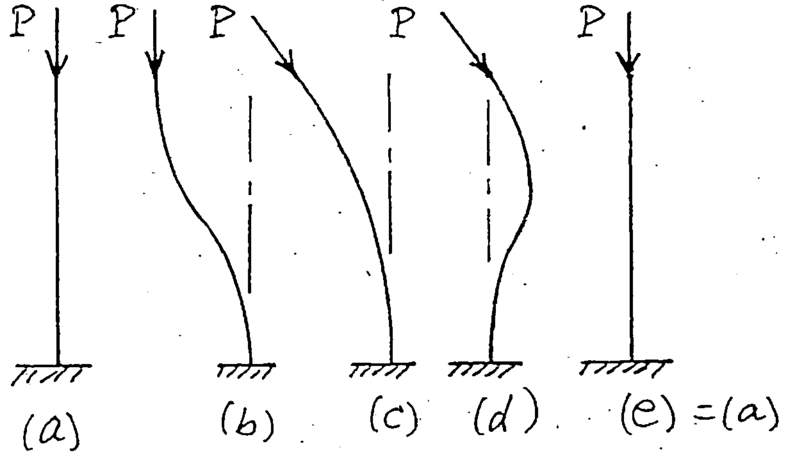
4.2-3

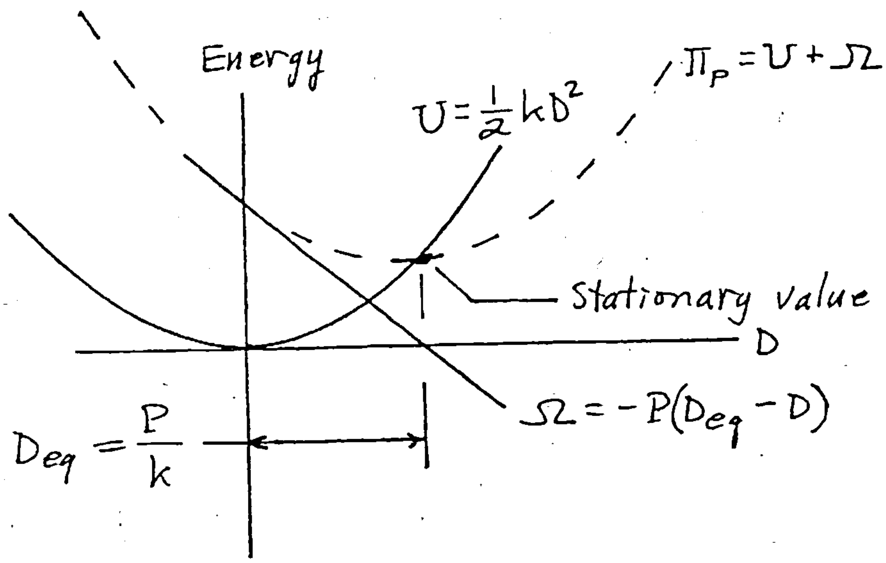
Consider the displacements of load P shown.



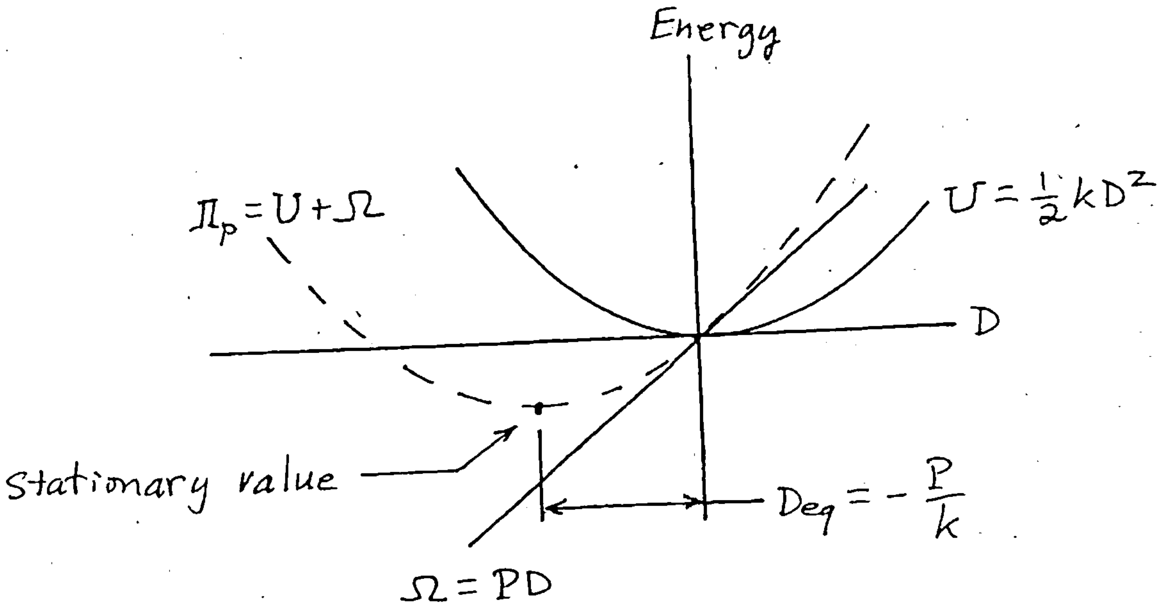
(a) Original state, then
(b) Translation, then
(c) Rotation, then
(d) Translation, then
(e) Rotation (original state)

Only the passage from (c) to (d) involves work, here done by the horizontal component of P. Nonzero net work is done when the original state is restored. Hence, not conservative.

4.2-4



4.2-5

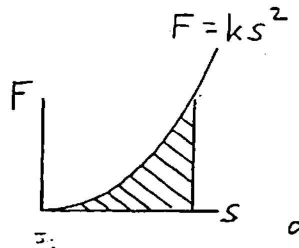


$$\Pi_p = \frac{1}{2}kD^2 + PD$$

$$d\Pi_p = (kD_{eq} + P)\,dD = 0$$

$$D_{eq} = -\frac{P}{k}$$

4.2-6



$$U = \int_0^D F\,ds = \frac{kD^3}{3}$$

$$\Pi_p = U + \Omega = \frac{kD^3}{3} - PD$$

$$d\Pi_p = 0 = (kD_{eq}^2 - P)\,dD$$

$$D_{eq} = \sqrt{P/k}$$

(Checks $F = ks^2$ for $F = P$ and $s = D_{eq}$.)

4.3-1

Let overbars denote relative displacements. Thus $D_1 = \bar{D}_1$,
$D_2 = D_1 + \bar{D}_2$, $D_3 = D_2 + \bar{D}_3$
$D_3 = D_1 + \bar{D}_2 + \bar{D}_3$

$$\Pi_p = \tfrac{1}{2} k_1 D_1^2 + \tfrac{1}{2} k_2 \bar{D}_2^2 + \tfrac{1}{2} k_3 \bar{D}_3^2 - P_1 D_1 - P_2 (D_1 + \bar{D}_2) - P_3 (D_1 + \bar{D}_2 + \bar{D}_3)$$

$$\left.\begin{aligned} \frac{\partial \Pi_p}{\partial D_1} &= 0 = k D_1 - 3P, \quad D_1 = \frac{3P}{k} \\ \frac{\partial \Pi_p}{\partial \bar{D}_2} &= 0 = k\bar{D}_2 - 2P, \quad \bar{D}_2 = \frac{2P}{k} \\ \frac{\partial \Pi_p}{\partial \bar{D}_3} &= 0 = k\bar{D}_3 - P, \quad \bar{D}_3 = \frac{P}{k} \end{aligned}\right\} \begin{aligned} &\text{For } k_1 = \\ &k_2 = k_3 = k \\ &\text{and } P_1 = \\ &P_2 = P_3 = P \end{aligned}$$

$$D_2 = \frac{3P}{k} + \frac{2P}{k} = \frac{5P}{k}, \quad D_3 = \frac{5P}{k} + \frac{P}{k} = \frac{6P}{k}$$

Eq. 4.3-5:

$$k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \frac{P}{k} \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = P \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \checkmark$$

4.3-2

Rows of $[\underset{\sim}{k}]$ given by $\partial U / \partial d_i = 0$.

$$U = \tfrac{1}{2} k e^2 = \tfrac{1}{2} k \left[d_3 - (c d_1 + s d_2) \right]^2$$

where $c = \cos\beta$ & $s = \sin\beta$.

$$\frac{\partial U}{\partial d_1} = ke(-c)$$

$$\frac{\partial U}{\partial d_2} = ke(-s) \qquad [\underset{\sim}{k}] = k \begin{bmatrix} c^2 & cs & -c & 0 \\ cs & s^2 & -s & 0 \\ -c & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial U}{\partial d_3} = ke$$

$$\frac{\partial U}{\partial d_4} = 0$$

$\boxed{4.3\text{-}3}$ <u>Part (a)</u>

(a) of Problem 2.2-2

$$\Pi_p = \frac{1}{2}\left[k_1 u_1^2 + k_2 u_3^2 + k_3(u_3-u_1)^2 + u_4(u_2-u_1)^2\right] - F_1 u_1 - F_2 u_2 - F_3 u_3$$

$$\frac{\partial \Pi_p}{\partial u_1} = 0 = k_1 u_1 - k_3(u_3-u_1) - k_4(u_2-u_1) - F_1$$

$$\frac{\partial \Pi_p}{\partial u_2} = 0 = k_4(u_2-u_1) - F_2$$

$$\frac{\partial \Pi_p}{\partial u_3} = 0 = k_2 u_3 + k_3(u_3-u_1) - F_3$$

$$\begin{bmatrix} k_1+k_3+k_4 & -k_4 & -k_3 \\ -k_4 & k_4 & 0 \\ -k_3 & 0 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

(b) of Problem 2.2-2

$$\Pi_p = \frac{1}{2}\Big[k_1(u_4-u_2)^2 + k_2(u_2-u_1)^2 + k_3(u_3-u_2)^2 + k_4(u_4-u_3)^2$$
$$+ k_5 u_1^2 + k_6(u_3-u_1)^2\Big] - F_1 u_1 - F_2 u_2 - F_3 u_3 - F_4 u_4$$

$$\frac{\partial \Pi_p}{\partial u_1} = 0 = -k_2(u_2-u_1) + k_5 u_1 - k_6(u_3-u_1) - F_1$$

$$\frac{\partial \Pi_p}{\partial u_2} = 0 = k_1(u_4-u_2) + k_2(u_2-u_1) - k_3(u_3-u_2) - F_2$$

$$\frac{\partial \Pi_p}{\partial u_3} = 0 = k_3(u_3-u_2) - k_4(u_4-u_3) + k_6(u_3-u_1) - F_3$$

$$\frac{\partial \Pi_p}{\partial u_4} = 0 = k_1(u_4-u_2) + k_4(u_4-u_3) - F_4$$

$$\begin{bmatrix} k_2+k_5+k_6 & -k_2 & -k_6 & 0 \\ -k_2 & k_1+k_2+k_3 & -k_3 & -k_1 \\ -k_6 & -k_3 & k_3+k_4+k_6 & -k_4 \\ 0 & -k_1 & -k_4 & k_1+k_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

(continues)

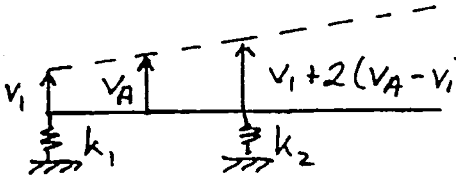
$\boxed{4.3\text{-}3}$ (continued) <u>Part (b)</u>

(a) of Problem 2.2-3

$$\Pi_p = \frac{1}{2}\left[k_1 v_1^2 + k_2 v_2^2\right] \qquad \frac{\partial \Pi_p}{\partial v_1} = 0 = k_1 v_1 \qquad \frac{\partial \Pi_p}{\partial v_2} = 0 = k_2 v_2$$

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(b) of Problem 2.2-3



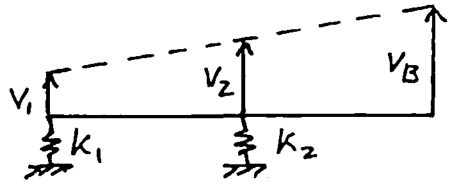
$$\Pi_p = \frac{1}{2}\left[k_1 v_1^2 + k_2 (2v_A - v_1)^2\right]$$

$$\frac{\partial \Pi_p}{\partial v_1} = 0 = k_1 - k_2 (2v_A - v_1)$$

$$\frac{\partial \Pi_p}{\partial v_A} = 0 = 2k_2 (2v_A - v_1)$$

$$\begin{bmatrix} k_1 + k_2 & -2k_2 \\ -2k_2 & 4k_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(c) of Problem 2.2-3



$$v_1 = v_2 - (v_B - v_2) = 2v_2 - v_B$$

$$\Pi_p = \frac{1}{2}\left[k_1 (2v_2 - v_B)^2 + k_2 v_2^2\right]$$

$$\frac{\partial \Pi_p}{\partial v_2} = 0 = 2k_1 (2v_2 - v_B) + k_2 v_2$$

$$\frac{\partial \Pi_p}{\partial v_B} = -k_1 (2v_2 - v_B)$$

$$\begin{bmatrix} 4k_1 + k_2 & -2k_1 \\ -2k_1 & k_1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(continues)

$\boxed{4.3\text{-}3}$ (continued) Part (c)

(a) of Problem 2.3-2

$$\Pi_p = \frac{k}{2}\Big[(u_2 - b\theta_2) - (u_1 - b\theta_1)\Big]^2 - F_1 u_1 - M_1\theta_1 - F_2 u_2 - M_2\theta_2$$

$$\frac{\partial \Pi_p}{\partial u_1} = 0 = -k[---] - F_1$$

$$\frac{\partial \Pi_p}{\partial \theta_1} = 0 = kb[---] - M_1$$

$$\frac{\partial \Pi_p}{\partial u_2} = 0 = k[---] - F_2$$

$$\frac{\partial \Pi_p}{\partial \theta_2} = 0 = -kb[---] - M_2$$

$$\begin{bmatrix} k & -kb & -k & kb \\ -kb & kb^2 & kb & -kb^2 \\ -k & kb & k & -kb \\ kb & -kb^2 & -kb & kb^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

(b) of Problem 2.3-2

$$\Pi_p = \frac{k}{2}\Big\langle u_1 - (u_2 - b\theta_2)\Big\rangle^2 + \frac{k}{2}\Big[u_2 - (u_1 + b\theta_1)\Big]^2 - F_1 u_1 - M_1\theta_1 - F_2 u_2 - M_2\theta_2$$

$$\frac{\partial \Pi_p}{\partial u_1} = 0 = k\langle --- \rangle - k[---] - F_1 = k(u_1 - u_2 + b\theta_2 - u_2 + u_1 + b\theta_1) - F_1$$

$$\frac{\partial \Pi_p}{\partial \theta_1} = 0 = -kb[---] - M_1 = kb(u_1 - u_2 + b\theta_1) - M_1$$

$$\frac{\partial \Pi_p}{\partial u_2} = 0 = -k\langle --- \rangle + k[---] - F_2 = k(-u_1 + u_2 - b\theta_2 + u_2 - u_1 - b\theta_1) - F_2$$

$$\frac{\partial \Pi_p}{\partial \theta_2} = 0 = kb\langle --- \rangle - M_2 = kb(u_1 - u_2 + b\theta_2) - M_2$$

$$\begin{bmatrix} 2k & kb & -2k & kb \\ kb & kb^2 & -kb & 0 \\ -2k & -kb & 2k & -kb \\ kb & 0 & -kb & kb^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

(continues)

$\boxed{4.3\text{-}3}$ (concluded) <u>Part (d)</u>

(a) of Problem 2.3-3

$$\Pi_p = \frac{1}{2}k_A u_A^2 + \frac{1}{2}k_B\left(\frac{u_c - u_A}{4L}\right)^2$$

$$\frac{\partial \Pi_p}{\partial u_A} = 0 = k_A u_A - \frac{k_B}{4L}\left(\frac{u_c - u_A}{4L}\right)$$

$$\frac{\partial \Pi_p}{\partial u_c} = 0 = \frac{k_B}{4L}\left(\frac{u_c - u_A}{4L}\right)$$

$$\begin{bmatrix} k_A + \frac{k_B}{16L^2} & -\frac{k_B}{16L^2} \\ -\frac{k_B}{16L^2} & \frac{k_B}{16L^2} \end{bmatrix} \begin{Bmatrix} u_A \\ u_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(b) of Problem 2.3-3

$$\Pi_p = \frac{1}{2}k(u - \theta b)^2 + \frac{1}{2}k(2a\theta)^2 - Fu - M\theta$$

$$\frac{\partial \Pi_p}{\partial u} = 0 = k(u - \theta b) - F$$

$$\frac{\partial \Pi_p}{\partial \theta} = 0 = -kb(u - \theta b) + k(2a\theta)(2a) - M$$

$$\begin{bmatrix} k & -kb \\ -kb & kb^2 + 4ka^2 \end{bmatrix} \begin{Bmatrix} u \\ \theta \end{Bmatrix} = \begin{Bmatrix} F \\ M \end{Bmatrix}$$

$\boxed{4.3\text{-}4}$

$$[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \begin{Bmatrix} k_1 D_1 + k_2(D_1 - D_2) \\ k_2(D_2 - D_1) + k_3(D_2 - D_3) \\ k_3(D_3 - D_2) \end{Bmatrix}$$

$$\frac{1}{2}\{\underset{\sim}{D}\}^T([\underset{\sim}{K}]\{\underset{\sim}{D}\}) = \frac{1}{2}\Big[k_1 D_1^2 + k_2(D_1^2 - D_1 D_2 -$$

$$D_1 D_2 + D_2^2) + k_3(D_2^2 - D_2 D_3 - D_2 D_3 + D_3^2)\Big]$$

$$\frac{1}{2}\{\underset{\sim}{D}\}^T[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \frac{1}{2}k_1 D_1^2 + \frac{1}{2}k_2(D_1^2 - 2D_1 D_2 + D_2^2)$$
$$+ \frac{1}{2}k_3(D_2^2 - 2D_2 D_3 + D_3^2) \quad \checkmark$$

$$\Omega = -\{\underset{\sim}{D}\}^T\{\underset{\sim}{P}\} = -D_1 P_1 - D_2 P_2 - D_3 P_3$$

$\boxed{4.4\text{-}1}$

Uniaxial stress: $\{\underset{\sim}{\epsilon}\} = \epsilon_x \lfloor 1 \ \ -\nu \ \ 0 \rfloor^T$.
Then

$$[\underset{\sim}{E}]\{\underset{\sim}{\epsilon}\} = \frac{E\epsilon_x}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} 1 \\ -\nu \\ 0 \end{Bmatrix}$$

$$\frac{1}{2}\{\underset{\sim}{\epsilon}\}^T\left([\underset{\sim}{E}]\{\underset{\sim}{\epsilon}\}\right) = \frac{\epsilon_x}{2}\lfloor 1 \ \ -\nu \ \ 0 \rfloor \begin{Bmatrix} E\epsilon_x \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{2}\epsilon_x^2$$

Checks Eq. 4.4-9.

$\boxed{4.4\text{-}2}$

Add to 1^{st} integral the terms

$\int [-E\epsilon_x\epsilon_{xo} + \epsilon_x\sigma_{xo}]\,dV$, where

$\epsilon_{xo} = -zK_o$, $\sigma_{xo} = -m_o z/I$, $dV = dA\,dx$,

and $\epsilon_x = -z w_{,xx}$. Thus

$$\iint\left[-Ez^2K_o w_{,xx} - z w_{,xx}\left(-\frac{m_o z}{I}\right)\right] dA\,dx =$$

$$\int I\left[-EK_o w_{,xx} + w_{,xx}\frac{m_o}{I}\right]dx = \int w_{,xx}\left[-EIK_o + m_o\right]dx$$

$\boxed{4.4\text{-}3}$

Now $\sigma_o = 0$, $\epsilon_o = \alpha T$, so Eq. 4.4-12 becomes

$$\Pi_p = \int_0^L \left[\frac{1}{2}E\left(\frac{D}{L}\right)^2 - \frac{D}{L}E(\alpha T)\right]A\,dx - DP = \frac{EAD^2}{2L} - DEA\alpha T - DP$$

Same as latter form of Eq. 4.4-12, so same result is obtained.

$\boxed{4.5\text{-}1}$
$\boxed{4.5\text{-}2}$

Set up using 3 terms, then drop a_3 to do Problem 4.5-1.

$$u = a_1 x + a_2 x^2 + a_3 x^3$$
$$u_{,x} = a_1 + 2a_2 x + 3a_3 x^2$$
$$\int_0^{L_T} u_{,x}^2\, dx = a_1^2 L_T + \frac{4}{3} a_2^2 L_T^3 + \frac{9}{5} a_3^2 L_T^5 + 2a_1 a_2 L_T^2 + 2a_1 a_3 L_T^3 + 3a_2 a_3 L_T^4$$
$$\int_0^{L_T} c x u\, dx = c\left(\frac{1}{3} a_1 L_T^3 + \frac{1}{4} a_2 L_T^4 + \frac{1}{5} a_3 L_T^5\right)$$
$$\Pi_p = \frac{AE}{2}(1^{st} \text{ integral}) - (2^{nd} \text{ integral})$$

4.5-1

Set $a_3 = 0$ to do Problem 4.5-1. Then

$$0 = \frac{\partial \Pi_p}{\partial a_1} = AE\left(a_1 L_T + a_2 L_T^2\right) - \frac{1}{3} c L_T^3$$
$$0 = \frac{\partial \Pi_p}{\partial a_2} = AE\left(\frac{4}{3} a_2 L_T^3 + a_1 L_T^2\right) - \frac{1}{4} c L_T^4$$

Checks the first of Eqs. 4.5-6a.

4.5-2

Now retain the a_3 term in Π_p.

$$\frac{\partial \Pi_p}{\partial a_1} = 0 = AE\left(a_1 L_T + a_2 L_T^2 + a_3 L_T^3\right) - \frac{1}{3} c L_T^3$$
$$\frac{\partial \Pi_p}{\partial a_2} = 0 = AE\left(\frac{4}{3} a_2 L_T^3 + a_1 L_T^2 + \frac{3}{2} a_3 L_T^4\right) - \frac{c L_T^4}{4}$$
$$\frac{\partial \Pi_p}{\partial a_3} = 0 = AE\left(\frac{9}{5} a_3 L_T^5 + a_1 L_T^3 + \frac{3}{2} a_2 L_T^4\right) - \frac{c L_T^5}{5}$$

In Matrix format,

$$A\ldots\begin{bmatrix} 1 & L_T & L_T^2 \\ L_T & \frac{4}{3} L_T^2 & \frac{3}{2} L_T^3 \\ L_T^2 & \frac{3}{2} L_T^3 & \frac{9}{5} L_T^4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{c L_T^2}{60} \begin{Bmatrix} 20 \\ 15 L_T \\ 12 L_T^2 \end{Bmatrix}$$

Satisfied by Eqs. 4.5-9.

$\boxed{4.5\text{-}3}$

Differential equation, Eq. 4.5-7 ; $AE u_{,xx} + cx = 0$

Eq. 4.5-5c : $u = \frac{cL_T^2}{3AE}x$, $AE(0) + cx = 0$

Satisfied only at $x = 0$

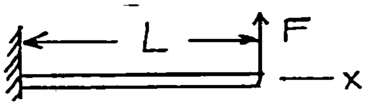
Eq. 4.5-6b : $u = \frac{cL_T}{12AE}(7L_T x - 3x^2)$, $AE\left[\frac{cL_T}{12AE}(-6)\right] + cx = 0$

Satisfied only at $x = \frac{L_T}{2}$

Eq. 4.5-8: $u = \frac{c}{6AE}(3L_T^2 x - x^3)$, $AE\left[\frac{c}{6AE}(-6x)\right] + cx = 0$

Satisfied for all x.

4.5-4



(a) $v = a_1 x^3, \quad v_{,x} = 3a_1 x^2$

Admissible, as $v = v_{,x} = 0$ @ $x = 0$.
But poor, as it omits lowest-order admissible term, namely x^2.

(b) $v = a_1 x^2 + a_2 x^3 + a_3 x^4$

(c) Will be exact, since exact v is cubic in x for this load. $\therefore a_3 = 0$.

(d) $\Pi_p = \int_0^L \frac{EI}{2} (6a_1 x)^2 dx - F(a_1 L^3)$

$\frac{d\Pi_p}{da_1} = \frac{d}{da_1}\left(6EIL^3 a_1^2 - FL^3 a_1\right) = 0; \; a_1 = F/12EI$

At $x = L$, gives $v = FL^3/12EI$

$\boxed{4.5\text{-}5}$

Assume that uniform q acts <u>up</u>.

(a) $v = a_1 x(L-x)$ admissible

$$\Pi_p = \frac{EI}{2}\int_0^L (2a_1)^2 dx - \int_0^L q\, a_1 x(L-x)\, dx$$

$$\frac{d\Pi_p}{da_1} = 0 = \frac{d}{da_1}\left(2EIL a_1^2 - \frac{qL^3}{6} a_1\right), \quad a_1 = \frac{qL^2}{24EI}$$

At center, $x = L/2$,

$$v = a_1(L^2/4) = 0.010417\,(qL^4/EI)$$

$$M = EI\, v_{,xx} = -0.08333\, qL^2$$

(b) $v = a_1 \sin\frac{\pi x}{L}$, $v_{,xx} = -a_1 \frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$

Substitute $\theta = \frac{\pi x}{L}$, $dx = \frac{L}{\pi} d\theta$

$$\Pi_p = \frac{EI}{2}\frac{a_1^2 \pi^4}{L^4}\frac{L}{\pi}\int_0^\pi \sin^2\theta\, d\theta - q a_1 \frac{L}{\pi}\int_0^\pi \sin\theta\, d\theta$$

$$\Pi_p = \frac{EI\pi^4}{4L^3} a_1^2 - \frac{2qL}{\pi} a_1$$

$d\Pi_p / da_1 = 0$ yields $a_1 = \frac{4qL^4}{\pi^5 EI}$

At center, $x = L/2$,

$$v = \frac{4qL^4}{\pi^5 EI} = 0.013071\,(qL^4/EI)$$

$$M = EI\, v_{,xx} = EI\left(-\frac{\pi^2 a_1}{L^2}\right) = -0.12901\, qL^2$$

(c) Exact values at center:

$$v = \frac{5qL^4}{384EI} = 0.013021\frac{qL^4}{EI}, \quad M = -\frac{qL^2}{8}$$

Assumption (a) satisfies only essential BC's; assumption (b) also satisfies non-essential BC's, so of course works better.

$\boxed{4.5\text{-}6}$

Exact answers are

$$v_L = \frac{P_L L^3}{3EI} - \frac{M_L L^2}{2EI} - \frac{q L^4}{8EI}$$

$$(v_{,x})_L = \frac{P_L L^2}{2EI} - \frac{M_L L}{EI} - \frac{q L^3}{6EI}$$

Set-up for (a) and (b):

$$v = a_1 x^2 + a_2 x^3, \quad v_{,x} = 2a_1 x + 3a_2 x^2$$

$$v_{,xx} = 2a_1 + 6a_2 x, \quad \Pi_p = U + \Omega$$

$$U = \int_0^L \frac{EI}{2} v_{,xx}^2 \, dx = 2EIL(a_1^2 + 3a_1 a_2 L + 3a_2^2 L^2)$$

$$\Omega = -P_L v_L + M_L (v_{,x})_L + \int_0^L q \, v \, dx$$

$$\Omega = -P_L(a_1 L^2 + a_2 L^3) + M_L(2a_1 L + 3a_2 L^2) + q\left(\frac{a_1 L^3}{3} + \frac{a_2 L^4}{4}\right)$$

$$\frac{\partial \Pi_p}{\partial a_1} = 0 = 2EIL(2a_1 + 3a_2 L) - P_L L^2 + M_L(2L) + \frac{qL^3}{3}$$

$$\frac{\partial \Pi_p}{\partial a_2} = 0 = 2EIL(3a_1 L + 6a_2 L^2) - P_L L^3 + M_L(3L^2) + qL^4/4$$

(a) Set $a_2 = 0$; from 1st eq.

$$a_1 = \frac{P_L L}{4EI} - \frac{M_L}{2EI} - \frac{qL^2}{12EI}$$

$$v_L = a_1 L^2 = \frac{P_L L^3}{4EI} - \frac{M_L L^2}{2EI} - \frac{qL^4}{12EI}$$

$$(v_{,x})_L = 2a_1 L = \frac{P_L L^2}{2EI} - \frac{M_L L}{EI} - \frac{qL^3}{6EI}$$

(b) $$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{6EIL^2} \begin{Bmatrix} 3P_L L^3 - 3M_L L^2 + \frac{5}{4} q L^4 \\ -P_L L^2 \qquad + \frac{1}{2} q L^3 \end{Bmatrix}$$

$$v_L = a_1 L^2 + a_2 L^3 = \frac{P_L L^3}{3EI} - \frac{M_L L^2}{2EI} - \frac{qL^4}{8EI}$$

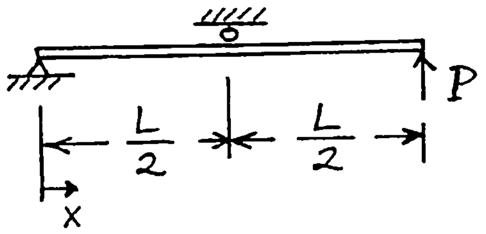
$$(v_{,x})_L = 2a_1 L + 3a_2 L^2 = \frac{P_L L^2}{2EI} - \frac{M_L L}{EI} - \frac{qL^3}{6EI}$$

(c) Only when $P_L = q = 0$, since $v = a_1 x^2$ only for M_L loading.

$\boxed{4,5\text{-}7}$

An infinite number. We are trying to model, by one polynomial series, two different polynomials that meet with the same v & same $v_{,x}$ at $x = \frac{L}{2}$.

4.5-8



$v = a_0 + a_1 x + a_2 x^2$

$v = 0$ at $x = 0$; $a_0 = 0$

$v = 0$ at $x = \frac{L}{2}$; $a_1 \frac{L}{2} + a_2 \frac{L^2}{4}$; $a_2 = -\frac{2}{L} a_1$

$\therefore \; v = a_1 \left(x - \frac{2}{L} x^2 \right)$

$v_{,x} = a_1 \left(1 - \frac{4}{L} x \right)$

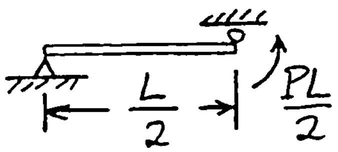
$v_{,xx} = -a_1 \frac{4}{L}$

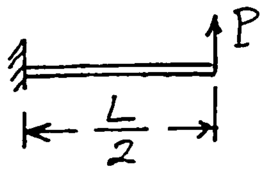
$$\Pi_p = \int_0^L \frac{EI}{2} \left(-a_1 \frac{4}{L} \right)^2 dx - a_1 \left(L - \frac{2}{L} L^2 \right) = \frac{EI}{2} a_1^2 \frac{16}{L^2} L + a_1 L P$$

$$\frac{d\Pi_p}{da_1} = 0 = \frac{16EI}{L} a_1 + PL \; , \quad a_1 = -\frac{PL^2}{16EI}$$

At $x = L$, $v = -a_1 L = \frac{PL^3}{16EI}$

Elementary beam theory:





$$v = \quad \Delta \frac{L}{2} \quad + \quad \frac{P(L/2)^3}{3EI}$$

$$v = \frac{(PL/2)(L/2)}{3EI} \frac{L}{2} + \frac{PL^3}{24EI} = \frac{PL^3}{3EI} \left(\frac{1}{8} + \frac{1}{8} \right) = \frac{PL^3}{12EI}$$

$$\frac{1/16 - 1/12}{1/12} 100\% = \left(\frac{3}{4} - 1 \right) 100\% = -25\%$$

$$M = EI v_{,xx} = EI \left(-\frac{4}{L} \right) \left(-\frac{PL^2}{16EI} \right) = \frac{PL}{4}$$

(for all x)

error is -50% at $x = \frac{L}{2}$

$\boxed{4.5\text{-}9}$

$$v = a_0 + a_1 x + a_2 x^2 \quad , \quad v_{,x} = a_1 + 2a_2 x$$
$$v = 0 \text{ at } x = 0; \quad a_0 = 0$$
$$v_{,x} = 0 \text{ at } x = L; \quad a_1 = -2a_2 L$$

$$\left.\right\} \quad v = a_2(-2Lx + x^2), \quad v_{,x} = a_2(-2L + 2x), \quad v_{,xx} = 2a_2$$

$$\Pi_p = \int_0^L \frac{EI}{2}(v_{,xx})^2 dx + Pv_L = \frac{EIL}{2}(4a_2^2) + P(-L^2)a_2$$

$$\frac{d\Pi_p}{da_2} = 0 = 4EILa_2 - PL^2 \quad ; \quad a_2 = \frac{PL}{4EI}$$

At $x = L$, $v = -L^2 a_2 = -\dfrac{PL^3}{4EI}$

Beam theory: at $x = L$, $v = -\dfrac{PL^3}{3EI}$

$$\frac{1/4 - 1/3}{1/3} 100\% = \left(\frac{3}{4} - 1\right) 100\% = -25\%$$

$$M = EI v_{,xx} = EI(2a_2) = \frac{PL}{2}$$
(for all x)

error is -50%
at $x = L$

$\boxed{4.5-10}$

$$u = a_0 + a_1 x + a_2 x^2$$

$$\left.\begin{array}{l} u=0 \text{ at } x=0;\ a_0 = 0 \\ u=0 \text{ at } x=L;\ a_2 = -\dfrac{a_1}{L} \end{array}\right\} \begin{array}{l} u = a_1\left(x - \dfrac{x^2}{L}\right) \\ u_{,x} = a_1\left(1 - \dfrac{2x}{L}\right) \end{array}$$

Eq. 4.4-9:

$$\Pi_p = \int_0^L \left(\frac{1}{2} E u_{,x}^2 - u_{,x} E \epsilon_{x0}\right) A\,dx \qquad \text{where } \epsilon_{x0} = \alpha \frac{T_0 x}{L}$$

$$\Pi_p = \int_0^L \left[\frac{E}{2} a_1^2 \left(1 - \frac{4x}{L} + \frac{4x^2}{L^2}\right) - \frac{E\alpha T_0}{L} a_1 \left(x - \frac{2x^2}{L}\right)\right] A\,dx$$

$$\frac{d\Pi_p}{da_1} = 0, \quad 0 = \left[a_1\left(x - \frac{2x^2}{L} + \frac{4x^3}{3L^2}\right) - \frac{\alpha T_0}{L}\left(\frac{x^2}{L} - \frac{2x^3}{3L}\right)\right]_0^L$$

$$0 = a_1 \frac{L}{3} - \frac{\alpha T_0}{L}\left(-\frac{L^2}{6}\right), \qquad a_1 = -\frac{\alpha T_0}{2}$$

$$u = -\frac{\alpha T_0}{2}\left(x - \frac{x^2}{L}\right)$$

Stress field:

$$\sigma = E\epsilon_x + \sigma_0 = E u_{,x} + \left(-E\alpha \frac{T_0 x}{L}\right)$$

$$\sigma = E\left[-\frac{\alpha T_0}{2}\left(1 - \frac{2x}{L}\right) - \frac{\alpha T_0 x}{L}\right]$$

$$\sigma = -\frac{E\alpha T_0}{2} \quad ✓$$

$\boxed{4.5-11}$

$$u = a_1 x \qquad v = b_1 x^2$$

$$u_{,x} = a_1 \qquad v_{,x} = 2b_1 x$$

$$v_{,xx} = 2b_1$$

$$\Pi_p = \int_0^L \frac{AE}{L} a_1^2 \, dx + \int_0^L \frac{P}{2} 4b_1^2 x^2 \, dx + \int_0^L \frac{EI}{2} 4b_1^2 \, dx - P(a_1 L)$$

$$= \frac{AEL}{2} a_1^2 + \frac{2PL^3}{3} b_1^2 + 2EIL\, b_1^2 - PLa_1$$

$$\frac{\partial \Pi_p}{\partial a_1} = 0 = AELa_1 - PL \longrightarrow a_1 = \frac{P}{AE}, \therefore \sigma_x = Eu_{,x} = \frac{P}{A} \checkmark$$

$$\frac{\partial \Pi_p}{\partial b_1} = 0 = \frac{4PL^3}{3} b_1 + 4EILb_1$$

$$0 = b_1 \underbrace{\left(\frac{4PL^3}{3} + 4EIL\right)}_{\text{must vanish if } b_1 \neq 0} \longrightarrow P = -\frac{3EI}{L^2}$$

Exact P_{cr} is $P_{cr} = -\frac{\pi^2 EI}{4L^2} = -2.467\frac{EI}{L^2}$

$\boxed{4.5\text{-}12}$

$$v = a_1 x(L-x) \qquad q = q_0\left(1-\frac{x}{L}\right) \qquad v_{L/2} = \frac{a_1 L^2}{4}$$

$$v_{,x} = a_1(L-2x) \qquad (v_{,x})_L = -La_1$$

$$v_{,xx} = -2a_1$$

$$\Pi_p = \int_0^L \frac{EI}{2} v_{,xx}^2\,dx + \int_0^L q\,v\,dx + \frac{1}{2}k\,v_{L/2}^2 - M_L(v_{,x})_L$$

$$\Pi_p = \int_0^L \frac{EI}{2} 4a_1^2\,dx + \int_0^L q_0 a_1\left(Lx - 2x^2 + \frac{x^3}{L}\right)dx + \frac{1}{2}k\frac{a_1^2 L^4}{16} - M_L(-La_1)$$

$$\Pi_p = 2EILa_1^2 + q_0 a_1\left(\frac{Lx^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4L}\right)_0^L + \frac{kL^4}{32}a_1^2 + M_L L a_1$$

$$\Pi_p = 2EILa_1^2 + \frac{q_0 L^3}{12}a_1 + \frac{kL^4}{32}a_1^2 + M_L L a_1$$

$$\frac{d\Pi_p}{da_1} = 0 = 4EILa_1 + \frac{q_0 L^3}{12} + \frac{kL^4}{16}a_1 + M_L L$$

$$a_1 = -\frac{\frac{q_0 L^3}{12} + M_L L}{4EIL + \frac{kL^4}{12}} = -\frac{\frac{q_0}{L} + \frac{12M_L}{L^3}}{\frac{48EI}{L^3} + k}$$

$$v = a_1 x(L-x) = -\frac{\frac{q_0}{L} + \frac{12M_L}{L^3}}{\frac{48EI}{L^3} + k}\,x(L-x)$$

$\boxed{4.5\text{-}13}$

℄ symm: need i odd only.

$$v = \sum a_i \sin\frac{i\pi x}{L} \quad , \; v_{,xx} = \sum -a_i\left(\frac{i\pi}{L}\right)^2 \sin\frac{i\pi x}{L}$$

$$U = \sum \frac{EI}{2} a_i^2 \left(\frac{i\pi}{L}\right)^4 \frac{L}{\pi}\int_0^\pi \sin^2\theta \, d\theta = \sum \frac{EI\, i^4 \pi^4}{4L^3} a_i^2$$

$$\Omega = +P\, v_{L/2} = P\sum a_i \sin\frac{i\pi}{2} \quad , \; \Pi_p = U + \Omega$$

$$\frac{d\Pi_p}{da_i} = 0 = \frac{EI\, i^4\pi^4}{2L^3} a_i + P\sin\frac{i\pi}{2}$$

$$a_i = -\frac{2PL^3}{EI\pi^4}\sum\frac{1}{i^4}\sin\frac{i\pi}{2} \, , \; v_{\frac{L}{2}} = -\frac{2PL^3}{EI\pi^4}\sum\frac{1}{i^4}$$

$$M = EI\, v_{,xx} = EI\frac{2PL^3}{EI\pi^4}\sum\left(\frac{i\pi}{L}\right)^2\frac{1}{i^4}\sin^2\frac{i\pi}{2}$$

$$M_{L/2} = \frac{2PL}{\pi^2}\sum\frac{1}{i^2}$$. At center, $x = L/2$,

$v_{L/2} = -C_D \dfrac{PL^3}{EI}$ & $M_{\frac{L}{2}} = C_M PL$, where

	$n=1$	$n=2$	$n=3$	$n=4$	exact
C_D	.020532	.020785	.020818	.020827	.020833
C_M	.20264	.22516	.23326	.23740	.25000

4.5-14

℄ symm: need i odd only.

U is the same as in Problem 4.5-13.

$$\Omega = \int_0^L q\, v\, dx = q \sum a_i \int_0^L \sin \frac{i\pi x}{L}\, dx = a_i \frac{2qL}{\pi} \sum \frac{1}{i}$$

$$\frac{d\Pi_p}{da_i} = 0 = \frac{EI i^4 \pi^4}{2L^3} a_i + \frac{2qL}{i\pi}, \quad a_i = -\frac{4qL^4}{EI\pi^5 i^5}$$

At the center, $x = L/2$,

$$v_{\frac{L}{2}} = -\frac{4qL^4}{EI\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots\right)$$

$$M_{\frac{L}{2}} = \left(EI\, v_{,xx}\right)_{\frac{L}{2}} = \frac{4qL^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right)$$

Let $v_{\frac{L}{2}} = -c_D \frac{qL^4}{EI}$ & $M_{\frac{L}{2}} = c_M (qL^2)$

	$n=1$	$n=2$	$n=3$	$n=4$	exact
c_D	.013071	.013017	.013021	.013021	.013021
c_M	.12901	.12423	.12526	.12488	.12500

$\boxed{4.6\text{-}1}$

(a) $W_e = \int_0^{L_T} qu\,dx = \dfrac{c^2}{6AE}\int_0^{L_T}(3L_T^2x^2 - x^4)\,dx$

$$W_e = \frac{c^2}{6AE}\left(L_T^2x^3 - \frac{x^5}{5}\right)_0^{L_T} = 0.13333\,\frac{c^2L_T^5}{AE}$$

(b) Eq. 4.5-5

$$W_1 = \int_0^{L_T} qu\,dx = \frac{c^2L_T^2}{3AE}\int_0^{L_T} x^2\,dx = 0.11111\,\frac{c^2L_T^5}{AE}$$

Eq. 4.5-6

$$W_2 = \int_0^{L_T} qu\,dx = \frac{c^2L_T}{12AE}\int_0^{L_T}(7L_Tx^2 - 3x^3)\,dx$$

$$W_2 = 0.13194\,(c^2L_T^5/AE)$$

$W_e > W_2 > W_1$. Displacement u is underestimated (in an integral sense) but gets more exact as more terms used.

$\boxed{4.6-2}$

(a) $\epsilon_x = \frac{c}{6AE}(3L_T^2 - 3x^2)$

$U_e = \int_0^{L_T} \frac{1}{2} E\epsilon_x^2 A dx = \frac{EAc^2}{8A^2E^2}\int_0^{L_T}(L_T^2 - x^2)^2 dx$

$U_e = 0.06667 \frac{c^2 L_T^5}{AE} = \frac{1}{2} W_e$ from 3.26(a)

As expected: work done by <u>gradually</u> applied load is $\frac{1}{2}W_e$ & is stored as strain energy

(b) $U_1 = \int_0^{L_T} \frac{1}{2} E\epsilon_x^2 A dx = \frac{EA}{2}\left(\frac{cL_T^2}{3AE}\right)^2 \int_0^{L_T} dx$

$U_1 = 0.05556 \frac{c^2 L_T^5}{AE} = \frac{1}{2} W_1$ from 3.26(b)

$U_2 = \int_0^{L_T} \frac{1}{2} E\epsilon_x^2 A dx = \frac{EA}{2}\left(\frac{cL_T}{12AE}\right)^2 \int_0^{L_T} (7L_T - 6x)^2 dx$

$U_2 = 0.065972 \frac{c^2 L_T^5}{AE} = \frac{1}{2} W_2$ from 3.26(b)

$\boxed{4.7\text{-}1}$

(a) Apply Eq. 4.7-2a: $\dfrac{\partial F}{\partial \phi} - \dfrac{d}{dx}\dfrac{\partial F}{\partial \phi_{,x}} + \dfrac{d^2}{dx^2}\dfrac{\partial F}{\partial \phi_{,xx}} = 0$

$$(2c_3\phi + c_4) - \frac{d}{dx}(2c_2\phi_{,x}) + \frac{d^2}{dx^2}(2c_1\phi_{,xx}) = 0$$

$$2c_1\phi_{,xxxx} - 2c_2\phi_{,xx} + 2c_3\phi + c_4 = 0$$

(b) $\Pi = \int\left[c_1\phi_{,xx}^2 + c_2\phi_{,x}^2 + c_3\phi^2 + c_4\phi + c_5\right]dx$

Let $\phi + \delta\phi = \phi + e\eta$

$$\Pi + \delta\Pi = \int\Big[c_1(\phi_{,xx} + e\eta_{,xx})^2 + c_2(\phi_{,x} + e\eta_{,x})^2 + c_3(\phi + e\eta)^2 + c_4(\phi + e\eta) + c_5\Big]dx$$

$$\int(\phi_{,xx} + e\eta_{,xx})^2 dx = \int(\phi_{,xx}^2 + 2e\phi_{,xx}\eta_{,xx} + \underbrace{e^2\eta_{,xx}^2}_{\text{discard}})\,dx$$

$$= \int(\phi_{,xx}^2 + 2e\phi_{,xx}\eta_{,xx})dx$$

$$= \int\phi_{,xx}^2\,dx - 2e\int\phi_{,xxx}\eta_{,x}\,dx + \text{B.C. term}$$

$$= \int\phi_{,xx}^2\,dx + 2e\int\phi_{,xxxx}\eta\,dx + \text{B.C. terms}$$

$$\int(\phi_{,x} + e\eta_{,x})^2 dx = \int(\phi_{,x}^2 + 2e\phi_{,x}\eta_{,x} + \underbrace{e^2\eta_{,x}^2}_{\text{discard}})\,dx$$

$$= \int\phi_{,x}^2 dx + 2e\int\phi_{,x}\eta_{,x}dx = \int\phi_{,x}^2 dx - 2e\int\phi_{,xx}\eta\,dx + \text{B.C. term}$$

$$\int(\phi + e\eta)^2 dx = \int(\phi^2 + 2e\eta\phi + \underbrace{e^2\eta^2}_{\text{discard}})dx$$

$$\delta\Pi = (\Pi + \delta\Pi) - \Pi = e\int\underbrace{(2c_1\phi_{,xxxx} - 2c_2\phi_{,xx} + 2c_3\phi + c_4)}_{\text{must vanish}}\eta\,dx + \text{B.C. terms}$$

$\boxed{4.7\text{-}2}$

Eq. 4.7-2a : $\frac{\partial F}{\partial \phi} - \frac{d}{dx}\frac{\partial F}{\partial \phi_{,x}} = 0$

$-50 - \frac{d}{dx}\phi_{,x} = 0$ i.e. $\phi_{,xx} + 50 = 0$

$\therefore \phi_{,x} = -50x + C_1$ & $\phi = -25x^2 + C_1 x + C_2$

Impose B.C.'s : $\phi = 0$ @ $x=0$, so $C_2 = 0$

$20 = -25L^2 + C_1 L$, so $C_1 = \frac{20}{L} + 25L$

$\phi = -25x^2 + \left(\frac{20}{L} + 25L\right)x$

$\boxed{4.7\text{-}3}$

Use Eq. 4.7-2a with $F = \frac{EI_z}{2} v_{,xx}^2 - qv$

$$\frac{\partial F}{\partial v} = -q \quad \text{and} \quad \frac{\partial F}{\partial v_{,xx}} = EI_z v_{,xx}$$

$$\frac{\partial F}{\partial v} + \frac{d^2}{dx^2}\frac{\partial F}{\partial v_{,xx}} \quad \text{yields} \quad (EI_z v_{,xx})_{,xx} - q = 0$$

If EI_z is constant, $EI_z v_{,xxxx} - q = 0$

$\boxed{4.7\text{-}4}$

$$\Pi_p = \int_0^L \left[\tfrac{1}{2} EI_z v_{,xx} - qv\right] dx \qquad \text{Let } v + \delta v = v + e\eta$$

$$\Pi_p + \delta\Pi_p = \int_0^L \left[\tfrac{1}{2} EI_z \left(v_{,xx} + e\eta_{,xx}\right)^2 - q\left(v + e\eta\right)\right] dx$$

$$= \int_0^L \left[\tfrac{1}{2} EI_z \left(v_{,xx}^2 + 2e\, v_{,xx}\eta_{,xx} + e^2\eta_{,xx}^2\right) - q(v + e\eta)\right] dx$$

discard

Two integrations by parts:

$$\int_0^L EI_z v_{,xx}\eta_{,xx}\, dx = -\int_0^L (EI_z v_{,xx})_{,x}\, \eta_{,x}\, dx \;+ \text{B.C. term}$$

$$= \int_0^L (EI_z v_{,xx})_{,xx}\, \eta\, dx \;+ \text{B.C. terms}$$

$$\Pi_p + \delta\Pi_p = \int_0^L \left[\tfrac{1}{2} EI_z v_{,xx}^2 + e\,(EI_z v_{,xx})_{,xx}\,\eta - q(v + e\eta)\right] dx \;+ \text{B.C. terms}$$

$$\delta\Pi_p = (\Pi_p + \delta\Pi_p) - \Pi_p = \int_0^L e\, \underbrace{\left[(EI_z v_{,xx})_{,xx} - q\right]}_{\text{must vanish}} \eta\, dx \;+ \text{B.C. terms}$$

$\boxed{4.7\text{-}5}$ $$F = \frac{D}{2}\left[w_{,xx}^2 + 2w_{,xx}w_{,yy} + w_{,yy}^2 - 2(1-\nu)(w_{,xx}w_{,yy} - w_{,xy}^2) - \frac{2q}{D}w\right]$$

$$\frac{\partial F}{\partial w} = -q \qquad \frac{\partial F}{\partial w_{,x}} = 0 \qquad \frac{\partial F}{\partial w_{,y}} = 0$$

$$\frac{\partial F}{\partial w_{,xx}} = D\left[w_{,xx} + w_{,yy} - (1-\nu)w_{,yy}\right]$$

$$\frac{\partial F}{\partial w_{,yy}} = D\left[w_{,xx} + w_{,yy} - (1-\nu)w_{,xx}\right]$$

$$\frac{\partial F}{\partial w_{,xy}} = 2D(1-\nu)w_{,xy}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{,xx}} = D\left[w_{,xxxx} + w_{,xxyy} - (1-\nu)w_{,xxyy}\right] = D\left[w_{,xxxx} + \nu w_{,xxyy}\right]$$

$$\frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial w_{,yy}} = D\left[w_{,xxyy} + w_{,yyyy} - (1-\nu)w_{,xxyy}\right] = D\left[w_{,yyyy} + \nu w_{,xxyy}\right]$$

$$\frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial w_{,xy}} = 2D(1-\nu)w_{,xxyy}$$

Substitute into $\dfrac{\partial F}{\partial w} + \dfrac{\partial^2}{\partial x^2}\dfrac{\partial F}{\partial w_{,xx}} + \dfrac{\partial^2}{\partial y^2}\dfrac{\partial F}{\partial w_{,yy}} + \dfrac{\partial^2}{\partial x \partial y}\dfrac{\partial F}{\partial w_{,xy}} = 0$

$$-q + D\left[w_{,xxxx} + \nu w_{,xxyy} + w_{,yyyy} + \nu w_{,xxyy} + 2(1-\nu)w_{,xxyy}\right] = 0$$

$$-q + D\left(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right) = 0$$

$$\nabla^4 w = 0$$

$$\nabla^4 w = \frac{q}{D}$$

$\boxed{4.7\text{-}6}$

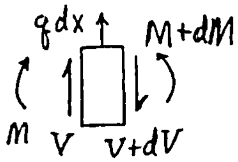
$$\frac{\partial F}{\partial M} = \frac{M}{EI}, \quad \frac{\partial F}{\partial M_{,x}} = v_{,x}$$

$$\frac{d}{dx}\frac{\partial F}{\partial M_{,x}} - \frac{\partial F}{\partial M} = v_{,xx} - \frac{M}{EI} = 0$$

This is the moment-curvature relation.

$$\frac{\partial F}{\partial v} = q, \quad \frac{\partial F}{\partial v_{,x}} = M_{,x}$$

$$\frac{d}{dx}\frac{\partial F}{\partial v_{,x}} - \frac{\partial F}{\partial v} = M_{,xx} - q = 0 \quad \text{(equil. equation)}$$



$$\left.\begin{array}{l} \dfrac{dV}{dx} = q \\ \dfrac{dM}{dx} = V \end{array}\right\} \frac{d^2M}{dx^2} = q$$

$\boxed{4.7\text{-}7}$

For a rigid bar, use a linear displacement field.

$$v = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{d}\} = \left\lfloor \frac{L-x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix} v_i \\ v_j \end{Bmatrix}$$

As with a linear spring, strain energy (here the increment dU) is

$$dU = \frac{1}{2} v\, dF = \frac{1}{2} v\,(k v\, dx) = \frac{1}{2} k v^2 dx = \frac{1}{2} k v^T v\, dx$$

$$U = \frac{1}{2}\int_0^L v^T v\, k\, dx = \frac{1}{2}\begin{Bmatrix} v_i \\ v_j \end{Bmatrix}^T \underbrace{\int_0^L \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor k\, dx}_{[\underset{\sim}{k}]} \begin{Bmatrix} v_i \\ v_j \end{Bmatrix}$$

$$[\underset{\sim}{k}] = \frac{k}{L^2}\int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} dx = \frac{k}{L^2}\begin{bmatrix} L^3/3 & L^3/6 \\ L^3/6 & L^3/3 \end{bmatrix} = \frac{kL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\boxed{4.8-1}$

(a) $\Pi_p = \int_0^L \frac{1}{2} EI\, v_{,xx}^2\, dx - \int_0^L q\, v\, dx$

$v = \lfloor N \rfloor \{d\}, \quad v_{,xx} = \lfloor N_{,xx} \rfloor \{d\}$

$$\Pi_p = \frac{1}{2}\{d\}^T \underbrace{\int_0^L \lfloor N_{,xx} \rfloor^T EI \lfloor N_{,xx} \rfloor dx}_{[k]} \{d\} - \{d\}^T \underbrace{\int_0^L \lfloor N \rfloor^T q\, dx}_{\{r_q\}}$$

$\left\{\frac{\partial \Pi_p}{\partial d}\right\} = \{0\}$ yields $\quad [k]\{d\} = \{r_q\}$

(b) Henceforth $\sim$ denotes a matrix.

$$\Pi = \int_0^L \left(\frac{M^2}{2EI} + M_{,x} v_{,x} + q v\right) dx \qquad \begin{matrix} M = N_m\, m_e \\ v = N_v\, v_e \end{matrix}$$

$$\Pi = \frac{1}{2} m_e^T \underbrace{\int_0^L N_m^T N_m \frac{dx}{EI}}_{H_{11}} m_e + m_e^T \underbrace{\int_0^L N_{m,x} N_{v,x} dx}_{H_{12}} v_e$$

$$+ v_e^T \underbrace{\int_0^L N_v^T q\, dx}_{r_q}$$

Same as $v_e^T H_{12}^T m_e$

$$\Pi = \frac{1}{2} m_e^T H_{11} m_e + m_e^T H_{12} v_e + v_e^T r_q$$

$\left\{\frac{\partial \Pi}{\partial m_e}\right\} = \{0\} = H_{11} m_e + H_{12} v_e = 0$

$\left\{\frac{\partial \Pi}{\partial v_e}\right\} = \{0\} = H_{12}^T m_e + r_q = 0$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & 0 \end{bmatrix} \begin{Bmatrix} m_e \\ v_e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -r_q \end{Bmatrix}$$

$\ldots_e \;, \; p_{,x} = N_{,x}\, p_e$, etc.

$$\Pi = p_e^T \int_0^L \left(N_{,x}^T N_{,x} + N_{,y}^T N_{,y} + N_{,z}^T N_{,z}\right) dV\, p_e - \omega^2 p_e^T \int_0^L \frac{1}{c^2} N^T N\, dV\, p_e$$

$\Pi = p_e^T H\, p_e - \omega^2 p_e Q\, p_e$

$\left\{\frac{\partial \Pi}{\partial p_e}\right\} = \{0\}$ yields $\quad (H - \omega^2 Q)\, p_e = 0$

(d) $\Pi_p = \iint \left(\frac{1}{2} \kappa^T D \kappa - q w\right) dx\, dy \qquad \kappa = \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{Bmatrix} \qquad D = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$

$w = N d \qquad \kappa = \begin{Bmatrix} N_{,xx} \\ N_{,yy} \\ 2N_{,xy} \end{Bmatrix} d = B d$

$$\Pi_p = \frac{1}{2} d^T \underbrace{\iint B^T D B\, dx\, dy}_{k}\, d - d^T \underbrace{\int N^T q\, dx\, dy}_{r}$$

$\frac{\partial \Pi_p}{\partial d} = 0$ yields

$k d = r$

$\boxed{4.9\text{-}1}$

$$\theta = 30^\circ \quad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} a_1 & -0.134 & -0.5 \\ a_4 & 0.5 & -0.134 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

Node 1: $x = y = 0$ Node 2: $x=1,\ y=0$ Node 3: $x=0,\ y=1$

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_4 \end{Bmatrix} \qquad \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} a_1 - 0.134 \\ a_4 + 0.5 \end{Bmatrix} \qquad \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} a_1 - 0.5 \\ a_4 - 0.134 \end{Bmatrix}$$

Final positions:

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_4 \end{Bmatrix}, \qquad \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} a_1 + 0.866 \\ a_4 + 0.5 \end{Bmatrix}, \qquad \begin{Bmatrix} x_3 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} a_1 - 0.5 \\ a_4 + 0.866 \end{Bmatrix}$$

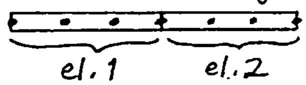
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{0.866^2 + 0.5^2} = 1 \checkmark$$

$$\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = \sqrt{1.366^2 + 0.366^2} = 1.414 = \sqrt{2} \checkmark$$

$$\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} = \sqrt{0.5^2 + 0.866^2} = 1 \checkmark$$

$\boxed{4.9\text{-}2}$

Π_p contains 2^{nd} derivatives of v so $m = 2$ and continuity of first derivatives ($v_{,x}$) is required between elements (requirement 2). This the proposed element cannot do. Also, imagine two adjacent els.



If all d.o.f. of el. 1 suppressed, but not all d.o.f. of el. 2, then $v_{,x} = 0$ in el. 1 but not in el. 2 @ juncture.

$\boxed{4.9\text{-}3}$

Do not want to favor one coord. direction over another.

Total d.o.f.	Cubic d.o.f.	
11	1	xyz
13	3	x^3, y^3, z^3
14	4	x^3, y^3, z^3, xyz
16	6	$x^2y, xy^2, y^2z, yz^2, z^2x, zx^2$
17	7	$x^2y, xy^2, y^2z, yz^2, z^2x, zx^2,$ xyz
19	9	$x^3, y^3, z^3,$ $x^2y, xy^2, y^2z, yz^2, z^2x, zx^2$ (i.e. all but xyz)

4.9-4

(a) If consistent nodal loads are used, both meshes give exact displacements at nodes (see Ref. 2.5). Between nodes, both meshes give approximate displacements (and stresses). But since linear elements give only straight-line plots of displacement versus axial coordinate, quadratic elements should be able to provide a better fit to displacements produced by smoothly-varying loads (note that if loads do not vary smoothly, it is not obvious which kind of element will be better — for example, a concentrated load applied to the interior node of a quadratic element will produce poor results). Similar remarks can be made for stress fields calculated from element displacement gradients.

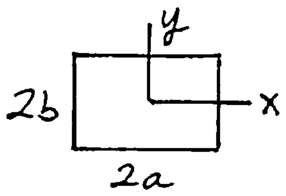
(b) Apply Eqs. 4.9-2, 4.9-3, 4.9-5. Let h = element length, whether linear or quadratic. Orders of error:

	Linear element	Quadratic element
Displacement error	$O(h^2)$	$O(h^3)$
Stress error	$O(h)$	$O(h^2)$

Therefore, when element lengths are halved, approximate reduction factors for error are

	Linear element	Quadratic element
Displacement	$\frac{1}{4}$	$\frac{1}{8}$
Stress	$\frac{1}{2}$	$\frac{1}{4}$

$\boxed{4.10\text{-}1}$



Uniform t, five β's, $\nu = 0$

$$[\underset{\sim}{E}] = E \lceil 1 \quad 1 \quad 1/2 \rfloor, \quad [\underset{\sim}{E}]^{-1} = \frac{1}{E} \lceil 1 \quad 1 \quad 2 \rfloor$$

$$[\underset{\sim}{P}]^T [\underset{\sim}{E}]^{-1} [\underset{\sim}{P}] = \frac{1}{E} \begin{bmatrix} 1 & 0 & 0 \\ y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{E} \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ y & y^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & x & x^2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\int dA = 4ab, \quad \int x\, dA = \int y\, dA = 0, \quad \int x^2 dA = \frac{2b(2a)^3}{12} = \frac{4a^3 b}{3}$$

$$\int y^2 dA = \frac{2a(2b)^3}{12} = \frac{4ab^3}{3}$$

$$[H] = \int [P]^T [\underset{\sim}{E}]^{-1} [\underset{\sim}{P}]\, dV = \frac{4abt}{3} \left\lceil 1 \quad \frac{b^3}{3} \quad 1 \quad \frac{a^3}{3} \quad 2 \right\rfloor$$

$\boxed{4.10-2}$

$$M = \lfloor 1 \quad x \rfloor \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \quad \text{or} \quad M = [\underset{\sim}{P}]\{\underset{\sim}{\beta}\} \quad \text{where} \quad [\underset{\sim}{P}] = \lfloor 1 \quad x \rfloor$$

$$[\underset{\sim}{H}] = \frac{1}{EI_z}\int_0^L \begin{Bmatrix} 1 \\ x \end{Bmatrix} \lfloor 1 \quad x \rfloor \, dx = \frac{1}{EI_z}\begin{bmatrix} L & L^2/2 \\ L^2/2 & L^3/3 \end{bmatrix}$$

$$[\underset{\sim}{H}]^{-1} = \frac{12EI_z}{L^4}\begin{bmatrix} L^3/3 & -L^2/2 \\ -L^2/2 & L \end{bmatrix} = \frac{12EI_z}{L^3}\begin{bmatrix} L^2/3 & -L/2 \\ -L/2 & 1 \end{bmatrix}$$

As in Eq. 4.10-19, $[\underset{\sim}{R}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & L \end{bmatrix}$, $[\underset{\sim}{G}] = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & L \end{bmatrix}$

$$[\underset{\sim}{H}]^{-1}[\underset{\sim}{G}] = \frac{12EI_z}{L^3}\begin{bmatrix} -L/2 & -L^2/3 & L/2 & -L^2/6 \\ 1 & L/2 & -1 & L/2 \end{bmatrix}$$

$$[\underset{\sim}{k}] = [\underset{\sim}{G}]^T[\underset{\sim}{H}]^{-1}[\underset{\sim}{G}] = \frac{12EI_z}{L^3}\begin{bmatrix} 1 & L/2 & -1 & L/2 \\ L/2 & L^2/3 & -L/2 & L^2/6 \\ -1 & -L/2 & 1 & -L/2 \\ L/2 & L^2/6 & -L/2 & L^2/3 \end{bmatrix}$$