INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

Chapter 15

LAPLACE'S EQUATION REVISITED

15.1. Use of Möbius Transformations

Recall that Laplace's equation involves finding a harmonic function in a given region and which satisfies given boundary conditions. In this chapter, we shall illustrate very briefly the use of transformations to simplify this problem. Note, however, that we are not discussing the general problem of the solution of Laplace's equation; that is a topic in partial differential equations. Here we shall satisfy ourselves on how to use a few simple cases of Laplace's equation to obtain solutions in more complicated situations. We first discuss an example which uses Möbius transformations.

Example 15.5.1. Consider the lens region formed by the intersection of the two discs

$$\{z \in \mathbb{C} : |z+1|^2 < 2\} \cap \{z \in \mathbb{C} : |z-1|^2 < 2\}.$$

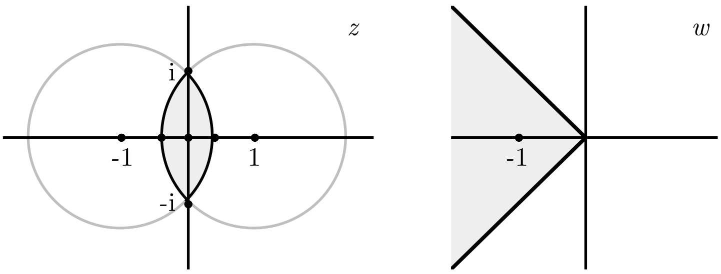
Here the two discs both have radius $\sqrt{2}$ and are centred at $z = -1$ and $z = 1$ respectively (see the picture on the next page). Suppose that we are required to find a harmonic function ϕ in this region with $\phi = 1$ on the right hand boundary and $\phi = 0$ on the left hand boundary. Note that both the right hand and left hand boundaries are parts of circles and intersect at $z = \pm\mathrm{i}$. If we use a Möbius transformation with pole at $z = \mathrm{i}$, then both boundaries are transformed into straight lines. Let us try the transformation

$$w = f(z) = \frac{z+\mathrm{i}}{z-\mathrm{i}}.$$

Then $f(\mathrm{i}) = \infty$ and $f(-\mathrm{i}) = 0$. By considering, for example, $f(\sqrt{2}-1)$ and $f(1-\sqrt{2})$, it is not difficult to show that the right hand and left hand boundaries are transformed into the half lines $\arg w = 3\pi/4$ and $\arg w = 5\pi/4$ respectively. Note also that $f(0) = -1$. It follows that the lens region is transformed into the region

$$\left\{ w \in \mathbb{C} : \frac{3\pi}{4} < \arg w < \frac{5\pi}{4} \right\}.$$

We summarize the above discussion in the pictures below:



It is easy to check that the function

$$\psi(w) = \frac{2}{\pi}\left(\frac{5\pi}{4} - \arg w\right)$$

is harmonic in this region and satisfies $\psi(w) = 1$ when $\arg w = 3\pi/4$ and $\psi(w) = 0$ when $\arg w = 5\pi/4$. It follows that our required harmonic function is given by

$$\phi(z) = \frac{2}{\pi}\left(\frac{5\pi}{4} - \arg\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)\right).$$

15.2. Use of Schwarz-Christoffel Transformations

We now discuss three examples which use Schwarz-Christoffel transformations.

Example 15.2.1. We wish to find a non-constant harmonic function in the region above the polygonal path given in Example 14.4.2, with boundary condition $\phi = 0$ on the polygonal path. Here $\phi =$ const can be interpreted as lines of flow on a river over a step on the river bed. Recall that the Schwarz-Christoffel transformation

$$f(z) = \frac{1}{\pi}\left((z^2-1)^{1/2} + \log(z + (z^2-1)^{1/2})\right)$$

maps the upper half plane onto the region in question. We now need to find a non-constant harmonic function ψ on the upper half plane with boundary condition $\psi = 0$ on the real line. For example, the function

$$\psi(z) = \mathfrak{Im} z$$

satisfies the requirements. We now need to invert the function $f(z)$ to obtain a harmonic function

$$\phi(w) = \mathfrak{Im}(f^{-1}(w))$$

in the original region.

Example 15.2.2. We wish to find a harmonic function in the slit plane given in Example 14.4.4, with boundary conditions $\phi = 1$ on the upper slit and $\phi = -1$ on the lower slit. Here $\phi =$ const can be interpreted as equipotential lines in a region around two semi-infinite conducting plates with opposite charges. Recall that the Schwarz-Christoffel transformation

$$f(z) = -\frac{2}{\pi}\left(\frac{z^2}{2} - \log z\right) + \left(\frac{1}{\pi} - \mathrm{i}\right)$$

maps the upper half plane onto the region in question. Furthermore, it maps the negative and positive real axis onto the upper and lower slits respectively. We now need to find a harmonic function ψ on the upper half plane with boundary conditions $\psi = 1$ on the negative real axis and $\psi = -1$ on the positive real axis. For example, the function

$$\psi(z) = \frac{2}{\pi} \arg z - 1$$

satisfies the requirements (here we take the principal value of the argument). We now need to invert the function $f(z)$ to obtain a harmonic function

$$\phi(w) = \frac{2}{\pi} \arg(f^{-1}(w)) - 1$$

in the original region.

EXAMPLE 15.2.3. We wish to find a non-constant harmonic function in the slit upper half plane given in Example 14.4.5, with boundary condition $\phi = 0$ on the slit and the real axis. Here $\phi = \text{const}$ can be interpreted as lines of flow past an obstacle. Recall that the Schwarz-Christoffel transformation

$$f(z) = (z^2 - 1)^{1/2}$$

maps the upper half plane onto the region in question. We now need to find a harmonic function ψ on the upper half plane with boundary conditions $\psi = 0$ on the real line. For example, the function

$$\psi(z) = \mathfrak{Im} z$$

satisfies the requirements. Note now that

$$f^{-1}(w) = (w^2 + 1)^{1/2}.$$

We therefore obtain the harmonic function

$$\phi(w) = \mathfrak{Im}((w^2 + 1)^{1/2})$$

in the original region. Here we choose a branch of the square root that is positive for large positive w.