Solving 1st order ode's using Laplace Transforms

Worked Example (b)

Using the Laplace Transform method, solve the following 1st order initial value problem

$$\frac{dy}{dt} + y = \sin t \quad , \quad y(0) = -1 .$$

Taking Laplace Transforms gives

$$s\bar{y} - y(0) + \bar{y} = \frac{1}{s^2+1} .$$

Using the initial condition gives

$$(1+s)\bar{y} = \frac{1}{s^2+1} - 1$$

or

$$\bar{y} = \frac{1}{(s+1)(s^2+1)} - \frac{1}{(s+1)} \qquad (A)$$

To put the first term in (A) into partial fraction form

$$\frac{1}{(s+1)(s^2+1)} \equiv \frac{A}{(s+1)} + \frac{Bs+C}{s^2+1}$$

or

$$1 \equiv A(s^2+1) + (Bs+C)(s+1)$$

To evaluate A, B and C, we use

$s=-1$: $\quad 1 = A(2) \quad , \quad A = 1/2$

$s=0$: $\quad 1 = A(1) + C(1)$

$$C = 1 - A = 1 - 1/2 = 1/2$$

$s=1$: $\quad 1 = A(2) + (B+C)(2)$

$$2B = 1 - 2A - 2C = 1-1-1 = -1$$

$$B = -1/2$$

So

$$\frac{1}{(s+1)(s^2+1)} \equiv \frac{1}{2}\cdot\frac{1}{(s+1)} + \frac{1}{2}\cdot\frac{(1-s)}{(s^2+1)}$$

and (A) becomes

$$\bar{y} = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\cdot\frac{1-s}{s^2+1} - \frac{1}{s+1}$$

$$\bar{y} = \frac{1}{2}\cdot\frac{1}{(s^2+1)} - \frac{1}{2}\cdot\frac{s}{(s^2+1)} - \frac{1}{2}\cdot\frac{1}{(s+1)}$$

$\bar{y}$ can now be transformed back to the t-domain using Inverse Laplace Transforms

$$y = \frac{1}{2}\sin t - \frac{1}{2}\cos t - \frac{1}{2}e^{-t}$$

Solving 2nd order ode's using Laplace Transforms

Worked Example (C)

Using the Laplace Transform method, solve the following 2nd order initial value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = te^t, \quad y(0)=1, \quad y'(0)=4$$

Taking Laplace Transforms gives

$$(s^2\bar{y} - sy(0) - y'(0)) + 4(s\bar{y} - y(0)) + 13\bar{y} = \frac{1!}{(s-1)^2}$$

Using the initial conditions gives

$$(s^2+4s+13)\bar{y} = \frac{1}{(s-1)^2} + (s+4) + 4 \qquad (A)$$

The quadratic term on the left-hand side is now expressed as follows by completing the square

$$s^2+4s+13 = (s+2)^2 - 4 + 13$$
$$= (s+2)^2 + 9$$

$\bar{y}$ in equation (A) now becomes:

$$\bar{y} = \frac{1}{(s-1)^2((s+2)^2+9)} + \frac{s+8}{((s+2)^2+9)}. \qquad (B)$$

The first term in (B) is expressed using partial fractions, whereas the second term can be written as

$$\frac{s+8}{((s+2)^2+9)} = \frac{s+2}{((s+2)^2+9)} + 2\cdot\frac{3}{(s+2)^2+9}. \qquad (C)$$

The right-hand side of (C) is now readily amenable to Inverse Laplace Transform and can be dealt with at once.

The first term in (B) is now expressed as

$$\frac{1}{(s-1)^2((s+2)^2+9)} \equiv \frac{A(s-1)+B}{(s-1)^2} + \frac{C(s+2)+D}{(s+2)^2+9}. \qquad (D)$$

(Note that the partial fractions must contain an appropriate linear function in each term.)

So

$$1 \equiv (A(s-1)+B)((s+2)^2+9) + (s-1)^2(C(s+2)+D)$$

We now use appropriate values of s to find the constants A, B, C and D.

Worked Example (c) (continued)

$s=1$: $1 = B(18)$, $\underline{B = 1/18}$

$s=-2$: $1 = (A(-3)+B)(9) + (-3)^2(D)$

$s=0$: $1 = (A(-1)+B)(2^2+9) + (-1)^2(C(2)+D)$

$s=-1$: $1 = (A(-2)+B)(1^2+9) + (-2)^2(C(1)+D)$

Simplifying these equations now gives

$$1 = -27A + 9B + 9D$$

$$1 = -13A + 13B + 2C + D$$

$$1 = -20A + 10B + 4C + 4D$$

with $\underline{B = 1/18}$ already determined.

Substituting for B and rearranging gives

$$-27A + 9D = 9/18$$

$$-13A + 2C + D = 5/18$$

$$-20A + 4C + 4D = 8/18$$

With a view to Gaussian elimination this becomes:

$$\begin{aligned} 2C + 2D - 10A &= 4/18 \\ D - 3A &= 1/18 \\ 2C + D - 13A &= 5/18 \end{aligned}$$

Subtracting row 1 from row 3 now gives

$$\begin{aligned} 2C + 2D - 10A &= 4/18 \\ D - 3A &= 1/18 \\ -D - 3A &= 1/18 \end{aligned}$$

Adding row 3 and row 2 now gives

$$\begin{aligned} 2C + 2D - 10A &= 4/18 \\ D - 3A &= 1/18 \\ -6A &= 2/18 \end{aligned}$$

By back substitution, we have

$\underline{A = -1/54}$, $\underline{D} = 1/18 + 3(-1/54) \underline{= 0}$

and $2C = 4/18 + 10(-1/54) = 2/54$, $\underline{C = 1/54}$

Returning to equation (D), we can now write

$$\frac{1}{(s-1)^2((s+2)^2+9)} \equiv -\frac{1}{54}\cdot\frac{1}{(s-1)} + \frac{1}{18}\cdot\frac{1}{(s-1)^2} + \frac{1}{54}\cdot\frac{(s+2)}{(s+2)^2+9} \qquad (E)$$

Combining (E), (C) and (B) now gives

$$\bar{y} = -\frac{1}{54}\cdot\frac{1}{(s-1)} + \frac{1}{18}\cdot\frac{1}{(s-1)^2} + \frac{55}{54}\cdot\frac{(s+2)}{(s+2)^2+9} + 2\cdot\frac{3}{(s+2)^2+9}$$

Applying Inverse Laplace Transforms gives the answer:

$$\boxed{y = -\frac{1}{54}e^t + \frac{1}{18}te^t + \frac{55}{54}e^{-2t}\cos 3t + 2e^{-2t}\sin 3t}$$