...n finding $y_P$ in Q1, Application Problems 3

After finding $y_1$, as $y_1 = e^{-t/2}\cos(\sqrt{7}\,t/2)$, and $W$, as $W = e^{-t}$, and then applying the formula for $y_P$, the following integral is required:

$$\boxed{I = 8\int \cos\tfrac{\sqrt{7}}{2}t \,.\, \sin t \, dt}$$

Apply integration by parts twice to find $I$

$$I = 8\left[\left(\tfrac{2}{\sqrt{7}}\sin\tfrac{\sqrt{7}}{2}t\right)\sin t - \int \cos t\left(\tfrac{2}{\sqrt{7}}\sin\tfrac{\sqrt{7}}{2}t\right)dt\right]$$

$$I = \frac{16}{\sqrt{7}}\left[\sin\tfrac{\sqrt{7}}{2}t \sin t - J\right] \qquad (*)$$

where

$$J = \int \sin\tfrac{\sqrt{7}}{2}t \,.\, \cos t \, dt$$

Integrating by parts again

$$J = \left(-\tfrac{2}{\sqrt{7}}\cos\tfrac{\sqrt{7}}{2}t\right)\cos t - \int(-\sin t)\left(-\tfrac{2}{\sqrt{7}}\cos\tfrac{\sqrt{7}}{2}t\right)dt$$

$$= -\frac{2}{\sqrt{7}}\cos\tfrac{\sqrt{7}}{2}t\cos t - \frac{2}{\sqrt{7}}\int\cos\tfrac{\sqrt{7}}{2}t \sin t \, dt$$

$$J = -\frac{2}{\sqrt{7}}\left[\cos\tfrac{\sqrt{7}}{2}\cos t + I/8\right]$$

Putting $J$ back into $(*)$ above gives

$$I = \frac{16}{\sqrt{7}}\left[\sin\tfrac{\sqrt{7}}{2}t \sin t + \tfrac{2}{\sqrt{7}}\cos\tfrac{\sqrt{7}}{2}\cos t + \frac{I}{4\sqrt{7}}\right]$$

$$I - \frac{4I}{7} = \frac{16}{\sqrt{7}}\sin\tfrac{\sqrt{7}}{2}t \sin t + \frac{32}{7}\cos\tfrac{\sqrt{7}}{2}\cos t$$

or

$$\boxed{I = \frac{32}{3}\left[\tfrac{\sqrt{7}}{2}\sin\tfrac{\sqrt{7}}{2}t \,.\, \sin t + \cos\tfrac{\sqrt{7}}{2}t \,.\, \cos t\right]}$$

Now find $y_P$ as

$$y_P = y_1\int\frac{e^{-t}}{e^{-t}\cos^2\frac{\sqrt{7}}{2}t} \,.\, I \, dt$$

Using $I$ above gives

$$\boxed{y_P = \frac{32}{3}y_1\int\frac{\frac{\sqrt{7}}{2}\sin\frac{\sqrt{7}}{2}t \,.\, \sin t + \cos\frac{\sqrt{7}}{2}t\cos t}{\cos^2\frac{\sqrt{7}}{2}t}\,dt}$$

Using the quotient rule (in reverse) gives

$$\boxed{y_P = \frac{32}{3}y_1\int\frac{d}{dt}\left[\frac{\sin t}{\cos\frac{\sqrt{7}}{2}t}\right]dt}$$

Try it and see!

So

$$y_P = \frac{32}{3}y_1\,\frac{\sin t}{\cos\frac{\sqrt{7}}{2}t}$$

But $y_1 = e^{-t/2}\cos\frac{\sqrt{7}}{2}t$

So finally

$$\boxed{y_P = \frac{32}{3}e^{-t/2}\sin t}$$

...on to Q8, Example Sheet 5

$$y'' + \frac{2x-1}{x}y' - \frac{2}{x}y = x$$

$$p(x) = 2 - \frac{1}{x}$$

$$W = \exp\left[-\int\left(2 - \frac{1}{x}\right)dx\right]$$

$$= \exp\left[-2x + \ln x\right]$$

$$\underline{W = xe^{-2x}}$$

$$y_p = y_1 \int \left[\frac{W}{y_1^2} \int \frac{y_1 g}{W} dx\right] dx \quad , \quad y_1 = 2x-1$$
$$g = x$$

$$= (2x-1)\int\left[\frac{xe^{-2x}}{(2x-1)^2}\int\frac{(2x-1)x}{xe^{-2x}}dx\right]dx$$

$$= (2x-1)\int\frac{xe^{-2x}}{(2x-1)^2}\, I \, dx$$

where $I = \int (2x-1)e^{2x}\,dx$

$$= (2x-1)\frac{e^{2x}}{2} - \int 2\frac{e^{2x}}{2}dx$$

$$I = (2x-1)\frac{e^{2x}}{2} - \frac{e^{2x}}{2} = \frac{e^{2x}}{2}(2x-2)$$

$$\underline{I = e^{2x}(x-1)}$$

So substituting $I$ into $y_p$ gives

$$\underline{y_p = (2x-1)\int\frac{x(x-1)}{(2x-1)^2}dx}$$

Let $u = 2x-1$, $du = 2dx$, $dx = \frac{1}{2}du$

$$x = (u+1)/2$$

$$y_p = u\int\frac{\frac{(u+1)}{2}\left(\frac{u+1}{2}-1\right)}{u^2}\frac{1}{2}du$$

$$= \frac{u}{8}\int\frac{(u+1)(u-1)}{u^2}du$$

$$= \frac{u}{8}\int\frac{u^2-1}{u^2}du = \frac{u}{8}\int 1 - \frac{1}{u^2}du$$

$$= \frac{u}{8}\left[u + \frac{1}{u}\right] = \frac{u^2}{8} + \frac{1}{8}$$

or

$$y_p = \frac{(2x-1)^2}{8} + \frac{1}{8}$$

$$= \frac{4x^2-4x+1}{8} + \frac{1}{8}$$

$$= \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}$$

$$= \frac{x^2}{2} - \frac{1}{4}(2x-1)$$

This is part of $y_1$ the first solution

So strictly correct form for $y_p$ is:

$$y_p = \frac{x^2}{2}$$

## Written solution to Q2, Example Sheet 3

$$y_p = y_1 \int \frac{W}{y_1^2} I \, dx$$

where $\quad I = \int y_1 g / W \, dx \quad , \quad I' = \frac{y_1 g}{W}$

$$y_p' = y_1' \int \frac{W}{y_1^2} I \, dx + y_1 \frac{WI}{y_1^2} = y_1' \int \frac{W}{y_1^2} I \, dx + \frac{WI}{y_1}$$

$$y_p'' = y_1'' \int \frac{W}{y_1^2} I \, dx + y_1' \frac{WI}{y_1^2} + \frac{(WI)'}{y_1} - WI \frac{y_1'}{y_1^2}$$

$$= y_1'' \int \frac{W}{y_1^2} I \, dx + \frac{W'I + WI'}{y_1}$$

Put $y_p$, $y_p'$ and $y_p''$ into LHS of equation (A)

$$LHS(A) = y_1'' \int \frac{W}{y_1^2} I \, dx + \frac{W'I + WI'}{y_1}$$

$$+ p\left(y_1' \int \frac{W}{y_1^2} I \, dx + \frac{WI}{y_1}\right) + q y_1 \int \frac{W}{y_1^2} I \, dx$$

$$= \int \frac{W}{y_1^2} I \, dx \left[ y_1'' + p y_1' + q y_1 \right]$$

$$+ \frac{W'I}{y_1} + \frac{WI'}{y_1} + \frac{pWI}{y_1}$$

Now $[y_1'' + p y_1' + q y_1] = 0$, $\{y_1$ satisfies homogeneous form of (A)$\}$

$I' = y_1 g / W$

$W' = -pW$, $\{W = e^{-\int p \, dx}, W' = e^{-\int p \, dx}(-p) = -pW\}$

So $LHS(A) = g$ , as required.

## Written solution to Q1, Example Sheet 4

$$y'' - \frac{x(x+2)}{x^2} y' + \frac{(x+2)}{x^2} y = 0$$

$$p(x) = -\left(1 + \frac{2}{x}\right)$$

$$W = \exp\left[-\int -\left(1 + \frac{2}{x}\right) dx\right] = \exp\left[x + 2 \ln x\right]$$

$$\underline{W = x^2 e^x}$$

$$y_2 = y_1 \int W / y_1^2 \, dx \quad , \quad y_1 = x$$

$$y_2 = x \int \frac{x^2 e^x}{x^2} dx = x \int e^x dx = \underline{x e^x}$$

## Written solution to Q3, Example Sheet 5

$$y'' + 2y' + y = e^{-x}$$

$p(x) = 2$ , $\quad W = \exp\left[-\int 2 \, dx\right] = \underline{e^{-2x}}$

$$y_p = y_1 \int \left[ \frac{W}{y_1^2} \int \frac{y_1 g}{W} dx \right] dx, \quad y_1 = e^{-x}, \quad g = e^{-x}$$

$$y_p = e^{-x} \int \left[ \frac{e^{-2x}}{e^{-2x}} \int \frac{e^{-x} e^{-x}}{e^{-2x}} dx \right] dx$$

$$= e^{-x} \int \left[ \int dx \right] dx = e^{-x} \int x \, dx$$

$$\underline{y_p = \frac{x^2}{2} e^{-x}}$$