Exercises 10.1

1. $\{1\}$ and the whole group are always normal subgroups and so need to be added to the subgroups below in each case.

   a) Non-trivial normal subgroups of Q_4 are

   $\{1, a, a^2, a^3\}$, $\{1, b, a^2, a^2b\}$, $\{1, ab, a^2, a^3b\}$, $\{1, a^2\}$

   b) $\{1, a, b^2ab, bab^2\}$

   c) $\{1, a, a^2, a^3, a^4, a^5\}$, $\{1, a^3\}$, $\{1, a^2, a^4\}$, $\{1, a^2, a^4, b, a^2b, a^4b\}$, $\{1, a^2, a^4, ab, a^3b, a^5b\}$,

   d) All subgroups of an abelian group are normal. Hence normal subgroups of $\{1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ where $a^6 = b^2 = 1$, $ab=ba$ are

   $\{1, a, a^2, a^3, a^4, a^5\}$, $\{1, a^2, a^4\}$, $\{1, a^3\}$, $\{1, b\}$, $\{1, ab, a^2, a^3b, a^4, a^5b\}$ $\{1, a^2b, a^4, b, a^2, a^4b\}$, $\{1, a^3b\}$.

2. Have seen in a previous exercise (7.3) that $A\cap B$ is a subgroup so need to show that the definition of a normal subgroup applies.

   Suppose $x\in A\cap B$ and $g\in G$. Then

   $gxg^{-1}\in A$ as A is a normal subgroup;
   $gxg^{-1}\in B$ as B is a normal subgroup.

   Hence $A\cap B$ is a normal subgroup.

3. Suppose $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ are two elements of H. Then

   $AB^{-1} = \begin{pmatrix} a/c & 0 \\ 0 & b/d \end{pmatrix}$. This is an element of H so H is a subgroup.

   Suppose $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is in G and that $X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is in H. Thus

$$AXA^{-1} = \frac{1}{ps - rq} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$$

$$= \frac{1}{ps - qr} \begin{pmatrix} pas - qbr & pq(b-a) \\ rs(a-b) & sbp - raq \end{pmatrix}$$

Since this is not (in general) an element of H it follows that H is not a normal subgroup.

Exercises 10.2

1. a) Consider the subgroup H = {1, a, a^2, a^3} of Q_4. Suppose α = H1, β = Hb. Then the operation table is

	α	β
α	α	β
β	β	α

If H = {1, b, a^2, a^2b} with α = H1, β = Ha then the induced operation is

	α	β
α	α	β
β	β	α

If H = {1, ab, a^2, a^3b} with α = H1, β = Ha then the induced operation is

	α	β
α	α	β
β	β	α

If H = {1, a^2} with α = H1, β = Ha, γ = Hb, δ = Hab the induced operation table is

	α	β	γ	δ
α	α	β	γ	δ
β	β	α	δ	γ
γ	γ	δ	α	β
δ	δ	γ	β	α

b) If G = A_4 and H = {1, a, b^2ab, bab^2} then there are three cosets α = H1, β = Hb, γ = Hb^2. The operation table is

	α	β	γ
α	α	β	γ
β	β	γ	α
γ	γ	α	β

c) If G = D_6 and H={1, a, a^2, a^3, a^4, a^5} then there are two cosets α = H1 and β = Hb. The operation table is

	α	β
α	α	β
β	β	α

If H = {1, a^3} there are 6 cosets α= H1, β=Ha, γ= Ha^2, δ = Hb, ε = Hab, ϕ = Ha^2b. The operation table is

	α	β	γ	δ	ε	ϕ
α	α	β	γ	δ	ε	ϕ
β	β	γ	α	ε	ϕ	δ
γ	γ	α	β	ϕ	δ	ε
δ	δ	ϕ	ε	α	γ	β
ε	ε	δ	ϕ	β	α	γ
ϕ	ϕ	ε	δ	γ	β	α

If H = {1, a^2, a^4} there are 4 cosets α = H1, β = Ha, γ = Hb, δ = Hab and the operation table is

	α	β	γ	δ
α	α	β	γ	δ
β	β	α	δ	γ
γ	γ	δ	α	β
δ	δ	γ	β	α

If H = {1, a^2, a^4, b, a^2b, a^4b} there are 2 cosets α = H1, β = Ha and the operation table is

	α	β
α	α	β
β	β	α

If H = {1, a^2b, a^4, b, a^2, a^4b} there are 2 cosets α = H1, β = Ha and the operation table is

	α	β
α	α	β
β	β	α

d) If H={1, a, a^2, a^3, a^4, a^5} is a subgroup of Z_6 x Z_2 there are two cosets α = H1 and β = Hb. The operation table is

	α	β
α	α	β
β	β	α

If H = {1, a^2, a^4} there are 4 cosets α = H1, β = Ha, γ = Hb, δ = Hab and the operation table is

	α	β	γ	δ
α	α	β	γ	δ
β	β	α	δ	γ
γ	γ	δ	α	β
δ	δ	γ	β	α

If H = {1, a^3} there are 6 cosets α= H1, β=Ha, γ= Ha^2, δ = Hb, ε = Hab, ϕ = Ha^2b. The operation table is

	α	β	γ	δ	ε	ϕ
α	α	β	γ	δ	ε	ϕ
β	β	γ	α	ε	ϕ	δ
γ	γ	α	β	ϕ	δ	ε
δ	δ	ε	ϕ	α	β	γ
ε	ε	ϕ	δ	β	γ	α
ϕ	ϕ	δ	ε	γ	α	β

If H = {1, b} there are 6 cosets α= H1, β=Ha, γ= Ha^2, δ = Ha^3, ε = Ha^4, ϕ = Ha^5. The operation table is

	α	β	γ	δ	ε	ϕ
α	α	β	γ	δ	ε	ϕ
β	β	γ	δ	ε	ϕ	α
γ	γ	δ	ε	ϕ	α	β
δ	δ	ε	ϕ	α	β	γ
ε	ε	ϕ	α	β	γ	δ
ϕ	ϕ	α	β	γ	δ	ε

If H = {1, ab, a^2, a^3b, a^4, a^5b} there are 2 cosets α = H1, β = Ha and the operation table is

	α	β
α	α	β
β	β	α

If H = {1, a^2b, a^4, b, a^2, a^4b} there are 2 cosets α = H1, β = Ha and the operation table is

	α	β
α	α	β
β	β	α

If H = {1, a^3b} there are 6 cosets α= H1, β=Ha, γ= Ha^2, δ = Ha^3, ε = Ha^4, ϕ = Ha^5. The operation table is

	α	β	γ	δ	ε	ϕ
α	α	β	γ	δ	ε	ϕ
β	β	γ	δ	ε	ϕ	α
γ	γ	δ	ε	ϕ	α	β
δ	δ	ε	ϕ	α	β	γ
ε	ε	ϕ	α	β	γ	δ
ϕ	ϕ	α	β	γ	δ	ε

2. a) True. If G is an abelian group and H is a normal subgroup then aHbH = abH = baH = bHaH for all a,b in G.

b) False. For example {1, a, a^2, a^3} in Q_4 has induced operation table that is cyclic of order 2 and so is abelian.

c) True. If G has order n and H is a normal subgroup of order m then G/H has order n/m.

d) False. For example if G is the integers under addition and H is the set of even integers then G/H has order 2.

d) False. For example a has order 6 in the group {1, a, a^2, a^3, a^4, a^5} where $a^6 = 1$. But if H = {1, a^3} then a has order 3 in G/H. (Note the order of Ha in G/H always divides the order of a in G but is not necessarily equal to it).

2. The problem is with the choice of elements of G/H. The elements of the quotient group G/H are cosets and so if we want to select two elements we should say "Suppose Ha and Hb are elements of G/H. We aim to show that HaHb = HbHa".