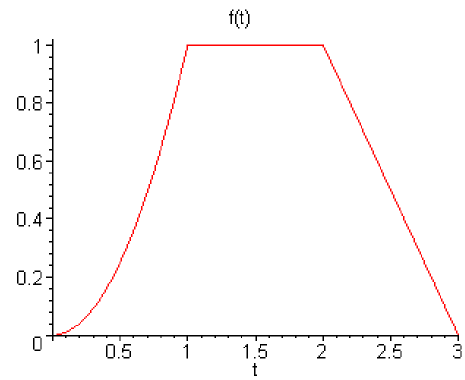
Chapter Six

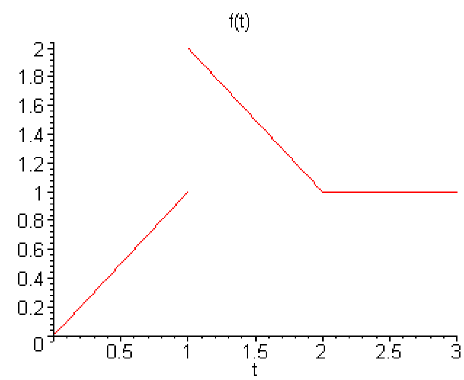
Section 6.1

3.



The function $f(t)$ is *continuous*.

4.



The function $f(t)$ has a *jump discontinuity* at $t = 1$.

7. Integration is a linear operation. It follows that

$$\int_0^A cosh\, bt \cdot e^{-st} dt = \frac{1}{2}\int_0^A e^{bt} \cdot e^{-st} dt + \frac{1}{2}\int_0^A e^{-bt} \cdot e^{-st} dt$$
$$= \frac{1}{2}\int_0^A e^{(b-s)t} dt + \frac{1}{2}\int_0^A e^{-(b+s)t} dt\,.$$

Hence

$$\int_0^A cosh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] + \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty cosh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-b}\right] + \frac{1}{2}\left[\frac{1}{s+b}\right]$$
$$= \frac{s}{s^2 - b^2}\,.$$

Note that the above is valid for $s > |b|$.

8. Proceeding as in Prob. 7,

$$\int_0^A sinh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty sinh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-b}\right] - \frac{1}{2}\left[\frac{1}{s+b}\right]$$
$$= \frac{b}{s^2 - b^2}\,.$$

The limit exists as long as $s > |b|$.

10. Observe that $e^{at} sinh\, bt = \left(e^{(a+b)t} - e^{(a-b)t}\right)/2$. It follows that

$$\int_0^A e^{at}\, sinh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(a+b-s)A}}{s-a+b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b-a+s)A}}{s+b-a}\right].$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty e^{at}\, sinh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1}{s-a+b}\right] - \frac{1}{2}\left[\frac{1}{s+b-a}\right]$$
$$= \frac{b}{(s-a)^2 - b^2}\,.$$

The limit exists as long as $s - a > |b|$.

11. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\, sin\, bt] = \frac{1}{2i}\mathcal{L}\left[e^{ibt}\right] - \frac{1}{2i}\mathcal{L}\left[e^{-ibt}\right].$$

Since

$$\int_0^\infty e^{(a+ib)t} e^{-st} dt = \frac{1}{s-a-ib}\,,$$

we have

$$\int_0^\infty e^{\pm ibt}\, e^{-st} dt = \frac{1}{s \mp ib} \, .$$

Therefore

$$\begin{aligned} \mathcal{L}[sin\, bt] &= \frac{1}{2i}\left[\frac{1}{s\, -\, ib} - \frac{1}{s\, +\, ib}\right] \\ &= \frac{b}{s^2 + b^2} \, . \end{aligned}$$

12. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\, cos\, bt] = \frac{1}{2}\mathcal{L}\left[e^{ibt}\right] + \frac{1}{2}\mathcal{L}\left[e^{-ibt}\right].$$

From Prob. 11 , we have

$$\int_0^\infty e^{\pm ibt}\, e^{-st} dt = \frac{1}{s \mp ib} \, .$$

Therefore

$$\begin{aligned} \mathcal{L}[cos\, bt] &= \frac{1}{2}\left[\frac{1}{s\, -\, ib} + \frac{1}{s\, +\, ib}\right] \\ &= \frac{s}{s^2 + b^2} \, . \end{aligned}$$

14. Using the *linearity* of the Laplace transform,

$$\mathcal{L}\left[\, e^{at} cos\, bt\right] = \frac{1}{2}\mathcal{L}\left[e^{(a+ib)t}\right] + \frac{1}{2}\mathcal{L}\left[e^{(a-ib)t}\right].$$

Based on the integration in Prob. 11 ,

$$\int_0^\infty e^{(a \pm ib)t} e^{-st} dt = \frac{1}{s - a \mp ib} \, .$$

Therefore

$$\begin{aligned} \mathcal{L}\left[e^{at} cos\, bt\right] &= \frac{1}{2}\left[\frac{1}{s\, - a -\, ib} + \frac{1}{s\, - a +\, ib}\right] \\ &= \frac{s - a}{(s-a)^2 + b^2} \, . \end{aligned}$$

The above is valid for $s > a$.

15. Integrating *by parts*,

$$\int_0^A te^{at}\cdot e^{-st}dt \;=\; -\left.\frac{t\,e^{(a-s)t}}{s-a}\right|_0^A + \int_0^A \frac{1}{s-a}e^{(a-s)t}dt$$
$$= \frac{1-e^{A(a-s)}+A(a-s)e^{A(a-s)}}{(s-a)^2}\,.$$

Taking a *limit*, as $A\to\infty$,

$$\int_0^\infty te^{at}\cdot e^{-st}dt = \frac{1}{(s-a)^2}\,.$$

Note that the limit exists as long as $s>a$.

17. Observe that $t\,cosh\,at = (t\,e^{at}+t\,e^{-at})/2$. For any value of c,

$$\int_0^A t\,e^{ct}\cdot e^{-st}dt \;=\; -\left.\frac{t\,e^{(c-s)t}}{s-c}\right|_0^A + \int_0^A \frac{1}{s-c}e^{(c-s)t}dt$$
$$= \frac{1-e^{A(c-s)}+A(c-s)e^{A(c-s)}}{(s-c)^2}\,.$$

Taking a *limit*, as $A\to\infty$,

$$\int_0^\infty te^{ct}\cdot e^{-st}dt = \frac{1}{(s-c)^2}\,.$$

Note that the limit exists as long as $s>|c|$. Therefore,

$$\int_0^\infty t\,cosh\,at\cdot e^{-st}dt \;=\; \frac{1}{2}\left[\frac{1}{(s-a)^2}+\frac{1}{(s+a)^2}\right]$$
$$= \frac{s^2+a^2}{(s-a)^2(s+a)^2}\,.$$

18. Integrating *by parts*,

$$\int_0^A t^n e^{at}\cdot e^{-st}dt \;=\; -\left.\frac{t^n\,e^{(a-s)t}}{s-a}\right|_0^A + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt$$
$$= -\frac{A^n e^{-(s-a)A}}{s-a} + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt\,.$$

Continuing to integrate by parts, it follows that

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = -\frac{A^n e^{(a-s)A}}{s-a} - \frac{nA^{n-1}e^{(a-s)A}}{(s-a)^2} - \\ - \frac{n!Ae^{(a-s)A}}{(n-2)!(s-a)^3} - \cdots - \frac{n!\left(e^{(a-s)A} - 1\right)}{(s-a)^{n+1}} \,.$$

That is,

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = p_n(A) \cdot e^{(a-s)A} + \frac{n!}{(s-a)^{n+1}},$$

in which $p_n(\xi)$ is a *polynomial* of degree n. For any *given* polynomial,

$$\lim_{A \to \infty} p_n(A) \cdot e^{-(s-a)A} = 0 \,,$$

as long as $s > a$. Therefore,

$$\int_0^\infty t^n e^{at} \cdot e^{-st} dt = \frac{n!}{(s-a)^{n+1}} \,.$$

20. Observe that $t^2 sinh\, at = (t^2 e^{at} - t^2 e^{-at})/2$. Using the result in Prob. 18,

$$\int_0^\infty t^2 sinh\, at \cdot e^{-st} dt = \frac{1}{2}\left[\frac{2!}{(s-a)^3} - \frac{2!}{(s+a)^3}\right] \\ = \frac{2a(3s^2 + a^2)}{(s^2 - a^2)^3} \,.$$

The above is valid for $s > |a|$.

22. Integrating *by parts*,

$$\int_0^A t\, e^{-t} dt = -\, t\, e^{-t}\Big|_0^A + \int_0^A e^{-t} dt \\ = 1 - e^{-A} - Ae^{-A}.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty t\, e^{-t} dt = 1 - e^{-A} \,.$$

Hence the integral *converges*.

23. Based on a series expansion, note that for $t > 0$,

$$e^t > 1 + t + t^2/2 > t^2/2 \,.$$

It follows that for $t > 0$,

$$t^{-2}e^t > \frac{1}{2}.$$

Hence for any finite $A > 1$,

$$\int_1^A t^{-2}e^t dt > \frac{A-1}{2}.$$

It is evident that the limit as $A \to \infty$ does not exist.

24. Using the fact that $|cos\, t| \leq 1$, and the fact that

$$\int_0^\infty e^{-t}dt = 1,$$

it follows that the given integral *converges*.

25(a). Let $p > 0$. Integrating *by parts*,

$$\begin{aligned}\int_0^A e^{-x}x^p dx &= -e^{-x}x^p\Big|_0^A + p\int_0^A e^{-x}x^{p-1}dx \\ &= -A^p e^{-A} + p\int_0^A e^{-x}x^{p-1}dx.\end{aligned}$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty e^{-x}x^p dx = p\int_0^\infty e^{-x}x^{p-1}dx.$$

That is, $\Gamma(p+1) = p\,\Gamma(p)$.

(b). Setting $p = 0$,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

(c). Let $p = n$. Using the result in Part (b),

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &\;\;\vdots \\ &= n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1).\end{aligned}$$

Since $\Gamma(1) = 1$, $\Gamma(n+1) = n!$.

(d). Using the result in Part (b),

$$\begin{aligned}\Gamma(p+n) &= (p+n-1)\,\Gamma(p+n-1) \\ &= (p+n-1)(p+n-2)\Gamma(p+n-2) \\ &\ \vdots \\ &= (p+n-1)(p+n-2)\cdots(p+1)p\,\Gamma(p)\,.\end{aligned}$$

Hence

$$\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+1)\cdots(p+n-1)\,.$$

Given that $\Gamma(1/2) = \sqrt{\pi}$, it follows that

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{945\sqrt{\pi}}{32}\,.$$

Section 6.2

1. Write the function as

$$\frac{3}{s^2+4}=\frac{3}{2}\,\frac{2}{s^2+4}\,.$$

Hence $\mathcal{L}^{-1}[Y(s)]=\frac{3}{2}\,sin\,2t\,.$

3. Using *partial fractions*,

$$\frac{2}{s^2+3s-4}=\frac{2}{5}\left[\frac{1}{s-1}-\frac{1}{s+4}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)]=\frac{2}{5}\left(e^t-e^{-4t}\right).$

5. Note that the denominator s^2+2s+5 is *irreducible* over the reals. Completing the square, $s^2+2s+5=(s+1)^2+4$. Now convert the function to a *rational function* of the variable $\xi=s+1$. That is,

$$\frac{2s+2}{s^2+2s+5}=\frac{2(s+1)}{(s+1)^2+4}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4}\right]=2\,cos\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)]=\mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right]=2e^{-t}cos\,2t\,.$$

6. Using *partial fractions*,

$$\frac{2s-3}{s^2-4}=\frac{1}{4}\left[\frac{1}{s-2}+\frac{7}{s+2}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)]=\frac{1}{4}(e^{2t}+7e^{-2t})$. Note that we can also write

$$\frac{2s-3}{s^2-4}=2\,\frac{s}{s^2-4}-\frac{3}{2}\,\frac{2}{s^2-4}.$$

8. Using *partial fractions*,

$$\frac{8s^2-4s+12}{s(s^2+4)}=3\,\frac{1}{s}+5\,\frac{s}{s^2+4}-2\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}[Y(s)] = 3 + 5\,cos\,2t - 2\,sin\,2t$.

9. The denominator $s^2 + 4s + 5$ is *irreducible* over the reals. Completing the square, $s^2 + 4s + 5 = (s+2)^2 + 1$. Now convert the function to a *rational function* of the variable $\xi = s + 2$. That is,

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1} .$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2+1} - \frac{2\,\xi}{\xi^2+1}\right] = 5\,sin\,t - 2\,cos\,t .$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1-2s}{s^2+4s+5}\right] = e^{-2t}(5\,sin\,t - 2\,cos\,t) .$$

10. Note that the denominator $s^2 + 2s + 10$ is *irreducible* over the reals. Completing the square, $s^2 + 2s + 10 = (s+1)^2 + 9$. Now convert the function to a *rational function* of the variable $\xi = s + 1$. That is,

$$\frac{2s-3}{s^2+2s+10} = \frac{2(s+1)-5}{(s+1)^2+9} .$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+9} - \frac{5}{\xi^2+9}\right] = 2\,cos\,3t - \frac{5}{3}\,sin\,3t .$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] = e^{-t}\left(2\,cos\,3t - \frac{5}{3}\,sin\,3t\right) .$$

12. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = 0 .$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = 0 .$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s+3}{s^2+3s+2} .$$

Using *partial fractions*,

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence $y(t) = \mathcal{L}^{-1}[Y(s)] = 2\,e^{-t} - e^{-2t}$.

13. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2-2s+2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. That is,

$$\frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{1}{\xi^2+1}\right] = sin\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+2}\right] = e^t sin\,t\,.$$

Hence $y(t) = e^t sin\,t$.

15. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] - 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) - 2\,Y(s) - 2s + 4 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s-4}{s^2-2s-2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. Completing the square,

$$\frac{2s-4}{s^2-2s-2}=\frac{2(s-1)-2}{(s-1)^2-3}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2-3}-\frac{2}{\xi^2-3}\right]=2\,cosh\,\sqrt{3}\,t-\frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)]=\mathcal{L}[f(t)]_{s\to s-a}$, the solution of the IVP is

$$y(t)=\mathcal{L}^{-1}\left[\frac{2s-4}{s^2-2s-2}\right]=e^{t}\left(2\,cosh\,\sqrt{3}\,t-\frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\right).$$

16. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s)-s\,y(0)-y'(0)+2[s\,Y(s)-y(0)]+5\,Y(s)=0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s)+2s\,Y(s)+5\,Y(s)-2s-3=0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s)=\frac{2s+3}{s^2+2s+5}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi=s+1$. That is,

$$\frac{2s+3}{s^2+2s+5}=\frac{2(s+1)+1}{(s+1)^2+4}.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4}+\frac{1}{\xi^2+4}\right]=2\,cos\,2t+\frac{1}{2}\,sin\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)]=\mathcal{L}[f(t)]_{s\to s-a}$, the solution of the IVP is

$$y(t)=\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+2s+5}\right]=e^{-t}\left(2\,cos\,2t+\frac{1}{2}\,sin\,2t\right).$$

17. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s)-s^3y(0)-s^2y'(0)-s\,y''(0)-y'''(0)-4\left[s^3Y(s)-s^2\,y(0)-s\,y'(0)-y''(0)\right]+$$
$$+\,6\left[s^2\,Y(s)-s\,y(0)-y'(0)\right]-4[s\,Y(s)-y(0)]+\,Y(s)=0$$

Applying the *initial conditions*,

$$s^4Y(s) - 4s^3Y(s) + 6s^2\,Y(s) - 4s\,Y(s) + \,Y(s) - s^2 + 4s - 7 = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2\, - 4s\, + 1} = \frac{s^2 - 4s + 7}{(s-1)^4}\,.$$

Using *partial fractions*,

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}.$$

Note that $\mathcal{L}[\,t^n\,] = (n!)/s^{n+1}$ and $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$. Hence the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2 - 4s + 7}{(s-1)^4}\right] = \frac{2}{3}\,t^3e^t - t^2e^t + t\,e^t.$$

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - \,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 - 1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right] = cosh\,t\,.$

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 + 2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right] = cos\,\sqrt{2}\,t\,.$

20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \omega^2\,Y(s) = \frac{s}{s^2+4}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + \omega^2\,Y(s) - s = \frac{s}{s^2+4}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right].$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = cos\,\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = cos\,2t\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t + cos\,\omega t \\ &= \frac{5-\omega^2}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t\,. \end{aligned}$$

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 2 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-2}{s^2-2s+2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{2}{5}\,\frac{2s-3}{s^2-2s+2}\,.$$

For the *last term*, we note that $s^2 - 2s + 2 = (s-1)^2 + 1$. So that

$$\frac{2s-3}{s^2-2s+2} = \frac{2(s-1)-1}{(s-1)^2+1}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+1} - \frac{1}{\xi^2+1}\right] = 2\,cos\,t - sin\,t\,.$$

Based on the *translation property* of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-2s+2}\right] = e^t(2\,cos\,t - sin\,t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,cos\,t - \frac{2}{5}\,sin\,t + \frac{2}{5}\,e^t(2\,cos\,t - sin\,t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + Y(s) = \frac{4}{s+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the *translation property* of the Laplace transform, the solution of the IVP is

$$y(t) = 2\,t^2 e^{-t} + t\,e^{-t} + 2\,e^{-t}.$$

25. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform
of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\,Y(s) + Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \int_0^\infty f(t)\,e^{-st}dt \\ &= \int_0^1 t\,e^{-st}dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\,.\end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\frac{s+1}{s^2(s^2+1)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

and

$$\frac{s}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

We find, by inspection, that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - sin\,t\,.$$

Referring to *Line* 13 , in Table 6.2.1 ,

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Let

$$\mathcal{L}[\,g(t)\,] = \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\,.$$

Then $g(t) = 1 + t - cos\,t - sin\,t$. It follows, therefore, that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{s+1}{s^2(s^2+1)}\right] = u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,]\,.$$

Combining the above, the solution of the IVP is

$$y(t) = t - sin\,t - u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,]\,.$$

26. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform
of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 4\,Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \int_0^\infty f(t)\,e^{-st}dt \\ &= \int_0^1 t\,e^{-st}dt + \int_1^\infty e^{-st}dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}\,.\end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

We find that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}\,t - \frac{1}{8}\,sin\,t\,.$$

Referring to *Line* 13, in Table 6.2.1,

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

It follows that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{1}{s^2(s^2+4)}\right]=u_1(t)\left[\frac{1}{4}(t-1)-\frac{1}{8}\,sin(t-1)\right].$$

Combining the above, the solution of the IVP is

$$y(t)=\frac{1}{4}\,t-\frac{1}{8}\,sin\,t-u_1(t)\left[\frac{1}{4}(t-1)-\frac{1}{8}\,sin(t-1)\right].$$

$28(a)$. Assuming that the conditions of Theorem 6.2.1 are satisfied,

$$\begin{aligned}F'(s)\;&=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt\\&=\int_0^\infty\frac{\partial}{\partial s}\big[e^{-st}f(t)\big]dt\\&=\int_0^\infty\big[-t\,e^{-st}f(t)\big]dt\\&=\int_0^\infty e^{-st}[-tf(t)]dt\,.\end{aligned}$$

(b). Using *mathematical induction*, suppose that for some $k\geq 1$,

$$F^{(k)}(s)=\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt\,.$$

Differentiating both sides,

$$\begin{aligned}F^{(k+1)}(s)\;&=\frac{d}{ds}\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt\\&=\int_0^\infty\frac{\partial}{\partial s}\Big[e^{-st}(-t)^k f(t)\Big]dt\\&=\int_0^\infty\Big[-t\,e^{-st}(-t)^k f(t)\Big]dt\\&=\int_0^\infty e^{-st}\Big[(-t)^{k+1}f(t)\Big]dt\,.\end{aligned}$$

29. We know that

$$\mathcal{L}\big[e^{at}\big]=\frac{1}{s-a}\,.$$

Based on Prob. 28,

$$\mathcal{L}\big[-t\,e^{at}\big]=\frac{d}{ds}\left[\frac{1}{s-a}\right].$$

Therefore,

$$\mathcal{L}\big[t\,e^{at}\big] = \frac{1}{(s-a)^2}.$$

31. Based on Prob. 28,

$$\begin{aligned}\mathcal{L}[(-t)^n] &= \frac{d^n}{ds^n}\mathcal{L}[1] \\ &= \frac{d^n}{ds^n}\left[\frac{1}{s}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}[t^n] &= (-1)^n\frac{(-1)^n n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}.\end{aligned}$$

33. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,sin\,bt\big] = \frac{b}{(s-a)^2+b^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}\big[t\,e^{at}\,sin\,bt\big] &= -\frac{d}{ds}\left[\frac{b}{(s-a)^2+b^2}\right] \\ &= \frac{2b(s-a)}{(s^2-2as+a^2+b^2)^2}.\end{aligned}$$

34. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,cos\,bt\big] = \frac{s-a}{(s-a)^2+b^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}\big[t\,e^{at}\,cos\,bt\big] &= -\frac{d}{ds}\left[\frac{s-a}{(s-a)^2+b^2}\right] \\ &= \frac{(s-a)^2-b^2}{(s^2-2as+a^2+b^2)^2}.\end{aligned}$$

35(a). Taking the Laplace transform of the given *Bessel equation*,

$$\mathcal{L}[\,t\,y''] + \mathcal{L}[\;y'] + \mathcal{L}[\,t\,y\,] = 0\,.$$

Using the *differentiation property* of the transform,

$$-\,\frac{d}{ds}\mathcal{L}[\,y''] + \mathcal{L}[\;y'] - \frac{d}{ds}\mathcal{L}[\,y\,] = 0\,.$$

That is,

$$-\,\frac{d}{ds}\big[s^2Y(s) - s\,y(0) - y'(0)\big] + s\,Y(s) - \,y(0) - \frac{d}{ds}Y(s) = 0\,.$$

It follows that

$$\big(1+s^2\big)Y'(s) + s\,Y(s) = 0\,.$$

(b). We obtain a *first-order linear* ODE in $Y(s)$:

$$Y'(s) + \frac{s}{s^2+1}Y(s) = 0\,,$$

with *integrating factor*

$$\mu(s) = exp\bigg(\int \frac{s}{s^2+1}ds\bigg) = \sqrt{s^2+1}\;.$$

The first-order ODE can be written as

$$\frac{d}{ds}\Big[\sqrt{s^2+1}\cdot Y(s)\Big] = 0\,,$$

with solution

$$Y(s) = \frac{c}{\sqrt{s^2+1}}\;.$$

(c). In order to obtain *negative* powers of s, first write

$$\frac{1}{\sqrt{s^2+1}} = \frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}.$$

Expanding $\left(1+\frac{1}{s^2}\right)^{-1/2}$ in a *binomial series*,

$$\frac{1}{\sqrt{1+(1/s^2)}} = 1 - \frac{1}{2}\,s^{-2} + \frac{1\cdot 3}{2\cdot 4}\,s^{-4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,s^{-6} + \cdots,$$

valid for $s^{-2} < 1$. Hence, we can formally express $Y(s)$ as

$$Y(s) = c\left[\frac{1}{s} - \frac{1}{2}\,\frac{1}{s^3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{s^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{s^7} + \cdots\right].$$

Assuming that *term-by-term* inversion is valid,

$$y(t) = c\left[1 - \frac{1}{2}\frac{t^2}{2!} + \frac{1\cdot 3}{2\cdot 4}\frac{t^4}{4!} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^6}{6!} + \cdots\right]$$
$$= c\left[1 - \frac{2!}{2^2}\frac{t^2}{2!} + \frac{4!}{2^2\cdot 4^2}\frac{t^4}{4!} - \frac{6!}{2^2\cdot 4^2\cdot 6^2}\frac{t^6}{6!} + \cdots\right].$$

It follows that

$$y(t) = c\left[1 - \frac{1}{2^2}t^2 + \frac{1}{2^2\cdot 4^2}t^4 - \frac{1}{2^2\cdot 4^2\cdot 6^2}t^6 + \cdots\right]$$
$$= c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}t^{2n}.$$

The series is evidently the expansion, about $x = 0$, of $J_0(t)$.

36(b). Taking the Laplace transform of the given *Legendre equation*,

$$\mathcal{L}[\,y''] - \mathcal{L}\left[t^2 y''\right] - 2\,\mathcal{L}[t\,y'] + \alpha(\alpha+1)\mathcal{L}[\,y\,] = 0\,.$$

Using the *differentiation property* of the transform,

$$\mathcal{L}[\,y''] - \frac{d^2}{ds^2}\mathcal{L}[\,y''] + 2\frac{d}{ds}\mathcal{L}[\,y'] + \alpha(\alpha+1)\mathcal{L}[\,y\,] = 0\,.$$

That is,

$$\left[s^2Y(s) - s\,y(0) - y'(0)\right] - \frac{d^2}{ds^2}\left[s^2Y(s) - s\,y(0) - y'(0)\right] +$$
$$+ 2\frac{d}{ds}[s\,Y(s) - y(0)] + \alpha(\alpha+1)Y(s) = 0\,.$$

Invoking the *initial conditions*, we have

$$s^2Y(s) - 1 - \frac{d^2}{ds^2}\left[s^2Y(s) - 1\right] + 2\frac{d}{ds}[s\,Y(s)] + \alpha(\alpha+1)Y(s) = 0\,.$$

After carrying out the differentiation, the equation simplifies to

$$\frac{d^2}{ds^2}\left[s^2Y(s)\right] - 2\frac{d}{ds}[s\,Y(s)] - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

That is,

$$s^2\frac{d^2}{ds^2}Y(s) + 2s\,\frac{d}{ds}Y(s) - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

37. By definition of the Laplace transform, given the appropriate conditions,

$$\begin{aligned}\mathcal{L}[\,g(t)] &= \int_0^\infty e^{-st}\left[\int_0^t f(\tau)d\tau\right]dt \\ &= \int_0^\infty \int_0^t e^{-st} f(\tau)d\tau dt\,.\end{aligned}$$

Assuming that the order of integration can be exchanged,

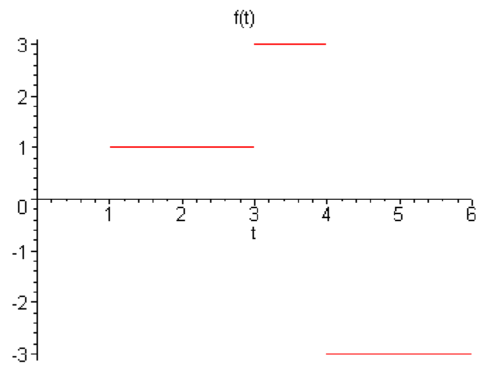
$$\begin{aligned}\mathcal{L}[\,g(t)] &= \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}dt\right]d\tau \\ &= \int_0^\infty f(\tau)\left[\frac{e^{-s\tau}}{s}\right]d\tau.\end{aligned}$$

[Note the *region* of integration is the area between the lines $\tau(t) = t$ and $\tau(t) = 0$.]
Hence

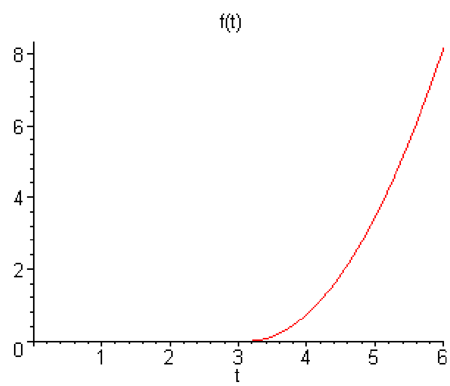
$$\begin{aligned}\mathcal{L}[\,g(t)] &= \frac{1}{s}\int_0^\infty f(\tau)\, e^{-s\tau}d\tau \\ &= \frac{1}{s}\,\mathcal{L}[\,f(t)].\end{aligned}$$

Section 6.3

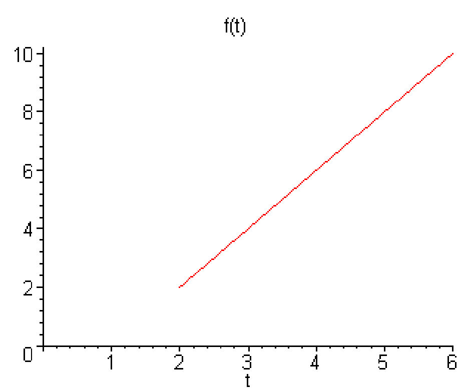
1.



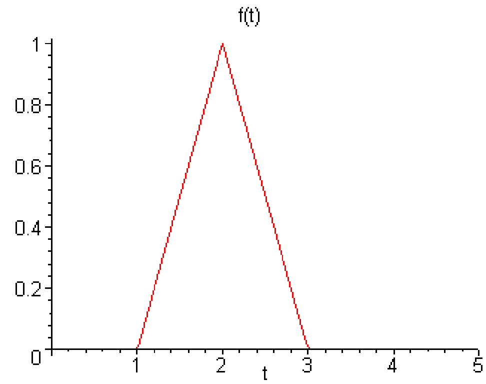
3.



5.



6.



7. Using the Heaviside function, we can write

$$f(t) = (t-2)^2\, u_2(t)\,.$$

The Laplace transform has the property that

$$\mathcal{L}[\, u_c(t) f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Hence

$$\mathcal{L}\big[\,(t-2)^2\, u_2(t)\big] = \frac{2\,e^{-2s}}{s^2}.$$

9. The function can be expressed as

$$f(t) = (t-\pi)[u_\pi(t) - u_{2\pi}(t)].$$

Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-\pi)\, u_\pi(t) - (t-2\pi)\, u_{2\pi}(t) - \pi\, u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}[\, f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

10. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[\, f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}.$$

11. Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-2)\, u_2(t) - \, u_2(t) - (t-3)\, u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}[\,f(t)] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

12. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[\,f(t)] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

13. Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}.$$

15. First consider the function

$$G(s) = \frac{2(s-1)}{s^2-2s+2}.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2+1}.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2\,e^t \cos t\,.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s}G(s)\right] = 2\,e^{(t-2)}\cos\,(t-2)\,u_2(t)\,.$$

16. The *inverse transform* of the function $2/(s^2-4)$ is $f(t) = \sinh 2t$. Using the *translation property* of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\,e^{-2s}}{s^2-4}\right] = \sinh 2(t-2)\cdot\, u_2(t)\,.$$

17. First consider the function

$$G(s) = \frac{(s-2)}{s^2-4s+3}.$$

Completing the square in the denominator,

$$G(s) = \frac{(s-2)}{(s-2)^2 - 1} \,.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = e^{2t}\, cosh\, t \,.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{(s-2)e^{-s}}{s^2 - 4s + 3}\right] = e^{2(t-1)} cosh\,(t-1)\, u_1(t) \,.$$

18. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} \,.$$

It follows from the $translation\, property$ of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) - u_4(t) \,.$$

19(a). By definition of the Laplace transform,

$$\mathcal{L}[\,f(ct)] = \int_0^\infty e^{-st} f(ct) dt \,.$$

Making a change of variable, $\tau = ct$, we have

$$\begin{aligned} \mathcal{L}[\,f(ct)] &= \frac{1}{c}\int_0^\infty e^{-s(\tau/c)} f(\tau) d\tau \\ &= \frac{1}{c}\int_0^\infty e^{-(s/c)\tau} f(\tau) d\tau \,. \end{aligned}$$

Hence $\mathcal{L}[\,f(ct)] = \frac{1}{c}\, F\left(\frac{s}{c}\right)$, where $s/c > a$.

(b). Using the result in Part (a),

$$\mathcal{L}\left[\,f\left(\frac{t}{k}\right)\right] = k\, F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k}\, f\left(\frac{t}{k}\right).$$

(c). From Part (b),

$$\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)]_{s \to s-c}$,

$$\mathcal{L}^{-1}[F(as + b)] = e^{-bt/a} \, \frac{1}{a} f\left(\frac{t}{a}\right).$$

20. First write

$$F(s) = \frac{n!}{\left(\frac{s}{2}\right)^{n+1}}.$$

Let $G(s) = n!/s^{n+1}$. Based on the results in Prob. 19,

$$\frac{1}{2}\,\mathcal{L}^{-1}\left[G\left(\frac{s}{2}\right)\right] = g(2t),$$

in which $g(t) = t^n$. Hence

$$\mathcal{L}^{-1}[F(s)] = 2\,(2t)^n = 2^{n+1} t^n.$$

23. First write

$$F(s) = \frac{e^{-4(s-1/2)}}{2(s-1/2)}.$$

Now consider

$$G(s) = \frac{e^{-2s}}{s}.$$

Using the result in Prob. $19(b)$,

$$\mathcal{L}^{-1}[G(2s)] = \frac{1}{2} g\left(\frac{t}{2}\right),$$

in which $g(t) = u_2(t)$. Hence $\mathcal{L}^{-1}[G(2s)] = \frac{1}{2}\, u_2(t/2) = \frac{1}{2}\, u_4(t)$. It follows that

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2} e^{t/2}\, u_4(t).$$

24. By definition of the Laplace transform,

$$\mathcal{L}[\,f(t)] = \int_0^\infty e^{-st}u_1(t)dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[\,f(t)] &= \int_0^1 e^{-st}\,dt \\ &= \frac{1-e^{-s}}{s}\,.\end{aligned}$$

25. First write the function as $f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$. It follows that

$$\mathcal{L}[\,f(t)] = \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[\,f(t)] &= \frac{1-e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} \\ &= \frac{1-e^{-s}+e^{-2s}-e^{-3s}}{s}\,.\end{aligned}$$

26. The transform may be computed directly. On the other hand, using the *translation property* of the transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{1}{s} + \sum_{k=1}^{2n+1}(-1)^k\,\frac{e^{-ks}}{s} \\ &= \frac{1}{s}\left[\sum_{k=0}^{2n+1}(-e^{-s})^k\right] \\ &= \frac{1}{s}\,\frac{1-(-e^{-s})^{2n+2}}{1+e^{-s}}\,.\end{aligned}$$

That is,

$$\mathcal{L}[f(t)] = \frac{1-(e^{-2s})^{n+1}}{s(1+e^{-s})}\,.$$

29. The given function is *periodic*, with $T = 2$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}dt\,.$$

That is,

$$\mathcal{L}[f(t)] = \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1}{s(1+e^{-s})}.$$

31. The function is *periodic*, with $T = 1$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-s}} \int_0^1 t\, e^{-st} dt.$$

It follows that

$$\mathcal{L}[f(t)] = \frac{1-e^{-s}(1+s)}{s^2(1-e^{-s})}.$$

32. The function is *periodic*, with $T = \pi$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\pi s}} \int_0^\pi sin\, t \cdot e^{-st} dt.$$

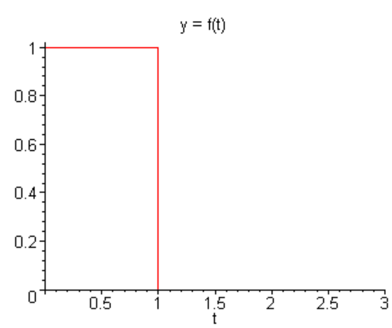
We first calculate

$$\int_0^\pi sin\, t \cdot e^{-st} dt = \frac{1+e^{-\pi s}}{1+s^2}.$$

Hence

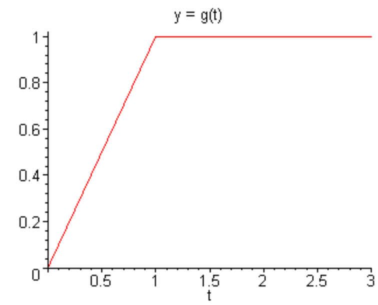
$$\mathcal{L}[f(t)] = \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(1+s^2)}.$$

33(a).



$$\mathcal{L}[f(t)] = \mathcal{L}[1] - \mathcal{L}[u_1(t)] = \frac{1}{s} - \frac{e^{-s}}{s}.$$

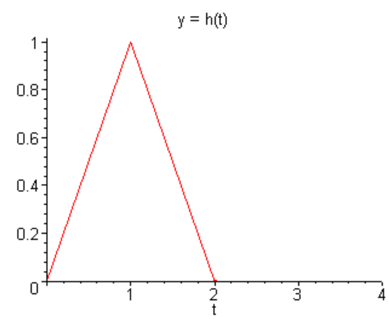
(b).



Let $F(s) = \mathcal{L}[1 - u_1(t)]$. Then

$$\mathcal{L}\left[\int_0^t [1 - u_1(\tau)]d\tau\right] = \frac{1}{s}\,F(s) = \frac{1 - e^{-s}}{s^2}\,.$$

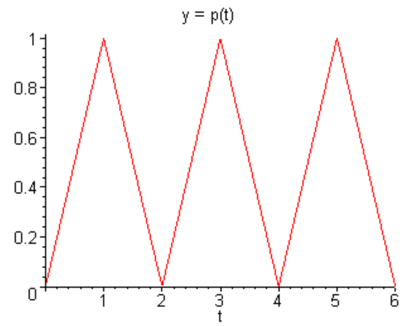
(c).



Let $G(s) = \mathcal{L}[g(t)]$. Then

$$\begin{aligned}\mathcal{L}[h(t)] &= G(s) - e^{-s}\,G(s) \\ &= \frac{1 - e^{-s}}{s^2} - e^{-s}\,\frac{1 - e^{-s}}{s^2} \\ &= \frac{(1 - e^{-s})^2}{s^2}\,.\end{aligned}$$

$34(a)$.



(b). The given function is *periodic*, with $T = 2$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} p(t) dt \,.$$

Based on the piecewise definition of $p(t)$,

$$\begin{aligned} \int_0^2 e^{-st} p(t) dt &= \int_0^1 t\, e^{-st} dt + \int_1^2 (2-t) e^{-st} dt \\ &= \frac{1}{s^2}(1-e^{-s})^2. \end{aligned}$$

Hence

$$\mathcal{L}[p(t)] = \frac{(1-e^{-s})}{s^2(1+e^{-s})} \,.$$

(c). Since $p(t)$ satisfies the hypotheses of Theorem 6.2.1 ,

$$\mathcal{L}[p'(t)] = s\,\mathcal{L}[p(t)] - p(0) \,.$$

Using the result of Prob. 30,

$$\mathcal{L}[p'(t)] = \frac{(1-e^{-s})}{s(1+e^{-s})} \,.$$

We note the $p(0) = 0$, hence

$$\mathcal{L}[p(t)] = \frac{1}{s}\left[\frac{(1-e^{-s})}{s(1+e^{-s})}\right].$$

Section 6.4

2. Let $h(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - \,y(0)] + 2\,Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2\,Y(s) + 2s\,Y(s) + 2\,Y(s) - 1 = \mathcal{L}[h(t)].$$

The forcing function can be written as $h(t) = u_\pi(t) - u_{2\pi}(t)$. Its transform is

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s} - e^{-2\pi s}}{s}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}\,.$$

First note that

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}.$$

Using partial fractions,

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{(s+1)+1}{(s+1)^2+1}\,.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}sin\,t\,.$$

Now let

$$G(s) = \frac{1}{s(s^2+2s+2)}\,.$$

Then

$$\mathcal{L}^{-1}[G(s)] = \frac{1}{2} - \frac{1}{2}e^{-t}cos\,t - \frac{1}{2}e^{-t}sin\,t\,.$$

Using Theorem 6.3.1 ,

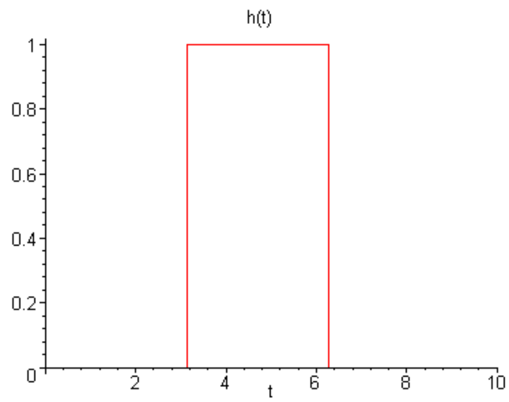
$$\mathcal{L}^{-1}[e^{-cs}G(s)] = \frac{1}{2}u_c(t) - \frac{1}{2}e^{-(t-c)}[cos(t-c) + sin(t-c)]u_c(t)\,.$$

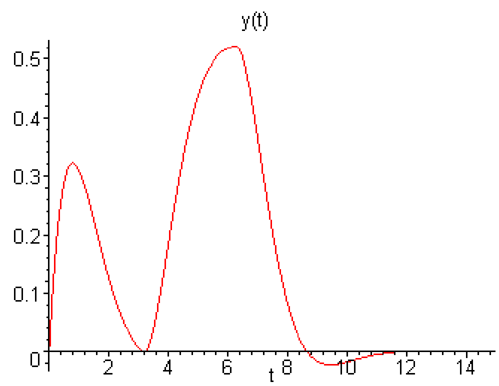
Hence the solution of the IVP is

$$
\begin{aligned}
y(t) &= e^{-t} sin\, t + \frac{1}{2} u_{\pi}(t) - \frac{1}{2} e^{-(t-\pi)} [cos(t-\pi) + sin(t-\pi)] u_{\pi}(t) - \\
&\quad - \frac{1}{2} u_{2\pi}(t) + \frac{1}{2} e^{-(t-2\pi)} [cos(t-2\pi) + sin(t-2\pi)] u_{2\pi}(t)\,.
\end{aligned}
$$

That is,

$$
\begin{aligned}
y(t) &= e^{-t} sin\, t + \frac{1}{2}[u_{\pi}(t) - u_{2\pi}(t)] + \frac{1}{2} e^{-(t-\pi)} [cos\, t + sin\, t]\, u_{\pi}(t) + \\
&\quad + \frac{1}{2} e^{-(t-2\pi)} [cos\, t + sin\, t] u_{2\pi}(t)\,.
\end{aligned}
$$





The solution starts out as free oscillation, due to the initial conditions. The amplitude increases, as long as the forcing is present. Thereafter, the solution rapidly decays.

4. Let $h(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 4\, Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 4Y(s) = \mathcal{L}[h(t)].$$

The transform of the forcing function is

$$\mathcal{L}[h(t)] = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+4)(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

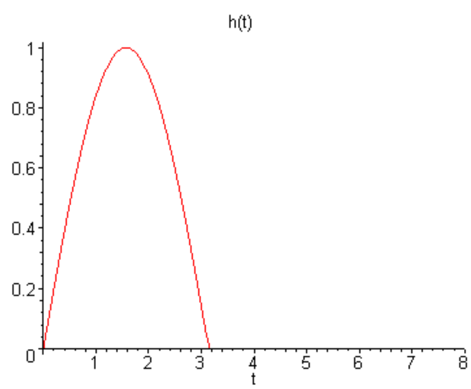
$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[sin\, t - \frac{1}{2}\, sin\, 2t\right].$$

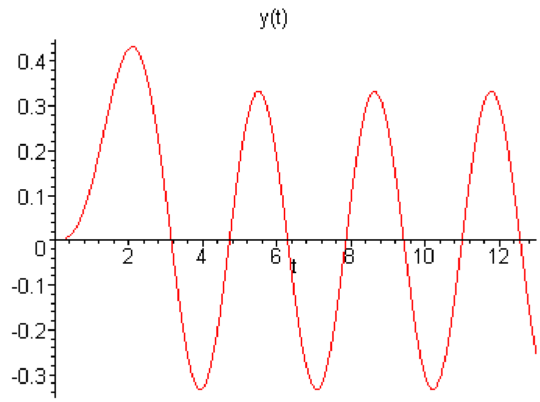
Based on Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[sin(t-\pi) - \frac{1}{2}\, sin(2t-2\pi)\right]u_\pi(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{3}\left[sin\, t - \frac{1}{2}\, sin\, 2t\right] - \frac{1}{3}\left[sin\, t + \frac{1}{2}\, sin\, 2t\right]u_\pi(t)\,.$$





Since there is no *damping term*, the solution follows the forcing function, after which the response is a steady oscillation about $y = 0$.

5. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = \mathcal{L}[f(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 3s\,Y(s) + 2\,Y(s) = \mathcal{L}[f(t)].$$

The transform of the forcing function is

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}\,.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Hence

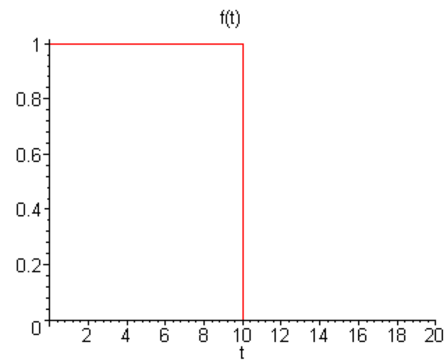
$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\,.$$

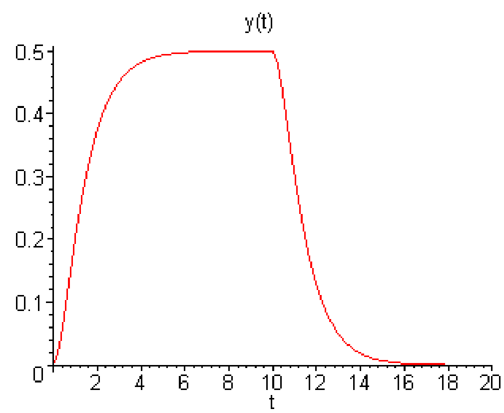
Based on Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left[1 + e^{-2(t-10)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$





The solution increases to a *temporary* steady value of $y = 1/2$. After the forcing ceases, the response decays exponentially to $y = 0$.

6. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - \,y(0)] + 2\,Y(s) = \frac{e^{-2s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - 1 = \frac{e^{-2s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s(s^2 + 3s + 2)}\,.$$

Using partial fractions,

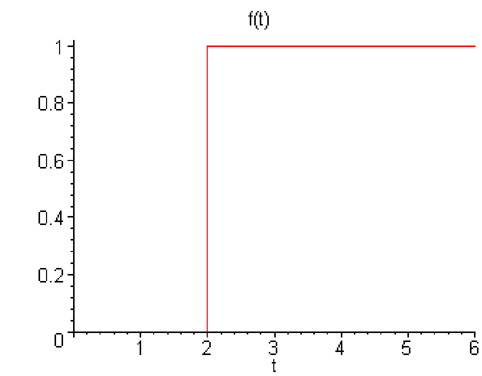
$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

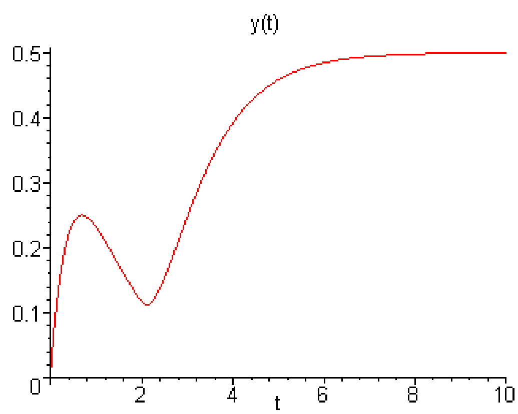
and

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Taking the inverse transform. term-by-term, the solution of the IVP is

$$y(t) = e^{-t} - e^{-2t} + \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u_2(t)\,.$$





Due to the initial conditions, the response has a transient *overshoot*, followed by an exponential convergence to a steady value of $y_s = 1/2$.

7. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) - s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

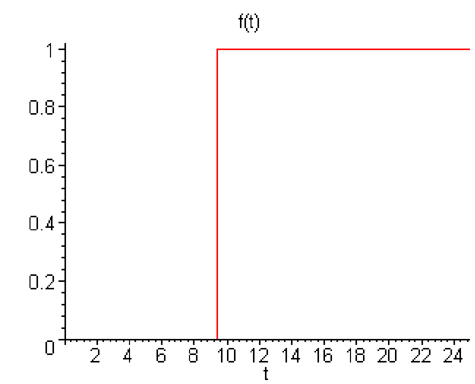
$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

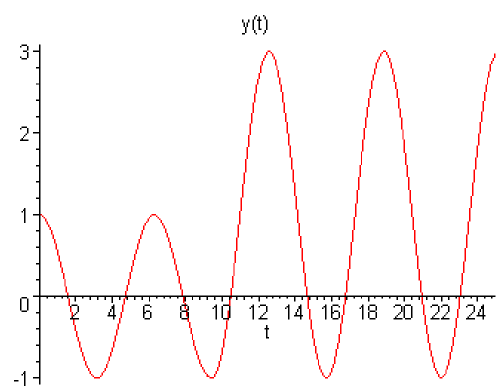
Hence

$$Y(s) = \frac{s}{s^2+1} + e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

Taking the inverse transform, the solution of the IVP is

$$\begin{aligned} y(t) &= cos\,t + [1 - cos(t-3\pi)]u_{3\pi}(t) \\ &= cos\,t + [1 + cos\,t\,]u_{3\pi}(t)\,. \end{aligned}$$





Due to initial conditions, the solution temporarily oscillates about $y = 0$. After the forcing is applied, the response is a steady oscillation about $y_m = 1$.

9. Let $g(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}[g(t)].$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) - 1 = \mathcal{L}[g(t)].$$

The forcing function can be written as

$$\begin{aligned} g(t) &= \frac{t}{2}[1 - u_6(t)] + 3\,u_6(t) \\ &= \frac{t}{2} - \frac{1}{2}(t-6)u_6(t) \end{aligned}$$

with Laplace transform

$$\mathcal{L}[g(t)] = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}\,.$$

Solving for the transform,

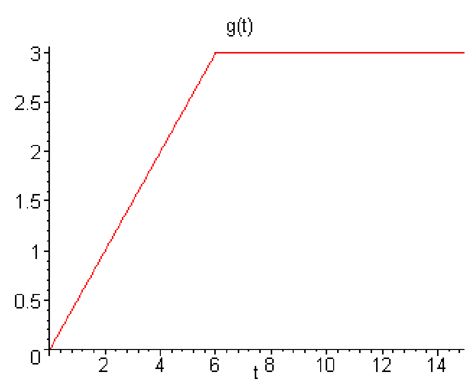
$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}\,.$$

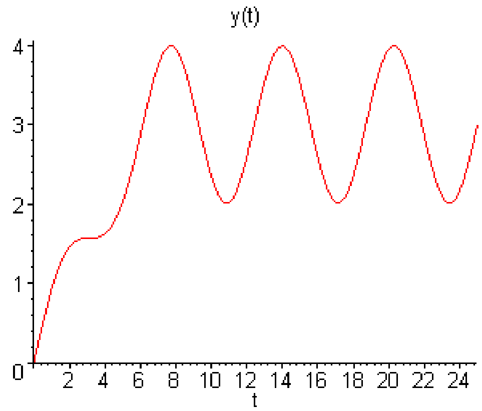
Using partial fractions,

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right].$$

Taking the inverse transform, and using Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= sin\,t + \frac{1}{2}[t - sin\,t] - \frac{1}{2}[(t-6) - sin(t-6)]u_6(t) \\ &= \frac{1}{2}[t + sin\,t] - \frac{1}{2}[(t-6) - sin(t-6)]u_6(t). \end{aligned}$$





The solution increases, in response to the *ramp input*, and thereafter oscillates about a mean value of $y_m = 3$.

11. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

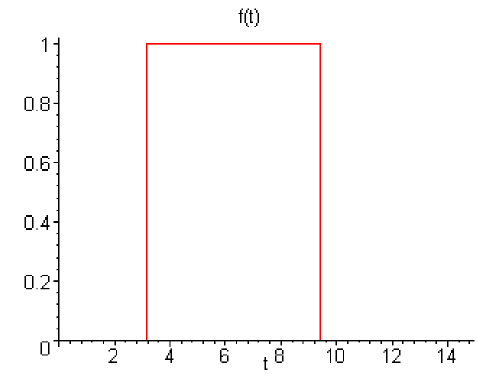
$$Y(s) = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-3\pi s}}{s(s^2+4)}.$$

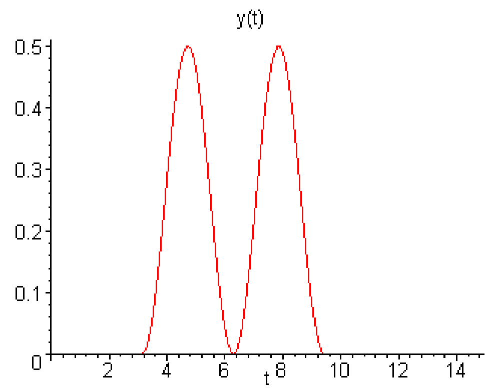
Using partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking the inverse transform, and applying Theorem 6.3.1 ,

$$\begin{aligned} y(t) &= \frac{1}{4}[1 - cos(2t - 2\pi)]u_\pi(t) - \frac{1}{4}[1 - cos(2t - 6\pi)]u_{3\pi}(t) \\ &= \frac{1}{4}[u_\pi(t) - u_{3\pi}(t)] - \frac{1}{4}cos\,2t \cdot [u_\pi(t) - u_{3\pi}(t)]\,. \end{aligned}$$





Since there is no damping term, the solution responds immediately to the forcing input. There is a temporary oscillation about $y = 1/4$.

12. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Applying the *initial conditions*,

$$s^4Y(s) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{e^{-s}}{s(s^4-1)} - \frac{e^{-2s}}{s(s^4-1)}\,.$$

Using partial fractions,

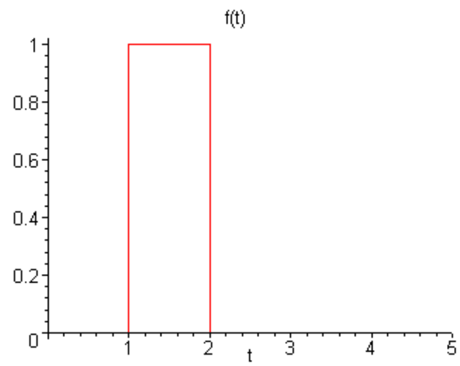
$$\frac{1}{s(s^4-1)} = \frac{1}{4}\left[-\frac{4}{s} + \frac{1}{s+1} + \frac{1}{s-1} + \frac{2s}{s^2+1}\right].$$

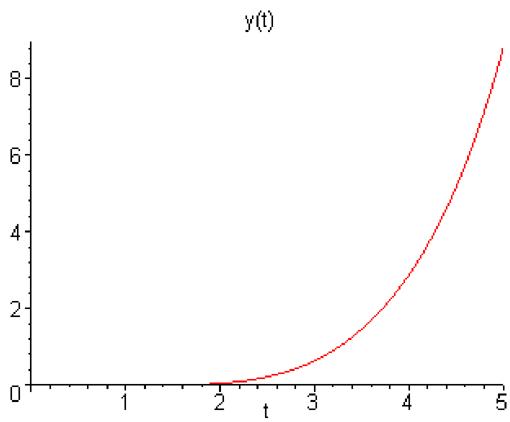
It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4-1)}\right] = \frac{1}{4}\big[-4+e^{-t}+e^{t}+2\,cos\,t\big].$$

Based on Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) \;=\; & -[u_1(t)-u_2(t)] + \frac{1}{4}\big[e^{-(t-1)}+e^{(t-1)}+2\,cos(t-1)\big]u_1(t) - \\ & -\frac{1}{4}\big[e^{-(t-2)}+e^{(t-2)}+2\,cos(t-2)\big]u_2(t)\,. \end{aligned}$$





The solution increases without bound, exponentially.

13. Taking the Laplace transform of the ODE, we obtain

$$\begin{aligned} & s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) + \\ & +5\big[s^2Y(s) - s\,y(0) - y'(0)\big] + 4\,Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}. \end{aligned}$$

Applying the *initial conditions*,

$$s^4Y(s) + 5s^2Y(s) + \; 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s(s^4+5s^2+4)} - \frac{e^{-\pi s}}{s(s^4+5s^2+4)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^4+5s^2+4)} = \frac{1}{12}\left[\frac{3}{s} + \frac{s}{s^2+4} - \frac{4s}{s^2+1}\right].$$

It follows that

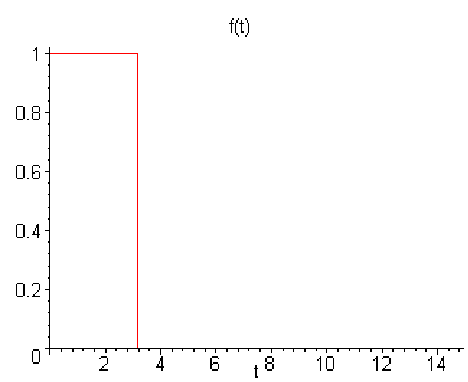
$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4+5s^2+4)}\right] = \frac{1}{12}[3 + cos\,2t - 4\,cos\,t].$$

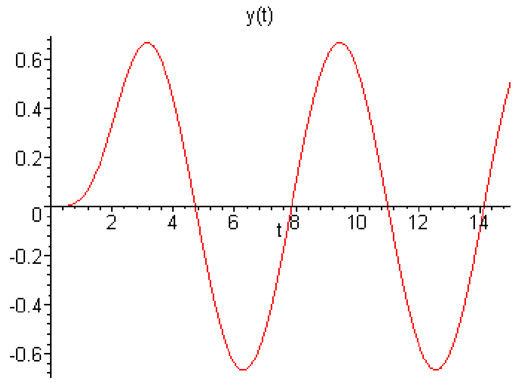
Based on Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4}[1-u_\pi(t)] + \frac{1}{12}[cos\,2t - 4\,cos\,t] - \\ &- \frac{1}{12}[cos\,2(t-\pi) - 4\,cos(t-\pi)]u_\pi(t)\,. \end{aligned}$$

That is,

$$\begin{aligned} y(t) &= \frac{1}{4}[1-u_\pi(t)] + \frac{1}{12}[cos\,2t - 4\,cos\,t] - \\ &- \frac{1}{12}[cos\,2t + 4\,cos\,t]u_\pi(t)\,. \end{aligned}$$





After an initial transient, the solution oscillates about $y_m = 0$.

14. The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \le t < t_0 + k \\ h, & t \ge t_0 + k \end{cases}$$

which can conveniently be expressed as

$$f(t) = \frac{h}{k}(t - t_0)\, u_{t_0}(t) - \frac{h}{k}(t - t_0 - k)\, u_{t_0+k}(t)\,.$$

15. The function is defined by

$$g(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \le t < t_0 + k \\ -\frac{h}{k}(t - t_0 - 2k), & t_0 + k \le t < t_0 + 2k \\ 0, & t \ge t_0 + 2k \end{cases}$$

which can also be written as

$$g(t) = \frac{h}{k}(t - t_0)\, u_{t_0}(t) - \frac{2h}{k}(t - t_0 - k)\, u_{t_0+k}(t) + \frac{h}{k}(t - t_0 - 2k)\, u_{t_0+2k}(t).$$

16(d). From Part (c), the solution is

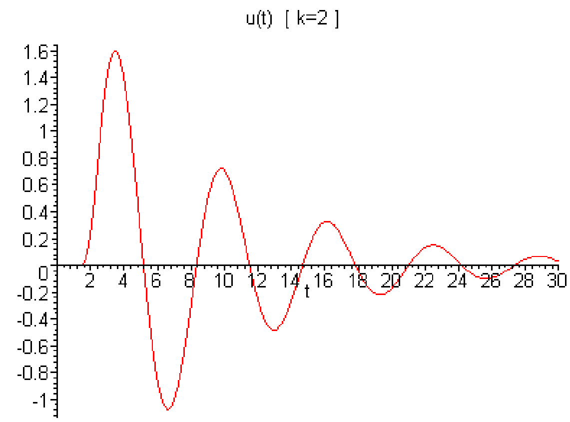
$$u(t) = 4k\, u_{3/2}(t)\, h\left(t - \frac{3}{2}\right) - 4k\, u_{5/2}(t)\, h\left(t - \frac{5}{2}\right),$$

where

$$h(t) = \frac{1}{4} - \frac{\sqrt{7}}{84}\, e^{-t/8} sin\left(\frac{3\sqrt{7}\, t}{8}\right) - \frac{1}{4}\, e^{-t/8} cos\left(\frac{3\sqrt{7}\, t}{8}\right).$$

Due to the *damping term*, the solution will decay to *zero*. The maximum will occur shortly after the forcing ceases. By plotting the various solutions, it appears that the solution will reach a value of $y = 2$, as long as $k > 2.51$.

(e).



Based on the graph, and numerical calculation, $|u(t)| < 0.1$ for $t > 25.6773$.

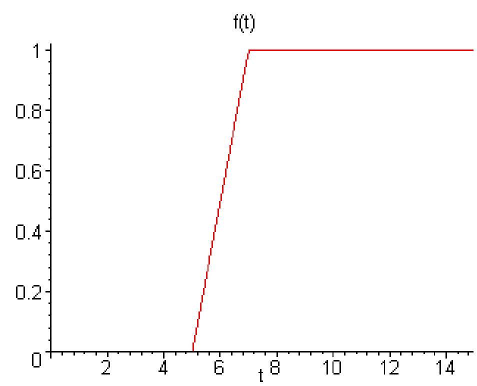
17. We consider the initial value problem

$$y'' + 4y = \frac{1}{k}[(t-5)\,u_5(t) - (t-5-k)\,u_{5+k}(t)],$$

with $y(0) = y'(0) = 0$.

(a). The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{1}{k}(t-5), & 5 \le t < 5+k \\ 1, & t \ge 5+k \end{cases}$$



(b). Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-5s}}{ks^2(s^2+4)} - \frac{e^{-(5+k)s}}{ks^2(s^2+4)}\,.$$

Using partial fractions,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}t - \frac{1}{8}sin\,2t\,.$$

Using Theorem 6.3.1 , the solution of the IVP is

$$y(t) = \frac{1}{k}[h(t-5)\,u_5(t) - h(t-5-k)\,u_{5+k}(t)]\,,$$

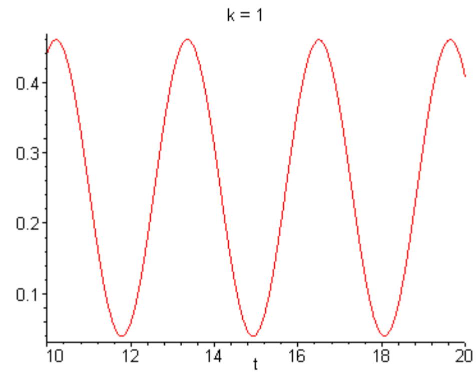
in which $h(t) = \frac{1}{4}t - \frac{1}{8}sin\,2t$.

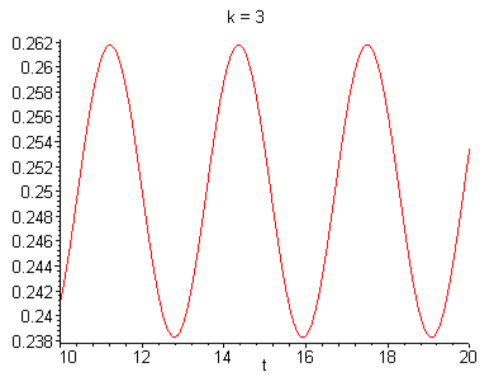
(c). Note that for $t > 5+k$, the solution is given by

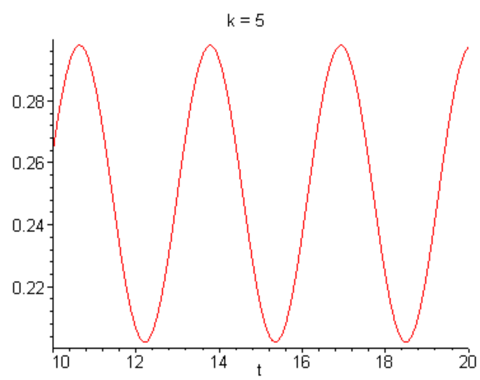
$$\begin{aligned} y(t) &= \frac{1}{4} - \frac{1}{8k}sin(2t-10) + \frac{1}{8k}\,sin(2t-10-2k) \\ &= \frac{1}{4} - \frac{sin\,k}{4k}\,cos(2t-10-k)\,. \end{aligned}$$

So for $t > 5+k$, the solution oscillates about $y_m = 1/4$, with an amplitude of

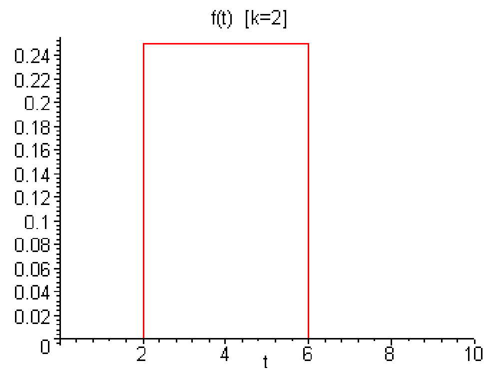
$$A = \frac{|sin(k)|}{4k}\,.$$


k = 1
0.4
0.3
0.2
0.1
10
12
14
t
16
18
20


k = 3
0.262
0.26
0.258
0.256
0.254
0.252
0.25
0.248
0.246
0.244
0.242
0.24
0.238
10
12
14
t
16
18
20


k = 5
0.28
0.26
0.24
0.22
10
12
14
t
16
18
20

$18(a)$.



(b). The forcing function can be expressed as

$$f_k(t) = \frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)].$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \frac{1}{3}[s\,Y(s) -\, y(0)] + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Applying the initial conditions,

$$s^2\,Y(s) + \frac{1}{3}s\,Y(s) + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Solving for the transform,

$$Y(s) = \frac{3\,e^{-(4-k)s}}{2ks(3s^2+s+12)} - \frac{3\,e^{-(4+k)s}}{2ks(3s^2+s+12)}\,.$$

Using partial fractions,

$$\begin{aligned}\frac{1}{s(3s^2+s+12)} &= \frac{1}{12}\left[\frac{1}{s} - \frac{1+3s}{3s^2+s+12}\right] \\ &= \frac{1}{12}\left[\frac{1}{s} - \frac{1}{6}\,\frac{1+6\left(s+\frac{1}{6}\right)}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right].\end{aligned}$$

Let

$$H(s) = \frac{1}{8k}\left[\frac{1}{s} - \frac{\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right].$$

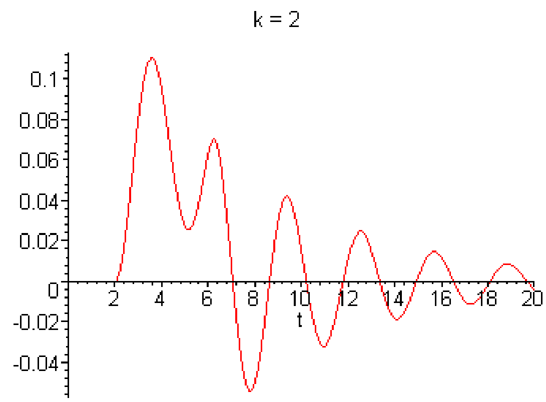
It follows that

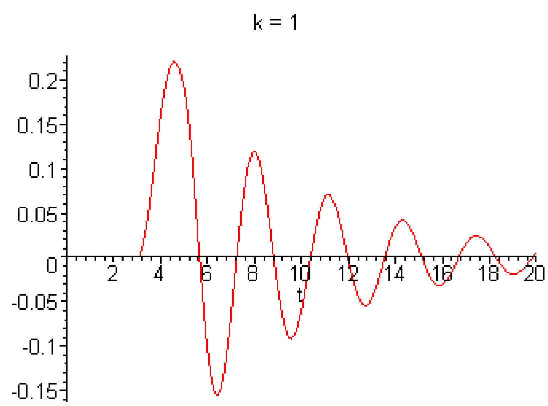
$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{1}{8k} - \frac{e^{-t/6}}{8k}\left[\frac{1}{\sqrt{143}}sin\left(\frac{\sqrt{143}\,t}{6}\right) + cos\left(\frac{\sqrt{143}\,t}{6}\right)\right].$$

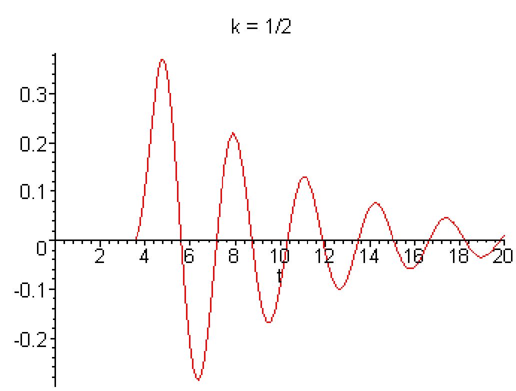
Based on Theorem 6.3.1 , the solution of the IVP is

$$y(t) = h(t-4+k)\,u_{4-k}(t) - h(t-4-k)\,u_{4+k}(t)\,.$$

(c).



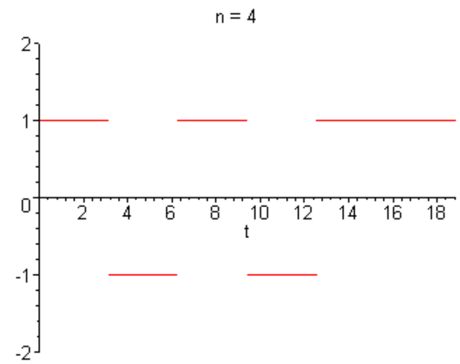




As the parameter k decreases, the solution remains *null* for a longer period of time.

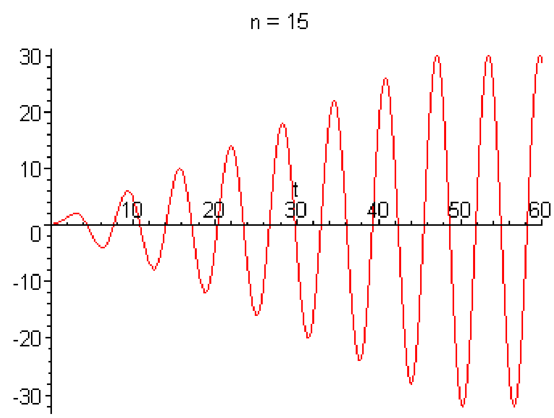
Since the *magnitude* of the impulsive force *increases*, the initial *overshoot* of the response also increases. The *duration* of the impulse decreases. All solutions eventually decay to $y = 0$.

19(a).

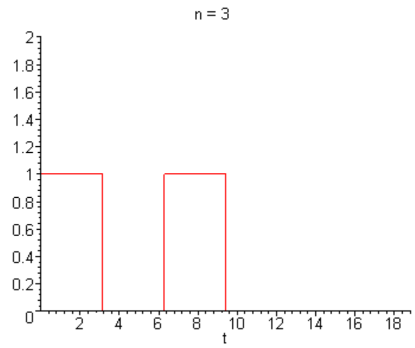


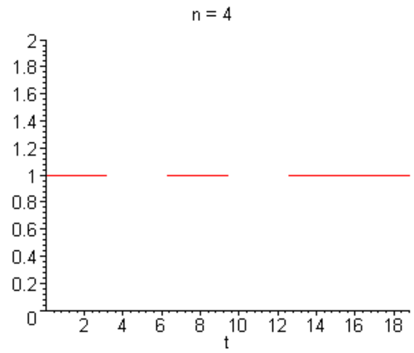
(c). From Part (b),

$$u(t) = 1 - cos\, t + 2\sum_{k=1}^{n}(-1)^k[1 - cos(t - k\pi)]u_{k\pi}(t)\,.$$



$21(a)$.





(b). Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,U(s) - s\,u(0) - u'(0) + U(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,U(s) + U(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2+1)} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - cos\, t\,.$$

Applying Theorem 6.3.1 , term-by-term, the solution of the IVP is

$$u(t) = h(t) + \sum_{k=1}^{n}(-1)^k h(t-k\pi)\, u_{k\pi}(t)\,.$$

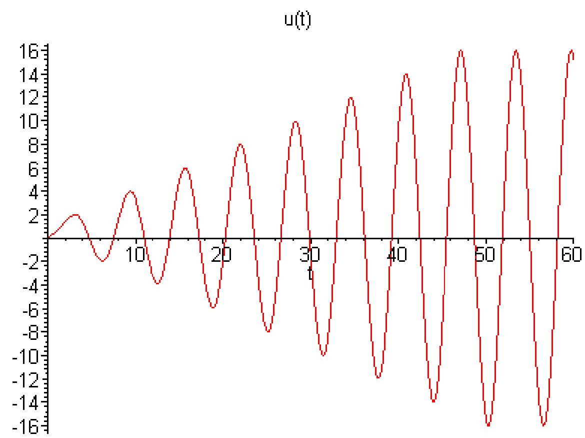
Note that

$$\begin{aligned} h(t-k\pi) &= u_0(t-k\pi) - cos(t-k\pi) \\ &= u_{k\pi}(t) - (-1)^k cos\, t\,. \end{aligned}$$

Hence

$$u(t) = 1 - cos\, t + \sum_{k=1}^{n}(-1)^k\, u_{k\pi}(t) - (cos\, t)\sum_{k=1}^{n} u_{k\pi}(t)\,.$$

(c).



The ODE has no *damping term*. Each interval of forcing adds to the energy of the system.
Hence the amplitude will increase. For $n = 15$, $g(t) = 0$ when $t > 15\pi$. Therefore the oscillation will eventually become *steady*, with an amplitude depending on the values of $u(15\pi)$ and $u'(15\pi)$.

(d). As n increases, the interval of forcing also increases. Hence the amplitude of the transient will increase with n . Eventually, the forcing function will be *constant*. In fact, for *large* values of t ,

$$g(t) = \begin{cases} 1\,, & n \text{ even} \\ 0\,, & n \text{ odd} \end{cases}$$

Further, for $t > n\pi$,

$$u(t) = 1 - cos\, t - n\, cos\, t - \frac{1 - (-1)^n}{2}\,.$$

Hence the steady state solution will oscillate about 0 or 1 , depending on n, with an amplitude of $A = n + 1$.

In the limit, as $n \to \infty$, the forcing function will be a periodic function, with period 2π. From Prob. 27, in Section 6.3 ,

$$\mathcal{L}[g(t)] = \frac{1}{s(1 + e^{-s})}\,.$$

As n increases, the duration and magnitude of the transient will increase without bound.

$22(a)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, U(s) + 0.1\, s\, U(s) + U(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2 + 0.1s + 1)} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s(s^2 + 0.1s + 1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 0.1s + 1)} = \frac{1}{s} - \frac{s + 0.1}{s^2 + 0.1s + 1}\,.$$

Since the denominator in the second term is irreducible, write

$$\frac{s + 0.1}{s^2 + 0.1s + 1} = \frac{(s + 0.05) + 0.05}{(s + 0.05)^2 + (399/400)}\,.$$

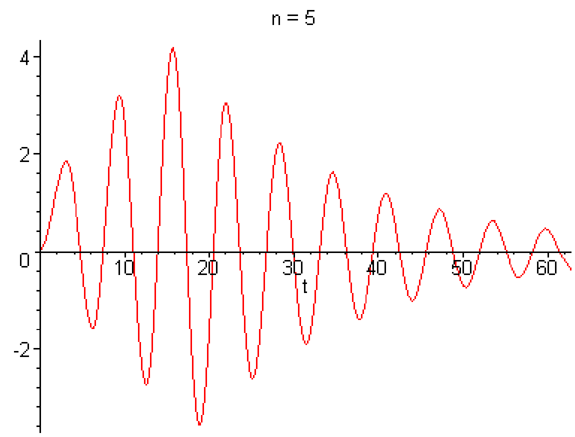
Let

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s + 0.05)}{(s + 0.05)^2 + (399/400)} - \frac{0.05}{(s + 0.05)^2 + (399/400)}\right] \\ &= 1 - e^{-t/20}\left[cos\left(\frac{\sqrt{399}}{20}\, t\right) + \frac{1}{\sqrt{399}}\, sin\left(\frac{\sqrt{399}}{20}\, t\right)\right]. \end{aligned}$$

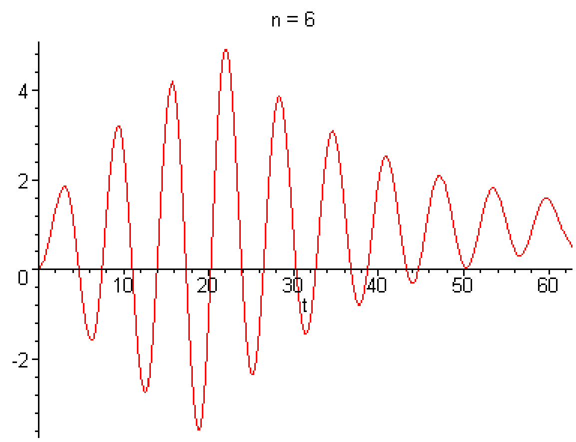
Applying Theorem 6.3.1 , term-by-term, the solution of the IVP is

$$u(t) = h(t) + \sum_{k=1}^{n} (-1)^k h(t - k\pi)\, u_{k\pi}(t)\,.$$

For *odd* values of n, the solution approaches $y = 0$.



For *even* values of n, the solution approaches $y = 1$.



(b). The solution is a sum of *damped sinusoids*, each of frequency $\omega = \sqrt{399}\,/20 \approx 1$. Each term has an 'initial' amplitude of approximately 1 . For any given n , the solution contains $n + 1$ such terms. Although the amplitude will *increase* with n, the amplitude will also be bounded by $n + 1$.

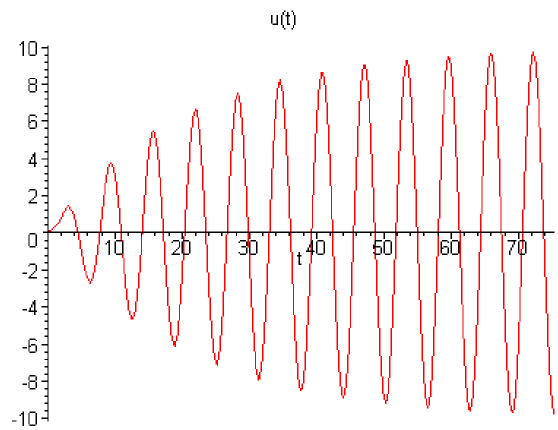
(c). Suppose that the forcing function is replaced by $g(t) = sin\, t$. Based on the methods
in Chapter 3, the general solution of the differential equation is

$$u(t) = e^{-t/20}\left[c_1\, cos\left(\frac{\sqrt{399}}{20}\,t\right) + c_2\, sin\left(\frac{\sqrt{399}}{20}\,t\right)\right] + u_p(t)\,.$$

Note that $u_p(t) = A\, cos\, t + B\, sin\, t$. Using the method of *undetermined coefficients*, $A = -10$ and $B = 0$. Based on the initial conditions, the solution of the IVP is

$$u(t) = 10\,e^{-t/20}\left[cos\left(\frac{\sqrt{399}}{20}\,t\right) + \frac{1}{\sqrt{399}}sin\left(\frac{\sqrt{399}}{20}\,t\right)\right] - 10\,cos\,t\,.$$

Observe that both solutions have the same frequency, $\omega = \sqrt{399}/20 \approx 1$.



$23(a)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,U(s) + U(s) = \frac{1}{s} + 2\sum_{k=1}^{n}\frac{(-1)^k e^{-(11k/4)s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}\frac{(-1)^k e^{-(11k/4)s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - cos\,t\,.$$

Applying Theorem 6.3.1 , term-by-term, the solution of the IVP is

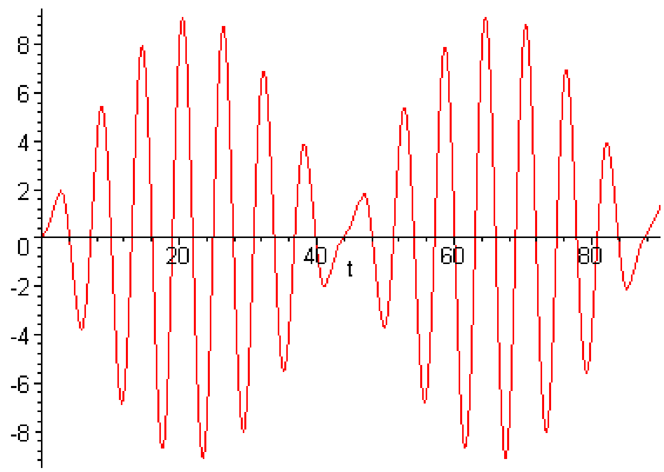
$$u(t) = h(t) + 2\sum_{k=1}^{n}(-1)^k h\left(t - \frac{11k}{4}\right)u_{11k/4}(t)\,.$$

That is,

$$u(t) = 1 - cos\, t + 2\sum_{k=1}^{n}(-1)^k\left[1 - cos\left(t - \frac{11k}{4}\right)\right] u_{11k/4}(t)\,.$$

(b).

n = 35



(c). Based on the plot, the '*slow period*' appears to be 88. The '*fast period*' appears to be about 6. These values correspond to a '*slow frequency*' of $\omega_s = 0.0714$ and a '*fast frequency*' $\omega_f = 1.0472$.

(d). The natural frequency of the system is $\omega_0 = 1$. The forcing function is initially periodic, with period $T = 11/2 = 5.5$. Hence the corresponding forcing frequency is $w = 1.1424$. Using the results in Section 3.9, the '*slow frequency*' is given by

$$\omega_s = \frac{|\omega - \omega_0|}{2} = 0.0712$$

and the '*fast frequency*' is given by

$$\omega_f = \frac{|\omega + \omega_0|}{2} = 1.0712\,.$$

Based on theses values, the '*slow period*' is predicted as 88.247 and the '*fast period*' is given as 5.8656.

Section 6.5

2. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \,4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Applying the initial conditions,

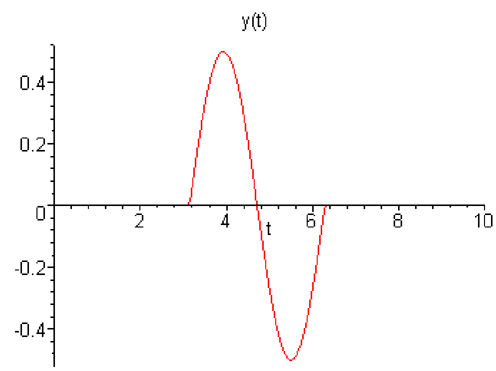
$$s^2\,Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}\,.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) \;&= \frac{1}{2}sin(2t-2\pi)u_{\pi}(t) - \frac{1}{2}sin(2t-4\pi)u_{2\pi}(t) \\ &= \frac{1}{2}sin(2t)[\,u_{\pi}(t) - u_{2\pi}(t)\,]\,. \end{aligned}$$



4. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - Y(s) = \,-\,20\,e^{-3s}.$$

Applying the initial conditions,

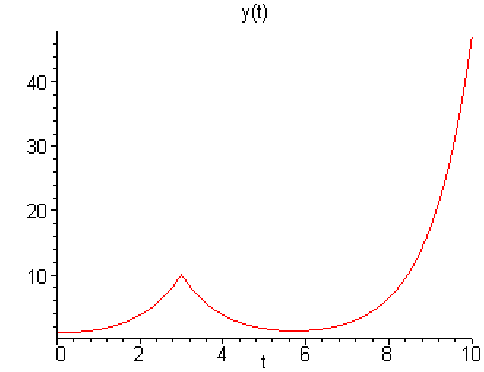
$$s^2\,Y(s) - Y(s) - s = \,-\,20\,e^{-3s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2-1} - \frac{20\,e^{-3s}}{s^2-1}\,.$$

Using a *table of transforms*, and Theorem 6.3.1 , the solution of the IVP is

$$y(t) = cosh\,t - 20\,sinh(t-3)u_3(t)\,.$$



6. Taking the initial conditions into consideration, the transform of the ODE is

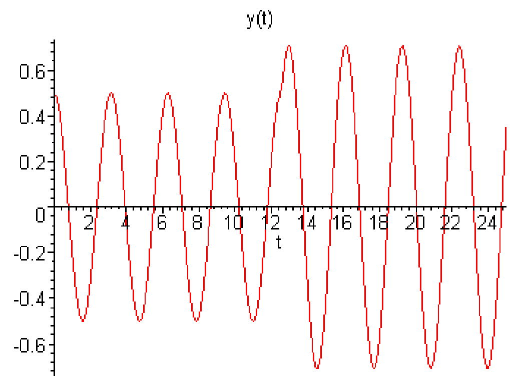
$$s^2\, Y(s) + 4Y(s) - s/2 = \, e^{-4\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{s/2}{s^2+4} + \frac{e^{-4\pi s}}{s^2+4}\,.$$

Using a *table of transforms*, and Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) \; &= \frac{1}{2}cos\,2t + \frac{1}{2}sin(2t - 8\pi)u_{4\pi}(t) \\ &= \frac{1}{2}cos\,2t + \frac{1}{2}sin(2t)\,u_{4\pi}(t)\,. \end{aligned}$$



8. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + \; 4Y(s) = 2\, e^{-(\pi/4)s}.$$

Applying the initial conditions,

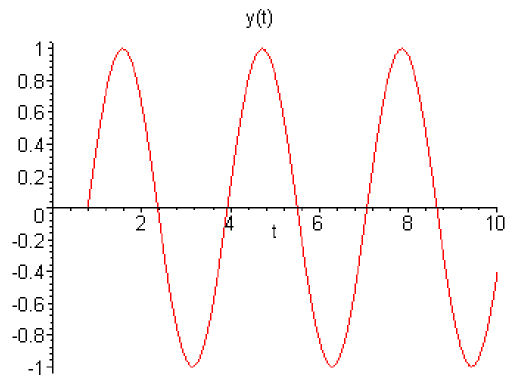
$$s^2\, Y(s) + 4Y(s) = 2\, e^{-(\pi/4)s}.$$

Solving for the transform,

$$Y(s) = \frac{2\,e^{-(\pi/4)s}}{s^2+4}\,.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$y(t) = sin\Big(2t - \frac{\pi}{2}\Big)u_{\pi/4}(t) = -\,cos(2t)\,u_{\pi/4}(t)\,.$$



9. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + Y(s) = \frac{e^{-(\pi/2)s}}{s} + 3\,e^{-(3\pi/2)s} - \frac{e^{-2\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-(\pi/2)s}}{s(s^2+1)} + \frac{3\,e^{-(3\pi/2)s}}{s^2+1} - \frac{e^{-2\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Hence

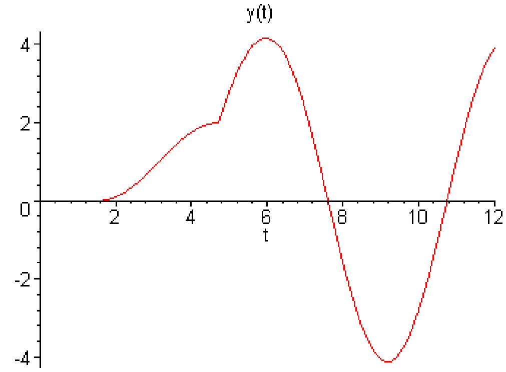
$$Y(s) = \frac{e^{-(\pi/2)s}}{s} - \frac{s\,e^{-(\pi/2)s}}{s^2+1} + \frac{3\,e^{-(3\pi/2)s}}{s^2+1} - \frac{e^{-2\pi s}}{s} + \frac{s\,e^{-2\pi s}}{s^2+1}\,.$$

Based on Theorem 6.3.1 , the solution of the IVP is

$$y(t) = u_{\pi/2}(t) - cos\Big(t - \frac{\pi}{2}\Big)u_{\pi/2}(t) + 3\,sin\Big(t - \frac{3\pi}{2}\Big)u_{3\pi/2}(t) - \\ - u_{2\pi}(t) + cos(t - 2\pi)u_{2\pi}(t)\,.$$

That is,

$$y(t) = [1 - sin(t)]\,u_{\pi/2}(t) + 3\,cos(t)\,u_{3\pi/2}(t) - [1 - cos(t)]\,u_{2\pi}(t)\,.$$



10. Taking the transform of both sides of the ODE,

$$\begin{aligned} 2s^2Y(s) + sY(s) + 4Y(s) &= \int_0^\infty e^{-st}\,\delta\Big(t - \frac{\pi}{6}\Big)\,sin\,t\,dt \\ &= \frac{1}{2}\,e^{-(\pi/6)s}. \end{aligned}$$

Solving for the transform,

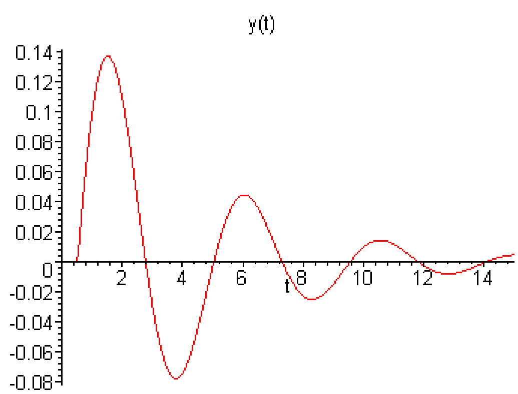
$$Y(s) = \frac{e^{-(\pi/6)s}}{2(2s^2 + s + 4)}\,.$$

First write

$$\frac{1}{2(2s^2 + s + 4)} = \frac{\frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{31}{16}}\,.$$

It follows that

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{\sqrt{31}}e^{-(t-\pi/6)/4}\cdot\,sin\frac{\sqrt{31}}{4}\Big(t - \frac{\pi}{6}\Big)u_{\pi/6}(t)\,.$$



11. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + 2s\, Y(s) + 2\, Y(s) = \frac{s}{s^2+1} + e^{-(\pi/2)s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-(\pi/2)s}}{s^2+2s+2}\,.$$

Using partial fractions,

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right].$$

We can also write

$$\frac{s+4}{s^2+2s+2} = \frac{(s+1)+3}{(s+1)^2+1}\,.$$

Let

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+2)}\,.$$

Then

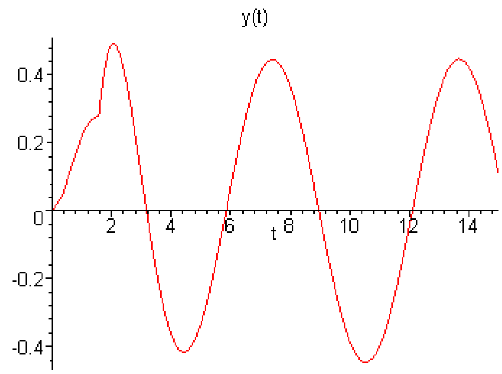
$$\mathcal{L}^{-1}[Y_1(s)] = \frac{1}{5}\, cos\, t + \frac{2}{5}\, sin\, t - \frac{1}{5}\, e^{-t}[cos\, t + 3\, sin\, t].$$

Applying Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-(\pi/2)s}}{s^2+2s+2}\right] = e^{-\left(t-\frac{\pi}{2}\right)}\, sin\left(t - \frac{\pi}{2}\right) u_{\pi/2}(t)\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) \ &= \frac{1}{5}\, cos\, t + \frac{2}{5}\, sin\, t - \frac{1}{5}\, e^{-t}[cos\, t + 3\, sin\, t] - \\ &- e^{-\left(t-\frac{\pi}{2}\right)}\, cos(t)\, u_{\pi/2}(t)\,. \end{aligned}$$



12. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4 Y(s) - Y(s) = e^{-s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-s}}{s^4 - 1}.$$

Using partial fractions,

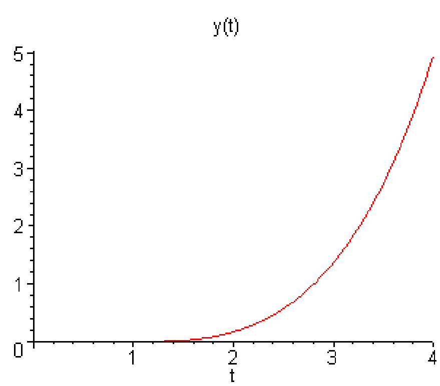
$$\frac{1}{s^4 - 1} = \frac{1}{2}\left[\frac{1}{s^2 - 1} - \frac{1}{s^2 + 1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4 - 1}\right] = \frac{1}{2}\, sinh\, t - \frac{1}{2}\, sin\, t\,.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$y(t) = \frac{1}{2}[sinh(t-1) - sin(t-1)]u_1(t)\,.$$



$14(a)$. The Laplace transform of the ODE is

$$s^2\,Y(s) + \frac{1}{2}s\,Y(s) + \,Y(s) = e^{-s}.$$

Solving for the transform of the solution,

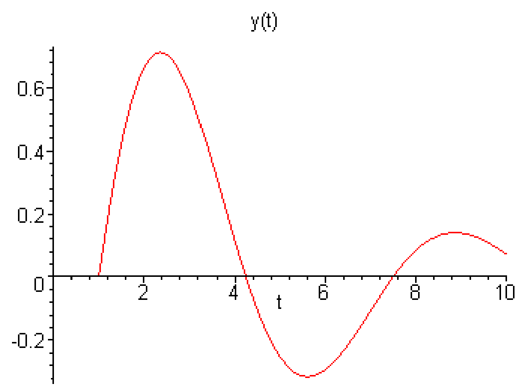
$$Y(s) = \frac{e^{-s}}{s^2 + s/2 + 1}\,.$$

First write

$$\frac{1}{s^2 + s/2 + 1} = \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}\,.$$

Taking the inverse transform and applying both *shifting theorems*,

$$y(t) = \frac{4}{\sqrt{15}}e^{-(t-1)/4}\,sin\frac{\sqrt{15}}{4}(t-1)\,u_1(t)\,.$$



(b). As shown on the graph, the maximum is attained at some $t_1 > 2$. Note that for $t > 2$,

$$y(t) = \frac{4}{\sqrt{15}}e^{-(t-1)/4}\,sin\frac{\sqrt{15}}{4}(t-1)\,.$$

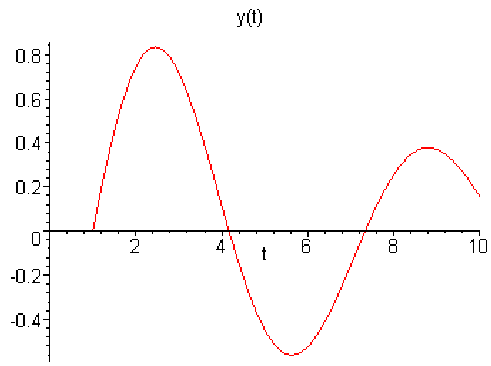
Setting $y'(t) = 0$, we find that $t_1 \approx 2.3613$. The maximum value is calculated as $y(2.3613) \approx 0.71153$.

(c). Setting $\gamma = 1/4$, the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + s/4 + 1}\,.$$

Following the same steps, it follows that

$$y(t) = \frac{8}{3\sqrt{7}} e^{-(t-1)/8} \, sin \frac{3\sqrt{7}}{8}(t-1)\, u_1(t)\,.$$



Once again, the maximum is attained at some $t_1 > 2$. Setting $y'(t) = 0$, we find that $t_1 \approx 2.4569$, with $y(t_1) \approx 0.8335$.

(d). Now suppose that $0 < \gamma < 1$. Then the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + \gamma s + 1}\,.$$

First write

$$\frac{1}{s^2 + \gamma s + 1} = \frac{1}{(s + \gamma/2)^2 + (1 - \gamma^2/4)}\,.$$

It follows that

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + \gamma s + 1}\right] = \frac{2}{\sqrt{4 - \gamma^2}}\, e^{-\gamma t/2} sin\left(\sqrt{1 - \gamma^2/4} \cdot t\right).$$

Hence the solution is

$$y(t) = h(t-1)\, u_1(t)\,.$$

The solution is nonzero only if $t > 1$, in which case $y(t) = h(t-1)$. Setting $y'(t) = 0$
,
we obtain

$$tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right] = \frac{1}{\gamma}\sqrt{4 - \gamma^2}\,,$$

that is,

$$\frac{tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right]}{\sqrt{1 - \gamma^2/4}} = \frac{2}{\gamma}\,.$$

As $\gamma \to 0$, we obtain the *formal* equation $tan(t-1) = \infty$. Hence $t_1 \to 1 + \frac{\pi}{2}$. Setting $t = \pi/2$ in $h(t)$, and letting $\gamma \to 0$, we find that $y_1 \to 1$. These conclusions agree with the case $\gamma = 0$, for which it is easy to show that the solution is

$$y(t) = sin(t-1)\, u_1(t)\,.$$

$15(a)$. See Prob. 14. It follows that the solution of the IVP is

$$y(t) = \frac{4k}{\sqrt{15}} e^{-(t-1)/4}\, sin\frac{\sqrt{15}}{4}(t-1)\, u_1(t)\,.$$

This function is a *multiple* of the answer in Prob. $14(a)$. Hence the peak value occurs at $t_1 \approx 2.3613$. The maximum value is calculated as $y(2.3613) \approx 0.71153\, k$. We find that the appropriate value of k is $k_1 = 2/0.71153 \approx 2.8108$.

(b). Based on Prob. $14(c)$, the solution is

$$y(t) = \frac{8\,k}{3\sqrt{7}} e^{-(t-1)/8}\, sin\frac{3\sqrt{7}}{8}(t-1)\, u_1(t)\,.$$

Since this function is a *multiple* of the solution in Prob. $14(c)$, we have $t_1 \approx 2.4569$, with $y(t_1) \approx 0.8335\, k$. The solution attains a value of $y = 2$, for $k_1 = 2/0.8335$, that is, $k_1 \approx 2.3995$.

(c). Similar to Prob. $14(d)$, for $0 < \gamma < 1$, the solution is

$$y(t) = h(t-1)\, u_1(t)\,,$$

in which

$$h(t) = \frac{2\,k}{\sqrt{4-\gamma^2}}\; e^{-\gamma t/2} sin\Big(\sqrt{1-\gamma^2/4}\cdot t\Big).$$

It follows that $t_1 - 1 \to \pi/2$. Setting $t = \pi/2$ in $h(t)$, and letting $\gamma \to 0$, we find that $y_1 \to k$. Requiring that the *peak value* remains at $y = 2$, the limiting value of k is $k_1 = 2$. These conclusions agree with the case $\gamma = 0$, for which it is easy to show that the solution is

$$y(t) = k\, sin(t-1)\, u_1(t)\,.$$

$16(a)$. Taking the initial conditions into consideration, the transformation of the ODE is

$$s^2\, Y(s) + \, Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s} - \frac{e^{-(4+k)s}}{s}\right].$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s(s^2+1)} - \frac{e^{-(4+k)s}}{s(s^2+1)}\right].$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Now let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - cos\,t\,.$$

Applying Theorem 6.3.1 , the solution is

$$\phi(t\,,k) = \frac{1}{2k}[h(t-4+k)\,u_{4-k}(t) - h(t-4-k)\,u_{4+k}(t)].$$

That is,

$$\begin{aligned}\phi(t\,,k) \;&= \frac{1}{2k}[\,u_{4-k}(t) - \,u_{4+k}(t)] - \\ &\quad - \frac{1}{2k}[cos(t-4+k)\,u_{4-k}(t) - cos(t-4-k)\,u_{4+k}(t)].\end{aligned}$$

(b). Consider various values of t. For any fixed $t < 4$, $\phi(t\,,k) = 0$, as long as $4-k > t$. If $t \geq 4$, then for $4+k < t$,

$$\phi(t\,,k) = \,-\frac{1}{2k}[cos(t-4+k) - cos(t-4-k)\,].$$

It follows that

$$\begin{aligned}\lim_{k\to 0}\phi(t\,,k) \;&=\; \lim_{k\to 0}\; -\frac{cos(t-4+k) - cos(t-4-k)}{2k} \\ &= sin(t-4)\,.\end{aligned}$$

Hence

$$\lim_{k\to 0}\phi(t\,,k) = sin(t-4)\,u_4(t)\,.$$

(c). The Laplace transform of the differential equation

$$y'' + y = \delta(t-4),$$

with $y(0) = y'(0) = 0$, is

$$s^2\,Y(s) + \,Y(s) = e^{-4s}.$$

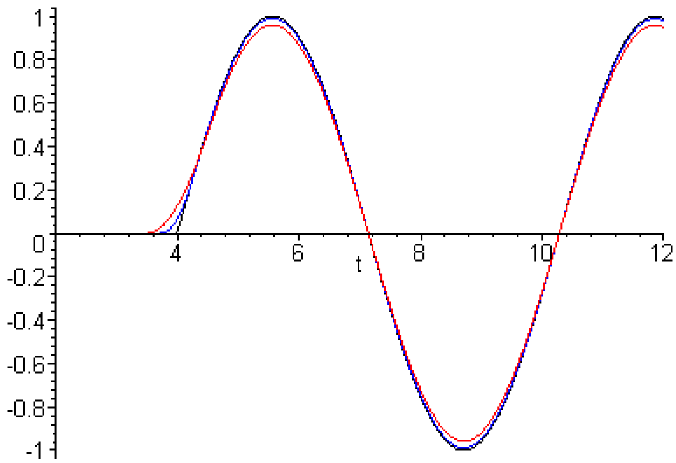
Solving for the transform of the solution,

$$Y(s) = \frac{e^{-4s}}{s^2+1}\,.$$

It follows that the solution is

$$\phi_0(t) = sin(t-4)\,u_4(t)\,.$$

(d).



$18(b)$. The transform of the ODE (given the specified initial conditions) is

$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20}(-1)^{k+1}e^{-k\pi s}.$$

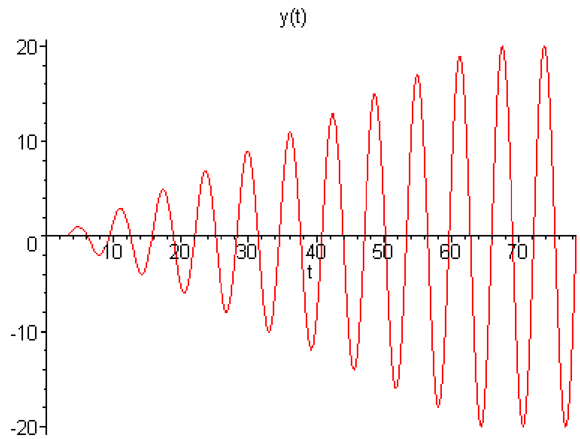
Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+1}\;\sum_{k=1}^{20}(-1)^{k+1}e^{-k\pi s}.$$

Applying Theorem 6.3.1 , term-by-term,

$$\begin{aligned} y(t) &= \sum_{k=1}^{20}(-1)^{k+1}sin(t-k\pi)\,u_{k\pi}(t) \\ &= -\,sin(t)\cdot\sum_{k=1}^{20}u_{k\pi}(t)\,. \end{aligned}$$

(c).



$19(b)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, Y(s) + Y(s) = \sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

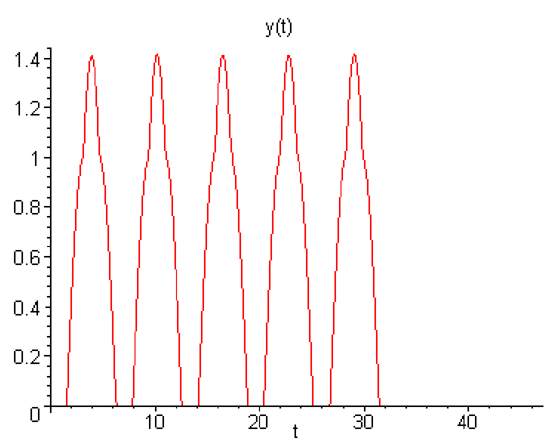
Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+1} \sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20} sin\left(t - \frac{k\pi}{2}\right) u_{k\pi/2}(t)\,.$$

(c).



$20(b)$. The transform of the ODE (given the specified initial conditions) is

$$s^2 Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-(k\pi/2)s}.$$

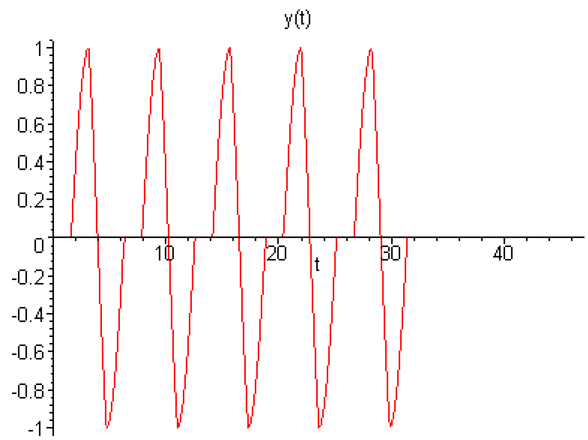
Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} (-1)^{k+1} \frac{e^{-(k\pi/2)s}}{s^2+1}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} sin\left(t - \frac{k\pi}{2}\right) u_{k\pi/2}(t) .$$

(c).



$22(b)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + Y(s) = \sum_{k=1}^{40} (-1)^{k+1} e^{-(11k/4)s}.$$

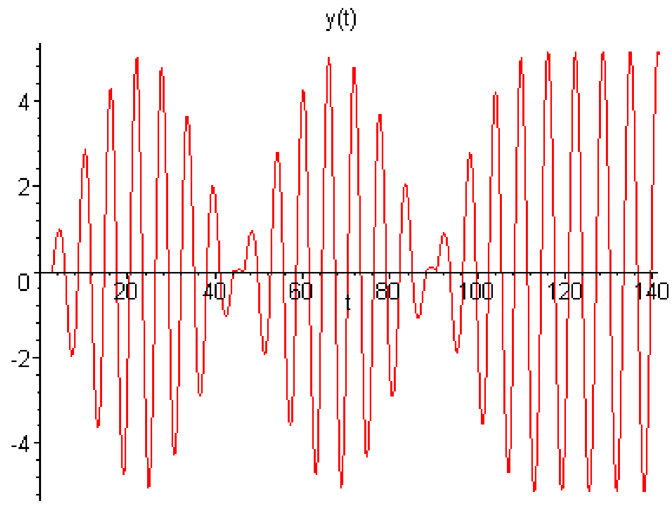
Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{40} (-1)^{k+1} \frac{e^{-(11k/4)s}}{s^2+1}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{40} (-1)^{k+1} sin\left(t - \frac{11k}{4}\right) u_{11k/4}(t)\,.$$

(c).



$23(b)$. The transform of the ODE (given the specified initial conditions) is

$$s^2\,Y(s) + 0.1s\,Y(s) + \,Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2 + 0.1s + 1}.$$

First write

$$\frac{1}{s^2 + 0.1s + 1} = \frac{1}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\,t\right).$$

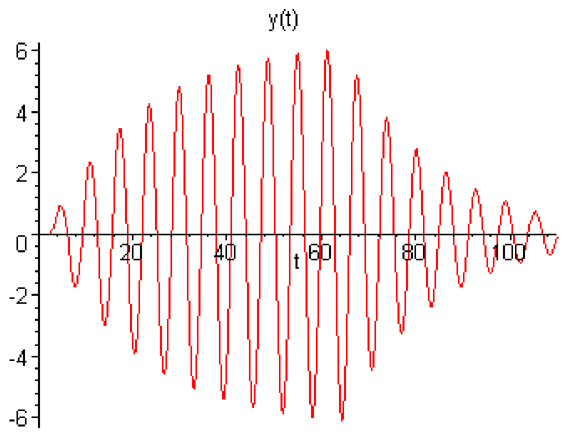
Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} h(t - k\pi)\, u_{k\pi}(t)\,,$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\, t\right).$$

(c).



$24(b)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, Y(s) + 0.1s\, Y(s) + \, Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi s}}{s^2 + 0.1s + 1}.$$

As shown in Prob. 23 ,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\, t\right).$$

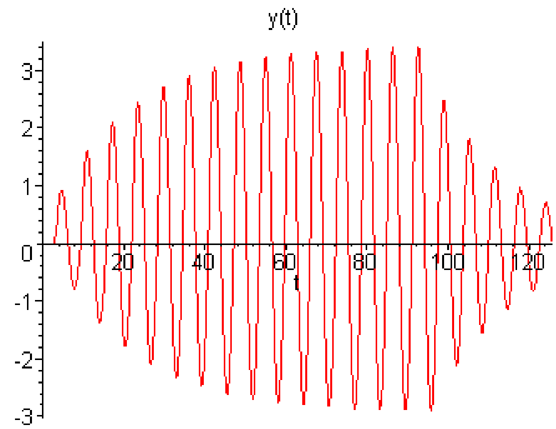
Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{15} h[t - (2k-1)\pi]\, u_{(2k-1)\pi}(t)\,,$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\, t\right).$$

(c).



$25(a)$. A fundamental set of solutions is $y_1(t) = e^{-t} cos\, t$ and $y_2(t) = e^{-t} sin\, t$. Based on Prob. 22 , in Section 3.7 , a particular solution is given by

$$y_p(t) = \int_0^t \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(y_1\,, y_2)(s)} f(s)ds\,.$$

In the given problem,

$$\begin{aligned} y_p(t) &= \int_0^t \frac{e^{-s-t}[cos(s)sin(t) - sin(s)cos(t)]}{exp(\,-2s)} f(s)ds\,. \\ &= \int_0^t e^{-(t-s)} sin(t-s)f(s)ds\,. \end{aligned}$$

Given the specified initial conditions,

$$y(t) = \int_0^t e^{-(t-s)} sin(t-s)f(s)ds\,.$$

(b). Let $f(t) = \delta(t-\pi)$. It is easy to see that if $t < \pi$, $y(t) = 0$. If $t > \pi$,

$$\int_0^t e^{-(t-s)} sin(t-s)\delta(s-\pi)ds = e^{-(t-\pi)} sin(t-\pi)\,.$$

Setting $t = \pi + \varepsilon$, and letting $\varepsilon \to 0$, we find that $y(\pi) = 0$. Hence

$$y(t) = e^{-(t-\pi)} sin(t-\pi)\, u_\pi(t)\,.$$

(c). The Laplace transform of the solution is

$$\begin{aligned} Y(s) &= \frac{e^{-\pi s}}{s^2+2s+2} \\ &= \frac{e^{-\pi s}}{(s+1)^2+1}\,. \end{aligned}$$

Hence the solutions agree.

Section 6.6

$1(a)$. The *convolution integral* is defined as

$$f*g\,(t)=\int_0^t f(t-\tau)g(\tau)d\tau\,.$$

Consider the change of variable $u=t-\tau$. It follows that

$$\begin{aligned}\int_0^t f(t-\tau)g(\tau)d\tau &= \int_t^0 f(u)g(t-u)(-du)\\ &= \int_0^t g(t-u)f(u)du\\ &= g*f\,(t)\,.\end{aligned}$$

(b). Based on the distributive property of the *real numbers*, the convolution is also distributive.

(c). By definition,

$$\begin{aligned}f*(g*h)(t) &= \int_0^t f(t-\tau)[g*h\,(\tau)]d\tau\\ &= \int_0^t f(t-\tau)\left[\int_0^\tau g(\tau-\eta)h(\eta)d\eta\right]d\tau\\ &= \int_0^t\int_0^\tau f(t-\tau)g(\tau-\eta)h(\eta)\,d\eta d\tau\,.\end{aligned}$$

The region of integration, in the double integral is the area between the straight lines $\eta=0\,,\eta=\tau$ and $\tau=t$. Interchanging the order of integration,

$$\begin{aligned}\int_0^t\int_0^\tau f(t-\tau)g(\tau-\eta)h(\eta)\,d\eta d\tau &= \int_0^t\int_\eta^t f(t-\tau)g(\tau-\eta)h(\eta)\,d\tau d\eta\\ &= \int_0^t\left[\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau\right]h(\eta)\,d\eta\,.\end{aligned}$$

Now let $\tau-\eta=u$. Then

$$\begin{aligned}\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau &= \int_0^{t-\eta} f(t-\eta-u)g(u)du\\ &= f*g\,(t-\eta)\,.\end{aligned}$$

Hence

$$\int_0^t f(t-\tau)[g*h\,(\tau)]d\tau=\int_0^t[f*g\,(t-\tau)]h(\tau)\,d\tau\,.$$

2. Let $f(t) = e^t$. Then

$$\begin{aligned} f * 1\,(t) &= \int_0^t e^{t-\tau} \cdot 1\, d\tau \\ &= e^t \int_0^t e^{-\tau} d\tau \\ &= e^t - 1\,. \end{aligned}$$

3. It follows directly that

$$\begin{aligned} f * f\,(t) &= \int_0^t sin(t-\tau)\, sin(\tau)\, d\tau \\ &= \frac{1}{2}\int_0^t [cos(t-2\tau) - cos(t)] d\tau \\ &= \frac{1}{2}[sin(t) - t\,cos(t)]. \end{aligned}$$

The *range* of the resulting function is $\mathbb{R}$.

5. We have $\mathcal{L}[e^{-t}] = 1/(s+1)$ and $\mathcal{L}[sin\, t] = 1/(s^2+1)$. Based on Theorem 6.6.1,

$$\begin{aligned} \mathcal{L}\left[\int_0^t e^{-(t-\tau)} sin(\tau)\, d\tau\right] &= \frac{1}{s+1} \cdot \frac{1}{s^2+1} \\ &= \frac{1}{(s+1)(s^2+1)}\,. \end{aligned}$$

6. Let $g(t) = t$ and $h(t) = e^t$. Then $f(t) = g * h\,(t)$. Applying Theorem 6.6.1 ,

$$\begin{aligned} \mathcal{L}\left[\int_0^t g(t-\tau) h(\tau)\, d\tau\right] &= \frac{1}{s^2} \cdot \frac{1}{s-1} \\ &= \frac{1}{s^2(s-1)}\,. \end{aligned}$$

7. We have $f(t) = g * h\,(t)$, in which $g(t) = sin\, t$ and $h(t) = cos\, t$. The transform of the convolution integral is

$$\begin{aligned} \mathcal{L}\left[\int_0^t g(t-\tau) h(\tau)\, d\tau\right] &= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}\,. \end{aligned}$$

9. It is easy to see that

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = cos\,2t\,.$$

Applying Theorem 6.6.1 ,

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1)(s^2+4)}\right] = \int_0^t e^{-(t-\tau)} cos\,2\tau\,d\tau\,.$$

10. We first note that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = t\,e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2} sin\,2t\,.$$

Based on the *convolution theorem* ,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] &= \frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)} sin\,2\tau\,d\tau \\ &= \frac{1}{2}\int_0^t \tau\,e^{-\tau} sin(2t-2\tau)\,d\tau\,.\end{aligned}$$

11. Let $g(t) = \mathcal{L}^{-1}[G(s)]$. Since $\mathcal{L}^{-1}[1/(s^2+1)] = sin\,t$, the inverse transform of the product is

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right] &= \int_0^t g(t-\tau)\,sin\,\tau\,d\tau \\ &= \int_0^t sin(t-\tau)\,g(\tau)\,d\tau\,.\end{aligned}$$

12. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) - 1 + \omega^2\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+\omega^2} + \frac{G(s)}{s^2+\omega^2}.$$

As shown in a related situation, Prob. 11 ,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+\omega^2}\right] = \frac{1}{\omega}\int_0^t sin\,\omega(t-\tau)\,g(\tau)\,d\tau\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{\omega}\,sin\,\omega t + \frac{1}{\omega}\int_0^t sin\,\omega(t-\tau)\,g(\tau)\,d\tau\,.$$

14. The transform of the ODE (given the specified initial conditions) is

$$4s^2\,Y(s) + 4s\,Y(s) + \,17\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{4s^2 + 4s + \,17}.$$

First write

$$\frac{1}{4s^2 + 4s + \,17} = \frac{\frac{1}{4}}{\left(s+\frac{1}{2}\right)^2 + 4}\,.$$

Based on the elementary properties of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{1}{4s^2 + 4s + \,17}\right] = \frac{1}{8}e^{-t/2}sin\,2t\,.$$

Applying the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{8}\int_0^t e^{-(t-\tau)/2}\,sin\,2(t-\tau)\,g(\tau)\,d\tau\,.$$

16. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) - 2s + 3 + 4[s\,Y(s) - 2] + 4\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.$$

We can write

$$\frac{2s+5}{(s+2)^2} = \frac{2}{s+2} + \frac{1}{(s+2)^2}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{2}{s+2}\right] = 2e^{-2t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = t\,e^{-2t}\,.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = 2e^{-2t} + t\,e^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\,d\tau\,.$$

18. The transform of the ODE (given the specified initial conditions) is

$$s^4\,Y(s) - \,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{s^4-1}.$$

First write

$$\frac{1}{s^4-1} = \frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4-1}\right] = \frac{1}{2}[sinh\,t - sin\,t]\,.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{2}\int_0^t [sinh(t-\tau) - sin(t-\tau)]g(\tau)\,d\tau\,.$$

19. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4\,Y(s) - s^3 + 5s^2\,Y(s) - 5s + 4\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^3+5s}{(s^2+1)(s^2+4)} + \frac{G(s)}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we find that

$$\frac{s^3+5s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{4s}{s^2+1} - \frac{s}{s^2+4}\right],$$

and

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{s(s^2+5)}{(s^2+1)(s^2+4)}\right] = \frac{4}{3}\,cos\,t - \frac{1}{3}\,cos\,2t,$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\,sin\,t - \frac{1}{6}\,sin\,2t\,.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{4}{3}\,cos\,t - \frac{1}{3}\,cos\,2t + \frac{1}{6}\int_0^t [2\,sin(t-\tau) - sin\,2(t-\tau)]g(\tau)\,d\tau\,.$$

$21(a)$. Let $\phi(t) = u''(t)$. Substitution into the *integral equation* results in

$$u''(t) + \int_0^t (t-\xi)\,u''(\xi)\,d\xi = sin\,2t\,.$$

Integrating by parts,

$$\begin{aligned}\int_0^t (t-\xi)\,u''(\xi)\,d\xi &= (t-\xi)\,u'(\xi)\Big|_{\xi=0}^{\xi=t} + \int_0^t u'(\xi)\,d\xi \\ &= -\,t\,u'(0) + u(t) - u(0)\,.\end{aligned}$$

Hence

$$u''(t) + u(t) - t\,u'(0) - u(0) = sin\,2t\,.$$

(b). Substituting the given *initial conditions* for the function $u(t)$,

$$u''(t) + u(t) = sin\,2t\,.$$

Hence the solution of the IVP is equivalent to solving the integral equation in Part (a).

(c). Taking the Laplace transform of the integral equation, with $\Phi(s) = \mathcal{L}[\phi(t)]$,

$$\Phi(s) + \frac{1}{s^2}\cdot\Phi(s) = \frac{2}{s^2+4}\,.$$

Note that the *convolution theorem* was applied. Solving for the transform $\Phi(s)$,

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)}\,.$$

Using partial fractions, we can write

$$\frac{2s^2}{(s^2+1)(s^2+4)} = \frac{2}{3}\left[\frac{4}{s^2+4} - \frac{1}{s^2+1}\right].$$

Therefore the solution of the *integral equation* is

$$\phi(t) = \frac{4}{3}\, sin\, 2t - \frac{2}{3}\, sin\, t\,.$$

(d). Taking the Laplace transform of the ODE, with $U(s) = \mathcal{L}[u(t)]$,

$$s^2U(s) + U(s) = \frac{2}{s^2+4}\,.$$

Solving for the transform of the solution,

$$U(s) = \frac{2}{(s^2+1)(s^2+4)}\,.$$

Using partial fractions, we can write

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{2}{s^2+1} - \frac{2}{s^2+4}\right].$$

It follows that the solution of the IVP is

$$u(t) = \frac{2}{3} sin\, t - \frac{1}{3}\, sin\, 2t\,.$$

We find that $u''(t) = -\frac{2}{3} sin\, t + \frac{4}{3}\, sin\, 2t\,.$

$22(a)$. First note that

$$\int_0^b \frac{f(y)}{\sqrt{b-y}}\, dy = \left(\frac{1}{\sqrt{y}} * f\right)(b).$$

Take the Laplace transformation of both sides of the equation. Using the *convolution theorem*, with $F(s) = \mathcal{L}[f(y)]$,

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}} F(s) \cdot \mathcal{L}\left[\frac{1}{\sqrt{y}}\right].$$

It was shown in Prob. $27(c)$, Section 6.1 , that

$$\mathcal{L}\left[\frac{1}{\sqrt{y}}\right] = \sqrt{\frac{\pi}{s}}\,.$$

Hence

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}} F(s) \cdot \sqrt{\frac{\pi}{s}} \,,$$

with

$$F(s) = \sqrt{\frac{2g}{\pi}} \cdot \frac{T_0}{\sqrt{s}} \,.$$

Taking the inverse transform, we obtain

$$f(y) = \frac{T_0}{\pi} \sqrt{\frac{2g}{y}} \,.$$

(b). Combining equations (i) and (iv),

$$\frac{2g\,T_0^2}{\pi^2\,y} = 1 + \left(\frac{dx}{dy}\right)^2 .$$

Solving for the derivative dx/dy,

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}} \,,$$

in which $\alpha = gT_0^2/\pi^2$.

(c). Consider the *change of variable* $y = 2\alpha\, sin^2(\theta/2)$. Using the chain rule,

$$\frac{dy}{dx} = 2\alpha\, sin(\theta/2) cos(\theta/2) \cdot \frac{d\theta}{dx}$$

and

$$\frac{dx}{dy} = \frac{1}{2\alpha\, sin(\theta/2) cos(\theta/2)} \cdot \frac{dx}{d\theta} \,.$$

It follows that

$$\begin{aligned} \frac{dx}{d\theta} &= 2\alpha\, sin(\theta/2) cos(\theta/2) \sqrt{\frac{cos^2(\theta/2)}{sin^2(\theta/2)}} \\ &= 2\alpha\, cos^2(\theta/2) \\ &= \alpha + \alpha\, cos\,\theta \,. \end{aligned}$$

Direct integration results in

$$x(\theta) = \alpha\,\theta + \alpha\, sin\,\theta + C \,.$$

Since the curve passes through the *origin*, we require $y(0) = x(0) = 0$. Hence $C = 0$, and $x(\theta) = \alpha\,\theta + \alpha\, sin\,\theta$. We also have

$$\begin{aligned} y(\theta) &= 2\alpha\, sin^2(\theta/2) \\ &= \alpha - \alpha\, cos\,\theta\,. \end{aligned}$$