Lecture Note
2301308 Functions of A Complex Variable

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Contents

CHAPTER 1

Complex Numbers and Functions

A **complex number** is an expression of the form $a+bi$ (or $a+ib$) where $a, b \in \mathbb{R}$ and i is a symbol satisfying the relation $i^2=-1$. The set of all complex numbers will be denoted by $\mathbb{C}$ and we also let $\mathbb{C}^*=\mathbb{C}-\{0\}$. Notice that $\mathbb{R} \subseteq \mathbb{C}$ (by letting $b=0$) and the function $\phi: \mathbb{C} \rightarrow \mathbb{R}^2$ given by $\phi(a+bi)=(a, b)$ is clearly a bijection. We usually use this bijection to identify $\mathbb{C}$ with $\mathbb{R}^2$ (i.e., $a+bi=(a, b)$) while $\mathbb{R}$ can be viewed as the X-axis of $\mathbb{R}^2$. In this note, we will regard $a+bi$ as the **standard form** and (a, b) as the **vector form** of a complex number.

Some Notations : For a complex number $z=a+bi$,

- the **real part** of z is $\operatorname{Re}(z)=a$,
- the **imaginary part** of z is $\operatorname{Im}(z)=b$,
- the **conjugate** of z is $\bar{z}=a-bi$,
- the **modulus** of z is $|z|=\sqrt{a^2+b^2}$,
- an **argument** of $z \neq 0$ is an angle between the vectors (0,1) and (a, b) (viewing in $\mathbb{R}^2$) measured in the counter-clockwise direction. Notice that $\arg(z)$ is multivalued. We usually call the argument that lies in the interval $(-\pi, \pi]$ the **principal argument** of z and denote it by $\operatorname{Arg}(z)$.

Other Forms of Complex Numbers: For a complex number $z=a+bi$, let $r=|z|$ and $\theta=\arg(z)$. [Euler formula : $e^{i\theta}=\cos\theta+i\sin\theta$.]

- The **polar form** of z is $z=r(\cos\theta+i\sin\theta)$.
- The **exponential form** of z is $z=re^{i\theta}$.

Notice that both polar form and exponential form of a complex number z is not unique and we always have $|\cos\theta+i\sin\theta|=|e^{i\theta}|=1$.

Complex Algebra : For two complex numbers $z=a+bi$ and $w=c+di$, we define

$$z+w=(a+c)+(b+d)i$$

and

$$z \cdot w=(ac-bd)+(ad+bc)i.$$

It is straightforward to verify that $(\mathbb{C},+,\cdot)$ is a field with 0 as the additive identity, 1 as the multiplicative identity, $-a-bi$ as the additive inverse of $a+bi$ and $\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$ as the multiplicative inverse of $a+bi \neq 0$. As usual, we will denote the additive inverse and the multiplicative inverse (if exists) of z by $-z$ and z^{-1} (or $\frac{1}{z}$) respectively.

In terms of polar and exponential forms, if $z_1=r_1(\cos\theta_1+i\sin\theta_1)=r_1e^{i\theta_1}$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)=r_2e^{i\theta_2}$, one can show that

$$z_1z_2=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))=r_1r_2e^{in(\theta_1+\theta_2)}$$

and, when $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) = \frac{r_1}{r_2}e^{in(\theta_1 - \theta_2)}.$$

EXERCISE 1.1. *Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and $n \in \mathbb{N}$. Prove that*

(1) $z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = re^{in\theta}$.
(2) *There are exactly n solutions of the equation $w^n = z$ which are*

$$w = \sqrt[n]{r}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}) = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}},$$

for $k = 0, 1, \ldots, n-1$.

EXERCISE 1.2. *Let $z, w \in \mathbb{C}$. Prove that*

(1) $\overline{\overline{z}} = z$.
(2) $|-z| = |z| = |\overline{z}|$.
(3) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$.
(4) $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.
(5) $z\overline{z} = |z|^2$.
(6) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
(7) $\overline{zw} = \overline{zw}$.
(8) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ *whenever $w \neq 0$.*
(9) $|z + w| \leq |z| + |w|$. *(triangle inequality).*
(10) $||z| - |w|| \leq |z - w| \leq |z| + |w|$.
(11) $|zw| = |z||w|$.
(12) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ *whenever $w \neq 0$.*

Complex Function : A **complex function** is simply a function $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$. For examples,

(1) $id : \mathbb{C} \to \mathbb{C}$ defined by $id(z) = z$.
(2) $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$.
(3) $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \overline{z}$.
(4) $h : \mathbb{C} \to \mathbb{C}$ defined by $h(z) = \mathrm{Re}(z) + \mathrm{Im}(z)$.
(5) $i : \mathbb{C}^* \to \mathbb{C}$ defined by $i(z) = \frac{1}{z}$.
(6) a **complex polynomial function** which is a complex function of the form

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n = \sum_{i=0}^{n} c_i z^i,$$

for some $n \in \mathbb{N} \cup \{0\}$ and $c_0, c_1, \ldots, c_n \in \mathbb{C}$.

Using various forms of a complex number, we may represent a complex function f as follows :

$$f(x + iy) = u(x, y) + iv(x, y)$$

or

$$f(re^i\theta) = \rho(r, \theta)e^{i\phi(r,\theta)}$$

where u, v, ρ, ϕ are real-valued functions of two real variables.

CHAPTER 2

Topology of $\mathbb{C}$: A Fast Glimpse

DEFINITION 2.1. For each $z_0 \in \mathbb{C}$ and $r > 0$, we define the **open ball centered at** z_0 **of radius** r to be the set

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We also let $B^*(z_0, r) = B(z_0, r) - \{z_0\}$.

DEFINITION 2.2. Let $\Omega \subseteq \mathbb{C}$. We say that Ω is

- **open** if for each $z \in \Omega$, there is $r > 0$ such that $B(z, r) \subseteq \Omega$.
- **closed** if $\mathbb{C} - \Omega$ is open.
- **bounded** if $\Omega \subseteq B(0, R)$ for some $R > 0$.
- **compact** if it is closed and bounded.

EXAMPLE 2.3. $\emptyset$ and $\mathbb{C}$ are both open and closed at the same time. All open balls are clearly open as well as the following sets : $\{z : \mathrm{Re}(z) > 0\}$, $\{z : \mathrm{Re}(z) < 0\}$, $\{z : \mathrm{Im}(z) > 0\}$, $\{z : \mathrm{Im}(z) < 0\}$ and $\mathbb{C}^*$.

EXERCISE 2.4. *For $z_0 \in \mathbb{C}$ and $0 < r < R$, we define*

- *an **open annulus** to be $A(z_0, r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$.*
- *a **closed annulus** to be $A[z_0, r, R] = \{z \in \mathbb{C} : r \le |z - z_0| \le R\}$.*
- *a **closed ball** to be $B[z_0, r] = \{z \in \mathbb{C} : |z - z_0| \le r\}$.*

Prove that an open annulus is always open while a closed ball and a closed annulus are always closed (in fact, compact).

EXAMPLE 2.5. $\mathbb{C}$, $\mathbb{C}^*$ and $B(0, 1)$ are not compact.

DEFINITION 2.6. A **neighborhood** of a point $z_0 \in \mathbb{C}$ is simply an open subset of $\mathbb{C}$ containing z_0.

DEFINITION 2.7. Let $\Omega \subseteq \mathbb{C}$.

- The **interior** of Ω, denoted by Ω°, is the largest open subset of $\mathbb{C}$ contained in Ω.
- The **closure** of Ω, denoted by $\overline{\Omega}$, is the smallest closed subset of $\mathbb{C}$ containing Ω.
- The **boundary** of Ω is defined to be $\partial\Omega = \overline{\Omega} \cap \overline{\mathbb{C} - \Omega}$.

DEFINITION 2.8. A point $z \in \mathbb{C}$ is a **limit point** of $\Omega \subseteq \mathbb{C}$ if for each $r > 0$,

$$B^*(z, r) \cap \Omega \neq \emptyset.$$

The set of all limit points of Ω is called the **derived set** of Ω and it is usually denoted by Ω'; that is

$$\Omega' = \{z \in \mathbb{C} : \forall r > 0, B^*(z, r) \cap \Omega \neq \emptyset\}.$$

EXAMPLE 2.9. From the definitions above, we clearly have the followings.

(1) $B(0,1)^\circ = B(0,1), \overline{B(0,1)} = B[0,1]$ and $\partial B(0,1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$.
(2) $B[0,1]^\circ = B(0,1), \overline{B[0,1]} = B[0,1]$ and $\partial B[0,1] = \{e^{i\theta} : \theta \in \mathbb{R}\}$.
(3) $A(0,0,1)^\circ = A(0,0,1), \overline{A(0,0,1)} = A[0,0,1]$ and $\partial A(0,0,1) = \{0\} \cup \partial B(0,1)$.

Theorem 2.10. *For any* $\Omega \subseteq \mathbb{C}$, *we have*

$$\overline{\Omega} = \Omega^\circ \cup \partial\Omega = \Omega \cup \Omega',$$

and hence $\overline{\Omega} = \{z \in \mathbb{C} : \forall r > 0, B(z,r) \cap \Omega \neq \emptyset\}$.

Definition 2.11. Let $f : \Omega \to \mathbb{C}$, $z_0 \in \Omega'$ and $L \in \mathbb{C}$. We say that the **limit of** $f(z)$ **as** z **approaches** z_0 **is** L, written as $\lim\limits_{z\to z_0} f(z) = L$, if for each $\epsilon > 0$, there is $\delta > 0$ such that for any $z \in \Omega$ with $0 < |z - z_0| < \delta$, we must have $|f(z) - L| < \epsilon$ [or equivalently, $f(B^*(z_0,\delta) \cap \Omega) \subseteq B(L,\epsilon)$].

Remark 2.12. If $\lim\limits_{z\to z_0} f(z) = L$, then $f(z)$ approaches L as z approaches z_0 in any direction. In other words, if we can find one direction in which the limit does not exist, or at least two directions in which the limits exist but not equal, then we can conclude that $\lim\limits_{z\to z_0} f(z)$ does not exist.

Theorem 2.13 (Limit Theorem). *Let* $f, g : \Omega \to \mathbb{C}$ *and* $z_0 \in \Omega'$. *If* $\lim\limits_{z\to z_0} f(z)$ *and* $\lim\limits_{z\to z_0} g(z)$ *exist, we have*

(1) $\lim\limits_{z\to z_0} (f(z) \pm g(z)) = \lim\limits_{z\to z_0} f(z) \pm \lim\limits_{z\to z_0} g(z)$.
(2) $\lim\limits_{z\to z_0} f(z)g(z) = \lim\limits_{z\to z_0} f(z) \lim\limits_{z\to z_0} g(z)$.
(3) $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim\limits_{z\to z_0} f(z)}{\lim\limits_{z\to z_0} g(z)}$ *provided that* $\lim\limits_{z\to z_0} g(z) \neq 0$.

Proof. Exercise. □

Example 2.14. It is easy to see that $\lim\limits_{z\to 0} z = \lim\limits_{z\to 0} \overline{z} = \lim\limits_{z\to 0}(z+\overline{z}) = \lim\limits_{z\to 0}(z\overline{z}) = 0$. However, $\lim\limits_{z\to 0} \dfrac{\overline{z}}{z}$ does not exist because the limit along the real axis is 1 while the limit along the imaginary axis is -1.

Theorem 2.15. *Let* $f, g : \Omega \to \mathbb{C}$ *with* $z_0 \in \Omega'$. *Suppose further that*

(1) f *is bounded on* $B(z_0, r)$ *for some* $r > 0$ *and*
(2) $\lim\limits_{z\to z_0} g(z) = 0$.

Then $\lim\limits_{z\to z_0} f(z)g(z) = 0$.

Exercise 2.16. *Find the following limits (if exist).*

(1) $\lim\limits_{z\to 0} \dfrac{\operatorname{Re}(z)}{|z|}$.
(2) $\lim\limits_{z\to 0} \dfrac{\operatorname{Re}(z^2)}{|z|^2}$.
(3) $\lim\limits_{z\to 0} \dfrac{z \operatorname{Re}(z)}{|z|}$.

Definition 2.17 (Limit at Infinity). Let $f : \mathbb{C} \to \mathbb{C}$ and $L \in \mathbb{C}$. We will write $\lim_{|z|\to\infty} f(z) = L$, or simply $\lim_{z\to\infty} f(z) = L$, if for each $\epsilon > 0$, there exists $M \in \mathbb{R}^+$ such that $|f(z) - L| < \epsilon$ whenever $|z| > M$.

Remark 2.18. The limit theorem above also holds for the case $z_0 = \infty$.

Definition 2.19 (Infinite Limit). Let $f : \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C} \cup \{\infty\}$. We will write $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0} |f(z)| = \infty$.

Definition 2.20. Let $f : \Omega \to \mathbb{C}$ and $z_0 \in \Omega$. We say that f is **continuous at** z_0, if for each $\epsilon > 0$, there is $\delta > 0$ such that for any $z \in \Omega$ with $0 < |z - z_0| < \delta$, we must have $|f(z) - f(z_0)| < \epsilon$ [or equivalently, $f(B(z_0, \delta) \cap \Omega) \subseteq B(f(z_0), \epsilon)$]. We simply say that f is continuous if it is continuous at each point of Ω.

Example 2.21. The function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \frac{\bar{z}}{z} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

is not continuous at 0. However, its restriction $f|_{\mathbb{C}^*}$ is clearly continuous.

Exercise 2.22. *Let $f : \Omega \to \mathbb{C}$ and $z_0 \in \Omega$. Prove that f is continuous at z_0 if and only if $\lim_{z\to z_0} |f(z) - f(z_0)| = 0$.*

Theorem 2.23. *Let $f, g : \Omega \to \mathbb{C}$ and $h : \Delta \to \mathbb{C}$ be complex functions with $g(\Omega) \subseteq \Delta$, and $z_0 \in \Omega$.*

(1) *If both f and g are continuous at z_0, then so are $f \pm g$ and fg.*
(2) *If both f and g are continuous at z_0 and $g(z_0) \neq 0$, then so is $\frac{f}{g}$.*
(3) *If g is continuous at z_0 and h is continuous at $g(z_0)$, then $g \circ h$ is continuous at z_0.*

Proof. Exercise. □

Example 2.24. A complex polynomial function is always continuous as well as the function $i : \mathbb{C}^* \to \mathbb{C}$ defined by $i(z) = \frac{1}{z}$.

Exercise 2.25. *Prove or disprove : if $f : \Omega \to \mathbb{C}$ is a continuous complex function and A is an open [closed] subset of Ω, then $f(A)$ is open [closed] in $\mathbb{C}$.*

Theorem 2.26. *Let $f : \Omega \to \mathbb{C}$ be a continuous complex function and $A \subseteq \Omega$.*

(1) *If A is compact [path-connected], then so is $f(A)$.*
(2) *If A is compact, then $|f| : A \to \mathbb{R}$ attains its minimum and maximum.*

Definition 2.27. A **path** in $\mathbb{C}$ is simply a continuous function $\gamma : [a, b] \to \mathbb{C}$. The image of a path γ is called a **curve represented by** γ. Two paths are said to be **equivalent** if they represents the same curve in the same direction.

Exercise 2.28. *Draw the curve represented by each one of the following paths in $\mathbb{C}$.*

(1) $\gamma_1(t) = t + it;\ 0 \leq t \leq 1$.
(2) $\gamma_2(t) = t^2 + it^2;\ 0 \leq t \leq 1$.
(3) $\gamma_3(t) = t + it^2;\ 0 \leq t \leq 1$.
(4) $\gamma_4(t) = \cos t + i \sin t = e^{it};\ 0 \leq t \leq 2\pi$.

(5) $\gamma_5(t) = \begin{cases} t + it \ ; 0 \leq t \leq 1 \\ 1 + it \ ; 1 \leq t \leq 2 \end{cases}$.

DEFINITION 2.29. A curve C represented by $\gamma : [a, b] \to \mathbb{C}$ is said to be

- **a closed curve** if $\gamma(a) = \gamma(b)$.
- **a simple closed curve** if it is closed and γ is 1-1 on (a, b).
- **a smooth curve** if γ is continuously differentiable on $[a, b]$ (i.e., γ is smooth).
- **a piecewise smooth curve or a contour** if γ can be decomposed into finitely many smooth curves.

DEFINITION 2.30. Let $\Omega \subseteq \mathbb{C}$. We say that Ω is

- **path-connected** if each pair of points in Ω can be joined by a path in Ω.
- **a domain** if it is nonempty, open and path-connected.

THEOREM 2.31. *If Ω is a domain, then each pair of points in Ω can be joined by a polygonal path (= a path consisting of straight lines) in Ω.*

DEFINITION 2.32. A **sequence** of complex numbers is a function $f : \mathbb{N} \to \mathbb{C}$. We will usually denote a sequence by (z_n) where $z_n = f(n)$. We will call (z_n) **Cauchy** if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|z_m - z_n| < \epsilon$, for all $m, n \geq N$.

DEFINITION 2.33. A sequence (z_n) of complex numbers is said to **converge** to $z_0 \in \mathbb{C}$, written as $(z_n) \to z_0$ or $\lim\limits_{n\to\infty} z_n = z_0$, if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $z_n \in B(z_0, \epsilon)$ for all $n \geq N$, and we will call z_0 the limit of (z_n).

EXERCISE 2.34. *Let (z_n) be a sequence of complex numbers and $z_0 \in \mathbb{C}$. Use the above definition to prove the following statements :*

(1) $(z_0) \to z_0$.
(2) $(z_n) \to z_0$ *if and only if* $(\overline{z}_n) \to \overline{z}_0$.
(3) $(z_n) \to 0$ *if and only if* $(|z_n|) \to 0$.

THEOREM 2.35. *Let (z_n) and (w_n) be sequences of complex numbers. If $(z_n) \to z_0$ and $(w_n) \to w_0$, then*

(1) $(z_n \pm w_n) \to z_0 \pm w_0$.
(2) $(z_n w_n) \to z_0 w_0$.
(3) $(\frac{z_n}{w_n}) \to \frac{z_0}{w_0}$ *provided that* $w_n \neq 0$ *for all* $n = 0, 1, 2, \ldots$

EXAMPLE 2.36. $\lim\limits_{n\to\infty} \dfrac{n}{n+i} = \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{i}{n}} = \dfrac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \dfrac{i}{n}} = 1.$

THEOREM 2.37. *Let $f : \Omega \to \mathbb{C}$ be a complex function and $z_0 \in \Omega$. Then f is continuous at z_0 if and only if, for any sequence (z_n) converging to z_0, the sequence $f(z_n)$ converges to $f(z_0)$.*

EXAMPLE 2.38. Since the complex sine function and the complex conjugation are continuous on $\mathbb{C}$, we have $\lim\limits_{n\to\infty} \overline{\sin(\dfrac{i}{n})} = \overline{\sin(\lim\limits_{n\to\infty} \dfrac{i}{n})} = \overline{\sin(0)} = 0.$

THEOREM 2.39 (Completeness of $\mathbb{C}$). *Let (z_n) be a sequence of complex numbers. Then (z_n) converges if and only if (z_n) is Cauchy.*

THEOREM 2.40 (Sequence Lemma and its converse). *Let $\Omega \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$. Then $z_0 \in \overline{\Omega}$ if and only if there exists a sequence (z_n) in Ω such that $(z_n) \to z_0$.*

DEFINITION 2.41. For a sequence (z_n) of complex numbers, we call the sequence (s_n), where $s_n = \sum_{i=1}^{n} z_n$, a **series** of complex numbers and denote it by $\sum z_n$ or $\sum_{n=1}^{\infty} z_n$ or $z_1 + z_2 + z_3 + \dots$.

The convergence of $\sum z_n$ is simply the convergence of the sequence (s_n). When the series $\sum z_n$ converges, we will also denote its limit by $\sum_{n=1}^{\infty} z_n$.

The series $\sum z_n$ is said to **converge absolutely** if the series $\Sigma|z_n|$ of real numbers converges.

THEOREM 2.42. *Let $\sum z_n$ be a series of complex numbers.*

(1) *If $\sum z_n$ converges absolutely, then $\sum z_n$ converges.*
(2) *If $\sum z_n$ converges, then $\lim\limits_{n\to\infty} z_n = 0$.*

REMARK 2.43. The converse of (2) above is not true as we can take $z_n = \frac{1}{n}$.

THEOREM 2.44 (Convergence Tests). *Let $\sum z_n$ be a sequence of complex numbers and $\sum a_n$ a sequence of real numbers. Then the series $\sum z_n$ converges abosolutely if one of the following statements hold :*

(1) *Comparison Test : There exists $N \in \mathbb{N}$ such that $|z_n| \leq a_n$ for all $n \geq N$, and $\sum a_n$ converges.*
(2) *Ratio Test : For each $n \in \mathbb{N}$ $z_n \neq 0$, and $\lim\limits_{n\to\infty} \left|\dfrac{z_{n+1}}{z_n}\right| < 1$.*
(3) *Root Test : $\lim\limits_{n\to\infty} |z_n|^{1/n} < 1$.*

EXAMPLE 2.45. The series $\sum \frac{e^{in}}{2^n}$ converges absolutely because $\left|\frac{e^{in}}{2^n}\right| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$, and $\sum \frac{1}{2^n}$ converges.

EXAMPLE 2.46. The series $\sum \frac{(1+i)^n}{n!}$ converges absolutely because $\frac{(1+i)^n}{n!} \neq 0$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} \left|\dfrac{(1+i)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{(1+i)^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{1+i}{n+1}\right| = 0 < 1$.

EXAMPLE 2.47. The series $\sum \frac{z^{2n}}{(n+1)^n}$ converges absolutely for any $z \in \mathbb{C}$ because for each $z \in \mathbb{C}$, we have $\lim\limits_{n\to\infty} \left|\dfrac{(z+i)^{2n}}{(n+1)^n}\right|^{1/n} = \lim\limits_{n\to\infty} \dfrac{|z+i|^2}{|n+1|} = 0 < 1$.

CHAPTER 3

Differentiability and Analyticity

Let Ω be a domain, $f : \Omega \to \mathbb{C}$ be a complex function and $z_0 \in \mathbb{C}$.

DEFINITION 3.1. We say that f is **differentiable** at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, we will call such a limit the **derivative** of f at z_0 and denote it by $f'(z_0)$ or $\frac{df}{dz}(z_0)$. We simply call f differentiable if it is differentiable at each point of Ω.

THEOREM 3.2. *If f, g are differentiable at z_0, then we have*

(1) $(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$.
(2) $(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$.
(3) $(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$ *provided that* $g(z_0) \neq 0$.

DEFINITION 3.3. We will say that f is **analytic** at z_0 if f is differentiable on some neighborhood of z_0, and simply call f analytic if it is analytic at each point of Ω.

In general, analyticity is stronger than differentiability. However, since we are assuming Ω is open, both terms are equivalent.

DEFINITION 3.4. A function f is said to be **entire** if it is analytic on $\mathbb{C}$.

THEOREM 3.5. *If f is differentiable at z_0, it is continuous at z_0.*

PROOF. Since f is differentiable at z_0, we have

$$\lim_{z \to z_0} |f(z) - f(z_0)| = \lim_{z \to z_0} |\frac{f(z) - f(z_0)}{z - z_0}| \lim_{z \to z_0} |z - z_0| = 0.$$

$\square$

THEOREM 3.6. *If $f(z) = f(x, y) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = (x_0, y_0)$, then u_x, u_y, v_x, v_y exist and satisfy the so-called Cauchy-Riemann equations* $\underline{\text{at } z_0}$ *:*

$$u_x = v_y \text{ and } u_y = -v_x.$$

Moreover, we have

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

PROOF. Since $f(z) = f(x, y) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = (x_0, y_0)$, the limits in following directions must exist and are equal to $f'(z_0$.

(1) Along the set $S_1 = \{z \in \Omega : \operatorname{Im}(z) = y_0\}$, we have

$$\lim_{S_1} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \frac{(u(x, y_0) - u(x_0, y_0)) + i(v(x, y_0) - v(x_0, y_0))}{x - x_0} = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

(2) Along the set $S_2 = \{z \in \Omega : \operatorname{Re}(z) = x_0\}$, we have

$$\lim_{S_2} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{y \to y_0} \frac{-i(u(x_0, y) - u(x_0, y_0)) + (v(x_0, y) - v(x_0, y_0))}{y - y_0} = -iu_y(x_0, y_0) + v_y(x_0, y_0).$$

Therefore, we have

$$u_x(x_0, y_0) = v_y(x_0, y_0),$$
$$v_x(x_0, y_0) = -u_y(x_0, y_0),$$

and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

□

THEOREM 3.7. *If u_x, u_y, v_x, v_y exist in a neighborhood of z_0, continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0, then f is differentiable at z_0.*

EXAMPLE 3.8. The function $f(z) = \overline{z}$ is not differentiable at any point in $\mathbb{C}$. Let $z = (x, y) \in \mathbb{C}$. By Cauchy-Riemann equations at z, we have

$$\frac{\partial u}{\partial x}(x, y) = 1 \neq -1 = \frac{\partial v}{\partial y}(x, y).$$

Hence, f is not differentiable at z.

EXERCISE 3.9. *Prove that $f(z) = |z|$ is continuous, but not differentiable at any point in $\mathbb{C}$.*

EXERCISE 3.10. *Prove that the function $f(z) = \overline{z}^2$ is differentiable only at 0.*

EXAMPLE 3.11. Let $f = u + iv$ be an analytic function on Ω. Suppose u is constant on Ω. Then, by Cauchy-Riemann equations, we have $\frac{\partial v}{\partial y} = 0 = -\frac{\partial v}{\partial x}$ on Ω which clearly imples that v is also constant on Ω (since Ω is path-connected). Hence, f is constant on Ω.

EXAMPLE 3.12. Let $f = u + iv$ be an analytic function on Ω. Suppose $\overline{f}$ is analytic on Ω. Then, by Cauchy-Riemann equations of f and $\overline{f}$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \text{ on } \Omega.$$

It follows that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ on Ω which implies that u, v and f are constant on Ω (since Ω is path-connected).

EXERCISE 3.13. *Let $f = u + iv$ be an analytic function on Ω. Prove that :*

(1) *If v is constant on Ω, so if f.*
(2) *If $|f|$ is constant on Ω, so if f.*
(3) *If $|f|$ are analytic on Ω, then f is constant on Ω.*
(4) *If $\operatorname{Re}(f) = \operatorname{Im}(f)$ on Ω, then f is constant on Ω.*

CHAPTER 4

Elementary Functions

Definition 4.1. The **complex exponential function** is $\exp : \mathbb{C} \to \mathbb{C}^*$ defined by

$$\exp(z) = e^x(\cos y + i \sin y),$$

where $z = x + iy$.

The complex exponential function is clearly an extension of the real exponential function. Moreover, for $z, w \in \mathbb{C}$ and $\theta \in \mathbb{R}$, we have

- $\exp(z + w) = \exp(z)\exp(w)$,
- $\exp(-z) = \frac{1}{\exp(z)}$,
- $|\exp(z)| = e^x > 0$,
- $\exp(i\theta) = \cos\theta + i \sin\theta$.

We usually write e^z instead of $\exp(z)$.

Using the complex exponential function defined above, we can define the complex sine and cosine functions as follows.

Definition 4.2. The **complex sine function** $\sin : \mathbb{C} \to \mathbb{C}$ and the **complex cosine function** $\cos : \mathbb{C} \to \mathbb{C}$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

for any $z \in \mathbb{C}$.

It is clear that the complex sine and cosine functions are extensions of the real sine and cosine functions, respectively. Moreover, for $z, w \in \mathbb{C}$, we have

- $\sin(-z) = -\sin z$,
- $\cos(-z) = \cos z$,
- $\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w$,
- $\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w$,
- $\sin^2 z + \cos^2 z = 1$.

Exercise 4.3. *Let $z = x + iy$. Prove that*

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

and use the results to conclude that $\sin z$ and $\cos z$ are not bounded as functions of z.

Theorem 4.4. *The functions $\exp, \sin$ and $\cos$ are entire with*

- $\frac{de^z}{dz} = e^z$,
- $\frac{d \sin z}{dz} = \cos z$,
- $\frac{d \cos z}{dz} = -\sin z$.

PROOF. Since $e^z = u + iv$ with $u = e^x \cos y$ and $v = e^x \sin y$, and all partial derivatives of u and v exist and continuous on $\mathbb{C}$, it follows that e^z is differentable on $\mathbb{C}$ and $\frac{de^z}{dz} = u_x + iu_y = e^x \cos y + ie^x \sin y = e^z$. The other two formulas can be shown similarly. $\square$

EXAMPLE 4.5. $\sin i = \frac{i}{2}(e - \frac{1}{e})$ and $\cos i = \frac{1}{2}(e + \frac{1}{e})$.

DEFINITION 4.6. For each $\alpha \in [0, 2\pi)$ and $z \in \mathbb{C} - \{0\}$, we define

$$\log_{(\alpha)}(z) = \ln(r) + i(\theta + 2n\pi) \ ; \ n \in \mathbb{Z}$$

where $z = re^{i\theta}$, $r > 0$ and $\alpha \le \theta < \alpha + 2\pi$. That is $\log_{(\alpha)}$ is multi-valued. By letting $n = 0$ in the above formula, we obtain the **principal value** of $\log_{(\alpha)}$:

$$\mathrm{Log}_{(\alpha)}(z) = \ln(r) + i\theta.$$

Now, $\mathrm{Log}_{(\alpha)} : \mathbb{C}^* \to \mathbb{C}$ now becomes a function, but unfortunately, it is not continuous at each point of $R_\alpha = \{z : z = re^{i\alpha}, r \ge 0\}$. Therefore, we can avoid this problem by restricting the domain of $\mathrm{Log}_{(\alpha)}$ to $\mathbb{C} - R_\alpha$.

THEOREM 4.7. *For each $\alpha \in [0, 2\pi)$, the function $\mathrm{Log}_{(\alpha)} : \mathbb{C} - R_\alpha \to \mathbb{C}$ defined by*

$$\mathrm{Log}_{(\alpha)}(z) = \ln(r) + i\theta$$

where $z = re^{i\theta}, \alpha < \theta < \alpha + 2\pi$ is analytic and

$$\mathrm{Log}'_{(\alpha)}(z) = \frac{1}{z}.$$

DEFINITION 4.8. For each $\alpha \in [0, 2\pi)$, the analytic function $\mathrm{Log}_{(\alpha)}$ defined as above is called a **branch** of the complex logarithm function. We will call $\mathrm{Log}_{(-\pi)}$ the **principal logarithm function** and simply denote it by Log; i.e.,

$$\mathrm{Log}(z) = \ln(|z|) + i\,\mathrm{Arg}(z),$$

for all $z \in \mathbb{C}$ such that $z \neq 0$ and $\mathrm{Arg}(z) \neq -\pi$.

EXAMPLE 4.9. For each $a \in \mathbb{R}^+$, $\mathrm{Log}(a) = \ln(a)$. Also, $\mathrm{Log}(i) = \frac{\pi}{2}i$ and $\mathrm{Log}(1 + i) = \ln(\sqrt{2}) + \frac{\pi}{4}i$.

REMARK 4.10. Notice that exp maps $\mathbb{C}$ onto $\mathbb{C}^*$ while $\mathrm{Log}_{(\alpha)}$ maps $\mathbb{C} - R_\alpha$ onto the strip $S_\alpha := \{x + iy : \alpha < y < \alpha + 2\pi\}$. Moreover, $\exp \circ \mathrm{Log}_{(\alpha)} = id_{\mathbb{C} - R_\alpha}$ and $(\mathrm{Log}_{(\alpha)} \circ \exp)|_{S_\alpha} = id_{S_\alpha}$.

CHAPTER 5

Line Integrals

Let Ω denote a domain throughout.

DEFINITION 5.1. For a path $\gamma : [a, b] \to \mathbb{C}$, we define

$$\int_a^b \gamma(t)dt = \int_a^b \mathrm{Re}(\gamma(t))dt + i\int_a^b \mathrm{Im}(\gamma(t))dt.$$

From the definition above, we have the followings :

(1) $\mathrm{Re}(\int_a^b \gamma(t)dt) = \int_a^b \mathrm{Re}(\gamma(t))dt$ and $\mathrm{Im}(\int_a^b \gamma(t)dt) = \int_a^b \mathrm{Im}(\gamma(t))dt$.
(2) $\int_a^b \gamma'(t)dt = \gamma(b) - \gamma(a)$.
(3) $|\int_a^b \gamma(t)dt| = e^{-i\theta}\int_a^b \gamma(t)dt = \mathrm{Re}(e^{-i\theta}\int_a^b \gamma(t)dt) = \int_a^b \mathrm{Re}(e^{-i\theta}\gamma(t))dt \leq \int_a^b |e^{-i\theta}\gamma(t)|dt = \int_a^b |\gamma(t)|dt$ where $\theta = \arg(\int_a^b \gamma(t)dt)$.

THEOREM 5.2. *If C be a smooth curve represented by a smooth path $\gamma : [a, b] \to \mathbb{C}$, then the length of C is $\int_a^b |\gamma'(t)|dt$.*

THEOREM 5.3. *Let $f : \Omega \to \mathbb{C}$ be a continuous function. If $\gamma : [a, b] \to \Omega$ and $\omega : [c, d] \to \Omega$ are equivalent smooth paths, then*

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(\omega(t))\omega'(t)dt.$$

DEFINITION 5.4. For a continuous function $f : \Omega \to \mathbb{C}$ and a smooth curve C in Ω, we define

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt,$$

where $z : [a, b] \to \Omega$ is a smooth path representing C. Note that this definition is independent of a path representing C by the previous theorem.

If C is a contour, says $C = C_1 \cup C_2 \cup \cdots \cup C_n$ where each C_i is smooth, we simply let

$$\int_C f(z)dz = \sum_{i=1}^n \int_{C_i} f(z)dz.$$

When C is closed contour in the counterclockwise direction, we usually write the integral as $\oint_C f(z)dz$.

THEOREM 5.5. *If $-C$ is the contour in the opposite direction of C, we have*

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

Example 5.6. Let C be a curve represented by $\gamma(t) = t + it$; $0 \le t \le 1$. Then

$$\int_C z^2 dz = \int_0^1 (t+it)^2(1+i)dt = (1+i)^3 \int_0^1 t^2 dt = (1+i)^3 \left[\frac{t^3}{3}\right]_0^1 = \frac{(1+i)^3}{3},$$

and hence, $\int_{-C} z^2 dz = -\frac{(1+i)^3}{3}$.

Exercise 5.7. *Let* $C = \{e^{it} : 0 \le t \le 2\pi\}$. *Show that*

$$\oint_C \frac{1}{z} dz = 2\pi i.$$

Theorem 5.8. *For a continuous function* $f : \Omega \to \mathbb{C}$ *and a contour* C *in* Ω, *we have*

$$\left| \int_C f(z)dz \right| \le ML,$$

where let L *is the length of* C *and* $M = \max\{|f(z)| : z \in C\}$.

Proof. WLOG, we may assume that C is smooth. Let $z : [a,b] \to \mathbb{C}$ be a smooth path representing C. Note that

$$M = \max\{|f(z)| : z \in C\} = \max\{|f(z(t))| : t \in [a,b]\}$$

exists since $|f \circ z| : [a,b] \to \mathbb{R}$ is continuous and $[a,b]$ is compact. We also have $L = \int_a^b |z'(t)|dt$ and hence

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \le \int_a^b |f(z(t))z'(t)|dt \le M \int_a^b |z'(t)|dt = ML.$$

$\square$

Definition 5.9. Let $f : \Omega \to \mathbb{C}$ be a complex function. We say that $F : \Omega \to \mathbb{C}$ is an **antiderivative** of f on Ω if $F'(z) = f(z)$ for all $z \in \Omega$. Note that F must be analytic on Ω.

Theorem 5.10. *Let* $f : \Omega \to \mathbb{C}$ *be a continuous function. TFAE :*

(1) *For any contour* C *in* Ω, $\int_C f(z)dz$ *depends only on the endpoints of* C.
(2) *For any closed contour* C *in* Ω, $\oint_C f(z)dz = 0$.
(3) f *has an antiderivative on* Ω.

Moreover, if F *is an antidervative of* f *on* Ω *and* $C \subseteq \Omega$ *is a contour from* z_1 *to* z_2, *we have*

$$\oint_C f(z)dz = F(z_2) - F(z_1).$$

Proof. (1) $\Leftrightarrow$ (2) : Easy.

(3) $\Rightarrow$ (1) : Assume (3) and let F be an antiderivative of f on Ω; i.e., $F'(z) = f(z)$. Let C be a contour in Ω. WLOG, assume that C is smooth and represented by $z : [a,b] \to \Omega$. It follows that

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b)) - F(z(a)).$$

This also proves the last statement of the theorem.

(1) $\Rightarrow$ (3) : Assume (1), fix $z_0 \in \Omega$ and define $F : \Omega \to \mathbb{C}$ by

$$F(z) = \int_{z_0}^z f(w)dw$$

along any polygonal path in Ω from z_0 to z. This is certainly well-defined by (1). Now, we will show that for each $z \in \Omega$,

$$f(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z}.$$

Let $z \in \Omega$ and $\epsilon > 0$. By the continuity of f, there exists $\delta > 0$ such that $|f(\xi) - f(z)| < \epsilon$ whenenver $|\xi - z| < \delta$. Since Ω is open, we may assume that δ is small enough so that $B(z, \delta) \subseteq \Omega$. Let $\Delta z \in \mathbb{C}$ be such that $|\Delta z| < \delta$. Therefore there is a straight line $C \subseteq \Omega$ joining z and $z + \Delta z$. Hence, along C, we clearly have

$$M = \max\{|f(w) - f(z)| : w \in C\} < \epsilon, \quad L_C = |\Delta z|,$$

and

$$\left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(w) - f(z))dw \right| \leq \frac{1}{|\Delta z|} M L_C < \epsilon.$$

It follows that

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(w)dw - f(z) \right| = \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(w) - f(z))dw \right| < \epsilon.$$

Therefore, $f(z) = \lim_{\Delta z \to 0} \dfrac{F(z + \Delta z) - F(z)}{\Delta z} = F'(z)$. □

Example 5.11. From the above theorem, we clearly have $\oint_C z^2 dz = 0$ for any closed contour $C \subseteq \mathbb{C}$ because $f(z) = \frac{z^3}{3}$ is an antidervative of $f(z) = z^2$ on $\mathbb{C}$. Hence, if C is any contour from 0 to $1 + i$, we immediately have $\int_C z^2 dz = \frac{(1+i)^3}{3}$.

Example 5.12. Let $\Omega = \mathbb{C}^*$ and consider $f(z) = \frac{1}{z}$ for all $z \in \Omega$. Clearly, f is continuous on Ω. For each $r > 0$, let C_r be the closed curve in Ω represented by $z(t) = re^{it}$ where $t \in [0, 2\pi]$. It is also easy to verify that $\oint_{C_r} f(z)dz = 2\pi i \neq 0$. Hence, by the above theorem, f does not have an antiderivative on $\mathbb{C}^*$ which is possible since a branch of the complex logarithm function cannot be defined on $\mathbb{C}^*$. However, if C is any contour in $\mathbb{C} - \mathbb{R}_0^-$, we immediately have $\oint_{C_r} f(z)dz = 0$.

CHAPTER 6

Cauchy Integral Theorem and Applications

THEOREM 6.1 (Green's Theorem). *Let C be a simple closed curve in $\mathbb{R}^2$ and R the closed region inside and on C. If $P(x,y)$ and $Q(x,y)$ are two real-valued functions whose all their first-ordered partial derivatives exist and are continuous on R, then we have*

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dxdy.$$

THEOREM 6.2 (Cauchy Integral Theorem). *Let C be a simple closed contour in $\mathbb{C}$ and R the closed region inside and on C. If f is analytic on R and f' is continuous, then*

$$\oint_C f(z)dz = 0.$$

PROOF. WLOG, assume that C is smooth. Write $f(z) = f(x,y) = u(x,y) + iv(x,y)$ and let $z(t) = x(t) + iy(t); a \leq t \leq b$ be a smooth path representing C. Then, by Cauchy-Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$ on R, and hence

$$\begin{aligned}
\oint_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b f(x(t) + iy(t))(x'(t) + iy'(t))dt \\
&= \int_a^b (u(x,y) + iv(x,y))(x' + iy')dt \\
&= \int_a^b [(ux' - vy') + i(vx' + uy')]dt \\
&= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \\
&= \iint_R (v_x + u_y)dxdy + i\iint_R (u_x - v_y)dxdy \\
&= 0.
\end{aligned}$$

□

REMARK 6.3. From the theorem above, we can replace the simple closed contour by any closed countour and drop the continuity of f' (See [**1**] for details) to obtain a more general theorem.

Theorem 6.4 (Cauchy-Goursat Theorem). *Let C be a closed contour in $\mathbb{C}$ and R the closed region inside and on C. If f is analytic on R, then*

$$\oint_C f(z)dz = 0.$$

Corollary 6.5. *If f is analytic on a simply connected domain Ω and C is a simple closed contour in Ω, then*

$$\oint_C f(z)dz = 0.$$

Corollary 6.6. *Let C be a simple closed contour and let $C_1, C_2, \ldots, C_n$ be simple closed contours in the region interior to C such that the regions interior to each C_i have no points in common. Let R be the closed region inside and on C except for points interior to each C_i. If f is analytic on R, then*

$$\oint_C f(z)dz = \Sigma_{i=1}^{n} \oint_{C_i} f(z)dz.$$

Proof. See [**1**] Section 36. □

Theorem 6.7 (Cauchy Integral Formula). *Let C be a simple closed contour in $\mathbb{C}$ and R the closed region inside and on C. If f is analytic on R, then for any $w \in Int(R)$, then for each $n = 0, 1, 2, \ldots,$ we have*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-w)^{n+1}} dz.$$

In particular,

$$f(w) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-w} dz$$

and

$$f'(w) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-w)^2} dz.$$

Proof. We will prove only the case $n = 0$. Let $\epsilon > 0$. Since $w \in Int(R)$ and f is continuous at w, there exists $r > 0$ such that $\overline{B(w,r)} \subseteq R$ and $|f(w) - f(z)| < \frac{\epsilon}{2\pi}$ for all $z \in \overline{B(w,r)}$ (can you see why?).

Let $C' = \partial \overline{B(w,r)}$ be represented by $z(t) = w + re^{it}, 0 \leq t \leq 2\pi$. Then we have

$$\oint_{C'} \frac{f(w)}{z-w} dz = f(w) \oint_{C'} \frac{1}{z-w} dz = f(w) \int_0^{2\pi} \left(\frac{1}{re^{it}} \right) ire^{it} dt = 2\pi i f(w),$$

and hence

$$\begin{aligned}\left|\oint_C \frac{f(z)}{z-w}dz - 2\pi i f(w)\right| &= \left|\oint_{C'} \frac{f(z)}{z-w}dz - 2\pi i f(w)\right| \text{ (by the previous corollary)}\\ &= \left|\oint_{C'} \frac{f(z)}{z-w}dz - \oint_{C'} \frac{f(w)}{z-w}dz\right|\\ &= \left|\oint_{C'} \left(\frac{f(z)-f(w)}{z-w}\right)dz\right|\\ &= \left|\int_0^{2\pi} \left(\frac{f(w+re^{it})-f(w)}{re^{it}}\right) ire^{it}dt\right|\\ &\leq \int_0^{2\pi} |(f(w+re^{it})-f(w))i|dt\\ &< \frac{\epsilon}{2\pi}\int_0^{2\pi} dt = \epsilon.\end{aligned}$$

Since ϵ is arbitrary, we must have $\oint_C \frac{f(z)}{z-w}dz = 2\pi i f(w)$. □

REMARK 6.8. From the previous theorem, we observe that if f is analytic on a domain Ω, then $f \in C^\infty(\Omega)$.

EXAMPLE 6.9. By letting $f(z) = 1$ for all $z \in \mathbb{C}$, $w = 0$ and $C = \partial B(0,1)$ in the above theorem, we immediately obtain $\oint_C \frac{1}{z}dz = 2\pi i$ and $\oint_C \frac{1}{z^n}dz = 0$ for all $n > 1$.

EXERCISE 6.10. *Show, by example, that Cauchy Integral Formula is not generally true for any closed contour.*

EXERCISE 6.11. *Let C be a simple closed curve such that 0 is inside C and 1 is outside C. Compute the following integrals.*

(1) $\oint_C \frac{e^z}{z}dz$.
(2) $\oint_C \frac{\sin z}{z^2(z-1)}dz$
(3) $\oint_C \frac{z^7-z^5+3z+1}{z^2-2z+1}dz$.

COROLLARY 6.12 (Mean Value Theorem). *If f is analytic on the closed disk $\overline{B(z_0,r)}$, we have*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})dt.$$

PROOF. Let $C = \partial B(z_0,r)$ be represented by $z(t) = z_0 + re^{it}$; $0 \leq t \leq 2\pi$. Then, by the above theorem, we have

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}} ire^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})dt.$$

□

COROLLARY 6.13 (Cauchy Estimate). *Let $f : \Omega \to \mathbb{C}$ be an analytic function, $z \in \Omega$ and $r > 0$ be such that $\overline{B(z,r)} \subseteq \Omega$. If f is bounded by M on $\partial B(z,r)$, then for any $n = 0, 1, 2, \ldots$, we have*

$$|f^{(n)}(z)| \leq \frac{n!M}{r^n}.$$

In particular, $|f'(z)| \leq \frac{M}{r}$.

Proof. Let $C = \partial B(z,r) \subseteq \Omega$. Then by CIF, we have

$$|f^{(n)}(z)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi\right| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(\xi)}{r^{n+1}}d\xi\right| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}}(2\pi r) = \frac{n!M}{r^n}.$$

□

Example 6.14. If $f : B(0,1) \to B(0,1)$ is analytic, then f is bounded by $M = 1$, $\overline{B(0,\frac{1}{2})} \subseteq B(0,1)$ and hence $|f'(0)| \le \frac{1!1}{(1/2)} \le 2$.

Exercise 6.15. *If $f : B(0,1) \to B(0,1)$ is analytic, prove that $|f'(\frac{1}{2})| \le 4$.*

Corollary 6.16 (Liouville's Theorem). *A bounded entire function must be constant.*

Proof. Suppose f is bounded by M on $\Omega = \mathbb{C}$. Then for any $r > 0$, the Cauchy estimate implies that

$$|f'(z)| \le \frac{M}{r}$$

for any $z \in \mathbb{C}$. Since r can be arbitrarily large, we must $f' = 0$ on $\mathbb{C}$; i.e., f is constant. □

Example 6.17. sin, cos and exp are all unbounded.

Example 6.18. Is the function $f(z) = z^2 \sin z$ bounded on $\mathbb{C}$? Justify your answer.

Corollary 6.19 (Fundamental Theorem of Algebra). *Every non-constant complex polynomial $P(z)$ must have a root in $\mathbb{C}$.*

Proof. Suppose $P(z)$ is a non-constant complex polynomial that has no root in $\mathbb{C}$. Then $f(z) = \frac{1}{P(z)}$ is entire. Moreover, it is easy to see that f is bounded on $\mathbb{C}$ because $\lim\limits_{|z|\to\infty} f(z) = 0$. It follows from Liouville's Theorem that f must be constant and so is P. This is a contradiction. □

Corollary 6.20 (Maximum Modulus Principle). *Let f be an analytic function on a domain Ω. If f attains the maximum modulus at some point in Ω, then f must be constant on Ω.*

Fact : If an analytic function is not constant on a domain Ω, then it is not constant on any open disk in Ω. [See [**1**] Section 103 Corollary 2]

Proof. Let f be an analytic function on a domain Ω. Suppose $|f|$ attains its maximum at $z_0 \in \Omega$. From the above fact, it suffices to find $r > 0$ so that f is constant on $B(z_0, r)$. Since Ω is open, there is $r > 0$ such that $B(z_0, r) \subseteq \Omega$.

Now, for $0 < \rho < r$, from $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})dt$, we have

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{it})|dt \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|dt = |f(z_0)|,$$

and hence

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{it})|dt.$$

It follows that

$$\int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{it})|)dt = 0.$$

Since $|f(z_0)|$ is maximum, the integrand is nonnegative and we must have

$$|f(z_0)| = |f(z_0 + \rho e^{it})|$$

for all $0 \leq t \leq 2\pi$. Therefore, $|f(z)| = |f(z_0)|$ for all $z \in B(z_0, r)$. Finally, the analyticity of f implies that $f(z)$ is the constant $f(z_0)$ for all $z \in B(z_0, r)$ as desired. $\square$

Corollary 6.21. *Let K be a compact subset of $\mathbb{C}$. If $f : K \to \mathbb{C}$ is a non-constant analytic function, then f attains the maximum modulus on ∂K.*

Proof. Since $|f|$ is continuous on a compact set K, it attains a maximum. However, since f is non-constant and analytic on K°, the maximum cannot occur in (any path component of) K° and hence it must be on the boundary of K. $\square$

Corollary 6.22 (Minimum Modulus Principle). *Let f be an analytic function on a domain Ω. Suppose further that $f(z) \neq 0$ for all $z \in \Omega$. If f attains the minimum modulus at some point in Ω, then f must be constant on Ω.*

Proof. Apply the maximum modulus principle to $\frac{1}{f}$. $\square$

Corollary 6.23. *Let K be a compact subset of $\mathbb{C}$. If $f : K \to \mathbb{C}$ is a non-constant analytic function with $f(z) \neq 0$ for all $z \in K^\circ$, then f attains the minimum modulus on ∂K.*

Example 6.24. Consider $f : A[0, 1, 2] \to \mathbb{C}$ defined by $f(z) = \frac{e^z}{z}$. Then f attains its maximum and minimum modulus (can you see why?) on $\partial A[0, \frac{1}{2}, 1]$. Let $z \in \partial A[0, \frac{1}{2}, 1] = \{re^{i\theta} : (r = 1 \text{ or } 2) \text{ and } 0 \leq \theta \leq 2\pi\}$, then

$$|f(z)| = |f(re^{i\theta})| = \left| \frac{e^{r(\cos\theta + i\sin\theta)}}{re^{i\theta}} \right| = \left| \frac{e^{rcos\theta}}{r} \right|,$$

where $(r = 1$ or $2)$ and $0 \leq \theta \leq 2\pi$. Clearly, $|f(z)|$ attains its maximum when $\theta = 0$ $(z = 1$ or $2)$ and its minimum when $\theta = -1$ $(z = -1$ or -2). Since

$$|f(-2)| = \frac{1}{2e^2} < |f(-1)| = \frac{1}{e} < |f(1)| = e < |f(2)| = \frac{e^2}{2},$$

then $|f|$ has the maximum value at $z = 2$ and the minimum value at $z = -2$.

Example 6.25. Find the maximum and minimum moduli of $f(z) = \frac{e^z}{z}$ on $A[0, \frac{1}{2}, 1]$.

CHAPTER 7

Sequences and Series of Complex Functions

In this chapter, we extend the notions of sequences and series to complex functions. However, for convenience, we will start our sequence from 0-th term.

DEFINITION 7.1. A sequence of complex functions is a function from $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to the set of all complex functions. As usual, a sequence will be written as $f_0, f_1, f_2, \dots$ or (f_n).

DEFINITION 7.2. A sequence (f_n) of complex functions **converges (pointwise)** to a complex function f on $A \subseteq \mathbb{C}$, written as $(f_n) \to f$, if $(f_n(z)) \to f(z)$ for all $z \in A$. Also, we say that (f_n) **converges uniformly** to f on A if for any $\epsilon > 0$, there is $N \in \mathbb{N}_0$ such that $|f(z) - f_n(z)| < \epsilon$ for any $n \geq N$ and $z \in A$.

EXAMPLE 7.3. For each $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, let $f_n(z) = \frac{z}{n+1}$ and $f(z) = 0$. Then the sequence (f_n) clearly converges pointwise to f. However, the convergence is not uniform on $\mathbb{C}$ because, for example when $\epsilon = 1$, $|f_n(n+1)| \geq \epsilon$ each $n \in \mathbb{N}$.

EXERCISE 7.4. *With (f_n) and f as above, prove that (f_n) converges uniformly to f on $B[0,1]$.*

THEOREM 7.5. *Suppose (f_n) is a sequence of continuous complex functions converging uniformly to f on $A \subseteq \mathbb{C}$. Then f is also continuous on A.*

PROOF. Let $z \in A$ and $\epsilon > 0$. By uniform convergence, there is $N \in \mathbb{N}_0$ such that $|f_n(w) - f(w)| < \frac{\epsilon}{3}$ for any $n \geq N$ and $w \in A$. In particular, $|f_N(w) - f(w)| < \frac{\epsilon}{3}$ for any $w \in A$. Now, since f^N is continuous at z, there exists $\delta > 0$ such that $|f^N(z) - f^N(w)| < \frac{\epsilon}{3}$ whenever $|z - w| < \delta$. Hence, for any $w \in A$ such that $|z - w| < \delta$, we have

$$|f(z) - f(w)| \leq |f(z) - f^N(z)| + |f^N(z) - f^N(w)| + |f^N(w) - f(w)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Therefore, f is continuous at z. $\square$

EXAMPLE 7.6. For each $n \in \mathbb{N}_0$ and $z \in [0,1]$, let $f_0(z) = 1$, $f_n(z) = z^n$ for $n \geq 1$ and

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z < 1, \\ 1 & \text{if } z = 1. \end{cases}$$

Then (f_n) clearly converges to f on $[0,1]$. However, the convergence is not uniform since f is not continuous.

THEOREM 7.7. *Let C be contour in $\mathbb{C}$ and suppose (f_n) is a sequence of continuous complex functions converging uniformly to f on C. Then*

$$\lim_{n\to\infty} \int_C f_n(z)dz = \int_C \lim_{n\to\infty} f_n(z)dz = \int_C f(z)dz.$$

Proof. Let $L > 0$ be the length of C and $\epsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}_0$ such that $|f(z) - f_n(z)| < \frac{\epsilon}{L}$ for all $n \geq N$ and $z \in C$; i.e., $M = \max\{|f(z) - f_n(z)| : z \in C\} < \frac{\epsilon}{L}$ whenever $n \geq N$. Thus, for each $n \geq N$, we have

$$\left|\int_C f(z)dz - \int_C f_n(z)dz\right| = \left|\int_C (f(z) - f_n(z))dz\right| \leq ML < \epsilon.$$

Therefore, $(\int_C f_n(z)dz) \to \int_C f(z)dz$ as desired. □

Theorem 7.8. *Suppose (f_n) be a sequence of analytic functions converging uniformly to f on any compact subset of a domain Ω. Then f is analytic on Ω and*

$$f^{(k)}(z) = \lim_{n\to\infty} f_n^{(k)}(z),$$

for each $k \geq 0$ and $z \in \Omega$.

Proof. Let $z_0 \in \Omega$. Since Ω is open, there is $r > 0$ such that $B(z_0, r) \subseteq \Omega$. Let C be any closed contour in $B(z_0, r)$. Since C is compact, (f_n) converges uniformly to f on C by assumption. Now for each n, since f_n is analytic on Ω, we also have $\oint_C f_n(z)dz = 0$ by CIF. Hence, it follows from the previous theorem that

$$\oint_C f(z)dz = \lim_{n\to\infty} \oint_C f_n(z)dz = 0.$$

Since C is arbitrary, f has an antiderivative on $B(z_0, r)$, says F, by Theorem 5.10. Therefore, f must be analytic at z_0.

Now, for each $k \geq 0$, it is not difficult to verify that the sequence $\left(\frac{f_n(z)}{(z-z_0)^{k+1}}\right)$ converges uniformly to $\frac{f(z)}{(z-z_0)^{k+1}}$ on the simple closed curve $\partial B(z_0, \frac{r}{2})$, and hence by CIF and the previous theorem, we have

$$\lim_{n\to\infty} f_n^{(k)}(z_0) = \frac{k!}{2\pi i} \lim_{n\to\infty} \oint_{\partial B(z_0,\frac{r}{2})} \frac{f_n(z)}{(z-z_0)^{k+1}}dz = \oint_{\partial B(z_0,\frac{r}{2})} \frac{f(z)}{(z-z_0)^{k+1}}dz = f^{(k)}(z_0)$$

as desired. □

Definition 7.9. For a sequence (g_n) of complex functions, we call the sequence (s_n), where $s_n = \sum_{i=0}^n g_n$, a **series** of complex functions and denote it by $\sum g_n$ or $\sum_{n=0}^\infty g_n$ or $g_0 + g_1 + g_2 + \dots$. The convergence of $\sum g_n$ is simply the convergence of the sequence (s_n).

Theorem 7.10. *Let (g_n) be a sequence of continuous complex functions on a domain Ω.*

(1) *If the series Σg_n converges uniformly to a complex function g on a contour $C \subseteq \Omega$, then g is continuous on C and*

$$\sum_{n=0}^\infty \int_C g_n(z)dz = \int_C g(z)dz.$$

(2) *If each g_n is analytic and the series Σg_n converges uniformly to a complex function g on each compact subset of Ω, then g is analytic and*

$$\sum_{n=0}^\infty g_n^{(k)}(z) = g^{(k)}(z)$$

for any $z \in \Omega$ and $k \in \mathbb{N}$.

Proof. Follows directly from the previous theorems. □

Theorem 7.11 (Weierstrass M-test). *Let (g_n) be a sequence of complex functions on A. If there is a sequence (M_n) of nonnegative real numbers such that*

(1) *$|g_n(z)| \le M_n$ for all $z \in A$ and $n \in \mathbb{N}$ and*
(2) *the series $\sum M_n$ converges,*

then $\sum g_n$ converges uniformly on A. Moreover, for each $z \in A$, the series $\sum g_n(z)$ converges absolutely.

Proof. Let $\epsilon > 0$. Since ΣM_n converges, it is Cauchy and there is $N \in \mathbb{N}$ such that for any $n \ge m \ge N$,

$$\sum_{i=m+1}^{n} M_i = \left|\sum_{i=m+1}^{n} M_i\right| = \left|\sum_{i=0}^{n} M_i - \sum_{i=0}^{m} M_i\right| < \epsilon.$$

Then for any $z \in A$ and $n \ge m \ge N$, we clearly have

$$\left|\sum_{i=0}^{n} g_i(z) - \sum_{i=0}^{m} g_i(z)\right| = \left|\sum_{i=m+1}^{n} g_i(z)\right| \le \sum_{i=m+1}^{n} |g_i(z)| \le \sum_{i=m+1}^{n} M_i < \epsilon \quad \dots(*)$$

It follows that $(\sum g_n(z))$ is Cauchy and hence converges, says to $g(z)$. With ϵ and N as above, by letting $m \to \infty$ in (*), we also have

$$\left|\sum_{i=0}^{n} g_i(z) - g(z)\right| < \epsilon$$

for all $z \in A$ and $n \ge N$. Therefore, the convergence is uniform. Moreover, for each $z \in A$, $\sum g_n(z)$ converges absolutely by both conditions above and the comparison test. □

Definition 7.12 (Power Series). A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

where $z_0, a_0, a_1, a_2, \dots \in \mathbb{C}$. We will call z_0 the **center** and $a_0, a_1, a_2, \dots$ the **coefficients** of the series.

Clearly $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at $z = z_0$. In fact, two obvious possibilities for the convergence of the power series are

(1) the series converges only at $z = z_0$, or
(2) the series converges for all $z \in \mathbb{C}$.

Lemma 7.13. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series.*

(1) *If the series converges at z_1, it also converges at each $z \in B(z_0, |z_1 - z_0|)$.*
(2) *If the series diverges at z_2, it also diverges at each $z \notin B[z_0, |z_2 - z_0|]$.*

Proof. For (1), suppose the series converges at z_1. Let $z \in B(z_0, |z_1 - z_0|)$ and $r = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$. Since $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges, we have $(|a_n(z_1-z_0)^n|) \to 0$ and hence is bounded, says by M. Then,

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \le Mr^n$$

for all n. Since $\sum Mr^n$ converges, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges by comparison test.

For (2), suppose the series diverges at z_2. If the series converges at some $w \notin B[z_0, |z_2 - z_0|]$, then by (1), the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z \in B(z_0, |w-z_0|)$. This clearly implies the convergence of the series at z_2 which is a contradiction. $\square$

THEOREM 7.14. *For a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, if it does not satisfy any of the above possibilities of convergence, then there exists $R > 0$ such that the series converges for each $z \in B(z_0, R)$ and diverges for each $z \notin B[z_0, R]$.*

PROOF. By assumption, there exist $z_1 \in \mathbb{C} - \{z_0\}$ and $z_2 \in \mathbb{C}$ such that the series converges at z_1 and diverges at z_2. Then, by the previous lemma, the series also converges at each $z \in B(z_0, |z_1 - z_0|)$ and diverges at each $z \notin B[z_0, |z_2 - z_0|]$.

Now, the set

$$S = \{s : \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges for all } z \in B(z_0, s)\} \subseteq \mathbb{R}_0^+$$

is a nonempty (since $|z_1 - z_0| \in S$). It is also easy to see that S is bounded above (since $s \notin S$ for all $s > |z_2 - z_0|$). Let $R = \sup S$. It is easy to verify that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z \in B(z_0, R)$. Moreover, for $z \notin B[z_0, R]$, we must have $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges because otherwise $\sum_{n=0}^{\infty} a_n(\xi - z_0)^n$ converges for all $\xi \in B(z_0, |z-z_0|)$ and hence $|z - z_0| \in S$. This is certainly a contradiction since $|z - z_0| > R$. $\square$

DEFINITION 7.15. The real number R in the above theorem is called the **radius of convergence** of the power series. We can extend the definition of R to include the other 2 possibilities of convergence as well by letting

- $R = 0$ if the series converges only at $z = z_0$, and
- $R = \infty$ if the series converges for all $z \in \mathbb{C}$.

EXAMPLE 7.16. The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is ∞.

EXERCISE 7.17. *Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} z^n$ is 1 and*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

for all $z \in B(0, 1)$.

THEOREM 7.18. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series converging to a function f on some open ball $B(z_0, r)$. Then for each $0 < r' < r$, the series converges uniformly to f on $B[z_0, r']$.*

PROOF. Pick $z_1 \in B(z_0, r)$ such that $|z_1 - z_0| > r'$. Since $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges, the sequence $(a_n(z_1 - z_0)^n)$ is bounded and hence there exist $M \in \mathbb{R}^+$ and $N \in \mathbb{N}_0$ such that $|a_n(z_1 - z_0)^n| \le M$ whenever $n \ge N$. Let $\rho = \left|\frac{r'}{z_1 - z_0}\right| < 1$. Now, for each $z \in B[z_0, r']$, we clearly have

(1) $|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \le M \left|\frac{r'}{z_1-z_0}\right|^n = M\rho^n$ for all $n \ge N$,

(2) $\sum_{n=0}^{\infty} M\rho^n$ converges.

Then, by Weierstrass M-test, the series converges uniformly on $B[z_0, r']$. $\square$

Exercise 7.19. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series converging to a function f on some open ball $B(z_0, r)$. Prove that f also is analytic on $B(z_0, r)$ and for each $k \in \mathbb{N}_0$ and $z \in B(z_0, r)$, we have*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k}.$$

Corollary 7.20. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series converging to a function f on some open ball $B(z_0, r)$. Then, for each $n \in \mathbb{N}_0$, we have*

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proof. By the previous exercise, for each $k \in \mathbb{N}_0$, we have

$$f^{(k)}(z_0) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z_0-z_0)^{n-k} = k!a_k.$$

Hence, $a_n = \frac{f^(n)(z_0)}{n!}$ for all $n \in \mathbb{N}_0$ as desired. $\square$

Corollary 7.21 (Uniqueness). *If the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge to the same function f some open ball $B(z_0, r)$, then the two series must be the same; i.e., $a_n = b_n$ for all $n \in \mathbb{N}_0$.*

Proof. By the previous corollary, we immediately have $a_n = \frac{f^(n)(z_0)}{n!} = b_n$ for all $n \in \mathbb{N}_0$. $\square$

Exercise 7.22. *Let $z_0, z, \xi \in \mathbb{C}$ be such that $|z-z_0| < |\xi - z_0|$. Then we have*

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}.$$

Theorem 7.23 (Taylor Theorem). *If f is an analytic function on some open ball $B(z_0, r)$, then f can be uniquely represented by the power series (so-called the Taylor series of f around z_0)*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

on $B(z_0, r)$.

Proof. Suppose f is an analytic function on some open ball $B(z_0, r)$. Let $z \in B(z_0, r)$ and $r' > 0$ such that $z \in B[z_0, r'] \subseteq B(z_0, r)$. Then, by using the previous exercise, it is not difficult to verify that the series $\sum_{n=0}^{\infty} \frac{f(\xi)(z-z_0)^n}{(\xi-z_0)^{n+1}}$ converges uniformly to $\frac{f(\xi)}{\xi - z}$ for $\xi \in \partial B(z_0, r')$. Hence, by CIF and Theorem 7.10(1), we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{\partial B(z_0,r')} \frac{f(\xi)}{\xi - z} d\xi \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_{\partial B(z_0,r')} \frac{f(\xi)(z-z_0)^n}{(\xi - z_0)^{n+1}} d\xi \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\partial B(z_0,r')} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi\right)(z-z_0)^n \\
&= \sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n.
\end{aligned}$$

□

REMARK 7.24. If $f(z) = \sum_{i=0}^{n} a_i z^i$ is a complex polynomail, then $\sum_{i=0}^{n} a_i z^i$ itself is the Taylor series of $f(z)$ on any open ball around 0.

EXERCISE 7.25. *Find the Taylor series around 1 of* $f(z) = z$.

EXAMPLE 7.26. Some well-known Taylor series around $z_0 = 0$:

(1) $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\ ;\ z \in \mathbb{C}$.
(2) $\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\ ;\ z \in \mathbb{C}$.
(3) $\mathrm{Log}(z+1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} z^{n+1}\ ;\ |z| < 1$.
(4) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots\ ;\ |z| < 1$.

EXAMPLE 7.27. Find the Taylor series of $f(z) = \frac{1}{z^2}$ around 1.

First, notice that $f(z) = \frac{1}{z^2}$ is analytic on $B(1,1)$ and hence it be represented by a unique Taylor series on that ball.

Since

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

for all $z \in B(1,1)$, then by differentiation, we obtain

$$f(z) = \frac{1}{z^2} = -\frac{d}{dz}(\frac{1}{z}) = \sum_{n=1}^{\infty}(-1)^{n-1}n(z-1)^{n-1}$$

for all $z \in B(1,1)$, which is certainly the desired Taylor series of $f(z)$ around 1.

EXAMPLE 7.28. Find the Taylor series of $f(z) = \frac{z}{z^2-z-2}$ around 1.

First, notice that $f(z) = \frac{z}{z^2-z-2} = \frac{1/3}{z+1} + \frac{2/3}{z-2}$ is analytic on $B(1,1)$ and hence it be represented by a unique Taylor series on that ball.

Since

$$\frac{1}{z+1} = \frac{1}{2-(1-z)} = \frac{1}{2}\left(\frac{1}{1-(\frac{1-z}{2})}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{1-z}{2})^n = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{1}{2})^n(z-1)^n$$

for all $z \in B(1,2)$, and

$$\frac{1}{z-2} = \frac{1}{-1-(1-z)} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n$$

for all $z \in B(1,1)$, then we have

$$f(z) = \frac{z}{z^2 - z + 2} = \sum_{n=0}^{\infty} \left[(\frac{1}{6})(-\frac{1}{2})^n - \frac{2}{3} \right] (z-1)^n$$

for all $z \in B(1,1)$, which is certainly the desired Taylor series of $f(z)$ around 1.

EXERCISE 7.29. *Find the Taylor series of the following functions :*

(1) $f(z) = e^z$ *around 1.*
(2) $f(z) = \sin z$ *around* i.
(3) $f(z) = \frac{1}{z^2+z+1}$ *around 0.*
(4) $f(z) = \frac{e^z}{1-z} \sin z$ *around 0.*

DEFINITION 7.30. A **Laurent series** is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

The above series converges if both $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ and $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges.

THEOREM 7.31 (Laurent Theorem). *If f is an analytic function on some open annulus $A(z_0, r_1, r_2)$, then f can be uniquely represented by a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

on $A(z_0, r_1, r_2)$, where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \quad ; \; n \in \mathbb{Z}$$

and C is any simple closed contour (positively oriented) around z_0 in $A(z_0, r_1, r_2)$. Moreover, the convergence is uniform on any closed annulus $A[z_0, r_1', r_2']$ where $r_1 < r_1' < r_2' < r_2$. Hence, on $A(z_0, r_1, r_2)$, differentiation and contour integration of the Laurent series of f can be done term by term.

PROOF. See [**1**]. □

EXAMPLE 7.32. The Laurent series of $f(z)$ on A :

(1) $f(z) = \frac{z}{z-1}$ on $A = A(1, 0, \infty)$.

Notice that

$$f(z) = \frac{z}{z-1} = \frac{z-1+1}{z-1} = 1 + \frac{1}{z-1}$$

for all $z \in A(0, 0, \infty)$. Then $1 + \frac{1}{z-1}$ is the Laurent series of $f(z)$ on $A(1, 0, \infty)$.

(2) $f(z) = e^{\frac{1}{z}}$ on $A = A(0, 0, \infty) = \mathbb{C}^*$.

Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for all $z \in \mathbb{C}$, then

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$$

is the Laurent series of $f(z)$ on $A(0, 0, \infty)$.

(3) $f(z) = \frac{1}{1-z}$ on $A = A(0, 1, \infty)$.

Notice that

$$f(z) = \frac{1}{1-z} = \frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{-1}{z^n}$$

whenever $\left|\frac{1}{z}\right| < 1$. Then $\sum_{n=1}^{\infty}\frac{-1}{z^n}$ is the Laurent series of $f(z)$ on $A(0, 1, \infty)$.

(4) $f(z) = \frac{1}{(1-z)^2}$ on $A = A(0, 1, \infty)$.

By differentiating the Laurent series of $\frac{1}{1-z}$ from the previous example, we have

$$f(z) = \frac{1}{(1-z)^2} = -\frac{d}{dz}(\frac{1}{1-z}) = -\sum_{n=1}^{\infty}\frac{-n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{n}{z^{n+1}}$$

on $A(0, 1, \infty)$.

Example 7.33. Find the Laurent series of $f(z) = \frac{3}{(1+z)(2-z)} = \frac{1}{1+z} + \frac{1}{2-z}$ on each of the following annuli : $A(0, 0, 1)$, $A(0, 1, 2)$ and $A(0, 2, \infty)$.

First, notice that

(1) On $B(0, 1)$: $\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-1)^n z^n$.

(2) On $A(0, 1, \infty)$: $\frac{1}{1+z} = \frac{1}{z}\left(\frac{1}{1-(-\frac{1}{z})}\right) = \frac{1}{z}\sum_{n=0}^{\infty}(-\frac{1}{z})^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$.

(3) On $B(0, 2)$: $\frac{1}{2-z} = \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{2})^n z^n = \sum_{n=0}^{\infty}(\frac{1}{2})^{n+1} z^n$.

(4) On $A(0, 2, \infty)$: $\frac{1}{2-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) = -\frac{1}{z}\sum_{n=0}^{\infty}(\frac{2}{z})^n = \sum_{n=0}^{\infty}\frac{-2^n}{z^{n+1}}$.

Hence,

(1) On $A(0, 0, 1) \subseteq B(0, 1)$:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^n + \sum_{n=0}^{\infty}(\frac{1}{2})^{n+1} z^n = \sum_{n=0}^{\infty}[(-1)^n + (\frac{1}{2})^{n+1}]z^n.$$

(2) On $A(0, 1, 2)$:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}(\frac{1}{2})^{n+1} z^n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{z^n} + \sum_{n=0}^{\infty}(\frac{1}{2})^{n+1} z^n.$$

(3) On $A(0, 2, \infty)$:

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{-2^n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1} - 2^{n-1}}{z^n}.$$

Exercise 7.34. *Find the Laurent series of the following functions :*

(1) $f(z) = \frac{e^{\frac{1}{z}}}{z^2}$ *on* $A(0, 0, \infty)$.

(2) $f(z) = \mathrm{Log}(1 + \frac{1}{z})$ *on* $A(0, 1, \infty)$.

(3) $f(z) = \frac{1}{(z-3)(z-4)}$ *on* $A(0, 0, 3)$.

(4) $f(z) = \frac{1}{(z-3)(z-4)}$ *on* $A(0, 3, 4)$.

(5) $f(z) = \frac{1}{(z-3)(z-4)}$ *on* $A(0, 4, \infty)$.

CHAPTER 8

Singularity

DEFINITION 8.1. Let $z_0 \in \mathbb{C}$ and f a complex function. We say that z_0 is a **singularity** of f if f is not defined or not differentiable at z_0. A singularity z_0 is said to be **isolated** if f is analytic on an annulus $A(z_0, 0, r)$ for some $r > 0$.

DEFINITION 8.2. Let z_0 be an isolated singularity of f. Suppose the Laurent series $f(z)$ on some annulus $A(z_0, 0, r)$ around z_0 is $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$. We say that z_0 is

- **a removable singularity** if $a_n = 0$ for all $n < 0$,
- **a pole of order** $N \in \mathbb{N}$ if $a_{-N} \neq 0$ and $a_n = 0$ for all $n < -N$,
- **an essential singularity** if for any $n < 0$, there is $m < n$ such that $a_m \neq 0$.

A pole of order 1 is always called a **simple pole**.

EXAMPLE 8.3. Each of the following functions has only one isolated singularity at 0, however

(1) $f(z) = \frac{\sin z}{z}$ has a removable singularity at 0,
(2) $f(z) = \frac{1}{z}$ has a pole of order 1 at 0,
(3) $f(z) = e^{\frac{1}{z}}$ has an essential singularity at 0.

LEMMA 8.4. *If f has a removable singularity at z_0, then f can be extended to an analytic function on some open ball around z_0.*

PROOF. Suppose f has a removable singularity at z_0. Then, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

on some annulus $A(z_0, 0, r)$. By defining $g : B(0, r) \to \mathbb{C}$ by

$$g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

it is clear that g is the desired analytic extension of f on $B(0, r)$. $\square$

THEOREM 8.5. *Let z_0 be an isolated singularity of f. TFAE :*

(1) *z_0 is a removable singularity of f.*
(2) *f is bounded on some annulus $A(z_0, 0, r)$.*
(3) *$\lim_{z \to z_0} f(z)$ exists.*

DEFINITION 8.6. Suppose f is analytic at z_0 and $f(z_0) = 0$. We say that z_0 is a **zero of order** N of f if the Taylor series of f around z_0 is of the form $\sum_{n=N}^{\infty} a_n(z - z_0)^n$ with $a_N \neq 0$.

Example 8.7. 0 is a zero of order 2 of $f(z) = \sin^2 z$ because

$$f(z) = (z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots) = z^2 - \frac{2}{3!}z^3 + \dots$$

for all $z \in \mathbb{C}$.

Theorem 8.8. *Let z_0 be an isolated singularity of f. TFAE :*

(1) *z_0 is a pole of order $N > 0$ of f.*
(2) *$f(z) = \frac{g(z)}{(z-z_0)^N}$ where g is analytic at z_0 and $g(z_0) \neq 0$.*
(3) *$\lim_{z\to z_0} f(z) = \infty$ and $\lim_{z\to z_0} (z-z_0)^N f(z) \neq 0$.*
(4) *z_0 is a zero of order N of $\frac{1}{f}$.*

Example 8.9. Let $f(z) = \frac{e^z}{(z-1)^2(z-3)^4}$. Then 2 is a pole of order 2 while 3 is a pole of order 4 of f. Also note that $f(z) = \frac{g(z)}{(z-1)^2}$ where $g(z) = \frac{e^z}{(z-3)^4}$ is analytic at 1 and $g(1) \neq 0$.

Example 8.10. Since 0 is a zero of order 2 of $\sin^2 z$, then by the previous theorem, $f(z) = \frac{1}{\sin^2 z}$ has a pole of order 2 at 0.

Theorem 8.11. *Let z_0 be an isolated singularity of f. If z_0 is an essential singularity of f, then for any $w \in \mathbb{C}$, there exists a sequence $(z_n) \to z_0$ such that*

$$\lim_{n\to\infty} f(z_n) = w.$$

Definition 8.12. Let z_0 be an isolated singularity of f, says f is analytic on some annulus $A(z_0, 0, r)$. Suppose the Laurent series of f on $A(z_0, 0, r)$ is $\sum_{n=0}^{\infty} a_n (z-z_0)^n$. We define the **residue of f at** z_0 by

$$Res(f, z_0) = a_{-1}.$$

Theorem 8.13. *If z_0 is a pole of order N of f, then*

$$Res(f, z_0) = \frac{1}{(N-1)!} \lim_{z\to z_0} \frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)].$$

In particular, if z_0 is a simple pole of f, we have

$$Res(f, z_0) = \lim_{z\to z_0} (z-z_0) f(z).$$

Proof. Suppose $f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n$ where $a_N \neq 0$ on some annulus $A(z_0, 0, r)$. Then, it is easy to verify that

$$\lim_{z\to z_0} \frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)] = (N-1)! a_{-1}$$

and hence

$$Res(f, z_0) = a_{-1} = \frac{1}{(N-1)!} \lim_{z\to z_0} \frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)].$$

□

Example 8.14. Let $f(z) = \frac{e^z}{(z^2+1)z^2}$. Clearly, $0, i$ and $-i$ are poles of order 2,1 and 1, respectively, of f. Hence,

$$Res(f, 0) = \lim_{z\to 0} \frac{d}{dz}[z^2 f(z)] = \lim_{z\to 0} \frac{d}{dz}[\frac{e^z}{z^2+1}] = \lim_{z\to 0} \frac{(z^2+1)e^z - 2ze^z}{(z^2+1)^2} = 1,$$

$$Res(f,i) = \lim_{z\to i}[(z-i)f(z)] = \lim_{z\to i}\frac{e^z}{(z+i)z^2} = -\frac{e^i}{2i}$$

and

$$Res(f,-i) = \lim_{z\to -i}[(z+i)f(z)] = \lim_{z\to -i}\frac{e^z}{(z-i)z^2} = \frac{e^{-i}}{2i}.$$

EXERCISE 8.15. *Show that* $f(z) = \frac{e^z}{\sin^2 z}$ *has poles of order 2 at* $0, \pm\pi, \pm 2\pi, \ldots$ *and* $Res(f,\pi) = e^\pi$.

Suppose f has an isolated singularity at z_0. Then Laurent theorem guarantees that f has the Laurent series, says $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, on some annulus $A(z_0,0,r)$ where f is analytic. Then, for any (positively oriented) simple closed contour $C \subseteq A(z_0,0,r)$ around z_0, we clearly have

$$\oint_C f(z)dz = 2\pi i a_{-1} = 2\pi i Res(f,z_0).$$

In fact, we have a more general theorem whose proof is straightforward.

THEOREM 8.16 (Residue Theorem). *Let C be a (positively oriented) simple closed contour in the domain of f and R the region inside and on C. Suppose f has singularities at $z_1, z_2, \ldots, z_n \in Int(R)$ and f is analytic on $R - \{z_1, z_2, \ldots, z_n\}$. Then we have*

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f,z_k).$$

PROOF. By assumption, we can write

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$

where C_i is a (positively oriented) simple closed contour in $Int(R)$ such that z_k is the only singularity of f inside C_k. Then, by the previous observation, we have $\oint_{C_k} f(z)dz = 2\pi i Res(f,z_k)$ and hence the theorem follows. □

EXAMPLE 8.17. Let $C = \partial B(\pi,1)$ (positively oriented). Since π is the only singularity of $f(z) = \frac{e^z}{\sin^2 z}$ in $B(\pi,1)$, we have

$$\oint_C f(z)dz = 2\pi i Res(f,0) = 2\pi i e^\pi.$$

EXERCISE 8.18. *Find the following integrals :*

(1) $\oint_{\partial B(0,10)} \frac{e^z}{\sin^2 z}dz$.
(2) $\oint_{\partial B(0,2)} \frac{e^z}{(z^2+1)z^2}dz$.

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