INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 14

SCHWARZ-CHRISTOFFEL TRANSFORMATIONS

14.1. Introduction

Recall that a function $f(z)$ is conformal at every point where it is analytic and has non-zero derivative. In this chapter, we shall study the situation at points where $f(z)$ is not conformal.

Suppose that $x_0 \in \mathbb{R}$ is fixed. Consider a function $f(z)$ with derivative

$$f'(z) = (z - x_0)^\alpha,$$

where $-1 < \alpha < 1$. Here we have chosen the branch of the argument so that

$$-\frac{\pi}{2} < \arg(z - x_0) \le \frac{3\pi}{2},$$

introducing a branch cut along the axis $\{x_0 + \mathrm{i}y : y \le 0\}$. We shall study the image of the real axis under this mapping f.

Suppose first of all that z lies on the real axis and $z > x_0$. Then $f(z)$ is conformal at such a point z, since $f'(z) \ne 0$. Note also that

$$\arg f'(z) = \alpha \arg(z - x_0) = 0$$

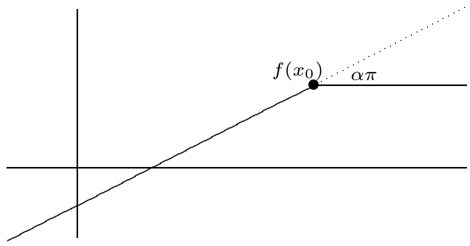
for all such points z, ignoring multiples of 2π. Since the tangent at every point of the half line (x_0, ∞) has slope 0, it follows that the tangent at every point of the image curve $f((x_0, \infty))$ has slope $\arg f'(z) = 0$. Hence $f((x_0, \infty))$ is a half line parallel to the real axis and has left hand end point $f(x_0)$.

Suppose next that z lies on the real axis and $z < x_0$. Again $f(z)$ is conformal at such a point z, since $f'(z) \ne 0$. Note also that

$$\arg f'(z) = \alpha \arg(z - x_0) = \alpha\pi$$

for all such points z, again ignoring multiples of 2π. It follows easily that $f((-\infty, x_0))$ is a half line making an angle $\alpha\pi$ with the horizontal axis.

Summarizing the above, we have the following diagram which describes the image of the real axis under f.



14.2. A Generalization

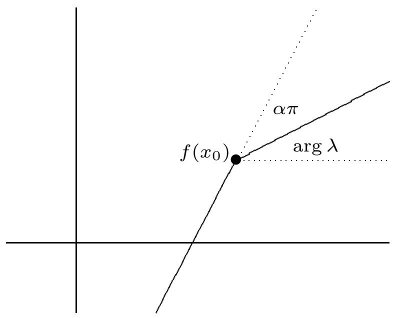
Again, suppose that $x_0 \in \mathbb{R}$ is fixed. Consider a function $f(z)$ with derivative

$$f'(z) = \lambda(z - x_0)^\alpha,$$

where $\lambda \in \mathbb{C}$ is non-zero and $-1 < \alpha < 1$. Then

$$\arg f'(z) = \arg \lambda + \alpha \arg(z - x_0).$$

In other words, there is an extra rotation by $\arg \lambda$ from the case in the previous section. This leads to the following diagram which describes the image of the real axis under f.



Suppose now that $x_1, \ldots, x_k \in \mathbb{R}$ are fixed, and that $x_1 < \ldots < x_k$. Consider a function $f(z)$ with derivative

$$f'(z) = \lambda(z - x_1)^{\alpha_1} \ldots (z - x_k)^{\alpha_k}, \tag{1}$$

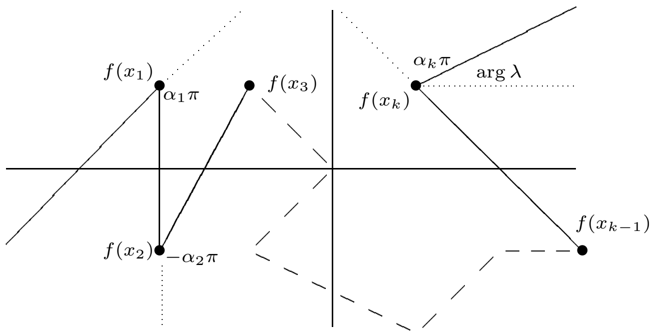
where $\lambda \in \mathbb{C}$ is non-zero and $-1 < \alpha_1, \ldots, \alpha_k < 1$. Then

$$\arg f'(z) = \arg \lambda + \alpha_1 \arg(z - x_1) + \ldots + \alpha_k \arg(z - x_k).$$

It is easy to see that if z is on the real axis, then

$$\arg f'(z) = \begin{cases} \arg \lambda & \text{if } z > x_k, \\ \arg \lambda + \alpha_k \pi & \text{if } x_{k-1} < z < x_k, \\ \vdots & \\ \arg \lambda + \alpha_2 \pi + \ldots + \alpha_k \pi & \text{if } x_1 < z < x_2, \\ \arg \lambda + \alpha_1 \pi + \ldots + \alpha_k \pi & \text{if } z < x_1. \end{cases}$$

This leads to the following diagram which describes the image of the real axis under f.



Suppose now that a function $f(z)$ satisfies (1). Then it is analytic on the complex plane $\mathbb{C}$ with a few branch cuts at $x_1, \ldots, x_k$. More precisely, it is analytic in the domain

$$\mathbb{C} \setminus (\{x_1 + \mathrm{i}y : y \le 0\} \cup \ldots \cup \{x_k + \mathrm{i}y : y \le 0\}).$$

It follows that for any $z \in \mathcal{H}$, where $\mathcal{H}$ denotes the upper half plane, we can write

$$f(z) = \int_{[z_0,z]} f'(\zeta)\,\mathrm{d}\zeta + B = \lambda \int_{[z_0,z]} (\zeta - x_1)^{\alpha_1} \ldots (\zeta - x_k)^{\alpha_k}\,\mathrm{d}\zeta + B. \tag{2}$$

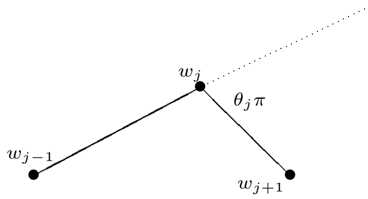
Here, z_0 is a suitably chosen point in $\mathcal{H}$ or its boundary. Also, for every $z \in \mathcal{H}$, $[z_0, z]$ denotes the straight line segment from z_0 to z.

Definition. A function $f(z)$ of the form (2) is called a Schwarz-Christoffel transformation.

14.3. Polygons

Note that the function (2) maps the real axis onto a polygonal path. We now wish to construct a one-to-one analytic function that maps the upper half plane $\mathcal{H}$ onto the interior of a given polygon P. The idea is to tailor a Schwarz-Christoffel transformation to achieve this.

Suppose that the vertices of the polygon P are given by $w_1, \ldots, w_k$ in the anticlockwise direction. Let us follow the edges of the polygon P. At vertex w_j, suppose that we make a right turn of angle $\theta_j\pi$, where $-1 < \theta_j < 1$, with the convention that $\theta_j < 0$ denotes a left turn.



Since P is a polygon and its vertices are given in the anticlockwise direction, we must have

$$\theta_1\pi + \ldots + \theta_k\pi = -2\pi.$$

It is an elementary fact in geometry that if we know the vertices $w_1, \ldots, w_{k-1}$ and angles $\theta_1\pi, \ldots, \theta_{k-1}\pi$ of the polygon P, then the last vertex w_k and angle $\theta_k\pi$ are uniquely determined. The idea is therefore to find real numbers $x_1 < \ldots < x_{k-1}$ to act as preimages of the vertices $w_1, \ldots, w_{k-1}$, and to assume that $x = \infty$ is the preimage of the vertex w_k.

Suppose that $x_1 < \ldots < x_{k-1}$. Clearly the function

$$g(z) = \int_{[z_0,z]} (\zeta - x_1)^{\theta_1} \ldots (\zeta - x_{k-1})^{\theta_{k-1}} \, d\zeta$$

maps the real line onto some polygon Q of k sides. However, the polygon Q may not be the polygon P, but at least it has the required right hand turn angles $\theta_1, \ldots, \theta_{k-1}$ at the vertices $g(x_1), \ldots, g(x_{k-1})$. We can adjust the lengths of the sides of the polygon Q by choosing $x_1, \ldots, x_{k-1}$ carefully, so that Q is similar to the polygon P. Once this is achieved, we can then map the polygon Q to the polygon P by a linear transformation.

We state, without proof, the following important result.

THEOREM 14A. *Suppose that P is a polygon with vertices $w_1, \ldots, w_k$ in the anticlockwise direction, with corresponding right turns of angles $\theta_1\pi, \ldots, \theta_k\pi$ respectively, where $-1 < \theta_1, \ldots, \theta_k < 1$. Then there exists a function of the form*

$$f(z) = A \int_{[z_0,z]} (\zeta - x_1)^{\theta_1} \ldots (\zeta - x_{k-1})^{\theta_{k-1}} \, d\zeta + B,$$

where $A, B \in \mathbb{C}$, that maps the upper half plane $\mathcal{H}$ one-to-one and conformally onto the interior of P, with

$$f(x_1) = w_1, \qquad \ldots, \qquad f(x_{k-1}) = w_{k-1}, \qquad f(\infty) = w_k.$$

Remarks. (1) Note that we do not even need to have very precise information on w_k and θ_k.

(2) Certain infinite regions can sometimes be thought of as infinite polygons. In this case, it is sometimes convenient to take w_k as the point at infinity, as we need no information on the angle θ_k when we use Theorem 14A.

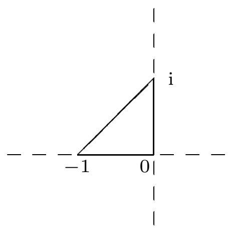
(3) It can be shown that a Schwarz-Christoffel transformation can be uniquely determined by three points, as is the case for Möbius transformations. This can be interpreted as three degrees of freedom in our construction of the transformation. One of these is used by taking $f(\infty) = w_k$. We can therefore afford to choose x_1 and x_2 freely, subject to the restriction that $-\infty < x_1 < x_2 < \infty$.

(4) Occasionally, we may choose extra points apart from x_1 and x_2 due to symmetry properties of the polygon P. We shall illustrate this point in Examples 14.4.3–14.4.5 below.

(5) Note that the integrals involved may be impossible to calculate in practice. Numerical techniques are often used. However, we shall not discuss these here.

14.4. Examples

Example 14.4.1. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane $\mathcal{H}$ to the inside of the triangle with vertices at -1, 0 and i. The boundary of the triangle is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = \mathrm{i}, \qquad w_2 = -1, \qquad w_3 = 0,$$

so that

$$\theta_1 = -3/4, \qquad \theta_2 = -3/4, \qquad \theta_3 = -1/2.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0,z]} (\zeta - x_1)^{-3/4} (\zeta - x_2)^{-3/4} \, \mathrm{d}\zeta + B.$$

We may choose $x_1 = -1$ and $x_2 = 1$, and obtain, using $z_0 = 0$,

$$f(z) = A \int_{[0,z]} (\zeta + 1)^{-3/4} (\zeta - 1)^{-3/4} \, \mathrm{d}\zeta + B = A \int_{[0,z]} (\zeta^2 - 1)^{-3/4} \, \mathrm{d}\zeta + B.$$

We need $f(-1) = \mathrm{i}$ and $f(1) = -1$. It follows that

$$A \int_0^{-1} (\zeta^2 - 1)^{-3/4} \, \mathrm{d}\zeta + B = \mathrm{i} \qquad \text{and} \qquad A \int_0^{1} (\zeta^2 - 1)^{-3/4} \, \mathrm{d}\zeta + B = -1.$$

Writing

$$\kappa = \int_0^1 (\zeta^2 - 1)^{-3/4} \, \mathrm{d}\zeta,$$

we have

$$-A\kappa + B = \mathrm{i} \qquad \text{and} \qquad A\kappa + B = -1,$$

so that

$$A = \frac{-1 - \mathrm{i}}{2\kappa} \qquad \text{and} \qquad B = \frac{\mathrm{i} - 1}{2}.$$

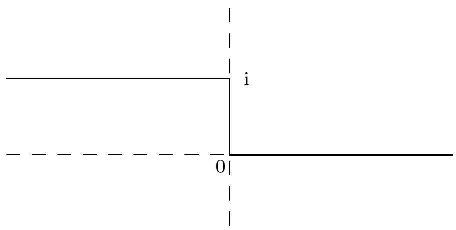
Hence

$$f(z) = \frac{-1 - \mathrm{i}}{2\kappa} \int_{[0,z]} (\zeta^2 - 1)^{-3/4} \, \mathrm{d}\zeta + \frac{\mathrm{i} - 1}{2}.$$

EXAMPLE 14.4.2. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane $\mathcal{H}$ to the set

$$P = \{z = x + \mathrm{i}y : x > 0 \text{ and } y > 0\} \cup \{z = x + \mathrm{i}y : x \leq 0 \text{ and } y > 1\}.$$

The boundary of P is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = \mathrm{i}, \qquad w_2 = 0, \qquad w_3 = \infty,$$

so that

$$\theta_1 = 1/2 \qquad \text{and} \qquad \theta_2 = -1/2.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0,z]} (\zeta - x_1)^{1/2} (\zeta - x_2)^{-1/2} \, \mathrm{d}\zeta + B'.$$

We may choose $x_1 = -1$ and $x_2 = 1$, and obtain

$$\begin{aligned} f(z) &= A \int_{[z_0,z]} (\zeta + 1)^{1/2} (\zeta - 1)^{-1/2} \, \mathrm{d}\zeta + B' = A \int_{[z_0,z]} \left(\frac{\zeta + 1}{\zeta - 1} \right)^{1/2} \mathrm{d}\zeta + B' \\ &= A \left((z^2 - 1)^{1/2} + \log(z + (z^2 - 1)^{1/2}) \right) + B. \end{aligned}$$

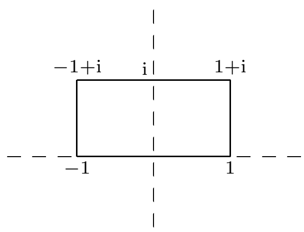
We shall omit some of the painful analysis, and claim that we can choose a branch of the function which is analytic in the upper half plane $\mathcal{H}$. We need $f(-1) = \mathrm{i}$ and $f(1) = -1$. It follows that by choosing a suitable branch of the logarithm, we have

$$A \log(-1) + B = \mathrm{i} \qquad \text{and} \qquad A \log 1 + B = 0,$$

so that $A = 1/\pi$ and $B = 0$. Hence

$$f(z) = \frac{1}{\pi} \left((z^2 - 1)^{1/2} + \log(z + (z^2 - 1)^{1/2}) \right).$$

Example 14.4.3. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane $\mathcal{H}$ to the inside of the rectangle with vertices at ± 1 and $\pm 1 + \mathrm{i}$. The boundary of the rectangle is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = -1 + \mathrm{i}, \qquad w_2 = -1, \qquad w_3 = 1, \qquad w_4 = 1 + \mathrm{i}, \qquad w_5 = \mathrm{i}$$

(here we have used an extra point w_5 in order to create some symmetry; see Remark (4) in the previous section), so that

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = -1/2 \qquad \text{and} \qquad \theta_5 = 0.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0,z]} (\zeta - x_1)^{-1/2} (\zeta - x_2)^{-1/2} (\zeta - x_3)^{-1/2} (\zeta - x_4)^{-1/2} \, \mathrm{d}\zeta + B.$$

We shall choose

$$x_1 = -\alpha, \qquad x_2 = -1, \qquad x_3 = 1, \qquad x_4 = \alpha,$$

where $\alpha > 1$ will be determined later. Note that we are attempting to benefit from the symmetry here. With such a choice, we obtain, using $z_0 = 0$,

$$\begin{aligned} f(z) &= A \int_{[0,z]} (\zeta+\alpha)^{-1/2}(\zeta+1)^{-1/2}(\zeta-1)^{-1/2}(\zeta-\alpha)^{-1/2}\,\mathrm{d}\zeta + B \\ &= A \int_{[0,z]} (\zeta^2-1)^{-1/2}(\zeta^2-\alpha^2)^{-1/2}\,\mathrm{d}\zeta + B = A \int_{[0,z]} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} + B. \end{aligned}$$

We need

$$f(-\alpha) = -1+\mathrm{i}, \qquad f(-1) = -1, \qquad f(1) = 1, \qquad f(\alpha) = 1+\mathrm{i}.$$

It follows that

(3) $$A \int_0^{-\alpha} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} + B = -1+\mathrm{i},$$

(4) $$A \int_0^{-1} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} + B = -1,$$

(5) $$A \int_0^{1} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} + B = 1,$$

(6) $$A \int_0^{\alpha} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} + B = 1+\mathrm{i}.$$

Subtracting (4) from (3) and subtracting (5) from (6), we obtain respectively

$$A \int_{-1}^{-\alpha} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} = \mathrm{i} \qquad \text{and} \qquad A \int_{1}^{\alpha} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} = \mathrm{i},$$

which are in fact the same equation (note that symmetry is at work here). Multiplying the denominator by i, we obtain

(7) $$A \int_1^{\alpha} \frac{\mathrm{d}\zeta}{\sqrt{(\zeta^2-1)(\alpha^2-\zeta^2)}} = 1.$$

On the other hand, if $B = 0$, then (4) and (5) are the same, and can be represented by

(8) $$A \int_0^{1} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} = 1.$$

It follows that our choice of α should be made so that

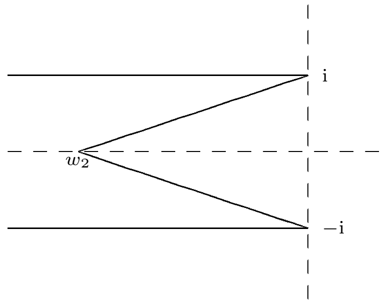
$$\int_0^1 \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(\alpha^2-\zeta^2)}} = \int_1^{\alpha} \frac{\mathrm{d}\zeta}{\sqrt{(\zeta^2-1)(\alpha^2-\zeta^2)}}.$$

We can then take A to be the reciprocal of the common value of these two integrals.

EXAMPLE 14.4.4. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane $\mathcal{H}$ to the domain

$$P = \mathbb{C} \setminus \{z = x \pm \mathrm{i} : x \leq 0\}.$$

The boundary of the set P is described by the solid edges in the picture below when the point w_2 is taken to infinity along the negative real axis.



Let us write, in our notation,

$$w_1 = \mathrm{i}, \qquad w_2 = \infty, \qquad w_3 = -\mathrm{i}, \qquad w_4 = \infty$$

(note again the symmetry; see Remark (4) in the previous section), so that

$$\theta_1 = 1, \qquad \theta_2 = -1, \qquad \theta_3 = 1.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0,z]} (\zeta - x_1)(\zeta - x_2)^{-1}(\zeta - x_3)\,\mathrm{d}\zeta + B'.$$

We shall choose

$$x_1 = -1, \qquad x_2 = 0, \qquad x_3 = 1,$$

and note that we are attempting to benefit from the symmetry here. We obtain

$$\begin{aligned} f(z) &= A \int_{[z_0,z]} (\zeta + 1)\zeta^{-1}(\zeta - 1)\,\mathrm{d}\zeta + B' = A \int_{[z_0,z]} (\zeta^2 - 1)\zeta^{-1}\,\mathrm{d}\zeta + B' \\ &= A \int_{[z_0,z]} \left(\zeta - \frac{1}{\zeta}\right) \mathrm{d}\zeta + B' = A\left(\frac{z^2}{2} - \log z\right) + B. \end{aligned}$$

We need

$$f(-1) = \mathrm{i}, \qquad f(0) = -\infty, \qquad f(1) = -\mathrm{i}.$$

It follows that by choosing a suitable branch of the logarithm, we have

$$A\left(\frac{1}{2} - \mathrm{i}\pi\right) + B = \mathrm{i} \qquad \text{and} \qquad A\left(\frac{1}{2} - 0\right) + B = -\mathrm{i},$$

so that $A = -2/\pi$ and $B = 1/\pi - \mathrm{i}$. Hence

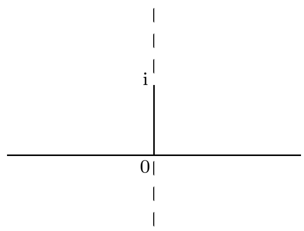
$$f(z) = -\frac{2}{\pi}\left(\frac{z^2}{2} - \log z\right) + \left(\frac{1}{\pi} - \mathrm{i}\right).$$

Note that $|f(z)| \to \infty$ as $z \to 0$.

EXAMPLE 14.4.5. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane $\mathcal{H}$ to the domain

$$P = \mathcal{H} \setminus \{z = y\mathrm{i} : y \le 1\}.$$

The boundary of the set P is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = 0, \qquad w_2 = \mathrm{i}, \qquad w_3 = 0, \qquad w_4 = \infty$$

(note again the symmetry as well as the use of the point 0 twice), so that

$$\theta_1 = -1/2, \qquad \theta_2 = 1, \qquad \theta_3 = -1/2.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0,z]} (\zeta - x_1)^{-1/2}(\zeta - x_2)(\zeta - x_3)^{-1/2}\,\mathrm{d}\zeta + B'.$$

We shall choose

$$x_1 = -1, \qquad x_2 = 0, \qquad x_3 = 1,$$

and note again that we are attempting to benefit from the symmetry here. We obtain

$$f(z) = A \int_{[z_0,z]} (\zeta + 1)^{-1/2}\zeta(\zeta - 1)^{-1/2}\,\mathrm{d}\zeta + B' = A \int_{[z_0,z]} (\zeta^2 - 1)^{-1/2}\zeta\,\mathrm{d}\zeta + B' = A(z^2 - 1)^{1/2} + B.$$

We need

$$f(-1) = 0, \qquad f(0) = \mathrm{i}, \qquad f(1) = 0.$$

It follows that by choosing a suitable branch of the function which is positive for large positive z, we have

$$A\mathrm{i} + B = \mathrm{i} \qquad \text{and} \qquad B = 0,$$

so that $A = 1$ and $B = 0$. Hence

$$f(z) = (z^2 - 1)^{1/2}.$$

PROBLEMS FOR CHAPTER 14

1. Use these notes and without reproducing proofs, find a transformation that maps the unit disc $D = \{z : |z| < 1\}$ onto the domain $D' = \mathcal{H} \setminus \{z = y\mathrm{i} : y \le 1\}$, where $\mathcal{H}$ denotes the upper half plane.

2. For each of the sets A below, find a Schwarz-Christoffel transformation that maps the upper half plane $\mathcal{H}$ onto the set A:
   a) A is an open triangular region with vertices ± 1 and $\mathrm{i}\sqrt{3}$.
   b) A is the region above the polygonal path

$$\{z = x + \mathrm{i} : x \leq 0\} \cup \{z = x + (1-x)\mathrm{i} : 0 \leq x \leq 1\} \cup \{z = x : x \geq 1\}.$$

   c) $A = \{z = x + \mathrm{i}y : y > 0 \text{ or } |x| < 1\}$.