Answers to Exercises

Linear Algebra

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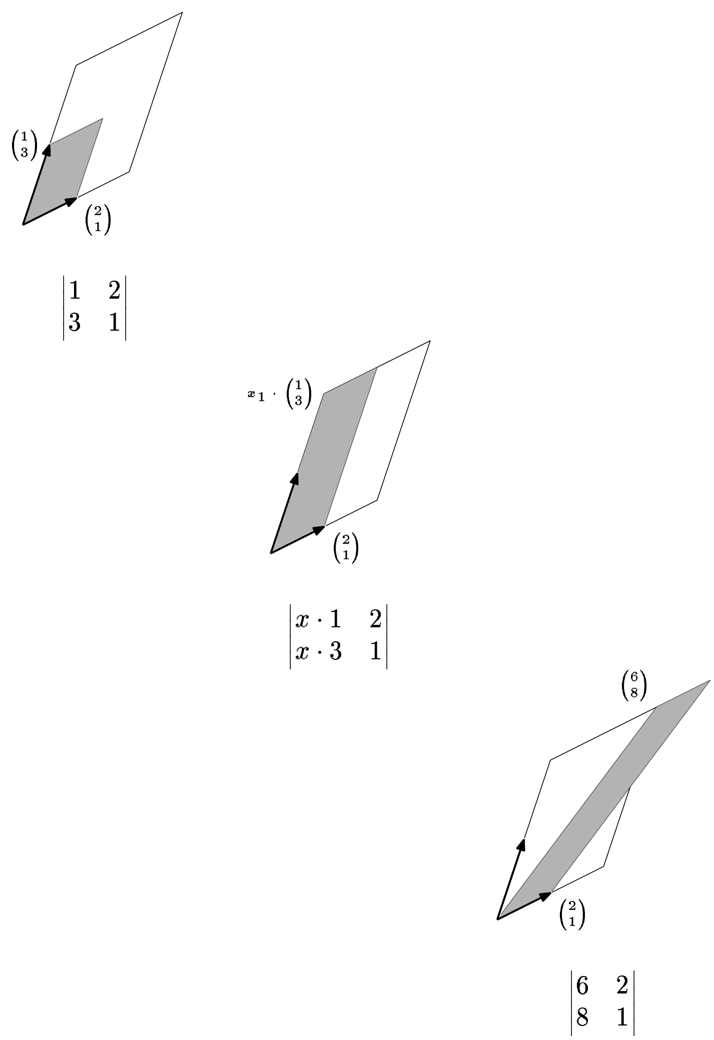


Notation

$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$	real numbers, reals greater than 0, n-tuples of reals
$\mathbb{N}$	natural numbers: $\{0, 1, 2, \ldots\}$
$\mathbb{C}$	complex numbers
$\{\ldots \mid \ldots\}$	set of ... such that ...
$(a\,..\,b)$, $[a\,..\,b]$	interval (open or closed) of reals between a and b
$\langle \ldots \rangle$	sequence; like a set but order matters
V, W, U	vector spaces
$\vec{v}, \vec{w}$	vectors
$\vec{0}$, $\vec{0}_V$	zero vector, zero vector of V
B, D	bases
$\mathcal{E}_n = \langle \vec{e}_1, \ldots, \vec{e}_n \rangle$	standard basis for $\mathbb{R}^n$
$\vec{\beta}, \vec{\delta}$	basis vectors
$\mathrm{Rep}_B(\vec{v})$	matrix representing the vector
$\mathcal{P}_n$	set of n-th degree polynomials
$\mathcal{M}_{n\times m}$	set of $n\times m$ matrices
$[S]$	span of the set S
$M \oplus N$	direct sum of subspaces
$V \cong W$	isomorphic spaces
h, g	homomorphisms, linear maps
H, G	matrices
t, s	transformations; maps from a space to itself
T, S	square matrices
$\mathrm{Rep}_{B,D}(h)$	matrix representing the map h
$h_{i,j}$	matrix entry from row i, column j
$\lvert T\rvert$	determinant of the matrix T
$\mathscr{R}(h), \mathscr{N}(h)$	rangespace and nullspace of the map h
$\mathscr{R}_\infty(h), \mathscr{N}_\infty(h)$	generalized rangespace and nullspace

Lower case Greek alphabet

name	character	name	character	name	character
alpha	α	iota	ι	rho	ρ
beta	β	kappa	κ	sigma	σ
gamma	γ	lambda	λ	tau	τ
delta	δ	mu	μ	upsilon	υ
epsilon	ϵ	nu	ν	phi	ϕ
zeta	ζ	xi	ξ	chi	χ
eta	η	omicron	o	psi	ψ
theta	θ	pi	π	omega	ω

Cover. This is Cramer's Rule for the system $x_1 + 2x_2 = 6$, $3x_1 + x_2 = 8$. The size of the first box is the determinant shown (the absolute value of the size is the area). The size of the second box is x_1 times that, and equals the size of the final box. Hence, x_1 is the final determinant divided by the first determinant.

These are answers to the exercises in Linear Algebra *by J. Hefferon. Corrections or comments are very welcome, email to jimjoshua.smcvt.edu*

An answer labeled here as, for instance, One.II.3.4, matches the question numbered 4 from the first chapter, second section, and third subsection. The Topics are numbered separately.

Contents

Chapter Two: Vector Spaces

Subsection Two.I.1: Definition and Examples

Two.I.1.18 **(a)** $0 + 0x + 0x^2 + 0x^3$
(b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(c) The constant function $f(x) = 0$
(d) The constant function $f(n) = 0$

Two.I.1.19 **(a)** $3 + 2x - x^2$ **(b)** $\begin{pmatrix} -1 & +1 \\ 0 & -3 \end{pmatrix}$ **(c)** $-3e^x + 2e^{-x}$

Two.I.1.20 Most of the conditions are easy to check; use Example 1.3 as a guide. Here are some comments.
(a) This is just like Example 1.3; the zero element is $0 + 0x$.
(b) The zero element of this space is the 2×2 matrix of zeroes.
(c) The zero element is the vector of zeroes.
(d) Closure of addition involves noting that the sum

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}$$

is in L because $(x_1+x_2)+(y_1+y_2)-(z_1+z_2)+(w_1+w_2) = (x_1+y_1-z_1+w_1)+(x_2+y_2-z_2+w_2) = 0+0$. Closure of scalar multiplication is similar. Note that the zero element, the vector of zeroes, is in L.

Two.I.1.21 In each item the set is called Q. For some items, there are other correct ways to show that Q is not a vector space.
(a) It is not closed under addition; it fails to meet condition (1).

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

(b) It is not closed under addition.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

(c) It is not closed under addition.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \notin Q$$

(d) It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

(e) It is empty, violating condition (4).

Two.I.1.22 The usual operations $(v_0 + v_1 i) + (w_0 + w_1 i) = (v_0 + w_0) + (v_1 + w_1)i$ and $r(v_0 + v_1 i) = (rv_0) + (rv_1)i$ suffice. The check is easy.

Two.I.1.23 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

Two.I.1.24 The natural operations are $(v_1 x + v_2 y + v_3 z) + (w_1 x + w_2 y + w_3 z) = (v_1 + w_1)x + (v_2 + w_2)y + (v_3 + w_3)z$ and $r \cdot (v_1 x + v_2 y + v_3 z) = (rv_1)x + (rv_2)y + (rv_3)z$. The check that this is a vector space is easy; use Example 1.3 as a guide.

Two.I.1.25 The '+' operation is not commutative (that is, condition (2) is not met); producing two members of the set witnessing this assertion is easy.

Two.I.1.26 **(a)** It is not a vector space.

$$(1+1)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\neq\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}$$

(b) It is not a vector space.

$$1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\neq\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Two.I.1.27 For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions.

(a) Yes.
(b) Yes.
(c) No, it is not closed under addition. The vector of all 1/4's, when added to itself, makes a nonmember.
(d) Yes.
(e) No, $f(x)=e^{-2x}+(1/2)$ is in the set but $2\cdot f$ is not (that is, condition (6) fails).

Two.I.1.28 It is a vector space. Most conditions of the definition of vector space are routine; we here check only closure. For addition, $(f_1+f_2)\,(7)=f_1(7)+f_2(7)=0+0=0$. For scalar multiplication, $(r\cdot f)\,(7)=rf(7)=r0=0$.

Two.I.1.29 We check Definition 1.1.

First, closure under '+' holds because the product of two positive reals is a positive real. The second condition is satisfied because real multiplication commutes. Similarly, as real multiplication associates, the third checks. For the fourth condition, observe that multiplying a number by $1\in\mathbb{R}^+$ won't change the number. Fifth, any positive real has a reciprocal that is a positive real.

The sixth, closure under '$\cdot$', holds because any power of a positive real is a positive real. The seventh condition is just the rule that v^{r+s} equals the product of v^r and v^s. The eight condition says that $(vw)^r=v^rw^r$. The ninth condition asserts that $(v^r)^s=v^{rs}$. The final condition says that $v^1=v$.

Two.I.1.30 **(a)** No: $1\cdot(0,1)+1\cdot(0,1)\neq(1+1)\cdot(0,1)$.

(b) No; the same calculation as the prior answer shows a contition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1\cdot(0,1)\neq(0,1)$.

Two.I.1.31 It is not a vector space since it is not closed under addition, as $(x^2)+(1+x-x^2)$ is not in the set.

Two.I.1.32 **(a)** 6
(b) nm
(c) 3
(d) To see that the answer is 2, rewrite it as

$$\{\begin{pmatrix}a&0\\b&-a-b\end{pmatrix}\mid a,b\in\mathbb{R}\}$$

so that there are two parameters.

Two.I.1.33 A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '$\vec{+}$' and '$\vec{\cdot}$' subject to these conditions. Where $\vec{v},\vec{w}\in V$, (1) their *vector sum* $\vec{v}\,\vec{+}\,\vec{w}$ is an element of V. If $\vec{u},\vec{v},\vec{w}\in V$ then (2) $\vec{v}\,\vec{+}\,\vec{w}=\vec{w}\,\vec{+}\,\vec{v}$ and (3) $(\vec{v}\,\vec{+}\,\vec{w})\,\vec{+}\,\vec{u}=\vec{v}\,\vec{+}\,(\vec{w}\,\vec{+}\,\vec{u})$. (4) There is a *zero vector* $\vec{0}\in V$ such that $\vec{v}\,\vec{+}\,\vec{0}=\vec{v}$ for all $\vec{v}\in V$. (5) Each $\vec{v}\in V$ has an *additive inverse* $\vec{w}\in V$ such that $\vec{w}\,\vec{+}\,\vec{v}=\vec{0}$. If r,s are *scalars*, that is, members of $\mathbb{R}$), and $\vec{v},\vec{w}\in V$ then (6) each *scalar multiple* $r\cdot\vec{v}$ is in V. If $r,s\in\mathbb{R}$ and $\vec{v},\vec{w}\in V$ then (7) $(r+s)\cdot\vec{v}=r\cdot\vec{v}\,\vec{+}\,s\cdot\vec{v}$, and (8) $r\,\vec{\cdot}\,(\vec{v}+\vec{w})=r\,\vec{\cdot}\,\vec{v}+r\,\vec{\cdot}\,\vec{w}$, and (9) $(rs)\,\vec{\cdot}\,\vec{v}=r\,\vec{\cdot}\,(s\,\vec{\cdot}\,\vec{v})$, and (10) $1\,\vec{\cdot}\,\vec{v}=\vec{v}$.

Two.I.1.34 **(a)** Let V be a vector space, assume that $\vec{v}\in V$, and assume that $\vec{w}\in V$ is the additive inverse of $\vec{v}$ so that $\vec{w}+\vec{v}=\vec{0}$. Because addition is commutative, $\vec{0}=\vec{w}+\vec{v}=\vec{v}+\vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

(b) Let V be a vector space and suppose $\vec{v},\vec{s},\vec{t}\in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v}+\vec{s}=\vec{v}+\vec{t}$ gives that $-\vec{v}+\vec{v}+\vec{s}=-\vec{v}+\vec{v}+\vec{t}$, which says that $\vec{0}+\vec{s}=\vec{0}+\vec{t}$ and so $\vec{s}=\vec{t}$.

Two.I.1.35 Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v}=\vec{v}+\vec{0}=\vec{0}+\vec{v}$.

Two.I.1.36 It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

Two.I.1.37 Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + \vec{0} = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = \vec{0} + \vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

Two.I.1.38 **(a)** Every such set has the form $\{r \cdot \vec{v} + s \cdot \vec{w} \mid r, s \in \mathbb{R}\}$ where either or both of $\vec{v}, \vec{w}$ may be $\vec{0}$. With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ are easy. The other conditions are also routine.

(b) No such set can be a vector space under the inherited operations because it does not have a zero element.

Two.I.1.39 Assume that $\vec{v} \in V$ is not $\vec{0}$.

(a) One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

(b) Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

(c) A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

(d) The solution set is either trivial, or nontrivial. In the second case, it is infinite.

Two.I.1.40 Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f+g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\, f'$.

Two.I.1.41 The check is routine. Note that '1' is $1 + 0i$ and the zero elements are these.

(a) $(0 + 0i) + (0 + 0i)x + (0 + 0i)x^2$

(b) $\begin{pmatrix} 0 + 0i & 0 + 0i \\ 0 + 0i & 0 + 0i \end{pmatrix}$

Two.I.1.42 Notably absent from the definition of a vector space is a distance measure.

Two.I.1.43 **(a)** A small rearrangement does the trick.

$$\begin{aligned} (\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) + \vec{v}_4 &= ((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \vec{v}_4 \\ &= (\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) \\ &= \vec{v}_1 + (\vec{v}_2 + (\vec{v}_3 + \vec{v}_4)) \\ &= \vec{v}_1 + ((\vec{v}_2 + \vec{v}_3) + \vec{v}_4) \end{aligned}$$

Each equality above follows from the associativity of three vectors that is given as a condition in the definition of a vector space. For instance, the second '=' applies the rule $(\vec{w}_1 + \vec{w}_2) + \vec{w}_3 = \vec{w}_1 + (\vec{w}_2 + \vec{w}_3)$ by taking $\vec{w}_1$ to be $\vec{v}_1 + \vec{v}_2$, taking $\vec{w}_2$ to be $\vec{v}_3$, and taking $\vec{w}_3$ to be $\vec{v}_4$.

(b) The base case for induction is the three vector case. This case $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$ is required of any triple of vectors by the definition of a vector space.

For the inductive step, assume that any two sums of three vectors, any two sums of four vectors, ..., any two sums of k vectors are equal no matter how the sums are parenthesized. We will show that any sum of $k + 1$ vectors equals this one $((\cdots((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \cdots) + \vec{v}_k) + \vec{v}_{k+1}$.

Any parenthesized sum has an outermost '+'. Assume that it lies between $\vec{v}_m$ and $\vec{v}_{m+1}$ so the sum looks like this.

$$(\cdots \vec{v}_1 \cdots \vec{v}_m \cdots) + (\cdots \vec{v}_{m+1} \cdots \vec{v}_{k+1} \cdots)$$

The second half involves fewer than $k + 1$ additions, so by the inductive hypothesis we can re-parenthesize it so that it reads left to right from the inside out, and in particular, so that its outermost '+' occurs right before $\vec{v}_{k+1}$.

$$= (\cdots \vec{v}_1 \cdots \vec{v}_m \cdots) + ((\cdots(\vec{v}_{m+1} + \vec{v}_{m+2}) + \cdots + \vec{v}_k) + \vec{v}_{k+1})$$

Apply the associativity of the sum of three things

$$= ((\cdots \vec{v}_1 \cdots \vec{v}_m \cdots) + (\cdots (\vec{v}_{m+1} + \vec{v}_{m+2}) + \cdots \vec{v}_k)) + \vec{v}_{k+1}$$

and finish by applying the inductive hypothesis inside these outermost parenthesis.

Two.I.1.44 **(a)** We outline the check of the conditions from Definition 1.1.

Additive closure holds because if $a_0 + a_1 + a_2 = 0$ and $b_0 + b_1 + b_2 = 0$ then

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

is in the set since $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) = (a_0 + a_1 + a_2) + (b_0 + b_1 + b_2)$ is zero. The second through fifth conditions are easy.

Closure under scalar multiplication holds because if $a_0 + a_1 + a_2 = 0$ then

$$r \cdot (a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

is in the set as $ra_0 + ra_1 + ra_2 = r(a_0 + a_1 + a_2)$ is zero. The remaining conditions here are also easy.

(b) This is similar to the prior answer.

(c) Call the vector space V. We have two implications: left to right, if S is a subspace then it is closed under linear combinations of pairs of vectors and, right to left, if a nonempty subset is closed under linear combinations of pairs of vectors then it is a subspace. The left to right implication is easy; we here sketch the other one by assuming S is nonempty and closed, and checking the conditions of Definition 1.1.

First, to show closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$ as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, suppose that $\vec{s}$ is in the nonempty set S and note that $0 \cdot \vec{s} = \vec{0} \in S$; showing that the $\vec{0}$ of V acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$ closure under linear combinations shows that the vector $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is in S; showing that it is the additive inverse of $\vec{s}$ under the inherited operations is routine.

The proofs for the remaining conditions are similar.

Subsection Two.I.2: Subspaces and Spanning Sets

Two.I.2.20 By Lemma 2.9, to see if each subset of $\mathcal{M}_{2\times2}$ is a subspace, we need only check if it is nonempty and closed.

(a) Yes, it is easily checked to be nonempty and closed. This is a parametrization.

$$\{a\begin{pmatrix}1&0\\0&0\end{pmatrix} + b\begin{pmatrix}0&0\\0&1\end{pmatrix} \mid a, b \in \mathbb{R}\}$$

By the way, the parametrization also shows that it is a subspace, it is given as the span of the two-matrix set, and any span is a subspace.

(b) Yes; it is easily checked to be nonempty and closed. Alternatively, as mentioned in the prior answer, the existence of a parametrization shows that it is a subspace. For the parametrization, the condition $a + b = 0$ can be rewritten as $a = -b$. Then we have this.

$$\{\begin{pmatrix}-b&0\\0&b\end{pmatrix} \mid b \in \mathbb{R}\} = \{b\begin{pmatrix}-1&0\\0&1\end{pmatrix} \mid b \in \mathbb{R}\}$$

(c) No. It is not closed under addition. For instance,

$$\begin{pmatrix}5&0\\0&0\end{pmatrix} + \begin{pmatrix}5&0\\0&0\end{pmatrix} = \begin{pmatrix}10&0\\0&0\end{pmatrix}$$

is not in the set. (This set is also not closed under scalar multiplication, for instance, it does not contain the zero matrix.)

(d) Yes.

$$\{b\begin{pmatrix}-1&0\\0&1\end{pmatrix} + c\begin{pmatrix}0&1\\0&0\end{pmatrix} \mid b, c \in \mathbb{R}\}$$

Two.I.2.21 No, it is not closed. In particular, it is not closed under scalar multiplication because it does not contain the zero polynomial.

Two.I.2.22 **(a)** Yes, solving the linear system arising from

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix} + r_2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$$

gives $r_1 = 2$ and $r_2 = 1$.

(b) Yes; the linear system arising from $r_1(x^2)+r_2(2x+x^2)+r_3(x+x^3)=x-x^3$

$$\begin{aligned} 2r_2+r_3&=1\\ r_1+\;r_2&=0\\ r_3&=-1 \end{aligned}$$

gives that $-1(x^2)+1(2x+x^2)-1(x+x^3)=x-x^3$.

(c) No; any combination of the two given matrices has a zero in the upper right.

Two.I.2.23 **(a)** Yes; it is in that span since $1\cdot\cos^2 x+1\cdot\sin^2 x=f(x)$.

(b) No, since $r_1\cos^2 x+r_2\sin^2 x=3+x^2$ has no scalar solutions that work for all x. For instance, setting x to be 0 and π gives the two equations $r_1\cdot 1+r_2\cdot 0=3$ and $r_1\cdot 1+r_2\cdot 0=3+\pi^2$, which are not consistent with each other.

(c) No; consider what happens on setting x to be $\pi/2$ and $3\pi/2$.

(d) Yes, $\cos(2x)=1\cdot\cos^2(x)-1\cdot\sin^2(x)$.

Two.I.2.24 **(a)** Yes, for any $x,y,z\in\mathbb{R}$ this equation

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix}+r_2\begin{pmatrix}0\\2\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

has the solution $r_1=x$, $r_2=y/2$, and $r_3=z/3$.

(b) Yes, the equation

$$r_1\begin{pmatrix}2\\0\\1\end{pmatrix}+r_2\begin{pmatrix}1\\1\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives rise to this

$$\begin{aligned} 2r_1+r_2&=x\\ r_2&=y\\ r_1+r_3&=z \end{aligned} \;\xrightarrow{-(1/2)\rho_1+\rho_3}\;\xrightarrow{(1/2)\rho_2+\rho_3}\; \begin{aligned} 2r_1+r_2&=x\\ r_2&=y\\ r_3&=-(1/2)x+(1/2)y+z \end{aligned}$$

so that, given any x, y, and z, we can compute that $r_3=(-1/2)x+(1/2)y+z$, $r_2=y$, and $r_1=(1/2)x-(1/2)y$.

(c) No. In particular, the vector

$$\begin{pmatrix}0\\0\\1\end{pmatrix}$$

cannot be gotten as a linear combination since the two given vectors both have a third component of zero.

(d) Yes. The equation

$$r_1\begin{pmatrix}1\\0\\1\end{pmatrix}+r_2\begin{pmatrix}3\\1\\0\end{pmatrix}+r_3\begin{pmatrix}-1\\0\\0\end{pmatrix}+r_4\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left(\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\1&0&0&5&z\end{array}\right)\xrightarrow{-\rho_1+\rho_3}\;\xrightarrow{3\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\0&0&1&6&-x+3y+z\end{array}\right)$$

We have infinitely many solutions. We can, for example, set r_4 to be zero and solve for r_3, r_2, and r_1 in terms of x, y, and z by the usual methods of back-substitution.

(e) No. The equation

$$r_1\begin{pmatrix}2\\1\\1\end{pmatrix}+r_2\begin{pmatrix}3\\0\\1\end{pmatrix}+r_3\begin{pmatrix}5\\1\\2\end{pmatrix}+r_4\begin{pmatrix}6\\0\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left(\begin{array}{cccc|c}2&3&5&6&x\\1&0&1&0&y\\1&1&2&2&z\end{array}\right)\xrightarrow[-(1/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\;\xrightarrow{-(1/3)\rho_2+\rho_3}\left(\begin{array}{cccc|c}2&3&5&6&x\\0&-3/2&-3/2&-3&-(1/2)x+y\\0&0&0&0&-(1/3)x-(1/3)y+z\end{array}\right)$$

This shows that not every three-tall vector can be so expressed. Only the vectors satisfying the restriction that $-(1/3)x-(1/3)y+z=0$ are in the span. (To see that any such vector is indeed expressible, take r_3 and r_4 to be zero and solve for r_1 and r_2 in terms of x, y, and z by back-substitution.)

Two.I.2.25 (a) $\{(c\ \ b\ \ c)\mid b,c\in\mathbb{R}\}=\{b\,(0\ \ 1\ \ 0)+c\,(1\ \ 0\ \ 1)\mid b,c\in\mathbb{R}\}$ The obvious choice for the set that spans is $\{(0\ \ 1\ \ 0),(1\ \ 0\ \ 1)\}$.

(b) $\{\begin{pmatrix}-d&b\\c&d\end{pmatrix}\mid b,c,d\in\mathbb{R}\}=\{b\begin{pmatrix}0&1\\0&0\end{pmatrix}+c\begin{pmatrix}0&0\\1&0\end{pmatrix}+d\begin{pmatrix}-1&0\\0&1\end{pmatrix}\mid b,c,d\in\mathbb{R}\}$ One set that spans this space consists of those three matrices.

(c) The system

$$\begin{aligned}a+3b\qquad\quad&=0\\2a\qquad -c-d&=0\end{aligned}$$

gives $b=-(c+d)/6$ and $a=(c+d)/2$. So one description is this.

$$\{c\begin{pmatrix}1/2&-1/6\\1&0\end{pmatrix}+d\begin{pmatrix}1/2&-1/6\\0&1\end{pmatrix}\mid c,d\in\mathbb{R}\}$$

That shows that a set spanning this subspace consists of those two matrices.

(d) The $a=2b-c$ gives $\{(2b-c)+bx+cx^3\mid b,c\in\mathbb{R}\}=\{b(2+x)+c(-1+x^3)\mid b,c\in\mathbb{R}\}$. So the subspace is the span of the set $\{2+x,-1+x^3\}$.

(e) The set $\{a+bx+cx^2\mid a+7b+49c=0\}$ parametrized as $\{b(-7+x)+c(-49+x^2)\mid b,c\in\mathbb{R}\}$ has the spanning set $\{-7+x,-49+x^2\}$.

Two.I.2.26 Each answer given is only one out of many possible.

(a) We can parametrize in this way

$$\{\begin{pmatrix}x\\0\\z\end{pmatrix}\mid x,z\in\mathbb{R}\}=\{x\begin{pmatrix}1\\0\\0\end{pmatrix}+z\begin{pmatrix}0\\0\\1\end{pmatrix}\mid x,z\in\mathbb{R}\}$$

giving this for a spanning set.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

(b) Parametrize it with $\{y\begin{pmatrix}-2/3\\1\\0\end{pmatrix}+z\begin{pmatrix}-1/3\\0\\1\end{pmatrix}\mid y,z\in\mathbb{R}\}$ to get $\{\begin{pmatrix}-2/3\\1\\0\end{pmatrix},\begin{pmatrix}-1/3\\0\\1\end{pmatrix}\}$.

(c) $\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}-1/2\\0\\0\\1\end{pmatrix}\}$

(d) Parametrize the description as $\{-a_1+a_1x+a_3x^2+a_3x^3\mid a_1,a_3\in\mathbb{R}\}$ to get $\{-1+x,x^2+x^3\}$.

(e) $\{1,x,x^2,x^3,x^4\}$

(f) $\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\}$

Two.I.2.27 Technically, no. Subspaces of $\mathbb{R}^3$ are sets of three-tall vectors, while $\mathbb{R}^2$ is a set of two-tall vectors. Clearly though, $\mathbb{R}^2$ is "just like" this subspace of $\mathbb{R}^3$.

$$\{\begin{pmatrix}x\\y\\0\end{pmatrix}\mid x,y\in\mathbb{R}\}$$

Two.I.2.28 Of course, the addition and scalar multiplication operations are the ones inherited from the enclosing space.

(a) This is a subspace. It is not empty as it contains at least the two example functions given. It is closed because if f_1,f_2 are even and c_1,c_2 are scalars then we have this.

$$(c_1f_1+c_2f_2)\,(-x)=c_1\,f_1(-x)+c_2\,f_2(-x)=c_1\,f_1(x)+c_2\,f_2(x)=(c_1f_1+c_2f_2)\,(x)$$

(b) This is also a subspace; the check is similar to the prior one.

Two.I.2.29 It can be improper. If $\vec{v}=\vec{0}$ then this is a trivial subspace. At the opposite extreme, if the vector space is $\mathbb{R}^1$ and $\vec{v}\neq\vec{0}$ then the subspace is all of $\mathbb{R}^1$.

Two.I.2.30 No, such a set is not closed. For one thing, it does not contain the zero vector.

Two.I.2.31 No. The only subspaces of $\mathbb{R}^1$ are the space itself and its trivial subspace. Any subspace S of $\mathbb{R}$ that contains a nonzero member $\vec{v}$ must contain the set of all of its scalar multiples $\{r\cdot\vec{v}\mid r\in\mathbb{R}\}$. But this set is all of $\mathbb{R}$.

Two.I.2.32 Item (1) is checked in the text.

Item (2) has five conditions. First, for closure, if $c \in \mathbb{R}$ and $\vec{s} \in S$ then $c \cdot \vec{s} \in S$ as $c \cdot \vec{s} = c \cdot \vec{s} + 0 \cdot \vec{0}$. Second, because the operations in S are inherited from V, for $c, d \in \mathbb{R}$ and $\vec{s} \in S$, the scalar product $(c+d) \cdot \vec{s}$ in S equals the product $(c+d) \cdot \vec{s}$ in V, and that equals $c \cdot \vec{s} + d \cdot \vec{s}$ in V, which equals $c \cdot \vec{s} + d \cdot \vec{s}$ in S.

The check for the third, fourth, and fifth conditions are similar to the second conditions's check just given.

Two.I.2.33 An exercise in the prior subsection shows that every vector space has only one zero vector (that is, there is only one vector that is the additive identity element of the space). But a trivial space has only one element and that element must be this (unique) zero vector.

Two.I.2.34 As the hint suggests, the basic reason is the Linear Combination Lemma from the first chapter. For the full proof, we will show mutual containment between the two sets.

The first containment $[[S]] \supseteq [S]$ is an instance of the more general, and obvious, fact that for any subset T of a vector space, $[T] \supseteq T$.

For the other containment, that $[[S]] \subseteq [S]$, take m vectors from $[S]$, namely $c_{1,1}\vec{s}_{1,1} + \cdots + c_{1,n_1}\vec{s}_{1,n_1}$, $\ldots$, $c_{1,m}\vec{s}_{1,m} + \cdots + c_{1,n_m}\vec{s}_{1,n_m}$, and note that any linear combination of those

$$r_1(c_{1,1}\vec{s}_{1,1} + \cdots + c_{1,n_1}\vec{s}_{1,n_1}) + \cdots + r_m(c_{1,m}\vec{s}_{1,m} + \cdots + c_{1,n_m}\vec{s}_{1,n_m})$$

is a linear combination of elements of S

$$= (r_1c_{1,1})\vec{s}_{1,1} + \cdots + (r_1c_{1,n_1})\vec{s}_{1,n_1} + \cdots + (r_mc_{1,m})\vec{s}_{1,m} + \cdots + (r_mc_{1,n_m})\vec{s}_{1,n_m}$$

and so is in $[S]$. That is, simply recall that a linear combination of linear combinations (of members of S) is a linear combination (again of members of S).

Two.I.2.35 **(a)** It is not a subspace because these are not the inherited operations. For one thing, in this space,

$$0 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

while this does not, of course, hold in $\mathbb{R}^3$.

(b) We can combine the argument showing closure under addition with the argument showing closure under scalar multiplication into one single argument showing closure under linear combinations of two vectors. If $r_1, r_2, x_1, x_2, y_1, y_2, z_1, z_2$ are in $\mathbb{R}$ then

$$r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + 1 \\ r_1y_1 \\ r_1z_1 \end{pmatrix} + \begin{pmatrix} r_2x_2 - r_2 + 1 \\ r_2y_2 \\ r_2z_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 - r_1 + r_2x_2 - r_2 + 1 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}$$

(note that the definition of addition in this space is that the first components combine as $(r_1x_1 - r_1 + 1) + (r_2x_2 - r_2 + 1) - 1$, so the first component of the last vector does not say '$+2$'). Adding the three components of the last vector gives $r_1(x_1 - 1 + y_1 + z_1) + r_2(x_2 - 1 + y_2 + z_2) + 1 = r_1 \cdot 0 + r_2 \cdot 0 + 1 = 1$.

Most of the other checks of the conditions are easy (although the oddness of the operations keeps them from being routine). Commutativity of addition goes like this.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 - 1 \\ y_2 + y_1 \\ z_2 + z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Associativity of addition has

$$(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 - 1) + x_3 - 1 \\ (y_1 + y_2) + y_3 \\ (z_1 + z_2) + z_3 \end{pmatrix}$$

while

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}) = \begin{pmatrix} x_1 + (x_2 + x_3 - 1) - 1 \\ y_1 + (y_2 + y_3) \\ z_1 + (z_2 + z_3) \end{pmatrix}$$

and they are equal. The identity element with respect to this addition operation works this way

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x + 1 - 1 \\ y + 0 \\ z + 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the additive inverse is similar.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -x+2 \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} x+(-x+2)-1 \\ y-y \\ z-z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The conditions on scalar multiplication are also easy. For the first condition,

$$(r+s)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (r+s)x-(r+s)+1 \\ (r+s)y \\ (r+s)z \end{pmatrix}$$

while

$$r\begin{pmatrix} x \\ y \\ z \end{pmatrix} + s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx-r+1 \\ ry \\ rz \end{pmatrix} + \begin{pmatrix} sx-s+1 \\ sy \\ sz \end{pmatrix} = \begin{pmatrix} (rx-r+1)+(sx-s+1)-1 \\ ry+sy \\ rz+sz \end{pmatrix}$$

and the two are equal. The second condition compares

$$r\cdot(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = r\cdot\begin{pmatrix} x_1+x_2-1 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} r(x_1+x_2-1)-r+1 \\ r(y_1+y_2) \\ r(z_1+z_2) \end{pmatrix}$$

with

$$r\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} rx_1-r+1 \\ ry_1 \\ rz_1 \end{pmatrix} + \begin{pmatrix} rx_2-r+1 \\ ry_2 \\ rz_2 \end{pmatrix} = \begin{pmatrix} (rx_1-r+1)+(rx_2-r+1)-1 \\ ry_1+ry_2 \\ rz_1+rz_2 \end{pmatrix}$$

and they are equal. For the third condition,

$$(rs)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rsx-rs+1 \\ rsy \\ rsz \end{pmatrix}$$

while

$$r(s\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = r(\begin{pmatrix} sx-s+1 \\ sy \\ sz \end{pmatrix}) = \begin{pmatrix} r(sx-s+1)-r+1 \\ rsy \\ rsz \end{pmatrix}$$

and the two are equal. For scalar multiplication by 1 we have this.

$$1\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x-1+1 \\ 1y \\ 1z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus all the conditions on a vector space are met by these two operations.

Remark. A way to understand this vector space is to think of it as the plane in $\mathbb{R}^3$

$$P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bigm| x+y+z=0\}$$

displaced away from the origin by 1 along the x-axis. Then addition becomes: to add two members of this space,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},\ \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

(such that $x_1+y_1+z_1=1$ and $x_2+y_2+z_2=1$) move them back by 1 to place them in P and add as usual,

$$\begin{pmatrix} x_1-1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2-1 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2-2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} \qquad \text{(in } P\text{)}$$

and then move the result back out by 1 along the x-axis.

$$\begin{pmatrix} x_1+x_2-1 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}.$$

Scalar multiplication is similar.

(c) For the subspace to be closed under the inherited scalar multiplication, where $\vec{v}$ is a member of that subspace,

$$0 \cdot \vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

must also be a member.

The converse does not hold. Here is a subset of $\mathbb{R}^3$ that contains the origin

$$\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$$

(this subset has only two elements) but is not a subspace.

Two.I.2.36 **(a)** $(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) - (\vec{v}_1 + \vec{v}_2) = \vec{v}_3$
(b) $(\vec{v}_1 + \vec{v}_2) - (\vec{v}_1) = \vec{v}_2$
(c) Surely, $\vec{v}_1$.
(d) Taking the one-long sum and subtracting gives $(\vec{v}_1) - \vec{v}_1 = \vec{0}$.

Two.I.2.37 Yes; any space is a subspace of itself, so each space contains the other.

Two.I.2.38 **(a)** The union of the x-axis and the y-axis in $\mathbb{R}^2$ is one.
(b) The set of integers, as a subset of $\mathbb{R}^1$, is one.
(c) The subset $\{\vec{v}\}$ of $\mathbb{R}^2$ is one, where $\vec{v}$ is any nonzero vector.

Two.I.2.39 Because vector space addition is commutative, a reordering of summands leaves a linear combination unchanged.

Two.I.2.40 We always consider that span in the context of an enclosing space.

Two.I.2.41 It is both 'if' and 'only if'.

For 'if', let S be a subset of a vector space V and assume $\vec{v} \in S$ satisfies $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ where $c_1, \ldots, c_n$ are scalars and $\vec{s}_1, \ldots, \vec{s}_n \in S$. We must show that $[S \cup \{\vec{v}\}] = [S]$.

Containment one way, $[S] \subseteq [S \cup \{\vec{v}\}]$ is obvious. For the other direction, $[S \cup \{\vec{v}\}] \subseteq [S]$, note that if a vector is in the set on the left then it has the form $d_0\vec{v} + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ where the d's are scalars and the $\vec{t}$'s are in S. Rewrite that as $d_0(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ and note that the result is a member of the span of S.

The 'only if' is clearly true — adding $\vec{v}$ enlarges the span to include at least $\vec{v}$.

Two.I.2.42 **(a)** Always.

Assume that A, B are subspaces of V. Note that their intersection is not empty as both contain the zero vector. If $\vec{w}, \vec{s} \in A \cap B$ and r, s are scalars then $r\vec{v} + s\vec{w} \in A$ because each vector is in A and so a linear combination is in A, and $r\vec{v} + s\vec{w} \in B$ for the same reason. Thus the intersection is closed. Now Lemma 2.9 applies.

(b) Sometimes (more precisely, only if $A \subseteq B$ or $B \subseteq A$).

To see the answer is not 'always', take V to be $\mathbb{R}^3$, take A to be the x-axis, and B to be the y-axis. Note that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in A \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in B \quad \text{but} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin A \cup B$$

as the sum is in neither A nor B.

The answer is not 'never' because if $A \subseteq B$ or $B \subseteq A$ then clearly $A \cup B$ is a subspace.

To show that $A \cup B$ is a subspace only if one subspace contains the other, we assume that $A \not\subseteq B$ and $B \not\subseteq A$ and prove that the union is not a subspace. The assumption that A is not a subset of B means that there is an $\vec{a} \in A$ with $\vec{a} \notin B$. The other assumption gives a $\vec{b} \in B$ with $\vec{b} \notin A$. Consider $\vec{a} + \vec{b}$. Note that sum is not an element of A or else $(\vec{a} + \vec{b}) - \vec{a}$ would be in A, which it is not. Similarly the sum is not an element of B. Hence the sum is not an element of $A \cup B$, and so the union is not a subspace.

(c) Never. As A is a subspace, it contains the zero vector, and therefore the set that is A's complement does not. Without the zero vector, the complement cannot be a vector space.

Two.I.2.43 The span of a set does not depend on the enclosing space. A linear combination of vectors from S gives the same sum whether we regard the operations as those of W or as those of V, because the operations of W are inherited from V.

Two.I.2.44 It is; apply Lemma 2.9. (You must consider the following. Suppose B is a subspace of a vector space V and suppose $A \subseteq B \subseteq V$ is a subspace. From which space does A inherit its operations? The answer is that it doesn't matter — A will inherit the same operations in either case.)

Two.I.2.45 **(a)** Always; if $S \subseteq T$ then a linear combination of elements of S is also a linear combination of elements of T.

(b) Sometimes (more precisely, if and only if $S \subseteq T$ or $T \subseteq S$).

The answer is not 'always' as is shown by this example from $\mathbb{R}^3$

$$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}, \quad T = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

because of this.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \in [S \cup T] \qquad \begin{pmatrix}1\\1\\1\end{pmatrix} \not\in [S] \cup [T]$$

The answer is not 'never' because if either set contains the other then equality is clear. We can characterize equality as happening only when either set contains the other by assuming $S \not\subseteq T$ (implying the existence of a vector $\vec{s} \in S$ with $\vec{s} \not\in T$) and $T \not\subseteq S$ (giving a $\vec{t} \in T$ with $\vec{t} \not\in S$), noting $\vec{s} + \vec{t} \in [S \cup T]$, and showing that $\vec{s} + \vec{t} \not\in [S] \cup [T]$.

(c) Sometimes.

Clearly $[S \cap T] \subseteq [S] \cap [T]$ because any linear combination of vectors from $S \cap T$ is a combination of vectors from S and also a combination of vectors from T.

Containment the other way does not always hold. For instance, in $\mathbb{R}^2$, take

$$S = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}, \quad T = \{\begin{pmatrix}2\\0\end{pmatrix}\}$$

so that $[S] \cap [T]$ is the x-axis but $[S \cap T]$ is the trivial subspace.

Characterizing exactly when equality holds is tough. Clearly equality holds if either set contains the other, but that is not 'only if' by this example in $\mathbb{R}^3$.

$$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}, \quad T = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

(d) Never, as the span of the complement is a subspace, while the complement of the span is not (it does not contain the zero vector).

Two.I.2.46 Call the subset S. By Lemma 2.9, we need to check that $[S]$ is closed under linear combinations. If $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n, c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m \in [S]$ then for any $p, r \in \mathbb{R}$ we have

$$p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m$$

which is an element of $[S]$. (*Remark.* If the set S is empty, then that 'if ... then ...' statement is vacuously true.)

Two.I.2.47 For this to happen, one of the conditions giving the sensibleness of the addition and scalar multiplication operations must be violated. Consider $\mathbb{R}^2$ with these operations.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad r\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

The set $\mathbb{R}^2$ is closed under these operations. But it is not a vector space.

$$1 \cdot \begin{pmatrix}1\\1\end{pmatrix} \neq \begin{pmatrix}1\\1\end{pmatrix}$$

Subsection Two.II.1: Definition and Examples

Two.II.1.18 For each of these, when the subset is independent it must be proved, and when the subset is dependent an example of a dependence must be given.

(a) It is dependent. Considering

$$c_1\begin{pmatrix}1\\-3\\5\end{pmatrix}+c_2\begin{pmatrix}2\\2\\4\end{pmatrix}+c_3\begin{pmatrix}4\\-4\\14\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

gives rise to this linear system.

$$\begin{aligned}c_1+2c_2+\ \ 4c_3&=0\\-3c_1+2c_2-\ \ 4c_3&=0\\5c_1+4c_2+14c_3&=0\end{aligned}$$

Gauss' method

$$\left(\begin{array}{ccc|c}1&2&4&0\\-3&2&-4&0\\5&4&14&0\end{array}\right)\xrightarrow[-5\rho_1+\rho_3]{3\rho_1+\rho_2}\xrightarrow{(3/4)\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&2&4&0\\0&8&8&0\\0&0&0&0\end{array}\right)$$

yields a free variable, so there are infinitely many solutions. For an example of a particular dependence we can set c_3 to be, say, 1. Then we get $c_2=-1$ and $c_1=-2$.

(b) It is dependent. The linear system that arises here

$$\left(\begin{array}{ccc|c}1&2&3&0\\7&7&7&0\\7&7&7&0\end{array}\right)\xrightarrow[-7\rho_1+\rho_3]{-7\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&2&3&0\\0&-7&-14&0\\0&0&0&0\end{array}\right)$$

has infinitely many solutions. We can get a particular solution by taking c_3 to be, say, 1, and back-substituting to get the resulting c_2 and c_1.

(c) It is linearly independent. The system

$$\left(\begin{array}{cc|c}0&1&0\\0&0&0\\-1&4&0\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_2}\xrightarrow{\rho_3\leftrightarrow\rho_1}\left(\begin{array}{cc|c}-1&4&0\\0&1&0\\0&0&0\end{array}\right)$$

has only the solution $c_1=0$ and $c_2=0$. (We could also have gotten the answer by inspection — the second vector is obviously not a multiple of the first, and vice versa.)

(d) It is linearly dependent. The linear system

$$\left(\begin{array}{cccc|c}9&2&3&12&0\\9&0&5&12&0\\0&1&-4&-1&0\end{array}\right)$$

has more unknowns than equations, and so Gauss' method must end with at least one variable free (there can't be a contradictory equation because the system is homogeneous, and so has at least the solution of all zeroes). To exhibit a combination, we can do the reduction

$$\xrightarrow{-\rho_1+\rho_2}\xrightarrow{(1/2)\rho_2+\rho_3}\left(\begin{array}{cccc|c}9&2&3&12&0\\0&-2&2&0&0\\0&0&-3&-1&0\end{array}\right)$$

and take, say, $c_4=1$. Then we have that $c_3=-1/3$, $c_2=-1/3$, and $c_1=-31/27$.

Two.II.1.19 In the cases of independence, that must be proved. Otherwise, a specific dependence must be produced. (Of course, dependences other than the ones exhibited here are possible.)

(a) This set is independent. Setting up the relation $c_1(3-x+9x^2)+c_2(5-6x+3x^2)+c_3(1+1x-5x^2)=0+0x+0x^2$ gives a linear system

$$\left(\begin{array}{ccc|c}3&5&1&0\\-1&-6&1&0\\9&3&-5&0\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{(1/3)\rho_1+\rho_2}\xrightarrow{3\rho_2}\xrightarrow{-(12/13)\rho_2+\rho_3}\left(\begin{array}{ccc|c}3&5&1&0\\0&-13&4&0\\0&0&-128/13&0\end{array}\right)$$

with only one solution: $c_1=0$, $c_2=0$, and $c_3=0$.

(b) This set is independent. We can see this by inspection, straight from the definition of linear independence. Obviously neither is a multiple of the other.

(c) This set is linearly independent. The linear system reduces in this way

$$\left(\begin{array}{ccc|c}2&3&4&0\\1&-1&0&0\\7&2&-3&0\end{array}\right)\xrightarrow[-(7/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-(17/5)\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&3&4&0\\0&-5/2&-2&0\\0&0&-51/5&0\end{array}\right)$$

to show that there is only the solution $c_1=0$, $c_2=0$, and $c_3=0$.

(d) This set is linearly dependent. The linear system

$$\left(\begin{array}{cccc|c} 8 & 0 & 2 & 8 & 0 \\ 3 & 1 & 2 & -2 & 0 \\ 3 & 2 & 2 & 5 & 0 \end{array}\right)$$

must, after reduction, end with at least one variable free (there are more variables than equations, and there is no possibility of a contradictory equation because the system is homogeneous). We can take the free variables as parameters to describe the solution set. We can then set the parameter to a nonzero value to get a nontrivial linear relation.

Two.II.1.20 Let Z be the zero function $Z(x) = 0$, which is the additive identity in the vector space under discussion.

(a) This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$. Plugging in $x = 1$ and $x = 2$ gives a linear system

$$\begin{aligned} c_1 \cdot 1 + \quad c_2 \cdot 1 &= 0 \\ c_1 \cdot 2 + c_2 \cdot (1/2) &= 0 \end{aligned}$$

with the unique solution $c_1 = 0$, $c_2 = 0$.

(b) This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plug in $x = 0$ and $x = \pi/2$ to get

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= 0 \\ c_1 \cdot 0 + c_2 \cdot 1 &= 0 \end{aligned}$$

which obviously gives that $c_1 = 0$, $c_2 = 0$.

(c) This set is also linearly independent. Considering $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plugging in $x = 1$ and $x = e$

$$\begin{aligned} c_1 \cdot e + c_2 \cdot 0 &= 0 \\ c_1 \cdot e^e + c_2 \cdot 1 &= 0 \end{aligned}$$

gives that $c_1 = 0$ and $c_2 = 0$.

Two.II.1.21 In each case, that the set is independent must be proved, and that it is dependent must be shown by exihibiting a specific dependence.

(a) This set is dependent. The familiar relation $\sin^2(x) + \cos^2(x) = 1$ shows that $2 = c_1 \cdot (4\sin^2(x)) + c_2 \cdot (\cos^2(x))$ is satisfied by $c_1 = 1/2$ and $c_2 = 2$.

(b) This set is independent. Consider the relationship $c_1 \cdot 1 + c_2 \cdot \sin(x) + c_3 \cdot \sin(2x) = 0$ (that '0' is the zero function). Taking $x = 0$, $x = \pi/2$ and $x = \pi/4$ gives this system.

$$\begin{aligned} c_1 \qquad\qquad\qquad &= 0 \\ c_1 + \qquad c_2 \qquad &= 0 \\ c_1 + (\sqrt{2}/2)c_2 + c_3 &= 0 \end{aligned}$$

whose only solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

(c) By inspection, this set is independent. Any dependence $\cos(x) = c \cdot x$ is not possible since the cosine function is not a multiple of the identity function (we are applying Corollary 1.17).

(d) By inspection, we spot that there is a dependence. Because $(1 + x)^2 = x^2 + 2x + 1$, we get that $c_1 \cdot (1 + x)^2 + c_2 \cdot (x^2 + 2x) = 3$ is satisfied by $c_1 = 3$ and $c_2 = -3$.

(e) This set is dependent. The easiest way to see that is to recall the triginometric relationship $\cos^2(x) - \sin^2(x) = \cos(2x)$. (*Remark.* A person who doesn't recall this, and tries some x's, simply never gets a system leading to a unique solution, and never gets to conclude that the set is independent. Of course, this person might wonder if they simply never tried the right set of x's, but a few tries will lead most people to look instead for a dependence.)

(f) This set is dependent, because it contains the zero object in the vector space, the zero polynomial.

Two.II.1.22 No, that equation is not a linear relationship. In fact this set is independent, as the system arising from taking x to be 0, $\pi/6$ and $\pi/4$ shows.

Two.II.1.23 To emphasize that the equation $1 \cdot \vec{s} + (-1) \cdot \vec{s} = \vec{0}$ does not make the set dependent.

Two.II.1.24 We have already showed this: the Linear Combination Lemma and its corollary state that in an echelon form matrix, no nonzero row is a linear combination of the others.

Two.II.1.25 **(a)** Assume that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so that any relationship $d_0\vec{u} + d_1\vec{v} + d_2\vec{w} = \vec{0}$ leads to the conclusion that $d_0 = 0$, $d_1 = 0$, and $d_2 = 0$.

Consider the relationship $c_1(\vec{u}) + c_2(\vec{u} + \vec{v}) + c_3(\vec{u} + \vec{v} + \vec{w}) = \vec{0}$. Rewrite it to get $(c_1 + c_2 + c_3)\vec{u} + (c_2 + c_3)\vec{v} + (c_3)\vec{w} = \vec{0}$. Taking d_0 to be $c_1 + c_2 + c_3$, taking d_1 to be $c_2 + c_3$, and taking d_2

to be c_3 we have this system.

$$\begin{aligned} c_1+c_2+c_3&=0\\ c_2+c_3&=0\\ c_3&=0 \end{aligned}$$

Conclusion: the c's are all zero, and so the set is linearly independent.

(b) The second set is dependent

$$1\cdot(\vec{u}-\vec{v})+1\cdot(\vec{v}-\vec{w})+1\cdot(\vec{w}-\vec{u})=\vec{0}$$

whether or not the first set is independent.

Two.II.1.26 **(a)** A singleton set $\{\vec{v}\}$ is linearly independent if and only if $\vec{v}\neq\vec{0}$. For the 'if' direction, with $\vec{v}\neq\vec{0}$, we can apply Lemma 1.4 by considering the relationship $c\cdot\vec{v}=\vec{0}$ and noting that the only solution is the trivial one: $c=0$. For the 'only if' direction, just recall that Example 1.11 shows that $\{\vec{0}\}$ is linearly dependent, and so if the set $\{\vec{v}\}$ is linearly independent then $\vec{v}\neq\vec{0}$.

(*Remark.* Another answer is to say that this is the special case of Lemma 1.16 where $S=\varnothing$.)

(b) A set with two elements is linearly independent if and only if neither member is a multiple of the other (note that if one is the zero vector then it is a multiple of the other, so this case is covered). This is an equivalent statement: a set is linearly dependent if and only if one element is a multiple of the other.

The proof is easy. A set $\{\vec{v}_1,\vec{v}_2\}$ is linearly dependent if and only if there is a relationship $c_1\vec{v}_1+c_2\vec{v}_2=\vec{0}$ with either $c_1\neq 0$ or $c_2\neq 0$ (or both). That holds if and only if $\vec{v}_1=(-c_2/c_1)\vec{v}_2$ or $\vec{v}_2=(-c_1/c_2)\vec{v}_1$ (or both).

Two.II.1.27 This set is linearly dependent set because it contains the zero vector.

Two.II.1.28 The 'if' half is given by Lemma 1.14. The converse (the 'only if' statement) does not hold. An example is to consider the vector space $\mathbb{R}^2$ and these vectors.

$$\vec{x}=\begin{pmatrix}1\\0\end{pmatrix},\quad \vec{y}=\begin{pmatrix}0\\1\end{pmatrix},\quad \vec{z}=\begin{pmatrix}1\\1\end{pmatrix}$$

Two.II.1.29 **(a)** The linear system arising from

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has the unique solution $c_1=0$ and $c_2=0$.

(b) The linear system arising from

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}3\\2\\0\end{pmatrix}$$

has the unique solution $c_1=8/3$ and $c_2=-1/3$.

(c) Suppose that S is linearly independent. Suppose that we have both $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n$ and $\vec{v}=d_1\vec{t}_1+\cdots+d_m\vec{t}_m$ (where the vectors are members of S). Now,

$$c_1\vec{s}_1+\cdots+c_n\vec{s}_n=\vec{v}=d_1\vec{t}_1+\cdots+d_m\vec{t}_m$$

can be rewritten in this way.

$$c_1\vec{s}_1+\cdots+c_n\vec{s}_n-d_1\vec{t}_1-\cdots-d_m\vec{t}_m=\vec{0}$$

Possibly some of the $\vec{s}$'s equal some of the $\vec{t}$'s; we can combine the associated coefficients (i.e., if $\vec{s}_i=\vec{t}_j$ then $\cdots+c_i\vec{s}_i+\cdots-d_j\vec{t}_j-\cdots$ can be rewritten as $\cdots+(c_i-d_j)\vec{s}_i+\cdots$). That equation is a linear relationship among distinct (after the combining is done) members of the set S. We've assumed that S is linearly independent, so all of the coefficients are zero. If i is such that $\vec{s}_i$ does not equal any $\vec{t}_j$ then c_i is zero. If j is such that $\vec{t}_j$ does not equal any $\vec{s}_i$ then d_j is zero. In the final case, we have that $c_i-d_j=0$ and so $c_i=d_j$.

Therefore, the original two sums are the same, except perhaps for some $0\cdot\vec{s}_i$ or $0\cdot\vec{t}_j$ terms that we can neglect.

(d) This set is not linearly independent:

$$S=\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\}\subset\mathbb{R}^2$$

and these two linear combinations give the same result

$$\begin{pmatrix}0\\0\end{pmatrix}=2\cdot\begin{pmatrix}1\\0\end{pmatrix}-1\cdot\begin{pmatrix}2\\0\end{pmatrix}=4\cdot\begin{pmatrix}1\\0\end{pmatrix}-2\cdot\begin{pmatrix}2\\0\end{pmatrix}$$

Thus, a linearly dependent set might have indistinct sums.

In fact, this stronger statement holds: if a set is linearly dependent then it must have the property that there are two distinct linear combinations that sum to the same vector. Briefly, where $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ then multiplying both sides of the relationship by two gives another relationship. If the first relationship is nontrivial then the second is also.

Two.II.1.30 In this 'if and only if' statement, the 'if' half is clear — if the polynomial is the zero polynomial then the function that arises from the action of the polynomial must be the zero function $x \mapsto 0$. For 'only if' we write $p(x) = c_nx^n + \cdots + c_0$. Plugging in zero $p(0) = 0$ gives that $c_0 = 0$. Taking the derivative and plugging in zero $p'(0) = 0$ gives that $c_1 = 0$. Similarly we get that each c_i is zero, and p is the zero polynomial.

Two.II.1.31 The work in this section suggests that an n-dimensional non-degenerate linear surface should be defined as the span of a linearly independent set of n vectors.

Two.II.1.32 **(a)** For any $a_{1,1}, \ldots, a_{2,4}$,

$$c_1\begin{pmatrix}a_{1,1}\\a_{2,1}\end{pmatrix} + c_2\begin{pmatrix}a_{1,2}\\a_{2,2}\end{pmatrix} + c_3\begin{pmatrix}a_{1,3}\\a_{2,3}\end{pmatrix} + c_4\begin{pmatrix}a_{1,4}\\a_{2,4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

yields a linear system

$$\begin{aligned}a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 &= 0\\ a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 &= 0\end{aligned}$$

that has infinitely many solutions (Gauss' method leaves at least two variables free). Hence there are nontrivial linear relationships among the given members of $\mathbb{R}^2$.

(b) Any set five vectors is a superset of a set of four vectors, and so is linearly dependent.

With three vectors from $\mathbb{R}^2$, the argument from the prior item still applies, with the slight change that Gauss' method now only leaves at least one variable free (but that still gives infintely many solutions).

(c) The prior item shows that no three-element subset of $\mathbb{R}^2$ is independent. We know that there are two-element subsets of $\mathbb{R}^2$ that are independent — one is

$$\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$$

and so the answer is two.

Two.II.1.33 Yes; here is one.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\}$$

Two.II.1.34 Yes. The two improper subsets, the entire set and the empty subset, serve as examples.

Two.II.1.35 In $\mathbb{R}^4$ the biggest linearly independent set has four vectors. There are many examples of such sets, this is one.

$$\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}$$

To see that no set with five or more vectors can be independent, set up

$$c_1\begin{pmatrix}a_{1,1}\\a_{2,1}\\a_{3,1}\\a_{4,1}\end{pmatrix} + c_2\begin{pmatrix}a_{1,2}\\a_{2,2}\\a_{3,2}\\a_{4,2}\end{pmatrix} + c_3\begin{pmatrix}a_{1,3}\\a_{2,3}\\a_{3,3}\\a_{4,3}\end{pmatrix} + c_4\begin{pmatrix}a_{1,4}\\a_{2,4}\\a_{3,4}\\a_{4,4}\end{pmatrix} + c_5\begin{pmatrix}a_{1,5}\\a_{2,5}\\a_{3,5}\\a_{4,5}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

and note that the resulting linear system

$$\begin{aligned}a_{1,1}c_1 + a_{1,2}c_2 + a_{1,3}c_3 + a_{1,4}c_4 + a_{1,5}c_5 &= 0\\ a_{2,1}c_1 + a_{2,2}c_2 + a_{2,3}c_3 + a_{2,4}c_4 + a_{2,5}c_5 &= 0\\ a_{3,1}c_1 + a_{3,2}c_2 + a_{3,3}c_3 + a_{3,4}c_4 + a_{3,5}c_5 &= 0\\ a_{4,1}c_1 + a_{4,2}c_2 + a_{4,3}c_3 + a_{4,4}c_4 + a_{4,5}c_5 &= 0\end{aligned}$$

has four equations and five unknowns, so Gauss' method must end with at least one c variable free, so there are infinitely many solutions, and so the above linear relationship among the four-tall vectors has more solutions than just the trivial solution.

The smallest linearly independent set is the empty set.

The biggest linearly dependent set is $\mathbb{R}^4$. The smallest is $\{\vec{0}\}$.

Two.II.1.36 **(a)** The intersection of two linearly independent sets $S \cap T$ must be linearly independent as it is a subset of the linearly independent set S (as well as the linearly independent set T also, of course).

(b) The complement of a linearly independent set is linearly dependent as it contains the zero vector.

(c) We must produce an example. One, in $\mathbb{R}^2$, is

$$S = \{\begin{pmatrix}1\\0\end{pmatrix}\} \quad \text{and} \quad T = \{\begin{pmatrix}2\\0\end{pmatrix}\}$$

since the linear dependence of $S_1 \cup S_2$ is easily seen.

(d) The union of two linearly independent sets $S \cup T$ is linearly independent if and only if their spans have a trivial intersection $[S] \cap [T] = \{\vec{0}\}$. To prove that, assume that S and T are linearly independent subsets of some vector space.

For the 'only if' direction, assume that the intersection of the spans is trivial $[S] \cap [T] = \{\vec{0}\}$. Consider the set $S \cup T$. Any linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ gives $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = -d_1\vec{t}_1 - \cdots - d_m\vec{t}_m$. The left side of that equation sums to a vector in $[S]$, and the right side is a vector in $[T]$. Therefore, since the intersection of the spans is trivial, both sides equal the zero vector. Because S is linearly independent, all of the c's are zero. Because T is linearly independent, all of the d's are zero. Thus, the original linear relationship among members of $S \cup T$ only holds if all of the coefficients are zero. That shows that $S \cup T$ is linearly independent.

For the 'if' half we can make the same argument in reverse. If the union $S \cup T$ is linearly independent, that is, if the only solution to $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ is the trivial solution $c_1 = 0, \ldots, d_m = 0$, then any vector $\vec{v}$ in the intersection of the spans $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = -d_1\vec{t}_1 - \cdots = d_m\vec{t}_m$ must be the zero vector because each scalar is zero.

Two.II.1.37 **(a)** We do induction on the number of vectors in the finite set S.

The base case is that S has no elements. In this case S is linearly independent and there is nothing to check — a subset of S that has the same span as S is S itself.

For the inductive step assume that the theorem is true for all sets of size $n = 0$, $n = 1$, …, $n = k$ in order to prove that it holds when S has $n = k+1$ elements. If the $k+1$-element set $S = \{\vec{s}_0, \ldots, \vec{s}_k\}$ is linearly independent then the theorem is trivial, so assume that it is dependent. By Corollary 1.17 there is an $\vec{s}_i$ that is a linear combination of other vectors in S. Define $S_1 = S - \{\vec{s}_i\}$ and note that S_1 has the same span as S by Lemma 1.1. The set S_1 has k elements and so the inductive hypothesis applies to give that it has a linearly independent subset with the same span. That subset of S_1 is the desired subset of S.

(b) Here is a sketch of the argument. The induction argument details have been left out.

If the finite set S is empty then there is nothing to prove. If $S = \{\vec{0}\}$ then the empty subset will do.

Otherwise, take some nonzero vector $\vec{s}_1 \in S$ and define $S_1 = \{\vec{s}_1\}$. If $[S_1] = [S]$ then this proof is finished by noting that S_1 is linearly independent.

If not, then there is a nonzero vector $\vec{s}_2 \in S - [S_1]$ (if every $\vec{s} \in S$ is in $[S_1]$ then $[S_1] = [S]$). Define $S_2 = S_1 \cup \{\vec{s}_2\}$. If $[S_2] = [S]$ then this proof is finished by using Theorem 1.17 to show that S_2 is linearly independent.

Repeat the last paragraph until a set with a big enough span appears. That must eventually happen because S is finite, and $[S]$ will be reached at worst when every vector from S has been used.

Two.II.1.38 **(a)** Assuming first that $a \neq 0$,

$$x\begin{pmatrix}a\\c\end{pmatrix} + y\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

gives

$$\begin{array}{rl} ax + by = 0 \\ cx + dy = 0 \end{array} \xrightarrow{-(c/a)\rho_1+\rho_2} \begin{array}{rr} ax + & by = 0 \\ & (-(c/a)b + d)y = 0 \end{array}$$

which has a solution if and only if $0 \neq -(c/a)b + d = (-cb + ad)/d$ (we've assumed in this case that $a \neq 0$, and so back substitution yields a unique solution).

The $a = 0$ case is also not hard — break it into the $c \neq 0$ and $c = 0$ subcases and note that in these cases $ad - bc = 0 \cdot d - bc$.

Comment. An earlier exercise showed that a two-vector set is linearly dependent if and only if either vector is a scalar multiple of the other. That can also be used to make the calculation.

(b) The equation

$$c_1\begin{pmatrix}a\\d\\g\end{pmatrix}+c_2\begin{pmatrix}b\\e\\h\end{pmatrix}+c_3\begin{pmatrix}c\\f\\i\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

gives rise to a homogeneous linear system. We proceed by writing it in matrix form and applying Gauss' method.

We first reduce the matrix to upper-triangular. Assume that $a\neq 0$.

$$\xrightarrow{(1/a)\rho_1}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\d&e&f&0\\g&h&i&0\end{array}\right)\xrightarrow[-g\rho_1+\rho_3]{-d\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ae-bd)/a&(af-cd)/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right)$$

$$\xrightarrow{(a/(ae-bd))\rho_2}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&1&(af-cd)/(ae-bd)&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right)$$

(where we've assumed for the moment that $ae-bd\neq 0$ in order to do the row reduction step). Then, under the assumptions, we get this.

$$\xrightarrow{((ah-bg)/a)\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&\frac{b}{a}&\frac{c}{a}&0\\0&1&\frac{af-cd}{ae-bd}&0\\0&0&\frac{aei+bgf+cdh-hfa-idb-gec}{ae-bd}&0\end{array}\right)$$

shows that the original system is nonsingular if and only if the $3,3$ entry is nonzero. This fraction is defined because of the $ae-bd\neq 0$ assumption, and it will equal zero if and only if its numerator equals zero.

We next worry about the assumptions. First, if $a\neq 0$ but $ae-bd=0$ then we swap

$$\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&0&(af-cd)/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\end{array}\right)\xrightarrow{\rho_2\leftrightarrow\rho_3}\left(\begin{array}{ccc|c}1&b/a&c/a&0\\0&(ah-bg)/a&(ai-cg)/a&0\\0&0&(af-cd)/a&0\end{array}\right)$$

and conclude that the system is nonsingular if and only if either $ah-bg=0$ or $af-cd=0$. That's the same as asking that their product be zero:

$$\begin{aligned}ahaf-ahcd-bgaf+bgcd&=0\\ahaf-ahcd-bgaf+aegc&=0\\a(haf-hcd-bgf+egc)&=0\end{aligned}$$

(in going from the first line to the second we've applied the case assumption that $ae-bd=0$ by substituting ae for bd). Since we are assuming that $a\neq 0$, we have that $haf-hcd-bgf+egc=0$. With $ae-bd=0$ we can rewrite this to fit the form we need: in this $a\neq 0$ and $ae-bd=0$ case, the given system is nonsingular when $haf-hcd-bgf+egc-i(ae-bd)=0$, as required.

The remaining cases have the same character. Do the $a=0$ but $d\neq 0$ case and the $a=0$ and $d=0$ but $g\neq 0$ case by first swapping rows and then going on as above. The $a=0$, $d=0$, and $g=0$ case is easy — a set with a zero vector is linearly dependent, and the formula comes out to equal zero.

(c) It is linearly dependent if and only if either vector is a multiple of the other. That is, it is not independent iff

$$\begin{pmatrix}a\\d\\g\end{pmatrix}=r\cdot\begin{pmatrix}b\\e\\h\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}b\\e\\h\end{pmatrix}=s\cdot\begin{pmatrix}a\\d\\g\end{pmatrix}$$

(or both) for some scalars r and s. Eliminating r and s in order to restate this condition only in terms of the given letters a, b, d, e, g, h, we have that it is not independent — it is dependent — iff $ae-bd=ah-gb=dh-ge$.

(d) Dependence or independence is a function of the indices, so there is indeed a formula (although at first glance a person might think the formula involves cases: "if the first component of the first vector is zero then ...", this guess turns out not to be correct).

Two.II.1.39 Recall that two vectors from $\mathbb{R}^n$ are perpendicular if and only if their dot product is zero.

(a) Assume that $\vec{v}$ and $\vec{w}$ are perpendicular nonzero vectors in $\mathbb{R}^n$, with $n>1$. With the linear relationship $c\vec{v}+d\vec{w}=\vec{0}$, apply $\vec{v}$ to both sides to conclude that $c\cdot\|\vec{v}\|^2+d\cdot 0=0$. Because $\vec{v}\neq\vec{0}$ we have that $c=0$. A similar application of $\vec{w}$ shows that $d=0$.

(b) Two vectors in $\mathbb{R}^1$ are perpendicular if and only if at least one of them is zero. We define $\mathbb{R}^0$ to be a trivial space, and so both $\vec{v}$ and $\vec{w}$ are the zero vector.

(c) The right generalization is to look at a set $\{\vec{v}_1,\ldots,\vec{v}_n\}\subseteq\mathbb{R}^k$ of vectors that are *mutually orthogonal* (also called *pairwise perpendicular*): if $i\neq j$ then $\vec{v}_i$ is perpendicular to $\vec{v}_j$. Mimicing the proof of the first item above shows that such a set of nonzero vectors is linearly independent.

Two.II.1.40 **(a)** This check is routine.

(b) The summation is infinite (has infinitely many summands). The definition of linear combination involves only finite sums.

(c) No nontrivial finite sum of members of $\{g, f_0, f_1, \ldots\}$ adds to the zero object: assume that

$$c_0\cdot(1/(1-x))+c_1\cdot 1+\cdots+c_n\cdot x^n=0$$

(any finite sum uses a highest power, here n). Multiply both sides by $1-x$ to conclude that each coefficient is zero, because a polynomial describes the zero function only when it is the zero polynomial.

Two.II.1.41 It is both 'if' and 'only if'.

Let T be a subset of the subspace S of the vector space V. The assertion that any linear relationship $c_1\vec{t}_1+\cdots+c_n\vec{t}_n=\vec{0}$ among members of T must be the trivial relationship $c_1=0,\ \ldots,\ c_n=0$ is a statement that holds in S if and only if it holds in V, because the subspace S inherits its addition and scalar multiplication operations from V.

Subsection Two.III.1: Basis

Two.III.1.16 By Theorem 1.12, each is a basis if and only if each vector in the space can be given in a unique way as a linear combination of the given vectors.

(a) Yes this is a basis. The relation

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}3\\2\\1\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives

$$\left(\begin{array}{ccc|c}1&3&0&x\\2&2&0&y\\3&1&1&z\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\xrightarrow{-2\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&3&0&x\\0&-4&0&-2x+y\\0&0&1&x-2y+z\end{array}\right)$$

which has the unique solution $c_3=x-2y+z$, $c_2=x/2-y/4$, and $c_1=-x/2+3y/4$.

(b) This is not a basis. Setting it up as in the prior item

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives a linear system whose solution

$$\left(\begin{array}{cc|c}1&3&x\\2&2&y\\3&1&z\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\xrightarrow{-2\rho_2+\rho_3}\left(\begin{array}{cc|c}1&3&x\\0&-4&-2x+y\\0&0&x-2y+z\end{array}\right)$$

is possible if and only if the three-tall vector's components x, y, and z satisfy $x-2y+z=0$. For instance, we can find the coefficients c_1 and c_2 that work when $x=1$, $y=1$, and $z=1$. However, there are no c's that work for $x=1$, $y=1$, and $z=2$. Thus this is not a basis; it does not span the space.

(c) Yes, this is a basis. Setting up the relationship leads to this reduction

$$\left(\begin{array}{ccc|c}0&1&2&x\\2&1&5&y\\-1&1&0&z\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_3}\xrightarrow{2\rho_1+\rho_2}\xrightarrow{-(1/3)\rho_2+\rho_3}\left(\begin{array}{ccc|c}-1&1&0&z\\0&3&5&y+2z\\0&0&1/3&x-y/3-2z/3\end{array}\right)$$

which has a unique solution for each triple of components x, y, and z.

(d) No, this is not a basis. The reduction

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & x \\ 2 & 1 & 3 & y \\ -1 & 1 & 0 & z \end{array}\right) \xrightarrow{\rho_1\leftrightarrow\rho_3} \xrightarrow{2\rho_1+\rho_2} \xrightarrow{(-1/3)\rho_2+\rho_3} \left(\begin{array}{ccc|c} -1 & 1 & 0 & z \\ 0 & 3 & 3 & y+2z \\ 0 & 0 & 0 & x-y/3-2z/3 \end{array}\right)$$

which does not have a solution for each triple x, y, and z. Instead, the span of the given set includes only those three-tall vectors where $x = y/3 + 2z/3$.

Two.III.1.17 **(a)** We solve

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

with

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \end{array}\right)$$

and conclude that $c_2 = 1/2$ and so $c_1 = 3/2$. Thus, the representation is this.

$$\text{Rep}_B(\begin{pmatrix}1\\2\end{pmatrix}) = \begin{pmatrix}3/2\\1/2\end{pmatrix}_B$$

(b) The relationship $c_1\cdot(1) + c_2\cdot(1+x) + c_3\cdot(1+x+x^2) + c_4\cdot(1+x+x^2+x^3) = x^2+x^3$ is easily solved by eye to give that $c_4 = 1$, $c_3 = 0$, $c_2 = -1$, and $c_1 = 0$.

$$\text{Rep}_D(x^2+x^3) = \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}_D$$

(c) $\text{Rep}_{\mathcal{E}_4}(\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}) = \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}_{\mathcal{E}_4}$

Two.III.1.18 A natural basis is $\langle 1, x, x^2\rangle$. There are bases for $\mathcal{P}_2$ that do not contain any polynomials of degree one or degree zero. One is $\langle 1+x+x^2, x+x^2, x^2\rangle$. (Every basis has at least one polynomial of degree two, though.)

Two.III.1.19 The reduction

$$\left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

gives that the only condition is that $x_1 = 4x_2 - 3x_3 + x_4$. The solution set is

$$\{\begin{pmatrix}4x_2-3x_3+x_4\\x_2\\x_3\\x_4\end{pmatrix} \mid x_2,x_3,x_4\in\mathbb{R}\} = \{x_2\begin{pmatrix}4\\1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}-3\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \mid x_2,x_3,x_4\in\mathbb{R}\}$$

and so the obvious candidate for the basis is this.

$$\langle\begin{pmatrix}4\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\rangle$$

We've shown that this spans the space, and showing it is also linearly independent is routine.

Two.III.1.20 There are many bases. This is a natural one.

$$\langle\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\rangle$$

Two.III.1.21 For each item, many answers are possible.

(a) One way to proceed is to parametrize by expressing the a_2 as a combination of the other two $a_2 = 2a_1 + a_0$. Then $a_2x^2 + a_1x + a_0$ is $(2a_1+a_0)x^2 + a_1x + a_0$ and

$$\{(2a_1+a_0)x^2 + a_1x + a_0 \mid a_1,a_0\in\mathbb{R}\} = \{a_1\cdot(2x^2+x) + a_0\cdot(x^2+1) \mid a_1,a_0\in\mathbb{R}\}$$

suggests $\langle 2x^2+x, x^2+1\rangle$. This only shows that it spans, but checking that it is linearly independent is routine.

(b) Parametrize $\{\begin{pmatrix}a&b&c\end{pmatrix} \mid a+b=0\}$ to get $\{\begin{pmatrix}-b&b&c\end{pmatrix} \mid b,c\in\mathbb{R}\}$, which suggests using the sequence $\langle\begin{pmatrix}-1&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\end{pmatrix}\rangle$. We've shown that it spans, and checking that it is linearly independent is easy.

(c) Rewriting

$$\{\begin{pmatrix} a & b \\ 0 & 2b \end{pmatrix} \mid a,b\in\mathbb{R}\} = \{a\cdot\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\cdot\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \mid a,b\in\mathbb{R}\}$$

suggests this for the basis.

$$\langle\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\rangle$$

Two.III.1.22 We will show that the second is a basis; the first is similar. We will show this straight from the definition of a basis, because this example appears before Theorem 1.12.

To see that it is linearly independent, we set up $c_1\cdot(\cos\theta-\sin\theta)+c_2\cdot(2\cos\theta+3\sin\theta) = 0\cos\theta+0\sin\theta$. Taking $\theta=0$ and $\theta=\pi/2$ gives this system

$$\begin{array}{rl} c_1\cdot 1+c_2\cdot 2=0 \\ c_1\cdot(-1)+c_2\cdot 3=0 \end{array} \xrightarrow{\rho_1+\rho_2} \begin{array}{rl} c_1+2c_2=0 \\ +5c_2=0 \end{array}$$

which shows that $c_1=0$ and $c_2=0$.

The calculation for span is also easy; for any $x,y\in\mathbb{R}$, we have that $c_1\cdot(\cos\theta-\sin\theta)+c_2\cdot(2\cos\theta+3\sin\theta)=x\cos\theta+y\sin\theta$ gives that $c_2=x/5+y/5$ and that $c_1=3x/5-2y/5$, and so the span is the entire space.

Two.III.1.23 **(a)** Asking which $a_0+a_1x+a_2x^2$ can be expressed as $c_1\cdot(1+x)+c_2\cdot(1+2x)$ gives rise to three linear equations, describing the coefficients of x^2, x, and the constants.

$$\begin{array}{rl} c_1+c_2&=a_0 \\ c_1+2c_2&=a_1 \\ 0&=a_2 \end{array}$$

Gauss' method with back-substitution shows, provided that $a_2=0$, that $c_2=-a_0+a_1$ and $c_1=2a_0-a_1$. Thus, with $a_2=0$, we can compute appropriate c_1 and c_2 for any a_0 and a_1. So the span is the entire set of linear polynomials $\{a_0+a_1x \mid a_0,a_1\in\mathbb{R}\}$. Parametrizing that set $\{a_0\cdot 1+a_1\cdot x \mid a_0,a_1\in\mathbb{R}\}$ suggests a basis $\langle 1,x\rangle$ (we've shown that it spans; checking linear independence is easy).

(b) With

$$a_0+a_1x+a_2x^2=c_1\cdot(2-2x)+c_2\cdot(3+4x^2)=(2c_1+3c_2)+(-2c_1)x+(4c_2)x^2$$

we get this system.

$$\begin{array}{rl} 2c_1+3c_2&=a_0 \\ -2c_1&=a_1 \\ 4c_2&=a_2 \end{array} \xrightarrow{\rho_1+\rho_2}\xrightarrow{(-4/3)\rho_2+\rho_3} \begin{array}{rl} 2c_1+3c_2&=a_0 \\ 3c_2&=a_0+a_1 \\ 0&=(-4/3)a_0-(4/3)a_1+a_2 \end{array}$$

Thus, the only quadratic polynomials $a_0+a_1x+a_2x^2$ with associated c's are the ones such that $0=(-4/3)a_0-(4/3)a_1+a_2$. Hence the span is $\{(-a_1+(3/4)a_2)+a_1x+a_2x^2 \mid a_1,a_2\in\mathbb{R}\}$. Parametrizing gives $\{a_1\cdot(-1+x)+a_2\cdot((3/4)+x^2) \mid a_1,a_2\in\mathbb{R}\}$, which suggests $\langle -1+x,(3/4)+x^2\rangle$ (checking that it is linearly independent is routine).

Two.III.1.24 **(a)** The subspace is $\{a_0+a_1x+a_2x^2+a_3x^3 \mid a_0+7a_1+49a_2+343a_3=0\}$. Rewriting $a_0=-7a_1-49a_2-343a_3$ gives $\{(-7a_1-49a_2-343a_3)+a_1x+a_2x^2+a_3x^3 \mid a_1,a_2,a_3\in\mathbb{R}\}$, which, on breaking out the parameters, suggests $\langle -7+x,-49+x^2,-343+x^3\rangle$ for the basis (it is easily verified).

(b) The given subspace is the collection of cubics $p(x)=a_0+a_1x+a_2x^2+a_3x^3$ such that $a_0+7a_1+49a_2+343a_3=0$ and $a_0+5a_1+25a_2+125a_3=0$. Gauss' method

$$\begin{array}{rl} a_0+7a_1+49a_2+343a_3=0 \\ a_0+5a_1+25a_2+125a_3=0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} a_0+7a_1+49a_2+343a_3=0 \\ -2a_1-24a_2-218a_3=0 \end{array}$$

gives that $a_1=-12a_2-109a_3$ and that $a_0=35a_2+420a_3$. Rewriting $(35a_2+420a_3)+(-12a_2-109a_3)x+a_2x^2+a_3x^3$ as $a_2\cdot(35-12x+x^2)+a_3\cdot(420-109x+x^3)$ suggests this for a basis $\langle 35-12x+x^2,420-109x+x^3\rangle$. The above shows that it spans the space. Checking it is linearly independent is routine. (*Comment.* A worthwhile check is to verify that both polynomials in the basis have both seven and five as roots.)

(c) Here there are three conditions on the cubics, that $a_0+7a_1+49a_2+343a_3=0$, that $a_0+5a_1+25a_2+125a_3=0$, and that $a_0+3a_1+9a_2+27a_3=0$. Gauss' method

$$\begin{array}{rl} a_0+7a_1+49a_2+343a_3=0 \\ a_0+5a_1+25a_2+125a_3=0 \\ a_0+3a_1+9a_2+27a_3=0 \end{array} \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{-2\rho_2+\rho_3} \begin{array}{rl} a_0+7a_1+49a_2+343a_3=0 \\ -2a_1-24a_2-218a_3=0 \\ 8a_2+120a_3=0 \end{array}$$

yields the single free variable a_3, with $a_2 = -15a_3$, $a_1 = 71a_3$, and $a_0 = -105a_3$. The parametrization is this.

$$\{(-105a_3) + (71a_3)x + (-15a_3)x^2 + (a_3)x^3 \bigm| a_3 \in \mathbb{R}\} = \{a_3 \cdot (-105 + 71x - 15x^2 + x^3) \bigm| a_3 \in \mathbb{R}\}$$

Therefore, a natural candidate for the basis is $\langle -105 + 71x - 15x^2 + x^3 \rangle$. It spans the space by the work above. It is clearly linearly independent because it is a one-element set (with that single element not the zero object of the space). Thus, any cubic through the three points $(7,0)$, $(5,0)$, and $(3,0)$ is a multiple of this one. (*Comment.* As in the prior question, a worthwhile check is to verify that plugging seven, five, and three into this polynomial yields zero each time.)

(d) This is the trivial subspace of $\mathcal{P}_3$. Thus, the basis is empty $\langle\rangle$.

Remark. The polynomial in the third item could alternatively have been derived by multiplying out $(x-7)(x-5)(x-3)$.

Two.III.1.25 Yes. Linear independence and span are unchanged by reordering.

Two.III.1.26 No linearly independent set contains a zero vector.

Two.III.1.27 **(a)** To show that it is linearly independent, note that $d_1(c_1\vec{\beta}_1)+d_2(c_2\vec{\beta}_2)+d_3(c_3\vec{\beta}_3) = \vec{0}$ gives that $(d_1c_1)\vec{\beta}_1 + (d_2c_2)\vec{\beta}_2 + (d_3c_3)\vec{\beta}_3 = \vec{0}$, which in turn implies that each d_ic_i is zero. But with $c_i \neq 0$ that means that each d_i is zero. Showing that it spans the space is much the same; because $\langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ is a basis, and so spans the space, we can for any $\vec{v}$ write $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$, and then $\vec{v} = (d_1/c_1)(c_1\vec{\beta}_1) + (d_2/c_2)(c_2\vec{\beta}_2) + (d_3/c_3)(c_3\vec{\beta}_3)$.

If any of the scalars are zero then the result is not a basis, because it is not linearly independent.

(b) Showing that $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ is linearly independent is easy. To show that it spans the space, assume that $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$. Then, we can represent the same $\vec{v}$ with respect to $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3\rangle$ in this way $\vec{v} = (1/2)(d_1 - d_2 - d_3)(2\vec{\beta}_1) + d_2(\vec{\beta}_1 + \vec{\beta}_2) + d_3(\vec{\beta}_1 + \vec{\beta}_3)$.

Two.III.1.28 Each forms a linearly independent set if $\vec{v}$ is ommitted. To preserve linear independence, we must expand the span of each. That is, we must determine the span of each (leaving $\vec{v}$ out), and then pick a $\vec{v}$ lying outside of that span. Then to finish, we must check that the result spans the entire given space. Those checks are routine.

(a) Any vector that is not a multiple of the given one, that is, any vector that is not on the line $y = x$ will do here. One is $\vec{v} = \vec{e}_1$.

(b) By inspection, we notice that the vector $\vec{e}_3$ is not in the span of the set of the two given vectors. The check that the resulting set is a basis for $\mathbb{R}^3$ is routine.

(c) For any member of the span $\{c_1 \cdot (x) + c_2 \cdot (1 + x^2) \bigm| c_1, c_2 \in \mathbb{R}\}$, the coefficient of x^2 equals the constant term. So we expand the span if we add a quadratic without this property, say, $\vec{v} = 1 - x^2$. The check that the result is a basis for $\mathcal{P}_2$ is easy.

Two.III.1.29 To show that each scalar is zero, simply subtract $c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k - c_{k+1}\vec{\beta}_{k+1} - \cdots - c_n\vec{\beta}_n = \vec{0}$. The obvious generalization is that in any equation involving only the $\vec{\beta}$'s, and in which each $\vec{\beta}$ appears only once, each scalar is zero. For instance, an equation with a combination of the even-indexed basis vectors (i.e., $\vec{\beta}_2$, $\vec{\beta}_4$, etc.) on the right and the odd-indexed basis vectors on the left also gives the conclusion that all of the coefficients are zero.

Two.III.1.30 No; no linearly independent set contains the zero vector.

Two.III.1.31 Here is a subset of $\mathbb{R}^2$ that is not a basis, and two different linear combinations of its elements that sum to the same vector.

$$\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\} \qquad 2 \cdot \begin{pmatrix}1\\2\end{pmatrix} + 0 \cdot \begin{pmatrix}2\\4\end{pmatrix} = 0 \cdot \begin{pmatrix}1\\2\end{pmatrix} + 1 \cdot \begin{pmatrix}2\\4\end{pmatrix}$$

Thus, when a subset is not a basis, it can be the case that its linear combinations are not unique.

But just because a subset is not a basis does not imply that its combinations must be not unique. For instance, this set

$$\{\begin{pmatrix}1\\2\end{pmatrix}\}$$

does have the property that

$$c_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} = c_2 \cdot \begin{pmatrix}1\\2\end{pmatrix}$$

implies that $c_1 = c_2$. The idea here is that this subset fails to be a basis because it fails to span the space; the proof of the theorem establishes that linear combinations are unique if and only if the subset is linearly independent.

Two.III.1.32 **(a)** Describing the vector space as

$$\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

suggests this for a basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$$

Verification is easy.

(b) This is one possible basis.

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rangle$$

(c) As in the prior two questions, we can form a basis from two kinds of matrices. First are the matrices with a single one on the diagonal and all other entries zero (there are n of those matrices). Second are the matrices with two opposed off-diagonal entries are ones and all other entries are zeros. (That is, all entries in M are zero except that $m_{i,j}$ and $m_{j,i}$ are one.)

Two.III.1.33 **(a)** Any four vectors from $\mathbb{R}^3$ are linearly related because the vector equation

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + c_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} + c_4 \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to a linear system

$$\begin{aligned} x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 &= 0 \\ y_1c_1 + y_2c_2 + y_3c_3 + y_4c_4 &= 0 \\ z_1c_1 + z_2c_2 + z_3c_3 + z_4c_4 &= 0 \end{aligned}$$

that is homogeneous (and so has a solution) and has four unknowns but only three equations, and therefore has nontrivial solutions. (Of course, this argument applies to any subset of $\mathbb{R}^3$ with four or more vectors.)

(b) Given $x_1, \ldots, z_2$,

$$S = \{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\}$$

to decide which vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

are in the span of S, set up

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and row reduce the resulting system.

$$\begin{aligned} x_1c_1 + x_2c_2 &= x \\ y_1c_1 + y_2c_2 &= y \\ z_1c_1 + z_2c_2 &= z \end{aligned}$$

There are two variables c_1 and c_2 but three equations, so when Gauss' method finishes, on the bottom row there will be some relationship of the form $0 = m_1x + m_2y + m_3z$. Hence, vectors in the span of the two-element set S must satisfy some restriction. Hence the span is not all of $\mathbb{R}^3$.

Two.III.1.34 We have (using these peculiar operations with care)

$$\{\begin{pmatrix} 1-y-z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R}\} = \{\begin{pmatrix} -y+1 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z+1 \\ 0 \\ z \end{pmatrix} \mid y, z \in \mathbb{R}\} = \{y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R}\}$$

and so a natural candidate for a basis is this.

$$\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

To check linear independence we set up

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(the vector on the right is the zero object in this space). That yields the linear system

$$\begin{aligned} (-c_1 + 1) + (-c_2 + 1) - 1 &= 1 \\ c_1 \qquad\qquad &= 0 \\ c_2 &= 0 \end{aligned}$$

with only the solution $c_1 = 0$ and $c_2 = 0$. Checking the span is similar.

Subsection Two.III.2: Dimension

Two.III.2.14 One basis is $\langle 1, x, x^2 \rangle$, and so the dimension is three.

Two.III.2.15 The solution set is

$$\{ \begin{pmatrix} 4x_2 - 3x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid x_2, x_3, x_4 \in \mathbb{R} \}$$

so a natural basis is this

$$\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

(checking linear independence is easy). Thus the dimension is three.

Two.III.2.16 For this space

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \} = \{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \cdots + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \}$$

this is a natural basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The dimension is four.

Two.III.2.17 **(a)** As in the prior exercise, the space $\mathcal{M}_{2\times 2}$ of matrices without restriction has this basis

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

and so the dimension is four.

(b) For this space

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = b - 2c \text{ and } d \in \mathbb{R} \} = \{ b \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid b, c, d \in \mathbb{R} \}$$

this is a natural basis.

$$\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The dimension is three.

(c) Gauss' method applied to the two-equation linear system gives that $c = 0$ and that $a = -b$. Thus, we have this description

$$\{ \begin{pmatrix} -b & b \\ 0 & d \end{pmatrix} \mid b, d \in \mathbb{R} \} = \{ b \cdot \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid b, d \in \mathbb{R} \}$$

and so this is a natural basis.

$$\langle \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The dimension is two.

Two.III.2.18 The bases for these spaces are developed in the answer set of the prior subsection.

(a) One basis is $\langle -7+x, -49+x^2, -343+x^3\rangle$. The dimension is three.
(b) One basis is $\langle 35-12x+x^2, 420-109x+x^3\rangle$ so the dimension is two.
(c) A basis is $\{-105+71x-15x^2+x^3\}$. The dimension is one.
(d) This is the trivial subspace of $\mathcal{P}_3$ and so the basis is empty. The dimension is zero.

Two.III.2.19 First recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and so deletion of $\cos 2\theta$ from this set leaves the span unchanged. What's left, the set $\{\cos^2\theta, \sin^2\theta, \sin 2\theta\}$, is linearly independent (consider the relationship $c_1\cos^2\theta + c_2\sin^2\theta + c_3\sin 2\theta = Z(\theta)$ where Z is the zero function, and then take $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$ to conclude that each c is zero). It is therefore a basis for its span. That shows that the span is a dimension three vector space.

Two.III.2.20 Here is a basis

$$\langle (1+0i, 0+0i, \ldots, 0+0i),\ (0+1i, 0+0i, \ldots, 0+0i), (0+0i, 1+0i, \ldots, 0+0i), \ldots\rangle$$

and so the dimension is $2\cdot 47 = 94$.

Two.III.2.21 A basis is

$$\langle \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \ldots, \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{pmatrix} \rangle$$

and thus the dimension is $3\cdot 5 = 15$.

Two.III.2.22 In a four-dimensional space a set of four vectors is linearly independent if and only if it spans the space. The form of these vectors makes linear independence easy to show (look at the equation of fourth components, then at the equation of third components, etc.).

Two.III.2.23 **(a)** The diagram for $\mathcal{P}_2$ has four levels. The top level has the only three-dimensional subspace, $\mathcal{P}_2$ itself. The next level contains the two-dimensional subspaces (*not* just the linear polynomials; any two-dimensional subspace, like those polynomials of the form ax^2+b). Below that are the one-dimensional subspaces. Finally, of course, is the only zero-dimensional subspace, the trivial subspace.
(b) For $\mathcal{M}_{2\times 2}$, the diagram has five levels, including subspaces of dimension four through zero.

Two.III.2.24 **(a)** One **(b)** Two **(c)** n

Two.III.2.25 We need only produce an infinite linearly independent set. One is $\langle f_1, f_2, \ldots\rangle$ where $f_i\colon \mathbb{R}\to\mathbb{R}$ is

$$f_i(x) = \begin{cases} 1 & \text{if } x = i \\ 0 & \text{otherwise} \end{cases}$$

the function that has value 1 only at $x = i$.

Two.III.2.26 Considering a function to be a set, specifically, a set of ordered pairs $(x, f(x))$, then the only function with an empty domain is the empty set. Thus this is a trivial vector space, and has dimension zero.

Two.III.2.27 Apply Corollary 2.8.

Two.III.2.28 A plane has the form $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 \mid t_1, t_2 \in \mathbb{R}\}$. (The first chapter also calls this a '2-flat', and contains a discussion of why this is equivalent to the description often taken in Calculus as the set of points (x, y, z) subject to a condition of the form $ax + by + cz = d$). When the plane passes through the origin we can take the particular vector $\vec{p}$ to be $\vec{0}$. Thus, in the language we have developed in this chapter, a plane through the origin is the span of a set of two vectors.

Now for the statement. Asserting that the three are not coplanar is the same as asserting that no vector lies in the span of the other two — no vector is a linear combination of the other two. That's simply an assertion that the three-element set is linearly independent. By Corollary 2.12, that's equivalent to an assertion that the set is a basis for $\mathbb{R}^3$.

Two.III.2.29 Let the space V be finite dimensional. Let S be a subspace of V.
(a) The empty set is a linearly independent subset of S. By Corollary 2.10, it can be expanded to a basis for the vector space S.
(b) Any basis for the subspace S is a linearly independent set in the superspace V. Hence it can be expanded to a basis for the superspace, which is finite dimensional. Therefore it has only finitely many members.

Two.III.2.30 It ensures that we exhaust the $\vec{\beta}$'s. That is, it justifies the first sentence of the last paragraph.

Two.III.2.31 Let B_U be a basis for U and let B_W be a basis for W. The set $B_U \cup B_W$ is linearly dependent as it is a six member subset of the five-dimensional space $\mathbb{R}^5$. Thus some member of B_W is in the span of B_U, and thus $U \cap W$ is more than just the trivial space $\{\vec{0}\}$.

Generalization: if U, W are subspaces of a vector space of dimension n and if $\dim(U)+\dim(W) > n$ then they have a nontrivial intersection.

Two.III.2.32 First, note that a set is a basis for some space if and only if it is linearly independent, because in that case it is a basis for its own span.

(a) The answer to the question in the second paragraph is "yes" (implying "yes" answers for both questions in the first paragraph). If B_U is a basis for U then B_U is a linearly independent subset of W. Apply Corollary 2.10 to expand it to a basis for W. That is the desired B_W.

The answer to the question in the third paragraph is "no", which implies a "no" answer to the question of the fourth paragraph. Here is an example of a basis for a superspace with no sub-basis forming a basis for a subspace: in $W = \mathbb{R}^2$, consider the standard basis $\mathcal{E}_2$. No sub-basis of $\mathcal{E}_2$ forms a basis for the subspace U of $\mathbb{R}^2$ that is the line $y = x$.

(b) It is a basis (for its span) because the intersection of linearly independent sets is linearly independent (the intersection is a subset of each of the linearly independent sets).

It is not, however, a basis for the intersection of the spaces. For instance, these are bases for $\mathbb{R}^2$:

$$B_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle \quad \text{and} \quad B_2 = \langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

and $\mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$, but $B_1 \cap B_2$ is empty. All we can say is that the intersection of the bases is a basis for a subset of the intersection of the spaces.

(c) The union of bases need not be a basis: in $\mathbb{R}^2$

$$B_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \quad \text{and} \quad B_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

have a union $B_1 \cup B_2$ that is not linearly independent. A necessary and sufficient condition for a union of two bases to be a basis

$$B_1 \cup B_2 \text{ is linearly independent} \iff [B_1 \cap B_2] = [B_1] \cap [B_2]$$

it is easy enough to prove (but perhaps hard to apply).

(d) The complement of a basis cannot be a basis because it contains the zero vector.

Two.III.2.33 **(a)** A basis for U is a linearly independent set in W and so can be expanded via Corollary 2.10 to a basis for W. The second basis has at least as many members as the first.

(b) One direction is clear: if $V = W$ then they have the same dimension. For the converse, let B_U be a basis for U. It is a linearly independent subset of W and so can be expanded to a basis for W. If $\dim(U) = \dim(W)$ then this basis for W has no more members than does B_U and so equals B_U. Since U and W have the same bases, they are equal.

(c) Let W be the space of finite-degree polynomials and let U be the subspace of polynomails that have only even-powered terms $\{a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n} \mid a_0, \ldots, a_n \in \mathbb{R}\}$. Both spaces have infinite dimension, but U is a proper subspace.

Two.III.2.34 The possibilities for the dimension of V are 0, 1, $n-1$, and n.

To see this, first consider the case when all the coordinates of $\vec{v}$ are equal.

$$\vec{v} = \begin{pmatrix} z \\ z \\ \vdots \\ z \end{pmatrix}$$

Then $\sigma(\vec{v}) = \vec{v}$ for every permutation σ, so V is just the span of $\vec{v}$, which has dimension 0 or 1 according to whether $\vec{v}$ is $\vec{0}$ or not.

Now suppose not all the coordinates of $\vec{v}$ are equal; let x and y with $x \neq y$ be among the coordinates of $\vec{v}$. Then we can find permutations σ_1 and σ_2 such that

$$\sigma_1(\vec{v}) = \begin{pmatrix} x \\ y \\ a_3 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \sigma_2(\vec{v}) = \begin{pmatrix} y \\ x \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$$

for some $a_3,\ldots,a_n \in \mathbb{R}$. Therefore,

$$\frac{1}{y-x}\left(\sigma_1(\vec{v})-\sigma_2(\vec{v})\right)=\begin{pmatrix}-1\\1\\0\\\vdots\\0\end{pmatrix}$$

is in V. That is, $\vec{e}_2-\vec{e}_1 \in V$, where $\vec{e}_1$, $\vec{e}_2$, …, $\vec{e}_n$ is the standard basis for $\mathbb{R}^n$. Similarly, $\vec{e}_3-\vec{e}_2$, …, $\vec{e}_n-\vec{e}_1$ are all in V. It is easy to see that the vectors $\vec{e}_2-\vec{e}_1$, $\vec{e}_3-\vec{e}_2$, …, $\vec{e}_n-\vec{e}_1$ are linearly independent (that is, form a linearly independent set), so $\dim V \geq n-1$.

Finally, we can write

$$\begin{aligned}\vec{v}&=x_1\vec{e}_1+x_2\vec{e}_2+\cdots+x_n\vec{e}_n\\&=(x_1+x_2+\cdots+x_n)\vec{e}_1+x_2(\vec{e}_2-\vec{e}_1)+\cdots+x_n(\vec{e}_n-\vec{e}_1)\end{aligned}$$

This shows that if $x_1+x_2+\cdots+x_n=0$ then $\vec{v}$ is in the span of $\vec{e}_2-\vec{e}_1$, …, $\vec{e_n}-\vec{e}_1$ (that is, is in the span of the set of those vectors); similarly, each $\sigma(\vec{v})$ will be in this span, so V will equal this span and $\dim V=n-1$. On the other hand, if $x_1+x_2+\cdots+x_n\neq 0$ then the above equation shows that $\vec{e}_1\in V$ and thus $\vec{e}_1,\ldots,\vec{e}_n\in V$, so $V=\mathbb{R}^n$ and $\dim V=n$.

Subsection Two.III.3: Vector Spaces and Linear Systems

Two.III.3.16 **(a)** $\begin{pmatrix}2&3\\1&1\end{pmatrix}$ **(b)** $\begin{pmatrix}2&1\\1&3\end{pmatrix}$ **(c)** $\begin{pmatrix}1&6\\4&7\\3&8\end{pmatrix}$ **(d)** $\begin{pmatrix}0&0&0\end{pmatrix}$ **(e)** $\begin{pmatrix}-1\\-2\end{pmatrix}$

Two.III.3.17 **(a)** Yes. To see if there are c_1 and c_2 such that $c_1\cdot\begin{pmatrix}2&1\end{pmatrix}+c_2\cdot\begin{pmatrix}3&1\end{pmatrix}=\begin{pmatrix}1&0\end{pmatrix}$ we solve

$$\begin{aligned}2c_1+3c_2&=1\\c_1+c_2&=0\end{aligned}$$

and get $c_1=-1$ and $c_2=1$. Thus the vector is in the row space.
(b) No. The equation $c_1\begin{pmatrix}0&1&3\end{pmatrix}+c_2\begin{pmatrix}-1&0&1\end{pmatrix}+c_3\begin{pmatrix}-1&2&7\end{pmatrix}=\begin{pmatrix}1&1&1\end{pmatrix}$ has no solution.

$$\left(\begin{array}{ccc|c}0&-1&-1&1\\1&0&2&1\\3&1&7&1\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_2}\xrightarrow{-3\rho_1+\rho_2}\xrightarrow{\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&0&2&1\\0&-1&-1&1\\0&0&0&-1\end{array}\right)$$

Thus, the vector is not in the row space.

Two.III.3.18 **(a)** No. To see if there are $c_1,c_2\in\mathbb{R}$ such that

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

we can use Gauss' method on the resulting linear system.

$$\begin{aligned}c_1+c_2&=1\\c_1+c_2&=3\end{aligned}\;\xrightarrow{-\rho_1+\rho_2}\;\begin{aligned}c_1+c_2&=1\\0&=2\end{aligned}$$

There is no solution and so the vector is not in the column space.
(b) Yes. From this relationship

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2\begin{pmatrix}3\\0\\-3\end{pmatrix}+c_3\begin{pmatrix}1\\4\\3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

we get a linear system that, when Gauss' method is applied,

$$\left(\begin{array}{ccc|c}1&3&1&1\\2&0&4&0\\1&-3&-3&0\end{array}\right)\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&3&1&1\\0&-6&2&-2\\0&0&-6&1\end{array}\right)$$

yields a solution. Thus, the vector is in the column space.

Two.III.3.19 A routine Gaussian reduction

$$\begin{pmatrix}2&0&3&4\\0&1&1&-1\\3&1&0&2\\1&0&-4&1\end{pmatrix}\xrightarrow[-(1/2)\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3}\xrightarrow{-\rho_2+\rho_3}\xrightarrow{-\rho_3+\rho_4}\begin{pmatrix}2&0&3&4\\0&1&1&-1\\0&0&-11/2&-3\\0&0&0&0\end{pmatrix}$$

suggests this basis $\langle\begin{pmatrix}2&0&3&4\end{pmatrix},\begin{pmatrix}0&1&1&-1\end{pmatrix},\begin{pmatrix}0&0&-11/2&-3\end{pmatrix}\rangle$.

Another, perhaps more convenient procedure, is to swap rows first,

$$\xrightarrow{\rho_1\leftrightarrow\rho_4}\xrightarrow[-2\rho_1+\rho_4]{-3\rho_1+\rho_3}\xrightarrow{-\rho_2+\rho_3}\xrightarrow{-\rho_3+\rho_4}\begin{pmatrix}1&0&-4&-1\\0&1&1&-1\\0&0&11&6\\0&0&0&0\end{pmatrix}$$

leading to the basis $\langle\begin{pmatrix}1&0&-4&-1\end{pmatrix},\begin{pmatrix}0&1&1&-1\end{pmatrix},\begin{pmatrix}0&0&11&6\end{pmatrix}\rangle$.

Two.III.3.20 **(a)** This reduction

$$\xrightarrow[-(1/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-(1/3)\rho_2+\rho_3}\begin{pmatrix}2&1&3\\0&-3/2&1/2\\0&0&4/3\end{pmatrix}$$

shows that the row rank, and hence the rank, is three.

(b) Inspection of the columns shows that that the others are multiples of the first (inspection of the rows shows the same thing). Thus the rank is one.

Alternatively, the reduction

$$\begin{pmatrix}1&-1&2\\3&-3&6\\-2&2&-4\end{pmatrix}\xrightarrow[2\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}1&-1&2\\0&0&0\\0&0&0\end{pmatrix}$$

shows the same thing.

(c) This calculation

$$\begin{pmatrix}1&3&2\\5&1&1\\6&4&3\end{pmatrix}\xrightarrow[-6\rho_1+\rho_3]{-5\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&3&2\\0&-14&-9\\0&0&0\end{pmatrix}$$

shows that the rank is two.

(d) The rank is zero.

Two.III.3.21 **(a)** This reduction

$$\begin{pmatrix}1&3\\-1&3\\1&4\\2&1\end{pmatrix}\xrightarrow[\substack{-\rho_1+\rho_3\\-2\rho_1+\rho_4}]{\rho_1+\rho_2}\xrightarrow[(5/6)\rho_2+\rho_4]{-(1/6)\rho_2+\rho_3}\begin{pmatrix}1&3\\0&6\\0&0\\0&0\end{pmatrix}$$

gives $\langle\begin{pmatrix}1&3\end{pmatrix},\begin{pmatrix}0&6\end{pmatrix}\rangle$.

(b) Transposing and reducing

$$\begin{pmatrix}1&2&1\\3&1&-1\\1&-3&-3\end{pmatrix}\xrightarrow[-\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}1&2&1\\0&-5&-4\\0&-5&-4\end{pmatrix}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&2&1\\0&-5&-4\\0&0&0\end{pmatrix}$$

and then transposing back gives this basis.

$$\langle\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\-5\\-4\end{pmatrix}\rangle$$

(c) Notice first that the surrounding space is given as $\mathcal{P}_3$, not $\mathcal{P}_2$. Then, taking the first polynomial $1+1\cdot x+0\cdot x^2+0\cdot x^3$ to be "the same" as the row vector $\begin{pmatrix}1&1&0&0\end{pmatrix}$, etc., leads to

$$\begin{pmatrix}1&1&0&0\\1&0&-1&0\\3&2&-1&0\end{pmatrix}\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&1&0&0\\0&-1&-1&0\\0&0&0&0\end{pmatrix}$$

which yields the basis $\langle 1+x,-x-x^2\rangle$.

(d) Here "the same" gives

$$\begin{pmatrix}1&0&1&3&1&-1\\1&0&3&2&1&4\\-1&0&-5&-1&-1&-9\end{pmatrix}\xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{2\rho_2+\rho_3}\begin{pmatrix}1&0&1&3&1&-1\\0&0&2&-1&0&5\\0&0&0&0&0&0\end{pmatrix}$$

leading to this basis.

$$\langle\begin{pmatrix}1&0&1\\3&1&-1\end{pmatrix},\begin{pmatrix}0&0&2\\-1&0&5\end{pmatrix}\rangle$$

Two.III.3.22 Only the zero matrices have rank of zero. The only matrices of rank one have the form

$$\begin{pmatrix}k_1\cdot\rho\\ \vdots\\ k_m\cdot\rho\end{pmatrix}$$

where ρ is some nonzero row vector, and not all of the k_i's are zero. (*Remark.* We can't simply say that all of the rows are multiples of the first because the first row might be the zero row. *Another Remark.* The above also applies with 'column' replacing 'row'.)

Two.III.3.23 If $a\neq 0$ then a choice of $d=(c/a)b$ will make the second row be a multiple of the first, specifically, c/a times the first. If $a=0$ and $b=0$ then any non-0 choice for d will ensure that the second row is nonzero. If $a=0$ and $b\neq 0$ and $c=0$ then any choice for d will do, since the matrix will automatically have rank one (even with the choice of $d=0$). Finally, if $a=0$ and $b\neq 0$ and $c\neq 0$ then no choice for d will suffice because the matrix is sure to have rank two.

Two.III.3.24 The column rank is two. One way to see this is by inspection — the column space consists of two-tall columns and so can have a dimension of at least two, and we can easily find two columns that together form a linearly independent set (the fourth and fifth columns, for instance). Another way to see this is to recall that the column rank equals the row rank, and to perform Gauss' method, which leaves two nonzero rows.

Two.III.3.25 We apply Theorem 3.13. The number of columns of a matrix of coefficients A of a linear system equals the number n of unknowns. A linear system with at least one solution has at most one solution if and only if the space of solutions of the associated homogeneous system has dimension zero (recall: in the 'General = Particular + Homogeneous' equation $\vec{v}=\vec{p}+\vec{h}$, provided that such a $\vec{p}$ exists, the solution $\vec{v}$ is unique if and only if the vector $\vec{h}$ is unique, namely $\vec{h}=\vec{0}$). But that means, by the theorem, that $n=r$.

Two.III.3.26 The set of columns must be dependent because the rank of the matrix is at most five while there are nine columns.

Two.III.3.27 There is little danger of their being equal since the row space is a set of row vectors while the column space is a set of columns (unless the matrix is 1×1, in which case the two spaces must be equal).

Remark. Consider

$$A=\begin{pmatrix}1&3\\2&6\end{pmatrix}$$

and note that the row space is the set of all multiples of $\begin{pmatrix}1&3\end{pmatrix}$ while the column space consists of multiples of

$$\begin{pmatrix}1\\2\end{pmatrix}$$

so we also cannot argue that the two spaces must be simply transposes of each other.

Two.III.3.28 First, the vector space is the set of four-tuples of real numbers, under the natural operations. Although this is not the set of four-wide row vectors, the difference is slight — it is "the same" as that set. So we will treat the four-tuples like four-wide vectors.

With that, one way to see that $(1,0,1,0)$ is not in the span of the first set is to note that this reduction

$$\begin{pmatrix}1&-1&2&-3\\1&1&2&0\\3&-1&6&-6\end{pmatrix}\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&-1&2&-3\\0&2&0&3\\0&0&0&0\end{pmatrix}$$

and this one

$$\begin{pmatrix}1&-1&2&-3\\1&1&2&0\\3&-1&6&-6\\1&0&1&0\end{pmatrix}\xrightarrow[\substack{-3\rho_1+\rho_3\\-\rho_1+\rho_4}]{-\rho_1+\rho_2}\xrightarrow[-(1/2)\rho_2+\rho_4]{-\rho_2+\rho_3}\xrightarrow{\rho_3\leftrightarrow\rho_4}\begin{pmatrix}1&-1&2&-3\\0&2&0&3\\0&0&-1&3/2\\0&0&0&0\end{pmatrix}$$

yield matrices differing in rank. This means that addition of $(1,0,1,0)$ to the set of the first three four-tuples increases the rank, and hence the span, of that set. Therefore $(1,0,1,0)$ is not already in the span.

Two.III.3.29 It is a subspace because it is the column space of the matrix

$$\begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

of coefficients. To find a basis for the column space,

$$\{c_1\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} \bigm| c_1, c_2, c_3 \in \mathbb{R}\}$$

we take the three vectors from the spanning set, transpose, reduce,

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \\ 4 & -1 & 5 \end{pmatrix} \xrightarrow[-(4/3)\rho_1+\rho_3]{-(2/3)\rho_1+\rho_2} \xrightarrow{-(7/2)\rho_2+\rho_3} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -2/3 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

and transpose back to get this.

$$\langle \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2/3 \\ 2/3 \end{pmatrix} \rangle$$

Two.III.3.30 This can be done as a straightforward calculation.

$$\begin{aligned}
(rA+sB)^{\text{trans}} &= \begin{pmatrix} ra_{1,1}+sb_{1,1} & \ldots & ra_{1,n}+sb_{1,n} \\ \vdots & & \vdots \\ ra_{m,1}+sb_{m,1} & \ldots & ra_{m,n}+sb_{m,n} \end{pmatrix}^{\text{trans}} \\
&= \begin{pmatrix} ra_{1,1}+sb_{1,1} & \ldots & ra_{m,1}+sb_{m,1} \\ \vdots & & \\ ra_{1,n}+sb_{1,n} & \ldots & ra_{m,n}+sb_{m,n} \end{pmatrix} \\
&= \begin{pmatrix} ra_{1,1} & \ldots & ra_{m,1} \\ \vdots & & \\ ra_{1,n} & \ldots & ra_{m,n} \end{pmatrix} + \begin{pmatrix} sb_{1,1} & \ldots & sb_{m,1} \\ \vdots & & \\ sb_{1,n} & \ldots & sb_{m,n} \end{pmatrix} \\
&= rA^{\text{trans}} + sB^{\text{trans}}
\end{aligned}$$

Two.III.3.31 **(a)** These reductions give different bases.

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \xrightarrow{2\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) An easy example is this.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

This is a less simplistic example.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 4 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

(c) Assume that A and B are matrices with equal row spaces. Construct a matrix C with the rows of A above the rows of B, and another matrix D with the rows of B above the rows of A.

$$C = \begin{pmatrix} A \\ B \end{pmatrix} \qquad D = \begin{pmatrix} B \\ A \end{pmatrix}$$

Observe that C and D are row-equivalent (via a sequence of row-swaps) and so Gauss-Jordan reduce to the same reduced echelon form matrix.

Because the row spaces are equal, the rows of B are linear combinations of the rows of A so Gauss-Jordan reduction on C simply turns the rows of B to zero rows and thus the nonzero rows of C are just the nonzero rows obtained by Gauss-Jordan reducing A. The same can be said for the matrix D — Gauss-Jordan reduction on D gives the same non-zero rows as are produced by reduction on B alone. Therefore, A yields the same nonzero rows as C, which yields the same nonzero rows as D, which yields the same nonzero rows as B.

Two.III.3.32 It cannot be bigger.

Two.III.3.33 The number of rows in a maximal linearly independent set cannot exceed the number of rows. A better bound (the bound that is, in general, the best possible) is the minimum of m and n, because the row rank equals the column rank.

Two.III.3.34 Because the rows of a matrix A are turned into the columns of A^{trans} the dimension of the row space of A equals the dimension of the column space of A^{trans}. But the dimension of the row space of A is the rank of A and the dimension of the column space of A^{trans} is the rank of A^{trans}. Thus the two ranks are equal.

Two.III.3.35 False. The first is a set of columns while the second is a set of rows.

This example, however,

$$A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix},\qquad A^{\text{trans}}=\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$$

indicates that as soon as we have a formal meaning for "the same", we can apply it here:

$$\text{Columnspace}(A)=[\{\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}2\\5\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}\}]$$

while

$$\text{Rowspace}(A^{\text{trans}})=[\{\begin{pmatrix}1&4\end{pmatrix},\begin{pmatrix}2&5\end{pmatrix},\begin{pmatrix}3&6\end{pmatrix}\}]$$

are "the same" as each other.

Two.III.3.36 No. Here, Gauss' method does not change the column space.

$$\begin{pmatrix}1&0\\3&1\end{pmatrix}\xrightarrow{-3\rho_1+\rho_2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Two.III.3.37 A linear system

$$c_1\vec{a}_1+\cdots+c_n\vec{a}_n=\vec{d}$$

has a solution if and only if $\vec{d}$ is in the span of the set $\{\vec{a}_1,\ldots,\vec{a}_n\}$. That's true if and only if the column rank of the augmented matrix equals the column rank of the matrix of coefficients. Since rank equals the column rank, the system has a solution if and only if the rank of its augmented matrix equals the rank of its matrix of coefficients.

Two.III.3.38 **(a)** Row rank equals column rank so each is at most the minimum of the number of rows and columns. Hence both can be full only if the number of rows equals the number of columns. (Of course, the converse does not hold: a square matrix need not have full row rank or full column rank.)

(b) If A has full row rank then, no matter what the right-hand side, Gauss' method on the augmented matrix ends with a leading one in each row and none of those leading ones in the furthest right column (the "augmenting" column). Back substitution then gives a solution.

On the other hand, if the linear system lacks a solution for some right-hand side it can only be because Gauss' method leaves some row so that it is all zeroes to the left of the "augmenting" bar and has a nonzero entry on the right. Thus, if A does not have a solution for some right-hand sides, then A does not have full row rank because some of its rows have been eliminated.

(c) The matrix A has full column rank if and only if its columns form a linearly independent set. That's equivalent to the existence of only the trivial linear relationship.

(d) The matrix A has full column rank if and only if the set of its columns is linearly independent, and so forms a basis for its span. That's equivalent to the existence of a unique linear representation of all vectors in that span.

Two.III.3.39 Instead of the row spaces being the same, the row space of B would be a subspace (possibly equal to) the row space of A.

Two.III.3.40 Clearly $\text{rank}(A)=\text{rank}(-A)$ as Gauss' method allows us to multiply all rows of a matrix by -1. In the same way, when $k\neq 0$ we have $\text{rank}(A)=\text{rank}(kA)$.

Addition is more interesting. The rank of a sum can be smaller than the rank of the summands.

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}-1&-2\\-3&-4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

The rank of a sum can be bigger than the rank of the summands.

$$\begin{pmatrix}1&2\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

But there is an upper bound (other than the size of the matrices). In general, $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

To prove this, note that Gaussian elimination can be performed on $A+B$ in either of two ways: we can first add A to B and then apply the appropriate sequence of reduction steps

$$(A+B) \xrightarrow{\text{step}_1} \cdots \xrightarrow{\text{step}_k} \text{echelon form}$$

or we can get the same results by performing step_1 through step_k separately on A and B, and then adding. The largest rank that we can end with in the second case is clearly the sum of the ranks. (The matrices above give examples of both possibilities, $\text{rank}(A+B) < \text{rank}(A) + \text{rank}(B)$ and $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$, happening.)

Subsection Two.III.4: Combining Subspaces

Two.III.4.20 With each of these we can apply Lemma 4.15.

(a) Yes. The plane is the sum of this W_1 and W_2 because for any scalars a and b

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ 0 \end{pmatrix} + \begin{pmatrix} b \\ b \end{pmatrix}$$

shows that the general vector is a sum of vectors from the two parts. And, these two subspaces are (different) lines through the origin, and so have a trivial intersection.

(b) Yes. To see that any vector in the plane is a combination of vectors from these parts, consider this relationship.

$$\begin{pmatrix} a \\ b \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1.1 \end{pmatrix}$$

We could now simply note that the set

$$\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1.1 \end{pmatrix}\}$$

is a basis for the space (because it is clearly linearly independent, and has size two in $\mathbb{R}^2$), and thus ther is one and only one solution to the above equation, implying that all decompositions are unique. Alternatively, we can solve

$$\begin{array}{rl} c_1 + c_2 = a \\ c_1 + 1.1c_2 = b \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1 + c_2 = & a \\ 0.1c_2 = & -a+b \end{array}$$

to get that $c_2 = 10(-a+b)$ and $c_1 = 11a - 10b$, and so we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11a-10b \\ 11a-10b \end{pmatrix} + \begin{pmatrix} -10a+10b \\ 1.1\cdot(-10a+10b) \end{pmatrix}$$

as required. As with the prior answer, each of the two subspaces is a line through the origin, and their intersection is trivial.

(c) Yes. Each vector in the plane is a sum in this way

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the intersection of the two subspaces is trivial.

(d) No. The intersection is not trivial.

(e) No. These are not subspaces.

Two.III.4.21 With each of these we can use Lemma 4.15.

(a) Any vector in $\mathbb{R}^3$ can be decomposed as this sum.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

And, the intersection of the xy-plane and the z-axis is the trivial subspace.

(b) Any vector in $\mathbb{R}^3$ can be decomposed as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-z \\ y-z \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ z \\ z \end{pmatrix}$$

and the intersection of the two spaces is trivial.

Two.III.4.22 It is. Showing that these two are subspaces is routine. To see that the space is the direct sum of these two, just note that each member of $\mathcal{P}_2$ has the unique decomposition $m + nx + px^2 = (m + px^2) + (nx)$.

Two.III.4.23 To show that they are subspaces is routine. We will argue they are complements with Lemma 4.15. The intersection $\mathcal{E} \cap \mathcal{O}$ is trivial because the only polynomial satisfying both conditions $p(-x) = p(x)$ and $p(-x) = -p(x)$ is the zero polynomial. To see that the entire space is the sum of the subspaces $\mathcal{E} + \mathcal{O} = \mathcal{P}_n$, note that the polynomials $p_0(x) = 1$, $p_2(x) = x^2$, $p_4(x) = x^4$, etc., are in $\mathcal{E}$ and also note that the polynomials $p_1(x) = x$, $p_3(x) = x^3$, etc., are in $\mathcal{O}$. Hence any member of $\mathcal{P}_n$ is a combination of members of $\mathcal{E}$ and $\mathcal{O}$.

Two.III.4.24 Each of these is $\mathbb{R}^3$.

(a) These are broken into lines for legibility.

$W_1 + W_2 + W_3$, $W_1 + W_2 + W_3 + W_4$, $W_1 + W_2 + W_3 + W_5$, $W_1 + W_2 + W_3 + W_4 + W_5$,
$W_1 + W_2 + W_4$, $W_1 + W_2 + W_4 + W_5$, $W_1 + W_2 + W_5$,
$W_1 + W_3 + W_4$, $W_1 + W_3 + W_5$, $W_1 + W_3 + W_4 + W_5$,
$W_1 + W_4$, $W_1 + W_4 + W_5$,
$W_1 + W_5$,
$W_2 + W_3 + W_4$, $W_2 + W_3 + W_4 + W_5$,
$W_2 + W_4$, $W_2 + W_4 + W_5$,
$W_3 + W_4$, $W_3 + W_4 + W_5$,
$W_4 + W_5$

(b) $W_1 \oplus W_2 \oplus W_3$, $W_1 \oplus W_4$, $W_1 \oplus W_5$, $W_2 \oplus W_4$, $W_3 \oplus W_4$

Two.III.4.25 Clearly each is a subspace. The bases $B_i = \langle x^i \rangle$ for the subspaces, when concatenated, form a basis for the whole space.

Two.III.4.26 It is W_2.

Two.III.4.27 True by Lemma 4.8.

Two.III.4.28 Two distinct direct sum decompositions of $\mathbb{R}^4$ are easy to find. Two such are $W_1 = [\{\vec{e}_1, \vec{e}_2\}]$ and $W_2 = [\{\vec{e}_3, \vec{e}_4\}]$, and also $U_1 = [\{\vec{e}_1\}]$ and $U_2 = [\{\vec{e}_2, \vec{e}_3, \vec{e}_4\}]$. (Many more are possible, for example $\mathbb{R}^4$ and its trivial subspace.)

In contrast, any partition of $\mathbb{R}^1$'s single-vector basis will give one basis with no elements and another with a single element. Thus any decomposition involves $\mathbb{R}^1$ and its trivial subspace.

Two.III.4.29 Set inclusion one way is easy: $\{\vec{w}_1 + \cdots + \vec{w}_k \mid \vec{w}_i \in W_i\}$ is a subset of $[W_1 \cup \ldots \cup W_k]$ because each $\vec{w}_1 + \cdots + \vec{w}_k$ is a sum of vectors from the union.

For the other inclusion, to any linear combination of vectors from the union apply commutativity of vector addition to put vectors from W_1 first, followed by vectors from W_2, etc. Add the vectors from W_1 to get a $\vec{w}_1 \in W_1$, add the vectors from W_2 to get a $\vec{w}_2 \in W_2$, etc. The result has the desired form.

Two.III.4.30 One example is to take the space to be $\mathbb{R}^3$, and to take the subspaces to be the xy-plane, the xz-plane, and the yz-plane.

Two.III.4.31 Of course, the zero vector is in all of the subspaces, so the intersection contains at least that one vector.. By the definition of direct sum the set $\{W_1, \ldots, W_k\}$ is independent and so no nonzero vector of W_i is a multiple of a member of W_j, when $i \neq j$. In particular, no nonzero vector from W_i equals a member of W_j.

Two.III.4.32 It can contain a trivial subspace; this set of subspaces of $\mathbb{R}^3$ is independent: $\{\{\vec{0}\}, x\text{-axis}\}$. No nonzero vector from the trivial space $\{\vec{0}\}$ is a multiple of a vector from the x-axis, simply because the trivial space has no nonzero vectors to be candidates for such a multiple (and also no nonzero vector from the x-axis is a multiple of the zero vector from the trivial subspace).

Two.III.4.33 Yes. For any subspace of a vector space we can take any basis $\langle \vec{\omega}_1, \ldots, \vec{\omega}_k \rangle$ for that subspace and extend it to a basis $\langle \vec{\omega}_1, \ldots, \vec{\omega}_k, \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$ for the whole space. Then the complemen of the original subspace has this for a basis: $\langle \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$.

Two.III.4.34 **(a)** It must. Any member of $W_1 + W_2$ can be written $\vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$. As S_1 spans W_1, the vector $\vec{w}_1$ is a combination of members of S_1. Similarly $\vec{w}_2$ is a combination of members of S_2.

(b) An easy way to see that it can be linearly independent is to take each to be the empty set. On the other hand, in the space $\mathbb{R}^1$, if $W_1 = \mathbb{R}^1$ and $W_2 = \mathbb{R}^1$ and $S_1 = \{1\}$ and $S_2 = \{2\}$, then their union $S_1 \cup S_2$ is not independent.

Two.III.4.35 **(a)** The intersection and sum are

$$\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \bigm| c \in \mathbb{R}\} \qquad \{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \bigm| b, c, d \in \mathbb{R}\}$$

which have dimensions one and three.

(b) We write $B_{U\cap W}$ for the basis for $U \cap W$, we write B_U for the basis for U, we write B_W for the basis for W, and we write B_{U+W} for the basis under consideration.

To see that that B_{U+W} spans $U + W$, observe that any vector $c\vec{u} + d\vec{w}$ from $U + W$ can be written as a linear combination of the vectors in B_{U+W}, simply by expressing $\vec{u}$ in terms of B_U and expressing $\vec{w}$ in terms of B_W.

We finish by showing that B_{U+W} is linearly independent. Consider

$$c_1\vec{\mu}_1 + \cdots + c_{j+1}\vec{\beta}_1 + \cdots + c_{j+k+p}\vec{\omega}_p = \vec{0}$$

which can be rewritten in this way.

$$c_1\vec{\mu}_1 + \cdots + c_j\vec{\mu}_j = -c_{j+1}\vec{\beta}_1 - \cdots - c_{j+k+p}\vec{\omega}_p$$

Note that the left side sums to a vector in U while right side sums to a vector in W, and thus both sides sum to a member of $U \cap W$. Since the left side is a member of $U \cap W$, it is expressible in terms of the members of $B_{U\cap W}$, which gives the combination of $\vec{\mu}$'s from the left side above as equal to a combination of $\vec{\beta}$'s. But, the fact that the basis B_U is linearly independent shows that any such combination is trivial, and in particular, the coefficients c_1, ..., c_j from the left side above are all zero. Similarly, the coefficients of the $\vec{\omega}$'s are all zero. This leaves the above equation as a linear relationship among the $\vec{\beta}$'s, but $B_{U\cap W}$ is linearly independent, and therefore all of the coefficients of the $\vec{\beta}$'s are also zero.

(c) Just count the basis vectors in the prior item: $\dim(U + W) = j + k + p$, and $\dim(U) = j + k$, and $\dim(W) = k + p$, and $\dim(U \cap W) = k$.

(d) We know that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. Because $W_1 \subseteq W_1 + W_2$, we know that $W_1 + W_2$ must have dimension greater than that of W_1, that is, must have dimension eight, nine, or ten. Substituting gives us three possibilities $8 = 8 + 8 - \dim(W_1 \cap W_2)$ or $9 = 8 + 8 - \dim(W_1 \cap W_2)$ or $10 = 8 + 8 - \dim(W_1 \cap W_2)$. Thus $\dim(W_1 \cap W_2)$ must be either eight, seven, or six. (Giving examples to show that each of these three cases is possible is easy, for instance in $\mathbb{R}^{10}$.)

Two.III.4.36 Expand each S_i to a basis B_i for W_i. The concatenation of those bases $B_1 \frown \cdots \frown B_k$ is a basis for V and thus its members form a linearly independent set. But the union $S_1 \cup \cdots \cup S_k$ is a subset of that linearly independent set, and thus is itself linearly independent.

Two.III.4.37 **(a)** Two such are these.

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For the antisymmetric one, entries on the diagonal must be zero.

(b) A square symmetric matrix equals its transpose. A square antisymmetric matrix equals the negative of its transpose.

(c) Showing that the two sets are subspaces is easy. Suppose that $A \in \mathcal{M}_{n\times n}$. To express A as a sum of a symmetric and an antisymmetric matrix, we observe that

$$A = (1/2)(A + A^{\text{trans}}) + (1/2)(A - A^{\text{trans}})$$

and note the first summand is symmetric while the second is antisymmetric. Thus $\mathcal{M}_{n\times n}$ is the sum of the two subspaces. To show that the sum is direct, assume a matrix A is both symmetric $A = A^{\text{trans}}$ and antisymmetric $A = -A^{\text{trans}}$. Then $A = -A$ and so all of A's entries are zeroes.

Two.III.4.38 Assume that $\vec{v} \in (W_1 \cap W_2) + (W_1 \cap W_3)$. Then $\vec{v} = \vec{w}_2 + \vec{w}_3$ where $\vec{w}_2 \in W_1 \cap W_2$ and $\vec{w}_3 \in W_1 \cap W_3$. Note that $\vec{w}_2, \vec{w}_3 \in W_1$ and, as a subspace is closed under addition, $\vec{w}_2 + \vec{w}_3 \in W_1$. Thus $\vec{v} = \vec{w}_2 + \vec{w}_3 \in W_1 \cap (W_2 + W_3)$.

This example proves that the inclusion may be strict: in $\mathbb{R}^2$ take W_1 to be the x-axis, take W_2 to be the y-axis, and take W_3 to be the line $y = x$. Then $W_1 \cap W_2$ and $W_1 \cap W_3$ are trivial and so their sum is trivial. But $W_2 + W_3$ is all of $\mathbb{R}^2$ so $W_1 \cap (W_2 + W_3)$ is the x-axis.

Two.III.4.39 It happens when at least one of W_1, W_2 is trivial. But that is the only way it can happen.

To prove this, assume that both are non-trivial, select nonzero vectors $\vec{w}_1, \vec{w}_2$ from each, and consider $\vec{w}_1 + \vec{w}_2$. This sum is not in W_1 because $\vec{w}_1 + \vec{w}_2 = \vec{v} \in W_1$ would imply that $\vec{w}_2 = \vec{v} - \vec{w}_1$ is in W_1, which violates the assumption of the independence of the subspaces. Similarly, $\vec{w}_1 + \vec{w}_2$ is not in W_2. Thus there is an element of V that is not in $W_1 \cup W_2$.

Two.III.4.40 **(a)** The set

$$\{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = 0 \text{ for all } x \in \mathbb{R}\}$$

is easily seen to be the y-axis.

(b) The yz-plane.

(c) The z-axis.

(d) Assume that U is a subspace of some $\mathbb{R}^n$. Because $U^\perp$ contains the zero vector, since that vector is perpendicular to everything, we need only show that the orthocomplement is closed under linear combinations of two elements. If $\vec{w}_1, \vec{w}_2 \in U^\perp$ then $\vec{w}_1 \cdot \vec{u} = 0$ and $\vec{w}_2 \cdot \vec{u} = 0$ for all $\vec{u} \in U$. Thus $(c_1\vec{w}_1 + c_2\vec{w}_2) \cdot \vec{u} = c_1(\vec{w}_1 \cdot \vec{u}) + c_2(\vec{w}_2 \cdot \vec{u}) = 0$ for all $\vec{u} \in U$ and so $U^\perp$ is closed under linear combinations.

(e) The only vector orthogonal to itself is the zero vector.

(f) This is immediate.

(g) To prove that the dimensions add, it suffices by Corollary 4.13 and Lemma 4.15 to show that $U \cap U^\perp$ is the trivial subspace $\{\vec{0}\}$. But this is one of the prior items in this problem.

Two.III.4.41 Yes. The left-to-right implication is Corollary 4.13. For the other direction, assume that $\dim(V) = \dim(W_1) + \cdots + \dim(W_k)$. Let $B_1, \ldots, B_k$ be bases for $W_1, \ldots, W_k$. As V is the sum of the subspaces, any $\vec{v} \in V$ can be written $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$ and expressing each $\vec{w}_i$ as a combination of vectors from the associated basis B_i shows that the concatenation $B_1 {}^\frown \cdots {}^\frown B_k$ spans V. Now, that concatenation has $\dim(W_1) + \cdots + \dim(W_k)$ members, and so it is a spanning set of size $\dim(V)$. The concatenation is therefore a basis for V. Thus V is the direct sum.

Two.III.4.42 No. The standard basis for $\mathbb{R}^2$ does not split into bases for the complementary subspaces the line $x = y$ and the line $x = -y$.

Two.III.4.43 **(a)** Yes, $W_1 + W_2 = W_2 + W_1$ for all subspaces W_1, W_2 because each side is the span of $W_1 \cup W_2 = W_2 \cup W_1$.

(b) This one is similar to the prior one — each side of that equation is the span of $(W_1 \cup W_2) \cup W_3 = W_1 \cup (W_2 \cup W_3)$.

(c) Because this is an equality between sets, we can show that it holds by mutual inclusion. Clearly $W \subseteq W + W$. For $W + W \subseteq W$ just recall that every subset is closed under addition so any sum of the form $\vec{w}_1 + \vec{w}_2$ is in W.

(d) In each vector space, the identity element with respect to subspace addition is the trivial subspace.

(e) Neither of left or right cancelation needs to hold. For an example, in $\mathbb{R}^3$ take W_1 to be the xy-plane, take W_2 to be the x-axis, and take W_3 to be the y-axis.

Two.III.4.44 **(a)** They are equal because for each, V is the direct sum if and only if each $\vec{v} \in V$ can be written in a unique way as a sum $\vec{v} = \vec{w}_1 + \vec{w}_2$ and $\vec{v} = \vec{w}_2 + \vec{w}_1$.

(b) They are equal because for each, V is the direct sum if and only if each $\vec{v} \in V$ can be written in a unique way as a sum of a vector from each $\vec{v} = (\vec{w}_1 + \vec{w}_2) + \vec{w}_3$ and $\vec{v} = \vec{w}_1 + (\vec{w}_2 + \vec{w}_3)$.

(c) Any vector in $\mathbb{R}^3$ can be decomposed uniquely into the sum of a vector from each axis.

(d) No. For an example, in $\mathbb{R}^2$ take W_1 to be the x-axis, take W_2 to be the y-axis, and take W_3 to be the line $y = x$.

(e) In any vector space the trivial subspace acts as the identity element with respect to direct sum.

(f) In any vector space, only the trivial subspace has a direct-sum inverse (namely, itself). One way to see this is that dimensions add, and so increase.

Chapter Three: Linear Maps between Vector Spaces

Subsection Three.I.1: Definition

Three.I.1.17 **(a)** Yes. The verification is straightforward.

$$h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})=h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix})$$
$$=\begin{pmatrix}c_1x_1+c_2x_2\\c_1x_1+c_2x_2+c_1y_1+c_2y_2+c_1z_1+c_2z_2\end{pmatrix}$$
$$=c_1\cdot\begin{pmatrix}x_1\\x_1+y_1+z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\c_2+y_2+z_2\end{pmatrix}$$
$$=c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})$$

(b) Yes. The verification is easy.

$$h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})=h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix})$$
$$=\begin{pmatrix}0\\0\end{pmatrix}$$
$$=c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})$$

(c) No. An example of an addition that is not respected is this.

$$h(\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix})=\begin{pmatrix}1\\1\end{pmatrix}\neq h(\begin{pmatrix}0\\0\\0\end{pmatrix})+h(\begin{pmatrix}0\\0\\0\end{pmatrix})$$

(d) Yes. The verification is straightforward.

$$h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})=h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix})$$
$$=\begin{pmatrix}2(c_1x_1+c_2x_2)+(c_1y_1+c_2y_2)\\3(c_1y_1+c_2y_2)-4(c_1z_1+c_2z_2)\end{pmatrix}$$
$$=c_1\cdot\begin{pmatrix}2x_1+y_1\\3y_1-4z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}2x_2+y_2\\3y_2-4z_2\end{pmatrix}$$
$$=c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})$$

Three.I.1.18 For each, we must either check that linear combinations are preserved, or give an example of a linear combination that is not.

(a) Yes. The check that it preserves combinations is routine.

$$\begin{aligned}h(r_1\cdot\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})&=h(\begin{pmatrix}r_1a_1+r_2a_2&r_1b_1+r_2b_2\\r_1c_1+r_2c_2&r_1d_1+r_2d_2\end{pmatrix})\\&=(r_1a_1+r_2a_2)+(r_1d_1+r_2d_2)\\&=r_1(a_1+d_1)+r_2(a_2+d_2)\\&=r_1\cdot h(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})+r_2\cdot h(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})\end{aligned}$$

(b) No. For instance, not preserved is multiplication by the scalar 2.

$$h(2\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix})=h(\begin{pmatrix}2&0\\0&2\end{pmatrix})=4\quad\text{while}\quad 2\cdot h(\begin{pmatrix}1&0\\0&1\end{pmatrix})=2\cdot 1=2$$

(c) Yes. This is the check that it preserves combinations of two members of the domain.

$$\begin{aligned}h(r_1\cdot\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})&=h(\begin{pmatrix}r_1a_1+r_2a_2&r_1b_1+r_2b_2\\r_1c_1+r_2c_2&r_1d_1+r_2d_2\end{pmatrix})\\&=2(r_1a_1+r_2a_2)+3(r_1b_1+r_2b_2)+(r_1c_1+r_2c_2)-(r_1d_1+r_2d_2)\\&=r_1(2a_1+3b_1+c_1-d_1)+r_2(2a_2+3b_2+c_2-d_2)\\&=r_1\cdot h(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot h(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})\end{aligned}$$

(d) No. An example of a combination that is not preserved is this.

$$h(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix})=h(\begin{pmatrix}2&0\\0&0\end{pmatrix})=4\quad\text{while}\quad h(\begin{pmatrix}1&0\\0&0\end{pmatrix})+h(\begin{pmatrix}1&0\\0&0\end{pmatrix})=1+1=2$$

Three.I.1.19 The check that each is a homomorphisms is routine. Here is the check for the differentiation map.

$$\begin{aligned}&\frac{d}{dx}(r\cdot(a_0+a_1x+a_2x^2+a_3x^3)+s\cdot(b_0+b_1x+b_2x^2+b_3x^3))\\&\qquad=\frac{d}{dx}((ra_0+sb_0)+(ra_1+sb_1)x+(ra_2+sb_2)x^2+(ra_3+sb_3)x^3)\\&\qquad=(ra_1+sb_1)+2(ra_2+sb_2)x+3(ra_3+sb_3)x^2\\&\qquad=r\cdot(a_1+2a_2x+3a_3x^2)+s\cdot(b_1+2b_2x+3b_3x^2)\\&\qquad=r\cdot\frac{d}{dx}(a_0+a_1x+a_2x^2+a_3x^3)+s\cdot\frac{d}{dx}(b_0+b_1x+b_2x^2+b_3x^3)\end{aligned}$$

(An alternate proof is to simply note that this is a property of differentiation that is familar from calculus.)

These two maps are not inverses as this composition does not act as the identity map on this element of the domain.

$$1\in\mathcal{P}_3\overset{d/dx}{\longmapsto}0\in\mathcal{P}_2\overset{\int}{\longmapsto}0\in\mathcal{P}_3$$

Three.I.1.20 Each of these projections is a homomorphism. Projection to the xz-plane and to the yz-plane are these maps.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x\\0\\z\end{pmatrix}\qquad\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}0\\y\\z\end{pmatrix}$$

Projection to the x-axis, to the y-axis, and to the z-axis are these maps.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x\\0\\0\end{pmatrix}\qquad\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}0\\y\\0\end{pmatrix}\qquad\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\z\end{pmatrix}$$

And projection to the origin is this map.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Verification that each is a homomorphism is straightforward. (The last one, of course, is the zero transformation on $\mathbb{R}^3$.)

Three.I.1.21 The first is not onto; for instance, there is no polynomial that is sent the constant polynomial $p(x)=1$. The second is not one-to-one; both of these members of the domain

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

are mapped to the same member of the codomain, $1\in\mathbb{R}$.

Three.I.1.22 Yes; in any space $\mathrm{id}(c\cdot\vec{v}+d\cdot\vec{w})=c\cdot\vec{v}+d\cdot\vec{w}=c\cdot\mathrm{id}(\vec{v})+d\cdot\mathrm{id}(\vec{w})$.

Three.I.1.23 **(a)** This map does not preserve structure since $f(1+1)=3$, while $f(1)+f(1)=2$.
(b) The check is routine.

$$\begin{aligned}f(r_1\cdot\begin{pmatrix}x_1\\y_1\end{pmatrix}+r_2\cdot\begin{pmatrix}x_2\\y_2\end{pmatrix})&=f(\begin{pmatrix}r_1x_1+r_2x_2\\r_1y_1+r_2y_2\end{pmatrix})\\&=(r_1x_1+r_2x_2)+2(r_1y_1+r_2y_2)\\&=r_1\cdot(x_1+2y_1)+r_2\cdot(x_2+2y_2)\\&=r_1\cdot f(\begin{pmatrix}x_1\\y_1\end{pmatrix})+r_2\cdot f(\begin{pmatrix}x_2\\y_2\end{pmatrix})\end{aligned}$$

Three.I.1.24 Yes. Where $h\colon V\to W$ is linear, $h(\vec{u}-\vec{v})=h(\vec{u}+(-1)\cdot\vec{v})=h(\vec{u})+(-1)\cdot h(\vec{v})=h(\vec{u})-h(\vec{v})$.

Three.I.1.25 **(a)** Let $\vec{v}\in V$ be represented with respect to the basis as $\vec{v}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$. Then $h(\vec{v})=h(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)=c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)=c_1\cdot\vec{0}+\cdots+c_n\cdot\vec{0}=\vec{0}$.
(b) This argument is similar to the prior one. Let $\vec{v}\in V$ be represented with respect to the basis as $\vec{v}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$. Then $h(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)=c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n=\vec{v}$.
(c) As above, only $c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)=c_1r\vec{\beta}_1+\cdots+c_nr\vec{\beta}_n=r(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)=r\vec{v}$.

Three.I.1.26 That it is a homomorphism follows from the familiar rules that the logarithm of a product is the sum of the logarithms $\ln(ab)=\ln(a)+\ln(b)$ and that the logarithm of a power is the multiple of the logarithm $\ln(a^r)=r\ln(a)$. This map is an isomorphism because it has an inverse, namely, the exponential map, so it is a correspondence, and therefore it is an isomorphism.

Three.I.1.27 Where $\hat{x}=x/2$ and $\hat{y}=y/3$, the image set is

$$\{\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\mid\frac{(2\hat{x})^2}{4}+\frac{(3\hat{y})^2}{9}=1\}=\{\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\mid\hat{x}^2+\hat{y}^2=1\}$$

the unit circle in the $\hat{x}\hat{y}$-plane.

Three.I.1.28 The circumference function $r\mapsto 2\pi r$ is linear. Thus we have $2\pi\cdot(r_{\text{earth}}+6)-2\pi\cdot(r_{\text{earth}})=12\pi$. Observe that it takes the same amount of extra rope to raise the circle from tightly wound around a basketball to six feet above that basketball as it does to raise it from tightly wound around the earth to six feet above the earth.

Three.I.1.29 Verifying that it is linear is routine.

$$\begin{aligned}h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})&=h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix})\\&=3(c_1x_1+c_2x_2)-(c_1y_1+c_2y_2)-(c_1z_1+c_2z_2)\\&=c_1\cdot(3x_1-y_1-z_1)+c_2\cdot(3x_2-y_2-z_2)\\&=c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})\end{aligned}$$

The natural guess at a generalization is that for any fixed $\vec{k}\in\mathbb{R}^3$ the map $\vec{v}\mapsto\vec{v}\cdot\vec{k}$ is linear. This statement is true. It follows from properties of the dot product we have seen earlier: $(\vec{v}+\vec{u})\cdot\vec{k}=\vec{v}\cdot\vec{k}+\vec{u}\cdot\vec{k}$ and $(r\vec{v})\cdot\vec{k}=r(\vec{v}\cdot\vec{k})$. (The natural guess at a generalization of this generalization, that the map from $\mathbb{R}^n$ to $\mathbb{R}$ whose action consists of taking the dot product of its argument with a fixed vector $\vec{k}\in\mathbb{R}^n$ is linear, is also true.)

Three.I.1.30 Let $h\colon\mathbb{R}^1\to\mathbb{R}^1$ be linear. A linear map is determined by its action on a basis, so fix the basis $\langle 1\rangle$ for $\mathbb{R}^1$. For any $r\in\mathbb{R}^1$ we have that $h(r)=h(r\cdot 1)=r\cdot h(1)$ and so h acts on any argument r by multiplying it by the constant $h(1)$. If $h(1)$ is not zero then the map is a correspondence — its inverse is division by $h(1)$ — so any nontrivial transformation of $\mathbb{R}^1$ is an isomorphism.

This projection map is an example that shows that not every transformation of $\mathbb{R}^n$ acts via multiplication by a constant when $n>1$, including when $n=2$.

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\mapsto\begin{pmatrix}x_1\\0\\\vdots\\0\end{pmatrix}$$

Three.I.1.31 **(a)** Where c and d are scalars, we have this.

$$\begin{aligned}h(c\cdot\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}+d\cdot\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix})&=h(\begin{pmatrix}cx_1+dy_1\\\vdots\\cx_n+dy_n\end{pmatrix})\\&=\begin{pmatrix}a_{1,1}(cx_1+dy_1)+\cdots+a_{1,n}(cx_n+dy_n)\\\vdots\\a_{m,1}(cx_1+dy_1)+\cdots+a_{m,n}(cx_n+dy_n)\end{pmatrix}\\&=c\cdot\begin{pmatrix}a_{1,1}x_1+\cdots+a_{1,n}x_n\\\vdots\\a_{m,1}x_1+\cdots+a_{m,n}x_n\end{pmatrix}+d\cdot\begin{pmatrix}a_{1,1}y_1+\cdots+a_{1,n}y_n\\\vdots\\a_{m,1}y_1+\cdots+a_{m,n}y_n\end{pmatrix}\\&=c\cdot h(\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix})+d\cdot h(\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix})\end{aligned}$$

(b) Each power i of the derivative operator is linear because of these rules familiar from calculus.

$$\frac{d^i}{dx^i}(\,f(x)+g(x)\,)=\frac{d^i}{dx^i}f(x)+\frac{d^i}{dx^i}g(x)\quad\text{and}\quad\frac{d^i}{dx^i}r\cdot f(x)=r\cdot\frac{d^i}{dx^i}f(x)$$

Thus the given map is a linear transformation of $\mathcal{P}_n$ because any linear combination of linear maps is also a linear map.

Three.I.1.32 (This argument has already appeared, as part of the proof that isomorphism is an equivalence.) Let $f\colon U\to V$ and $g\colon V\to W$ be linear. For any $\vec{u}_1,\vec{u}_2\in U$ and scalars c_1,c_2 combinations are preserved.

$$\begin{aligned}g\circ f(c_1\vec{u}_1+c_2\vec{u}_2)&=g(\,f(c_1\vec{u}_1+c_2\vec{u}_2)\,)=g(\,c_1f(\vec{u}_1)+c_2f(\vec{u}_2)\,)\\&=c_1\cdot g(f(\vec{u}_1))+c_2\cdot g(f(\vec{u}_2))=c_1\cdot g\circ f(\vec{u}_1)+c_2\cdot g\circ f(\vec{u}_2)\end{aligned}$$

Three.I.1.33 **(a)** Yes. The set of $\vec{w}$'s cannot be linearly independent if the set of $\vec{v}$'s is linearly dependent because any nontrivial relationship in the domain $\vec{0}_V=c_1\vec{v}_1+\cdots+c_n\vec{v}_n$ would give a nontrivial relationship in the range $f(\vec{0}_V)=\vec{0}_W=f(c_1\vec{v}_1+\cdots+c_n\vec{v}_n)=c_1f(\vec{v}_1)+\cdots+c_nf(\vec{v}_n)=c_1\vec{w}+\cdots+c_n\vec{w}_n$.

(b) Not necessarily. For instance, the transformation of $\mathbb{R}^2$ given by

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x+y\\x+y\end{pmatrix}$$

sends this linearly independent set in the domain to a linearly dependent image.

$$\{\vec{v}_1,\vec{v}_2\}=\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\}\;\mapsto\;\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\}=\{\vec{w}_1,\vec{w}_2\}$$

(c) Not necessarily. An example is the projection map $\pi\colon\mathbb{R}^3\to\mathbb{R}^2$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x\\y\end{pmatrix}$$

and this set that does not span the domain but maps to a set that does span the codomain.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\}\stackrel{\pi}{\longmapsto}\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\}$$

(d) Not necessarily. For instance, the injection map $\iota\colon\mathbb{R}^2\to\mathbb{R}^3$ sends the standard basis $\mathcal{E}_2$ for the domain to a set that does not span the codomain. (*Remark.* However, the set of $\vec{w}$'s does span the range. A proof is easy.)

Three.I.1.34 Recall that the entry in row i and column j of the transpose of M is the entry $m_{j,i}$ from row j and column i of M. Now, the check is routine.

$$\begin{aligned}
[r\cdot\begin{pmatrix} & \vdots & \\ \cdots & a_{i,j} & \cdots \\ & \vdots & \end{pmatrix}+s\cdot\begin{pmatrix} & \vdots & \\ \cdots & b_{i,j} & \cdots \\ & \vdots & \end{pmatrix}]^{\text{trans}} &= \begin{pmatrix} & \vdots & \\ \cdots & ra_{i,j}+sb_{i,j} & \cdots \\ & \vdots & \end{pmatrix}^{\text{trans}} \\
&= \begin{pmatrix} & \vdots & \\ \cdots & ra_{j,i}+sb_{j,i} & \cdots \\ & \vdots & \end{pmatrix} \\
&= r\cdot\begin{pmatrix} & \vdots & \\ \cdots & a_{j,i} & \cdots \\ & \vdots & \end{pmatrix}+s\cdot\begin{pmatrix} & \vdots & \\ \cdots & b_{j,i} & \cdots \\ & \vdots & \end{pmatrix} \\
&= r\cdot\begin{pmatrix} & \vdots & \\ \cdots & a_{j,i} & \cdots \\ & \vdots & \end{pmatrix}^{\text{trans}}+s\cdot\begin{pmatrix} & \vdots & \\ \cdots & b_{j,i} & \cdots \\ & \vdots & \end{pmatrix}^{\text{trans}}
\end{aligned}$$

The domain is $\mathcal{M}_{m\times n}$ while the codomain is $\mathcal{M}_{n\times m}$.

Three.I.1.35 **(a)** For any homomorphism $h\colon\mathbb{R}^n\to\mathbb{R}^m$ we have

$$h(\ell)=\{h(t\cdot\vec{u}+(1-t)\cdot\vec{v})\bigm| t\in[0..1]\}=\{t\cdot h(\vec{u})+(1-t)\cdot h(\vec{v})\bigm| t\in[0..1]\}$$

which is the line segment from $h(\vec{u})$ to $h(\vec{v})$.

(b) We must show that if a subset of the domain is convex then its image, as a subset of the range, is also convex. Suppose that $C\subseteq\mathbb{R}^n$ is convex and consider its image $h(C)$. To show $h(C)$ is convex we must show that for any two of its members, $\vec{d}_1$ and $\vec{d}_2$, the line segment connecting them

$$\ell=\{t\cdot\vec{d}_1+(1-t)\cdot\vec{d}_2\bigm| t\in[0..1]\}$$

is a subset of $h(C)$.

Fix any member $\hat{t}\cdot\vec{d}_1+(1-\hat{t})\cdot\vec{d}_2$ of that line segment. Because the endpoints of ℓ are in the image of C, there are members of C that map to them, say $h(\vec{c}_1)=\vec{d}_1$ and $h(\vec{c}_2)=\vec{d}_2$. Now, where $\hat{t}$ is the scalar that is fixed in the first sentence of this paragraph, observe that $h(\hat{t}\cdot\vec{c}_1+(1-\hat{t})\cdot\vec{c}_2)=\hat{t}\cdot h(\vec{c}_1)+(1-\hat{t})\cdot h(\vec{c}_2)=\hat{t}\cdot\vec{d}_1+(1-\hat{t})\cdot\vec{d}_2$ Thus, any member of ℓ is a member of $h(C)$, and so $h(C)$ is convex.

Three.I.1.36 **(a)** For $\vec{v}_0,\vec{v}_1\in\mathbb{R}^n$, the line through $\vec{v}_0$ with direction $\vec{v}_1$ is the set $\{\vec{v}_0+t\cdot\vec{v}_1\bigm| t\in\mathbb{R}\}$. The image under h of that line $\{h(\vec{v}_0+t\cdot\vec{v}_1)\bigm| t\in\mathbb{R}\}=\{h(\vec{v}_0)+t\cdot h(\vec{v}_1)\bigm| t\in\mathbb{R}\}$ is the line through $h(\vec{v}_0)$ with direction $h(\vec{v}_1)$. If $h(\vec{v}_1)$ is the zero vector then this line is degenerate.

(b) A k-dimensional linear surface in $\mathbb{R}^n$ maps to a (possibly degenerate) k-dimensional linear surface in $\mathbb{R}^m$. The proof is just like that the one for the line.

Three.I.1.37 Suppose that $h\colon V\to W$ is a homomorphism and suppose that S is a subspace of V. Consider the map $\hat{h}\colon S\to W$ defined by $\hat{h}(\vec{s})=h(\vec{s})$. (The only difference between $\hat{h}$ and h is the difference in domain.) Then this new map is linear: $\hat{h}(c_1\cdot\vec{s}_1+c_2\cdot\vec{s}_2)=h(c_1\vec{s}_1+c_2\vec{s}_2)=c_1h(\vec{s}_1)+c_2h(\vec{s}_2)=c_1\cdot\hat{h}(\vec{s}_1)+c_2\cdot\hat{h}(\vec{s}_2)$.

Three.I.1.38 This will appear as a lemma in the next subsection.

(a) The range is nonempty because V is nonempty. To finish we need to show that it is closed under combinations. A combination of range vectors has the form, where $\vec{v}_1,\dots,\vec{v}_n\in V$,

$$c_1\cdot h(\vec{v}_1)+\dots+c_n\cdot h(\vec{v}_n)=h(c_1\vec{v}_1)+\dots+h(c_n\vec{v}_n)=h(c_1\cdot\vec{v}_1+\dots+c_n\cdot\vec{v}_n),$$

which is itself in the range as $c_1\cdot\vec{v}_1+\dots+c_n\cdot\vec{v}_n$ is a member of domain V. Therefore the range is a subspace.

(b) The nullspace is nonempty since it contains $\vec{0}_V$, as $\vec{0}_V$ maps to $\vec{0}_W$. It is closed under linear combinations because, where $\vec{v}_1,\dots,\vec{v}_n\in V$ are elements of the inverse image set $\{\vec{v}\in V\bigm| h(\vec{v})=\vec{0}_W\}$, for $c_1,\dots,c_n\in\mathbb{R}$

$$\vec{0}_W=c_1\cdot h(\vec{v}_1)+\dots+c_n\cdot h(\vec{v}_n)=h(c_1\cdot\vec{v}_1+\dots+c_n\cdot\vec{v}_n)$$

and so $c_1\cdot\vec{v}_1+\dots+c_n\cdot\vec{v}_n$ is also in the inverse image of $\vec{0}_W$.

(c) This image of U nonempty because U is nonempty. For closure under combinations, where $\vec{u}_1,\ldots,\vec{u}_n \in U$,

$$c_1 \cdot h(\vec{u}_1) + \cdots + c_n \cdot h(\vec{u}_n) = h(c_1 \cdot \vec{u}_1) + \cdots + h(c_n \cdot \vec{u}_n) = h(c_1 \cdot \vec{u}_1 + \cdots + c_n \cdot \vec{u}_n)$$

which is itself in $h(U)$ as $c_1 \cdot \vec{u}_1 + \cdots + c_n \cdot \vec{u}_n$ is in U. Thus this set is a subspace.

(d) The natural generalization is that the inverse image of a subspace of is a subspace.

Suppose that X is a subspace of W. Note that $\vec{0}_W \in X$ so the set $\{\vec{v} \in V \mid h(\vec{v}) \in X\}$ is not empty. To show that this set is closed under combinations, let $\vec{v}_1,\ldots,\vec{v}_n$ be elements of V such that $h(\vec{v}_1) = \vec{x}_1, \ldots, h(\vec{v}_n) = \vec{x}_n$ and note that

$$h(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot h(\vec{v}_1) + \cdots + c_n \cdot h(\vec{v}_n) = c_1 \cdot \vec{x}_1 + \cdots + c_n \cdot \vec{x}_n$$

so a linear combination of elements of $h^{-1}(X)$ is also in $h^{-1}(X)$.

Three.I.1.39 No; the set of isomorphisms does not contain the zero map (unless the space is trivial).

Three.I.1.40 If $\langle \vec{\beta}_1,\ldots,\vec{\beta}_n \rangle$ doesn't span the space then the map needn't be unique. For instance, if we try to define a map from $\mathbb{R}^2$ to itself by specifying only that $\vec{e}_1$ is sent to itself, then there is more than one homomorphism possible; both the identity map and the projection map onto the first component fit this condition.

If we drop the condition that $\langle \vec{\beta}_1,\ldots,\vec{\beta}_n \rangle$ is linearly independent then we risk an inconsistent specification (i.e, there could be no such map). An example is if we consider $\langle \vec{e}_2, \vec{e}_1, 2\vec{e}_1 \rangle$, and try to define a map from $\mathbb{R}^2$ to itself that sends $\vec{e}_2$ to itself, and sends both $\vec{e}_1$ and $2\vec{e}_1$ to $\vec{e}_1$. No homomorphism can satisfy these three conditions.

Three.I.1.41 **(a)** Briefly, the check of linearity is this.

$$F(r_1 \cdot \vec{v}_1 + r_2 \cdot \vec{v}_2) = \begin{pmatrix} f_1(r_1\vec{v}_1 + r_2\vec{v}_2) \\ f_2(r_1\vec{v}_1 + r_2\vec{v}_2) \end{pmatrix} = r_1 \begin{pmatrix} f_1(\vec{v}_1) \\ f_2(\vec{v}_1) \end{pmatrix} + r_2 \begin{pmatrix} f_1(\vec{v}_2) \\ f_2(\vec{v}_2) \end{pmatrix} = r_1 \cdot F(\vec{v}_1) + r_2 \cdot F(\vec{v}_2)$$

(b) Yes. Let $\pi_1 \colon \mathbb{R}^2 \to \mathbb{R}^1$ and $\pi_2 \colon \mathbb{R}^2 \to \mathbb{R}^1$ be the projections

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\pi_1}{\longmapsto} x \quad\text{and}\quad \begin{pmatrix} x \\ y \end{pmatrix} \overset{\pi_2}{\longmapsto} y$$

onto the two axes. Now, where $f_1(\vec{v}) = \pi_1(F(\vec{v}))$ and $f_2(\vec{v}) = \pi_2(F(\vec{v}))$ we have the desired component functions.

$$F(\vec{v}) = \begin{pmatrix} f_1(\vec{v}) \\ f_2(\vec{v}) \end{pmatrix}$$

They are linear because they are the composition of linear functions, and the fact that the composition of linear functions is linear was shown as part of the proof that isomorphism is an equivalence relation (alternatively, the check that they are linear is straightforward).

(c) In general, a map from a vector space V to an $\mathbb{R}^n$ is linear if and only if each of the component functions is linear. The verification is as in the prior item.

Subsection Three.I.2: Rangespace and Nullspace

Three.I.2.22 First, to answer whether a polynomial is in the nullspace, we have to consider it as a member of the domain $\mathcal{P}_3$. To answer whether it is in the rangespace, we consider it as a member of the codomain $\mathcal{P}_4$. That is, for $p(x) = x^4$, the question of whether it is in the rangespace is sensible but the question of whether it is in the nullspace is not because it is not even in the domain.

(a) The polynomial $x^3 \in \mathcal{P}_3$ is not in the nullspace because $h(x^3) = x^4$ is not the zero polynomial in $\mathcal{P}_4$. The polynomial $x^3 \in \mathcal{P}_4$ is in the rangespace because $x^2 \in \mathcal{P}_3$ is mapped by h to x^3.

(b) The answer to both questions is, "Yes, because $h(0) = 0$." The polynomial $0 \in \mathcal{P}_3$ is in the nullspace because it is mapped by h to the zero polynomial in $\mathcal{P}_4$. The polynomial $0 \in \mathcal{P}_4$ is in the rangespace because it is the image, under h, of $0 \in \mathcal{P}_3$.

(c) The polynomial $7 \in \mathcal{P}_3$ is not in the nullspace because $h(7) = 7x$ is not the zero polynomial in $\mathcal{P}_4$. The polynomial $x^3 \in \mathcal{P}_4$ is not in the rangespace because there is no member of the domain that when multiplied by x gives the constant polynomial $p(x) = 7$.

(d) The polynomial $12x - 0.5x^3 \in \mathcal{P}_3$ is not in the nullspace because $h(12x - 0.5x^3) = 12x^2 - 0.5x^4$. The polynomial $12x - 0.5x^3 \in \mathcal{P}_4$ is in the rangespace because it is the image of $12 - 0.5x^2$.

(e) The polynomial $1+3x^2-x^3 \in \mathcal{P}_3$ is not in the nullspace because $h(1+3x^2-x^3) = x+3x^3-x^4$. The polynomial $1+3x^2-x^3 \in \mathcal{P}_4$ is not in the rangespace because of the constant term.

Three.I.2.23 **(a)** The nullspace is

$$\mathscr{N}(h) = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid a + ax + ax^2 + 0x^3 = 0 + 0x + 0x^2 + 0x^3\} = \{\begin{pmatrix} 0 \\ b \end{pmatrix} \mid b \in \mathbb{R}\}$$

while the rangespace is

$$\mathscr{R}(h) = \{a + ax + ax^2 \in \mathcal{P}_3 \mid a, b \in \mathbb{R}\} = \{a \cdot (1 + x + x^2) \mid a \in \mathbb{R}\}$$

and so the nullity is one and the rank is one.

(b) The nullspace is this.

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0\} = \{\begin{pmatrix} -d & b \\ c & d \end{pmatrix} \mid b, c, d \in \mathbb{R}\}$$

The rangespace

$$\mathscr{R}(h)\{a + d \mid a, b, c, d \in \mathbb{R}\}$$

is all of $\mathbb{R}$ (we can get any real number by taking d to be 0 and taking a to be the desired number). Thus, the nullity is three and the rank is one.

(c) The nullspace is

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b + c = 0 \text{ and } d = 0\} = \{\begin{pmatrix} -b-c & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathbb{R}\}$$

while the rangespace is $\mathscr{R}(h) = \{r + sx^2 \mid r, s \in \mathbb{R}\}$. Thus, the nullity is two and the rank is two.

(d) The nullspace is all of $\mathbb{R}^3$ so the nullity is three. The rangespace is the trivial subspace of $\mathbb{R}^4$ so the rank is zero.

Three.I.2.24 For each, use the result that the rank plus the nullity equals the dimension of the domain.

(a) 0 **(b)** 3 **(c)** 3 **(d)** 0

Three.I.2.25 Because

$$\frac{d}{dx}(a_0 + a_1x + \cdots + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

we have this.

$$\begin{aligned}\mathscr{N}(\frac{d}{dx}) &= \{a_0 + \cdots + a_nx^n \mid a_1 + 2a_2x + \cdots + na_nx^{n-1} = 0 + 0x + \cdots + 0x^{n-1}\} \\ &= \{a_0 + \cdots + a_nx^n \mid a_1 = 0, \text{ and } a_2 = 0, \ldots, a_n = 0\} \\ &= \{a_0 + 0x + 0x^2 + \cdots + 0x^n \mid a_0 \in \mathbb{R}\}\end{aligned}$$

In the same way,

$$\mathscr{N}(\frac{d^k}{dx^k}) = \{a_0 + a_1x + \cdots + a_nx^n \mid a_0, \ldots, a_{k-1} \in \mathbb{R}\}$$

for $k \leq n$.

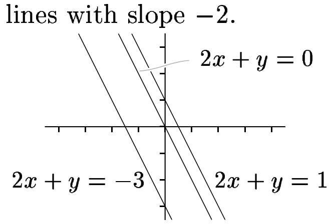
Three.I.2.26 The shadow of a scalar multiple is the scalar multiple of the shadow.

Three.I.2.27 **(a)** Setting $a_0 + (a_0 + a_1)x + (a_2 + a_3)x^3 = 0 + 0x + 0x^2 + 0x^3$ gives $a_0 = 0$ and $a_0 + a_1 = 0$ and $a_2 + a_3 = 0$, so the nullspace is $\{-a_3x^2 + a_3x^3 \mid a_3 \in \mathbb{R}\}$.

(b) Setting $a_0 + (a_0 + a_1)x + (a_2 + a_3)x^3 = 2 + 0x + 0x^2 - x^3$ gives that $a_0 = 2$, and $a_1 = -2$, and $a_2 + a_3 = -1$. Taking a_3 as a parameter, and renaming it $a_3 = a$ gives this set description $\{2 - 2x + (-1-a)x^2 + ax^3 \mid a \in \mathbb{R}\} = \{(2 - 2x - x^2) + a \cdot (-x^2 + x^3) \mid a \in \mathbb{R}\}$.

(c) This set is empty because the range of h includes only those polynomials with a $0x^2$ term.

Three.I.2.28 All inverse images are lines with slope -2.



Three.I.2.29 These are the inverses.

(a) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_1x + (a_2/2)x^2 + (a_3/3)x^3$

(b) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_2x + a_1x^2 + a_3x^3$

(c) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_3 + a_0x + a_1x^2 + a_2x^3$
(d) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3$

For instance, for the second one, the map given in the question sends $0 + 1x + 2x^2 + 3x^3 \mapsto 0 + 2x + 1x^2 + 3x^3$ and then the inverse above sends $0 + 2x + 1x^2 + 3x^3 \mapsto 0 + 1x + 2x^2 + 3x^3$. So this map is actually self-inverse.

Three.I.2.30 For any vector space V, the nullspace

$$\{\vec{v} \in V \mid 2\vec{v} = \vec{0}\}$$

is trivial, while the rangespace

$$\{\vec{w} \in V \mid \vec{w} = 2\vec{v} \text{ for some } \vec{v} \in V\}$$

is all of V, because every vector $\vec{w}$ is twice some other vector, specifically, it is twice $(1/2)\vec{w}$. (Thus, this transformation is actually an automorphism.)

Three.I.2.31 Because the rank plus the nullity equals the dimension of the domain (here, five), and the rank is at most three, the possible pairs are: $(3, 2)$, $(2, 3)$, $(1, 4)$, and $(0, 5)$. Coming up with linear maps that show that each pair is indeed possible is easy.

Three.I.2.32 No (unless $\mathcal{P}_n$ is trivial), because the two polynomials $f_0(x) = 0$ and $f_1(x) = 1$ have the same derivative; a map must be one-to-one to have an inverse.

Three.I.2.33 The nullspace is this.

$$\begin{aligned}\{a_0 + a_1x + \cdots + a_nx^n \mid a_0(1) + \frac{a_1}{2}(1^2) + \cdots + \frac{a_n}{n+1}(1^{n+1}) = 0\}\\ = \{a_0 + a_1x + \cdots + a_nx^n \mid a_0 + (a_1/2) + \cdots + (a_{n+1}/n + 1) = 0\}\end{aligned}$$

Thus the nullity is n.

Three.I.2.34 (a) One direction is obvious: if the homomorphism is onto then its range is the codomain and so its rank equals the dimension of its codomain. For the other direction assume that the map's rank equals the dimension of the codomain. Then the map's range is a subspace of the codomain, and has dimension equal to the dimension of the codomain. Therefore, the map's range must equal the codomain, and the map is onto. (The 'therefore' is because there is a linearly independent subset of the range that is of size equal to the dimension of the codomain, but any such linearly independent subset of the codomain must be a basis for the codomain, and so the range equals the codomain.)
(b) By Theorem 2.21, a homomorphism is one-to-one if and only if its nullity is zero. Because rank plus nullity equals the dimension of the domain, it follows that a homomorphism is one-to-one if and only if its rank equals the dimension of its domain. But this domain and codomain have the same dimension, so the map is one-to-one if and only if it is onto.

Three.I.2.35 We are proving that $h\colon V \to W$ is nonsingular if and only if for every linearly independent subset S of V the subset $h(S) = \{h(\vec{s}) \mid \vec{s} \in S\}$ of W is linearly independent.

One half is easy — by Theorem 2.21, if h is singular then its nullspace is nontrivial (contains more than just the zero vector). So, where $\vec{v} \neq \vec{0}_V$ is in that nullspace, the singleton set $\{\vec{v}\}$ is independent while its image $\{h(\vec{v})\} = \{\vec{0}_W\}$ is not.

For the other half, assume that h is nonsingular and so by Theorem 2.21 has a trivial nullspace. Then for any $\vec{v}_1, \ldots, \vec{v}_n \in V$, the relation

$$\vec{0}_W = c_1 \cdot h(\vec{v}_1) + \cdots + c_n \cdot h(\vec{v}_n) = h(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n)$$

implies the relation $c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n = \vec{0}_V$. Hence, if a subset of V is independent then so is its image in W.

Remark. The statement is that a linear map is nonsingular if and only if it preserves independence for *all* sets (that is, if a set is independent then its image is also independent). A singular map may well preserve some independent sets. An example is this singular map from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + y + z \\ 0 \end{pmatrix}$$

Linear independence is preserved for this set

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\} \mapsto \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

and (in a somewhat more tricky example) also for this set

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\}\mapsto\{\begin{pmatrix}1\\0\end{pmatrix}\}$$

(recall that in a set, repeated elements do not appear twice). However, there are sets whose independence is not preserved under this map;

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\}\mapsto\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\}$$

and so not all sets have independence preserved.

Three.I.2.36 (We use the notation from Theorem 1.9.) Fix a basis $\langle\vec{\beta}_1,\dots,\vec{\beta}_n\rangle$ for V and a basis $\langle\vec{w}_1,\dots,\vec{w}_k\rangle$ for W. If the dimension k of W is less than or equal to the dimension n of V then the theorem gives a linear map from V to W determined in this way.

$$\vec{\beta}_1\mapsto\vec{w}_1,\ \dots,\ \vec{\beta}_k\mapsto\vec{w}_k\quad\text{and}\quad\vec{\beta}_{k+1}\mapsto\vec{w}_k,\ \dots,\ \vec{\beta}_n\mapsto\vec{w}_k$$

We need only to verify that this map is onto.

Any member of W can be written as a linear combination of basis elements $c_1\cdot\vec{w}_1+\dots+c_k\cdot\vec{w}_k$. This vector is the image, under the map described above, of $c_1\cdot\vec{\beta}_1+\dots+c_k\cdot\vec{\beta}_k+0\cdot\vec{\beta}_{k+1}\dots+0\cdot\vec{\beta}_n$. Thus the map is onto.

Three.I.2.37 By assumption, h is not the zero map and so a vector $\vec{v}\in V$ exists that is not in the nullspace. Note that $\langle h(\vec{v})\rangle$ is a basis for $\mathbb{R}$, because it is a size one linearly independent subset of $\mathbb{R}$. Consequently h is onto, as for any $r\in\mathbb{R}$ we have $r=c\cdot h(\vec{v})$ for some scalar c, and so $r=h(c\vec{v})$.

Thus the rank of h is one. Because the nullity is given as n, the dimension of the domain of h (the vector space V) is $n+1$. We can finish by showing $\{\vec{v},\vec{\beta}_1,\dots,\vec{\beta}_n\}$ is linearly independent, as it is a size $n+1$ subset of a dimension $n+1$ space. Because $\{\vec{\beta}_1,\dots,\vec{\beta}_n\}$ is linearly independent we need only show that $\vec{v}$ is not a linear combination of the other vectors. But $c_1\vec{\beta}_1+\dots+c_n\vec{\beta}_n=\vec{v}$ would give $-\vec{v}+c_1\vec{\beta}_1+\dots+c_n\vec{\beta}_n=\vec{0}$ and applying h to both sides would give a contradiction.

Three.I.2.38 Yes. For the transformation of $\mathbb{R}^2$ given by

$$\begin{pmatrix}x\\y\end{pmatrix}\stackrel{h}{\longmapsto}\begin{pmatrix}0\\x\end{pmatrix}$$

we have this.

$$\mathscr{N}(h)=\{\begin{pmatrix}0\\y\end{pmatrix}\bigm| y\in\mathbb{R}\}=\mathscr{R}(h)$$

Remark. We will see more of this in the fifth chapter.

Three.I.2.39 This is a simple calculation.

$$\begin{aligned}h([S])&=\{h(c_1\vec{s}_1+\dots+c_n\vec{s}_n)\bigm| c_1,\dots,c_n\in\mathbb{R}\text{ and }\vec{s}_1,\dots,\vec{s}_n\in S\}\\&=\{c_1h(\vec{s}_1)+\dots+c_nh(\vec{s}_n)\bigm| c_1,\dots,c_n\in\mathbb{R}\text{ and }\vec{s}_1,\dots,\vec{s}_n\in S\}\\&=[h(S)]\end{aligned}$$

Three.I.2.40 **(a)** We will show that the two sets are equal $h^{-1}(\vec{w})=\{\vec{v}+\vec{n}\bigm|\vec{n}\in\mathscr{N}(h)\}$ by mutual inclusion. For the $\{\vec{v}+\vec{n}\bigm|\vec{n}\in\mathscr{N}(h)\}\subseteq h^{-1}(\vec{w})$ direction, just note that $h(\vec{v}+\vec{n})=h(\vec{v})+h(\vec{n})$ equals $\vec{w}$, and so any member of the first set is a member of the second. For the $h^{-1}(\vec{w})\subseteq\{\vec{v}+\vec{n}\bigm|\vec{n}\in\mathscr{N}(h)\}$ direction, consider $\vec{u}\in h^{-1}(\vec{w})$. Because h is linear, $h(\vec{u})=h(\vec{v})$ implies that $h(\vec{u}-\vec{v})=\vec{0}$. We can write $\vec{u}-\vec{v}$ as $\vec{n}$, and then we have that $\vec{u}\in\{\vec{v}+\vec{n}\bigm|\vec{n}\in\mathscr{N}(h)\}$, as desired, because $\vec{u}=\vec{v}+(\vec{u}-\vec{v})$.

(b) This check is routine.

(c) This is immediate.

(d) For the linearity check, briefly, where c,d are scalars and $\vec{x},\vec{y}\in\mathbb{R}^n$ have components $x_1,\dots,x_n$

and $y_1,\ldots,y_n$, we have this.

$$\begin{aligned}
h(c\cdot\vec{x}+d\cdot\vec{y}) &= \begin{pmatrix} a_{1,1}(cx_1+dy_1)+\cdots+a_{1,n}(cx_n+dy_n) \\ \vdots \\ a_{m,1}(cx_1+dy_1)+\cdots+a_{m,n}(cx_n+dy_n) \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}cx_1+\cdots+a_{1,n}cx_n \\ \vdots \\ a_{m,1}cx_1+\cdots+a_{m,n}cx_n \end{pmatrix} + \begin{pmatrix} a_{1,1}dy_1+\cdots+a_{1,n}dy_n \\ \vdots \\ a_{m,1}dy_1+\cdots+a_{m,n}dy_n \end{pmatrix} \\
&= c\cdot h(\vec{x})+d\cdot h(\vec{y})
\end{aligned}$$

The appropriate conclusion is that General = Particular + Homogeneous.

(e) Each power of the derivative is linear because of the rules

$$\frac{d^k}{dx^k}(f(x)+g(x)) = \frac{d^k}{dx^k}f(x)+\frac{d^k}{dx^k}g(x) \quad\text{and}\quad \frac{d^k}{dx^k}rf(x) = r\frac{d^k}{dx^k}f(x)$$

from calculus. Thus the given map is a linear transformation of the space because any linear combination of linear maps is also a linear map by Lemma 1.16. The appropriate conclusion is General = Particular + Homogeneous, where the associated homogeneous differential equation has a constant of 0.

Three.I.2.41 Because the rank of t is one, the rangespace of t is a one-dimensional set. Taking $\langle h(\vec{v})\rangle$ as a basis (for some appropriate $\vec{v}$), we have that for every $\vec{w}\in V$, the image $h(\vec{w})\in V$ is a multiple of this basis vector — associated with each $\vec{w}$ there is a scalar $c_{\vec{w}}$ such that $t(\vec{w})=c_{\vec{w}}t(\vec{v})$. Apply t to both sides of that equation and take r to be $c_{t(\vec{v})}$

$$t\circ t(\vec{w}) = t(c_{\vec{w}}\cdot t(\vec{v})) = c_{\vec{w}}\cdot t\circ t(\vec{v}) = c_{\vec{w}}\cdot c_{t(\vec{v})}\cdot t(\vec{v}) = c_{\vec{w}}\cdot r\cdot t(\vec{v}) = r\cdot c_{\vec{w}}\cdot t(\vec{v}) = r\cdot t(\vec{w})$$

to get the desired conclusion.

Three.I.2.42 Fix a basis $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ for V. We shall prove that this map

$$h \stackrel{\Phi}{\longmapsto} \begin{pmatrix} h(\vec{\beta}_1) \\ \vdots \\ h(\vec{\beta}_n) \end{pmatrix}$$

is an isomorphism from V^* to $\mathbb{R}^n$.

To see that Φ is one-to-one, assume that h_1 and h_2 are members of V^* such that $\Phi(h_1)=\Phi(h_2)$. Then

$$\begin{pmatrix} h_1(\vec{\beta}_1) \\ \vdots \\ h_1(\vec{\beta}_n) \end{pmatrix} = \begin{pmatrix} h_2(\vec{\beta}_1) \\ \vdots \\ h_2(\vec{\beta}_n) \end{pmatrix}$$

and consequently, $h_1(\vec{\beta}_1)=h_2(\vec{\beta}_1)$, etc. But a homomorphism is determined by its action on a basis, so $h_1=h_2$, and therefore Φ is one-to-one.

To see that Φ is onto, consider

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

for $x_1,\ldots,x_n\in\mathbb{R}$. This function h from V to $\mathbb{R}$

$$c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n \stackrel{h}{\longmapsto} c_1x_1+\cdots+c_nx_n$$

is easily seen to be linear, and to be mapped by Φ to the given vector in $\mathbb{R}^n$, so Φ is onto.

The map Φ also preserves structure: where

$$c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n \stackrel{h_1}{\longmapsto} c_1h_1(\vec{\beta}_1)+\cdots+c_nh_1(\vec{\beta}_n)$$
$$c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n \stackrel{h_2}{\longmapsto} c_1h_2(\vec{\beta}_1)+\cdots+c_nh_2(\vec{\beta}_n)$$

we have

$$\begin{aligned}
(r_1h_1+r_2h_2)(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n) &= c_1(r_1h_1(\vec{\beta}_1)+r_2h_2(\vec{\beta}_1))+\cdots+c_n(r_1h_1(\vec{\beta}_n)+r_2h_2(\vec{\beta}_n)) \\
&= r_1(c_1h_1(\vec{\beta}_1)+\cdots+c_nh_1(\vec{\beta}_n))+r_2(c_1h_2(\vec{\beta}_1)+\cdots+c_nh_2(\vec{\beta}_n))
\end{aligned}$$

so $\Phi(r_1h_1+r_2h_2)=r_1\Phi(h_1)+r_2\Phi(h_2)$.

Three.I.2.43 Let $h\colon V\to W$ be linear and fix a basis $\langle\vec{\beta}_1,\dots,\vec{\beta}_n\rangle$ for V. Consider these n maps from V to W

$$h_1(\vec{v})=c_1\cdot h(\vec{\beta}_1),\quad h_2(\vec{v})=c_2\cdot h(\vec{\beta}_2),\quad\ldots\quad,h_n(\vec{v})=c_n\cdot h(\vec{\beta}_n)$$

for any $\vec{v}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$. Clearly h is the sum of the h_i's. We need only check that each h_i is linear: where $\vec{u}=d_1\vec{\beta}_1+\cdots+d_n\vec{\beta}_n$ we have $h_i(r\vec{v}+s\vec{u})=rc_i+sd_i=rh_i(\vec{v})+sh_i(\vec{u})$.

Three.I.2.44 Either yes (trivially) or no (nearly trivially).

If V 'is homomorphic to' W is taken to mean there is a homomorphism from V into (but not necessarily onto) W, then every space is homomorphic to every other space as a zero map always exists.

If V 'is homomorphic to' W is taken to mean there is an onto homomorphism from V to W then the relation is not an equivalence. For instance, there is an onto homomorphism from $\mathbb{R}^3$ to $\mathbb{R}^2$ (projection is one) but no homomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^3$ by Corollary 2.17, so the relation is not reflexive.*

Three.I.2.45 That they form the chains is obvious. For the rest, we show here that $\mathscr{R}(t^{j+1})=\mathscr{R}(t^j)$ implies that $\mathscr{R}(t^{j+2})=\mathscr{R}(t^{j+1})$. Induction then applies.

Assume that $\mathscr{R}(t^{j+1})=\mathscr{R}(t^j)$. Then $t\colon\mathscr{R}(t^{j+1})\to\mathscr{R}(t^{j+2})$ is the same map, with the same domain, as $t\colon\mathscr{R}(t^j)\to\mathscr{R}(t^{j+1})$. Thus it has the same range: $\mathscr{R}(t^{j+2})=\mathscr{R}(t^{j+1})$.

Subsection Three.II.1: Representing Linear Maps with Matrices

Three.II.1.11 **(a)** $\begin{pmatrix}1\cdot2+3\cdot1+1\cdot0\\0\cdot2+(-1)\cdot1+2\cdot0\\1\cdot2+1\cdot1+0\cdot0\end{pmatrix}=\begin{pmatrix}5\\-1\\3\end{pmatrix}$ **(b)** Not defined. **(c)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$

Three.II.1.12 **(a)** $\begin{pmatrix}2\cdot4+1\cdot2\\3\cdot4-(1/2)\cdot2\end{pmatrix}=\begin{pmatrix}10\\11\end{pmatrix}$ **(b)** $\begin{pmatrix}4\\1\end{pmatrix}$ **(c)** Not defined.

Three.II.1.13 Matrix-vector multiplication gives rise to a linear system.

$$\begin{aligned}2x+y+z&=8\\y+3z&=4\\x-y+2z&=4\end{aligned}$$

Gaussian reduction shows that $z=1$, $y=1$, and $x=3$.

Three.II.1.14 Here are two ways to get the answer.

First, obviously $1-3x+2x^2=1\cdot1-3\cdot x+2\cdot x^2$, and so we can apply the general property of preservation of combinations to get $h(1-3x+2x^2)=h(1\cdot1-3\cdot x+2\cdot x^2)=1\cdot h(1)-3\cdot h(x)+2\cdot h(x^2)=1\cdot(1+x)-3\cdot(1+2x)+2\cdot(x-x^3)=-2-3x-2x^3$.

The other way uses the computation scheme developed in this subsection. Because we know where these elements of the space go, we consider this basis $B=\langle1,x,x^2\rangle$ for the domain. Arbitrarily, we can take $D=\langle1,x,x^2,x^3\rangle$ as a basis for the codomain. With those choices, we have that

$$\text{Rep}_{B,D}(h)=\begin{pmatrix}1&1&0\\1&2&1\\0&0&0\\0&0&-1\end{pmatrix}_{B,D}$$

and, as

$$\text{Rep}_B(1-3x+2x^2)=\begin{pmatrix}1\\-3\\2\end{pmatrix}_B$$

the matrix-vector multiplication calculation gives this.

$$\text{Rep}_D(h(1-3x+2x^2))=\begin{pmatrix}1&1&0\\1&2&1\\0&0&0\\0&0&-1\end{pmatrix}_{B,D}\begin{pmatrix}1\\-3\\2\end{pmatrix}_B=\begin{pmatrix}-2\\-3\\0\\-2\end{pmatrix}_D$$

Thus, $h(1-3x+2x^2)=-2\cdot1-3\cdot x+0\cdot x^2-2\cdot x^3=-2-3x-2x^3$, as above.

*More information on equivalence relations is in the appendix.

Three.II.1.15 Again, as recalled in the subsection, with respect to $\mathcal{E}_i$, a column vector represents itself.

(a) To represent h with respect to $\mathcal{E}_2, \mathcal{E}_3$ we take the images of the basis vectors from the domain, and represent them with respect to the basis for the codomain.

$$\operatorname{Rep}_{\mathcal{E}_3}(\,h(\vec{e}_1)\,) = \operatorname{Rep}_{\mathcal{E}_3}(\begin{pmatrix}2\\2\\0\end{pmatrix}) = \begin{pmatrix}2\\2\\0\end{pmatrix} \qquad \operatorname{Rep}_{\mathcal{E}_3}(\,h(\vec{e}_2)\,) = \operatorname{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\1\\-1\end{pmatrix}) = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

These are adjoined to make the matrix.

$$\operatorname{Rep}_{\mathcal{E}_2,\mathcal{E}_3}(h) = \begin{pmatrix}2&0\\2&1\\0&-1\end{pmatrix}$$

(b) For any $\vec{v}$ in the domain $\mathbb{R}^2$,

$$\operatorname{Rep}_{\mathcal{E}_2}(\vec{v}) = \operatorname{Rep}_{\mathcal{E}_2}(\begin{pmatrix}v_1\\v_2\end{pmatrix}) = \begin{pmatrix}v_1\\v_2\end{pmatrix}$$

and so

$$\operatorname{Rep}_{\mathcal{E}_3}(\,h(\vec{v})\,) = \begin{pmatrix}2&0\\2&1\\0&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}2v_1\\2v_1+v_2\\-v_2\end{pmatrix}$$

is the desired representation.

Three.II.1.16 **(a)** We must first find the image of each vector from the domain's basis, and then represent that image with respect to the codomain's basis.

$$\operatorname{Rep}_B(\frac{d\,1}{dx}) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad \operatorname{Rep}_B(\frac{d\,x}{dx}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \operatorname{Rep}_B(\frac{d\,x^2}{dx}) = \begin{pmatrix}0\\2\\0\\0\end{pmatrix} \quad \operatorname{Rep}_B(\frac{d\,x^3}{dx}) = \begin{pmatrix}0\\0\\3\\0\end{pmatrix}$$

Those representations are then adjoined to make the matrix representing the map.

$$\operatorname{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}$$

(b) Proceeding as in the prior item, we represent the images of the domain's basis vectors

$$\operatorname{Rep}_B(\frac{d\,1}{dx}) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad \operatorname{Rep}_B(\frac{d\,x}{dx}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \operatorname{Rep}_B(\frac{d\,x^2}{dx}) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \operatorname{Rep}_B(\frac{d\,x^3}{dx}) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

and adjoin to make the matrix.

$$\operatorname{Rep}_{B,D}(\frac{d}{dx}) = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

Three.II.1.17 For each, we must find the image of each of the domain's basis vectors, represent each image with respect to the codomain's basis, and then adjoin those representations to get the matrix.

(a) The basis vectors from the domain have these images

$$1 \mapsto 0 \quad x \mapsto 1 \quad x^2 \mapsto 2x \quad \ldots$$

and these images are represented with respect to the codomain's basis in this way.

$$\operatorname{Rep}_B(0) = \begin{pmatrix}0\\0\\0\\\vdots\\ \\ \end{pmatrix} \quad \operatorname{Rep}_B(1) = \begin{pmatrix}1\\0\\0\\\vdots\\ \\ \end{pmatrix} \quad \operatorname{Rep}_B(2x) = \begin{pmatrix}0\\2\\0\\\vdots\\ \\ \end{pmatrix} \quad \ldots \quad \operatorname{Rep}_B(nx^{n-1}) = \begin{pmatrix}0\\0\\0\\\vdots\\n\\0\end{pmatrix}$$

The matrix

$$\text{Rep}_{B,B}(\frac{d}{dx})=\begin{pmatrix}0&1&0&\ldots&0\\0&0&2&\ldots&0\\&\vdots\\0&0&0&\ldots&n\\0&0&0&\ldots&0\end{pmatrix}$$

has $n+1$ rows and columns.

(b) Once the images under this map of the domain's basis vectors are determined

$$1\mapsto x\quad x\mapsto x^2/2\quad x^2\mapsto x^3/3\quad\ldots$$

then they can be represented with respect to the codomain's basis

$$\text{Rep}_{B_{n+1}}(x)=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\quad\text{Rep}_{B_{n+1}}(x^2/2)=\begin{pmatrix}0\\0\\1/2\\\vdots\end{pmatrix}\quad\ldots\quad\text{Rep}_{B_{n+1}}(x^{n+1}/(n+1))=\begin{pmatrix}0\\0\\0\\\vdots\\1/(n+1)\end{pmatrix}$$

and put together to make the matrix.

$$\text{Rep}_{B_n,B_{n+1}}(\int)=\begin{pmatrix}0&0&\ldots&0&0\\1&0&\ldots&0&0\\0&1/2&\ldots&0&0\\&\vdots\\0&0&\ldots&0&1/(n+1)\end{pmatrix}$$

(c) The images of the basis vectors of the domain are

$$1\mapsto 1\quad x\mapsto 1/2\quad x^2\mapsto 1/3\quad\ldots$$

and they are represented with respect to the codomain's basis as

$$\text{Rep}_{\mathcal{E}_1}(1)=1\quad\text{Rep}_{\mathcal{E}_1}(1/2)=1/2\quad\ldots$$

so the matrix is

$$\text{Rep}_{B,\mathcal{E}_1}(\int)=\begin{pmatrix}1&1/2&\cdots&1/n&1/(n+1)\end{pmatrix}$$

(this is an $1\times(n+1)$ matrix).

(d) Here, the images of the domain's basis vectors are

$$1\mapsto 1\quad x\mapsto 3\quad x^2\mapsto 9\quad\ldots$$

and they are represented in the codomain as

$$\text{Rep}_{\mathcal{E}_1}(1)=1\quad\text{Rep}_{\mathcal{E}_1}(3)=3\quad\text{Rep}_{\mathcal{E}_1}(9)=9\quad\ldots$$

and so the matrix is this.

$$\text{Rep}_{B,\mathcal{E}_1}(\int_0^1)=\begin{pmatrix}1&3&9&\cdots&3^n\end{pmatrix}$$

(e) The images of the basis vectors from the domain are

$$1\mapsto 1\quad x\mapsto x+1=1+x\quad x^2\mapsto(x+1)^2=1+2x+x^2\quad x^3\mapsto(x+1)^3=1+3x+3x^2+x^3\quad\ldots$$

which are represented as

$$\text{Rep}_B(1)=\begin{pmatrix}1\\0\\0\\0\\\vdots\\0\end{pmatrix}\quad\text{Rep}_B(1+x)=\begin{pmatrix}1\\1\\0\\0\\\vdots\\0\end{pmatrix}\quad\text{Rep}_B(1+2x+x^2)=\begin{pmatrix}1\\2\\1\\0\\\vdots\\0\end{pmatrix}\quad\ldots$$

The resulting matrix

$$\text{Rep}_{B,B}(\int)=\begin{pmatrix}1&1&1&1&\ldots&1\\0&1&2&3&\ldots&\binom{n}{2}\\0&0&1&3&\ldots&\binom{n}{3}\\&\vdots\\0&0&0&&\ldots&1\end{pmatrix}$$

is *Pascal's triangle* (recall that $\binom{n}{r}$ is the number of ways to choose r things, without order and without repetition, from a set of size n).

Three.II.1.18 Where the space is n-dimensional,

$$\text{Rep}_{B,B}(\text{id}) = \begin{pmatrix} 1 & 0 \ldots & 0 \\ 0 & 1 \ldots & 0 \\ & \vdots & \\ 0 & 0 \ldots & 1 \end{pmatrix}_{B,B}$$

is the $n \times n$ identity matrix.

Three.II.1.19 Taking this as the natural basis

$$B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_4 \rangle = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

the transpose map acts in this way

$$\vec{\beta}_1 \mapsto \vec{\beta}_1 \quad \vec{\beta}_2 \mapsto \vec{\beta}_3 \quad \vec{\beta}_3 \mapsto \vec{\beta}_2 \quad \vec{\beta}_4 \mapsto \vec{\beta}_4$$

so that representing the images with respect to the codomain's basis and adjoining those column vectors together gives this.

$$\text{Rep}_{B,B}(\text{trans}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B,B}$$

Three.II.1.20 **(a)** With respect to the basis of the codomain, the images of the members of the basis of the domain are represented as

$$\text{Rep}_B(\vec{\beta}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{Rep}_B(\vec{\beta}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{Rep}_B(\vec{\beta}_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{Rep}_B(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

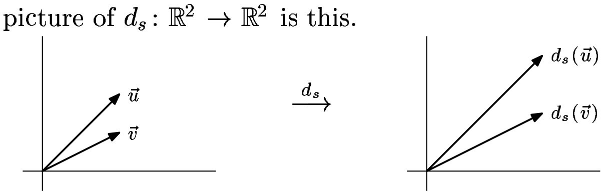
and consequently, the matrix representing the transformation is this.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Three.II.1.21 **(a)** The picture of $d_s \colon \mathbb{R}^2 \to \mathbb{R}^2$ is this.



This map's effect on the vectors in the standard basis for the domain is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{d_s}{\longmapsto} \begin{pmatrix} s \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \overset{d_s}{\longmapsto} \begin{pmatrix} 0 \\ s \end{pmatrix}$$

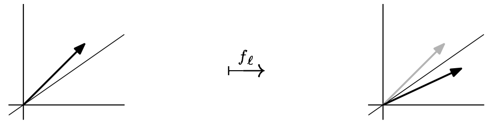
and those images are represented with respect to the codomain's basis (again, the standard basis) by themselves.

$$\text{Rep}_{\mathcal{E}_2}(\begin{pmatrix} s \\ 0 \end{pmatrix}) = \begin{pmatrix} s \\ 0 \end{pmatrix} \qquad \text{Rep}_{\mathcal{E}_2}(\begin{pmatrix} 0 \\ s \end{pmatrix}) = \begin{pmatrix} 0 \\ s \end{pmatrix}$$

Thus the representation of the dilation map is this.

$$\text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(d_s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

(b) The picture of $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ is this.



Some calculation (see Exercise I.??) shows that when the line has slope k

$$\begin{pmatrix}1\\0\end{pmatrix} \stackrel{f_\ell}{\longmapsto} \begin{pmatrix}(1-k^2)/(1+k^2)\\2k/(1+k^2)\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \stackrel{f_\ell}{\longmapsto} \begin{pmatrix}2k/(1+k^2)\\-(1-k^2)/(1+k^2)\end{pmatrix}$$

(the case of a line with undefined slope is separate but easy) and so the matrix representing reflection is this.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_\ell) = \frac{1}{1+k^2}\cdot\begin{pmatrix}1-k^2 & 2k\\2k & -(1-k^2)\end{pmatrix}$$

Three.II.1.22 Call the map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$.

(a) To represent this map with respect to the standard bases, we must find, and then represent, the images of the vectors $\vec{e}_1$ and $\vec{e}_2$ from the domain's basis. The image of $\vec{e}_1$ is given.

One way to find the image of $\vec{e}_2$ is by eye — we can see this.

$$\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix}2\\0\end{pmatrix}-\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix}$$

A more systemmatic way to find the image of $\vec{e}_2$ is to use the given information to represent the transformation, and then use that representation to determine the image. Taking this for a basis,

$$C = \langle\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\rangle$$

the given information says this.

$$\mathrm{Rep}_{C,\mathcal{E}_2}(t)\begin{pmatrix}2 & -1\\0 & 0\end{pmatrix}$$

As

$$\mathrm{Rep}_C(\vec{e}_2) = \begin{pmatrix}1\\-1\end{pmatrix}_C$$

we have that

$$\mathrm{Rep}_{\mathcal{E}_2}(t(\vec{e}_2)) = \begin{pmatrix}2 & -1\\0 & 0\end{pmatrix}_{C,\mathcal{E}_2}\begin{pmatrix}1\\-1\end{pmatrix}_C = \begin{pmatrix}3\\0\end{pmatrix}_{\mathcal{E}_2}$$

and consequently we know that $t(\vec{e}_2) = 3\cdot\vec{e}_1$ (since, with respect to the standard basis, this vector is represented by itself). Therefore, this is the representation of t with respect to $\mathcal{E}_2, \mathcal{E}_2$.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) = \begin{pmatrix}-1 & 3\\0 & 0\end{pmatrix}_{\mathcal{E}_2,\mathcal{E}_2}$$

(b) To use the matrix developed in the prior item, note that

$$\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix}0\\5\end{pmatrix}) = \begin{pmatrix}0\\5\end{pmatrix}_{\mathcal{E}_2}$$

and so we have this is the representation, with respect to the codomain's basis, of the image of the given vector.

$$\mathrm{Rep}_{\mathcal{E}_2}(t(\begin{pmatrix}0\\5\end{pmatrix})) = \begin{pmatrix}-1 & 3\\0 & 0\end{pmatrix}_{\mathcal{E}_2,\mathcal{E}_2}\begin{pmatrix}0\\5\end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix}15\\0\end{pmatrix}_{\mathcal{E}_2}$$

Because the codomain's basis is the standard one, and so vectors in the codomain are represented by themselves, we have this.

$$t(\begin{pmatrix}0\\5\end{pmatrix}) = \begin{pmatrix}15\\0\end{pmatrix}$$

(c) We first find the image of each member of B, and then represent those images with respect to D. For the first step, we can use the matrix developed earlier.

$$\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix}1\\-1\end{pmatrix}) = \begin{pmatrix}-1 & 3\\0 & 0\end{pmatrix}_{\mathcal{E}_2,\mathcal{E}_2}\begin{pmatrix}1\\-1\end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix}-4\\0\end{pmatrix}_{\mathcal{E}_2} \quad\text{so}\quad t(\begin{pmatrix}1\\-1\end{pmatrix}) = \begin{pmatrix}-4\\0\end{pmatrix}$$

Actually, for the second member of B there is no need to apply the matrix because the problem statement gives its image.

$$t(\begin{pmatrix}1\\1\end{pmatrix}) = \begin{pmatrix}2\\0\end{pmatrix}$$

Now representing those images with respect to D is routine.

$$\text{Rep}_D(\begin{pmatrix} -4 \\ 0 \end{pmatrix}) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}_D \quad \text{and} \quad \text{Rep}_D(\begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}_D$$

Thus, the matrix is this.

$$\text{Rep}_{B,D}(t) = \begin{pmatrix} -1 & 1/2 \\ 2 & -1 \end{pmatrix}_{B,D}$$

(d) We know the images of the members of the domain's basis from the prior item.

$$t(\begin{pmatrix} 1 \\ -1 \end{pmatrix}) = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \qquad t(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

We can compute the representation of those images with respect to the codomain's basis.

$$\text{Rep}_B(\begin{pmatrix} -4 \\ 0 \end{pmatrix}) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}_B \quad \text{and} \quad \text{Rep}_B(\begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_B$$

Thus this is the matrix.

$$\text{Rep}_{B,B}(t) = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}_{B,B}$$

Three.II.1.23 **(a)** The images of the members of the domain's basis are

$$\vec{\beta}_1 \mapsto h(\vec{\beta}_1) \quad \vec{\beta}_2 \mapsto h(\vec{\beta}_2) \quad \ldots \quad \vec{\beta}_n \mapsto h(\vec{\beta}_n)$$

and those images are represented with respect to the codomain's basis in this way.

$$\text{Rep}_{h(B)}(\,h(\vec{\beta}_1)\,) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{Rep}_{h(B)}(\,h(\vec{\beta}_2)\,) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots \quad \text{Rep}_{h(B)}(\,h(\vec{\beta}_n)\,) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Hence, the matrix is the identity.

$$\text{Rep}_{B,h(B)}(h) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

(b) Using the matrix in the prior item, the representation is this.

$$\text{Rep}_{h(B)}(\,h(\vec{v})\,) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_{h(B)}$$

Three.II.1.24 The product

$$\begin{pmatrix} h_{1,1} & \ldots & h_{1,i} & \ldots & h_{1,n} \\ h_{2,1} & \ldots & h_{2,i} & \ldots & h_{2,n} \\ & \vdots & & & \\ h_{m,1} & \ldots & h_{m,i} & \ldots & h_{1,n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} h_{1,i} \\ h_{2,i} \\ \vdots \\ h_{m,i} \end{pmatrix}$$

gives the i-th column of the matrix.

Three.II.1.25 **(a)** The images of the basis vectors for the domain are $\cos x \stackrel{d/dx}{\longmapsto} -\sin x$ and $\sin x \stackrel{d/dx}{\longmapsto} \cos x$. Representing those with respect to the codomain's basis (again, B) and adjoining the representations gives this matrix.

$$\text{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{B,B}$$

(b) The images of the vectors in the domain's basis are $e^x \stackrel{d/dx}{\longmapsto} e^x$ and $e^{2x} \stackrel{d/dx}{\longmapsto} 2e^{2x}$. Representing with respect to the codomain's basis and adjoining gives this matrix.

$$\text{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}_{B,B}$$

(c) The images of the members of the domain's basis are $1 \stackrel{d/dx}{\longmapsto} 0$, $x \stackrel{d/dx}{\longmapsto} 1$, $e^x \stackrel{d/dx}{\longmapsto} e^x$, and $xe^x \stackrel{d/dx}{\longmapsto} e^x + xe^x$. Representing these images with respect to B and adjoining gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{B,B}$$

Three.II.1.26 **(a)** It is the set of vectors of the codomain represented with respect to the codomain's basis in this way.

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R}\} = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x, y \in \mathbb{R}\}$$

As the codomain's basis is $\mathcal{E}_2$, and so each vector is represented by itself, the range of this transformation is the x-axis.

(b) It is the set of vectors of the codomain represented in this way.

$$\{\begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R}\} = \{\begin{pmatrix} 0 \\ 3x + 2y \end{pmatrix} \mid x, y \in \mathbb{R}\}$$

With respect to $\mathcal{E}_2$ vectors represent themselves so this range is the y axis.

(c) The set of vectors represented with respect to $\mathcal{E}_2$ as

$$\{\begin{pmatrix} a & b \\ 2a & 2b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R}\} = \{\begin{pmatrix} ax + by \\ 2ax + 2by \end{pmatrix} \mid x, y \in \mathbb{R}\} = \{(ax + by) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid x, y \in \mathbb{R}\}$$

is the line $y = 2x$, provided either a or b is not zero, and is the set consisting of just the origin if both are zero.

Three.II.1.27 Yes, for two reasons.

First, the two maps h and $\hat{h}$ need not have the same domain and codomain. For instance,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represents a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the standard bases that sends

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

and also represents a $\hat{h}\colon \mathcal{P}_1 \to \mathbb{R}^2$ with respect to $\langle 1, x\rangle$ and $\mathcal{E}_2$ that acts in this way.

$$1 \mapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

The second reason is that, even if the domain and codomain of h and $\hat{h}$ coincide, different bases produce different maps. An example is the 2×2 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which represents the identity map on $\mathbb{R}^2$ with respect to $\mathcal{E}_2, \mathcal{E}_2$. However, with respect to $\mathcal{E}_2$ for the domain but the basis $D = \langle \vec{e}_2, \vec{e}_1\rangle$ for the codomain, the same matrix I represents the map that swaps the first and second components

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

(that is, reflection about the line $y = x$).

Three.II.1.28 We mimic Example 1.1, just replacing the numbers with letters.

Write B as $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ and D as $\langle \vec{\delta}_1, \ldots, \vec{\delta}_m\rangle$. By definition of representation of a map with respect to bases, the assumption that

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & \ldots & h_{1,n} \\ \vdots & & \vdots \\ h_{m,1} & \ldots & h_{m,n} \end{pmatrix}$$

means that $h(\vec{\beta}_i) = h_{i,1}\vec{\delta}_1 + \cdots + h_{i,n}\vec{\delta}_n$. And, by the definition of the representation of a vector with respect to a basis, the assumption that

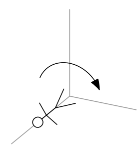
$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

means that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Substituting gives

$$\begin{aligned} h(\vec{v}) &= h(c_1 \cdot \vec{\beta}_1 + \cdots + c_n \cdot \vec{\beta}_n) \\ &= c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot \vec{\beta}_n \\ &= c_1 \cdot (h_{1,1}\vec{\delta}_1 + \cdots + h_{m,1}\vec{\delta}_m) + \cdots + c_n \cdot (h_{1,n}\vec{\delta}_1 + \cdots + h_{m,n}\vec{\delta}_m) \\ &= (h_{1,1}c_1 + \cdots + h_{1,n}c_n) \cdot \vec{\delta}_1 + \cdots + (h_{m,1}c_1 + \cdots + h_{m,n}c_n) \cdot \vec{\delta}_m \end{aligned}$$

and so $h(\vec{v})$ is represented as required.

Three.II.1.29 **(a)** The picture is this.



The images of the vectors from the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\\cos\theta\\-\sin\theta\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\\sin\theta\\\cos\theta\end{pmatrix}$$

are represented with respect to the codomain's basis (again, $\mathcal{E}_3$) by themselves, so adjoining the representations to make the matrix gives this.

$$\text{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(r_\theta) = \begin{pmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{pmatrix}$$

(b) The picture is similar to the one in the prior answer. The images of the vectors from the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}\cos\theta\\0\\\sin\theta\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}-\sin\theta\\0\\\cos\theta\end{pmatrix}$$

are represented with respect to the codomain's basis $\mathcal{E}_3$ by themselves, so this is the matrix.

$$\begin{pmatrix}\cos\theta&0&-\sin\theta\\0&1&0\\\sin\theta&0&\cos\theta\end{pmatrix}$$

(c) To a person standing up, with the vertical z-axis, a rotation of the xy-plane that is clockwise proceeds from the positive y-axis to the positive x-axis. That is, it rotates opposite to the direction in Example 1.8. The images of the vectors from the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}\cos\theta\\-\sin\theta\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}\sin\theta\\\cos\theta\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\0\\1\end{pmatrix}$$

are represented with respect to $\mathcal{E}_3$ by themselves, so the matrix is this.

$$\begin{pmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}$$

(d) $\begin{pmatrix}\cos\theta&\sin\theta&0&0\\-\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

Three.II.1.30 **(a)** Write B_U as $\langle\vec{\beta}_1,\ldots,\vec{\beta}_k\rangle$ and then B_V as $\langle\vec{\beta}_1,\ldots,\vec{\beta}_k,\vec{\beta}_{k+1},\ldots,\vec{\beta}_n\rangle$. If

$$\text{Rep}_{B_U}(\vec{v}) = \begin{pmatrix}c_1\\\vdots\\c_k\end{pmatrix} \qquad \text{so that } \vec{v} = c_1 \cdot \vec{\beta}_1 + \cdots + c_k \cdot \vec{\beta}_k$$

then,

$$\mathrm{Rep}_{B_V}(\vec{v})=\begin{pmatrix}c_1\\ \vdots\\ c_k\\ 0\\ \vdots\\ 0\end{pmatrix}$$

because $\vec{v}=c_1\cdot\vec{\beta}_1+\cdots+c_k\cdot\vec{\beta}_k+0\cdot\vec{\beta}_{k+1}+\cdots+0\cdot\vec{\beta}_n$.

(b) We must first decide what the question means. Compare $h\colon V\to W$ with its restriction to the subspace $h\restriction_U\colon U\to W$. The rangespace of the restriction is a subspace of W, so fix a basis $D_{h(U)}$ for this rangespace and extend it to a basis D_V for W. We want the relationship between these two.

$$\mathrm{Rep}_{B_V,D_V}(h)\quad\text{and}\quad\mathrm{Rep}_{B_U,D_{h(U)}}(h\restriction_U)$$

The answer falls right out of the prior item: if

$$\mathrm{Rep}_{B_U,D_{h(U)}}(h\restriction_U)=\begin{pmatrix}h_{1,1}&\ldots&h_{1,k}\\ \vdots&&\vdots\\ h_{p,1}&\ldots&h_{p,k}\end{pmatrix}$$

then the extension is represented in this way.

$$\mathrm{Rep}_{B_V,D_V}(h)=\begin{pmatrix}h_{1,1}&\ldots&h_{1,k}&h_{1,k+1}&\ldots&h_{1,n}\\ \vdots&&&&&\vdots\\ h_{p,1}&\ldots&h_{p,k}&h_{p,k+1}&\ldots&h_{p,n}\\ 0&\ldots&0&h_{p+1,k+1}&\ldots&h_{p+1,n}\\ \vdots&&&&&\vdots\\ 0&\ldots&0&h_{m,k+1}&\ldots&h_{m,n}\end{pmatrix}$$

(c) Take W_i to be the span of $\{h(\vec{\beta}_1),\ldots,h(\vec{\beta}_i)\}$.

(d) Apply the answer from the second item to the third item.

(e) No. For instance $\pi_x\colon\mathbb{R}^2\to\mathbb{R}^2$, projection onto the x axis, is represented by these two upper-triangular matrices

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi_x)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad\text{and}\quad\mathrm{Rep}_{C,\mathcal{E}_2}(\pi_x)=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

where $C=\langle\vec{e}_2,\vec{e}_1\rangle$.

Subsection Three.II.2: Any Matrix Represents a Linear Map

Three.II.2.9 **(a)** Yes; we are asking if there are scalars c_1 and c_2 such that

$$c_1\begin{pmatrix}2\\2\end{pmatrix}+c_2\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}1\\-3\end{pmatrix}$$

which gives rise to a linear system

$$\begin{array}{rl}2c_1+c_2=&1\\2c_1+5c_2=&-3\end{array}\ \xrightarrow{-\rho_1+\rho_2}\ \begin{array}{rl}2c_1+c_2=&1\\4c_2=&-4\end{array}$$

and Gauss' method produces $c_2=-1$ and $c_1=1$. That is, there is indeed such a pair of scalars and so the vector is indeed in the column space of the matrix.

(b) No; we are asking if there are scalars c_1 and c_2 such that

$$c_1\begin{pmatrix}4\\2\end{pmatrix}+c_2\begin{pmatrix}-8\\-4\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

and one way to proceed is to consider the resulting linear system

$$\begin{array}{l}4c_1-8c_2=0\\2c_1-4c_2=1\end{array}$$

that is easily seen to have no solution. Another way to proceed is to note that any linear combination of the columns on the left has a second component half as big as its first component, but the vector on the right does not meet that criterion.

(c) Yes; we can simply observe that the vector is the first column minus the second. Or, failing that, setting up the relationship among the columns

$$c_1\begin{pmatrix}1\\1\\-1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\\-1\end{pmatrix}+c_3\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$$

and considering the resulting linear system

$$\begin{array}{rl}c_1-c_2+c_3=2\\ c_1+c_2-c_3=0\\ -c_1-c_2+c_3=0\end{array}\xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2}\begin{array}{rl}c_1-c_2+c_3=2\\ 2c_2-2c_3=-2\\ -2c_2+2c_3=2\end{array}\xrightarrow{\rho_2+\rho_3}\begin{array}{rl}c_1-c_2+c_3=2\\ 2c_2-2c_3=-2\\ 0=0\end{array}$$

gives the additional information (beyond that there is at least one solution) that there are infinitely many solutions. Parametizing gives $c_2=-1+c_3$ and $c_1=1$, and so taking c_3 to be zero gives a particular solution of $c_1=1$, $c_2=-1$, and $c_3=0$ (which is, of course, the observation made at the start).

Three.II.2.10 As described in the subsection, with respect to the standard bases, representations are transparent, and so, for instance, the first matrix describes this map.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{E}_3}\mapsto\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{E}_2}=\begin{pmatrix}1\\0\end{pmatrix}\qquad\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\1\end{pmatrix}\qquad\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}3\\4\end{pmatrix}$$

So, for this first one, we are asking whether thare are scalars such that

$$c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}+c_3\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

that is, whether the vector is in the column space of the matrix.

(a) Yes. We can get this conclusion by setting up the resulting linear system and applying Gauss' method, as usual. Another way to get it is to note by inspection of the equation of columns that taking $c_3=3/4$, and $c_1=-5/4$, and $c_2=0$ will do. Still a third way to get this conclusion is to note that the rank of the matrix is two, which equals the dimension of the codomain, and so the map is onto — the range is all of $\mathbb{R}^2$ and in particular includes the given vector.

(b) No; note that all of the columns in the matrix have a second component that is twice the first, while the vector does not. Alternatively, the column space of the matrix is

$$\{c_1\begin{pmatrix}2\\4\end{pmatrix}+c_2\begin{pmatrix}0\\0\end{pmatrix}+c_3\begin{pmatrix}3\\6\end{pmatrix}\mid c_1,c_2,c_3\in\mathbb{R}\}=\{c\begin{pmatrix}1\\2\end{pmatrix}\mid c\in\mathbb{R}\}$$

(which is the fact already noted, but was arrived at by calculation rather than inspiration), and the given vector is not in this set.

Three.II.2.11 **(a)** The first member of the basis

$$\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}_B$$

is mapped to

$$\begin{pmatrix}1/2\\-1/2\end{pmatrix}_D$$

which is this member of the codomain.

$$\frac{1}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

(b) The second member of the basis is mapped

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}_B\mapsto\begin{pmatrix}(1/2\\1/2\end{pmatrix}_D$$

to this member of the codomain.

$$\frac{1}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

(c) Because the map that the matrix represents is the identity map on the basis, it must be the identity on all members of the domain. We can come to the same conclusion in another way by considering

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}_B$$

which is mapped to
$$\begin{pmatrix}(x+y)/2\\(x-y)/2\end{pmatrix}_D$$
which represents this member of $\mathbb{R}^2$.
$$\frac{x+y}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}+\frac{x-y}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

Three.II.2.12 A general member of the domain, represented with respect to the domain's basis as
$$a\cos\theta+b\sin\theta=\begin{pmatrix}a\\a+b\end{pmatrix}_B$$
is mapped to
$$\begin{pmatrix}0\\a\end{pmatrix}_D\quad\text{representing}\quad 0\cdot(\cos\theta+\sin\theta)+a\cdot(\cos\theta)$$
and so the linear map represented by the matrix with respect to these bases
$$a\cos\theta+b\sin\theta\mapsto a\cos\theta$$
is projection onto the first component.

Three.II.2.13 Denote the given basis of $\mathcal{P}_2$ by B. Then application of the linear map is represented by matrix-vector addition. Thus, the first vector in $\mathcal{E}_3$ is mapped to the element of $\mathcal{P}_2$ represented with respect to B by
$$\begin{pmatrix}1&3&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$$
and that element is $1+x$. The other two images of basis vectors are calculated similarly.
$$\begin{pmatrix}1&3&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}=\text{Rep}_B(4+x^2)\qquad\begin{pmatrix}1&3&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}=\text{Rep}_B(x)$$
We can thus decide if $1+2x$ is in the range of the map by looking for scalars c_1, c_2, and c_3 such that
$$c_1\cdot(1)+c_2\cdot(1+x^2)+c_3\cdot(x)=1+2x$$
and obviously $c_1=1$, $c_2=0$, and $c_3=1$ suffice. Thus it is in the range, and in fact it is the image of this vector.
$$1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Three.II.2.14 Let the matrix be G, and suppose that it rperesents $g\colon V\to W$ with respect to bases B and D. Because G has two columns, V is two-dimensional. Because G has two rows, W is two-dimensional. The action of g on a general member of the domain is this.
$$\begin{pmatrix}x\\y\end{pmatrix}_B\mapsto\begin{pmatrix}x+2y\\3x+6y\end{pmatrix}_D$$

(a) The only representation of the zero vector in the codomain is
$$\text{Rep}_D(\vec{0})=\begin{pmatrix}0\\0\end{pmatrix}_D$$
and so the set of representations of members of the nullspace is this.
$$\{\begin{pmatrix}x\\y\end{pmatrix}_B\bigm| x+2y=0\text{ and }3x+6y=0\}=\{y\cdot\begin{pmatrix}-1/2\\1\end{pmatrix}_D\bigm| y\in\mathbb{R}\}$$

(b) The representation map $\text{Rep}_D\colon W\to\mathbb{R}^2$ and its inverse are isomorphisms, and so preserve the dimension of subspaces. The subspace of $\mathbb{R}^2$ that is in the prior item is one-dimensional. Therefore, the image of that subspace under the inverse of the representation map—the nullspace of G, is also one-dimensional.

(c) The set of representations of members of the rangespace is this.
$$\{\begin{pmatrix}x+2y\\3x+6y\end{pmatrix}_D\bigm| x,y\in\mathbb{R}\}=\{k\cdot\begin{pmatrix}1\\3\end{pmatrix}_D\bigm| k\in\mathbb{R}\}$$

(d) Of course, Theorem 2.3 gives that the rank of the map equals the rank of the matrix, which is one. Alternatively, the same argument that was used above for the nullspace gives here that the dimension of the rangespace is one.

(e) One plus one equals two.

Three.II.2.15 No, the rangespaces may differ. Example 2.2 shows this.

Three.II.2.16 Recall that the represention map

$$V \stackrel{\text{Rep}_B}{\longmapsto} \mathbb{R}^n$$

is an isomorphism. Thus, its inverse map $\text{Rep}_B^{-1}\colon \mathbb{R}^n \to V$ is also an isomorphism. The desired transformation of $\mathbb{R}^n$ is then this composition.

$$\mathbb{R}^n \stackrel{\text{Rep}_B^{-1}}{\longmapsto} V \stackrel{\text{Rep}_D}{\longmapsto} \mathbb{R}^n$$

Because a composition of isomorphisms is also an isomorphism, this map $\text{Rep}_D \circ \text{Rep}_B^{-1}$ is an isomorphism.

Three.II.2.17 Yes. Consider

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

representing a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. With respect to the standard bases $B_1 = \mathcal{E}_2, D_1 = \mathcal{E}_2$ this matrix represents the identity map. With respect to

$$B_2 = D_2 = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

this matrix again represents the identity. In fact, as long as the starting and ending bases are equal — as long as $B_i = D_i$ — then the map represented by H is the identity.

Three.II.2.18 This is immediate from Corollary 2.6.

Three.II.2.19 The first map

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{E}_2} \mapsto \begin{pmatrix} 3x \\ 2y \end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix} 3x \\ 2y \end{pmatrix}$$

stretches vectors by a factor of three in the x direction and by a factor of two in the y direction. The second map

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{E}_2} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

projects vectors onto the x axis. The third

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{E}_2} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix} y \\ x \end{pmatrix}$$

interchanges first and second components (that is, it is a reflection about the line $y = x$). The last

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{E}_2} \mapsto \begin{pmatrix} x+3y \\ y \end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix} x+3y \\ y \end{pmatrix}$$

stretches vectors parallel to the y axis, by an amount equal to three times their distance from that axis (this is a *skew.*)

Three.II.2.20 **(a)** This is immediate from Theorem 2.3.

(b) Yes. This is immediate from the prior item.

To give a specific example, we can start with $\mathcal{E}_3$ as the basis for the domain, and then we require a basis D for the codomain $\mathbb{R}^3$. The matrix H gives the action of the map as this

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{E}_3} \mapsto \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}_D \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{E}_3} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_D \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{E}_3} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_D$$

and there is no harm in finding a basis D so that

$$\text{Rep}_D(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}_D \quad \text{and} \quad \text{Rep}_D(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_D$$

that is, so that the map represented by H with respect to $\mathcal{E}_3, D$ is projection down onto the xy plane. The second condition gives that the third member of D is $\vec{e}_2$. The first condition gives that the first member of D plus twice the second equals $\vec{e}_1$, and so this basis will do.

$$D = \langle \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$$

Three.II.2.21 (a) Recall that the representation map $\mathrm{Rep}_B\colon V\to\mathbb{R}^n$ is linear (it is actually an isomorphism, but we do not need that it is one-to-one or onto here). Considering the column vector x to be a $n\times 1$ matrix gives that the map from $\mathbb{R}^n$ to $\mathbb{R}$ that takes a column vector to its dot product with $\vec{x}$ is linear (this is a matrix-vector product and so Theorem 2.1 applies). Thus the map under consideration $h_{\vec{x}}$ is linear because it is the composistion of two linear maps.

$$\vec{v}\mapsto \mathrm{Rep}_B(\vec{v})\mapsto \vec{x}\cdot\mathrm{Rep}_B(\vec{v})$$

(b) Any linear map $g\colon V\to\mathbb{R}$ is represented by some matrix

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n\end{pmatrix}$$

(the matrix has n columns because V is n-dimensional and it has only one row because $\mathbb{R}$ is one-dimensional). Then taking $\vec{x}$ to be the column vector that is the transpose of this matrix

$$\vec{x}=\begin{pmatrix} g_1\\ \vdots\\ g_n\end{pmatrix}$$

has the desired action.

$$\vec{v}=\begin{pmatrix} v_1\\ \vdots\\ v_n\end{pmatrix}\mapsto\begin{pmatrix} g_1\\ \vdots\\ g_n\end{pmatrix}\cdot\begin{pmatrix} v_1\\ \vdots\\ v_n\end{pmatrix}=g_1v_1+\cdots+g_nv_n$$

(c) No. If $\vec{x}$ has any nonzero entries then $h_{\vec{x}}$ cannot be the zero map (and if $\vec{x}$ is the zero vector then $h_{\vec{x}}$ can only be the zero map).

Three.II.2.22 See the following section.

Subsection Three.III.1: Changing Representations of Vectors

Three.III.1.6 For the matrix to change bases from D to $\mathcal{E}_2$ we need that $\mathrm{Rep}_{\mathcal{E}_2}(\mathrm{id}(\vec{\delta}_1))=\mathrm{Rep}_{\mathcal{E}_2}(\vec{\delta}_1)$ and that $\mathrm{Rep}_{\mathcal{E}_2}(\mathrm{id}(\vec{\delta}_2))=\mathrm{Rep}_{\mathcal{E}_2}(\vec{\delta}_2)$. Of course, the representation of a vector in $\mathbb{R}^2$ with respect to the standard basis is easy.

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{\delta}_1)=\begin{pmatrix}2\\1\end{pmatrix}\qquad \mathrm{Rep}_{\mathcal{E}_2}(\vec{\delta}_2)=\begin{pmatrix}-2\\4\end{pmatrix}$$

Concatenating those two together to make the columns of the change of basis matrix gives this.

$$\mathrm{Rep}_{D,\mathcal{E}_2}(\mathrm{id})=\begin{pmatrix}2&-2\\1&4\end{pmatrix}$$

The change of basis matrix in the other direction can be gotten by calculating $\mathrm{Rep}_D(\mathrm{id}(\vec{e}_1))=\mathrm{Rep}_D(\vec{e}_1)$ and $\mathrm{Rep}_D(\mathrm{id}(\vec{e}_2))=\mathrm{Rep}_D(\vec{e}_2)$ (this job is routine) or it can be found by taking the inverse of the above matrix. Because of the formula for the inverse of a 2×2 matrix, this is easy.

$$\mathrm{Rep}_{\mathcal{E}_2,D}(\mathrm{id})=\frac{1}{10}\cdot\begin{pmatrix}4&2\\-1&2\end{pmatrix}=\begin{pmatrix}4/10&2/10\\-1/10&2/10\end{pmatrix}$$

Three.III.1.7 In each case, the columns $\mathrm{Rep}_D(\mathrm{id}(\vec{\beta}_1))=\mathrm{Rep}_D(\vec{\beta}_1)$ and $\mathrm{Rep}_D(\mathrm{id}(\vec{\beta}_2))=\mathrm{Rep}_D(\vec{\beta}_2)$ are concatenated to make the change of basis matrix $\mathrm{Rep}_{B,D}(\mathrm{id})$.

(a) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ (b) $\begin{pmatrix}2&-1/2\\-1&1/2\end{pmatrix}$ (c) $\begin{pmatrix}1&1\\2&4\end{pmatrix}$ (d) $\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$

Three.III.1.8 One way to go is to find $\mathrm{Rep}_B(\vec{\delta}_1)$ and $\mathrm{Rep}_B(\vec{\delta}_2)$, and then concatenate them into the columns of the desired change of basis matrix. Another way is to find the inverse of the matrices that answer Exercise 7.

(a) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ (b) $\begin{pmatrix}1&1\\2&4\end{pmatrix}$ (c) $\begin{pmatrix}2&-1/2\\-1&1/2\end{pmatrix}$ (d) $\begin{pmatrix}2&1\\1&1\end{pmatrix}$

Three.III.1.9 The columns vector representations $\mathrm{Rep}_D(\mathrm{id}(\vec{\beta}_1))=\mathrm{Rep}_D(\vec{\beta}_1)$, and $\mathrm{Rep}_D(\mathrm{id}(\vec{\beta}_2))=\mathrm{Rep}_D(\vec{\beta}_2)$, and $\mathrm{Rep}_D(\mathrm{id}(\vec{\beta}_3))=\mathrm{Rep}_D(\vec{\beta}_3)$ make the change of basis matrix $\mathrm{Rep}_{B,D}(\mathrm{id})$.

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & -1 & 1/2 \\ 1 & 1 & -1/2 \\ 0 & 2 & 0 \end{pmatrix}$

E.g., for the first column of the first matrix, $1 = 0 \cdot x^2 + 1 \cdot 1 + 0 \cdot x$.

Three.III.1.10 A matrix changes bases if and only if it is nonsingular.

(a) This matrix is nonsingular and so changes bases. Finding to what basis $\mathcal{E}_2$ is changed means finding D such that

$$\text{Rep}_{\mathcal{E}_2,D}(\text{id}) = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

and by the definition of how a matrix represents a linear map, we have this.

$$\text{Rep}_D(\text{id}(\vec{e}_1)) = \text{Rep}_D(\vec{e}_1) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \qquad \text{Rep}_D(\text{id}(\vec{e}_2)) = \text{Rep}_D(\vec{e}_2) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Where

$$D = \langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle$$

we can either solve the system

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 0 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 4 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix}$$

or else just spot the answer (thinking of the proof of Lemma 1.4).

$$D = \langle \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/4 \end{pmatrix} \rangle$$

(b) Yes, this matrix is nonsingular and so changes bases. To calculate D, we proceed as above with

$$D = \langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle$$

to solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 3 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 1 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix}$$

and get this.

$$D = \langle \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle$$

(c) No, this matrix does not change bases because it is nonsingular.

(d) Yes, this matrix changes bases because it is nonsingular. The calculation of the changed-to basis is as above.

$$D = \langle \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \rangle$$

Three.III.1.11 This question has many different solutions. One way to proceed is to make up any basis B for any space, and then compute the appropriate D (necessarily for the same space, of course). Another, easier, way to proceed is to fix the codomain as $\mathbb{R}^3$ and the codomain basis as $\mathcal{E}_3$. This way (recall that the representation of any vector with respect to the standard basis is just the vector itself), we have this.

$$B = \langle \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} \rangle \qquad D = \mathcal{E}_3$$

Three.III.1.12 Checking that $B = \langle 2\sin(x) + \cos(x), 3\cos(x) \rangle$ is a basis is routine. Call the natural basis D. To compute the change of basis matrix $\text{Rep}_{B,D}(\text{id})$ we must find $\text{Rep}_D(2\sin(x) + \cos(x))$ and $\text{Rep}_D(3\cos(x))$, that is, we need x_1, y_1, x_2, y_2 such that these equations hold.

$$\begin{aligned} x_1 \cdot \sin(x) + y_1 \cdot \cos(x) &= 2\sin(x) + \cos(x) \\ x_2 \cdot \sin(x) + y_2 \cdot \cos(x) &= 3\cos(x) \end{aligned}$$

Obviously this is the answer.

$$\text{Rep}_{B,D}(\text{id}) = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

For the change of basis matrix in the other direction we could look for $\text{Rep}_B(\sin(x))$ and $\text{Rep}_B(\cos(x))$ by solving these.

$$\begin{aligned} w_1 \cdot (2\sin(x) + \cos(x)) + z_1 \cdot (3\cos(x)) &= \sin(x) \\ w_2 \cdot (2\sin(x) + \cos(x)) + z_2 \cdot (3\cos(x)) &= \cos(x) \end{aligned}$$

An easier method is to find the inverse of the matrix found above.

$$\mathrm{Rep}_{D,B}(\mathrm{id})=\begin{pmatrix}2&0\\1&3\end{pmatrix}^{-1}=\frac{1}{6}\cdot\begin{pmatrix}3&0\\-1&2\end{pmatrix}=\begin{pmatrix}1/2&0\\-1/6&1/3\end{pmatrix}$$

Three.III.1.13 We start by taking the inverse of the matrix, that is, by deciding what is the inverse to the map of interest.

$$\mathrm{Rep}_{D,\mathcal{E}_2}(\mathrm{id})\mathrm{Rep}_{D,\mathcal{E}_2}(\mathrm{id})^{-1}=\frac{1}{-\cos^2(2\theta)-\sin^2(2\theta)}\cdot\begin{pmatrix}-\cos(2\theta)&-\sin(2\theta)\\-\sin(2\theta)&\cos(2\theta)\end{pmatrix}=\begin{pmatrix}\cos(2\theta)&\sin(2\theta)\\\sin(2\theta)&-\cos(2\theta)\end{pmatrix}$$

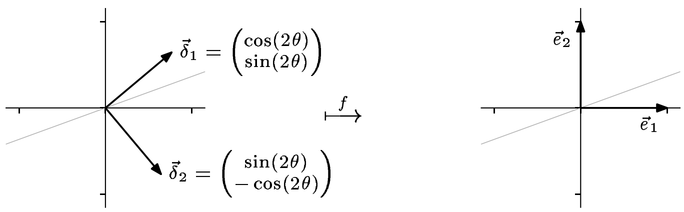
This is more tractable than the representation the other way because this matrix is the concatenation of these two column vectors

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{\delta}_1)=\begin{pmatrix}\cos(2\theta)\\\sin(2\theta)\end{pmatrix}\qquad\mathrm{Rep}_{\mathcal{E}_2}(\vec{\delta}_2)=\begin{pmatrix}\sin(2\theta)\\-\cos(2\theta)\end{pmatrix}$$

and representations with respect to $\mathcal{E}_2$ are transparent.

$$\vec{\delta}_1=\begin{pmatrix}\cos(2\theta)\\\sin(2\theta)\end{pmatrix}\qquad\vec{\delta}_2=\begin{pmatrix}\sin(2\theta)\\-\cos(2\theta)\end{pmatrix}$$

This pictures the action of the map that transforms D to $\mathcal{E}_2$ (it is, again, the inverse of the map that is the answer to this question). The line lies at an angle θ to the x axis.



This map reflects vectors over that line. Since reflections are self-inverse, the answer to the question is: the original map reflects about the line through the origin with angle of elevation θ. (Of course, it does this to any basis.)

Three.III.1.14 The appropriately-sized identity matrix.

Three.III.1.15 Each is true if and only if the matrix is nonsingular.

Three.III.1.16 What remains to be shown is that left multiplication by a reduction matrix represents a change from another basis to $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$.

Application of a row-multiplication matrix $M_i(k)$ translates a representation with respect to the basis $\langle\vec{\beta}_1,\ldots,k\vec{\beta}_i,\ldots,\vec{\beta}_n\rangle$ to one with respect to B, as here.

$$\vec{v}=c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot(k\vec{\beta}_i)+\cdots+c_n\cdot\vec{\beta}_n\ \mapsto\ c_1\cdot\vec{\beta}_1+\cdots+(kc_i)\cdot\vec{\beta}_i+\cdots+c_n\cdot\vec{\beta}_n=\vec{v}$$

Applying a row-swap matrix $P_{i,j}$ translates a representation with respect to $\langle\vec{\beta}_1,\ldots,\vec{\beta}_j,\ldots,\vec{\beta}_i,\ldots,\vec{\beta}_n\rangle$ to one with respect to $\langle\vec{\beta}_1,\ldots,\vec{\beta}_i,\ldots,\vec{\beta}_j,\ldots,\vec{\beta}_n\rangle$. Finally, applying a row-combination matrix $C_{i,j}(k)$ changes a representation with respect to $\langle\vec{\beta}_1,\ldots,\vec{\beta}_i+k\vec{\beta}_j,\ldots,\vec{\beta}_j,\ldots,\vec{\beta}_n\rangle$ to one with respect to B.

$$\begin{aligned}\vec{v}=c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot(\vec{\beta}_i+k\vec{\beta}_j)+\cdots+c_j\vec{\beta}_j+\cdots+c_n\cdot\vec{\beta}_n&\\ \mapsto\ c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot\vec{\beta}_i+\cdots+(kc_i+c_j)\cdot\vec{\beta}_j+\cdots+c_n\cdot\vec{\beta}_n=\vec{v}&\end{aligned}$$

(As in the part of the proof in the body of this subsection, the various conditions on the row operations, e.g., that the scalar k is nonzero, assure that these are all bases.)

Three.III.1.17 Taking H as a change of basis matrix $H=\mathrm{Rep}_{B,\mathcal{E}_n}(\mathrm{id})$, its columns are

$$\begin{pmatrix}h_{1,i}\\\vdots\\h_{n,i}\end{pmatrix}=\mathrm{Rep}_{\mathcal{E}_n}(\mathrm{id}(\vec{\beta}_i))=\mathrm{Rep}_{\mathcal{E}_n}(\vec{\beta}_i)$$

and, because representations with respect to the standard basis are transparent, we have this.

$$\begin{pmatrix}h_{1,i}\\\vdots\\h_{n,i}\end{pmatrix}=\vec{\beta}_i$$

That is, the basis is the one composed of the columns of H.

Three.III.1.18 **(a)** We can change the starting vector representation to the ending one through a sequence of row operations. The proof tells us what how the bases change. We start by swapping the first and second rows of the representation with respect to B to get a representation with resepect to a new basis B_1.

$$\mathrm{Rep}_{B_1}(1-x+3x^2-x^3)=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}_{B_1}\qquad B_1=\langle 1-x,1+x,x^2+x^3,x^2-x^3\rangle$$

We next add -2 times the third row of the vector representation to the fourth row.

$$\mathrm{Rep}_{B_3}(1-x+3x^2-x^3)=\begin{pmatrix}1\\0\\1\\0\end{pmatrix}_{B_2}\qquad B_2=\langle 1-x,1+x,3x^2-x^3,x^2-x^3\rangle$$

(The third element of B_2 is the third element of B_1 minus -2 times the fourth element of B_1.) Now we can finish by doubling the third row.

$$\mathrm{Rep}_{D}(1-x+3x^2-x^3)=\begin{pmatrix}1\\0\\2\\0\end{pmatrix}_{D}\qquad D=\langle 1-x,1+x,(3x^2-x^3)/2,x^2-x^3\rangle$$

(b) Here are three different approaches to stating such a result. The first is the assertion: where V is a vector space with basis B and $\vec{v}\in V$ is nonzero, for any nonzero column vector $\vec{z}$ (whose number of components equals the dimension of V) there is a change of basis matrix M such that $M\cdot\mathrm{Rep}_B(\vec{v})=\vec{z}$. The second possible statement: for any (n-dimensional) vector space V and any nonzero vector $\vec{v}\in V$, where $\vec{z}_1,\vec{z}_2\in\mathbb{R}^n$ are nonzero, there are bases $B,D\subset V$ such that $\mathrm{Rep}_B(\vec{v})=\vec{z}_1$ and $\mathrm{Rep}_D(\vec{v})=\vec{z}_2$. The third is: for any nonzero $\vec{v}$ member of any vector space (of dimension n) and any nonzero column vector (with n components) there is a basis such that $\vec{v}$ is represented with respect to that basis by that column vector.

The first and second statements follow easily from the third. The first follows because the third statement gives a basis D such that $\mathrm{Rep}_D(\vec{v})=\vec{z}$ and then $\mathrm{Rep}_{B,D}(\mathrm{id})$ is the desired M. The second follows from the third because it is just a doubled application of it.

A way to prove the third is as in the answer to the first part of this question. Here is a sketch. Represent $\vec{v}$ with respect to any basis B with a column vector $\vec{z}_1$. This column vector must have a nonzero component because $\vec{v}$ is a nonzero vector. Use that component in a sequence of row operations to convert $\vec{z}_1$ to $\vec{z}$. (This sketch could be filled out as an induction argument on the dimension of V.)

Three.III.1.19 This is the topic of the next subsection.

Three.III.1.20 A change of basis matrix is nonsingular and thus has rank equal to the number of its columns. Therefore its set of columns is a linearly independent subset of size n in $\mathbb{R}^n$ and it is thus a basis. The answer to the second half is also 'yes'; all implications in the prior sentence reverse (that is, all of the 'if ... then ...' parts of the prior sentence convert to 'if and only if' parts).

Three.III.1.21 In response to the first half of the question, there are infinitely many such matrices. One of them represents with respect to $\mathcal{E}_2$ the transformation of $\mathbb{R}^2$ with this action.

$$\begin{pmatrix}1\\0\end{pmatrix}\mapsto\begin{pmatrix}4\\0\end{pmatrix}\qquad\begin{pmatrix}0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\-1/3\end{pmatrix}$$

The problem of specifying two distinct input/output pairs is a bit trickier. The fact that matrices have a linear action precludes some possibilities.

(a) Yes, there is such a matrix. These conditions

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\qquad\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$$

can be solved

$$\begin{array}{rl}a+3b&=1\\ c+3d&=1\\ 2a-b&=-1\\ 2c-d&=-1\end{array}$$

to give this matrix.

$$\begin{pmatrix} -2/7 & 3/7 \\ -2/7 & 3/7 \end{pmatrix}$$

(b) No, because

$$2 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \text{but} \quad 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

no linear action can produce this effect.

(c) A sufficient condition is that $\{\vec{v}_1, \vec{v}_2\}$ be linearly independent, but that's not a necessary condition. A necessary and sufficient condition is that any linear dependences among the starting vectors appear also among the ending vectors. That is,

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \quad \text{implies} \quad c_1\vec{w}_1 + c_2\vec{w}_2 = \vec{0}.$$

The proof of this condition is routine.

Subsection Three.III.2: Changing Map Representations

Three.III.2.10 **(a)** Yes, each has rank two.
(b) Yes, they have the same rank.
(c) No, they have different ranks.

Three.III.2.11 We need only decide what the rank of each is.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Three.III.2.12 Recall the diagram and the formula.

$$\begin{array}{ccc} \mathbb{R}^2_{\text{w.r.t. } B} & \xrightarrow[T]{t} & \mathbb{R}^2_{\text{w.r.t. } D} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ \mathbb{R}^2_{\text{w.r.t. } \hat{B}} & \xrightarrow[\hat{T}]{t} & \mathbb{R}^2_{\text{w.r.t. } \hat{D}} \end{array} \qquad \hat{T} = \text{Rep}_{D,\hat{D}}(\text{id}) \cdot T \cdot \text{Rep}_{\hat{B},B}(\text{id})$$

(a) These two

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-3) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

show that

$$\text{Rep}_{D,\hat{D}}(\text{id}) = \begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix}$$

and similarly these two

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

give the other nonsinguar matrix.

$$\text{Rep}_{\hat{B},B}(\text{id}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Then the answer is this.

$$\hat{T} = \begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -10 & -18 \\ -2 & -4 \end{pmatrix}$$

Although not strictly necessary, a check is reassuring. Arbitrarily fixing

$$\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

we have that

$$\text{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}_B \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{B,D} \begin{pmatrix} 3 \\ 2 \end{pmatrix}_B = \begin{pmatrix} 7 \\ 17 \end{pmatrix}_D$$

and so $t(\vec{v})$ is this.

$$7 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 17 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ -10 \end{pmatrix}$$

Doing the calculation with respect to $\hat{B},\hat{D}$ starts with

$$\mathrm{Rep}_{\hat{B}}(\vec{v})=\begin{pmatrix}-1\\3\end{pmatrix}_{\hat{B}}\qquad\begin{pmatrix}-10&-18\\-2&-4\end{pmatrix}_{\hat{B},\hat{D}}\begin{pmatrix}-1\\3\end{pmatrix}_{\hat{B}}=\begin{pmatrix}-44\\-10\end{pmatrix}_{\hat{D}}$$

and then checks that this is the same result.

$$-44\cdot\begin{pmatrix}-1\\0\end{pmatrix}-10\cdot\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}24\\-10\end{pmatrix}$$

(b) These two

$$\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{3}\cdot\begin{pmatrix}1\\2\end{pmatrix}+\frac{1}{3}\cdot\begin{pmatrix}2\\1\end{pmatrix}\qquad\begin{pmatrix}1\\-1\end{pmatrix}=-1\cdot\begin{pmatrix}1\\2\end{pmatrix}+1\cdot\begin{pmatrix}2\\1\end{pmatrix}$$

show that

$$\mathrm{Rep}_{D,\hat{D}}(\mathrm{id})=\begin{pmatrix}1/3&-1\\1/3&1\end{pmatrix}$$

and these two

$$\begin{pmatrix}1\\2\end{pmatrix}=1\cdot\begin{pmatrix}1\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\end{pmatrix}\qquad\begin{pmatrix}1\\0\end{pmatrix}=-1\cdot\begin{pmatrix}1\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

show this.

$$\mathrm{Rep}_{\hat{B},B}(\mathrm{id})=\begin{pmatrix}1&1\\2&0\end{pmatrix}$$

With those, the conversion goes in this way.

$$\hat{T}=\begin{pmatrix}1/3&-1\\1/3&1\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&1\\2&0\end{pmatrix}=\begin{pmatrix}-28/3&-8/3\\38/3&10/3\end{pmatrix}$$

As in the prior item, a check provides some confidence that this calculation was performed without mistakes. We can for instance, fix the vector

$$\vec{v}=\begin{pmatrix}-1\\2\end{pmatrix}$$

(this is selected for no reason, out of thin air). Now we have

$$\mathrm{Rep}_B(\vec{v})=\begin{pmatrix}-1\\2\end{pmatrix}\qquad\begin{pmatrix}1&2\\3&4\end{pmatrix}_{B,D}\begin{pmatrix}-1\\2\end{pmatrix}_B=\begin{pmatrix}3\\5\end{pmatrix}_D$$

and so $t(\vec{v})$ is this vector.

$$3\cdot\begin{pmatrix}1\\1\end{pmatrix}+5\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}8\\-2\end{pmatrix}$$

With respect to $\hat{B},\hat{D}$ we first calculate

$$\mathrm{Rep}_{\hat{B}}(\vec{v})=\begin{pmatrix}1\\-2\end{pmatrix}\qquad\begin{pmatrix}-28/3&-8/3\\38/3&10/3\end{pmatrix}_{\hat{B},\hat{D}}\begin{pmatrix}1\\-2\end{pmatrix}_{\hat{B}}=\begin{pmatrix}-4\\6\end{pmatrix}_{\hat{D}}$$

and, sure enough, that is the same result for $t(\vec{v})$.

$$-4\cdot\begin{pmatrix}1\\2\end{pmatrix}+6\cdot\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}8\\-2\end{pmatrix}$$

Three.III.2.13 Where H and $\hat{H}$ are $m\times n$, the matrix P is $m\times m$ while Q is $n\times n$.

Three.III.2.14 Any $n\times n$ matrix is nonsingular if and only if it has rank n, that is, by Theorem 2.6, if and only if it is matrix equivalent to the $n\times n$ matrix whose diagonal is all ones.

Three.III.2.15 If $PAQ=I$ then $QPAQ=Q$, so $QPA=I$, and so $QP=A^{-1}$.

Three.III.2.16 By the definition following Example 2.2, a matrix M is diagonalizable if it represents $M=\mathrm{Rep}_{B,D}(t)$ a transformation with the property that there is some basis $\hat{B}$ such that $\mathrm{Rep}_{\hat{B},\hat{B}}(t)$ is a diagonal matrix — the starting and ending bases must be equal. But Theorem 2.6 says only that there are $\hat{B}$ and $\hat{D}$ such that we can change to a representation $\mathrm{Rep}_{\hat{B},\hat{D}}(t)$ and get a diagonal matrix. We have no reason to suspect that we could pick the two $\hat{B}$ and $\hat{D}$ so that they are equal.

Three.III.2.17 Yes. Row rank equals column rank, so the rank of the transpose equals the rank of the matrix. Same-sized matrices with equal ranks are matrix equivalent.

Three.III.2.18 Only a zero matrix has rank zero.

Three.III.2.19 For reflexivity, to show that any matrix is matrix equivalent to itself, take P and Q to be identity matrices. For symmetry, if $H_1 = PH_2Q$ then $H_2 = P^{-1}H_1Q^{-1}$ (inverses exist because P and Q are nonsingular). Finally, for transitivity, assume that $H_1 = P_2H_2Q_2$ and that $H_2 = P_3H_3Q_3$. Then substitution gives $H_1 = P_2(P_3H_3Q_3)Q_2 = (P_2P_3)H_3(Q_3Q_2)$. A product of nonsingular matrices is nonsingular (we've shown that the product of invertible matrices is invertible; in fact, we've shown how to calculate the inverse) and so H_1 is therefore matrix equivalent to H_3.

Three.III.2.20 By Theorem 2.6, a zero matrix is alone in its class because it is the only $m\times n$ of rank zero. No other matrix is alone in its class; any nonzero scalar product of a matrix has the same rank as that matrix.

Three.III.2.21 There are two matrix-equivalence classes of 1×1 matrices — those of rank zero and those of rank one. The 3×3 matrices fall into four matrix equivalence classes.

Three.III.2.22 For $m\times n$ matrices there are classes for each possible rank: where k is the minimum of m and n there are classes for the matrices of rank 0, 1, ..., k. That's $k+1$ classes. (Of course, totaling over all sizes of matrices we get infinitely many classes.)

Three.III.2.23 They are closed under nonzero scalar multiplication, since a nonzero scalar multiple of a matrix has the same rank as does the matrix. They are not closed under addition, for instance, $H + (-H)$ has rank zero.

Three.III.2.24 **(a)** We have

$$\mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id}) = \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id})^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$$

and thus the answer is this.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -5 & 2 \end{pmatrix}$$

As a quick check, we can take a vector at random

$$\vec{v} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

giving

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} = t(\vec{v})$$

while the calculation with respect to B, B

$$\mathrm{Rep}_{B}(\vec{v}) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 \\ -5 & 2 \end{pmatrix}_{B,B} \begin{pmatrix} 1 \\ -3 \end{pmatrix}_B = \begin{pmatrix} -2 \\ -11 \end{pmatrix}_B$$

yields the same result.

$$-2\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 11\cdot\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

(b) We have

$$\begin{array}{ccc} \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \mathrm{id}\big\downarrow & & \mathrm{id}\big\downarrow \\ \mathbb{R}^2_{\text{w.r.t. } B} & \xrightarrow[\hat{T}]{t} & \mathbb{R}^2_{\text{w.r.t. } B} \end{array} \qquad \mathrm{Rep}_{B,B}(t) = \mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id}) \cdot T \cdot \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id})$$

and, as in the first item of this question

$$\mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} \vec{\beta}_1 & \big| & \cdots & \big| & \vec{\beta}_n \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id}) = \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id})^{-1}$$

so, writing Q for the matrix whose columns are the basis vectors, we have that $\mathrm{Rep}_{B,B}(t) = Q^{-1}TQ$.

Three.III.2.25 **(a)** The adapted form of the arrow diagram is this.

$$\begin{array}{ccc} V_{\text{w.r.t. } B_1} & \xrightarrow[H]{h} & W_{\text{w.r.t. } D} \\ \mathrm{id}\big\downarrow Q & & \mathrm{id}\big\downarrow P \\ V_{\text{w.r.t. } B_2} & \xrightarrow[\hat{H}]{h} & W_{\text{w.r.t. } D} \end{array}$$

Since there is no need to change bases in W (or we can say that the change of basis matrix P is the identity), we have $\mathrm{Rep}_{B_2,D}(h) = \mathrm{Rep}_{B_1,D}(h) \cdot Q$ where $Q = \mathrm{Rep}_{B_2,B_1}(\mathrm{id})$.

(b) Here, this is the arrow diagram.

$$\begin{array}{ccc} V_{\text{w.r.t. } B} & \xrightarrow[H]{h} & W_{\text{w.r.t. } D_1} \\ \text{id}\big\downarrow Q & & \text{id}\big\downarrow P \\ V_{\text{w.r.t. } B} & \xrightarrow[\hat{H}]{h} & W_{\text{w.r.t. } D_2} \end{array}$$

We have that $\text{Rep}_{B,D_2}(h) = P \cdot \text{Rep}_{B,D_1}(h)$ where $P = \text{Rep}_{D_1,D_2}(\text{id})$.

Three.III.2.26 **(a)** Here is the arrow diagram, and a version of that diagram for inverse functions.

$$\begin{array}{ccc} V_{\text{w.r.t. } B} & \xrightarrow[H]{h} & W_{\text{w.r.t. } D} \\ \text{id}\big\downarrow Q & & \text{id}\big\downarrow P \\ V_{\text{w.r.t. } \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\text{w.r.t. } \hat{D}} \end{array} \qquad \begin{array}{ccc} V_{\text{w.r.t. } B} & \xleftarrow[H^{-1}]{h^{-1}} & W_{\text{w.r.t. } D} \\ \text{id}\big\downarrow Q & & \text{id}\big\downarrow P \\ V_{\text{w.r.t. } \hat{B}} & \xleftarrow[\hat{H}^{-1}]{h^{-1}} & W_{\text{w.r.t. } \hat{D}} \end{array}$$

Yes, the inverses of the matrices represent the inverses of the maps. That is, we can move from the lower right to the lower left by moving up, then left, then down. In other words, where $\hat{H} = PHQ$ (and P, Q invertible) and $H, \hat{H}$ are invertible then $\hat{H}^{-1} = Q^{-1}H^{-1}P^{-1}$.

(b) Yes; this is the prior part repeated in different terms.

(c) No, we need another assumption: if H represents h with respect to the same starting as ending bases B, B, for some B then H^2 represents $h \circ h$. As a specific example, these two matrices are both rank one and so they are matrix equivalent

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

but the squares are not matrix equivalent — the square of the first has rank one while the square of the second has rank zero.

(d) No. These two are not matrix equivalent but have matrix equivalent squares.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Three.III.2.27 **(a)** The definition is suggested by the appropriate arrow diagram.

$$\begin{array}{ccc} V_{\text{w.r.t. } B_1} & \xrightarrow[T]{t} & V_{\text{w.r.t. } B_1} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ V_{\text{w.r.t. } B_2} & \xrightarrow[\hat{T}]{t} & V_{\text{w.r.t. } B_2} \end{array}$$

Call matrices $T, \hat{T}$ *similar* if there is a nonsingular matrix P such that $\hat{T} = P^{-1}TP$.

(b) Take P^{-1} to be P and take P to be Q.

(c) *This is as in Exercise 19.* Reflexivity is obvious: $T = I^{-1}TI$. Symmetry is also easy: $\hat{T} = P^{-1}TP$ implies that $T = P\hat{T}P^{-1}$ (multiply the first equation from the right by P^{-1} and from the left by P). For transitivity, assume that $T_1 = {P_2}^{-1}T_2P_2$ and that $T_2 = {P_3}^{-1}T_3P_3$. Then $T_1 = {P_2}^{-1}({P_3}^{-1}T_3P_3)P_2 = ({P_2}^{-1}{P_3}^{-1})T_3(P_3P_2)$ and we are finished on noting that P_3P_2 is an invertible matrix with inverse ${P_2}^{-1}{P_3}^{-1}$.

(d) Assume that $\hat{T} = P^{-1}TP$. For the squares: $\hat{T}^2 = (P^{-1}TP)(P^{-1}TP) = P^{-1}T(PP^{-1})TP = P^{-1}T^2P$. Higher powers follow by induction.

(e) These two are matrix equivalent but their squares are not matrix equivalent.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

By the prior item, matrix similarity and matrix equivalence are thus different.

Chapter Five: Similarity and Diagonalization

Subsection Five.I.1: Definition and Examples

Five.I.1.4 One way to proceed is left to right.

$$PSP^{-1}=\begin{pmatrix}4&2\\-3&2\end{pmatrix}\begin{pmatrix}1&3\\-2&-6\end{pmatrix}\begin{pmatrix}2/14&-2/14\\3/14&4/14\end{pmatrix}=\begin{pmatrix}0&0\\-7&-21\end{pmatrix}\begin{pmatrix}2/14&-2/14\\3/14&4/14\end{pmatrix}=\begin{pmatrix}0&0\\-11/2&-5\end{pmatrix}$$

Five.I.1.5 **(a)** Because the matrix (2) is 1×1, the matrices P and P^{-1} are also 1×1 and so where $P=(p)$ the inverse is $P^{-1}=(1/p)$. Thus $P(2)P^{-1}=(p)(2)(1/p)=(2)$.

(b) Yes: recall that scalar multiples can be brought out of a matrix $P(cI)P^{-1}=cPIP^{-1}=cI$. By the way, the zero and identity matrices are the special cases $c=0$ and $c=1$.

(c) No, as this example shows.

$$\begin{pmatrix}1&-2\\-1&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-3\end{pmatrix}\begin{pmatrix}-1&-2\\-1&-1\end{pmatrix}=\begin{pmatrix}-5&-4\\2&1\end{pmatrix}$$

Five.I.1.6 Gauss' method shows that the first matrix represents maps of rank two while the second matrix represents maps of rank three.

Five.I.1.7 **(a)** Because t is described with the members of B, finding the matrix representation is easy:

$$\mathrm{Rep}_B(t(x^2))=\begin{pmatrix}0\\1\\1\end{pmatrix}_B\qquad\mathrm{Rep}_B(t(x))=\begin{pmatrix}1\\0\\-1\end{pmatrix}_B\qquad\mathrm{Rep}_B(t(1))=\begin{pmatrix}0\\0\\3\end{pmatrix}_B$$

gives this.

$$\mathrm{Rep}_{B,B}(t)\begin{pmatrix}0&1&0\\1&0&0\\1&-1&3\end{pmatrix}$$

(b) We will find $t(1)$, $t(1+x)$, and $t(1+x+x^2$, to find how each is represented with respect to D. We are given that $t(1)=3$, and the other two are easy to see: $t(1+x)=x^2+2$ and $t(1+x+x^2)=x^2+x+3$. By eye, we get the representation of each vector

$$\mathrm{Rep}_D(t(1))=\begin{pmatrix}3\\0\\0\end{pmatrix}_D\qquad\mathrm{Rep}_D(t(1+x))=\begin{pmatrix}2\\-1\\1\end{pmatrix}_D\qquad\mathrm{Rep}_D(t(1+x+x^2))=\begin{pmatrix}2\\0\\1\end{pmatrix}_D$$

and thus the representation of the map.

$$\mathrm{Rep}_{D,D}(t)=\begin{pmatrix}3&2&2\\0&-1&0\\0&1&1\end{pmatrix}$$

(c) The diagram, adapted for this T and S,

$$\begin{array}{ccc}V_{\text{w.r.t. }D}&\xrightarrow[S]{t}&V_{\text{w.r.t. }D}\\ \mathrm{id}\big\downarrow P&&\mathrm{id}\big\downarrow P\\ V_{\text{w.r.t. }B}&\xrightarrow[T]{t}&V_{\text{w.r.t. }B}\end{array}$$

shows that $P=\mathrm{Rep}_{D,B}(\mathrm{id})$.

$$P=\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}$$

Five.I.1.8 One possible choice of the bases is

$$B = \langle \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \rangle \qquad D = \mathcal{E}_2 = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle$$

(this B is suggested by the map description). To find the matrix $T = \mathrm{Rep}_{B,B}(t)$, solve the relations

$$c_1 \begin{pmatrix}1\\2\end{pmatrix} + c_2 \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} \qquad \hat{c}_1 \begin{pmatrix}1\\2\end{pmatrix} + \hat{c}_2 \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

to get $c_1 = 1$, $c_2 = -2$, $\hat{c}_1 = 1/3$ and $\hat{c}_2 = 4/3$.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix}1 & 1/3\\-2 & 4/3\end{pmatrix}$$

Finding $\mathrm{Rep}_{D,D}(t)$ involves a bit more computation. We first find $t(\vec{e}_1)$. The relation

$$c_1 \begin{pmatrix}1\\2\end{pmatrix} + c_2 \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

gives $c_1 = 1/3$ and $c_2 = -2/3$, and so

$$\mathrm{Rep}_B(\vec{e}_1) = \begin{pmatrix}1/3\\-2/3\end{pmatrix}_B$$

making

$$\mathrm{Rep}_B(t(\vec{e}_1)) = \begin{pmatrix}1 & 1/3\\-2 & 4/3\end{pmatrix}_{B,B} \begin{pmatrix}1/3\\-2/3\end{pmatrix}_B = \begin{pmatrix}1/9\\-14/9\end{pmatrix}_B$$

and hence t acts on the first basis vector $\vec{e}_1$ in this way.

$$t(\vec{e}_1) = (1/9) \cdot \begin{pmatrix}1\\2\end{pmatrix} - (14/9) \cdot \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}5/3\\-4/3\end{pmatrix}$$

The computation for $t(\vec{e}_2)$ is similar. The relation

$$c_1 \begin{pmatrix}1\\2\end{pmatrix} + c_2 \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

gives $c_1 = 1/3$ and $c_2 = 1/3$, so

$$\mathrm{Rep}_B(\vec{e}_1) = \begin{pmatrix}1/3\\1/3\end{pmatrix}_B$$

making

$$\mathrm{Rep}_B(t(\vec{e}_1)) = \begin{pmatrix}1 & 1/3\\-2 & 4/3\end{pmatrix}_{B,B} \begin{pmatrix}1/3\\1/3\end{pmatrix}_B = \begin{pmatrix}4/9\\-2/9\end{pmatrix}_B$$

and hence t acts on the second basis vector $\vec{e}_2$ in this way.

$$t(\vec{e}_2) = (4/9) \cdot \begin{pmatrix}1\\2\end{pmatrix} - (2/9) \cdot \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2/3\\2/3\end{pmatrix}$$

Therefore

$$\mathrm{Rep}_{D,D}(t) = \begin{pmatrix}5/3 & 2/3\\-4/3 & 2/3\end{pmatrix}$$

and these are the change of basis matrices.

$$P = \mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix} \qquad P^{-1} = \bigl(\mathrm{Rep}_{B,D}(\mathrm{id})\bigr)^{-1} = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1/3 & 1/3\\-2/3 & 1/3\end{pmatrix}$$

The check of these computations is routine.

$$\begin{pmatrix}1 & -1\\2 & 1\end{pmatrix} \begin{pmatrix}1 & 1/3\\-2 & 4/3\end{pmatrix} \begin{pmatrix}1/3 & 1/3\\-2/3 & 1/3\end{pmatrix} = \begin{pmatrix}5/3 & 2/3\\-4/3 & 2/3\end{pmatrix}$$

Five.I.1.9 The only representation of a zero map is a zero matrix, no matter what the pair of bases $\mathrm{Rep}_{B,D}(z) = Z$, and so in particular for any single basis B we have $\mathrm{Rep}_{B,B}(z) = Z$. The case of the identity is related, but slightly different: the only representation of the identity map, with respect to any B, B, is the identity $\mathrm{Rep}_{B,B}(\mathrm{id}) = I$. (*Remark:* of course, we have seen examples where $B \neq D$ and $\mathrm{Rep}_{B,D}(\mathrm{id}) \neq I$ — in fact, we have seen that any nonsingular matrix is a representation of the identity map with respect to some B, D.)

Five.I.1.10 No. If $A = PBP^{-1}$ then $A^2 = (PBP^{-1})(PBP^{-1}) = PB^2P^{-1}$.

Five.I.1.11 Matrix similarity is a special case of matrix equivalence (if matrices are similar then they are matrix equivalent) and matrix equivalence preserves nonsingularity.

Five.I.1.12 A matrix is similar to itself; take P to be the identity matrix: $IPI^{-1} = IPI = P$.

If T is similar to S then $T = PSP^{-1}$ and so $P^{-1}TP = S$. Rewrite this as $S = (P^{-1})T(P^{-1})^{-1}$ to conclude that S is similar to T.

If T is similar to S and S is similar to U then $T = PSP^{-1}$ and $S = QUQ^{-1}$. Then $T = PQUQ^{-1}P^{-1} = (PQ)U(PQ)^{-1}$, showing that T is similar to U.

Five.I.1.13 Let f_x and f_y be the reflection maps (sometimes called 'flip's). For any bases B and D, the matrices $\text{Rep}_{B,B}(f_x)$ and $\text{Rep}_{D,D}(f_y)$ are similar. First note that

$$S = \text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T = \text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are similar because the second matrix is the representation of f_x with respect to the basis $A = \langle \vec{e}_2, \vec{e}_1 \rangle$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$$

where $P = \text{Rep}_{A,\mathcal{E}_2}(\text{id})$.

$$\begin{array}{ccc} \mathbb{R}^2_{\text{w.r.t. } A} & \xrightarrow[T]{f_x} & V\mathbb{R}^2_{\text{w.r.t. } A} \\ \text{id}\big\downarrow P & & \text{id}\big\downarrow P \\ \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[S]{f_x} & \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \end{array}$$

Now the conclusion follows from the transitivity part of Exercise 12.

To finish without relying on that exercise, write $\text{Rep}_{B,B}(f_x) = QTQ^{-1} = Q\text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_x)Q^{-1}$ and $\text{Rep}_{D,D}(f_y) = RSR^{-1} = R\text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_y)R^{-1}$. Using the equation in the first paragraph, the first of these two becomes $\text{Rep}_{B,B}(f_x) = QP\text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_y)P^{-1}Q^{-1}$ and rewriting the second of these two as $R^{-1} \cdot \text{Rep}_{D,D}(f_y) \cdot R = \text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_y)$ and substituting gives the desired relationship

$$\begin{aligned} \text{Rep}_{B,B}(f_x) &= QP\text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(f_y)P^{-1}Q^{-1} \\ &= QPR^{-1} \cdot \text{Rep}_{D,D}(f_y) \cdot RP^{-1}Q^{-1} = (QPR^{-1}) \cdot \text{Rep}_{D,D}(f_y) \cdot (QPR^{-1})^{-1} \end{aligned}$$

Thus the matrices $\text{Rep}_{B,B}(f_x)$ and $\text{Rep}_{D,D}(f_y)$ are similar.

Five.I.1.14 We must show that if two matrices are similar then they have the same determinant and the same rank. Both determinant and rank are properties of matrices that we have already shown to be preserved by matrix equivalence. They are therefore preserved by similarity (which is a special case of matrix equivalence: if two matrices are similar then they are matrix equivalent).

To prove the statement without quoting the results about matrix equivalence, note first that rank is a property of the map (it is the dimension of the rangespace) and since we've shown that the rank of a map is the rank of a representation, it must be the same for all representations. As for determinants, $|PSP^{-1}| = |P| \cdot |S| \cdot |P^{-1}| = |P| \cdot |S| \cdot |P|^{-1} = |S|$.

The converse of the statement does not hold; for instance, there are matrices with the same determinant that are not similar. To check this, consider a nonzero matrix with a determinant of zero. It is not similar to the zero matrix, the zero matrix is similar only to itself, but they have they same determinant. The argument for rank is much the same.

Five.I.1.15 The matrix equivalence class containing all $n \times n$ rank zero matrices contains only a single matrix, the zero matrix. Therefore it has as a subset only one similarity class.

In contrast, the matrix equivalence class of 1×1 matrices of rank one consists of those 1×1 matrices (k) where $k \neq 0$. For any basis B, the representation of multiplication by the scalar k is $\text{Rep}_{B,B}(t_k) = (k)$, so each such matrix is alone in its similarity class. So this is a case where a matrix equivalence class splits into infinitely many similarity classes.

Five.I.1.16 Yes, these are similar

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

since, where the first matrix is $\text{Rep}_{B,B}(t)$ for $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$, the second matrix is $\text{Rep}_{D,D}(t)$ for $D = \langle \vec{\beta}_2, \vec{\beta}_1 \rangle$.

Five.I.1.17 The k-th powers are similar because, where each matrix represents the map t, the k-th powers represent t^k, the composition of k-many t's. (For instance, if $T = reptB, B$ then $T^2 = \text{Rep}_{B,B}(t \circ t)$.)

Restated more computationally, if $T = PSP^{-1}$ then $T^2 = (PSP^{-1})(PSP^{-1}) = PS^2P^{-1}$. Induction extends that to all powers.

For the $k \leq 0$ case, suppose that S is invertible and that $T = PSP^{-1}$. Note that T is invertible: $T^{-1} = (PSP^{-1})^{-1} = PS^{-1}P^{-1}$, and that same equation shows that T^{-1} is similar to S^{-1}. Other negative powers are now given by the first paragraph.

Five.I.1.18 In conceptual terms, both represent $p(t)$ for some transformation t. In computational terms, we have this.

$$\begin{aligned} p(T) &= c_n(PSP^{-1})^n + \cdots + c_1(PSP^{-1}) + c_0I \\ &= c_nPS^nP^{-1} + \cdots + c_1PSP^{-1} + c_0I \\ &= Pc_nS^nP^{-1} + \cdots + Pc_1SP^{-1} + Pc_0P^{-1} \\ &= P(c_nS^n + \cdots + c_1S + c_0)P^{-1} \end{aligned}$$

Five.I.1.19 There are two equivalence classes, (i) the class of rank zero matrices, of which there is one: $\mathscr{C}_1 = \{(0)\}$, and (2) the class of rank one matrices, of which there are infinitely many: $\mathscr{C}_2 = \{(k) \mid k \neq 0\}$.

Each 1×1 matrix is alone in its similarity class. That's because any transformation of a one-dimensional space is multiplication by a scalar $t_k\colon V \to V$ given by $\vec{v} \mapsto k \cdot \vec{v}$. Thus, for any basis $B = \langle\vec{\beta}\rangle$, the matrix representing a transformation t_k with respect to B, B is $(\text{Rep}_B(t_k(\vec{\beta}))) = (k)$.

So, contained in the matrix equivalence class $\mathscr{C}_1$ is (obviously) the single similarity class consisting of the matrix (0). And, contained in the matrix equivalence class $\mathscr{C}_2$ are the infinitely many, one-member-each, similarity classes consisting of (k) for $k \neq 0$.

Five.I.1.20 No. Here is an example that has two pairs, each of two similar matrices:

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 5/3 & -2/3 \\ -4/3 & 7/3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -5 & -4 \\ 2 & 1 \end{pmatrix}$$

(this example is mostly arbitrary, but not entirely, because the the center matrices on the two left sides add to the zero matrix). Note that the sums of these similar matrices are not similar

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 5/3 & -2/3 \\ -4/3 & 7/3 \end{pmatrix} + \begin{pmatrix} -5 & -4 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

since the zero matrix is similar only to itself.

Five.I.1.21 If $N = P(T - \lambda I)P^{-1}$ then $N = PTP^{-1} - P(\lambda I)P^{-1}$. The diagonal matrix λI commutes with anything, so $P(\lambda I)P^{-1} = PP^{-1}(\lambda I) = \lambda I$. Thus $N = PTP^{-1} - \lambda I$ and consequently $N + \lambda I = PTP^{-1}$. (So not only are they similar, in fact they are similar via the same P.)

Subsection Five.I.2: Diagonalizability

Five.I.2.6 Because the basis vectors are chosen arbitrarily, many different answers are possible. However, here is one way to go; to diagonalize

$$T = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

take it as the representation of a transformation with respect to the standard basis $T = \text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ and look for $B = \langle\vec{\beta}_1, \vec{\beta}_2\rangle$ such that

$$\text{Rep}_{B,B}(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

that is, such that $t(\vec{\beta}_1) = \lambda_1$ and $t(\vec{\beta}_2) = \lambda_2$.

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \vec{\beta}_1 = \lambda_1 \cdot \vec{\beta}_1 \qquad \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \vec{\beta}_2 = \lambda_2 \cdot \vec{\beta}_2$$

We are looking for scalars x such that this equation

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has solutions b_1 and b_2, which are not both zero. Rewrite that as a linear system

$$\begin{array}{rl} (4-x)\cdot b_1 + & -2\cdot b_2 = 0 \\ 1\cdot b_1 + & (1-x)\cdot b_2 = 0 \end{array}$$

If $x = 4$ then the first equation gives that $b_2 = 0$, and then the second equation gives that $b_1 = 0$. The case where both b's are zero is disallowed so we can assume that $x \neq 4$.

$$\xrightarrow{(-1/(4-x))\rho_1+\rho_2} \begin{array}{rl} (4-x)\cdot b_1 + & -2\cdot b_2 = 0 \\ & ((x^2-5x+6)/(4-x))\cdot b_2 = 0 \end{array}$$

Consider the bottom equation. If $b_2 = 0$ then the first equation gives $b_1 = 0$ or $x = 4$. The $b_1 = b_2 = 0$ case is disallowed. The other possibility for the bottom equation is that the numerator of the fraction $x^2 - 5x + 6 = (x-2)(x-3)$ is zero. The $x = 2$ case gives a first equation of $2b_1 - 2b_2 = 0$, and so associated with $x = 2$ we have vectors whose first and second components are equal:

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (\text{so } \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ and } \lambda_1 = 2).$$

If $x = 3$ then the first equation is $b_1 - 2b_2 = 0$ and so the associated vectors are those whose first component is twice their second:

$$\vec{\beta}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad (\text{so } \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and so } \lambda_2 = 3).$$

This picture

$$\begin{array}{ccc} \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ \mathbb{R}^2_{\text{w.r.t. } B} & \xrightarrow[D]{t} & \mathbb{R}^2_{\text{w.r.t. } B} \end{array}$$

shows how to get the diagonalization.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Comment. This equation matches the $T = PSP^{-1}$ definition under this renaming.

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \quad P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

Five.I.2.7 **(a)** Setting up

$$\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x\cdot\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{array}{rl} (-2-x)\cdot b_1 + & b_2 = 0 \\ & (2-x)\cdot b_2 = 0 \end{array}$$

gives the two possibilities that $b_2 = 0$ and $x = 2$. Following the $b_2 = 0$ possibility leads to the first equation $(-2-x)b_1 = 0$ with the two cases that $b_1 = 0$ and that $x = -2$. Thus, under this first possibility, we find $x = -2$ and the associated vectors whose second component is zero, and whose first component is free.

$$\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ 0 \end{pmatrix} = -2\cdot\begin{pmatrix} b_1 \\ 0 \end{pmatrix} \qquad \vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Following the other possibility leads to a first equation of $-4b_1 + b_2 = 0$ and so the vectors associated with this solution have a second component that is four times their first component.

$$\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ 4b_1 \end{pmatrix} = 2\cdot\begin{pmatrix} b_1 \\ 4b_1 \end{pmatrix} \qquad \vec{\beta}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

The diagonalization is this.

$$\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^{-1}\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}^{-1}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

(b) The calculations are like those in the prior part.

$$\begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x\cdot\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{array}{rl} (5-x)\cdot b_1 + & 4\cdot b_2 = 0 \\ & (1-x)\cdot b_2 = 0 \end{array}$$

The bottom equation gives the two possibilities that $b_2 = 0$ and $x = 1$. Following the $b_2 = 0$ possibility, and discarding the case where both b_2 and b_1 are zero, gives that $x = 5$, associated with vectors whose second component is zero and whose first component is free.

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The $x = 1$ possibility gives a first equation of $4b_1 + 4b_2 = 0$ and so the associated vectors have a second component that is the negative of their first component.

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We thus have this diagonalization.

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

Five.I.2.8 For any integer p,

$$\begin{pmatrix} d_1 & 0 & \\ 0 & \ddots & \\ & & d_n \end{pmatrix}^p = \begin{pmatrix} d_1^p & 0 & \\ 0 & \ddots & \\ & & d_n^p \end{pmatrix}.$$

Five.I.2.9 These two are not similar

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

because each is alone in its similarity class.

For the second half, these

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

are similar via the matrix that changes bases from $\langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ to $\langle \vec{\beta}_2, \vec{\beta}_1 \rangle$. (*Question.* Are two diagonal matrices similar if and only if their diagonal entries are permutations of each other's?)

Five.I.2.10 Contrast these two.

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

The first is nonsingular, the second is singular.

Five.I.2.11 To check that the inverse of a diagonal matrix is the diagonal matrix of the inverses, just multiply.

$$\begin{pmatrix} a_{1,1} & 0 & & \\ 0 & a_{2,2} & & \\ & & \ddots & \\ & & & a_{n,n} \end{pmatrix} \begin{pmatrix} 1/a_{1,1} & 0 & & \\ 0 & 1/a_{2,2} & & \\ & & \ddots & \\ & & & 1/a_{n,n} \end{pmatrix}$$

(Showing that it is a left inverse is just as easy.)

If a diagonal entry is zero then the diagonal matrix is singular; it has a zero determinant.

Five.I.2.12 **(a)** The check is easy.

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) It is a coincidence, in the sense that if $T = PSP^{-1}$ then T need not equal $P^{-1}SP$. Even in the case of a diagonal matrix D, the condition that $D = PTP^{-1}$ does not imply that D equals $P^{-1}TP$. The matrices from Example 2.2 show this.

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 5 & -1 \end{pmatrix} \qquad \begin{pmatrix} 6 & 0 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & 12 \\ -6 & 11 \end{pmatrix}$$

Five.I.2.13 The columns of the matrix are chosen as the vectors associated with the x's. The exact choice, and the order of the choice was arbitrary. We could, for instance, get a different matrix by swapping the two columns.

Five.I.2.14 Diagonalizing and then taking powers of the diagonal matrix shows that

$$\begin{pmatrix} -3 & 1 \\ -4 & 2 \end{pmatrix}^k = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} + (\frac{-2}{3})^k \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}.$$

Five.I.2.15 **(a)** $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Five.I.2.16 Yes, ct is diagonalizable by the final theorem of this subsection.

No, $t+s$ need not be diagonalizable. Intuitively, the problem arises when the two maps diagonalize with respect to different bases (that is, when they are not *simultaneously diagonalizable*). Specifically, these two are diagonalizable but their sum is not:

$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\qquad\begin{pmatrix}-1&0\\0&0\end{pmatrix}$$

(the second is already diagonal; for the first, see Exercise 15). The sum is not diagonalizable because its square is the zero matrix.

The same intuition suggests that $t\circ s$ is not be diagonalizable. These two are diagonalizable but their product is not:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\qquad\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

(for the second, see Exercise 15).

Five.I.2.17 If

$$P\begin{pmatrix}1&c\\0&1\end{pmatrix}P^{-1}=\begin{pmatrix}a&0\\0&b\end{pmatrix}$$

then

$$P\begin{pmatrix}1&c\\0&1\end{pmatrix}=\begin{pmatrix}a&0\\0&b\end{pmatrix}P$$

so

$$\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}1&c\\0&1\end{pmatrix}=\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix}$$
$$\begin{pmatrix}p&cp+q\\r&cr+s\end{pmatrix}=\begin{pmatrix}ap&aq\\br&bs\end{pmatrix}$$

The $1,1$ entries show that $a=1$ and the $1,2$ entries then show that $pc=0$. Since $c\neq 0$ this means that $p=0$. The $2,1$ entries show that $b=1$ and the $2,2$ entries then show that $rc=0$. Since $c\neq 0$ this means that $r=0$. But if both p and r are 0 then P is not invertible.

Five.I.2.18 **(a)** Using the formula for the inverse of a 2×2 matrix gives this.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\cdot\frac{1}{ad-bc}\cdot\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix}ad+2bd-2ac-bc&-ab-2b^2+2a^2+ab\\cd+2d^2-2c^2-cd&-bc-2bd+2ac+ad\end{pmatrix}$$

Now pick scalars $a,\ldots,d$ so that $ad-bc\neq 0$ and $2d^2-2c^2=0$ and $2a^2-2b^2=0$. For example, these will do.

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\cdot\frac{1}{-2}\cdot\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}-6&0\\0&2\end{pmatrix}$$

(b) As above,

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x&y\\y&z\end{pmatrix}\cdot\frac{1}{ad-bc}\cdot\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix}adx+bdy-acy-bcz&-abx-b^2y+a^2y+abz\\cdx+d^2y-c^2y-cdz&-bcx-bdy+acy+adz\end{pmatrix}$$

we are looking for scalars $a,\ldots,d$ so that $ad-bc\neq 0$ and $-abx-b^2y+a^2y+abz=0$ and $cdx+d^2y-c^2y-cdz=0$, no matter what values x, y, and z have.

For starters, we assume that $y\neq 0$, else the given matrix is already diagonal. We shall use that assumption because if we (arbitrarily) let $a=1$ then we get

$$\begin{aligned}-bx-b^2y+y+bz&=0\\(-y)b^2+(z-x)b+y&=0\end{aligned}$$

and the quadratic formula gives

$$b=\frac{-(z-x)\pm\sqrt{(z-x)^2-4(-y)(y)}}{-2y}\qquad y\neq 0$$

(note that if x, y, and z are real then these two b's are real as the discriminant is positive). By the same token, if we (arbitrarily) let $c=1$ then

$$\begin{aligned}dx+d^2y-y-dz&=0\\(y)d^2+(x-z)d-y&=0\end{aligned}$$

and we get here

$$d=\frac{-(x-z)\pm\sqrt{(x-z)^2-4(y)(-y)}}{2y}\qquad y\neq 0$$

(as above, if $x, y, z \in \mathbb{R}$ then this discriminant is positive so a symmetric, real, 2×2 matrix is similar to a real diagonal matrix).

For a check we try $x = 1$, $y = 2$, $z = 1$.

$$b = \frac{0 \pm \sqrt{0+16}}{-4} = \mp 1 \qquad d = \frac{0 \pm \sqrt{0+16}}{4} = \pm 1$$

Note that not all four choices $(b, d) = (+1, +1), \ldots, (-1, -1)$ satisfy $ad - bc \neq 0$.

Subsection Five.I.3: Eigenvalues and Eigenvectors

Five.I.3.20 **(a)** This

$$0 = \begin{vmatrix} 10-x & -9 \\ 4 & -2-x \end{vmatrix} = (10-x)(-2-x) - (-36)$$

simplifies to the characteristic equation $x^2 - 8x + 16 = 0$. Because the equation factors into $(x-4)^2$ there is only one eigenvalue $\lambda_1 = 4$.

(b) $0 = (1-x)(3-x) - 8 = x^2 - 4x - 5$; $\lambda_1 = 5$, $\lambda_2 = -1$

(c) $x^2 - 21 = 0$; $\lambda_1 = \sqrt{21}$, $\lambda_2 = -\sqrt{21}$

(d) $x^2 = 0$; $\lambda_1 = 0$

(e) $x^2 - 2x + 1 = 0$; $\lambda_1 = 1$

Five.I.3.21 **(a)** The characteristic equation is $(3-x)(-1-x) = 0$. Its roots, the eigenvalues, are $\lambda_1 = 3$ and $\lambda_2 = -1$. For the eigenvectors we consider this equation.

$$\begin{pmatrix} 3-x & 0 \\ 8 & -1-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the eigenvector associated with $\lambda_1 = 3$, we consider the resulting linear system.

$$\begin{aligned} 0 \cdot b_1 + 0 \cdot b_2 &= 0 \\ 8 \cdot b_1 + -4 \cdot b_2 &= 0 \end{aligned}$$

The eigenspace is the set of vectors whose second component is twice the first component.

$$\{\begin{pmatrix} b_2/2 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} b_2/2 \\ b_2 \end{pmatrix} = 3 \cdot \begin{pmatrix} b_2/2 \\ b_2 \end{pmatrix}$$

(Here, the parameter is b_2 only because that is the variable that is free in the above system.) Hence, this is an eigenvector associated with the eigenvalue 3.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Finding an eigenvector associated with $\lambda_2 = -1$ is similar. This system

$$\begin{aligned} 4 \cdot b_1 + 0 \cdot b_2 &= 0 \\ 8 \cdot b_1 + 0 \cdot b_2 &= 0 \end{aligned}$$

leads to the set of vectors whose first component is zero.

$$\{\begin{pmatrix} 0 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ b_2 \end{pmatrix} = -1 \cdot \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$$

And so this is an eigenvector associated with λ_2.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) The characteristic equation is

$$0 = \begin{vmatrix} 3-x & 2 \\ -1 & -x \end{vmatrix} = x^2 - 3x + 2 = (x-2)(x-1)$$

and so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$. To find eigenvectors, consider this system.

$$\begin{aligned} (3-x) \cdot b_1 + 2 \cdot b_2 &= 0 \\ -1 \cdot b_1 - x \cdot b_2 &= 0 \end{aligned}$$

For $\lambda_1 = 2$ we get

$$\begin{aligned} 1 \cdot b_1 + 2 \cdot b_2 &= 0 \\ -1 \cdot b_1 - 2 \cdot b_2 &= 0 \end{aligned}$$

leading to this eigenspace and eigenvector.

$$\{\begin{pmatrix}-2b_2\\ b_2\end{pmatrix} \mid b_2\in\mathbb{C}\} \qquad \begin{pmatrix}-2\\ 1\end{pmatrix}$$

For $\lambda_2 = 1$ the system is

$$\begin{aligned} 2\cdot b_1 + 2\cdot b_2 &= 0\\ -1\cdot b_1 - 1\cdot b_2 &= 0 \end{aligned}$$

leading to this.

$$\{\begin{pmatrix}-b_2\\ b_2\end{pmatrix} \mid b_2\in\mathbb{C}\} \qquad \begin{pmatrix}-1\\ 1\end{pmatrix}$$

Five.I.3.22 The characteristic equation

$$0 = \begin{vmatrix}-2-x & -1\\ 5 & 2-x\end{vmatrix} = x^2+1$$

has the complex roots $\lambda_1 = i$ and $\lambda_2 = -i$. This system

$$\begin{aligned} (-2-x)\cdot b_1 - \quad 1\cdot b_2 &= 0\\ 5\cdot b_1 \quad (2-x)\cdot b_2 &= 0 \end{aligned}$$

For $\lambda_1 = i$ Gauss' method gives this reduction.

$$\begin{aligned} (-2-i)\cdot b_1 - \quad 1\cdot b_2 &= 0\\ 5\cdot b_1 - (2-i)\cdot b_2 &= 0 \end{aligned} \xrightarrow{(-5/(-2-i))\rho_1+\rho_2} \begin{aligned} (-2-i)\cdot b_1 - 1\cdot b_2 &= 0\\ 0 &= 0 \end{aligned}$$

(For the calculation in the lower right get a common denominator

$$\frac{5}{-2-i} - (2-i) = \frac{5}{-2-i} - \frac{-2-i}{-2-i}\cdot(2-i) = \frac{5-(-5)}{-2-i}$$

to see that it gives a $0=0$ equation.) These are the resulting eigenspace and eigenvector.

$$\{\begin{pmatrix}(1/(-2-i))b_2\\ b_2\end{pmatrix} \mid b_2\in\mathbb{C}\} \qquad \begin{pmatrix}1/(-2-i)\\ 1\end{pmatrix}$$

For $\lambda_2 = -i$ the system

$$\begin{aligned} (-2+i)\cdot b_1 - \quad 1\cdot b_2 &= 0\\ 5\cdot b_1 - (2+i)\cdot b_2 &= 0 \end{aligned} \xrightarrow{(-5/(-2+i))\rho_1+\rho_2} \begin{aligned} (-2+i)\cdot b_1 - 1\cdot b_2 &= 0\\ 0 &= 0 \end{aligned}$$

leads to this.

$$\{\begin{pmatrix}(1/(-2+i))b_2\\ b_2\end{pmatrix} \mid b_2\in\mathbb{C}\} \qquad \begin{pmatrix}1/(-2+i)\\ 1\end{pmatrix}$$

Five.I.3.23 The characteristic equation is

$$0 = \begin{vmatrix}1-x & 1 & 1\\ 0 & -x & 1\\ 0 & 0 & 1-x\end{vmatrix} = (1-x)^2(-x)$$

and so the eigenvalues are $\lambda_1 = 1$ (this is a repeated root of the equation) and $\lambda_2 = 0$. For the rest, consider this system.

$$\begin{aligned} (1-x)\cdot b_1 + \quad b_2 + \quad\quad b_3 &= 0\\ -x\cdot b_2 + \quad\quad b_3 &= 0\\ (1-x)\cdot b_3 &= 0 \end{aligned}$$

When $x = \lambda_1 = 1$ then the solution set is this eigenspace.

$$\{\begin{pmatrix}b_1\\ 0\\ 0\end{pmatrix} \mid b_1\in\mathbb{C}\}$$

When $x = \lambda_2 = 0$ then the solution set is this eigenspace.

$$\{\begin{pmatrix}-b_2\\ b_2\\ 0\end{pmatrix} \mid b_2\in\mathbb{C}\}$$

So these are eigenvectors associated with $\lambda_1 = 1$ and $\lambda_2 = 0$.

$$\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \qquad \begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$$

Five.I.3.24 **(a)** The characteristic equation is

$$0 = \begin{vmatrix} 3-x & -2 & 0 \\ -2 & 3-x & 0 \\ 0 & 0 & 5-x \end{vmatrix} = x^3 - 11x^2 + 35x - 25 = (x-1)(x-5)^2$$

and so the eigenvalues are $\lambda_1 = 1$ and also the repeated eigenvalue $\lambda_2 = 5$. To find eigenvectors, consider this system.

$$\begin{aligned} (3-x)\cdot b_1 - \quad 2\cdot b_2 \qquad &= 0 \\ -2\cdot b_1 + (3-x)\cdot b_2 \qquad &= 0 \\ (5-x)\cdot b_3 &= 0 \end{aligned}$$

For $\lambda_1 = 1$ we get

$$\begin{aligned} 2\cdot b_1 - 2\cdot b_2 \qquad &= 0 \\ -2\cdot b_1 + 2\cdot b_2 \qquad &= 0 \\ 4\cdot b_3 &= 0 \end{aligned}$$

leading to this eigenspace and eigenvector.

$$\{\begin{pmatrix} b_2 \\ b_2 \\ 0 \end{pmatrix} \bigm| b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 1$ the system is

$$\begin{aligned} -2\cdot b_1 - 2\cdot b_2 \qquad &= 0 \\ -2\cdot b_1 - 2\cdot b_2 \qquad &= 0 \\ 0\cdot b_3 &= 0 \end{aligned}$$

leading to this.

$$\{\begin{pmatrix} -b_2 \\ b_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix} \bigm| b_2, b_3 \in \mathbb{C}\} \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) The characteristic equation is

$$0 = \begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 4 & -17 & 8-x \end{vmatrix} = -x^3 + 8x^2 - 17x + 4 = -1\cdot(x-4)(x^2-4x+1)$$

and the eigenvalues are $\lambda_1 = 4$ and (by using the quadratic equation) $\lambda_2 = 2+\sqrt{3}$ and $\lambda_3 = 2-\sqrt{3}$. To find eigenvectors, consider this system.

$$\begin{aligned} -x\cdot b_1 + \quad b_2 \qquad &= 0 \\ -x\cdot b_2 + \quad b_3 &= 0 \\ 4\cdot b_1 - \ 17\cdot b_2 + (8-x)\cdot b_3 &= 0 \end{aligned}$$

Substituting $x = \lambda_1 = 4$ gives the system

$$\begin{aligned} -4\cdot b_1 + \quad b_2 \qquad &= 0 \\ -4\cdot b_2 + \quad b_3 &= 0 \\ 4\cdot b_1 - \ 17\cdot b_2 + 4\cdot b_3 &= 0 \end{aligned} \ \xrightarrow{\rho_1+\rho_3} \ \begin{aligned} -4\cdot b_1 + \quad b_2 \qquad &= 0 \\ -4\cdot b_2 + \quad b_3 &= 0 \\ -16\cdot b_2 + 4\cdot b_3 &= 0 \end{aligned} \ \xrightarrow{-4\rho_2+\rho_3} \ \begin{aligned} -4\cdot b_1 + \quad b_2 \qquad &= 0 \\ -4\cdot b_2 + b_3 &= 0 \\ 0 &= 0 \end{aligned}$$

leading to this eigenspace and eigenvector.

$$V_4 = \{\begin{pmatrix} (1/16)\cdot b_3 \\ (1/4)\cdot b_3 \\ b_3 \end{pmatrix} \bigm| b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$$

Substituting $x = \lambda_2 = 2+\sqrt{3}$ gives the system

$$\begin{aligned} (-2-\sqrt{3})\cdot b_1 + \qquad b_2 \qquad &= 0 \\ (-2-\sqrt{3})\cdot b_2 + \qquad b_3 &= 0 \\ 4\cdot b_1 - \qquad 17\cdot b_2 + (6-\sqrt{3})\cdot b_3 &= 0 \end{aligned}$$

$$\xrightarrow{(-4/(-2-\sqrt{3}))\rho_1+\rho_3} \ \begin{aligned} (-2-\sqrt{3})\cdot b_1 + \qquad b_2 \qquad &= 0 \\ (-2-\sqrt{3})\cdot b_2 + \qquad b_3 &= 0 \\ +\,(-9-4\sqrt{3})\cdot b_2 + (6-\sqrt{3})\cdot b_3 &= 0 \end{aligned}$$

(the middle coefficient in the third equation equals the number $(-4/(-2-\sqrt{3}))-17$; find a common denominator of $-2-\sqrt{3}$ and then rationalize the denominator by multiplying the top and bottom of the frsction by $-2+\sqrt{3}$)

$$\xrightarrow{((9+4\sqrt{3})/(-2-\sqrt{3}))\rho_2+\rho_3} \ \begin{aligned} (-2-\sqrt{3})\cdot b_1 + \qquad b_2 \qquad &= 0 \\ (-2-\sqrt{3})\cdot b_2 + b_3 &= 0 \\ 0 &= 0 \end{aligned}$$

which leads to this eigenspace and eigenvector.

$$V_{2+\sqrt{3}}=\{\begin{pmatrix}(1/(2+\sqrt{3})^2)\cdot b_3\\(1/(2+\sqrt{3}))\cdot b_3\\b_3\end{pmatrix}\bigm| b_3\in\mathbb{C}\}\qquad\begin{pmatrix}(1/(2+\sqrt{3})^2)\\(1/(2+\sqrt{3}))\\1\end{pmatrix}$$

Finally, substituting $x=\lambda_3=2-\sqrt{3}$ gives the system

$$\begin{aligned}(-2+\sqrt{3})\cdot b_1+\qquad\qquad b_2\qquad\qquad&=0\\(-2+\sqrt{3})\cdot b_2+\qquad b_3&=0\\4\cdot b_1-\qquad 17\cdot b_2+(6+\sqrt{3})\cdot b_3&=0\end{aligned}$$

$$\xrightarrow{(-4/(-2+\sqrt{3}))\rho_1+\rho_3}\begin{aligned}(-2+\sqrt{3})\cdot b_1+\qquad\qquad b_2\qquad\qquad&=0\\(-2+\sqrt{3})\cdot b_2+\qquad b_3&=0\\(-9+4\sqrt{3})\cdot b_2+(6+\sqrt{3})\cdot b_3&=0\end{aligned}$$

$$\xrightarrow{((9-4\sqrt{3})/(-2+\sqrt{3}))\rho_2+\rho_3}\begin{aligned}(-2+\sqrt{3})\cdot b_1+\qquad b_2\qquad&=0\\(-2+\sqrt{3})\cdot b_2+b_3&=0\\0&=0\end{aligned}$$

which gives this eigenspace and eigenvector.

$$V_{2-\sqrt{3}}=\{\begin{pmatrix}(1/(2+\sqrt{3})^2)\cdot b_3\\(1/(2-\sqrt{3}))\cdot b_3\\b_3\end{pmatrix}\bigm| b_3\in\mathbb{C}\}\qquad\begin{pmatrix}(1/(-2+\sqrt{3})^2)\\(1/(-2+\sqrt{3}))\\1\end{pmatrix}$$

Five.I.3.25 With respect to the natural basis $B=\langle 1,x,x^2\rangle$ the matrix representation is this.

$$\text{Rep}_{B,B}(t)=\begin{pmatrix}5&6&2\\0&-1&-8\\1&0&-2\end{pmatrix}$$

Thus the characteristic equation

$$0=\begin{pmatrix}5-x&6&2\\0&-1-x&-8\\1&0&-2-x\end{pmatrix}=(5-x)(-1-x)(-2-x)-48-2\cdot(-1-x)$$

is $0=-x^3+2x^2+15x-36=-1\cdot(x+4)(x-3)^2$. To find the associated eigenvectors, consider this system.

$$\begin{aligned}(5-x)\cdot b_1+\qquad 6\cdot b_2+\qquad 2\cdot b_3&=0\\(-1-x)\cdot b_2-\qquad 8\cdot b_3&=0\\b_1\qquad\qquad+(-2-x)\cdot b_3&=0\end{aligned}$$

Plugging in $x=\lambda_1=4$ gives

$$\begin{aligned}b_1+\quad 6\cdot b_2+2\cdot b_3&=0\\-5\cdot b_2-8\cdot b_3&=0\\b_1\qquad\quad-6\cdot b_3&=0\end{aligned}\xrightarrow{-\rho_1+\rho_2}\begin{aligned}b_1+\quad 6\cdot b_2+2\cdot b_3&=0\\-5\cdot b_2-8\cdot b_3&=0\\-6\cdot b_2-8\cdot b_3&=0\end{aligned}\xrightarrow{-\rho_1+\rho_2}\begin{aligned}b_1+\quad 6\cdot b_2+2\cdot b_3&=0\\-5\cdot b_2-8\cdot b_3&=0\\-6\cdot b_2-8\cdot b_3&=0\end{aligned}$$

Five.I.3.26 $\lambda=1,\begin{pmatrix}0&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}2&3\\1&0\end{pmatrix}$, $\lambda=-2,\begin{pmatrix}-1&0\\1&0\end{pmatrix}$, $\lambda=-1,\begin{pmatrix}-2&1\\1&0\end{pmatrix}$

Five.I.3.27 Fix the natural basis $B=\langle 1,x,x^2,x^3\rangle$. The map's action is $1\mapsto 0$, $x\mapsto 1$, $x^2\mapsto 2x$, and $x^3\mapsto 3x^2$ and its representation is easy to compute.

$$T=\text{Rep}_{B,B}(d/dx)=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}_{B,B}$$

We find the eigenvalues with this computation.

$$0=|T-xI|=\begin{vmatrix}-x&1&0&0\\0&-x&2&0\\0&0&-x&3\\0&0&0&-x\end{vmatrix}=x^4$$

Thus the map has the single eigenvalue $\lambda=0$. To find the associated eigenvectors, we solve

$$\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}_{B,B}\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}_B=0\cdot\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}_B\qquad\Longrightarrow\qquad b_2=0,\ b_3=0,\ b_4=0$$

to get this eigenspace.

$$\{\begin{pmatrix}b_1\\0\\0\\0\end{pmatrix}_B \bigm| b_1\in\mathbb{C}\} = \{b_1+0\cdot x+0\cdot x^2+0\cdot x^3 \bigm| b_1\in\mathbb{C}\} = \{b_1 \bigm| b_1\in\mathbb{C}\}$$

Five.I.3.28 The determinant of the triangular matrix $T-xI$ is the product down the diagonal, and so it factors into the product of the terms $t_{i,i}-x$.

Five.I.3.29 Just expand the determinant of $T-xI$.

$$\begin{vmatrix}a-x & c\\ b & d-x\end{vmatrix} = (a-x)(d-x)-bc = x^2+(-a-d)\cdot x+(ad-bc)$$

Five.I.3.30 Any two representations of that transformation are similar, and similar matrices have the same characteristic polynomial.

Five.I.3.31 **(a)** Yes, use $\lambda=1$ and the identity map.
(b) Yes, use the transformation that multiplies by λ.

Five.I.3.32 If $t(\vec{v})=\lambda\cdot\vec{v}$ then $\vec{v}\mapsto\vec{0}$ under the map $t-\lambda\cdot\mathrm{id}$.

Five.I.3.33 The characteristic equation

$$0=\begin{vmatrix}a-x & b\\ c & d-x\end{vmatrix} = (a-x)(d-x)-bc$$

simplifies to $x^2+(-a-d)\cdot x+(ad-bc)$. Checking that the values $x=a+b$ and $x=a-c$ satisfy the equation (under the $a+b=c+d$ condition) is routine.

Five.I.3.34 Consider an eigenspace V_λ. Any $\vec{w}\in V_\lambda$ is the image $\vec{w}=\lambda\cdot\vec{v}$ of some $\vec{v}\in V_\lambda$ (namely, $\vec{v}=(1/\lambda)\cdot\vec{w}$). Thus, on V_λ (which is a nontrivial subspace) the action of t^{-1} is $t^{-1}(\vec{w})=\vec{v}=(1/\lambda)\cdot\vec{w}$, and so $1/\lambda$ is an eigenvalue of t^{-1}.

Five.I.3.35 **(a)** We have $(cT+dI)\vec{v}=cT\vec{v}+dI\vec{v}=c\lambda\vec{v}+d\vec{v}=(c\lambda+d)\cdot\vec{v}$.
(b) Suppose that $S=PTP^{-1}$ is diagonal. Then $P(cT+dI)P^{-1}=P(cT)P^{-1}+P(dI)P^{-1}=cPTP^{-1}+dI=cS+dI$ is also diagonal.

Five.I.3.36 The scalar λ is an eigenvalue if and only if the transformation $t-\lambda\mathrm{id}$ is singular. A transformation is singular if and only if it is not an isomorphism (that is, a transformation is an isomorphism if and only if it is nonsingular).

Five.I.3.37 **(a)** Where the eigenvalue λ is associated with the eigenvector $\vec{x}$ then $A^k\vec{x}=A\cdots A\vec{x}=A^{k-1}\lambda\vec{x}=\lambda A^{k-1}\vec{x}=\cdots=\lambda^k\vec{x}$. (The full details can be put in by doing induction on k.)
(b) The eigenvector associated wih λ might not be an eigenvector associated with μ.

Five.I.3.38 No. These are two same-sized, equal rank, matrices with different eigenvalues.

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad\begin{pmatrix}1&0\\0&2\end{pmatrix}$$

Five.I.3.39 The characteristic polynomial has an odd power and so has at least one real root.

Five.I.3.40 The characteristic polynomial x^3-5x^2+6x has distinct roots $\lambda_1=0$, $\lambda_2=-2$, and $\lambda_3=-3$. Thus the matrix can be diagonalized into this form.

$$\begin{pmatrix}0&0&0\\0&-2&0\\0&0&-3\end{pmatrix}$$

Five.I.3.41 We must show that it is one-to-one and onto, and that it respects the operations of matrix addition and scalar multiplication.

To show that it is one-to-one, suppose that $t_P(T)=t_P(S)$, that is, suppose that $PTP^{-1}=PSP^{-1}$, and note that multiplying both sides on the left by P^{-1} and on the right by P gives that $T=S$. To show that it is onto, consider $S\in\mathcal{M}_{n\times n}$ and observe that $S=t_P(P^{-1}SP)$.

The map t_P preserves matrix addition since $t_P(T+S)=P(T+S)P^{-1}=(PT+PS)P^{-1}=PTP^{-1}+PSP^{-1}=t_P(T+S)$ follows from properties of matrix multiplication and addition that we have seen. Scalar multiplication is similar: $t_P(cT)=P(c\cdot T)P^{-1}=c\cdot(PTP^{-1})=c\cdot t_P(T)$.

Five.I.3.42 *This is how the answer was given in the cited source.* If the argument of the characteristic function of A is set equal to c, adding the first $(n-1)$ rows (columns) to the nth row (column) yields a determinant whose nth row (column) is zero. Thus c is a characteristic root of A.