Chapter Six

More Integration

6.1. Cauchy's Integral Formula. Suppose f is analytic in a region containing a simple closed contour C with the usual positive orientation and its inside , and suppose z_0 is inside C. Then it turns out that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

This is the famous **Cauchy Integral Formula**. Let's see why it's true.

Let $\varepsilon > 0$ be any positive number. We know that f is continuous at z_0 and so there is a number δ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Now let $\rho > 0$ be a number such that $\rho < \delta$ and the circle $C_0 = \{z : |z - z_0| = \rho\}$ is also inside C. Now, the function $\frac{f(z)}{z-z_0}$ is analytic in the region between C and C_0; thus

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_0} \frac{f(z)}{z - z_0} dz.$$

We know that $\int_{C_0} \frac{1}{z-z_0} dz = 2\pi i$, so we can write

$$\begin{aligned} \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) &= \int_{C_0} \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_0} \frac{1}{z - z_0} dz \\ &= \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz. \end{aligned}$$

For $z \epsilon C_0$ we have

$$\begin{aligned} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| &= \frac{|f(z) - f(z_0)|}{|z - z_0|} \\ &\leq \frac{\varepsilon}{\rho}. \end{aligned}$$

Thus,

$$\left| \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| = \left| \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz \right|$$

$$\leq \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon.$$

But ε is *any* positive number, and so

$$\left| \int_{C_0} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| = 0,$$

or,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} dz,$$

which is exactly what we set out to show.

Meditate on this result. It says that if f is analytic on and inside a simple closed curve and we know the values $f(z)$ for every z on the simple closed curve, then we know the value for the function at every point inside the curve—quite remarkable indeed.

Example

Let C be the circle $|z| = 4$ traversed once in the counterclockwise direction. Let's evaluate the integral

$$\int_C \frac{\cos z}{z^2 - 6z + 5} dz.$$

We simply write the integrand as

$$\frac{\cos z}{z^2 - 6z + 5} = \frac{\cos z}{(z-5)(z-1)} = \frac{f(z)}{z-1},$$

where

$$f(z) = \frac{\cos z}{z - 5}.$$

Observe that f is analytic on and inside C, and so,

$$\int_C \frac{\cos z}{z^2 - 6z + 5} dz = \int_C \frac{f(z)}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i \frac{\cos 1}{1-5} = -\frac{i\pi}{2}\cos 1$$

Exercises

1. Suppose f and g are analytic on and inside the simple closed curve C, and suppose moreover that $f(z) = g(z)$ for all z on C. Prove that $f(z) = g(z)$ for all z inside C.

2. Let C be the ellipse $9x^2 + 4y^2 = 36$ traversed once in the counterclockwise direction. Define the function g by

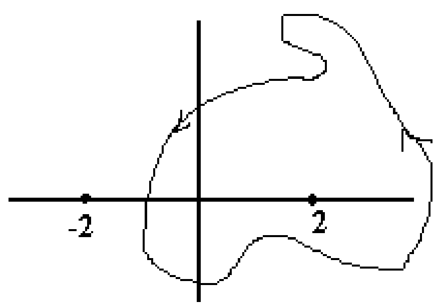
$$g(z) = \int_C \frac{s^2 + s + 1}{s - z} ds.$$

Find a) $g(i)$ b) $g(4i)$

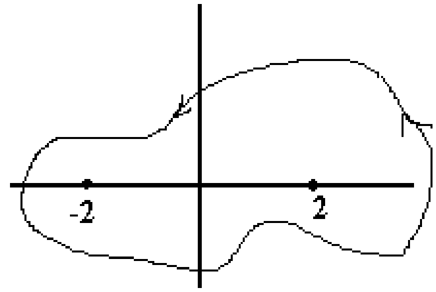
3. Find

$$\int_C \frac{e^{2z}}{z^2 - 4} dz,$$

where C is the closed curve in the picture:



4. Find $\int_\Gamma \frac{e^{2z}}{z^2-4} dz$, where Γ is the contour in the picture:



6.2. Functions defined by integrals. Suppose C is a curve (not necessarily a simple closed curve, just a curve) and suppose the function g is continuous on C (not necessarily analytic, just continuous). Let the function G be defined by

$$G(z) = \int_C \frac{g(s)}{s-z} ds$$

for all $z \notin C$. We shall show that G is analytic. Here we go.

Consider,

$$\begin{aligned} \frac{G(z+\Delta z) - G(z)}{\Delta z} &= \frac{1}{\Delta z} \int_C \left[\frac{1}{s-z-\Delta z} - \frac{1}{s-z} \right] g(s) ds \\ &= \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)} ds. \end{aligned}$$

Next,

$$\begin{aligned} \frac{G(z+\Delta z) - G(z)}{\Delta z} - \int_C \frac{g(s)}{(s-z)^2} ds &= \int_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2} \right] g(s) ds \\ &= \int_C \left[\frac{(s-z) - (s-z-\Delta z)}{(s-z-\Delta z)(s-z)^2} \right] g(s) ds \\ &= \Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds. \end{aligned}$$

Now we want to show that

$$\lim_{\Delta z \to 0} \left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds \right] = 0.$$

To that end, let $M = \max\{|g(s)| : s \in C\}$, and let d be the shortest distance from z to C. Thus, for $s \in C$, we have $|s - z| \geq d > 0$ and also

$$|s - z - \Delta z| \geq |s - z| - |\Delta z| \geq d - |\Delta z|.$$

Putting this all together, we can estimate the integrand above:

$$\left| \frac{g(s)}{(s-z-\Delta z)(s-z)^2} \right| \leq \frac{M}{(d - |\Delta z|)d^2}$$

for all $s \in C$. Finally,

$$\left| \Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds \right| \leq |\Delta z| \frac{M}{(d - |\Delta z|)d^2} \text{length}(C),$$

and it is clear that

$$\lim_{\Delta z \to 0} \left[\Delta z \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)^2} ds \right] = 0,$$

just as we set out to show. Hence G has a derivative at z, and

$$G'(z) = \int_C \frac{g(s)}{(s-z)^2} ds.$$

Truly a miracle!

Next we see that G' has a derivative and it is just what you think it should be. Consider

$$\begin{aligned}\frac{G'(z+\Delta z)-G'(z)}{\Delta z} &= \frac{1}{\Delta z}\int_C\left[\frac{1}{(s-z-\Delta z)^2}-\frac{1}{(s-z)^2}\right]g(s)ds \\ &= \frac{1}{\Delta z}\int_C\left[\frac{(s-z)^2-(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds \\ &= \frac{1}{\Delta z}\int_C\left[\frac{2(s-z)\Delta z-(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds \\ &= \int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}\right]g(s)ds\end{aligned}$$

Next,

$$\begin{aligned}&\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C\frac{g(s)}{(s-z)^3}ds \\ &= \int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right]g(s)ds \\ &= \int_C\left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z-\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds \\ &= \int_C\left[\frac{2(s-z)^2-\Delta z(s-z)-2(s-z)^2+4\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds \\ &= \int_C\left[\frac{3\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds\end{aligned}$$

Hence,

$$\begin{aligned}\left|\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C\frac{g(s)}{(s-z)^3}ds\right| &= \left|\int_C\left[\frac{3\Delta z(s-z)-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}\right]g(s)ds\right| \\ &\leq |\Delta z|\frac{(|3m|+2|\Delta z|)M}{(d-\Delta z)^2d^3},\end{aligned}$$

where $m = \max\{|s-z| : s \in C\}$. It should be clear then that

$$\lim_{\Delta z\to 0}\left|\frac{G'(z+\Delta z)-G'(z)}{\Delta z}-2\int_C\frac{g(s)}{(s-z)^3}ds\right| = 0,$$

or in other words,

$$G''(z) = 2\int_C \frac{g(s)}{(s-z)^3}ds.$$

Suppose f is analytic in a region D and suppose C is a positively oriented simple closed curve in D. Suppose also the inside of C is in D. Then from the Cauchy Integral formula, we know that

$$2\pi i f(z) = \int_C \frac{f(s)}{s-z}ds$$

and so with $g = f$ in the formulas just derived, we have

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds, \text{ and } f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}ds$$

for all z inside the closed curve C. Meditate on these results. They say that the derivative of an analytic function is also analytic. Now suppose f is continuous on a domain D in which every point of D is an interior point and suppose that $\int_C f(z)dz = 0$ for every closed curve in D. Then we know that f has an antiderivative in D—in other words f is the derivative of an analytic function. We now know this means that f is itself analytic. We thus have the celebrated **Morera's Theorem:**

If $f{:}D \to \mathbf{C}$ is continuous and such that $\int_C f(z)dz = 0$ for every closed curve in D, then f is analytic in D.

Example

Let's evaluate the integral

$$\int_C \frac{e^z}{z^3}dz,$$

where C is any positively oriented closed curve around the origin. We simply use the equation

$$f''(z) = \frac{2}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}ds$$

with $z = 0$ and $f(s) = e^s$. Thus,

$$\pi i e^0 = \pi i = \int_C \frac{e^z}{z^3} dz.$$

Exercises

5. Evaluate

$$\int_C \frac{\sin z}{z^2} dz$$

where C is a positively oriented closed curve around the origin.

6. Let C be the circle $|z - i| = 2$ with the positive orientation. Evaluate

a) $\int_C \frac{1}{z^2+4} dz$ b) $\int_C \frac{1}{(z^2+4)^2} dz$

7. Suppose f is analytic inside and on the simple closed curve C. Show that

$$\int_C \frac{f'(z)}{z-w} dz = \int_C \frac{f(z)}{(z-w)^2} dz$$

for every $w \notin C$.

8. a) Let α be a real constant, and let C be the circle $\gamma(t) = e^{it}$, $-\pi \le t \le \pi$. Evaluate

$$\int_C \frac{e^{\alpha z}}{z} dz.$$

b) Use your answer in part a) to show that

$$\int_0^\pi e^{\alpha \cos t} \cos(\alpha \sin t) dt = \pi.$$

6.3. Liouville's Theorem. Suppose f is entire and bounded; that is, f is analytic in the entire plane and there is a constant M such that $|f(z)| \le M$ for all z. Then it must be true that $f'(z) = 0$ identically. To see this, suppose that $f'(w) \ne 0$ for some w. Choose R large enough to insure that $\frac{M}{R} < |f'(w)|$. Now let C be a circle centered at 0 and with radius

$\rho > \max\{R, |w|\}$. Then we have :

$$\frac{M}{\rho} < |f'(w)| \leq \left| \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-w)^2} ds \right|$$
$$\leq \frac{1}{2\pi} \frac{M}{\rho^2} 2\pi\rho = \frac{M}{\rho},$$

a contradiction. It must therefore be true that there is no w for which $f'(w) \neq 0$; or, in other words, $f'(z) = 0$ for all z. This, of course, means that f is a constant function. What we have shown has a name, **Liouville's Theorem:**

The only bounded entire functions are the constant functions.

Let's put this theorem to some good use. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial. Then

$$p(z) = \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right) z^n.$$

Now choose R large enough to insure that for each $j = 1, 2, \ldots, n$, we have $\left| \frac{a_{n-j}}{z^j} \right| < \frac{|a_n|}{2n}$ whenever $|z| > R$. (We are assuming that $a_n \neq 0$.) Hence, for $|z| > R$, we know that

$$|p(z)| \geq \left| |a_n| - \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right| \right| |z|^n$$
$$\geq \left| |a_n| - \left| \frac{a_{n-1}}{z} \right| - \left| \frac{a_{n-2}}{z^2} \right| - \ldots - \left| \frac{a_0}{z^n} \right| \right| |z|^n$$
$$> \left| |a_n| - \frac{|a_n|}{2n} - \frac{|a_n|}{2n} - \ldots - \frac{|a_n|}{2n} \right| |z|^n$$
$$> \frac{|a_n|}{2} |z|^n.$$

Hence, for $|z| > R$,

$$\frac{1}{|p(z)|} < \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n| R^n}.$$

Now suppose $p(z) \neq 0$ for all z. Then $\frac{1}{p(z)}$ is also bounded on the disk $|z| \leq R$. Thus, $\frac{1}{p(z)}$ is a bounded entire function, and hence, by Liouville's Theorem, constant! Hence the polynomial is constant if it has no zeros. In other words, if $p(z)$ is of degree at least one, there must be at least one z_0 for which $p(z_0) = 0$. This is, of course, the celebrated

Fundamental Theorem of Algebra.

Exercises

9. Suppose f is an entire function, and suppose there is an M such that $\mathrm{Re} f(z) \leq M$ for all z. Prove that f is a constant function.

10. Suppose w is a solution of $5z^4 + z^3 + z^2 - 7z + 14 = 0$. Prove that $|w| \leq 3$.

11. Prove that if p is a polynomial of degree n, and if $p(a) = 0$, then $p(z) = (z-a)q(z)$, where q is a polynomial of degree $n-1$.

12. Prove that if p is a polynomial of degree $n \geq 1$, then

$$p(z) = c(z-z_1)^{k_1}(z-z_2)^{k_2}\ldots(z-z_j)^{k_j},$$

where $k_1, k_2, \ldots, k_j$ are positive integers such that $n = k_1 + k_2 + \ldots + k_j$.

13. Suppose p is a polynomial with real coefficients. Prove that p can be expressed as a product of linear and quadratic factors, each with real coefficients.

6.4. Maximum moduli. Suppose f is analytic on a closed domain D. Then, being continuous, $|f(z)|$ must attain its maximum value somewhere in this domain. Suppose this happens at an interior point. That is, suppose $|f(z)| \leq M$ for all $z \in D$ and suppose that $|f(z_0)| = M$ for some z_0 in the interior of D. Now z_0 is an interior point of D, so there is a number R such that the disk Λ centered at z_0 having radius R is included in D. Let C be a positively oriented circle of radius $\rho \leq R$ centered at z_0. From Cauchy's formula, we know

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z_0}ds.$$

Hence,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})dt,$$

and so,

$$M = |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + \rho e^{it})|dt \leq M.$$

since $|f(z_0 + \rho e^{it})| \leq M$. This means

$$M = \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + \rho e^{it})|dt.$$

Thus,

$$M - \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + \rho e^{it})|dt = \frac{1}{2\pi}\int_0^{2\pi}[M - |f(z_0 + \rho e^{it})|]dt = 0.$$

This integrand is continuous and non-negative, and so must be zero. In other words, $|f(z)| = M$ for all $z \in C$. There was nothing special about C except its radius $\rho \leq R$, and so we have shown that f must be constant on the disk Λ.

I hope it is easy to see that if D is a region (=connected and open), then the only way in which the modulus $|f(z)|$ of the analytic function f can attain a maximum on D is for f to be constant.

Exercises

14. Suppose f is analytic and not constant on a region D and suppose $f(z) \neq 0$ for all $z \in D$. Explain why $|f(z)|$ does not have a minimum in D.

15. Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic on a region D. Prove that if $u(x,y)$ attains a maximum value in D, then u must be constant.