INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 16

UNIFORM CONVERGENCE

16.1. Uniform Convergence of Sequences

Recall that if a sequence a_n of complex numbers converges to a, then, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a| < \epsilon$ whenever $n > N$.

We can extend this to pointwise convergence in a region $D \subseteq \mathbb{C}$ in a natural way. A sequence of complex valued functions $a_n(z)$ defined on D converges pointwise to a function $a(z)$ defined on D if, given any $\epsilon > 0$ and any $z \in D$, there exists $N \in \mathbb{R}$ such that $|a_n(z) - a(z)| < \epsilon$ whenever $n > N$. Here the value of N may depend on the choice of $z \in D$. Indeed, for any fixed $z \in D$, we simply consider the convergence of the sequence $a_n(z)$ of complex numbers to the complex number $a(z)$. The region D does not play any essential part in the argument apart from providing the complex numbers z in question.

In this chapter, we introduce the idea of uniformity to the question of convergence. Put simply, uniformity transfers the dependence of N on z to dependence of N only on the region D containing the complex numbers z in question. More precisely, we have the following definition.

Definition. Suppose that $D \subseteq \mathbb{C}$ is a region. We say that a sequence of complex valued functions $a_n(z)$ converges uniformly in D to a function $a(z)$, denoted by $a_n(z) \to a(z)$ as $n \to \infty$ uniformly in D, if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a(z)| < \epsilon$ whenever $n > N$.

Remark. Note that N no longer depends on the choice of $z \in D$. Note also that a precise definition can be given by requiring $N \in \mathbb{R}$ to satisfy

$$\sup_{z \in D} |a_n(z) - a(z)| < \epsilon$$

whenever $n > N$.

EXAMPLE 16.1.1. Consider the sequence

$$a_n(z) = \frac{z}{n}$$

in the region $D = \{z : |z| < 1\}$. Note first of all that for every fixed $z \in D$, we have $a_n(z) \to 0$ as $n \to \infty$. On the other hand, given any $\epsilon > 0$, we have, for every $z \in D$, that

$$|a_n(z) - 0| = \frac{|z|}{n} < \frac{1}{n} < \epsilon$$

whenever $n > 1/\epsilon$. Hence $a_n(z) \to 0$ as $n \to \infty$ uniformly in D. Now consider the same sequence in the region $D = \mathbb{C}$. Note that

$$|a_n(z) - 0| < \epsilon \qquad \text{if and only if} \qquad n > \frac{|z|}{\epsilon}.$$

It is therefore impossible to find a suitable N independent of the choice of $z \in \mathbb{C}$. Hence $a_n(z)$ converges to 0, but not uniformly, in $\mathbb{C}$.

16.2. Consequences of Uniform Convergence

In this section, we show that uniform convergence carries a number of properties of the sequence over to the limit function. The following three results concern respectively continuity, integrability and differentiability.

THEOREM 16A. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that $a_n(z) \to a(z)$ as $n \to \infty$ uniformly in D. Then $a(z)$ is continuous in D.*

PROOF. Suppose that $z_0 \in D$ is fixed. For every $z \in D$, we have

$$a(z) - a(z_0) = a(z) - a_n(z) + a_n(z) - a_n(z_0) + a_n(z_0) - a(z_0),$$

so that

$$|a(z) - a(z_0)| \le |a_n(z) - a(z)| + |a_n(z) - a_n(z_0)| + |a_n(z_0) - a(z_0)|. \tag{1}$$

Given any $\epsilon > 0$, there exists N (independent of the choice of $z \in D$) such that

$$|a_n(z) - a(z)| < \frac{\epsilon}{3} \qquad \text{and} \qquad |a_n(z_0) - a(z_0)| < \frac{\epsilon}{3} \tag{2}$$

whenever $n > N$. We now choose any $n > N$ and consider the function $a_n(z)$. Clearly this function is continuous at z_0. Hence given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|a_n(z) - a_n(z_0)| < \frac{\epsilon}{3} \qquad \text{whenever } |z - z_0| < \delta. \tag{3}$$

Combining (1)–(3), we conclude that $|a(z) - a(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$, so that $a(z)$ is continuous at z_0. Since $z_0 \in D$ is arbitrary, the result follows. ○

EXAMPLE 16.2.1. Consider the sequence $a_n(z) = z^n$ on the real interval $[0, 1]$. Each function $a_n(z)$ is clearly continuous in $[0, 1]$. Also $a_n(z) \to 0$ as $n \to \infty$ if $z \in [0, 1)$ and $a_n(1) \to 1$ as $n \to \infty$, so that the limit function is not continuous in $[0, 1]$. In view of Theorem 16A, it is clear that this discontinuity is caused by the lack of uniform convergence of $a_n(z)$ in $[0, 1]$.

THEOREM 16B. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that $a_n(z) \to a(z)$ as $n \to \infty$ uniformly in D. Then for any contour C lying in D, we have*

$$\lim_{n\to\infty} \int_C a_n(z)\,\mathrm{d}z = \int_C a(z)\,\mathrm{d}z.$$

PROOF. Note first of all that the integrals exist, since integrability over C is a consequence of continuity in D. Suppose now that the contour C has length L. Given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a(z)| < \epsilon/L$ whenever $n > N$. Then

$$\left| \int_C a_n(z)\,\mathrm{d}z - \int_C a(z)\,\mathrm{d}z \right| \le L \sup_{z\in C} |a_n(z) - a(z)| \le \epsilon$$

whenever $n > N$. ◯

THEOREM 16C. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is analytic in a disc $D = \{z : |z - z_0| < R\}$. Suppose further that $a_n(z) \to a(z)$ as $n \to \infty$ uniformly in $D_r = \{z : |z - z_0| \le r\}$ for every $r \in [0, R)$. Then $a(z)$ is analytic in D, and $a_n'(z) \to a'(z)$ as $n \to \infty$ uniformly in D_r for every $r \in [0, R)$.*

PROOF. Suppose that T is any triangular path in D. We now choose $r \in [0, R)$ so that $T \subseteq D_r$. Then

$$\int_T a(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_T a_n(z)\,\mathrm{d}z = 0.$$

Here the second equality follows from Cauchy's integral theorem, while the first equality follows from Theorem 16B, in view of uniform convergence in D_r. The assertion that $a(z)$ is analytic in D now follows from Morera's theorem (Theorem 6G). Suppose next that $r \in [0, R)$ is fixed. We now choose $\rho = (r + R)/2$, so that $r < \rho < R$, and let C_ρ denote the circle $\{\zeta : |\zeta - z_0| = \rho\}$, followed in the positive (anticlockwise) direction (the reader is advised to draw a picture). For every $z \in D_r$, we have, by Cauchy's integral formula, that

$$a_n'(z) - a'(z) = \frac{1}{2\pi\mathrm{i}} \int_{C_\rho} \frac{a_n(\zeta) - a(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta.$$

Note that for every $\zeta \in C_\rho$, we have $|\zeta - z| \ge \rho - r$. Also, in view of the uniform convergence of the sequence $a_n(z)$ in D_ρ, we have, given any $\epsilon > 0$, there exists N such that for every $z \in D_\rho$,

$$|a_n(z) - a(z)| < \frac{(\rho - r)^2 \epsilon}{\rho}$$

whenever $n > N$. It follows that for every $z \in D_r$, we have

$$|a_n'(z) - a'(z)| < \rho \sup_{\zeta \in C_\rho} \left| \frac{a_n(\zeta) - a(\zeta)}{(\zeta - z)^2} \right| \le \epsilon$$

whenever $n > N$. Hence $a_n'(z) \to a'(z)$ as $n \to \infty$ uniformly in D_r. ◯

Note that Theorem 16C is restricted to discs. However, as far as application is concerned, this is not a serious restriction. For any point z in an arbitrary domain $D \subseteq \mathbb{C}$, we can always find an open disc D' such that $z \in D' \subseteq D$, and so we can apply Theorem 16C to the disc D'. We immediately have the following result.

THEOREM 16D. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is analytic in a domain $D \subseteq \mathbb{C}$. Suppose further that $a_n(z) \to a(z)$ as $n \to \infty$ uniformly in D. Then $a(z)$ is analytic in D. Furthermore, for every $z \in D$ and every $k \in \mathbb{N}$, we have $a_n^{(k)}(z) \to a^{(k)}(z)$ as $n \to \infty$.*

16.3. Cauchy Sequences

Suppose that a sequence of complex numbers a_n converges to a. Then given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a| < \epsilon/2$ whenever $n > N$. It follows that

$$|a_n - a_m| \le |a_n - a| + |a_m - a| < \epsilon$$

whenever $m, n > N$.

DEFINITION. We say that a sequence of complex numbers a_n is a Cauchy sequence if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$.

In the last section of this chapter, we shall prove the following result.

THEOREM 16E. (GENERAL PRINCIPLE OF CONVERGENCE) *A sequence of complex numbers a_n is convergent if and only if it is a Cauchy sequence. In other words, a sequence a_n of complex numbers is convergent if and only if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$.*

DEFINITION. Suppose that $D \subseteq \mathbb{C}$ is a region. We say that a sequence of complex valued functions $a_n(z)$ is a uniform Cauchy sequence in D, if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a_m(z)| < \epsilon$ whenever $m, n > N$.

We have the following important result.

THEOREM 16F. (GENERAL PRINCIPLE OF UNIFORM CONVERGENCE) *Suppose that $D \subseteq \mathbb{C}$ is a region. A sequence of complex valued functions $a_n(z)$ converges uniformly in D if and only if it is a uniform Cauchy sequence in D.*

PROOF. It is simple to show that uniform convergence implies uniform Cauchy. To prove the converse, note that for every fixed $z \in D$, the sequence of complex numbers $a_n(z)$ is a Cauchy sequence. It follows from Theorem 16E that $a_n(z)$ converges to $a(z)$, say. Since $a_n(z)$ is a uniform Cauchy sequence in D, it follows that, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a_m(z)| < \epsilon$ whenever $m, n > N$. Letting $m \to \infty$, we conclude that $|a_n(z) - a(z)| \le \epsilon$ whenever $n > N$. ◯

16.4. Uniform Convergence of Series

Recall that the convergence of a series depends on the convergence of the sequence of partial sums.

DEFINITION. Suppose that $D \subseteq \mathbb{C}$ is a region. We say that a series of complex valued functions

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly in D if the sequence of partial sums

$$s_N(z) = \sum_{n=1}^{N} a_n(z)$$

converges uniformly in D.

We immediately have the following analogues of Theorems 16A, 16B, 16D, 16E and 16F. They can be established by applying the earlier results to the sequence of partial sums.

THEOREM 16G. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that the series*

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly to a function $s(z)$ in D. Then $s(z)$ is continuous in D.

THEOREM 16H. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that the series*

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly to a function $s(z)$ in D. Then for any contour C lying in D, we have

$$\sum_{n=1}^{\infty} \int_C a_n(z)\,\mathrm{d}z = \int_C s(z)\,\mathrm{d}z.$$

In other words, we can interchange the order of summation and integration.

THEOREM 16J. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is analytic in a domain $D \subseteq \mathbb{C}$. Suppose further that the series*

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly to a function $s(z)$ in D. Then $s(z)$ is analytic in D. Furthermore, for every $z \in D$ and every $k \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} a_n^{(k)}(z) = s^{(k)}(z).$$

In other words, we can interchange the order of summation and differentiation.

THEOREM 16K. (GENERAL PRINCIPLE OF CONVERGENCE) *A series*

$$\sum_{n=1}^{\infty} a_n$$

of complex numbers converges if and only if, given any $\epsilon > 0$, there exists $N_0 \in \mathbb{R}$ such that

$$\left| \sum_{n=N_1+1}^{N_2} a_n \right| < \epsilon$$

whenever $N_2 > N_1 > N_0$.

THEOREM 16L. (GENERAL PRINCIPLE OF UNIFORM CONVERGENCE) *Suppose that $D \subseteq \mathbb{C}$ is a region. A series*

$$\sum_{n=1}^{\infty} a_n(z)$$

of complex valued functions converges uniformly in D if and only if, given any $\epsilon > 0$, there exists $N_0 \in \mathbb{R}$ such that for every $z \in D$,

$$\left| \sum_{n=N_1+1}^{N_2} a_n(z) \right| < \epsilon$$

whenever $N_2 > N_1 > N_0$.

We can also establish the following uniform versions of the Comparison test and the Ratio test.

THEOREM 16M. (WEIERSTRASS M-TEST) *Suppose that $D \subseteq \mathbb{C}$ is a region. Suppose further that $a_n(z)$ is a sequence of complex valued functions such that $|a_n(z)| \leq M_n$ for every $z \in D$, where the real series*

$$\sum_{n=1}^{\infty} M_n$$

of non-negative terms is convergent. Then the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly (and absolutely) in D.

PROOF. Using the Triangle inequality, we have

$$\left| \sum_{n=N_1+1}^{N_2} a_n(z) \right| \leq \sum_{n=N_1+1}^{N_2} |a_n(z)| \leq \sum_{n=N_1+1}^{N_2} M_n.$$

Given any $\epsilon > 0$, it follows from Theorem 16K that there exists N_0 such that

$$\sum_{n=N_1+1}^{N_2} M_n < \epsilon$$

whenever $N_2 > N_1 > N_0$. It follows that for every $z \in D$,

$$\left| \sum_{n=N_1+1}^{N_2} a_n(z) \right| < \epsilon$$

whenever $N_2 > N_1 > N_0$. The result now follows from Theorem 16L. ◯

THEOREM 16N. (RATIO TEST) *Suppose that $D \subseteq \mathbb{C}$ is a region. Suppose further that $a_n(z)$ is a sequence of complex valued functions such that $a_1(z)$ is bounded in D, and*

$$\left|\frac{a_{n+1}(z)}{a_n(z)}\right| \le R < 1 \tag{4}$$

for every $z \in D$, where R is constant. Then the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly (and absolutely) in D.

PROOF. Note that the condition (4) implies $|a_n(z)| \le R^{n-1}|a_1(z)|$ for every $n \in \mathbb{N}$. On the other hand, there exists $M \in \mathbb{R}$ such that $|a_1(z)| \le M$ for every $z \in D$. It follows that for every $z \in D$ and every $n \in \mathbb{N}$, we have $|a_n(z)| \le MR^{n-1}$. The result now follows from the Weierstrass M-test, noting that the geometric series

$$\sum_{n=1}^{\infty} MR^{n-1}$$

converges. ◯

EXAMPLE 16.4.1. The series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{5}$$

converges absolutely for every z satisfying $\mathfrak{Re}\, z > 1$. To see this, note that writing $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, we have

$$\frac{1}{n^z} = \frac{1}{n^{x+\mathrm{i}y}} = \frac{1}{n^x} n^{-\mathrm{i}y} = \frac{1}{n^x}\mathrm{e}^{-\mathrm{i}y\log n} = \frac{1}{n^x}(\cos(y\log n) - \mathrm{i}\sin(y\log n)),$$

so that

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^x}.$$

Since $x > 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^x}$$

of non-negative terms is convergent. It follows from the Comparison test that the series (5) converges absolutely. Suppose now that $\delta > 0$ is fixed. Consider the region $D = \{z : \mathfrak{Re}\, z > 1 + \delta\}$. Then for every $z \in D$, we have

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^x} < \frac{1}{n^{1+\delta}}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$$

of non-negative terms is convergent. It follows from the Weierstrass M-test that the series (5) converges uniformly in D. We comment here that the series (5) is called the Riemann zeta function, and is crucial in the study of the distribution of prime numbers. Indeed, the study of this function has led to much of the development in complex analysis.

EXAMPLE 16.4.2. In Chapter 10, we discussed the function $\pi \cot \pi z$, and showed that it has simple poles at the (real) integers with residue 1. Here we shall make a more detailed study. Consider the function

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Let us first of all study this function in the region $D_R = \{z : |z| < R\}$, where $R > 0$ is fixed. Let $N \in \mathbb{N}$ satisfy $N > 2R$, and write $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2} \qquad \text{and} \qquad f_2(z) = \sum_{n=N+1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Clearly the function $f_1(z)$ is analytic in D_R, with the exception of simple poles at the (real) integers in D_R. Consider next the function $f_2(z)$ in D_R. For every $z \in D_R$ and every $n > N > 2R$, we have

$$\left| \frac{2z}{z^2 - n^2} \right| \le \frac{2R}{n^2 - R^2} = \frac{1}{n^2} \frac{2R}{1 - (R/n)^2} < \frac{8R}{3n^2}.$$

It follows from the Weierstrass M-test that the series for $f_2(z)$ converges uniformly in D_R, and is analytic in D_R in view of Theorem 16J. Hence $f(z)$ is analytic in D_R, with the exception of simple poles at the (real) integers in D_R. It follows that $f(z)$ is meromorphic in $\mathbb{C}$, with simple poles at the (real) integers. It is easy to check that all these simple poles have residue 1. Note also that we can write

$$f(z) = z \sum_{n \in \mathbb{Z}} \frac{1}{z^2 - n^2}.$$

We shall show that $f(z) = \pi \cot \pi z$. For convenience, we shall change notation, and show that

$$\sum_{n \in \mathbb{Z}} \frac{1}{a^2 - n^2} = \frac{\pi \cot \pi a}{a} \tag{6}$$

whenever $a \notin \mathbb{Z}$. Consider the function

$$g(z) = \frac{\pi \cot \pi z}{a^2 - z^2}.$$

Since the function $\pi \cot \pi z$ has simple poles at every $n \in \mathbb{Z}$ with residue 1, and since $a \notin \mathbb{Z}$, it follows that $g(z)$ has simple poles at every $n \in \mathbb{Z}$ and at $z = \pm a$, with residues

$$\operatorname{res}(g, n) = \frac{1}{a^2 - n^2} \qquad \text{and} \qquad \operatorname{res}(g, \pm a) = -\frac{\pi \cot \pi a}{2a}.$$

For every $N \in \mathbb{N}$, let C_N denote the boundary of the rectangular domain

$$\left\{ z = x + \mathrm{i}y : |x| < N + \frac{1}{2} \text{ and } |y| < N \right\},$$

followed in the positive (anticlockwise) direction. If $N > |a|$, then we have

$$\frac{1}{2\pi \mathrm{i}} \int_{C_N} \frac{\pi \cot \pi z}{a^2 - z^2} \, \mathrm{d}z = \sum_{-N \le n \le N} \frac{1}{a^2 - n^2} - \frac{\pi \cot \pi a}{a}.$$

Clearly (6) will follow if we show that the integral on the left hand side converges to 0 as $N \to \infty$. It can be shown that $|\cot \pi z| \le \coth \pi$ for every $z \in C_N$. Hence for every $N > |a|$, we have

$$\left| \int_{C_N} \frac{\pi \cot \pi z}{a^2 - z^2} \, \mathrm{d}z \right| \le (8N+2) \sup_{z \in C_N} \left| \frac{\pi \cot \pi z}{a^2 - z^2} \right| \le \frac{(8N+2)\pi \coth \pi}{N^2 - |a|^2} \to 0 \quad \text{as } N \to \infty.$$

16.5. Application to Power Series

Let $z, \alpha \in \mathbb{C}$. In this section, we shall study series of the type

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \qquad (a_0, a_1, a_2, \ldots \in \mathbb{C}), \tag{7}$$

known commonly as power series.

THEOREM 16P. *Suppose that the series given by* (7) *converges for a particular value $z = z_0$. Then, for every $r < |z_0 - \alpha|$, the series converges uniformly (and absolutely) in the disc $D_r = \{z : |z - \alpha| \le r\}$.*

PROOF. Suppose that

$$\sum_{n=0}^{\infty} a_n (z_0 - \alpha)^n$$

converges. Then $a_n(z_0 - \alpha)^n \to 0$ as $n \to \infty$, and so there exists $M \in \mathbb{R}$ such that $|a_n(z_0 - \alpha)^n| \le M$ for every $n \in \mathbb{N} \cup \{0\}$. For every $z \in D_r$, we have

$$|a_n(z - \alpha)^n| \le M \left| \frac{z - \alpha}{z_0 - \alpha} \right|^n \le M \left| \frac{r}{z_0 - \alpha} \right|^n$$

for every $n \in \mathbb{N} \cup \{0\}$. The result now follows from the Weierstrass M-test, noting that the geometric series

$$\sum_{n=0}^{\infty} M \left| \frac{r}{z_0 - \alpha} \right|^n$$

converges. ◯

THEOREM 16Q. (CONVERGENCE THEOREM FOR POWER SERIES) *For the power series given by* (7)*, exactly one of the following holds:*
(a) The series converges absolutely for every $z \in \mathbb{C}$.
(b) There exists a positive real number R such that the series converges absolutely for every $z \in \mathbb{C}$ satisfying $|z - \alpha| < R$ and diverges for every $z \in \mathbb{C}$ satisfying $|z - \alpha| > R$.
(c) The series diverges for every $z \ne \alpha$.

SKETCH OF PROOF. In the notation of Theorem 16P, consider

$$S = \{r \ge 0 : (7) \text{ converges absolutely in } D_r\}.$$

Then S contains the number 0. In view of Theorem 16P, S must be an interval with lower end-point 0, so that $S = [0, \infty)$, $S = \{0\}$ or there exists some positive number R such that $S = [0, R)$ or $S = [0, R]$. The first two possibilities correspond to (a) and (c) respectively, while the last possibility corresponds to (b). ◯

DEFINITION. The number R in Theorem 16Q is called the radius of convergence of the series (7). We also say that $R = 0$ if case (c) occurs, and that $R = \infty$ if case (a) occurs.

We now show that differentiation of a power series can be carried out term by term, and that the series so obtained converges to the derivative.

THEOREM 16R. *Suppose that the power series given by (7) has radius of convergence $R > 0$. Then it represents an analytic function $f(z)$ in the open disc $D = \{z : |z - \alpha| < R\}$. Furthermore, the derivatives of $f(z)$ can be obtained by differentiating the series term by term.*

PROOF. For every $r < R$, it follows from Theorem 16P that the series converges uniformly in the disc $D_r = \{z : |z-\alpha| < r\}$. It now follows from Theorem 16J that the series converges to an analytic function $f(z)$ in D_r, and the derivatives of $f(z)$ can be obtained by differentiating the series term by term. Since the above holds for any $r < R$, the result follows. ○

EXAMPLE 16.5.1. Suppose that $f(t)$ is a complex valued function continuous (and so bounded) on the closed real interval $[0, 1]$. Consider the function

$$F(z) = \int_0^1 \mathrm{e}^{-zt} f(t)\,\mathrm{d}t.$$

For any fixed $z \in \mathbb{C}$, we have the power series (here t is the variable)

$$\mathrm{e}^{-zt} = \sum_{n=0}^{\infty} \frac{(-zt)^n}{n!}, \tag{8}$$

with infinite radius of convergence. It follows from Theorem 16P that the series (8) converges uniformly in $[0, 1]$, and so can be multiplied by the bounded function $f(t)$ and integrated term by term, in view of Theorem 16H. Hence

$$F(z) = \sum_{n=0}^{\infty} \int_0^1 \frac{(-zt)^n}{n!} f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \int_0^1 t^n f(t)\,\mathrm{d}t. \tag{9}$$

Furthermore, if $|f(t)| \le M$, where M is a fixed positive number, then

$$\left| \int_0^1 t^n f(t)\,\mathrm{d}t \right| \le M \int_0^1 t^n\,\mathrm{d}t = \frac{M}{n+1}.$$

Suppose now that $R > 0$ is fixed. If $|z| < R$, then

$$\left| \frac{(-z)^n}{n!} \int_0^1 t^n f(t)\,\mathrm{d}t \right| \le \frac{MR^n}{(n+1)!}.$$

Note that the series

$$\sum_{n=0}^{\infty} \frac{MR^n}{(n+1)!}$$

converges, so it follows from the Weierstrass M-test that the series in (9) converges uniformly in the disc $\{z : |z| < R\}$. By Theorem 16J, the function $F(z)$ is analytic in $\{z : |z| < R\}$. Since $R > 0$ is arbitrary, it follows that $F(z)$ is entire.

16.6. Cauchy Sequences

In this section, we shall prove Theorem 16E. Clearly a convergent sequence of complex numbers is Cauchy. It remains to show that a Cauchy sequence of complex numbers is convergent.

The proof of this result usually involves the Bolzano-Weierstrass theorem which states that every bounded sequence of complex numbers has a convergent subsequence. Here, we shall give a proof without using the Bolzano-Weierstrass theorem.

Assume, first of all, that the sequence a_n is real. Since a_n is a Cauchy sequence, it follows that there exists an increasing sequence of natural numbers

$$N_1 < N_2 < \ldots < N_p < \ldots$$

such that

$$|a_n - a_m| < \frac{1}{2^p}$$

whenever $n, m \geq N_p$ (we simply take $\epsilon = 2^{-p}$ for every $p \in \mathbb{N}$). In particular, we have

$$|a_{N_{p+1}} - a_{N_p}| < \frac{1}{2^p}$$

for every $p \in \mathbb{N}$. For every $p \in \mathbb{N}$, let

$$b_p = a_{N_p} - \frac{1}{2^{p-1}}.$$

Then

$$b_{p+1} - b_p = a_{N_{p+1}} - a_{N_p} + \frac{1}{2^p} \geq \frac{1}{2^p} - |a_{N_{p+1}} - a_{N_p}| > 0,$$

so that the sequence b_p is increasing. Note next that

$$|b_p| = \left| a_{N_p} - \frac{1}{2^{p-1}} \right| \leq |a_{N_p} - a_{N_1}| + |a_{N_1}| + \frac{1}{2^{p-1}} \leq \frac{1}{2} + |a_{N_1}| + \frac{1}{2^{p-1}},$$

so that the sequence b_p is bounded. Hence the sequence b_p converges to L, say, as $p \to \infty$.

We now show that $a_n \to L$ as $n \to \infty$. Given any $\epsilon > 0$, we now choose $p \in \mathbb{N}$ so large that

$$\frac{1}{2^p} < \frac{\epsilon}{4} \qquad \text{and} \qquad |b_p - L| < \frac{\epsilon}{4}.$$

Suppose that $n \geq N_p$. Then

$$|a_n - L| \leq |a_n - a_{N_p}| + |a_{N_p} - b_p| + |b_p - L| < \frac{1}{2^p} + \frac{1}{2^{p-1}} + \frac{\epsilon}{4} < \epsilon$$

as required.

Suppose now that the sequence a_n is complex valued. Then we can write $a_n = x_n + \mathrm{i}y_n$, where $x_n, y_n \in \mathbb{R}$. If a_n is a Cauchy sequence, then it is easy to see that the real sequences x_n and y_n are real Cauchy sequences. It follows that both x_n and y_n converge, and so a_n converges.

Problems for Chapter 16

1. Suppose that $a_n(z) \to a(z)$ and $b_n(z) \to b(z)$ as $n \to \infty$ uniformly in a region D.
   a) Show that $a_n(z) + b_n(z) \to a(z) + b(z)$ as $n \to \infty$ uniformly in D.
   b) Suppose that $f(z)$ is bounded in D. Show that $a_n(z)f(z) \to a(z)f(z)$ as $n \to \infty$ uniformly in D.
   c) Write $f(z) = 1/z$ and $a_n(z) = 1/n$. Find a region D such that $a_n(z)$ converges uniformly in D but $a_n(z)f(z)$ does not converge uniformly in D.

2. For each of the following power series, find a number R such that the series converges for $|z| < R$ and diverges for $|z| > R$:
   a) $\displaystyle\sum_{n=0}^{\infty} 2^n z^n$
   b) $\displaystyle\sum_{n=1}^{\infty} n^2 z^n$
   c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n z^{2n}}{n^2+n}$
   d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n z^n}{4^n+5^n}$

3. Show that each of the following represents an entire function:
   a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{1/2}}$
   b) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{2^{n^2}}$
   c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n^z}$

4. Show that each of the following functions is meromorphic in $\mathbb{C}$, and find the residues at the poles:
   a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)}$
   b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$

5. Show that for every $z \notin \mathbb{Z}$, we have $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2} = \left(\frac{\pi}{\sin \pi z}\right)^2.$

6. a) Show that except at the poles, we have $\displaystyle\sum_{n=-\infty}^{\infty} \frac{z}{n^2+z^2} = \frac{\pi}{\tanh \pi z}.$
   b) By writing the series as $1/z$ plus a sum over all natural numbers, evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}.$
   c) By letting $z \to 0$, show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

7. Consider the exponential series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

   which converges for every $z \in \mathbb{C}$. Suppose further that $e(z)$ is the sum of the series.
   a) Show that the series converges uniformly in the disc $D_R = \{z : |z| < R\}$ for every real number $R > 0$.
   b) Suppose that D is a bounded region in $\mathbb{C}$. Explain why the series converges uniformly in D.
   c) Show that for every $z \in \mathbb{C}$ satisfying $|z| = R$, we have

$$\left|\sum_{n=N+1}^{M} \frac{z^n}{n!}\right| \geq \frac{R^M}{M!} - R^M\left(\frac{1}{R} + \frac{1}{R^2} + \ldots + \frac{1}{R^{M-N-1}}\right) \geq R^M\left(\frac{1}{M!} - \frac{1}{R-1}\right).$$

d) Use (c) to show that the series does not converge uniformly in $\mathbb{C}$.
e) Explain carefully why $e(z)$ is an entire function in $\mathbb{C}$.
[REMARK: In view of the unfavourable conclusion of (d), you should take extra care here.]
f) Show that $e'(z) = e(z)$ for every $z \in \mathbb{C}$ and $e(0) = 1$.
g) Let $g(z) = e(-z)e(z)$. Show that $g'(z) = 0$ for every $z \in \mathbb{C}$, and deduce that $e(-z)e(z) = 1$ for every $z \in \mathbb{C}$.
h) Suppose that $a \in \mathbb{C}$ is fixed. By studying the function $g_a(z) = e(-z)e(z + a)$, show that $e(z + a) = e(z)e(a)$ for every $z \in \mathbb{C}$.

8. This question makes use of the function $e(z)$ discussed in Problem 7. Suppose that for every $z \in \mathbb{C}$, we write

$$c(z) = \frac{e(\mathrm{i}z) + e(-\mathrm{i}z)}{2} \qquad \text{and} \qquad s(z) = \frac{e(\mathrm{i}z) - e(-\mathrm{i}z)}{2\mathrm{i}}.$$

a) By using the Taylor series for $e(\mathrm{i}z)$ and $e(-\mathrm{i}z)$, find the Taylor series for $c(z)$ and $s(z)$.
b) Show that $c'(z) = -s(z)$ and $s'(z) = c(z)$ for every $z \in \mathbb{C}$.
c) By studying the function $h(z) = c^2(z) + s^2(z)$, show that $c^2(z) + s^2(z) = 1$ for every $z \in \mathbb{C}$.