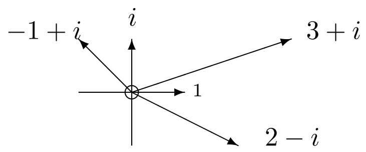
# 1 Complex numbers

The complex numbers $\mathbb{C}$ are the real numbers $\mathbb{R}$ *extended* by an *imaginary* unit $i$ with $\boxed{i^2 = -1}$.

$$\mathbb{C} \;=\; \{\, z \,:\, z = a + ib,\;\; a, b \in \mathbb{R} \,\} \qquad\qquad z = a + ib \;\Leftrightarrow\; \mathrm{Re}(z) = a,\; \mathrm{Im}(z) = b.$$

$a$ is the *real part* of $z = a + ib$, and $b$ the *imaginary part*. On the *Argand diagram*:




Real axis: horizontal
Imaginary axis: vertical

Addition, subtraction and multiplication also extend from real numbers to complex numbers:

$$\begin{aligned} (a+ib) \;\pm\; (c+id) \;&=\; (a \pm c) \;+\; i(b \pm d) \\ (a+ib)(c+id) \;&=\; ac + ibc + iad + i^2 bd \;=\; (ac - bd) \;+\; i(bc + ad) \end{aligned}$$

**Examples:** $(2+4i) + (2-i) = 4 + 3i, \quad (2+4i)(2-i) = 8 + 6i$

**Exercises:** If $z = 1 + 2i$ and $w = 1 - 4i$, calculate $z + w$, $z - 2w$, $wz$ and $(w+1)z$.

The *complex conjugate* of $z = a + ib$ is $\boxed{\overline{z} = a - ib}$. The *modulus* is $\boxed{\|z\| = \sqrt{a^2 + b^2}}$.

Note that $\boxed{z \cdot \overline{z} = \|z\|^2}$ because $(a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2$.

Using the conjugate and modulus we can *divide* by any non-zero complex number:

$$\boxed{\text{If } z \neq 0: \qquad \frac{1}{z} \;=\; \frac{1}{z}\cdot\frac{\overline{z}}{\overline{z}} \;=\; \frac{\overline{z}}{\|z\|^2}\,, \qquad\qquad \frac{1}{a+ib} \;=\; \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}}$$

**Example:** If $z = 3 - 4i$ then $\overline{z} = 3 + 4i$ and $\|z\| = \sqrt{3^2 + 4^2} = 5$, and so $z^{-1} = \frac{3}{25} + \frac{4}{25}\,i$.

Arithmetic with complex numbers satisfies

| | | |
|---|---|---|
| **Distributive law:** | $u(w+z) = uw + uz$ | |
| **Commutative laws:** | $w + z = z + w$ | $zw = wz$ |
| **Associative laws:** | $u + (w+z) = (u+w) + z$ | $(uw)z = u(wz)$ |
| **and:** | $\overline{w+z} = \overline{w} + \overline{z}$ | $\overline{wz} = \overline{w}\,\overline{z}$ |
| | $\|wz\| = \|w\|\,\|z\|$ | $\overline{\overline{z}} = z$ |
| | $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ | $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$ |

**Exercises:** Check the last 6 laws in the case $w = 5 + 12i$ and $z = 3 - 4i$.
Show also that $\|w + z\| \neq \|w\| + \|z\|$.

## 1.1 Factorising polynomials and solving equations

A *polynomial* of *degree* $n$ is an expression of the form

$$f(z) \;=\; \lambda_n z^n \;+\; \lambda_{n-1} z^{n-1} \;+\; \cdots \;+\; \lambda_2 z^2 \;+\; \lambda_1 z \;+\; \lambda_0$$

for some constant coefficients $\lambda_i$. We are interested in factorising $f(z)$, and in solving $f(z) = 0$.

**Theorem:** A constant $\alpha$ is a solution of $f(z) = 0$ if and only if $(z - \alpha)$ is a factor of $f(z)$.

A solution of $f(z) = 0$ is sometimes called a *root* of $f(z) = 0$ or a *zero* of $f(z)$.

**Examples:** The polynomial $f(z) = z^2 - 3z + 2$ factorises into $(z - 1)(z - 2)$, and the equation $f(z) = 0$ has solutions $z = 1$ and $z = 2$. Note that $f(z) = z^2 - 1 = (z + 1)(z - 1)$, but the polynomial $f(z) = z^2 + 1$ does not factorise *unless* we use complex numbers:

$$z^2 + 1 \;=\; (z - i)(z + i)\,, \qquad\qquad z^2 + 1 = 0 \text{ has solutions } z = \pm i\,.$$

We can factorise *any* quadratic polynomial over the complex numbers

$$a\,z^2 + b\,z + c \;=\; a(z - \alpha_1)(z - \alpha_2) \quad\Leftrightarrow\quad \alpha_1, \alpha_2 \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \;=\; -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

**Example:** The roots of $z^2 - 4z + 5 = 0$ are $2 \pm \sqrt{2^2 - 5} = 2 \pm i$, that is,

$$z^2 - 4z + 5 \;=\; (z - 2 + i)(z - 2 - i)\,.$$

The *Fundamental Theorem of Algebra* says this isn't just true for quadratic polynomials:

**Theorem:** Any polynomial of degree $n$ factorises as a product of $n$ linear factors over $\mathbb{C}$.

That is, any polynomial of degree $n$ has $n$ possible complex solutions [some may be repeated!]

**Example:** Find all the zeros of the polynomials (i) $f(z) = z^3 + z^2 + z + 1$, (ii) $f(z) = z^4 - 1$.

(i) We can see one solution of $f(z) = 0$ is $z = -1$, so $(z + 1)$ is a factor of $f(z)$. In fact

$$f(z) \;=\; z^3 + z^2 + z + 1 \;=\; (z + 1)(z^2 + 1) \;=\; (z + 1)(z - i)(z + i)$$

so the roots of $f(z) = 0$ are $-1, i, -i$. (ii) Both 1 and $-1$ are zeros, and so are $i$ and $-i$.

**Theorem:** Suppose all the coefficients of the polynomial $f(z)$ are real numbers.
Then if $z = \alpha$ is a solution of $f(z) = 0$, so is the conjugate $z = \overline{\alpha}$.

*Proof:* Because all the coefficients $\lambda_i$ are real, the complex conjugate of $f(z)$ is just $f(\overline{z})$, so if $f(\alpha) = 0$ then $f(\overline{\alpha}) = \overline{f(\alpha)} = \overline{0} = 0$ also.

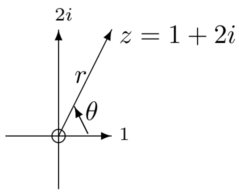
**Example:** If $1 + i$ is one root of $f(z) = z^3 + 4z^2 - 10z + 12 = 0$, find the others.

*Solution:* If $1 + i$ is a root, so is $1 - i$. Suppose $\alpha$ is the other. Then

$$f(z) \;=\; z^3 + 4z^2 - 10z + 12 \;=\; (z - 1 - i)(z - 1 + i)(z - \alpha) \;=\; (z^2 - 2z + 2)(z - \alpha) \;\Rightarrow\; \alpha = -6\,.$$

### 1.2 Modulus–argument form

A complex number $z = x + iy$ may be given in terms of **polar coordinates** $(r, \theta)$ instead of the usual rectangular cartesian coordinates $(x, y)$. On the Argand diagram one can see that $r$ is just the **modulus** $\|z\|$, and $\theta$ is the **argument**: the angle from the positive real axis to $0z$.




$$z = x + iy$$
$$(x, y) \leftrightarrow (r, \theta)$$
$$r = \|z\| = +\sqrt{x^2 + y^2}$$
$$x = r\cos\theta\,, \quad y = r\sin\theta$$
$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

From the equations $x = r\cos\theta$ and $y = r\sin\theta$ we see that the argument satisfies $\boxed{y/x = \tan\theta}$. We can add multiples of $2\pi$ (that is, 360°) to the argument without changing the complex number, so we usually take the **principal value:** an angle between −180° and +180°, or between $-\pi$ and $\pi$ if we are using radians instead of degrees.

**Examples:** (i) Convert $z = 1 + 2i$ to the form $r(\cos\theta + i\sin\theta)$.
(ii) If $z = x + iy$ has polar coordinates $r = \sqrt{2}$ and $\theta = \pi/4$, find $x$ and $y$.
**(i)** $r = \|z\| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $\tan(\theta) = 2/1$, so $\theta = \tan^{-1}(2) = 1.1$ radians or 63.4°.
**(ii)** $x = r\cos\theta = \sqrt{2}\cos(\pi/4) = 1$ and $y = r\sin\theta = \sqrt{2}\sin(\pi/4) = 1$.

Knowing the tangent is not *quite* enough to know the argument $\theta$: we must take care especially if $x$ or $y$ are negative. Note that $\tan^{-1}(y/x) = \tan^{-1}(-y/-x)$ and that $\tan\theta = \tan(\theta \pm \pi)$.

**Exercise:** If $z_1 = 1 + i\sqrt{3} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, express $z_2 = -1 + i\sqrt{3}$, $z_3 = -1 - i\sqrt{3}$, $z_4 = 1 - i\sqrt{3}$ in modulus-argument form. Sketch these 4 points on the Argand diagram.

If complex numbers are expressed in polar coordinates, multiplication becomes much simpler:

**Theorem:** If $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$ then $zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi))$

In other words: to multiply complex numbers, just *multiply the moduli and add the arguments.*

*Proof:* $zw = r(\cos\theta + i\sin\theta)\, s(\cos\phi + i\sin\phi) = rs\,(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) =$
$= rs\,((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) = rs\,(\cos(\theta + \phi) + i\sin(\theta + \phi))$

Not only is multiplication easier when we know the modulus-argument form, so are inverses, division and conjugation: if $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$ then

$$\overline{z} = r\,(\cos(-\theta) + i\sin(-\theta)), \quad z^{-1} = \frac{1}{r}\,(\cos(-\theta) + i\sin(-\theta)), \quad \frac{z}{w} = \frac{r}{s}\,(\cos(\theta - \phi) + i\sin(\theta - \phi)).$$

To find quotients one divides the moduli and subtracts the arguments. Note that the argument of the imaginary unit $i$ is $\pi/2$ (or 90°), so multiplying or dividing by $i$ is the same as adding or subtracting $\pi/2$ to the angle and converting to the principal value.

### 1.3 Exponential form and de Moivre's theorem

In the same way that multiplication is achieved by multiplying moduli and adding arguments, and division by dividing moduli and subtracting arguments, it turns out that any **power** of a complex number $z = x + iy = r(\cos\theta + i\sin\theta)$ may be calculated by multiplying the argument, and taking the power of the modulus.

**Theorem:** If $z = x + iy = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

If $\|z\| = 1$ then we have the special case: $\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$.

**Exercise:** If $z = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$, sketch the values of $z^n$ on the Argand diagram, for $n = 0, \pm1, \pm2, \pm3, \pm4$.

One of the easiest ways of proving the theorem is to use a trick involving series expansions.
For any $x$,

$$\left.\begin{array}{rcl} e^x &=& 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots \\ \cos(x) &=& 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin(x) &=& x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \end{array}\right\} \Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Therefore $z = r(\cos\theta + i\sin\theta)$ can be written in exponential form $\boxed{z = re^{i\theta}}$ and $z^n = r^n e^{in\theta}$.

**Exercise:** Give the rules for multiplication and division of complex numbers in exponential form.

### 1.4 Applications

**Trigonometric identities:** If $z = \cos\theta + i\sin\theta$ then $z^4$ can be calculated in two ways:

$$\cos(4\theta) + i\sin(4\theta) = \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$$

and so $\boxed{\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ and $\boxed{\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$.

*Exercise:* Using the relation $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$, prove that $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$.

**Solutions of equations of the form $z^n = w$:** The obvious solution is $z = \sqrt[n]{w}$. Why is this not good enough? Because we know that a polynomial of degree $n$ should have $n$ solutions.

The answer is to write $w$ as a complex number and allow non-principal values for the argument. Instead of writing $w = se^{i\phi}$ we write $w = se^{i(\phi+2k\pi)}$ for all the values $k = 0, 1, \ldots, n-1$.

Then $z^n = w$ can be written as $z = w^{1/n}$, so we have to divide these arguments $\phi + 2k\pi$ by $n$,

$$z^n = w \iff z = \sqrt[n]{s}\,e^{i\phi/n},\ \sqrt[n]{s}\,e^{i(\phi+2\pi)/n},\ \sqrt[n]{s}\,e^{i(\phi+4\pi)/n}, \ldots\ \sqrt[n]{s}\,e^{i(\phi+2(n-1)\pi)/n}.$$

*Exercise:* Find all the solutions of $z^6 = 1$ and of $z^4 = 16i$, and show them on the Argand diagram.

## Tutorial exercises

1. Calculate:

| | | | | | |
|---|---|---|---|---|---|
| (i) | $(3+4i)+(8-2i)$ | (ii) | $(3+4i)-(8-2i)$ | (iii) | $(3+4i)-2(8-2i)$ |
| (iv) | $(3+4i)i$ | (v) | $i^3$ | (vi) | $i^4-i^{-1}$ |
| (vii) | $i^5-i^6$ | (viii) | $i^{1001}$ | (ix) | $(1+i)(1-i)$ |
| (x) | $(1+i)^{-1}$ | (xi) | $(1+5i)/(1+i)$ | (xii) | $(4-7i)/(3-i)$ |

2. Indicate the the answers to the previous question on an Argand diagram.

3. If $z=4-3i, \quad w=-1+2i, \quad u=1+i,$ calculate:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (i) | $\overline{z}$ | (ii) | $1/\overline{z}$ | (iii) | $z/\Vert z\Vert$ | (iv) | $\overline{z}/z$ |
| (v) | $z/(z+\overline{z})$ | (vi) | $\overline{z+u}$ | (vii) | $w/\overline{z}$ | (viii) | $\overline{w}/z$ |
| (ix) | $\overline{\overline{w}+z}/\overline{z}$ | (x) | $u^2$ | (xi) | $u^2-w^2$ | (xii) | $(u-w)(u+w)$ |

4. Draw an Argand diagram for the answers of the previous question, together with the complex numbers $z$, $w$, $u$. What is the relation between $z$ and $\overline{z}$ in the diagram?

5. Solve the following equations:

| | | | | | |
|---|---|---|---|---|---|
| (i) | $z^2-2z+5=0$ | (ii) | $z^2-6z+25=0$ | (iii) | $z^2+2z+10=0$ |
| (iv) | $z^4-3z^2-4=0$ | (v) | $5z^2+6z+2=0$ | | |
| (vi) | $z^3-6z^2+13z-10=0$ | : | (try $z=2$) | | |
| (vii) | $z^4-z^3+z+35=0$ | : | (try $2+i\sqrt{}3)$ | | |

6. Prove that if $z=a+ib$ is a solution of a polynomial equation $f(z)=0$ with real coefficients, then $z^2-2az+(a^2+b^2)$ is a factor of $f(z)$.

7. Using the fact that the zeros of any real polynomial occur in conjugate pairs, prove that all real polynomial equations of *odd* degree have at least one real root. Give an example to show this is not true for equations of even degree.

8. Solve the equations (i) $z^2+2iz+2=0$ and (ii) $z^2=i$.

9. Find the equation whose roots are 1, 2, $-2$, $1+i$ and $1-i$.

10. If $a$ and $b$ are real numbers and $1 + 3i$ is a root of the equation $z^4 - 6z^3 + az^2 + bz + 70 = 0$, find the values of $a$ and $b$ and the other 3 roots of the equation.

11. Convert the following to polar form $(r, \theta)$, with $\theta$ in radians:
(i) $-\sqrt{8} + i\sqrt{8}$ (ii) $\sqrt{3} - 3i$ (iii) $-4\sqrt{3} + 4i$

12. Convert the following to cartesian form, $x + iy$, and draw them on the Argand diagram:
(i) $(2, \pi/3)$ (ii) $(2, -\pi/3)$ (iii) $(-2, \pi/3)$ (iv) $\sqrt{3}(\cos(2\pi/3) + i\sin(2\pi/3))$

13. Convert the following to the form $r(\cos\theta + i\sin\theta)$, with $\theta$ in radians:
(i) $3 + i\sqrt{3}$ (ii) $4 - 4i$ (iii) $-4i$ (iv) $-3 + i4$

14. Find the modulus and principal argument of (a) $z = 1 + i$ and (b) $w = \sqrt{3} + i$.
Use these answers to find the moduli and principal arguments of
(i) $zw$ (ii) $z/w$ (iii) $w/z$ (iv) $w^3$ (v) $1/z$ (vi) $iz$

15. Given a point $z$ on the Argand diagram, explain how to construct the points
(i) $2z$ (ii) $-z$ (iii) $|z|$ (iv) $iz$ (v) $1 + iz$ (vi) $(1 + i)z$ (vii) $z^2$

16. Use modulus-argument form to simplify the following:
(i) $(\sqrt{3} - i)^2/(2i)^3$ (ii) $(1 + i)^3/(1 - i)^3$ (iii) $(-\sqrt{3} + i)^4(-1 + i)^3$

17. Express $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

18. Express $\sin^3\theta$ and $\sin^3\theta\cos^4\theta$ in terms of sines of multiples of $\theta$.

19. Solve: (i) $z^3 + 27i = 0$, (ii) $z^4 - 1 = i\sqrt{3}$, (iii) $(z - 1)^4 = (z + 1)^4$.
Factorise completely the polynomials $f(z) = z^3 + 27i$ and $g(z) = z^4 - 1 - i\sqrt{3}$.

20. Find all the fifth-roots of unity, that is, find the roots $\alpha_1, \alpha_2, \ldots, \alpha_5$ of the equation $z^5 = 1$.
If $\alpha_1 = 1$ and none of the other solutions are real, expand the following expressions:
(i) $(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)(x - \alpha_5)$
(ii) $(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)(x - \alpha_5)$

## Tutorial Exercise Solutions

1. (i) $11+2i$; (ii) $-5+6i$; (iii) $-13+8i$; (iv) $-4+3i$; (v) $-i$; (vi) $1+i$;
   (vii) $1+i$; (viii) $i$; (ix) $2$; (x) $\frac{1}{2}-\frac{i}{2}$; (xi) $3+2i$; (xii) $(4-7i)(3+i)/10=\frac{19}{10}-\frac{17}{10}i$.

3. (i) $4+3i$; (ii) $\frac{4}{25}-\frac{3}{25}i$ ; (iii) $\frac{4}{5}-\frac{3}{5}i$; (iv) $\frac{\overline{z}}{z}=\frac{\overline{z}^2}{|z|^2}=\frac{7}{25}+\frac{24}{25}i$; (v) $\frac{z}{z+\overline{z}}=\frac{z}{8}=\frac{1}{2}-\frac{3}{8}i$;
   (vi) $5+2i$; (vii) $(-1+2i)(\frac{4}{25}-\frac{3}{25}i)=\frac{2}{25}+\frac{11}{25}i$; (viii) $\frac{2}{25}-\frac{11}{25}i$; (ix) $=\frac{w}{\overline{z}}+1=\frac{27}{25}+\frac{11}{25}i$;
   (x) $2i$; (xi) $2i-(-3-4i)=3+6i$; (xii) $3+6i$.

5. i) $z=\frac{1}{2}(2\pm\sqrt{4-20})=\frac{1}{2}(2\pm\sqrt{-16})=\frac{1}{2}(2\pm 4i)=1\pm 2i$.
   ii) $z=\frac{1}{2}(6\pm\sqrt{36-100})=\frac{1}{2}(6\pm\sqrt{-64})=\frac{1}{2}(6\pm 8i)=3\pm 4i$.
   iii) $z=\frac{1}{2}(-2\pm\sqrt{4-40})=\frac{1}{2}(-2\pm\sqrt{-36})=\frac{1}{2}(-2\pm 6i)=-1\pm 3i$.
   iv) Put $w=z^2$, then $w^2-3w-4=0$ is $(w-4)(w+1)=0$ so $w=4$ or $-1$; $z=\pm 2$ or $\pm i$.
   v) $z=\frac{1}{10}(-6\pm\sqrt{36-40})=\frac{1}{10}(-6\pm\sqrt{-4})=\frac{1}{10}(-6\pm 2i)=-\frac{3}{5}\pm\frac{i}{5}$.
   vi) $2^3-6\times 2^2+13\times 2-10=0$ so 2 is a solution, so $(z-2)$ is a factor. Writing the polynomial $z^3-6z^2+13z-10=(z-2)(z^2+az+5)$ we need $-6z^2=-2z^2+az^2$ so $a=-4$.

   Solving $z^2-4z+5=0$ in the usual way gives the other two solutions $z=2\pm i$.

   vii) If you check that $2+i\sqrt{3}$ is a root, then $2-i\sqrt{3}$ is too, so we have factors $(z-2-i\sqrt{3})$ and $(z-2+i\sqrt{3})$. But $(z-2-i\sqrt{3})(z-2+i\sqrt{3})=(z^2-4z+7)$ so we write the polynomial $z^4-z^3+z+35=(z^2-4z+7)(z^2+az+5)$ in which $-z^3=az^3-4z^3$ so $a=3$.

   Finally the equation $z^2+3z+5=0$ gives the other two solutions $z=-\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$.

6. If $a+ib$ is a root, so is $a-ib$, and we have a factor $(z-a-ib)(z-a+ib)=z^2-2az+a^2+b^2$.

7. A real polynomial $f(z)$ has a factor $(z^2-2az+a^2+b^2)$ for each pair of complex conjugate zeros. If $f(z)$ has odd degree there will therefore be at least one linear factor $(z-\alpha)$ left.

   This is not true for even degree polynomials: $z^2+1=0$ has no real roots.

8. i) $z=\frac{1}{2}(-2i\pm\sqrt{-4-8})=-i\pm\sqrt{-3}=(-1\pm\sqrt{3})i$
   ii) Recall $(1+i)^2=1^2+2i-1=2i$ so $\sqrt{2}\sqrt{i}=\pm(1+i)$, and $z^2=i$ has solutions $z=\pm\frac{\sqrt{2}}{2}(1+i)$.

9. Each root is a factor: $(z-1)(z-2)(z+2)(z-1-i)(z-1+i)=(z-1)(z^2-4)(z^2-2z+2)=(z^3-z^2-4z+4)(z^2-2z+2)=z^5-3z^4+10z^2-16z+8$, so equation is $z^5-3z^4+10z^2-16z+8=0$.

10. If equation is real and $1+3i$ is a root then $1-3i$ is another root and we have a factor $(z-1-3i)(z-1+3i)=z^2-2z+10$. We must have $z^4-6z^3+az^2+bz+70=(z^2-2z+10)(z^2+cz+7)$, in which $-6z^3=cz^3-2z^3$ so $c=-4$, and so $a=7-2c+10=25$, $b=10c-14=-54$.

    Now the other two roots are the roots of $z^2-4z+7=0$, which are $2\pm i\sqrt{3}$.

11. i) $r=|-\sqrt{8}+i\sqrt{8}|=4$, $\theta=3\pi/4$; ii) $r=|\sqrt{3}-3i|=2\sqrt{3}$, $\theta=-pi/3$;
    iii) $r=|-4\sqrt{3}+4i|=8$, $\theta=5\pi/6$.

12. i) $2(\cos(\pi/3) + i\sin(\pi/3)) = 1 + i\sqrt{3}$; ii) $2(\cos(-\pi/3) + i\sin(-\pi/3)) = 1 - i\sqrt{3}$;
iii) $-2(\cos(\pi/3)+i\sin(\pi/3)) = -1-i\sqrt{3}$; iv) $\sqrt{3}(\cos(2\pi/3)+i\sin(2\pi/3)) = -\sqrt{3}/2+3i/2$.

13. i) $r = |3 + i\sqrt{3}| = 2\sqrt{3}$, $\theta = \pi/6$; ii) $r = |4 - 4i| = 4\sqrt{2}$, $\theta = -\pi/4$;
iii) $r = |-4i| = 4$, $\theta = -\pi/2$; iv) $r = |-3 + 4i| = 5$, $\theta = 2.214$ rad.

14. To find the *argument* of a product, quotient or power, just add, subtract or multiply angles.

| | $z = 1 + i$ | $w = \sqrt{3} + i$ | $zw$ | $z/w$ | $w/z$ | $w^3$ | $1/z$ | $iz$ |
|---|---|---|---|---|---|---|---|---|
| modulus: | $\sqrt{2}$ | 2 | $2\sqrt{2}$ | $\sqrt{2}/2$ | $\sqrt{2}$ | 8 | $\sqrt{2}/2$ | $\sqrt{2}$ |
| argument: | $\pi/4$ | $\pi/6$ | $5\pi/12$ | $\pi/12$ | $-\pi/12$ | $\pi/2$ | $-\pi/4$ | $3\pi/4$ |

16. i) $\dfrac{(\sqrt{3}-i)^2}{(2i)^3} = \dfrac{(2(\cos(-\pi/6) + i\sin(-\pi/6)))^2}{(2(\cos(\pi/2) + i\sin(\pi/2)))^3} = \dfrac{4(\cos(-\pi/3) + i\sin(-\pi/3))}{8(\cos(3\pi/2) + i\sin(3\pi/2))} =$
$= \frac{1}{2}(\cos(-\pi/3 - 3\pi/2) + i\sin(-\pi/3 - 3\pi/2)) = \frac{1}{2}(\cos(\pi/6) + i\sin(\pi/6)) = \sqrt{3}/4 + i/4$.
ii) $\dfrac{(1+i)^3}{(1-i)^3} = \dfrac{(\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)))^3}{(\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)))^3} = \cos(3\pi/2) + i\sin(3\pi/2) = -i$
iii) $(-\sqrt{3}+i)^4(-1+i)^3 = (2(\cos(5\pi/6) + i\sin(5\pi/6)))^4(\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4)))^3$
$= 32\sqrt{2}(\cos(67\pi/12) + i\sin(67\pi/12)) = 32\sqrt{2}(\cos(-5\pi/12) + i\sin(-5\pi/12))$, or $11.7 - 43.7i$.

17. $\cos(3\theta) + i\sin(3\theta) = (\cos(\theta) + i\sin(\theta))^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$.
Compare imaginary and real parts: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

18. If $z = \cos(\theta) + i\sin(\theta) = e^{i\theta}$ then $z - \overline{z} = z - z^{-1} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ and we can write:

$$-8i\sin^3\theta = (e^{i\theta} - e^{-i\theta})^3 = e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta} = 2i\sin 3\theta - 6i\sin\theta$$

so that $\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$.
Using also $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ we have

$$\begin{aligned}
-8i\sin^3\theta \times 16\cos^4\theta &= (e^{i\theta} - e^{-i\theta})^3(e^{i\theta} + e^{-i\theta})^4 = (e^{2i\theta} - e^{-2i\theta})^3(e^{i\theta} + e^{-i\theta}) \\
&= (e^{6i\theta} - 3e^{2i\theta} + 3e^{-2i\theta} - e^{-6i\theta})(e^{i\theta} + e^{-i\theta}) \\
&= e^{7i\theta} - e^{-7i\theta} + e^{5i\theta} - e^{-5i\theta} - 3(e^{3i\theta} - e^{-3i\theta} + e^{i\theta} - e^{-i\theta}) \\
\Longrightarrow \quad \sin^3\theta\cos^4\theta &= -\frac{1}{64}(\sin 7\theta + \sin 5\theta) + \frac{3}{64}(\sin 3\theta + \sin\theta)
\end{aligned}$$

19. i) $z^3 = -27i = 3^3e^{3i\pi/2}$, $3^3e^{-i\pi/2}$ or $3^3e^{-5i\pi/2}$ $\Rightarrow$ $z = 3e^{i\pi/2}$, $3e^{-i\pi/6}$ or $3e^{-5i\pi/6}$.
ii) $z^4 = 1 + i\sqrt{3} = 2e^{i\pi/3} = 2e^{(6k+1)i\pi/3}$ $\Rightarrow$ $z = \sqrt[4]{2}\,e^{i(6k+1)\pi/12}$ $(k = -2, -1, 0, 1)$.
iii) $(z+1)^4 - (z-1)^4 = ((z+1)^2 - (z-1)^2)((z+1)^2 + (z-1)^2) = 4z(2z^2+2) = 0 \Leftrightarrow z = 0, \pm i$.

20. $z^5 = 1 = e^{2ki\pi}$ $\Rightarrow$ $z = e^{2ki/5}$ $(k = 0, \pm1, \pm2)$, that is, $z = 1$, $\cos\frac{2\pi}{5} \pm \sin\frac{2\pi}{5}$, $\cos\frac{4\pi}{5} \pm \sin\frac{4\pi}{5}$.
$(x - e^{2i\pi/5})(x - e^{-2i\pi/5})(x - e^{4i\pi/5})(x - e^{-4i\pi/5}) = (x^2 - 2x\cos\frac{2\pi}{5} + 1)(x^2 - 2x\cos\frac{4\pi}{5} + 1) = x^4+x^3+x^2+x+1$ and obviously $(x-1)(x-e^{2i\pi/5})(x-e^{-2i\pi/5})(x-e^{4i\pi/5})(x-e^{-4i\pi/5}) = x^5-1$.