Series Solution about an ordinary point

Worked Example (b)

Solve the equation

$$y'' + xy' + y = 0, \qquad (A)$$

in series expansion about the ordinary point $x_0 = 0$.

1. **Assume** a series expansion

Memorize → $$y = \sum_{n=0}^{\infty} a_n x^n \qquad (B)$$ { formula not given in exam }

Differentiate to give

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad (C)$$

and

$$y'' = \sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} \qquad (D)$$

2. **Substitute** (B), (C) and (D) into (A) to give

$$\sum_{n=2}^{\infty} (n-1)n a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad (E)$$

take x inside Σ, then shifting index

3. **Align terms** so that each is x^n.
This is done by shifting the index of summation in the first term; and taking the factor x inside the second summation.
Equation (E) becomes

$$\underbrace{\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n}_{n \text{ in (E) has been replaced by } (n+2).} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0. \qquad (F)$$

lower limit has been changed to $n=0$ } why

4. **Equation (F)** can now be written as a single summation because all terms are in x^n and all lower limits are the same, $n=0$.
Equation (F) becomes

$$\sum_{n=0}^{\infty} \{ (n+1)(n+2) a_{n+2} + n a_n + a_n \} x^n = 0$$

or

$$\sum_{n=0}^{\infty} (n+1)\{ (n+2) a_{n+2} + a_n \} x^n = 0. \qquad (G)$$

5. <u>To ensure</u> (G) is zero for all x, each and every coefficient of x must be zero. The recurrence relation becomes

$$a_{n+2} = -\frac{a_n}{(n+2)}, \quad n = 0, 1, \ldots$$

Replacing n by $(n-2)$ gives

$$a_n = -\frac{a_{n-2}}{n}, \quad n = 2, 3, \ldots \qquad \text{(H)}$$

Split (H) into even and odd terms.

6. <u>Even terms</u> $n = 2, 4,$ etc in (H).

$n=2$: $a_2 = -\frac{a_0}{2} = -\frac{a_0}{2.1} = -\frac{a_0}{2.1!}$

$n=4$: $a_4 = -\frac{a_2}{4} = +\frac{a_0}{4.2} = +\frac{a_0}{2^2.2!}$

$n=6$: $a_6 = -\frac{a_4}{6} = -\frac{a_0}{6.4.2} = -\frac{a_0}{2^3.3!}$

$n=2k$: $a_{2k} = \frac{(-1)^k a_0}{2^k k!}$

So the general even coefficient is

$$a_{2k} = \frac{(-1)^k a_0}{2^k k!}, \quad k = 1, 2, \ldots \qquad \text{(I)}$$

7. <u>Odd terms</u> $n = 3, 5,$ etc in (H).

$n=3$: $a_3 = -\frac{a_1}{3} = -\frac{a_1}{3.1}.\left[\frac{2}{2}\right] = -\frac{a_1 2.1!}{3!}$

$n=5$: $a_5 = -\frac{a_3}{5} = +\frac{a_1}{5.3.1}.\left[\frac{4.2}{4.2}\right] = \frac{+a_1 2^2 2!}{5!}$

$n=7$: $a_7 = -\frac{a_5}{7} = -\frac{a_1}{7.5.3.1}.\left[\frac{6.4.2}{6.4.2}\right] = -\frac{a_1 2^3 3!}{7!}$

$n=2k+1$: $a_{2k+1} = \frac{(-1)^k a_1 2^k.k!}{(2k+1)!}$

So the general odd coefficient is

$$a_{2k+1} = \frac{(-1)^k 2^k k!}{(2k+1)!} a_1, \quad k = 1, 2, \ldots \qquad \text{(J)}$$

8. <u>Substitute</u> (I) and (J) into (B) to give solution.

First

$$y = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

and then

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}$$

This is the general solution of (A).

Notes on substituting series expansions

Equation (98) is:

$$y'' + y = 0$$

Using (101) for y'' and (99) for y and arranging in columns gives:

$$2a_2 + 2.3a_3 x + 3.4a_4 x^2 + \dots + (n-1)n\,a_n x^{n-2} + n(n+1)a_{n+1}x^{n-1} + (n+1)(n+2)a_{n+2}x^n + \dots$$

$$+ a_0 + a_1 x + a_2 x^2 + \dots \qquad \dots + a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots$$

$$= 0$$

$\boxed{\text{Now add the columns}}$

$$\underbrace{(a_0 + 2a_2)}_{\text{All constants}} + \underbrace{(a_1 + 2.3a_3)x}_{\text{All terms in } x} + \underbrace{(a_2 + 3.4a_4)x^2}_{\text{All terms in } x^2} + \dots \qquad \dots + \underbrace{(a_n + (n+1)(n+2)a_{n+2})x^n}_{\text{All terms in } x^n} + \dots = 0$$

$\boxed{\text{Note that the general term can be used to generate all the others. Try } n=2.}$

$\boxed{\text{In summation form: } \sum_{n=0}^{\infty} \left(a_n + (n+1)(n+2)a_{n+2}\right)x^n = 0.}$