Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my "class notes" they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don't have good Calculus I skills and you are constantly getting stuck on the Calculus I portion of the problem you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone it is simply an acknowledgement that you can't just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general and you will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you've got it memorized doesn't mean that you can use it. You'll need to be able to look at an integral and realize that integration by parts can be used (which isn't always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you'll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often

don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Improper Integrals

In this section we need to take a look at a couple of different kinds of integrals. Both of these are examples of integrals that are called Improper Integrals.

Let's start with the first kind of improper integrals that we're going to take a look at.

Infinite Interval
In this kind of integrals we are going to take a look at integrals that in which one or both of the limits of integration are infinity. In these cases the interval of integration is said to be over an infinite interval.

Let's take a look at an example that will also show us how we are going to deal with these integrals.

Example 1 Evaluate the following integral.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Solution
This is an innocent enough looking integral. However, because infinity is not a real number we can't just integrate as normal and then "plug in" the infinity to get an answer.

To see how we're going to do this integral let's think of this as an area problem. So instead of asking what the integral is, let's instead ask what the area under $f\left(x\right)=\frac{1}{x^2}$ on the interval $\left[1,\infty\right)$ is.

We still aren't able to do this, however, let's step back a little and instead ask what the area under $f\left(x\right)$ is on the interval $\left[1,t\right]$ were $t>1$ and t is finite. This is a problem that we can do.

$$A_t=\int_1^t \frac{1}{x^2}\,dx=-\frac{1}{x}\bigg|_1^t=1-\frac{1}{t}$$

Now, we can get the area under $f\left(x\right)$ on $\left[1,\infty\right)$ simply by taking the limit of A_t as t goes to infinity.

$$A=\lim_{t\to\infty}A_t=\lim_{t\to\infty}\left(1-\frac{1}{t}\right)=1$$

This is then how we well do the integral itself.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx&=\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx\\&=\lim_{t\to\infty}\left(-\frac{1}{x}\right)\bigg|_1^t\\&=\lim_{t\to\infty}\left(1-\frac{1}{t}\right)=1\end{aligned}$$

So, this is how we will deal with these kinds of integrals in general. We will replace the infinity with a variable (usually *t*), do the integral and then take the limit of the result as *t* goes to infinity.

On a side note, notice that the area under a curve on an infinite interval was not infinity as we might have suspected it to be. In fact, it was a surprisingly small number. Of course this won't always be the case, but it is important enough to point out that not all areas on an infinite interval will yield infinite areas.

Let's now get some definitions out of the way. We will call these integrals **convergent** if the associated limit exists and is a finite number (*i.e.* it's not plus or minus infinity) and **divergent** if the associated limits either doesn't exist or is (plus or minus) infinity.

Let's now formalize up the method for dealing with infinite intervals. There are essentially three cases that we'll need to look at.

1. If $\int_a^t f(x)\,dx$ exists for every $t > a$ then,

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists and is finite.

2. If $\int_t^b f(x)\,dx$ exists for every $t < b$ then,

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided the limits exists and is finite.

3. If $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ are both convergent then,

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Where *c* is any number. Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

Let's take a look at a couple more examples.

Example 2 Determine if the follow integral is convergent or divergent and if it's convergent find its value.

$$\int_1^\infty \frac{1}{x}\,dx$$

Solution

So, the first thing we do is convert the integral to a limit.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

Now, do the integral and the limit.

$$\begin{aligned}\int_1^{\infty}\frac{1}{x}\,dx &= \lim_{t\to\infty}\ln(x)\Big|_1^t\\ &= \lim_{t\to\infty}\left(\ln(t)-\ln 1\right)\\ &= \infty\end{aligned}$$

So, the limit is infinite and so the integral is divergent.

If we go back to thinking in terms of area notice that the area under $g(x)=\frac{1}{x}$ on the interval $[1,\infty)$ is infinite. This is in contrast to the area under $f(x)=\frac{1}{x^2}$ which was quite small. There really isn't all that much difference between these two functions and yet there is a large difference in the area under them. We can actually extend this out to the following fact.

Fact

If $a>0$ then

$$\int_a^{\infty}\frac{1}{x^p}\,dx$$

is convergent if $p>1$ and divergent if $p\le 1$.

One thing to note about this fact is that it's in essence saying that if an integrand goes to zero fast enough then the integral will converge. How fast is fast enough? If we use this fact as a guide it looks like integrands that go to zero faster than $\frac{1}{x}$ goes to zero will probably converge.

Let's take a look at a couple more examples.

Example 3 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{0}\frac{1}{\sqrt{3-x}}\,dx$$

Solution

There really isn't much to do with these problems once you know how to do them. We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$$\begin{aligned}\int_{-\infty}^{0}\frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to-\infty}\int_t^0\frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to-\infty}-2\sqrt{3-x}\Big|_t^0\\ &= \lim_{t\to-\infty}\left(-2\sqrt{3}+2\sqrt{3-t}\right)\\ &= -2\sqrt{3}+\infty\\ &= \infty\end{aligned}$$

So, the limit is infinite and so this integral is divergent.

Example 4 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

Solution

In this case we've got infinities in both limits and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose $a = 0$ since this will be a convenient point for the evaluate process. The integral is then,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

We've now got to look at each of the individual limits.

$$\begin{aligned}\int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} x\mathbf{e}^{-x^2}\,dx \\ &= \lim_{t\to-\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_{t}^{0} \\ &= \lim_{t\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}\mathbf{e}^{-t^2}\right) \\ &= -\frac{1}{2}\end{aligned}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\begin{aligned}\int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to\infty}\int_{0}^{t} x\mathbf{e}^{-x^2}\,dx \\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_{0}^{t} \\ &= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-t^2}+\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

This integral is convergent and so since they are both convergent the integral we were actually asked to deal with is also convergent and its value is,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx = -\frac{1}{2}+\frac{1}{2} = 0$$

Example 5 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{\infty} \sin x\,dx$$

Solution

First convert to a limit.

$$\int_{-2}^{\infty} \sin x \, dx = \lim_{t \to \infty} \int_{-2}^{t} \sin x \, dx$$
$$= \lim_{t \to \infty} \left(-\cos x \right) \Big|_{-2}^{t}$$
$$= \lim_{t \to \infty} \left(\cos 2 - \cos t \right)$$

This limit doesn't exist and so the integral is divergent.

In most examples in a Calculus II class that are worked over infinite intervals the limit either exists or is infinite. However, there are limits that don't exist, as the previous example showed, so don't forget about those.

Discontinuous Integrand

We now need to look at the second type of improper integrals that we'll be looking at in this section. These are integrals that have discontinuous integrands. The process here is basically the same with one on subtle difference. Here are the general cases that we'll look at for these integrals.

1. If $f(x)$ is continuous on the interval $[a,b)$ and not continuous at $x = b$ then,

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

provided the limit exists and is finite. Note as well that we do need to use a left hand limit here since the interval of integration is entirely on the left side of the upper limit.

2. If $f(x)$ is continuous on the interval $(a,b]$ and not continuous at $x = a$ then,

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided the limit exists and is finite. In this case we need to use a right hand limit here since the interval of integration is entirely on the right side of the lower limit.

3. If $f(x)$ is not continuous at $x = c$ where $a < c < b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

As with the infinite interval case this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

4. If $f(x)$ is not continuous at $x = a$ and $x = b$ and if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Where *c* is any number. Again, this requires BOTH of the integrals to be convergent in order for this integral to also be convergent.

Note that the limits in these cases really do need to be right or left handed limits. Since we will be working inside the interval of integration we will need to make sure that we stay inside that interval. This means that we'll used one-sided limits to make sure we stay inside the interval.

Let's do a couple of examples of these kinds of integrals.

Example 6 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

Solution

The problem point is the upper limit so we are in the first case above.

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to 3^-}\left(-2\sqrt{3-x}\right)\Big|_0^t\\ &= \lim_{t\to 3^-}\left(2\sqrt{3}-2\sqrt{3-t}\right)\\ &= 2\sqrt{3}\end{aligned}$$

The limit exists and is finite and so the integral converges and the integrals value is $2\sqrt{3}$.

Example 7 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^3 \frac{1}{x^3}\,dx$$

Solution

This integrand is not continuous at $x=0$ and so we'll need to split the integral up at that point.

$$\int_{-2}^3 \frac{1}{x^3}\,dx = \int_{-2}^0 \frac{1}{x^3}\,dx + \int_0^3 \frac{1}{x^3}\,dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\begin{aligned}\int_{-2}^0 \frac{1}{x^3}\,dx &= \lim_{t\to 0^-}\int_{-2}^t \frac{1}{x^3}\,dx\\ &= \lim_{t\to 0^-}\left(-\frac{1}{2x^2}\right)\Big|_{-2}^t\\ &= \lim_{t\to 0^-}\left(-\frac{1}{2t^2}+\frac{1}{8}\right)\\ &= -\infty\end{aligned}$$

At this point we're done. One of the integrals is divergent that means the integral that we were asked to look at is divergent. We don't even need to bother with the second integral.

Before leaving this section lets note that we can also have integrals that involve both of these cases. Consider the following integral.

Example 8 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{\infty} \frac{1}{x^2}\,dx$$

Solution

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals. We can split it up anywhere, but pick a value that will be convenient for evaluation purposes.

$$\int_0^{\infty} \frac{1}{x^2}\,dx = \int_0^{1} \frac{1}{x^2}\,dx + \int_1^{\infty} \frac{1}{x^2}\,dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent.

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\begin{aligned}\int_0^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^+}\int_t^{1} \frac{1}{x^2}\,dx \\ &= \lim_{t\to 0^+}\left.\left(-\frac{1}{x}\right)\right|_t^1 \\ &= \lim_{t\to 0^+}\left(-1+\frac{1}{t}\right) \\ &= \infty\end{aligned}$$

So, the first integral is divergent and so the whole integral is divergent.