

# INTRODUCTION TO COMPLEX ANALYSIS

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## Chapter 12

### HARMONIC FUNCTIONS AND CONFORMAL MAPPINGS

#### 12.1. A Local Property of Analytic Functions

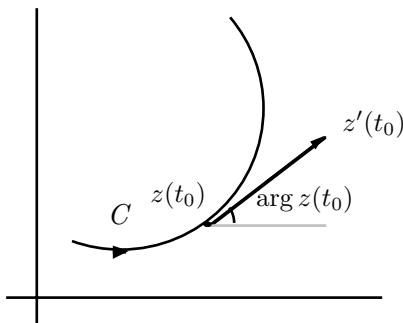
Consider an arc  $C$  given by  $z(t)$ , where  $t \in [A, B]$ . For every  $t \in [A, B]$ , we can write

$$z(t) = x(t) + iy(t),$$

where  $x(t), y(t) \in \mathbb{R}$ . Then the vector

$$z'(t) = x'(t) + iy'(t)$$

has slope  $dy/dx$ , which is also the slope of the arc  $C$ . It follows that if  $z'(t_0) \neq 0$ , then the vector  $z'(t_0)$  is tangent to the arc at the point  $z_0 = z(t_0)$ , and  $\arg z'(t_0)$  is the angle this directed tangent makes with the positive  $x$ -axis.



Suppose now that  $C$  lies in a domain  $D$ , and that a function  $f(z)$  is analytic in  $D$ . Consider the arc  $f(C)$  given by  $w(t) = f(z(t))$ , where  $t \in [A, B]$ . By the Chain rule,

$$w'(t) = f'(z(t))z'(t).$$

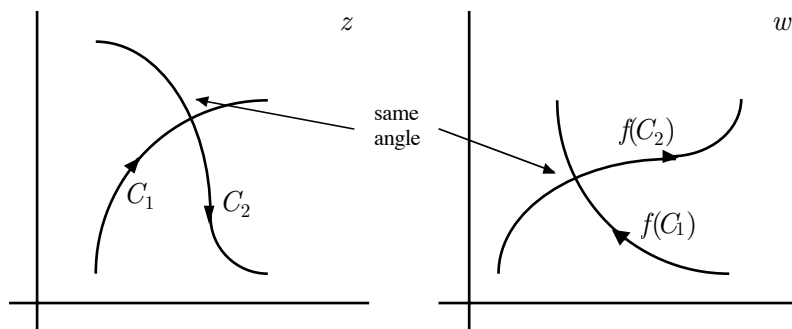
Suppose now that  $z'(t_0) \neq 0$  and  $f'(z_0) \neq 0$ , where  $z_0 = z(t_0)$ . Then  $w'(t_0) \neq 0$ , and

$$\arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0).$$

We can interpret this geometrically in the following way: The angle between the directed tangent to  $C$  at  $z_0 = z(t_0)$  and the directed tangent to  $f(C)$  at  $f(z_0)$  is  $\arg f'(z_0)$ .

In other words, under the mapping  $f$ , the directed tangent to any arc through  $z_0$  is rotated by an angle  $\arg f'(z_0)$ , independent of the choice of the arc through  $z_0$ . This also means that if two arcs  $C_1$  and  $C_2$  intersect at  $z_0$  at an angle, then the two arcs  $f(C_1)$  and  $f(C_2)$  intersect at  $f(z_0)$  at the same angle.

**DEFINITION.** An analytic function  $f$  is said to be conformal at  $z_0$  if the following condition is satisfied: If two arcs  $C_1$  and  $C_2$  meet at  $z_0$ , then the angle from  $f(C_1)$  to  $f(C_2)$  at  $f(z_0)$  is the same as the angle from  $C_1$  to  $C_2$  at  $z_0$ .



We have in fact proved the following result.

**THEOREM 12A.** Suppose that a function  $f$  is analytic in a domain  $D$ , and that  $z_0 \in D$ . Suppose further that  $f'(z_0) \neq 0$ . Then  $f$  is conformal at  $z_0$ .

**REMARK.** Conformality is considered a local property of analytic functions. Note also that

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|.$$

This shows that  $|f'(z_0)|$  is a local scaling factor of the function  $f$  at  $z_0$ , and is independent of the direction of  $z$  from  $z_0$ .

To say that an analytic function is conformal usually means that it is locally one-to-one. In particular, we have the following result.

**THEOREM 12B.** Suppose that a function  $f$  is analytic and one-to-one in a domain  $D$ . Then  $f$  is conformal at every point in  $D$ .

This follows immediately from the result below and Theorem 12A.

**THEOREM 12C.** Suppose that a non-constant function  $f$  is analytic in a domain  $D$ , and that  $z_0 \in D$ . Suppose further that  $f'(z_0) = 0$ . Then  $f$  cannot be one-to-one in any disc containing  $z_0$ .

PROOF. Write  $w_0 = f(z_0)$ . Since  $f(z)$  is not identically constant, the function  $g(z) = f(z) - w_0$  is not identically zero. If  $f'(z_0) = 0$ , then  $g(z)$  has a zero of finite order at least 2 at  $z_0$ . Since the zeros of an analytic function are isolated, we can choose  $r > 0$  so small that both  $g(z)$  and  $f'(z)$  have no zeros in the punctured disc  $\{z : 0 < |z - z_0| \leq r\}$ . Then

$$m = \min_{z \in C} |g(z)| > 0,$$

where  $C = \{z : |z - z_0| = r\}$  denotes the boundary of the disc. Let  $w \in \mathbb{C}$  satisfy  $0 < |w - w_0| < m$ . Then

$$|w_0 - w| < |g(z)|$$

on  $C$ . It follows from Rouché's theorem that the functions  $g(z)$  and  $g(z) + (w_0 - w)$  have the same number of zeros inside  $C$ . Hence

$$g(z) + (w_0 - w) = f(z) - w$$

has at least two zeros inside  $C$ . Clearly none of these zeros can be  $z_0$ . Since  $f'(z) \neq 0$  inside the punctured disc, it follows that these zeros must be simple, and so distinct.  $\circ$

EXAMPLE 12.1.1. The exponential function  $f(z) = e^z$  has non-zero derivative at every  $z \in \mathbb{C}$ , and is therefore conformal at every  $z \in \mathbb{C}$ . Note, however, that  $f : \mathbb{C} \rightarrow \mathbb{C}$  is not one-to-one. On the other hand, the function is one-to-one if we restrict its domain of definition to any strip of the form

$$\{z = x + iy : a \leq y \leq b\},$$

where  $0 < b - a < 2\pi$ . We therefore say that the exponential function is locally one-to-one.

## 12.2. Laplace's Equation

Recall that a continuous function  $\phi(x, y)$  that satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in a domain  $D \subseteq \mathbb{C}$  is said to be harmonic in  $D$ . In Theorem 3E, we have shown that such a function can be written as the real part of an analytic function  $f$  in  $D$ .

Our main task in this section is to show that a harmonic function can be carried from one domain to another by analytic functions. More precisely, we prove the following result.

**THEOREM 12D.** *Suppose that  $D, D' \subseteq \mathbb{C}$  are domains, and that  $f : D \rightarrow D'$  is a one-to-one and onto analytic function. Suppose further that for every  $z \in D$ , we write  $w = f(z)$ , where  $z = x + iy$  and  $w = u + iv$ , with  $x, y, u, v \in \mathbb{R}$ . Then for every function  $\phi(x, y)$  harmonic in  $D$ , the function  $\psi(u, v)$ , defined by*

$$(1) \quad \psi(u, v) = \phi(x(u, v), y(u, v)),$$

*is harmonic in  $D'$ .*

Theorem 12D is particularly useful in applications which involve the solution of the Dirichlet problem concerning the question of finding a harmonic function in a domain  $D$  which takes specified values on the boundary of  $D$ . Once we solve this problem for a particular domain, we can use Theorem 12D to find solutions on all domains which can be obtained from  $D$  by a one-to-one and onto analytic function, so

long as the boundary values correspond. We can therefore select the domain which makes the problem simplest.

Note that (1) can be written in the form

$$\psi(w) = \phi(f^{-1}(w)).$$

Our first task is therefore to establish the analyticity of the inverse function  $f^{-1} : D' \rightarrow D$ .

**THEOREM 12E.** *Suppose that  $D, D' \subseteq \mathbb{C}$  are domains, and that  $f : D \rightarrow D'$  is a one-to-one and onto analytic function. Then the inverse function  $f^{-1} : D' \rightarrow D$  is analytic in  $D'$ . Suppose further that for every  $z \in D$ , we write  $w = f(z)$ , where  $z = x + iy$  and  $w = u + iv$ , with  $x, y, u, v \in \mathbb{R}$ . Then*

$$(2) \quad \frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \quad \text{and} \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

PROOF. To prove the first assertion, it suffices to prove that for every  $w_0 = f(z_0) \in D'$ , the limit

$$\lim_{w \rightarrow w_0} \frac{z - z_0}{w - w_0}$$

exists. To establish this, note first of all that by Theorem 12C, we have

$$\lim_{z \rightarrow z_0} \frac{w - w_0}{z - z_0} \neq 0.$$

Since  $f$  is continuous in  $D$ , we clearly have  $w \rightarrow w_0$  as  $z \rightarrow z_0$ . Since  $f$  is one-to-one in  $D$ , we clearly have  $w \neq w_0$  when  $z \neq z_0$ . It follows that

$$\lim_{w \rightarrow w_0} \frac{z - z_0}{w - w_0} = \left( \lim_{z \rightarrow z_0} \frac{w - w_0}{z - z_0} \right)^{-1}.$$

To complete the proof of the theorem, note that (2) are the Cauchy-Riemann equations of the analytic function  $f^{-1}$ .  $\circ$

PROOF OF THEOREM 12D. Note that

$$\frac{\partial \psi}{\partial u} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial x}{\partial u} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial \phi}{\partial y},$$

so that

$$(3) \quad \frac{\partial^2 \psi}{\partial u^2} = \frac{\partial}{\partial u} \left( \frac{\partial x}{\partial u} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial \phi}{\partial y} \right) = \frac{\partial^2 x}{\partial u^2} \frac{\partial \phi}{\partial x} + \frac{\partial x}{\partial u} \frac{\partial}{\partial u} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial^2 y}{\partial u^2} \frac{\partial \phi}{\partial y} + \frac{\partial y}{\partial u} \frac{\partial}{\partial u} \left( \frac{\partial \phi}{\partial y} \right).$$

On the other hand,

$$(4) \quad \frac{\partial}{\partial u} \left( \frac{\partial \phi}{\partial x} \right) = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial x} \right) = \frac{\partial x}{\partial u} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial y}{\partial u} \frac{\partial^2 \phi}{\partial x \partial y}.$$

Similarly,

$$(5) \quad \frac{\partial}{\partial u} \left( \frac{\partial \phi}{\partial y} \right) = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial y} \right) + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right) = \frac{\partial x}{\partial u} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial y}{\partial u} \frac{\partial^2 \phi}{\partial y^2}.$$

Combining (3)–(5), we obtain

$$(6) \quad \frac{\partial^2 \psi}{\partial u^2} = \frac{\partial^2 x}{\partial u^2} \frac{\partial \phi}{\partial x} + \frac{\partial^2 y}{\partial u^2} \frac{\partial \phi}{\partial y} + \left( \frac{\partial x}{\partial u} \right)^2 \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \frac{\partial^2 \phi}{\partial x \partial y} + \left( \frac{\partial y}{\partial u} \right)^2 \frac{\partial^2 \phi}{\partial y^2}.$$

A similar argument gives

$$(7) \quad \frac{\partial^2 \psi}{\partial v^2} = \frac{\partial^2 x}{\partial v^2} \frac{\partial \phi}{\partial x} + \frac{\partial^2 y}{\partial v^2} \frac{\partial \phi}{\partial y} + \left(\frac{\partial x}{\partial v}\right)^2 \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} \frac{\partial^2 \phi}{\partial x \partial y} + \left(\frac{\partial y}{\partial v}\right)^2 \frac{\partial^2 \phi}{\partial y^2}.$$

Adding (6) and (7), we have

$$(8) \quad \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = \left(\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right) \frac{\partial \phi}{\partial x} + \left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2}\right) \frac{\partial \phi}{\partial y} + 2 \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial x}{\partial v} \frac{\partial y}{\partial v}\right) \frac{\partial^2 \phi}{\partial x \partial y} \\ + \left(\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right) \frac{\partial^2 \phi}{\partial x^2} + \left(\left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\right) \frac{\partial^2 \phi}{\partial y^2}.$$

Suppose now that  $\phi(x, y)$  harmonic in  $D$ . Then

$$(9) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in  $D$ . On the other hand, the Cauchy-Riemann equations (2) give

$$(10) \quad \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = 0 \quad \text{and} \quad \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} = 0,$$

as well as

$$(11) \quad \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} = 0 \quad \text{and} \quad \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 = \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2$$

in  $D'$ . Combining (8)–(11), it is easily seen that

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0$$

in  $D'$ , so that  $\psi(u, v)$  is harmonic in  $D'$ .  $\circ$

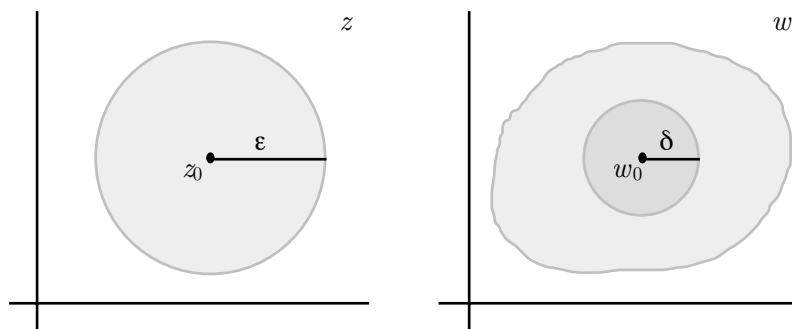
### 12.3. Global Properties of Analytic Functions

We begin by studying the following result which can be considered both local and global. It can be proved by means of Rouché's theorem in the same spirit as the proof of Theorem 12C.

**THEOREM 12F.** (OPEN MAPPING THEOREM) *Suppose that a non-constant function  $f$  is analytic in a domain  $D$ . Then  $f(S)$  is open for any open set  $S \subseteq D$ . More specifically, suppose that  $z_0 \in D$ , and that  $w_0 = f(z_0)$ . Then for all sufficiently small  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$\{w : |w - w_0| < \delta\} \subseteq f(\{z : |z - z_0| < \epsilon\}).$$

*In other words,  $w_0$  is an interior point of  $f(D)$ .*



PROOF. Since  $f(z)$  is not identically constant, the function  $g(z) = f(z) - w_0$  is not identically zero, and has a zero of finite order at  $z_0$ . Since the zeros of an analytic function are isolated, we can choose  $r < \epsilon$  so small that  $g(z)$  has no zeros in the punctured disc  $\{z : 0 < |z - z_0| \leq r\}$ . Then

$$\delta = \min_{z \in C} |g(z)| > 0,$$

where  $C = \{z : |z - z_0| = r\}$  denotes the boundary of the disc. Let  $w \in \mathbb{C}$  satisfy  $|w - w_0| < \delta$ . Then

$$|w_0 - w| < |g(z)|$$

on  $C$ . It follows from Rouché's theorem that the functions  $g(z)$  and  $g(z) + (w_0 - w)$  have the same number of zeros inside  $C$ . Hence

$$g(z) + (w_0 - w) = f(z) - w$$

has a solution inside  $C$ , so that  $w \in f(\{z : |z - z_0| < r\})$ . It follows that

$$\{w : |w - w_0| < \delta\} \subseteq f(\{z : |z - z_0| < r\}).$$

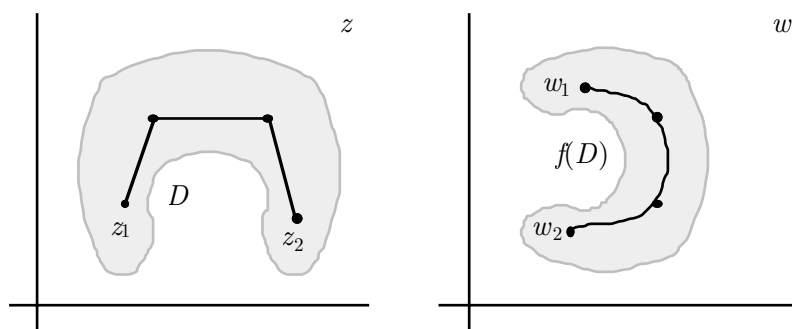
The result follows.  $\circ$

Let us examine Theorem 12D again. We have assumed that both  $D$  and  $D'$  are domains. The following result allows us to somewhat relax our assumptions.

**THEOREM 12G.** *Suppose that a non-constant function  $f$  is analytic in a domain  $D$ . Then  $f(D)$  is a domain.*

REMARK. Recall that an open set  $S$  is connected if every two points in  $S$  can be joined by the union of a finite number of line segments lying in  $S$ . An easy theorem in real analysis states that any contour can be approximated arbitrarily well by the union of a finite number of line segments. It follows that  $S$  is connected if every two points in  $S$  can be joined by a contour lying in  $S$ .

PROOF OF THEOREM 12G. To show that  $f(D)$  is a domain, we need to show that it is open and connected. In view of Theorem 12F, it remains to show that  $f(D)$  is connected. Suppose that  $w_1, w_2 \in f(D)$ . Then there exist  $z_1, z_2 \in D$  such that  $f(z_1) = w_1$  and  $f(z_2) = w_2$ . Since  $D$  is connected,  $z_1$  and  $z_2$  can be joined by the union of a finite number of line segments lying in  $S$ . The image of each line segment under  $f$  is an arc in  $f(D)$ , since  $f$  is differentiable in  $D$ .



It follows that  $w_1$  and  $w_2$  can be joined by a contour lying in  $f(D)$ .  $\circ$

We conclude this section by stating the following result.

**THEOREM 12H.** (RIEMANN MAPPING THEOREM) *Suppose that  $D$  is a simply connected domain in  $\mathbb{C}$  which is different from  $\mathbb{C}$ . Then there exists a one-to-one and onto analytic function of the type  $f : D \rightarrow \mathcal{U}$ , where  $\mathcal{U} = \{w : |w| < 1\}$  denotes the unit open disc.*

REMARKS. (1) If we prescribe a point in  $D$  and a direction through this point, then there is a unique function of the type described which maps this point and direction to the origin and the positive  $x$ -axis respectively.

(2) The proof can be split into three steps. Let  $z_0 \in D$  be fixed. One begins by showing that the collection  $\mathcal{S}$  of one-to-one analytic functions of the type  $f : D \rightarrow \mathcal{U}$  and satisfying the conditions  $f(z_0) = 0$  and  $f'(z_0) > 0$  is non-empty. One then shows that there is an extremal member in  $\mathcal{S}$  with greatest  $f'(z_0)$ . Finally, one shows that if a member in  $\mathcal{S}$  is not an onto function, then it cannot be this extremal member. It follows that the extremal member satisfies the requirements of the theorem.

(3) Unfortunately, Theorem 12H is a purely existence theorem, and so cannot be used in conjunction with Theorem 12D. In Chapters 13–14, we shall study some techniques which may enable us to construct such a function.

#### PROBLEMS FOR CHAPTER 12

1. Let  $C_1$  and  $C_2$  be two straight lines that meet at the origin at an angle  $\phi$ . Consider the function  $f(z) = z^3$ .
  - a) At what angle do the two lines  $f(C_1)$  and  $f(C_2)$  meet?
  - b) Comment on the solution in (a).
2. Discuss angles at the origin under the mapping  $f(z) = z^\alpha$ , where  $0 < \alpha < 1$ .
3. Use the Open mapping theorem to prove the Maximum principle.
4. Explain why the conclusion of the Riemann mapping theorem cannot hold when  $D = \mathbb{C}$ .