CHAPTER

7 Transcendental Functions

7.1 Concepts Review

1. $\int_1^x \frac{1}{t}\,dt$; $(0,\infty)$; $(-\infty,\infty)$

2. $\frac{1}{x}$

3. $\frac{1}{x}$; $\ln|x|+C$

4. $\ln x+\ln y$; $\ln x-\ln y$; $r\ln x$

Problem Set 7.1

1. a. $\ln 6=\ln(2\cdot 3)=\ln 2+\ln 3$
$=0.693+1.099=1.792$

b. $\ln 1.5=\ln\left(\frac{3}{2}\right)=\ln 3-\ln 2=1.099-0.693$
$=0.406$

c. $\ln 81=\ln 3^4=4\ln 3=4(1.099)=4.396$

d. $\ln\sqrt{2}=\ln 2^{1/2}=\frac{1}{2}\ln 2=\frac{1}{2}(0.693)=0.3465$

e. $\ln\left(\frac{1}{36}\right)=-\ln 36=-\ln(2^2\cdot 3^2)$
$=-2\ln 2-2\ln 3=-2(0.693)-2(1.099)$
$=-3.584$

f. $\ln 48=\ln(2^4\cdot 3)=4\ln 2+\ln 3$
$=4(0.693)+1.099=3.871$

2. a. 1.792 b. 0.405

c. 4.394 d. 0.3466

e. −3.584 f. 3.871

3. $D_x\ln(x^2+3x+\pi)$
$=\frac{1}{x^2+3x+\pi}\cdot D_x(x^2+3x+\pi)$
$=\frac{2x+3}{x^2+3x+\pi}$

4. $D_x\ln(3x^3+2x)=\frac{1}{3x^3+2x}D_x(3x^3+2x)$
$=\frac{9x^2+2}{3x^3+2x}$

5. $D_x\ln(x-4)^3=D_x 3\ln(x-4)$
$=3\cdot\frac{1}{x-4}D_x(x-4)=\frac{3}{x-4}$

6. $D_x\ln\sqrt{3x-2}=D_x\frac{1}{2}\ln(3x-2)$
$=\frac{1}{2}\cdot\frac{1}{3x-2}D_x(3x-2)=\frac{3}{2(3x-2)}$

7. $\frac{dy}{dx}=3\cdot\frac{1}{x}=\frac{3}{x}$

8. $\frac{dy}{dx}=x^2\cdot\frac{1}{x}+2x\cdot\ln x=x(1+2\ln x)$

9. $z=x^2\ln x^2+(\ln x)^3=x^2\cdot 2\ln x+(\ln x)^3$
$\frac{dz}{dx}=x^2\cdot\frac{2}{x}+2x\cdot 2\ln x+3(\ln x)^2\cdot\frac{1}{x}$
$=2x+4x\ln x+\frac{3}{x}(\ln x)^2$

10. $r=\frac{\ln x}{x^2\ln x^2}+\left(\ln\frac{1}{x}\right)^3=\frac{\ln x}{x^2\cdot 2\ln x}+(-\ln x)^3$
$=\frac{1}{2}x^{-2}-(\ln x)^3$
$\frac{dr}{dx}=\frac{-2}{2}x^{-3}-3(\ln x)^2\cdot\frac{1}{x}=-\frac{1}{x^3}-\frac{3(\ln x)^2}{x}$

11. $g'(x) = \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x\right]$

$= \frac{1}{\sqrt{x^2+1}}$

12. $h'(x) = \frac{1}{x+\sqrt{x^2-1}}\left[1+\frac{1}{2}(x^2-1)^{-1/2}\cdot 2x\right]$

$= \frac{1}{\sqrt{x^2-1}}$

13. $f(x) = \ln\sqrt[3]{x} = \frac{1}{3}\ln x$

$f'(x) = \frac{1}{3}\cdot\frac{1}{x} = \frac{1}{3x}$

$f'(81) = \frac{1}{3\cdot 81} = \frac{1}{243}$

14. $f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

$f'\left(\frac{\pi}{2}\right) = -\tan\left(\frac{\pi}{2}\right)$, which is undefined.

15. Let $u = 2x + 1$ so $du = 2\,dx$.

$\int\frac{1}{2x+1}dx = \frac{1}{2}\int\frac{1}{u}du$

$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x+1| + C$

16. Let $u = 1 - 2x$ so $du = -2dx$.

$\int\frac{1}{1-2x}dx = -\frac{1}{2}\int\frac{1}{u}du$

$= -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|1-2x| + C$

17. Let $u = 3v^2 + 9v$ so $du = 6v + 9$.

$\int\frac{6v+9}{3v^2+9v}dv = \int\frac{1}{u}du = \ln|u| + C$

$= \ln\left|3v^2+9v\right| + C$

18. Let $u = 2z^2 + 8$ so $du = 4z\,dz$.

$\int\frac{z}{2z^2+8}dz = \frac{1}{4}\int\frac{1}{u}du$

$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left|2z^2+8\right| + C$

19. Let $u = \ln x$ so $du = \frac{1}{x}dx$

$\int\frac{2\ln x}{x}dx = 2\int u\,du$

$= u^2 + C = (\ln x)^2 + C$

20. Let $u = \ln x$, so $du = \frac{1}{x}dx$.

$\int\frac{-1}{x(\ln x)^2}dx = -\int u^{-2}du$

$= \frac{1}{u} + C = \frac{1}{\ln x} + C$

21. Let $u = 2x^5 + \pi$ so $du = 10x^4dx$.

$\int\frac{x^4}{2x^5+\pi}dx = \frac{1}{10}\int\frac{1}{u}du$

$= \frac{1}{10}\ln|u| + C = \frac{1}{10}\ln\left|2x^5+\pi\right| + C$

$\int_0^3\frac{x^4}{2x^5+\pi}dx = \left[\frac{1}{10}\ln\left|2x^5+\pi\right|\right]_0^3$

$= \frac{1}{10}[\ln(486+\pi) - \ln\pi] = \ln\sqrt[10]{\frac{486+\pi}{\pi}}$

22. Let $u = 2t^2 + 4t + 3$ so $du = (4t+4)dt$.

$\int\frac{t+1}{2t^2+4t+3}dt = \frac{1}{4}\int\frac{1}{u}du$

$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left|2t^2+4t+3\right| + C$

$\int_0^1\frac{t+1}{2t^2+4t+3}dt = \left[\frac{1}{4}\ln\left|2t^2+4t+3\right|\right]_0^1$

$= \frac{1}{4}\ln 9 - \frac{1}{4}\ln 3 = \ln\sqrt[4]{\frac{9}{3}} = \ln\sqrt[4]{3} = \frac{1}{4}\ln 3$

23. $2\ln(x+1) - \ln x = \ln(x+1)^2 - \ln x$

$= \ln\frac{(x+1)^2}{x}$

24. $\frac{1}{2}\ln(x-9) + \frac{1}{2}\ln x = \ln\sqrt{x-9} - \ln\sqrt{x}$

$= \ln\frac{\sqrt{x-9}}{\sqrt{x}} = \ln\sqrt{\frac{x-9}{x}}$

25. $\ln(x-2) - \ln(x+2) + 2\ln x$

$= \ln(x-2) - \ln(x+2) + \ln x^2$

$= \ln\frac{x^2(x-2)}{x+2}$

26. $\ln(x^2-9)-2\ln(x-3)-\ln(x+3)$

$=\ln(x^2-9)-\ln(x-3)^2-\ln(x+3)$

$=\ln\dfrac{x^2-9}{(x-3)^2(x+3)}=\ln\dfrac{1}{x-3}$

27. $\ln y=\ln(x+11)-\frac{1}{2}\ln(x^3-4)$

$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x+11}\cdot 1-\frac{1}{2}\cdot\frac{1}{x^3-4}\cdot 3x^2$

$=\frac{1}{x+11}-\frac{3x^2}{2(x^3-4)}$

$\frac{dy}{dx}=y\cdot\left[\frac{1}{x+11}-\frac{3x^2}{2(x^3-4)}\right]$

$=\frac{x+11}{\sqrt{x^3-4}}\left[\frac{1}{x+11}-\frac{3x^2}{2(x^3-4)}\right]$

$=-\frac{x^3+33x^2+8}{2(x^3-4)^{3/2}}$

28. $\ln y=\ln(x^2+3x)+\ln(x-2)+\ln(x^2+1)$

$\frac{1}{y}\frac{dy}{dx}=\frac{2x+3}{x^2+3x}+\frac{1}{x-2}+\frac{2x}{x^2+1}$

$\frac{dy}{dx}=(x^2+3x)(x-2)(x^2+1)\left(\frac{2x+3}{x^2+3x}+\frac{1}{x-2}+\frac{2x}{x^2+1}\right)=5x^4+4x^3-15x^2+2x-6$

29. $\ln y=\frac{1}{2}\ln(x+13)-\ln(x-4)-\frac{1}{3}\ln(2x+1)$

$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2(x+13)}-\frac{1}{x-4}-\frac{2}{3(2x+1)}$

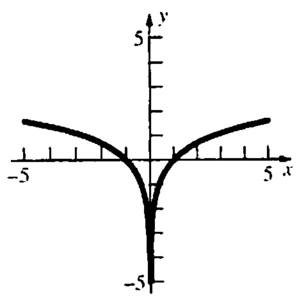
$\frac{dy}{dx}=\frac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}}\left[\frac{1}{2(x+13)}-\frac{1}{x-4}-\frac{2}{3(2x+1)}\right]=-\frac{10x^2+219x-118}{6(x-4)^2(x+13)^{1/2}(2x+1)^{4/3}}$

30. $\ln y=\frac{2}{3}\ln(x^2+3)+2\ln(3x+2)-\frac{1}{2}\ln(x+1)$

$\frac{1}{y}\frac{dy}{dx}=\frac{2}{3}\cdot\frac{2x}{x^2+3}+\frac{2\cdot 3}{3x+2}-\frac{1}{2(x+1)}$

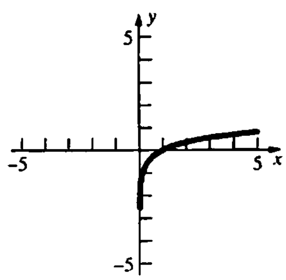
$\frac{dy}{dx}=\frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\left[\frac{4x}{3(x^2+3)}+\frac{6}{3x+2}-\frac{1}{2(x+1)}\right]=\frac{(3x+2)(51x^3+70x^2+97x+90)}{6(x^2+3)^{1/3}(x+1)^{3/2}}$

31.



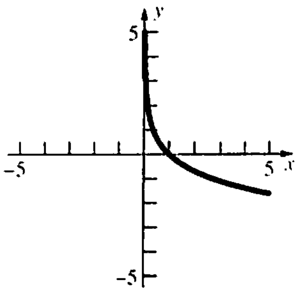
$y=\ln x$ is reflected across the y-axis.

32.



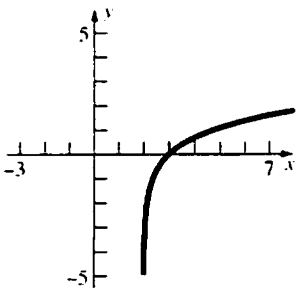
The y-values of $y=\ln x$ are multiplied by $\frac{1}{2}$, since $\ln\sqrt{x}=\frac{1}{2}\ln x$.

33.



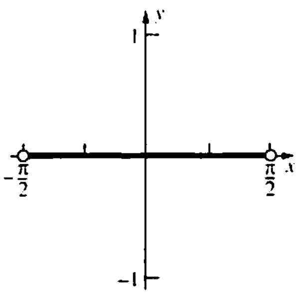
$y = \ln x$ is reflected across the x-axis since $\ln\left(\frac{1}{x}\right) = -\ln x$.

34.



$y = \ln x$ is shifted two units to the right.

35.



$$y = \ln \cos x + \ln \sec x$$
$$= \ln \cos x + \ln \frac{1}{\cos x}$$
$$= \ln \cos x - \ln \cos x = 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

36. Since ln is continuous,

$$\lim_{x\to 0} \ln \frac{\sin x}{x} = \ln \lim_{x\to 0} \frac{\sin x}{x} = \ln 1 = 0$$

37. $f'(x) = 4x \ln x + 2x^2\left(\frac{1}{x}\right) - 2x = 4x \ln x$

so $f(1) = -1$ is a minimum.

38. Let $r(x)$ = rate of transmission

$$= kx^2 \ln \frac{1}{x} = -kx^2 \ln x.$$

$$r'(x) = -2kx \ln x - kx^2\left(\frac{1}{x}\right) = -kx(2 \ln x + 1)$$

$r'(x) = 0$ if $\ln x = -\frac{1}{2}$, or $-\ln x = \frac{1}{2}$, so

$$\ln \frac{1}{x} = \frac{1}{2}.$$

$$\ln 1.65 \approx \frac{1}{2}, \text{ so } x \approx \frac{1}{1.65} \approx 0.606.$$

$$r''(x) = -k(2 \ln x + 1) - kx\left(2 \cdot \frac{1}{x}\right) = -k(2 \ln x + 3)$$

$r''(0.606) \approx -2k < 0$ since $k > 0$, so $x \approx 0.606$ gives the maximum rate of transmission.

39. $\ln 4 > 1$

so $\ln 4^m = m \ln 4 > m \cdot 1 = m$

Thus $x > 4^m \Rightarrow \ln x > m$

so $\lim_{x\to\infty} \ln x = \infty$

40. Let $z = \frac{1}{x}$ so $z \to \infty$ as $x \to 0^+$

Then $\lim_{x\to 0^+} \ln x = \lim_{z\to\infty} \ln\left(\frac{1}{z}\right) = \lim_{z\to\infty} (-\ln z)$

$= -\lim_{z\to\infty} \ln z = -\infty$

41. $\int_{1/3}^{x} \frac{1}{t} dt = 2\int_{1}^{x} \frac{1}{t} dt$

$\int_{1/3}^{1} \frac{1}{t} dt + \int_{1}^{x} \frac{1}{t} dt = 2\int_{1}^{x} \frac{1}{t} dt$

$\int_{1/3}^{1} \frac{1}{t} dt = \int_{1}^{x} \frac{1}{t} dt$

$-\int_{1}^{1/3} \frac{1}{t} dt = \int_{1}^{x} \frac{1}{t} dt$

$-\ln \frac{1}{3} = \ln x$

$\ln 3 = \ln x$

$x = 3$

42. a. $\frac{1}{t} < \frac{1}{\sqrt{t}}$ for $t > 1$.

so $\ln x = \int_1^x \frac{1}{t} dt < \int_1^x \frac{1}{\sqrt{t}} dt = \int_1^x t^{-1/2} dt$

$= \left[2\sqrt{t}\right]_1^x = 2(\sqrt{x} - 1)$

so $\ln x < 2(\sqrt{x} - 1)$

b. If $x > 1$, $0 < \ln x < 2(\sqrt{x} - 1)$,

so $0 < \frac{\ln x}{x} < \frac{2(\sqrt{x} - 1)}{x}$.

Hence $0 \le \lim_{x\to\infty} \frac{\ln x}{x} \le \lim_{x\to\infty} \frac{2(\sqrt{x}+1)}{x} = 0$

and $\lim_{x\to\infty} \frac{\ln x}{x} = 0.$

43. $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$

$= \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]\cdot\frac{1}{n}$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\cdot\frac{1}{n} = \int_1^2 \frac{1}{x}dx = \ln 2 \approx 0.693$

44. $\frac{1,000,000}{\ln 1,000,000} \approx 72,382$

45. a. $f(x) = \ln\left(\frac{ax-b}{ax+b}\right)^c = c\ln\left(\frac{ax-b}{ax+b}\right)$

$= \frac{a^2-b^2}{2ab}[\ln(ax-b)-\ln(ax+b)]$

$f'(x) = \frac{a^2-b^2}{2ab}\left[\frac{a}{ax-b}-\frac{a}{ax+b}\right]$

$= \frac{a^2-b^2}{2ab}\left[\frac{2ab}{(ax-b)(ax+b)}\right] = \frac{a^2-b^2}{a^2x^2-b^2}$

$f'(1) = \frac{a^2-b^2}{a^2-b^2} = 1$

b. $f'(x) = \cos^2 u \cdot \frac{du}{dx}$

$= \cos^2[\ln(x^2+x-1)]\cdot\frac{2x+1}{x^2+x-1}$

$f'(1) = \cos^2[\ln(1^2+1-1)]\cdot\frac{2\cdot 1+1}{1^2+1-1}$

$= 3\cos^2(0) = 3$

46. $\int_0^{\pi/3} \tan x\,dx = \int_0^{\pi/3}\frac{\sin x}{\cos x}dx$

Let $u = \cos x$ so $du = -\sin x\,dx$.

$\int \frac{\sin x}{\cos x}dx = -\int\frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$

$\int_0^{\pi/3}\tan x\,dx = [-\ln|\cos x|]_0^{\frac{\pi}{3}}$

$= -\ln\left|\cos\frac{\pi}{3}\right| + \ln|\cos 0| = -\ln\frac{1}{2} + \ln 1 = \ln 2 \approx 0.693$

47. $V = 2\pi\int_1^4 xf(x)dx = \int_1^4 \frac{2\pi x}{x^2+4}dx$

Let $u = x^2+4$ so $du = 2x\,dx$.

$\int \frac{2\pi x}{x^2+4}dx = \pi\int\frac{1}{u}du = \pi\ln|u| + C$

$= \pi\ln\left|x^2+4\right| + C$

$\int_1^4 \frac{2\pi x}{x^2+4}dx = \left[\pi\ln\left|x^2+4\right|\right]_1^4$

$= \pi\ln 20 - \pi\ln 5 = \pi\ln 4 \approx 4.355$

48. $y = \frac{x^2}{4} - \ln\sqrt{x} = \frac{x^2}{4} - \frac{1}{2}\ln x$

$\frac{dy}{dx} = \frac{2x}{4} - \frac{1}{2}\cdot\frac{1}{x} = \frac{x}{2} - \frac{1}{2x}$

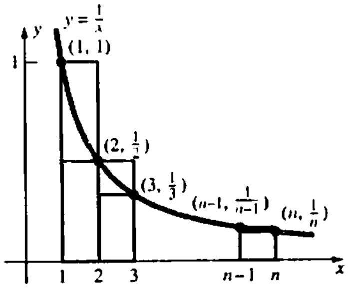
$L = \int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_1^2\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}dx$

$= \int_1^2\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}dx = \int_1^2\left(\frac{x}{2}+\frac{1}{2x}\right)dx$

$= \frac{1}{2}\left[\frac{x^2}{2}+\ln|x|\right]_1^2 = \frac{1}{2}\left[2+\ln 2-\left(\frac{1}{2}+\ln 1\right)\right]$

$= \frac{3}{4}+\frac{1}{2}\ln 2 \approx 1.097$

49.



$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} =$ the lower approximate area

$1+\frac{1}{2}+\cdots+\frac{1}{n-1} =$ the upper approximate area

$\ln n =$ the exact area under the curve

Thus,

$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{6} < \ln n < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}.$

50. $\frac{\ln y - \ln x}{y-x} = \frac{\int_1^y \frac{1}{t}dt - \int_1^x \frac{1}{t}dt}{y-x} = \frac{\int_x^y \frac{1}{t}dt}{y-x}$

= the average value of $\frac{1}{t}$ on $[x, y]$.

Since $\frac{1}{t}$ is decreasing on the interval $[x, y]$, the average value is between the minimum value of $\frac{1}{y}$ and the maximum value of $\frac{1}{x}$.

51. a. $f'(x)=\frac{1}{1.5+\sin x}\cdot\cos x=\frac{\cos x}{1.5+\sin x}$

$f'(x)=0$ when $\cos x=0$.

Critical points: $0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, 3\pi$

$f(0)\approx 0.405$,

$f\left(\frac{\pi}{2}\right)\approx 0.916, f\left(\frac{3\pi}{2}\right)\approx -0.693$,

$f\left(\frac{5\pi}{2}\right)\approx 0.916, f(3\pi)\approx 0.405$.

On $[0,3\pi]$, the maximum value points are $\left(\frac{\pi}{2},0.916\right),\left(\frac{5\pi}{2},0.916\right)$ and the minimum value point is $\left(\frac{3\pi}{2},-0.693\right)$.

b. $f''(x)=-\frac{1+1.5\sin x}{(1.5+\sin x)^2}$

On $[0,3\pi]$, $f''(x)=0$ when $x\approx 3.871$, 5.553.

Inflection points are (3.871, –0.182), (5.553, –0.183).

c. $\int_0^{3\pi}\ln(1.5+\sin x)dx\approx 4.042$

52. a. $f'(x)=-\frac{\sin(\ln x)}{x}$

On [0.1, 20], $f'(x)=0$ when $x=1$.

Critical points: 0.1, 1, 20

$f(0.1)\approx -0.668, f(1)=1, f(20)\approx -0.989$

On [0.1, 20], the maximum value point is (1, 1) and minimum value point is (20, –0.989).

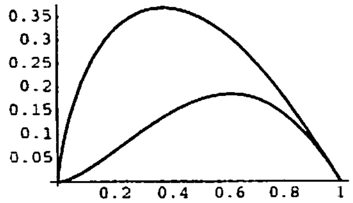
b. On [0.01, 0.1], $f'(x)=0$ when $x\approx 0.043$.

$f(0.01)\approx -0.107, f(0.043)\approx -1$

On [0.01, 20], the maximum value point is (1, 1) and the minimum value point is (0.043, –1).

c. $\int_{0.1}^{20}\cos(\ln x)dx=-8.37$

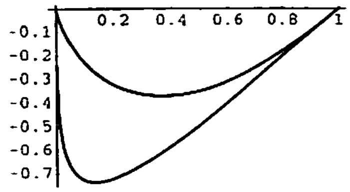
53.



a. $\int_0^1\left[x\ln\left(\frac{1}{x}\right)-x^2\ln\left(\frac{1}{x}\right)\right]dx\approx 0.139$

b. Maximum of ≈ 0.260 at $x\approx 0.236$

54.



a. $\int_0^1[x\ln x-\sqrt{x}\ln x]dx\approx 0.194$

b. Maximum of ≈ 0.521 at $x\approx 0.0555$

7.2 Concepts Review

1. $f(x_1)\neq f(x_2)$

2. x; $f^{-1}(y)$

3. monotonic; increasing; decreasing

4. $(f^{-1})'(y)=\frac{1}{f'(x)}$

Problem Set 7.2

1. $f(x)$ is one-to-one, so it has an inverse.
Since $f(4)=2, f^{-1}(2)=4$.

2. $f(x)$ is one-to-one, so it has an inverse.
Since $f(1)=2$, $f^{-1}(2)=1$.

3. $f(x)$ is not one-to-one, so it does not have an inverse.

4. $f(x)$ is not one-to-one, so it does not have an inverse.

5. $f(x)$ is one-to-one, so it has an inverse.
Since $f(-1)=2$, $f^{-1}(2)=-1$.

6. $f(x)$ is one-to-one, so it has an inverse. Since $f\left(\frac{1}{2}\right)=2, f^{-1}(2)=\frac{1}{2}$.

7. $f'(x) = -5x^4 - 3x^2 = -(5x^4 + 3x^2) < 0$ for all $x \neq 0$. $f(x)$ is strictly decreasing at $x = 0$ because $f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$. Therefore $f(x)$ is strictly decreasing for x and so it has an inverse.

8. $f'(x) = 7x^6 + 15x^2 > 0$ for all $x \neq 0$.
$f(x)$ is strictly increasing at $x = 0$ because $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Therefore $f(x)$ is strictly increasing for all x and so it has an inverse.

9. $f'(\theta) = -\sin\theta < 0$ for $0 < \theta < \pi$
$f(\theta)$ is decreasing at $\theta = 0$ because $f(0) = 1$ and $f(\theta) < 1$ for $0 < \theta < \pi$. $f(\theta)$ is decreasing at $\theta = \pi$ because $f(\pi) = -1$ and $f(\theta) > -1$ for $0 < \theta < \pi$. Therefore $f(\theta)$ is strictly decreasing on $0 \le \theta \le \pi$ and so it has an inverse.

10. $f'(x) = -\csc^2 x < 0$ for $0 < x < \frac{\pi}{2}$

$f(x)$ is decreasing on $0 < x < \frac{\pi}{2}$ and so it has an inverse.

11. $f'(z) = 2(z-1) > 0$ for $z > 1$
$f(z)$ is increasing at $z = 1$ because $f(1) = 0$ and $f(z) > 0$ for $z > 1$. Therefore, $f(z)$ is strictly increasing on $z \ge 1$ and so it has an inverse.

12. $f'(x) = 2x + 1 > 0$ for $x \ge 2$. $f(x)$ is strictly increasing on $x \ge 2$ and so it has an inverse.

13. $f'(x) = \sqrt{x^4 + x^2 + 10} > 0$ for all real x. $f(x)$ is strictly increasing and so it has an inverse.

14. $f(r) = \int_r^1 \cos^4 t\,dt = -\int_1^r \cos^4 t\,dt$

$f'(r) = -\cos^4 r < 0$ for all $r \neq k\pi + \frac{\pi}{2}$, k any integer.

$f(r)$ is decreasing at $r = k\pi + \frac{\pi}{2}$ since $f'(r) < 0$ on the deleted neighborhood $\left(k\pi + \frac{\pi}{2} - \varepsilon, k\pi + \frac{\pi}{2} + \varepsilon\right)$. Therefore, $f(r)$ is strictly decreasing for all r and so it has an inverse.

15. Step 1:
$y = x + 1$
$x = y - 1$
Step 2: $f^{-1}(y) = y - 1$
Step 3: $f^{-1}(x) = x - 1$
Check:
$f^{-1}(f(x)) = (x+1) - 1 = x$
$f(f^{-1}(x)) = (x-1) + 1 = x$

16. Step 1:
$y = -\frac{x}{3} + 1$
$-\frac{x}{3} = y - 1$
$x = -3(y-1) = 3 - 3y$
Step 2: $f^{-1}(y) = 3 - 3y$
Step 3: $f^{-1}(x) = 3 - 3x$
Check:
$f^{-1}(f(x)) = 3 - 3\left(-\frac{x}{3} + 1\right)$
$= 3 + (x - 3) = x$
$f(f^{-1}(x)) = \frac{-(3-3x)}{3} + 1$
$= (-1 + x) + 1 = x$

17. Step 1:
$y = \sqrt{x+1}$ (note that $y \ge 0$)
$x + 1 = y^2$
$x = y^2 - 1, y \ge 0$
Step 2: $f^{-1}(y) = y^2 - 1, y \ge 0$
Step 3: $f^{-1}(x) = x^2 - 1, x \ge 0$
Check:
$f^{-1}(f(x)) = (\sqrt{x+1})^2 - 1 = (x+1) - 1 = x$
$f(f^{-1}(x)) = \sqrt{(x^2-1)+1} = \sqrt{x^2} = |x| = x$

18. Step 1:
$y = -\sqrt{1-x}$ (note that $y \le 0$)
$\sqrt{1-x} = -y$
$1 - x = (-y)^2 = y^2$
$x = 1 - y^2, y \le 0$
Step 2: $f^{-1}(y) = 1 - y^2, y \le 0$
Step 3: $f^{-1}(x) = 1 - x^2, x \le 0$
Check:
$f^{-1}(f(x)) = 1 - (-\sqrt{1-x})^2 = 1 - (1-x) = x$
$f(f^{-1}(x)) = -\sqrt{1-(1-x^2)} = -\sqrt{x^2} = -|x|$
$= -(-x) = x$

19. Step 1:
$y = -\frac{1}{x-3}$

$x-3=-\frac{1}{y}$

$x=3-\frac{1}{y}$

Step 2: $f^{-1}(y)=3-\frac{1}{y}$

Step 3: $f^{-1}(x)=3-\frac{1}{x}$

Check:

$f^{-1}(f(x))=3-\frac{1}{-\frac{1}{x-3}}=3+(x-3)=x$

$f(f^{-1}(x))=-\frac{1}{\left(3-\frac{1}{x}\right)-3}=-\frac{1}{-\frac{1}{x}}=x$

20. Step 1:

$y=\sqrt{\frac{1}{x-2}}$ (note that $y>0$)

$y^2=\frac{1}{x-2}$

$x-2=\frac{1}{y^2}$

$x=2+\frac{1}{y^2}, y>0$

Step 2: $f^{-1}(y)=2+\frac{1}{y^2}, y>0$

Step 3: $f^{-1}(x)=2+\frac{1}{x^2}, x>0$

Check:

$f^{-1}(f(x))=2+\frac{1}{\left(\sqrt{\frac{1}{x-2}}\right)^2}=2+\frac{1}{\left(\frac{1}{x-2}\right)}$

$=2+(x-2)=x$

$f(f^{-1}(x))=\sqrt{\frac{1}{\left(2+\frac{1}{x^2}\right)-2}}=\sqrt{\frac{1}{\left(\frac{1}{x^2}\right)}}=\sqrt{x^2}$

$=|x|=x$

21. Step 1:

$y=4x^2, x\le 0$ (note that $y\ge 0$)

$x^2=\frac{y}{4}$

$x=-\sqrt{\frac{y}{4}}=-\frac{\sqrt{y}}{2}$, negative since $x\le 0$

Step 2: $f^{-1}(y)=-\frac{\sqrt{y}}{2}$

Step 3: $f^{-1}(x)=-\frac{\sqrt{x}}{2}$

Check:

$f^{-1}(f(x))=-\frac{\sqrt{4x^2}}{2}=-\sqrt{x^2}=-|x|=-(-x)=x$

$f(f^{-1}(x))=4\left(-\frac{\sqrt{x}}{2}\right)^2=4\cdot\frac{x}{4}=x$

22. Step 1:

$y=(x-3)^2, x\ge 3$ (note that $y\ge 0$)

$x-3=\sqrt{y}$

$x=3+\sqrt{y}$

Step 2: $f^{-1}(y)=3+\sqrt{y}$

Step 3: $f^{-1}(x)=3+\sqrt{x}$

Check:

$f^{-1}(f(x))=3+\sqrt{(x-3)^2}=3+|x-3|$

$=3+(x-3)=x$

$f(f^{-1}(x))=[(3+\sqrt{x})-3]^2=(\sqrt{x})^2=x$

23. Step 1:

$y=(x-1)^3$

$x-1=\sqrt[3]{y}$

$x=1+\sqrt[3]{y}$

Step 2: $f^{-1}(y)=1+\sqrt[3]{y}$

Step 3: $f^{-1}(x)=1+\sqrt[3]{x}$

Check: $f^{-1}(f(x))=1+\sqrt[3]{(x-1)^3}=1+(x-1)=x$

$f(f^{-1}(x))=[(1+\sqrt[3]{x})-1]^3=(\sqrt[3]{x})^3=x$

24. Step 1:

$y=x^{5/2}, x\ge 0$

$x=y^{2/5}$

Step 2: $f^{-1}(y)=y^{2/5}$

Step 3: $f^{-1}(x)=x^{2/5}$

Check:

$f^{-1}(f(x))=(x^{5/2})^{2/5}=x$

$f(f^{-1}(x))=(x^{2/5})^{5/2}=x$

25. Step 1:

$y=\frac{x-1}{x+1}$

$xy+y=x-1$

$x-xy=1+y$

$x=\frac{1+y}{1-y}$

Step 2: $f^{-1}(y)=\frac{1+y}{1-y}$

Step 3: $f^{-1}(x) = \frac{1+x}{1-x}$

Check:

$$f^{-1}(f(x)) = \frac{1+\frac{x-1}{x+1}}{1-\frac{x-1}{x+1}} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1} = \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x$$

26. Step 1:

$$y = \left(\frac{x-1}{x+1}\right)^3$$

$$y^{1/3} = \frac{x-1}{x+1}$$

$$xy^{1/3} + y^{1/3} = x-1$$

$$x - xy^{1/3} = 1 + y^{1/3}$$

$$x = \frac{1+y^{1/3}}{1-y^{1/3}}$$

Step 2: $f^{-1}(y) = \frac{1+y^{1/3}}{1-y^{1/3}}$

Step 3: $f^{-1}(x) = \frac{1+x^{1/3}}{1-x^{1/3}}$

Check:

$$f^{-1}(f(x)) = \frac{1+\left[\left(\frac{x-1}{x+1}\right)^3\right]^{1/3}}{1-\left[\left(\frac{x-1}{x+1}\right)^3\right]^{1/3}} = \frac{1+\frac{x-1}{x+1}}{1-\frac{x-1}{x+1}}$$

$$= \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = \left(\frac{\frac{1+x^{1/3}}{1-x^{1/3}}-1}{\frac{1+x^{1/3}}{1-x^{1/3}}+1}\right)^3 = \left(\frac{1+x^{1/3}-1+x^{1/3}}{1+x^{1/3}+1-x^{1/3}}\right)^3$$

$$= \left(\frac{2x^{1/3}}{2}\right)^3 = (x^{1/3})^3 = x$$

27. Step 1:

$$y = \frac{x^3+2}{x^3+1}$$

$$x^3y + y = x^3 + 2$$

$$x^3y - x^3 = 2 - y$$

$$x^3 = \frac{2-y}{y-1}$$

$$x = \left(\frac{2-y}{y-1}\right)^{1/3}$$

Step 2: $f^{-1}(y) = \left(\frac{2-y}{y-1}\right)^{1/3}$

Step 3: $f^{-1}(x) = \left(\frac{2-x}{x-1}\right)^{1/3}$

Check:

$$f^{-1}(f(x)) = \left(\frac{2-\frac{x^3+2}{x^3+1}}{\frac{x^3+2}{x^3+1}-1}\right)^{1/3} = \left(\frac{2x^3+2-x^3-2}{x^3+2-x^3-1}\right)^{1/3}$$

$$= \left(\frac{x^3}{1}\right)^{1/3} = x$$

$$f(f^{-1}(x)) = \frac{\left[\left(\frac{2-x}{x-1}\right)^{1/3}\right]^3+2}{\left[\left(\frac{2-x}{x-1}\right)^{1/3}\right]^3+1} = \frac{\frac{2-x}{x-1}+2}{\frac{2-x}{x-1}+1}$$

$$= \frac{2-x+2x-2}{2-x+x-1} = \frac{x}{1} = x$$

28. Step 1:

$$y = \left(\frac{x^3+2}{x^3+1}\right)^5$$

$$y^{1/5} = \frac{x^3+2}{x^3+1}$$

$$x^3y^{1/5} + y^{1/5} = x^3 + 2$$

$$x^3y^{1/5} - x^3 = 2 - y^{1/5}$$

$$x^3 = \frac{2-y^{1/5}}{y^{1/5}-1}$$

$$x = \left(\frac{2-y^{1/5}}{y^{1/5}-1}\right)^{1/3}$$

Step 2: $f^{-1}(y) = \left(\frac{2-y^{1/5}}{y^{1/5}-1}\right)^{1/3}$

Step 3: $f^{-1}(x) = \left(\frac{2-x^{1/5}}{x^{1/5}-1}\right)^{1/3}$

Check:

$$f^{-1}(f(x)) = \left\{\frac{2-\left[\left(\frac{x^3+2}{x^3+1}\right)^5\right]^{1/5}}{\left[\left(\frac{x^3+2}{x^3+1}\right)^5\right]^{1/5}-1}\right\}^{1/3}$$

$$= \left(\frac{2-\frac{x^3+2}{x^3+1}}{\frac{x^3+2}{x^3+1}-1}\right)^{1/3} = \left(\frac{2x^3+2-x^3-2}{x^3+2-x^3-1}\right)^{1/3}$$

$$= \left(\frac{x^3}{1}\right)^{1/3} = x$$

$$f(f^{-1}(x)) = \left\{\frac{\left[\left(\frac{2-x^{1/5}}{x^{1/5}-1}\right)^{1/3}\right]^3 + 2}{\left[\left(\frac{2-x^{1/5}}{x^{1/5}-1}\right)^{1/3}\right]^3 + 1}\right\}^5$$

$$= \left(\frac{\frac{2-x^{1/5}}{x^{1/5}-1} + 2}{\frac{2-x^{1/5}}{x^{1/5}-1} + 1}\right)^5 = \left(\frac{2 - x^{1/5} + 2x^{1/5} - 2}{2 - x^{1/5} + x^{1/5} - 1}\right)^5$$

$$= \left(\frac{x^{1/5}}{1}\right)^5 = x$$

29. By similar triangles $\frac{r}{h} = \frac{4}{6}$. Thus, $r = \frac{2h}{3}$

This gives

$$V = \frac{\pi r^2 h}{3} = \frac{\pi\left(4h^2/9\right)h}{3} = \frac{4\pi h^3}{27}$$

$$h^3 = \frac{27V}{4\pi}$$

$$h = 3\sqrt[3]{\frac{V}{4\pi}}$$

30. $v = v_0 - 32t$

$v = 0$ when $v_0 = 32t$, that is, when

$t = \frac{v_0}{32}$. The position function is

$s(t) = v_0 t - 16t^2$. The ball then reaches a height of

$$H = s(v_0/32) = v_0\frac{v_0}{32} - 16\frac{v_0^2}{32^2} = \frac{v_0^2}{64}$$

$$v_0^2 = 64H$$

$$v_0 = 8\sqrt{H}$$

31. $f'(x) = 4x + 1$; $f'(x) > 0$ when $x > -\frac{1}{4}$ and

$f'(x) < 0$ when $x < -\frac{1}{4}$.

The function is decreasing on $\left(-\infty, -\frac{1}{4}\right]$ and increasing on $\left[-\frac{1}{4}, \infty\right)$. Restrict the domain to $\left(-\infty, -\frac{1}{4}\right]$ or restrict it to $\left[-\frac{1}{4}, \infty\right)$.

Then $f^{-1}(x) = \frac{1}{4}(-1 - \sqrt{8x+33})$ or

$f^{-1}(x) = \frac{1}{4}(-1 + \sqrt{8x+33})$.

32. $f'(x) = 2x - 3$; $f'(x) > 0$ when $x > \frac{3}{2}$

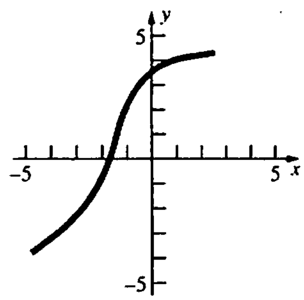
and $f'(x) < 0$ when $x < \frac{3}{2}$.

The function is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$. Restrict the domain to $\left(-\infty, \frac{3}{2}\right]$ or restrict it to $\left[\frac{3}{2}, \infty\right)$. Then

$f^{-1}(x) = \frac{1}{2}(3 - \sqrt{4x+5})$ or

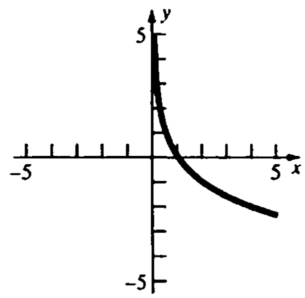
$f^{-1}(x) = \frac{1}{2}(3 + \sqrt{4x+5})$.

33.

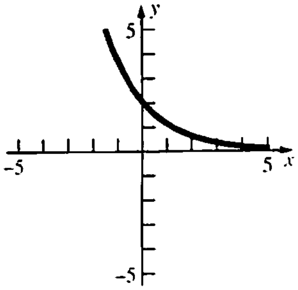


$(f^{-1})'(3) \approx \frac{1}{10}$

34. $(f^{-1})'(3) \approx -\frac{1}{2}$

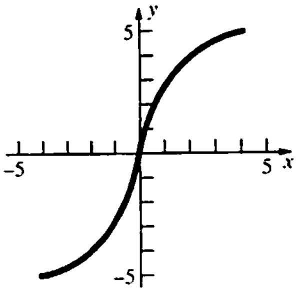


35.



$(f^{-1})'(3) \approx -\frac{1}{3}$

36. $(f^{-1})'(3) \approx \frac{1}{2}$



37. $f'(x) = 15x^4 + 1$ and $y = 2$ corresponds to $x = 1$,

so $(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{15+1} = \frac{1}{16}$.

38. $f'(x) = 5x^4 + 5$ and $y = 2$ corresponds to $x = 1$,

so $(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{5+5} = \frac{1}{10}$

39. $f'(x) = 2\sec^2 x$ and $y = 2$ corresponds to $x = \frac{\pi}{4}$,

so $(f^{-1})'(2) = \frac{1}{f'\left(\frac{\pi}{4}\right)} = \frac{1}{2\sec^2\left(\frac{\pi}{4}\right)} = \frac{1}{2}\cos^2\left(\frac{\pi}{4}\right)$

$= \frac{1}{4}$.

40. $f'(x) = \frac{1}{2\sqrt{x+1}}$ and $y = 2$ corresponds to $x = 3$,

so $(f^{-1})'(2) = \frac{1}{f'(3)} = 2\sqrt{3+1} = 4$.

41. $(g^{-1} \circ f^{-1})(h(x)) = (g^{-1} \circ f^{-1})(f(g(x)))$

$= g^{-1} \circ [f^{-1}(f(g(x)))] = g^{-1} \circ [g(x)] = x$

Similarly,

$h((g^{-1} \circ f^{-1})(x)) = f(g((g^{-1} \circ f^{-1})(x)))$

$= f(g(g^{-1}(f^{-1}(x)))) = f(f^{-1}(x)) = x$

Thus $h^{-1} = g^{-1} \circ f^{-1}$

42. Find $f^{-1}(x)$:

$y = \frac{1}{x}$, $x = \frac{1}{y}$

$f^{-1}(y) = \frac{1}{y}$

$f^{-1}(x) = \frac{1}{x}$

Find $g^{-1}(x)$:

$y = 3x + 2$

$x = \frac{y-2}{3}$

$g^{-1}(y) = \frac{y-2}{3}$

$g^{-1}(x) = \frac{x-2}{3}$

$h(x) = f(g(x)) = f(3x+2) = \frac{1}{3x+2}$

$h^{-1}(x) = g^{-1}(f^{-1}(x)) = g^{-1}\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)-2}{3}$

$h^{-1}(h(x)) = h^{-1}\left(\frac{1}{3x+2}\right) = \frac{(3x+2)-2}{3} = \frac{3x}{3} = x$

$h(h^{-1}(x)) = h\left(\frac{\left(\frac{1}{x}\right)-2}{3}\right) = \frac{1}{\left[\left(\frac{1}{x}\right)-2\right]+2} = \frac{1}{\left(\frac{1}{x}\right)} = x$

43. f has an inverse because it is monotonic (increasing):

$f'(x) = \sqrt{1+\cos^2 x} > 0$

a. $(f^{-1})'(A) = \frac{1}{f'\left(\frac{\pi}{2}\right)} = \frac{1}{\sqrt{1+\cos^2\left(\frac{\pi}{2}\right)}} = 1$

b. $(f^{-1})'(B) = \frac{1}{f'\left(\frac{5\pi}{6}\right)} = \frac{1}{\sqrt{1+\cos^2\left(\frac{5\pi}{6}\right)}} = \frac{1}{\sqrt{\frac{7}{4}}}$

$= \frac{2}{\sqrt{7}}$

c. $(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{\sqrt{1+\cos^2(0)}} = \frac{1}{\sqrt{2}}$

44. a. $y = \frac{ax+b}{cx+d}$

$cxy + dy = ax + b$

$(cy - a)x = b - dy$

$$x = \frac{b-dy}{cy-a} = -\frac{dy-b}{cy-a}$$

$$f^{-1}(y) = -\frac{dy-b}{cy-a}$$

$$f^{-1}(x) = -\frac{dx-b}{cx-a}$$

b. If $bc - ad = 0$, then $f(x)$ is either a constant function or undefined.

c. If $f = f^{-1}$, then for all x in the domain we have:

$$\frac{ax+b}{cx+d} + \frac{dx-b}{cx-a} = 0$$

$$(ax + b)(cx - a) + (dx - b)(cx + d) = 0$$

$$acx^2 + (bc - a^2)x - ab + dcx^2$$

$$+(d^2 - bc)x - bd = 0$$

$$(ac + dc)x^2 + (d^2 - a^2)x + (-ab - bd) = 0$$

Setting the coefficients equal to 0 gives three requirements:

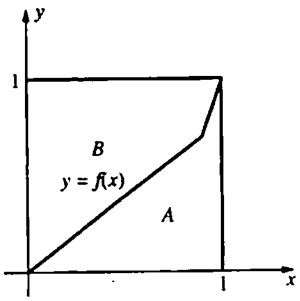
(1) $a = -d$ or $c = 0$

(2) $a = \pm d$

(3) $a = -d$ or $b = 0$

If $a = d$, then $f = f^{-1}$ requires $b = 0$ and $c = 0$, so $f(x) = \frac{ax}{d} = x$. If $a = -d$, there are no requirements on b and c (other than $bc - ad \neq 0$). Therefore, $f = f^{-1}$ if $a = -d$ or if f is the identity function.

45.



$$\int_0^1 f^{-1}(y)\,dy = \text{(Area of region } B)$$

$$= 1 - \text{(Area of region } A)$$

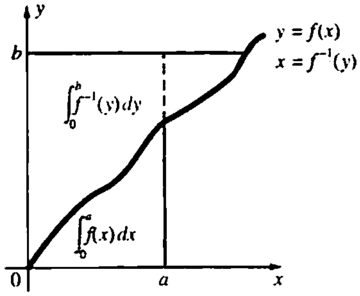
$$= 1 - \int_0^1 f(x)\,dx = 1 - \frac{2}{5} = \frac{3}{5}$$

46. $\int_0^a f(x)dx =$ the area bounded by $y = f(x)$, $y = 0$, and $x = a$ [the area under the curve].

$\int_0^b f^{-1}(y)dy =$ the area bounded by $x = f^{-1}(y)$ $x = 0$, and $y = b$.

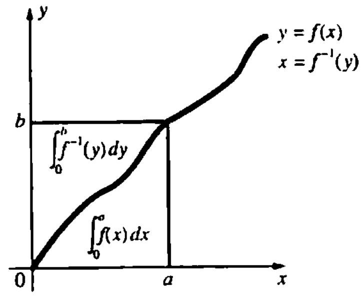
$ab =$ the area of the rectangle bounded by $x = 0$, $x = a$, $y = 0$, and $y = b$.

Case 1: $b > f(a)$



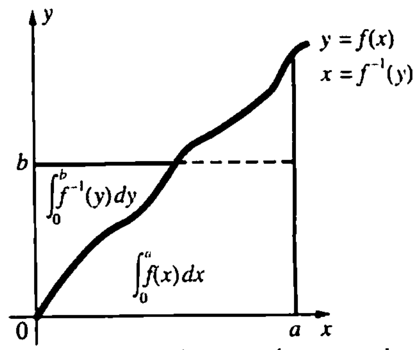
The area above the curve is greater than the area of the part of the rectangle above the curve, so the total area represented by the sum of the two integrals is greater than the area ab of the rectangle.

Case 2: $b = f(a)$



The area represented by the sum of the two integrals = the area ab of the rectangle.

Case 3: $b < f(a)$



The area below the curve is greater than the area of the part of the rectangle which is below the curve, so the total area represented by the sum of the two integrals is greater than the area ab of the rectangle.

$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$ with equality holding when $b = f(a)$.

47. Given $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f(x) = x^{p-1}$,

solving $\frac{1}{p} + \frac{1}{q} = 1$ for p gives $p = \frac{q}{q-1}$, so

$$\frac{1}{p-1}=\frac{1}{\frac{q}{q-1}-1}=\frac{1}{\left[\frac{q-(q-1)}{q-1}\right]}=\frac{q-1}{1}=q-1.$$

Thus, if $y=x^{p-1}$ then $x=y^{\frac{1}{p-1}}=y^{q-1}$, so $f^{-1}(y)=y^{q-1}$.

By Problem 44, since $f(x)=x^{p-1}$ is strictly increasing for $p>1$, $ab\le\int_0^a x^{p-1}dx+\int_0^b y^{q-1}dy$

$$ab\le\left[\frac{x^p}{p}\right]_0^a+\left[\frac{y^q}{q}\right]_0^b$$

$$ab\le\frac{a^p}{p}+\frac{b^q}{q}$$

7.3 Concepts Review

1. increasing; exp

2. $\ln e=1$; 2.72

3. x; x

4. e^x; e^x+C

Problem Set 7.3

1. a. 20.086

   b. 8.1662

   c. $e^{\sqrt{2}}\approx e^{1.41}=e^{1.4}\cdot e^{0.01}\approx 4.1$

   d. $e^{\cos(\ln 4)}\approx e^{0.18}\approx 1.20$

2. a. $e^{3\ln 2}=e^{\ln(2^3)}=e^{\ln 8}=8$

   b. $e^{\frac{\ln 64}{2}}=e^{\ln(64^{1/2})}=e^{\ln 8}=8$

3. $e^{3\ln x}=e^{\ln x^3}=x^3$

4. $e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}=\frac{1}{x^2}$

5. $\ln e^{\cos x}=\cos x$

6. $\ln e^{-2x-3}=-2x-3$

7. $\ln(x^3e^{-3x})=\ln x^3+\ln e^{-3x}=3\ln x-3x$

8. $e^{x-\ln x}=\frac{e^x}{e^{\ln x}}=\frac{e^x}{x}$

9. $e^{\ln 3+2\ln x}=e^{\ln 3}\cdot e^{2\ln x}=3\cdot e^{\ln x^2}=3x^2$

10. $e^{\ln x^2-y\ln x}=\frac{e^{\ln x^2}}{e^{y\ln x}}=\frac{x^2}{e^{\ln x^y}}=\frac{x^2}{x^y}=x^{2-y}$

11. $D_xe^{x+2}=e^{x+2}D_x(x+2)=e^{x+2}$

12. $D_xe^{2x^2-x}=e^{2x^2-x}D_x(2x^2-x)$
$=e^{2x^2-x}(4x-1)$

13. $D_xe^{\sqrt{x+2}}=e^{\sqrt{x+2}}D_x\sqrt{x+2}=\frac{e^{\sqrt{x+2}}}{2\sqrt{x+2}}$

14. $D_xe^{-\frac{1}{x^2}}=e^{-\frac{1}{x^2}}D_x\left(-\frac{1}{x^2}\right)$
$=e^{-\frac{1}{x^2}}\cdot 2x^{-3}=\frac{2e^{-\frac{1}{x^2}}}{x^3}$

15. $D_xe^{2\ln x}=D_xe^{\ln x^2}=D_xx^2=2x$

16. $D_xe^{\frac{x}{\ln x}}=e^{\frac{x}{\ln x}}D_x\frac{x}{\ln x}=e^{\frac{x}{\ln x}}\cdot\frac{(\ln x)\cdot 1-x\cdot\frac{1}{x}}{(\ln x)^2}$
$=\frac{e^{\frac{x}{\ln x}}(\ln x-1)}{(\ln x)^2}$

17. $D_x(x^3e^x)=x^3D_xe^x+e^xD_x(x^3)$
$=x^3e^x+e^x\cdot 3x^2=x^2e^x(x+3)$

18. $D_xe^{x^3\ln x}=e^{x^3\ln x}D_x(x^3\ln x)$
$=e^{x^3\ln x}\left(x^3\cdot\frac{1}{x}+\ln x\cdot 3x^2\right)$
$=e^{x^3\ln x}(x^2+3x^2\ln x)$
$=x^2e^{x^3\ln x}(1+3\ln x)$

19. $D_x[\sqrt{e^{x^2}} + e^{\sqrt{x^2}}] = D_x(e^{x^2})^{1/2} + D_x e^{\sqrt{x^2}}$

$= \frac{1}{2}(e^{x^2})^{-1/2} D_x e^{x^2} + e^{\sqrt{x^2}} D_x\sqrt{x^2}$

$= \frac{1}{2}(e^{x^2})^{-1/2} e^{x^2} D_x x^2 + e^{\sqrt{x^2}} \cdot \frac{x}{\sqrt{x^2}}$

$= \frac{1}{2}(e^{x^2})^{1/2} 2x + e^{\sqrt{x^2}} \cdot \frac{x}{|x|}$

$= x\sqrt{e^{x^2}} + \frac{xe^{\sqrt{x^2}}}{|x|}$

20. $D_x\left[e^{1/x^2} + \frac{1}{e^{x^2}}\right] = D_x e^{x^{-2}} + D_x e^{-x^2}$

$= e^{x^{-2}} D_x x^{-2} + e^{-x^2} D_x[-x^2]$

$= e^{x^{-2}} \cdot (-2x^{-3}) + e^{-x^2} \cdot (-2x)$

$= -\frac{2e^{1/x^2}}{x^3} - \frac{2x}{e^{x^2}}$

21. $D_x[e^{xy} + xy] = D_x[2]$

$e^{xy}(xD_x y + y) + (xD_x y + y) = 0$

$xe^{xy} D_x y + ye^{xy} + xD_x y + y = 0$

$xe^{xy} D_x y + xD_x y = -ye^{xy} - y$

$D_x y = \frac{-ye^{xy} - y}{xe^{xy} + x} = -\frac{y(e^{xy}+1)}{x(e^{xy}+1)} = -\frac{y}{x}$

22. $D_x[e^{x+y}] = D_x[x+y]$

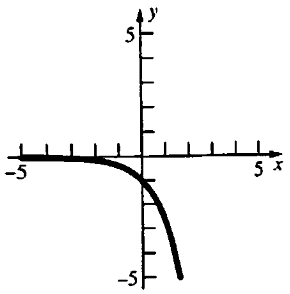
$e^{x+y}(1 + D_x y) = 1 + D_x y$

$e^{x+y} + e^{x+y} D_x y = 1 + D_x y$

$e^{x+y} D_x y - D_x y = 1 - e^{x+y}$

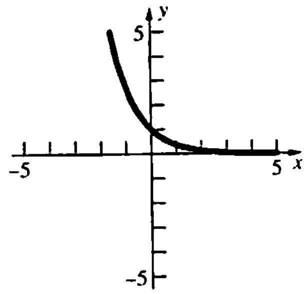
$D_x y = \frac{1 - e^{x+y}}{e^{x+y} - 1} = -1$

23. a.



The graph of $y = e^x$ is reflected across the x-axis.

b.



The graph of $y = e^x$ is reflected across the y-axis.

24. $a < b \Rightarrow -a > -b \Rightarrow e^{-a} > e^{-b}$, since e^x is an increasing function.

25. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$;

$f'(x) = 0$ when $x = 1$

$f'(x) > 0$ on $(-\infty, 1)$

$f''(x) = xe^{-x} - 2e^{-x} = e^{-x}(x-2)$;

$f''(x) = 0$ when $x = 2$

$f''(x) > 0$ on $(2, \infty)$

Increasing on $(-\infty, 1]$

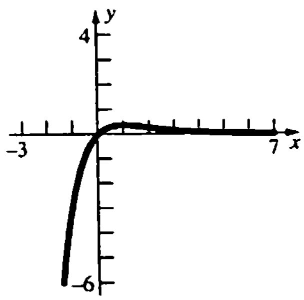
Decreasing on $[1, \infty)$

Maximum at $\left(1, \frac{1}{e}\right) \approx (1, 0.4)$

Concave up on $(2, \infty)$

Concave down on $(-\infty, 2)$

Inflection point at $\left(2, \frac{2}{e^2}\right) \approx (2, 0.3)$



26. $f'(x) = e^x + 1$

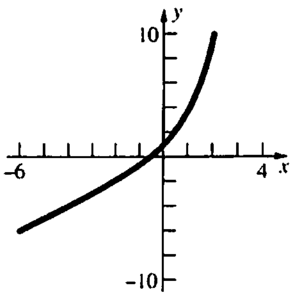
$f'(x) > 0$ for all real x.

$f''(x) = e^x$

$f''(x) > 0$ for all real x.

Increasing on $(-\infty, \infty)$

Concave up on $(-\infty, \infty)$



27. $f'(x) = -2(x-2)e^{-(x-2)^2}$;

$f'(x) = 0$ when $x = 2$

$f'(x) > 0$ on $(-\infty, 2)$

$f''(x) = 4(x-2)^2 e^{-(x-2)^2} - 2e^{-(x-2)^2}$

$= 2e^{-(x-2)^2}[2(x-2)^2 - 1]$;

$f''(x) = 0$ when $x = 2 \pm \frac{1}{2}\sqrt{2}$

$f''(x) > 0$ on $\left(-\infty, 2 - \frac{1}{2}\sqrt{2}\right)$ and

$\left(2 + \frac{1}{2}\sqrt{2}, \infty\right)$

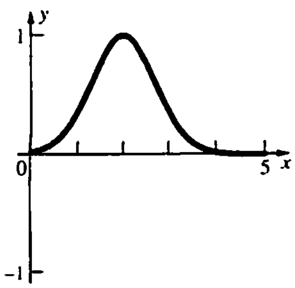
Increasing on $(-\infty, 2]$
Decreasing on $[2, \infty)$
Maximum at $(2, 1)$

Concave up on $\left(-\infty, 2 - \frac{1}{2}\sqrt{2}\right) \cup \left(2 + \frac{1}{2}\sqrt{2}, \infty\right)$

Concave down on $\left(2 - \frac{1}{2}\sqrt{2}, 2 + \frac{1}{2}\sqrt{2}\right)$

Inflection points at $\left(2 - \frac{1}{2}\sqrt{2}, \frac{1}{\sqrt{e}}\right) \approx (1.3, 0.6)$

and $\left(2 + \frac{1}{2}\sqrt{2}, \frac{1}{\sqrt{e}}\right) \approx (2.7, 0.6)$



28. $f'(x) = e^x + e^{-x}$

$f'(x) > 0$ for all x

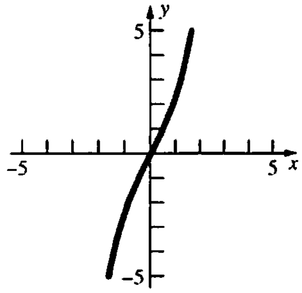
$f''(x) = e^x - e^{-x} = e^{-x}(e^{2x} - 1)$;

$f''(x) = 0$ when $e^{2x} = 1$ or $x = 0$.

$f''(x) > 0$ when $x > 0$

Increasing for all x

Concave up on $(0, \infty)$
Concave down on $(-\infty, 0)$
Inflection point at $(0, 0)$



29. Let $u = 3x + 1$, so $du = 3dx$.

$\int e^{3x+1}dx = \frac{1}{3}\int e^{3x+1}3dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C$

$= \frac{1}{3}e^{3x+1} + C$

30. Let $u = x^2 - 3$, so $du = 2x\,dx$.

$\int xe^{x^2-3}dx = \frac{1}{2}\int e^{x^2-3}2x\,dx = \frac{1}{2}\int e^u du$

$= \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2-3} + C$

31. Let $u = x^2 + 6x$, so $du = (2x + 6)dx$.

$\int (x+3)e^{x^2+6x}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2+6x} + C$

32. Let $u = e^x - 1$, so $du = e^x dx$.

$\int \frac{e^x}{e^x - 1}dx = \int \frac{1}{u}du = \ln|u| + C = \ln\left|e^x - 1\right| + C$

33. Let $u = -\frac{1}{x}$, so $du = \frac{1}{x^2}dx$.

$\int \frac{e^{-1/x}}{x^2}dx = \int e^u du = e^u + C = e^{-1/x} + C$

34. $\int e^{x+e^x}dx = \int e^x \cdot e^{e^x}dx$

Let $u = e^x$, so $du = e^x dx$.

$\int e^x \cdot e^{e^x}dx = \int e^u du = e^u + C = e^{e^x} + C$

35. Let $u = 2x + 3$, so $du = 2dx$

$\int e^{2x+3}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x+3} + C$

$$\int_0^1 e^{2x+3}dx = \left[\frac{1}{2}e^{2x+3}\right]_0^1 = \frac{1}{2}e^5 - \frac{1}{2}e^3$$
$$= \frac{1}{2}e^3(e^2-1) \approx 64.2$$

36. Let $u = \frac{3}{x}$, so $du = -\frac{3}{x^2}dx$.

$$\int \frac{e^{3/x}}{x^2}dx = -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C$$
$$= -\frac{1}{3}e^{3/x} + C$$
$$\int_1^2 \frac{e^{3/x}}{x^2}dx = \left[-\frac{1}{3}e^{3/x}\right]_1^2 = -\frac{1}{3}e^{3/2} + \frac{1}{3}e^3 \approx 5.2$$

37. $V = \pi\int_0^{\ln 3}(e^x)^2 dx = \pi\int_0^{\ln 3} e^{2x}dx$

$$= \pi\left[\frac{1}{2}e^{2x}\right]_0^{\ln 3} = \pi\left(\frac{1}{2}e^{2\ln 3} - \frac{1}{2}e^0\right) \approx 12.57$$

38. $V = \int_0^1 2\pi x e^{-x^2}dx$.

Let $u = -x^2$, so $du = -2x\, dx$.

$$\int 2\pi x e^{-x^2}dx = -\pi\int e^{-x^2}(-2x)dx = -\pi\int e^u du$$
$$= -\pi e^u + C = -\pi e^{-x^2} + C$$
$$\int_0^1 2\pi x e^{-x^2}dx = -\pi\left[e^{-x^2}\right]_0^1 = -\pi(e^{-1} - e^0)$$
$$= \pi(1 - e^{-1}) \approx 1.99$$

39. The line through (0, 1) and $\left(1, \frac{1}{e}\right)$ has slope

$$\frac{\frac{1}{e}-1}{1-0} = \frac{1}{e} - 1 = \frac{1-e}{e} \Rightarrow y - 1 = \frac{1-e}{e}(x-0);$$
$$y = \frac{1-e}{e}x + 1$$
$$\int_0^1\left[\left(\frac{1-e}{e}x+1\right) - e^{-x}\right]dx = \left[\frac{1-e}{2e}x^2 + x + e^{-x}\right]_0^1$$
$$= \frac{1-e}{2e} + 1 + \frac{1}{e} - 1 = \frac{3-e}{2e} \approx 0.052$$

40. $f'(x) = \frac{(e^x-1)(1) - x(e^x)}{(e^x-1)^2} - \frac{1}{1-e^{-x}}(-e^{-x})(-1)$

$$= \frac{e^x - 1 - xe^x}{(e^x-1)^2} - \frac{1}{1-e^{-x}}\left(\frac{1}{e^x}\right)$$
$$= \frac{e^x - 1 - xe^x}{(e^x-1)^2} - \frac{1}{e^x - 1} = \frac{e^x - 1 - xe^x - (e^x - 1)}{(e^x-1)^2}$$
$$= -\frac{xe^x}{(e^x-1)^2}$$

When $x > 0$, $f'(x) < 0$, so $f(x)$ is decreasing for $x > 0$.

41. a. Exact:

$$10! = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$$
$$= 3,628,800$$

Approximate:

$$10! \approx \sqrt{20\pi}\left(\frac{10}{e}\right)^{10} \approx 3,598,696$$

b. $60! \approx \sqrt{120\pi}\left(\frac{60}{e}\right)^{60} \approx 8.31\times 10^{81}$

42. $e^{0.3} \approx \left\{\left[\left(\frac{0.3}{4}+1\right)\frac{0.3}{3}+1\right]\frac{0.3}{2}+1\right\}(0.3)+1$

$$= 1.3498375$$

$e^{0.3} \approx 1.3498588$ by direct calculation

43. $x = e^t \sin t$, so $dx = (e^t \sin t + e^t \cos t)dt$

$y = e^t \cos t$, so $dy = (e^t \cos t - e^t \sin t)dt$

$$ds = \sqrt{dx^2 + dy^2}$$
$$= e^t\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2}\,dt$$
$$= e^t\sqrt{2\sin^2 t + 2\cos^2 t}\,dt = \sqrt{2}e^t dt$$

The length of the curve is

$$\int_0^\pi \sqrt{2}e^t dt = \sqrt{2}\left[e^t\right]_0^\pi = \sqrt{2}(e^\pi - 1) \approx 31.312$$

44. Use $x = 30$, $n = 8$, and $k = 0.25$.

$$P_n(x) = \frac{(kx)^n e^{-kx}}{n!} = \frac{(0.25\cdot 30)^8 e^{-0.25\cdot 30}}{8!} \approx 0.14$$

45. a. $\lim_{x\to 0^+}\frac{\ln x}{1+(\ln x)^2}$ is of the form $\frac{\infty}{\infty}$.

$$= \lim_{x\to 0^+}\frac{D_x \ln x}{D_x[1+(\ln x)^2]} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{2\ln x\cdot\frac{1}{x}}$$
$$= \lim_{x\to 0^+}\frac{1}{2\ln x} = 0$$
$$\lim_{x\to\infty}\frac{\ln x}{1+(\ln x)^2} = \lim_{x\to\infty}\frac{1}{2\ln x} = 0$$

b. $f'(x) = \frac{[1+(\ln x)^2]\cdot\frac{1}{x} - \ln x\cdot 2\ln x\cdot\frac{1}{x}}{[1+(\ln x)^2]^2}$

$$= \frac{1-(\ln x)^2}{x[1+(\ln x)^2]^2}$$

$f'(x) = 0$ when $\ln x = \pm 1$ so $x = e^1 = e$

or $x = e^{-1} = \frac{1}{e}$

$$f(e) = \frac{\ln e}{1+(\ln e)^2} = \frac{1}{1+1^2} = \frac{1}{2}$$

$$f\left(\frac{1}{e}\right) = \frac{\ln \frac{1}{e}}{1+\left(\ln \frac{1}{e}\right)^2} = \frac{-1}{1+(-1)^2} = -\frac{1}{2}$$

Maximum value of $\frac{1}{2}$ at $x = e$; minimum value of $-\frac{1}{2}$ at $x = e^{-1}$.

c. $F(x) = \int_1^{x^2} \frac{\ln t}{1+(\ln t)^2}\,dt$

$$F'(x) = \frac{\ln x^2}{1+(\ln x^2)^2} \cdot 2x$$

$$F'(\sqrt{e}) = \frac{\ln(\sqrt{e})^2}{1+[\ln(\sqrt{e})^2]^2} \cdot 2\sqrt{e} = \frac{1}{1+1^2} \cdot 2\sqrt{e}$$

$= \sqrt{e} \approx 1.65$

46. Let (x_0, e^{x_0}) be the point of tangency. Then

$$\frac{e^{x_0} - 0}{x_0 - 0} = f'(x_0) = e^{x_0} \Rightarrow e^{x_0} = x_0 e^{x_0} \Rightarrow x_0 = 1$$

so the line is $y = e^{x_0}x$ or $y = ex$.

a. $A = \int_0^1 (e^x - ex)\,dx = \left[e^x - \frac{ex^2}{2}\right]_0^1$

$= e - \frac{e}{2} - (e^0 - 0) = \frac{e}{2} - 1 \approx 0.36$

b. $V = \pi \int_0^1 [(e^x)^2 - (ex)^2]\,dx$

$= \pi \int_0^1 (e^{2x} - e^2x^2)\,dx = \pi \left[\frac{1}{2}e^{2x} - \frac{e^2x^3}{3}\right]_0^1$

$= \pi \left[\frac{1}{2}e^2 - \frac{e^2}{3} - \left(\frac{1}{2}e^0\right)\right] = \frac{\pi}{6}(e^2 - 3) \approx 2.30$

47. $\lim_{n\to\infty} \left(e^{1/n} + e^{2/n} + \cdots + e^{n/n}\right)\left(\frac{1}{n}\right) = \int_0^1 e^x\,dx$

$= \left[e^x\right]_0^1 = e - 1 \approx 1.718$

48. a. The reflection of the point $(x, f(x))$ through the line $x = \mu$ is $(2\mu - x, f(x))$.
Thus we want to show that $f(2\mu - x) = f(x)$

$$f(2\mu - x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{2\mu - x - \mu}{\sigma}\right)^2\right] = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\mu - x}{\sigma}\right)^2\right] = f(x) \text{ since } (\mu - x)^2 = (x-\mu)^2$$

b. $f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\left[-\frac{1}{2}\cdot 2\left(\frac{x-\mu}{\sigma}\right)\cdot\frac{1}{\sigma}\right] = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

$f'(x) = 0$ when $x = \mu$

$$f''(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\left[-\frac{1}{2}\cdot 2\left(\frac{x-\mu}{\sigma}\right)\cdot\frac{1}{\sigma}\right] + \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\left(-\frac{1}{\sigma^3\sqrt{2\pi}}\right)$$

$$= \frac{1}{\sigma^5\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right][(x-\mu)^2 - \sigma^2]$$

$f''(\mu) = -\frac{1}{\sigma^3\sqrt{2\pi}} < 0$ so $f(x)$ has a maximum at $x = \mu$.

$f''(x) = 0$ when $(x-\mu)^2 = \sigma^2$ so $f(x)$ has inflection points at $x = \mu \pm \sigma$.

49. a. $\int_{-3}^3 \exp\left(-\frac{1}{x^2}\right)dx = 2\int_0^3 \exp\left(-\frac{1}{x^2}\right)dx \approx 3.11$

b. $\int_0^{8\pi} e^{-0.1x} \sin x\,dx \approx 0.910$

50. a. $\lim_{x\to 0}(1+x)^{1/x} = e \approx 2.72$

b. $\lim_{x\to 0}(1+x)^{-1/x} = \frac{1}{e} \approx 0.368$

51. $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}(2x^2-1)$

$y = f(x)$ and $y = f''(x)$ intersect when

$e^{-x^2} = 2e^{-x^2}(2x^2-1); 1 = 4x^2-2;$

$4x^2 - 3 = 0, x = \pm\frac{\sqrt{3}}{2}$

Both graphs are symmetric with respect to the y-axis so the area is

$$2\left\{\int_0^{\frac{\sqrt{3}}{2}}[e^{-x^2} - 2e^{x^2}(2x^2-1)]dx + \int_{\frac{\sqrt{3}}{2}}^{3}[2e^{-x^2}(2x^2-1) - e^{-x^2}]dx\right\}$$

≈ 4.2614

52. a. $\lim_{x\to\infty} x^p e^{-x} = 0$

b. $f'(x) = x^p e^{-x}(-1) + e^{-x}\cdot px^{p-1}$

$= x^{p-1}e^{-x}(p-x)$

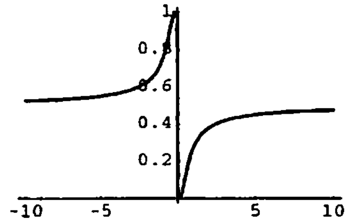
$f'(x) = 0$ when $x = p$

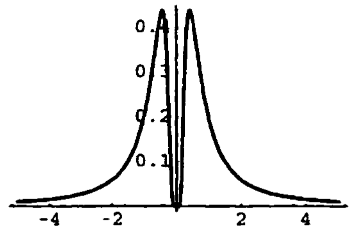
53. $\lim_{x\to-\infty} \ln(x^2 + e^{-x}) = \infty$ (behaves like $-x$)

$\lim_{x\to\infty} \ln(x^2 + e^{-x}) = \infty$ (behaves like $2\ln x$)

54. $f'(x) = -(1+e^{x^{-1}})^{-2}\cdot e^{x^{-1}}(-x^{-2})$

$= \frac{e^{1/x}}{x^2(1+e^{1/x})^2}$





a. $\lim_{x\to 0^+} f(x) = 0$

b. $\lim_{x\to 0^-} f(x) = 1$

c. $\lim_{x\to\pm\infty} f(x) = \frac{1}{2}$

d. $\lim_{x\to 0} f'(x) = 0$

e. f has no minimum or maximum values.

7.4 Concepts Review

1. $e^{\sqrt{3}\ln\pi}$; $e^{x\ln a}$

2. e

3. $\frac{\ln x}{\ln a}$

4. ax^{a-1}; $a^x \ln a$

Problem Set 7.4

1. $2^x = 8 = 2^3$; $x = 3$

2. $x = 5^2 = 25$

3. $x = 4^{3/2} = 8$

4. $x^4 = 64$

$x = \sqrt[4]{64} = 2\sqrt{2}$

5. $\log_9\left(\frac{x}{3}\right)=\frac{1}{2}$

$\frac{x}{3}=9^{1/2}=3$

$x=9$

6. $4^3=\frac{1}{2x}$

$x=\frac{1}{2\cdot 4^3}=\frac{1}{128}$

7. $\log_2(x+3)-\log_2 x=2$

$\log_2\frac{x+3}{x}=2$

$\frac{x+3}{x}=2^2=4$

$x+3=4x$

$x=1$

8. $\log_5(x+3)-\log_5 x=1$

$\log_5\frac{x+3}{x}=1$

$\frac{x+3}{x}=5^1=5$

$x+3=5x$

$x=\frac{3}{4}$

9. $\log_5 12=\frac{\ln 12}{\ln 5}\approx 1.544$

10. $\log_7 0.11=\frac{\ln 0.11}{\ln 7}\approx -1.1343$

11. $\log_{11}(8.12)^{1/5}=\frac{1}{5}\frac{\ln 8.12}{\ln 11}\approx 0.1747$

12. $\log_{10}(8.57)^7=7\frac{\ln 8.57}{\ln 10}\approx 6.5309$

13. $x\ln 2=\ln 17$

$x=\frac{\ln 17}{\ln 2}\approx 4.08746$

14. $x\ln 5=\ln 13$

$x=\frac{\ln 13}{\ln 5}\approx 1.5937$

15. $(2s-3)\ln 5=\ln 4$

$2s-3=\frac{\ln 4}{\ln 5}$

$s=\frac{1}{2}\left(3+\frac{\ln 4}{\ln 5}\right)\approx 1.9307$

16. $\frac{1}{\theta-1}\ln 12=\ln 4$

$\frac{\ln 12}{\ln 4}=\theta-1$

$\theta=1+\frac{\ln 12}{\ln 4}\approx 2.7925$

17. $D_x(6^{2x})=6^{2x}\ln 6\cdot D_x(2x)=2\cdot 6^{2x}\ln 6$

18. $D_x(3^{2x^2-3x})=3^{2x^2-3x}\ln 3\cdot D_x(2x^2-3x)$

$=(4x-3)\cdot 3^{2x^2-3x}\ln 3$

19. $D_x\log_3 e^x=\frac{1}{e^x\ln 3}\cdot D_x e^x$

$=\frac{e^x}{e^x\ln 3}=\frac{1}{\ln 3}\approx 0.9102$

Alternate method:

$D_x\log_3 e^x=D_x(x\log_3 e)=\log_3 e$

$=\frac{\ln e}{\ln 3}=\frac{1}{\ln 3}\approx 0.9102$

20. $D_x\log_{10}(x^3+9)=\frac{1}{(x^3+9)\ln 10}\cdot D_x(x^3+9)$

$=\frac{3x^2}{(x^3+9)\ln 10}$

21. $D_z[3^z\ln(z+5)]$

$=3^z\cdot\frac{1}{z+5}(1)+\ln(z+5)\cdot 3^z\ln 3$

$=3^z\left[\frac{1}{z+5}+\ln(z+5)\ln 3\right]$

22. $D_\theta\sqrt{\log_{10}(3^{\theta^2-\theta})}=D_\theta\sqrt{(\theta^2-\theta)\log_{10}3}$

$=D_\theta\sqrt{\frac{(\theta^2-\theta)\ln 3}{\ln 10}}=\sqrt{\frac{\ln 3}{\ln 10}}\cdot D_\theta\sqrt{\theta^2-\theta}$

$=\sqrt{\frac{\ln 3}{\ln 10}}\cdot\frac{1}{2}(\theta^2-\theta)^{-1/2}(2\theta-1)$

$=\frac{2\theta-1}{2\sqrt{\theta^2-\theta}}\sqrt{\frac{\ln 3}{\ln 10}}$

23. Let $u = x^2$ so $du = 2xdx$.

$$\int x \cdot 2^{x^2}\,dx = \frac{1}{2}\int 2^u\,du = \frac{1}{2}\cdot\frac{2^u}{\ln 2} + C$$

$$= \frac{2^{x^2}}{2\ln 2} + C \;=\; \frac{2^{x^2-1}}{\ln 2} + C$$

24. Let $u = 5x - 1$, so $du = 5\,dx$.

$$\int 10^{5x-1}dx = \frac{1}{5}\int 10^u\,du = \frac{1}{5}\cdot\frac{10^u}{\ln 10} + C$$

$$= \frac{10^{5x-1}}{5\ln 10} + C$$

25. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}dx$.

$$\int \frac{5^{\sqrt{x}}}{\sqrt{x}}dx = 2\int 5^u\,du = 2\cdot\frac{5^u}{\ln 5} + C$$

$$= \frac{2\cdot 5^{\sqrt{x}}}{\ln 5} + C$$

$$\int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}}dx = 2\left[\frac{5^{\sqrt{x}}}{\ln 5}\right]_1^4 = 2\left(\frac{25}{\ln 5} - \frac{5}{\ln 5}\right)$$

$$= \frac{40}{\ln 5} \approx 24.85$$

26. $\int_0^1 (10^{3x} + 10^{-3x})dx = \int_0^1 10^{3x}dx + \int_0^1 10^{-3x}dx$

Let $u = 3x$, so $du = 3dx$.

$$\int 10^{3x}dx = \frac{1}{3}\int 10^u\,du = \frac{1}{3}\cdot\frac{10^u}{\ln 10} + C$$

$$= \frac{10^{3x}}{3\ln 10} + C$$

Now let $u = -3x$, so $du = -3dx$.

$$\int 10^{-3x}dx = -\frac{1}{3}\int 10^u\,du = -\frac{1}{3}\cdot\frac{10^u}{\ln 10} + C$$

$$= -\frac{10^{-3x}}{3\ln 10} + C$$

Thus, $\int_0^1 (10^{3x} + 10^{-3x})dx = \left[\frac{10^{3x} - 10^{-3x}}{3\ln 10}\right]_0^1$

$$= \frac{1}{3\ln 10}\left(1000 - \frac{1}{1000}\right) = \frac{999{,}999}{3000\ln 10}$$

≈ 144.76

27. $\frac{d}{dx}10^{(x^2)} = 10^{(x^2)}\ln 10\frac{d}{dx}x^2 = 10^{(x^2)}2x\ln 10$

$\frac{d}{dx}(x^2)^{10} = \frac{d}{dx}x^{20} = 20x^{19}$

$\frac{dy}{dx} = \frac{d}{dx}[10^{(x^2)} + (x^2)^{10}]$

$= 10^{(x^2)}2x\ln 10 + 20x^{19}$

28. $\frac{d}{dx}\sin^2 x = 2\sin x\frac{d}{dx}\sin x = 2\sin x\cos x$

$\frac{d}{dx}2^{\sin x} = 2^{\sin x}\ln 2\frac{d}{dx}\sin x = 2^{\sin x}\ln 2\cos x$

$\frac{dy}{dx} = \frac{d}{dx}(\sin^2 x + 2^{\sin x})$

$= 2\sin x\cos x + 2^{\sin x}\cos x\ln 2$

29. $\frac{d}{dx}x^{\pi+1} = (\pi+1)x^{\pi}$

$\frac{d}{dx}(\pi+1)^x = (\pi+1)^x\ln(\pi+1)$

$\frac{dy}{dx} = \frac{d}{dx}[x^{\pi+1} + (\pi+1)^x]$

$= (\pi+1)x^{\pi} + (\pi+1)^x\ln(\pi+1)$

30. $\frac{d}{dx}2^{(e^x)} = 2^{(e^x)}\ln 2\frac{d}{dx}e^x = 2^{(e^x)}e^x\ln 2$

$\frac{d}{dx}(2^e)^x = (2^e)^x\ln 2^e = (2^e)^x e\ln 2$

$\frac{dy}{dx} = \frac{d}{dx}[2^{(e^x)} + (2^e)^x]$

$= 2^{(e^x)}e^x\ln 2 + (2^e)^x e\ln 2$

31. $y = (x^2+1)^{\ln x} = e^{(\ln x)\ln(x^2+1)}$

$\frac{dy}{dx} = e^{(\ln x)\ln(x^2+1)}\frac{d}{dx}[(\ln x)\ln(x^2+1)]$

$= e^{(\ln x)\ln(x^2+1)}\left[\frac{1}{x}\ln(x^2+1) + \ln x\frac{2x}{x^2+1}\right]$

$= (x^2+1)^{\ln x}\left(\frac{\ln(x^2+1)}{x} + \frac{2x\ln x}{x^2+1}\right)$

32. $y = (\ln x^2)^{2x+3} = e^{(2x+3)\ln(\ln x^2)}$

$\frac{dy}{dx} = e^{(2x+3)\ln(\ln x^2)}\frac{d}{dx}[(2x+3)\ln(\ln x^2)]$

$= e^{(2x+3)\ln(\ln x^2)}\left[2\ln(\ln x^2) + (2x+3)\frac{1}{\ln x^2}\frac{1}{x^2}(2x)\right]$

$= \underbrace{(2\ln x)}_{\ln x^2}{}^{2x+3}\left[2\ln\underbrace{(2\ln x)}_{\ln x^2} + \frac{2x+3}{x\ln x}\right]$

33. $f(x) = x^{\sin x} = e^{\sin x \ln x}$

$f'(x) = e^{\sin x \ln x} \frac{d}{dx}(\sin x \ln x)$

$= e^{\sin x \ln x}\left[(\sin x)\left(\frac{1}{x}\right) + (\cos x)(\ln x)\right]$

$= x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$

$f'(1) = 1^{\sin 1}\left(\frac{\sin 1}{1} + \cos 1 \ln 1\right) = \sin 1 \approx 0.8415$

34. $f(e) = \pi^e \approx 22.46$

$g(e) = e^{\pi} \approx 23.14$

$g(e)$ is larger than $f(e)$.

$f'(x) = \frac{d}{dx}\pi^x = \pi^x \ln \pi$

$f'(e) = \pi^e \ln \pi \approx 25.71$

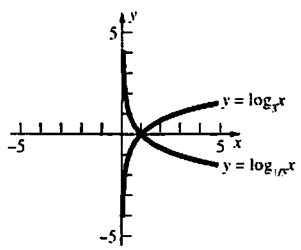
$g'(x) = \frac{d}{dx}x^{\pi} = \pi x^{\pi-1}$

$g'(e) = \pi e^{\pi-1} \approx 26.74$

$g'(e)$ is larger than $f'(e)$.

35. $\log_{1/2} x = \frac{\ln x}{\ln \frac{1}{2}} = \frac{\ln x}{-\ln 2} = -\log_2 x$

36.



37. $M = 0.67\log_{10}(0.37E) + 1.46$

$\log_{10}(0.37E) = \frac{M - 1.46}{0.67}$

$E = \frac{10^{\frac{M-1.46}{0.67}}}{0.37}$

Evaluating this expression for $M = 7$ and $M = 8$ gives $E \approx 5.017 \times 10^8$ kW-h and $E \approx 1.560 \times 10^{10}$ kW-h, respectively.

38. $115 = 20\log_{10}(121.3P)$

$\log_{10}(121.3P) = 5.75$

$P = \frac{10^{5.75}}{121.3} \approx 4636$ lb/in.^{2}

39. If r is the ratio between the frequencies of successive notes, then the frequency of $\bar{C} = r^{12}$ (the frequency of C). Since $\bar{C}$ has twice the frequency of C, $r = 2^{1/12} \approx 1.0595$

Frequency of $\bar{C} = 440(2^{1/12})^3 = 440\sqrt[4]{2} \approx 523.25$

40. Assume $\log_2 3 = \frac{p}{q}$ where p and q are integers, $q \neq 0$. Then $2^{p/q} = 3$ or $2^p = 3^q$. But $2^p = 2 \cdot 2 \ldots 2$ (p times) and has only powers of 2 as factors and $3^q = 3 \cdot 3 \ldots 3$ (q times) and has only powers of 3 as factors.

$2^p = 3^q$ only for $p = q = 0$ which contradicts our assumption, so $\log_2 3$ cannot be rational.

41. If $y = A \cdot b^x$, then $\ln y = \ln A + x \ln b$, so the $\ln y$ vs. x plot will be linear.

If $y = C \cdot x^d$, then $\ln y = \ln C + d \ln x$, so the $\ln y$ vs. $\ln x$ plot will be linear.

42. WRONG 1:

$y = f(x)^{g(x)}$

$y' = g(x)f(x)^{g(x)-1}f'(x)$

WRONG 2:

$y = f(x)^{g(x)}$

$y' = f(x)^{g(x)}(\ln f(x)) \cdot g'(x) = f(x)^{g(x)}g'(x)\ln f(x)$

RIGHT:

$y = f(x)^{g(x)} = e^{g(x)\ln f(x)}$

$y' = e^{g(x)\ln f(x)}\frac{d}{dx}[g(x)\ln f(x)]$

$= f(x)^{g(x)}\left[g'(x)\ln f(x) + g(x)\frac{1}{f(x)}f'(x)\right]$

$= f(x)^{g(x)}g'(x)\ln f(x) + f(x)^{g(x)-1}g(x)f'(x)$

Note that RIGHT = WRONG 2 + WRONG 1.

43. $f(x) = (x^x)^x = x^{(x^2)} \neq x^{(x^x)} = g(x)$

$f(x) = x^{(x^2)} = e^{x^2 \ln x}$

$f'(x) = e^{x^2 \ln x}\frac{d}{dx}(x^2 \ln x) = e^{x^2 \ln x}\left(2x\ln x + x^2 \cdot \frac{1}{x}\right)$

$= x^{(x^2)}(2x\ln x + x)$

$g(x) = x^{(x^x)} = e^{x^x \ln x}$

Using the result from Example 5 $\left(\frac{d}{dx}x^x = x^x(1 + \ln x)\right)$:

$g'(x) = e^{x^x \ln x}\frac{d}{dx}(x^x \ln x)$

$$= e^{x^x \ln x}\left[x^x(1+\ln x)\ln x + x^x \cdot \frac{1}{x}\right]$$

$$= x^{(x^x)}x^x\left[(1+\ln x)\ln x + \frac{1}{x}\right]$$

$$= x^{x^x+x}\left[\ln x + (\ln x)^2 + \frac{1}{x}\right]$$

44. $f(x) = \frac{a^x - 1}{a^x + 1}$

$$f'(x) = \frac{(a^x+1)a^x \ln a - (a^x - 1)a^x \ln a}{(a^x+1)^2} = \frac{2a^x \ln a}{(a^x+1)^2}$$

Since a is positive, a^x is always positive. $(a^x+1)^2$ is also always positive, thus $f'(x) > 0$ if $\ln a > 0$ and $f'(x) < 0$ if $\ln a < 0$. $f(x)$ is either always increasing or always decreasing, depending on a, so $f(x)$ has an inverse.

$$y = \frac{a^x - 1}{a^x + 1}$$

$$y(a^x + 1) = a^x - 1$$

$$a^x(y - 1) = -1 - y$$

$$a^x = \frac{1+y}{1-y}$$

$$x \ln a = \ln \frac{1+y}{1-y}$$

$$x = \frac{\ln \frac{1+y}{1-y}}{\ln a} = \log_a \frac{1+y}{1-y}$$

$$f^{-1}(y) = \log_a \frac{1+y}{1-y}$$

$$f^{-1}(x) = \log_a \frac{1+x}{1-x}$$

45. a. Let $g(x) = \ln f(x) = \ln\left(\frac{x^a}{a^x}\right) = a \ln x - x \ln a$.

$$g'(x) = \left(\frac{a}{x}\right) - \ln a$$

$g'(x) < 0$ when $x > \frac{a}{\ln a}$, so as $x \to \infty$ $g(x)$ is decreasing. $g''(x) = -\frac{a}{x^2}$, so $g(x)$ is concave down. Thus, $\lim_{x\to\infty} g(x) = -\infty$, so $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{g(x)} = 0$.

b. Again let $g(x) = \ln f(x) = a \ln x - x \ln a$. Since $y = \ln x$ is an increasing function, $f(x)$ is maximized when $g(x)$ is maximized.

$g'(x) = \left(\frac{a}{x}\right) - \ln a$, so $g'(x) > 0$ on $\left(0, \frac{a}{\ln a}\right)$ and $g'(x) < 0$ on $\left(\frac{a}{\ln a}, \infty\right)$.

Therefore, $g(x)$ (and hence $f(x)$) is maximized at $x_0 = \frac{a}{\ln a}$.

c. Note that $x^a = a^x$ is equivalent to $g(x) = 0$.

By part b., $g(x)$ is maximized at $x_0 = \frac{a}{\ln a}$.

If $a = e$, then

$$g(x_0) = g\left(\frac{e}{\ln e}\right) = g(e) = e \ln e - e \ln e = 0.$$

Since $g(x) < g(x_0) = 0$ for all $x \neq x_0$, the equation $g(x) = 0$ (and hence $x^a = a^x$) has just one positive solution. If $a \neq e$, then

$$g(x_0) = g\left(\frac{a}{\ln a}\right) = a \ln\left(\frac{a}{\ln a}\right) - \frac{a}{\ln a}(\ln a)$$

$$= a\left[\ln\left(\frac{a}{\ln a}\right) - 1\right].$$

Now $\frac{a}{\ln a} > e$ (justified below), so

$g(x_0) = a\left[\ln \frac{a}{\ln a} - 1\right] > a(\ln e - 1) = 0$. Since $g'(x) > 0$ on $(0, x_0)$, $g(x_0) > 0$, and $\lim_{x\to 0} g(x) = -\infty$, $g(x) = 0$ has exactly one solution on $(0, x_0)$.

Since $g'(x) < 0$ on (x_0, ∞), $g(x_0) > 0$, and $\lim_{x\to\infty} g(x) = -\infty$, $g(x) = 0$ has exactly one solution on (x_0, ∞). Therefore, the equation $g(x) = 0$ (and hence $x^a = a^x$) has exactly two positive solutions.

To show that $\frac{a}{\ln a} > e$ when $a \neq e$:

Consider the function $h(x) = \frac{x}{\ln x}$, for $x > 1$.

$$h'(x) = \frac{\ln(x)(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

Note that $h'(x) < 0$ on $(1, e)$ and $h'(x) > 0$ on (e, ∞), so $h(x)$ has its minimum at (e, e).

Therefore $\frac{x}{\ln x} > e$ for all $x \neq e, x > 1$.

d. For the case $a = e$, part c. shows that $g(x) = e\ln x - x\ln e < 0$ for $x \neq e$. Therefore, when $x \neq e$, $\ln x^e < \ln e^x$, which implies $x^e < e^x$. In particular, $\pi^e < e^\pi$.

46. a. $f_u(x) = x^u e^{-x}$

$f_u'(x) = ux^{u-1}e^{-x} - x^u e^{-x} = (u-x)x^{u-1}e^{-x}$

Since $f_u'(x) > 0$ on $(0, u)$ and $f_u'(x) < 0$ on (u, ∞), $f_u(x)$ attains its maximum at $x_0 = u$.

b. $f_u(u) > f_u(u+1)$ means

$u^u e^{-u} > (u+1)^u e^{-(u+1)}$.

Multiplying by $\frac{e^{u+1}}{u^u}$ gives $e > \left(\frac{u+1}{u}\right)^u$.

$f_{u+1}(u+1) > f_{u+1}(u)$ means

$(u+1)^{u+1}e^{-(u+1)} > u^{u+1}e^{-u}$.

Multiplying by $\frac{e^{u+1}}{u^{u+1}}$ gives $\left(\frac{u+1}{u}\right)^{u+1} > e$.

Combining the two inequalities,

$\left(\frac{u+1}{u}\right)^u < e < \left(\frac{u+1}{u}\right)^{u+1}$.

c. From part b., $e < \left(\frac{u+1}{u}\right)^{u+1}$.

Multiplying by $\frac{u}{u+1}$ gives $\frac{u}{u+1}e < \left(\frac{u+1}{u}\right)^u$.

We showed $\left(\frac{u+1}{u}\right)^u < e$ in part b., so

$\frac{u}{u+1}e < \left(\frac{u+1}{u}\right)^u < e$.

Since $\lim_{u\to\infty} \frac{u}{u+1}e = e$, this implies that

$\lim_{u\to\infty}\left(\frac{u+1}{u}\right)^u = e$, i.e., $\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u = e$.

47. $f(x) = x^x = e^{x\ln x}$

Let $g(x) = x\ln x$.

Using L'Hôpital's Rule,

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$

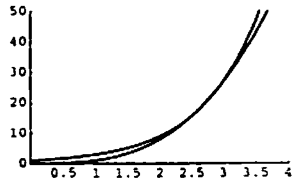
$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$

Therefore, $\lim_{x\to 0^+} x^x = e^0 = 1$.

$g'(x) = \ln x - 1$

Since $g'(x) < 0$ on $(0, e)$ and $g'(x) > 0$ on (e, ∞), $g(x)$ has its minimum at (e, e). Therefore, $f(x)$ has its minimum at (e, e^e).

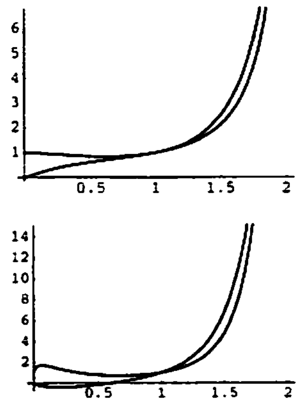
48.



(2.4781, 15.2171), (3, 27)

49. $\int_0^{4\pi} x^{\sin x}dx \approx 20.2259$

50.



7.5 Concepts Review

1. ky; $ky(L-y)$
2. $2^3 = 8$
3. half-life
4. $(1+h)^{1/h}$

Problem Set 7.5

1. $k = -6$, $y_0 = 4$, so $y = 4e^{-6t}$

2. $k = 6$, $y_0 = 1$, so $y = e^{6t}$

3. $k = 0.005$, so $y = y_0e^{0.005t}$

$y(10) = y_0e^{0.005(10)} = y_0e^{0.05}$

$y(10) = 2 \Rightarrow y_0 = \frac{2}{e^{0.05}}$

$y = \frac{2}{e^{0.05}}e^{0.005t} = 2e^{0.005t-0.05} = 2e^{0.005(t-10)}$

4. $k=-0.003$, so $y=y_0e^{-0.003t}$

$$y(-2)=y_0e^{(-0.003)(-2)}=y_0e^{0.006}$$

$$y(-2)=3\Rightarrow y_0=\frac{3}{e^{0.006}}$$

$$y=\frac{3}{e^{0.006}}e^{-0.003t}=3e^{-0.003t-0.006}=3e^{-0.003(t+2)}$$

5. $y_0=10{,}000,\ y(10)=20{,}000$

$$20{,}000=10{,}000e^{k(10)}$$

$$2=e^{10k}$$

$$\ln 2=10k$$

$$k=\frac{\ln 2}{10}\approx 0.069$$

$$y=10{,}000e^{0.069t}$$

After 25 days, $y=10{,}000e^{0.069\cdot 25}\approx 56{,}125$.

6. Since the growth is exponential and it doubles in 10 days (from $t = 0$ to $t = 10$), it will always double in 10 days.

7. $3y_0=y_0e^{0.069t}$

$$3=e^{0.069}$$

$$\ln 3=0.069t$$

$$t=\frac{\ln 3}{0.069}\approx 15.9\text{ days}$$

8. Let $P(t)$ = population (in millions) in year $1790 + t$.
In 1960, $t = 170$.

$$P(t)=P_0e^{kt}$$

$$178=3.9e^{170k}$$

$$45.64=e^{170k}$$

$$k=\frac{\ln 45.64}{170}\approx 0.02248$$

In 2000, $t = 210$

$$P(210)\approx 3.9e^{0.02248\cdot 210}\approx 438$$

The model predicts that the population will be about 438 million. The actual number, 275 million, is quite a bit smaller because the rate of growth has declined in recent decades.

9. 1 year: (4.5 million) (1.032) $\approx$ 4.64 million

2 years: (4.5 million)$(1.032)^2\approx$ 4.79 million

10 years: (4.5 million)$(1.032)^{10}\approx$ 6.17 million

100 years: (4.5 million)$(1.032)^{100}\approx$ 105 million

10. $y=y_0e^{kt}$

$$1.032A=Ae^{k(1)}$$

$$k=\ln 1.032\approx 0.03150$$

At $t = 100$, $y=4.5e^{(0.03150)(100)}\approx 105$.
After 100 years, the population will be about 105 million.

11. $\frac{1}{2}=e^{k(700)}$ and $y_0=10$

$$-\ln 2=700k$$

$$k=-\frac{\ln 2}{700}\approx -0.00099$$

$$y=10e^{-0.00099t}$$

At $t = 300$, $y=10e^{-0.00099\cdot 300}\approx 7.43$.
After 10 years there will be about 7.43 g.

12. $0.85=e^{k(2)}$

$$\ln 0.85=2k$$

$$k=\frac{\ln 0.85}{2}\approx -0.0813$$

$$\frac{1}{2}=e^{-0.0813t}$$

$$-\ln 2=-0.0813t$$

$$t=\frac{\ln 2}{0.0813}\approx 8.53$$

The half-life is about 8.53 days.

13. $\frac{1}{2}=e^{5730k}$

$$k=\frac{\ln\left(\frac{1}{2}\right)}{5730}\approx -1.210\times 10^{-4}$$

$$0.7y_0=y_0e^{(-1.210\times 10^{-4})t}$$

$$t=\frac{\ln 0.7}{-1.210\times 10^{-4}}\approx 2950$$

The fort burned down about 2950 years ago.

14. $\frac{1}{2}=e^{5730k}$

$$k=\frac{\ln\left(\frac{1}{2}\right)}{5730}\approx -1.210\times 10^{-4}$$

$$0.51y_0=y_0e^{(-1.210\times 10^{-4})t}$$

$$t=\frac{\ln 0.51}{-1.210\times 10^{-4}}\approx 5565$$

The body was buried about 5565 years ago.

15. $\frac{dT}{dt}=k(T-75)$

$$\int\frac{1}{T-75}dT=\int k\,dt$$

$$\ln|T-75|=kt+C$$

$$|T-75|=e^{kt+C}$$

$$T=75+Ae^{kt}$$

$T(0) = 75 + A$
$T(0) = 300 \Rightarrow A = 225$
$T(0.5) = 200 \Rightarrow 200 = 75 + 225e^{0.5k};$
$\frac{125}{225} = e^{0.5k}; k = 2\ln\frac{5}{9} \approx -1.176$
$T = 75 + 225e^{-1.176t}$
At $t = 3$, $T = 75 + 225e^{-1.176 \cdot 3} \approx 81.6$.
After 3 hours, the temperature will be about 81.6°F.

16. $\frac{dT}{dt} = k(T-24)$

$\int \frac{dT}{T-24} = \int k\,dt$
$\ln|T-24| = kt + C$
$|T-24| = e^{kt+C}$
$T = 24 + Ae^{kt}$
$T(0) = -20 \Rightarrow A = -44.$
$T = 24 - 44e^{kt}$
$T(5) = 0 \Rightarrow 0 = 24 - 44e^{5k}.$
$k = \frac{\ln\left(\frac{24}{44}\right)}{5} \approx -0.1212$
$T = 24 - 44e^{-0.1212t}$
$20 = 24 - 44e^{-0.1212t}$
$e^{-0.1212t} = \frac{1}{11}$
$t = \frac{\ln\left(\frac{1}{11}\right)}{-0.1212} \approx 19.78$
It will register 20°C after about 19.78 minutes.

17. a. $(\$375)(1.095)^2 \approx \$449.63$

b. $(\$375)\left(1+\frac{0.095}{12}\right)^{24} \approx \$453.13$

c. $(\$375)\left(1+\frac{0.095}{365}\right)^{730} \approx \$453.46$

d. $(\$375)e^{0.095 \cdot 2} \approx \$453.47$

18. a. $(\$375)(1.144)^2 = \$490.78$

b. $(\$375)\left(1+\frac{0.144}{12}\right)^{24} \approx \$499.30$

c. $(\$375)\left(1+\frac{0.144}{365}\right)^{730} \approx \$500.13$

d. $(\$375)e^{0.144 \cdot 2} \approx \$500.16$

$T = 75 + 225e^{kt}$

19. a. $\left(1+\frac{0.12}{12}\right)^{12t} = 2$

$1.01^{12t} = 2$
$12t = \log_{1.01} 2$
$t = \frac{1}{12}\log_{1.01} 2 = \frac{\ln 2}{12\ln 1.01} \approx 5.805$
It will take about 5.805 years or 5 years, 10 months.

b. $e^{0.12t} = 2 \Rightarrow t = \frac{\ln 2}{0.12} \approx 5.776$
It will take about 5.776 years or 5 years, 9 months, and 9 days.

20. $\$4000(1.115)^4 \approx \$6182$

21. 1626 to 2000 is 374 years.
$y = 24e^{0.06 \cdot 374} \approx \$133.6$ billion

22. $\$100(1.08)^{969} \approx \$2.441 \times 10^{34}$

23. If t is the doubling time, then
$\left(1+\frac{p}{100}\right)^t = 2$
$t\ln\left(1+\frac{p}{100}\right) = \ln 2$
$t = \frac{\ln 2}{\ln\left(1+\frac{p}{100}\right)} \approx \frac{\ln 2}{\frac{p}{100}} = \frac{100\ln 2}{p} \approx \frac{70}{p}$

24. $\frac{dy}{dt} = ky(L-y)$

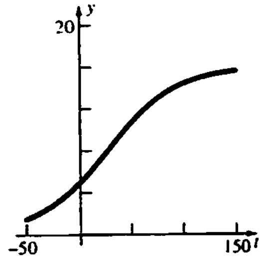
$\frac{1}{y(L-y)}dy = kdt$
$\left[\frac{1}{Ly} + \frac{1}{L(L-y)}\right]dy = kdt$
$\frac{1}{L}\int\left(\frac{1}{y} + \frac{1}{L-y}\right)dy = \int kdt$
$\frac{1}{L}[\ln|y| - \ln|L-y|] = kt + C_1$
$\ln\left|\frac{y}{L-y}\right| = Lkt + LC_1$
$\left|\frac{y}{L-y}\right| = e^{Lkt+LC_1} = e^{LC_1} \cdot e^{Lkt}$, so $\frac{y}{L-y} = Ce^{Lkt}$
$\left(\text{Note that: } C = Ce^0 = Ce^{Lk \cdot 0} = \frac{y(0)}{L-y(0)} = \frac{y_0}{L-y_0}.\right)$
$y = LCe^{Lkt} - yCe^{Lkt}$

$$y + yCe^{Lkt} = LCe^{Lkt}$$

$$y = \frac{LCe^{Lkt}}{1+Ce^{Lkt}} = \frac{LC}{\frac{1}{e^{Lkt}}+C} = \frac{LC}{C+e^{-Lkt}}$$

$$= \frac{L\cdot\frac{y_0}{L-y_0}}{\frac{y_0}{L-y_0}+e^{-Lkt}} = \frac{Ly_0}{y_0+(L-y_0)e^{-Lkt}}$$

25. $y = \dfrac{16\cdot 5}{5+(16-5)e^{-16(0.00186)t}} = \dfrac{80}{5+11e^{-0.02976t}}$



26. a. $\lim_{x\to 0}(1+x)^{1000} = 1^{1000} = 1$

b. $\lim_{x\to 0} 1^{1/x} = \lim_{x\to 0} 1 = 1$

c. $\lim_{x\to 0^+}(1+\varepsilon)^{1/x} = \lim_{n\to\infty}(1+\varepsilon)^n = \infty$

d. $\lim_{x\to 0^-}(1+\varepsilon)^{1/x} = \lim_{n\to\infty}\dfrac{1}{(1+\varepsilon)^n} = 0$

e. $\lim_{x\to 0}(1+x)^{1/x} = e$

27. a. $\lim_{x\to 0}(1-x)^{1/x} = \lim_{x\to 0}\dfrac{1}{[1+(-x)]^{1/(-x)}} = \dfrac{1}{e}$

b. $\lim_{x\to 0}(1+3x)^{1/x} = \lim_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^3 = e^3$

c. $\lim_{n\to\infty}\left(\dfrac{n+2}{n}\right)^n = \lim_{n\to\infty}\left(1+\dfrac{2}{n}\right)^n$

$= \lim_{x\to 0^+}(1+2x)^{1/x}$

$= \lim_{x\to 0^+}\left[(1+2x)^{\frac{1}{2x}}\right]^2 = e^2$

d. $\lim_{n\to\infty}\left(\dfrac{n-1}{n}\right)^{2n} = \lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{2n}$

$= \lim_{x\to 0^+}(1-x)^{2/x}$

$= \lim_{x\to 0^+}\left[(1-x)^{\frac{1}{-x}}\right]^{-2} = \dfrac{1}{e^2}$

28. $\dfrac{dy}{dt} = ay+b$

$\int \dfrac{dy}{y+\frac{b}{a}} = \int a\,dt$

$\ln\left|y+\dfrac{b}{a}\right| = at + C$

$\left|y+\dfrac{b}{a}\right| = e^{at+C};\ y+\dfrac{b}{a} = Ae^{at}$

$y = Ae^{at} - \dfrac{b}{a}$

$y_0 = A - \dfrac{b}{a} \Rightarrow A = y_0 + \dfrac{b}{a}$

$y = \left(y_0+\dfrac{b}{a}\right)e^{at} - \dfrac{b}{a}$

29. Let y = population in millions, $t = 0$ in 1985, $a = 0.012$, $b = 0.06$, $y_0 = 10$

$\dfrac{dy}{dt} = 0.012y + 0.06$

$y = \left(10+\dfrac{0.06}{0.012}\right)e^{0.012t} - \dfrac{0.06}{0.012} = 15e^{0.012t} - 5$

From 1985 to 2010 is 25 years. At $t = 25$, $y = 15e^{0.012\cdot 25} - 5 \approx 15.25$. The population in 2010 will be about 15.25 million.

30. Let $N(t)$ be the number of people who have heard the news after t days. Then $\dfrac{dN}{dt} = k(L-N)$.

$\int \dfrac{1}{L-N}dN = \int k\,dt$

$-\ln(L-N) = kt + C$

$L - N = e^{-kt-C}$

$N = L - Ae^{-kt}$

$N(0) = 0,\ \Rightarrow\ A = L$

$N(t) = L(1-e^{-kt})$.

$N(5) = \dfrac{L}{2}\ \Rightarrow\ \dfrac{L}{2} = L(1-e^{-5k})$

$\dfrac{1}{2} = e^{-5k}$

$k = \dfrac{\ln\frac{1}{2}}{-5} \approx 0.1386$

$N(t) = L(1 - e^{-0.1386t})$

$0.99L = L(1 - e^{-0.1386t})$

$0.01 = e^{-0.1386t}$

$t = \frac{\ln 0.01}{-0.1386} \approx 33$

99% of the people will have heard about the scandal after 33 days.

31. Maximum population:

$13{,}500{,}000 \text{ mi}^2 \cdot \frac{640 \text{ acres}}{1 \text{ mi}^2} \cdot \frac{1 \text{ person}}{\frac{1}{2} \text{ acre}}$

$= 1.728 \times 10^{10}$ people

Let $t = 0$ be in 1998.

$(5.9 \times 10^9)e^{0.0132t} = 1.728 \times 10^{10}$

$t = \frac{\ln\left(\frac{1.728 \cdot 10^{10}}{5.9 \cdot 10^9}\right)}{0.0132} \approx 81.4$ years from 1998, or sometime in the year 2079.

32. a. $k = 0.0132 - 0.0002t$

b. $y' = (0.0132 - 0.0002t)y$

c. $\frac{dy}{dt} = (0.0132 - 0.0002t)y$

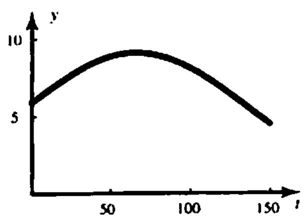
$\frac{dy}{y} = (0.0132 - 0.0002t)dt$

$\ln y = 0.0132t - 0.0001t^2 + C_0$

$y = C_1 e^{0.0132t - 0.0001t^2}$

The initial condition $y(0) = 5.9$ implies that $C_1 = 5.9$. Thus $y = 5.9e^{0.0132t - 0.0001t^2}$

d.



e. The maximum population will occur when

$\frac{d}{dt}\left(0.0132t - 0.0001t^2\right) = 0$

$0.0132 = 0.0002t$

$t = 0.0132 / 0.0002 = 66$

$t = 66$, which is year 2064.

The population will equal the 1998 value of 5.9 billion when $0.0132t - 0.0001t^2 = 0$

$t = 0$ or $t = 132$.

The model predicts that the population will return to the 1998 level in year 2130.

33. a. $k = 0.0132 - 0.0001t$

b. $y' = (0.0132 - 0.0001t)y$

c. $\frac{dy}{dt} = (0.0132 - 0.0001t)y$

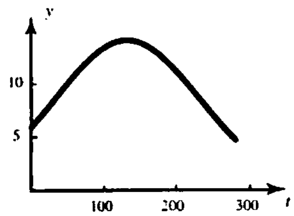
$\frac{dy}{y} = (0.0132 - 0.0001t)dt$

$\ln y = 0.0132t - 0.00005t^2 + C_0$

$y = C_1 e^{0.0132t - 0.00005t^2}$

The initial condition $y(0) = 5.9$ implies that $C_1 = 5.9$. Thus $y = 5.9e^{0.0132t - 0.00005t^2}$

d.



e. The maximum population will occur when

$\frac{d}{dt}\left(0.0132t - 0.00005t^2\right) = 0$

$0.0132 = 0.0001t$

$t = 0.0132 / 0.0001 = 132$

$t = 132$, which is year 2130.

The population will equal the 1998 value of 5.9 billion when $0.0132t - 0.00005t^2 = 0$

$t = 0$ or $t = 264$.

The model predicts that the population will return to the 1998 level in year 2262.

34. $E'(x) = \lim_{h \to 0} \frac{E(x+h) - E(x)}{h}$

$= \lim_{h \to 0} \frac{E(x)E(h) - E(x)}{h}$

$= \lim_{h \to 0} E(x) \cdot \frac{E(h) - 1}{h} = E(x) \lim_{h \to 0} \frac{E(h) - 1}{h}$

$E(x) = E(x + 0) = E(x) \cdot E(0)$

so $E(0) = 1$.

Thus, $E'(x) = E(x) \lim_{h \to 0} \frac{E(h) - E(0)}{h}$

$= E(x) \lim_{h \to 0} \frac{E(0+h) - E(0)}{h} = E(x) \cdot E'(0)$

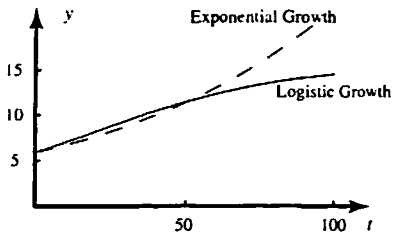
$= kE(x)$ where $k = E'(0)$.

Hence, $E(x) = E_0 e^{kx} = E(0)e^{kx} = 1 \cdot e^{kx} = e^{kx}$.

Check: $E(u+v) = e^{k(u+v)} = e^{ku+kv}$

$= e^{ku} \cdot e^{kv} = E(u) \cdot E(v)$

35.



Exponential growth:
In 2010 ($t = 12$): 6.91 billion
In 2040 ($t = 42$): 10.27 billion
In 2090 ($t = 92$): 19.87 billion
Logistic growth:
In 2010 ($t = 12$): 7.29 billion
In 2040 ($t = 42$): 10.74 billion
In 2090 ($t = 92$): 14.28 billion

36. a. $\lim_{x\to 0}(1+x)^{1/x} = e$

b. $\lim_{x\to 0}(1-x)^{1/x} = \frac{1}{e}$

7.6 Concepts Review

1. $\exp\left(\int P(x)dx\right)$

2. $y\exp\left(\int P(x)dx\right)$

3. $\frac{1}{x}; \frac{d}{dx}\left(\frac{y}{x}\right) = 1; x^2 + Cx$

4. particular

Problem Set 7.6

1. Integrating factor is e^x.

$D(ye^x) = 1$

$y = e^{-x}(x + C)$

2. The left-hand side is already an exact derivative.

$D[y(x+1)] = x^2 - 1$

$y = \frac{x^3 - 3x + C}{3(x+1)}$

3. $y' + \frac{x}{1-x^2}y = \frac{ax}{1-x^2}$

Integrating factor:

$\exp\int \frac{x}{1-x^2}dx = \exp\left[\ln(1-x^2)^{-1/2}\right]$

$= (1-x^2)^{-1/2}$

$D[y(1-x^2)^{-1/2}] = ax(1-x^2)^{-3/2}$ (see note 2 above.)

Then $y(1-x^2)^{-1/2} = a(1-x^2)^{-1/2} + C$, so

$y = a + C(1-x^2)^{1/2}$.

4. Integrating factor is $\sec x$.

$D[y\sec x] = \sec^2 x$

$y = \sin x + C\cos x$

5. Integrating factor is $\frac{1}{x}$.

$D\left[\frac{y}{x}\right] = e^x$

$y = xe^x + Cx$

6. $y' - ay = f(x)$

Integrating factor: $e^{\int -a dx} = e^{-ax}$

$D[ye^{-ax}] = e^{-ax}f(x)$

Then $ye^{-ax} = \int e^{-ax}f(x)dx$, so

$y = e^{ax}\int e^{-ax}f(x)dx$.

7. Integrating factor is x. $D[yx] = 1$; $y = 1 + Cx^{-1}$

8. Integrating factor is $(x+1)^2$.

$D[y(x+1)^2] = (x+1)^5$

$y = \left(\frac{1}{6}\right)(x+1)^4 + C(x+1)^{-2}$

9. $y' + f(x)y = f(x)$

Integrating factor: $e^{\int f(x)dx}$

$D\left[ye^{\int f(x)dx}\right] = f(x)e^{\int f(x)dx}$

Then $ye^{\int f(x)dx} = e^{\int f(x)dx} + C$, so

$y = 1 + Ce^{-\int f(x)dx}$.

10. Integrating factor is e^{2x}.

$D[ye^{2x}] = xe^{2x}$

$y = \left(\frac{1}{2}\right)x - \left(\frac{1}{4}\right) + Ce^{-2x}$

11. Integrating factor is $\frac{1}{x}$. $D\left[\frac{y}{x}\right] = 3x^2$; $y = x^4 + Cx$

$y = x^4 + 2x$ goes through (1, 3).

12. $y' + 3y = e^{2x}$

Integrating factor: $e^{\int 3dx} = e^{3x}$

$D[ye^{3x}] = e^{5x}$

Then $ye^{3x} = \frac{e^{5x}}{5} + C$. $x = 0, y = 1 \Rightarrow C = \frac{4}{5}$, so

$ye^{3x} = \frac{e^{5x}}{5} + \frac{4}{5}$.

Therefore, $y = \frac{e^{2x} + 4e^{-3x}}{5}$ is the particular solution through (0, 1).

13. Integrating factor: xe^x

$d[yxe^x] = 1$: $y = e^{-x}(1 + Cx^{-1})$; $y = e^{-x}(1 - x^{-1})$ through (1, 0).

14. Integrating factor is $\sin^2 x$.

$D[y\sin^2 x] = 2\sin^2 x\cos x$

$y\sin^2 x = \frac{2}{3}\sin^3 x + C$

$y = \frac{2}{3}\sin x + \frac{C}{\sin^2 x}$

$y = \frac{2}{3}\sin x + \frac{5}{12}\csc^2 x$

goes through $\left(\frac{\pi}{6}, 2\right)$.

15. Let y denote the number of pounds of chemical A after t minutes.

$\frac{dy}{dt} = \left(2\frac{\text{lbs}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y\text{ lbs}}{20\text{ gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right)$

$= 6 - \frac{3y}{20}$ lb/min

$y' + \frac{3}{20}y = 6$

Integrating factor: $e^{\int(3/20)dt} = e^{3t/20}$

$D[ye^{3t/20}] = 6e^{3t/20}$

Then $ye^{3t/20} = 40e^{3t/20} + C$. $t = 0, y = 10$
$\Rightarrow C = -30$.

Therefore, $y(t) = 40 - 30e^{-3t/20}$, so

$y(20) = 40 - 30e^{-3} \approx 38.506$ lb.

16. $\frac{dy}{dt} = (2)(4) - \left(\frac{y}{200}\right)(4)$ or $y' + \frac{y}{50} = 8$

Integrating factor is $e^{t/50}$.

$D[ye^{t/50}] = 8e^{t/50}$

$y(t) = 400 + Ce^{-t/50}$

$y(t) = 400 - 350e^{-t/50}$ goes through (0, 50).

$y(40) = 400 - 350e^{-0.8} \approx 242.735$ lb of salt

17. $\frac{dy}{dt} = 4 - \left[\frac{y}{(120 - 2t)}\right](6)$ or $y' + \left[\frac{3}{(60 - t)}\right]y = 4$

Integrating factor is $(60 - t)^{-3}$.

$D[y(60 - t)^{-3}] = 4(60 - t)^{-3}$

$y(t) = 2(60 - t) + C(60 - t)^3$

$y(t) = 2(60 - t) - \left(\frac{1}{1800}\right)(60 - t)^3$ goes through (0, 0).

18. $\frac{dy}{dt} = \frac{-2y}{50 + t}$ or $y' + \frac{2}{50 + t}y = 0$.

Integrating factor:

$\exp\left(\int\frac{2}{50 + t}dt\right) = e^{2\ln(50+t)} = (50 + t)^2$

$D[y(50 + t)^2] = 0$

Then $y(50 + t)^2 = C$. $t = 0, y = 30 \Rightarrow C = 75000$

Thus, $y(50 + t)^2 = 75,000$.

If $y = 25$, $25(50 + t)^2 = 75,000$, so

$t = \sqrt{3000} - 50 \approx 4.772$ min.

19. $I' + 10^6 I = 1$

Integrating factor $= \exp(10^6 t)$

$D[I\exp(10^6 t)] = \exp(10^6 t)$

$I(t) = 10^{-6} + C\exp(-10^6 t)$

$I(t) = 10^{-6}[1 - \exp(-10^6 t)]$ goes through (0, 0).

20. $3.5I' = 120\sin 377t$

$I' = \left(\frac{240}{7}\right)\sin 377t$

$I = \left(-\frac{240}{2639}\right)\cos 377t + C$

$I(t) = \left(\frac{240}{2639}\right)(1 - \cos 377t)$ through (0, 0).

21. $1000\,I = 120 \sin 377t$
$I(t) = 0.12 \sin 377t$

22. $\frac{dx}{dt} = -\frac{2x}{100}$
$x' + \left(\frac{1}{50}\right)x = 0$
Integrating factor is $e^{t/50}$.
$D[xe^{t/50}] = 0$
$x = Ce^{-t/50}$
$x(t) = 50e^{-t/50}$ satisfies $t = 0, x = 50$.
$\frac{dy}{dt} = 2\left(\frac{50e^{-t/50}}{100}\right) - 2\left(\frac{y}{200}\right)$
$y' + \left(\frac{1}{100}\right)y = e^{-t/50}$
Integrating factor is $e^{t/100}$.
$D[ye^{t/100}] = e^{-t/100}$
$y(t) = e^{-t/100}(C - 100e^{-t/100})$
$y(t) = e^{-t/100}(250 - 100e^{-t/100})$ satisfies $t = 0$, $y = 150$.

23. Let y be the number of gallons of pure alcohol in the tank at time t.

a. $y' = \frac{dy}{dt} = 5(0.25) - \left(\frac{5}{100}\right)y = 1.25 - 0.05y$
Integrating factor is $e^{0.05t}$.
$y(t) = 25 + Ce^{-0.05t}$; $y = 100, t = 0, C = 75$
$y(t) = 25 + 75e^{-0.05t}$; $y = 50$, $t = T$,
$T = 20(\ln 3) \approx 21.97$ min

b. Let A be the number of gallons of pure alcohol drained away.
$(100 - A) + 0.25A = 50 \Rightarrow A = \frac{200}{3}$
It took $\frac{\frac{200}{3}}{5}$ minutes for the draining and the same amount of time to refill, so
$T = \frac{2\left(\frac{200}{3}\right)}{5} = \frac{80}{3} \approx 26.67$ min.

c. c would need to satisfy
$\frac{\frac{200}{3}}{5} + \frac{\frac{200}{3}}{c} < 20(\ln 3).$
$c > \frac{10}{(3\ln 3 - 2)} \approx 7.7170$

d. $y' = 4(0.25) - 0.05y = 1 - 0.05y$
Solving for y, as in part a, yields $y = 20 + 80e^{-0.05t}$. The drain is closed when $t = 0.8T$. We require that
$(20 + 80e^{-0.05 \cdot 0.8T}) + 4 \cdot 0.25 \cdot 0.25 \cdot 0.8T = 50$,
or $400e^{-0.04T} + T = 150$.

24. a. $v' + av = -g$
Integrating factor: e^{at}
$e^{at}(v' + av) = -ge^{at}; \frac{d}{dt}(ve^{at}) = -ge^{at}$
$ve^{at} = \int -ge^{at}dt = \frac{-g}{a}e^{at} + C; v = \frac{-g}{a} + Ce^{-at}$
$v = v_0, t = 0$
$v_0 = \frac{-g}{a} + C \Rightarrow C = v_0 + \frac{g}{a}$
Therefore, $v = \frac{-g}{a} + \left(v_0 + \frac{g}{a}\right)e^{-at}$, so
$v(t) = v_\infty + (v_0 - v_\infty)e^{-at}$.

b. $\frac{dy}{dt} = v_\infty + (v_0 - v_\infty)e^{-at}$, so
$y = v_\infty - \frac{(v_0 - v_\infty)e^{-at}}{a} + C.$
$y = y_0, t = 0 \Rightarrow y_0 = \frac{-(v_0 - v_\infty)}{a} + C$
$\Rightarrow C = y_0 + \frac{v_0 - v_\infty}{a}$
$y = v_\infty t - \frac{(v_0 - v_\infty)e^{-at}}{a} + \left(y_0 + \frac{v_0 - v_\infty}{a}\right)$
$= y_0 + v_\infty t + \frac{v_0 - v_\infty}{a}(1 - e^{-at})$

25. a. $v_\infty = -\frac{32}{0.05} = -640$
$v(t) = [120 - (-640)]e^{-0.05t} + (-640) = 0$ if
$t = 20\ln\left(\frac{19}{16}\right)$.
$y(t) = 0 + (-640)t$
$+ \left(\frac{1}{0.05}\right)[120 - (-640)](1 - e^{-0.05t})$
$= -640t + 15{,}200(1 - e^{-0.05t})$
Therefore, the maximum altitude is
$y\left(20\ln\left(\frac{19}{16}\right)\right) = -12{,}800\ln\left(\frac{19}{16}\right) + \frac{45{,}600}{19}$
≈ 200.32

b. $-640T + 15,200(1 - e^{-0.05T}) = 0;$

$95 - 4T - 95e^{-0.05T} = 0$

26. For t in [0, 15],

$v_\infty = \frac{-32}{0.10} = -320.$

$v(t) = (0+320)e^{-0.1t} - 320 = 320(e^{-0.1t} - 1);$

$v(15) = 320(e^{-1.5} - 1) \approx -248.6$

$y(t) = 8000 - 320t + 10(320)(1 - e^{-0.1t});$

$y(15) = 3200(2 - e^{-1.5}) \approx 5686$

Let t be the number of seconds after the parachute opens that it takes Megan to reach the ground.

For t in [15, 15+T], $v_\infty = -\frac{32}{1.6} = -20.$

$0 = y(T+15)$

$= [3200(2 - e^{-1.5})]$

$-20T + (0.625)[320(e^{-1.5} - 1) + 20](1 - e^{-1.6T})$

$\approx 5543 - 20T - 142.9e^{-1.6T} \approx 5543 - 20T$ [since $T > 50$, so $e^{-1.6T} < 10^{-35}$ (very small)]

Therefore, $T \approx 277$, so it takes Megan about 292 s (4 min, 52 s) to reach the ground.

27. a. $e^{-\ln x + C}\left(\frac{dy}{dx} - \frac{y}{x}\right) = x^2 e^{-\ln x + C}$

$e^{-\ln x}e^C\left(\frac{dy}{dx} - \frac{y}{x}\right) = x^2 e^C e^{-\ln x}$

$\frac{1}{x}e^C \frac{dy}{dx} - ye^C \frac{1}{x^2} = x^2 e^C \frac{1}{x}$

$\frac{d}{dx}\left(e^C \frac{1}{x} y\right) = xe^C$

b. $e^C \frac{y}{x} = e^C \int x\,dx$

$\frac{y}{x} = \frac{x^2}{2} + C_1$

$y = \frac{x^3}{2} + C_1 x$

28. $e^{\int P(x)dx + C}\frac{dy}{dx} + P(x)e^{\int P(x)dx + C}y$

$= Q(x)e^{\int P(x)dx + C}$

$\frac{d}{dx}\left(e^{\int P(x)dx + C}y\right) = Q(x)e^{\int P(x)dx + C}$

$ye^{\int P(x)dx + C} = \int Q(x)e^{\int P(x)dx}e^C\,dx + C_1$

$y = e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$

$+ C_2 e^{-\int P(x)dx}$

7.7 Concepts Review

1. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; arcsin

2. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; arctan

3. 1

4. π

Problem Set 7.7

1. $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

3. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

5. $\arctan(\sqrt{3}) = \frac{\pi}{3}$ since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

6. $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so $\sec\left(\frac{\pi}{3}\right) = 2$

7. $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

8. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$ since $\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

9. $\sin(\sin^{-1} 0.4567) = 0.4567$ by definition

10. $\cos(\sin^{-1} 0.56) = \sqrt{1-\sin^2(\sin^{-1} 0.56)}$
$= \sqrt{1-(0.56)^2} \approx 0.828$

11. $\sin^{-1}(0.1113) \approx 0.1115$

12. $\arccos(0.6341) \approx 0.8840$

13. $\cos(\operatorname{arccot} 3.212) = \cos\left(\arctan\frac{1}{3.212}\right)$
$\approx \cos 0.3018 \approx 0.9548$

14. $\sec(\arccos 0.5111) = \frac{1}{\cos(\arccos 0.5111)}$
$= \frac{1}{0.5111} \approx 1.957$

15. $\sec^{-1}(-2.222) = \cos^{-1}\left(\frac{1}{-2.222}\right) \approx 2.038$

16. $\tan^{-1}(-60.11) \approx -1.554$

17. $\cos(\sin(\tan^{-1} 2.001)) \approx 0.6259$

18. $\sin^2(\ln(\cos 0.5555)) \approx 0.02632$

19. $\theta = \sin^{-1}\frac{x}{8}$

20. $\theta = \tan^{-1}\frac{x}{6}$

21. $\theta = \sin^{-1}\frac{5}{x}$

22. $\theta = \cos^{-1}\frac{9}{x}$ or $\theta = \sec^{-1}\frac{x}{9}$

23. Let θ_1 be the angle opposite the side of length 3, and $\theta_2 = \theta_1 - \theta$, so $\theta = \theta_1 - \theta_2$. Then $\tan\theta_1 = \frac{3}{x}$ and $\tan\theta_2 = \frac{1}{x}$. $\theta = \tan^{-1}\frac{3}{x} - \tan^{-1}\frac{1}{x}$.

24. Let θ_1 be the angle opposite the side of length 5, and $\theta_2 = \theta_1 - \theta$, and y the length of the unlabeled side. Then $\theta = \theta_1 - \theta_2$ and $y = \sqrt{x^2-25}$.
$\tan\theta_1 = \frac{5}{y} = \frac{5}{\sqrt{x^2-25}}$, $\tan\theta_2 = \frac{2}{y} = \frac{2}{\sqrt{x^2-25}}$,
$\theta = \tan^{-1}\frac{5}{\sqrt{x^2-25}} - \tan^{-1}\frac{2}{\sqrt{x^2-25}}$

25. $\cos\left[2\sin^{-1}\left(-\frac{2}{3}\right)\right] = 1-2\sin^2\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$
$= 1-2\left(-\frac{2}{3}\right)^2 = \frac{1}{9}$

26. $\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] = \frac{2\tan\left[\tan^{-1}\left(\frac{1}{3}\right)\right]}{1-\tan^2\left[\tan^{-1}\left(\frac{1}{3}\right)\right]}$
$= \frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

27. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{5}{13}\right)\right] = \sin\left[\cos^{-1}\left(\frac{3}{5}\right)\right]\cos\left[\cos^{-1}\left(\frac{5}{13}\right)\right]+\cos\left[\cos^{-1}\left(\frac{3}{5}\right)\right]\sin\left[\cos^{-1}\left(\frac{5}{13}\right)\right]$
$= \sqrt{1-\left(\frac{3}{5}\right)^2}\cdot\frac{5}{13}+\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{56}{65}$

28. $\cos\left[\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{12}{13}\right)\right] = \cos\left[\cos^{-1}\left(\frac{4}{5}\right)\right]\cos\left[\sin^{-1}\left(\frac{12}{13}\right)\right]-\sin\left[\cos^{-1}\left(\frac{4}{5}\right)\right]\sin\left[\sin^{-1}\left(\frac{12}{13}\right)\right]$
$= \frac{4}{5}\cdot\sqrt{1-\left(\frac{12}{13}\right)^2}-\sqrt{1-\left(\frac{4}{5}\right)^2}\cdot\frac{12}{13} = -\frac{16}{65}$

29. $\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}}$

30. $\sin(\tan^{-1} x) = \frac{1}{\csc(\tan^{-1} x)} = \frac{1}{\sqrt{1+\cot^2(\tan^{-1} x)}}$

$= \frac{1}{\sqrt{1+\frac{1}{\tan^2(\tan^{-1} x)}}} = \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{x}{\sqrt{x^2+1}}$

31. $\cos(2\sin^{-1} x) = 1 - 2\sin^2(\sin^{-1} x) = 1 - 2x^2$

32. $\tan(2\tan^{-1} x) = \frac{2\tan(\tan^{-1} x)}{1-\tan^2(\tan^{-1} x)} = \frac{2x}{1-x^2}$

33. a. $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ since $\lim_{\theta\to\pi/2^-} \tan\theta = \infty$

b. $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$ since $\lim_{\theta\to-\pi/2^+} \tan\theta = -\infty$

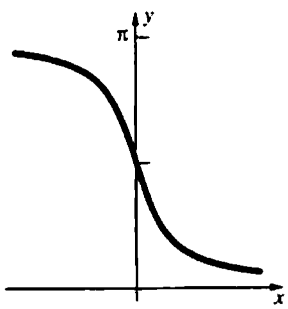
34. a. $\lim_{x\to\infty} \sec^{-1} x = \lim_{x\to\infty} \cos^{-1}\left(\frac{1}{x}\right)$

$= \lim_{z\to 0^+} \cos^{-1} z = \frac{\pi}{2}$

b. $\lim_{x\to-\infty} \sec^{-1} x = \lim_{x\to-\infty} \cos^{-1}\left(\frac{1}{x}\right)$

$= \lim_{z\to 0^-} \cos^{-1} z = \frac{\pi}{2}$

35.



36. $\frac{d}{dx} e^{\tan x} = e^{\tan x} \frac{d}{dx} \tan x = e^{\tan x} \sec^2 x$

37. $\frac{d}{dx} \ln(\sec x + \tan x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{(\sec x)(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

38. $\frac{d}{dx}[-\ln(\csc x + \cot x)] = -\frac{-\csc x \cot x - \csc^2 x}{\csc x + \cot x}$

$= \frac{\csc x(\cot x + \csc x)}{\cot x + \csc x} = \csc x$

39. $\frac{d}{dx} \sin^{-1}(2x^2) = \frac{1}{\sqrt{1-(2x^2)^2}} \cdot 4x = \frac{4x}{\sqrt{1-4x^4}}$

40. $\frac{d}{dx} \arccos(e^x) = -\frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = -\frac{e^x}{\sqrt{1-e^{2x}}}$

41. $\frac{d}{dx}[x^3 \tan^{-1}(e^x)] = x^3 \cdot \frac{e^x}{1+(e^x)^2} + 3x^2 \tan^{-1}(e^x)$

$= x^2\left[\frac{xe^x}{1+e^{2x}} + 3\tan^{-1}(e^x)\right]$

42. $\frac{d}{dx}(e^x \arcsin x^2) = e^x \cdot \frac{2x}{\sqrt{1-(x^2)^2}} + e^x \arcsin x^2$

$= e^x\left(\frac{2x}{\sqrt{1-x^4}} + \arcsin x^2\right)$

43. $\frac{d}{dx}(\tan^{-1} x)^3 = 3(\tan^{-1} x)^2 \cdot \frac{1}{1+x^2} = \frac{3(\tan^{-1} x)^2}{1+x^2}$

44. $\frac{d}{dx} \tan(\cos^{-1} x) = \frac{d}{dx} \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x)} = \frac{d}{dx} \frac{\sqrt{1-x^2}}{x}$

$= \frac{x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}}(-2x) - \sqrt{1-x^2} \cdot 1}{x^2}$

$= \frac{-x^2 - (1-x^2)}{x^2\sqrt{1-x^2}} = -\frac{1}{x^2\sqrt{1-x^2}}$

45. $\frac{d}{dx} \sec^{-1}(x^3) = \frac{1}{|x^3|\sqrt{(x^3)^2 - 1}} \cdot 3x^2 = \frac{3}{|x|\sqrt{x^6 - 1}}$

46. $\frac{d}{dx}(\sec^{-1} x)^3 = 3(\sec^{-1} x)^2 \cdot \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{3(\sec^{-1} x)^2}{|x|\sqrt{x^2-1}}$

47. $\frac{d}{dx}(1+\sin^{-1} x)^3 = 3(1+\sin^{-1} x)^2 \cdot \frac{1}{\sqrt{1-x^2}}$

$= \frac{3(1+\sin^{-1} x)^2}{\sqrt{1-x^2}}$

48. Let $u = x^2$, so $du = 2x\,dx$.

$\int x\sin(x^2)dx = \frac{1}{2}\int \sin(x^2) \cdot 2x\,dx$

$$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}\cos(x^2) + C$$

49. Let $u = \sin 2x$, so $du = 2\cos 2x\,dx$.

$$\int \sin 2x \cos 2x\,dx = \frac{1}{2}\int \sin 2x(2\cos 2x)dx$$

$$= \frac{1}{2}\int u\,du$$

$$= \frac{u^2}{4} + C = \frac{1}{4}\sin^2 2x + C$$

50. Let $u = \cos x$, so $du = -\sin x\,dx$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{\cos x}(-\sin x)dx$$

$$= -\int \frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$$

51. Let $u = e^{2x}$, so $du = 2e^{2x}dx$.

$$\int e^{2x}\cos(e^{2x})dx = \frac{1}{2}\int \cos(e^{2x})(2e^{2x})dx$$

$$= \frac{1}{2}\int \cos u\,du$$

$$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(e^{2x}) + C$$

$$\int_0^1 e^{2x}\cos(e^{2x})\,dx = \left[\frac{1}{2}\sin(e^{2x})\right]_0^1$$

$$= \left[\frac{1}{2}\sin(e^2) - \frac{1}{2}\sin(e^0)\right]$$

$$= \frac{\sin e^2 - \sin 1}{2} \approx 0.0262$$

52. Let $u = \sin x$, so $du = \cos x\,dx$.

$$\int \sin^2 x\cos x\,dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

$$\int_0^{\pi/2} \sin^2 x\cos x\,dx = \left[\frac{\sin^3 x}{3}\right]_0^{\pi/2} = \frac{1}{3} - 0 = \frac{1}{3}$$

53. $\displaystyle\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}dx = [\arcsin x]_0^{\sqrt{2}/2}$

$$= \arcsin\frac{\sqrt{2}}{2} - \arcsin 0 = \frac{\pi}{4}$$

54. $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^2 \frac{dx}{|x|\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_{\sqrt{2}}^2$

$$= \sec^{-1} 2 - \sec^{-1}\sqrt{2}$$

$$= \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

55. $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}dx = \left[\tan^{-1} x\right]_{-1}^1 = \tan^{-1} 1 - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

56. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$.

$$\int \frac{\sin\theta}{1+\cos^2\theta}d\theta = -\int \frac{1}{1+\cos^2\theta}(-\sin\theta)d\theta$$

$$= -\int \frac{1}{1+u^2}du = -\tan^{-1} u + C$$

$$= -\tan^{-1}(\cos\theta) + C$$

$$\int_0^{\pi/2} \frac{\sin\theta}{1+\cos^2\theta}d\theta = \left[-\tan^{-1}(\cos\theta)\right]_0^{\pi/2}$$

$$= -\tan^{-1} 0 + \tan^{-1} 1 = -0 + \frac{\pi}{4} = \frac{\pi}{4}$$

57. Let $u = 2x$, so $du = 2\,dx$.

$$\int \frac{1}{1+4x^2}dx = \frac{1}{2}\int \frac{1}{1+(2x)^2}2dx$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C$$

$$= \frac{1}{2}\arctan 2x + C$$

58. Let $u = e^x$, so $du = e^x dx$.

$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{e^x}{1+(e^x)^2}dx = \int \frac{1}{1+u^2}du$$

$= \arctan u + C = \arctan e^x + C$

59. The top of the picture is 7.6 ft above eye level, and the bottom of the picture is 2.6 ft above eye level. Let θ_1 be the angle between the viewer's line of sight to the top of the picture and the horizontal. Then call $\theta_2 = \theta_1 - \theta$, so $\theta = \theta_1 - \theta_2$.

$$\tan\theta_1 = \frac{7.6}{b};\ \tan\theta_2 = \frac{2.6}{b};$$

$$\theta = \tan^{-1}\frac{7.6}{b} - \tan^{-1}\frac{2.6}{b}$$

If $b = 12.9$, $\theta \approx 0.3335$ or $19.1°$.

60. **a.** Restrict $2x$ to $[0, \pi]$, i.e., restrict x to $\left[0, \frac{\pi}{2}\right]$.

Then $y = 3\cos 2x$

$$\frac{y}{3} = \cos 2x$$

$$2x = \arccos \frac{y}{3}$$

$$x = f^{-1}(y) = \frac{1}{2} \arccos \frac{y}{3}$$

$$f^{-1}(x) = \frac{1}{2} \arccos \frac{x}{3}$$

b. Restrict $3x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e., restrict x to $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

Then $y = 2 \sin 3x$

$$\frac{y}{2} = \sin 3x$$

$$3x = \arcsin \frac{y}{2}$$

$$x = f^{-1}(y) = \frac{1}{3} \arcsin \frac{y}{2}$$

$$f^{-1}(x) = \frac{1}{3} \arcsin \frac{x}{2}$$

c. Restrict x to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$y = \frac{1}{2} \tan x$$

$$2y = \tan x$$

$$x = f^{-1}(y) = \arctan 2y$$

$$f^{-1}(x) = \arctan 2x$$

d. Restrict x to $\left(-\infty, -\frac{2}{\pi}\right) \cup \left(\frac{2}{\pi}, \infty\right)$ so $\frac{1}{x}$ is restricted to $\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$

then $y = \sin \frac{1}{x}$

$$\frac{1}{x} = \arcsin y$$

$$x = f^{-1}(y) = \frac{1}{\arcsin y}$$

$$f^{-1}(x) = \frac{1}{\arcsin x}$$

61. $\tan\left[2\tan^{-1}\left(\frac{1}{4}\right)\right] = \frac{2\tan\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}{1-\tan^2\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}$

$$= \frac{2 \cdot \frac{1}{4}}{1-\left(\frac{1}{4}\right)^2} = \frac{8}{15}$$

$$\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)\right] = \tan\left[2\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{4}\right)\right]$$

$$= \frac{\tan\left[2\tan^{-1}\left(\frac{1}{4}\right)\right] + \tan\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}{1-\tan\left[2\tan^{-1}\left(\frac{1}{4}\right)\right]\tan\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}$$

$$= \frac{\frac{8}{15}+\frac{1}{4}}{1-\frac{8}{15}\cdot\frac{1}{4}} = \frac{47}{52}$$

$$\tan\left[3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{5}{99}\right)\right]$$

$$= \frac{\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)\right] + \tan\left[\tan^{-1}\left(\frac{5}{99}\right)\right]}{1-\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)\right]\tan\left[\tan^{-1}\left(\frac{5}{99}\right)\right]}$$

$$= \frac{\frac{47}{52}+\frac{5}{99}}{1-\frac{47}{52}\cdot\frac{5}{99}} = \frac{4913}{4913} = 1 = \tan\frac{\pi}{4}$$

Thus, $3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{5}{99}\right) = \tan^{-1}(1) = \frac{\pi}{4}$.

62. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right] = \frac{2\tan\left[\tan^{-1}\left(\frac{1}{5}\right)\right]}{1-\tan^2\left[\tan^{-1}\left(\frac{1}{5}\right)\right]}$

$$= \frac{2 \cdot \frac{1}{5}}{1-\left(\frac{1}{5}\right)^2} = \frac{5}{12}$$

$$\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right] = \tan\left[2 \cdot 2\tan^{-1}\left(\frac{1}{5}\right)\right]$$

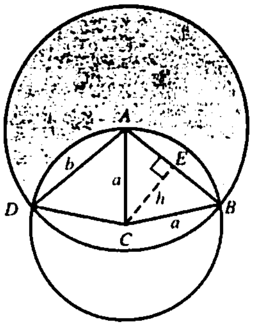
$$= \frac{2\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right]}{1-\tan^2\left[2\tan^{-1}\left(\frac{1}{5}\right)\right]} = \frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} = \frac{120}{119}$$

$$\tan\left[4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right]$$

$$= \frac{\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right] - \tan\left[\tan^{-1}\left(\frac{1}{239}\right)\right]}{1+\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right]\tan\left[\tan^{-1}\left(\frac{1}{239}\right)\right]}$$

$$= \frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} = \frac{28{,}561}{28{,}561} = 1 = \tan\frac{\pi}{4}$$

Thus, $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

63. 

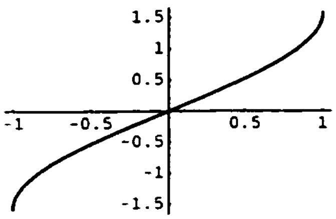
Let θ represent $\angle DAB$, then $\angle CAB$ is $\frac{\theta}{2}$. Since ΔABC is isosceles, $|AE| = \frac{b}{2}$, $\cos\frac{\theta}{2} = \frac{\frac{b}{2}}{a} = \frac{b}{2a}$ and $\theta = 2\cos^{-1}\frac{b}{2a}$. Thus sector ADB has area $\frac{1}{2}\left(2\cos^{-1}\frac{b}{2a}\right)b^2 = b^2\cos^{-1}\frac{b}{2a}$. Let ϕ represent $\angle DCB$, then $\angle ACB$ is $\frac{\phi}{2}$ and $\angle ECA$ is $\frac{\phi}{4}$, so $\sin\frac{\phi}{4} = \frac{\frac{b}{2}}{a} = \frac{b}{2a}$ and $\phi = 4\sin^{-1}\frac{b}{2a}$. Thus sector DCB has area $\frac{1}{2}\left(4\sin^{-1}\frac{b}{2a}\right)a^2 = 2a^2\sin^{-1}\frac{b}{2a}$.

These sectors overlap on the triangles ΔDAC and ΔCAB, each of which has area

$$\frac{1}{2}|AB|h = \frac{1}{2}b\sqrt{a^2 - \left(\frac{b}{2}\right)^2} = \frac{1}{2}b\frac{\sqrt{4a^2 - b^2}}{2}.$$

The large circle has area πb^2, hence the shaded region has area

$$\pi b^2 - b^2\cos^{-1}\frac{b}{2a} - 2a^2\sin^{-1}\frac{b}{2a} + \frac{1}{2}b\sqrt{4a^2 - b^2}$$

64. 

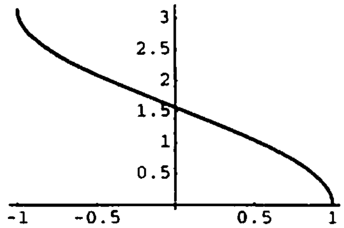
They have the same graph.

Conjecture: $\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

Proof: Let $\theta = \arcsin x$, so $x = \sin\theta$.

Then $\frac{x}{\sqrt{1-x^2}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

so $\theta = \arctan\frac{x}{\sqrt{1-x^2}}$.

65. 

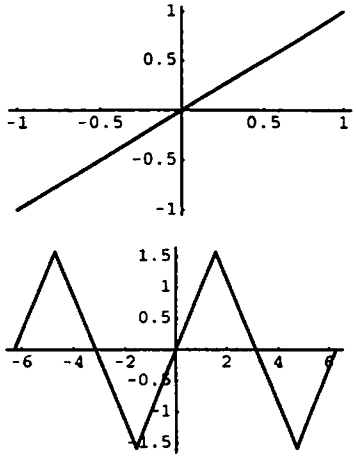
It is the same graph as $y = \arccos x$.

Conjecture: $\frac{\pi}{2} - \arcsin x = \arccos x$

Proof: Let $\theta = \frac{\pi}{2} - \arcsin x$

Then $x = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

so $\theta = \arccos x$.

66. 

$y = \sin(\arcsin x)$ is the line $y = x$, but only defined for $-1 \le x \le 1$.

$y = \arcsin(\sin x)$ is defined for all x, but only the portion for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is the line $y = x$.

67. $\int\frac{dx}{\sqrt{a^2 - x^2}} = \int\frac{dx}{\sqrt{a^2\left[1 - \left(\frac{x}{a}\right)^2\right]}}$

$= \int\frac{1}{|a|}\cdot\frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int\frac{1}{a}\cdot\frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$ since $a > 0$

Let $u = \frac{x}{a}$, so $du = \frac{1}{a}dx$.

$\int\frac{1}{a}\cdot\frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int\frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C$

$= \sin^{-1}\frac{x}{a} + C$

68. $D_x \sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{1}{a}$

$= \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a} = \frac{|a|}{\sqrt{a^2-x^2}}\cdot\frac{1}{a}$

$= \frac{a}{\sqrt{a^2-x^2}}\cdot\frac{1}{a}$, since $a > 0$

$= \frac{1}{\sqrt{a^2-x^2}}$

69. Let $u = \frac{x}{a}$, so $du = \frac{1}{a}dx$

$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\int\frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{1}{a}dx$

$= \frac{1}{a}\int\frac{1}{1+u^2}du = \frac{1}{a}\tan^{-1}u + C$

$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

70. Let $u = \frac{x}{a}$, so $du = (1/a)dx$. Since $a > 0$,

$\int\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\int\frac{1}{\left(\frac{x}{a}\right)\sqrt{\left(\frac{x}{a}\right)^2-1}}\frac{1}{a}dx$

$= \frac{1}{a}\int\frac{1}{u\sqrt{u^2-1}}du$

$= \frac{1}{a}\sec^{-1}|u| + C = \frac{1}{a}\sec^{-1}\frac{|x|}{a} + C$

71. Note that $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$ (See Problem 67).

$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C\right]$

$= \frac{1}{2}\sqrt{a^2-x^2} + \frac{x}{2}\frac{1}{2\sqrt{a^2-x^2}}(-2x)$

$+ \frac{a^2}{2}\frac{1}{\sqrt{a^2-x^2}} + 0$

$= \frac{1}{2}\sqrt{a^2-x^2} + \frac{1}{2}\frac{-x^2+a^2}{\sqrt{a^2-x^2}} = \sqrt{a^2-x^2}$

72. $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = 2\int_0^a\sqrt{a^2-x^2}\,dx$

$= 2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= 2\left[\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1) - \frac{0}{2}\sqrt{a^2} - \frac{a^2}{2}\sin^{-1}(0)\right]$

$= a^2\sin^{-1}(1) = \frac{\pi a^2}{2}$

This result is expected because the integral should be half the area of a circle with radius a.

73. Let θ be the angle subtended by the viewer's eye.

$\theta = \tan^{-1}\left(\frac{12}{b}\right) - \tan^{-1}\left(\frac{2}{b}\right)$

$\frac{d\theta}{db} = \frac{1}{1+\left(\frac{12}{b}\right)^2}\left(-\frac{12}{b^2}\right) - \frac{1}{1+\left(\frac{2}{b}\right)^2}\left(-\frac{2}{b^2}\right)$

$= \frac{2}{b^2+4} - \frac{12}{b^2+144} = \frac{10(24-b^2)}{(b^2+4)(b^2+144)}$

Since $\frac{d\theta}{db} > 0$ for b in $\left[0, 2\sqrt{6}\right)$

and $\frac{d\theta}{db} < 0$ for b in $\left(2\sqrt{6}, \infty\right)$, the angle is maximized for $b = 2\sqrt{6} \approx 4.899$.

The ideal distance is about 4.9 ft from the wall.

74. a. $\theta = \cos^{-1}\left(\frac{x}{b}\right) - \cos^{-1}\left(\frac{x}{a}\right)$ $\frac{d\theta}{dt} = \left(\frac{-1}{\sqrt{1-\left(\frac{x}{b}\right)^2}}\right)\left(\frac{1}{b}\right)\left(\frac{dx}{dt}\right) - \left(\frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)\left(\frac{1}{a}\right)\left(\frac{dx}{dt}\right)$

$= \left(\frac{1}{\sqrt{a^2-x^2}} - \frac{1}{\sqrt{b^2-x^2}}\right)\frac{dx}{dt}$

b. $\theta = \tan^{-1}\left(\frac{a+x}{\sqrt{b^2-x^2}}\right) - \sin^{-1}\left(\frac{x}{b}\right)$

$$\frac{d\theta}{dt}=\left(\frac{1}{1+\left(\frac{a+x}{\sqrt{b^2-x^2}}\right)^2}\right)\left(\frac{\sqrt{b^2-x^2}+\frac{(a+x)x}{\sqrt{b^2-x^2}}}{b^2-x^2}\right)\left(\frac{dx}{dt}\right)-\left(\frac{1}{\sqrt{1-\left(\frac{x}{b}\right)^2}}\right)\left(\frac{1}{b}\right)\left(\frac{dx}{dt}\right)$$

$$=\left[\left(\frac{b^2-x^2}{b^2-x^2+(a+x)^2}\right)\left(\frac{b^2+ax}{(b^2-x^2)^{3/2}}\right)-\frac{1}{\sqrt{b^2-x^2}}\right]\frac{dx}{dt}$$

$$=\left[\frac{b^2+ax}{(b^2+a^2+2ax)\sqrt{b^2-x^2}}-\frac{1}{\sqrt{b^2-x^2}}\right]\frac{dx}{dt}$$

$$=\left[-\frac{a^2+ax}{(b^2+a^2+2ax)\sqrt{b^2-x^2}}\right]\frac{dx}{dt}$$

75. Let $h(t)$ represent the height of the elevator (the number of feet above the spectator's line of sight) t seconds after the line of sight passes horizontal, and let $\theta(t)$ denote the angle of elevation.

Then $h(t) = 15t$, so $\theta(t)=\tan^{-1}\left(\frac{15t}{60}\right)=\tan^{-1}\left(\frac{t}{4}\right)$.

$$\frac{d\theta}{dt}=\frac{1}{1+\left(\frac{t}{4}\right)^2}\left(\frac{1}{4}\right)=\frac{4}{16+t^2}$$

At $t = 6$, $\frac{d\theta}{dt}=\frac{4}{16+6^2}=\frac{1}{13}$ radians per second or about 4.41° per second.

76. Let $x(t)$ be the *horizontal* distance from the observer to the plane, in miles, t minutes *before* the plane is overhead. Let $t = 0$ when the distance to the plane is 3 miles. Then

$x(0)=\sqrt{3^2-2^2}=\sqrt{5}$. The speed of the plane is 10 miles per minute, so $x(t)=\sqrt{5}-10t$. The angle of elevation is $\theta(t)=\tan^{-1}\left(\frac{2}{x(t)}\right)=\tan^{-1}\left(\frac{2}{\sqrt{5}-10t}\right)$,

so $\frac{d\theta}{dt}=\frac{1}{1+\left(2/\left(\sqrt{5}-10t\right)\right)^2}\left(\frac{-2}{(\sqrt{5}-10t)^2}\right)(-10)$

$$=\frac{20}{(\sqrt{5}-10t)^2+4}.$$

When $t = 0$, $\frac{d\theta}{dt}=\frac{20}{9}\approx 2.22$ radians per minute.

77. Let x represent the position on the shoreline and let θ represent the angle of the beam ($x = 0$ and $\theta = 0$ when the light is pointed at P). Then

$\theta=\tan^{-1}\left(\frac{x}{2}\right)$, so $\frac{d\theta}{dt}=\frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{1}{2}\frac{dx}{dt}=\frac{2}{4+x^2}\frac{dx}{dt}$

When $x = 1$,

$\frac{dx}{dt}=5\pi$, so $\frac{d\theta}{dt}=\frac{2}{4+1^2}(5\pi)=2\pi$ The beacon revolves at a rate of 2π radians per minute or 1 revolution per minute.

78. Let x represent the length of the rope and let θ represent the angle of depression of the rope.

Then $\theta=\sin^{-1}\left(\frac{8}{x}\right)$, so

$$\frac{d\theta}{dt}=\frac{1}{\sqrt{1-\left(\frac{8}{x}\right)^2}}\cdot-\frac{8}{x^2}\frac{dx}{dt}=-\frac{8}{x\sqrt{x^2-64}}\frac{dx}{dt}.$$

When $x = 17$ and $\frac{dx}{dt}=-5$, we obtain

$$\frac{d\theta}{dt}=-\frac{8}{17\sqrt{17^2-64}}(-5)=\frac{8}{51}.$$

The angle of depression is increasing at a rate of $8/51\approx 0.16$ radians per second.

79. Let x represent the distance to the *center* of the earth and let θ represent the angle subtended by the earth. Then $\theta=2\sin^{-1}\left(\frac{6376}{x}\right)$, so

$$\frac{d\theta}{dt}=2\frac{1}{\sqrt{1-\left(\frac{6376}{x}\right)^2}}\left(-\frac{6376}{x^2}\right)\frac{dx}{dt}$$

$$=-\frac{12,752}{x\sqrt{x^2-6376^2}}\frac{dx}{dt}$$

When she is 3000 km from the surface $x = 3000 + 6376 = 9376$ and $\frac{dx}{dt}=-2$. Substituting these values, we obtain $\frac{d\theta}{dt}\approx 3.96\times 10^{-4}$ radians per second.

7.8 Concepts Review

1. $\frac{e^x - e^{-x}}{2}; \frac{e^x + e^{-x}}{2}$

2. $\cosh^2 x - \sinh^2 x = 1$

3. the graph of $x^2 - y^2 = 1$, a hyperbola

4. catenary; a hanging chain

Problem Set 7.8

1. $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$

$= \frac{2e^x}{2} = e^x$

2. $\cosh 2x + \sinh 2x = \frac{e^{2x} + e^{-2x}}{2} + \frac{e^{2x} - e^{-2x}}{2}$

$= \frac{2e^{2x}}{2} = e^{2x}$

3. $\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}$

$= \frac{2e^{-x}}{2} = e^{-x}$

4. $\cosh 2x - \sinh 2x = \frac{e^{2x} + e^{-2x}}{2} - \frac{e^{2x} - e^{-2x}}{2} = \frac{2e^{-2x}}{2} = e^{-2x}$

5. $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4}$

$= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$

6. $\sinh x \cosh y - \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} - \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} - \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4}$

$= \frac{2e^{x-y} - 2e^{-x+y}}{4} = \frac{e^{x-y} - e^{-(x-y)}}{2} = \sinh(x-y)$

7. $\cosh x\cosh y+\sinh x\sinh y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}}{4}+\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}$

$=\frac{2e^{x+y}+2e^{-x-y}}{4}=\frac{e^{x+y}+e^{-(x+y)}}{2}=\cosh(x+y)$

8. $\cosh x\cosh y-\sinh x\sinh y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}-\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}}{4}-\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}$

$=\frac{2e^{x-y}+2e^{-x+y}}{4}=\frac{e^{x-y}+e^{-(x-y)}}{2}=\cosh(x-y)$

9. $\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}=\frac{\frac{\sinh x}{\cosh x}+\frac{\sinh y}{\cosh y}}{1+\frac{\sinh x}{\cosh x}\cdot\frac{\sinh y}{\cosh y}}$

$=\frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y}=\frac{\sinh(x+y)}{\cosh(x+y)}$

$=\tanh(x+y)$

10. $\frac{\tanh x-\tanh y}{1-\tanh x\tanh y}=\frac{\frac{\sinh x}{\cosh x}-\frac{\sinh y}{\cosh y}}{1-\frac{\sinh x}{\cosh x}\cdot\frac{\sinh y}{\cosh y}}$

$=\frac{\sinh x\cosh y-\cosh x\sinh y}{\cosh x\cosh y-\sinh x\sinh y}=\frac{\sinh(x-y)}{\cosh(x-y)}$

$=\tanh(x-y)$

11. $2\sinh x\cosh x=\sinh x\cosh x+\cosh x\sinh x$
$=\sinh(x+x)=\sinh 2x$

12. $\cosh^2 x+\sinh^2 x=\cosh x\cosh x+\sinh x\sinh x$
$=\cosh(x+x)=\cosh 2x$

13. $D_x\sinh^2 x=2\sinh x\cosh x=\sinh 2x$

14. $D_x\cosh^2 x=2\cosh x\sinh x=\sinh 2x$

15. $D_x(5\sinh^2 x)=10\sinh x\cdot\cosh x=5\sinh 2x$

16. $D_x\cosh^3 x=3\cosh^2 x\sinh x$

17. $D_x\cosh(3x+1)=\sinh(3x+1)\cdot 3=3\sinh(3x+1)$

18. $D_x\sinh(x^2+x)=\cosh(x^2+x)\cdot(2x+1)$
$=(2x+1)\cosh(x^2+x)$

19. $D_x\ln(\sinh x)=\frac{1}{\sinh x}\cdot\cosh x=\frac{\cosh x}{\sinh x}$
$=\coth x$

20. $D_x\ln(\coth x)=\frac{1}{\coth x}(-\operatorname{csch}^2 x)$
$=-\frac{\sinh x}{\cosh x}\cdot\frac{1}{\sinh^2 x}=-\frac{1}{\sinh x\cosh x}$
$=-\operatorname{csch} x\operatorname{sech} x$

21. $D_x(x^2\cosh x)=x^2\cdot\sinh x+\cosh x\cdot 2x$
$=x^2\sinh x+2x\cosh x$

22. $D_x(x^{-2}\sinh x)=x^{-2}\cdot\cosh x+\sinh x\cdot(-2x^{-3})$
$=x^{-2}\cosh x-2x^{-3}\sinh x$

23. $D_x(\cosh 3x\sinh x)=\cosh 3x\cdot\cosh x+\sinh x\cdot\sinh 3x\cdot 3=\cosh 3x\cosh x+3\sinh 3x\sinh x$

24. $D_x(\sinh x\cosh 4x)=\sinh x\cdot\sinh 4x\cdot 4+\cosh 4x\cdot\cosh x=4\sinh x\sinh 4x+\cosh x\cosh 4x$

25. $D_x(\tanh x\sinh 2x)=\tanh x\cdot\cosh 2x\cdot 2+\sinh 2x\cdot\operatorname{sech}^2 x=2\tanh x\cosh 2x+\sinh 2x\operatorname{sech}^2 x$

26. $D_x(\coth 4x\sinh x)=\coth 4x\cdot\cosh x+\sinh x(-\operatorname{csch}^2 4x)\cdot 4=\cosh x\coth 4x-4\sinh x\operatorname{csch}^2 4x$

27. $D_x \sinh^{-1}(x^2) = \frac{1}{\sqrt{(x^2)^2+1}} \cdot 2x = \frac{2x}{\sqrt{x^4+1}}$

28. $D_x \cosh^{-1}(x^3) = \frac{1}{\sqrt{(x^3)^2-1}} \cdot 3x^2 = \frac{3x^2}{\sqrt{x^6-1}}$

29. $D_x \tanh^{-1}(2x-3) = \frac{1}{1-(2x-3)^2} \cdot 2 = \frac{2}{1-(4x^2-12x+9)} = \frac{2}{-4x^2+12x-8} = -\frac{1}{2(x^2-3x+2)}$

30. $D_x \coth^{-1}(x^5) = D_x \tanh^{-1}\left(\frac{1}{x^5}\right) = \frac{1}{1-\left(\frac{1}{x^5}\right)^2} \cdot \left(-\frac{5}{x^6}\right) = \frac{x^{10}}{x^{10}-1} \cdot \left(-\frac{5}{x^6}\right) = -\frac{5x^4}{x^{10}-1}$

31. $D_x[x\cosh^{-1}(3x)] = x \cdot \frac{1}{\sqrt{(3x)^2-1}} \cdot 3 + \cosh^{-1}(3x) \cdot 1 = \frac{3x}{\sqrt{9x^2-1}} + \cosh^{-1} 3x$

32. $D_x(x^2 \sinh^{-1} x^5) = x^2 \cdot \frac{1}{\sqrt{(x^5)^2+1}} \cdot 5x^4 + \sinh^{-1} x^5 \cdot 2x = \frac{5x^6}{\sqrt{x^{10}+1}} + 2x\sinh^{-1} x^5$

33. $D_x \ln(\cosh^{-1} x) = \frac{1}{\cosh^{-1} x} \cdot \frac{1}{\sqrt{x^2-1}}$

$= \frac{1}{\sqrt{x^2-1}\cosh^{-1} x}$

34. $\cosh^{-1}(\cos x)$ does not have a derivative, since $D_u \cosh^{-1} u$ is only defined for $u > 1$ while $\cos x \le 1$ for all x.

35. $D_x \tanh(\cot x) = \text{sech}^2(\cot x) \cdot (-\csc^2 x)$

$= -\csc^2 x \,\text{sech}^2(\cot x)$

36. $D_x \coth^{-1}(\tanh x) = D_x \tanh^{-1}\left(\frac{1}{\tanh x}\right)$

$= D_x \tanh^{-1}(\coth x)$

$= \frac{1}{1-(\coth x)^2}(-\text{csch}^2 x) = \frac{-\text{csch}^2 x}{-\text{csch}^2 x} = 1$

37. Area $= \int_0^{\ln 3} \cosh 2x\,dx = \left[\frac{1}{2}\sinh 2x\right]_0^{\ln 3}$

$= \frac{1}{2}\left(\frac{e^{2\ln 3} - e^{-2\ln 3}}{2} - \frac{e^0 - e^{-0}}{2}\right)$

$= \frac{1}{4}(e^{\ln 9} - e^{\ln\frac{1}{9}}) = \frac{1}{4}\left(9 - \frac{1}{9}\right) = \frac{20}{9}$

38. Let $u = 3x + 2$, so $du = 3\,dx$.

$\int \sinh(3x+2)dx = \frac{1}{3}\int \sinh u\,du = \frac{1}{3}\cosh u + C$

$= \frac{1}{3}\cosh(3x+2) + C$

39. Let $u = \pi x^2 + 5$, so $du = 2\pi x dx$.

$\int x\cosh(\pi x^2+5)dx = \frac{1}{2\pi}\int \cosh u\,du$

$= \frac{1}{2\pi}\sinh u + C = \frac{1}{2\pi}\sinh(\pi x^2+5) + C$

40. Let $u = \sqrt{z}$, so $du = \frac{1}{2\sqrt{z}}dz$.

$\int \frac{\cosh\sqrt{z}}{\sqrt{z}}dz = 2\int \cosh u\,du = 2\sinh u + C$

$= 2\sinh\sqrt{z} + C$

41. Let $u = 2z^{1/4}$, so $du = \frac{1}{4} \cdot 2z^{-3/4}dz = \frac{1}{2\sqrt[4]{z^3}}dz$.

$\int \frac{\sinh(2z^{1/4})}{\sqrt[4]{z^3}}dz = 2\int \sinh u\,du = 2\cosh u + C$

$= 2\cosh(2z^{1/4}) + C$

42. Let $u = e^x$, so $du = e^x dx$.

$\int e^x \sinh e^x dx = \int \sinh u\, du = \cosh u + C$

$= \cosh e^x + C$

43. Let $u = \sin x$, so $du = \cos x\, dx$

$\int \cos x \sinh(\sin x) dx = \int \sinh u\, du = \cosh u + C$

$= \cosh(\sin x) + C$

44. Let $u = \ln(\cosh x)$, so

$du = \frac{1}{\cosh x} \cdot \sinh x = \tanh x\, dx$.

$\int \tanh x \ln(\cosh x) dx = \int u\, du = \frac{u^2}{2} + C$

$= \frac{1}{2}[\ln(\cosh x)]^2 + C$

45. Let $u = \ln(\sinh x^2)$, so

$du = \frac{1}{\sinh x^2} \cdot \cosh x^2 \cdot 2x dx = 2x \coth x^2 dx$.

$\int x \coth x^2 \ln(\sinh x^2) dx = \frac{1}{2}\int u\, du = \frac{1}{2} \cdot \frac{u^2}{2} + C$

$= \frac{1}{4}[\ln(\sinh x^2)]^2 + C$

46. Area $= \int_{-\ln 5}^{\ln 5} \cosh 2x\, dx = 2\int_0^{\ln 5} \cosh 2x\, dx$

$= 2\left[\frac{1}{2}\sinh 2x\right]_0^{\ln 5}$

$= \sinh(2\ln 5) = \frac{1}{2}(e^{2\ln 5} - e^{-2\ln 5})$

$= \frac{1}{2}(e^{\ln 25} - e^{\ln \frac{1}{25}}) = \frac{1}{2}\left(25 - \frac{1}{25}\right)$

$= \frac{312}{25} = 12.48$

47. Note that the graphs of $y = \sinh x$ and $y = 0$ intersect at the origin.

Area $= \int_0^{\ln 2} \sinh x\, dx = [\cosh x]_0^{\ln 2}$

$= \frac{e^{\ln 2} + e^{-\ln 2}}{2} - \frac{e^0 + e^0}{2} = \frac{1}{2}\left(2 + \frac{1}{2}\right) - 1 = \frac{1}{4}$

48. $\tanh x = 0$ when $\sinh x = 0$, which is when $x = 0$.

Area $= \int_{-8}^{0} (-\tanh x)\, dx + \int_0^8 \tanh x\, dx$

$= 2\int_0^8 \tanh x\, dx = 2\int_0^8 \frac{\sinh x}{\cosh x} dx$

Let $u = \cosh x$, so $du = \sinh x dx$.

$2\int \frac{\sinh x}{\cosh x} dx = 2\int \frac{1}{u} du = 2\ln|u| + C$

$2\int_0^8 \frac{\sinh x}{\cosh x} dx = \left[2\ln|\cosh x|\right]_0^8$

$= 2(\ln|\cosh 8| - \ln 1) = 2\ln(\cosh 8) \approx 14.61$

49. Volume $= \int_0^1 \pi \cosh^2 x\, dx = \frac{\pi}{2}\int_0^1 (1 + \cosh 2x) dx$

$= \frac{\pi}{2}\left[x + \frac{\sinh 2x}{2}\right]_0^1$

$= \frac{\pi}{2}\left(1 + \frac{\sinh 2}{2} - 0\right)$

$= \frac{\pi}{2} + \frac{\pi \sinh 2}{4} \approx 4.42$

50. Volume $= \int_0^{\ln 10} \pi \sinh^2 x dx$

$= \pi \int_0^{\ln 10} \left(\frac{e^x - e^{-x}}{2}\right)^2 dx$

$= \pi \int_0^{\ln 10} \frac{e^{2x} - 2 + e^{-2x}}{4} dx = \frac{\pi}{4}\int_0^{\ln 10} (e^{2x} - 2 + e^{-2x}) dx$

$= \frac{\pi}{4}\left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_0^{\ln 10}$

$= \frac{\pi}{8}[e^{2x} - 4x - e^{-2x}]_0^{\ln 10}$

$= \frac{\pi}{8}\left(100 - 4\ln 10 - \frac{1}{100}\right) \approx 35.65$

51. Note that $1 + \sinh^2 x = \cosh^2 x$ and

$\cosh^2 x = \frac{1 + \cosh 2x}{2}$

Surface area $= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$= \int_0^1 2\pi \cosh x \sqrt{1 + \sinh^2 x}\, dx$

$= \int_0^1 2\pi \cosh x \cosh x\, dx$

$= \int_0^1 \pi(1 + \cosh 2x) dx$

$= \left[\pi x + \frac{\pi}{2}\sinh 2x\right]_0^1 = \pi + \frac{\pi}{2}\sinh 2 \approx 8.84$

52. Surface area = $\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi \sinh x\sqrt{1+\cosh^2 x}\,dx$

Let $u = \cosh x$, so $du = \sinh x\, dx$

$\int 2\pi \sinh x\sqrt{1+\cosh^2 x}\,dx = 2\pi\int\sqrt{1+u^2}\,du = 2\pi\left[\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right|+C\right]$

$= \pi\cosh x\sqrt{1+\cosh^2 x}+\pi\ln\left|\cosh x+\sqrt{1+\cosh^2 x}\right|+C$ (The integration of $\int\sqrt{1+u^2}\,du$ is shown in Formula 44 of the Tables in the back of the text, which is covered in Chapter 8.)

$\int_0^1 2\pi\sinh x\sqrt{1+\cosh^2 x}\,dx = \pi\left[\cosh x\sqrt{1+\cosh^2 x}+\ln\left|\cosh x+\sqrt{1+\cosh^2 x}\right|\right]_0^1$

$= \pi\left[\cosh 1\sqrt{1+\cosh^2 1}+\ln\left|\cosh 1+\sqrt{1+\cos^2 1}\right|-\left(\sqrt{2}+\ln\left|1+\sqrt{2}\right|\right)\right] \approx 5.53$

53. $y = a\cosh\left(\frac{x}{a}\right)+C$

$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$

$\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$

We need to show that $\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Note that $1+\sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$ and $\cosh\left(\frac{x}{a}\right) > 0$. Therefore,

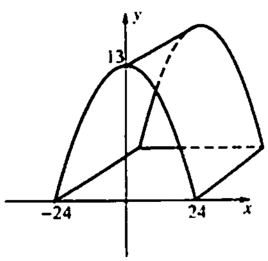
$\frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{a}\sqrt{1+\sinh^2\left(\frac{x}{a}\right)} = \frac{1}{a}\sqrt{\cosh^2\left(\frac{x}{a}\right)} = \frac{1}{a}\cosh\left(\frac{x}{a}\right) = \frac{d^2y}{dx^2}$

54. a. The graph of $y = b - a\cosh\left(\frac{x}{a}\right)$ is symmetric about the y-axis, so if its width along the x-axis is $2a$, its x-intercepts are ($\pm a$, 0). Therefore, $y(a) = b - a\cosh\left(\frac{a}{a}\right) = 0$, so $b = a\cosh 1 \approx 1.54308a$.

b. The height is $y(0) \approx 1.54308a - a\cosh 0 = 0.54308a$.

c. If $2a = 48$, the height is about $0.54308a = (0.54308)(24) \approx 13$.

55. a.



b. Area under the curve is

$\int_{-24}^{24}\left[37-24\cosh\left(\frac{x}{24}\right)\right]dx = \left[37x-576\sinh\left(\frac{x}{24}\right)\right]_{-24}^{24} \approx 422$

Volume is about $(422)(100) = 42{,}200$ ft^3.

c. Length of the curve is

$$\int_{-24}^{24}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-24}^{24}\sqrt{1+\sinh^2\left(\frac{x}{24}\right)}dx = \int_{-24}^{24}\cosh\left(\frac{x}{24}\right)dx = \left[24\sinh\left(\frac{x}{24}\right)\right]_{-24}^{24} = 48\sinh 1 \approx 56.4$$

Surface area $\approx (56.4)(100) = 5640\ \text{ft}^2$

56. $\text{Area} = \frac{1}{2}\cosh t\sinh t - \int_1^{\cosh t}\sqrt{x^2-1}\,dx = \frac{1}{2}\cosh t\sinh t - \left[\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|\right]_1^{\cosh t}$

$= \frac{1}{2}\cosh t\sinh t - \left[\frac{1}{2}\cosh t\sqrt{\cosh^2 t-1} - \frac{1}{2}\ln\left|\cosh t+\sqrt{\cosh^2 t-1}\right| - 0\right]$

$= \frac{1}{2}\cosh t\sinh t - \frac{1}{2}\cosh t\sinh t + \frac{1}{2}\ln\left|\cosh t+\sinh t\right| = \frac{1}{2}\ln e^t = \frac{t}{2}$

57. a. $(\sinh x+\cosh x)^r = \left(\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}\right)^r = \left(\frac{2e^x}{2}\right)^r = e^{rx}$

$\sinh rx+\cosh rx = \frac{e^{rx}-e^{-rx}}{2}+\frac{e^{rx}+e^{-rx}}{2} = \frac{2e^{rx}}{2} = e^{rx}$

b. $(\cosh x-\sinh x)^r = \left(\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right)^r = \left(\frac{2e^{-x}}{2}\right)^r = e^{-rx}$

$\cosh rx-\sinh rx = \frac{e^{rx}+e^{-rx}}{2}-\frac{e^{rx}-e^{-rx}}{2} = \frac{2e^{-rx}}{2} = e^{-rx}$

c. $(\cos x+i\sin x)^r = \left(\frac{e^{ix}+e^{-ix}}{2}+i\frac{e^{ix}-e^{-ix}}{2i}\right)^r = \left(\frac{2e^{ix}}{2}\right)^r = e^{irx}$

$\cos rx+i\sin rx = \frac{e^{irx}+e^{-irx}}{2}+i\frac{e^{irx}-e^{-irx}}{2i} = \frac{2e^{irx}}{2} = e^{irx}$

d. $(\cos x-i\sin x)^r = \left(\frac{e^{ix}+e^{-ix}}{2}-i\frac{e^{ix}-e^{-ix}}{2i}\right)^r = \left(\frac{2e^{-ix}}{2}\right)^r = e^{-irx}$

$\cos rx-i\sin rx = \frac{e^{irx}+e^{-irx}}{2}-i\frac{e^{irx}-e^{-irx}}{2i} = \frac{2e^{-irx}}{2} = e^{-irx}$

58. a. $gd(-t) = \tan^{-1}[\sinh(-t)]$

$= \tan^{-1}(-\sinh t) = -\tan^{-1}(\sinh t) = -gd(t)$

so gd is odd.

$D_t[gd(t)] = \frac{1}{1+\sinh^2 t}\cdot\cosh t = \frac{\cosh t}{\cosh^2 t}$

$= \text{sech}\ t > 0$ for all t, so gd is increasing.

$D_t^2[gd(t)] = D_t(\text{sech}\ t) = -\text{sech}\ t\tanh t$

$D_t^2[gd(t)] = 0$ when $\tanh t = 0$, since $\text{sech}\ t > 0$ for all t. $\tanh t = 0$ at $t = 0$ and $\tanh t < 0$ for $t < 0$, thus $D_t^2[gd(t)] > 0$ for $t < 0$ and $D_t^2[gd(t)] < 0$ for $t > 0$. Hence $gd(t)$ has an inflection point at $(0, gd(0)) = (0, \tan^{-1} 0) = (0, 0)$.

b. If $y = \tan^{-1}(\sinh t)$ then $\tan y = \sinh t$ so

$\sin y = \frac{\tan y}{\sqrt{\tan^2 y+1}} = \frac{\sinh t}{\sqrt{\sinh^2 t+1}}$

$= \frac{\sinh t}{\cosh t} = \tanh t$ so $y = \sin^{-1}(\tanh t)$

Also, $D_t y = \frac{1}{1+\sinh^2 t}\cdot\cosh t$

$= \frac{\cosh t}{\cosh^2 t} = \frac{1}{\cosh t} = \text{sech}\ t,$

so $y = \int_0^t \operatorname{sech} u\, du$ by the Fundamental Theorem of Calculus.

59. Area = $\int_0^x \cosh t\, dt = [\sinh t]_0^x = \sinh x$

Arc length =

$\int_0^x \sqrt{1+[D_t \cosh t]^2}\, dt = \int_0^x \sqrt{1+\sinh^2 t}\, dt$

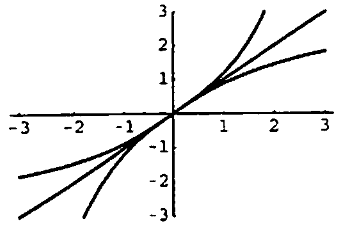
$= \int_0^x \cosh t\, dt = [\sinh t]_0^x = \sinh x$

60. From Problem 54, the equation of an inverted catenary is $y = b - a\cosh\frac{x}{a}$. Given the information about the Gateway Arch, the curve passes through the points (±315, 0) and (0, 630). Thus, $b = a\cosh\frac{315}{a}$ and $630 = b - a$, so $b = a + 630$.

$a + 630 = a\cosh\frac{315}{a} \Rightarrow a \approx 128$, so $b \approx 758$.

The equation is $y = 758 - 128\cosh\frac{x}{128}$.

61.



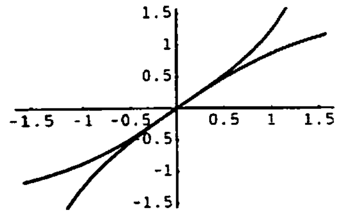
The functions $y = \sinh x$ and $y = \ln(x + \sqrt{x^2+1})$ are inverse functions.

62. $y = gd(x) = \tan^{-1}(\sinh x)$

$\tan y = \sinh x$

$x = gd^{-1}(y) = \sinh^{-1}(\tan y)$

Thus, $y = gd^{-1}(x) = \sinh^{-1}(\tan x)$



7.9 Chapter Review

Concepts Test

1. False: $\ln 0$ is undefined.
2. True: $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ for all $x > 0$.
3. True: $\int_1^{e^3} \frac{1}{t} dt = \left[\ln|t|\right]_1^{e^3} = \ln e^3 - \ln 1 = 3$
4. False: The graph is intersected *at most* once by every horizontal line.
5. True: The range of $y = \ln x$ is the set of all real numbers.
6. False: $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$
7. False: $4\ln x = \ln(x^4)$
8. True: $\ln(2e^{x+1}) - \ln(2e^x) = \ln\frac{2e^{x+1}}{2e^x}$ $= \ln e = 1$
9. True: $f(g(x)) = 4 + e^{\ln(x-4)}$ $= 4 + (x-4) = x$ and $g(f(x)) = \ln(4 + e^x - 4) = \ln e^x = x$
10. False: $\exp(x+y) = \exp x \exp y$
11. True: $\ln x$ is an increasing function.
12. False: Only true for $x > 1$, or $\ln x > 0$.
13. True: $e^z > 0$ for all z.
14. True: e^x is an increasing function.
15. True: $\lim_{x\to 0^+} (\ln \sin x - \ln x)$ $= \lim_{x\to 0^+} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0$
16. True: $\pi^{\sqrt{2}} = e^{\sqrt{2}\ln \pi}$
17. False: $\ln \pi$ is a constant so $\frac{d}{dx}\ln \pi = 0$.

18. True: $\frac{d}{dx}(\ln 3|x| + C)$
$= \frac{d}{dx}(\ln|x| + \ln 3 + C) = \frac{1}{x}$

19. True: e is a number.

20. True: $\exp[g(x)] \neq 0$ because 0 is not in the range of the function $y = e^x$.

21. False: $D_x(x^x) = x^x(1 + \ln x)$

22. True: $2(\tan x + \sec x)' - (\tan x + \sec x)^2$
$= 2(\sec^2 x + \sec x \tan x)$
$- \tan^2 x - 2\tan x \sec x - \sec^2 x$
$= \sec^2 x - \tan^2 x = 1$

23. True: The integrating factor is
$e^{\int 4/x\, dx} = e^{4\ln x} = (e^{\ln x})^4 = x^4$

24. False: $\sin(\arcsin(2))$ is undefined

25. False: $\arcsin(\sin 2\pi) = \arcsin 0 = 0$

26. True: $\sinh x$ is increasing.

27. False: $\cosh x$ is not increasing.

28. True: $\cosh(0) = 1 = e^0$
If $x > 0$, $e^x > 1$ while $e^{-x} < 1 < e^x$ so
$\cosh x = \frac{1}{2}(e^x + e^{-x}) < \frac{1}{2}(2e^x)$
$= e^x = e^{|x|}$. If $x < 0$, $-x > 0$ and
$e^{-x} > 1$ while $e^x < 1 < e^{-x}$ so
$\cosh x = \frac{1}{2}(e^x + e^{-x}) < \frac{1}{2}(2e^{-x})$
$= e^{-x} = e^{|x|}$.

29. True: $|\sinh x| \leq \frac{1}{2}e^{|x|}$ is equivalent to
$|e^x - e^{-x}| \leq e^{|x|}$. When $x = 0$,
$\sinh x = 0 < \frac{1}{2}e^0 = \frac{1}{2}$. If $x > 0$,
$e^x > 1$ and $e^{-x} < 1 < e^x$, thus
$|e^x - e^{-x}| = e^x - e^{-x} < e^x = e^{|x|}$.
If $x < 0$, $e^{-x} > 1$ and $e^x < 1 < e^{-x}$, thus
$|e^x - e^{-x}| = -(e^x - e^{-x})$
$= e^{-x} - e^x < e^{-x} = e^{|x|}$.

30. False: $\tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636$
but $\frac{\sin^{-1}\left(\frac{1}{2}\right)}{\cos^{-1}\left(\frac{1}{2}\right)} = \frac{1}{2}$

31. False: $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2}$
$= \frac{1}{2}\left(3 + \frac{1}{3}\right) = \frac{5}{3}$

32. False: $\lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0$

33. True: $\lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}$, since
$\lim_{x\to -\frac{\pi}{2}^+} \tan x = -\infty$.

34. False: $\cosh x > 1$ for $x \neq 0$, while $\sin^{-1} u$ is only defined for $-1 \leq u \leq 1$.

35. True: $\tanh x = \frac{\sinh x}{\cosh x}$; $\sinh x$ is an odd function and $\cosh x$ is an even function.

36. False: Both functions satisfy $y' - y = 0$.

37. True: $\ln 3^{100} = 100\ln 3 > 100 \cdot 1$ since $\ln 3 > 1$.

38. False: $\ln(x - 3)$ is not defined for $x < 3$.

39. True: y triples every time t increases by t_1.

40. False: $x(0) = C$; $\frac{1}{2}C = Ce^{-kt}$ when
$\frac{1}{2} = e^{-kt}$, so $\ln\frac{1}{2} = -kt$ or
$t = \frac{\ln\frac{1}{2}}{-k} = \frac{-\ln 2}{-k} = \frac{\ln 2}{k}$

41. True: $(y(t) + z(t))' = y'(t) + z'(t)$
$= ky(t) + kz(t) = k(y(t) + z(t))$

42. False: Only true if $C = 0$;
$(y_1(t) + y_2(t))' = y_1'(t) + y_2'(t)$

$= ky_1(t) + C + ky_2(t) + C$
$= k(y_1(t) + y_2(t)) + 2C$.

43. False: Use the substitution $u = -h$.
$\lim_{h\to 0}(1-h)^{-1/h} = \lim_{u\to 0}(1+u)^{1/u} = e$
by Theorem 7.5.A.

44. False: $e^{0.05} \approx 1.051 < \left(1+\frac{0.06}{12}\right)^{12} \approx 1.062$

45. True: If $D_x(a^x) = a^x \ln a = a^x$, then $\ln a = 1$, so $a = e$.

Sample Test Problems

1. $\ln\frac{x^4}{2} = 4\ln x - \ln 2$

$\frac{d}{dx}\ln\frac{x^4}{2} = \frac{d}{dx}(4\ln x - \ln 2) = \frac{4}{x}$

2. $\frac{d}{dx}\sin^2(x^3) = 2\sin(x^3)\frac{d}{dx}\sin(x^3)$

$= 2\sin(x^3)\cos(x^3)\frac{d}{dx}x^3 = 6x^2\sin(x^3)\cos(x^3)$

3. $\frac{d}{dx}e^{x^2-4x} = e^{x^2-4x}\frac{d}{dx}(x^2-4x)$

$= (2x-4)e^{x^2-4x}$

4. $\frac{d}{dx}\log_{10}(x^5-1) = \frac{1}{(x^5-1)\ln 10}\frac{d}{dx}(x^5-1)$

$= \frac{5x^4}{(x^5-1)\ln 10}$

5. $\frac{d}{dx}\tan(\ln e^x) = \frac{d}{dx}\tan x = \sec^2 x$

6. $\frac{d}{dx}e^{\ln\cot x} = \frac{d}{dx}\cot x = -\csc^2 x$

7. $\frac{d}{dx}2\tanh\sqrt{x} = 2\text{sech}^2\sqrt{x}\frac{d}{dx}\sqrt{x} = \frac{\text{sech}^2\sqrt{x}}{\sqrt{x}}$

8. $\frac{d}{dx}\tanh^{-1}(\sin x) = \frac{1}{1-\sin^2 x}\frac{d}{dx}\sin x = \frac{\cos x}{1-\sin^2 x}$

$= \frac{\cos x}{\cos^2 x} = \sec x$

9. $\frac{d}{dx}\sinh^{-1}(\tan x) = \frac{1}{\sqrt{\tan^2 x+1}}\frac{d}{dx}\tan x$

$= \frac{\sec^2 x}{\sqrt{\tan^2 x+1}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = |\sec x|$

10. $\frac{d}{dx}2\sin^{-1}\sqrt{3x} = \frac{2}{\sqrt{1-\left(\sqrt{3x}\right)^2}}\frac{d}{dx}\sqrt{3x}$

$= \frac{2}{\sqrt{1-3x}}\frac{3}{2\sqrt{3x}} = \frac{3}{\sqrt{3x-9x^2}}$

11. $\frac{d}{dx}\sec^{-1}e^x = \frac{1}{|e^x|\sqrt{(e^x)^2-1}}\frac{d}{dx}e^x$

$= \frac{e^x}{e^x\sqrt{e^{2x}-1}} = \frac{1}{\sqrt{e^{2x}-1}}$

12. $\frac{d}{dx}\ln\sin^2\left(\frac{x}{2}\right) = \frac{1}{\sin^2\left(\frac{x}{2}\right)}\frac{d}{dx}\sin^2\left(\frac{x}{2}\right)$

$= \frac{1}{\sin^2\left(\frac{x}{2}\right)}2\sin\left(\frac{x}{2}\right)\frac{d}{dx}\sin\left(\frac{x}{2}\right)$

$= \frac{1}{\sin^2\left(\frac{x}{2}\right)}\left[2\sin\left(\frac{x}{2}\right)\right]\frac{1}{2}\cos\left(\frac{x}{2}\right) = \cot\left(\frac{x}{2}\right)$

13. $\frac{d}{dx}3\ln(e^{5x}+1) = \frac{3}{e^{5x}+1}(5e^{5x}) = \frac{15e^{5x}}{e^{5x}+1}$

14. $\frac{d}{dx}\ln(2x^3-4x+5)$

$= \frac{1}{2x^3-4x+5}\frac{d}{dx}(2x^3-4x+5) = \frac{6x^2-4}{2x^3-4x+5}$

15. $\frac{d}{dx}\cos e^{\sqrt{x}} = -\sin e^{\sqrt{x}}\frac{d}{dx}e^{\sqrt{x}}$

$= (-\sin e^{\sqrt{x}})e^{\sqrt{x}}\frac{d}{dx}\sqrt{x}$

$= -\frac{e^{\sqrt{x}}\sin e^{\sqrt{x}}}{2\sqrt{x}}$

16. $\frac{d}{dx}\ln(\tanh x) = \frac{1}{\tanh x}\frac{d}{dx}\tanh x$

$= \frac{1}{\tanh x}\text{sech}^2 x = \text{csch}\, x\,\text{sech}\, x$

17. $\frac{d}{dx}2\cos^{-1}\sqrt{x} = \frac{-2}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x}$

$= \frac{-2}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} = -\frac{1}{\sqrt{x-x^2}}$

18. $\frac{d}{dx}\left[4^{3x}+(3x)^4\right] = \frac{d}{dx}(64^x+81x^4)$

$= 64^x \ln 64 + 324x^3$

19. $\frac{d}{dx}2\csc e^{\ln\sqrt{x}} = \frac{d}{dx}2\csc\sqrt{x}$

$= -2\csc\sqrt{x}\cot\sqrt{x}\frac{d}{dx}\sqrt{x}$

$= -\frac{\csc\sqrt{x}\cot\sqrt{x}}{\sqrt{x}}$

20. $\frac{d}{dx}(\log_{10}2x)^{2/3}$

$= \frac{2}{3}(\log_{10}2x)^{-1/3}\frac{d}{dx}(\log_{10}2+\log_{10}x)$

$= \frac{2}{3}(\log_{10}2x)^{-1/3}\frac{1}{x\ln 10}$

$= \frac{2}{3x\ln 10\sqrt[3]{\log_{10}2x}}$

21. $\frac{d}{dx}4\tan 5x\sec 5x$

$= 20\sec^2 5x\sec 5x + 20\tan 5x\sec 5x\tan 5x$

$= 20\sec 5x(\sec^2 5x+\tan^2 5x)$

$= 20\sec 5x(2\sec^2 5x-1)$

22 $\frac{d}{dx}\tan^{-1}\left(\frac{x^2}{2}\right) = \frac{1}{\left(\frac{x^2}{2}\right)^2+1}\frac{d}{dx}\left(\frac{x^2}{2}\right)$

$= \frac{x}{\left(\frac{x^4}{4}\right)+1} = \frac{4x}{x^4+4}$

$\frac{d}{dx}\left[x\tan^{-1}\left(\frac{x^2}{2}\right)\right]$

$= (1)\tan^{-1}\left(\frac{x^2}{2}\right)+(x)\left(\frac{4x}{x^4+4}\right)$

$= \tan^{-1}\left(\frac{x^2}{2}\right)+\frac{4x^2}{x^4+4}$

23. $\frac{d}{dx}x^{1+x} = \frac{d}{dx}e^{(1+x)\ln x}$

$= e^{(1+x)\ln x}\frac{d}{dx}[(1+x)\ln x]$

$= x^{1+x}\left[(1)(\ln x)+(1+x)\left(\frac{1}{x}\right)\right]$

$= x^{1+x}\left(\ln x+1+\frac{1}{x}\right)$

24. $\frac{d}{dx}(1+x^2)^e = e(1+x^2)^{e-1}\frac{d}{dx}(1+x^2)$

$= 2xe(1+x^2)^{e-1}$

25. Let $u = 3x-1$, so $du = 3\,dx$.

$\int e^{3x-1}dx = \frac{1}{3}\int e^{3x-1}3dx = \frac{1}{3}\int e^u du$

$= \frac{1}{3}e^u + C = \frac{1}{3}e^{3x-1}+C$

Check:

$\frac{d}{dx}\left(\frac{1}{3}e^{3x-1}+C\right) = \frac{1}{3}e^{3x-1}\frac{d}{dx}(3x-1) = e^{3x-1}$

26. Let $u = \sin 3x$, so $du = 3\cos 3x\,dx$.

$\int 6\cot 3x\,dx = 2\int\frac{1}{\sin 3x}3\cos 3x\,dx = 2\int\frac{1}{u}du$

$= 2\ln|u|+C = 2\ln|\sin 3x|+C$

Check:

$\frac{d}{dx}(2\ln|\sin 3x|+C) = \frac{2}{\sin 3x}\frac{d}{dx}\sin 3x$

$= \frac{2(3\cos 3x)}{\sin 3x} = 6\cot 3x$

27. Let $u = e^x$, so $du = e^x dx$.

$\int e^x\sin e^x dx = \int\sin u\,du = -\cos u + C$

$= -\cos e^x + C$

Check:

$\frac{d}{dx}(-\cos e^x + C) = (\sin e^x)\frac{d}{dx}e^x = e^x\sin e^x$

28. Let $u = x^2+x-5$, so $du = (2x+1)dx$.

$\int\frac{6x+3}{x^2+x-5}dx = 3\int\frac{1}{x^2+x-5}(2x+1)dx$

$= 3\int\frac{1}{u}du = 3\ln|u|+C = 3\ln\left|x^2+x-5\right|+C$

Check:

$\frac{d}{dx}\left(3\ln\left|x^2+x-5\right|+C\right)$

$$= \frac{3}{x^2+x-5}\frac{d}{dx}(x^2+x-5)$$
$$= \frac{6x+3}{x^2+x-5}$$

29. Let $u = e^{x+3}+1$, so $du = e^{x+3}dx$.

$$\int \frac{e^{x+2}}{e^{x+3}+1}dx = \frac{1}{e}\int \frac{1}{e^{x+3}+1}e^{x+3}dx = \frac{1}{e}\int \frac{1}{u}du$$
$$= \frac{1}{e}\ln|u| + C = \frac{\ln(e^{x+3}+1)}{e} + C$$

Check:

$$\frac{d}{dx}\left(\frac{\ln(e^{x+3}+1)}{e} + C\right) = \frac{1}{e}\frac{1}{e^{x+3}+1}\frac{d}{dx}(e^{x+3}+1)$$
$$= \frac{e^{x+3}e^{-1}}{e^{x+3}+1} = \frac{e^{x+2}}{e^{x+3}+1}$$

30. Let $u = x^2$, so $du = 2x\,dx$.

$$\int 4x\cos x^2 dx = 2\int (\cos x^2)2x\,dx = 2\int \cos u\,du$$
$$= 2\sin u + C = 2\sin x^2 + C$$

Check:

$$\frac{d}{dx}(2\sin x^2 + C) = 2\cos x^2\frac{d}{dx}x^2 = 4x\cos x^2$$

31. Let $u = 2x$, so $du = 2\,dx$.

$$\int \frac{4}{\sqrt{1-4x^2}}dx = 2\int \frac{1}{\sqrt{1-(2x)^2}}2dx$$
$$= 2\int \frac{1}{\sqrt{1-u^2}}du$$
$$= 2\sin^{-1}u + C = 2\sin^{-1}2x + C$$

Check:

$$\frac{d}{dx}(2\sin^{-1}2x + C) = 2\left(\frac{1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}2x$$
$$= \frac{4}{\sqrt{1-4x^2}}$$

32. Let $u = \sin x$, so $du = \cos x\,dx$.

$$\int \frac{\cos x}{1+\sin^2 x}dx = \int \frac{1}{1+u^2}du = \tan^{-1}u + C$$
$$= \tan^{-1}(\sin x) + C$$

Check:

$$\frac{d}{dx}\left[\tan^{-1}(\sin x) + C\right] = \frac{1}{1+\sin^2 x}\frac{d}{dx}\sin x$$
$$= \frac{\cos x}{1+\sin^2 x}$$

33. Let $u = \ln x$, so $du = \frac{1}{x}dx$.

$$\int \frac{-1}{x+x(\ln x)^2}dx = -\int \frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}dx$$
$$= -\int \frac{1}{1+u^2}du = -\tan^{-1}u + C = -\tan^{-1}(\ln x) + C$$

Check:

$$\frac{d}{dx}[-\tan^{-1}(\ln x) + C] = -\frac{1}{1+(\ln x)^2}\frac{d}{dx}\ln x$$
$$= \frac{-1}{x+x(\ln x)^2}$$

34. Let $u = x-3$, so $du = dx$.

$$\int \text{sech}^2(x-3)dx = \int \text{sech}^2 u\,du = \tanh u + C$$
$$= \tanh(x-3) + C$$

Check:

$$\frac{d}{dx}[\tanh(x-3)] = \text{sech}^2(x-3)\frac{d}{dx}(x-3)$$
$$= \text{sech}^2(x-3)$$

35. $f'(x) = \cos x - \sin x$; $f'(x) = 0$ when $\tan x = 1$, $x = \frac{\pi}{4}$

$f'(x) > 0$ when $\cos x > \sin x$ which occurs when $-\frac{\pi}{2} \le x < \frac{\pi}{4}$.

$f''(x) = -\sin x - \cos x$; $f''(x) = 0$ when $\tan x = -1$, $x = -\frac{\pi}{4}$

$f''(x) > 0$ when $\cos x < -\sin x$ which occurs when $-\frac{\pi}{2} \le x < -\frac{\pi}{4}$.

Increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$

Decreasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

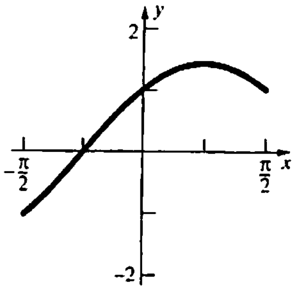
Concave up on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$

Concave down on $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$

Inflection point at $\left(-\frac{\pi}{4}, 0\right)$

Global maximum at $\left(\frac{\pi}{4}, \sqrt{2}\right)$

Global minimum at $\left(-\frac{\pi}{2}, -1\right)$



36. $f(x) = \frac{x^2}{e^x}$

$$f'(x) = \frac{e^x(2x) - x^2(e^x)}{(e^x)^2} = \frac{2x - x^2}{e^x}$$

f is increasing on [0, 2] because $f'(x) > 0$ on (0, 2).

f is decreasing on $(-\infty, 0] \cup [2, \infty)$ because $f'(x) < 0$ on $(-\infty, 0) \cup (2, \infty)$.

$$f''(x) = \frac{e^x(2-2x) - (2x - x^2)e^x}{(e^x)^2} = \frac{x^2 - 4x + 2}{e^x}$$

Inflection points are at

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2} = 2 \pm \sqrt{2}.$$

The graph of f is concave up on $(-\infty, 2 - \sqrt{2}) \cup (2 + \sqrt{2}, \infty)$ because $f''(x) > 0$ on these intervals.

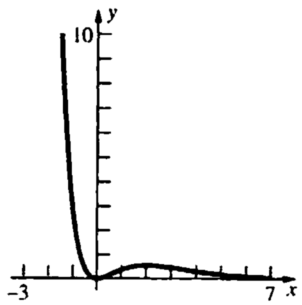
The graph of f is concave down on $(2 - \sqrt{2}, 2 + \sqrt{2})$ because $f''(x) < 0$ on this interval.

The absolute minimum value is $f(0) = 0$.

The relative maximum value is $f(2) = \frac{4}{e^2}$.

The inflection points are

$$\left(2 - \sqrt{2}, \frac{6 - 4\sqrt{2}}{e^{2-\sqrt{2}}}\right) \text{ and } \left(2 + \sqrt{2}, \frac{6 + 4\sqrt{2}}{e^{2+\sqrt{2}}}\right).$$



37. a. $f'(x) = 5x^4 + 6x^2 + 4 \ge 4 > 0$ for all x, so $f(x)$ is increasing.

b. $f(1) = 7$, so $g(7) = f^{-1}(7) = 1$.

c. $g'(7) = \frac{1}{f'(1)} = \frac{1}{15}$

38. $\frac{1}{2} = e^{10k}$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{10} \approx -0.06931$$

$$y = 100e^{-0.06931t}$$

$$1 = 100e^{-0.06931t}$$

$$t = \frac{\ln\left(\frac{1}{100}\right)}{-0.06931} \approx 66.44$$

It will take about 66.44 years.

39. a. $(\$100)(1.12) = \$112$

b. $(\$100)\left(1 + \frac{0.12}{12}\right)^{12} \approx \$112.68$

c. $(\$100)\left(1 + \frac{0.12}{365}\right)^{365} \approx \$112.75$

d. $(\$100)e^{(0.12)(1)} \approx \$112.75$

40. Let x be the horizontal distance from the airplane to the searchlight, $\frac{dx}{dt} = 300$.

$$\tan\theta = \frac{500}{x}, \text{ so } \theta = \tan^{-1}\frac{500}{x}.$$

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{500}{x}\right)^2}\left(-\frac{500}{x^2}\right)\frac{dx}{dt}$$

$$= -\frac{500}{x^2 + 250{,}000}\frac{dx}{dt}$$

When $\theta = 30°$, $x = \frac{500}{\tan 30°} = 500\sqrt{3}$ and

$$\frac{d\theta}{dt} = -\frac{500}{(500\sqrt{3})^2 + (500)^2}(300)$$

$= -\frac{300}{2000} = -\frac{3}{20}$. The angle is decreasing at the rate of 0.15 rad/s $\approx$ 8.59°/s.

41. $y = (\cos x)^{\sin x} = e^{\sin x \ln(\cos x)}$

$$\frac{dy}{dx} = e^{\sin x \ln(\cos x)}\frac{d}{dx}[\sin x \ln(\cos x)]$$

$$= e^{\sin x \ln(\cos x)}\left[\cos x \ln(\cos x) + (\sin x)\left(\frac{1}{\cos x}\right)(-\sin x)\right]$$

$= (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$

At $x = 0$, $\frac{dy}{dx} = 1^0(1\ln 1 - 0) = 0$.

The tangent line has slope 0, so it is horizontal: $y = 1$.

42. Let t represent the number of years since 1990.

$14,000 = 10,000e^{10k}$

$k = \frac{\ln(1.4)}{10} \approx 0.03365$

$y = 10,000e^{0.03365t}$

$y(20) = 10,000e^{(0.03365)(20)} \approx 19,601$

The population will be about 19,600.

43. Integrating factor is $|x|$. $D[y|x|] = 0; y = Cx^{-1}$

44. Integrating factor is x^2.

$D[yx^2] = x^3; y = \left(\frac{1}{4}\right)x^2 + Cx^{-2}$

45. (Linear first-order) $y' + 2xy = 2x$

Integrating factor: $e^{\int 2x dx} = e^{x^2}$

$D[ye^{x^2}] = 2xe^{x^2}; ye^{x^2} = e^{x^2} + C;$

$y = 1 + Ce^{-x^2}$

If $x = 0, y = 3$, then $3 = 1 + C$, so $C = 2$.

Therefore, $y = 1 + 2e^{-x^2}$.

46. Integrating factor is e^{-ax}.

$D[ye^{-ax}] = 1; y = e^{ax}(x + C)$

47. Integrating factor is e^{-2x}.

$D[ye^{-2x}] = e^{-x}; y = -e^x + Ce^{2x}$

48. **a.** $Q'(t) = 3 - 0.02Q$

b. $Q'(t) + 0.02Q = 3$

Integrating factor is $e^{0.02t}$

$D[Qe^{0.02t}] = 3e^{0.02t}$

$Q(t) = 150 + Ce^{-0.02t}$

$Q(t) = 150 - 30e^{-0.02t}$ goes through (0, 120).

c. $Q \to 150$ g, as $t \to \infty$.

7.10 Additional Problem Set

1. $1000e^{(0.05)(1)} = \$1051.27$

2. $A_0e^{(0.05)(1)} = 1000$

$A_0 = 1000e^{-0.05} \approx \$951.23$

3. **a.**

$$100e^{0.05(360/365)} + 100e^{0.05(330/365)} + 100e^{0.05\cdot(300/365)} + 100e^{0.05\cdot(270/365)} + 100e^{0.05\cdot(240/365)} + 100e^{0.05\cdot(210/365)} + 100e^{0.05\cdot(180/365)} + 100e^{0.05(150/365)} + 100e^{0.05\cdot(120/365)} + 100e^{0.05\cdot(90/365)} + 100e^{0.05\cdot(60/365)} + 100e^{0.05\cdot(30/365)} \approx \$1232.61$$

b. The formula in part a. is a geometric series $\sum_{k=1}^{n} ar^{k-1}$ with $a = 100e^{0.05\cdot(30/365)}$, $r = e^{0.05\cdot(30/365)}$, and $n = 12$. The sum is $\frac{100e^{0.05\cdot(30/365)} - 100e^{0.05\cdot(390/365)}}{1 - e^{0.05\cdot(30/365)}}$.

4. **a.** Let $q = 1 + 0.05\cdot\left(\frac{30}{365}\right)$.

The amount of money after 30 days is $A_1 = A_0q - 144$.

The amount of money after 60 days is $A_2 = A_1q - 144 = A_0q^2 - 144q - 144$.

The amount of money after 90 days is $A_3 = A_2q - 144 = A_0q^3 - 144q^2 - 144q - 144$. Thus, the amount of money after $30n$ days is $A_n = A_0q^n - 144\sum_{k=0}^{n-1} q^k = A_0q^n - 144\left(\frac{q^n - 1}{q-1}\right)$. If the amount of money after $3600 = 30 \cdot 120$ days is 0, we have $0 = A_0q^{120} - 144\left(\frac{q^{120}-1}{q-1}\right)$.

$$A_0q^{120} = 144\left(\frac{q^{120}-1}{q-1}\right)$$

$$A_0 = 144\left(\frac{1-q^{-120}}{q-1}\right) = 144\left[\frac{1-\left(1+0.05\cdot\left(\frac{30}{365}\right)\right)^{-120}}{0.05\cdot\left(\frac{30}{365}\right)}\right]$$

Note that $-10\cdot\left(\frac{365}{30}\right) \approx -120$.

$\Bigg($Or ten years $= 10\cdot\frac{365}{30}\cdot 30$ days, so $0 = A_0q^{10\cdot(365/30)} - 144\left(\frac{q^{10\cdot(365/30)}-1}{q-1}\right)$.

Thus, $A_0 = 144\left[\frac{1-\left(1+0.05\cdot\left(\frac{30}{365}\right)\right)^{-10\cdot(365/30)}}{0.05\cdot\left(\frac{30}{365}\right)}\right]\Bigg)$.

b. Let $q = e^{-0.05(30/365)}$. Following the same reasoning as in part a., the amount of money after $30n$ days is $A_n = A_0q^n - 144\left(\frac{q^n-1}{q-1}\right)$, and if the amount of money after 3600 days is 0,

$$A_0 = 144\left(\frac{1-q^{-120}}{q-1}\right) = 144\left[\frac{1-e^{0.05(3600/365)}}{e^{-0.05(30/365)}-1}\right].$$

Or, by using 10 years, $A_0 = 144\left[\frac{1-e^{0.05\cdot 10}}{e^{-0.05(30/365)}-1}\right]$.

5. a. If you deposit A_0 into an account earning an interest rate of $100r$ percent, compounded continuously, then the amount you have at time t is $A(t) = A_0e^{rt}$. The money has doubled when $A(t) = 2A_0$ or $A_0e^{rt} = 2A_0$, so T satisfies the equation $A_0e^{rT} = 2A_0$.

$$A_0e^{r(70/100r)} = A_0e^{0.7} \approx 2A_0$$

b. Solving $A_0e^{rT} = 2A_0$ for T to get $T = \frac{1}{r}\ln 2 \approx \frac{1}{r}(0.7) = \frac{70}{100r}$

c. $T \approx \frac{70}{100(0.07)} = 10$

Ten years

6. If $f(t) = e^{kt}$, then $\frac{f'(t)}{f(t)} = \frac{ke^{kt}}{e^{kt}} = k$.

7. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$$\lim_{x\to\infty} \frac{f'(x)}{f(x)} = \lim_{x\to\infty} \frac{na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1}{a_n x^n + a_{n-1}x + \cdots + a_1 x + a_0} = \lim_{x\to\infty} \frac{\frac{na_n}{x} + \frac{(n-1)a_{n-1}}{x^2} + \cdots + \frac{a_1}{x^n}}{a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}} = 0$$

8. $\frac{f'(x)}{f(x)} = k > 0$ can be written as

$\frac{1}{y}\frac{dy}{dx} = k$ where $y = f(x)$.

$\frac{dy}{y} = k\,dx$ has the solution $y = Ce^{kx}$.

Thus, the equation $f(x) = Ce^{kx}$ represents exponential growth since $k > 0$.

9. $\frac{f'(x)}{f(x)} = k < 0$ can be written as $\frac{1}{y}\frac{dy}{dx} = k$ where $y = f(x)$. $\frac{dy}{y} = k\,dx$ has the solution $y = Ce^{kx}$.

Thus, $f(x) = Ce^{kx}$ which represents exponential decay since $k < 0$.

10. a. The error in $f(x)$ due to a relative error in the input of size Δx is $f(x + \Delta x) - f(x)$, thus the relative error in $f(x)$ due to this relative error in input is

$$\frac{f(x+\Delta x) - f(x)}{f(x)} = \frac{f(x+\Delta x) - f(x)}{f(x)} \cdot \frac{\Delta x}{\Delta x} \cdot \frac{x}{x} = \frac{f(x+\Delta x) - f(x)}{\Delta x \cdot f(x)} \cdot \frac{\Delta x}{x} \cdot x = \frac{f(x+\Delta x) - f(x)}{\Delta x \cdot f(x)} \cdot \frac{(x+\Delta x) - x}{x} \cdot x$$

b. $\rho_x = \lim_{\Delta x\to 0} \frac{(x+\Delta x) - x}{x}$, while $\frac{d\ln(f(x))}{dx} = \frac{f'(x)}{f(x)}$, so $\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{f(x)}$

$$= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x \cdot f(x)} \cdot \frac{(x+\Delta x) - x}{x} \cdot x\right] = \left[\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \frac{1}{f(x)}\right] \cdot \left[\lim_{\Delta x\to 0} \frac{(x+\Delta x) - x}{x}\right] \cdot x$$

$$= f'(x) \cdot \frac{1}{f(x)} \cdot \rho_x \cdot x = x \cdot \frac{d\ln(f(x))}{dx} \cdot \rho_x$$

c. $x \cdot \frac{d}{dx}[\ln(e^x)] = x \cdot \frac{d}{dx}(x) = x \cdot 1 = x$

11. a. In order of increasing slope, the graphs represent the curves $y = 2^x$, $y = 3^x$, and $y = 4^x$.

b. $\ln y$ is linear with respect to x, and at $x = 0$,
$y = 1$ since $C = 1$.

c. The graph passes through the points (0.2, 4) and (0.6, 8). Thus, $4 = Cb^{0.2}$ and $8 = Cb^{0.6}$.
Dividing the second equation by the first, gets $2 = b^{0.4}$ so $b = 2^{5/2}$.
Therefore $C = 2^{3/2}$.

12. The graph of the equation whose log-log plot has negative slope contains the points (2, 7) and (7, 2).

Thus, $7 = C2^r$ and $2 = C7^r$, so $\frac{7}{2} = \left(\frac{2}{7}\right)^r$. $\ln\frac{7}{2} = r\ln\frac{2}{7} \Rightarrow r = \frac{\ln 7 - \ln 2}{\ln 2 - \ln 7} = -1$ and $C = 14$.

Hence, one equation is $y = 14x^{-1}$.
The graph of one equation contains the points

(7, 30) and (10, 70). Thus, $30 = C7^r$ and $70 = C10^r$, so $\frac{3}{7} = \left(\frac{7}{10}\right)^r$

$\ln\frac{3}{7} = r\ln\frac{7}{10} \Rightarrow r = \frac{\ln 3 - \ln 7}{\ln 7 - \ln 10} \approx 2.38$ and $C \approx 30 \cdot 7^{-2.38} \approx 0.29$. Hence, another equation is $y = 0.29x^{2.38}$.

The graph of another equation contains the points (1, 2) and (7, 5). Thus, $2 = C1^r$ and $5 = C7^r$, so $C = 2$ and

$\ln 5 - \ln 2 = r\ln 7 \Rightarrow r = \frac{\ln 5 - \ln 2}{\ln 7} \approx 0.47$. Hence, the last equation is $y = x^{0.47}$. Other answers are possible depending on selection of points.

13. **a.** $\eta = 3\operatorname{sech}^2\frac{x}{2}$

$$\eta' = -3\operatorname{sech}^2\frac{x}{2}\tanh\frac{x}{2}$$

$$3\eta^2 = 3 \cdot 9\operatorname{sech}^4\frac{x}{2} = 27\operatorname{sech}^4\frac{x}{2}$$

$$\eta^3 + 3(\eta')^2 = 27\operatorname{sech}^6\frac{x}{2} + 3\left(9\operatorname{sech}^4\frac{x}{2}\tanh^2\frac{x}{2}\right)$$

$$= 27\operatorname{sech}^4\frac{x}{2}\left(\operatorname{sech}^2\frac{x}{2} + \tanh^2\frac{x}{2}\right)$$

$$= 27\operatorname{sech}^4\frac{x}{2} = 3\eta^2$$

Thus, $\eta = 3\operatorname{sech}^2\frac{x}{2}$ satisfies the equation.

b.