INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 4

COMPLEX INTEGRALS

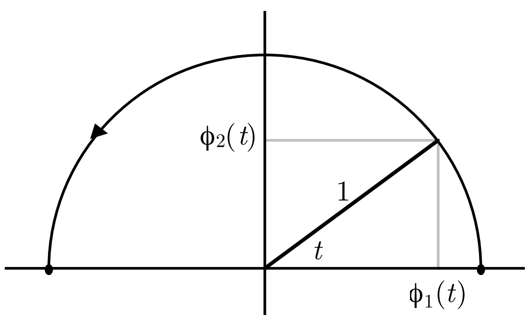
4.1. Curves in the Complex Plane

Integration of functions of a complex variable is carried out over curves in $\mathbb{C}$ and leads to many important results useful in pure and applied mathematics. In this section, we give a brief introduction to curves in $\mathbb{C}$. We are interested in complex valued functions of the form

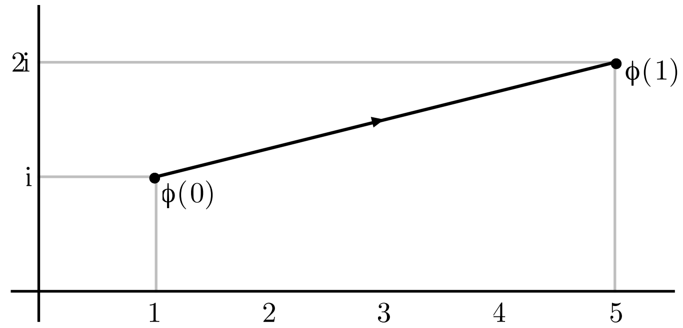
$$\phi(t) = \phi_1(t) + \mathrm{i}\phi_2(t),$$

where the functions ϕ_1 and ϕ_2 are real valued and defined on some closed interval $[A,B]$ in $\mathbb{R}$. The functions ϕ_1 and ϕ_2 are called the real and imaginary parts of the function ϕ respectively.

EXAMPLE 4.1.1. The function $\phi(t) = \mathrm{e}^{\mathrm{i}t}$, defined on the interval $[0,\pi]$, represents the upper half of a circle centred at the origin and of radius 1. As t varies from 0 to π, $\phi(t)$ follows this half-circle in an anticlockwise direction. Also $\phi_1(t) = \cos t$ and $\phi_2(t) = \sin t$.



EXAMPLE 4.1.2. The function $\phi(t) = (4t+1) + \mathrm{i}(t+1)$, defined on the interval $[0,1]$, represents a line segment from the point $1+\mathrm{i}$ to the point $5+2\mathrm{i}$.



The functions ϕ_1 and ϕ_2 are real valued functions of a real variable, and we have already studied continuity, differentiability and integrability of such functions. We can now extend these definitions to the function ϕ.

We say that ϕ is continuous at t_0 if both ϕ_1 and ϕ_2 are continuous at t_0. We also say that ϕ is continuous in an interval if both ϕ_1 and ϕ_2 are continuous in the interval. It is simple to show that the arithmetic of limits, as applied to continuity, holds.

We say that ϕ is differentiable at t_0 if both ϕ_1 and ϕ_2 are differentiable at t_0, and write

$$\phi'(t_0) = \phi_1'(t_0) + \mathrm{i}\phi_2'(t_0).$$

It is simple to show that the arithmetic of derivatives holds.

We also have the Chain rule: Suppose that f is analytic at the point $z_0 = \phi(t_0)$, and that ϕ is differentiable at t_0. Then the complex valued function $\psi(t) = f(\phi(t))$ is differentiable at t_0, and

$$\psi'(t_0) = f'(z_0)\phi'(t_0).$$

We say that ϕ is integrable over the interval $[A,B]$ if both ϕ_1 and ϕ_2 are integrable over $[A,B]$, and write

$$\int_A^B \phi(t)\,\mathrm{d}t = \int_A^B \phi_1(t)\,\mathrm{d}t + \mathrm{i}\int_A^B \phi_2(t)\,\mathrm{d}t.$$

Many rules of integration for real valued functions can be carried over to this case. For example, if ϕ is continuous in $[A,B]$, then there exists a function Φ satisfying $\Phi' = \phi$, and the Fundamental theorem of integral calculus can be generalized to

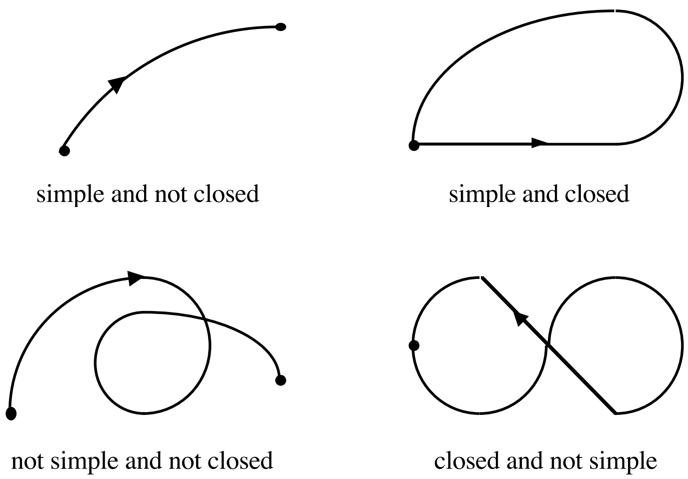
$$\int_A^B \phi(t)\,\mathrm{d}t = \Phi(B) - \Phi(A) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_A^t \phi(\tau)\,\mathrm{d}\tau = \phi(t).$$

DEFINITION. A complex valued function $\zeta : [A,B] \to \mathbb{C}$ is called a curve. The curve is said to be continuous if ζ is continuous in $[A,B]$, and differentiable if ζ is also differentiable in $[A,B]$. The set $\zeta([A,B])$ is called the trace of the curve. The point $\zeta(A)$ is called the initial point of the curve, and the point $\zeta(B)$ is called the terminal point of the curve.

REMARKS. (1) Of course, we can only have continuity and differentiability from the right at A and from the left at B.

(2) Usually, we do not distinguish between the curve and the function ζ, and simply refer to the curve ζ.

Definition. A curve $\zeta : [A, B] \to \mathbb{C}$ is said to be simple if $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, with the possible exception that $\zeta(A) = \zeta(B)$. A curve $\zeta : [A, B] \to \mathbb{C}$ is said to be closed if $\zeta(A) = \zeta(B)$.



4.2. Contour Integrals

Definition. A curve $\zeta : [A, B] \to \mathbb{C}$ is said to be an arc if ζ is differentiable in $[A, B]$ and ζ' is continuous in $[A, B]$.

Example 4.2.1. The unit circle is a simple closed arc, since we can described it by $\zeta : [0, 2\pi] \to \mathbb{C}$, given by $\zeta(t) = \mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i} \sin t$. It is easy to check that $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, the only exception being $\zeta(0) = \zeta(2\pi)$. Furthermore, $\zeta'(t) = -\sin t + \mathrm{i} \cos t$ is continuous in $[0, 2\pi]$.

Definition. Suppose that C is an arc given by the function $\zeta : [A, B] \to \mathbb{C}$. A complex valued function f is said to be continuous on the arc C if the function $\psi(t) = f(\zeta(t))$ is continuous in $[A, B]$. In this case, the integral of f on C is defined to be

$$\int_C f(z)\,\mathrm{d}z = \int_A^B f(\zeta(t))\zeta'(t)\,\mathrm{d}t. \tag{1}$$

Remarks. (1) Note that (1) can be obtained by the formal substitution $z = \zeta(t)$ and $\mathrm{d}z = \zeta'(t)\,\mathrm{d}t$.

(2) If we describe the arc C in the opposite direction from $t = B$ to $t = A$, this opposite arc can be designated by $-C$. Since

$$\int_B^A f(\zeta(t))\zeta'(t)\,\mathrm{d}t = -\int_A^B f(\zeta(t))\zeta'(t)\,\mathrm{d}t,$$

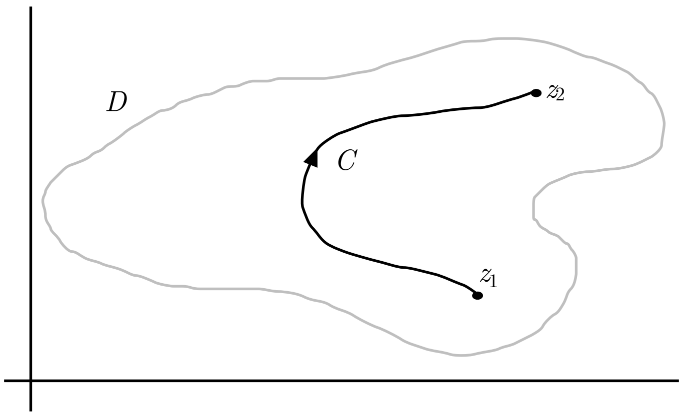
we have

$$\int_{-C} f(z)\,\mathrm{d}z = -\int_C f(z)\,\mathrm{d}z.$$

(3) Integration of functions on arcs is a linear operation. More precisely, suppose that f and g are continuous on the arc C. Then for any $\alpha, \beta \in \mathbb{C}$, we have

$$\int_C (\alpha f(z) + \beta g(z))\,\mathrm{d}z = \alpha \int_C f(z)\,\mathrm{d}z + \beta \int_C g(z)\,\mathrm{d}z.$$

(4) Suppose that F is analytic in a domain D and has a continuous derivative $f = F'$ in D. Suppose further that C, defined by $\zeta : [A, B] \to \mathbb{C}$, is an arc lying in D, with initial point z_1 and terminal point z_2.



Define $\psi : [A, B] \to \mathbb{C}$ by $\psi(t) = F(\zeta(t))$. Then by the Fundamental theorem of integral calculus applied to the function $\psi'(t) = F'(\zeta(t))\zeta'(t) = f(\zeta(t))\zeta'(t)$, we have

$$\int_C f(z)\,\mathrm{d}z = \int_A^B f(\zeta(t))\zeta'(t)\,\mathrm{d}t = F(\zeta(B)) - F(\zeta(A)) = F(z_2) - F(z_1).$$

We can extend the case of one arc to the case of a finite number of arcs joined together.

Definition. Suppose that $A_1 < B_1 = A_2 < B_2 = \ldots = A_k < B_k$ are real numbers, and that for every $j = 1, \ldots, k$, C_j is an arc given by the function $\zeta_j : [A_j, B_j] \to \mathbb{C}$. Suppose further that $\zeta_j(B_j) = \zeta_{j+1}(A_{j+1})$ for every $j = 1, \ldots, k-1$. Then

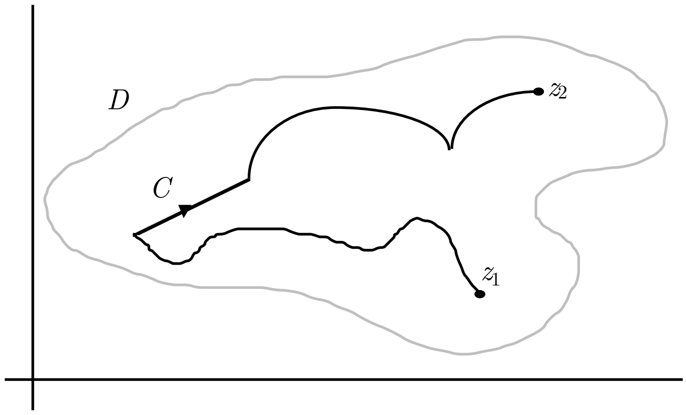
$$C = C_1 \cup C_2 \cup \ldots \cup C_k$$

is called a contour. The point $\zeta_1(A_1)$ is called the initial point of the contour C, and the point $\zeta_k(B_k)$ is called the terminal point of the contour C. A complex valued function f is said to be continuous on the contour C if it is continuous on the arc C_j for every $j = 1, \ldots, k$. In this case, the integral of f on C is defined to be

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z + \ldots + \int_{C_k} f(z)\,\mathrm{d}z.$$

The following result follows immediately from this definition and Remark (4) above.

THEOREM 4A. *Suppose that F is analytic in a domain D and has a continuous derivative $f = F'$ in D. Suppose further that C is a contour lying in D, with initial point z_1 and terminal point z_2.*



Then

$$\int_C f(z)\,\mathrm{d}z = F(z_2) - F(z_1). \tag{2}$$

REMARKS. (1) Note that the right hand side of (2) is independent of the contour C. It follows that under the hypotheses of Theorem 4A, we have

$$\int_C f(z)\,\mathrm{d}z = 0 \tag{3}$$

for any closed contour C in D.

(2) Naturally, we would like to extend (3) to all analytic functions f in D and all closed contours C in D. Note, however, the restrictive nature of the hypotheses of Theorem 4A in this case. In many situations, no analytic functions F satisfying $F' = f$ may be at hand. Consider, for example, the function $f(z) = \cos z^2$.

EXAMPLE 4.2.2. Consider the contour C given by $\zeta : [A, B] \to \mathbb{C}$, where $\zeta(t) = \mathrm{e}^{\mathrm{i}t}$. Then

$$\frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{d}z}{z} = \frac{1}{2\pi\mathrm{i}} \int_A^B \frac{\zeta'(t)}{\zeta(t)}\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}} \int_A^B \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}}\,\mathrm{d}t = \frac{B-A}{2\pi}.$$

Suppose that $A = 0$ and $B = 2k\pi$, so that the contour "winds" round the origin k times in the anticlockwise direction. In this case, we have

$$\frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{d}z}{z} = k.$$

Note, however, that in this case, the initial point and terminal point of the contour are the same. Yet, (3) does not hold. Clearly the function $1/z$ is analytic in the domain $D = \{z : 1/2 < |z| < 3/2\}$ and the contour C lies in D. However, we cannot find an analytic function F in D such that $F'(z) = 1/z$ in D. The logarithmic function $\log z$ appears to be a candidate; however, it is not possible to define $\log z$ to be continuous in this annulus D. See Example 3.5.3.

EXAMPLE 4.2.3. Suppose that C is any contour in $\mathbb{C}$ with initial point z_1 and terminal point z_2. Then

$$\int_C \mathrm{e}^z\,\mathrm{d}z = \mathrm{e}^{z_2} - \mathrm{e}^{z_1} \qquad \text{and} \qquad \int_C \cos z\,\mathrm{d}z = \sin z_2 - \sin z_1.$$

These follow from Theorem 4A since the entire functions e^z and $\sin z$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \mathrm{e}^z \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z.$$

Example 4.2.4. Suppose that f is a polynomial in z with coefficients in $\mathbb{C}$. Then

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C in $\mathbb{C}$. This is a special case of (3). For any such polynomial f, it is easy to find a polynomial F in z with coefficients in $\mathbb{C}$ and such that $F'(z) = f(z)$ in $\mathbb{C}$.

4.3. Inequalities for Contour Integrals

Suppose that $\phi : [A, B] \to \mathbb{C}$ is continuous in $[A, B]$. Let

$$I = \left| \int_A^B \phi(t)\,\mathrm{d}t \right|.$$

If $I = 0$, then clearly

$$\left| \int_A^B \phi(t)\,\mathrm{d}t \right| \le \int_A^B |\phi(t)|\,\mathrm{d}t. \tag{4}$$

If $I > 0$, then there exists a real number θ such that

$$\int_A^B \phi(t)\,\mathrm{d}t = I\mathrm{e}^{\mathrm{i}\theta},$$

so that

$$I = \int_A^B \mathrm{e}^{-\mathrm{i}\theta}\phi(t)\,\mathrm{d}t = \int_A^B \mathfrak{Re}[\mathrm{e}^{-\mathrm{i}\theta}\phi(t)]\,\mathrm{d}t + \mathrm{i}\int_A^B \mathfrak{Im}[\mathrm{e}^{-\mathrm{i}\theta}\phi(t)]\,\mathrm{d}t. \tag{5}$$

Since I is real, the last integral on the right hand side of (5) must be 0. It follows that

$$I = \int_A^B \mathfrak{Re}[\mathrm{e}^{-\mathrm{i}\theta}\phi(t)]\,\mathrm{d}t.$$

On the other hand, clearly

$$\mathfrak{Re}[\mathrm{e}^{-\mathrm{i}\theta}\phi(t)] \le |\mathrm{e}^{-\mathrm{i}\theta}\phi(t)| = |\phi(t)|$$

for every $t \in [A, B]$, and so it follows from the theory of real integration that

$$I \le \int_A^B |\phi(t)|\,\mathrm{d}t.$$

Hence the inequality (4) always holds.

Consider now an arc C given by the function $\zeta : [A, B] \to \mathbb{C}$. Suppose that the function f is continuous on C. Then it follows from (4) that

$$\left| \int_C f(z)\,\mathrm{d}z \right| = \left| \int_A^B f(\zeta(t))\zeta'(t)\,\mathrm{d}t \right| \le \int_A^B |f(\zeta(t))||\zeta'(t)|\,\mathrm{d}t. \tag{6}$$

Suppose that $|f(z)| \leq M$ on C, where M is a real constant, then we have

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leq M \int_A^B |\zeta'(t)|\,\mathrm{d}t. \tag{7}$$

Let us investigate the integral on the right hand side of (7) more closely. If $\zeta(t) = x(t) + \mathrm{i}y(t)$, then

$$|\zeta'(t)| = \left|\frac{\mathrm{d}x}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}t}\right| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \frac{\mathrm{d}s}{\mathrm{d}t},$$

where $s(t)$ is the length of the arc C between $\zeta(A)$ and $\zeta(t)$. It follows that the integral

$$\int_A^B |\zeta'(t)|\,\mathrm{d}t$$

is the length of the arc C. We have proved the following result.

THEOREM 4B. *Suppose that a function f is continuous on a contour C. Then*

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leq ML,$$

where L is the length of the contour C and where M is a real constant such that $|f(z)| \leq M$ on C.

REMARK. We usually write

$$\int_C f(z)|\,\mathrm{d}z| = \int_A^B f(\zeta(t))|\zeta'(t)|\,\mathrm{d}t.$$

In this notation, we have

$$L = \int_C |\mathrm{d}z|,$$

and (6) can be represented by

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leq \int_C |f(z)|\,|\mathrm{d}z|.$$

4.4. Equivalent Curves

EXAMPLE 4.4.1. Consider the arc C' given by the function $\zeta : [0,1] \to \mathbb{C}$ where $\zeta(t) = (1+\mathrm{i})t^2$. Consider also the arc C'' given by the function $\xi : [0,\pi/2] \to \mathbb{C}$ where $\xi(\tau) = (1+\mathrm{i})\sin\tau$. Clearly $\zeta(0) = \xi(0)$, so that the two arcs have the same initial point. Also, $\zeta(1) = \xi(\pi/2)$, so that the two arcs have the same terminal point. Furthermore, $\zeta([0,1]) = \xi([0,\pi/2])$, so that the two arcs have the same trace. Suppose that the function f is continuous on C' and C''. On the one hand, we have

$$\int_{C'} f(z)\,\mathrm{d}z = \int_0^1 f(\zeta(t))\zeta'(t)\,\mathrm{d}t = \int_0^1 f((1+\mathrm{i})t^2)2(1+\mathrm{i})t\,\mathrm{d}t.$$

On the other hand, we have

$$\int_{C''} f(z)\,\mathrm{d}z = \int_0^{\pi/2} f(\xi(\tau))\xi'(\tau)\,\mathrm{d}\tau = \int_0^{\pi/2} f((1+\mathrm{i})\sin\tau)(1+\mathrm{i})\cos\tau\,\mathrm{d}\tau.$$

If we perform a formal change of variables

$$t^2 = \sin\tau \qquad \text{and} \qquad 2t\,\mathrm{d}t = \cos\tau\,\mathrm{d}\tau,$$

then we see in fact that

$$\int_{C'} f(z)\,\mathrm{d}z = \int_{C''} f(z)\,\mathrm{d}z.$$

This is not surprising, considering that basically the two arcs "are the same".

Definition. Two curves C' and C'' are said to be equivalent if they have the same trace and if

$$\int_{C'} f(z)\,\mathrm{d}z = \int_{C''} f(z)\,\mathrm{d}z$$

holds for all functions f which are continuous in a region containing this trace.

Remarks. (1) It can be shown that two simple arcs are equivalent if they have the same initial points, the same terminal points and the same trace, with the convention that in the case of closed arcs, the arcs must be followed in the same direction.

(2) In fact, the definition

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z + \ldots + \int_{C_k} f(z)\,\mathrm{d}z$$

of a contour integral in terms of integrals over arcs as discussed earlier is made in the same spirit.

(3) The practical importance of these considerations is that when we consider integrals over a simple contour, we may choose the most convenient parameterization of the given contour.

Example 4.4.2. Consider the integral

$$\int_C \frac{\mathrm{d}z}{z},$$

where C is a contour that avoids the origin. Suppose first of all that C is an arc $\zeta : [A, B] \to \mathbb{C}$. Then

$$\int_C \frac{\mathrm{d}z}{z} = \int_A^B \frac{\zeta'(t)}{\zeta(t)}\,\mathrm{d}t = \mathrm{var}(\log z, C).$$

Here the variation function $\mathrm{var}(\log z, C)$ is interpreted in the following way: We choose a branch of the logarithmic function at the initial point $z_1 = \zeta(A)$ of the arc C and then let $\log z$ vary continuously as z follows C to the terminal point $z_2 = \zeta(B)$ of the arc C. In other words, the function $\log\zeta(t)$ must be continuous in $[A, B]$. Then we calculate $\log\zeta(B) - \log\zeta(A)$. This result is then extended to contours by addition. Let us interpret this geometrically. Note that $\log z = \log|z| + \mathrm{i}\arg z$. Since $\log|z|$ is single valued, its variation around a closed contour C is 0. In this case, we have

$$\mathrm{var}(\log z, C) = \mathrm{var}(\mathrm{i}\arg z, C),$$

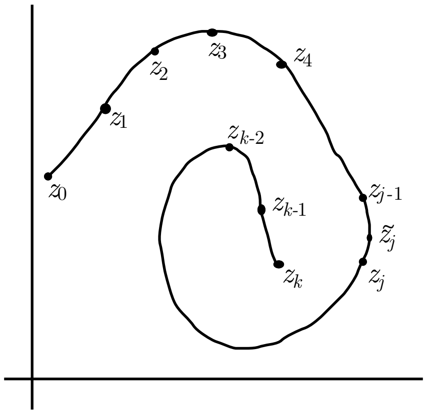
and this gives a value $2\pi\mathrm{i}$ every time the closed contour winds round the origin.

4.5. Riemann Sums

The following brief discussion of complex integrals in terms of Riemann sums will further demonstrate the independence of the integral from the parameterization of the arcs in question. The discussion is heuristic, as we only want to illustrate ideas.

Suppose that C is an arc, with initial point z_0 and terminal point z_k. Suppose further that we divide the arc C into subarcs $C_1, \ldots, C_k$ in the following way. The points $z_0, z_1, \ldots, z_k$ are points on the arc C, and they occur in the given order as we follow the arc C from z_0 to z_k. For every $j = 1, \ldots, k$, the subarc C_j is then the part of C between z_{j-1} and z_j, with initial point z_{j-1} and terminal point z_j.

For every $j = 1, \ldots, k$, we write $\Delta z_j = z_j - z_{j-1}$, and we let $\tilde{z}_j$ denote a point on the subarc C_j.



As in real variables, we can then construct the Riemann sum

$$S = \sum_{j=1}^{k} f(\tilde{z}_j)\Delta z_j.$$

We now consider subdivisions of the arc C which are made finer and finer by subdivision into more and more subarcs. The precise requirement will be

$$k \to \infty \qquad \text{and} \qquad \max_{1 \le j \le k} |\Delta z_j| \to 0.$$

When the subdivision becomes arbitrarily fine, the Riemann sum S has a unique limit, independent of the manner of subdivision. This limit is the integral

$$\int_C f(z)\,\mathrm{d}z.$$

Problems for Chapter 4

1. Consider the integral $\displaystyle\int_C z^n\,\mathrm{d}z$, where $n \in \mathbb{Z}$ and C is a closed contour on the complex plane.
   a) Suppose that $n \ge 0$. Use Theorem 4A to explain why the integral is equal to zero.
   b) Suppose that $n < -1$, and that the contour C does not pass through the origin $z = 0$. Use Theorem 4A to explain why the integral is equal to zero.
   c) What is the value of the integral if $n = -1$ and C is the unit circle $\{z : |z| = 1\}$, followed in the positive (anticlockwise) direction?

2. a) Sketch each of the arcs $z = 2 + \mathrm{i}t$, $z = \mathrm{e}^{-\pi \mathrm{i}t}$, $z = \mathrm{e}^{4\pi \mathrm{i}t}$ and $z = 1 + \mathrm{i}t + t^2$ for $t \in [0, 1]$.
   b) Using Theorem 4A if appropriate, integrate each of the functions $f(z) = 4z^3$, $f(z) = \overline{z}$ and $f(z) = 1/z$ over each of the arcs in part (a).

3. Suppose that a function $f(z)$ satisfies $f'(z) = 0$ throughout a domain $D \subseteq \mathbb{C}$. Use Theorem 4A to prove that $f(z)$ is constant in D.

4. Suppose that C_1 is the semicircle from 1 to -1 through i, followed in the positive (anticlockwise) direction. Suppose also that C_2 is the semicircle from 1 to -1 through $-$i, followed in the negative (clockwise) direction. Show that

$$\int_{C_1} z^3 \,\mathrm{d}z = \int_{C_2} z^3 \,\mathrm{d}z \qquad \text{and} \qquad \int_{C_1} \overline{z} \,\mathrm{d}z \neq \int_{C_2} \overline{z} \,\mathrm{d}z.$$

Use Theorem 4A to comment on the two results.

5. Suppose that $\alpha = a + \mathrm{i}b$, where $a, b \in \mathbb{R}$ are fixed. By integrating the function $\mathrm{e}^{\alpha t}$ over an interval $[0, T]$ and equating real parts, show that

$$(a^2 + b^2) \int_0^T \mathrm{e}^{at} \cos bt \,\mathrm{d}t = \mathrm{e}^{aT}(a \cos bT + b \sin bT) - a.$$

6. Suppose that $f(t) = f_1(t) + \mathrm{i}f_2(t)$ and $g(t) = g_1(t) + \mathrm{i}g_2(t)$ are differentiable complex valued functions of a real variable t. Deduce the formulas

$$(f + g)'(t) = f'(t) + g'(t) \qquad \text{and} \qquad (fg)'(t) = f(t)g'(t) + f'(t)g(t)$$

from known results of these types for real valued functions $f_1(t)$, $f_2(t)$, $g_1(t)$ and $g_2(t)$.

7. Consider an arc $z(t) = x(t) + \mathrm{i}y(t)$, where $t \in [A, B]$. Use

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}t}$$

to interpret $z'(t)$ as the complex representation of a vector tangent to the arc at any point where $z'(t)$ is non-zero.

8. Suppose that $\zeta(t) = t^2$ for $t \in [-1, 0]$ and $\zeta(t) = \mathrm{i}t^2$ for $t \in [0, 1]$. Show that the curve $\zeta : [-1, 1] \to \mathbb{C}$ is an arc (although its trace has a corner).

9. Consider the curve $\zeta : [-1, 1] \to \mathbb{C}$, given by $\zeta(t) = -t$ for $t \in [-1, 0]$ and $\zeta(t) = t + \mathrm{i}t^3 \sin(1/t)$ for $t \in (0, 1]$.
   a) Show that $\zeta : [-1, 1] \to \mathbb{C}$ is an arc.
   b) Determine all points of self-intersection of this arc; in other words, find all points $z \in \mathbb{C}$ such that there exist $t_1, t_2 \in [-1, 1]$ satisfying $t_1 \neq t_2$ and $z = \zeta(t_1) = \zeta(t_2)$.

10. Suppose that $C = \{z : |z| = 1\}$ is the unit circle, followed in the positive (anticlockwise) direction. Evaluate each of the following integrals:

a) $\displaystyle\int_C \frac{\mathrm{d}z}{z}$ b) $\displaystyle\int_C \frac{\mathrm{d}z}{|z|}$ c) $\displaystyle\int_C \frac{|\mathrm{d}z|}{z}$

d) $\displaystyle\int_C \frac{\mathrm{d}z}{z^2}$ e) $\displaystyle\int_C \frac{\mathrm{d}z}{|z^2|}$ f) $\displaystyle\int_C \frac{|\mathrm{d}z|}{z^2}$

11. Suppose that $C = \{z : |z| = 1\}$ is the unit circle, followed in the positive (anticlockwise) direction.
    a) Use Theorem 4B to show that
$$\left| \int_C \frac{\mathrm{d}z}{4+3z} \right| \leq 2\pi.$$
    b) By dividing the circle C into its left half and its right half and applying Theorem 4B to each half, establish the better bound
$$\left| \int_C \frac{\mathrm{d}z}{4+3z} \right| \leq \frac{6\pi}{5}.$$

12. Consider the circle $C = \{z : |z-1| = 1/2\}$, followed in the positive (anticlockwise) direction with initial point $z = 1/2$. Evaluate the integral
$$\int_C \frac{\mathrm{d}z}{(z^2-1)^{1/2}},$$
given that the integrand is equal to the derivative of the function $\log(z + (z^2-1)^{1/2})$.

13. Writing $f = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$, computation suggests the identity
$$\int_C f(z)\,\mathrm{d}z = \int_C (u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y) + \mathrm{i}\int_C (u(x,y)\,\mathrm{d}y + v(x,y)\,\mathrm{d}x).$$
Suppose now that the arc C is given by $\zeta(t) = \xi(t) + \mathrm{i}\eta(t)$ for $t \in [A, B]$. Show from first definition that the identity holds.

14. Suppose that the arc C_1 is given by $z = \zeta_1(t)$ for $t \in [A, B]$, and that the arc C_2 is given by $z = \zeta_2(\tau)$ for $\tau \in [\alpha, \beta]$. Suppose further that there is a differentiable function $\phi : [A, B] \to [\alpha, \beta]$ such that $\phi(A) = \alpha$, $\phi(B) = \beta$. Show that the two arcs C_1 and C_2 are equivalent.

15. Suppose that C denotes the ellipse $x = a\cos t$ and $y = b\sin t$, where $a, b \in \mathbb{R}$ are positive and fixed, and $t \in [0, 2\pi]$, so that C is followed in the positive (anticlockwise) direction.
    a) By referring to Example 4.4.2 if necessary, explain why $\displaystyle\frac{1}{2\pi\mathrm{i}}\int_C \frac{\mathrm{d}z}{z} = 1$.
    b) Hence show that $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}t}{a^2\cos^2 t + b^2\sin^2 t} = \frac{1}{ab}$.