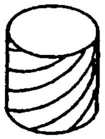
14.1-1



Imagine stiff fibers in rubber. Uniform σ_z on ends makes it unwind; i.e. $v \neq 0$. Can make $v = 0$ by adding torque, but then (by statics) $\tau_{\theta z} = 0$. Need both $v = 0$ and $\tau_{\theta z} = 0$ for axisymmetric behavior.

$\boxed{14.2\text{-}1}$

		zero ϵ energy	nonzero ϵ energy
(a) Plane,	1 Gauss pt.	1, 2, 3, 7, 8	4, 5, 6
(b) Plane,	4 Gauss pts.	1, 2, 3	4, 5, 6, 7, 8
(c) Axisym.,	1 Gauss pt.	1, 3, 7, 8	2, 3, 4, 5
(d) Axisym.,	4 Gauss pts.	1	2 through 8

$\boxed{14.2-2}$

For a one-radian segment, with $r = r_7 + J\xi$ and $J = \frac{r_3 - r_4}{2}$,

$$\{\underset{\sim}{r}_e\} = \int [\underset{\sim}{N}]^T \{\underset{\sim}{\Phi}\}\, dS = \int_{-1}^{1} \begin{Bmatrix} (\xi^2 - \xi)/2 \\ 1 - \xi^2 \\ (\xi^2 + \xi)/2 \end{Bmatrix} p r J \, d\xi$$

(a) p is constant:

$$\{\underset{\sim}{r}_e\} = \frac{pJr_7}{3} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \frac{pJ^2}{3} \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$$

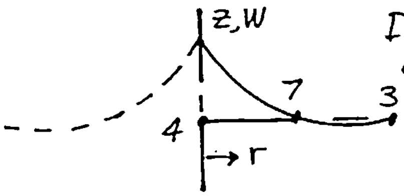
(b)

$$p = \frac{\xi^2 - \xi}{2} p_4 + (1 - \xi^2) p_7 + \frac{\xi^2 + \xi}{2} p_3$$

Algebra is straightforward but tedious

$$\{\underset{\sim}{r}_e\} = \frac{r_7 J}{30} \begin{Bmatrix} 8p_4 + 4p_7 - 2p_3 \\ 4p_4 + 32p_7 + 4p_3 \\ -2p_4 + 4p_7 + 8p_3 \end{Bmatrix} + \frac{J^2}{30} \begin{Bmatrix} -6p_4 - 4p_7 \\ -4p_4 + 4p_3 \\ 4p_7 + 6p_3 \end{Bmatrix}$$

$\boxed{14.2\text{-}3}$



Implies γ_{rz} discontinuous across $r=0$; not reasonable; not compatible.

Need $\frac{dw}{dz}=0$ at $r=0$, but need such a nodal d.o.f. to achieve it.

$\boxed{14.2\text{-}4}$

For compatibility, $u=0$ at $r=0$ (no hole appears). Expand u:

$$u = c_1 r + c_2 z + c_3 r^2 + c_4 r z + c_5 z^2 + \cdots$$

Hence

$$\epsilon_\theta = \frac{u}{r} = c_1 + c_2 \frac{z}{r} + c_3 r + c_4 z + c_5 \frac{z^2}{r} + \cdots$$

$$\epsilon_r = \frac{du}{dr} = c_1 + 2c_3 r + c_4 z + 3c_6 r^2 + \cdots$$

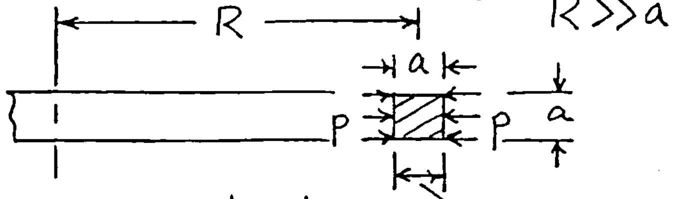
Strains cannot be infinite, so $c_2 = 0$, $c_5 = 0, \ldots$ Hence at $r=0$

$$\left.\begin{aligned} \epsilon_\theta &= c_1 + c_4 z + \cdots \\ \epsilon_r &= c_1 + c_4 z + \cdots \end{aligned}\right\} \text{same}$$

One could also say that directions r and θ lack separate meaning at $r=0$, so $\epsilon_\theta = \epsilon_r$ there.

14.2-5

Consider ϵ_r in a ring.



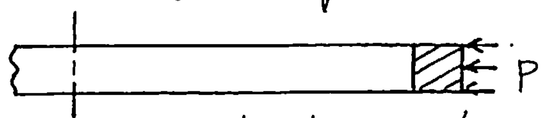
δ = change in this dimension

$$\delta = \frac{PL}{AE} = \frac{(paR\,d\theta)a}{(R\,d\theta)aE} = \frac{pa}{E}$$

Per radian, the associated stiffness is

$$k_{r1} = \frac{P}{\delta} = \frac{(pa)R}{\delta} = ER$$

Consider ϵ_θ in ring of same dimensions.



Δ = radial displacement

$$\sigma_\theta = \frac{pR}{a}, \quad \epsilon_\theta = \frac{\sigma_\theta}{E}, \quad \Delta = R\epsilon_\theta = \frac{pR^2}{Ea}$$

Per radian, the associated stiffness is

$$k_{r2} = \frac{P}{\Delta} = \frac{(pa)R}{\Delta} = ER\left(\frac{a}{R}\right)^2$$

Stiffness ratio: $\dfrac{k_{r1}}{k_{r2}} = \left(\dfrac{R}{a}\right)^2$

Becomes large if R/a is large.

$\boxed{14.2\text{-}6}$

(a) Let $u = a_1 + a_2 r$. Formal process or trial leads to
$$u = \begin{bmatrix} \frac{r_2 - r}{r_2 - r_1} & \frac{r - r_1}{r_2 - r_1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \end{Bmatrix} = \begin{Bmatrix} \partial/\partial r \\ 1/r \end{Bmatrix} u = \underbrace{\frac{1}{r_2 - r_1} \begin{bmatrix} -1 & 1 \\ \frac{r_2}{r} - 1 & 1 - \frac{r_1}{r} \end{bmatrix}}_{[\underset{\sim}{B}]} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

(b) $[\underset{\sim}{E}] = E \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, hence $[\underset{\sim}{B}]^T [\underset{\sim}{E}] [\underset{\sim}{B}]$ is
$$\frac{E}{(r_2 - r_1)^2} \begin{bmatrix} 2 + \frac{r_2^2}{r^2} - 2\frac{r_2}{r} & -2 + \frac{r_1 + r_2}{r} - \frac{r_1 r_2}{r^2} \\ \text{symm.} & 2 + \frac{r_1^2}{r^2} - 2\frac{r_1}{r} \end{bmatrix}$$

$$[\underset{\sim}{k}] = \int_0^t \int_0^1 \int_{r_1}^{r_2} [\underset{\sim}{B}]^T [\underset{\sim}{E}] [\underset{\sim}{B}]\, r\,dr\,d\theta\,dz$$

$$[\underset{\sim}{k}] = \frac{Et}{(r_2 - r_1)^2} \begin{bmatrix} -(r_2 - r_1)^2 + r_2^2 \ln\frac{r_2}{r_1} & -r_1 r_2 \ln\frac{r_2}{r_1} \\ -r_1 r_2 \ln\frac{r_2}{r_1} & (r_2 - r_1)^2 + r_1^2 \ln\frac{r_2}{r_1} \end{bmatrix}$$

(c) Let $r_m = \frac{1}{2}(r_1 + r_2)$, $L = r_2 - r_1$. With $r = r_m$,
$$[\underset{\sim}{B}] = \begin{bmatrix} -1/L & 1/L \\ 1/2r_m & 1/2r_m \end{bmatrix}, \quad [\underset{\sim}{E}] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

$[\underset{\sim}{k}] = [\underset{\sim}{B}]^T [\underset{\sim}{E}] [\underset{\sim}{B}]\, t\, r_m (1) L$, i.e.
$$\frac{E t r_m L}{1 - \nu^2} \begin{bmatrix} \frac{1}{L^2} - \frac{\nu}{L r_m} + \frac{1}{4 r_m^2} & -\frac{1}{L^2} + \frac{1}{4 r_m^2} \\ -\frac{1}{L^2} + \frac{1}{4 r_m^2} & \frac{1}{L^2} + \frac{\nu}{L r_m} + \frac{1}{4 r_m^2} \end{bmatrix}$$

(d) For $r_m \to \infty$, part (c) yields
$$[\underset{\sim}{k}] = \frac{E t r_m}{(1 - \nu^2) L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (A)$$

In $[\underset{\sim}{k}]$ of part (b),
$$\ln\frac{r_2}{r_1} = \left(\frac{r_2}{r_1} - 1\right) - \frac{1}{2}\left(\frac{r_2}{r_1} - 1\right)^2 + \cdots \approx \frac{r_2}{r_1} - 1 = \frac{r_2 - r_1}{r_1} = L/r_1$$

$$[\underset{\sim}{k}] = \frac{Et}{L^2} \begin{bmatrix} r_2^2 \frac{L}{r_1} - L^2 & -r_1 r_2 \frac{L}{r_1} \\ -r_1 r_2 \frac{L}{r_1} & r_1^2 \frac{L}{r_1} + L^2 \end{bmatrix} \qquad (B)$$

$$\frac{r_2^2}{r_1} L - L^2 = \frac{\left(r_m + \frac{L}{2}\right)^2}{\left(r_m - \frac{L}{2}\right)} L - L^2 \approx r_m L - L^2 \approx r_m L$$

$$r_1 r_2 \frac{L}{r_1} = r_2 L \approx r_m L, \qquad r_1^2 \frac{L}{r_1} + L^2 = r_1 L + L^2 \approx r_m L$$

Thus (A) & (B) agree if $\nu = 0$.

(e) With $u_1 = 0$, only k_{22} remains, which for $r_1 = 0$ is $k_{22} = \frac{Et}{r_2^2}\left[r_2^2 + \lim_{r_1 \to 0}\left(r_1^2 \ln\frac{r_2}{r_1}\right)\right]$

$$\lim_{r_1 \to 0}\left(r_1^2 \ln\frac{r_2}{r_1}\right) = \lim_{r_1 \to 0} \frac{\ln r_2 - \ln r_1}{r_1^{-2}} = \lim_{r_1 \to 0} \frac{-1/r_1}{-2/r_1^3} = 0$$

So k_{22} reduces to Et.

(f) $r_m = \frac{L}{2}$, so $k_{22} = \frac{E t L^2}{2(1 - \nu^2)}\left(\frac{1}{L^2} + \frac{2\nu}{L^2} + \frac{1}{L^2}\right)$

which reduces to $k_{22} = Et$ if $\nu = 0$.

14.2-7

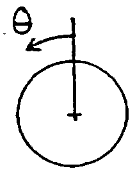
$$\{\underset{\sim}{r}_e\} = \int [\underset{\sim}{N}]^T \begin{Bmatrix} F_r \\ 0 \end{Bmatrix} dV, \quad F_r = \rho r \omega^2$$

$$\{\underset{\sim}{r}_e\} = \frac{\rho\omega^2}{4ab} \int_{-b}^{b} \int_{-a}^{a} \int_{-\pi}^{\pi} \begin{Bmatrix} (a-x)(b-y) \\ (a+x)(b-y) \\ (a+x)(b+y) \\ (a-x)(b+y) \end{Bmatrix} r^2 \, d\theta \, dx \, dy$$

where $r = r_m + x$.

$\boxed{14.4\text{-}1}$

(a)



$$\text{old } \theta = \text{new } \theta + \frac{\pi}{2}$$

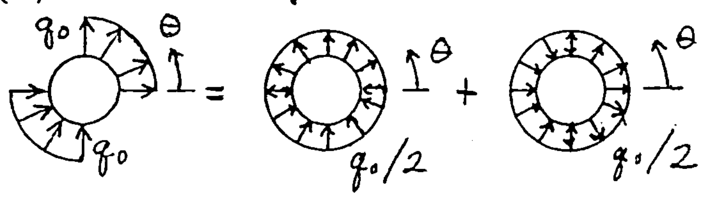
$$\sin n\left(\theta + \frac{\pi}{2}\right) = \sin n\theta \cos\frac{n\pi}{2} + \cos n\theta \sin\frac{n\pi}{2}$$

$$= \cos n\theta \sin\frac{n\pi}{2} \quad \text{for } n \text{ odd}$$

Formula in Fig. 6.6-2 becomes

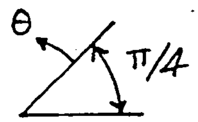
$$q = \frac{4q_0}{\pi}\left(\cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \cdots\right)$$

(b) Fig. 6.6-2 Part (a)



$$q = \frac{2q_0}{\pi}\left(\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right) + \frac{2q_0}{\pi}\left(\cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \cdots\right)$$

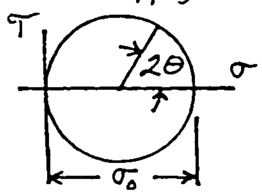
(c) Relocate the $\theta = 0$ position to where $\theta = \pi/4$ in part (b), i.e.



14.4-2

Must determine tractions Φ to be applied to the dashed circular boundary.

Can apply Mohr circle analysis.



$$\sigma_r = \frac{\sigma_0}{2}(1+\cos 2\theta)$$

$$\tau_{r\theta} = -\frac{\sigma_0}{2}\sin 2\theta$$

(a) $\sigma_0 = P/ht$, so

$$\Phi_r = \frac{P}{2ht}(1+\cos 2\theta),\quad \Phi_\theta = -\frac{P}{2ht}\sin 2\theta,\quad \Phi_z = 0$$

Symmetric terms, zeroth & second harmonics. Independent of r.

(b) $$\sigma_0 = -\frac{My}{I} = -\frac{M(r\sin\theta)}{\frac{1}{12}th^3} = -\frac{12M}{th^3}r\sin\theta$$

$$(1+\cos 2\theta)\sin\theta = (2-2\sin^2\theta)\sin\theta$$
$$= 2(\sin\theta - \sin^3\theta)$$

$$\sin 2\theta\sin\theta = 2\sin\theta\cos\theta\sin\theta$$
$$= 2\cos\theta(1-\cos^2\theta)$$

$$\Phi_r = -\frac{12M}{th^3}r\sin\theta\,\frac{1+\cos 2\theta}{2} = \frac{12M}{th^3}(-r\sin\theta + r\sin^3\theta)$$

$$\Phi_\theta = -\frac{12M}{th^3}r\sin\theta\left(-\frac{\sin 2\theta}{2}\right) = \frac{12M}{th^3}(r\cos\theta - r\cos^3\theta)$$

$$\Phi_z = 0$$

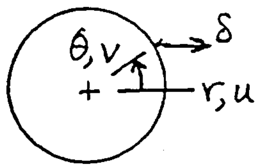
Antisymmetric terms, 1^{st} & 3^{rd} harmonics.

Φ's on boundary of radius c.

$\boxed{14.4\text{-}3}$

(a) $u = v = 0$, $w = w_0$

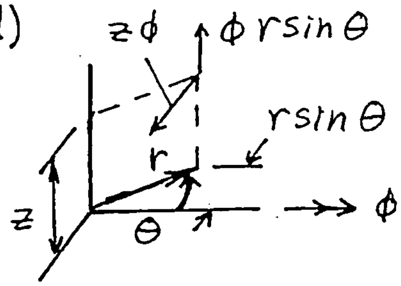
(b)



$u = \delta \cos\theta$
$v = -\delta \sin\theta$
$w = 0$

(c) Orient δ vertically in part (b); then
$u = \delta \sin\theta$, $v = \delta \cos\theta$, $w = 0$

(d)



ϕ = small rotation

$u = -z\phi \sin\theta$
$v = -z\phi \cos\theta$
$w = \phi r \sin\theta$

14.4-4

(a) Shown by direct substitution.

(b) 1^{st} 3 cols., single-barred
2^{nd} 3 cols., double-barred

(c) Consider a_1 thru a_6 in turn. Let x = radial axis along $\theta = 0$, y = radial axis along $\theta = \pi/2$, both in $z = 0$ plane.

a_1 — axial (z-dir.) translation.
a_2 — x direction translation
a_3 — rotation about y axis.
a_4 — rotation about z axis.
a_5 — y direction translation.
a_6 — rotation about x axis.

14.4-5

(a) Exact, at center $x = \frac{L}{2}$:

$$v = \frac{5 q_o L^4}{384 EI} \quad , \quad M = -\frac{q_o L^2}{8}$$

$$v_n = \frac{4 q_o}{n\pi} \frac{L^4}{EI n^4 \pi^4} \sin\frac{n\pi x}{L} = \frac{4 q_o L^4}{EI \pi^5 n^5} \sin\frac{n\pi x}{L}$$

$$M_n = EI \frac{d^2 v_n}{dx^2} = -EI \frac{n^2\pi^2}{L^2} v_n = -\frac{4 q_o L^2}{\pi^3 n^3} \sin\frac{n\pi x}{L}$$

At $x = \frac{L}{2}$ and with n odd,

$$v = \sum v_n = \frac{4 q_o L^4}{EI \pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \cdots\right)$$

$$M = \sum M_n = \frac{4 q_o L^2}{\pi^3}\left(-1 + \frac{1}{3^3} - \frac{1}{5^3} + \cdots\right)$$

Numerical multipliers in ans. are

	exact	1 term	2 terms	3 terms
v	0.013021	0.013071 (+0.38%)	0.013017 (-0.03%)	0.013021 —
|M|	0.125	0.1290 (+3.2%)	0.1242 (-0.6%)	0.1253 (+0.2%)

(b) Exact, at center $x = \frac{L}{2}$:

$$v = \frac{PL^3}{48 EI} \quad , \quad M = -\frac{PL}{4}$$

Proceding as in part (a),

$$v_n = \frac{2PL^3}{EI n^4 \pi^4} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{L}$$

$$M_n \;\; \frac{2PL}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{L}$$

At $x = \frac{L}{2}$, only odd terms are nonzero.

$$v = \sum v_n = \frac{2PL^3}{EI \pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$$

$$M = \sum M_n = \frac{2PL}{\pi^2}\left(-1 - \frac{1}{3^2} - \frac{1}{5^2} - \cdots\right)$$

Numerical multipliers in ans. are

	exact	1 term	2 terms	3 terms
v	0.02083	0.02053 (-1.45%)	0.02079 (-0.23%)	0.2082 (-0.07%)
|M|	0.2500	0.2026 (-18.9%)	0.2252 (-9.94%)	0.2333 (-6.69%)

14.5-1 A slender ring of square cross section:

Consider the lowest load harmonic that is statically equivalent to zero:

$q = q_2^c \cos 2\theta$ and the associated radial displacement $u = \bar{u}_2 \cos 2\theta$

Compute $\bar{u}_2$ by equating strain energy U to work W done by load.

$$U = \frac{EI}{2}\int_0^{2\pi}\left[\frac{d^2u}{(R\,d\theta)^2}\right]^2 R\,d\theta = \frac{E}{2}\frac{a^4}{12}\frac{\bar{u}_2^{\,2}}{R^3}4^2\int_0^{2\pi}\cos^2 2\theta\,d\theta$$

Substitute $\phi = 2\theta$: $$U = \frac{2Ea^4\bar{u}_2^{\,2}}{3R^3}\int_0^{4\pi}\frac{\cos^2\phi}{2}\,d\phi = \frac{2\pi E a^4 \bar{u}_2^{\,2}}{3R^3}$$

$$W = \frac{1}{2}\int_0^{2\pi} u\,(p\,a\,R\,d\theta) = \frac{q_2^c\,\bar{u}_2\,aR}{2}\int_0^{2\pi}\cos^2 2\theta\,d\theta$$

$$W = \frac{q_2^c\,\bar{u}_2\,aR}{2}\int_0^{4\pi}\frac{\cos^2\phi}{2}\,d\phi = \frac{\pi a R q_2^c \bar{u}_2}{2}$$

$U = W$ gives $\bar{u}_2 = \dfrac{3R^4}{4Ea^3}q_2^c$

Flexural stiffness per radian might be defined as

$$k_{f2} = \frac{R(q_2^c a)}{\bar{u}_2} = \frac{4ER}{3}\left(\frac{a}{R}\right)^4$$

Circumferential stiffness, from Problem 14.2-5, is

$$k_{r2} = ER\left(\frac{R}{a}\right)^2$$

$\therefore \dfrac{k_{r2}}{k_{f2}} = \dfrac{3}{4}\left(\dfrac{R}{a}\right)^2$ Very large if $\frac{R}{a}$ is large.

$\boxed{14.5-2}$

(a) First partition, analogous to that in Eq. 14.5-5, is, with $[\underset{\sim}{\partial}]$ from Eq. 14.4-4,

$$[\underset{\sim}{\partial}]\begin{bmatrix} N_1 \sin n\theta & 0 & 0 \\ 0 & -N_1 \cos n\theta & 0 \\ 0 & 0 & N_1 \sin n\theta \end{bmatrix} =$$

$$\begin{bmatrix} N_{1,r}\sin n\theta & 0 & 0 \\ \frac{N_1}{r}\sin n\theta & \frac{nN_1}{r}\sin n\theta & 0 \\ 0 & 0 & N_{1,z}\sin n\theta \\ N_{1,z}\cos n\theta & 0 & N_{1,r}\sin n\theta \\ \frac{nN_1}{r}\cos n\theta & -\left(N_{1,r}-\frac{N_1}{r}\right)\cos n\theta & 0 \\ 0 & -N_{1,z}\cos n\theta & \frac{nN_1}{r}\cos n\theta \end{bmatrix}$$

Differs from Eq. 14.5-5 in algebraic signs and sine & cosine terms.

(b) To see form of $[\underset{\sim}{K}_n]$, part (a) as compared with that provided by Eq. 14.5-5, need consider only sine & cosine terms. Let $c = \cos n\theta$, $s = \sin n\theta$. With $[\underset{\sim}{E}]$ populated as in Eq. 14.4-3, part (a) yields

$$\underbrace{\begin{bmatrix} s & s & 0 & s & nc & 0 \\ 0 & ns & 0 & 0 & -c & -c \\ 0 & 0 & s & s & 0 & nc \end{bmatrix}}_{[\underset{\sim}{B}]^T} \quad \underbrace{\begin{bmatrix} s & ns & s \\ s & ns & s \\ s & ns & s \\ s & ns & s \\ nc & -c & nc \\ nc & -c & nc \end{bmatrix}}_{[\underset{\sim}{E}][\underset{\sim}{B}]}$$

Form of $[\underset{\sim}{B}]^T[\underset{\sim}{E}][\underset{\sim}{B}]$:

$$\begin{bmatrix} s^2+n^2c^2 & n(s^2-c^2) & s^2+n^2c^2 \\ n(s^2-c^2) & n^2s^2+c^2 & n(s^2-c^2) \\ \cdots & n(s^2-c^2) & s^2+n^2c^2 \end{bmatrix}$$

Off-diag. blocks contain $s_i s_j$ & $c_i c_j$ with $i \neq j$; integrate to zero over $\theta = -\pi$ to $\theta = +\pi$. On-diag. blocks integrate to π (or to 2π for $n = 0$), for both s^2 and c^2 terms.

In similar fashion, from the first partition in Eq. 14.5-5,

$$\underbrace{\begin{bmatrix} c & c & 0 & c & -ns & 0 \\ 0 & nc & 0 & 0 & s & s \\ 0 & 0 & c & c & 0 & -ns \end{bmatrix}}_{[\underset{\sim}{B}]^T} \quad \underbrace{\begin{bmatrix} c & nc & c \\ c & nc & c \\ c & nc & c \\ c & nc & c \\ -ns & s & -ns \\ -ns & s & -ns \end{bmatrix}}_{[\underset{\sim}{E}][\underset{\sim}{B}]}$$

Form of $[\underset{\sim}{B}]^T[\underset{\sim}{E}][\underset{\sim}{B}]$:

$$\begin{bmatrix} c^2+n^2s^2 & n(c^2-s^2) & c^2+ns^2 \\ n(c^2-s^2) & nc^2+s^2 & n(c^2-s^2) \\ c^2+ns^2 & n(c^2-s^2) & c^2+ns^2 \end{bmatrix}$$

Since s^2 and c^2 both integrate to π (with limits $-\pi$ to $+\pi$), the result is the same as produced by part (a).

(c) The middle column of the 6 by 3 result matrix in part (a) changes sign. Then in part (b), row 2 of $[\underset{\sim}{B}]^T$ and column 2 of $[\underset{\sim}{E}][\underset{\sim}{B}]$ change sign. Hence the form of sine and cosine terms in $[\underset{\sim}{B}]^T[\underset{\sim}{E}][\underset{\sim}{B}]$ becomes

$$\begin{bmatrix} s^2+n^2c^2 & -n(s^2-c^2) & s^2+n^2c^2 \\ -n(s^2-c^2) & n^2s^2+c^2 & -n(s^2-c^2) \\ s^2+n^2c^2 & -n(s^2-c^2) & s^2+n^2c^2 \end{bmatrix}$$

Signs of some terms differ, so $[\underset{\sim}{K}_n]$ changes.

14.5-3

(a) If $n=6$, $\sin n\theta$ goes to 12π when $\theta = 2\pi$, i.e. 12 half-waves, so $m = 3(12) = 36$

(b) For solid-of-revolution analysis, 4-noded

$$\underset{12\times 12}{[\underset{\sim}{k}]} = \underset{12\times 6}{[\underset{\sim}{B}]^T}\underset{6\times 6}{[\underset{\sim}{E}]}\underset{6\times 12}{[\underset{\sim}{B}]}$$

. Count multiplications, ignoring symmetry of $[\underset{\sim}{k}]$: 4 Gauss pts./el., 20 els., 6 analyses:

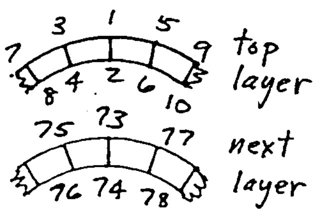
$$4*20*6*(6*12*6+6*12*12) = 622{,}000$$

For 3-D els., 8 node els., $[\underset{\sim}{B}]$ is $6*24$.
8 Gauss pts./el., $20*36$ els.:

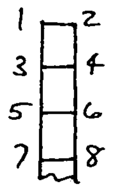
$$8*20*36(6*24*6+6*24*24) = 24.9(10)^6$$

$$\text{ratio} = 24.9(10)^6/622{,}000 = 40$$

(c) 3-D mesh numbering:



Solid of rev. mesh numbers:



3-D: $NB^2 = (72*21)(78)^2 = 9.2(10)^6$

Solid of rev.: $NB^2 = 42(4)^2 = 672$

(exclusive of 3 d.o.f. per node in each case).

With 6 solid-of-rev. analyses,

$$\text{ratio} = 9.2(10)^6/(6*672) = 2300$$

$\boxed{14.5\text{-}4}$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{1}{r_2 - r_1}\begin{bmatrix} (r_2 - r)\cos n\theta & 0 & (r - r_1)\cos n\theta & 0 \\ 0 & (r_2 - r)\sin n\theta & 0 & (r - r_1)\sin n\theta \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

From Eq. 10.5-4,

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial r} - \frac{1}{r} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}. \quad \text{Hence}$$

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \frac{1}{r_2 - r_1}\begin{bmatrix} -\cos n\theta & 0 & \cos n\theta & 0 \\ \frac{r_2 - r}{r}\cos n\theta & \frac{r_2 - r}{r} n\cos n\theta & \frac{r - r_1}{r}\cos n\theta & \frac{r - r_1}{r} n\cos n\theta \\ -\frac{r_2 - r}{r} n\sin n\theta & -\frac{r_2}{r}\sin n\theta & -\frac{r - r_1}{r} n\sin n\theta & \frac{r_1}{r}\sin n\theta \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$