$\boxed{13.1\text{-}1}$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Yields bilinear element.

Check:

$$\begin{aligned} u = {} & \left[\frac{1}{4}(1-\xi)(1-\eta) - \frac{N_5}{2} - \frac{N_8}{2}\right]u_1 + N_5\left(\frac{u_1}{2} + \frac{u_2}{2}\right) + \\ & \left[\frac{1}{4}(1+\xi)(1-\eta) - \frac{N_5}{2} - \frac{N_6}{2}\right]u_2 + N_6\left(\frac{u_2}{2} + \frac{u_3}{2}\right) + \\ & \left[\frac{1}{4}(1+\xi)(1+\eta) - \frac{N_6}{2} - \frac{N_7}{2}\right]u_3 + N_7\left(\frac{u_3}{2} + \frac{u_4}{2}\right) + \\ & \left[\frac{1}{4}(1-\xi)(1+\eta) - \frac{N_7}{2} - \frac{N_8}{2}\right]u_4 + N_8\left(\frac{u_4}{2} + \frac{u_1}{2}\right) \end{aligned}$$

$$\begin{aligned} u = \frac{1}{4}\big[& (1-\xi)(1-\eta)u_1 + (1+\xi)(1-\eta)u_2 + \\ & (1+\xi)(1+\eta)u_3 + (1-\xi)(1+\eta)u_4\big] \quad \checkmark \end{aligned}$$

$\boxed{13.1\text{-}2}$

If $[\underset{\sim}{K}]$ operates on $\lfloor u_A \ \theta_A \ u_B \ \theta_B \rfloor^T$, $[\underset{\sim}{K}] =$

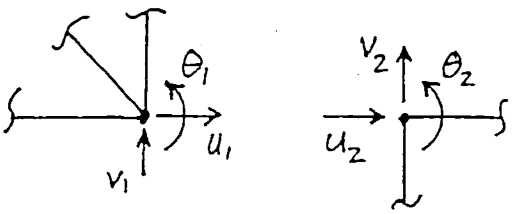
$$\frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & 0 & 0 \\ 3L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{2EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 3L \\ 0 & 0 & 3L & 2L^2 \end{bmatrix} +$$

$$\frac{E}{L}\begin{bmatrix} A & 0 & -A & 0 \\ 0 & 4I & 0 & 2I \\ -A & 0 & A & 0 \\ 0 & 2I & 0 & 4I \end{bmatrix}, \qquad \begin{Bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\underset{\sim}{T}]} \begin{Bmatrix} u_A \\ \theta_A \\ \theta_B \end{Bmatrix}$$

$[\underset{\sim}{T}]^T[\underset{\sim}{K}][\underset{\sim}{T}] =$

$$EI\begin{bmatrix} 24/L^3 & 6/L^2 & 6/L^2 \\ 6/L^2 & 8/L & 2/L \\ 6/L^2 & 2/L & 8/L \end{bmatrix}$$

$\boxed{13.1\text{-}3}$



$$\left[\underbrace{\begin{matrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{matrix}}_{\substack{[\underset{\sim}{C}_r]\\ 2\times 4}}\;\underbrace{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}}_{\substack{[\underset{\sim}{C}_c]\\ 2\times 2}}\right]\begin{Bmatrix}u_1\\ v_1\\ \theta_1\\ \theta_2\\ u_2\\ v_2\end{Bmatrix} = \{\underset{\sim}{0}\}$$

$\boxed{13.1\text{-}4}$

$$\{\underset{\sim}{D}_r\} = \begin{Bmatrix}u_1\\ v_1\\ u_2\end{Bmatrix}, \quad \{\underset{\sim}{D}_c\} = \begin{Bmatrix}v_2\\ u_3\\ v_3\end{Bmatrix}. \text{ Eq. 8.5-6 yields:}$$

$$v_2 = -\frac{a}{b}u_1 + v_1 + \frac{a}{b}u_2$$
$$u_3 = u_1$$
$$v_3 = -\frac{a}{b}u_1 + v_1 + \frac{a}{b}u_2$$

Now write these eqs. in homogeneous form:

$$\begin{bmatrix}a/b & -1 & -a/b & 1 & 0 & 0\\ -1 & 0 & 0 & 0 & 1 & 0\\ a/b & -1 & -a/b & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\end{Bmatrix} = \{\underset{\sim}{0}\}$$

Three eqs. of constraint.

$\boxed{13.1\text{-}5}$

(a) $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$, $k = \dfrac{AE}{L}$

$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} u_1$, $\lfloor 1 \quad 0 \rfloor \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = k$

$k u_1 = P$, $u_1 = \dfrac{P}{k}$

(b) $[0 \quad 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \bar{u}$, $\underset{\sim}{T} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\underset{\sim}{Q} = \bar{u}$,

$\underset{\sim}{C}_c^{-1} = 1$

$\underset{\sim}{T}^T \underset{\sim}{K}' \underset{\sim}{T} = \lfloor 1 \quad 0 \rfloor \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = k$

$\underset{\sim}{K}' \begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{C}_c^{-1} \underset{\sim}{Q} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} 0 \\ \bar{u} \end{Bmatrix} = \begin{Bmatrix} -k\bar{u} \\ k\bar{u} \end{Bmatrix}$,

$-T^T K' \begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{C}_c^{-1} \underset{\sim}{Q} \end{Bmatrix} = k\bar{u}$. Final eq. is

$k u_1 = P + k\bar{u}$, from which $u_1 = \dfrac{P}{k} + \bar{u}$

$\boxed{13.1\text{-}6}$

$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}\begin{Bmatrix} v_A \\ v_B \\ v_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}$$

Enforce

$v_C = v_A + \dfrac{v_B - v_A}{L} 2L$

$v_C = 2v_B - v_A$

$$\begin{Bmatrix} v_A \\ v_B \\ v_C \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} v_A \\ v_B \end{Bmatrix} = [\underset{\sim}{T}]\begin{Bmatrix} v_A \\ v_B \end{Bmatrix} \qquad (a)$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{K}][\underset{\sim}{T}]) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} k \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} = k\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$$[\underset{\sim}{T}]^T\{\underset{\sim}{R}\} = [\underset{\sim}{T}]^T\begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix}$$

$k\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{Bmatrix} v_A \\ v_B \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix}$ yields $\begin{Bmatrix} v_A \\ v_B \end{Bmatrix} = \dfrac{P}{k}\begin{Bmatrix} 1/6 \\ -1/3 \end{Bmatrix}$

Hence Eq. (a) gives $v_c = -\dfrac{5P}{6k}$

$\boxed{13.1\text{-}7}$

$u = cx$, where c = constant. But $u_1 = cx_1$, so $c = \dfrac{u_1}{x_1}$ and $u = \dfrac{u_1}{x_1} x$.

Then

$$u_2 = \frac{x_2}{x_1} u_1 \qquad \begin{bmatrix} \dfrac{x_2}{x_1} & -1 & 0 \\ \dfrac{x_3}{x_1} & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{ \underset{\sim}{0} \}$$

$$u_3 = \frac{x_3}{x_1} u_1$$

$\boxed{13.1\text{-}8}$

$$\frac{EI}{a^3}\begin{bmatrix} 24 & 0 & -12 & 6a \\ 0 & 8a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix} \quad \text{or} \quad [\underset{\sim}{K}]\{\underset{\sim}{D}\} = \{\underset{\sim}{R}\}$$

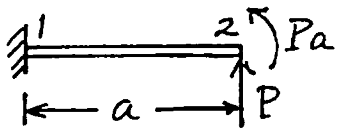
$$\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & a \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = [\underset{\sim}{T}]\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{K}][\underset{\sim}{T}]) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & a & 1 \end{bmatrix}\frac{EI}{a^3}\begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \frac{EI}{a^3}\begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix}$$

$$[\underset{\sim}{T}]^T\{\underset{\sim}{R}\} = \begin{Bmatrix} P \\ Pa \end{Bmatrix}$$

$$\frac{EI}{a^3}\begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ Pa \end{Bmatrix} \quad \text{yields} \quad \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 5Pa^3/6EI \\ 3Pa^2/2EI \end{Bmatrix}$$

Beam theory:



$$v_2 = \frac{Pa^3}{3EI} + \frac{(Pa)a^2}{2EI} = \frac{5Pa^3}{6EI}$$

$$\theta_2 = \frac{Pa^2}{2EI} + \frac{(Pa)a}{EI} = \frac{3Pa^2}{2EI}$$

$\boxed{13.1\text{-}9}$

For one el., $$[\underset{\sim}{k}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Or, if v_1 and v_2 are suppressed,

$$[\underset{\sim}{k}] = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For the present two-element structure,

$$\frac{2EI}{L}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_0 \end{Bmatrix}$$

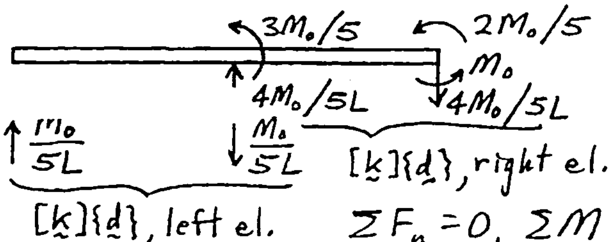
Here $\lfloor \theta_1 \;\; \theta_2 \;\; \theta_3 \rfloor^T = [\underset{\sim}{T}]\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$, where

$[\underset{\sim}{T}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. Transformations, as in Eqs. 13.1-4, yield

$$\frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}, \quad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{M_0 L}{30EI}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$$

(b) $$\frac{2EI}{L}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\frac{M_0 L}{30EI}\begin{Bmatrix} -1 \\ 2 \\ 2 \end{Bmatrix} = M_0\begin{Bmatrix} 0 \\ 3/5 \\ 2/5 \end{Bmatrix}$$

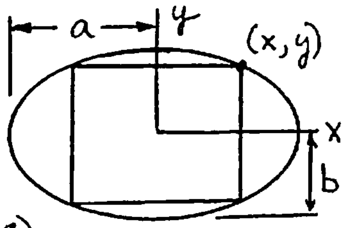
Nodal forces from separate elements can be found from $[\underset{\sim}{k}]\{\underset{\sim}{d}\}$ of each element, using rows 1 and 3 of $\underset{4\times 4}{[\underset{\sim}{k}]}$.



$\Sigma F_y = 0, \; \Sigma M \neq 0$

$\boxed{13.2\text{-}1}$

$$A = 4xy + \lambda\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right]$$



$$\frac{\partial A}{\partial x} = 0 = 4y + \lambda\frac{2x}{a^2} \quad (a)$$

$$\frac{\partial A}{\partial y} = 0 = 4x + \lambda\frac{2y}{b^2} \quad (b)$$

$$\frac{\partial A}{\partial \lambda} = 0 = \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right] \quad (c)$$

(a) times x + (b) times y is

$$0 = 4xy + 4xy + 2\lambda\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right]$$

So, in view of (c), $0 = 4xy + \lambda(1)$

$$\lambda = -4xy$$

From (a), $0 = 4y - \frac{8x^2y}{a^2}$, $x = a/\sqrt{2}$

From (b), $0 = 4x - \frac{8xy^2}{b^2}$, $y = b/\sqrt{2}$

$A = 4xy = 2ab$

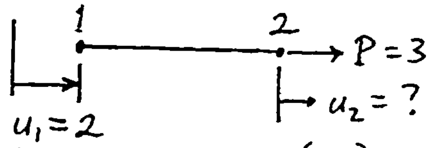
$\boxed{13.2\text{-}2}$

$$[\underset{\sim}{K}] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \qquad \underbrace{\lfloor 1 \quad -1 \rfloor}_{[\underset{\sim}{C}]} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$$

Eq. 13.2-2 becomes

$$\begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}, \qquad \begin{Bmatrix} v_1 \\ v_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P/2k \\ P/2k \\ P/2 \end{Bmatrix}$$

$\boxed{13.2\text{-}3}$



$$k = \frac{AE}{L}$$

$$[\underset{\sim}{K}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad \underbrace{\lfloor 1 \quad 0 \rfloor}_{[\underset{\sim}{C}]} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 2$$

Eq. 13.2-2 becomes

$$\begin{bmatrix} k & -k & 1 \\ -k & k & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \\ 2 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 + 3/k \\ 3 \end{Bmatrix}$$

$\boxed{13.2\text{-}4}$

(a) $\theta_2 = \dfrac{v_2}{L}$, $\underbrace{\begin{bmatrix} \frac{1}{L} & -1 \end{bmatrix}}_{[C]} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = 0$

$$\begin{bmatrix} 12EI/L^3 & -6EI/L^2 & 1/L \\ -6EI/L^2 & 4EI/L & -1 \\ 1/L & -1 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

$v_2 = PL^3/4EI$, $\theta_2 = PL^2/4EI$, $\lambda = -P/2$

(b) $\theta = \lfloor N_{,x} \rfloor \{d\}$ where $\{d\} = \lfloor 0 \;\; 0 \;\; v_2 \;\; \theta_2 \rfloor^T$

$\theta = \left(\dfrac{6x}{L^2} - \dfrac{6x^2}{L^3}\right) v_2 + \left(-\dfrac{2x}{L} + \dfrac{3x^2}{L^2}\right)\theta_2$. At $x = \dfrac{L}{2}$

$\theta = \theta_c = \dfrac{1.5}{L} v_2 - \dfrac{1}{4}\theta_2$. Want $\theta_2 = -\dfrac{1}{2}\theta_c$

or $\theta_c = -2\theta_2$

Hence $-2\theta_2 = \dfrac{1.5}{L} v_2 - \dfrac{1}{4}\theta_2$

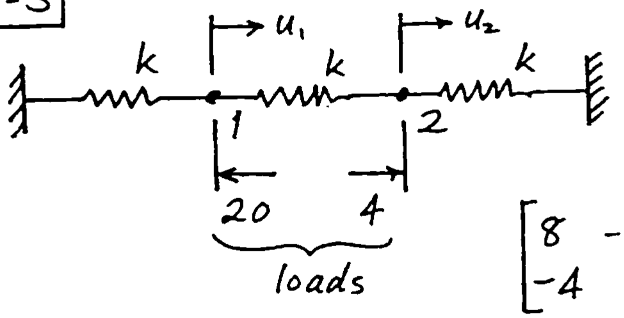
or $\underbrace{\begin{bmatrix} \frac{1.5}{L} & 1.75 \end{bmatrix}}_{[C]} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = 0$

$$\begin{bmatrix} 12EI/L^3 & -6EI/L^2 & 1.5/L \\ -6EI/L^2 & 4EI/L & 1.75 \\ 1.5/L & 1.75 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0.03964\, PL^3/EI \\ -0.03398\, PL^2/EI \\ 0.21363\, PL \end{Bmatrix}$$

13.2-5

(a)



$k = 4$

$$\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 4 \end{Bmatrix}$$

(b) $[\underset{\sim}{C}]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$ where $[\underset{\sim}{C}] = \lfloor 1 \quad 0 \rfloor$

$$\begin{bmatrix} 8 & -4 & 1 \\ -4 & 8 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -20 \\ 4 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ -18 \end{Bmatrix}$$

$\lambda = -18$; $|-18|$ is the magnitude of axial constraint force needed at node 1.

(c) $[\underset{\sim}{C}]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -1$ where $[\underset{\sim}{C}] = \lfloor 1 \quad 0 \rfloor$

$$\begin{bmatrix} 8 & -4 & 1 \\ -4 & 8 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -20 \\ 4 \\ -1 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ -12 \end{Bmatrix}$$

$\lambda = -12$; $|-12|$ is the magnitude of axial constraint force needed at node 1.

$\boxed{13.3\text{-}1}$

(a) $$\Pi_p = \frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D} - \mathbf{D}^T\mathbf{R} + \frac{1}{2}(\mathbf{D}^T\mathbf{C}^T - \mathbf{Q}^T)\boldsymbol{\alpha}(\mathbf{C}\mathbf{D} - \mathbf{Q})$$

$$\Pi_p = \frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D} - \mathbf{D}^T\mathbf{R} + \frac{1}{2}(\mathbf{D}^T\mathbf{C}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{D} - \mathbf{D}^T\mathbf{C}^T\boldsymbol{\alpha}\mathbf{Q} - \mathbf{Q}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{D} + \mathbf{Q}^T\mathbf{Q})$$

$$\Pi_p = \frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D} - \mathbf{D}^T\mathbf{R} + \frac{1}{2}\mathbf{D}^T\mathbf{C}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{D} - \mathbf{D}^T\mathbf{C}^T\boldsymbol{\alpha}\mathbf{Q} + \frac{\mathbf{Q}^T\mathbf{Q}}{2}$$

$$\left\{\frac{\partial \Pi_p}{\partial \mathbf{D}}\right\} = \mathbf{K}\mathbf{D} - \mathbf{R} + \mathbf{C}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{D} - \mathbf{C}^T\boldsymbol{\alpha}\mathbf{Q} = 0$$

$$(\mathbf{K} + \mathbf{C}^T\boldsymbol{\alpha}\mathbf{C})\mathbf{D} = \mathbf{R} + \mathbf{C}^T\boldsymbol{\alpha}\mathbf{Q}$$

(b) Let $[\boldsymbol{\alpha}] = \alpha[\boldsymbol{\alpha}']$, where α is a magnitude and $[\boldsymbol{\alpha}']$ gives proportions of terms in $[\boldsymbol{\alpha}]$.

Then $$(\mathbf{K} + \alpha\mathbf{C}^T\boldsymbol{\alpha}'\mathbf{C})\mathbf{D} = \mathbf{R} + \alpha\mathbf{C}^T\boldsymbol{\alpha}'\mathbf{Q}$$

or $$\left(\frac{1}{\alpha}\mathbf{K} + \mathbf{C}^T\boldsymbol{\alpha}'\mathbf{C}\right)\mathbf{D} = \frac{1}{\alpha}\mathbf{R} + \mathbf{C}^T\boldsymbol{\alpha}'\mathbf{Q}$$

As $\alpha \to \infty$, get $(\mathbf{C}^T\boldsymbol{\alpha}'\mathbf{C})\mathbf{D} \approx \mathbf{C}^T\boldsymbol{\alpha}'\mathbf{Q}$

i.e. $\mathbf{Q}$ dictates response; response associated with $\mathbf{K}$ and $\mathbf{R}$ is lost.

If $\mathbf{Q} = \mathbf{0}$ then $\mathbf{D} = \mathbf{0}$; i.e. the mesh is locked (same conclusion as before).

$\boxed{13.3-2}$

Constraint $v_1 = v_2$ is $[\underset{\sim}{C}]\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$, where $[\underset{\sim}{C}] = [1 \quad -1]$. Eq. 13.3-3 becomes

$$\left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} + \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}. \text{ Solving,}$$

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{1}{(k+\alpha)^2 - \alpha^2}\begin{bmatrix} k+\alpha & \alpha \\ \alpha & k+\alpha \end{bmatrix}\begin{Bmatrix} P \\ 0 \end{Bmatrix} = \frac{P}{(k^2 + 2k\alpha)}\begin{Bmatrix} k+\alpha \\ \alpha \end{Bmatrix}$$

For $\alpha = 0$, $\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{P}{k^2}\begin{Bmatrix} k \\ 0 \end{Bmatrix} = \begin{Bmatrix} P/k \\ 0 \end{Bmatrix}$ ✓

For $\alpha \to \infty$,

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{P}{k\left(\frac{k}{\alpha} + 2\right)}\begin{Bmatrix} \frac{k}{\alpha} + 1 \\ 1 \end{Bmatrix} \to \frac{P}{2k}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$ ✓

$\boxed{13.3\text{-}3}$

$[\underset{\sim}{C}] = \lfloor 1 \quad 0 \rfloor$. Eq. 13.3-3 becomes

$$\left(\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \alpha \lfloor 1 \quad 0 \rfloor\right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \alpha (2)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{k\alpha} \begin{bmatrix} k & k \\ k & k+\alpha \end{bmatrix} \begin{Bmatrix} 2\alpha \\ 3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \left(\frac{1}{\alpha} + \frac{1}{k}\right) \end{bmatrix} \begin{Bmatrix} 2\alpha \\ 3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 2 + \frac{3}{\alpha} \\ 2 + \frac{3}{\alpha} + \frac{3}{k} \end{Bmatrix}.$$ For $k = 1$,

	$\alpha = 0$	$\alpha = 1$	$\alpha = 4$	$\alpha = 10$	$\alpha = 100$	$\alpha = \infty$
u_1	∞	5	2.75	2.30	2.03	2
u_2	∞	8	5.75	5.30	5.03	5

$\boxed{13.3\text{-}4}$

(a) $[\underset{\sim}{C}] = \lfloor \frac{1}{L} \quad -1 \rfloor$. Eq. 13.3-3 becomes

$$\left(\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} + \alpha \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L} \\ -\frac{1}{L} & 1 \end{bmatrix}\right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Let $\alpha = \alpha' EI/L$. Thus

$$\begin{bmatrix} 12+\alpha' & -6-\alpha' \\ -6-\alpha' & 4+\alpha' \end{bmatrix} \begin{Bmatrix} v_2 \\ L\theta_2 \end{Bmatrix} = \begin{Bmatrix} PL^3/EI \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{12+4\alpha'} \begin{bmatrix} 4+\alpha' & 6+\alpha' \\ 6+\alpha' & 12+\alpha' \end{bmatrix} \begin{Bmatrix} PL^3/EI \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{PL^3/EI}{\frac{12}{\alpha'}+4} \begin{Bmatrix} \frac{4}{\alpha'}+1 \\ \text{——} \end{Bmatrix} \xrightarrow[\text{if } \alpha'=\infty]{} \begin{Bmatrix} PL^3/4EI \\ \text{——} \end{Bmatrix}$$

(b) $[\underset{\sim}{C}] = \lfloor \frac{1.5}{L} \quad 1.75 \rfloor$. Eq. 13.3-3 becomes

$$\left(\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} + \alpha \begin{bmatrix} \frac{2.25}{L^2} & \frac{2.625}{L} \\ \frac{2.625}{L} & 3.0625 \end{bmatrix}\right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Let $\alpha = \alpha' EI/L$. Thus

$$\begin{bmatrix} 12+2.25\alpha' & -6+2.625\alpha' \\ -6+2.625\alpha' & 4+3.0625\alpha' \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} PL^3/EI \\ 0 \end{Bmatrix}$$

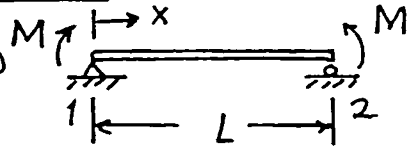
$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{12+77.25\alpha'} \begin{bmatrix} 4+3.0625\alpha' & 6-2.625\alpha' \\ 6-2.625\alpha' & 12+2.25\alpha' \end{bmatrix} \begin{Bmatrix} \frac{PL^3}{EI} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{PL^3/EI}{\frac{12}{\alpha'}+77.25} \begin{Bmatrix} \frac{4}{\alpha'}+3.0625 \\ \text{——} \end{Bmatrix}$$

As $\alpha' \to \infty$, $v_2 \to \frac{3.0625}{77.25}\frac{PL^3}{EI} = 0.03964\frac{PL^3}{EI}$

$\boxed{13.4\text{-}1}$

(a)



$w_1 = w_2 = 0$

$-\psi_1 = \psi_2$

For Mindlin element, $w = 0$ throughout,

and $\psi = \frac{L-x}{L}\psi_1 + \frac{x}{L}\psi_2 = \frac{L-2x}{L}\psi_1$

From Eqs. 13.4-2,

$$U = U_b + U_s = \frac{1}{2}\frac{Ebt^3}{12}\int_0^L \left(-\frac{2\psi_1}{L}\right)^2 dx + \frac{1}{2}\frac{Gbt}{1.2}\int_0^L \left(-\frac{L-2x}{L}\psi_1\right)^2 dx$$

$$U = \frac{1}{2}\left[\frac{Ebt^3}{12L}4\psi_1^2 + \frac{Gbt}{1.2}\frac{L\psi_1^2}{3}\right] = \frac{2\psi_1^2}{L}\left[\frac{Ebt^3}{12} + \frac{GbtL^2}{12(1.2)}\right]$$

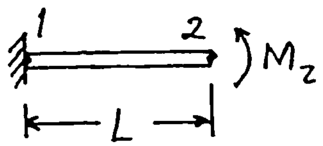
Let $I = \frac{bt^3}{12}$; $U = \frac{2\psi_1^2}{L}EI\left[1 + \frac{GL^2}{1.2Et^2}\right] = \frac{2\psi_1^2}{L}EI_e$

(b) One-point quadrature: evaluate integrals at $x = \frac{L}{2}$

The second integral (for U_s) vanishes, and

$U = \frac{2\psi_1^2}{L}\frac{Ebt^3}{12}$; exact; $I = \frac{bt^3}{12}$; $U = \frac{2\psi_1^2}{L}EI$

13.4-2



Exact: $w_2 = \dfrac{M_2 L^2}{2EI}$, $\psi_2 = \dfrac{M_2 L}{EI}$

Mindlin beam element, with exact integration:

$$\left(EI \begin{bmatrix} 0 & 0 \\ 0 & 1/L \end{bmatrix} + GA_s \begin{bmatrix} 1/L & -1/2 \\ -1/2 & L/3 \end{bmatrix} \right) \begin{Bmatrix} w_2 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_2 \end{Bmatrix}$$

1st eq. gives $\psi_2 = \frac{2}{L} w_2$; then 2nd eq. becomes

$$\frac{EI}{L}\frac{2w_2}{L} + GA_s\left(-\frac{w_2}{2} + \frac{L}{3}\frac{2w_2}{L}\right) = M_2$$

$$w_2 = \frac{M_2}{\left(\frac{2EI}{L^2} + \frac{GA_s}{6}\right)}$$

Set this w_2 equal to $0.9\dfrac{M_2 L^2}{2EI}$, thus

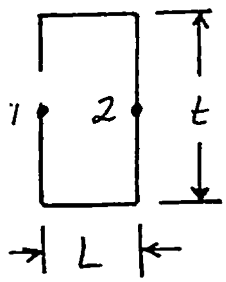
$$0.9\left(1 + \frac{GA_s L^2}{12EI}\right) = 1$$

$$\frac{GA_s L^2}{12EI} = 0.1111$$

$$\frac{(E/2)(5bt/6)L^2}{Ebt^3} = 0.1111$$

$$\left(\frac{L}{t}\right)^2 = 0.2667$$

$\frac{L}{t} = 0.516$, i.e., about to scale,



$\boxed{13.4\text{-}3}$

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad u = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{u}\} \qquad v = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{v}\}$$

where the N_i for a rectangular element are given in Eqs. 3.6-4. Thus

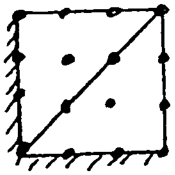
$$\epsilon_x + \epsilon_y = \frac{1}{4ab}\left\lfloor -(b-y), -(a-x), (b-y), -(a+x), (b+y), (a+x), -(b+y), (a-x) \right\rfloor \{\underset{\sim}{d}\}$$

Odd powers of x and y integrate to zero, so

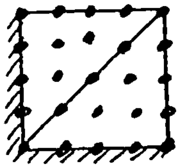
$$\int_{-b}^{b}\int_{-a}^{a} (\epsilon_x + \epsilon_y)\, dx\, dy = \lfloor -b \quad -a \quad b \quad -a \quad b \quad a \quad -b \quad a \rfloor \{\underset{\sim}{d}\}$$

With one Gauss point at $x = y = 0$, the product of Gauss weights is 4, and the Jacobian determinant is $J = ab$. Therefore $(\epsilon_x + \epsilon_y)_0 (4)(ab)$ is the same result.

$\boxed{13.5-1}$



$$r = \frac{2(9)}{2(6)} = \frac{3}{2}$$



$$r = \frac{2(16)}{2(10)} = \frac{8}{5}$$

$\boxed{13.6\text{-}1}$

(a) From Eq. 13.4-5,

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{Bmatrix} = \left(\frac{G}{3}\begin{bmatrix}4 & -2 & -2\\ -2 & 4 & -2\\ -2 & -2 & 4\end{bmatrix} + B\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\right)\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\end{Bmatrix}$$

From which $\sigma_x + \sigma_y + \sigma_z = 3B\epsilon_v$

where $\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$. Hence

$$B\epsilon_v = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \lambda$$

(b) From Eq. 13.4-10,

$$\beta\alpha H = \frac{1}{2}\underbrace{\frac{E}{3(1-2\nu)}}_{B}\{\underset{\sim}{\epsilon}\}^T[\underset{\sim}{E}_B]\{\underset{\sim}{\epsilon}\}$$

where $[\underset{\sim}{E}_B]$ is the second square matrix in Eq. 13.4-5. Since $[\underset{\sim}{E}_B]\{\underset{\sim}{\epsilon}\}$ yields volumetric strains ϵ_v,

$$\beta\alpha H = \frac{B}{2}\lfloor \epsilon_x \;\; \epsilon_y \;\; \epsilon_z \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{zx} \rfloor \begin{Bmatrix}\epsilon_v\\ \epsilon_v\\ \epsilon_v\\ 0\\ 0\\ 0\end{Bmatrix}$$

where $\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$

$$\beta\alpha H = \frac{B}{2}\epsilon_v(\epsilon_x + \epsilon_y + \epsilon_z) = \frac{B}{2}\epsilon_v^2$$