1) Closed with respect to multiplication

if $k, k' \in \mathbb{Z}$ then $k+k' \in \mathbb{Z}$, & $\frac{3^k}{2^{2k}} \times \frac{3^{k'}}{2^{2k'}} = \frac{3^{(k+k')}}{2^{2(k+k')}}$

Identity : if $k = 0 \in \mathbb{Z}$ then $\frac{3^k}{2^{2k}} = \frac{3^0}{2^0} = \frac{1}{1} = 1$,

& $1 \cdot \frac{3^k}{2^{2k}} = \frac{3^k}{2^{2k}} \cdot 1 = \frac{3^k}{2^{2k}}$ $\forall k \in \mathbb{Z}$.

Inverses : if $\frac{3^k}{2^{2k}} \in G$, $k \in \mathbb{Z}$, then $\frac{3^{-k}}{2^{-2k}} \in G$ since $-k \in \mathbb{Z}$

& $\frac{3^k}{2^{2k}} \times \frac{3^{-k}}{2^{-2k}} = \frac{3^{-k}}{2^{-2k}} \times \frac{3^k}{2^{2k}} = 1$

Associativity : can assume mult^n is associative.

$\therefore$ G is a group.

[<u>may</u> be proved by other methods, e.g. using subgroup tests on $G \subseteq (\mathbb{R}^*, \times)$]

2) a) $[13]_{14} = [-1]_{14}$ so $[13^2]_{14} = [(-1)^2]_{14} = 1$

$\Rightarrow$ order of $[13]_{14}$ is $\underline{\underline{2}}$

b) $([3]_{14})^2 = [9]_{14}$

$([3]_{14})^3 = [27]_{14} = [13]_{14}$ which has order 2

$\Rightarrow$ $[3]_{14}$ has order $\underline{\underline{6}}$

c) inverse of $[9]_{14} = ([3]_{14})^2$

is $([3]_{14})^4 = [39]_{14} = \underline{\underline{[11]_{14}}}$

3) $$(x^2yx)^N = \overbrace{x^2yx \cdot x^2yx \cdot \ldots \cdot x^2yx}^{N \text{ times}}$$
$$= x^2y\cancel{x}^3y\cancel{x}^3 \ldots \cancel{x}^3yx$$
$$= x^2 y^N x$$
$$= x^2 \cdot 1 \cdot x$$
$$= x^3 = 1$$

4) a) $\langle b^2 \rangle = \{1, b^2, b^4 = b\}$
$= \{1, b, b^2\}$

b) $H1 = H = \{1, ab, a^2b^2\} = Hab = Ha^2b^2$
$Ha = \{a, a^2b, b^2\} = Hb^2 = Ha^2b$
$Hb = \{b, ab^2, a^2\} = Hab^2 = Ha^2$
i.e. 3 right cosets.

5) $\underline{(gg')\theta} = (gg')^{-1} = g'^{-1}g^{-1}$
$\underline{g\theta \cdot g'\theta} = g^{-1}g'^{-1}$
Since G is abelian, $g^{-1}g'^{-1} = g'^{-1}g^{-1}$ & these are equal so θ is a homomorphism

Q6. check that for u,v in this set u+v and λu are still of the form $\begin{pmatrix} a+b \\ a-b \\ a \end{pmatrix}$. $\begin{matrix} a=1 \\ b=0 \end{matrix}$ gives basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{matrix} a=0 \\ b=1 \end{matrix}$ gives $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, dim 2.

Q7. $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Basis: $(1\ 0\ -1\ 0)$, $(0\ 0\ 3\ 1)$

Q.8. Must check that

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in W$ $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \in W$ & $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \in V$ $\begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \in V$

Q.9. a) No $f(0) \neq 0$

b) YES

c) YES

Q10. If $f: V \to W$ has rank r and nullity n
(dim of image(f)) (dim of Kernel(f))

then $n + r = \dim V$

$f: M_{2\times 2}\mathbb{R} \mapsto \mathbb{R}^2$: $r \leq 2$ So $n \geq \dim V - 2 = 4 - 2 = 2$

Q11

a) i) order of G = number of elements in G

ii) order of g = smallest positive integer k s.t. $g^k = 1$

b) The elements in the set

$$H = \{1, g, g^2, \dots, g^{k-1}\}$$

are all different if k is the order of g.

But H is a <u>subgroup</u> : [has identity

not required $\longrightarrow$ & $g^i g^j = g^{(i+j \bmod k)}$

$(g^i)^{-1} = g^{k-i}$ if $1 \leq i \leq k-1$]

So Lagrange's theorem says $|H|$ divides $|G|$

i.e. k divides $|G|$

i.e. order of g divides order of G.

(c) Enough to show that $x^k = 1$, $y^k = 1$

$$\Rightarrow (xy^{-1})^k = 1$$

(other methods possible!)

This holds since

$$(xy^{-1})^k = x^k (y^{-1})^k \quad \text{because } G \text{ is abelian}$$

$$= x^k (y^k)^{-1}$$

$$= 1 . 1^{-1} = 1$$

d) i) $1^4 = 1$, $a^4 = 1$, $(a^2)^4 = 1$, $(a^3)^4 = 1$, $\boxed{b^4 \neq 1}$, $(a^2b)^4 \neq 1$, $(b^2)^4 \neq 1$

$(ab)^4 = 1$, $(a^3b)^4 = 1$, $(ab^2)^4 = 1$, $(a^3b^2)^4 = 1$, $(a^2b^2)^4 \neq 1$

So $H_4 = \{1, a, a^2, a^3, ab, a^3b, ab^2, a^3b^2\}$

ii) No : H_4 is not a subgroup because its order 8 doesn't divide $|G| = 12$.

Q.8 <u>OR</u> $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \end{pmatrix}$

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & -1 \\ 5 & 7 & 9 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & -1 \\ 0 & 7 & 14 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

So both $V = W = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \right\}$

12a) Define $a\theta = c^2$, $b\theta = 1$ so that θ: $\boxed{4}$

$1 \mapsto 1$
$a \mapsto c^2$
$a^2 \mapsto 1$
$a^3 \mapsto c^2$
$b \mapsto 1$
$ab \mapsto c^2$
$a^2 b \mapsto 1$
$a^3 b \mapsto c^2$

Check relations:

('wrong convention!) $\theta(a^4) = c^8 = 1 = \theta(1)$ ok

! $\theta(a^2) = c^4 = 1 = \theta(b^2)$ ok

$\theta(ba) = \theta(a) = \theta(a^3 b)$ ok
$= \theta(a^3)$ $\boxed{3}$

Image is $\{1, c^2\}$

Kernel is $\{1, a^2, b, a^2 b\}$

(12) (b) $f(1) = 1$ $[a=1, b=c=0]$

$f(x) = 2x+1$ $[a=0, b=1, c=0]$ $\Rightarrow C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

$f(x^2) = x + 3x^2$ $[a=b=0, c=1]$

Characteristic polynomial of f = charac. poly$^{\underline{L}}$ of $C = (-1)^3 \det(C - \lambda I)$

$$= - \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)$$

$\Rightarrow$ eigenvectors $\lambda = 1, 2, 3$

Eigenvectors

$\lambda = 1$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0}$ eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ie 1.

$\lambda = 2$ $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0}$ eigenvector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\underline{1+x}$.

$\lambda = 3$ $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0}$ eigenvector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\underline{1+2x+2x^2}$.

So $B = \{1, 1+x, 1+2x+2x^2\}$ $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

and f is diagonal w.r.t this basis.

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ check: $f(1+2x+2x^2) = 3+6x+6x^2$ ✓

Relationship between these matrices:

$$D = P^{-1}CP$$

Q13. a) $f(a+bx+cx^2) = 0 \Leftrightarrow \begin{cases} a+b=0 \\ a-c=0 \\ b+c=0 \end{cases} \Leftrightarrow a=-b=c$

So nullspace is spanned by $1-x+x^2$ & has dim 1.

b) $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad f(x^2) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$

$a=0,\ b=1,\ c=0 \qquad a=0,\ b=0,\ c=1$

These matrices are linearly independent.
They span the image because

$$\text{rank} = \dim(P_2) - \dim(\text{nullspace}) = 2$$

[rank-nullity theorem]

So they are a basis for the image.

c) $\begin{array}{c} \begin{array}{ccc} f(1) & f(x) & f(x^2) \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$ column space has basis

eg. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

d) $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ for example

i) $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ ii) $Q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. So $P^{-1}AQ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$v \mapsto f(v) = 0$
$x \mapsto f(x)$
$x^2 \mapsto f(x^2)$