12 Homomorphisms

We have seen that two groups G and G' are identical in structure if there is a one-one and onto function $\theta : G \to G'$ such that

$$(xy)\theta = x\theta y\theta \qquad (\#)$$

In this section we consider what happens if we relax the condition that θ is one-one and onto and concentrate only on the property (#). We call a mapping satisfying this condition a *homomorphism* and note that a homomorphism is an isomorphism if and only if θ is one-one and onto. Homomorphisms are important because much of the structure of G is retained (they are called structure preserving maps and the word homomorphism comes from the Greek words meaning same or similar shape). They are also closely connected with quotient groups of the sort we saw in the previous section.

Example 12.1.1

Suppose we take the groups **Z** with operation addition and $\mathbf{Z}_n$ with operation addition modulo n. We define the map $\theta:\mathbf{Z} \to \mathbf{Z}_n$ by mapping the integer m to the natural number r ($0 \leq r < n$) where [r] is the name of the congruence of m modulo n. Clearly this map is not an isomorphism since θ is not one-one. It is however a homomorphism since we have seen in the section earlier on congruence classes that

$$(m_1+m_2)\theta = [m_1 + m_2] = [m_1] \oplus [m_2] = m_1\theta \oplus m_2\theta$$

Example 12.1.2

Suppose $G = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7\}$ where $x^8 = 1$

$G' = \{1, y, y^2, y^3\}$ where $y^4 = 1$

Then the mapping

$$\begin{array}{llll} 1_G\theta = 1_{G'}, & x\theta = y, & x^2\theta = y^2, & x^3\theta = y^3 \\ x^4\theta = 1_{G'}, & x^5\theta = y, & x^6\theta = y^2, & x^7\theta = y^3 \end{array}$$

is a homomorphism. We will not check every pair of elements (as we will see shortly that this is not necessary) but for example we have

$$(x^3 x^6)\theta = x^9\theta = x\theta = y = y^3y^2 = (x^3)\theta\,(x^6)\theta$$

and so property(*) (which we will call the *homomorphism property*) is satisfied for this pair of elements.

Exercise 12.1

1. θ defined above is an onto homomorphism from G to G'. Try to find another onto homomorphism.

12.2 How to find a homomorphism between two groups

Given two groups G and G' we will always be able to find at least one homomorphism from G to G' since the mapping θ that sends every element of G to the identity element of G' will always satisfy the homomorphism property since

$$(xy)\theta = 1_{G'} = 1_{G'}1_{G'} = (x\theta)(y\theta)$$

However, this mapping, which we call the trivial homomorphism, is not often of much interest; what we want to find are non-trivial homomorphisms. For example, suppose

$$G = \{1, a, a^2\} \text{ where } a^3 = 1,\ G' = \{1, b\} \text{ where } b^2 = 1$$

We check the possibilities:

1. if $a\theta = b$ we have $a^2\theta = (a\theta)(a\theta) = b^2 = 1$. But then

   $$a^3\theta = (a\theta)(a^2\theta) = b.1 = b$$
   $$a^4\theta = (a^2\theta)(a^2\theta) = 1.1 = 1$$

   But $a^4 = a$ and $a\theta = b$ so this mapping leads to inconsistencies.

2. if $a\theta = 1$ then $a^2\theta = (a\theta)(a\theta) = 1.1 = 1$, $1\theta = a^3\theta = (a\theta)^3 = 1.1.1 = 1$

So the only consistent possibility is the trivial homomorphism.

Clearly this is not a process we want to go through when our groups are any larger than this so we need a little theory. We begin by reviewing our ideas for isomorphism:

- an isomorphism is completely determined by its action on a generating set.

This result was really just a consequence of the homomorphism property - it didn't rely on the fact that θ was one-one and onto. Hence it also applies to homomorpmisms.

Hence in finding homomorphisms we begin by deciding what happens to a generating set and use that to decide what happens to all other elements.

- in an isomorphism the identity gets mapped to the identity and inverses get mapped to inverses. Again this was just a consequence of the homomorphism property and so also applies here.

Hence if $\theta : G \rightarrow G'$ is a homomorphism then $1_G\theta = 1_{G'}$ $(a^{-1})\theta = (a\theta)^{-1}$

- in an isomorphism the order of a is equal to the order of $a\theta$. The proof of this result did use the fact that θ was one-one so we can't apply it directly to general homomorphisms. However a similar result is true:

Theorem 12.2.1

Suppose θ: G→G' is a homomorphism. Then the order of aθ divides the order of a for every a in G.

Proof

Suppose a has order n (so that $a^n = 1_G$) and that aθ has order m. Hence

$$1_{G'} = (1_G)\theta = (a^n)\theta = (a\theta)^n$$

Hence by theorem *.* it follows that n must be a multiple of m.

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> Hence when we construct a homomorphism we choose elements from a generating set for G, calculate their orders, and map them to elements of G' that have orders that divide them.

- the defining equations of the group G must map onto equations of G'. Again this is a consequence of the homomorphism property and applies even if θ is not an isomorphism.

Exercise 12.2

1. Explain why the following mapping from $Z_3(\oplus)$ to $Z_2(\oplus)$ is not a homomorphism:
   $$[0] \to [0], \qquad [1] \to [1], \qquad [2] \to [0]$$
2. Suppose $G = Z_6(\oplus)$ and that G' is the cyclic group with three elements. Construct all possible homomorphisms from G to G'
3. Suppose $G = \{Z_5 - [0]\}(\otimes)$ and G' is the Klien 4-group. Construct all possible homomorphisms from G to G'
4. Prove that G is abelian if and only if the mapping $\theta : G \to G$ defined by $a\theta = a^{-1}$ is a homomorphism.
5. If $\theta : G \to G'$ is a homomorphism show that the set $G\theta = \{x\theta : x \in G\}$ is a subgroup of G'.
6. Suppose that G is the Klein 4-group and that G' is the group $\{1, -1, i, -i\}$ where $i^2 = -1$. Show by construction that iti is not possible to find an isomorphism from G to G'. Is it possible to find a non-trivial homomorphism?
7. Prove that the mapping $\theta : Z(\oplus)$ to $Z_m(\oplus)$ defined by $x\theta = [x]$ is a homomorphism for every $m \geq 2$. What is the kernel of θ?
8. If $G = Z(\oplus)$ and G' is the group $\{1, -1, i, -i\}$ where $i^2 = -1$, show that the mapping $\theta : G \to G'$ given by $n\theta = i^n$ is a homomorphism. What is the kernel of θ?
9. Show that $G = \{1, a, b, a^2, ab, ba\}$ where $a^3 = b^2 = 1$, $a^2b = ba$ and $G' = \{1, c, d, cd, dc, cdc\}$ where $c^2 = d^2 = 1$, $cdc = dcd$ are isomorphic.