16.1-1

(a) Cylindrical part; x-direction axial:

$$\sigma_x = \frac{pR}{2t} \quad \sigma_y = \frac{pR}{t} \quad \tau_{xy} = 0$$

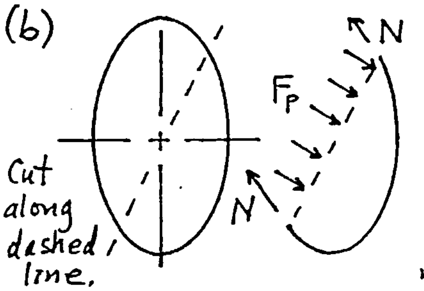
$$N_x = \sigma_x t = \frac{pR}{2}, \; N_y = \sigma_y t = pR, \; N_{xy} = 0$$

Hemispherical cap: if n and s are any two surface-tangent directions,

$$\sigma_n = \sigma_s = \frac{pR}{2t}, \; \tau_{ns} = 0$$

$$N_n = N_s = \frac{pR}{2}, \; N_{ns} = 0$$

(b)

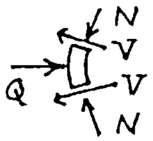


Membrane forces N not parallel to force F_p from pressure. Transverse shear forces must also appear.

Transverse shear forces, and consequent bending, vanish only for circular cylinder.

16.1-2

(a)

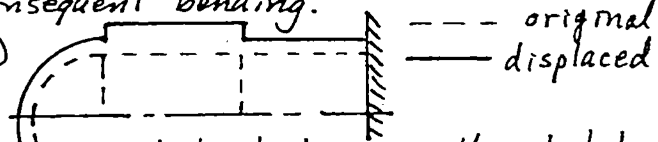


Forces N: membrane theory

Forces V: bending theory

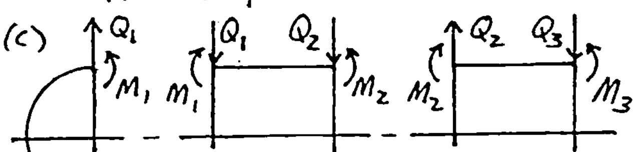
Shrink slice shown to zero size to span only point load Q. Then forces N are parallel and must become infinite to carry Q. Can't happen. Q carried by V, with consequent bending.

(b)



Also, but not shown in the sketch, is axial disp. associated with axial ϵ.

(c)



Q_i & M_i are line loads uniformly distributed around parallels. Direction of M_2 quessed at. Analysis will show $M_1 = 0$.

$\boxed{16.2\text{-}1}$

$$\epsilon_s = \frac{d}{ds}(\delta_a + \delta_c) + \frac{w}{R} = \frac{d}{ds}\left[u\left(1+\frac{z}{R}\right) - z w_{,s}\right] + \frac{w}{R}$$

$$\epsilon_s = u_{,s} + z\frac{u_{,s}}{R} - z\frac{u}{R^2}R_{,s} - z w_{,ss} + \frac{w}{R}$$

$$\epsilon_s = u_{,s} + \frac{w}{R} + z\left(\frac{u_{,s}}{R} - w_{,ss} - \frac{u}{R^2}R_{,s}\right)$$

Term not in Eq. 16.2-2

$\boxed{16.2\text{-}2}$

(a) $\epsilon_m = 0$ gives $u_{,s} = -\frac{w}{R}$ or $\frac{u_{,s}}{R} = -\frac{w}{R^2}$

$\kappa = 0$ gives $\frac{u_{,s}}{R} - w_{,ss} = 0$ or $w_{,ss} + \frac{w}{R^2} = 0$

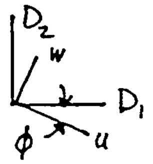
Integrate; let $\phi = \frac{s}{R}$:

$w = b_1 \sin\phi - b_2 \cos\phi$; hence $u_{,s} = \frac{-b_1 \sin\phi + b_2 \cos\phi}{R}$

or $u_{,\phi} = -b_1 \sin\phi + b_2 \cos\phi$

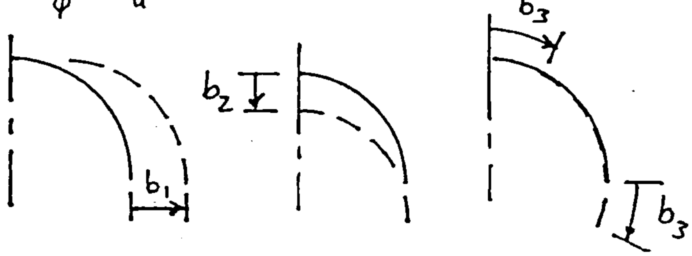
$u = b_1 \cos\phi + b_2 \sin\phi + b_3$

(b)

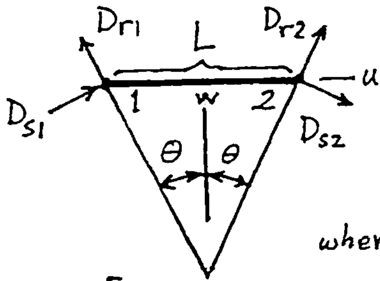


$D_1 = u\cos\phi + w\sin\phi = b_1 + b_3 \cos\phi$

$D_2 = -u\sin\phi + w\cos\phi = -b_2 - b_3 \sin\phi$



$\boxed{16.2\text{-}3}$



$$\theta = \arcsin \frac{L/2}{R}$$

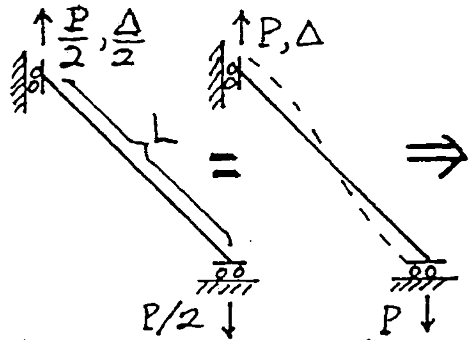
$$[\underset{\sim}{k}_{global}] = [\underset{\sim}{T}]^T [\underset{\sim}{k}_{Eq. 16.2\text{-}6}] [\underset{\sim}{T}]$$

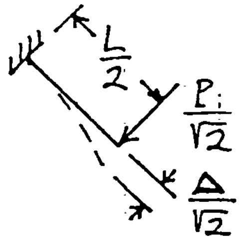
where $[\underset{\sim}{T}]$ is

$$\begin{Bmatrix} u_1 \\ u_2 \\ w_1 \\ \psi_1 \\ w_2 \\ \psi_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_{s1} \\ D_{r1} \\ \psi_1 \\ D_{s2} \\ D_{r2} \\ \psi_2 \end{Bmatrix}$$

$\boxed{16.2\text{-}4}$

Exact: $\Delta = 0.1488\, PR^3/EI$





$$\frac{\Delta}{\sqrt{2}} = \frac{\frac{P}{\sqrt{2}}\left(\frac{L}{2}\right)^3}{3EI}$$

(Element bends as cubic, so beam theory applies.)

$$\Delta = \frac{PL^3}{24EI}$$

(a) $L = \sqrt{2}R$, $\Delta = 0.1179\, PR^3/EI$ 20.8% low

(b) $L = \frac{\pi R}{2}$, $\Delta = 0.1615\, PR^3/EI$ 8.5% high

$\boxed{16.2\text{-}5}$

$$u = a_1 + a_2 s \qquad \epsilon_m = a_2 + \frac{a_3 + a_4 s + a_5 s^2 + a_6 s^3}{R}$$

$$w = a_3 + a_4 s + a_5 s^2 + a_6 s^3 \qquad \text{or}$$

$$\epsilon_m = b_1 + b_2 s + b_3 s^3 + b_4 s^4$$

where

$$b_1 = a_2 + \frac{a_3}{R}, \quad b_2 = \frac{a_4}{R}, \quad b_3 = \frac{a_5}{R}, \quad b_4 = \frac{a_6}{R}$$

$$\epsilon_m^2 = b_1^2 + b_2^2 s^2 + b_3^2 s^4 + b_4^2 s^6 + 2b_1 b_3 s^2 + 2b_2 b_4 s^4$$

$$+ (\text{terms with odd powers of } s)$$

Odd powers will integrate to zero, so ignore.

Let β_i = constants; thus

$$\int_{-L/2}^{L/2} \epsilon_m^2 \, ds = \beta_1 b_1^2 + \beta_2 b_2^2 + \beta_3 b_3^2 + \beta_4 b_4^2 + \beta_5 b_1 b_3 + \beta_6 b_2 b_4$$

$$\int_{-L/2}^{L/2} \epsilon_m^2 \, ds = 0 \text{ implies } b_1 = b_2 = b_3 = b_4 = 0$$

i.e. that $a_2 + \frac{a_3}{R} = a_4 = a_5 = a_6 = 0$

$\boxed{16.2\text{-}6}$

Eqs. 16.2-9.

$$\psi = w_{,s} = a_4 + 2a_5 s + 3a_6 s^2$$

node	s
1	$-L/2$
2	$+L/2$

$$\begin{Bmatrix} u_1 \\ w_1 \\ \psi_1 \\ u_2 \\ w_2 \\ \psi_2 \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{L}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{L}{2} & \frac{L^2}{4} & -\frac{L^3}{8} \\ 0 & 0 & 0 & 1 & -L & -\frac{3L^2}{2} \\ 1 & \frac{L}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{L}{2} & \frac{L^2}{4} & \frac{L^3}{8} \\ 0 & 0 & 0 & 1 & L & \frac{3L^2}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

(b) Eqs. 16.2-12

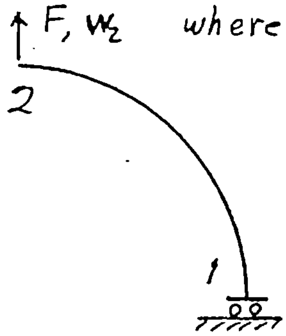
$$\psi = w_{,s} = -\frac{1}{R}\left(2a_3 + 6a_4\phi + 12a_5\phi^2 + 20a_6\phi^3\right)$$

Node 1: $\phi = -L/2R$; Node 2: $\phi = +L/2R$

Let $\alpha = L/2R$

$$\begin{Bmatrix} u_1 \\ w_1 \\ \psi_1 \\ u_2 \\ w_2 \\ \psi_2 \end{Bmatrix} = \begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 & \alpha^4 & -\alpha^5 \\ 0 & -1 & 2\alpha & -3\alpha^2 & 4\alpha^3 & -5\alpha^4 \\ 0 & 0 & -\frac{2}{R} & \frac{6\alpha}{R} & -\frac{12\alpha^2}{R} & \frac{20\alpha^3}{R} \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \\ 0 & -1 & -2\alpha & -3\alpha^2 & -4\alpha^3 & -5\alpha^4 \\ 0 & 0 & -\frac{2}{R} & -\frac{6\alpha}{R} & -\frac{12\alpha^2}{R} & -\frac{20\alpha^3}{R} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

16.2-7



where $F = \frac{P}{2}$ (P = a diametral force on the complete ring)

$u_1 = u_2 = \psi_1 = \psi_2 = 0$

$$\frac{EI}{R^3}\begin{bmatrix} k_{22} & k_{25} \\ k_{52} & k_{55} \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix} \qquad k_{52} = k_{25},\ k_{22} = k_{55}$$

First eq. gives $w_1 = -\frac{k_{25}}{k_{22}} w_2$

Second eq. becomes

$$\left(-\frac{k_{25}^2}{k_{22}} + k_{55}\right) w_2 = F\frac{R^3}{EI}$$

$\beta = \frac{\pi R/2}{2R} = \frac{\pi}{4}$ hence $k_{22} = k_{25} = 42.9128$

$k_{25} = 39.3948$

Therefore $w_2 = 0.1482\, F\frac{R^3}{EI}$

Change in dia. of ring $= 2w_2 = 2\left(0.1482 \frac{P}{2}\frac{R^3}{EI}\right) = 0.1482\frac{PR^3}{EI}$

(0.4% low)

exact: $0.1488\frac{PR^3}{EI}$

$\boxed{16.2\text{-}8}$

In elements of Eqs. 16.2-9, constraint $\epsilon_m = 0$ is not explicit; it is implicitly imposed (e.g. Eq. 16.2-11). If ϵ_m were ignored, membrane stiffness would be zero, not infinite ($\epsilon_m = 0$); singular $[\underset{\sim}{k}]$.

$\boxed{16.2\text{-}9}$

(a) $$\epsilon_m = u_{,s} + \frac{w}{R} = a_2 + \frac{a_3 + a_4 s}{R}$$

$$\gamma_{zs} = w_{,s} - \beta = a_4 - (a_5 + a_6 s)$$

$\epsilon_m = 0$ for all s implies

$$\left.\begin{array}{l} a_2 + \dfrac{a_3}{R} = a_4 = 0 \\ \gamma_{zs} = 0 \text{ for all } s \text{ implies} \\ a_4 - a_5 = a_6 = 0 \end{array}\right\} \therefore a_5 = 0$$

For a straight el., $R \to \infty$, and

$$a_2 = a_4 - a_5 = a_6 = 0$$

(b) Evaluate at $s = 0$:

$$\left.\begin{array}{l} \epsilon_m = a_2 + \dfrac{a_3}{R} \\ \gamma_{zs} = a_4 - a_5 \end{array}\right\} \text{constraints } a_2 + \frac{a_3}{R} = a_4 - a_5 = 0$$

$\boxed{16.2\text{-}10}$

$$\gamma_{zs} = w_{,s} - \beta = \frac{2a_5}{L} + \frac{4a_6}{L}\xi - a_7 - a_8\xi - a_9\xi^2$$

$$\gamma_{zs} = \left(\frac{2a_5}{L} - a_7\right) + \left(\frac{4a_6}{L} - a_8\right)\xi - (a_9)\xi^2 \quad (A)$$

$\gamma_{zs} = 0$ for all ξ implies that all three parenthetic expressions in Eq. (A) vanish.
Note; $a_9 = 0$ implies $\beta_{,ss} = 0$.

Rewrite Eq. (A):

$$0 = \left(\frac{2a_5}{L} - a_7 - \frac{a_9}{3}\right) + \left(\frac{4a_6}{L} - a_8\right)\xi - \left[\xi^2 - \frac{1}{3}\right]a_9$$

At Gauss pts, $\xi = \pm\frac{1}{\sqrt{3}}$, $\left[\xi^2 - \frac{1}{3}\right] = 0$.

Constraints are then

$$\frac{2a_5}{L} - a_7 - \frac{a_9}{3} = 0$$

$$\frac{4a_6}{L} - a_8 = 0$$

$\boxed{16.3\text{-}1}$

$$U_m = \frac{1}{2}\int_{-L/2}^{L/2} \frac{Et}{1-\nu^2} \begin{Bmatrix} \epsilon_{ms} \\ \epsilon_{m\theta} \end{Bmatrix}^T \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{Bmatrix} \epsilon_{ms} \\ \epsilon_{m\theta} \end{Bmatrix} 2\pi R\, ds$$

Unrestrained axially: $\epsilon_{ms} = -\nu\epsilon_{m\theta} = -\nu \frac{W}{R}$

$$U_m = \frac{1}{2}\int_{-L/2}^{L/2} \frac{Et}{1-\nu^2} \frac{w^2}{R^2} \begin{Bmatrix} -\nu \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{Bmatrix} -\nu \\ 1 \end{Bmatrix} 2\pi R\, ds$$

$$U_m = \frac{1}{2}\int_{-L/2}^{L/2} \frac{Et}{1-\nu^2} \frac{w^2}{R^2}(1-\nu^2)\, 2\pi R\, ds = \frac{1}{2}\int \frac{Et}{R^2} w^2\, dA$$

Compare with Eq. 8.8-1:
foundation modulus is $\beta = \frac{Et}{R^2}$

16.3-2

$$U = \frac{1}{2}\int_{-L/2}^{L/2} E\epsilon_{m\theta}^2 (2\pi R t)\,ds + \frac{1}{2}\int_{-L/2}^{L/2} D\kappa_s^2 (2\pi R)\,ds \qquad \text{(see Fig. 16.3-1b)}$$

where, from Eq. 16.3-3, $\epsilon_{m\theta} = \frac{w}{R}$, $\kappa_s = \frac{d^2w}{ds^2}$. Also $D = \frac{Et^3}{12(1-\nu^2)}$

$$[\underset{\sim}{k}] = [\underset{\sim}{k}_m] + [\underset{\sim}{k}_b] = \frac{2\pi E t}{R}\int_{-L/2}^{L/2} \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor\, ds + 2\pi D R \int_{-L/2}^{L/2} \lfloor \underset{\sim}{N}_{,ss} \rfloor^T \lfloor \underset{\sim}{N}_{,ss} \rfloor\, ds$$

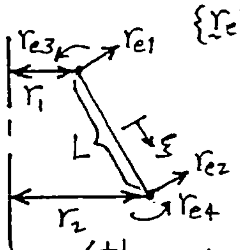
$[\underset{\sim}{k}_m]$ has the form of a mass matrix (Eq. 11.2-6) with $2\pi E t/R$ in place of m; thus

$$[\underset{\sim}{k}_m] = \frac{2\pi E t L}{420 R}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{symm.} & & & 4L^2 \end{bmatrix}$$

$[\underset{\sim}{k}_b]$ has the form of a standard beam element matrix with $2\pi R D$ in place of EI; thus

$$[\underset{\sim}{k}_b] = \frac{2\pi R D}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm.} & & & 4L^2 \end{bmatrix}$$

16.3-3



$$\{\underset{\sim}{r_e}\} = \int [\underset{\sim}{N}]^T p \, dA \qquad p = \text{constant}$$

$$dA = 2\pi r \, ds = 2\pi r \frac{L}{2} d\xi = \pi r L \, d\xi$$

$$r = \frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2}\xi$$

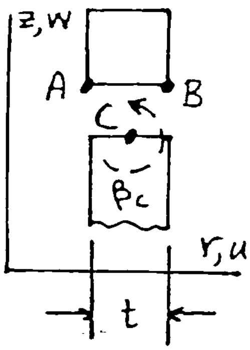
$$\{\underset{\sim}{r_e}\} = \int_{-1}^{+1} \frac{p\pi L}{2} \begin{Bmatrix} (1-\xi)/2 \\ (1+\xi)/2 \\ (1-\xi^2)L/8 \\ -(1-\xi^2)L/8 \end{Bmatrix} \left[(r_1+r_2) + (r_2-r_1)\xi\right] d\xi$$

$$\{\underset{\sim}{r_e}\} = \frac{p\pi L}{2}(r_1+r_2) \begin{Bmatrix} 1 \\ 1 \\ \frac{L}{8}\left(\frac{4}{3}\right) \\ -\frac{L}{8}\left(\frac{4}{3}\right) \end{Bmatrix} + \frac{p\pi L}{2}(r_2-r_1) \begin{Bmatrix} -\frac{1}{3} \\ +\frac{1}{3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\underset{\sim}{r_e}\} = \frac{p\pi L}{2}(r_1+r_2) \begin{Bmatrix} 1 \\ 1 \\ L/6 \\ -L/6 \end{Bmatrix} + \frac{p\pi L(r_2-r_1)}{6} \begin{Bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$\boxed{16.3-4}$

(a)



Axially symmetric:

$$\begin{Bmatrix} w_c \\ u_c \\ \beta_c \end{Bmatrix} = \underset{3\times 4}{[T]} \begin{Bmatrix} w_A \\ u_A \\ w_B \\ u_B \end{Bmatrix}$$

where, following arguments in Section 8.5, [T] is

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{t} & 0 & \frac{1}{t} & 0 \end{bmatrix}$$

(b) Circumferential displacement v enters.

$$\begin{Bmatrix} w_c \\ u_c \\ v_c \\ \beta_c \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{t} & 0 & 0 & \frac{1}{t} & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_A \\ u_A \\ v_A \\ w_B \\ u_B \\ v_B \end{Bmatrix}$$

(c) For the case without axial symmetry, we write

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 \\ -\frac{1}{t} & 0 & 0 & \frac{1}{t} & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} w_A \\ u_A \\ v_A \\ w_B \\ u_B \\ v_B \\ w_c \\ u_c \\ v_c \\ \beta_c \end{Bmatrix} = \{\underset{\sim}{0}\}$$

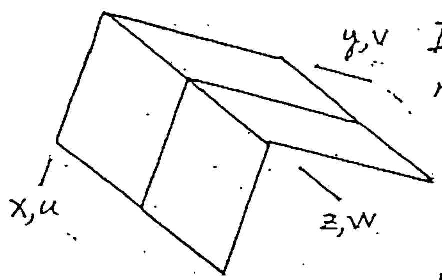
To account for other d.o.f. in the structure above & below section [illegible], expand the rectangular matrix left & right by adding zeros, & expand the column vector up & down by adding d.o.f.

If there is axial symmetry, we can omit the information content associated with v_A, v_B, and v_c.

16.4-1

Let s be a coordinate parallel to an element edge. Element-normal displacement is cubic or quadratic in s (depending on element type), while element-tangent (but edge-normal) displacement is linear in s. Thus there is incompatibility, most strongly if adjacent elements meet at a right angle. There is no incompatibility if elements are coplanar, and such a condition is approached as the mesh is indefinitely refined.

16.4-2



In-plane stiffnesses much greater than bending stiffnesses. Therefore bending response of each side is modeled well if nodal d.o.f. ψ_y and ψ_x set to zero (retain ψ_z). Indeed, each side could be analyzed separately, except that ψ_z provides elastic support to element edges that lie on the z axis.

16.5-1

Let j and k refer to upper and lower surface nodes respectively. The N's are given by Eqs. 6.4-1.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{k=1}^{8} N_k \frac{1-\zeta}{2} \begin{Bmatrix} x_k \\ y_k \\ z_k \end{Bmatrix} + \sum_{j=1}^{8} N_j \frac{1+\zeta}{2} \begin{Bmatrix} x_j \\ y_j \\ z_j \end{Bmatrix}$$

$\boxed{16.5-2}$

(a) $$[\underset{\sim}{J}] = \begin{bmatrix} x_{,\xi} & y_{,\xi} & z_{,\xi} \\ x_{,\eta} & y_{,\eta} & z_{,\eta} \\ x_{,\zeta} & y_{,\zeta} & z_{,\zeta} \end{bmatrix}$$ From Eq. 16.5-2,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum \begin{bmatrix} N_i & 0 & 0 & N_i\zeta/2 & 0 & 0 \\ 0 & N_i & 0 & 0 & N_i\zeta/2 & 0 \\ 0 & 0 & N_i & 0 & 0 & N_i\zeta/2 \end{bmatrix} \underset{6\times 1}{\begin{Bmatrix} x_i \\ y_i \\ z_i \\ t_i l_{3i} \\ t_i m_{3i} \\ t_i n_{3i} \end{Bmatrix}}$$

Terms in (e.g.) col. 1 of $[\underset{\sim}{J}]$ are

$$\frac{\partial x}{\partial \xi} = \sum \lfloor N_{i,\xi} \quad 0 \quad 0 \quad \tfrac{\zeta}{2}N_{i,\xi} \quad 0 \quad 0 \rfloor \underset{6\times 1}{\{\ \}}$$

$$\frac{\partial x}{\partial \eta} = \sum \lfloor N_{i,\eta} \quad 0 \quad 0 \quad \tfrac{\zeta}{2}N_{i,\eta} \quad 0 \quad 0 \rfloor \underset{6\times 1}{\{\ \}}$$

$$\frac{\partial x}{\partial \zeta} = \sum \lfloor 0 \quad 0 \quad 0 \quad \tfrac{N_i}{2} \quad 0 \quad 0 \rfloor \underset{6\times 1}{\{\ \}}$$

(the N_i are functions of ξ and η but not of ζ). Sum over no. of nodes in element.

(b) Here $z_i = l_{3i} = m_{3i} = 0$, $n_{3i} = 1$, $t_i = t$.

Note that $\sum N_i = 1$ and $\sum N_{i,\xi} = \sum N_{i,\eta} = \sum N_{i,\zeta} = 0$.

$$\frac{\partial x}{\partial \xi} = \sum N_{i,\xi} x_i \qquad \frac{\partial y}{\partial \xi} = \sum N_{i,\xi} y_i \qquad \frac{\partial z}{\partial \xi} = 0$$

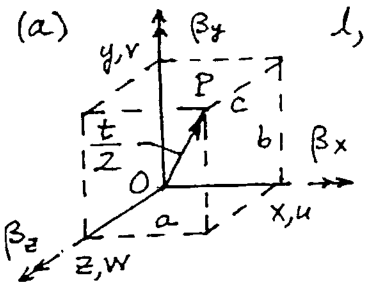
$$\frac{\partial x}{\partial \eta} = \sum N_{i,\eta} x_i \qquad \frac{\partial y}{\partial \eta} = \sum N_{i,\eta} y_i \qquad \frac{\partial z}{\partial \eta} = 0$$

$$\frac{\partial x}{\partial \zeta} = 0 \qquad \frac{\partial y}{\partial \zeta} = 0 \qquad \frac{\partial z}{\partial \zeta} = \frac{t}{2}$$

Thus $[\underset{\sim}{J}]$ becomes as for a flat plate.

16.5-3

(a)



l, m, n = dir. cosines of line OP

$a = l\frac{t}{2}$, $b = m\frac{t}{2}$, $c = n\frac{t}{2}$

$u_P = c\beta_y - b\beta_z$

$v_P = -c\beta_x + a\beta_z$

$w_P = b\beta_x - a\beta_y$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \sum N_i \zeta \frac{t_i}{2} \begin{bmatrix} 0 & n_i & -m_i \\ -n_i & 0 & l_i \\ m_i & -l_i & 0 \end{bmatrix} \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{Bmatrix}$$

(b) E.g., if $\underset{\sim}{V}_3$ is x-parallel, then $l_i = 1$, $m_i = n_i = 0$, β_y becomes α and β_z becomes β (see Fig. 16.5-2). And, in Fig. 16.5-2b, $\underset{\sim}{V}_1$ becomes y-parallel & $\underset{\sim}{V}_2$ becomes z-parallel. Thus, $[\underset{\sim}{\mu}]\{\beta\}$ products in above eq. & in Eq. 16.5-6 become respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_x \\ \alpha \\ \beta \end{Bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix}$$

These agree.

$\boxed{16.5\text{-}4}$

$$\begin{Bmatrix} \epsilon_x \\ \vdots \\ \gamma_{zx} \end{Bmatrix} = \underset{6\times 9}{[\underset{\sim}{H}]} \begin{Bmatrix} u_{,x} \\ \vdots \\ w_{,z} \end{Bmatrix} = [\underset{\sim}{H}] \begin{bmatrix} \underset{\sim}{\Gamma} & 0 & 0 \\ 0 & \underset{\sim}{\Gamma} & 0 \\ 0 & 0 & \underset{\sim}{\Gamma} \end{bmatrix} \begin{Bmatrix} u_{,\xi} \\ \vdots \\ w_{,\zeta} \end{Bmatrix}$$

where $\underset{3\times 3}{[\underset{\sim}{J}]} = \underset{3\times 3}{[\underset{\sim}{\Gamma}]}^{-1}$ is given by Eq. 6.5-2, $[H]$ is given by Eq. 6.5-3, and the column vector $\lfloor u_{,\xi} \; \cdots \; w_{,\zeta} \rfloor^T$ by Eq. 16.5-9. We want

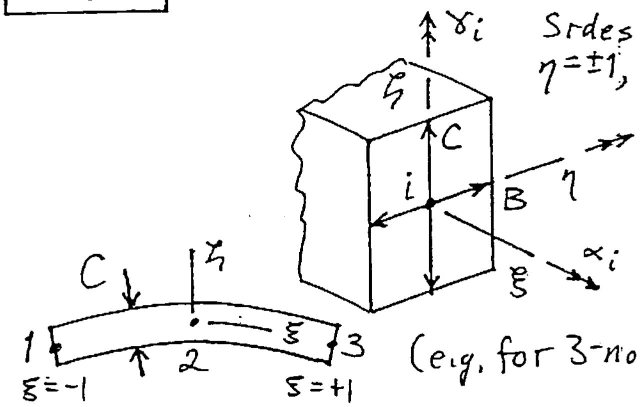
$$[\underset{\sim}{\bar{B}}_i] = \begin{bmatrix} \underset{\sim}{\Gamma} & 0 & 0 \\ 0 & \underset{\sim}{\Gamma} & 0 \\ 0 & 0 & \underset{\sim}{\Gamma} \end{bmatrix} [\text{square array in Eq. 16.5-9}]$$

Let $A_i = \Gamma_{11} N_{i,\xi} + \Gamma_{12} N_{i,\eta}$ $\quad B_i = \Gamma_{21} N_{i,\xi} + \Gamma_{22} N_{i,\eta}$ $\quad C_i = \Gamma_{31} N_{i,\xi} + \Gamma_{32} N_{i,\eta}$. Then $[\underset{\sim}{\bar{B}}_i] =$

$$\begin{bmatrix}
A_i & 0 & 0 & -(A_i\zeta_i + \Gamma_{13}N_i)\ell_{2i}t_i/2 & (A_i\zeta_i + \Gamma_{13}N_i)\ell_{1i}t_i/2 \\
B_i & 0 & 0 & -(B_i\zeta_i + \Gamma_{23}N_i)\ell_{2i}t_i/2 & (B_i\zeta_i + \Gamma_{23}N_i)\ell_{1i}t_i/2 \\
C_i & 0 & 0 & -(C_i\zeta_i + \Gamma_{33}N_i)\ell_{2i}t_i/2 & (C_i\zeta_i + \Gamma_{33}N_i)\ell_{1i}t_i/2 \\
0 & A_i & 0 & -(A_i\zeta_i + \Gamma_{13}N_i)m_{2i}t_i/2 & (A_i\zeta_i + \Gamma_{13}N_i)m_{1i}t_i/2 \\
0 & B_i & 0 & -(B_i\zeta_i + \Gamma_{23}N_i)m_{2i}t_i/2 & (B_i\zeta_i + \Gamma_{23}N_i)m_{1i}t_i/2 \\
0 & C_i & 0 & -(C_i\zeta_i + \Gamma_{33}N_i)m_{2i}t_i/2 & (C_i\zeta_i + \Gamma_{33}N_i)m_{1i}t_i/2 \\
0 & 0 & A_i & -(A_i\zeta_i + \Gamma_{13}N_i)n_{2i}t_i/2 & (A_i\zeta_i + \Gamma_{13}N_i)n_{1i}t_i/2 \\
0 & 0 & B_i & -(B_i\zeta_i + \Gamma_{23}N_i)n_{2i}t_i/2 & (B_i\zeta_i + \Gamma_{23}N_i)n_{1i}t_i/2 \\
0 & 0 & C_i & -(C_i\zeta_i + \Gamma_{33}N_i)n_{2i}t_i/2 & (C_i\zeta_i + \Gamma_{33}N_i)n_{1i}t_i/2
\end{bmatrix}$$

16.5-5

Sides at $\eta = \pm 1,\ \zeta = \pm 1.$



(e.g. for 3-node el.)

(a)
$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum N_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \sum \frac{B}{2} \eta N_i \begin{Bmatrix} l_{2i} \\ m_{2i} \\ n_{2i} \end{Bmatrix} + \sum \frac{C}{2} \zeta N_i \begin{Bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{Bmatrix}$$

$\left(|\underset{\sim}{V}_{2i}| = B; \quad |\underset{\sim}{V}_{3i}| = C \right)$

(b)

Consider e.g. motion of corner $\eta = \zeta = +1$; due to rotations, it is:

$\beta_i C/2 - \gamma_i B/2$ in ξ direction
$-\alpha_i C/2$ in η direction
$\alpha_i B/2$ in ζ direction

In this way, and resolving into xyz components, for an arbitrary point,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}$$

$$\sum \cdots \frac{B}{2} \begin{bmatrix} l_{3i} & 0 & -l_{1i} \\ m_{3i} & 0 & -m_{1i} \\ n_{3i} & 0 & -n_{1i} \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{Bmatrix}$$

$$+ \sum N_i \zeta \frac{C}{2} \begin{bmatrix} -l_{2i} & l_{1i} & 0 \\ -m_{2i} & m_{1i} & 0 \\ -n_{2i} & n_{1i} & 0 \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{Bmatrix}$$