INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 11

EVALUATION OF DEFINITE INTEGRALS

11.1. Introduction

The calculus of residues often provides an efficient method for evaluating certain real and complex integrals. This is particularly important when it is not possible to find indefinite integrals explicitly. Even in cases when ordinary methods of calculus can be applied, the use of residues often proves to be a labour saving device.

Naturally, the calculus of residues gives rise to complex integrals, and this suggests that we may be at a disadvantage if we want to evaluate real integrals. In practice, this is seldom the case, since a complex integral is equivalent to two real integrals.

However, there are limitations to this approach. The integrand must be closely associated with some analytic function. We usually want to integrate some elementary functions, and these can be extended to the complex domain. Also, the techniques of complex integration applies to closed curves while a real integral is over an interval. It follows that we need a device to reduce our problem to one which concerns integration over closed curves. There are a number of ways to achieve this, depending on circumstances. The technique is best learned by studying typical examples, and complete mastery does not guarantee success.

11.2. Rational Functions over the Unit Circle

We shall be concerned mainly with integrals of the type

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta,$$

where $f(x, y)$ is a real valued rational function in the real variables x and y.

If we use the substitution

$$z = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \qquad \text{and} \qquad \mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta,$$

then

$$\frac{1}{z} = \cos\theta - \mathrm{i}\sin\theta \qquad \text{and} \qquad \mathrm{d}\theta = -\mathrm{i}\frac{\mathrm{d}z}{z}.$$

We can therefore write

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \sin\theta = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right),$$

so that

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,\mathrm{d}\theta = -\mathrm{i}\int_C f\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)\right)\frac{\mathrm{d}z}{z},$$

where C is the unit circle $\{z : |z| = 1\}$, followed in the positive (anticlockwise) direction.

EXAMPLE 11.2.1. Suppose that the real number $a > 1$ is fixed. Consider the integral

$$\int_0^\pi \frac{\mathrm{d}\theta}{a + \cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}.$$

Using the substitution $z = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, we have

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta} = -\mathrm{i}\int_C \frac{\mathrm{d}z}{z(a + \frac{1}{2}(z + \frac{1}{z}))} = -\mathrm{i}\int_C \frac{2\mathrm{d}z}{z^2 + 2az + 1},$$

where C is the unit circle $\{z : |z| = 1\}$, followed in the positive (anticlockwise) direction. It follows that

$$\int_0^\pi \frac{\mathrm{d}\theta}{a + \cos\theta} = -\mathrm{i}\int_C \frac{\mathrm{d}z}{z^2 + 2az + 1}.$$

If we factorize the denominator $z^2 + 2az + 1$, we obtain roots

$$\alpha = -a + \sqrt{a^2 - 1} \qquad \text{and} \qquad \beta = -a - \sqrt{a^2 - 1}.$$
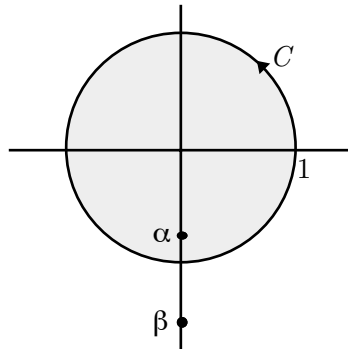
Clearly $|\beta| > 1$. Since $\alpha\beta = 1$, it follows that $|\alpha| < 1$. Hence the function

$$\frac{1}{z^2 + 2az + 1}$$

is analytic in some simply connected domain containing the unit circle C, except for a simple pole at $z = \alpha$ inside C, with residue

$$\mathrm{res}\left(\frac{1}{z^2 + 2az + 1}, \alpha\right) = \lim_{z \to \alpha}(z - \alpha)\frac{1}{z^2 + 2az + 1} = \lim_{z \to \alpha}\frac{1}{z - \beta} = \frac{1}{\alpha - \beta} = \frac{1}{2\sqrt{a^2 - 1}}.$$

It follows from Cauchy's residue theorem that

$$\int_C \frac{\mathrm{d}z}{z^2 + 2az + 1} = 2\pi\mathrm{i}\,\mathrm{res}\left(\frac{1}{z^2 + 2az + 1}, \alpha\right) = \frac{2\pi\mathrm{i}}{2\sqrt{a^2 - 1}},$$

and so

$$\int_0^\pi \frac{\mathrm{d}\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}.$$

EXAMPLE 11.2.2. Now let $w \in \mathbb{C} \setminus [-1, 1]$, and consider the integral

$$F(w) = \int_0^\pi \frac{\mathrm{d}\theta}{w + \cos\theta}.$$

Note that we have excluded the closed interval $[-1, 1]$ to ensure that the denominator of the integrand does not vanish. One can show that $F'(w)$ exists in the domain $\mathbb{C} \setminus [-1, 1]$, so that $F(w)$ is analytic there. We know from Example 11.2.1 that

(1)
$$F(w) = \frac{\pi}{(w^2 - 1)^{1/2}}$$

on the real axis to the right of the point $w = 1$. It follows from Theorem 7H that (1) holds for every $w \in \mathbb{C} \setminus [-1, 1]$. Note, however, that the branch of the square root must be chosen so that it is positive for $w > 1$.

11.3. Rational Functions over the Real Line

We shall be concerned mainly with integrals of the type

(2)
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x,$$

where $f(x)$ is a real valued rational function in the real variable x. Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the real line, so that the integral (2) is convergent.
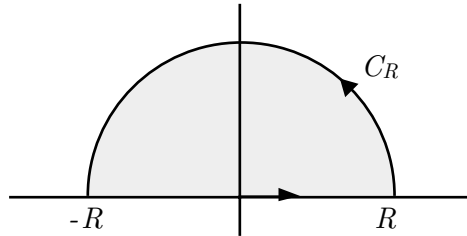
Consider first of all the integral

$$\int_{-R}^R f(x)\,\mathrm{d}x,$$

where $R > 0$. We then extend the definition of the rational function f to the complex domain, and consider also the integral

$$\int_{C_R} f(z)\,\mathrm{d}z,$$

where C_R is the semicircular arc given by $z = R\mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi]$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $[-R, R]$ denotes the line segment from $-R$ to R.

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_i \text{ inside } C} \mathrm{res}(f, z_i),$$

where the summation is taken over all the poles of f inside the Jordan contour C. It is easily shown that

$$\int_{C_R} f(z)\,\mathrm{d}z \to 0$$

as $R \to \infty$, so that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i} \sum_{\mathfrak{Im}\, z_i > 0} \mathrm{res}(f, z_i),$$

where the summation is taken over all the poles of f in the upper half plane.

EXAMPLE 11.3.1. Suppose that the real number $a > 0$ is fixed. Consider the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^3}\,\mathrm{d}x.$$
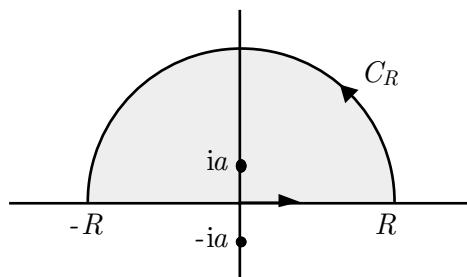
To evaluate this integral, note that the rational function

$$f(z) = \frac{z^2}{(z^2 + a^2)^3}$$

has poles of order 3 at $z = \pm\mathrm{i}a$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > a$.

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{res}(f,\mathrm{i}a).$$

Since

$$\mathrm{res}(f,\mathrm{i}a) = \frac{1}{2}\lim_{z\to\mathrm{i}a}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left((z-\mathrm{i}a)^3\frac{z^2}{(z^2+a^2)^3}\right) = \frac{1}{2}\lim_{z\to\mathrm{i}a}\left(\frac{2}{(z+\mathrm{i}a)^3} - \frac{12z}{(z+\mathrm{i}a)^4} + \frac{12z^2}{(z+\mathrm{i}a)^5}\right)$$

$$= \frac{1}{2}\left(\frac{2}{(2\mathrm{i}a)^3} - \frac{12\mathrm{i}a}{(2\mathrm{i}a)^4} - \frac{12a^2}{(2\mathrm{i}a)^5}\right) = -\frac{\mathrm{i}}{16a^3},$$

it follows that

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = \frac{\pi}{8a^3}.$$

Note now that

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \le \frac{R^2}{(R^2-a^2)^3}\pi R \to 0$$

as $R\to\infty$. Hence

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi}{8a^3}.$$

EXAMPLE 11.3.2.   Consider the integral

$$\int_{-\infty}^{\infty} \frac{x^2+3}{x^4+5x^2+4}\,\mathrm{d}x.$$

To evaluate this integral, note that the rational function

$$f(z) = \frac{z^2+3}{z^4+5z^2+4} = \frac{z^2+3}{(z^2+1)(z^2+4)}$$

has simple poles at $z=\pm\mathrm{i}$ and $z=\pm2\mathrm{i}$. Consider now the Jordan contour

$$C = [-R,R] \cup C_R,$$

where $R > 2$.
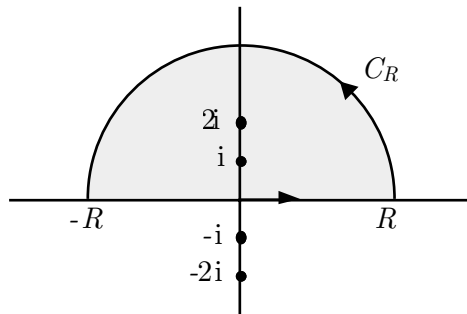
By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}(\mathrm{res}(f,\mathrm{i}) + \mathrm{res}(f,2\mathrm{i})).$$

By Problem 2 in Chapter 10, we have

$$\mathrm{res}(f,\mathrm{i}) = \lim_{z\to\mathrm{i}} \frac{z^2 + 3}{4z^3 + 10z} = \frac{1}{3\mathrm{i}} \qquad \text{and} \qquad \mathrm{res}(f,2\mathrm{i}) = \lim_{z\to 2\mathrm{i}} \frac{z^2 + 3}{4z^3 + 10z} = \frac{1}{12\mathrm{i}}.$$

It follows that

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = \frac{5\pi}{6}.$$

Note now that

$$\left| \int_{C_R} f(z)\,\mathrm{d}z \right| \le \frac{R^2 + 3}{R^4 - 5R^2 - 4}\pi R \to 0$$

as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{x^2 + 3}{x^4 + 5x^2 + 4}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{5\pi}{6}.$$

11.4. Rational and Trigonometric Functions over the Real Line

We shall be concerned mainly with integrals of the type

$$(3) \qquad \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x,$$

where $f(x)$ is a real valued rational function in the real variable x. Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the real line, so that the integral (3) is convergent. Note that the real and imaginary parts of the integral (3) are respectively

$$\int_{-\infty}^{\infty} f(x) \cos x\,\mathrm{d}x \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x) \sin x\,\mathrm{d}x.$$

Consider first of all the integral

$$\int_{-R}^{R} f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x,$$

where $R > 0$. We consider also the integral

$$\int_{C_R} f(z)\mathrm{e}^{\mathrm{i}z}\,\mathrm{d}z,$$

where C_R is the semicircular arc given by $z = R\mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi]$.
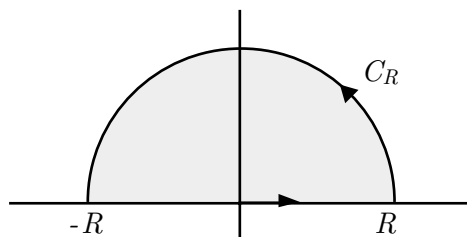
Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $[-R, R]$ denotes the line segment from $-R$ to R. By Cauchy's residue theorem, we have

$$(4) \qquad \int_{-R}^{R} f(x)e^{ix}\,dx + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(f(z)e^{iz}, z_i),$$

where the summation is taken over all the poles of $f(z)e^{iz}$ inside the Jordan contour C.

To study the second integral in (4), we prove the following estimate.

THEOREM 11A. (JORDAN'S LEMMA) *Suppose that $R > 0$. Suppose further that C_R is the semicircular arc given by $z = Re^{it}$, where $t \in [0, \pi]$. Then*

$$(5) \qquad \int_{C_R} |e^{iz}|\,|dz| < \pi.$$

PROOF. Note that

$$(6) \qquad \int_{C_R} |e^{iz}|\,|dz| = \int_0^\pi |e^{iRe^{it}}|\,|iRe^{it}|\,dt = R \int_0^\pi |e^{iR(\cos t + i \sin t)}|\,dt$$

$$= R \int_0^\pi e^{-R\sin t}\,dt = 2R \int_0^{\pi/2} e^{-R\sin t}\,dt.$$

Since

$$\sin t \geq \frac{2}{\pi}t \qquad \text{whenever } 0 \leq t \leq \frac{\pi}{2},$$

it follows that

$$(7) \qquad \int_0^{\pi/2} e^{-R\sin t}\,dt \leq \int_0^{\pi/2} e^{-2Rt/\pi}\,dt = \frac{\pi}{2R}(1 - e^{-R}) < \frac{\pi}{2R}.$$

The inequality (5) follows on combining (6) and (7). ◯

It follows easily from Theorem 11A that

$$\int_{C_R} f(z)e^{iz}\,dz \to 0$$

as $R \to \infty$, so that

$$\int_{-\infty}^{\infty} f(x)e^{ix}\,dx = 2\pi i \sum_{\Im m\, z_i > 0} \text{res}(f(z)e^{iz}, z_i),$$

where the summation is taken over all the poles of $f(z)e^{iz}$ in the upper half plane.

REMARK. In view of Jordan's lemma, we may consider integrals of the form (3) where the degree of the denominator of the rational function f exceeds the degree of the numerator of f by only 1. Note, however, that the argument in this case only establishes the existence of the integral (3) as

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)e^{ix}\,dx.$$

EXAMPLE 11.4.1. Suppose that the real number $a > 0$ is fixed. Consider the integral

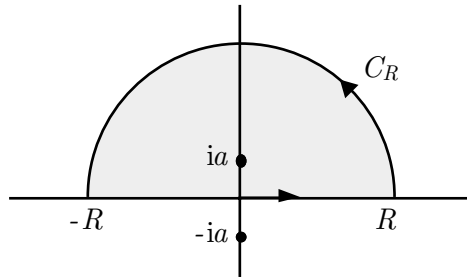$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx.$$

To evaluate this integral, note that the function

$$F(z) = \frac{e^{iz}}{z^2 + a^2}$$

has simple poles at $z = \pm ia$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > a$.

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} F(x) \, dx + \int_{C_R} F(z) \, dz = 2\pi i \operatorname{res}(F, ia).$$

Since

$$\operatorname{res}(F, ia) = \lim_{z \to ia} \left((z - ia) \frac{e^{iz}}{z^2 + a^2} \right) = \lim_{z \to ia} \frac{e^{iz}}{z + ia} = \frac{e^{-a}}{2ia},$$

it follows that

$$\int_{-R}^{R} F(x) \, dx + \int_{C_R} F(z) \, dz = \frac{\pi e^{-a}}{a}.$$

Note now that

$$\left| \int_{C_R} F(z) \, dz \right| \le \frac{1}{R^2 - a^2} \int_{C_R} |e^{iz}| |dz| < \frac{\pi}{R^2 - a^2} \to 0$$

as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} F(x) \, dx = \frac{\pi e^{-a}}{a},$$

so that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx = \mathfrak{Re} \int_{-\infty}^{\infty} F(x) \, dx = \frac{\pi e^{-a}}{a}.$$

EXAMPLE 11.4.2. Suppose that the real numbers $a > 0$ and $b > 0$ are fixed and different. Consider the integral

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)} \, dx.$$
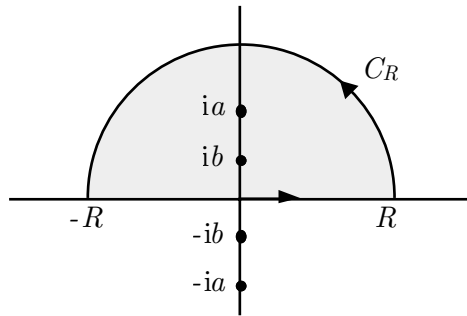
To evaluate this integral, note that the function

$$F(z) = \frac{z^3 \mathrm{e}^{\mathrm{i}z}}{(z^2 + a^2)(z^2 + b^2)}$$

has simple poles at $z = \pm \mathrm{i}a$ and $z = \pm \mathrm{i}b$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > \max\{a, b\}$.

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} F(x)\,\mathrm{d}x + \int_{C_R} F(z)\,\mathrm{d}z = 2\pi\mathrm{i}(\mathrm{res}(F, \mathrm{i}a) + \mathrm{res}(F, \mathrm{i}b)).$$

Since

$$\mathrm{res}(F, \mathrm{i}a) = \lim_{z \to \mathrm{i}a} \left((z - \mathrm{i}a)\frac{z^3 \mathrm{e}^{\mathrm{i}z}}{(z^2 + a^2)(z^2 + b^2)} \right) = \lim_{z \to \mathrm{i}a} \frac{z^3 \mathrm{e}^{\mathrm{i}z}}{(z + \mathrm{i}a)(z^2 + b^2)} = \frac{a^2 \mathrm{e}^{-a}}{2(a^2 - b^2)}$$

and

$$\mathrm{res}(F, \mathrm{i}b) = \lim_{z \to \mathrm{i}b} \left((z - \mathrm{i}b)\frac{z^3 \mathrm{e}^{\mathrm{i}z}}{(z^2 + a^2)(z^2 + b^2)} \right) = \lim_{z \to \mathrm{i}b} \frac{z^3 \mathrm{e}^{\mathrm{i}z}}{(z^2 + a^2)(z + \mathrm{i}b)} = \frac{b^2 \mathrm{e}^{-b}}{2(b^2 - a^2)},$$

it follows that

$$\int_{-R}^{R} F(x)\,\mathrm{d}x + \int_{C_R} F(z)\,\mathrm{d}z = \pi\mathrm{i}\left(\frac{a^2 \mathrm{e}^{-a}}{a^2 - b^2} + \frac{b^2 \mathrm{e}^{-b}}{b^2 - a^2} \right).$$

Note now that

$$\left| \int_{C_R} F(z)\,\mathrm{d}z \right| \leq \frac{R^3}{(R^2 - a^2)(R^2 - b^2)} \int_{C_R} |\mathrm{e}^{\mathrm{i}z}||\mathrm{d}z| < \frac{\pi R^3}{(R^2 - a^2)(R^2 - b^2)} \to 0$$

as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x = \frac{\pi\mathrm{i}(a^2 \mathrm{e}^{-a} - b^2 \mathrm{e}^{-b})}{a^2 - b^2},$$

so that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)}\,\mathrm{d}x = \mathfrak{Im} \int_{-\infty}^{\infty} F(x)\,\mathrm{d}x = \frac{\pi(a^2 \mathrm{e}^{-a} - b^2 \mathrm{e}^{-b})}{a^2 - b^2}.$$

EXAMPLE 11.4.3. Consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx.$$

To evaluate this integral, note that the function
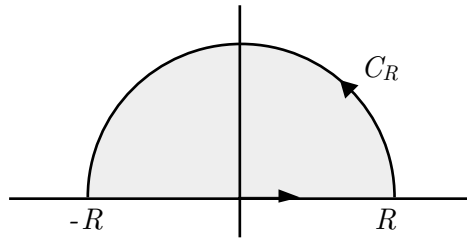
$$F(z) = \frac{e^{iz}}{z}$$

has a simple pole at $z = 0$. We consider instead the function

$$G(z) = \frac{e^{iz} - 1}{z}$$

which has a removable singularity at $z = 0$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > 0$.

By Cauchy's integral theorem, we have

$$\int_{-R}^{R} G(x) \, dx + \int_{C_R} G(z) \, dz = 0,$$

so that

$$\int_{-R}^{R} G(x) \, dx = \int_{C_R} \frac{dz}{z} - \int_{C_R} \frac{e^{iz}}{z} \, dz.$$

Note that

$$\int_{C_R} \frac{dz}{z} = \pi i \qquad \text{and} \qquad \left| \int_{C_R} \frac{e^{iz}}{z} \, dz \right| \leq \frac{1}{R} \int_{C_R} |e^{iz}| |dz| < \frac{\pi}{R}.$$

Hence

$$\left| \int_{-R}^{R} G(x) \, dx - \pi i \right| < \frac{\pi}{R}.$$

Since

$$\int_{-R}^{R} \frac{\sin x}{x} \, dx = \Im \int_{-R}^{R} G(x) \, dx,$$

it follows that

$$\left| \int_{-R}^{R} \frac{\sin x}{x} \, dx - \pi \right| < \frac{\pi}{R},$$

so that letting $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \pi.$$

We shall return to this example later.

Note that the previous two examples do not fit the discussion at the beginning of this section, since the degrees of the denominators of the rational functions in question do not exceed the degrees of the numerators by at least 2. In fact, we have a non-trivial convergence problem for the integral

$$\int_{-\infty}^{\infty} f(x) \sin x \, \mathrm{d}x.$$

The argument formally establishes the existence of this integral as

$$\lim_{R \to \infty} \int_{-R}^{R} f(x) \sin x \, \mathrm{d}x,$$

and not as

$$\lim_{\substack{X_1 \to \infty \\ X_2 \to \infty}} \int_{-X_1}^{X_2} f(x) \sin x \, \mathrm{d}x.$$

However, it turns out that this does not cause any difficulties, since the functions $f(x) \sin x$ in question turn out to be even functions of x, so that for $X_1, X_2 > 0$, we have

$$\int_{-X_1}^{X_2} f(x) \sin x \, \mathrm{d}x = \int_{0}^{X_1} f(x) \sin x \, \mathrm{d}x + \int_{0}^{X_2} f(x) \sin x \, \mathrm{d}x.$$

Let us examine the problem more carefully. Consider the integral

$$\int_{-\infty}^{\infty} f(x) \mathrm{e}^{\mathrm{i}x} \, \mathrm{d}x,$$

where $f(x)$ is a real valued rational function in the real variable x. Suppose now that the degree of the denominator of f exceeds the degree of the numerator of f by exactly 1, and that f has no poles on the real line. To establish the existence of the integral, we need to study the integral

$$\int_{-X_1}^{X_2} f(x) \mathrm{e}^{\mathrm{i}x} \, \mathrm{d}x,$$

where $-X_1 < 0 < X_2$, and consider the limit as $X_1 \to \infty$ and $X_2 \to \infty$. Clearly we cannot use the semicircular arc. We shall use instead a rectangular contour

$$C = [-X_1, X_2] \cup [X_2, X_2 + \mathrm{i}Y] \cup [X_2 + \mathrm{i}Y, -X_1 + \mathrm{i}Y] \cup [-X_1 + \mathrm{i}Y, -X_1],$$

where $Y > 0$. Here $[Z_1, Z_2]$, where $Z_1, Z_2 \in \mathbb{C}$, denotes the line segment from Z_1 to Z_2.
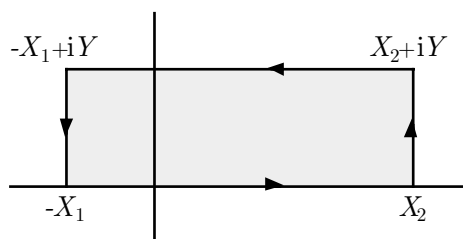
By Cauchy's residue theorem, we have

$$(8) \quad \int_{-X_1}^{X_2} f(x)e^{ix} \, dx + \int_{[X_2, X_2+iY]} f(z)e^{iz} \, dz + \int_{[X_2+iY, -X_1+iY]} f(z)e^{iz} \, dz + \int_{[-X_1+iY, -X_1]} f(z)e^{iz} \, dz$$

$$= 2\pi i \sum_{z_i \text{ inside } C} \text{res}(f(z)e^{iz}, z_i),$$

where the summation is taken over all the poles of $f(z)e^{iz}$ inside the rectangular contour C. When X_1, X_2 and Y are large, then all the poles of the function $f(z)e^{iz}$ in the upper half plane are inside the contour C.

Under our hypotheses, the function $zf(z)$ is bounded. Suppose that $|zf(z)| \le M$ for every $z \in \mathbb{C}$.

Note first of all that

$$\int_{[X_2, X_2+iY]} f(z)e^{iz} \, dz = i \int_0^Y f(X_2 + iy)e^{i(X_2+iy)} \, dy.$$

Since

$$|f(X_2 + iy)| \le \frac{M}{|X_2 + iy|} \le \frac{M}{X_2},$$

we have

$$(9) \quad \left| \int_{[X_2, X_2+iY]} f(z)e^{iz} \, dz \right| \le \frac{M}{X_2} \int_0^Y e^{-y} \, dy = \frac{M}{X_2}(1 - e^{-Y}) < \frac{M}{X_2}.$$

Similarly,

$$(10) \quad \left| \int_{[-X_1+iY, -X_1]} f(z)e^{iz} \, dz \right| < \frac{M}{X_1}.$$

Next, note that

$$\int_{[X_2+iY, -X_1+iY]} f(z)e^{iz} \, dz = - \int_{-X_1}^{X_2} f(x + iY)e^{i(x+iY)} \, dx.$$

Since

$$|f(x + iY)| \le \frac{M}{|x + iY|} \le \frac{M}{Y},$$

we have

$$(11) \quad \left| \int_{[X_2+iY, -X_1+iY]} f(z)e^{iz} \, dz \right| \le \frac{M}{Y} \int_{-X_1}^{X_2} e^{-Y} \, dx = \frac{Me^{-Y}}{Y}(X_1 + X_2).$$

Combining (8)–(11), we conclude that for sufficiently large X_1, X_2 and Y, we have

$$(12) \quad \left| \int_{-X_1}^{X_2} f(x)e^{ix} \, dx - 2\pi i \sum_{\Im z_i > 0} \text{res}(f(z)e^{iz}, z_i) \right| < \frac{M}{X_1} + \frac{M}{X_2} + \frac{Me^{-Y}}{Y}(X_1 + X_2).$$

Note that the left hand side of (12) is independent of Y. For fixed X_1 and X_2, we have

$$\frac{Me^{-Y}}{Y}(X_1 + X_2) \to 0$$

as $Y \to \infty$. It follows that

$$\left| \int_{-X_1}^{X_2} f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x - 2\pi\mathrm{i} \sum_{\mathfrak{Im}z_i > 0} \mathrm{res}(f(z)\mathrm{e}^{\mathrm{i}z}, z_i) \right| \le \frac{M}{X_1} + \frac{M}{X_2}.$$

Letting $X_1 \to \infty$ and $X_2 \to \infty$, we conclude that

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x = 2\pi\mathrm{i} \sum_{\mathfrak{Im}z_i > 0} \mathrm{res}(f(z)\mathrm{e}^{\mathrm{i}z}, z_i).$$

11.5. Bending Round a Singularity

We shall first indicate the ideas by two examples.

EXAMPLE 11.5.1. Recall Example 11.4.3, and consider again the integral
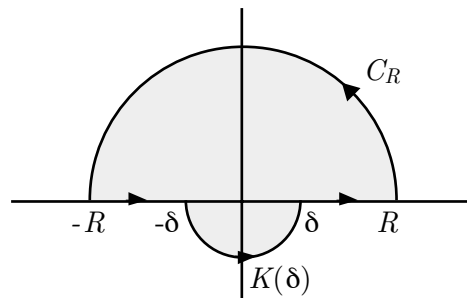
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$

If we use the function

$$F(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z}$$

to evaluate this integral, then the Jordan contour $C = [-R, R] \cup C_R$ discussed earlier is unsuitable, since the singular point $z = 0$ is on the contour. Let us consider instead the Jordan contour

$$C = [-R, -\delta] \cup K(\delta) \cup [\delta, R] \cup C_R,$$

where $R > \delta > 0$, and where $K(\delta)$ denotes the semicircular arc $z = \delta\mathrm{e}^{\mathrm{i}t}$, where $t \in [\pi, 2\pi]$.

By Cauchy's residue theorem, we have

$$\int_{-R}^{-\delta} F(x)\,\mathrm{d}x + \int_{K(\delta)} F(z)\,\mathrm{d}z + \int_{\delta}^{R} F(x)\,\mathrm{d}x + \int_{C_R} F(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{res}(F, 0).$$

Note that the function $F(z)$ in analytic in $\mathbb{C}$ apart from a simple pole at $z = 0$ with residue 1, so that $\mathrm{res}(F, 0) = 1$. It follows that

$$F(z) = \frac{1}{z} + G(z),$$

where $G(z)$ is entire. Furthermore, it is easy to show that

$$\int_{K(\delta)} \frac{\mathrm{d}z}{z} = \pi\mathrm{i}.$$

Hence

$$\int_{-R}^{-\delta} F(x)\,\mathrm{d}x + \int_{\delta}^{R} F(x)\,\mathrm{d}x + \int_{K(\delta)} G(z)\,\mathrm{d}z + \int_{C_R} F(z)\,\mathrm{d}z = \pi\mathrm{i}.$$

Since $G(z)$ is entire, there exists $M > 0$ such that $|G(z)| < M$ whenever $|z| \le 1$, so that for every $\delta < 1$, we have

$$\left| \int_{K(\delta)} G(z)\,\mathrm{d}z \right| \le M\pi\delta.$$

On the other hand, a simple application of Jordan's lemma gives

$$\left| \int_{C_R} F(z)\,\mathrm{d}z \right| < \frac{\pi}{R}.$$

It follows that if $\delta < 1$, then

$$\left| \int_{-R}^{-\delta} F(x)\,\mathrm{d}x + \int_{\delta}^{R} F(x)\,\mathrm{d}x - \pi\mathrm{i} \right| < M\pi\delta + \frac{\pi}{R}.$$

Letting $\delta \to 0$ and $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x = \pi\mathrm{i}.$$

Taking imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

EXAMPLE 11.5.2.   Suppose that the real number $a > 0$ is fixed. Consider the integral
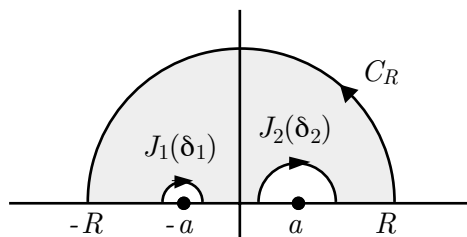
$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\,\mathrm{d}x.$$

To evaluate this integral, note that the function

$$F(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{a^2 - z^2}$$

has simple poles at $z = \pm a$. Consider now the Jordan contour

$$C = [-R, -a - \delta_1] \cup J_1(\delta_1) \cup [-a + \delta_1, a - \delta_2] \cup J_2(\delta_2) \cup [a + \delta_2, R] \cup C_R,$$

where $R > 2a$ and $0 < \delta_1, \delta_2 < a$.

Here $-J_1(\delta_1)$ denotes the semicircular arc $z = -a + \delta_1 e^{it}$, where $t \in [0, \pi]$, and $-J_2(\delta_2)$ denotes the semicircular arc $z = a + \delta_2 e^{it}$, where $t \in [0, \pi]$. By Cauchy's integral theorem, we have

$$\int_{-R}^{-a-\delta_1} F(x)\,\mathrm{d}x - \int_{-J_1(\delta_1)} F(z)\,\mathrm{d}z + \int_{-a+\delta_1}^{a-\delta_2} F(x)\,\mathrm{d}x$$

$$- \int_{-J_2(\delta_2)} F(z)\,\mathrm{d}z + \int_{a+\delta_2}^{R} F(x)\,\mathrm{d}x + \int_{C_R} F(z)\,\mathrm{d}z = 0.$$

Note that the function $F(z)$ in analytic in $\mathbb{C}$ apart from simple pole at $z = \pm a$ with residues

$$\mathrm{res}(F, -a) = \lim_{z \to -a} (z+a)F(z) = \frac{e^{-ia}}{2a} \qquad \text{and} \qquad \mathrm{res}(F, a) = \lim_{z \to a} (z-a)F(z) = -\frac{e^{ia}}{2a}.$$

It follows that

$$F(z) = \frac{e^{-ia}}{2a(z+a)} + G_1(z) = -\frac{e^{ia}}{2a(z-a)} + G_2(z),$$

where $G_1(z)$ is analytic in $\{z : |z+a| \le a\}$ and $G_2(z)$ is analytic in $\{z : |z-a| \le a\}$. Furthermore, it is easy to show that

$$\int_{-J_1(\delta_1)} \frac{e^{-ia}}{2a(z+a)}\,\mathrm{d}z = \frac{\pi i e^{-ia}}{2a} \qquad \text{and} \qquad \int_{-J_2(\delta_2)} \frac{e^{ia}}{2a(z-a)}\,\mathrm{d}z = \frac{\pi i e^{ia}}{2a}.$$

Hence

$$\int_{-R}^{-a-\delta_1} F(x)\,\mathrm{d}x + \int_{-a+\delta_1}^{a-\delta_2} F(x)\,\mathrm{d}x + \int_{a+\delta_2}^{R} F(x)\,\mathrm{d}x$$

$$- \int_{-J_1(\delta_1)} G_1(z)\,\mathrm{d}z - \int_{-J_2(\delta_2)} G_2(z)\,\mathrm{d}z + \int_{C_R} F(z)\,\mathrm{d}z$$

$$= \frac{\pi i e^{-ia}}{2a} - \frac{\pi i e^{ia}}{2a} = \frac{\pi i}{2a}(e^{-ia} - e^{ia}).$$

Since $G_1(z)$ is analytic in $\{z : |z+a| \le a\}$, there exists $M_1 > 0$ such that $|G_1(z)| < M_1$ whenever $|z+a| \le a$, so that for every $\delta_1 < a$, we have

$$\left| \int_{-J_1(\delta_1)} G_1(z)\,\mathrm{d}z \right| \le M_1 \pi \delta_1.$$

Since $G_2(z)$ is analytic in $\{z : |z-a| \le a\}$, there exists $M_2 > 0$ such that $|G_2(z)| < M_2$ whenever $|z-a| \le a$, so that for every $\delta_2 < a$, we have

$$\left| \int_{-J_2(\delta_2)} G_2(z)\,\mathrm{d}z \right| \le M_2 \pi \delta_2.$$

On the other hand, a simple application of Jordan's lemma gives

$$\left| \int_{C_R} F(z)\,\mathrm{d}z \right| < \frac{\pi}{R^2 - a^2}.$$

It follows that if $0 < \delta_1, \delta_2 < a$ and $R > 2a$, then

$$\left| \int_{-R}^{-a-\delta_1} F(x)\,\mathrm{d}x + \int_{-a+\delta_1}^{a-\delta_2} F(x)\,\mathrm{d}x + \int_{a+\delta_2}^{R} F(x)\,\mathrm{d}x - \frac{\pi i}{2a}(e^{-ia} - e^{ia}) \right|$$

$$< M_1 \pi \delta_1 + M_2 \pi \delta_2 + \frac{\pi}{R^2 - a^2}.$$

Letting $\delta_1, \delta_2 \to 0$ and $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} F(x) \, dx = \frac{\pi i}{2a} (e^{-ia} - e^{ia}).$$

Taking real parts gives

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} \, dx = \frac{\pi \sin a}{a}.$$

REMARK.   Note that in Example 11.5.1, we have bent round the singularity in question by using a contour with the singularity in the interior, whereas in Example 11.5.2, we have bent round the singularities in question by using a contour with the singularities in the exterior.

We have used the following general result.

THEOREM 11B.   *Suppose that a function $F(z)$ is analytic in an ϵ-neighbourhood of z_0, apart from a simple pole at z_0 with residue a_{-1}. Suppose further that $0 \le t_1 < t_2 \le 2\pi$. For every positive $\delta < \epsilon$, let $J(\delta)$ denote a circular arc of the form $z = z_0 + \delta e^{it}$, where $t \in [t_1, t_2]$.*
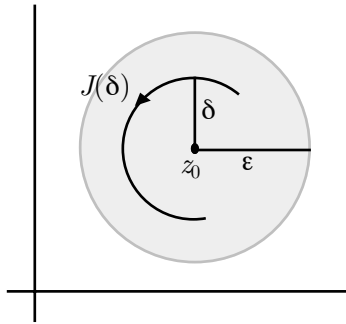
Then

$$\lim_{\delta \to 0} \int_{J(\delta)} F(z) \, dz = i a_{-1} (t_2 - t_1).$$

PROOF.   We can write

$$F(z) = \frac{a_{-1}}{z - z_0} + G(z),$$

where $G(z)$ is analytic in the closed disc $\{z : |z - z_0| \le \delta\}$. Then

$$\int_{J(\delta)} F(z) \, dz = \int_{J(\delta)} \frac{a_{-1}}{z - z_0} \, dz + \int_{J(\delta)} G(z) \, dz.$$

It is easy to check that

$$\int_{J(\delta)} \frac{a_{-1}}{z - z_0} \, dz = i a_{-1} (t_2 - t_1).$$

On the other hand, since $G(z)$ is analytic in the closed disc $\{z : |z - z_0| \le \delta\}$, it is bounded in this disc, and so there exists $M > 0$ such that $|G(z)| \le M$ whenever $|z - z_0| \le \delta$, whence

$$\left| \int_{J(\delta)} G(z) \, dz \right| \le M \int_{J(\delta)} |dz| \le 2\pi M \delta \to 0$$

as $\delta \to 0$. The result follows. ◯

11.6. Integrands with Branch Points

Consider an integral of the type

$$(13) \qquad \int_0^\infty x^\alpha f(x)\,\mathrm{d}x,$$

where $f(x)$ is a real valued rational function in the real variable x. Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the positive real axis and at most a simple pole at the origin. We shall also assume that $0 < \alpha < 1$.

The problem here is that the function $z^\alpha f(z)$ is not single valued. However, this is precisely the circumstance that makes it possible to find the integral. The simplest technique is to first of all make the substitution $x = u^2$, and note that

$$(14) \qquad \int_0^\infty x^\alpha f(x)\,\mathrm{d}x = 2\int_0^\infty u^{2\alpha+1} f(u^2)\,\mathrm{d}u.$$

We now consider the function

$$F(z) = z^{2\alpha+1} f(z^2).$$

For the function $z^{2\alpha}$, by choosing the branch so that the argument of $z^{2\alpha}$ lies between $-\pi\alpha$ and $3\pi\alpha$, it is easy to see that this is well defined and analytic in the region obtained from $\mathbb{C}$ by deleting the origin and the negative imaginary axis. It follows that as long as a Jordan contour avoids this cut, then we can use Cauchy's residue theorem on $F(z)$.

We shall consider the Jordan contour

$$C = [-R, -\delta] \cup J(\delta) \cup [\delta, R] \cup C_R,$$

where $R > \delta > 0$, $-J(\delta)$ denotes the semicircular arc of the form $z = \delta \mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi]$, and C_R denotes the semicircular arc of the form $z = R\mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi]$.
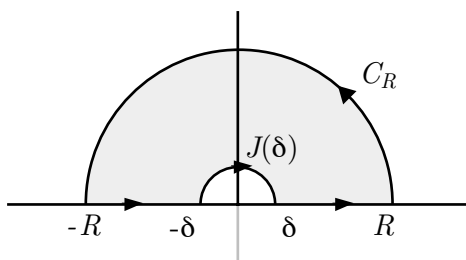
By Cauchy's residue theorem, we have

$$\int_{-R}^{-\delta} F(z)\,\mathrm{d}z + \int_{J(\delta)} F(z)\,\mathrm{d}z + \int_\delta^R F(z)\,\mathrm{d}z + \int_{C_R} F(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_i \text{ inside } C} \mathrm{res}(F, z_i),$$

where the summation is taken over all the poles of F inside the Jordan contour C. It is easily shown that

$$\int_{J(\delta)} F(z)\,\mathrm{d}z \to 0 \qquad \text{and} \qquad \int_{C_R} F(z)\,\mathrm{d}z \to 0$$

as $\delta \to 0$ and $R \to \infty$ respectively, so that

$$\int_{-\infty}^\infty F(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{\mathfrak{Im} z_i > 0} \mathrm{res}(F, z_i),$$

where the summation is taken over all the poles of F in the upper half plane. On the other hand, note that $(-z)^{2\alpha} = e^{2\pi i \alpha} z^{2\alpha}$, and so

$$\int_{-\infty}^{\infty} z^{2\alpha+1} f(z^2)\,dz = \int_{-\infty}^{\infty} (z^{2\alpha+1} + (-z)^{2\alpha+1}) f(z^2)\,dz = (1 - e^{2\pi i \alpha}) \int_{0}^{\infty} u^{2\alpha+1} f(u^2)\,du.$$

Since $e^{2\pi i \alpha} \neq 1$, this gives us a way of calculating the integral on the right hand side of (14).

EXAMPLE 11.6.1. Suppose that the real number $\alpha \in (0,1)$ is fixed. Consider the integral

$$\int_{0}^{\infty} \frac{x^{\alpha-1}}{1+x}\,dx.$$

The substitution $x = u^2$ gives

$$(15) \qquad \int_{0}^{\infty} \frac{x^{\alpha-1}}{1+x}\,dx = 2 \int_{0}^{\infty} \frac{u^{2\alpha+1}}{u^2 + u^4}\,du.$$
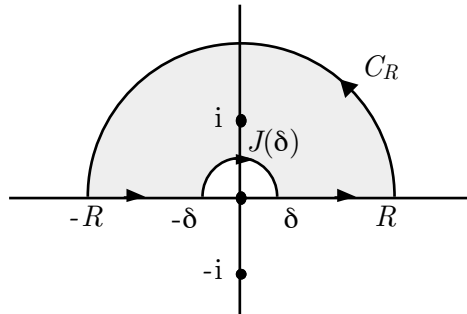
To evalaute this integral, note that the function

$$F(z) = \frac{z^{2\alpha+1}}{z^2 + z^4}$$

has a singularity at $z = 0$ and simple poles at $z = \pm i$. Consider now the Jordan contour

$$C = [-R, -\delta] \cup J(\delta) \cup [\delta, R] \cup C_R,$$

where $R > 1 > \delta > 0$.

By Cauchy's residue theorem, we have

$$\int_{-R}^{-\delta} F(z)\,dz + \int_{J(\delta)} F(z)\,dz + \int_{\delta}^{R} F(z)\,dz + \int_{C_R} F(z)\,dz = 2\pi i \operatorname{res}(F, i).$$

Since

$$\operatorname{res}(F, i) = \lim_{z \to i}(z - i)F(z) = -\frac{1}{2} e^{\pi i \alpha},$$

it follows that

$$\int_{-R}^{-\delta} F(z)\,dz + \int_{J(\delta)} F(z)\,dz + \int_{\delta}^{R} F(z)\,dz + \int_{C_R} F(z)\,dz = -\pi i e^{\pi i \alpha}.$$

Note now that

$$\left| \int_{J(\delta)} F(z)\,dz \right| \leq \frac{\delta^{2\alpha+1}}{\delta^2 - \delta^4} \pi \delta \to 0 \qquad \text{and} \qquad \left| \int_{C_R} F(z)\,dz \right| \leq \frac{R^{2\alpha+1}}{R^4 - R^2} \pi R \to 0$$

as $\delta \to 0$ and $R \to \infty$ respectively. Hence

$$(1 - e^{2\pi i \alpha}) \int_0^\infty \frac{u^{2\alpha+1}}{u^2 + u^4} \, du = \int_{-\infty}^\infty \frac{z^{2\alpha+1}}{z^2 + z^4} \, dz = -\pi i e^{\pi i \alpha},$$

so that

$$\int_0^\infty \frac{u^{2\alpha+1}}{u^2 + u^4} \, du = -\frac{\pi i e^{\pi i \alpha}}{1 - e^{2\pi i \alpha}} = -\frac{\pi i}{e^{-\pi i \alpha} - e^{\pi i \alpha}} = \frac{\pi}{2 \sin \pi \alpha}.$$

It now follows from (15) that

$$\int_0^\infty \frac{x^{\alpha-1}}{1 + x} \, dx = \frac{\pi}{\sin \pi \alpha}.$$

In fact, the integral (13) can be studied without the substitution $x = u^2$. However, the contour that we use will not be a Jordan contour. We shall consider the function
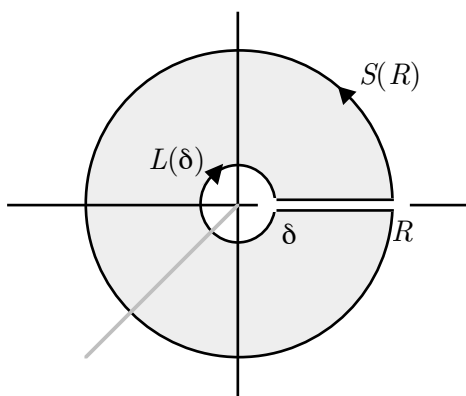
$$F(z) = z^\alpha f(z),$$

and choose a branch of z^α so that the argument lies between 0 and $2\pi\alpha$.

We now consider the contour

$$C = [\delta, R] \cup S(R) \cup [R, \delta] \cup L(\delta),$$

where $R > \delta > 0$, $S(R)$ denotes the circle of the form $z = Re^{it}$, where $t \in [0, 2\pi]$, and $-L(\delta)$ denotes the circle of the form $z = \delta e^{it}$, where $t \in [0, 2\pi]$.

Clearly C is not a Jordan contour. However, there clearly exists $t_0 \in (0, 2\pi)$ such that the line segment joining δe^{it_0} and Re^{it_0} does not pass through any singularities of $f(z)$ in $\{z : |z| \le R\}$. Now let

$$C_1 = [\delta, R] \cup S_1(R) \cup [Re^{it_0}, \delta e^{it_0}] \cup L_1(\delta),$$

where $S_1(R)$ denotes the circular arc of the form $z = Re^{it}$, where $t \in [0, t_0]$, and $-L_1(\delta)$ denotes the circular arc of the form $z = \delta e^{it}$, where $t \in [0, t_0]$. Also let

$$C_2 = [R, \delta] \cup L_2(\delta) \cup [\delta e^{it_0}, Re^{it_0}] \cup S_2(R),$$

where $S_2(R)$ denotes the circular arc of the form $z = Re^{it}$, where $t \in [t_0, 2\pi]$, and $-L_2(\delta)$ denotes the circular arc of the form $z = \delta e^{it}$, where $t \in [t_0, 2\pi]$.
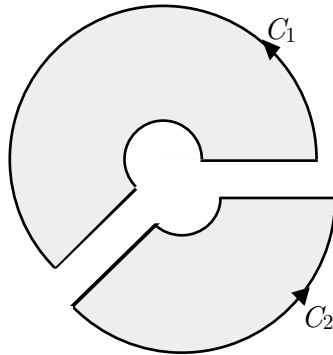
It is easy to see that

$$(16) \qquad \int_C F(z)\,\mathrm{d}z = \int_{C_1} F(z)\,\mathrm{d}z + \int_{C_2} F(z)\,\mathrm{d}z.$$

Clearly there exists $\epsilon_0 > 0$ such that C_1 is a Jordan contour in the simply connected domain

$$D_1 = \{z \neq 0 : -\epsilon_0 < \arg z < t_0 + \epsilon_0\}$$

and C_2 is a Jordan contour in the simply connected domain

$$D_2 = \{z \neq 0 : t_0 - \epsilon_0 < \arg z < 2\pi + \epsilon_0\}.$$

Applying Cauchy's residue theorem, we obtain

$$\int_{C_1} F(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_i \text{ inside } C_1} \mathrm{res}(F, z_i) \qquad \text{and} \qquad \int_{C_2} F(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_i \text{ inside } C_2} \mathrm{res}(F, z_i),$$

so that

$$(17) \qquad \int_C F(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_i \text{ inside } C} \mathrm{res}(F, z_i).$$

Note that

$$(18) \qquad \int_{C_1} F(z)\,\mathrm{d}z = \int_{[\delta,R]} F(z)\,\mathrm{d}z + \int_{S_1(R)} F(z)\,\mathrm{d}z + \int_{[R\mathrm{e}^{\mathrm{i}t_0},\delta\mathrm{e}^{\mathrm{i}t_0}]} F(z)\,\mathrm{d}z + \int_{L_1(\delta)} F(z)\,\mathrm{d}z,$$

and

$$(19) \qquad \int_{C_2} F(z)\,\mathrm{d}z = \int_{[R,\delta]} F(z)\,\mathrm{d}z + \int_{L_2(\delta)} F(z)\,\mathrm{d}z + \int_{[\delta\mathrm{e}^{\mathrm{i}t_0},R\mathrm{e}^{\mathrm{i}t_0}]} F(z)\,\mathrm{d}z + \int_{S_2(R)} F(z)\,\mathrm{d}z.$$

When we evaluate the integrals in (18), we need $0 \leq \arg z \leq t_0$. Hence

$$(20) \qquad \int_{[\delta,R]} F(z)\,\mathrm{d}z = \int_{\delta}^{R} F(x)\,\mathrm{d}x.$$

When we evaluate the integrals in (19), we need $t_0 \leq \arg z \leq 2\pi$. Hence

$$(21) \qquad \int_{[R,\delta]} F(z)\,\mathrm{d}z = \int_{[R,\delta]} z^\alpha f(z)\,\mathrm{d}z = -\int_{\delta}^{R} x^\alpha \mathrm{e}^{2\pi\mathrm{i}\alpha} f(x)\,\mathrm{d}x = -\mathrm{e}^{2\pi\mathrm{i}\alpha} \int_{\delta}^{R} F(x)\,\mathrm{d}x.$$

Also

(22)
$$\int_{[Re^{it_0}, \delta e^{it_0}]} F(z)\, dz + \int_{[\delta e^{it_0}, Re^{it_0}]} F(z)\, dz = 0.$$

It follows from (16), (18)–(22) that

(23)
$$\int_C F(z)\, dz = \int_{[\delta,R]} F(z)\, dz + \int_{[R,\delta]} F(z)\, dz + \int_{S_1(R)} F(z)\, dz + \int_{S_2(R)} F(z)\, dz$$

$$+ \int_{L_1(\delta)} F(z)\, dz + \int_{L_2(\delta)} F(z)\, dz$$

$$= (1 - e^{2\pi i\alpha}) \int_\delta^R F(x)\, dx + \int_{S(R)} F(z)\, dz + \int_{L(\delta)} F(z)\, dz.$$

It is easily shown that

$$\int_{L(\delta)} F(z)\, dz \to 0 \qquad \text{and} \qquad \int_{S(R)} F(z)\, dz \to 0$$

as $\delta \to 0$ and $R \to \infty$ respectively, so it follows from (17) and (23) that

$$(1 - e^{2\pi i\alpha}) \int_0^\infty F(x)\, dx = 2\pi i \sum_{z_i} \text{res}(f, z_i),$$

where the summation is taken over all the poles of F in $\mathbb{C} \setminus \{0\}$.

EXAMPLE 11.6.2. Suppose that the real number $\alpha \in (0,1)$ is fixed. Consider again the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx.$$

To evaluate this integral, note that the function
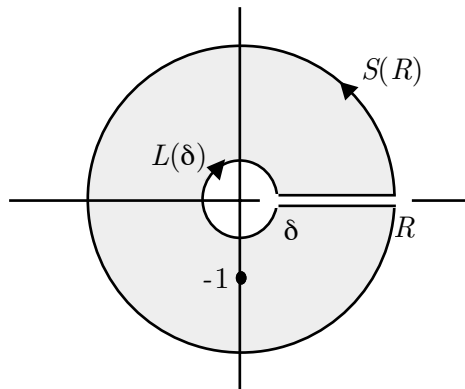
$$F(z) = \frac{z^{\alpha-1}}{1+z} = \frac{z^\alpha}{z(1+z)},$$

has a simple pole at $z = -1$, with residue

$$\text{res}(F, -1) = \lim_{z \to -1} (z+1) F(z) = \lim_{z \to -1} z^{\alpha-1} = e^{\pi i(\alpha-1)} = -e^{\pi i\alpha}.$$

Consider now the contour

$$C = [\delta, R] \cup S(R) \cup [R, \delta] \cup L(\delta),$$

where $R > 1 > \delta > 0$.

By Cauchy's residue theorem and our earlier observation, we have

$$(24) \qquad \int_C F(z)\,\mathrm{d}z = (1 - \mathrm{e}^{2\pi\mathrm{i}\alpha}) \int_\delta^R F(x)\,\mathrm{d}x + \int_{S(R)} F(z)\,\mathrm{d}z + \int_{L(\delta)} F(z)\,\mathrm{d}z = -2\pi\mathrm{i}\mathrm{e}^{\pi\mathrm{i}\alpha}.$$

Note now that

$$\left| \int_{L(\delta)} F(z)\,\mathrm{d}z \right| \le \frac{\delta^{\alpha-1}}{1-\delta} 2\pi\delta \to 0 \qquad \text{and} \qquad \left| \int_{S(R)} F(z)\,\mathrm{d}z \right| \le \frac{R^{\alpha-1}}{R-1} 2\pi R \to 0$$

as $\delta \to 0$ and $R \to \infty$ respectively, so it follows from (24) that

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = -\frac{2\pi\mathrm{i}\mathrm{e}^{\pi\mathrm{i}\alpha}}{1-\mathrm{e}^{2\pi\mathrm{i}\alpha}} = \frac{\pi}{\sin\pi\alpha}.$$

We next turn our attention to integrals of the types

$$(25) \qquad \int_0^\infty f(x)\log x\,\mathrm{d}x \qquad \text{and} \qquad \int_0^\infty f(x)\log^2 x\,\mathrm{d}x,$$

where $f(x)$ is a real valued rational function in the real variable x. Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the non-negative real axis, so that the integrals (25) are convergent. We shall also assume that f is an even function; in other words, $f(-x) = f(x)$ for every $x \in \mathbb{R} \setminus \{0\}$.

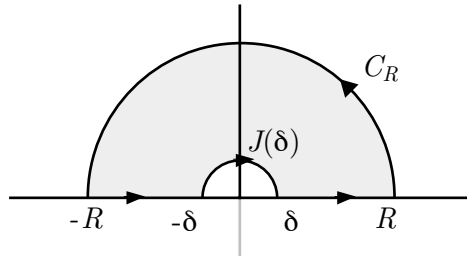We shall consider the function

$$F(z) = f(z)\log^2 z$$

and the Jordan contour

$$C = [\delta, R] \cup C_R \cup [-R, -\delta] \cup J(\delta),$$

where $R > \delta > 0$. Here C_R denotes the semicircular arc $z = R\mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi]$, and $-J(\delta)$ denotes the semicircular arc $z = \delta\mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi]$.

By Cauchy's residue theorem, we have

$$\int_{[\delta,R]} F(z)\,\mathrm{d}z + \int_{C_R} F(z)\,\mathrm{d}z + \int_{[-R,-\delta]} F(z)\,\mathrm{d}z + \int_{J(\delta)} F(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_i \text{ inside } C} \mathrm{res}(F, z_i).$$

If we impose the restriction $-\pi/2 \le \arg z < 3\pi/2$, then

$$\int_{[\delta,R]} F(z)\,\mathrm{d}z = \int_\delta^R f(x)\log^2 x\,\mathrm{d}x,$$

and

$$\int_{[-R,-\delta]} F(z)\,\mathrm{d}z = \int_{\delta}^{R} (\log x + \mathrm{i}\pi)^2 f(-x)\,\mathrm{d}x = \int_{\delta}^{R} (\log x + \mathrm{i}\pi)^2 f(x)\,\mathrm{d}x$$

$$= \int_{\delta}^{R} f(x)\log^2 x\,\mathrm{d}x + 2\pi\mathrm{i}\int_{\delta}^{R} f(x)\log x\,\mathrm{d}x - \pi^2\int_{\delta}^{R} f(x)\,\mathrm{d}x.$$

It is easily shown that

$$\int_{J(\delta)} F(z)\,\mathrm{d}z \to 0 \qquad \text{and} \qquad \int_{C_R} F(z)\,\mathrm{d}z \to 0$$

as $\delta \to 0$ and $R \to \infty$ respectively. It follows that

$$(26) \qquad 2\int_{0}^{\infty} f(x)\log^2 x\,\mathrm{d}x + 2\pi\mathrm{i}\int_{0}^{\infty} f(x)\log x\,\mathrm{d}x - \pi^2\int_{0}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{\mathfrak{Im}z_i>0} \mathrm{res}(F, z_i),$$

where the summation is taken over all the poles of F in the upper half plane. The integrals (25) can then be found on equating real and imaginary parts.

EXAMPLE 11.6.3.   Consider the integrals

$$\int_{0}^{\infty} \frac{\log x}{1 + x^2}\,\mathrm{d}x \qquad \text{and} \qquad \int_{0}^{\infty} \frac{\log^2 x}{1 + x^2}\,\mathrm{d}x.$$

To evaluate these integrals, note that the function
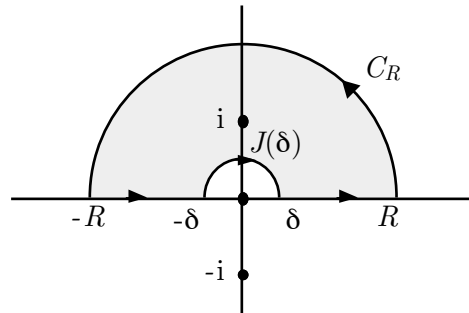
$$F(z) = \frac{\log^2 z}{1 + z^2}$$

has simple poles at $z = \pm\mathrm{i}$. In particular,

$$\mathrm{res}(F, \mathrm{i}) = \lim_{z\to\mathrm{i}}(z - \mathrm{i})F(z) = \lim_{z\to\mathrm{i}} \frac{\log^2 z}{z + \mathrm{i}} = -\frac{\pi^2}{8\mathrm{i}}.$$

Consider now the Jordan contour

$$C = [\delta, R] \cup C_R \cup [-R, -\delta] \cup J(\delta),$$

where $R > 1 > \delta > 0$.

By Cauchy's residue theorem, we have

$$\int_{[\delta,R]} F(z)\,\mathrm{d}z + \int_{C_R} F(z)\,\mathrm{d}z + \int_{[-R,-\delta]} F(z)\,\mathrm{d}z + \int_{J(\delta)} F(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{res}(F, \mathrm{i}) = -\frac{\pi^3}{4}.$$

On $J(\delta)$, we have $z = \delta e^{it}$, so that

$$\left|\frac{\log^2 z}{1+z^2}\right| \le \frac{|\log \delta + it|^2}{1-\delta^2} \le \frac{\log^2 \delta + \pi^2}{1-\delta^2}.$$

Hence

$$\left|\int_{J(\delta)} F(z)\,dz\right| \le \frac{\log^2 \delta + \pi^2}{1-\delta^2}\pi\delta \to 0$$

as $\delta \to 0$. On C_R, we have $z = Re^{it}$, so that

$$\left|\frac{\log^2 z}{1+z^2}\right| \le \frac{|\log R + it|^2}{R^2-1} \le \frac{\log^2 R + \pi^2}{R^2-1}.$$

Hence

$$\left|\int_{C_R} F(z)\,dz\right| \le \frac{\log^2 R + \pi^2}{R^2-1}\pi R \to 0$$

as $R \to \infty$. It follows from (26) that

$$2\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx + 2\pi i \int_0^\infty \frac{\log x}{1+x^2}\,dx - \pi^2 \int_0^\infty \frac{dx}{1+x^2} = -\frac{\pi^3}{4}.$$

It is well known that

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Hence

$$2\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx + 2\pi i \int_0^\infty \frac{\log x}{1+x^2}\,dx = \frac{\pi^3}{4}.$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx = \frac{\pi^3}{8} \qquad \text{and} \qquad \int_0^\infty \frac{\log x}{1+x^2}\,dx = 0.$$

We conclude this chapter by considering an example which does not easily fall into any general discussion. Some of the calculation is unpleasant, and we shall omit some details.

EXAMPLE 11.6.4. We wish to evaluate the integral

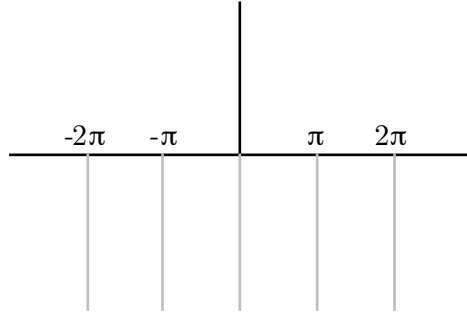$$\int_0^\pi \log \sin x\,dx.$$

To do this, consider the function

(27) $$1 - e^{2iz} = e^{iz}(e^{-iz} - e^{iz}) = -2ie^{iz}\sin z.$$

Note that if $z = x + iy$, where $x, y \in \mathbb{R}$, then

$$1 - e^{2iz} = 1 - e^{-2y}(\cos 2x + i\sin 2x),$$

so that the function is real and non-positive if and only if $x = n\pi$ and $y \leq 0$, where $n \in \mathbb{Z}$. Consider the simply connected domain D obtained from $\mathbb{C}$ by deleting all half lines of the form $\{z = n\pi + \mathrm{i}y : y \leq 0\}$, where $n \in \mathbb{Z}$.

In this domain, the principal branch of $\log(1 - \mathrm{e}^{2\mathrm{i}z})$, with imaginary part between $-\pi$ and π, is single valued and analytic. Consider the Jordan contour

$$C = [\delta, \pi - \delta] \cup T_1(\delta) \cup [\pi + \mathrm{i}\delta, \pi + \mathrm{i}Y] \cup [\pi + \mathrm{i}Y, \mathrm{i}Y] \cup [\mathrm{i}Y, \mathrm{i}\delta] \cup T_2(\delta),$$

where $\delta > 0$ is small and $Y > 0$ is large. Here $-T_1(\delta)$ denotes the circular arc $z = \pi + \delta\mathrm{e}^{\mathrm{i}t}$, where $t \in [\pi/2, \pi]$, and $-T_2(\delta)$ denotes the circular arc $z = \delta\mathrm{e}^{\mathrm{i}t}$, where $t \in [0, \pi/2]$.
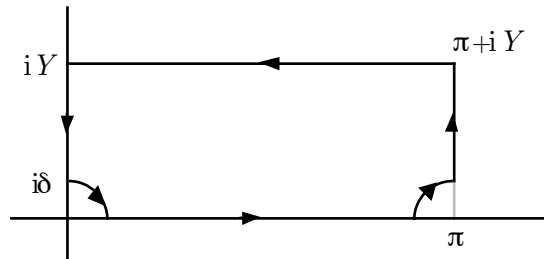
By Cauchy's integral theorem, we have

$$\int_{[\delta, \pi-\delta]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z + \int_{T_1(\delta)} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z + \int_{[\pi+\mathrm{i}\delta, \pi+\mathrm{i}Y]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z$$

$$+ \int_{[\pi+\mathrm{i}Y, \mathrm{i}Y]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z + \int_{[\mathrm{i}Y, \mathrm{i}\delta]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z + \int_{T_2(\delta)} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z = 0.$$

Using the periodicity of the integrand, we have

$$\int_{[\pi+\mathrm{i}\delta, \pi+\mathrm{i}Y]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z + \int_{[\mathrm{i}Y, \mathrm{i}\delta]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z = 0.$$

Furthermore, it can be shown that

$$\int_{T_1(\delta)} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z \to 0 \qquad \text{and} \qquad \int_{T_2(\delta)} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z \to 0$$

as $\delta \to 0$, and that

$$\int_{[\pi+\mathrm{i}Y, \mathrm{i}Y]} \log(1 - \mathrm{e}^{2\mathrm{i}z})\,\mathrm{d}z \to 0$$

as $Y \to \infty$. It follows that

(28)
$$\int_0^\pi \log(1 - \mathrm{e}^{2\mathrm{i}x})\,\mathrm{d}x = 0.$$

Next, consider the function $\log(-2\mathrm{i}\mathrm{e}^{\mathrm{i}x}\sin x)$. If we choose $\log \mathrm{e}^{\mathrm{i}x} = \mathrm{i}x$, then the imaginary part lies between 0 and π. To obtain the principal branch of the logarithm with imaginary part between $-\pi$ and π, we must choose $\log(-\mathrm{i}) = -\mathrm{i}\pi/2$. Hence

$$\log(-2\mathrm{i}\mathrm{e}^{\mathrm{i}x}\sin x) = \log 2 - \frac{\mathrm{i}\pi}{2} + \mathrm{i}x + \log\sin x,$$

so that

$$(29) \qquad \int_0^\pi \log(-2\mathrm{i}\mathrm{e}^{\mathrm{i}x}\sin x)\,\mathrm{d}x = \pi\log 2 - \frac{\mathrm{i}\pi^2}{2} + \frac{\mathrm{i}\pi^2}{2} + \int_0^\pi \log\sin x\,\mathrm{d}x.$$

Combining (27)–(29), we conclude that

$$\int_0^\pi \log\sin x\,\mathrm{d}x = -\pi\log 2.$$

PROBLEMS FOR CHAPTER 11

1. Show each of the following:

   a) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{2 + \sin\theta} = \frac{2\pi}{\sqrt{3}}$

   b) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi}{1 - a^2}$, where $a \in \mathbb{C}$ and $|a| < 1$

   c) $\displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}x}{1 + x^2} = \pi$

   d) $\displaystyle\int_0^\infty \frac{x^2}{(1 + x^2)^2}\,\mathrm{d}x = \frac{\pi}{4}$

   e) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{1 + x^6} = \frac{\pi}{3}$

   f) $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{(1 + x^2)^2}\,\mathrm{d}x = \frac{\pi}{\mathrm{e}}$

   g) $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x^2 + a^2}\,\mathrm{d}x = \frac{\pi}{a}\mathrm{e}^{-a}$, where $a \in \mathbb{R}$ and $a > 0$

   h) $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)}\,\mathrm{d}x = \frac{\pi}{a^2 - b^2}\left(\frac{\mathrm{e}^{-b}}{b} - \frac{\mathrm{e}^{-a}}{a}\right)$, where $a, b \in \mathbb{R}$ and $a > b > 0$

2. Suppose that $a \in \mathbb{R}$ and $0 < a < 1$. By integrating the function

   $$\frac{\mathrm{e}^{az}}{\mathrm{e}^z + 1}$$

   around a rectangle with vertices at $\pm R$ and $\pm R + 2\pi\mathrm{i}$, show that

   $$\int_{-\infty}^\infty \frac{\mathrm{e}^{ax}}{\mathrm{e}^x + 1}\,\mathrm{d}x = \frac{\pi}{\sin\pi a}.$$

3. Show each of the following:

   a) $\displaystyle\int_{-\infty}^\infty \frac{1 - \cos x}{x^2}\,\mathrm{d}x = \pi$

   b) $\displaystyle\int_0^\infty \frac{\sin\pi x}{x(1 - x^2)}\,\mathrm{d}x = \pi$

   c) $\displaystyle\int_0^\infty \frac{\log x}{x^4 + 1}\,\mathrm{d}x = -\frac{\pi^2\sqrt{2}}{16}$

   d) $\displaystyle\int_0^\infty \frac{\log x}{x^2 - 1}\,\mathrm{d}x = \frac{\pi^2}{4}$