MA208 Solutions Week 8

Exercise 12.1

$1_G\theta = 1_{G'}$, $x\theta = y^3$, $x^2\theta = y^2$, $x^3\theta = y$
$x^4\theta = 1_{G'}$, $x^5\theta = y^3$, $x^6\theta = y^2$, $x^7\theta = y$

Exercise 12.2

1 Let θ be the mapping from $Z_3(\oplus)$ to $Z_2(\oplus)$. For θ to be a homomorphism $(ab)\theta = a\theta.b\theta$
Then $([1] \oplus [2])\theta = [0]\theta = [0]$
But $[1]\theta \oplus [2]\theta = [1] \oplus [0] = [1]$ since these give different results θ is NOT a homorphism

2. $G = Z_6(\oplus) = \{[0],[1], [2], [3], [4], [5]\}$ and $G' = \{1, \varphi, \varphi^2\}$ where φ is a rotation of 120°
orders 1 6 3 2 3 6 1 3 3
there are 2 possible non-trivial homomorphisms:
$[0], [3] \rightarrow 1$, $[1], [4] \rightarrow \varphi$, $[2], [5] \rightarrow \varphi^2$
or $[0], [3] \rightarrow 1$, $[1], [4] \rightarrow \varphi^2$, $[2], [5] \rightarrow \varphi$

3. $G = \{Z_5 - [0]\}(\otimes) = \{[1], [2], [3], [4]\}$ $G' = \{1, a, b, ab\}$ (see page 5)
orders 1 4 4 2 1 2 2 2
there are 3 possible non-trivial homomorphisms:
$[1], [4] \rightarrow 1$, $[2], [3] \rightarrow a$
$[1], [4] \rightarrow 1$, $[2], [3] \rightarrow b$
$[1], [4] \rightarrow 1$, $[2], [3] \rightarrow ab$

4 If G is abelian then
$(ab)\theta = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = a\theta b\theta \Rightarrow \theta$ is a homomorphism
Conversely, if θ is a homomorphism then
$(ab)\theta = a\theta b\theta \Rightarrow (ab)^{-1} = a^{-1}b^{-1} \Rightarrow a(ab)^{-1} = a a^{-1}b^{-1} \Rightarrow a(ab)^{-1} = b^{-1} \Rightarrow ba(ab)^{-1} = bb^{-1}$
$\Rightarrow ba(ab)^{-1} = 1 \Rightarrow ba(ab)^{-1}(ab) = 1.ab \Rightarrow ba = ab \Rightarrow$ G is abelian

5. If $x, y, \in G\theta$ then $a\theta = x$ and $b\theta = y$ where $a, b \in G$ so
$xy^{-1} = a\theta.(b\theta)^{-1} = a\theta.(b^{-1})\theta = (ab^{-1})\theta \Rightarrow xy^{-1} \in G\theta$ and so $G\theta$ is a subgroup of G'

6. $G = \{1, a, b, ab\}$ $a^2 = 1, b^2 = 1$ $ab = ba$ and $G' = \{1, -1, i, -i\}$
orders 1 2 2 2 1 2 4 4
If two groups are isomorphic the corresponding elements have the same orders. There are no elements in G of order 4, ie there is nothing that will map onto i or –i. Thus G and G' are not isomorphic.
The possible homomorphisms are:
$1, ab \rightarrow 1$, $a, b \rightarrow -1$
$1, a \rightarrow 1$, $ab, b \rightarrow -1$
$1, b \rightarrow 1$, $a, ab \rightarrow -1$

7 $(x + y)\theta = [x + y] = [x] \oplus [y] = x\theta \oplus y\theta \Rightarrow \theta$ is a homomorphism
$\ker \theta = \{x : [x] = [0]\} = [0] \bmod m$

8. $(n + m)\theta = i^{n+m} = i^n i^m = n\theta.m\theta$
$\ker \theta = (n : n\theta = 1\} = \{n : i^n = 1\} = \{4k : k \in Z\}$

9. $1 \rightarrow 1$, $a \rightarrow cd$, $b \rightarrow c$, $a^2 \rightarrow dc$, $ab \rightarrow cdc$, $ba \rightarrow d$
is an isomorphism.