = -1, \quad \lambda_2 = 2$;

Solution of homogeneous equation: $y_h = c_1\,e^{-x} + c_2\,e^{2x}$.

For a particular solution, try

$$\begin{aligned} y_p &= A\,x^2 + B\,x + C + (Dx + E)e^x; \\ y_p' &= 2A\,x + B + De^x + (Dx + E)e^x; \\ y_p'' &= 2A + 2De^x + (Dx + E)e^x. \end{aligned}$$

Plug into the equation $y'' - y' - 2\,y = 2\,x^2 + x\,e^x$:

$$\begin{aligned} 2A + 2De^x + (Dx + E)e^x - 2A\,x - B - De^x - (Dx + E)e^x & \\ -2A\,x^2 - 2B\,x - 2C - 2(Dx + E)e^x &= 2\,x^2 + x\,e^x; \\ -2A\,x^2 + x(-2B - 2A) + 2A - B - 2C + (D - 2E)e^x - 2Dxe^x &= 2\,x^2 + x\,e^x; \\ -2A = 2 &\Rightarrow A = -1; \\ -2D = 1 &\Rightarrow D = -\frac{1}{2}; \\ -2B - 2A = 0 &\Rightarrow B = -A = 1; \\ 2A - 2C - B = 0 &\Rightarrow -2 - 1 - 2C = 0 \\ &\Rightarrow C = -\frac{3}{2} \\ D - 2E = 0 &\Rightarrow E = \frac{D}{2} = -\frac{1}{4}. \end{aligned}$$

Hence

$$y_p = -x^2 + x - \frac{3}{2}x + (-\frac{1}{2}x - \frac{1}{4})e^x;$$

and so the general solution

$$y_g = c_1\, e^{-x} + c_2\, e^{2x} - x^2 + x - \frac{3}{2}x + (-\frac{1}{2}x - \frac{1}{4})e^x.$$

39. Even though the equation is first order and we can use the integration factor method of the previous section, we will use the method of this section to illustrate another interesting approach.

Equation:	$y' + 2\,y$	$=$	$2\,x + \sin x;$
Homogeneous equation:	$y' + 2\,y$	$=$	$0;$
Characteristic equation:	$\lambda + 2$	$=$	0
Characteristic root:	$\lambda_1 = -2;$		
Solution of homogeneous equation:	$y_h = c_1\, e^{-2x}.$		

For a particular solution, try

$$\begin{aligned} y_p &= A\,x + B + C\cos x + D\sin x; \\ y_p' &= A - C\sin x + D\cos x; \end{aligned}$$

Plug into the equation $y' + 2\,y = 2x + \sin x$:

$$\begin{aligned} A - C\sin x + D\cos x + 2A\,x + 2B + 2C\cos x + 2D\sin x &= 2x + \sin x; \\ 2A = 2 &\Rightarrow A = 1; \\ A + 2B = 0 &\Rightarrow B = -\frac{1}{2}A = -\frac{1}{2}; \\ 2D - C = 1 \text{ and } D + 2C = 0 &\Rightarrow D = \frac{2}{5},\ C = -\frac{1}{5}. \end{aligned}$$

Hence

$$y_p = x - \frac{1}{2} - \frac{1}{5}\cos x + \frac{2}{5}\sin x;$$

and so the general solution

$$y_g = c_1 e^{-2x} + x - \frac{1}{2} - \frac{1}{5}\cos x + \frac{2}{5}\sin x.$$

40. Use preceding exercise:

Equation:	$y' + 2\,y$	$=$	$\sin x;$
Solution of homogeneous equation:	y_h	$=$	$c_1\, e^{-2x}.$

For a particular solution, try $y_p = C\cos x + D\sin x$, and as in the previous solution. it follows that

$$y_p = -\frac{1}{5}\cos x + \frac{2}{5}\sin x;$$

and so the general solution

$$y_g = c_1 e^{-2x} - \frac{1}{5}\cos x + \frac{2}{5}\sin x.$$

41. $2\,y' - y = e^{2x}$.

$$\begin{array}{lrcl} \text{Equation:} & 2\,y' - y & = & e^{2x}; \\ \text{Homogeneous equation:} & 2\,y' - y & = & 0; \\ \text{Characteristic equation:} & 2\,\lambda - 1 & = & 0 \\ \text{Characteristic root:} & \lambda_1 = \frac{1}{2}; & & \\ \text{Solution of homogeneous equation:} & y_h = c_1\, e^{x/2}. & & \end{array}$$

For a particular solution, try

$$\begin{aligned} y_p &= Ae^{2x}; \\ y_p' &= 2Ae^{2x}; \end{aligned}$$

Plug into the equation $2y' - y = e^{2x}$:

$$\begin{aligned} 4Ae^{2x} - Ae^{2x} &= e^{2x}; \\ 3A = 1 \quad &\Rightarrow \quad A = \frac{1}{3}. \end{aligned}$$

Hence

$$y_p = \frac{1}{3}e^{2x};$$

and so the general solution

$$y_g = c_1 e^{x/2} + \frac{1}{3}e^{2x}.$$

42. $y' - 7y = e^x + \cos x$.

$$\begin{array}{lrcl} \text{Equation:} & y' - 7y & = & e^x + \cos x; \\ \text{Homogeneous equation:} & y' - 7y & = & 0; \\ \text{Characteristic equation:} & \lambda - 7 & = & 0 \\ \text{Characteristic root:} & \lambda_1 = 7; & & \\ \text{Solution of homogeneous equation:} & y_h = c_1\, e^{7x}. & & \end{array}$$

For a particular solution, try

$$\begin{aligned} y_p &= Ae^x + B\cos x + C\sin x; \\ y_p' &= Ae^x - B\sin x + C\cos x; \end{aligned}$$

Plug into the equation $y' - 7y = e^x + \cos x$:

$$\begin{aligned} Ae^x - B\sin x + C\cos x - 7Ae^x - 7B\cos x - 7C\sin x &= e^x + \cos x; \\ -6A = 1 \quad &\Rightarrow \quad A = -\frac{1}{6}; \\ B + 7C = 0 \text{ and } 7B + 49C = 0 \quad &\Rightarrow \quad C = \frac{1}{50},\ B = -\frac{7}{50}. \end{aligned}$$

Hence

$$y_p = -\frac{1}{6}e^x - \frac{7}{50}\cos x + \frac{1}{50}\sin x;$$

and so the general solution

$$y_g = c_1\, e^{7x} + -\frac{1}{6}e^{x} - \frac{7}{50}\cos x + \frac{1}{50}\sin x.$$

43.

$$\begin{array}{lrcl}
\text{Equation:} & y'' + 9y & = & \sum_{n=1}^{6} \frac{\sin nx}{n}; \\
\text{Homogeneous equation:} & y'' + 9y & = & 0; \\
\text{Characteristic roots:} & \lambda_1 = 3i, & & \lambda_2 = -3i; \\
\text{Solution of homogeneous equation:} & y_h & = & c_1 \cos 3x + c_2 \sin 3x.
\end{array}$$

To find a particular solution, we note that the term $\frac{\sin 3x}{3}$, which appears in the nonhomogeneous term, is part of the homogeneous solution. So for a particular solution that corresponds to this term, we try $y_3 = A_3 x \sin 3x + B_3 x \cos 3x$. For the remaining terms of the nonhomogeneous part, use

$$\sum_{n=1,\ n\neq 3}^{6} B_n \sin nx.$$

Thus we try

$$\begin{aligned}
y_p &= A_3 x \sin 3x + B_3 x \cos 3x + \sum_{n=1,\ n\neq 3}^{6} B_n \sin nx; \\
y_p' &= A_3 \sin 3x + 3A_3 x \cos 3x + B_3 \cos 3x - 3B_3 x \sin 3x + \sum_{n=1,\ n\neq 3}^{6} nB_n \cos nx; \\
y_p'' &= 3A_3 \cos 3x + 3A_3 \cos 3x - 9A_3 x \sin 3x - 3B_3 \sin 3x - 3B_3 \sin 3x \\
&\quad -9B_3 x \cos 3x - \sum_{n=1,\ n\neq 3}^{6} n^2 B_n \sin nx;
\end{aligned}$$

Plug into the equation $y'' + 9y = \frac{\sin 3x}{3} + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n}$:

$$\begin{aligned}
6A_3 \cos 3x - 6B_3 \sin 3x + \sum_{n=1,\ n\neq 3}^{6} (9 - n^2) B_n \sin nx &= \frac{\sin 3x}{3} + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n} \\
6A_3 \cos 3x - 6B_3 \sin 3x &= \frac{\sin 3x}{3}; \\
\sum_{n=1,\ n\neq 3}^{6} (9 - n^2) B_n \sin nx &= \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n}; \\
6A_3 = 0,\ -6B_3 = \frac{1}{3} \quad &\Rightarrow \quad A_3 = 0,\ B_3 = -\frac{1}{18}; \\
(n^2 - 9) B_n = \frac{1}{n}\ (n \neq 3) \quad &\Rightarrow \quad B_n = \frac{1}{n(9 - n^2)}.
\end{aligned}$$

Hence

$$y_p = -\frac{1}{18} x \cos 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n(9 - n^2)};$$

and so the general solution

$$y_g = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{18} x \cos 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n(9-n^2)}.$$

44.

$$\begin{array}{lrcl} \text{Equation:} & y'' + y & = & \displaystyle\sum_{n=1}^{6} \frac{\sin 2nx}{n}; \\ \text{Homogeneous equation:} & y'' + y & = & 0; \\ \text{Characteristic roots:} & \lambda_1 = i, & & \lambda_2 = -i; \\ \text{Solution of homogeneous equation:} & y_h & = & c_1 \cos x + c_2 \sin x. \end{array}$$

To find a particular solution, try

$$\begin{aligned} y_p &= \sum_{n=1}^{6} B_n \sin 2nx; \\ y_p'' &= -\sum_{n=1}^{6} 4n^2 B_n \sin 2nx; \end{aligned}$$

Plug into the equation $y'' + y = \sum_{n=1}^{6} \frac{\sin 2nx}{n}$:

$$\begin{aligned} \sum_{n=1}^{6} (1-4n^2) B_n \sin 2nx &= \sum_{n=1}^{6} \frac{\sin 2nx}{n} \\ (1-4n^2) B_n &= \frac{1}{n}; \end{aligned}$$

$$B_n = \frac{1}{n(1-4n^2)}.$$

Hence

$$y_p = \sum_{n=1}^{6} \frac{\sin nx}{n(1-4n^2)};$$

and so the general solution

$$y_g = c_1 \cos x + c_2 \sin x + \sum_{n=1}^{6} \frac{\sin nx}{n(1-4n^2)}.$$

45. Write the equation in the form

$$\begin{aligned} y'' - 4y' + 3y &= e^{2x} \sinh x \\ &= e^{2x} \frac{1}{2} \left(e^{x} - e^{-x}\right) \\ &= \frac{1}{2} \left(e^{3x} - e^{x}\right). \end{aligned}$$

From Exercise 25, $y_h = c_1 e^x + c_2 e^{3x}$. For a particular solution try

$$y_p = Axe^{3x} + Bxe^{x}.$$

46. $y'' - 4\,y' + 3\,y = e^x \sinh 2x$. Write the equation in the form

$$\begin{aligned} y'' - 4\,y' + 3\,y &= e^x \sinh 2x \\ &= e^x \frac{1}{2}\big(e^{2x} - e^{-2x}\big) \\ &= \frac{1}{2}\big(e^{3x} - e^{-x}\big). \end{aligned}$$

From Exercise 25, $y_h = c_1 e^x + c_2 e^{3x}$. For a particular solution try

$$y_p = Axe^{3x} + Be^{-x}.$$

47. $y'' + 2y' + 2\,y = \cos x + 6x^2 - e^{-x} \sin x$. Characteristic equation

$$\lambda^2 + 2\,\lambda + 2 = 0 \;\Rightarrow\; \lambda = -1 \pm i.$$

So $y_h = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$. For a particular solution, try

$$y_p = A\cos x + B\sin x + Cx^2 + Dx + E + e^{-x}\big(Fx\sin x + Gx\cos x\big).$$

48. $y^{(4)} - y = \cosh x + \cosh 2x$. The equation is not of the type that is covered by the method of undetermined coefficients, but we can take a hint from this method. Characteristic equation

$$\lambda^4 - 1 = 0 \;\Rightarrow\; \lambda = \pm 1,\ \pm i.$$

So $y_h = c_1 e^{-x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$. For a particular solution, try

$$y_p = Ae^x + Be^{-x} + Ce^{2x} + De^{-2x}.$$

49. $y'' - 3\,y' + 2\,y = 3x^4 e^x + x\,e^{-2x} \cos 3x$. Characteristic equation

$$\lambda^2 - 3\,\lambda + 2 = 0 \;\Rightarrow\; \lambda_1 = 1,\ \lambda_2 = 2.$$

So $y_h = c_1 e^x + c_2 e^{2x}$. For a particular solution, try

$$y_p = x(Ax^4 + Bx^3 + Cx^2 + Dx + E)e^x + (Gx + H)\,e^{-2x}\cos 3x + (Kx + L)\,e^{-2x}\sin 3x.$$

50. $y'' - 3\,y' + 2\,y = x^4 e^{-x} + 7\,x^2\,e^{-2x}$. From Exercise 49, $y_h = c_1 e^x + c_2 e^{2x}$. For a particular solution, try

$$y_p = x(Ax^4 + Bx^3 + Cx^2 + Dx + E)e^{-x} + (Gx^2 + Hx + K)\,e^{-2x}.$$

51. $y'' + 4\,y = e^{2x}(\sin 2x + 2\cos 2x)$. We have $y_h = c_1 \cos 2x + c_2 \sin 2x$. For a particular solution, try

$$y_p = e^{2x}(A\sin 2x + B\cos 2x).$$

52. $y'' + 4\,y = x\sin 2x + 2\,e^{2x}\cos 2x$. We have $y_h = c_1 \cos 2x + c_2 \sin 2x$. For a particular solution, try

$$y_p = x(Ax + B)\sin 2x + x(Cx + D)\cos 2x + e^{2x}(E\sin 2x + F\cos 2x).$$

53. $y'' - 2\,y' + y = 6x - e^x$. Characteristic equation

$$\lambda^2 - 2\,\lambda + 1 = 0 \;\Rightarrow\; \lambda_1 = 1(\text{double root}).$$

So $y_h = c_1 e^x + c_2 x e^x$. For a particular solution, try

$$y_p = Ax + B + Cx^2 e^x.$$

54. $y'' - 3y' + 2y = e^x + 3e^{-2x}$. From Exercise 49, $y_h = c_1 e^x + c_2 e^{2x}$. For a particular solution, try

$$y_p = Axe^x + B\,e^{-2x}.$$

55. From Exercise 25, we have

$$y_h = c_1 e^x + c_2 e^{3x}.$$

If $\alpha \neq 1$ or 3 (the characteristic roots), a particular solution of

$$y'' - 4y' + 3y = e^{\alpha x}$$

is $y_p = Ae^{\alpha x}$. Plugging into the equation, we find

$$\begin{aligned} A\alpha^2 e^{\alpha x} - 4\alpha A e^{\alpha x} + 3Ae^{\alpha x} &= e^{\alpha x}; \\ A(\alpha^2 - 4\alpha + 3) &= 1 \quad (\text{divide by } e^{\alpha x} \neq 0); \\ A &= \frac{1}{\alpha^2 - 4\alpha + 3}. \end{aligned}$$

Note that $\alpha^2 - 4\alpha + 3 \neq 0$, because $\alpha \neq 1$ or 3. So the general solution in this case is of the form

$$y_g = c_1 e^x + c_2 e^{3x} + \frac{e^{\alpha x}}{\alpha^2 - 4\alpha + 3}.$$

If $\alpha = \lambda$ is one of the characteristic roots 1 or 3, then we modify the particular solution and use $y_p = Axe^{\lambda x}$. Then

$$\begin{aligned} y_p' &= Ae^{\lambda x} + Ax\lambda e^{\lambda x}, \\ y_p'' &= A\lambda e^{\lambda x} + A\lambda e^{\lambda x} + Ax\lambda^2 e^{\lambda x} \\ &= 2A\lambda e^{\lambda x} + Ax\lambda^2 e^{\lambda x}. \end{aligned}$$

Plug into the left side of the equation

$$\begin{aligned} & 2A\lambda e^{\lambda x} + Ax\lambda^2 e^{\lambda x} - 4\big(Ae^{\lambda x} + Ax\lambda e^{\lambda x}\big) + 3Ax\lambda e^{\lambda x} \\ =\; & Axe^{\lambda x} \overbrace{\big(\lambda^2 - 4\lambda + 3\big)}^{=0} - 2A\lambda e^{\lambda x} \\ =\; & -2A\lambda e^{\lambda x}. \end{aligned}$$

This should equal $e^{\lambda x}$. So $-2A = 1$ or $A = -1/2$ and hence

$$y_g = c_1 e^x + c_2 e^{3x} - \frac{1}{2} x e^{\lambda x}.$$

56. The characteristic equation of $y'' - 4y' + 4y = 0$ is $\lambda^2 - 4\lambda + 4 = 0$, with a double characteristic root $\lambda = 2$. Thus

$$y_h = c_1 e^{2x} + c_2 x e^{2x}.$$

If $\alpha \neq 2$, a particular solution of

$$y'' - 4y' + 4y = e^{\alpha x}$$

is $y_p = Ae^{\alpha x}$. Plugging into the equation, we find

$$\begin{aligned} A\alpha^2 e^{\alpha x} - 4\alpha A e^{\alpha x} + 4Ae^{\alpha x} &= e^{\alpha x}; \\ A(\alpha^2 - 4\alpha + 4) &= 1 \quad (\text{divide by } e^{\alpha x} \neq 0); \\ A &= \frac{1}{\alpha^2 - 4\alpha + 4}. \end{aligned}$$

Note that $\alpha^2 - 4\alpha + 4 \neq 0$, because $\alpha \neq 2$. So the general solution in this case is of the form

$$y_g = c_1 e^{2x} + c_2 x e^{2x} + \frac{e^{\alpha x}}{\alpha^2 - 4\alpha + 4}.$$

If $\alpha = 2$ is a characteristic root, then we modify the particular solution and use $y_p = Ax^2 e^{2x}$, since it is a double root. Then

$$\begin{aligned} y_p' &= 2Axe^{2x} + 2Ax^2e^{2x}, \\ y_p'' &= 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x} \\ &= 2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}. \end{aligned}$$

Plug into the left side of the equation $y'' - 4y' + 4y = e^{2x}$:

$$2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x} - 4\big(2Axe^{2x} + 2Ax^2e^{2x}\big) + 4Ax^2e^{2x} = 2Ae^{2x}.$$

This should equal e^{2x}. So $2A = 1$ or $A = 1/2$ and hence

$$y_g = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} x^2 e^{2x}.$$

57. $y'' + 4y = \cos \omega x$. We have

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

If $\omega \neq \pm 2$, a particular solution of

$$y'' + 4y = \cos \omega x$$

is $y_p = A \cos \omega x$. So $y_p'' = -A\omega^2 \cos \omega x$. Plugging into the equation, we find

$$\begin{aligned} A \cos \omega x (4 - \omega^2) &= \cos \omega x; \\ A(4 - \omega^2) &= 1; \\ A &= \frac{1}{4 - \omega^2}. \end{aligned}$$

Note that $4 - \omega^2 \neq 0$, because $\omega \neq \pm 2$. So the general solution in this case is of the form

$$y_g = c_1 \cos 2x + c_2 \sin 2x + \frac{\cos \omega x}{4 - \omega^2}.$$

If $\omega = \pm 2$, then we modify the particular solution and use $y_p = x\big(A \cos \omega x + B \sin \omega x\big)$. Then

$$\begin{aligned} y_p' &= \big(A \cos \omega x + B \sin \omega x\big) + x\omega\big(-A \sin \omega x + B \cos \omega x\big), \\ y_p'' &= x\omega^2\big(-A \cos \omega x - B \sin \omega x\big) + 2\omega\big(-A \sin \omega x + B \cos \omega x\big). \end{aligned}$$

Plug into the left side of the equation

$$x\omega^2\big(-A \cos \omega x - B \sin \omega x\big) + 2\omega\big(-A \sin \omega x + B \cos \omega x\big) + 4x\big(A \cos \omega x + B \sin \omega x\big).$$

Using $\omega^2 = 4$, this becomes

$$2\omega\big(-A \sin \omega x + B \cos \omega x\big).$$

This should equal $\cos \omega x$. So $A = 0$ and $2\omega B = 1$ or $B = 1/(2\omega)$.

$$y_g = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2\omega} x \sin \omega x \quad (\omega = \pm 2).$$

Note that if $\omega = 2$ or $\omega = -2$, the solution is

$$y_g = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} x \sin 2x.$$

58. We have

$$y_h = c_1 \cos 3x + c_2 \sin 3x.$$

If $\omega \neq \pm 3$, a particular solution of

$$y'' + 9y = \sin \omega x$$

is $y_p = A \sin \omega x$. So $y_p'' = -A\omega^2 \sin \omega x$. Plugging into the equation, we find

$$\begin{aligned} A \sin \omega x (9 - \omega^2) &= \sin \omega x; \\ A(9 - \omega^2) &= 1; \\ A &= \frac{1}{9 - \omega^2}. \end{aligned}$$

Note that $9 - \omega^2 \neq 0$, because $\omega \neq \pm 3$. So the general solution in this case is of the form

$$y_g = c_1 \cos 3x + c_2 \sin 3x + \frac{\sin \omega x}{9 - \omega^2}.$$

If $\omega = \pm 3$, then we modify the particular solution and use $y_p = Ax \cos \omega x$. Then

$$\begin{aligned} y_p' &= A \cos \omega x - A\omega x \sin \omega x, \\ y_p'' &= -2A\omega \sin \omega x - A\omega^2 x \cos \omega x. \end{aligned}$$

Plug into the equation and get

$$-2A\omega \sin \omega x = \sin \omega x.$$

So $-2\omega A = 1$ or $A = -1/(2\omega)$. The general solution is

$$y_g = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{2\omega} x \cos \omega x \quad (\omega = \pm 3).$$

If $\omega = 3$ the general solution is

$$y_g = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{6} x \cos 3x.$$

If $\omega = -3$ the general solution is

$$y_g = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{6} x \cos 3x.$$

59. We have

$$y_h = c_1 \cos \omega x + c_2 \sin \omega x.$$

If $\omega \neq \pm 2$, a particular solution of

$$y'' + \omega^2 y = \sin 2x$$

is $y_p = A \sin 2x$. So $y_p'' = -4A \sin 2x$. Plugging into the equation, we find

$$\begin{aligned} A \sin 2x(-4 + \omega^2) &= \sin 2; \\ A(\omega^2 - 4) &= 1; \\ A &= \frac{1}{\omega^2 - 4}. \end{aligned}$$

Note that $\omega^2 - 4 \neq 0$, because $\omega \neq \pm 2$. So the general solution in this case is of the form

$$y_g = c_1 \cos \omega x + c_2 \sin \omega x + \frac{\sin 2x}{\omega^2 - 4}.$$

If $\omega = \pm 2$, then we modify the particular solution and use $y_p = Ax \cos 2x$. Then

$$\begin{aligned} y_p' &= A \cos 2x - 2Ax \sin 2x, \\ y_p'' &= -4A \sin 2x - 4Ax \cos 2x. \end{aligned}$$

Plug into the equation and get

$$-4A \sin 2x = \sin 2x.$$

So $-4A = 1$ or $A = -1/4$. The general solution is

$$y_g = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{4} x \cos 2x.$$

60. The characteristic equation of $y'' + by' + y = 0$ is $\lambda^2 + b\lambda + 1 = 0$.
Case 1: $b^2 - 4 > 0$. In this case, we have two distinct roots,

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4}}{2} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4}}{2}.$$

Then

$$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

For a particular solution, we try $y_p = A \cos x + B \sin x$. Plugging into the equation, we find

$$\begin{aligned} -A \cos x - B \sin x + b(-A \sin x + B \cos x) + A \cos x + B \sin x &= \sin x; \\ b(-A \sin x + B \cos x) &= \sin x; \\ B = 0, \ -Ab = 1, \ A = -\frac{1}{b}. \end{aligned}$$

Note that $b \neq 0$. So the general solution in this case is of the form

$$y_g = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} - \frac{1}{b} \cos x.$$

Case 2: $b^2 - 4 = 0$. In this case, we have one double root, $\lambda = \frac{-b}{2}$. Then

$$y_h = c_1 e^{-\frac{b}{2} x} + c_2 x e^{-\frac{b}{2} x}.$$

A particular solution has the same form as the previous case. So

$$y_g = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} - \frac{1}{b} \cos x.$$

Case 3: $b^2 - 4 < 0$. In this case, we have two distinct roots,

$$\lambda_1 = \frac{-b + i\sqrt{-b^2 + 4}}{2} \quad \text{and} \quad \lambda_2 = \frac{-b - i\sqrt{-b^2 + 4}}{2}.$$

Then

$$y_h = c_1 e^{\frac{-b}{2} x} \cos\left(\frac{\sqrt{-b^2 + 4}}{2} x\right) + c_2 e^{\frac{-b}{2} x} \sin\left(\frac{\sqrt{-b^2 + 4}}{2} x\right).$$

Case 3 (a) If $b \neq 0$. Then we use the same form of the particular solution as in Case 1 and get

$$y_g = c_1 e^{\frac{-b}{2} x} \cos\left(\frac{\sqrt{-b^2 + 4}}{2} x\right) + c_2 e^{\frac{-b}{2} x} \sin\left(\frac{\sqrt{-b^2 + 4}}{2} x\right) - \frac{1}{b} \cos x.$$

Case 3 (b) If $b = 0$, the equation becomes $y'' + y = \sin x$. Then

$$y_h = c_1 \cos x + c_2 \sin x.$$

For a particular solution, we try $y_p = Ax \cos x$. Then

$$\begin{aligned} y_p' &= A\cos x - Ax \sin x \\ y_p'' &= -2A \sin x - Ax \cos x. \end{aligned}$$

Plugging into the equation, we find

$$\begin{aligned} -2A \sin x &= \sin x; \\ A &= -\frac{1}{2}. \end{aligned}$$

So the general solution in this case is of the form

$$y_g = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x.$$

61. To solve $y'' - 4\,y = 0,\ y(0) = 0,\ y'(0) = 3$, start with the general solution

$$y(x) = c_1 \cosh 2x + c_2 \sinh 2x.$$

Then

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 \cosh 0 + c_2 \sinh 0 = 0 \\ &\Rightarrow \quad c_1 = 0;\ \text{so } y(x) = c_2 \sinh 2x. \\ y'(0) = 3 \quad &\Rightarrow \quad 2c_2 \cosh 0 = 3 \\ &\Rightarrow \quad c_2 = \frac{3}{2} \\ &\Rightarrow \quad y(x) = \frac{3}{2} \sinh 2x. \end{aligned}$$

62. To solve $y'' + 2\,y' + y = 0, \quad y(0) = 2,\ y'(0) = -1$, recall the solution of the homogeneous equation from Exercise 35:

$$y = c_1 e^{-x} + c_2 x\, e^{-x}.$$

Then

$$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad c_1 = 2;\ \text{so } y(x) = 2e^{-x} + c_2 x\, e^{-x}. \\ y'(0) = -1 \quad &\Rightarrow \quad -2 + c_2 = -1 \\ &\Rightarrow \quad c_2 = 1 \\ &\Rightarrow \quad y(x) = 2e^{-x} + x\, e^{-x}. \end{aligned}$$

63. To solve $4y'' - 4\,y' + y = 0, \quad y(0) = -1,\ y'(0) = 1$, recall the solution of the homogeneous equation from Exercise 7:

$$y = c_1 e^{x/2} + c_2 x\, e^{x/2}.$$

Then

$$\begin{aligned} y(0) = -1 \quad &\Rightarrow \quad c_1 = -1;\ \text{so } y(x) = -e^{x/2} + c_2 x\, e^{x/2}. \\ y'(0) = 1 \quad &\Rightarrow \quad -\frac{1}{2} + c_2 = 1 \\ &\Rightarrow \quad c_2 = \frac{3}{2} \\ &\Rightarrow \quad y(x) = -e^{x/2} + \frac{3}{2} x\, e^{x/2}. \end{aligned}$$

64. To solve $y'' + y = 0, \quad y(\pi) = 1,\ y'(\pi) = 0$, use the general solution in the form

$$y = c_1 \cos(x - \pi) + c_2 \sin(x - \pi).$$

Then

$$\begin{aligned} y(\pi) = 1 \quad &\Rightarrow \quad c_1 = 1;\ \text{so } y(x) = \cos(x - \pi) + c_2 \sin(x - \pi).\\ y'(\pi) = 0 \quad &\Rightarrow \quad -\frac{1}{2} + c_2 = 1\\ &\Rightarrow \quad c_2 = 0\\ &\Rightarrow \quad y(x) = \cos(x - \pi) = -\cos x. \end{aligned}$$

65. To solve $y'' - 5\,y' + 6\,y = e^x, \quad y(0) = 0,\ y'(0) = 0$, use the general solution from Exercise 27 (modify it slightly):

$$y = c_1\, e^{2x} + c_2\, e^{3x} + \frac{1}{2}e^x.$$

Then

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 + c_2 = -\frac{1}{2};\\ y'(0) = 0 \quad &\Rightarrow \quad 2c_1 + 3c_2 = -\frac{1}{2}\\ &\Rightarrow \quad c_2 = \frac{1}{2}; c_1 = -1\\ &\Rightarrow \quad y(x) = -\,e^{2x} + \frac{1}{2}\,e^{3x} + \frac{1}{2}e^x. \end{aligned}$$

66. To solve $2\,y'' - 3\,y' + y = \sin x, \quad y(0) = 0,\ y'(0) = 0$, we first need the general solution of the differential equation. The solution of the homogeneous equation is

$$y_h = c_1 e^x + c_2 e^{x/2}$$

(see Exercise 4). For a particular solution, try

$$y = A\cos x + B\sin x.$$

Then $y'' = -A\cos x - B\sin x$, $y' = -A\sin x + B\cos x$. Plug into the equation

$$\begin{aligned} -2A\cos x - 2B\sin x + 3A\sin x - 3B\cos x + A\cos x + B\sin x &= \sin x\\ (-A - 3B)\cos x + (3A - B)\sin x &= \sin x;\\ -A - 3B &= 0\\ 3A - B &= 1. \end{aligned}$$

Solving for A and B, we find $A = \frac{3}{10}$ and $B = -\frac{1}{10}$. Thus the general solution is

$$y = c_1 e^x + c_2 e^{x/2} + \frac{3}{10}\cos x - \frac{1}{10}\sin x.$$

Using the initial conditions

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 + c_2 = -\frac{3}{10}\\ y'(0) = 0 \quad &\Rightarrow \quad c_1 + \frac{1}{2}c_2 = -\frac{1}{10};\\ &\Rightarrow \quad c_2 = -\frac{4}{5},\ c_1 = \frac{1}{2} \end{aligned}$$

$$y = \frac{1}{2}e^x - \frac{4}{5}e^{x/2} + \frac{3}{10}\cos x - \frac{1}{10}\sin x.$$

67. Start with the homogeneous equation $y'' - 4y' + 3y = 0$. Its characteristic equation is

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 3.$$

Thus

$$y_h = c_1 e^x + c_2 e^{3x}.$$

For a particular solution of $y'' - 4y' + 3y = x\,e^{-x}$, try

$$y = (Ax+b)e^{-x},\ y' = Ae^{-x} - (Ax+b)e^{-x},\ y'' = -2Ae^{-x} + (Ax+b)e^{-x}.$$

Plug into the equation and get

$$\begin{aligned}
-2Ae^{-x} + (Ax+b)e^{-x} - 4(Ae^{-x} - (Ax+b)e^{-x}) + 3(Ax+b)e^{-x} &= x\,e^{-x}\\
e^{-x}\big(-2A + b - 4A + 4b + 3b\big) + xe^{-x}\big(A + 4A + 3A\big) &= xe^{-x};\\
-6A + 8b &= 0\\
8A &= 1;
\end{aligned}$$

$$A = \frac{1}{8},\ b = \frac{6}{64} = \frac{3}{32}.$$

So the general solution is

$$y = c_1 e^x + c_2 e^{3x} + (\frac{1}{8}x + \frac{3}{32})e^{-x}.$$

We can now solve $y'' - 4y' + 3y = x\,e^{-x}$, $y(0) = 0$, $y'(0) = 1$. The boundary conditions imply that

$$\begin{aligned}
c_1 + c_2 + \frac{3}{32} &= 0\\
c_1 + 3c_2 + \frac{1}{8} - \frac{3}{32} &= 0;\\
c_1 + c_2 &= -\frac{3}{32}\\
c_1 + 3c_2 &= -\frac{1}{32};\\
c_2 = \frac{1}{32}; c_1 = -\frac{1}{8}
\end{aligned}$$

Hence

$$y = -\frac{1}{8}e^x + \frac{1}{32}e^{3x} + (\frac{1}{8}x + \frac{3}{32})e^{-x}.$$

68. The general solution of $y'' - 4y = 0$ is $y = c_1\cosh 2x + c_2\sinh 2x$. For a particular solution of $y'' - 4y = \cosh x$, we try $y = A\cosh x$. Then $y'' = A\cosh x$. Plugging into the equation:

$$\cosh x = y'' - 4y = A\cosh x - 4\cosh x = (A-4)\cosh x \Rightarrow 1 = A - 4,\ A = 5.$$

So $y = c_1\cosh 2x + c_2\sinh 2x + 5\cosh x$. Using the initial conditions

$$\begin{aligned}
y(0) = 1 &\Rightarrow c_1 + 5 = 1 \Rightarrow c_1 = -4\\
y'(0) = 0 &\Rightarrow c_2 = 0;
\end{aligned}$$

and so $y = -4\cosh 2x + 5\cosh x$.

69. Because of the initial conditions, it is more convenient to take

$$y = c_1 \cos[2(x - \frac{\pi}{2})] + c_2 \sin[2(x - \frac{\pi}{2})]$$

as a general solution of $y'' + 4y = 0$. For a particular solution of $y'' + 4y = \cos 2x$, we try

$$y = Ax \sin 2x,\ y' = A\sin 2x + 2Ax\cos 2x,\ y'' = 4A\cos 2x - 4Ax \sin 2x.$$

Plug into the equation,

$$4A\cos 2x = \cos 2x \ \Rightarrow\ A = \frac{1}{4}.$$

So the general solution is

$$y = c_1 \cos[2(x - \frac{\pi}{2})] + c_2 \sin[2(x - \frac{\pi}{2})] + \frac{1}{4}x\sin 2x.$$

Using the initial conditions

$$\begin{aligned} y(\pi/2) = 1 \quad &\Rightarrow \quad c_1 = 1 \\ y'(\pi/2) = 0 \quad &\Rightarrow \quad 2c_2 + \frac{\pi}{4}\cos\pi = 0 \\ &\Rightarrow \quad c_2 = \frac{\pi}{8}; \end{aligned}$$

and so

$$y = \cos[2(x - \frac{\pi}{2})] + \frac{\pi}{8}\sin[2(x - \frac{\pi}{2})] + \frac{1}{4}x\sin 2x.$$

using the addition formulas for the cosine and sine, we can write

$$\cos[2(x - \frac{\pi}{2})] = -\cos 2x \quad \text{and} \quad \sin[2(x - \frac{\pi}{2})] = -\sin 2x,$$

and so

$$y = -\cos 2x - \frac{\pi}{8}\sin 2x + \frac{1}{4}x\sin 2x = -\cos 2x + \big(-\frac{\pi}{8} + \frac{1}{4}x\big)\sin 2x.$$

70. From Exercise 43, the general solution of $y'' + 9\,y = \sum_{n=1}^{6} \frac{\sin nx}{n}$ is

$$y = c_1\ \cos 3x + c_2 \sin 3x - \frac{1}{18}x\cos 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n(9-n^2)}.$$

Using the initial condition $y(0) = 0$, we find $c_1 = 0$ and so

$$\begin{aligned} y &= c_2 \sin 3x - \frac{1}{18}x\cos 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n(9-n^2)} \\ y' &= 3c_2\cos 3x - \frac{1}{18}\cos 3x + \frac{1}{6}x\sin 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\cos nx}{9-n^2} \\ y'(0) = 2 \ \Rightarrow\ & 3c_2 - \frac{1}{18} + \sum_{n=1,\ n\neq 3}^{6} \frac{1}{9-n^2} = 2 \\ c_2 &= \frac{1}{3}\left[2 + \frac{1}{18} - \sum_{n=1,\ n\neq 3}^{6} \frac{1}{9-n^2}\right] \\ &= \frac{29831}{45360}, \end{aligned}$$

and so

$$y = \frac{29831}{45360}\sin 3x - \frac{1}{18}x\cos 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n(9-n^2)}.$$

71. An antiderivative of $g(x) = e^x \sin x$ is a solution of the differential equation

$$y' = e^x \sin x.$$

To solve this equation we used the method of undetermined coefficients. The solution of homogeneous equation $y' = 0$ is clearly $y = c$. To find a particular solution of $y' = e^x \sin x$ we try

$$\begin{aligned} y &= e^x(A\sin x + B\cos x) \\ y' &= e^x(A\sin x + B\cos x) + e^x(A\cos x - B\sin x) \\ &= e^x(A-B)\sin x + e^x(A+B); \end{aligned}$$

Plugging into the equation, we find

$$\begin{aligned} e^x(A-B)\sin x + e^x(A+B)\cos x &= e^x \sin x \\ A-B=1,\ A+B=0 \quad \Rightarrow \quad A &= \frac{1}{2},\ B = -\frac{1}{2}; \end{aligned}$$

so

$$\int e^x \sin x\, dx = \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C = \frac{e^x}{2}\big(\sin x - \cos x\big) + C.$$

72. See Exercise 73.

73. An antiderivative of $g(x) = e^{a\,x}\cos bx$ is a solution of the differential equation

$$y' = e^{a\,x}\cos bx.$$

We assume throughout this exercise that $a \neq 0$ and $b \neq 0$. For these special cases the integral is clear. To solve the differential equation we used the method of undetermined coefficients. The solution of homogeneous equation $y' = 0$ is $y = C$. To find a particular solution of $y' = e^{ax}\cos bx$ we try

$$\begin{aligned} y &= e^{ax}(A\cos bx + B\sin bx) \\ y' &= ae^{ax}(A\cos bx + B\sin bx) + e^{ax}(-bA\sin bx + bB\cos bx) \\ &= e^{ax}(Aa + bB)\cos bx + e^{ax}(aB - bA)\sin bx. \end{aligned}$$

Plugging into the equation, we find

$$\begin{aligned} e^{ax}(Aa+bB)\cos bx + e^{ax}(aB-bA)\sin bx &= e^{ax}\cos bx \\ Aa+bB=1,\ aB-bA=0 \quad \Rightarrow \quad A &= \frac{a}{a^2+b^2},\ B = \frac{b}{a^2+b^2}; \end{aligned}$$

so

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\big(a\cos x + b\sin bx\big) + C.$$

74. Repeat the steps in the previous exercise. To solve $y' = e^{ax}\sin bx$, try

$$\begin{aligned} y &= e^{ax}(A\cos bx + B\sin bx) \\ y' &= e^{ax}(Aa + bB)\cos bx + e^{ax}(aB - bA)\sin bx. \end{aligned}$$

Plugging into the equation, we find

$$\begin{aligned} e^{ax}(Aa+bB)\cos bx + e^{ax}(aB-bA)\sin bx &= e^{ax}\sin bx \\ Aa+bB=0,\ aB-bA=1 &\Rightarrow A=-\frac{b}{a^2+b^2},\ B=\frac{a}{a^2+b^2}; \end{aligned}$$

so

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\big(-b\cos x + a\sin bx\big) + C.$$

Concerning Exercises 75–78 and for information about the Vandermonde determinant, please refer to the book “Applied Linear Algebra,” by P. Olver and C. Shakiban, Prentice Hall, 2006.

Solutions to Exercises A.3

1. We apply the reduction of order formula and take all constants of integration equal to 0.

$$\begin{aligned} y'' + 2\,y' - 3\,y = 0, &\qquad y_1 = e^x; \\ p(x) = 2, \int p(x)\,dx = 2x, &\qquad e^{-\int p(x)\,dx} = e^{-2x}; \end{aligned}$$

$$\begin{aligned} y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= e^x \int \frac{e^{-2x}}{e^{2x}}\,dx \\ &= e^x \int e^{-4x}\,dx = e^x\left[-\frac{1}{4}e^{-4x}\right] = -\frac{1}{4}e^{-3x}. \end{aligned}$$

Thus the general solution is

$$y = c_1 e^x + c_2 e^{-3x}.$$

2. We apply the reduction of order formula and take all constants of integration equal to 0.

$$\begin{aligned} y'' - 5\,y' + 6\,y = 0, &\qquad y_1 = e^{3x}; \\ p(x) = -5, \int p(x)\,dx = -5x, &\qquad e^{-\int p(x)\,dx} = e^{5x}; \end{aligned}$$

$$\begin{aligned} y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= e^{3x} \int \frac{e^{5x}}{e^{6x}}\,dx \\ &= e^{3x} \int e^{-x}\,dx = e^{3x}\left[-e^{-x}\right] = -e^{2x}. \end{aligned}$$

Thus the general solution is

$$y = c_1 e^{3x} + c_2 e^{2x}.$$

3. $x\,y'' - (3+x)y' + 3y = 0,\quad y_1 = e^x$. We first put the equation in standard form.

$$y'' - \frac{3+x}{x}\,y' + \frac{3}{x}\,y = 0, \qquad y_1 = e^x;\ p(x) = -\frac{3+x}{x},$$

$$\begin{aligned} \int (-\frac{3}{x} - 1)\,dx &= -3\ln x - x, \\ e^{-\int p(x)\,dx} &= e^{3\ln x + x} = x^3 e^x; \\ y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= e^x \int \frac{x^3 e^x}{e^{2x}}\,dx \\ &= e^x \int x^3 e^{-x}\,dx \\ &= e^x\left[-x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x}\right] \\ &= -x^3 - 3x^2 - 6x - 6. \end{aligned}$$

Thus the general solution is

$$y = c_1 e^x + c_2(x^3 + 3x^2 + 6x + 6).$$

4. $xy'' - (2-x)y' - 2\,y = 0, \quad y_1 = e^{-x}.$

We first put the equation in standard form.

$$\begin{aligned}
y'' - \frac{2-x}{x}\,y' - \frac{2}{x}\,y = 0, &\qquad y_1 = e^{-x};\ p(x) = \frac{-2+x}{x},\\
\int (-\frac{2}{x} + 1)\,dx &= -2\ln x + x,\\
e^{-\int p(x)\,dx} &= e^{2\ln x - x} = x^2 e^{-x};\\
y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= e^{-x}\int \frac{x^2 e^{-x}}{e^{-2x}}\,dx\\
&= e^{-x}\int x^2 e^x\,dx\\
&= e^{-x}\left[x^2e^x - 2xe^x + 2e^x\right]\\
&= x^2 - 2x + 2.
\end{aligned}$$

Thus the general solution is

$$y = c_1 e^{-x} + c_2(x^2 - 2x + 2).$$

5. $y'' + 4\,y = 0, \quad y_1 = \cos 2x.$

$$\begin{aligned}
y'' + 4\,y = 0, &\qquad y_1 = \cos 2x;\\
p(x) = 0, \int p(x)\,dx = 0, &\qquad e^{-\int p(x)\,dx} = 1;\\
y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= \cos 2x \int \frac{1}{\cos^2 2x}\,dx\\
&= \cos 2x \int \sec^2 2x\,dx\\
&= \cos 2x \left[\frac{1}{2}\tan 2x\right] = \frac{1}{2}\sin 2x.
\end{aligned}$$

Thus the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

6.

$$
\begin{aligned}
y'' + 9\,y = 0, &\qquad y_1 = \sin 3x;\\
p(x) = 0,\ \int p(x)\,dx = 0, &\qquad e^{-\int p(x)\,dx} = 1;\\
y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= \sin 3x \int \frac{1}{\sin^2 3x}\,dx\\
&= \sin 3x \int \csc^2 3x\,dx\\
&= \sin 3x\left[-\frac{1}{3}\cot 3x\right] = -\frac{1}{3}\cos 3x.
\end{aligned}
$$

Thus the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x.$$

7.

$$
\begin{aligned}
y'' - y = 0, &\qquad y_1 = \cosh x;\\
p(x) = 0,\ \int p(x)\,dx = 0, &\qquad e^{-\int p(x)\,dx} = 1;\\
y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= \cosh x \int \frac{1}{\cosh^2 x}\,dx\\
&= \cosh x \int \operatorname{sech}^2 x\,dx\\
&= \cosh x\,[\tanh x] = \sinh x.
\end{aligned}
$$

Thus the general solution is

$$y = c_1 \cosh x + c_2 \sinh x.$$

8.

$$
\begin{aligned}
y'' + 2\,y' + y = 0, &\qquad y_1 = e^{-x};\\
p(x) = 2,\ \int p(x)\,dx = 2x, &\qquad e^{-\int p(x)\,dx} = e^{-2x};\\
y_2 = e^{-x} \int \frac{e^{-2x}}{e^{-2x}}\,dx &= e^{-x} \int dx = e^{-x}x.
\end{aligned}
$$

Thus the general solution is

$$y = c_1 e^{-x} + c_2\, x e^{-x}.$$

9. Put the equation in standard form:

$$(1-x^2)y'' - 2xy' + 2y = 0, \qquad y_1 = x;$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0, \qquad p(x) = -\frac{2x}{1-x^2};$$

$$\int p(x)\,dx = \int -\frac{2x}{1-x^2}\,dx = \ln(1-x^2)$$

$$e^{-\int p(x)\,dx} = e^{-\ln(1-x^2)} = \frac{1}{1-x^2};$$

$$y_2 = x\int \frac{1}{(1-x^2)x^2}\,dx.$$

To evaluate the last integral, we use the partial fractions decomposition

$$\begin{aligned}
\frac{1}{(1-x^2)x^2} &= \frac{1}{(1-x)(1+x)x^2}\\
&= \frac{A}{(1-x)} + \frac{B}{(1-x)} + \frac{C}{x} + \frac{D}{x^2};\\
&= \frac{A(1+x)x^2 + B(1-x)x^2 + C(1-x^2)x + D(1-x^2)}{(1-x)(1+x)x^2}\\
1 &= A(1+x)x^2 + B(1-x)x^2 + C(1-x^2)x + D(1-x^2).\\
\text{Take } x=0 &\Rightarrow D = 1.\\
\text{Take } x=1 &\Rightarrow 1 = 2A,\ A = \frac{1}{2}.\\
\text{Take } x=-1 &\Rightarrow 1 = 2B,\ B = \frac{1}{2}.
\end{aligned}$$

Checking the coefficient of x^3, we find $C = 0$. Thus

$$\begin{aligned}
\frac{1}{(1-x^2)x^2} &= \frac{1}{2(1-x)} + \frac{1}{2(1+x)} + + \frac{1}{x^2}\\
\int \frac{1}{(1-x^2)x^2}\,dx &= -\frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) - \frac{1}{x}\\
&= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}.
\end{aligned}$$

So

$$y_2 = x\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}\right] = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

Hence the general solution

$$y = c_1\,x + c_2\left[\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right].$$

10. Put the equation in standard form:

$$
\begin{aligned}
(1-x^2)y'' - 2xy' = 0, &\qquad y_1 = 1;\\
y'' - \frac{2x}{1-x^2}y' = 0, &\qquad p(x) = -\frac{2x}{1-x^2};
\end{aligned}
$$

$$
\begin{aligned}
\int p(x)\,dx &= \int -\frac{2x}{1-x^2}\,dx = \ln(1-x^2)\\
e^{-\int p(x)\,dx} &= e^{-\ln(1-x^2)} = \frac{1}{1-x^2};\\
y_2 &= \int \frac{1}{1-x^2}\,dx\\
&= \frac{1}{2}\int \left[\frac{1}{1-x} + \frac{1}{1+x}\right]\,dx\\
&= \frac{-1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)\\
&= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).
\end{aligned}
$$

Hence the general solution

$$
y = c_1 + c_2 \ln\left(\frac{1+x}{1-x}\right).
$$

11. Put the equation in standard form:

$$
\begin{aligned}
x^2y'' + xy' - y = 0, &\qquad y_1 = x;\\
y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0, &\qquad p(x) = \frac{1}{x};
\end{aligned}
$$

$$
\begin{aligned}
\int p(x)\,dx &= \int \frac{1}{x}\,dx = \ln x\\
e^{-\int p(x)\,dx} &= e^{-\ln x} = \frac{1}{x};\\
y_2 &= x\int \frac{1}{x^3}\,dx\\
&= x\left[-\frac{1}{2}x^{-2}\right]\\
&= \frac{-1}{2x}.
\end{aligned}
$$

Hence the general solution

$$
y = c_1 x + \frac{c_2}{x}.
$$

12. Put the equation in standard form:

$$x^2y'' - xy' + y = 0, \qquad y_1 = x;$$

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0, \qquad p(x) = -\frac{1}{x};$$

$$\begin{aligned}
\int p(x)\,dx &= \int -\frac{1}{x}\,dx = -\ln x \\
e^{-\int p(x)\,dx} &= e^{\ln x} = x; \\
y_2 &= x\int \frac{x}{x^2}\,dx \\
&= x\int \frac{1}{x}\,dx \\
&= x\,\ln x.
\end{aligned}$$

Hence the general solution

$$y = c_1x + c_2x\ln x.$$

13. Put the equation in standard form:

$$x^2y'' + xy' + y = 0, \qquad y_1 = \cos(\ln x);$$

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0, \qquad p(x) = \frac{1}{x};$$

$$\begin{aligned}
\int p(x)\,dx &= \int \frac{1}{x}\,dx = \ln x \\
e^{-\int p(x)\,dx} &= e^{-\ln x} = \frac{1}{x}; \\
y_2 &= \cos(\ln x)\int \frac{1}{x\,\cos^2(\ln x)}\,dx \\
&= \cos(\ln x)\int \frac{1}{\cos^2 u}\,du \quad \left(u = \ln x,\ du = \frac{1}{x}\,dx\right) \\
&= \cos u\,\tan u = \sin u = \sin(\ln u).
\end{aligned}$$

Hence the general solution

$$y = c_1\cos(\ln x) + c_2\sin(\ln u).$$

14. Put the equation in standard form:

$$x^2y'' + 2\,xy' + \frac{1}{4}\,y = 0, \qquad y_1 = \frac{1}{\sqrt{x}};$$

$$y'' + \frac{2}{x}y' + \frac{1}{4x^2}y = 0, \qquad p(x) = \frac{2}{x};$$

$$\begin{aligned}
\int p(x)\,dx &= \int \frac{2}{x}\,dx = 2\ln x \\
e^{-\int p(x)\,dx} &= e^{-2\ln x} = \frac{1}{x^2}; \\
y_2 &= \frac{1}{\sqrt{x}}\int \frac{x}{x^2}\,dx = \frac{\ln x}{\sqrt{x}}.
\end{aligned}$$

Hence the general solution

$$y = c_1 \frac{1}{\sqrt{x}} + c_2 \frac{\ln x}{\sqrt{x}}.$$

15. Put the equation in standard form:

$$x\,y'' + 2\,y' + 4x\,y = 0, \qquad y_1 = \frac{\sin 2x}{x};$$

$$y'' + \frac{2}{x}y' + 4y = 0, \qquad p(x) = \frac{2}{x};$$

$$\begin{aligned}
\int p(x)\,dx &= \int \frac{2}{x}\,dx = 2\ln x \\
e^{-\int p(x)\,dx} &= e^{-2\ln x} = \frac{1}{x^2}; \\
y_2 &= \frac{\sin 2x}{x} \int \frac{x^2}{x^2 \sin^2(2x)}\,dx \\
&= \frac{\sin 2x}{x} \int \frac{1}{\sin^2(2x)}\,dx = \frac{\sin 2x}{x} \frac{1}{-2} \cot(2x) \\
&= \frac{\cos 2x}{-2x}.
\end{aligned}$$

Hence the general solution

$$y = c_1 \frac{\sin 2x}{x} + c_2 \frac{\cos 2x}{x}.$$

16. Put the equation in standard form:

$$x\,y'' + 2\,y' - x\,y = 0, \qquad y_1 = \frac{e^x}{x};$$

$$y'' + \frac{2}{x}y' - y = 0, \qquad p(x) = \frac{2}{x};$$

$$\begin{aligned}
\int p(x)\,dx &= \int \frac{2}{x}\,dx = 2\ln x \\
e^{-\int p(x)\,dx} &= e^{-2\ln x} = \frac{1}{x^2}; \\
y_2 &= \frac{e^x}{x} \int \frac{x^2}{x^2} e^{-2x}\,dx \\
&= \frac{e^x}{x} \int e^{-2x}\,dx = \frac{e^x}{x} \frac{1}{-2} e^{-2x} \\
&= -\frac{e^{-x}}{2}.
\end{aligned}$$

Hence the general solution

$$y = c_1 \frac{e^x}{x} + c_2 \frac{e^{-x}}{x}.$$

17. Put the equation in standard form:

$$x\,y'' + 2\,(1-x)y' + (x-2)\,y = 0, \qquad y_1 = e^x;$$

$$y'' + \frac{2(1-x)}{x}y' + \frac{x-2}{x}y = 0, \qquad p(x) = \frac{2}{x} - 2;$$

$$\begin{aligned}
\int p(x)\,dx &= 2\ln x - 2x \\
e^{-\int p(x)\,dx} &= e^{-2\ln x + 2x} = \frac{e^{2x}}{x^2}; \\
y_2 &= e^x \int \frac{e^{2x}}{x^2 e^{2x}}\,dx \\
&= e^x \int \frac{1}{x^2}\,dx = -\frac{e^x}{x}.
\end{aligned}$$

Hence the general solution

$$y = c_1 e^x + c_2 \frac{e^x}{x}.$$

18. Put the equation in standard form:

$$(x-1)^2\,y'' - 3(x-1)\,y' + 4\,y = 0, \qquad y_1 = (x-1)^2;$$

$$y'' - \frac{3}{x-1}y' + \frac{4}{(x-1)^2}y = 0, \qquad p(x) = -\frac{3}{x-1};$$

$$\begin{aligned}
\int p(x)\,dx &= -3\ln(x-1) \\
e^{-\int p(x)\,dx} &= e^{3\ln(x-1)} = (x-1)^3; \\
y_2 &= (x-1)^2 \int \frac{(x-1)^3}{(x-1)^4}\,dx \\
&= (x-1)^2 \int \frac{1}{x-1}\,dx = (x-1)^2\ln(x-1).
\end{aligned}$$

Hence the general solution

$$y = c_1(x-1)^2 + c_2(x-1)^2\ln(x-1).$$

19. Put the equation in standard form:

$$x^2 y'' - 2x\,y' + 2\,y = 0, \qquad y_1 = x^2;$$

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0, \qquad p(x) = -\frac{2}{x};$$

$$\begin{aligned}
\int p(x)\,dx &= -2\ln x \\
e^{-\int p(x)\,dx} &= e^{2\ln x} = x^2; \\
y_2 &= x^2 \int \frac{x^2}{x^4}\,dx \\
&= x^2 \int \frac{1}{x^2}\,dx = -x.
\end{aligned}$$

Hence the general solution

$$y = c_1 x^2 + c_2 x.$$

20. Put the equation in standard form:

$(x^2-2x)y''-(x^2-2)\,y'+2(x-1)y=0, \qquad y_1=e^x;$

$$y''-\frac{x^2-2}{x^2-2x}y'+\frac{2(x-1)}{x^2-2x}y=0, \qquad p(x)=-\frac{x^2-2}{x^2-2x}=-\left[1+\frac{2x-2}{x^2-2x}\right];$$

$$\begin{aligned}\int p(x)\,dx &= -x-\ln(x^2-2x)\\ e^{-\int p(x)\,dx} &= e^{x+\ln(x^2-2x)}=(x^2-2x)e^x;\\ y_2 &= e^x\int\frac{(x^2-2x)e^x}{e^{2x}}\,dx\\ &= e^x\int(x^2-2x)e^{-x}\,dx\\ &= e^x[-e^{-x}x^2]=-x^2.\end{aligned}$$

Hence the general solution

$$y=c_1e^x+c_2x^2.$$

21. $y''-4y'+3\,y=e^{-x}$.

$$\begin{aligned}\lambda^2-4\lambda+3=0 \quad&\Rightarrow\quad (\lambda-1)(\lambda-3)=0\\ &\Rightarrow\quad \lambda=1 \text{ or } \lambda=3.\end{aligned}$$

Linearly independent solutions of the homogeneous equation:

$$y_1=e^x \quad\text{and}\quad y_2=e^{3x}.$$

Wronskian:

$$W(x)=\begin{vmatrix} e^x & e^{3x}\\ e^x & 3e^{3x}\end{vmatrix}=3e^{4x}-e^{4x}=2e^{4x}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}g(x) &= e^{-x};\\ y_p &= y_1\int\frac{-y_2g(x)}{W(x)}\,dx+y_2\int\frac{y_1g(x)}{W(x)}\,dx\\ &= e^x\int\frac{-e^{3x}e^{-x}}{2e^{4x}}\,dx+e^{3x}\int\frac{e^xe^{-x}}{2e^{4x}}\,dx\\ &= -\frac{e^x}{2}\int e^{-2x}\,dx+\frac{e^{3x}}{2}\int e^{-4x}\,dx\\ &= \frac{e^x}{4}e^{-2x}-\frac{e^{3x}}{8}e^{-4x}=\frac{e^{-x}}{4}-\frac{e^{-x}}{8}\\ &= \frac{e^{-x}}{8}.\end{aligned}$$

Thus the general solution is

$$y=c_1e^x+c_2e^{3x}+\frac{e^{-x}}{8}.$$

22. $y''-15\,y'+56\,y=e^{7x}+12\,x$.

$$\begin{aligned}\lambda^2-15\lambda+56=0 \quad&\Rightarrow\quad (\lambda-7)(\lambda-8)=0\\ &\Rightarrow\quad \lambda=7 \text{ or } \lambda=8.\end{aligned}$$

Linearly independent solutions of the homogeneous equation:

$$y_1 = e^{7x} \quad \text{and} \quad y_2 = e^{8x}.$$

Wronskian:

$$W(x) = \begin{vmatrix} e^{7x} & e^{8x} \\ 7e^{x} & 8e^{8x} \end{vmatrix} = 8e^{15x} - 7e^{15x} = e^{15x}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= e^{7x} + 12\,x; \\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx \\
&= e^{7x} \int \frac{-e^{8x}(e^{7x} + 12\,x)}{e^{15x}}\,dx + e^{8x} \int \frac{e^{7x}(e^{7x} + 12\,x)}{e^{15x}}\,dx \\
&= -e^{7x} \int (1 + 12xe^{-x})\,dx + e^{8x} \int e^{-x} + 12xe^{-8x})\,dx \\
&= -e^{7x} \left[x + 12 \int xe^{-8x}\,dx \right] + e^{8x} \left[-e^{-x} + 12 \int xe^{-8x}\,dx \right] \\
&= -e^{7x} \left[x - \frac{12}{7} xe^{-7x} - \frac{12}{49} e^{-7x} \right] + e^{8x} \left[-e^{-x} - \frac{12}{8} xe^{-8x} - \frac{12}{64} e^{-8x} \right] \\
&= -xe^{7x} - e^{7x} + \frac{12}{7}x + \frac{12}{49} - \frac{12}{8}x - \frac{12}{64} \\
&= -xe^{7x} - e^{7x} + \frac{3}{14}x + \frac{45}{784}.
\end{aligned}$$

Thus the general solution is

$$y = c_1 e^{7x} + c_2 e^{8x} - xe^{7x} - e^{7x} + \frac{3}{14}x + \frac{45}{784}.$$

23. $3\,y'' + 13\,y' + 10\,y = \sin x$.

$$\begin{aligned}
3\lambda^2 + 13\lambda + 10 = 0 \quad &\Rightarrow \quad \lambda = \frac{-13 \pm \sqrt{169 - 120}}{6} = \frac{-13 \pm 7}{6} \\
&\Rightarrow \quad \lambda = -\frac{20}{6} = -\frac{10}{3} \text{ or } \lambda = -1.
\end{aligned}$$

Two linearly independent solutions of the homogeneous equation:

$$y_1 = e^{-\frac{10}{3}x} \quad \text{and} \quad y_2 = e^{-x}.$$

Wronskian:

$$W(x) = \begin{vmatrix} e^{-\frac{10}{3}x} & e^{-x} \\ -\frac{10}{3} e^{-\frac{10}{3}x} & -e^{-x} \end{vmatrix} = -e^{-\frac{13}{3}x} + \frac{10}{3} e^{-\frac{13}{3}x} = \frac{7}{3} e^{-\frac{13}{3}x}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= \frac{\sin x}{3};\\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx\\
&= e^{-\frac{10}{3}x} \int \frac{-e^{-x}\sin x}{3\frac{7}{3}e^{-\frac{13}{3}x}}\,dx + e^{-x} \int \frac{e^{-\frac{10}{3}x}\sin x}{3\frac{7}{3}e^{-\frac{13}{3}x}}\,dx\\
&= -\frac{1}{7}e^{-\frac{10}{3}x} \int e^{\frac{10}{3}x}\sin x\,dx + \frac{1}{7}e^{-x} \int e^{x}\sin x\,dx.
\end{aligned}$$

Evaluate the integrals with the help of the formulas in the table at the end of the book and get

$$\begin{aligned}
y_p &= -\frac{1}{7}e^{-\frac{10}{3}x} \frac{e^{\frac{10}{3}x}}{(\frac{10}{3})^2 + 1^2}\left(\frac{10}{3}\sin x - \cos x\right) + \frac{1}{7}e^{-x}\frac{e^{x}}{1^2+1^2}(\sin x - \cos x)\\
&= -\frac{1}{7}\frac{1}{\frac{100}{9}+1}\left(\frac{10}{3}\sin x - \cos x\right) + \frac{1}{7}\frac{1}{2}(\sin x - \cos x)\\
&= -\frac{9}{763}\left(\frac{10}{3}\sin x - \cos x\right) + \frac{1}{14}(\sin x - \cos x)\\
&= \frac{7}{218}\sin x - \frac{13}{218}\cos x
\end{aligned}$$

Thus the general solution is

$$y = c_1 e^{-\frac{10}{3}x} + c_2 e^{-x} + \frac{7}{218}\sin x - \frac{13}{218}\cos x.$$

24. $y'' + 3\,y = x$.

Two linearly independent solutions of the homogeneous equation are:

$$y_1 = \cos\sqrt{3}x \quad \text{and} \quad y_2 = \sin\sqrt{3}x.$$

Wronskian:

$$W(x) = \begin{vmatrix} \cos\sqrt{3}x & \sin\sqrt{3}x \\ -\sqrt{3}\sin\sqrt{3}x & \sqrt{3}\cos\sqrt{3}x \end{vmatrix} = \sqrt{3}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= x; \\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx \\
&= -\cos\sqrt{3}x \int \frac{x\,\sin\sqrt{3}x}{\sqrt{3}}\,dx + \sin\sqrt{3}x \int \frac{x\,\cos\sqrt{3}x}{\sqrt{3}}\,dx \\
&= -\frac{\cos\sqrt{3}x}{\sqrt{3}}\left[-\frac{x}{\sqrt{3}}\cos\sqrt{3}x + \frac{1}{\sqrt{3}}\int \cos\sqrt{3}x\,dx\right] \\
&\quad + \frac{\sin\sqrt{3}x}{\sqrt{3}}\left[\frac{x}{\sqrt{3}}\sin\sqrt{3}x - \frac{1}{\sqrt{3}}\int \sin\sqrt{3}x\,dx\right] \\
&= -\frac{\cos\sqrt{3}x}{3}\left[-x\cos\sqrt{3}x + \frac{1}{\sqrt{3}}\sin\sqrt{3}x\right] \\
&\quad + \frac{\sin\sqrt{3}x}{3}\left[x\sin\sqrt{3}x + \frac{1}{\sqrt{3}}\cos\sqrt{3}x\right] \\
&= \frac{x}{3}.
\end{aligned}$$

Thus the general solution is

$$y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + \frac{x}{3}.$$

25. $y'' + y = \sec x$.
Two linearly independent solutions of the homogeneous equation are:

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x.$$

Wronskian:

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= \sec x = \frac{1}{\cos x}; \\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx \\
&= -\cos x \int \frac{\sin x}{\cos x}\,dx + \sin x \int dx \\
&= \cos x \cdot \ln(|\cos x|) + x\,\sin x.
\end{aligned}$$

Thus the general solution is

$$y = c_1\cos x + c_2\sin x + \cos x \cdot \ln(|\cos x|) + x\,\sin x.$$

26. $y'' + y = \sin x + \cos x$.
Two linearly independent solutions of the homogeneous equation are:

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x, \quad W(x) = 1$$

(as in previous exercise). We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= \sin x + \cos x;\\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx\\
&= -\cos x \int \sin x(\sin x + \cos x)\,dx + \sin x \int \cos x(\sin x + \cos x)\,dx\\
&= -\cos x \int (\sin^2 x + \sin x \cos x)\,dx + \sin x \int (\sin x \cos x + \cos^2 x)\,dx\\
&= -\cos x \left[\int \frac{1-\cos 2x}{2} dx + \frac{1}{2}\sin^2 x\right]\\
&\quad + \sin x \left[\frac{1}{2}\sin^2 x + \int \frac{1+\cos 2x}{2}\,dx\right]\\
&= -\cos x \left[\frac{x}{2} - \frac{\sin 2x}{4} + \frac{1}{2}\sin^2 x\right] + \sin x \left[\frac{1}{2}\sin^2 x + \frac{x}{2} - \frac{\sin 2x}{4}\right]\\
&= -\cos x \left[\frac{x}{2} - \frac{\sin 2x}{4} + \frac{1}{2}\sin^2 x\right] + \sin x \left[\frac{1}{2}\sin^2 x + \frac{x}{2} - \frac{\sin 2x}{4}\right]
\end{aligned}$$

Thus the general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \left[\frac{x}{2} - \frac{\sin 2x}{4} + \frac{1}{2}\sin^2 x\right] + \sin x \left[\frac{1}{2}\sin^2 x + \frac{x}{2} - \frac{\sin 2x}{4}\right].$$

27. $x\,y'' - (1+x)y' + y = x^3$.

Two linearly independent solutions of the homogeneous equation are given:

$$y_1 = 1 + x \quad \text{and} \quad y_2 = e^x.$$

Wronskian:

$$W(x) = \begin{vmatrix} 1+x & e^x \\ 1 & e^x x \end{vmatrix} = e^x + xe^x - xe^x = xe^x.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= \frac{x^3}{x} = x^2;\\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx\\
&= -(1+x) \int \frac{e^x}{xe^x} x^2\,dx + e^x \int (1+x)xe^{-x}\,dx\\
&= -(1+x)\frac{x^2}{2} + e^x \int (xe^{-x} + x^2 e^{-x})\,dx\\
&= -\frac{x^3}{2} - \frac{x^2}{2} + e^x\left[-xe^{-x} - e^{-x} - x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]\\
&= -\frac{x^3}{2} - \frac{3}{2}x^2 - 3x - 3.
\end{aligned}$$

Thus the general solution is

$$y = c_1(1+x) + c_2 e^x - \frac{x^3}{2} - \frac{3}{2}x^2 - 3x - 3.$$

This can also be written as

$$y = c_1(1+x) + c_2 e^x - \frac{x^3}{2} - \frac{3}{2}x^2.$$

28. $x\,y'' - (1+x)y' + y = x^4 e^x$.

Two linearly independent solutions of the homogeneous equation are given:

$$y_1 = 1 + x \quad \text{and} \quad y_2 = e^x.$$

Wronskian:

$$W(x) = \begin{vmatrix} 1+x & e^x \\ 1 & e^x x \end{vmatrix} = e^x + xe^x - xe^x = xe^x.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= x^3 e^x; \\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx \\
&= -(1+x)\int \frac{e^x}{xe^x} x^3 e^x\,dx + e^x \int \frac{(1+x)}{xe^x} x^3 e^x\,dx \\
&= -(1+x)\int x^2 e^x\,dx + e^x \int (1+x)x^2\,dx \\
&= -(1+x)\int x^2 e^x\,dx + e^x \int (1+x)x^2\,dx \\
&= -(1+x)\left[x^2e^x - 2xe^x + 2e^x\right] + e^x\left[\frac{x^3}{3} + \frac{x^4}{4}\right] \\
&= e^x\left[-(1+x)x^2 + 2x(1+x) - 2(1+x) + \frac{x^3}{3} + \frac{x^4}{4}\right] \\
&= e^x\left[\frac{x^4}{4} - \frac{2}{3}x^3 + x^2 - 2\right].
\end{aligned}$$

Thus the general solution is

$$y = c_1(1+x) + c_2 e^x + e^x\left[\frac{x^4}{4} - \frac{2}{3}x^3 + x^2 - 2\right].$$

29. $x^2y'' + 3\,xy' + y = \sqrt{x}$. The homogeneous equation is an Euler equation. The indicial equation is

$$r^2 + 2r + 1 = 0 \quad \Rightarrow (r+1)^2 = 0.$$

We have one double indicial root $r = -1$. Hence the solutions of the homogenous equation

$$y_1 = x^{-1} \quad \text{and} \quad y_2 = x^{-1}\ln x.$$

Wronskian:

$$W(x) = \begin{vmatrix} \frac{1}{x} & \frac{1}{x}\ln x \\ \frac{-1}{x^2} & \frac{1-\ln x}{x^2} \end{vmatrix} = \frac{1-\ln x}{x^3} + \frac{\ln x}{x^3} = \frac{1}{x^3}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= \frac{\sqrt{x}}{x^2} = x^{-\frac{3}{2}};\\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx\\
&= -\frac{1}{x}\int \frac{\ln x}{x} x^3 x^{-\frac{3}{2}}\,dx + \frac{\ln x}{x}\int \frac{1}{x} x^3 x^{-\frac{3}{2}}\,dx\\
&= -\frac{1}{x}\int \overbrace{\ln x}^{u}\,\overbrace{\sqrt{x}\,dx}^{dv} + \frac{\ln x}{x}\int x^{\frac{1}{2}}\,dx\\
&= -\frac{1}{x}\left[\frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\frac{1}{x}dx\right] + \frac{\ln x}{x}\frac{2}{3}x^{3/2}\\
&= \frac{1}{x}\frac{2}{3}\frac{2}{3}x^{3/2} = \frac{4}{9}x^{1/2}.
\end{aligned}$$

Thus the general solution is

$$y = c_1 x^{-1} + c_2 x^{-1}\ln x + \frac{4}{9}x^{1/2}.$$

30. The homogeneous equation, $x^2 y'' + x\,y' + y = 0$, is an Euler equation. The indicial equation is

$$r^2 + 1 = 0 \quad \Rightarrow r = \pm i.$$

Hence the solutions of the homogenous equation

$$y_1 = \cos(\ln x) \quad \text{and} \quad y_2 = \sin(\ln x).$$

Wronskian:

$$W(x) = \begin{vmatrix} \cos(\ln x) & \sin(\ln x)\ln x \\ \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}\left(\cos^2(\ln x) + \sin^2(\ln x)\right) = \frac{1}{x}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= \frac{1}{x};\\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx\\
&= -\cos(\ln x)\int \overbrace{\sin(\ln x)}^{u}\,\overbrace{dx}^{dv} + \sin(\ln x)\int \cos(\ln x)\,dx\\
\int \sin(\ln x)\,dx &= x\sin(\ln x) - \int \overbrace{\cos(\ln x)}^{u}\,\overbrace{dx}^{dv}\\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx;\\
2\int \sin(\ln x)\,dx &= x\sin(\ln x) - x\cos(\ln x);\\
\int \sin(\ln x)\,dx &= \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right].
\end{aligned}$$

Similarly,

$$\int \cos(\ln x)\,dx = \frac{x}{2}\left[\sin(\ln x) + \cos(\ln x)\right].$$

So

$$\begin{aligned} y_p &= -\cos(\ln x)\frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + \sin(\ln x)\frac{x}{2}\left[\sin(\ln x) + \cos(\ln x)\right] \\ &= \frac{x}{2}. \end{aligned}$$

Thus the general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + \frac{x}{2}.$$

31. $x^2y'' + 4xy' + 2\,y = 0$. Euler equation with $\alpha = 4$, $\beta = 2$, indicial equation

$$r^2 + 3r + 2 = 0 \qquad (r+1)(r+2) = 0;$$

indicial roots: $r_1 = -1$, $r_2 = -2$; hence the general solution

$$y = c_1 x^{-1} + c_2 x^{-2}.$$

32. $x^2y'' + x\,y' - 4y = 0$. Euler equation with $\alpha = 1$, $\beta = -4$, indicial equation

$$r^2 - 4 = 0 \qquad (r+2)(r-2) = 0;$$

indicial roots: $r_1 = -2$, $r_2 = 2$; hence the general solution

$$y = c_1 x^{-2} + c_2 x^2.$$

33. $x^2y'' + 3xy' + y = 0$. See Exercise 29.

34. $4\,x^2y'' + 8\,x\,y' + y = 0$ or $x^2y'' + 2\,x\,y' + \frac{1}{4}y = 0$. Euler equation with $\alpha = 2$, $\beta = \frac{1}{4}$, indicial equation

$$r^2 + r + \frac{1}{4} = 0 \qquad (r + \frac{1}{2})^2 = 0;$$

one double indicial root: $r = -\frac{1}{2}$; hence the general solution

$$y = c_1 x^{-1/2} + c_2 x^{-1/2}\ln x.$$

35. $x^2y'' + xy' + 4y = 0$. Euler equation with $\alpha = 1$, $\beta = 4$, indicial equation $r^2 + 4 = 0$; indicial roots: $r_1 = -2i$, $r_2 = 2i$; hence the general solution

$$y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x).$$

36. $4\,x^2y'' + 4x\,y' + y = 0$ or $x^2y'' + x\,y' + \frac{1}{4}y = 0$. Euler equation with $\alpha = 1$, $\beta = \frac{1}{4}$, indicial equation $r^2 + \frac{1}{4} = 0$; indicial roots: $r_1 = -i2$, $r_2 = i2$; hence the general solution

$$y = c_1 \cos(\frac{\ln x}{2}) + c_2 \sin(\frac{\ln x}{2}).$$

37. $x^2y'' + 7xy' + 13\,y = 0$. Euler equation with $\alpha = 7$, $\beta = 13$, indicial equation $r^2 + 6\,r + 13 = 0$; indicial roots:

$$r = -3 \pm \sqrt{-4}; \qquad r_1 = -3 - 2i, \quad r_2 = -3 + 2i.$$

Hence the general solution

$$y = x^{-3}\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right].$$

38. $x^2y'' - x\,y' + 5\,y = 0$. Euler equation with $\alpha = -1$, $\beta = 5$, indicial equation $r^2 - 2\,r + 5 = 0$; indicial roots:

$$r = 1 \pm \sqrt{-4}; \qquad r_1 = 1 - 2i, \quad r_2 = 1 + 2i.$$

Hence the general solution

$$y = x\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right].$$

39. $(x-2)^2y'' + 3(x-2)y' + y = 0$, $(x > 2)$. Let $t = x - 2$, $x = t + 2$, $y(x) = y(t+2) = Y(t)$,

$$y' = \frac{dy}{dx} = \frac{dY}{dx} = \frac{dY}{dt}\frac{dt}{dx} = \frac{dY}{dt} = Y';$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}Y' == \frac{dY'}{dt}\frac{dt}{dx} = \frac{d^2Y}{dt^2} = Y''.$$

The equation becomes $t^2Y'' + 3tY' + Y = 0$, $t > 0$. Euler equation with $\alpha = 3$, $\beta = 1$, indicial equation

$$r^2 + 2\,r + 1 = 0 \qquad (r+1)^2 = 0;$$

one double indicial root $r = -1$. Hence the general solution

$$Y = c_1 t^{-1} + c_2 t^{-1}\ln t,$$

and so

$$y = c_1(x-2)^{-1} + c_2(x-2)^{-1}\ln(x-2).$$

40. $(x+1)^2y'' + (x+1)\,y' + y = 0$, $(x > -1)$. Let $t = x + 1$, $x = t - 1$, $y(x) = y(t-1) = Y(t)$,

$$y' = \frac{dy}{dx} = \frac{dY}{dx} = \frac{dY}{dt}\frac{dt}{dx} = \frac{dY}{dt} = Y';$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}Y' == \frac{dY'}{dt}\frac{dt}{dx} = \frac{d^2Y}{dt^2} = Y''.$$

The equation becomes $t^2Y'' + tY' + Y = 0$, $t > 0$. Euler equation with $\alpha = 1$, $\beta = 1$, indicial equation

$$r^2 + 1 = 0;$$

indicial roots $r = \pm i$. Hence the general solution

$$Y = c_1 \cos(\ln t) + c_2 \sin(\ln t) \qquad (t > 0),$$

and so

$$y = c_1 \cos(\ln(x+1)) + c_2 \sin(\ln(x+1)) \qquad (x > -1).$$

41. We have

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

Using the product rule for differentiation

$$\begin{aligned} y_2' &= y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + y_1 \frac{e^{-\int p(x)\,dx}}{y_1^2} \\ &= y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + \frac{e^{-\int p(x)\,dx}}{y_1}. \end{aligned}$$

So

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \\ y_1' & y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + \frac{e^{-\int p(x)\,dx}}{y_1} \end{vmatrix} \\ &= y_1 y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + e^{-\int p(x)\,dx} - y_1' y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \\ &= e^{-\int p(x)\,dx} > 0. \end{aligned}$$

This also follows from Abel's formula, (4), Section A.1.

42. We have

$$\begin{aligned} y_1 &= x^a \cos(b \ln x) \\ y_1' &= a x^{a-1} \cos(b \ln x) - b x^{a-1} \sin(b \ln x); \\ y_2 &= x^a \sin(b \ln x) \\ y_2' &= a x^{a-1} \sin(b \ln x) + b x^{a-1} \cos(b \ln x). \end{aligned}$$

So

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} x^a \cos(b \ln x) & x^a \sin(b \ln x) \\ a x^{a-1} \cos(b \ln x) - b x^{a-1} \sin(b \ln x) & a x^{a-1} \sin(b \ln x) + b x^{a-1} \cos(b \ln x) \end{vmatrix} \\ &= a x^{2a-1} \cos(b \ln x) \sin(b \ln x) + b x^{2a-1} a \cos^2(b \ln x) \\ &\quad - a x^{2a-1} \sin(b \ln x) \cos(b \ln x) + b x^{2a-1} \sin^2(b \ln x) \\ &= b x^{2a-1} > 0. \end{aligned}$$

This also follows from Abel's formula, (4), Section A.1.

43 In the equation $x^2 y'' + \alpha y' + \beta y = 0$, let $t = \ln x$, $x = e^t$, $y(x) = y(e^t) = Y(t)$. Then,

$$\begin{aligned} \frac{dt}{dx} &= \frac{1}{x} = e^{-t}; \\ y' &= \frac{dy}{dx} = \frac{dY}{dx} = \frac{dY}{dt}\frac{dt}{dx} = Y' \cdot \frac{1}{x} = Y' \cdot e^{-t}; \\ y'' &= \frac{d}{dx}\left[Y' \cdot e^{-t}\right] = e^{-t}\frac{d}{dx}[Y'] + Y'\frac{d}{dx}[e^{-t}] \\ &= e^{-t}\frac{dY'}{dt}\frac{dt}{dx} + Y'[-e^{-t}\frac{dt}{dx}] \\ &= e^{-2t} Y'' - e^{-2t} Y'. \end{aligned}$$

Substituting into the equation and simplifying and using $x = e^t$, $x^2 = e^{2t}$, we obtain

$$\frac{d^2 Y}{dt^2} + (\alpha - 1)\frac{dY}{dt} + \beta Y = 0,$$

which is a second order linear differential equation in Y with constant coefficients.

44. Using the change of variables $t = \ln x$, we transformed Euler's equation into

$$Y'' + (\alpha - 1) Y' + \beta Y = 0.$$

Three cases can occur depending on the characteristic equation

$$r^2 + (1 - \alpha) r + \beta = 0$$

(we are using r instead of the traditional λ to denote the characteristic roots.
Case I: If $(1-\alpha)^2 - 4\beta > 0$, then we have two distinct roots

$$r_1 = \frac{-1+\alpha+\sqrt{(1-\alpha)^2-4}}{2} \quad r_2 = \frac{-1+\alpha-\sqrt{(1-\alpha)^2-4}}{2}.$$

The general solution in this case is

$$Y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Using $y(x) = Y(\ln x)$, we get

$$y(x) = Y(t) = c_1 e^{r_1 \ln x} + c_2 e^{r_2 \ln x} = c_1 x^{r_1} + c_2 x^{r_2}.$$

Case II: If $(1-\alpha)^2 - 4\beta = 0$, then we have one double root

$$r = \frac{-1+\alpha}{2}.$$

The general solution in this case is

$$Y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Using $y(x) = Y(\ln x)$, we get

$$y(x) = Y(t) = c_1 e^{r \ln x} + c_2 (\ln x) e^{r \ln x} = c_1 x^r + c_2 x^r \ln x.$$

Case III: If $(1-\alpha)^2 - 4\beta < 0$, then we have two complex conjugate roots

$$r_1 = \frac{-1+\alpha+i\sqrt{-(1-\alpha)^2+4}}{2} = a+ib \quad r_2 = \frac{-1+\alpha-i\sqrt{-(1-\alpha)^2+4}}{2} = a-ib.$$

The general solution in this case is

$$Y(t) = e^{at}\left[c_1 \cos bt + c_2 \sin bt\right].$$

Using $y(x) = Y(\ln x)$, we get

$$y(x) = Y(t) = e^{a \ln x}\left[c_1 \cos(b \ln x) + c_2 \sin(b \ln x)\right] = x^a \left[c_1 \cos(b \ln x) + c_2 \sin(b \ln x)\right].$$

45. (a) From Abel's formula (Theorem 2, Section A.1), the Wronskian is

$$y_1 y_2' - y_1' y_2 = C e^{-\int p(x)\,dx},$$

where y_1 and y_2 are any two solution of (2).
(b) Given y_1, set $C = 1$ in (a)

$$y_1 y_2' - y_1' y_2 = e^{-\int p(x)\,dx}.$$

This is a first-order differential equation in y_2 that we rewrite as

$$y_2' - \frac{y_1'}{y_1} y_2 = e^{-\int p(x)\,dx}.$$

The integrating factor is

$$e^{\int -\frac{y_1'}{y_1}\,dx} = e^{-\ln y_1} = \frac{1}{y_1}.$$

Multiply by the integrating factor:

$$\frac{y_2'}{y_1} - \frac{y_1'}{y_1^2} y_2 = \frac{1}{y_1} e^{-\int p(x)\,dx}$$

or

$$\frac{d}{dx}\left[\frac{y_2}{y_1}\right] = \frac{1}{y_1} e^{-\int p(x)\,dx}$$

Integrating both sides, we get

$$\begin{aligned} \frac{y_2}{y_1} &= \int \left[\frac{1}{y_1} e^{-\int p(x)\,dx}\right] dx; \\ y_2 &= y_1 \int \left[\frac{1}{y_1} e^{-\int p(x)\,dx}\right] dx, \end{aligned}$$

which implies (3).

46. (a) Following the proof of (3), put $y = vy_1$. Then,

$$\begin{aligned} y_1 v'' + [2y_1' + p(x)y_1]v' &= g(x) \\ v'' + \frac{[2y_1' + p(x)y_1]}{y_1} v' &= \frac{g(x)}{y_1} \\ v'' + \left[2\frac{y_1'}{y_1} + p(x)\right] v' &= \frac{g(x)}{y_1}. \end{aligned}$$

(b) To solve the first order differential equation in v' (Theorem 1, Section A.1), we use the integrating factor

$$e^{\int \left[2\frac{y_1'}{y_1} + p(x)\right] dx} = y_1^2 e^{\int p(x)\,dx}.$$

The equation becomes

$$\begin{aligned} y_1^2 e^{\int p(x)\,dx} v'' + y_1^2 e^{\int p(x)\,dx}\left[2\frac{y_1'}{y_1} + p(x)\right] v' &= y_1 e^{\int p(x)\,dx} g(x); \\ \frac{d}{dx}\left[y_1^2 e^{\int p(x)\,dx} v'\right] &= y_1 e^{\int p(x)\,dx} g(x); \\ y_1^2 e^{\int p(x)\,dx} v' &= \int \left[y_1 e^{\int p(x)\,dx} g(x)\right] dx + C_1; \\ v' &= \frac{1}{y_1^2} e^{-\int p(x)\,dx} \int \left[y_1 e^{\int p(x)\,dx} g(x)\right] dx \\ &\quad + c_1 \frac{1}{y_1^2} e^{-\int p(x)\,dx}. \end{aligned}$$

Integrating once more, we get

$$\begin{aligned} v &= \int \left[\frac{1}{y_1^2} e^{-\int p(x)\,dx} \int \left[y_1 e^{\int p(x)\,dx} g(x)\right] dx\right] \\ &\quad + c_1 \int \left[\frac{1}{y_1^2} e^{-\int p(x)\,dx}\right] dx; \end{aligned}$$

$$\begin{aligned} y_2 = c_1\, y_1 &\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\, dx \\ + y_1 &\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\, dx \left(\int y_1 e^{\int p(x)\,dx} g(x)\,dx\right) dx \end{aligned}$$

where the last two occurrences of $\int p(x)\,dx$ represent the same antiderivative of $p(x)$.

47. In solving the equation $y'' - 4y' + 3y = e^x, \quad y_1 = e^x$, we follow the steps in the proof of the previous exercise. Let

$$y_1 = e^x,\ y_2 = ve^x,\ y_2' = v'e^x + ve^x,\ y_2'' = v''e^x + 2v'e^x + ve^x.$$

Then

$$\begin{aligned} y'' - 4y' + 3y = e^x \quad &\Rightarrow \quad v''e^x + 2v'e^x + ve^x - 4(v'e^x + ve^x) + 3ve^x = e^x \\ &\Rightarrow \quad v'' + 2v' + v - 4(v' + v) + 3v = 1 \text{ (divide by } e^x) \\ &\Rightarrow \quad v'' - 2v' = 1. \end{aligned}$$

As expected, we arrive at a first order o.d.e. in v'. Its solution (using an integrating factor) is

$$\begin{aligned} e^{-2x}v'' - 2e^{-2x}v' &= e^{-2x} \\ \frac{d}{dx}\left[e^{-2x}v'\right] &= e^{-2x} \\ e^{-2x}v' &= \int e^{-2x}\,dx = \frac{-1}{2}e^{-2x} + C \\ v' &= \frac{-1}{2} + C\,e^{2x}. \end{aligned}$$

Integrating once more,

$$\begin{aligned} v &= \frac{-x}{2} + C_1\,e^{2x} + C_2 \\ y_2 &= vy_1 = \frac{-x}{2}e^x + C_1\,e^{3x} + C_2e^x. \end{aligned}$$

Note that the term C_2e^x is corresponds to the solution y_1.

48. $x^2y'' + 3xy' + y = \sqrt{x}, \quad y_1 = \frac{1}{x}$.

As in the previous exercise, let

$$y_1 = \frac{1}{x},\ y = v\frac{1}{x},\ y' = \frac{v'x - v}{x^2} = \frac{v'}{x} - \frac{v}{x^2},\ y'' = \frac{v''}{x} - 2\frac{v'}{x^2} + 2\frac{v}{x^3}.$$

Then

$$\begin{aligned} x^2y'' + 3xy' + y = \sqrt{x} \quad &\Rightarrow \quad xv'' - 2v' + 2\frac{v}{x} + 3v' - 3\frac{v}{x} + \frac{v}{x} = \sqrt{x} \\ &\Rightarrow \quad xv'' + v' = \sqrt{x}. \end{aligned}$$

We arrive at a first order o.d.e. in v'. Its solution (using an integrating factor) is

$$\begin{aligned} \frac{d}{dx}\left[xv'\right] &= \sqrt{x} \\ xv' &= \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C \\ v' &= \frac{2}{3}x^{1/2} + \frac{C}{x}. \end{aligned}$$

Integrating once more,

$$\begin{aligned} v &= \frac{4}{9}x^{3/2} + C\ln x + D \\ y &= vy_1 = \frac{4}{9}x^{1/2} + C\frac{\ln x}{x} + \frac{D}{x}. \end{aligned}$$

49. $3\,y'' + 13\,y' + 10\,y = \sin x, \quad y_1 = e^{-x}.$

As in the previous exercise, let

$$y_1 = e^{-x},\; y = ve^{-x},\; y' = v'e^{-x} - ve^{-x},\; y'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}.$$

Then

$$\begin{aligned} 3\,y'' + 13\,y' + 10\,y = \sin x \quad &\Rightarrow \quad 3(v''e^{-x} - 2v'e^{-x} + ve^{-x}) \\ &\qquad +13(v'e^{-x} - ve^{-x}) + 10ve^{-x} = \sin x \\ &\Rightarrow \quad 3v'' + 7v' = e^x \sin x \\ &\Rightarrow \quad v'' + \frac{7}{3}v' = \frac{1}{3}e^x \sin x. \end{aligned}$$

We now solve the first order o.d.e. in v':

$$\begin{aligned} e^{7x/3}v'' + \frac{7}{3}e^{7x/3}v' &= e^{7x/3}\frac{1}{3}e^x \sin x \\ \frac{d}{dx}\left[e^{7x/3}v'\right] &= \frac{1}{3}e^{10x/3}\sin x \\ e^{7x/3}v' &= \frac{1}{3}\int \frac{1}{3}e^{10x/3}\sin x\,dx \\ &= \frac{1}{3}\frac{e^{10x/3}}{(\frac{10}{3})^2+1}(\frac{10}{3}\sin x - \cos) + C \\ v' &= \frac{e^x}{109}(10\sin x - \frac{9}{3}\cos) + C. \end{aligned}$$

(We used the table of integrals to evaluate the preceding integral. We will use it again below.) Integrating once more,

$$\begin{aligned} v &= \frac{10}{109}\int e^x \sin x\,dx - \frac{9}{327}\int e^x \cos x\,dx \\ &= \frac{10}{109}\frac{e^x}{2}(\sin x - \cos x) - \frac{9}{327}\frac{e^x}{2}(\cos x + \sin x) + C \\ y &= vy_1 = \frac{10}{218}(\sin x - \cos x) - \frac{9}{654}(\cos x + \sin x) + Ce^{-x} \\ &= -\frac{13}{218}\cos x + \frac{7}{218}\sin x + Ce^{-x}. \end{aligned}$$

50. $x\,y'' - (1+x)y' + y = x^3, \quad y_1 = e^x.$ We have

$$y_1 = e^x,\; y = ve^x,\; y' = v'e^x + ve^x,\; y'' = v''e^x + 2v'e^x + ve^x.$$

Then

$$\begin{aligned}
x\left(v''e^x + 2v'e^x + ve^x\right) - (1+x)(v'e^x + ve^x) + ve^x = x^3 \quad &\Rightarrow \quad xe^x v'' + xe^x v' - e^x v' = x^3 \\
&\Rightarrow \quad \frac{e^x}{x}v'' + e^x\left(-\frac{1}{x^2} + \frac{1}{x}\right)v' = x \\
\frac{d}{dx}\left[\frac{e^x}{x}v'\right] \quad &= \quad x \\
\frac{e^x}{x}v' \quad &= \quad \int x\,dx = \frac{1}{2}x^2 + C \\
v' \quad &= \quad \frac{1}{2}x^3e^{-x} + Cxe^{-x}.
\end{aligned}$$

Integrating once more,

$$\begin{aligned}
v &= \frac{1}{2}\int x^3e^{-x}\,dx + +C\int xe^{-x}\,dx \\
&= \frac{1}{2}\left[-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right] + C\left[-xe^{-x} - e^{-x}\right] \\
y &= \frac{1}{2}\left[-x^3 - 3x^2 - 6x - 6\right] + C\left[-x - 1\right].
\end{aligned}$$

Solutions to Exercises A.4

1. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} \frac{x^m}{5m+1}$$

converges whenever the limit

$$\lim_{m\to\infty} \left| \frac{x^{m+1}}{5(m+1)+1} \Big/ \frac{x^m}{5m+1} \right| = \lim_{m\to\infty} \left(\frac{5m+1}{5m+6} \right) |x| = |x|$$

is less than 1. That is, $|x| < 1$. Thus, the interval of convergence is $|x| < 1$ or $(-1, 1)$. Since the series is centered at 0, the radius of convergence is 1.

2. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} \frac{3^m x^m}{m!}$$

converges whenever the following limit is less than 1:

$$\begin{aligned} \lim_{m\to\infty} \left| \frac{3^{m+1} x^{m+1}}{(m+1)!} \Big/ \frac{3^m x^m}{m!} \right| &= \lim_{m\to\infty} 3\,|x| \frac{m!}{(m+1)!} \\ &= 3\,|x| \lim_{m\to\infty} \frac{1}{m+1} = 3\,|x| \cdot 0 = 0. \end{aligned}$$

Since the limit is always less than 1, the series converges for all x. It is cantered at 0 and has radius of convergence $R = \infty$.

3. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{m+2}}{2^m}$$

converges whenever the limit

$$\lim_{m\to\infty} \left| \frac{(-1)^{m+1} x^{(m+1)+2}}{2^{m+1}} \Big/ \frac{(-1)^m x^{m+2}}{2^m} \right| = \frac{|x|}{2}$$

is less than 1. That is, $\frac{1}{2}|x| < 1$ or $|x| < 2$. Thus, the interval of convergence is $(-2, 2)$. Since the series is centered at 0, the radius of convergence is 2.

4. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} (x-2)^m$$

converges whenever the limit

$$\lim_{m\to\infty} \left| (x-2)^{m+1} \Big/ (x-2)^m \right| = |x-2|$$

is less than 1. That is, $|x-2| < 1$. This is an interval centered at 2 with radius 1. Thus interval of convergence is $(-1, 3)$.

5. Using the ratio test, we have that the series

$$\sum_{m=1}^{\infty} \frac{m^m x^m}{m!}$$

converges whenever the following limit is < 1:

$$\lim_{m\to\infty}\left|\frac{(m+1)^{m+1}\,x^{m+1}}{(m+1)!}\bigg/\frac{m^m\,x^m}{m!}\right| = \lim_{m\to\infty}\frac{(m+1)^m(m+1)}{(m+1)!}\cdot\frac{m!}{m^m\,x^m}\cdot|x|$$
$$= |x|\lim_{m\to\infty}\left(\frac{m+1}{m}\right)^m = e|x|.$$

We have used the limit

$$\lim_{m\to\infty}\left(\frac{m+1}{m}\right)^m = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$$

(see the remark at the end of the solution). From $|x|e < 1$ we get $|x| < 1/e$. Hence the interval of convergence is $(-1/e,\, 1/e)$. It is centered at 0 and has radius $1/e$.

One way to show

$$\lim_{m\to\infty}\left(\frac{m+1}{m}\right)^m = e$$

is to show that the natural logarithm of the limit is 1:

$$\ln\left(\frac{m+1}{m}\right)^m = m\ln\left(\frac{m+1}{m}\right) = m\left[\ln(m+1) - \ln m\right].$$

By the mean value theorem (applied to the function $f(x) = \ln x$ on the interval $[m,\, m+1]$), there is a real number c_m in $[m,\, m+1]$ such that

$$\ln(m+1) - \ln m = f'(c_m) = \frac{1}{c_m}$$

Note that

$$\frac{1}{m+1} \le \frac{1}{c_m} \le \frac{1}{m}.$$

So

$$\frac{m}{m+1} \le m\left[\ln(m+1) - \ln m\right] \le 1.$$

As $m\to\infty$, $\frac{m}{m+1}\to 1$, and so by the sandwich theorem,

$$m\left[\ln(m+1) - \ln m\right] \to 1.$$

Taking the exponential, we derive the desired limit.

6. Using the ratio test, we have that the series

$$\sum_{m=1}^{\infty}\frac{(m-1)(m+3)\,x^m}{m}$$

converges whenever the limit

$$\lim_{m\to\infty}\left|\frac{(m+1-1)(m+1+3)\,x^{m+1}}{m+1}\bigg/\frac{(m-1)(m+3)\,x^m}{m}\right|$$
$$= |x|\lim_{m\to\infty}\frac{m(m+4)}{m+1}\cdot\frac{m}{(m-1)(m+3)} = |x|$$

is less than 1. That is, $|x| < 1$. Thus the interval convergence is $(-1,\, 1)$. It is centered at 0 and has radius 1.

7. Using the ratio test, we have that the series

$$\sum_{m=2}^{\infty}\frac{x^m}{\ln m}$$

converges whenever the limit

$$\lim_{m\to\infty}\left|\frac{x^{m+1}}{\ln(m+1)}\Big/\frac{x^m}{\ln m}\right| \quad = \quad |x|\lim_{m\to\infty}\frac{\ln m}{\ln(m+1)}=|x|$$

is less than 1 (see the note at the end of the solution). That is, $|x|<1$. Thus the interval convergence is $(-1,\,1)$. It is centered at 0 and has radius 1. To show that

$$\lim_{m\to\infty}\frac{\ln m}{\ln(m+1)}=1,$$

you can use l'Hospital rule. Differentiating numerator and denominator, we obtain

$$\lim_{m\to\infty}\frac{\ln m}{\ln(m+1)}=\lim_{m\to\infty}\frac{1/m}{1/(m+1)}=\lim_{m\to\infty}\frac{m+1}{m}=1.$$

8. Using the ratio test, we have that the series

$$\sum_{m=1}^{\infty}\frac{(x+1)^m}{m^2}$$

converges whenever the limit

$$\lim_{m\to\infty}\left|\frac{(x+1)^{m+1}}{(m+1)^2}\Big/\frac{(x+1)^m}{m^2}\right| \quad = \quad |x+1|\lim_{m\to\infty}\frac{m^2}{(m+1)^2}=|x+1|$$

is less than 1. That is, $|x-(-1)|<1$. Thus the interval convergence is $(-2,\,0)$. It is centered at -1 and has radius 1.

9. Using the ratio test, we have that the series

$$\sum_{m=1}^{\infty}\frac{[10(x+1)]^{2m}}{(m!)^2}$$

converges whenever the following limit is <1:

$$\begin{aligned}\lim_{m\to\infty}\left|\frac{[10(x+1)]^{2(m+1)}}{((m+1)!)^2}\Big/\frac{[10(x+1)]^{2m}}{(m!)^2}\right| &= 10^2|x+1|^2\lim_{m\to\infty}\frac{(m!)^2}{((m+1)!)^2}\\ &= 10^2|x+1|^2\lim_{m\to\infty}\frac{(m!)^2}{(m+1)^2(m!)^2}\\ &= 10^2|x+1|^2\lim_{m\to\infty}\frac{1}{(m+1)^2}=0.\end{aligned}$$

Thus the series converges for all x, $R=\infty$.

10. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty}\frac{(9\,x)^{2m}}{(m!)^2}$$

converges whenever the following limit is <1:

$$\begin{aligned}\lim_{m\to\infty}\left|\frac{(9\,x)^{2(m+1)}}{((m+1)!)^2}\Big/\frac{(9\,x)^{2m}}{(m!)^2}\right| &= 9^2|x|^2\lim_{m\to\infty}\frac{(m!)^2}{((m+1)!)^2}\\ &= 9^2|x|^2\lim_{m\to\infty}\frac{(m!)^2}{(m+1)^2(m!)^2}\\ &= 9^2|x|^2\lim_{m\to\infty}\frac{1}{(m+1)^2}=0.\end{aligned}$$

Thus the series converges for all x, $R = \infty$.

11. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} \frac{2^m (x-2)^m}{10^m}$$

converges whenever the following limit is < 1:

$$\lim_{m\to\infty} \left| \frac{2^{m+1}(x-2)^{m+1}}{10^{m+1}} \Big/ \frac{2^m (x-2)^m}{10^m} \right| = \frac{2}{10}|x-2|.$$

Thus $|x-2| < 5$. Hence the interval of convergence is centered at 2 and has radius 5; that is, $(-3, 7)$

12. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} \frac{m!\,(x-2)^m}{10^m}$$

converges whenever the following limit is < 1:

$$\lim_{m\to\infty} \left| \frac{(m+1)!\,(x-2)^{m+1}}{10^{m+1}} \Big/ \frac{m!\,(x-2)^m}{10^m} \right| = |x-2| \lim_{m\to\infty} \frac{m+1}{10} = \infty$$

if $x \neq 2$. Hence the series converges only at its center 2 and diverges otherwise. Its radius of convergence is 0.

13. We use the geometric series. For $|x| < 1$,

$$\begin{aligned} \frac{3-x}{1+x} &= -\frac{1+x}{1+x} + \frac{4}{1+x} \\ &= -1 + \frac{4}{1-(-x)} \\ &= -1 + 4\sum_{n=0}^{\infty}(-1)^n x^n. \end{aligned}$$

14. Differentiate the geometric series

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n \qquad (|x| < 1)$$

term-by-term and you get

$$\frac{-1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^n n x^n \qquad (|x| < 1).$$

Multiply by $-x$ both sides,

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^{n+1} n x^{n+1} \qquad (|x| < 1).$$

Replace n by $n-1$ and shift the index up by 1:

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n (n-1) x^n \qquad (|x| < 1).$$

15. Start with the geometric series

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n \qquad (|x| < 1).$$

Replace x by x^2:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad (|x^2| < 1 \text{ or } |x| < 1).$$

Differentiate term-by-term:

$$\frac{-2x}{(1+x^2)^2} = \sum_{n=1}^{\infty}(-1)^n (2n)x^{2n-1} \qquad (|x| < 1).$$

Divide by -2 both sides,

$$\frac{x}{(1+x^2)^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{2n-1} \qquad (|x| < 1).$$

16. Using a geometric series,

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \qquad (|x| < 1).$$

So for $|x| < 1$,

$$\begin{aligned} \frac{x+2}{1-x^2} &= \frac{x}{1-x^2} + \frac{2}{1-x^2} \\ &= x\sum_{n=0}^{\infty} x^{2n} + 2\sum_{n=0}^{\infty} x^{2n} \\ &= \sum_{n=0}^{\infty} x^{2n+1} + 2\sum_{n=0}^{\infty} x^{2n} \\ &= \sum_{k=0}^{\infty} a_k x^k, \end{aligned}$$

where $a_0 = 3$, $a_{2k} = 2$, $a_{2k+1} = 1$.

17. Use the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty.$$

Then

$$e^{u^2} = \sum_{n=0}^{\infty} \frac{u^{2n}}{n!} \qquad -\infty < u < \infty.$$

Hence, for all x,

$$\begin{aligned} e^{3x^2+1} &= e \cdot e^{(\sqrt{3}x)^2} \\ &= e\sum_{n=0}^{\infty} \frac{(\sqrt{3}x)^{2n}}{n!} \\ &= e\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}. \end{aligned}$$

18. For $|x| < 1$,

$$\ln(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^m}{m};$$

so

$$x \ln(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{m+1}}{m} = \sum_{m=2}^{\infty} \frac{(-1)^m x^m}{m-1}.$$

19. Using the sine series expansion, we have for all x

$$x \sin x \, \cos x = \frac{x}{2} \sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}.$$

20. Using the sine and cosine series expansions, we have for all x

$$\begin{aligned} x\, \sin(x+1) &= x \left[\sin x \cos 1 + \cos x \sin 1\right] \\ &= x \cos 1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x \sin 1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= \cos 1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} + \sin 1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \\ &= \sum_{k=0}^{\infty} (-1)^k a_k x^k, \end{aligned}$$

where, for $k \geq 0$,

$$a_{2k+1} = \frac{\sin 1}{(2k)!} \quad \text{and} \quad a_{2k+2} = \frac{\cos 1}{(2k+1)!}.$$

21. We have

$$\frac{1}{2+3x} = \frac{1}{2(1-(-\frac{3x}{2}))} = \frac{1}{2(1-u)},$$

where $u = -\frac{3x}{2}$. So

$$\frac{1}{2+3x} = \frac{1}{2} \sum_{n=0}^{\infty} u^n = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{3x}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{n+1}} x^n.$$

The series converges if $|u| < 1$; that is $|x| < \frac{2}{3}$.

22. We have

$$\begin{aligned} \frac{1+x}{2-3x} &= -\frac{1}{3} \frac{2-3x}{2-3x} + \frac{1}{3} \frac{5}{2-3x} \\ &= -\frac{1}{3} + \frac{1}{3} \frac{5}{2(1-\frac{3x}{2})} \\ &= -\frac{1}{3} + \frac{5}{6} \frac{1}{1-u}, \end{aligned}$$

where $u = \frac{3x}{2}$. So

$$\frac{1}{2+3x} = -\frac{1}{3} + \frac{5}{6} \sum_{n=0}^{\infty} u^n = -\frac{1}{3} + \frac{5}{6} \sum_{n=0}^{\infty} \left(\frac{3x}{2}\right)^n.$$

The series converges if $|u| < 1$; that is $|x| < \frac{2}{3}$.

23. We have

$$\begin{aligned} 3\frac{x+x^3}{2+3x^2} &= x\frac{(3x^2+2+1)}{3x^2+2} \\ &= x+\frac{x}{3x^2+2} \\ &= x+\frac{x}{2(1-(-\frac{3}{2}x^2)} \\ &= x+\frac{x}{2(1-u)}, \end{aligned}$$

where $u=-\frac{3}{2}x^2$. So

$$\begin{aligned} 3\frac{x+x^3}{2+3x^2} &= x+\frac{x}{2}\sum_{n=0}^{\infty}u^n \\ &= x+\frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{3}{2}x^2\right)^n \\ &= x+\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^{n+1}}x^{2n+1}. \end{aligned}$$

The series converges if $|u|<1$; that is $x^2<\frac{2}{3}$ or $|x|<\sqrt{\frac{2}{3}}$.

24. To find the Taylor series expansion of $\frac{1}{x^2-2x+2}$ is not straightforward. We offer two solutions.

The first solution is based on partial fractions with complex roots. The final answer will be a series with real coefficients.

We start by finding the roots of

$$x^2-2x+2=0.$$

Using complex numbers, we find

$$x=\frac{2\pm\sqrt{-4}}{2} \quad\Rightarrow\quad x_1=1+i \quad\text{and}\quad x_2=1-i.$$

Write

$$\begin{aligned} \frac{1}{x^2-2x+2} &= \frac{A}{x-x_1}+\frac{B}{x-x_2} \\ &= \frac{A(x-x_2)+B(x-x_1)}{(x-x_1)(x-x_2)}; \\ 1 &= A(x-x_2)+B(x-x_1); \\ \text{Set } x=x_2 &\Rightarrow\ 1=B(x_2-x_1) \\ &\Rightarrow\ B=\frac{1}{x_2-x_1} \\ \text{Set } x=x_1 &\Rightarrow\ 1=A(x_1-x_2) \\ &\Rightarrow\ A=\frac{1}{x_1-x_2}=-B \end{aligned}$$

Note

$$x_1-x_2=2i, \quad x_1+x_2=-\frac{b}{a}=2, \quad x_1\cdot x_2=\frac{c}{a}=2.$$

$$\begin{aligned}
\frac{1}{x^2-2x+2} &= \frac{-i}{2(x-x_1)} + \frac{i}{2(x-x_2)} \\
&= \frac{-i}{2x_1(\frac{x}{x_1}-1)} + \frac{i}{2x_2(\frac{x}{x_2}-1)} \\
&= \frac{i}{2x_1(1-\frac{x}{x_1})} - \frac{i}{2x_2(1-\frac{x}{x_2})} \\
&= \frac{i}{2x_1}\sum_{n=0}^{\infty}(\frac{x}{x_1})^n - \frac{i}{2x_2}\sum_{n=0}^{\infty}(\frac{x}{x_2})^n \\
&= \sum_{n=0}^{\infty} x^n \left[\frac{i}{2x_1}\frac{1}{x_1^n} - \frac{i}{2x_2}\frac{1}{x_2^n}\right] \\
&= \frac{i}{2}\sum_{n=0}^{\infty} x^n \left[\frac{1}{x_1^{n+1}} - \frac{1}{x_2^{n+1}}\right].
\end{aligned}$$

Write $x_1 = \sqrt{2}e^{i\frac{\pi}{4}}$ and $x_2 = \sqrt{2}e^{-i\frac{\pi}{4}}$. Then

$$\begin{aligned}
\frac{1}{x_1^{n+1}} - \frac{1}{x_2^{n+1}} &= \frac{1}{2^{\frac{n+1}{2}}}\left[e^{-i(n+1)\frac{\pi}{4}} - e^{-i(n+1)\frac{\pi}{4}}\right] \\
&= \frac{1}{2^{\frac{n+1}{2}}}\left[-2i\sin((n+1)\frac{\pi}{4})\right],
\end{aligned}$$

by Euler identities. So

$$\begin{aligned}
\frac{1}{x^2-2x+2} &= \frac{i}{2}\sum_{n=0}^{\infty} x^n \frac{1}{2^{\frac{n+1}{2}}}\left[-2i\sin((n+1)\frac{\pi}{4})\right] \\
&= \sum_{n=0}^{\infty} \sin((n+1)\frac{\pi}{4})\frac{x^n}{2^{\frac{n+1}{2}}}.
\end{aligned}$$

Here are the first few terms of the series

$$\frac{1}{2} + \frac{x}{2} + \frac{x^2}{4} - \frac{x^4}{8} - \frac{x^5}{8} - \frac{x^6}{16} + \frac{x^8}{32} + \frac{x^9}{32} + \frac{x^{10}}{64} + \cdots$$

The series converges if $|x/x_1| < 1$ and $|x/x_2| < 1$. These were the conditions that we needed in order to apply the geometric series. Since $|x_1| = |x_2| = \sqrt{2}$, we find that the series converges if $|x| < \sqrt{2}$. Note that you can derive this from the series itself.

Professor Montgomery-Smith offered a faster but tricky solution. Notice that

$$x^4 + 4 = (x^2+2x+2)(x^2-2x+2).$$

So

$$\frac{1}{x^2-2x+2} = \frac{x^2+2x+2}{x^4+4}.$$

The rest is straightforward:

$$\begin{aligned}
\frac{x^2+2x+2}{x^4+4} &= \frac{x^2}{x^4+4}+\frac{2x}{x^4+4}+\frac{2}{x^4+4};\\
\frac{1}{x^4+4} &= \frac{1}{4\left(-(-(x/\sqrt{2})^4)+1\right)};\\
&= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{2^{n/2}};\\
\frac{2x}{x^4+4} &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{2^{n/2}};\\
\frac{x^2}{x^4+4} &= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2^{n/2}}.
\end{aligned}$$

The series converge if $|x|<\sqrt{2}$, as we found previously. If you combine the series, you should get the answer that we found previously. There are no terms in x^{4n+3}. So every 3rd term is 0–confirming what we found previously.

25. Let a be any real number $\neq 0$, then

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{a-a+x}\\
&= \frac{1}{a}\cdot\frac{1}{1-\left(\frac{a-x}{a}\right)}\\
&= \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a-x}{a}\right)^n\\
&= \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{(x-a)^n}{a^n}.
\end{aligned}$$

The series converges if

$$\left|\frac{a-x}{a}\right|<1 \quad \text{or} \quad |a-x|<|a|.$$

26. Let

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}} \quad \text{and} \quad g(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!(k+1)!2^{2k1}}.$$

(a) Using the ratio test, we have that the series for f converges if the following limit is <1:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2(k+1)}}{((k+1)!)^2 2^{2(k+1)}}\Big/\frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}}\right| &= x^2\lim_{k\to\infty}\frac{(k!)^2 2^{2k}}{((k+1)!)^2 2^{2(k+1)}}\\
&= x^2\lim_{k\to\infty}\frac{1}{(k+1)^2 2^2}=x^2\cdot 0=0.
\end{aligned}$$

Since the limit is always less than 1, the series converges for all x. The same works for the series for g.

(b) A converging power series can be differentiated term by term within its radius of convergence. So

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}};\\
f(x) &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{(k!)^2 2^{2k}};\\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (2(k+1)) x^{2(k+1)-1}}{((k+1)!)^2 2^{2(k+1)}};\\
&= -\sum_{k=0}^{\infty} \frac{(-1)^k (2(k+1)) x^{2k+1}}{(k+1)!(k+1)! 2^{2k+2}};\\
&= -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(k+1)! 2^{2k+1}} = -g(x).
\end{aligned}$$

(c) Integrating both sides of $f'(x) = -g(x)$, we get

$$\int g(x)\,dx = -f(x) + C.$$

(d) Integrating term by term,

$$\begin{aligned}
\int_0^x t\,f(t)\,dt &= \int_0^x t \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(k!)^2 2^{2k}} dt\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}} \int_0^x t^{2k+1} dt\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}} \frac{1}{2k+2} x^{2k+2}\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)(k!) 2^{2k}} \frac{1}{2(k+1)} x^{2k+2}\\
&= x \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)! k! 2^{2k+1}} x^{2k+1}\\
&= x\,g(x).
\end{aligned}$$

27. Let $f(x)$ be as in Exercise 26. From the formula for the Taylor coefficients, we know that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Comparing coefficients with

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}},$$

we find that

$$f(0) = \frac{(-1)^0}{(0!)^2 2^0} = 1;$$

$$f'(0) = 0;$$

$$\frac{f''(0)}{2!} = \frac{(-1)^1}{(1!)^2 2^2} \quad \Rightarrow \quad f''(0) = -\frac{1}{4}.$$

(b) In general, since the Taylor series has no terms with odd powers of x, we find that find $f^{(2k+1)}(0) = 0$. For the eve powers, we have

$$\frac{f^{(2k)}(0)}{(2k)!} = \frac{(-1)^k}{(k!)^2 2^{2k}} \quad \Rightarrow \quad f^{(2k)}(0) = \frac{(-1)^k (2k)!}{(k!)^2 2^{2k}}.$$

(c) Integrate term-by-term the Taylor series

$$\begin{aligned}\int_0^1 f(x)\,dx &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}} \int_0^1 x^{2k}\,dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}(2k+1)} x^{2k+1}\Big|_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}(2k+1)}.\end{aligned}$$

We estimate the alternating series (with terms decreasing to 0) by a partial sum. The error in stopping at the k-th term is less than or equal to the $(k+1)$-term. Thus

$$\left| \int_0^1 f(x)\,dx - \sum_{k=0}^{n} \frac{(-1)^k}{(k!)^2 2^{2k}(2k+1)} \right| \leq \frac{1}{((n+1)!)^2 2^{2n+2}(2n+3)}.$$

Taking $n = 3$, we find that the integral is equal to

$$\sum_{k=0}^{3} \frac{(-1)^k}{(k!)^2 2^{2k}(2k+1)} = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16128} = \frac{74167}{80640} \approx 0.91973$$

with an error less than

$$\frac{1}{(4!)^2 2^8 (9)} = \frac{1}{1327104} \approx 7.5 \times 10^{-7}.$$

With Mathematica, using the command **NIntegrate**, I found

$$\int_0^1 f(x)\,dx = \int_0^1 J_0(x)\,dx = 0.91973.$$

28. Let us find the Taylor series expansion around 0 of $f(x) = \frac{1+x}{1-x}$. We have

$$\begin{aligned}\frac{1+x}{1-x} &= -\frac{1-x}{1-x} + \frac{2}{1-x} \\ &= -1 + \frac{2}{1-x} \\ &= -1 + 2\sum_{n=0}^{\infty} x^n.\end{aligned}$$

Using

$$\frac{f^{(n)}(0)}{n!} = a_n \quad \text{or} \quad f^{(n)}(0) = a_n n!,$$

where a_n is the n-th Taylor coefficient, we conclude that

$$f(0) - 1 + 2 = 1 \qquad f'(0) = 2 \qquad f^{(100)}(0) = (2)100!.$$

29. Recall that changing m to $m-1$ in the terms of the series requires shifting the index of summation up by 1. This is what we will do in the second series:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{x^m}{m}-2\sum_{m=0}^{\infty}mx^{m+1} &= \sum_{m=1}^{\infty}\frac{x^m}{m}-2\sum_{m=1}^{\infty}(m-1)x^m\\ &= \sum_{m=1}^{\infty}x^m\left[\frac{1}{m}-2(m-1)\right]\\ &= \sum_{m=1}^{\infty}\frac{-2m^2+2m+1}{m}x^m.\end{aligned}$$

30. If k is a positive integer, changing m to $m-k$ in the terms of the series requires shifting the index of summation up by k; and changing m to $m+k$ in the terms of the series requires shifting the index of summation down by k:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{x^{m+1}}{m!}+\sum_{m=3}^{\infty}(m-1)x^{m-1} &= \sum_{m=2}^{\infty}\frac{x^m}{(m-1)!}+\sum_{m=2}^{\infty}mx^m\\ &= \sum_{m=2}^{\infty}\left[\frac{1}{(m-1)!}+m\right]x^m\\ &= \sum_{m=2}^{\infty}\frac{1+m!}{(m-1)!}x^m\end{aligned}$$

31.

$$\begin{aligned}&2x\sum_{m=2}^{\infty}2\sqrt{m+2}\,x^m+\sum_{m=2}^{\infty}\frac{x^{m-1}}{m+6}\\ &= \sum_{m=2}^{\infty}4\sqrt{m+2}\,x^{m+1}+\sum_{m=2}^{\infty}\frac{x^{m-1}}{m+6}\\ &= \sum_{m=3}^{\infty}4\sqrt{m+1}\,x^m+\sum_{m=1}^{\infty}\frac{x^m}{m+7}\\ &= \sum_{m=3}^{\infty}4\sqrt{m+1}\,x^m+\sum_{m=3}^{\infty}\left[\frac{x^m}{m+7}\right]+\frac{x}{8}+\frac{x^2}{9}\\ &= \frac{x}{8}+\frac{x^2}{9}+\sum_{m=3}^{\infty}\left[4\sqrt{m+1}+\frac{1}{m+7}\right]x^m.\end{aligned}$$

32.

$$\begin{aligned}(x+1)\sum_{m=2}^{\infty}2x^{m-1} &= x\sum_{m=2}^{\infty}2x^{m-1}+\sum_{m=2}^{\infty}2x^{m-1}\\ &= \sum_{m=2}^{\infty}2x^m+\sum_{m=2}^{\infty}2x^{m-1}=\sum_{m=2}^{\infty}2x^m+\sum_{m=1}^{\infty}2x^m\\ &= 2x+\sum_{m=2}^{\infty}2x^m+\sum_{m=2}^{\infty}2x^m\\ &= 2x+\sum_{m=2}^{\infty}4x^m.\end{aligned}$$

33. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned} y' + y &= \sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+1) a_{m+1} + a_m\right] x^m \end{aligned}$$

34. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned} y' + x^2 y &= \sum_{m=1}^{\infty} m a_m x^{m-1} + x^2 \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=0}^{\infty} a_m x^{m+2} \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=2}^{\infty} a_{m-2} x^m \\ &= a_1 + 2a_2 x + \sum_{m=2}^{\infty} \left[(m+1) a_{m+1} + a_{m-2}\right] x^m. \end{aligned}$$

35. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}.$$

Then

$$\begin{aligned}(1-x^2)y''+2xy' &= (1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + 2x\sum_{m=1}^{\infty} m a_m x^{m-1}\\ &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=1}^{\infty} 2m a_m x^m\\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=1}^{\infty} 2m a_m x^m\\ &= 2a_2 + 6a_3 x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m\\ &\quad - \sum_{m=2}^{\infty} m(m-1)a_m x^m + 2a_1 x + \sum_{m=2}^{\infty} 2m a_m x^m\\ &= 2a_2 + (2a_1 + 6a_3)x\\ &\quad + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} - m(m-1)a_m + 2m a_m\right]x^m.\end{aligned}$$

36. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned}(2+x)y' + x^3 y &= (2+x)\sum_{m=1}^{\infty} m a_m x^{m-1} + x^3\sum_{m=0}^{\infty} a_m x^m\\ &= 2\sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^{m+3}\\ &= \sum_{m=0}^{\infty} 2(m+1)a_{m+1}x^m + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=3}^{\infty} a_{m-3}x^m\\ &= 2a_1 + 4a_2 x + 6a_3 x^2\\ &\quad + \sum_{m=3}^{\infty} 2(m+1)a_{m+1}x^m + a_1 x + 2a_2 x^2 + \sum_{m=3}^{\infty} m a_m x^m + \sum_{m=3}^{\infty} a_{m-3}x^m\\ &= 2a_1 + 4a_2 x + 6a_3 x^2 + a_1 x + 2a_2 x^2\\ &\quad + \sum_{m=3}^{\infty}\left[2(m+1)a_{m+1} + m a_m + a_{m-3}\right]x^m\\ &= 2a_1 + (a_1 + 4a_2)x + (2a_2 + 6a_3)x^2\\ &\quad + \sum_{m=3}^{\infty}\left[2(m+1)a_{m+1} + m a_m + a_{m-3}\right]x^m.\end{aligned}$$

37. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned} x^2y''+y &= x^2\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}+\sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m}+\sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m}+a_0+a_1x+\sum_{m=2}^{\infty} a_m x^m \\ &= a_0+a_1x+\sum_{m=2}^{\infty}\left[m(m-1)a_m+a_m\right]x^m \\ &= a_0+a_1x+\sum_{m=2}^{\infty}(m^2-m+1)a_m x^m. \end{aligned}$$

38. Let

$$y=\sum_{m=0}^{\infty} a_m x^m \qquad y'=\sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y''=\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned} y''+y'+xy &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}+\sum_{m=1}^{\infty} m a_m x^{m-1}+\sum_{m=0}^{\infty} a_m x^{m+1} \\ &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m+\sum_{m=1}^{\infty}a_{m-1}x^m \\ &= 2a_2+\sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m \\ &\quad +a_1+\sum_{m=1}^{\infty}(m+1)a_{m+1}x^m+\sum_{m=1}^{\infty}a_{m-1}x^m \\ &= a_1+2a_2+\sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2}+(m+1)a_{m+1}+a_{m-1}\right]x^m. \end{aligned}$$

39. Let

$$y=\sum_{m=0}^{\infty} a_m x^m \qquad y'=\sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned} xy'+y-e^x &= \sum_{m=1}^{\infty} m a_m x^m+\sum_{m=0}^{\infty} a_m x^m-\sum_{m=0}^{\infty}\frac{x^m}{m!} \\ &= \sum_{m=0}^{\infty} m a_m x^m+\sum_{m=0}^{\infty} a_m x^m-\sum_{m=0}^{\infty}\frac{x^m}{m!} \\ &= \sum_{m=0}^{\infty}\left[(m+1)a_m-\frac{1}{m!}\right]x^m. \end{aligned}$$

40. Let

$$y=\sum_{m=0}^{\infty}a_mx^m \qquad y'=\sum_{m=1}^{\infty}ma_mx^{m-1} \qquad y''=\sum_{m=2}^{\infty}m(m-1)a_mx^{m-2}.$$

Then

$$\begin{aligned}(1-x)y''+\sin x &= \sum_{m=2}^{\infty}m(m-1)a_mx^{m-2}-\sum_{m=2}^{\infty}m(m-1)a_mx^{m-1}\\ &\quad+\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\\ &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m-\sum_{m=1}^{\infty}(m+1)ma_{m+1}x^m\\ &\quad+\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\\ &= 2a_2+\sum_{m=1}^{\infty}[(m+2)(m+1)a_{m+2}-(m+1)ma_{m+1}]x^m\\ &\quad+\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\\ &= 2a_2+\sum_{m=1}^{\infty}[(m+2)(m+1)a_{m+2}-(m+1)ma_{m+1}+b_m]x^m,\end{aligned}$$

where $b_{2k}=0$ and $b_{2k+1}=\frac{(-1)^k}{(2k+1)!}$.

Solutions to Exercises A.5

1. For the differential equation $y' + 2xy = 0$, $p(x) = 1$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned} y' + 2xy &= \sum_{m=1}^{\infty} m a_m x^{m-1} + 2x \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} 2 a_m x^{m+1} \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m \\ &= a_1 + \sum_{m=1}^{\infty} \left[(m+1) a_{m+1} + 2 a_{m-1}\right] x^m. \end{aligned}$$

So $y' + 2xy = 0$ implies that

$$\begin{aligned} a_1 + \sum_{m=1}^{\infty} \left[(m+1) a_{m+1} + 2 a_{m-1}\right] x^m &= 0; \\ a_1 &= 0 \\ (m+1) a_{m+1} + 2 a_{m-1} &= 0 \\ a_{m+1} &= -\frac{2}{m+1} a_{m-1}. \end{aligned}$$

From the recurrence relation,

$$a_1 = a_3 = a_5 = \cdots = a_{2k+1} = \cdots = 0;$$

a_0 is arbitrary;

$$\begin{aligned} a_2 &= -\frac{2}{2} a_0 = -a_0, \\ a_4 &= -\frac{2}{4} a_2 = \frac{1}{2!} a_0, \\ a_6 &= -\frac{2}{6} a_4 = -\frac{1}{3!} a_0, \\ a_8 &= -\frac{2}{8} a_6 = \frac{1}{4!} a_0, \\ &\vdots \\ a_{2k} &= \frac{(-1)^k}{k!} a_0. \end{aligned}$$

So

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = a_0 e^{-x^2}.$$

2. For $y' + y = 0$, $p(x) = 1$ is its own power series expansion about 0. So 0 is an ordinary point. To solve $y' + y = 0$, use the result of Exercise 33 or previous section. Then

$$
\begin{aligned}
y' + y = 0 \quad &\Rightarrow \quad \sum_{m=0}^{\infty} \left[(m+1)a_{m+1} + a_m\right] x^m = 0 \\
&\Rightarrow \quad (m+1)a_{m+1} + a_m = 0 \quad \text{for all } m \geq 0 \\
&\Rightarrow \quad a_{m+1} = \frac{a_m}{m+1} \quad \text{for all } m \geq 0.
\end{aligned}
$$

So a_0 is arbitrary;

$$
\begin{aligned}
a_1 &= -\frac{1}{1}a_0 = -a_0, \\
a_2 &= -\frac{1}{2}a_1 = \frac{1}{2!}a_0, \\
a_3 &= -\frac{1}{3}a_2 = -\frac{1}{3!}a_0, \\
a_4 &= -\frac{1}{4}a_3 = \frac{1}{4!}a_0, \\
&\vdots \\
a_m &= \frac{(-1)^m}{m!}a_0.
\end{aligned}
$$

So

$$
y = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m = a_0 \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} = a_0 e^{-x}.
$$

3. From the previous exercise

$$
\begin{aligned}
y' + y = x \quad &\Rightarrow \quad \sum_{m=0}^{\infty} \left[(m+1)a_{m+1} + a_m\right] x^m = x \\
&\Rightarrow \quad a_1 + a_0 = 0 \quad (m = 0) \\
&\qquad 2a_2 + a_1 = 1 \quad (m = 1) \\
&\qquad (m+1)a_{m+1} + a_m = 0 \quad \text{for all } m \geq 2 \\
&\Rightarrow \quad a_{m+1} = \frac{a_m}{m+1} \quad \text{for all } m \geq 2.
\end{aligned}
$$

So a_0 is arbitrary;

$$\begin{aligned} a_1 &= -a_0, \\ a_2 &= \frac{1+a_0}{2}, \\ a_3 &= -\frac{1}{3}a_2 = -\frac{1}{3!}a_0 - \frac{1}{3!}, \\ a_4 &= -\frac{1}{4}a_3 = \frac{1}{4!}a_0 + \frac{1}{4!}, \\ &\vdots \\ a_m &= \frac{(-1)^m}{m!}(a_0+1). \end{aligned}$$

So

$$y = -1 + x + (a_0+1)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}x^m = -1 + x + Ce^{-x}.$$

4. For the differential equation $y' + (\cos x)y = 0$, $p(x) = 1$ is its own power series expansion about 0. So $a = 0$ is an ordinary point. To solve, recall that $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ and let

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Then

$$\begin{aligned} y' + (\cos x)y &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\right)\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} c_n x^n, \end{aligned}$$

where, using the Cauchy product, we have

$$c_n = \sum_{k=0}^{n} a_n b_{n-k}, \qquad b_{2k+1} = 0,\ b_{2k} = \frac{(-1)^k}{(2k)!}.$$

So $y' + (\cos x)y = 0$ implies that

$$(n+1)a_{n+1} + c_n = 0 \quad \Rightarrow \quad a_{n+1} = -\frac{1}{n+1}\sum_{k=0}^{n} a_n b_{n-k}.$$

Take a_0 arbitrary, then

$$
\begin{aligned}
a_1 &= -a_0 b_0 = -a_0 \\
a_2 &= -\frac{1}{2}c_1 = -\frac{1}{2}(a_0 b_1 + a_1 b_0) \\
&= -\frac{1}{2}(-a_0) = \frac{1}{2}a_0 \\
a_3 &= -\frac{1}{3}c_2 = -\frac{1}{3}(a_0 b_2 + a_1 b_1 + a_2 b_0) \\
&= -\frac{1}{3}(a_0\frac{-1}{2} + a_2) = \frac{1}{6}a_0 - \frac{1}{3}a_2 \\
&= \frac{1}{6}a_0 - \frac{1}{3}\frac{1}{2}a_0 = 0 \\
&\vdots
\end{aligned}
$$

So

$$y = a_0 - a_0 x + \frac{a_0}{2}x^2 + 0 \cdot x^3 - \frac{a_0}{8}x^4 + \cdots = a_0(1 - x + \frac{x^2}{2} + -\frac{x^4}{8} + \cdots).$$

Another way to derive this solution is to use the method of Example 4.

5. For the differential equation $y''-y=0$, $p(x)=0$ is its own power series expansion about $a=0$. So $a=0$ is an ordinary point. To solve, let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$
\begin{aligned}
y'' - y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} - a_m\right] x^m.
\end{aligned}
$$

So $y''-y=0$ implies that

$$(m+2)(m+1)a_{m+2} - a_m = 0 \quad \Rightarrow \quad a_{m+2} = \frac{a_m}{(m+2)(m+1)} \text{ for all } m \geq 0.$$

So a_0 and a_1 are arbitrary;

$$
\begin{aligned}
a_2 &= \frac{a_0}{2}, \\
a_4 &= \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \\
a_6 &= \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!}, \\
&\vdots \\
a_{2n} &= \frac{a_0}{(2n)!}. \\
&.
\end{aligned}
$$

Similarly,

$$a_{2n+1} = \frac{a_1}{(2n+1)!},$$

and so

$$y = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = a_0 \cosh x + a_1 \sinh x.$$

6. As in the previous exercise, $y'' - y = x$ implies that

$$\begin{aligned}
\sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} - a_m\right] x^m &= x \\
2a_2 - a_0 &= 0 \quad \Rightarrow a_2 = \frac{a_0}{2} \\
3 \cdot 2a_3 - a_1 &= 1 \quad \Rightarrow a_3 = \frac{a_1}{3!} + \frac{1}{3!} \\
a_{m+2} &= \frac{a_m}{(m+2)(m+1)}, \quad m \geq 2.
\end{aligned}$$

Note that since the recurrence relation is the same as in the previous exercise and a_0 and a_2 are the same, all even-indexed coefficients will be the same. For the odd-indexed coefficients, we have

$$\begin{aligned}
a_5 &= \frac{1}{5 \cdot 4} a_3 = \frac{1}{5 \cdot 4}\left[\frac{a_1}{3!} + \frac{1}{3!}\right] \\
&= \frac{a_1}{5!} + \frac{1}{5!} \\
&\vdots \\
a_{2k+1} &= \frac{a_1}{(2k+1)!} + \frac{1}{(2k+1)!} \\
&\vdots
\end{aligned}$$

So

$$\begin{aligned}
y &= a_1 x + \left[\frac{a_1}{3!} + \frac{1}{3!}\right] x^3 + \cdots + \left[\frac{a_1}{(2k+1)!} + \frac{1}{(2k+1)!}\right] x^{2k+1} + \cdots \\
&= \sum_{k=0}^{\infty} \left[\frac{a_1}{(2k+1)!} + \frac{1}{(2k+1)!}\right] x^{2k+1} - x \\
&= (a_1 + 1) \sinh x - x.
\end{aligned}$$

Thus the general solution is

$$y c_1 \cosh x + c_2 \sinh x - x.$$

7. For the differential equation $y'' - x\,y' + y = 0$, $p(x) = -x$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}.$$

Then

$$\begin{aligned} y'' - xy' + y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - x\sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2}(1-m)a_m\right] x^m. \end{aligned}$$

So $y'' - x\,y' + y = 0$ implies that

$$(m+2)(m+1)a_{m+2}(1-m)a_m = 0 \quad \Rightarrow \quad a_{m+2} = \frac{(m-1)}{(m+2)(m+1)}a_m \text{ for all } m \geq 0.$$

So a_0 and a_1 are arbitrary;

$$\begin{aligned} a_2 &= -\frac{a_0}{2}, \\ a_4 &= \frac{a_2}{4\cdot 3} = -\frac{a_0}{4!}, \\ a_6 &= \frac{3a_4}{6\cdot 5} = -\frac{3}{6!}a_0, \\ &\vdots \end{aligned}$$

Similarly,

$$\begin{aligned} a_3 &= a_1 \cdot 0 = 0, \\ a_5 &= 0, \\ &\vdots \\ a_{2k+1} &= 0, \quad k \geq 1. \end{aligned}$$

So

$$y = a_1 x + a_0\left(1 - \frac{x^2}{2} - \frac{x^4}{4!} - \frac{3}{6!}x^6 \cdots\right).$$

8. For the differential equation $y'' + 2\,y' + 2\,y = 0$, $p(x) = 2$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned} y'' + 2\,y' + 2\,y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + 2\sum_{m=1}^{\infty} m a_m x^{m-1} + 2\sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} 2(m+1)a_{m+1}x^m + \sum_{m=0}^{\infty} 2a_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2}2(m+1)a_{m+1} + 2a_m\right] x^m. \end{aligned}$$

So $y'' + 2\,y' + 2\,y = 0$ implies that

$$(m+2)(m+1)a_{m+2}2(m+1)a_{m+1}+2a_m = 0 \quad \Rightarrow \quad a_{m+2} = -\frac{1}{(m+2)(m+1)}\,[2(m+1)a_{m+1} + 2$$

So a_0 and a_1 are arbitrary;

$$\begin{aligned} a_2 &= -a_0 + a_1, \\ a_3 &= a_1 - \frac{2}{3}a_0, \\ a_4 &= \frac{1}{2}a_0 - \frac{2}{3}a_1, \\ &\vdots \end{aligned}$$

So

$$y = a_0\left(1 + x^2 - \frac{2}{3}x^3 + \frac{x^4}{2} + \cdots\right) + a_1\left(x - x^2 + x^3 - \frac{2}{3}x^4 + \cdots\right).$$

9. For the differential equation $y'' + 2\,x\,y' + y = 0$, $p(x) = 2x$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned} y'' + 2\,x\,y' + y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} + (2m+1)a_m\right] x^m. \end{aligned}$$

So $y'' + 2\,x\,y' + \,y = 0$ implies that

$$\begin{aligned} &(m+2)(m+1)a_{m+2} + (2m+1)a_m = 0 \\ \Rightarrow \quad & a_{m+2} = -\frac{(2m+1)}{(m+2)(m+1)}a_m \text{ for all } m \geq 0. \end{aligned}$$

So a_0 and a_1 are arbitrary;

$$\begin{aligned}
a_2 &= -\frac{1}{2}a_0, \\
a_4 &= -\frac{5}{4\cdot 3}a_2 = \frac{5}{4!}a_0, \\
a_6 &= -\frac{9}{6\cdot 5}\frac{5}{4!}a_0 = \frac{-9\cdot 5}{6!}a_0, \\
&\vdots \\
a_3 &= -\frac{3}{3\cdot 2}a_1 \\
a_5 &= -\frac{7}{5\cdot 4}a_3 = \frac{7\cdot 3}{5!}a_1 \\
a_7 &= -\frac{11\cdot 7\cdot 3}{7!}a_1 \\
&\vdots
\end{aligned}$$

So

$$\begin{aligned}
y = a_0 &\left(1 - \frac{1}{2}x^2 + \frac{5}{4!}x^4 - \frac{9\cdot 5}{6!}x^6 + \cdots\right) \\
&+ a_1\left(x - \frac{3}{3!}x^3 + \frac{7\cdot 3}{5!}x^5 - \frac{11\cdot 7\cdot 3}{7!}x^7 + \cdots\right).
\end{aligned}$$

10. For the differential equation $y'' + x\,y' + y = 0$, $p(x) = x$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned}
y'' + x\,y' + y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} + (m+1)a_m\right] x^m.
\end{aligned}$$

So $y'' + x\,y' + y = 0$ implies that

$$\begin{aligned}
&(m+2)(m+1)a_{m+2} + (m+1)a_m = 0 \\
\Rightarrow\quad & a_{m+2} = -\frac{(m+1)}{(m+2)(m+1)}a_m = -\frac{a_m}{m+2} \text{ for all } m \geq 0.
\end{aligned}$$

So a_0 and a_1 are arbitrary;

$$\begin{aligned} a_2 &= -\frac{1}{2}a_0, \\ a_4 &= -\frac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0, \\ a_6 &= -\frac{1}{6}a_4 = -\frac{1}{6\cdot 4\cdot 2}a_0, \\ &\vdots \\ a_{2k} &= \frac{(-1)^k}{(2k)\cdot(2k-2)\cdot\dots\cdot 2}a_2 \\ &= \frac{(-1)^k}{2^k k!}a_0. \end{aligned}$$

Similarly,

$$\begin{aligned} a_3 &= -\frac{1}{3}a_1, \\ a_5 &= -\frac{1}{5}a_3 = \frac{1}{5\cdot 3}a_1, \\ a_7 &= -\frac{1}{7}a_5 = -\frac{1}{7\cdot 5\cdot 3}a_1, \\ &\vdots \\ a_{2k+1} &= \frac{(-1)^k}{(2k+1)\cdot(2k-1)\cdot\dots\cdot 3}a_2 \\ &= \frac{(-1)^k 2^k k!}{(2k+1)!}a_1. \end{aligned}$$

So

$$y = a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}x^{2k} + a_1\sum_{k=0}^{\infty}\frac{(-1)^k 2^k k!}{(2k+1)!}x^{2k+1}.$$

11. $y'' - 2y' + y = 0, \quad y(0) = 0,\ y'(0) = 1.$

For the differential equation $y'' - 2y' + y = 0$, $p(x) = -2$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

The initial conditions $y(0) = 0,\ y'(0) = 1$ tell us that $a_0 = 0$ and $a_1 = 1$. Then

$$\begin{aligned} y'' - 2y' + y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2\sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - 2\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m + \sum_{m=0}^{\infty}a_m x^m \\ &= \sum_{m=0}^{\infty}\left[(m+2)(m+1)a_{m+2} - 2(m+1)a_{m+1} + a_m\right]x^m. \end{aligned}$$

So $y'' - 2\,y' + y = 0$ implies that

$$(m+2)(m+1)a_{m+2} - 2(m+1)a_{m+1} + a_m = 0$$

$$\Rightarrow \quad a_{m+2} = \frac{2}{(m+2)}a_{m+1} - \frac{a_m}{(m+2)(m+1)} \text{ for all } m \geq 0.$$

Using $a_0 = 0$ and $a_1 = 1$, we find

$$\begin{aligned} a_2 &= 1, \\ a_3 &= \frac{2}{3} - \frac{1}{3\cdot 2} = \frac{1}{2}, \\ a_4 &= \frac{1}{2}\frac{1}{2} - \frac{1}{4\cdot 3} = \frac{1}{3!}, \\ &\vdots \\ a_k &= \frac{1}{(k-1)!}. \end{aligned}$$

(It is not difficult to derive this formula by induction.) So

$$y = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}x^k.$$

12. $y'' - 2y' + y = x, \quad y(0) = 2, y'(0) = 1$. From exercise 11, we have for all $m \neq 1$

$$(m+2)(m+1)a_{m+2} - 2(m+1)a_{m+1} + a_m = 0.$$

Now, $a_0 = 2$, $a_1 = 1$,

$$\begin{aligned} m = 0 \quad &\Rightarrow \quad 2a_2 - 2a_1 + a_0 = 0 \\ &\Rightarrow \quad a_2 = 0 \\ m = 1 \quad &\Rightarrow \quad 3\cdot 2a_3 - 2\cdot 2a_2 + a_1 = 1 \\ &\Rightarrow \quad a_3 = 0; \\ a_2 = a_3 = 0 \quad &\Rightarrow \quad a_k = 0 \text{ for all } k \geq 3. \end{aligned}$$

So $y = 2 + x$.

13. To solve $(1 - x^2)y'' - 2xy' + 2y = 0$, $y(0) = 0$, $y'(0) = 3$, follow the steps in Example 5 and you will arrive at the recurrence relation

$$a_{m+2} = \frac{m(m+1) - 2}{(m+2)(m+1)}a_m = \frac{(m+2)(m-1)}{(m+2)(m+1)}a_m = \frac{m-1}{m+1}a_m, \quad m \geq 0.$$

The initial conditions give you $a_0 = 0$ and $a_1 = 3$. So $a_2 = a_4 = \cdots = 0$ and, from the recurrence relation with $m = 1$,

$$a_3 = \frac{(1-1}{1+1}a_0 = 0.$$

So $a_5 = a_7 = \cdots = 0$ and hence $y = 3x$ is the solution.

14. Following the steps in Example 5, we find the recurrence relation for the coefficients of the solution of $(1 - x^2)y'' - 2xy' + 6y = 0$, $y(0) = 1$, $y'(0) = 1$, to be

$$a_{m+2} = \frac{m(m+1) - 6}{(m+2)(m+1)}a_m = \frac{(m+3)(m-2)}{(m+2)(m+1)}a_m, \quad m \geq 0.$$

The initial conditions give $a_0 = 1$ and $a_1 = 1$. So, from the recurrence relation,

$$\begin{aligned}
a_2 &= \frac{(0+3)(0-2)}{(0+2)(0+1)}a_0 = -3 \\
a_4 &= \frac{(2+3)(2-2)}{(2+2)(2+1)}a_2 = 0 \\
a_6 &= a_8 = \cdots = 0 \\
a_3 &= \frac{(1+3)(1-2)}{(1+2)(1+1)}a_1 = -\frac{2}{3} \\
a_5 &= \frac{(3+3)(3-2)}{(3+2)(3+1)}a_3 = -\frac{6}{20}\cdot\frac{2}{3} = -\frac{1}{5} \\
&\vdots
\end{aligned}$$

So

$$y = 1 - 3x^2 + x - \frac{2}{3}x^3 - \frac{1}{5}x^5 + \cdots$$

15. Following the steps in Example 5, we find the recurrence relation for the coefficients of the solution of $(1-x^2)y'' - 2xy' + 12y = 0$, $y(0) = 1$, $y'(0) = 0$, to be

$$a_{m+2} = \frac{m(m+1) - 12}{(m+2)(m+1)}a_m = \frac{(m-3)(m+4)}{(m+2)(m+1)}a_m, \quad m \geq 0.$$

The initial conditions give $a_0 = 1$ and $a_1 = 0$. So, $a_3 = a_5 = \cdots = 0$, and, from the recurrence relation,

$$\begin{aligned}
a_2 &= \frac{(0-3)(0+4)}{(0+2)(0+1)}a_0 = -6 \\
a_4 &= \frac{(2-3)(2+4)}{(2+2)(2+1)}a_2 = \frac{-6}{12}(-6) = 3 \\
a_6 &= \frac{(4-3)(4+4)}{(4+2)(4+1)}a_4 = \frac{8}{30}(3) = \frac{4}{5} \\
&\vdots
\end{aligned}$$

So

$$y = 1 - 6x^2 + 3x^4 + \frac{4}{5}x^6 + \cdots$$

16. We have the same recurrence relation as in Exercise 13 but here $a_0 = 1$ and $a_1 = 0$. So $a_3 = a_5 = \cdots = 0$ and

$$\begin{aligned}
a_2 &= \frac{0-1}{0+1}a_0 = -1 \\
a_4 &= \frac{2-1}{2+1}a_2 = \frac{1}{3}(-1) = -\frac{1}{3} \\
a_6 &= \frac{4-1}{4+1}a_4 = -\frac{3}{5}\frac{1}{3} = -\frac{1}{5} \\
&\vdots
\end{aligned}$$

So

$$y = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 + \cdots$$

17. Put the equation $(1-x^2)y''-2xy'+2y=0$ in the form

$$y''-\frac{2x}{1-x^2}+\frac{2}{1-x^2}y=0.$$

Apply the reduction of order formula with $y_1=x$ and $p(x)=-\frac{2x}{1-x^2}$. Then

$$\begin{aligned}
e^{-\int p(x)\,dx} &= e^{\int \frac{2x}{1-x^2}\,dx}\\
&= e^{-\ln(1-x^2)}=\frac{1}{1-x^2}\\
y_2 &= y_1\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx\\
&= x\int \frac{1}{x^2(1-x^2)}\,dx
\end{aligned}$$

Use a partial fractions decomposition

$$\begin{aligned}
\frac{1}{x^2(1-x^2)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{1-x}+\frac{D}{1+x}\\
&= \frac{1}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}
\end{aligned}$$

So

$$\begin{aligned}
y_2 &= x\int\left(\frac{1}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)dx\\
&= x\left[(-\frac{1}{x}-\frac{1}{2}\ln(1-x)+\frac{1}{2}\ln(1+x)\right]\\
&= -1+\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)
\end{aligned}$$

18. One solution of $(1-x^2)y''-2xy'+6y=0$ on the interval $-1<x<1$ is found in Exercise 14: $y_1=1-3x^2$. To find a second solution, we apply the reduction of order formula with $y_1=1-3x^2$ and $p(x)=-\frac{2x}{1-x^2}$. As in the previous exercise,

$$y_2 = (1-3x^2)\int \frac{1}{x^2(1-3x^2)^2}\,dx$$

Use a partial fractions decomposition

$$\frac{1}{(1-x^2)(1-3x^2)^2} = \frac{1}{8(x+1)}+\frac{3}{8\left(\sqrt{3}x-1\right)^2}+\frac{3}{8\left(\sqrt{3}x+1\right)^2}-\frac{1}{8(x-1)}$$

So

$$\begin{aligned}
y_2 &= (1-3x^2)\Big[\frac{1}{8}\ln(x+1)-\frac{3}{8\sqrt{3}}\frac{1}{\left(\sqrt{3}x-1\right)}\\
&\quad -\frac{3}{8\sqrt{3}}\frac{1}{\left(\sqrt{3}x+1\right)}-\frac{1}{8}\ln(x-1)\Big]\\
&= (1-3x^2)\Big[\frac{1}{8}\ln\left(\frac{x+1}{x-1}\right)-\frac{3}{8\sqrt{3}}\left(\frac{1}{\left(\sqrt{3}x-1\right)}+\frac{1}{\left(\sqrt{3}x+1\right)}\right)\Big]\\
&= (1-3x^2)\Big[\frac{1}{8}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x}{4\left(3x^2-1\right)}\Big]\\
&= \frac{1}{8}(1-3x^2)\ln\left(\frac{x+1}{x-1}\right)+\frac{3x}{4}.
\end{aligned}$$

The following notebook illustrates how we can use Mathematica to solve a differential equations with poser series.

The solution is y and we will solve for the first 10 coefficients.

Let's define a partial sum of the Taylor series solution (degree 3) and set y[0]=1:

In[82]:= **seriessol = Series[y[x], {x, 0, 3}] /. y[0] → 1**

Out[82]= $1 + y'[0]\,x + \frac{1}{2}\,y''[0]\,x^2 + \frac{1}{6}\,y^{(3)}[0]\,x^3 + O[x]^4$

Next we set equations based on the given differential equation y'+y=0.

In[83]:= **leftside = D[seriessol, x] + seriessol**
rightside = 0
equat = LogicalExpand[leftside == rightside]

Out[83]= $(1 + y'[0]) + (y'[0] + y''[0])\,x + \left(\frac{y''[0]}{2} + \frac{1}{2}\,y^{(3)}[0]\right) x^2 + O[x]^3$

Out[84]= 0

Out[85]= $1 + y'[0] == 0\ \&\&\ y'[0] + y''[0] == 0\ \&\&\ \frac{y''[0]}{2} + \frac{1}{2}\,y^{(3)}[0] == 0$

This gives you a set of equations in the coefficients that *Mathematica* can solve

In[86]:= **seriescoeff = Solve[equat]**

Out[86]= $\{\{y'[0] \to -1,\ y''[0] \to 1,\ y^{(3)}[0] \to -1\}\}$

Next, we substitute these coefficients in the series solution. This can be done as follows

In[87]:= **seriessol /. seriescoeff[[1]]**

Out[87]= $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + O[x]^4$

To get a partial sum without the Big O, use

In[88]:= **Normal[seriessol /. seriescoeff[[1]]]**

Out[88]= $1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

With the previous example in hand, we can solve Exercises 19-22 using Mathematica by repeating and modifying the commands. Here is an illustration with Exercise 19. We suppress some outcomes to save space.

19. $y'' - y' + 2y = e^x$, $y(0) = 0$, $y'(0) = 1$.

In[70]:=
```
Clear[y, seriessol, n, partsol]
n = 10
seriessol = Series[y[x], {x, 0, n}] /. {y[0] → 0, y'[0] → 1}
leftside = D[seriessol, {x, 2}] - D[seriessol, {x, 1}] + 2 seriessol;
rightside = Series[E^x, {x, 0, n}];
equat = LogicalExpand[leftside == rightside];
seriescoeff = Solve[equat];
partsol = Normal[seriessol /. seriescoeff[[1]]];
```

Out[71]= 10

Out[72]= $$x + \frac{1}{2} y''[0]\, x^2 + \frac{1}{6} y^{(3)}[0]\, x^3 + \frac{1}{24} y^{(4)}[0]\, x^4 + \frac{1}{120} y^{(5)}[0]\, x^5 + \frac{1}{720} y^{(6)}[0]\, x^6 + \frac{y^{(7)}[0]\, x^7}{5040} + \frac{y^{(8)}[0]\, x^8}{40320} + \frac{y^{(9)}[0]\, x^9}{362880} + \frac{y^{(10)}[0]\, x^{10}}{3628800} + O[x]^{11}$$

The equation can be solved using analytical methods (undetermined coefficients). The exact solution is

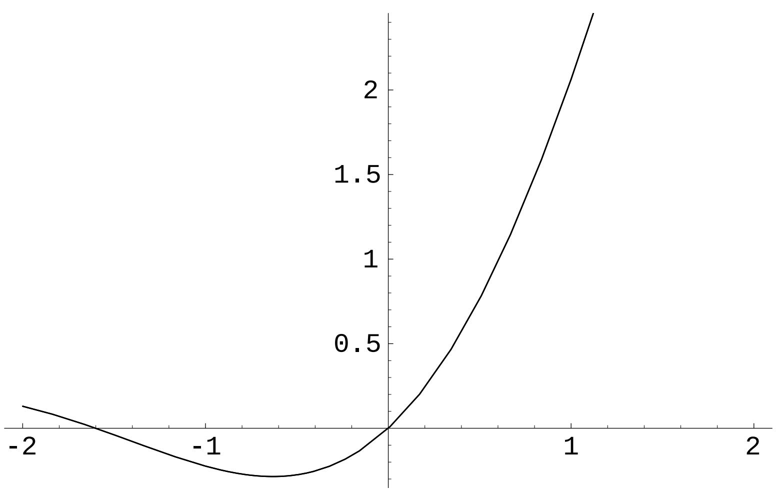
```
sol = DSolve[{ y''[x] - y'[x] + 2 y[x] == E^x, y[0] == 0, y'[0] == 1}, y[x], x];
```

In[46]:=
```
sss = sol[[1, 1, 2]]
```

$$\frac{1}{14} e^{x/2} \left(-7 \operatorname{Cos}\left[\frac{\sqrt{7}\, x}{2}\right] + 7 e^{x/2} \operatorname{Cos}\left[\frac{\sqrt{7}\, x}{2}\right]^2 + 3\sqrt{7} \operatorname{Sin}\left[\frac{\sqrt{7}\, x}{2}\right] + 7 e^{x/2} \operatorname{Sin}\left[\frac{\sqrt{7}\, x}{2}\right]^2\right)$$

Let's compare with the partial sum that we found earlier

In[67]:=
```
Plot[{sss, partsol}, {x, -2, 2}]
```



Out[67]= - Graphics -

We have a nice match on the interval [-2, 2]

Solutions to Exercises A.6

1. For the equation $y'' + (1 - x^2)y' + xy = 0$, $p(x) = 1 - x^2$ and $q(x) = x$ are both analytic at $a = 0$. So $a = 0$ is a ordinary point.

2. For the equation $xy'' + \sin x\, y' + \frac{1}{x}y = 0$,

$$p(x) = \frac{\sin x}{x}, \qquad x\,p(x) = \sin x;$$

$$q(x) = \frac{1}{x^2}, \qquad x^2 q(x) = 1.$$

$p(x)$ is analytic at $a = 0$ but $q(x)$ is not. So $a = 0$ is not an ordinary point. Since $xp(x)$ and $x^2 q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point.

3. For the equation $x^3 y'' + x^2 y' + y = 0$,

$$p(x) = \frac{1}{x}, \qquad x\,p(x) = 1;$$

$$q(x) = \frac{2}{x^3}, \qquad x^2 q(x) = \frac{2}{x}.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ is analytic but $x^2 q(x)$ is not analytic at $a = 0$, the point $a = 0$ is not a regular singular point.

4. For the equation $\sin x\, y'' + y' + \frac{1}{x}y = 0$,

$$p(x) = \frac{1}{\sin x}, \qquad x\,p(x) = \frac{x}{\sin x};$$

$$q(x) = \frac{1}{x \sin x}, \qquad x^2 q(x) = \frac{x}{\sin x}.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2 q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. To see that $\frac{x}{\sin x}$ is analytic at 0, note that $f(x) = \frac{\sin x}{x}$ is analytic and nonzero at 0. So $\frac{1}{f(x)}$ is analytic at 0.

5. For the equation $x^2 y'' + (1 - e^x)y' + xy = 0$,

$$p(x) = \frac{1 - e^x}{x^2}, \qquad x\,p(x) = \frac{1 - e^x}{x};$$

$$q(x) = \frac{1}{x}, \qquad x^2 q(x) = x.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2 q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. To see that $xp(x)$ is analytic at 0, derive its Taylor series as follows: for all x,

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
1 - e^x &= -x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\
&= x\left(-1 - \frac{x}{2!} - \frac{x^2}{3!} + \cdots\right) \\
\frac{1 - e^x}{x} &= -1 - \frac{x}{2!} - \frac{x^2}{3!} + \cdots .
\end{aligned}$$

Since $\frac{1-e^x}{x}$ has a Taylor series expansion about 0 (valid for all x), it is analytic at 0.

6. For the equation $3\,xy'' + 2\,y' - \frac{1}{3\,x}y = 0$,

$$p(x) = \frac{2}{3x}, \qquad x\,p(x) = \frac{2}{3};$$
$$q(x) = -\frac{1}{9x^2}, \qquad x^2q(x) = -\frac{1}{9}.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point.

7. For the equation $4\,xy'' + 6\,y' + y = 0$,

$$p(x) = \frac{3}{2x}, \qquad x\,p(x) = \frac{3}{2}, \qquad p_0 = \frac{3}{2};$$
$$q(x) = \frac{1}{4x}, \qquad x^2q(x) = \frac{1}{4}x, \qquad q_0 = 0.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$r(r-1) + \frac{3}{2}r = 0 \quad \Rightarrow \quad r^2 + \frac{1}{2}r = 0$$
$$= \quad r_1 = 0 \quad r_2 = -\frac{1}{2}.$$

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, we are in Case I. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^m \quad \text{and} \quad y_2 = \sum_{m=0}^{\infty} b_m x^{m-\frac{1}{2}},$$

with $a_0 \neq 0$ and $b_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^m; \quad y' = \sum_{m=0}^{\infty} m a_m x^{m-1}; \quad y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plug into $4\,xy'' + 6\,y' + y = 0$:

$$\begin{aligned}
\sum_{m=1}^{\infty} 4m(m-1)a_m x^{m-1} + \sum_{m=1}^{\infty} 6m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m &= 0\\
\sum_{m=0}^{\infty} 4(m+1)m a_{m+1} x^m + \sum_{m=0}^{\infty} 6(m+1) a_{m+1} x^m + \sum_{m=0}^{\infty} a_m x^m &= 0\\
\sum_{m=0}^{\infty} \left[4(m+1)m a_{m+1} + 6(m+1)a_{m+1} + a_m\right] x^m &= 0\\
4(m+1)m a_{m+1} + 6(m+1)a_{m+1} + a_m &= 0\\
(m+1)(4m+6)a_{m+1} + a_m &= 0.
\end{aligned}$$

This gives the recurrence relation: For all $m \geq 0$,

$$a_{m+1} = -\frac{1}{(m+1)(4m+6)} a_m.$$

Since a_0 is arbitrary, take $a_0 = 1$. Then

$$\begin{aligned}
a_1 &= -\frac{1}{6} = -\frac{1}{3!};\\
a_2 &= -\frac{1}{2(10)}(-\frac{1}{3!}) = \frac{1}{5!};\\
a_3 &= -\frac{1}{3\cdot 14}\frac{1}{5!} = -\frac{1}{7!};\\
&\vdots\\
y_1 &= a_0\left[1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \cdots\right]
\end{aligned}$$

We now turn to the second solution:

$$y = \sum_{m=0}^{\infty} b_m x^{m-\frac{1}{2}};\quad y' = \sum_{m=0}^{\infty}(m-\frac{1}{2})b_m x^{m-\frac{3}{2}};\quad y'' = \sum_{m=0}^{\infty}(m-\frac{1}{2})(m-\frac{3}{2})b_m x^{m-\frac{5}{2}}.$$

Plug into $4\,xy'' + 6\,y' + y = 0$:

$$\begin{aligned}
\sum_{m=0}^{\infty} 4(m-\frac{1}{2})(m-\frac{3}{2})b_m x^{m-\frac{3}{2}} + \sum_{m=0}^{\infty} 6(m-\frac{1}{2})b_m x^{m-\frac{3}{2}} & \\
+ \sum_{m=0}^{\infty} b_{m-1}x^{m-\frac{1}{2}} &= 0\\
\sum_{m=0}^{\infty} 4(m-\frac{1}{2})(m-\frac{3}{2})b_m x^{m-\frac{3}{2}} + \sum_{m=0}^{\infty} 6(m-\frac{1}{2})b_m x^{m-\frac{3}{2}} & \\
+ \sum_{m=1}^{\infty} b_{m-1}x^{m-\frac{3}{2}} &= 0\\
\overbrace{4(-\frac{1}{2})(-\frac{3}{2})b_0 x^{-\frac{3}{2}} + 6(-\frac{1}{2})b_0 x^{-\frac{3}{2}}}^{=0} & \\
+ \sum_{m=1}^{\infty}\left[4(m-\frac{1}{2})(m-\frac{3}{2})b_m + 6(m-\frac{1}{2})b_m + b_{m-1}\right]x^{m-\frac{3}{2}} &= 0\\
\sum_{m=1}^{\infty}\left[4b_m(m-\frac{1}{2})m + b_{m-1}\right]x^{m-\frac{3}{2}} &= 0\\
4b_m(m-\frac{1}{2})m + b_{m-1} &= 0.
\end{aligned}$$

This gives b_0 arbitrary and the recurrence relation: For all $m \geq 1$,

$$b_m = -\frac{b_{m-1}}{4m(m-\frac{1}{2})} = -\frac{b_{m-1}}{m(4m-2)}.$$

Since b_0 is arbitrary, take $b_0 = 1$. Then

$$\begin{aligned} b_1 &= -\frac{1}{2}; \\ b_2 &= -\frac{b_1}{2(6)} = \frac{1}{4!}; \\ b_3 &= -\frac{b_2}{3\cdot 10} = -\frac{1}{6!}; \\ &\vdots \\ y_2 &= b_0 x^{-\frac{1}{2}}\left[1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \cdots\right] \end{aligned}$$

9. For the equation $4\,x^2\,y'' - 14\,x\,y' + (20 - x)\,y = 0$,

$$p(x) = -\frac{7}{2x}, \qquad x\,p(x) = -\frac{7}{2}, \qquad p_0 = -\frac{7}{2};$$

$$q(x) = \frac{20 - x}{4x^2}, \qquad x^2 q(x) = 5 - \frac{x}{4}, \qquad q_0 = 5.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2 q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$r(r-1) - \frac{7}{2}r + 5 = 0 \quad \Rightarrow \quad 2r^2 - 9r + 10 = 0, \quad (r-2)(2r-5) = 0$$

$$= \quad r_1 = \frac{5}{2} \quad r_2 = 2.$$

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, we are in Case I. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m+2} \quad \text{and} \quad y_2 = \sum_{m=0}^{\infty} b_m x^{m+\frac{5}{2}},$$

with $a_0 \neq 0$ and $b_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^{m+2}; \quad y' = \sum_{m=0}^{\infty} (m+2) a_m x^{m+1}; \quad y'' = \sum_{m=0}^{\infty} (m+2)(m+1) a_m x^m.$$

Then

$4\,x^2 y'' - 14\,x\,y' + (20 - x)y$

$$\begin{aligned} &= \sum_{m=0}^{\infty} 4(m+2)(m+1)a_m x^{m+2} - 14\sum_{m=1}^{\infty}(m+2)a_m x^{m+2} + (20 - x)\sum_{m=0}^{\infty} a_m x^{m+2} \\ &= \sum_{m=0}^{\infty} [4(m+1) - 14](m+2)a_m x^{m+2} + 20\sum_{m=0}^{\infty} a_m x^{m+2} - \sum_{m=0}^{\infty} a_m x^{m+3} \\ &= \sum_{m=0}^{\infty} [(4m - 10)(m+2) + 20]a_m x^{m+2} - \sum_{m=0}^{\infty} a_m x^{m+3} \\ &= \sum_{m=0}^{\infty} [4m^2 - 2m]a_m x^{m+2} - \sum_{m=1}^{\infty} a_{m-1} x^{m+2} \\ &= \sum_{m=1}^{\infty} [4m^2 - 2m]a_m - a_{m-1}] x^{m+2} \end{aligned}$$

This gives the recurrence relation: For all $m \geq 0$,

$$a_m = \frac{a_{m-1}}{4m^2 - 2m} = \frac{a_{m-1}}{2m(2m-1)}.$$

Since a_0 is arbitrary, take $a_0 = 1$. Then

$$\begin{aligned}
a_1 &= \frac{1}{2};\\
a_2 &= \frac{1}{2(12)} = \frac{1}{4!};\\
a_3 &= \frac{1}{4!}\frac{1}{6 \cdot 5} = \frac{1}{6!};\\
&\vdots\\
y_1 &= a_0 x^2 \left[1 + \frac{1}{2!}x + \frac{1}{4!}x^2 + \frac{1}{6!}x^3 + \cdots\right]
\end{aligned}$$

We now turn to the second solution:

$$y = \sum_{m=0}^{\infty} b_m x^{m+\frac{5}{2}}; \quad y' = \sum_{m=0}^{\infty} (m+\frac{5}{2}) b_m x^{m+\frac{3}{2}}; \quad y'' = \sum_{m=0}^{\infty} (m+\frac{5}{2})(m+\frac{3}{2}) b_m x^{m+\frac{1}{2}}.$$

So

$$\begin{aligned}
&4\,x^2\,y'' - 14\,x\,y' + (20 - x)\,y\\
&= \sum_{m=0}^{\infty}\left[4(m+\frac{5}{2})(m+\frac{3}{2}) - 14(m+\frac{5}{2})\right] b_m x^{m+\frac{5}{2}} + (20 - x)\sum_{m=0}^{\infty} b_m x^{m+\frac{5}{2}}\\
&= \sum_{m=0}^{\infty}\left[4(m+\frac{5}{2})(m+\frac{3}{2}) - 14(m+\frac{5}{2}) + 20\right] b_m x^{m+\frac{5}{2}} - \sum_{m=0}^{\infty} b_m x^{m+\frac{7}{2}}\\
&= \sum_{m=0}^{\infty}\left[4m^2 + 2m\right] b_m x^{m+\frac{5}{2}} - \sum_{m=1}^{\infty} b_{m-1} x^{m+\frac{5}{2}}\\
&= \sum_{m=1}^{\infty}\left[(4m^2 + 2m) b_m - b_{m-1}\right] x^{m+\frac{5}{2}}
\end{aligned}$$

This gives b_0 arbitrary and the recurrence relation: For all $m \geq 1$,

$$b_m = \frac{b_{m-1}}{2m(2m+1)}.$$

Since b_0 is arbitrary, take $b_0 = 1$. Then

$$\begin{aligned}
b_1 &= \frac{1}{3!};\\
b_2 &= \frac{1}{3!}\frac{1}{4 \cdot 5} = \frac{1}{5!};\\
b_3 &= \frac{1}{5!}\frac{1}{6 \cdot 7} = \frac{1}{7!};\\
&\vdots\\
y_2 &= b_0 x^{5/2}\left[1 + \frac{1}{3!}x + \frac{1}{5!}x^2 + \frac{1}{7!}x^3 + \cdots\right]
\end{aligned}$$

13. For the equation $x\,y'' + (1-x)\,y' + y = 0$,

$$p(x) = \frac{1-x}{x}, \qquad x\,p(x) = 1-x, \qquad p_0 = 1;$$

$$q(x) = \frac{1}{x}, \qquad x^2 q(x) = x, \qquad q_0 = 0.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$r(r-1) + r = 0 \quad \Rightarrow \quad r = 0 \text{ (double root)}.$$

We are in Case II. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^m \quad \text{and} \quad y_2 = y_1 \ln x + \sum_{m=0}^{\infty} b_m x^m,$$

with $a_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^m; \quad y' = \sum_{m=0}^{\infty} m a_m x^{m-1}; \quad y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plug into $x\,y'' + (1-x)\,y' + y = 0$:

$$\begin{aligned}
\sum_{m=1}^{\infty} m(m-1)a_m x^{m-1} + \sum_{m=1}^{\infty} m a_m x^{m-1} & \\
- \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m &= 0 \\
\sum_{m=0}^{\infty} (m+1)m a_{m+1} x^m + \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m & \\
- \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m &= 0 \\
\sum_{m=1}^{\infty} (m+1)m a_{m+1} x^m + a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} x^m & \\
- \sum_{m=1}^{\infty} m a_m x^m + a_0 + \sum_{m=1}^{\infty} a_m x^m &= 0 \\
a_0 + a_1 + \sum_{m=1}^{\infty} \left[(m+1)m a_{m+1} + (m+1)a_{m+1} - m a_m + a_m\right] x^m &= 0 \\
a_0 + a_1 + \sum_{m=1}^{\infty} \left[(m+1)^2 a_{m+1} + (1-m)a_m\right] x^m &= 0
\end{aligned}$$

This gives $a_0 + a_1 = 0$ and the recurrence relation: For all $m \geq 1$,

$$a_{m+1} = -\frac{1-m}{(m+1)^2} a_m.$$

Take $a_0 = 1$. Then $a_1 = -1$ and $a_2 = a_3 = \cdots = 0$. So $y_1 = 1 - x$. We now turn to the second solution: (use $y_1 = 1 - x$, $y_1' = -1$, $y_1'' = 0$)

$$\begin{aligned}
y &= y_1 \ln x + \sum_{m=0}^{\infty} b_m x^m; \\
y' &= y_1' \ln x + \frac{y_1}{x} + \sum_{m=0}^{\infty} m b_m x^{m-1}; \\
y'' &= y_1'' \ln x + \frac{2}{x} y_1' - \frac{1}{x^2} y_1 + \sum_{m=0}^{\infty} m(m-1) b_m x^{m-2}.
\end{aligned}$$

Plug into $xy'' + (1-x)\,y' + y = 0$:

$$\begin{aligned}
xy_1'' \ln x + 2y_1' - \frac{1}{x} y_1 + \sum_{m=0}^{\infty} m(m-1) b_m x^{m-1} & \\
+(1-x) y_1' \ln x + \frac{y_1}{x}(1-x) + (1-x)\sum_{m=0}^{\infty} m b_m x^{m-1} + y_1 \ln x + \sum_{m=0}^{\infty} b_m x^m &= 0 \\
-2 - \frac{1}{x}(1-x) + \sum_{m=1}^{\infty} m(m-1) b_m x^{m-1} & \\
+\frac{(1-x)^2}{x} + \sum_{m=0}^{\infty} m b_m x^{m-1} - \sum_{m=0}^{\infty} m b_m x^m + \sum_{m=0}^{\infty} b_m x^m &= 0 \\
-3 + x + \sum_{m=0}^{\infty} [(m+1) m b_{m+1} + (m+1) b_{m+1} - m b_m + b_m] x^m &= 0 \\
-3 + x + \sum_{m=0}^{\infty} [(m+1)^2 b_{m+1} + (1-m) b_m] x^m &= 0
\end{aligned}$$

For the constant term, we get $b_1 + b_0 - 3 = 0$. Take $b_0 = 0$. Then $b_1 = 3$. For the term in x, we get

$$1 + 2b_2 + 2b_2 - b_1 + b_1 = 0 \quad \Rightarrow \quad b_2 = -\frac{1}{4}.$$

For all $m \geq 3$,

$$b_{m+1} = \frac{m-1}{(m+1)^2} b_m.$$

Then

$$\begin{aligned}
b_3 &= \frac{1}{9}\left(-\frac{1}{4}\right) = -\frac{1}{36}; \\
b_4 &= \frac{2}{16}\left(-\frac{1}{36}\right) = -\frac{1}{288}; \\
&\vdots \\
y_2 &= -3x - \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots
\end{aligned}$$

17. For the equation $x^2y'' + 4\,x\,y' + (2 - x^2)\,y = 0$,

$$p(x) = \frac{4}{x}, \qquad x\,p(x) = 4, \qquad p_0 = 4;$$

$$q(x) = \frac{2 - x^2}{x^2}, \qquad x^2 q(x) = 2 - x^2, \qquad q_0 = 2.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$r(r-1) + 4r + 2 = 0 \quad \Rightarrow \quad r^3 + 3r + 2 = 0$$
$$\Rightarrow \quad r_1 = -2 \quad r_2 = -1.$$

We are in Case III. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m-1} \quad \text{and} \quad y_2 = k y_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-2},$$

with $a_0 \neq 0$, $b_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^{m-1}; \quad y' = \sum_{m=0}^{\infty} (m-1) a_m x^{m-2}; \quad y'' = \sum_{m=0}^{\infty} (m-1)(m-2) a_m x^{m-3}.$$

Plug into $x^2y'' + 4\,x\,y' + (2 - x^2)\,y = 0$:

$$\begin{aligned}
\sum_{m=0}^{\infty} (m-1)(m-2) a_m x^{m-1} + \sum_{m=0}^{\infty} 4(m-1) a_m x^{m-1}& \\
\sum_{m=0}^{\infty} 2 a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^{m+1} &= 0 \\
(-1)(-2) a_0 x^{-1} + \sum_{m=2}^{\infty} (m-1)(m-2) a_m x^{m-1}& \\
+4(-1) a_0 x^{-1} + \sum_{m=2}^{\infty} 4(m-1) a_m x^{m-1}& \\
2 a_0 x^{-1} + 2 a_1 + \sum_{m=2}^{\infty} 2 a_m x^{m-1} - \sum_{m=2}^{\infty} a_{m-2} x^{m-1} &= 0 \\
2 a_1 + \sum_{m=2}^{\infty} [(m-1)(m-2) a_m + 4(m-1) a_m 2 a_m - a_{m-2}] x^{m-1} &= 0 \\
2 a_1 + \sum_{m=2}^{\infty} [(m^2 + m) a_m - a_{m-2}] x^{m-1} &= 0
\end{aligned}$$

This gives the recurrence relation: For all $m \geq 1$,

$$a_m = -\frac{1}{m^2 + m} a_{m-2}.$$

Take $a_0 = 1$ and $a_1 = 0$. Then $a_3 = a_5 = \cdots = 0$ and

$$\begin{aligned}
a_2 &= -\frac{1}{6} = -\frac{1}{3!}\\
a_4 &= -\frac{1}{4^2+4}a_2 = \frac{1}{20}\frac{1}{6} = \frac{1}{5!}\\
a_6 &= -\frac{1}{6^2+6}\frac{1}{5!} = -\frac{1}{7\cdot 6}\frac{1}{5!} = \frac{1}{7!}\\
&\vdots\\
y_1 &= a_0 x^{-1}\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots\right)
\end{aligned}$$

We now turn to the second solution:

$$\begin{aligned}
y &= k y_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-2};\\
y' &= k y_1' \ln x + k\frac{y_1}{x} + \sum_{m=0}^{\infty}(m-2) b_m x^{m-3};\\
y'' &= k y_1'' \ln x + \frac{2k}{x}y_1' - \frac{k}{x^2}y_1 + \sum_{m=0}^{\infty}(m-2)(m-3) b_m x^{m-4}.
\end{aligned}$$

Plug into $xy'' + (1-x)\,y' + y = 0$:

$$\begin{aligned}
2kxy_1' + ky_1''x^2\ln x - ky_1 + \sum_{m=0}^{\infty}(m-2)(m-3)b_m x^{m-2} + &\\
4ky_1' x\ln x + 4ky_1 + \sum_{m=0}^{\infty} 4(m-2) b_m x^{m-2} &\\
+(2-x^2)ky_1'\ln x + (2-x^2)\sum_{m=0}^{\infty} b_m x^{m-2} &= 0\\
2kxy_1' + 3ky_1 + \sum_{m=0}^{\infty}[(m+2)^2 - (m+2)b_{m+2} + b_m]x^m &= 0\\
(m+2)^2 - (m+2)b_{m+2} + b_m &= 0
\end{aligned}$$

Take $k = 0$ and for all $m \geq 0$,

$$b_{m+2} = -\frac{b_m}{(m+2)(m+1)}.$$

Take $b_0 = 1$ and $b_1 = 0$. (Note that by setting $b_1 = 1$ and $b_0 = 0$, you will get y_1.) Then

$$\begin{aligned}
b_2 &= -\frac{1}{2};\\
b_4 &= \frac{1}{4!};\\
&\vdots\\
y_2 &= b_0 x^{-2}\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots\right).
\end{aligned}$$