MA208 Solutions Week 7(A)

Exercise 11.1 Nos 1 - 2

Note in No 2 there are other possible isomorphisms, but in each part only one has been constructed

1 In each case we can find an inconsistency by showing $a^2\theta = (a\theta)^2 = x^2$ but in each case $b\theta = x^2$ which means the mapping is not one-one.

$a \to x^3$ so $a^2\theta = (a\theta)^2 = (x^3)^2 = x^6 = x^2$

$a \to y$ so $a^2\theta = (a\theta)^2 = (y)^2 = x^2$

$a \to xy$ so $a^2\theta = (a\theta)^2 = (xy)^2 = xyxy = xx^3y\, y = y^2 = x^2$

$a \to x^2y$ so $a^2\theta = (a\theta)^2 = (x^2y)^2 = x^2y\, x^2y = x^2y\, yx^2 = x^2y^2\, x^2 = x^2x^2x^2 = x^2$

$a \to x^3y$ so $a^2\theta = (a\theta)^2 = (x^3y)^2 = x^3y\, x^3y = x^3y\, yx = x^3y^2x = x^3x^2x = x^2$

a has now been mapped to all the possible elements of order 4, in each case the mapping is not one-one, so there is no isomorphism between D_4 and Q_4

2. a) $G = \{1, a, b, ab, ba, aba\}$ where $a^2 = b^2 = 1$, $aba = bab$

$G' = \{1, x, x^2, y, xy, x^2y\}$ where $x^3 = y^2 = 1$, $yx = x^2y$

Choose {a, b} as the generating set of G, Each element has order 2.

In G' y, xy and x^2y have order 2, so choose any two of these

Choose $a \to y$ and $b \to xy$ as a trial mapping

Then the other elements are mapped as follows:

$a^2 = 1$ so $1_G\theta = (a^2)\theta = (a\theta)^2 = y^2 = 1_{G'}$

$(ab)\theta = (a\theta)(b\theta) = y.xy = x^2yy = x^2$,

$(ba)\theta = (b\theta)(a\theta) = xyy = x$,

$(aba)\theta = (a\theta)(b\theta)(a\theta) = yxyy = yx = x^2y$

Check the defining relations: $a^2\theta = (a\theta)^2 = y^2 = 1_{G'} = 1_G\theta$

$b^2\theta = (b\theta)^2 = (xy)^2 = xyxy = x\,x^2yy = 1_{G'} = 1_G\theta$

$(aba)\theta = (a\theta)(b\theta)(a\theta) = yxyy = yx = x^2y$

and $(bab)\theta = (b\theta)(a\theta)(b\theta) = xyyxy = x^2y$ so $(aba)\theta = (bab)\theta$

The equations are consistent so θ is an isomorphism

b) $G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b, b^2, ab^2, a^2b^2, a^3b^2\}$, $a^4 = b^3 = 1$, $ba = ab$

$G' = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}\}$ where $x^{12} = 1$

(The easiest result is achieved if you spot ab generates G, then follow the example on page 60 of the notes, however if you don't spot this…)

Choose {a, b} as the generating set of G, a has order 4 and b has order 3

In G' x^3 and x^9 have order 4, and x^4 and x^8 have order 3

Choose $a \to x^3$ and $b \to x^4$ as a trial mapping

Then the other elements are mapped as follows:

$a^2\theta = (a\theta)^2 = x^3\, x^3 = x^6$, $a^3\theta = (a\theta)^3 = x^3\, x^3\, x^3 = x^9$,

$a^4 = 1_G$ so $1_G\theta = (a^4)\theta = (a\theta)^4 = x^3\, x^3\, x^3\, x^3 = x^{12} = 1_{G'}$

$b^2\theta = (b\theta)^2 = x^4\, x^4 = x^8$ $(ab)\theta = (a\theta)(b\theta) = x^3\, x^4 = x^7$

$a^2b\theta = (a\theta)^2(b\theta) = x^3\, x^3\, x^4 = x^{10}$, $a^3b\theta = (a\theta)^3(b\theta) = x^3\, x^3\, x^3\, x^4 = x^{13} = x$,

$ab^2\theta = (a\theta)(b\theta)^2 = x^3\, x^4\, x^4 = x^{11}$ $a^2b^2\theta = (a\theta)^2(b\theta)^2 = x^3\, x^3\, x^4\, x^4 = x^{14} = x^2$,

$a^3b^2\theta = (a\theta)^3(b\theta)^2 = x^3\, x^3\, x^3\, x^4\, x^4 = x^{17} = x^5$

Check the defining relations: $a^4\theta = (a\theta)^4 = (x^3)^4 = 1_{G'} = 1_G\theta$

$b^3\theta = (b\theta)^3 = (x^4)^3 = = 1_{G'} = 1_G\theta$

$(ab)\theta = (a\theta)(b\theta) = x^3\, x^4 = x^7$

and $(ba)\theta = (b\theta)(a\theta) = x^4\, x^3 = x^7$ so $(ab)\theta = (ba)\theta$

The equations are consistent so θ is an isomorphism

(c) $G = \{1, a, b, ab, ba, aba\}$ where $a^2 = b^2 = 1$, $aba = bab$

$$S_3 = \left\{ \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} \right\}$$

Choose $\{a, b\}$ as the generating set of G. Each element has order 2

In S_3 the last three elements listed have order 2, so choose any 2 of these

Choose $a \to \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$ and $b \to \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ as a trial mapping

Then the other elements are mapped as follows:

$a^2 = 1_G$ so $1_G\theta = (a^2)\theta = (a\theta)^2 = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$

$(ab)\theta = (a\theta)(b\theta) = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$

$(ba)\theta = (b\theta)(a\theta) = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$

$(aba)\theta = (a\theta)(b\theta)(a\theta) = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$

Check the defining relations: $a^2\theta = (a\theta)^2 = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = 1_G\theta$

$b^2\theta = (b\theta)^2 = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = 1_G\theta$

$(aba)\theta = (a\theta)(b\theta)(a\theta) = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$

and $(bab)\theta = (b\theta)(a\theta)(b\theta) = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$ so $(aba)\theta = (bab)\theta$

The equations are consistent so θ is an isomorphism

c) $C_4 = \{1, x, x^2, x^3\}$ and $x^4 = 1$ and $Z_5 = \{[0]\}(\otimes) = \{ [1], [2], [3], [4] \}$

Choose $\{x\}$ as the generating set of G. x has order 4

In G' [2] and [3] have order 4, so choose any 1 of these Choose $x \to [2]$ as a trial mapping

Then the other elements are mapped as follows:

$(x^2)\theta = (x\theta)^2 = [2] \otimes [2] = [4]$, $(x^3)\theta = (x\theta)^3 = (x\theta)^2 (x\theta) = [4] \otimes [2] = [3]$,

$x^4 = 1$ so $1\theta = (x^4)\theta = (x\theta)^4 = (x\theta)^3 (x\theta) = [3] \otimes [2] = [1]$,

The last mapping is consistent with the only defining relation $x^4 = 1$, so θ is an isomophism

d) $C_6 = \{1, x, x^2, x^3, x^4, x^5\}$ and $x^6 = 1$ $Z_7 = \{[0]\}(\otimes) = \{ [1], [2], [3], [4], [5], [6] \}$

Choose $\{x\}$ as the generating set of G. x has order 6

In G' [3] and [5] have order 6, so choose any 1 of these. Choose $x \to [3]$ as a trial mapping

Then the other elements are mapped as follows:

$(x^2)\theta = (x\theta)^2 = [3] \otimes [3] = [2]$, $(x^3)\theta = (x\theta)^3 = (x\theta)^2 (x\theta) = [2] \otimes [3] = [6]$,

$(x^4)\theta = (x\theta)^4 = (x\theta)^3 (x\theta) = [6] \otimes [3] = [4]$, $(x^5)\theta = (x\theta)^5 = (x\theta)^4 (x\theta) = [4] \otimes [3] = [5]$,

$x^6 = 1$ so $1\theta = (x^6)\theta = (x\theta)^6 = (x\theta)^5 (x\theta) = [5] \otimes [3] = [1]$

The last mapping is consistent with the only defining relation $x^6 = 1$, so θ is an isomophism