# Chapter Five

## Section 5.1

1. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|(x-3)^{n+1}\right|}{\left|(x-3)^{n}\right|}=\lim_{n\to\infty}|x-3|=|x-3|.$$

Hence the series converges absolutely for $|x-3|<1$ . The radius of convergence is $\rho=1$ . The series diverges for $x=2$ and $x=4$ , since the *n-th* term does not approach zero.

3. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!\,x^{2n+2}|}{|(n+1)!\,x^{2n}|}=\lim_{n\to\infty}\frac{x^2}{n+1}=0\,.$$

The series converges absolutely for *all* values of $x$ . Thus the radius of convergence is $\rho=\infty$ .

4. Apply the ratio test :

$$\lim_{n\to\infty}\frac{|2^{n+1}x^{n+1}|}{|2^{n}x^{n}|}=\lim_{n\to\infty}2|x|=2|x|.$$

Hence the series converges absolutely for $2|x|$, or $|x|<1/2$ . The radius of convergence is $\rho=1/2$ . The series diverges for $x=\pm 1/2$ , since the *n-th* term does not approach zero.

6. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n(x-x_0)^{n+1}\right|}{\left|(n+1)(x-x_0)^{n}\right|}=\lim_{n\to\infty}\frac{n}{n+1}|(x-x_0)|=|(x-x_0)|.$$

Hence the series converges absolutely for $|(x-x_0)|<1$ . The radius of convergence is $\rho=1$ . At $x=x_0+1$ , we obtain the *harmonic series*, which is *divergent*. At the other endpoint, $x=x_0-1$ , we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,,$$

which is *conditionally* convergent.

7. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|3^{n}(n+1)^2(x+2)^{n+1}\right|}{\left|3^{n+1}n^2(x+2)^{n}\right|}=\lim_{n\to\infty}\frac{(n+1)^2}{3\,n^2}|(x+2)|=\frac{1}{3}|(x+2)|.$$

Hence the series converges absolutely for $\frac{1}{3}|x+2|<1$, or $|x+2|<3$. The radius of convergence is $\rho=3$. At $x=-5$ and $x=+1$, the series diverges, since the *n-th* term does not approach zero.

8. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n^n(n+1)!\,x^{n+1}|}{\left|(n+1)^{n+1}n!\,x^n\right|}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}|x|=\frac{1}{e}|x|,$$

since

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=e^{-1}.$$

Hence the series converges absolutely for $|x|<e$. The radius of convergence is $\rho=e$. At $x=\pm e$, the series *diverges*, since the *n-th* term does not approach zero. This follows from the fact that

$$\lim_{n\to\infty}\frac{n!\,e^n}{n^n\sqrt{2\pi n}}=1\,.$$

10. We have $f(x)=e^x$, with $f^{(n)}(x)=e^x$, for $n=1,2,\cdots$. Therefore $f^{(n)}(0)=1$. Hence the Taylor expansion about $x_0=0$ is

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\,.$$

Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!x^{n+1}|}{|(n+1)!\,x^n|}=\lim_{n\to\infty}\frac{1}{n+1}|x|=0.$$

The radius of convergence is $\rho=\infty$.

11. We have $f(x)=x$, with $f'(x)=1$ and $f^{(n)}(x)=0$, for $n=2,\cdots$. Clearly, $f(1)=1$ and $f'(1)=1$, with all other derivatives equal to *zero*. Hence the Taylor expansion about $x_0=1$ is

$$x=1+(x-1).$$

Since the series has only a finite number of terms, the converges absolutely for all $x$.

14. We have $f(x)=1/(1+x)$, $f'(x)=-1/(1+x)^2$, $f''(x)=2/(1+x)^3,\cdots$ with $f^{(n)}(x)=(-1)^n n!/(1+x)^{n+1}$, for $n\geq 1$. It follows that $f^{(n)}(0)=(-1)^n n!$

for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

15. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1} n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n \to \infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

17. Applying the ratio test,

$$\lim_{n \to \infty} \frac{|(n+1)x^{n+1}|}{|n\,x^n|} = \lim_{n \to \infty} \frac{n+1}{n}|x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16\,x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n^2(n-1)\,x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \cdots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^2(n+1)\,x^n.$$

20. Shifting the index in the *second* series, that is, setting $n = k+1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\,.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^{k+1}.\end{aligned}$$

21. Shifting the index by $2$, that is, setting $m = n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2}\, x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n\,.\end{aligned}$$

22. Shift the index *down* by $2$, that is, set $m = n+2$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^{n+2} &= \sum_{m=2}^{\infty} a_{m-2}\, x^m \\ &= \sum_{n=2}^{\infty} a_{n-2}\, x^n\,.\end{aligned}$$

24. Clearly,

$$\left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

Shifting the index in the *first* series, that is, setting $k = n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}\, x^k \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.\end{aligned}$$

Hence

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=2}^{\infty} n(n-1)a_n\, x^n.$$

Note that when $n=0$ and $n=1$, the coefficients in the *second* series are *zero*. So that

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n] x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=1}^{\infty} n a_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^{n+1}.$$

Shifting the index in the *first* series, that is, setting $k = n-1$,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} = \sum_{k=0}^{\infty} (k+1)a_{k+1} x^k.$$

Shifting the index in the *second* series, that is, setting $k = n+1$,

$$\sum_{n=0}^{\infty} a_n\, x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Combining the series, and starting the summation at $n=1$,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}] x^n.$$

27. We note that

$$x\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^n.$$

Shifting the index in the *first* series, that is, setting $k = n - 1$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} &= \sum_{k=1}^{\infty} k(k+1)a_{k+1}x^k \\ &= \sum_{k=0}^{\infty} k(k+1)a_{k+1}x^k,\end{aligned}$$

since the coefficient of the term associated with $k = 0$ is *zero*. Combining the series,

$$x\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} [n(n+1)a_{n+1} + a_n]x^n.$$

**Section 5.2**

1. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0\,.$$

Equating all the coefficients to *zero*,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \cdots.$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \cdots.$$

The subscripts differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots = \frac{a_1}{(2k+1)!}\,.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\,.$$

The linearly independent solutions are

$$y_1 = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) = a_0 cosh\, x$$

$$y_2 = a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) = a_1 sinh\, x\,.$$

4. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Rewriting the *second* summation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + \sum_{n=2}^{\infty} k^2 a_{n-2}\, x^n = 0\,,$$

that is,

$$2a_2 + 3 \cdot 2\, a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + k^2 a_{n-2}\right] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2 a_{n-2} = 0\,, \quad \text{for } n = 2, 3, 4, \cdots.$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2 a_{n-2}}{(n+2)(n+1)}\,, \quad n = 2, 3, 4, \cdots.$$

The indices differ by *four*, so $a_4$, $a_8$, $a_{12}$, $\cdots$ are defined by

$$a_4 = -\frac{k^2 a_0}{4 \cdot 3}\,,\ a_8 = -\frac{k^2 a_4}{8 \cdot 7}\,, a_{12} = -\frac{k^2 a_8}{12 \cdot 11}\,, \cdots.$$

Similarly, $a_5$, $a_9$, $a_{13}$, $\cdots$ are defined by

$$a_5 = -\frac{k^2 a_1}{5 \cdot 4}\,,\ a_9 = -\frac{k^2 a_5}{9 \cdot 8}\,, a_{13} = -\frac{k^2 a_9}{13 \cdot 12}\,, \cdots.$$

The remaining coefficients are *zero*. Therefore the general solution is

$$y = a_0\left[1 - \frac{k^2}{4 \cdot 3}x^4 + \frac{k^4}{8 \cdot 7 \cdot 4 \cdot 3}x^8 - \frac{k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}x^{12} + \cdots\right] +$$
$$+ a_1\left[x - \frac{k^2}{5 \cdot 4}x^5 + \frac{k^4}{9 \cdot 8 \cdot 5 \cdot 4}x^9 - \frac{k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 4 \cdot 4}x^{13} + \cdots\right].$$

Note that for the *even* coefficients,

$$a_{4m} = -\frac{k^2 a_{4m-4}}{(4m-1)4m}, \quad m = 1, 2, 3, \cdots$$

and for the *odd* coefficients,

$$a_{4m+1} = -\frac{k^2 a_{4m-3}}{4m(4m+1)}, \quad m = 1, 2, 3, \cdots.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4m+3)(4m+4)}$$

$$y_2(x) = x\left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{4 \cdot 5 \cdot 8 \cdot 9 \cdots (4m+4)(4m+5)}\right].$$

6. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(2 + x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty} \left[2(n+2)(n+1)a_{n+2} + n(n-1)a_n - n\, a_n + 4a_n\right] x^n = 0\,.$$

Equating the coefficients to *zero*, we find that $a_2 = -a_0$ , $a_3 = -a_1/4$ , and

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}\,a_n\,, \quad n = 0, 1, 2, \cdots.$$

The indices differ by *two*, so for $k = 0, 1, 2, \cdots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)}\,a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)}\,a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots.$$

7. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + n\, a_n + 2a_n] x^n = 0\,.$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \cdots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \cdots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \cdots (2k)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}\,.$$

9. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(1 + x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4n\, a_n + 6a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\, a_n\,, \quad n = 0, 1, 2, \cdots.$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by *two*, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3\,.$$

10. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(4 - x^2\right)\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x + \sum_{n=2}^{\infty} \left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\right]x^n = 0\,.$$

We obtain $a_2 = -a_0/4$, $a_3 = -a_1/12$ and

$$4(n+2)a_{n+2} = (n-2)a_n\,, \quad n = 0, 1, 2, \cdots.$$

Note that for $n = 2$, $a_4 = 0$. Since the indices differ by *two*, we also have $a_{2k} = 0$ for $k = 2, 3, \cdots$. On the other hand, for $k = 1, 2, \cdots$,

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \cdots = \frac{-a_1}{4^k(2k-1)(2k+1)}\,.$$

Therefore the general solution is

$$y = a_0 + a_1 x - a_0 \frac{x^2}{4} - a_1 \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} \,.$$

Hence the linearly independent solutions are $y_1(x) = 1 - x^2/4$ and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \cdots = x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} \,.$$

11. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\left(3 - x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

It follows that

$$6a_2 - a_0 + (-4a_1 + 18a_3)x + \sum_{n=2}^{\infty} \left[ 3(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3n\, a_n - a_n \right] x^n = 0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2) a_{n+2} = (n+1) a_n \,, \quad n = 0, 1, 2, \cdots .$$

The indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{(2k-1) a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1) a_{2k-4}}{3^2 (2k-2)(2k)} = \cdots = \frac{3 \cdot 5 \cdots (2k-1)\, a_0}{3^k \cdot 2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \cdots = \frac{2\cdot 4\cdot 6\cdots(2k)\,a_1}{3^k\cdot 3\cdot 5\cdots(2k+1)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots = 1 + \sum_{n=1}^{\infty}\frac{3\cdot 5\cdots(2n-1)\,x^{2n}}{3^n\cdot 2\cdot 4\cdots(2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots = x + \sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)\,x^{2n+1}}{3^n\cdot 3\cdot 5\cdots(2n+1)}\,.$$

12. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Before proceeding, write

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

It follows that

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + n\,a_n - a_n\right]x^n = 0.$$

We obtain $a_2 = a_0/2$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + (n-1)a_n = 0$$

for $n = 0, 1, 2, \cdots$. Writing out the individual equations,

$$
\begin{aligned}
3\cdot 2\,a_3 - 2\cdot 1\,a_2 &= 0\\
4\cdot 3\,a_4 - 3\cdot 2\,a_3 + a_2 &= 0\\
5\cdot 4\,a_5 - 4\cdot 3\,a_4 + 2\,a_3 &= 0\\
6\cdot 5\,a_6 - 5\cdot 4\,a_5 + 3\,a_4 &= 0\\
&\vdots
\end{aligned}
$$

The coefficients can be calculated successively as $a_3 = a_0/(2\cdot 3)$, $a_4 = a_3/2 - a_2/12 = a_0/24$, $a_5 = 3a_4/5 - a_3/10 = a_0/120$, $\cdots$. We can now see that for $n \geq 2$, $a_n$ is proportional to $a_0$. In fact, for $n \geq 2$, $a_n = a_0/(n!)$. Therefore the general solution is

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \cdots.$$

Hence the linearly independent solutions are $y_2(x) = x$ and

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!}\,.$$

13. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + n\, a_n + 3a_n] x^n = 0\,.$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0$$

for $n = 0, 1, 2, \cdots$. The indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \cdots$$
$$= \frac{(-1)^k 3 \cdot 5 \cdots (2k+1)}{2^k\,(2k)!}\,a_0\,.$$

and

$$a_{2k+1} = -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \cdots$$
$$= \frac{(-1)^k\, 4 \cdot 6 \cdots (2k)(2k+2)}{2^k\,(2k+1)!}\,a_1\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \cdots (2n+1)}{2^n\,(2n)!}x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \cdots (2n+2)}{2^n\,(2n+1)!}x^{2n+1}.$$

$15(a)$. From Prob. 2 , we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\,n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n\,n!\,x^{2n+1}}{(2n+1)!}\ .$$

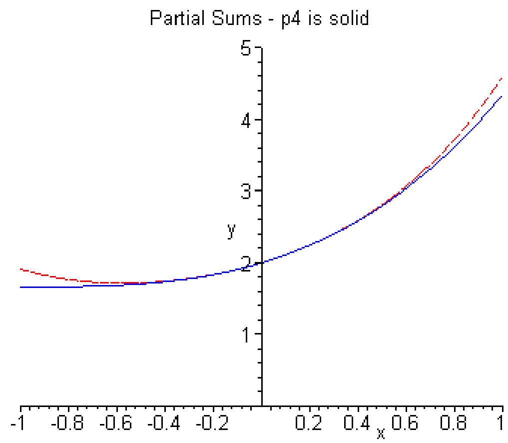
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2\,y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots.$$

The *four*- and *five*-term polynomial approximations are

$$p_4 = 2 + x + x^2 + x^3/3$$
$$p_5 = 2 + x + x^2 + x^3/3 + x^4/4\,.$$

$(b)$.




$(c)$. The *four-term* approximation $p_4$ appears to be reasonably accurate (within $10\%$) on the interval $|x| < 0.7$.

$17(a)$. From Prob. 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

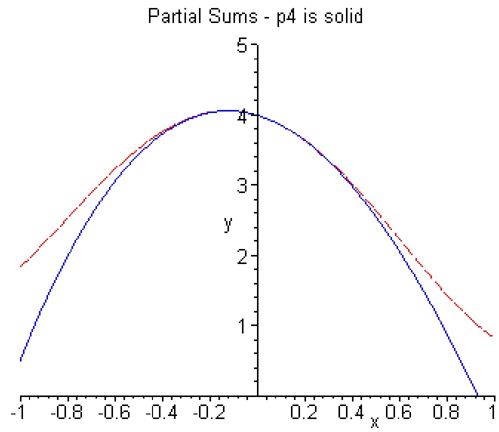
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4\, y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \cdots.$$

The *four-* and *five*-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3$$
$$p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

$(b)$.




$(c)$. The *four-term* approximation $p_4$ appears to be reasonably accurate (within $10\%$) on the interval $|x| < 0.5$ .

$18(a)$. From Prob. $12$ , we have

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad y_2(x) = x \,.$$

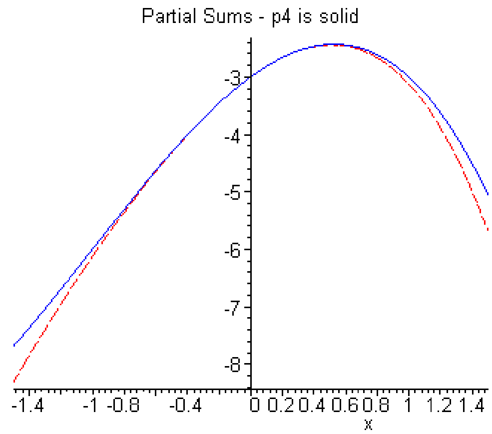
Since $a_0 = y(0)$ and $a_1 = y'(0)$ , we have $y(x) = -3\, y_1(x) + 2\, y_2(x)$ . That is,

$$y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 - \frac{1}{240}x^6 + \cdots .$$

The *four-* and *five-*term polynomial approximations are

$$p_4 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$p_5 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 \,.$$

$(b)$.




$(c)$. The *four-term* approximation $p_4$ appears to be reasonably accurate (within $10\%$) on the interval $|x| < 0.9$.

20. Two linearly independent solutions of *Airy's equation* (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}.$$

Applying the *ratio test* to the terms of $y_1(x)$,

$$\lim_{n\to\infty} \frac{|2 \cdot 3 \cdots (3n-1)(3n)\, x^{3n+3}|}{|2 \cdot 3 \cdots (3n+2)(3n+3)\, x^{3n}|} = \lim_{n\to\infty} \frac{1}{(3n+1)(3n+2)(3n+3)} |x|^3 = 0\,.$$

Similarly, applying the *ratio test* to the terms of $y_2(x)$,

$$\lim_{n\to\infty} \frac{|3 \cdot 4 \cdots (3n)(3n+1)\, x^{3n+4}|}{|3 \cdot 4 \cdots (3n+3)(3n+4)\, x^{3n+1}|} = \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)(3n+4)} |x|^3 = 0\,.$$

Hence both series converge *absolutely* for all $x$.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \lambda\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$2a_2 + \lambda\, a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - 2n\, a_n + \lambda\, a_n]x^n = 0\,.$$

Setting the coefficients equal to *zero*, it follows that

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n$$

for $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= \frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k} = \frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots \\ &= (-1)^k\, \frac{\lambda\cdots(\lambda-4k+8)(\lambda-4k+4)}{(2k)!}\, a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= \frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)} = \frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots \\ &= (-1)^k\, \frac{(\lambda-2)\cdots(\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!}\, a_1. \end{aligned}$$

Hence the linearly independent solutions of the *Hermite equation* (about $x_0 = 0$) are

$$y_1(x) = 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6\,!}x^6 + \cdots$$

$$y_2(x) = x - \frac{\lambda-2}{3\,!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5\,!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7\,!}x^7 + \cdots.$$

$(b)$. Based on the recurrence relation

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n \,,$$

the series solution will *terminate* as long as $\lambda$ is a *nonnegative* even integer. If $\lambda = 2m$, then *one or the other* of the solutions in Part $(b)$ will contain at most $m/2+1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda = 0, 2, 4, 6, 8, 10$ :

| | |
|---|---|
| $\lambda = 0$ | $y_1(x) = 1$ |
| $\lambda = 2$ | $y_2(x) = x$ |
| $\lambda = 4$ | $y_1(x) = 1 - 2x^2$ |
| $\lambda = 6$ | $y_2(x) = x - 2x^3/3$ |
| $\lambda = 8$ | $y_1(x) = 1 - 4x^2 + 4x^4/3$ |
| $\lambda = 10$ | $y_2(x) = x - 4x^3/3 + 4x^5/15$ |

$(c)$. Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (-1)^k \frac{2n \cdots (2n-4k+8)(2n-4k+4)}{(2k)!}$$

and

$$a_{2k+1} = (-1)^k \frac{(2n-2)\cdots(2n-4k+6)(2n-4k+2)}{(2k+1)!}\,.$$

for $k = 1, 2, \cdots [n/2]$. It follows that the *coefficient* of $x^n$, in $y_1$ and $y_2$, is

$$a_n = \begin{cases} (-1)^k \frac{4^k\, k!}{(2k)!} & \text{for } n = 2k \\ (-1)^k \frac{4^k\, k!}{(2k+1)!} & \text{for } n = 2k+1 \end{cases}$$

Then by definition,

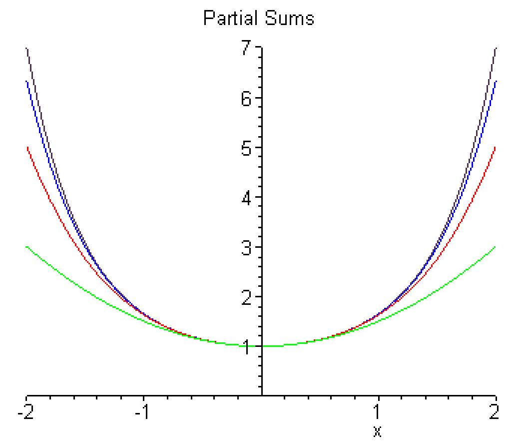
$$H_n(x) = \begin{cases} (-1)^k\, 2^n \frac{(2k)!}{4^k\, k!} y_1(x) = (-1)^k\, \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (-1)^k\, 2^n \frac{(2k+1)!}{4^k\, k!} y_2(x) = (-1)^k\, \frac{2\,(2k+1)!}{k!} y_2(x) & \text{for } n = 2k+1 \end{cases}$$

Therefore the first six *Hermite polynomials* are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^8 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned}$$

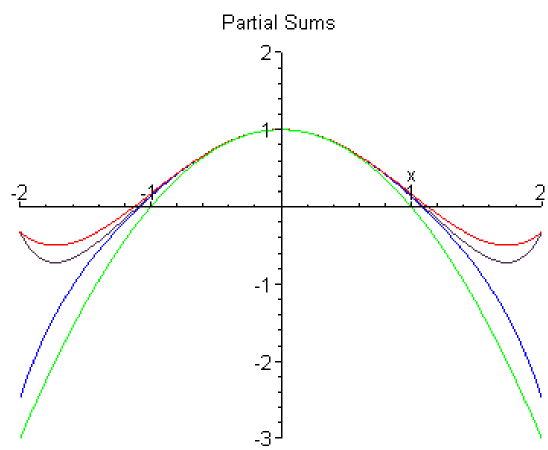
23. The series solution is given by

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2^2\,2!}x^4 + \frac{1}{2^3 3!}x^6 + \frac{1}{2^4 4!}x^8 + \cdots.$$



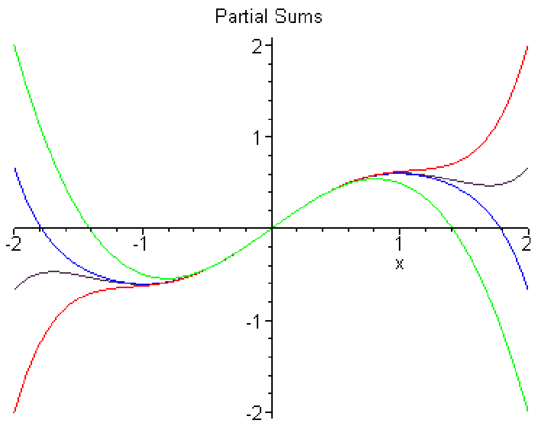

24. The series solution is given by

$$y(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \frac{x^8}{120} + \cdots.$$



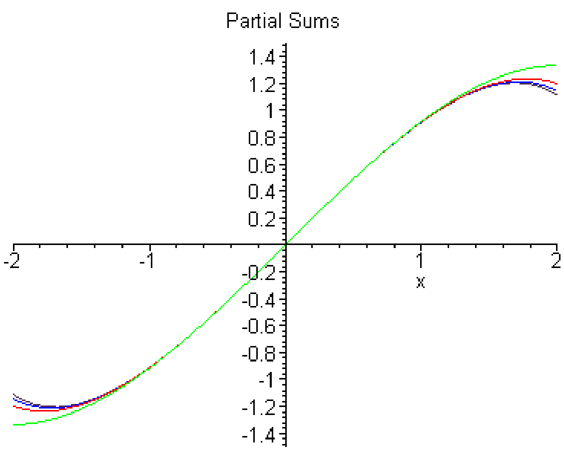

25. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \frac{x^9}{2 \cdot 4 \cdot 6 \cdot 8} - \cdots.$$



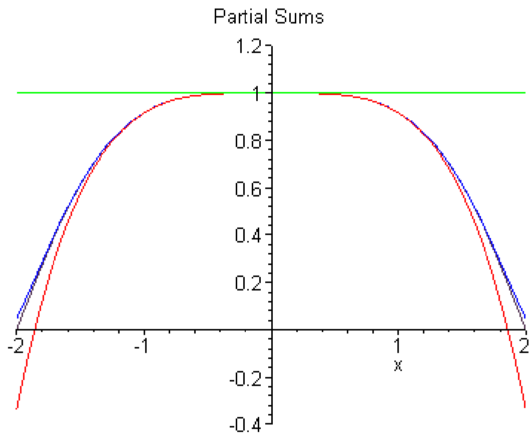

26. The series solution is given by

$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \frac{x^9}{16128} - \cdots.$$




27. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \cdots.$$




28. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 .$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - (n+1) n\, a_{n+1} + n\, a_n - 2\, a_n] x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1) a_{n+2} - (n+1) n\, a_{n+1} + (n-2) a_n = 0$$

for $n = 1, 2, \cdots$. Writing out the individual equations,

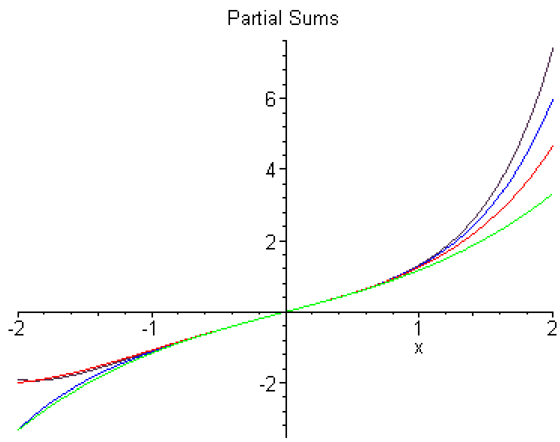
$$\begin{aligned} 3 \cdot 2\, a_3 - 2 \cdot 1\, a_2 - a_1 &= 0 \\ 4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 &= 0 \\ 5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + a_3 &= 0 \\ 6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$\begin{aligned} 3 \cdot 2\, a_3 - 1 &= 0 \\ 4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 &= 0 \\ 5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + a_3 &= 0 \\ 6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

That is, $a_3 = 1/6\,, a_4 = 1/12\,, a_5 = 1/24\,, a_6 = 1/45\,, \cdots$. Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \cdots.$$



**Section 5.3**

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(sin\, x)y' - (cos\, x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -(sin\, x)y'' - 2(cos\, x)y' + (sin\, x)y \\ y^{iv} &= -(sin\, x)y''' - 3(cos\, x)y'' + 3(sin\, x)y' + (cos\, x)y\,. \end{aligned}$$

Given that $\phi(0) = 0$ and $\phi'(0) = 1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -2$ and $\phi^{iv}(0) = 0$.

3. Let $y = \phi(x)$ be a solution of the initial value problem. First write

$$y'' = -\frac{1+x}{x^2}\,y' - \frac{3\,ln\,x}{x^2}\,y\,.$$

Differentiating twice,

$$y''' = \frac{-1}{x^3}\left[\left(x + x^2\right)y'' + (3x\,ln\,x - x - 2)y' + (3 - 6\,ln\,x)y\right].$$

$$\begin{aligned} y^{iv} = \frac{-1}{x^4}\Big[&\left(x^2 + x^3\right)y''' + \left(3x^2 ln\,x - 2x^2 - 4x\right)y'' + \\ &+ (6 + 8x - 12x\,ln\,x)y' + (18\,ln\,x - 15)y\Big]. \end{aligned}$$

Given that $\phi(1) = 2$ and $\phi'(1) = 0$, the *first* equation gives $\phi''(1) = 0$ and the last two equations give $\phi'''(0) = -6$ and $\phi^{iv}(0) = 42$.

4. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2\,y' - (sin\, x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -x^2\,y'' - (2x + sin\, x)y' - (cos\, x)y \\ y^{iv} &= -x^2\,y''' - (4x + sin\, x)y'' - (2 + 2cos\, x)y' + (sin\, x)y\,. \end{aligned}$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{iv}(0) = -4a_1$.

5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all $x$. Hence the series solutions converge *everywhere*.

7. The *zeroes* of $P(x) = 1 + x^3$ are the *three* cube roots of $-1$. They all lie on the unit circle in the complex plane. So for $x_0 = 0$, $\rho_{min} = 1$. For $x_0 = 2$, the *nearest*

root is $e^{i\pi/3} = \left(1 + i\sqrt{3}\right)/2$, hence $\rho_{min} = \sqrt{3}$ .

8. The only root of $P(x) = x$ is *zero*. Hence $\rho_{min} = 1$ .

$9(b)$. $p(x) = -x$ and $q(x) = -1$ are analytic for all $x$ .
$(c)$. $p(x) = -x$ and $q(x) = -1$ are analytic for all $x$ .
$(d)$. $p(x) = 0$ and $q(x) = kx^2$ are analytic for all $x$ .
$(e)$. The only root of $P(x) = 1 - x$ is $1$ . Hence $\rho_{min} = 1$ .
$(g)$. $p(x) = x$ and $q(x) = 2$ are analytic for all $x$ .
$(i)$. The *zeroes* of $P(x) = 1 + x^2$ are $\pm i$ . Hence $\rho_{min} = 1$ .
$(j)$. The *zeroes* of $P(x) = 4 - x^2$ are $\pm 2$ . Hence $\rho_{min} = 2$ .
$(k)$. The *zeroes* of $P(x) = 3 - x^2$ are $\pm\sqrt{3}$ . Hence $\rho_{min} = \sqrt{3}$ .
$(l)$. The only root of $P(x) = 1 - x$ is $1$ . Hence $\rho_{min} = 1$ .
$(m)$. $p(x) = x/2$ and $q(x) = 3/2$ are analytic for all $x$ .
$(n)$. $p(x) = (1 + x)/2$ and $q(x) = 3/2$ are analytic for all $x$ .

12. The Taylor series expansion of $e^x$, about $x_0 = 0$ , is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n\right] + x\sum_{n=0}^{\infty} a_n x^n = 0 .$$

First note that

$$x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n + \cdots .$$

The coefficient of $x^n$ in the *product* of the two series is

$$c_n = 2a_2 \frac{1}{n!} + 6a_3 \frac{1}{(n-1)!} + 12a_4 \frac{1}{(n-2)!} + \cdots + (n+1)n\, a_{n+1} + (n+2)(n+1)a_{n+2} .$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \cdots + \\ + a_0 x + a_1 x^2 + a_2 x^3 + \cdots = 0 .$$

Setting the coefficients equal to *zero*, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$ , $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0$ , $\cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x - a_0\frac{x^3}{6} + (a_0 - a_1)\frac{x^4}{12} + (2a_1 - a_0)\frac{x^5}{40} + \left(\tfrac{4}{3}a_0 - 2a_1\right)\frac{x^6}{120} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \cdots$$

$$y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$ .

13. The Taylor series expansion of $cos\, x$ , about $x_0 = 0$ , is

$$cos\, x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)\,!} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)\,!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n\right] + \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 .$$

The coefficient of $x^n$ in the *product* of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12a_4 b_{n-2} + \cdots + (n+1)n\, a_{n+1} b_1 + (n+2)(n+1)a_{n+2} b_0 ,$$

in which $cos\, x = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty}(n-2)a_n x^n = 0 .$$

Expanding the product of the series, it follows that

$$2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \cdots - \\ - a_1 x + a_3 x^3 + 2a_4 x^4 + \cdots = 0 .$$

Setting the coefficients equal to *zero*, $a_2 - a_0 = 0$ , $6a_3 - a_1 = 0$ , $-a_2 + 12a_4 = 0$ , $-3a_3 + 20a_5 + a_3 = 0$ , $\cdots$ . Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1\frac{x^3}{6} + a_0\frac{x^4}{12} + a_1\frac{x^5}{60} + a_0\frac{x^6}{120} + a_1\frac{x^7}{560} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \cdots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \cdots$$

The *nearest* zero of $P(x) = cos\, x$ is at $x = \pm\pi/2$ . Hence $\rho_{min} = \pi/2$ .

14. The Taylor series expansion of $ln(1+x)$ , about $x_0 = 0$ , is

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right] \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + \\ + \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\right] \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0 .$$

The *first* product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \cdots .$$

The *second* product is the series

$$a_1 x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^3 + \cdots .$$

Combining the series and equating the coefficients to *zero*, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1 \frac{7x^5}{120} + \left(\frac{5}{3}a_1 - a_0\right)\frac{x^6}{120} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \cdots$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \cdots$$

The coefficient $p(x) = e^x ln(1+x)$ is analytic at $x_0 = 0$ , but its power series has a radius of convergence $\rho = 1$ .

15. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting $y_2$ into the ODE results in

$$2\,P(x) + 2x\,Q(x) + x^2 R(x) = 0\,.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting $y_1$ into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that $y_1$ and $y_2$ were the *only* two solutions. But note that an *arbitrary* value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

16. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}\left[(n+1)a_{n+1} - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \cdots$. It is easy to see that $a_n = a_0/(n\,!)$. Therefore the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots\right] \\ &= a_0 e^x. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

17. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - x\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - a_{n-1}]\, x^n = 0\,.$$

Setting the coefficient equal to *zero*, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{a_0}{2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = 0\,.$$

Hence the general solution is

$$\begin{aligned} y(x) &= a_0 \left[ 1 + \frac{x^2}{2} + \frac{x^4}{2^2\, 2\,!} + \frac{x^6}{2^3\, 3\,!} + \cdots + \frac{x^{2n}}{2^n\, n\,!} + \cdots \right] \\ &= a_0 exp\big(x^2/2\big). \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

19. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$(1-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - n\, a_n - a_n]\, x^n = 0\,.$$

Setting the coefficients equal to *zero*, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \cdots$. Hence the general solution is

$$\begin{aligned} y(x) &= a_0 \left[ 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \right] \\ &= a_0 \frac{1}{1-x}\,. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + x\sum_{n=0}^{\infty}a_n x^n = 1 + x\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty}a_{n-1}x^n = 1 + x\,.$$

Combining the series, and the nonhomogeneous terms, we have

$$(a_1 - 1) + (2a_2 + a_0 - 1)x + \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_1 = 1$, $2a_2 + a_0 - 1 = 0$, and

$$a_n = -\,\frac{a_{n-2}}{n}\,, \quad n = 3, 4, \cdots.$$

The indices differ by *two*, so for $k = 2, 3, \cdots$

$$a_{2k} = -\,\frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{(-1)^{k-1}\,a_2}{4\cdot 6\cdots(2k)} = \frac{(-1)^k(a_0 - 1)}{2\cdot 4\cdot 6\cdots(2k)}\,,$$

and for $k = 1, 2, \cdots$

$$a_{2k+1} = = -\,\frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k-1)(2k+1)} = \cdots = \frac{(-1)^k}{3\cdot 5\cdots(2k+1)}\,.$$

Hence the general solution is

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{x^3}{3} + a_0\frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} - a_0\frac{x^6}{2^3\,3\,!} - \cdots$$

Collecting the terms containing $a_0$,

$$\begin{aligned} y(x) &= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} - \frac{x^6}{2^3\,3\,!} + \cdots\right] + \\ &+ \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right]. \end{aligned}$$

Upon inspection, we find that

$$y(x) = a_0 exp(-x^2/2) + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right].$$

Note that the given ODE is *first order linear*, with integrating factor $\mu(t) = e^{x^2/2}$. The general solution is given by

$$y(x) = e^{-x^2/2}\int_0^x e^{u^2/2}du + (y(0)-1)e^{-x^2/2} + 1\,.$$

23. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1 + \sum_{m=1}^{n} (-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+1)(2n+3)\cdots(2n+2m-1)}{(2m)!}\, x^{2m}.$$

| | |
|---|---|
| $\alpha = 0$ | $1$ |
| $\alpha = 2$ | $1 - 3x^2$ |
| $\alpha = 4$ | $1 - 10x^2 + \frac{35}{3}x^4$ |

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x + \sum_{m=1}^{n} (-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+3)(2n+5)\cdots(2n+2m+1)}{(2m+1)!}\, x^{2m+1}.$$

| | |
|---|---|
| $\alpha = 1$ | $x$ |
| $\alpha = 3$ | $x - \frac{5}{3}x^3$ |
| $\alpha = 5$ | $x - \frac{14}{3}x^3 + \frac{21}{5}x^5$ |

24$(a)$. Based on Prob. 23,

| | | |
|---|---|---|
| $\alpha = 0$ | $1$ | $y_1(1) = 1$ |
| $\alpha = 2$ | $1 - 3x^2$ | $y_1(1) = -2$ |
| $\alpha = 4$ | $1 - 10x^2 + \frac{35}{3}x^4$ | $y_1(1) = \frac{8}{3}$ |

Normalizing the polynomials, we obtain

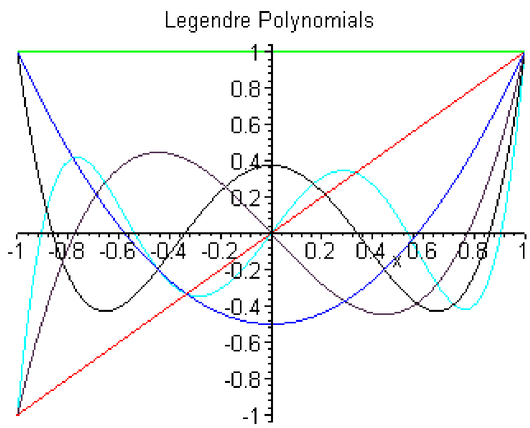
$$P_0(x) = 1$$
$$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$$
$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

| | | |
|---|---|---|
| $\alpha = 1$ | $x$ | $y_2(1) = 1$ |
| $\alpha = 3$ | $x - \frac{5}{3}x^3$ | $y_2(1) = -\frac{2}{3}$ |
| $\alpha = 5$ | $x - \frac{14}{3}x^3 + \frac{21}{5}x^5$ | $y_2(1) = \frac{8}{15}$ |

Similarly,

$$P_1(x) = x$$
$$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$
$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

$(b)$.




$(c)$. $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The *zeros* of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The *zeros* of $P_3(x)$ are $x = 0, \pm\sqrt{3/5}$. The roots of $P_4(x)$ are given by $x^2 = \left(15 + 2\sqrt{30}\right)/35$, $\left(15 - 2\sqrt{30}\right)/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = \left(35 + 2\sqrt{70}\right)/63$, $\left(35 - 2\sqrt{70}\right)/63$.

25. Observe that

$$\begin{aligned} P_n(-1) &= \frac{(-1)^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} \\ &= (-1)^n P_n(1). \end{aligned}$$

But $P_n(1) = 1$ for *all* nonnegative integers $n$.

27. We have

$$(x^2 - 1)^n = \sum_{k=0}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!} x^{2k},$$

which is a polynomial of *degree* $2n$. Differentiating *n times*,

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=\mu}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}(2k)(2k-1)\cdots(2k-n+1)x^{2k-n},$$

in which the *lower index* is $\mu = [n/2] + 1$. Note that if $n = 2m+1$, then $\mu = m+1$.

Now shift the index, by setting

$$k = n - j\,.$$

Hence

$$\begin{aligned}\frac{d^n}{dx^n}\left(x^2-1\right)^n &= \sum_{j=0}^{[n/2]} \frac{(-1)^j\, n\,!}{(n-j)!j\,!}(2n-2j)(2n-2j-1)\cdots(n-2j+1)x^{n-2j} \\ &= n! \sum_{j=0}^{[n/2]} \frac{(-1)^j (2n-2j)!}{(n-j)!j\,!(n-2j)!}\, x^{n-2j}.\end{aligned}$$

Based on Prob. 25,

$$\frac{d^n}{dx^n}\left(x^2-1\right)^n = n!\,2^n P_n(x).$$

29. Since the $n+1$ polynomials $P_0, P_1, \cdots, P_n$ are *linearly independent*, and the *degree* of $P_k$ is $k$ , any polynomial, $f$ , of degree $n$ can be expressed as a linear combination

$$f(x) = \sum_{k=0}^{n} a_k P_k(x)\,.$$

Multiplying both sides by $P_m$ and integrating,

$$\int_{-1}^{1} f(x) P_m(x) dx = \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x) P_m(x) dx\,.$$

Based on Prob. 28,

$$\int_{-1}^{1} P_k(x) P_m(x) dx = \frac{2}{2m+1}\delta_{km}.$$

Hence

$$\int_{-1}^{1} f(x) P_m(x) dx = \frac{2}{2m+1}\, a_m\,.$$

## Section 5.4

2. We see that $P(x)=0$ when $x=0$ and $1$. Since the three coefficients have no factors
in common, both of these points are singular points. Near $x=0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{2x}{x^2(1-x)^2} = 2\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{4}{x^2(1-x)^2} = 4\,.$$

The singular point $x=0$ is *regular*. Considering $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2x}{x^2(1-x)^2}\,.$$

The latter limit *does not exist*. Hence $x=1$ is an *irregular* singular point.

3. $P(x)=0$ when $x=0$ and $1$. Since the three coefficients have no common factors, both of these points are singular points. Near $x=0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{x-2}{x^2(1-x)}\,.$$

The limit *does not exist*, and so $x=0$ is an *irregular* singular point. Considering $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{x-2}{x^2(1-x)} = 1\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\frac{-3x}{x^2(1-x)} = 0\,.$$

Hence $x=1$ is a *regular* singular point.

4. $P(x)=0$ when $x=0$ and $\pm 1$. Since the three coefficients have no common factors, both of these points are singular points. Near $x=0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{2}{x^3(1-x^2)}\,.$$

The limit *does not exist*, and so $x=0$ is an *irregular* singular point. Near $x=-1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to-1}(x+1)^2 q(x) = \lim_{x\to-1}(x+1)^2\frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x = -1$ is a *regular* singular point. At $x = 1$,

$$\lim_{x\to1}(x-1)p(x) = \lim_{x\to1}(x-1)\frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to1}(x-1)^2 q(x) = \lim_{x\to1}(x-1)^2\frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

6. The only singular point is at $x = 0$. We find that

$$\lim_{x\to0} x\,p(x) = \lim_{x\to0} x\frac{x}{x^2} = 1\,.$$

$$\lim_{x\to0} x^2 q(x) = \lim_{x\to0} x^2\,\frac{x^2-\nu^2}{x^2} = -\nu^2.$$

Hence $x = 0$ is a *regular* singular point.

7. The only singular point is at $x = -3$. We find that

$$\lim_{x\to-3}(x+3)p(x) = \lim_{x\to-3}(x+3)\frac{-2x}{x+3} = 6\,.$$

$$\lim_{x\to-3}(x+3)^2 q(x) = \lim_{x\to-3}(x+3)^2\,\frac{1-x^2}{x+3} = 0.$$

Hence $x = -3$ is a *regular* singular point.

8. Dividing the ODE by $x(1-x^2)^3$, we find that

$$p(x) = \frac{1}{x(1-x^2)} \quad \text{and} \quad q(x) = \frac{2}{x(1+x)^2(1-x)^3}\,.$$

The singular points are at $x = 0$ and $\pm 1$. For $x = 0$,

$$\lim_{x\to0} x\,p(x) = \lim_{x\to0} x\frac{1}{x(1-x^2)} = 1\,.$$

$$\lim_{x\to0} x^2 q(x) = \lim_{x\to0} x^2\,\frac{2}{x(1+x)^2(1-x)^3} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. For $x = -1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2\,\frac{2}{x(1+x)^2(1-x)^3} = -\frac{1}{4}\,.$$

Hence $x = -1$ is a *regular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\,\frac{2}{x(1+x)^2(1-x)^3}\,.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

9. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x) = \frac{3}{(x+2)^2} \quad \text{and} \quad q(x) = \frac{-2}{(x+2)(x-1)}\,.$$

The singular points are at $x = -2$ and $1$. For $x = -2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}(x+2)\frac{3}{(x+2)^2}\,.$$

The limit *does not exist*. Hence $x = -2$ is an *irregular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{3}{(x+2)^2} = 0\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\,\frac{-2}{(x+2)(x-1)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

10. $P(x) = 0$ when $x = 0$ and $3$. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0}x\,p(x) = \lim_{x\to 0}x\frac{x+1}{x(3-x)} = \frac{1}{3}\,.$$

$$\lim_{x\to 0}x^2 q(x) = \lim_{x\to 0}x^2\frac{-2}{x(3-x)} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. For $x = 3$,

$$\lim_{x\to 3}(x-3)p(x) = \lim_{x\to 3}(x-3)\frac{x+1}{x(3-x)} = -\frac{4}{3}\,.$$

$$\lim_{x\to 3}(x-3)^2 q(x) = \lim_{x\to 3}(x-3)^2\frac{-2}{x(3-x)} = 0\,.$$

Hence $x = 3$ is a *regular* singular point.

11. Dividing the ODE by $(x^2+x-2)$, we find that

$$p(x) = \frac{x+1}{(x+2)(x-1)} \quad \text{and} \quad q(x) = \frac{2}{(x+2)(x-1)}\,.$$

The singular points are at $x = -2$ and $1$. For $x = -2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}\frac{x+1}{x-1} = \frac{1}{3}\,.$$

$$\lim_{x\to -2}(x+2)^2 q(x) = \lim_{x\to -2}\frac{2(x+2)}{x-1} = 0\,.$$

Hence $x = -2$ is a *regular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x+1}{x+2} = \frac{2}{3}\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{2(x-1)}{(x+2)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

13. Note that $p(x) = ln|x|$ and $q(x) = 3x$. Evidently, $p(x)$ is *not* analytic at $x_0 = 0$. Furthermore, the function $x\,p(x) = x\,ln|x|$ does *not* have a Taylor series about $x_0 = 0$. Hence $x = 0$ is an *irregular* singular point.

14. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $e^x - 1$, about $x = 0$, is

$$e^x - 1 = x + x^2/2 + x^3/6 + \cdots.$$

Hence the function $x\,p(x) = 2(e^x - 1)/x$ is analytic at $x = 0$. Similarly, the Taylor series of $e^{-x} cos\,x$, about $x = 0$, is

$$e^{-x} cos\,x = 1 - x + x^3/3 - x^4/6 + \cdots.$$

The function $x^2 q(x) = e^{-x} cos\,x$ is also analytic at $x = 0$. Hence $x = 0$ is a *regular* singular point.

15. $P(x)=0$ when $x=0$. Since the three coefficients have no common factors, $x=0$ is a singular point. The Taylor series of $sin\,x$, about $x=0$, is

$$sin\,x = x - x^3/3! + x^5/5! - \cdots .$$

Hence the function $x\,p(x) = -3sin\,x/x$ is analytic at $x=0$. On the other hand, $q(x)$ is a rational function, with

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1+x^2}{x^2} = 1\,.$$

Hence $x=0$ is a *regular* singular point.

16. $P(x)=0$ when $x=0$. Since the three coefficients have no common factors, $x=0$ is a singular point. We find that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{1}{x} = 1\,.$$

Although the function $R(x) = cot\,x$ does not have a Taylor series about $x=0$, note that $x^2q(x) = x\,cot\,x = 1 - x^2/3 - x^4/45 - 2x^6/945 - \cdots$. Hence $x=0$ is a *regular* singular point. Furthermore, $q(x) = cot\,x/x^2$ is undefined at $x = \pm\,n\pi$. Therefore the points $x = \pm\,n\pi$ are *also* singular points. First note that

$$\lim_{x\to \pm n\pi} (x \mp n\pi)p(x) = \lim_{x\to \pm n\pi} (x \mp n\pi)\frac{1}{x} = 0\,.$$

Furthermore, since $cot\,x$ has period $\pi$,

$$\begin{aligned} q(x) &= cot\,x/x = cot(x \mp n\pi)/x \\ &= cot(x \mp n\pi)\frac{1}{(x \mp n\pi) \pm n\pi}\,. \end{aligned}$$

Therefore

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi)cot(x \mp n\pi)\left[\frac{(x \mp n\pi)}{(x \mp n\pi) \pm n\pi}\right].$$

From above,

$$(x \mp n\pi)cot(x \mp n\pi) = 1 - (x \mp n\pi)^2/3 - (x \mp n\pi)^4/45 - \cdots .$$

Note that the function in *brackets* is analytic *near* $x = \pm\,n\pi$. It follows that the function $(x \mp n\pi)^2 q(x)$ is also analytic near $x = \pm\,n\pi$. Hence all the singular points are *regular*.

18. The singular points are located at $x = \pm\,n\pi$, $n = 0, 1, \cdots$. Dividing the ODE by $x\,sin\,x$, we find that $x\,p(x) = 3\,csc\,x$ and $x^2q(x) = x^2csc\,x$. Evidently, $x\,p(x)$ is not even defined at $x=0$. Hence $x=0$ is an *irregular* singular point. On the other hand, the Taylor series of $x\,csc\,x$, about $x=0$, is

$$x\,csc\,x = 1 + x^2/6 + 7x^4 360 + \cdots.$$

Noting that $csc(x \mp n\pi) = (-1)^n csc\,x$,

$$\begin{aligned}(x \mp n\pi)p(x) &= 3(-1)^n (x \mp n\pi)csc(x \mp n\pi)/x \\ &= 3(-1)^n (x \mp n\pi)csc(x \mp n\pi)\left[\frac{1}{(x \mp n\pi) \pm n\pi}\right].\end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$\begin{aligned}(x \mp n\pi)^2 q(x) &= (x \mp n\pi)^2 csc\,x \\ &= (-1)^n (x \mp n\pi)^2 csc(x \mp n\pi),\end{aligned}$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are *regular*.

20. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x \to 0} x\,p(x) = \lim_{x \to 0} x\frac{3}{2x} = \frac{3}{2},$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2}.$$

Hence $x = 0$ is a *regular* singular point. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substitution into the ODE results in

$$2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + 3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - (1+x)\sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$2\sum_{n=2}^{\infty} n(n-1)a_n\, x^n + 3\sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty}\left[2n(n-1)a_n + 3n\,a_n - a_n - a_{n-1}\right]x^n = 0.$$

Equating the coefficients to *zero*, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}, \quad n = 2, 3, \cdots.$$

We conclude that *all* the $a_n$ are *equal to zero*. Hence $y(x) = 0$ is the only solution that can be obtained.

22. Based on Prob. 21, the change of variable, $x = 1/\xi$, transforms the ODE into the

form

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0\,.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim_{\xi \to 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$, that is, $x = \infty$, is an *irregular* singular point.

24. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \left(1 - \frac{1}{\xi^2}\right) \frac{d^2y}{d\xi^2} + \left[2\xi^3 \left(1 - \frac{1}{\xi^2}\right) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,,$$

that is,

$$\left(\xi^4 - \xi^2\right) \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \text{ and } q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)}\,.$$

It follows that

$$\lim_{\xi \to 0} \xi\, p(\xi) = \lim_{\xi \to 0} \xi \frac{2\xi}{\xi^2 - 1} = 0\,,$$

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)} = -\alpha(\alpha+1)\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a *regular* singular point.

26. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \lambda\, y = 0\,,$$

that is,

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\left(\xi^3 + \xi\right) \frac{dy}{d\xi} + \lambda\, y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2+1)}{\xi^3} \text{ and } q(\xi) = \frac{\lambda}{\xi^4}\,.$$

It immediately follows that the limit $\lim_{\xi \to 0} \xi\, p(\xi)$ *does not exist*. Hence $\xi = 0$ $(x = \infty)$

is an *irregular* singular point.

27. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0 \, .$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi \to 0} \xi \, p(\xi) = \lim_{\xi \to 0} \xi \frac{2}{\xi} = 2 \, ,$$

but

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \, \frac{(-1)}{\xi^5} \, .$$

The latter limit *does not exist*. Hence $\xi = 0$ $(x = \infty)$ is an *irregular* singular point.

**Section 5.5**

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) + 4r + 2 \\ &= r^2 + 3r + 2\,. \end{aligned}$$

The roots are $r = -2\,,\; -1$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-2} + c_2\, x^{-1}.$$

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - 3r + 4 \\ &= r^2 - 4r + 4\,. \end{aligned}$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x^2.$$

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - r + 1 \\ &= r^2 - 2r + 1\,. \end{aligned}$$

The root is $r = 1$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x\,.$$

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 7r + 12\,.$$

The roots are $r = -3\,,\; -4$. Hence the general solution, for $x \neq 1$, is

$$y = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}.$$

7. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 5r - 1\,.$$

The roots are $r = -\left(5 \pm \sqrt{29}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{-\left(5+\sqrt{29}\right)/2} + c_2 |x|^{-\left(5-\sqrt{29}\right)/2}.$$

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 3r + 3\,.$$

The roots are complex, with $r = \left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{3/2} cos\left(\frac{\sqrt{3}}{2}\, ln|x|\right) + c_2|x|^{3/2} sin\left(\frac{\sqrt{3}}{2}\, ln|x|\right).$$

10. Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 4r + 8\,.$$

The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is

$$y = c_1\,(x-2)^{-2} cos(2\,ln|x-2|) + c_2(x-2)^{-2} sin(2\,ln|x-2|).$$

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + r + 4\,.$$

The roots are complex, with $r = -\left(1 \pm i\sqrt{15}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{-1/2} cos\left(\frac{\sqrt{15}}{2}\, ln|x|\right) + c_2|x|^{-1/2} sin\left(\frac{\sqrt{15}}{2}\, ln|x|\right).$$

12. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 5r + 4\,.$$

The roots are $r = 1\,, 4$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,x + c_2\,x^4.$$

14. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = 4r^2 + 4r + 17\,.$$

The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is

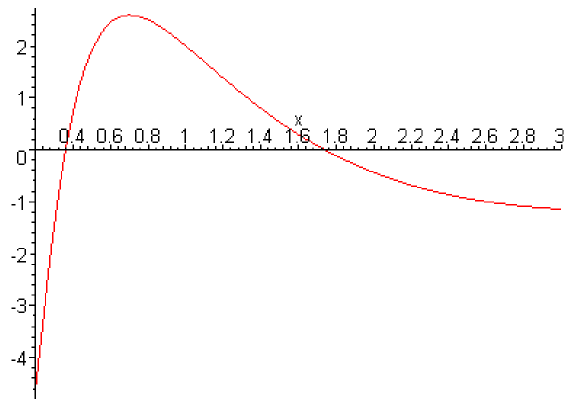
$$y = c_1\,x^{-1/2} cos(2\,ln\,x) + c_2\,x^{-1/2} sin(2\,ln\,x).$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-\frac{1}{2}c_1 + 2c_2 = -3$$

Hence the solution of the initial value problem is

$$y(x) = 2\,x^{-1/2}cos(2\,ln\,x) - x^{-1/2}sin(2\,ln\,x).$$




As $x \to 0^+$, the solution decreases without bound.

15. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 4r + 4\,.$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x < 0$, is

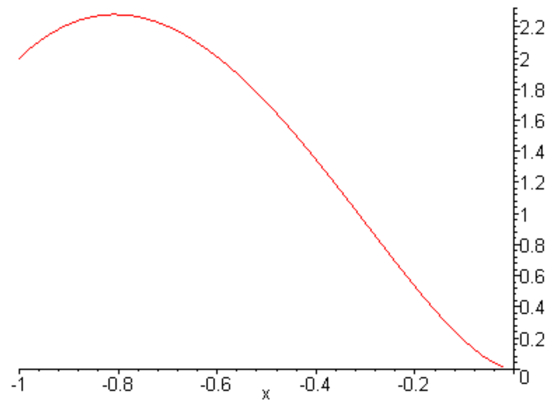
$$y = (c_1 + c_2\,ln\,|x|)\,x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-2c_1 - c_2 = 3$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7\,ln\,|x|)\,x^2.$$




We find that $y(x) \to 0$ as $x \to 0^-$.

18. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = \left(1 \pm i\sqrt{4\beta - 1}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2} cos\left(\frac{1}{2}\sqrt{4\beta - 1}\; ln|x|\right) + c_2 |x|^{1/2} sin\left(\frac{1}{2}\sqrt{4\beta - 1}\; ln|x|\right).$$

Since the trigonometric factors are *bounded*, $y(x) \to 0$ as $x \to 0$. If $\beta = 1/4$, the roots are *equal*, and

$$y = c_1 |x|^{1/2} + c_2 |x|^{1/2}\, ln\,|x|\,.$$

Since $\lim_{x \to 0} \sqrt{|x|}\, ln\,|x| = 0$, $y(x) \to 0$ as $x \to 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = \left(1 \pm \sqrt{1 - 4\beta}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2 + \sqrt{1-4\beta}/2} + c_2 |x|^{1/2 - \sqrt{1-4\beta}/2}.$$

Evidently, solutions approach *zero* as long as $1/2 - \sqrt{1 - 4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4\,.$$

Hence *all* solutions approach *zero*, for $\beta > 0$.

19. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1, 2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-1} + c_2\, x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 1$$
$$-c_1 + 2c_2 = \gamma$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3}x^{-1} + \frac{1+\gamma}{3}x^2.$$

The solution is *bounded*, as $x \to 0$, if $\gamma = 2$.

20. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$r = \frac{1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha - 9}}{2}$$
$$= \frac{1 - \alpha \pm \sqrt{\left(\alpha - 1 - \sqrt{10}\right)\left(\alpha - 1 + \sqrt{10}\right)}}{2}.$$

$(i)$ The roots $r_{1,2}$ will be *complex*, if $|1-\alpha| < \sqrt{10}$. For solutions to approach *zero*, as $x \to \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.
$(ii)$ The roots will be *equal*, if $|1-\alpha| = \sqrt{10}$. In this case, all solutions approach *zero* as long as $1 - \alpha = -\sqrt{10}$.
$(iii)$ The roots will be real and *distinct*, if $|1-\alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9}}{2}.$$

For solutions to approach *zero*, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$.

Hence all solutions approach *zero*, as $x \to \infty$, as long as $\alpha > 1$.

23$(a)$. Given that $x = e^z$, $y(x) = y(e^z) = w(z)$. By the chain rule,

$$\frac{dy}{dx} = \frac{d}{dx}w(z) = \frac{dw}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dw}{dz}.$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dw}{dz}\right] = -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x}\frac{d^2w}{dz^2}\frac{dz}{dx}$$
$$= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x^2}\frac{d^2w}{dz^2}.$$

$(b)$. Direct substitution results in

$$x^2\left[\frac{1}{x^2}\,\frac{d^2w}{dz^2}-\frac{1}{x^2}\,\frac{dw}{dz}\right]+\alpha\,x\left[\frac{1}{x}\,\frac{dw}{dz}\right]+\beta\,w=0\,,$$

that is,

$$\frac{d^2w}{dz^2}+(\alpha-1)\frac{dw}{dz}+\beta\,w=0\,.$$

The associated *characteristic equation* is $r^2+(\alpha-1)r+\beta=0$. Since $z=ln\,x$, it follows that $y(x)=w(ln\,x)$.

$(c)$. If the roots $r_{1,2}$ are real and *distinct*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2e^{r_2z}\\ &= c_1x^{r_1}+c_2x^{r_2}. \end{aligned}$$

$(d)$. If the roots $r_{1,2}$ are real and *equal*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2z\,e^{r_1z}\\ &= c_1x^{r_1}+c_2x^{r_1}ln\,x\,. \end{aligned}$$

$(e)$. If the roots are *complex conjugates*, then $r=\lambda\pm i\mu$, and

$$\begin{aligned} y &= e^{\lambda z}(c_1cos\,\mu z+c_2\,sin\,\mu z)\\ &= x^{\lambda}[c_1cos(\mu\,ln\,x)+c_2\,sin(\mu\,ln\,x)]. \end{aligned}$$

24. Based on Prob. 23, the change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}-\frac{dw}{dz}-2w=0\,.$$

The associated *characteristic equation* is $r^2-r-2=0$, with roots $r=\,-1\,,2\,.$ Hence $w(z)=c_1e^{-z}+c_2e^{2z}$, and $y(x)=c_1x^{-1}+c_2\,x^2$.

26. The change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}+6\frac{dw}{dz}+5w=e^z.$$

The associated *characteristic equation* is $r^2+6\,r+5=0$, with roots $r=\,-5\,,\,-1\,.$ Hence $w_c(z)=c_1e^{-z}+c_2e^{-5z}$. Since the right hand side is *not* a solution of the homogeneous equation, we can use the *method of undetermined coefficients* to show that a particular solution is $W=e^z/12$. Therefore the general solution is given by $w(z)=c_1e^{-z}+c_2e^{-5z}+e^z/12$, that is, $y(x)=c_1x^{-1}+c_2\,x^{-5}+x/12$.

27. The change of variable $x=e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} - 3\frac{dw}{dz} + 2w = 3e^{2z} + 2z\,.$$

The associated *characteristic equation* is $r^2 - 3\,r + 2 = 0$, with roots $r = 1\,,2\,.$ Hence $w_c(z) = c_1e^z + c_2e^{2z}$. Using the *method of undetermined coefficients*, let $W = Ae^{2z} + Bze^{2z} + Cz + D\,.$ It follows that the general solution is given by $w(z) = c_1e^z + c_2e^{2z} + 3ze^{2z} + z + 3/2\,$, that is,

$$y(x) = c_1x + c_2\,x^2 + 3\,x^2 ln\,x + ln\,x + 3/2\,.$$

28. The change of variable $x = e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} + 4w = sin\,z\,.$$

The solution of the homogeneous equation is $w_c(z) = c_1cos\,2z + c_2sin\,2z\,.$ The right hand side is *not* a solution of the homogeneous equation. We can use the *method of undetermined coefficients* to show that a particular solution is $W = \frac{1}{3}sin\,z\,.$ Hence the general solution is given by $w(z) = c_1cos\,2z + c_2sin\,2z + \frac{1}{3}sin\,z$, that is, $y(x) = c_1cos(2\,ln\,x) + c_2sin(2\,ln\,x) + \frac{1}{3}sin(ln\,x)\,.$

29. After dividing the equation by $3$, the change of variable $x = e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2} + 3\frac{dw}{dz} + 3w = 0\,.$$

The associated *characteristic equation* is $r^2 + 3\,r + 3 = 0$, with complex roots $r = -\left(3 \pm i\sqrt{3}\right)/2\,.$ Hence the general solution is

$$w(z) = e^{-3z/2}\left[c_1cos\left(\sqrt{3}\;z/2\right) + c_2sin\left(\sqrt{3}\;z/2\right)\right],$$

and therefore

$$y(x) = x^{-3/2}\left[c_1cos\left(\frac{\sqrt{3}}{2}\,ln\,x\right) + c_2sin\left(\frac{\sqrt{3}}{2}\,ln\,x\right)\right].$$

30. Let $x < 0$. Setting $y = (-x)^r$, successive differentiation gives $y' = -r(-x)^{r-1}$ and $y'' = r(r-1)(-x)^{r-2}$. It follows that

$$L[(-x)^r] = r(r-1)x^2(-x)^{r-2} - \alpha\,r\,x(-x)^{r-1} + \beta(-x)^r.$$

Since $x^2 = (-x)^2$, we find that

$$
\begin{aligned}
L[(-x)^r] &= r(r-1)(-x)^r + \alpha\, r(-x)^r + \beta(-x)^r \\
&= (-x)^r[r(r-1) + \alpha\, r + \beta].
\end{aligned}
$$

Given that $r_1$ and $r_2$ are roots of $F(r) = r(r-1) + \alpha\, r + \beta$, we have $L[(-x)^{r_i}] = 0$. Therefore $y_1 = (-x)^{r_1}$ and $y_2 = (-x)^{r_2}$ are *linearly independent* solutions of the differential equation, $L[y] = 0$, for $x < 0$, as long as $r_1 \neq r_2$.

**Section 5.6**

1. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{1}{2x} = \frac{1}{2}\,.$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{2} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. Let

$$y = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots\right) = \sum_{n=0}^{\infty} a_n\, x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n-1} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0\,.$$

That is,

$$2\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2}\, x^{r+n} = 0\,.$$

It follows that

$$a_0[2r(r-1) + r]x^r + a_1[2(r+1)r + r + 1]x^{r+1} + \\ + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-2}]x^{r+n} = 0\,.$$

Assuming that $a_0 \neq 0$, we obtain the *indicial equation* $2r^2 - r = 0$, with roots $r_1 = 1/2$

and $r_2 = 0$. It immediately follows that $a_1 = 0$. Setting the remaining coefficients equal
to *zero*, we have

$$a_n = \frac{-\,a_{n-2}}{(r+n)[2(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

For $r = 1/2$, the recurrence relation becomes

$$a_n = \frac{-\,a_{n-2}}{n(1+2n)}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*. Furthermore, for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(1+4k)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)} = \frac{(-1)^k a_0}{2^k\,k!\,5\cdot 9\cdot 13\cdots(4k+1)}\,.$$

For $r = 0$, the recurrence relation becomes

$$a_n = \frac{-\,a_{n-2}}{n(2n-1)}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*, and for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(4k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)} = \frac{(-1)^k a_0}{2^k\,k!\,3\cdot 7\cdot 11\cdots(4k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x) = \sqrt{x}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k\,x^{2k}}{2^k\,k!\,5\cdot 9\cdot 13\cdots(4k+1)}\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\,x^{2k}}{2^k\,k!\,3\cdot 7\cdot 11\cdots(4k-1)}\,.$$

3. Note that $x\,p(x) = 0$ and $x^2 q(x) = x$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\,x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,,$$

and after multiplying both sides of the equation by $x$,

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0\,.$$

It follows that

$$a_0[r(r-1)]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + a_{n-1}]x^{r+n} = 0\,.$$

Setting the coefficients equal to *zero*, the *indicial equation* is $r(r-1)=0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-\,a_{n-1}}{(r+n)(r+n-1)}\,,\quad n = 1, 2, \cdots.$$

For $r = 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)}\,,\quad n = 1, 2, \cdots.$$

Hence for $n \geq 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{n!(n+1)!}\,.$$

Therefore one solution is

$$y_1(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n\,x^n}{n!(n+1)!}\,.$$

5. Here $x\,p(x) = 2/3$ and $x^2q(x) = x^2/3$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$3\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+2} = 0\,.$$

It follows that

$$a_0[3r(r-1)+2r]x^r + a_1[3(r+1)r + 2(r+1)]x^{r+1} +$$
$$+\sum_{n=2}^{\infty}[3(r+n)(r+n-1)a_n + 2(r+n)a_n + a_{n-2}]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $3r^2 - r = 0$, with roots $r_1 = 1/3$, $r_2 = 0$. Setting the remaining coefficients equal to *zero*, we have $a_1 = 0$, and

$$a_n = \frac{-\,a_{n-2}}{(r+n)[3(r+n)-1]}\,,\quad n = 2, 3, \cdots.$$

It immediately follows that the *odd* coefficients are equal to *zero*. For $r = 1/3$,

$$a_n = \frac{-\,a_{n-2}}{n(1+3n)}\,,\quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(6k+1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-5)(6k+1)} = \frac{(-1)^k a_0}{2^k\, k!\, 7\cdot 13\cdots(6k+1)}\,.$$

For $r = 0$,

$$a_n = \frac{-\,a_{n-2}}{n(3n-1)}\,,\quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(6k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-7)(6k-1)} = \frac{(-1)^k a_0}{2^k\, k!\, 5\cdot 11\cdots(6k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x) = x^{1/3}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\, 7\cdot 13\cdots(6k+1)}\left(\frac{x^2}{2}\right)^k\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\, 5\cdot 11\cdots(6k-1)}\left(\frac{x^2}{2}\right)^k.$$

6. Note that $x\,p(x) = 1$ and $x^2 q(x) = x - 2$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} - 2\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1) + r - 2]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - 2\,a_n + a_{n-1}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{-\,a_{n-1}}{(r+n)^2 - 2}\,,\quad n = 1, 2, \cdots.$$

First note that $(r+n)^2 - 2 = \left(r + n + \sqrt{2}\right)\left(r + n - \sqrt{2}\right)$. So for $r = \sqrt{2}$,

$$a_n = \frac{-\,a_{n-1}}{n\left(n + 2\sqrt{2}\right)}\,,\quad n = 1, 2, \cdots.$$

It follows that

$$a_n = \frac{(-1)^n a_0}{n!\left(1+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\cdots\left(n+2\sqrt{2}\right)}\,, \quad n = 1, 2, \cdots.$$

For $r = -\sqrt{2}$ ,

$$a_n = \frac{-a_{n-1}}{n\left(n-2\sqrt{2}\right)}\,, \quad n = 1, 2, \cdots,$$

and therefore

$$a_n = \frac{(-1)^n a_0}{n!\left(1-2\sqrt{2}\right)\left(2-2\sqrt{2}\right)\cdots\left(n-2\sqrt{2}\right)}\,, \quad n = 1, 2, \cdots.$$

The two linearly independent solutions are

$$y_1(x) = x^{\sqrt{2}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!\left(1+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\cdots\left(n+2\sqrt{2}\right)}\right]$$

$$y_2(x) = x^{-\sqrt{2}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!\left(1-2\sqrt{2}\right)\left(2-2\sqrt{2}\right)\cdots\left(n-2\sqrt{2}\right)}\right].$$

7. Here $x\,p(x) = 1-x$ and $x^2 q(x) = -x$ , which are *both* analytic at $x = 0$ . Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n-1} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n-1} - \\ -\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After multiplying both sides by $x$ ,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} - \\ -\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n+1} - \sum_{n=0}^{\infty}a_n x^{r+n+1} = 0.$$

After adjusting the indices in the *last two* series, we obtain

$$a_0[r(r-1)+r]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - (r+n)a_{n-1}]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 = 0$, with roots $r_1 = r_2 = 0$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{a_{n-1}}{r+n}\,, \quad n = 1, 2, \cdots.$$

With $r = 0$,

$$a_n = \frac{a_{n-1}}{n}\,, \quad n = 1, 2, \cdots.$$

Hence one solution is

$$y_1(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x.$$

8. Note that $x\,p(x) = 3/2$ and $x^2 q(x) = x^2 - 1/2$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + 3\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \\ + 2\sum_{n=0}^{\infty}a_n x^{r+n+2} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[2r(r-1)+3r-1]x^r + a_1[2(r+1)r + 3(r+1) - 1] + \\ + \sum_{n=2}^{\infty}[2(r+n)(r+n-1)a_n + 3(r+n)a_n - a_n + 2\,a_{n-2}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $2r^2 + r - 1 = 0$, with roots $r_1 = 1/2$ and $r_2 = -1$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{-\,2a_{n-2}}{(r+n+1)[2(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

Setting the remaining coefficients equal to *zero*, we have $a_1 = 0$, which implies that all of the *odd* coefficients are *zero*. With $r = 1/2$,

$$a_n = \frac{-\,2a_{n-2}}{n(2n+3)}\,, \quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{k(4k+3)} = \frac{a_{2k-4}}{(k-1)k(4k-5)(4k+3)} = \frac{(\,-1)^k a_0}{k!\,7\cdot 11\cdots(4k+3)}\ .$$

With $r = \,-1$,

$$a_n = \frac{-\,2a_{n-2}}{n(2n-3)}\,,\quad n = 2,3,\cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{k(4k-3)} = \frac{a_{2k-4}}{(k-1)k(4k-11)(4k-3)} = \frac{(\,-1)^k a_0}{k!\,5\cdot 9\cdots(4k-3)}\ .$$

The two linearly independent solutions are

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(\,-1)^n\, x^{2n}}{n!\,7\cdot 11\cdots(4n+3)}\right]$$

$$y_2(x) = x^{-1}\left[1 + \sum_{n=1}^{\infty} \frac{(\,-1)^n\, x^{2n}}{n!\,5\cdot 9\cdots(4n-3)}\right].$$

9. Note that $x\,p(x) = \,-x-3$ and $x^2 q(x) = x+3$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n+1} - 3\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} + 3\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1) - 3r + 3]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n - (r+n-2)a_{n-1} - 3(r+n-1)a_n]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - 4r + 3 = 0$, with roots $r_1 = 3$ and $r_2 = 1$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{(r+n-2)a_{n-1}}{(r+n-1)(r+n-3)}\,,\quad n = 1,2,\cdots.$$

With $r = 3$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)}\,, \quad n = 1, 2, \cdots.$$

It follows that for $n \geq 1$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)} = \frac{a_{n-2}}{(n-1)(n+2)} = \cdots = \frac{2\,a_0}{n!\,(n+2)}\,.$$

Therefore one solution is

$$y_1(x) = x^3\left[1 + \sum_{n=1}^{\infty} \frac{2\,x^n}{n!\,(n+2)}\right].$$

10. Here $x\,p(x) = 0$ and $x^2 q(x) = x^2 + 1/4$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second* series, we obtain

$$a_0\left[r(r-1) + \frac{1}{4}\right]x^r + a_1\left[(r+1)r + \frac{1}{4}\right]x^{r+1} + \\ + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + \frac{1}{4}a_n + a_{n-2}\right]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - r + \frac{1}{4} = 0$, with roots $r_1 = r_2 = 1/2$. Setting the remaining coefficients equal to *zero*, we find that $a_1 = 0$. The recurrence relation is

$$a_n = \frac{-\,4a_{n-2}}{(2r+2n-1)^2}\,, \quad n = 2, 3, \cdots.$$

With $r = 1/2$,

$$a_n = \frac{-\,a_{n-2}}{n^2}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*. So for $k \geq 1$,

$$a_{2k} = \frac{-\,a_{2k-2}}{4k^2} = \frac{a_{2k-4}}{4^2(k-1)^2 k^2} = \cdots = \frac{(-1)^k a_0}{4^k (k!)^2}\,.$$

Therefore one solution is

$$y_1(x) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right].$$

$12(a)$. Dividing through by the leading coefficient, the ODE can be written as

$$y'' - \frac{x}{1-x^2}\, y' + \frac{\alpha^2}{1-x^2}\, y = 0\,.$$

For $x = 1$,

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1} \frac{\alpha^2(1-x)}{x+1} = 0\,.$$

For $x = -1$,

$$p_0 = \lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1} \frac{x}{x-1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1} \frac{\alpha^2(x+1)}{(1-x)} = 0\,.$$

Hence both $x = -1$ and $x = 1$ are *regular* singular points. As shown in Example **1**, the indicial equation is given by

$$r(r-1) + p_0 r + q_0 = 0\,.$$

In this case, *both* sets of roots are $r_1 = 1/2$ and $r_2 = 0$.

$(b)$. Let $t = x - 1$, and $u(t) = y(t+1)$. Under this change of variable, the differential equation becomes

$$\left(t^2 + 2t\right)u'' + (t+1)u' - \alpha^2 u = 0\,.$$

Based on Part $(a)$, $t = 0$ is a *regular* singular point. Set $u = \sum_{n=0}^{\infty} a_n\, t^{r+n}$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n-1} +$$
$$+ \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty} a_n t^{r+n} = 0\,.$$

Upon inspection, we can also write

$$\sum_{n=0}^{\infty}(r+n)^2 a_n\,t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)\left(r+n-\frac{1}{2}\right)a_n\,t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty}a_n t^{r+n} = 0.$$

After adjusting the indices in the *second* series, it follows that

$$a_0\left[2r\left(r-\frac{1}{2}\right)\right]t^{r-1} + \sum_{n=0}^{\infty}\left[(r+n)^2 a_n + 2(r+n+1)\left(r+n+\frac{1}{2}\right)a_{n+1} - \alpha^2 a_n\right]t^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, the *indicial equation* is $2r^2 - r = 0$, with roots $r = 0\,,\,1/2$. The recurrence relation is

$$(r+n)^2 a_n + 2(r+n+1)\left(r+n+\frac{1}{2}\right)a_{n+1} - \alpha^2 a_n = 0\,,\quad n = 0,1,2,\cdots.$$

With $r_1 = 1/2$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{4\alpha^2-(2n-1)^2}{4n(2n+1)}\,a_{n-1} \\ &= (-1)^n\frac{[1-4\alpha^2][9-4\alpha^2]\cdots\left[(2n-1)^2-4\alpha^2\right]}{2^n(2n+1)!}\,a_0\,. \end{aligned}$$

With $r_2 = 0$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{\alpha^2-(n-1)^2}{n(2n-1)}\,a_{n-1} \\ &= (-1)^n\frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2]\cdots\left[(n-1)^2-\alpha^2\right]}{n!\cdot 3\cdot 5\cdots(2n-1)}\,a_0\,. \end{aligned}$$

The two linearly independent solutions of the *Chebyshev equation* are

$$y_1(x) = |x-1|^{1/2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{[1-4\alpha^2][9-4\alpha^2]\cdots\left[(2n-1)^2-4\alpha^2\right]}{2^n(2n+1)!}(x-1)^n\right]$$

$$y_2(x) = 1+\sum_{n=1}^{\infty}(-1)^n\frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2]\cdots\left[(n-1)^2-\alpha^2\right]}{n!\cdot 3\cdot 5\cdots(2n-1)}(x-1)^n\,.$$

13. Here $x\,p(x) = 1-x$ and $x^2q(x) = \lambda\,x$, which are *both* analytic at $x=0$. In fact,

$$p_0 = \lim_{x\to 0} x\,p(x) = 1 \text{ and } q_0 = \lim_{x\to 0} x^2 q(x) = 0\,.$$

Hence the *indicial equation* is $r(r-1)+r=0$, with roots $r_{1,2}=0$. Set

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

Substitution into the ODE results in

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=1}^{\infty} na_n\, x^{n-1} - \\ - \sum_{n=0}^{\infty} na_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}\, x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}\, x^n - \\ - \sum_{n=1}^{\infty} na_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

It follows that

$$a_1 + \lambda\, a_0 + \sum_{n=1}^{\infty} \big[(n+1)^2 a_{n+1} - (n-\lambda)a_n\big] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we find that $a_1 = -\lambda a_0$, and

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1}\,, \quad n = 2, 3, \cdots.$$

That is, for $n \geq 2$,

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1} = \cdots = \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, a_0\,.$$

Therefore one solution of the *Laguerre equation* is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, x^n.$$

Note that if $\lambda = m$, a *positive integer*, then $a_n = 0$ for $n \geq m+1$. In that case, the solution is a *polynomial*

$$y_1(x) = 1 + \sum_{n=1}^{m} \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, x^n\,.$$

**Section 5.7**

2. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = -2 - x$ and $x^2 q(x) = 2 + x^2$. It follows that

$$p_0 = \lim_{x \to 0}(-2 - x) = -2$$
$$q_0 = \lim_{x \to 0}\left(2 + x^2\right) = 2$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - 2r + 2 = 0\,,$$

that is, $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$.

4. The coefficients $P(x)$, $Q(x)$, and $R(x)$ are analytic for all $x \in \mathbb{R}$. Hence there are *no* singular points.

5. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = 3\frac{sin\,x}{x}$ and $x^2 q(x) = -2$. It follows that

$$p_0 = \lim_{x \to 0} 3\,\frac{sin\,x}{x} = 3$$
$$q_0 = \lim_{x \to 0} -2 = -2$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + 3r - 2 = 0\,,$$

that is, $r^2 + 2r - 2 = 0$, with roots $r_1 = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$.

6. $P(x) = 0$ for $x = 0$ and $x = -2$. We note that $p(x) = x^{-1}(x+2)^{-1}/2$, and $q(x) = -(x+2)^{-1}/2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \to 0} \frac{1}{2(x+2)} = \frac{1}{4}$$
$$q_0 = \lim_{x \to 0} \frac{-x^2}{2(x+2)} = 0$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{3}{4}r = 0$, with roots $r_1 = \frac{3}{4}$ and $r_2 = 0$. For the singularity at $x = -2$,

$$p_0 = \lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$$
$$q_0 = \lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{-(x+2)}{2} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

7. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = \frac{1}{2} + \frac{\sin x}{2x}$ and $x^2 q(x) = 1$. It follows that

$$p_0 = \lim_{x \to 0} xp(x) = 1$$
$$q_0 = \lim_{x \to 0} x^2 q(x) = 1$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + r + 1 = 0\,,$$

that is, $r^2 + 1 = 0$, with *complex conjugate* roots $r = \pm\, i$.

8. Note that $P(x) = 0$ only for $x = -1$. We find that $p(x) = 3(x-1)/(x+1)$, and $q(x) = 3/(x+1)^2$. It follows that

$$p_0 = \lim_{x \to -1} (x+1)p(x) = \lim_{x \to -1} 3(x-1) = -6$$
$$q_0 = \lim_{x \to -1} (x+1)^2 q(x) = \lim_{x \to -1} 3 = 3$$

and therefore $x = -1$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - 6r + 3 = 0\,,$$

that is, $r^2 - 7r + 3 = 0$, with roots $r_1 = \left(7 + \sqrt{37}\right)/2$ and $r_2 = \left(7 - \sqrt{37}\right)/2$.

10. $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x(x-2)^{-2}(x+2)^{-1}$, and $q(x) = 3(x-2)^{-1}(x+2)^{-1}$. For the singularity at $x = 2$,

$$\lim_{x \to 2} (x-2)p(x) = \lim_{x \to 2} \frac{2x}{x^2 - 4}\,,$$

which is *undefined*. Therefore $x = 0$ is an *irregular* singular point. For the singularity at $x = -2$,

$$p_0 = \lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{2x}{(x-2)^2} = -\frac{1}{4}$$
$$q_0 = \lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{3(x+2)}{x-2} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

11. $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x/(4 - x^2)$, and $q(x) = 3/(4 - x^2)$. For the singularity at $x = 2$,

$$p_0 = \lim_{x \to 2} (x-2)p(x) = \lim_{x \to 2} \frac{-2x}{x+2} = -1$$
$$q_0 = \lim_{x \to 2} (x-2)^2 q(x) = \lim_{x \to 2} \frac{3(2-x)}{x+2} = 0$$

and therefore $x = 2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - r = 0\,,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$. For the singularity at $x = -2$,

$$p_0 = \lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{2x}{2-x} = -1$$
$$q_0 = \lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{3(x+2)}{2-x} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - r = 0\,,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

12. $P(x) = 0$ for $x = 0$ and $x = -3$. We note that $p(x) = -2x^{-1}(x+3)^{-1}$, and $q(x) = -1/(x+3)^2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \to 0} x\,p(x) = \lim_{x \to 0} \frac{-2}{x+3} = -\frac{2}{3}$$
$$q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} \frac{-x^2}{(x+3)^2} = 0$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1)-\frac{2}{3}r=0\,,$$

that is, $r^2-\frac{5}{3}r=0$, with roots $r_1=\frac{5}{3}$ and $r_2=0$. For the singularity at $x=-3$,

$$p_0=\lim_{x\to-3}\,(x+3)p(x)=\lim_{x\to-3}\,\frac{-2}{x}=\frac{2}{3}$$
$$q_0=\lim_{x\to-3}\,(x+3)^2q(x)=\lim_{x\to-3}\,(-1)=-1$$

and therefore $x=-3$ is a *regular* singular point. The indicial equation is given by

$$r(r-1)+\frac{2}{3}r-1=0\,,$$

that is, $r^2-\frac{1}{3}r-1=0$, with roots $r_1=\left(1+\sqrt{37}\right)/6$ and $r_2=\left(1-\sqrt{37}\right)/6$.

$13(a)$. Note the $p(x)=1/x$ and $q(x)=-1/x$. Furthermore, $x\,p(x)=1$ and $x^2q(x)=-x$. It follows that

$$p_0=\lim_{x\to0}(1)=1$$
$$q_0=\lim_{x\to0}(-x)=0$$

and therefore $x=0$ is a *regular* singular point.

$(b)$. The indicial equation is given by

$$r(r-1)+r=0\,,$$

that is, $r^2=0$, with roots $r_1=r_2=0$.

$(c)$. Let $y=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^{n+1}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0\,.$$

After adjusting the indices in the *first* series, we obtain

$$a_1-a_0+\sum_{n=1}^{\infty}[n(n+1)a_{n+1}+(n+1)a_{n+1}-a_n]x^n=0\,.$$

Setting the coefficients equal to *zero*, it follows that for $n\geq0$,

$$a_{n+1}=\frac{a_n}{(n+1)^2}\,.$$

So for $n\geq1$,

$$a_n = \frac{a_{n-1}}{n^2} = \frac{a_{n-2}}{n^2(n-1)^2} = \cdots = \frac{1}{(n!)^2}\, a_0\,.$$

With $a_0 = 1$, one solution is

$$y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \cdots + \frac{1}{(n!)^2}\, x^n + \cdots.$$

For a second solution, set $y_2(x) = y_1(x)\, ln\, x \, + \, b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$. Substituting into the ODE, we obtain

$$L[y_1(x)] \cdot ln\, x + 2\, y_1'(x) + L\left[\sum_{n=1}^{\infty} b_n\, x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[\sum_{n=1}^{\infty} b_n\, x^n\right] = \, - 2\, y_1'(x)\,.$$

More specifically,

$$b_1 + \sum_{n=1}^{\infty} [n(n+1)b_{n+1} + (n+1)b_{n+1} - b_n]x^n =$$
$$= \, - 2 - x - \frac{1}{6}x^2 - \frac{1}{72}x^3 - \frac{1}{1440}x^4 - \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} b_1 &= \, -2 \\ 4b_2 - b_1 &= \, -1 \\ 9b_3 - b_2 &= \, -1/6 \\ 16b_4 - b_3 &= \, -1/72 \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $b_1 = \, -2,\; b_2 = \, -3/4,\; b_3 = \, -11/108,$ $b_4 = \, -25/3456\,, \cdots$. Therefore a *second* solution is

$$y_2(x) = y_1(x)\, ln\, x + \left[\, -2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \cdots\right].$$

14$(a)$. Here $x\, p(x) = 2x$ and $x^2 q(x) = 6\, xe^x$. Both of these functions are *analytic* at $x = 0$, therefore $x = 0$ is a *regular* singular point. Note that $p_0 = q_0 = 0$.

$(b)$. The indicial equation is given by

$$r(r-1)=0\,,$$

that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

$(c)$. In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+1}\,x^{n+1} + 2\sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + 6\,e^x\sum_{n=0}^{\infty}a_n x^{n+1} = 0\,.$$

After adjusting the indices in the *first* two series, and expanding the *exponential* function,

$$\sum_{n=1}^{\infty}n(n+1)a_n\,x^n + 2\sum_{n=1}^{\infty}n\,a_{n-1}x^n + 6\,a_0 x + (6a_0+6a_1)x^2 + \\ + (6a_2+6a_1+3a_0)x^3 + (6a_3+6a_2+3a_1+a_0)x^4 + \cdots = 0\,.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 2a_1 + 2a_0 + 6a_0 &= 0 \\ 6a_2 + 4a_1 + 6a_0 + 6a_1 &= 0 \\ 12a_3 + 6a_2 + 6a_2 + 6a_1 + 3a_0 &= 0 \\ 20a_4 + 8a_3 + 6a_3 + 6a_2 + 3a_1 + a_0 &= 0 \\ &\vdots \end{aligned}$$

Setting $a_0 = 1$, solution of the system results in $a_1 = -4$, $a_2 = 17/3$, $a_3 = -47/12$, $a_4 = 191/120, \cdots$. Therefore one solution is

$$y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \cdots.$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x) = a\,y_1(x)\,ln\,x\; + 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

Substituting into the ODE, we obtain

$$a\,L[y_1(x)]\cdot ln\,x + 2a\,y_1'(x) + 2a\,y_1(x) - a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty}c_n\,x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[1 + \sum_{n=1}^{\infty}c_n\,x^n\right] = -2a\,y_1'(x) - 2a\,y_1(x) + a\frac{y_1(x)}{x}\,.$$

More specifically,

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + 2\sum_{n=1}^{\infty} n\,c_n x^n + 6\, + (6+6c_1)x +$$
$$+ (6c_2+6c_1+3)x^2 + \cdots = -a + 10ax - \frac{61}{3}ax^2 + \frac{193}{12}ax^3 + \cdots .$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 6 &= -a \\ 2c_2 + 8c_1 + 6 &= 10a \\ 6c_3 + 10c_2 + 6c_1 + 3 &= -\frac{61}{3}a \\ 12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 &= \frac{193}{12}a \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $a = -6$. In order to solve the remaining equations, set $c_1 = 0$. Then $c_2 = -33$, $c_3 = 449/6$, $c_4 = -1595/24, \cdots$. Therefore a *second* solution is

$$y_2(x) = -6\,y_1(x)\,ln\,x + \left[1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \cdots\right].$$

15$(a)$. Note the $p(x) = 6x/(x-1)$ and $q(x) = 3x^{-1}(x-1)^{-1}$. Furthermore, $x\,p(x) = 6x^2/(x-1)$ and $x^2q(x) = 3x/(x-1)$. It follows that

$$p_0 = \lim_{x\to 0} \frac{6x^2}{x-1} = 0$$
$$q_0 = \lim_{x\to 0} \frac{3x}{x-1} = 0$$

and therefore $x = 0$ is a *regular* singular point.

$(b)$. The indicial equation is given by

$$r(r-1) = 0\,,$$

that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

$(c)$. In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n\, x^{n+1} - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \\ + 6\sum_{n=0}^{\infty}(n+1)a_n x^{n+2} + 3\sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

After adjusting the indices, it follows that

$$\sum_{n=2}^{\infty} n(n-1)a_{n-1}\, x^n - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \\ + 6\sum_{n=2}^{\infty}(n-1)a_{n-2}x^n + 3\sum_{n=1}^{\infty} a_{n-1}x^n = 0\,.$$

That is,

$$-2a_1 + 3a_0 + \sum_{n=2}^{\infty}[-n(n+1)a_n + (n^2-n+3)a_{n-1} + 6(n-1)a_{n-2}]x^n = 0.$$

Setting the coefficients equal to *zero*, we have $a_1 = 3a_0/2$, and for $n \geq 2$,

$$n(n+1)a_n = \big(n^2-n+3\big)a_{n-1} + 6(n-1)a_{n-2}\,.$$

If we assign $a_0 = 1$, then we obtain $a_1 = 3/2$, $a_2 = 9/4$, $a_3 = 51/16$, $\cdots$. Hence one solution is

$$y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \frac{111}{40}x^5 + \cdots.$$

The exponents differ by an *integer*. So for a second solution, set

$$y_2(x) = a\,y_1(x)\,ln\,x + 1 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots.$$

Substituting into the ODE, we obtain

$$2ax\,y_1'(x) - 2a\,y_1'(x) + 6ax\,y_1(x) - a\,y_1(x) + a\frac{y_1(x)}{x} + L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right] = 0,$$

since $L[y_1(x)] = 0$. It follows that

$$L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right] = 2a\,y_1'(x) - 2ax\,y_1'(x) + a\,y_1(x) - 6ax\,y_1(x) - a\frac{y_1(x)}{x}\,.$$

Now

$$L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right] = 3 + (-2c_2+3c_1)x + (-6c_3+5c_2+6c_1)x^2 + \\ + (-12c_4+9c_3+12c_2)x^3 + (-20c_5+15c_4+18c_3)x^4 + \cdots.$$

Substituting for $y_1(x)$, the *right hand side* of the ODE is

$$a + \frac{7}{2}ax + \frac{3}{4}ax^2 + \frac{33}{16}ax^3 - \frac{867}{80}ax^4 - \frac{441}{10}ax^5 + \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 3 &= a \\ -2c_2 + 3c_1 &= \frac{7}{2}a \\ -6c_3 + 5c_2 + 6c_1 &= \frac{3}{4}a \\ -12c_4 + 9c_3 + 12c_2 &= \frac{33}{16}a \\ &\vdots \end{aligned}$$

We find that $a = 3$. In order to solve the second equation, set $c_1 = 0$. Solution of the remaining equations results in $c_2 = -21/4$, $c_3 = -19/4$, $c_4 = -597/64, \cdots$. Hence a second solution is

$$y_2(x) = 3\,y_1(x)\,ln\,x + \left[1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 - \frac{597}{64}x^4 + \cdots\right].$$

$16(a)$. After multiplying both sides of the ODE by $x$, we find that $x\,p(x) = 0$ and $x^2 q(x) = x$. Both of these functions are *analytic* at $x = 0$, hence $x = 0$ is a *regular* singular point.

$(b)$. Furthermore, $p_0 = q_0 = 0$. So the indicial equation is $r(r-1) = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

$(c)$. In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n\,x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

That is,

$$\sum_{n=1}^{\infty} \left[n(n+1)a_n + a_{n-1}\right] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we find that for $n \geq 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)}\,.$$

It follows that

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{(n!)^2(n+1)}\,.$$

Hence one solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \cdots.$$

The exponents differ by an *integer*. So for a second solution, set

$$y_2(x) = a\,y_1(x)\,ln\,x\, + 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

Substituting into the ODE, we obtain

$$a\,L[y_1(x)]\cdot ln\,x + 2a\,y_1'(x) - a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = -2a\,y_1'(x) + a\frac{y_1(x)}{x}\,.$$

Now

$$L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = 1 + (2c_2 + c_1)x + (6c_3 + c_2)x^2 + (12c_4 + c_3)x^3 + \\ + (20c_5 + c_4)x^4 + (30c_6 + c_5)x^5 + \cdots.$$

Substituting for $y_1(x)$, the *right hand side* of the ODE is

$$-a + \frac{3}{2}ax - \frac{5}{12}ax^2 + \frac{7}{144}ax^3 - \frac{1}{320}ax^4 + \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 1 &= -a \\ 2c_2 + c_1 &= \frac{3}{2}a \\ 6c_3 + c_2 &= -\frac{5}{12}a \\ 12c_4 + c_3 &= \frac{7}{144}a \\ &\vdots \end{aligned}$$

Evidently, $a = -1$. In order to solve the *second* equation, set $c_1 = 0$. We then find that $c_2 = -3/4$, $c_3 = 7/36$, $c_4 = -35/1728, \cdots$. Therefore a second solution is

$$y_2(x) = -y_1(x)\,ln\,x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \cdots\right].$$

19$(a)$. After dividing by the leading coefficient, we find that

$$p_0 = \lim_{x\to 0} x\, p(x) = \lim_{x\to 0} \frac{\gamma - (1+\alpha+\beta)x}{1-x} = \gamma\,.$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-\,\alpha\beta\, x}{1-x} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. The indicial equation is $r(r-1) + \gamma\, r = 0$, with roots $r_1 = 1-\gamma$ and $r_2 = 0$.

$(b)$. For $x = 1$,

$$p_0 = \lim_{x\to 1} (x-1)p(x) = \lim_{x\to 1} \frac{-\,\gamma + (1+\alpha+\beta)x}{x} = 1-\gamma+\alpha+\beta\,.$$

$$q_0 = \lim_{x\to 1} (x-1)^2 q(x) = \lim_{x\to 1} \frac{\alpha\beta(x-1)}{x} = 0\,.$$

Hence $x = 1$ is a *regular* singular point. The indicial equation is

$$r^2 - (\gamma-\alpha-\beta)\, r = 0\,,$$

with roots $r_1 = \gamma-\alpha-\beta$ and $r_2 = 0$.

$(c)$. Given that $r_1 - r_2$ is not a positive integer, we can set $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution into the ODE results in

$$x(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + [\gamma - (1+\alpha+\beta)x]\sum_{n=1}^{\infty} n\, a_n x^{n-1} - \alpha\beta\sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \gamma\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \\ -\,(1+\alpha+\beta)\sum_{n=1}^{\infty} n\, a_n x^n - \alpha\beta\sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we obtain

$$\gamma\, a_1 - \alpha\beta\, a_0 + [(2+2\gamma)a_2 - (1+\alpha+\beta+\alpha\beta)a_1]x + \sum_{n=2}^{\infty} A_n\, x^n = 0\,,$$

in which

$$A_n = (n+1)(n+\gamma)a_{n+1} - [n(n-1) + (1+\alpha+\beta)n + \alpha\beta]a_n \,.$$

Note that $n(n-1) + (1+\alpha+\beta)n + \alpha\beta = (n+\alpha)(n+\beta)$. Setting the coefficients equal to *zero*, we have $\gamma\, a_1 - \alpha\beta\, a_0 = 0$, and

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)}\, a_n$$

for $n \geq 1$. Hence one solution is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma\cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 2!}x^2 + \\ + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 3!}x^3 + \cdots .$$

Since the nearest other singularity is at $x = 1$, the radius of convergence of $y_1(x)$ will be *at least* $\rho = 1$.

$(d)$. Given that $r_1 - r_2$ is not a positive integer, we can set $y = x^{1-\gamma}\sum_{n=0}^{\infty} b_n x^n$. Then Substitution into the ODE results in

$$x(1-x)\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma-1} + \\ + [\gamma - (1+\alpha+\beta)x]\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0.$$

That is,

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma} + \\ + \gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n+1-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0.$$

After adjusting the indices,

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=1}^{\infty}(n-\gamma)(n-1-\gamma)a_{n-1} x^{n-\gamma} + \\ + \gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=1}^{\infty}(n-\gamma)a_{n-1} x^{n-\gamma} - \alpha\beta\sum_{n=1}^{\infty}a_{n-1} x^{n-\gamma} = 0.$$

Combining the series, we obtain

$$\sum_{n=1}^{\infty}B_n\, x^{n-\gamma} = 0\,,$$

in which

$$B_n = n(n+1-\gamma)b_n - [(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta]b_{n-1}.$$

Note that $(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta = (n+\alpha-\gamma)(n+\beta-\gamma)$. Setting $B_n = 0$, it follows that for $n \geq 1$,

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)}\, b_{n-1}.$$

Therefore a second solution is

$$y_2(x) = x^{1-\gamma}\left[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma)1!}x + \right.$$
$$\left. + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma)2\,!}x^2 + \cdots\right].$$

$(e)$. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4\frac{1}{\xi}\left(1-\frac{1}{\xi}\right)\frac{d^2y}{d\xi^2} + \left\{2\xi^3\frac{1}{\xi}\left(1-\frac{1}{\xi}\right) - \xi^2\left[\gamma-(1+\alpha+\beta)\frac{1}{\xi}\right]\right\}\frac{dy}{d\xi} - \alpha\beta\, y = 0\,.$$

That is,

$$\left(\xi^3-\xi^2\right)\frac{d^2y}{d\xi^2} + \left[2\xi^2 - \gamma\,\xi^2 + (\,-1+\alpha+\beta)\xi\right]\frac{dy}{d\xi} - \alpha\beta\; y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{(2-\gamma)\,\xi + (\,-1+\alpha+\beta)}{\xi^2-\xi} \text{ and } q(\xi) = \frac{-\,\alpha\beta}{\xi^3-\xi^2}\,.$$

It follows that

$$p_0 = \lim_{\xi\to 0}\xi\, p(\xi) = \lim_{\xi\to 0}\frac{(2-\gamma)\,\xi + (\,-1+\alpha+\beta)}{\xi-1} = 1-\alpha-\beta\,,$$

$$q_0 = \lim_{\xi\to 0}\xi^2 q(\xi) = \lim_{\xi\to 0}\frac{-\,\alpha\beta}{\xi-1} = \alpha\beta\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a *regular* singular point. The indicial equation is

$$r(r-1) + (1-\alpha-\beta)r + \alpha\beta = 0\,,$$

or $r^2 - (\alpha+\beta)r + \alpha\beta = 0$. Evidently, the roots are $r = \alpha$ and $r = \beta$.

21$(a)$. Note that

$$p(x) = \frac{\alpha}{x^s} \text{ and } q(\xi) = \frac{\beta}{x^t}\,.$$

It follows that

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} \alpha\, x^{1-s},$$

$$\lim_{\xi\to 0} \xi^2 q(\xi) = \lim_{\xi\to 0} \beta\, x^{2-s}.$$

Hence if $s > 1$ or $t > 2$, one or both of the limits does not exist. Therefore $x = 0$ is an *irregular* singular point.

$(c)$. Let $y = a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots$. Write the ODE as

$$x^3 y'' + \alpha\, x^2 y' + \beta\, y = 0\,.$$

Substitution of the assumed solution results in

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Adjusting the indices, we obtain

$$\sum_{n=1}^{\infty}(n-1+r)(n+r-2)a_{n-1}x^{n+r} + \alpha\sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Combining the series,

$$\beta\, a_0 + \sum_{n=1}^{\infty} A_n\, x^{n+r} = 0\,,$$

in which $A_n = \beta\, a_n + (n-1+r)(n+r+\alpha-2)a_{n-1}$. Setting the coefficients equal to *zero*, we have $a_0 = 0$. But for $n \geq 1$,

$$a_n = \frac{(n-1+r)(n+r+\alpha-2)}{\beta}\, a_{n-1}.$$

Therefore, regardless of the value of $r$, it follows that $a_n = 0$, for $n = 1, 2, \cdots$.

**Section 5.8**

3. Here $x\,p(x) = 1$ and $x^2 q(x) = 2x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}a_n x^{r+n+1} = 0\,.$$

After adjusting the indices in the *last* series, we obtain

$$a_0[r(r-1)+r]x^r + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n + 2\,a_{n-1}\right]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 = 0$, with *double* root $r = 0$. Setting the remaining coefficients equal to *zero*, we have for $n \geq 1$,

$$a_n(r) = -\frac{2}{(n+r)^2}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r) = \frac{(-1)^n\,2^n}{[(n+r)(n+r-1)\cdots(1+r)]^2}\,a_0\,,\quad n \geq 1\,.$$

Since $r = 0$, one solution is given by

$$y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\,2^n}{(n!)^2}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.7 . First
note that

$$\frac{a_n'(r)}{a_n(r)} = -2\left[\frac{1}{n+r} + \frac{1}{n+r-1} + \cdots + \frac{1}{1+r}\right].$$

Setting $r = 0$,

$$a_n'(0) = -2\,H_n\,a_n(0) = -2\,H_n\frac{(-1)^n\,2^n}{(n!)^2}\,.$$

Therefore,

$$y_2(x) = y_1(x)\,ln\,x - 2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^n H_n}{(n!)^2}\,x^n\,.$$

4. Here $x\,p(x) = 4$ and $x^2 q(x) = 2 + x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n}+\,4\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}+$$
$$+\sum_{n=0}^{\infty}a_n x^{r+n+1}+2\sum_{n=0}^{\infty}a_n x^{r+n}=0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+4r+2]x^r+\sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n+4(r+n)a_n+2\,a_n+a_{n-1}\right]x^{r+n}=0.$$

Assuming $a_0\neq 0$, the *indicial equation* is $r^2+3r+2=0$, with roots $r_1=\,-1$ and $r_2=\,-2$. Setting the remaining coefficients equal to *zero*, we have for $n\geq 1$,

$$a_n(r)=\,-\,\frac{1}{(n+r+1)(n+r+2)}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r)=\frac{(\,-1)^n}{[(n+r+1)(n+r)\cdots(2+r)][(n+r+2)(n+r)\cdots(3+r)]}\,a_0\,,\;n\geq 1\,.$$

Since $r_1=\,-1$, one solution is given by

$$y_1(x)=x^{-1}\sum_{n=0}^{\infty}\frac{(\,-1)^n}{(n)!(n+1)!}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.7 . Since $r_1-r_2=N=1$, we find that

$$a_1(r)=\,-\,\frac{1}{(r+2)(r+3)}\,,$$

with $a_0=1$. Hence the leading coefficient in the solution is

$$a=\lim_{r\to-2}(r+2)\,a_1(r)=\,-1\,.$$

Further,

$$(r+2)\,a_n(r)=\frac{(\,-1)^n}{(n+r+2)\,[(n+r+1)(n+r)\cdots(3+r)]^2}\,.$$

Let $A_n(r)=(r+2)\,a_n(r)$. It follows that

$$\frac{A_n'(r)}{A_n(r)}=\,-\,\frac{1}{n+r+2}-2\left[\frac{1}{n+r+1}+\frac{1}{n+r}+\cdots+\frac{1}{3+r}\right].$$

Setting $r=r_2=\,-2$,

$$\frac{A_n'(-2)}{A_n(-2)} = -\frac{1}{n} - 2\left[\frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1\right]$$
$$= -H_n - H_{n-1}\,.$$

Hence

$$c_n(-2) = -(H_n + H_{n-1})\,A_n(-2)$$
$$= -(H_n + H_{n-1})\frac{(-1)^n}{n!(n-1)!}\,.$$

Therefore,

$$y_2(x) = -y_1(x)\,ln\,x + x^{-2}\left[1 - \sum_{n=1}^{\infty}\frac{(-1)^n(H_n + H_{n-1})}{n!(n-1)!}\,x^n\right].$$

6. Let $y(x) = v(x)/\sqrt{x}$ . Then $y' = x^{-1/2}\,v' - x^{-3/2}\,v/2$ and $y'' = x^{-1/2}\,v'' -$ $- x^{-3/2}\,v' + 3\,x^{-5/2}\,v/4$ . Substitution into the ODE results in

$$\left[x^{3/2}\,v'' - x^{1/2}\,v' + 3\,x^{-1/2}\,v/4\right] + \left[x^{1/2}\,v' - x^{-1/2}\,v/2\right] + \left(x^2 - \frac{1}{4}\right)x^{-1/2}\,v = 0\,.$$

Simplifying, we find that

$$v'' + v = 0\,,$$

with *general solution* $v(x) = c_1 cos\,x + c_2\,sin\,x$ . Hence

$$y(x) = c_1 x^{-1/2} cos\,x + c_2\,x^{-1/2} sin\,x\,.$$

8. The absolute value of the ratio of consecutive terms is

$$\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \frac{|x|^{2m+2}\,2^{2m}(m+1)!\,m!}{|x|^{2m}\,2^{2m+2}(m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}\,.$$

Applying the *ratio test*,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\frac{|x|^2}{4(m+2)(m+1)} = 0\,.$$

Hence the series for $J_1(x)$ converges absolutely *for all* values of $x$ . Furthermore, since the series for $J_0(x)$ also converges absolutely for all $x$, term-by-term differentiation results in

$$\begin{aligned} J_0'(x) &= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}\, m!(m-1)!} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2m+1}}{2^{2m+1}(m+1)!\, m!} \\ &= -\frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m}(m+1)!\, m!} \,. \end{aligned}$$

Therefore, $J_0'(x) = -J_1(x)$.

9$(a)$. Note that $x\,p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are *both* analytic at $x = 0$. Thus $x = 0$ is a *regular* singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the *indicial equation* is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

$(b)$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n\, x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0\left[r(r-1) + r - \nu^2\right]x^r + a_1\left[(r+1)r + (r+1) - \nu^2\right] + \\ + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n - \nu^2 a_n + a_{n-2}\right]x^{r+n} = 0.$$

Setting the coefficients equal to *zero*, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2}\, a_{n-2}\,,$$

for $n \geq 2$. It follows that $a_3 = a_5 = \cdots = a_{2m+1} = \cdots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)}\, a_{n-2}\,.$$

So for $m = 1, 2, \cdots$,

$$\begin{aligned} a_{2m} &= \frac{-1}{2m(2m+2\nu)}\, a_{2m-2} \\ &= \frac{(-1)^m}{2^{2m}\, m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)}\, a_0\,. \end{aligned}$$

Hence one solution is

$$y_1(x) = x^{\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

$(c)$. Assuming that $r_1 - r_2 = 2\nu$ is *not* an integer, simply setting $r = -\nu$ in the above results in a second *linearly independent* solution

$$y_2(x) = x^{-\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\cdots(m-1-\nu)(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

$(d)$. The absolute value of the ratio of consecutive terms in $y_1(x)$ is

$$\begin{aligned}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| &= \frac{|x|^{2m+2}\,2^{2m}\,m!(1+v)\cdots(m+\nu)}{|x|^{2m}\,2^{2m+2}(m+1)!(1+v)\cdots(m+1+\nu)} \\ &= \frac{|x|^2}{4(m+1)(m+1+\nu)}\,.\end{aligned}$$

Applying the *ratio test*,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\frac{|x|^2}{4(m+1)(m+1+\nu)} = 0\,.$$

Hence the series for $y_1(x)$ converges absolutely *for all* values of $x$. The same can be shown for $y_2(x)$. Note also, that if $\nu$ is a *positive* integer, then the coefficients in the series for $y_2(x)$ are *undefined*.

$10(a)$. It suffices to calculate $L[J_0(x)\,ln\,x]$. Indeed,

$$[J_0(x)\,ln\,x]' = J_0'(x)\,ln\,x + \frac{J_0(x)}{x}$$

and

$$[J_0(x)\,ln\,x]'' = J_0''(x)\,ln\,x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}\,.$$

Hence

$$\begin{aligned}L[J_0(x)\,ln\,x] &= x^2 J_0''(x)\,ln\,x + 2x\,J_0'(x) - J_0(x) + \\ &\quad + x\,J_0'(x)\,ln\,x + J_0(x) + x^2 J_0(x)\,ln\,x\,.\end{aligned}$$

Since $x^2 J_0''(x) + x\,J_0'(x) + x^2 J_0(x) = 0$,

$$L[J_0(x)\,ln\,x] = 2x\,J_0'(x)\,.$$

$(b)$. Given that $L[y_2(x)] = 0$, after adjusting the indices in Part $(a)$, we have

$$b_1 x + 2^2 b_2\, x^2 + \sum_{n=3}^{\infty} \left(n^2 b_n + b_{n-2}\right) x^n = -2x\, J_0'(x)\,.$$

Using the series representation of $J_0'(x)$ in Problem 8,

$$b_1 x + 2^2 b_2\, x^2 + \sum_{n=3}^{\infty} \left(n^2 b_n + b_{n-2}\right) x^n = -2 \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n}}{2^{2n} (n!)^2}\,.$$

$(c)$. Equating the coefficients on both sides of the equation, we find that

$$b_1 = b_3 = \cdots = b_{2m+1} = \cdots = 0\,.$$

Also, with $n = 1$, $2^2 b_2 = 1/(1!)^2$, that is, $b_2 = 1/\left[2^2 (1!)^2\right]$. Furthermore, for $m \geq 2$,

$$(2m)^2 b_{2m} + b_{2m-2} = -2 \frac{(-1)^m (2m)}{2^{2m} (m!)^2}\,.$$

More explicitly,

$$\begin{aligned} b_4 &= -\frac{1}{2^2\, 4^2}\left(1 + \frac{1}{2}\right) \\ b_6 &= \frac{1}{2^2\, 4^2\, 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) \\ &\vdots \end{aligned}$$

It can be shown, in general, that

$$b_{2m} = (-1)^{m+1} \frac{H_m}{2^{2m} (m!)^2}\,.$$

11. Bessel's equation of *order one* is

$$x^2 y'' + x\, y' + \left(x^2 - 1\right) y = 0\,.$$

Based on Problem 9, the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Set $y = x^r (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n\, x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+r-1]x^r + a_1[(r+1)r+(r+1)-1] +$$
$$+\sum_{n=2}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - a_n + a_{n-2}]x^{r+n} = 0.$$

Setting the coefficients equal to *zero*, we find that $a_1 = 0$, and

$$\begin{aligned} a_n(r) &= \frac{-1}{(r+n)^2-1}\, a_{n-2}(r) \\ &= \frac{-1}{(n+r+1)(n+r-1)}\, a_{n-2}(r)\,, \end{aligned}$$

for $n \geq 2$. It follows that $a_3 = a_5 = \cdots = a_{2m+1} = \cdots = 0$. Solving the recurrence relation,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2 \cdots (r+3)^2(r+1)}\, a_0.$$

With $r = r_1 = 1$,

$$a_{2m}(1) = \frac{(-1)^m}{2^{2m}(m+1)!\, m!}\, a_0.$$

For a *second* linearly independent solution, we follow the discussion in Section 5.7 . Since $r_1 - r_2 = N = 2$, we find that

$$a_2(r) = -\frac{1}{(r+3)(r+1)}\,,$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a = \lim_{r \to -1}(r+1)\, a_2(r) = -\frac{1}{2}\,.$$

Further,

$$(r+1)\, a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)\,[(2m+r-1)\cdots(3+r)]^2}\,.$$

Let $A_n(r) = (r+1)\, a_n(r)$. It follows that

$$\frac{A_{2m}'(r)}{A_{2m}(r)} = -\frac{1}{2m+r+1} - 2\left[\frac{1}{2m+r-1} + \cdots + \frac{1}{3+r}\right].$$

Setting $r = r_2 = -1$, we calculate

$$
\begin{aligned}
c_{2m}(-1) &= -\frac{1}{2}(H_m + H_{m-1})A_{2m}(-1) \\
&= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2m\,[(2m-2)\cdots 2]^2} \\
&= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2^{2m-1}\,m!(m-1)!}\,.
\end{aligned}
$$

Note that $a_{2m+1}(r) = 0$ implies that $A_{2m+1}(r) = 0$, so

$$c_{2m+1}(-1) = \left[\frac{d}{dr}A_{2m+1}(r)\right]_{r=r_2} = 0\,.$$

Therefore,

$$y_2(x) = -\frac{1}{2}\left[x\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!\,m!}\left(\frac{x}{2}\right)^{2m}\right]ln\,x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m}\right].$$

Based on the definition of $J_1(x)$,

$$y_2(x) = -J_1(x)\,ln\,x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m}\right].$$

12. Consider a solution of the form

$$y(x) = \sqrt{x}\ f\big(\alpha x^{\beta}\big).$$

Then

$$y' = \frac{df}{d\xi}\cdot\frac{\alpha\beta\,x^{\beta}}{\sqrt{x}} + \frac{f(\xi)}{2\sqrt{x}}$$

in which $\xi = \alpha x^{\beta}$. Hence

$$y'' = \frac{d^2f}{d\xi^2}\cdot\frac{\alpha^2\beta^2\,x^{2\beta}}{x\sqrt{x}} + \frac{df}{d\xi}\cdot\frac{\alpha\beta^2\,x^{\beta}}{x\sqrt{x}} - \frac{f(\xi)}{4x\sqrt{x}}\,,$$

and

$$x^2\,y'' = \alpha^2\beta^2\,x^{2\beta}\sqrt{x}\,\frac{d^2f}{d\xi^2} + \alpha\beta^2\,x^{\beta}\sqrt{x}\,\frac{df}{d\xi} - \frac{1}{4}\sqrt{x}\,f(\xi)\,.$$

Substitution into the ODE results in

$$\alpha^2\beta^2\,x^{2\beta}\frac{d^2f}{d\xi^2} + \alpha\beta^2\,x^{\beta}\frac{df}{d\xi} - \frac{1}{4}f(\xi) + \left(\alpha^2\beta^2\,x^{2\beta} + \frac{1}{4} - \nu^2\beta^2\right)f(\xi) = 0\,.$$

Simplifying, and setting $\xi = \alpha x^{\beta}$, we find that

$$\xi^2 \frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + \left(\xi^2 - \nu^2\right) f(\xi) = 0\,, \quad (*)$$

which is a *Bessel* equation of *order* $\nu$. Therefore, the general solution of the given ODE is

$$y(x) = \sqrt{x}\,\left[c_1\, f_1\!\left(\alpha x^{\beta}\right) + c_2\, f_2\!\left(\alpha x^{\beta}\right)\right],$$

in which $f_1(\xi)$ and $f_2(\xi)$ are the linearly independent solutions of $(*)$.