

# Matrix Norms

## 30.4



### Introduction

A matrix norm is a number defined in terms of the entries of the matrix. The norm is a useful quantity which can give important information about a matrix.



### Prerequisites

Before starting this Section you should ...

- be familiar with matrices and their use in writing systems of equations
- revise material on matrix inverses, be able to find the inverse of a  $2 \times 2$  matrix, and know when no inverse exists
- revise Gaussian elimination and partial pivoting
- be aware of the discussion of ill-conditioned and well-conditioned problems earlier in Section 30.1



### Learning Outcomes

On completion you should be able to ...

- calculate norms and condition numbers of small matrices
- adjust certain systems of equations with a view to better conditioning

# 1. Matrix norms

The norm of a square matrix  $A$  is a non-negative real number denoted  $\|A\|$ . There are several different ways of defining a matrix norm, but they all share the following properties:

1.  $\|A\| \geq 0$  for any square matrix  $A$ .
2.  $\|A\| = 0$  if and only if the matrix  $A = 0$ .
3.  $\|kA\| = |k| \|A\|$ , for any scalar  $k$ .
4.  $\|A + B\| \leq \|A\| + \|B\|$ .
5.  $\|AB\| \leq \|A\| \|B\|$ .

The norm of a matrix is a measure of how large its elements are. It is a way of determining the “size” of a matrix that is not necessarily related to how many rows or columns the matrix has.



## Key Point 6

### Matrix Norm

The **norm** of a matrix is a real number which is a measure of the magnitude of the matrix.

Anticipating the places where we will use norms later, it is sufficient at this stage to restrict our attention to matrices with only real-valued entries. There is no need to consider complex numbers at this stage.

In the definitions of norms below we will use this notation for the elements of an  $n \times n$  matrix  $A$  where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

The subscripts on  $a$  have the row number first, then the column number. The fact that

$$a_{rc}$$

is reminiscent of the word “arc” may be a help in remembering how the notation goes.

In this Section we will define three commonly used norms. We distinguish them with a subscript. All three of them satisfy the five conditions listed above, but we will not concern ourselves with verifying that fact.

## The 1-norm

$$\|A\|_1 = \max_{1 \leq j \leq n} \left( \sum_{i=1}^n |a_{ij}| \right)$$

(the maximum absolute column sum). Put simply, we sum the absolute values down each column and then take the biggest answer.



### Example 9

Calculate the 1-norm of  $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$ .

#### Solution

The absolute column sums of  $A$  are  $1 + |-2| = 1 + 2 = 3$  and  $|-7| + |-3| = 7 + 3 = 10$ . The larger of these is 10 and therefore  $\|A\|_1 = 10$ .



### Example 10

Calculate the 1-norm of  $B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$ .

#### Solution

Summing down the columns of  $B$  we find that

$$\begin{aligned} \|B\|_1 &= \max(5 + 1 + 2, 4 + 2 + 1, 2 + 3 + 0) \\ &= \max(8, 7, 5) \\ &= 8 \end{aligned}$$



### Key Point 7

The 1-norm of a square matrix is the maximum of the absolute column sums.  
(A useful reminder is that “1” is a tall, thin character and a column is a tall, thin quantity.)

## The infinity-norm

$$\|A\|_{\infty} = \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |a_{ij}| \right)$$

(the maximum absolute row sum). Put simply, we sum the absolute values along each row and then take the biggest answer.



### Example 11

Calculate the infinity-norm of  $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$ .

#### Solution

The absolute row sums of  $A$  are  $1 + |-7| = 8$  and  $|-2| + |-3| = 5$ . The larger of these is 8 and therefore  $\|A\|_{\infty} = 8$ .



### Example 12

Calculate the infinity-norm of  $B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$ .

#### Solution

Summing along the rows of  $B$  we find that

$$\begin{aligned} \|B\|_{\infty} &= \max(5 + 4 + 2, 1 + 2 + 3, 2 + 1 + 0) \\ &= \max(11, 6, 3) \\ &= 11 \end{aligned}$$



### Key Point 8

The infinity-norm of a square matrix is the maximum of the absolute row sums.  
(A useful reminder is that “ $\infty$ ” is a short, wide character and a row is a short, wide quantity.)

## The Euclidean norm

$$\|A\|_E = \sqrt{\sum_{i=1}^n \sum_{j=1}^n (a_{ij})^2}$$

(the square root of the sum of all the squares). This is similar to ordinary “Pythagorean” length where the size of a vector is found by taking the square root of the sum of the squares of all the elements.



### Example 13

Calculate the Euclidean norm of  $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$ .

#### Solution

$$\begin{aligned} \|A\|_E &= \sqrt{1^2 + (-7)^2 + (-2)^2 + (-3)^2} \\ &= \sqrt{1 + 49 + 4 + 9} \\ &= \sqrt{63} \approx 7.937. \end{aligned}$$



### Example 14

Calculate the Euclidean norm of  $B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$ .

#### Solution

$$\begin{aligned} \|B\|_E &= \sqrt{25 + 16 + 4 + 1 + 4 + 9 + 4 + 1 + 0} \\ &= \sqrt{64} \\ &= 8. \end{aligned}$$



### Key Point 9

The Euclidean norm of a square matrix is the square root of the sum of all the squares of the elements.



Calculate the norms indicated of these matrices

$$A = \begin{bmatrix} 2 & -8 \\ 3 & 1 \end{bmatrix} \quad (1\text{-norm}), \quad B = \begin{bmatrix} 3 & 6 & -1 \\ 3 & 1 & 0 \\ 2 & 4 & -7 \end{bmatrix} \quad (\text{infinity-norm}),$$

$$C = \begin{bmatrix} 1 & 7 & 3 \\ 4 & -2 & -2 \\ -2 & -1 & 1 \end{bmatrix} \quad (\text{Euclidean-norm}).$$

### Your solution

### Answer

$$\|A\|_1 = \max(2 + 3, 8 + 1) = 9,$$

$$\|B\|_\infty = \max(3 + 6 + 1, 3 + 1 + 0, 2 + 4 + 7) = 13,$$

$$\begin{aligned} \|C\|_E &= \sqrt{1^2 + 7^2 + 3^2 + 4^2 + (-2)^2 + (-2)^2 + (-2)^2 + (-1)^2 + 1^2} \\ &= \sqrt{89} \approx 9.434 \end{aligned}$$

### Other norms

Any definition you can think of which satisfies the five conditions mentioned at the beginning of this Section is a definition of a norm. There are many many possibilities, but the three given above are among the most commonly used.

## 2. Condition numbers

The condition number of an invertible matrix  $A$  is defined to be

$$\kappa(A) = \|A\| \|A^{-1}\|.$$

This quantity is always bigger than (or equal to) 1.

We must use the same type of norm twice on the right-hand side of the above equation. Sometimes the notation is adjusted to make it clear which norm is being used, for example if we use the infinity norm we might write

$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty.$$



### Example 15

Use the norm indicated to calculate the condition number of the given matrices.

- (a)  $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ ; 1-norm.      (b)  $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ ; Euclidean norm.
- (c)  $B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ; infinity-norm.

#### Solution

$$(a) \|A\|_1 = \max(2 + 1, 3 + 1) = 4,$$

$$A^{-1} = \frac{1}{-2 - 3} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{-2}{5} \end{bmatrix}$$

$$\therefore \|A^{-1}\|_1 = \max\left(\frac{1}{5} + \frac{1}{5}, \frac{3}{5} + \frac{2}{5}\right) = 1.$$

$$\text{Therefore } \kappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = 4 \times 1 = 4.$$

$$(b) \|A\|_E = \sqrt{2^2 + 3^2 + 1^2 + (-1)^2} = \sqrt{15}. \text{ We can re-use } A^{-1} \text{ from above to see that}$$

$$\|A^{-1}\|_E = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{-2}{5}\right)^2} = \sqrt{\frac{15}{25}}.$$

$$\text{Therefore } \kappa_E(A) = \|A\|_E \|A^{-1}\|_E = \sqrt{15} \times \sqrt{\frac{15}{25}} = \frac{15}{\sqrt{25}} = \frac{15}{5} = 3.$$

$$(c) \|B\|_\infty = \max(3, 4, 2) = 4.$$

$$B^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\text{so } \|B^{-1}\|_\infty = \max\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2}. \quad \text{Therefore } \kappa_\infty(B) = \|B\|_\infty \|B^{-1}\|_\infty = 4 \times \frac{1}{2} = 2.$$



Calculate the condition numbers of these matrices, using the norm indicated

$$A = \begin{bmatrix} 2 & -8 \\ 3 & 1 \end{bmatrix} \quad (1\text{-norm}), \quad B = \begin{bmatrix} 3 & 6 \\ 1 & 0 \end{bmatrix} \quad (\text{infinity-norm}).$$

### Your solution

### Answer

$$A^{-1} = \frac{1}{2+24} \begin{bmatrix} 1 & 8 \\ -3 & 2 \end{bmatrix} \quad \text{so } \kappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = \max(5, 9) \times \max\left(\frac{4}{26}, \frac{10}{26}\right) = 9 \times \frac{10}{26} = \frac{45}{13}.$$

$$B^{-1} = \frac{1}{0-6} \begin{bmatrix} 0 & -6 \\ -1 & 3 \end{bmatrix} \quad \text{so } \kappa_\infty(B) = \|B\|_\infty \|B^{-1}\|_\infty = \max(9, 1) \times \max\left(1, \frac{4}{6}\right) = 9.$$

## Condition numbers and conditioning

As the name might suggest, the **condition number** gives us information regarding how well-conditioned a problem is. Consider this example

$$\begin{bmatrix} 1 & 10^4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10^4 \\ 1 \end{bmatrix}.$$

It is not hard to verify that the exact solution to this problem is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{10000}{10002} \\ \frac{10001}{10002} \end{bmatrix} = \begin{bmatrix} 0.999800\dots \\ 0.999900\dots \end{bmatrix}.$$



### Example 16

Using the 1-norm find the condition number of  $\begin{bmatrix} 1 & 10^4 \\ -1 & 2 \end{bmatrix}$ .

### Solution

Firstly,  $\|A\|_1 = 2 + 10^4$ . Also

$$A^{-1} = \frac{1}{2+10^4} \begin{bmatrix} 2 & -10^4 \\ 1 & 1 \end{bmatrix} \quad \therefore \quad \|A^{-1}\|_1 = \frac{1}{2+10^4} (1+10^4). \quad \text{Hence } \kappa_1(A) = 1+10^4 = 10001.$$



The fact that this number is large is the indication that the problem involving  $A$  is an ill-conditioned one. Suppose we consider finding its solution by Gaussian elimination, using 3 significant figures throughout. Eliminating the non-zero in the bottom left corner gives

$$\begin{bmatrix} 1 & 10^4 \\ 0 & 10^4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10^4 \\ 10^4 \end{bmatrix}.$$

which implies that  $x_2 = 1$  and  $x_1 = 0$ . This is a poor approximation to the true solution and partial pivoting will not help. We have altered the problem by a relatively tiny amount (that is, by neglecting the fourth significant figure) and the result  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  has changed by a large amount. In other words the problem is ill-conditioned.

One way that systems of equations can be made better conditioned is to fix things so that all the rows have largest elements that are about the same size. In the matrix  $A = \begin{bmatrix} 1 & 10^4 \\ -1 & 2 \end{bmatrix}$  the first row's largest element is  $10^4$ , the second row has largest element equal to 2. This is not a happy situation.

If we divide the first equation through by  $10^4$  then we have

$$\begin{bmatrix} 10^{-4} & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then the top row has largest entry equal to 1, and the bottom row still has 2 as its largest entry. These two values are of comparable size.

The solution to the system was found via pivoting (using 3 significant figures) in the Section concerning Gaussian elimination to be  $x_1 = x_2 = 1$ , a pretty good approximation to the exact values. The matrix in this second version of the problem is much better conditioned.



### Example 17

Using the 1-norm find the condition number of  $\begin{bmatrix} 10^{-4} & 1 \\ -1 & 2 \end{bmatrix}$ .

#### Solution

The 1-norm of  $A$  is easily seen to be  $\|A\|_1 = 3$ . We also need

$$A^{-1} = \frac{1}{2 \times 10^{-4} + 1} \begin{bmatrix} 2 & -1 \\ 1 & 10^{-4} \end{bmatrix} \quad \therefore \quad \|A^{-1}\|_1 = \frac{3}{2 \times 10^{-4} + 1}.$$

Hence

$$\kappa_1(A) = \frac{9}{2 \times 10^{-4} + 1} \approx 8.998$$

This condition number is much smaller than the earlier value of 10001, and this shows us that the second version of the system of equations is better conditioned.

## Exercises

1. Calculate the indicated norm of the following matrices

(a)  $A = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}$ ; 1-norm.

(b)  $A = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}$ ; infinity-norm.

(c)  $B = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ ; Euclidean norm.

(d)  $C = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 5 & 6 \\ 2 & -1 & 3 \end{bmatrix}$ ; infinity-norm.

(e)  $C = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 5 & 6 \\ 2 & -1 & 3 \end{bmatrix}$ ; 1-norm.

2. Use the norm indicated to calculate the condition number of the given matrices.

(a)  $D = \begin{bmatrix} 4 & -2 \\ 6 & 0 \end{bmatrix}$ ; 1-norm.

(b)  $E = \begin{bmatrix} -1 & 5 \\ 4 & 2 \end{bmatrix}$ ; Euclidean norm.

(c)  $F = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ; infinity-norm.

3. Why is it not sensible to ask what the condition number of  $\begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}$  is?

4. Verify that the inverse of  $G = \begin{bmatrix} 2 & 4 & -1 \\ 2 & 5 & 2 \\ -1 & -1 & 1 \end{bmatrix}$  is  $\frac{1}{5} \begin{bmatrix} -7 & 3 & -13 \\ 4 & -1 & 6 \\ -3 & 2 & -2 \end{bmatrix}$ .

Hence find the condition number of  $G$  using the 1-norm.

5. (a) Calculate the condition number (use any norm you choose) of the coefficient matrix of the system

$$\begin{bmatrix} 1 & 10^4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and hence conclude that the problem as stated is ill-conditioned.

(b) Multiply one of the equations through by a suitably chosen constant so as to make the system better conditioned. Calculate the condition number of the coefficient matrix in your new system of equations.

## Answers

- (a)  $\|A\|_1 = \max(2 + 1, 2 + 3) = 5.$

(b)  $\|A\|_\infty = \max(2 + -2, 1 + 3) = 4.$

(c)  $\|B\|_E = \sqrt{4 + 9 + 1 + 4} = \sqrt{18}$

(d)  $\|C\|_\infty = \max(1 + 2 + 3, 1 + 5 + 6, 2 + 1 + 3) = 12.$

(e)  $\|C\|_1 = \max(1 + 1 + 2, 2 + 5 + 1, 3 + 6 + 3) = 12.$
- (a) To work out the condition number we need to find

$$D^{-1} = \frac{1}{12} \begin{bmatrix} 0 & 2 \\ -6 & 4 \end{bmatrix}.$$

Given this we work out the condition number as the product of two norms as follows

$$\kappa_1(D) = \|D\|_1 \|D^{-1}\|_1 = 10 \times \frac{1}{2} = 5.$$

- (b) To work out the condition number we need to find

$$E^{-1} = \frac{1}{-22} \begin{bmatrix} 2 & -5 \\ -4 & -1 \end{bmatrix}.$$

Given this we work out the condition number as the product of two norms as follows

$$\kappa_E(E) = \|E\|_E \|E^{-1}\|_E = 6.782330 \times 0.308288 = 2.090909.$$

to 6 decimal places.

- (c) Here  $F^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$  so that  $\kappa_\infty(F) = \|F\|_\infty \|F^{-1}\|_\infty = 6 \times 1 = 6.$

3. The matrix is not invertible.

**Answers**

4. Verification is done by a direct multiplication to show that  $GG^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

Using the 1-norm we find that  $\kappa_1(G) = \|G\|_1 \|G^{-1}\|_1 = 10 \times \frac{21}{5} = 42$ .

5.

(a) The inverse of the coefficient matrix is

$$\frac{1}{3 - 2 \times 10^4} \begin{bmatrix} 3 & -10^4 \\ -2 & 1 \end{bmatrix} = \frac{-1}{19997} \begin{bmatrix} 3 & -10000 \\ -2 & 1 \end{bmatrix}.$$

Using the 1-norm the condition number of the coefficient matrix is

$$(3 + 10^4) \times \frac{1}{19997} (1 + 10^4) = 5002.75$$

to 6 significant figures. This is a large condition number, and the given problem is not well-conditioned.

(b) Now we multiply the top equation through by  $10^{-4}$  so that the system of equations becomes

$$\begin{bmatrix} 10^{-4} & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and the inverse of this new coefficient matrix is

$$\frac{1}{3 \times 10^{-4} - 2} \begin{bmatrix} 3 & -1 \\ -2 & 10^{-4} \end{bmatrix} = \frac{-1}{1.9997} \begin{bmatrix} 3 & -1 \\ -2 & .0001 \end{bmatrix}.$$

Using the 1-norm again we find that the condition number of the new coefficient matrix is

$$4 \times \frac{1}{1.9997} (5) = 10.0015$$

to 6 significant figures. This much smaller condition number implies that the second problem is better conditioned.