Chapter Ten

Poles, Residues, and All That

10.1. Residues. A point z_0 is a **singular point** of a function f if f not analytic at z_0, but is analytic at some point of each neighborhood of z_0. A singular point z_0 of f is said to be *isolated* if there is a neighborhood of z_0 which contains no singular points of f save z_0. In other words, f is analytic on some region $0 < |z - z_0| < \varepsilon$.

Examples

The function f given by

$$f(z) = \frac{1}{z(z^2+4)}$$

has isolated singular points at $z = 0$, $z = 2i$, and $z = -2i$.

Every point on the negative real axis and the origin is a singular point of $\operatorname{Log} z$, but there are *no* isolated singular points.

Suppose now that z_0 is an isolated singular point of f . Then there is a Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j$$

valid for $0 < |z - z_0| < R$, for some positive R. The coefficient c_{-1} of $(z - z_0)^{-1}$ is called the **residue** of f at z_0, and is frequently written

$$\operatorname*{Res}_{z=z_0} f.$$

Now, why do we care enough about c_{-1} to give it a special name? Well, observe that if C is any positively oriented simple closed curve in $0 < |z - z_0| < R$ and which contains z_0 inside, then

$$c_{-1} = \frac{1}{2\pi i} \int_C f(z)dz.$$

This provides the key to evaluating many complex integrals.

Example

We shall evaluate the integral

$$\int_C e^{1/z} dz$$

where C is the circle $|z| = 1$ with the usual positive orientation. Observe that the integrand has an isolated singularity at $z = 0$. We know then that the value of the integral is simply $2\pi i$ times the residue of $e^{1/z}$ at 0. Let's find the Laurent series about 0. We already know that

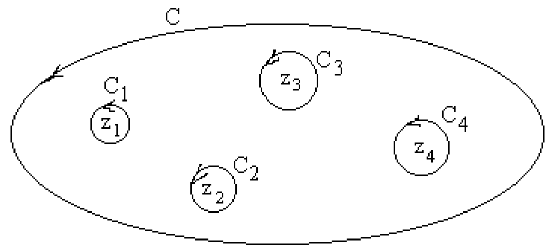
$$e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$$

for all z. Thus,

$$e^{1/z} = \sum_{j=0}^{\infty} \frac{1}{j!} z^{-j} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \ldots$$

The residue $c_{-1} = 1$, and so the value of the integral is simply $2\pi i$.

Now suppose we have a function f which is analytic everywhere except for isolated singularities, and let C be a simple closed curve (positively oriented) on which f is analytic. Then there will be only a finite number of singularities of f inside C (why?). Call them z_1, z_2, $\ldots$, z_n. For each $k = 1, 2, \ldots, n$, let C_k be a positively oriented circle centered at z_k and with radius small enough to insure that it is inside C and has no other singular points inside it.



Then,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz$$
$$= 2\pi i \operatorname*{Res}_{z=z_1} f + 2\pi i \operatorname*{Res}_{z=z_2} f + \ldots + 2\pi i \operatorname*{Res}_{z=z_n} f$$
$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f.$$

This is the celebrated **Residue Theorem**. It says that the integral of f is simply $2\pi i$ times the sum of the residues at the singular points enclosed by the contour C.

Exercises

Evaluate the integrals. In each case, C is the positively oriented circle $|z| = 2$.

1. $\int_C e^{1/z^2} dz.$

2. $\int_C \sin(\frac{1}{z}) dz.$

3. $\int_C \cos(\frac{1}{z}) dz.$

4. $\int_C \frac{1}{z} \sin(\frac{1}{z}) dz.$

5. $\int_C \frac{1}{z} \cos(\frac{1}{z}) dz.$

10.2. Poles and other singularities. In order for the Residue Theorem to be of much help in evaluating integrals, there needs to be some better way of computing the residue—finding the Laurent expansion about each isolated singular point is a chore. We shall now see that in the case of a special but commonly occurring type of singularity the residue is easy to find. Suppose z_0 is an isolated singularity of f and suppose that the Laurent series of f at z_0 contains only a finite number of terms involving negative powers of $z - z_0$. Thus,

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \ldots.$$

Multiply this expression by $(z-z_0)^n$:

$$\phi(z) = (z-z_0)^n f(z) = c_{-n} + c_{-n+1}(z-z_0) + \ldots + c_{-1}(z-z_0)^{n-1} + \ldots.$$

What we see is the Taylor series at z_0 for the function $\phi(z) = (z-z_0)^n f(z)$. The coefficient of $(z-z_0)^{n-1}$ is what we seek, and we know that this is

$$\frac{\phi^{(n-1)}(z_0)}{(n-1)!}$$

.

The sought after residue c_{-1} is thus

$$c_{-1} = \operatorname*{Res}_{z=z_0} f = \frac{\phi^{(n-1)}(z_0)}{(n-1)!},$$

where $\phi(z) = (z-z_0)^n f(z)$.

Example

We shall find all the residues of the function

$$f(z) = \frac{e^z}{z^2(z^2+1)}.$$

First, observe that f has isolated singularities at 0, and $\pm i$. Let's see about the residue at 0. Here we have

$$\phi(z) = z^2 f(z) = \frac{e^z}{(z^2+1)}.$$

The residue is simply $\phi'(0)$:

$$\phi'(z) = \frac{(z^2+1)e^z - 2ze^z}{(z^2+1)^2}.$$

Hence,

$$\operatorname*{Res}_{z=0} f = \phi'(0) = 1.$$

Next, let's see what we have at $z = i$:

$$\phi(z) = (z-i)f(z) = \frac{e^z}{z^2(z+i)},$$

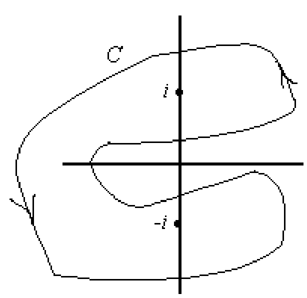
and so

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = -\frac{e^i}{2i}.$$

In the same way, we see that

$$\operatorname*{Res}_{z=-i} f = \frac{e^{-i}}{2i}.$$

Let's find the integral $\int_C \frac{e^z}{z^2(z^2+1)} dz$, where C is the contour pictured:



This is now easy. The contour is positive oriented and encloses two singularities of f; *viz,* i and $-i$. Hence,

$$\int_C \frac{e^z}{z^2(z^2+1)}dz = 2\pi i\left[\operatorname*{Res}_{z=i} f + \operatorname*{Res}_{z=-i} f\right]$$
$$= 2\pi i\left[-\frac{e^i}{2i} + \frac{e^{-i}}{2i}\right]$$
$$= -2\pi i \sin 1.$$

Miraculously easy!

There is some jargon that goes with all this. An isolated singular point z_0 of f such that the Laurent series at z_0 includes only a finite number of terms involving negative powers of $z - z_0$ is called a **pole**. Thus, if z_0 is a pole, there is an integer n so that $\phi(z) = (z-z_0)^n f(z)$ is analytic at z_0, and $f(z_0) \neq 0$. The number n is called the **order** of the pole. Thus, in the preceding example, 0 is a pole of order 2, while i and $-i$ are poles of order 1. (A pole of order 1 is frequently called a **simple pole.**) We must hedge just a bit here. If z_0 is an isolated singularity of f and there are *no* Laurent series terms involving negative powers of $z - z_0$, then we say z_0 is a **removable** singularity.

Example

Let

$$f(z) = \frac{\sin z}{z};$$

then the singularity $z = 0$ is a removable singularity:

$$f(z) = \frac{1}{z}\sin z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

and we see that in some sense f is ”really” analytic at $z = 0$ if we would just define it to be the right thing there.

A singularity that is neither a pole or removable is called an **essential** singularity.

Let’s look at one more labor-saving trick—or technique, if you prefer. Suppose f is a function:

$$f(z) = \frac{p(z)}{q(z)},$$

where p and q are analytic at z_0, and we have $q(z_0) = 0$, while $q'(z_0) \neq 0$, and $p(z_0) \neq 0$. Then

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z_0) + p'(z_0)(z - z_0) + \ldots}{q'(z_0)(z - z_0) + \frac{q''(z_0)}{2}(z - z_0)^2 \ldots},$$

and so

$$\phi(z) = (z - z_0)f(z) = \frac{p(z_0) + p'(z_0)(z - z_0) + \ldots}{q'(z_0) + \frac{q''(z_0)}{2}(z - z_0) + \ldots}.$$

Thus z_0 is a simple pole and

$$\operatorname*{Res}_{z=z_0} f = \phi(z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Example

Find the integral

$$\int_C \frac{\cos z}{e^z - 1} dz,$$

where C is the rectangle with sides $x = \pm 1$, $y = -\pi$, and $y = 3\pi$.
The singularities of the integrand are all the places at which $e^z = 1$, or in other words, the points $z = 0, \pm 2\pi i, \pm 4\pi i, \ldots$. The singularities enclosed by C are 0 and $2\pi i$. Thus,

$$\int_C \frac{\cos z}{e^z - 1} dz = 2\pi i \left[\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=2\pi i} f \right],$$

where

$$f(z) = \frac{\cos z}{e^z - 1}.$$

Observe this is precisely the situation just discussed: $f(z) = \frac{p(z)}{q(z)}$, where p and q are analytic, *etc.,etc.* Now,

$$\frac{p(z)}{q'(z)} = \frac{\cos z}{e^z}.$$

Thus,

$$\operatorname*{Res}_{z=0} f = \frac{\cos 0}{1} = 1, \text{and}$$
$$\operatorname*{Res}_{z=2\pi i} f = \frac{\cos 2\pi i}{e^{2\pi i}} = \frac{e^{-2\pi} + e^{2\pi}}{2} = \cosh 2\pi.$$

Finally,

$$\int_C \frac{\cos z}{e^z - 1} dz = 2\pi i \left[\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=2\pi i} f \right]$$
$$= 2\pi i(1 + \cosh 2\pi)$$

Exercises

6. Suppose f has an isolated singularity at z_0. Then, of course, the derivative f' also has an isolated singularity at z_0. Find the residue $\operatorname*{Res}_{z=z_0} f'$.

7. Given an example of a function f with a simple pole at z_0 such that $\operatorname*{Res}_{z=z_0} f = 0$, or explain carefully why there is no such function.

8. Given an example of a function f with a pole of order 2 at z_0 such that $\operatorname*{Res}_{z=z_0} f = 0$, or explain carefully why there is no such function.

9. Suppose g is analytic and has a zero of order n at z_0 (That is, $g(z) = (z - z_0)^n h(z)$, where $h(z_0) \neq 0$.). Show that the function f given by

$$f(z) = \frac{1}{g(z)}$$

has a pole of order n at z_0. What is $\operatorname*{Res}_{z=z_0} f$?

10. Suppose g is analytic and has a zero of order n at z_0. Show that the function f given by

$$f(z) = \frac{g'(z)}{g(z)}$$

has a simple pole at z_0, and $\operatorname*{Res}_{z=z_0} f = n$.

11. Find

$$\int_C \frac{\cos z}{z^2 - 4} dz,$$

where C is the positively oriented circle $|z| = 6$.

12. Find

$$\int_C \tan z dz,$$

where C is the positively oriented circle $|z| = 2\pi$.

13. Find

$$\int_C \frac{1}{z^2 + z + 1} dz,$$

where C is the positively oriented circle $|z| = 10$.