# CHAPTER 6

# Applications of the Integral

## 6.1 Concepts Review

1. $\int_a^b f(x)dx; -\int_a^b f(x)dx$

2. slice, approximate, integrate

3. $g(x)-f(x); f(x)=g(x)$

4. $\int_c^d [q(y)-p(y)]dy$

## Problem Set 6.1

1. Slice vertically.
$\Delta A \approx (x^2+1)\Delta x$
$A=\int_{-1}^{2}(x^2+1)dx=\left[\frac{1}{3}x^3+x\right]_{-1}^{2}$
$=\left(\frac{8}{3}+2\right)-\left(-\frac{1}{3}-1\right)=6$

2. Slice vertically.
$\Delta A \approx (x^3-x+2)\Delta x$
$A=\int_{-1}^{2}(x^3-x+2)dx=\left[\frac{1}{4}x^4-\frac{1}{2}x^2+2x\right]_{-1}^{2}$
$=(4-2+4)-\left(\frac{1}{4}-\frac{1}{2}-2\right)=\frac{33}{4}$

3. Slice vertically.
$\Delta A \approx \left[(x^2+2)-(-x)\right]\Delta x=(x^2+x+2)\Delta x$
$A=\int_{-2}^{2}(x^2+x+2)dx=\left[\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-2}^{2}$
$=\left(\frac{8}{3}+2+4\right)-\left(-\frac{8}{3}+2-4\right)=\frac{40}{3}$

4. Slice vertically.
$\Delta A \approx -(x^2+2x-3)\Delta x=(-x^2-2x+3)\Delta x$
$A=\int_{-3}^{1}(-x^2-2x+3)dx=\left[-\frac{1}{3}x^3-x^2+3x\right]_{-3}^{1}$
$=\left(-\frac{1}{3}-1+3\right)-(9-9-9)=\frac{32}{3}$

5. To find the intersection points, solve $2-x^2=x$.
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x=-2, 1$
Slice vertically.
$\Delta A \approx \left[(2-x^2)-x\right]\Delta x=(-x^2-x-2)\Delta x$
$A=\int_{-2}^{1}(-x^2-x+2)dx=\left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^{1}$
$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)=\frac{9}{2}$

6. To find the intersection points, solve
$x+4=x^2-2$.
$x^2-x-6=0$
$(x+2)(x-3)=0$
$x=-2, 3$
Slice vertically.
$\Delta A \approx \left[(x+4)-(x^2-2)\right]\Delta x=(-x^2+x+6)\Delta x$
$A=\int_{-2}^{3}(-x^2+x+6)dx=\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+6x\right]_{-2}^{3}$
$\left(-9+\frac{9}{2}+18\right)-\left(\frac{8}{3}+2-12\right)=\frac{125}{6}$

7. Solve $x^3-x^2-6x=0$.
$x(x^2-x-6)=0$
$x(x+2)(x-3)=0$
$x=-2, 0, 3$
Slice vertically.
$\Delta A_1 \approx (x^3-x^2-6x)\Delta x$
$\Delta A_2 \approx -(x^3-x^2-6x)\Delta x=(-x^3+x^2+6x)\Delta x$
$A=A_1+A_2$
$=\int_{-2}^{0}(x^3-x^2-6x)dx+\int_0^3(-x^3+x^2+6x)dx$

$$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\right]_{-2}^{0}$$

$$+ \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + 3x^2\right]_{0}^{3}$$

$$= \left[0 - \left(4 + \frac{8}{3} - 12\right)\right] + \left[-\frac{81}{4} + 9 + 27 - 0\right]$$

$$= \frac{16}{3} + \frac{63}{4} = \frac{253}{12}$$

**8.** To find the intersection points, solve

$-x + 2 = x^2$.

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, 1$

Slice vertically.

$$\Delta A \approx \left[(-x+2) - x^2\right]\Delta x = (-x^2 - x + 2)\Delta x$$

$$A = \int_{-2}^{1} (-x^2 - x + 2)dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$$

**9.** To find the intersection points, solve

$y + 1 = 3 - y^2$.

$y^2 + y - 2 = 0$

$(y + 2)(y - 1) = 0$

$y = -2, 1$

Slice horizontally.

$$\Delta A \approx \left[(3 - y^2) - (y + 1)\right]\Delta y = (-y^2 - y + 2)\Delta y$$

$$A = \int_{-2}^{1} (-y^2 - y + 2)dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-2}^{1}$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$$

**10.** To find the intersection point, solve $y^2 = 6 - y$.

$y^2 + y - 6 = 0$

$(y + 3)(y - 2) = 0$

$y = -3, 2$

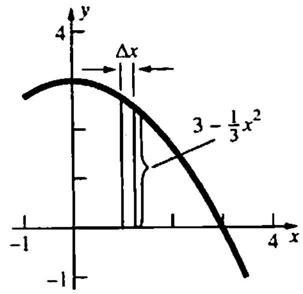
Slice horizontally.

$$\Delta A \approx \left[(6 - y) - y^2\right]\Delta y = (-y^2 - y + 6)\Delta y$$

$$A = \int_{0}^{2} (-y^2 - y + 6)dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 6y\right]_{0}^{2}$$

$$= -\frac{8}{3} - 2 + 12 = \frac{22}{3}$$

**11.**

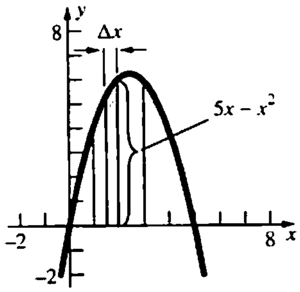



$$\Delta A \approx \left(3 - \frac{1}{3}x^2\right)\Delta x$$

$$A = \int_{0}^{3} \left(3 - \frac{1}{3}x^2\right)dx = \left[3x - \frac{1}{9}x^3\right]_{0}^{3} = 9 - 3 = 6$$

Estimate the area to be (3)(2) = 6.

**12.**



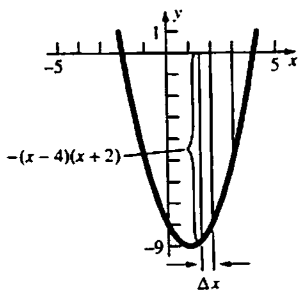

$$\Delta A \approx (5x - x^2)\Delta x$$

$$A = \int_{1}^{3} (5x - x^2)dx = \left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_{1}^{3}$$

$$= \left(\frac{45}{2} - 9\right) - \left(\frac{5}{2} - \frac{1}{3}\right) = \frac{34}{3} \approx 11.33$$

Estimate the area to be $(2)\left(5\frac{1}{2}\right) = 11$.

**13.**



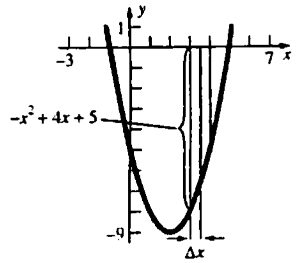

$$\Delta A \approx -(x-4)(x+2)\Delta x = (-x^2 + 2x + 8)\Delta x$$

$$A = \int_{0}^{3} (-x^2 + 2x + 8)dx = \left[-\frac{1}{3}x^3 + x^2 + 8x\right]_{0}^{3}$$

$= -9 + 9 + 24 = 24$

Estimate the area to be (3)(8) = 24.

14.




$\Delta A \approx -(x^2-4x-5)\Delta x = (-x^2+4x+5)\Delta x$

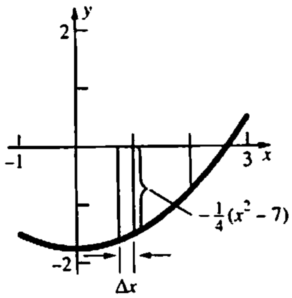
$A = \int_{-1}^{4}(-x^2+4x+5)dx$

$= \left[-\frac{1}{3}x^3+2x^2+5x\right]_{-1}^{4}$

$= \left(-\frac{64}{3}+32+20\right)-\left(\frac{1}{3}+2-5\right)=\frac{100}{3}\approx 33.33$

Estimate the area to be $(5)\left(6\frac{1}{2}\right)=32\frac{1}{2}$.

15.



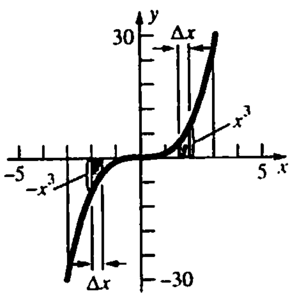

$\Delta A \approx -\frac{1}{4}(x^2-7)\Delta x$

$A = \int_0^2 -\frac{1}{4}(x^2-7)dx = -\frac{1}{4}\left[\frac{1}{3}x^3-7x\right]_0^2$

$= -\frac{1}{4}\left(\frac{8}{3}-14\right)=\frac{17}{6}\approx 2.83$

Estimate the area to be $(2)\left(1\frac{1}{2}\right)=3$.

16.




$\Delta A_1 \approx -x^3\Delta x$

$\Delta A_2 \approx x^3\Delta x$

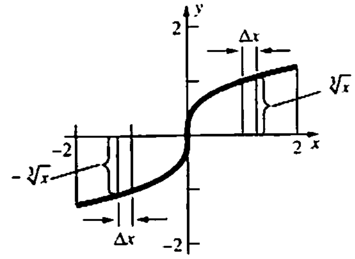
$A = A_1 + A_2 = \int_{-3}^{0} -x^3dx + \int_0^3 x^3dx$

$= \left[-\frac{1}{4}x^4\right]_{-3}^{0} + \left[\frac{1}{4}x^4\right]_0^3 = \left(\frac{81}{4}\right)+\left(\frac{81}{4}\right)=\frac{81}{2}$

$= 40.5$

Estimate the area to be (3)(7) + (3)(7) = 42.

17.




$\Delta A_1 \approx -\sqrt[3]{x}\,\Delta x$

$\Delta A_2 \approx \sqrt[3]{x}\,\Delta x$

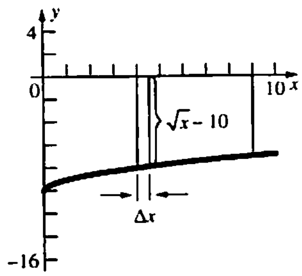
$A = A_1 + A_2 = \int_{-2}^{0} -\sqrt[3]{x}\,dx + \int_0^2 \sqrt[3]{x}\,dx$

$= \left[-\frac{3}{4}x^{4/3}\right]_{-2}^{0} + \left[\frac{3}{4}x^{4/3}\right]_0^2 = \left(\frac{3\sqrt[3]{2}}{2}\right)+\left(\frac{3\sqrt[3]{2}}{2}\right)$

$= 3\sqrt[3]{2}\approx 3.78$

Estimate the area to be (2)(1) + (2)(1) = 4.

18.



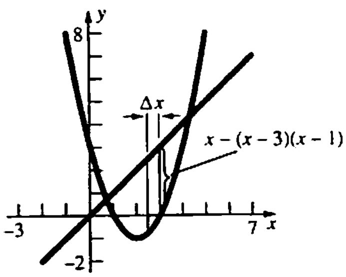

$\Delta A \approx -(\sqrt{x}-10)\Delta x = (10-\sqrt{x})\Delta x$

$A = \int_0^9 (10-\sqrt{x})dx = \left[10x-\frac{2}{3}x^{3/2}\right]_0^9$

$= 90 - 18 = 72$

Estimate the area to be $9 \cdot 8 = 72$.

19.




$\Delta A \approx [x-(x-3)(x-1)]\Delta x$

$= [x-(x^2-4x+3)]\Delta x = (-x^2+5x-3)\Delta x$

To find the intersection points, solve $x=(x-3)(x-1)$.

$x^2 - 5x + 3 = 0$

$x = \frac{5 \pm \sqrt{25-12}}{2}$

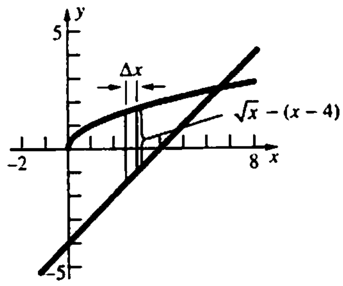
$x = \frac{5 \pm \sqrt{13}}{2}$

$A = \int_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} (-x^2 + 5x - 3)dx$

$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 3x\right]_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} = \frac{13\sqrt{13}}{6} \approx 7.81$

Estimate the area to be $\frac{1}{2}(4)(4) = 8$.

**20.**




$\Delta A \approx \left[\sqrt{x} - (x-4)\right]\Delta x = \left(\sqrt{x} - x + 4\right)\Delta x$

To find the intersection point, solve

$\sqrt{x} = (x-4)$.

$x = (x-4)^2$

$x^2 - 9x + 16 = 0$

$x = \frac{9 \pm \sqrt{81-64}}{2}$

$x = \frac{9 \pm \sqrt{17}}{2}$

$\left(x = \frac{9-\sqrt{17}}{2} \text{ is extraneous so } x = \frac{9+\sqrt{17}}{2}.\right)$

$A = \int_0^{\frac{9+\sqrt{17}}{2}} \left(\sqrt{x} - x + 4\right)dx$

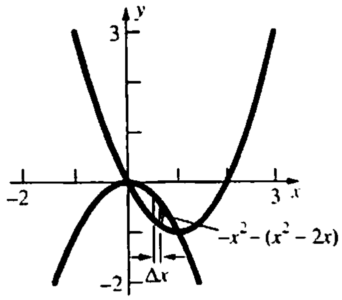
$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 4x\right]_0^{\frac{9+\sqrt{17}}{2}}$

$= \frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2} - \frac{1}{2}\left(\frac{9+\sqrt{17}}{2}\right)^2 + 4\left(\frac{9+\sqrt{17}}{2}\right)$

$= \frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2} + \frac{23}{4} - \frac{\sqrt{17}}{4} \approx 15.92$

Estimate the area to be $\frac{1}{2}\left(5\frac{1}{2}\right)\left(5\frac{1}{2}\right) = 15\frac{1}{8}$.

**21.**




$\Delta A \approx \left[-x^2 - (x^2 - 2x)\right]\Delta x = (-2x^2 + 2x)\Delta x$

To find the intersection points, solve

$-x^2 = x^2 - 2x$.

$2x^2 - 2x = 0$

$2x(x-1) = 0$

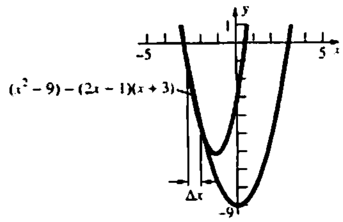
$x = 0, x = 1$

$A = \int_0^1 (-2x^2 + 2x)dx = \left[-\frac{2}{3}x^3 + x^2\right]_0^1$

$= -\frac{2}{3} + 1 = \frac{1}{3} \approx 0.33$

Estimate the area to be $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$.

**22.**




$\Delta A \approx \left[(x^2 - 9) - (2x-1)(x+3)\right]\Delta x$

$= \left[(x^2 - 9) - (2x^2 + 5x - 3)\right]\Delta x$

$= (-x^2 - 5x - 6)\Delta x$

To find the intersection points, solve

$(2x-1)(x+3) = x^2 - 9$.

$x^2 + 5x + 6 = 0$

$(x+3)(x+2) = 0$

$x = -3, -2$

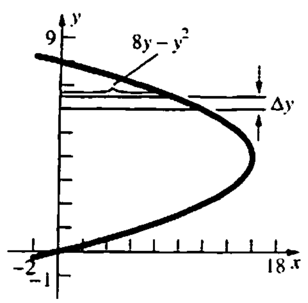
$A = \int_{-3}^{-2} (-x^2 - 5x - 6)dx$

$= \left[-\frac{1}{3}x^3 - \frac{5}{2}x^2 - 6x\right]_{-3}^{-2}$

$= \left(\frac{8}{3} - 10 + 12\right) - \left(9 - \frac{45}{2} + 18\right) = \frac{1}{6} \approx 0.17$

Estimate the area to be $\frac{1}{2}(1)\left(5 - 4\frac{2}{3}\right) = \frac{1}{6}$.

23.




$\Delta A \approx (8y - y^2)\Delta y$

To find the intersection points, solve

$8y - y^2 = 0.$

$y(8 - y) = 0$

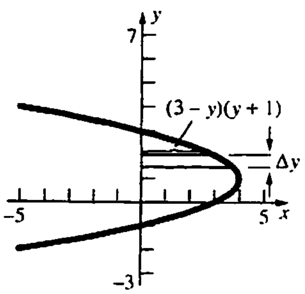
$y = 0, 8$

$$A = \int_0^8 (8y - y^2)\,dy = \left[4y^2 - \frac{1}{3}y^3\right]_0^8$$

$$= 256 - \frac{512}{3} = \frac{256}{3} \approx 85.33$$

Estimate the area to be (16)(5) = 80.

24.



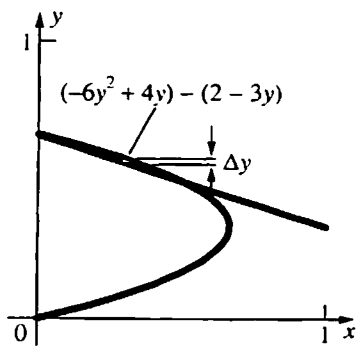

$\Delta A \approx (3 - y)(y + 1)\Delta y = (-y^2 + 2y + 3)\Delta y$

$$A = \int_{-1}^3 (-y^2 + 2y + 3)\,dy = \left[-\frac{1}{3}y^3 + y^2 + 3y\right]_{-1}^3$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = \frac{32}{3} \approx 10.67$$

Estimate the area to be $(4)\left(2\frac{1}{2}\right) = 10$.

25.




$\Delta A \approx \left[(-6y^2 + 4y) - (2 - 3y)\right]\Delta y$

$= (-6y^2 + 7y - 2)\Delta y$

To find the intersection points, solve

$-6y^2 + 4y = 2 - 3y.$

$6y^2 - 7y + 2 = 0$

$(2y - 1)(3y - 2) = 0$

$y = \frac{1}{2}, \frac{2}{3}$

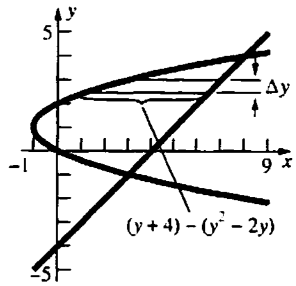
$$A = \int_{1/2}^{2/3} (-6y^2 + 7y - 2)\,dy = \left[-2y^3 + \frac{7}{2}y^2 - 2y\right]_{1/2}^{2/3}$$

$$= \left(-\frac{16}{27} + \frac{14}{9} - \frac{4}{3}\right) - \left(-\frac{1}{4} + \frac{7}{8} - 1\right) = \frac{1}{216} \approx 0.0046$$

Estimate the area to be

$$\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{5}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{120}.$$

26.




$\Delta A \approx \left[(y + 4) - (y^2 - 2y)\right]\Delta y = (-y^2 + 3y + 4)\Delta y$

To find the intersection points, solve

$y^2 - 2y = y + 4.$

$y^2 - 3y - 4 = 0$

$(y + 1)(y - 4) = 0$

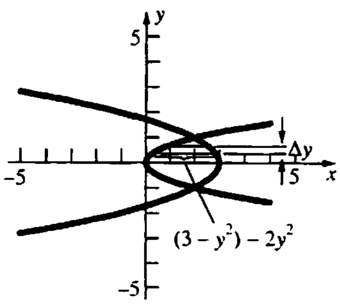
$y = -1, 4$

$$A = \int_{-1}^4 (-y^2 + 3y + 4)\,dy = \left[-\frac{1}{3}y^3 + \frac{3}{2}y^2 + 4y\right]_{-1}^4$$

$$= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = \frac{125}{6} \approx 20.83$$

Estimate the area to be (7)(3) = 21.

27.




$\Delta A \approx \left[(3 - y^2) - 2y^2\right]\Delta y = (-3y^2 + 3)\Delta y$

To find the intersection points, solve

$2y^2 = 3 - y^2.$

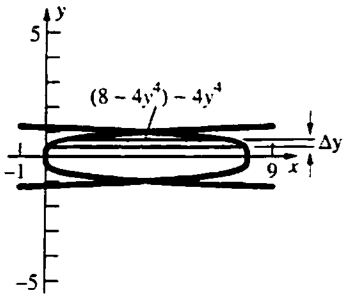
$3y^2 - 3 = 0$

$3(y + 1)(y - 1) = 0$

$y = -1, 1$

$A = \int_{-1}^{1}(-3y^2+3)dy = \left[-y^3+3y\right]_{-1}^{1}$
$= (-1+3)-(1-3) = 4$
Estimate the value to be (2)(2) = 4.

**28.**




$\Delta A \approx \left[(8-4y^4)-(4y^4)\right]\Delta y = (8-8y^4)\Delta y$

To find the intersection points, solve

$4y^4 = 8-4y^4$.

$8y^4 = 8$

$y^4 = 1$

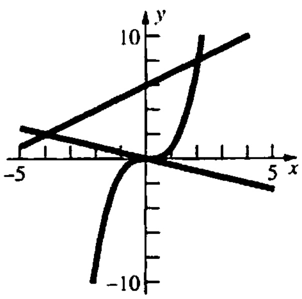
$y = \pm 1$

$A = \int_{-1}^{1}(8-8y^4)dy = \left[8y-\frac{8}{5}y^5\right]_{-1}^{1}$

$= \left(8-\frac{8}{5}\right)-\left(-8+\frac{8}{5}\right) = \frac{64}{5} = 12.8$

Estimate the area to be $(8)\left(1\frac{1}{2}\right) = 12$.

**29.**




Let $R_1$ be the region bounded by $2y+x=0$, $y=x+6$, and $x=0$.

$A(R_1) = \int_{-4}^{0}\left[(x+6)-\left(-\frac{1}{2}x\right)\right]dx$

$= \int_{-4}^{0}\left(\frac{3}{2}x+6\right)dx$

Let $R_2$ be the region bounded by $y = x + 6$, $y = x^3$, and $x = 0$.

$A(R_2) = \int_0^2\left[(x+6)-x^3\right]dx = \int_0^2(-x^3+x+6)dx$

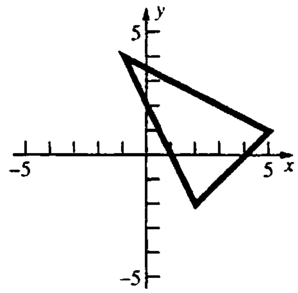
$A(R) = A(R_1)+A(R_2)$

$= \int_{-4}^{0}\left(\frac{3}{2}x+6\right)dx + \int_0^2(-x^3+x+6)dx$

$= \left[\frac{3}{4}x^2+6x\right]_{-4}^{0} + \left[-\frac{1}{4}x^4+\frac{1}{2}x^2+6x\right]_0^2$

$= 12 + 10 = 22$

**30.**




An equation of the line through (–1, 4) and (5, 1) is $y = -\frac{1}{2}x+\frac{7}{2}$. An equation of the line through (–1, 4) and (2, –2) is $y = -2x + 2$. An equation of the line through (2, –2) and (5, 1) is $y = x - 4$. Two integrals must be used. The left-hand part of the triangle has area

$\int_{-1}^{2}\left[-\frac{1}{2}x+\frac{7}{2}-(-2x+2)\right]dx = \int_{-1}^{2}\left(\frac{3}{2}x+\frac{3}{2}\right)dx$.

The right-hand part of the triangle has area

$\int_2^5\left[-\frac{1}{2}x+\frac{7}{2}-(x-4)\right]dx = \int_2^5\left(-\frac{3}{2}x+\frac{15}{2}\right)dx$.

The triangle has area

$\int_{-1}^{2}\left(\frac{3}{2}x+\frac{3}{2}\right)dx + \int_2^5\left(-\frac{3}{2}x+\frac{15}{2}\right)dx$

$= \left[\frac{3}{4}x^2+\frac{3}{2}x\right]_{-1}^{2} + \left[-\frac{3}{4}x^2+\frac{15}{2}x\right]_2^5$

$= \frac{27}{4}+\frac{27}{4} = \frac{27}{2} = 13.5$

**31.** $\int_{-1}^{9}(3t^2-24t+36)dt = \left[t^3-12t^2+36t\right]_{-1}^{9} = (729 - 972 + 324) - (-1 - 12 - 36) = 130$

The displacement is 130 ft. Solve $3t^2-24t+36=0$.
$3(t-2)(t-6) = 0$

$t = 2, 6$

$$|V(t)| = \begin{cases} 3t^2 - 24t + 36 & t \le 2, t \ge 6 \\ -3t^2 + 24t - 36 & 2 < t < 6 \end{cases}$$

$$\int_{-1}^{9} |3t^2 - 24t + 36|\,dt = \int_{-1}^{2} (3t^2 - 24t + 36)\,dt + \int_{2}^{6} (-3t^2 + 24t - 36)\,dt + \int_{6}^{9} (3t^2 - 24t + 36)\,dt$$

$$= \left[t^3 - 12t^2 + 36t\right]_{-1}^{2} + \left[-t^3 + 12t^2 - 36t\right]_{2}^{6} + \left[t^3 - 12t^2 + 36t\right]_{6}^{9} = 81 + 32 + 81 = 194$$

The total distance traveled is 194 feet.

32. $\int_0^{3\pi/2} \left(\frac{1}{2} + \sin 2t\right) dt = \left[\frac{1}{2}t - \frac{1}{2}\cos 2t\right]_0^{3\pi/2} = \left(\frac{3\pi}{4} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right) = \frac{3\pi}{4} + 1$

The displacement is $\frac{3\pi}{4} + 1 \approx 3.36$ feet. Solve $\frac{1}{2} + \sin 2t = 0$ for $0 \le t \le \frac{3\pi}{2}$.

$$\sin 2t = -\frac{1}{2} \Rightarrow 2t = \frac{7\pi}{6}, \frac{11\pi}{6} \Rightarrow t = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$\left|\frac{1}{2} + \sin 2t\right| = \begin{cases} \frac{1}{2} + \sin 2t & 0 \le t \le \frac{7\pi}{12}, \frac{11\pi}{12} \le t \le \frac{3\pi}{2} \\ -\frac{1}{2} - \sin 2t & \frac{7\pi}{12} < t < \frac{11\pi}{12} \end{cases}$$

$$\int_0^{3\pi/2} \left|\frac{1}{2} + \sin 2t\right| dt = \int_0^{7\pi/12} \left(\frac{1}{2} + \sin 2t\right) dt + \int_{7\pi/12}^{11\pi/12} \left(-\frac{1}{2} - \sin 2t\right) dt + \int_{11\pi/12}^{3\pi/2} \left(\frac{1}{2} + \sin 2t\right) dt$$

$$= \left[\frac{1}{2}t - \frac{1}{2}\cos 2t\right]_0^{7\pi/12} + \left[-\frac{1}{2}t + \frac{1}{2}\cos 2t\right]_{7\pi/12}^{11\pi/12} + \left[\frac{1}{2}t - \frac{1}{2}\cos 2t\right]_{11\pi/12}^{3\pi/2}$$

$$= \left(\frac{7\pi}{24} + \frac{\sqrt{3}}{4} + \frac{1}{2}\right) + \left(-\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) + \left(\frac{7\pi}{24} + \frac{\sqrt{3}}{4} + \frac{1}{2}\right) = \frac{5\pi}{12} + \sqrt{3} + 1$$

The total distance traveled is $\frac{5\pi}{12} + \sqrt{3} + 1 \approx 4.04$ feet.

33. $s(t) = \int v(t)dt = \int (2t - 4)dt = t^2 - 4t + C$

Since $s(0) = 0$, $C = 0$ and $s(t) = t^2 - 4t$. $s = 12$ when $t = 6$, so it takes the object 6 seconds to get $s = 12$.

$$|2t - 4| = \begin{cases} 4 - 2t & 0 \le t < 2 \\ 2t - 4 & 2 \le t \end{cases}$$

$\int_0^2 |2t - 4|\,dt = \left[-t^2 + 4t\right]_0^2 = 4$, so the object travels a distance of 4 cm in the first two seconds.

$$\int_2^x |2t - 4|\,dt = \left[t^2 - 4t\right]_2^x = x^2 - 4x + 4$$

$x^2 - 4x + 4 = 8$ when $x = 2 + 2\sqrt{2}$, so the object takes $2 + 2\sqrt{2} \approx 4.83$ seconds to travel a total distance of 12 centimeters.

34. a. $A = \int_1^6 x^{-2}dx = \left[-\frac{1}{x}\right]_1^6 = -\frac{1}{6} + 1 = \frac{5}{6}$

b. Find $c$ so that $\int_1^c x^{-2}dx = \frac{5}{12}$.

$$\int_1^c x^{-2}dx = \left[-\frac{1}{x}\right]_1^c = 1 - \frac{1}{c}$$

$$1 - \frac{1}{c} = \frac{5}{12}, c = \frac{12}{7}$$

The line $x = \frac{12}{7}$ bisects the area.

c. Slicing the region horizontally, the area is $\int_{1/36}^1 \frac{1}{\sqrt{y}}dy + \left(\frac{1}{36}\right)(5)$. Since $\frac{5}{36} < \frac{5}{12}$ the line that bisects the area is between $y = \frac{1}{36}$ and $y = 1$, so we find $d$ such that

$$\int_d^1 \frac{1}{\sqrt{y}}dy = \frac{5}{12};\ \int_d^1 \frac{1}{\sqrt{y}}dy = \left[2\sqrt{y}\right]_d^1$$

$$= 2 - 2\sqrt{d}\ ;\ 2 - 2\sqrt{d} = \frac{5}{12};$$

$d = \frac{361}{576} \approx 0.627$.

The line $y = 0.627$ approximately bisects the area.

**35.** Equation of line through (–2, 4) and (3, 9): $y = x + 6$

Equation of line through (2, 4) and (–3, 9): $y = -x + 6$

$$A(A) = \int_{-3}^{0}[9-(-x+6)]dx + \int_{0}^{3}[9-(x+6)]dx$$

$$= \int_{-3}^{0}(3+x)dx + \int_{0}^{3}(3-x)dx$$

$$= \left[3x + \frac{1}{2}x^2\right]_{-3}^{0} + \left[3x - \frac{1}{2}x^2\right]_{0}^{3} = \frac{9}{2} + \frac{9}{2} = 9$$

$$A(B) = \int_{-3}^{-2}[(-x+6)-x^2]dx + \int_{-2}^{0}[(-x+6)-(x+6)]dx$$

$$= \int_{-3}^{-2}(-x^2 - x + 6)dx + \int_{-2}^{0}(-2x)dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{-3}^{-2} + \left[-x^2\right]_{-2}^{0} = \frac{37}{6}$$

$A(C) = A(B) = \frac{37}{6}$ (by symmetry)

$$A(D) = \int_{-2}^{0}[(x+6)-x^2]dx + \int_{0}^{2}[(-x+6)-x^2]dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^{0} + \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{0}^{2}$$

$$= \frac{44}{3}$$

$A(A) + A(B) + A(C) + A(D) = 36$

$$A(A+B+C+D) = \int_{-3}^{3}(9-x^2)dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$$

$= 36$

**36.** Let $f(x)$ be the width of region 1 at every $x$.

$\Delta A_1 \approx f(x)\Delta x$, so $A_1 = \int_a^b f(x)dx$.

Let $g(x)$ be the width of region 2 at every $x$.

$\Delta A_2 \approx g(x)\Delta x$, so $A_2 = \int_a^b g(x)dx$.

Since $f(x) = g(x)$ at every $x$ in $[a, b]$,

$$A_1 = \int_a^b f(x)dx = \int_a^b g(x)dx = A_2.$$

**37.** The height of the triangular region is given by for $0 \le x \le 1$. We need only show that the height of the second region is the same in order to apply Cavalieri''s Principle. The height of the second region is

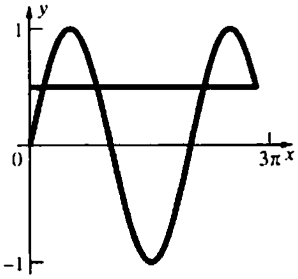
$$h_2 = (x^2 - 2x + 1) - (x^2 - 3x + 1)$$

$$= x^2 - 2x + 1 - x^2 + 3x - 1$$

$$= x \text{ for } 0 \le x \le 1.$$

Since $h_1 = h_2$ over the same closed interval, we can conclude that their areas are equal.

**38.** Sketch the graph.




Solve $\sin x = \frac{1}{2}$ for $0 \le x \le \frac{17\pi}{6}$.

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

The area of the trapped region is

$$\int_0^{\pi/6}\left(\frac{1}{2} - \sin x\right)dx + \int_{\pi/6}^{5\pi/6}\left(\sin x - \frac{1}{2}\right)dx$$

$$+ \int_{5\pi/6}^{13\pi/6}\left(\frac{1}{2} - \sin x\right)dx + \int_{13\pi/6}^{17\pi/6}\left(\sin x - \frac{1}{2}\right)dx$$

$$= \left[\frac{1}{2}x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \frac{1}{2}x\right]_{\pi/6}^{5\pi/6}$$

$$+ \left[\frac{1}{2}x + \cos x\right]_{5\pi/6}^{13\pi/6} + \left[-\cos x - \frac{1}{2}x\right]_{13\pi/6}^{17\pi/6}$$

$$= \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right) + \left(\sqrt{3} - \frac{\pi}{3}\right) + \left(\sqrt{3} + \frac{2\pi}{3}\right)$$

$$+ \left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{\pi}{12} + \frac{7\sqrt{3}}{2} - 1 \approx 5.32$$

## 6.2 Concepts Review

1. $\pi r^2 h$

2. $\pi(R^2 - r^2)h$

3. $\pi x^4 \Delta x$

4. $\pi[(x^2+2)^2 - 4]\Delta x$

## Problem Set 6.2

1. Slice vertically.

$\Delta V \approx \pi(x^2+1)^2 \Delta x = \pi(x^4+2x^2+1)\Delta x$

$V = \pi \int_0^2 (x^4+2x^2+1)dx$

$= \pi\left[\frac{1}{5}x^5+\frac{2}{3}x^3+x\right]_0^2 = \pi\left(\frac{32}{5}+\frac{16}{3}+2\right) = \frac{206\pi}{15}$

$\approx 43.14$

2. Slice vertically.

$\Delta V \approx \pi(-x^2+4x)^2 \Delta x = \pi(x^4-8x^3+16x^2)\Delta x$

$V = \pi \int_0^3 (x^4-8x^3+16x^2)dx$

$= \pi\left[\frac{1}{5}x^5-2x^4+\frac{16}{3}x^3\right]_0^3$

$= \pi\left(\frac{243}{5}-162+144\right)$

$= \frac{153\pi}{5} \approx 96.13$

3. a. Slice vertically.

$\Delta V \approx \pi(4-x^2)^2 \Delta x = \pi(16-8x^2+x^4)\Delta x$

$V = \pi \int_0^2 (16-8x^2+x^4)dx$

$= \frac{256\pi}{15} \approx 53.62$

b. Slice horizontally.

$x = \sqrt{4-y}$

Note that when $x = 0$, $y = 4$.

$\Delta V \approx \pi\left(\sqrt{4-y}\right)^2 \Delta y = \pi(4-y)\Delta y$

$V = \pi \int_0^4 (4-y)dy = \pi\left[4y-\frac{1}{2}y^2\right]_0^4$

$= \pi(16-8) = 8\pi \approx 25.13$

4. a. Slice vertically.

$\Delta V \approx \pi(4-2x)^2 \Delta x$

$0 \le x \le 2$

$V = \pi \int_0^2 (4-2x)^2 dx = \pi\left[-\frac{1}{6}(4-2x)^3\right]_0^2$

$= \frac{32\pi}{3} \approx 33.51$

b. Slice vertically.

$x = 2-\frac{y}{2}$

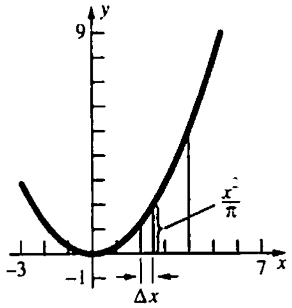
$\Delta V \approx \pi\left(2-\frac{y}{2}\right)^2 \Delta y$

$0 \le y \le 4$

$V = \pi \int_0^4 \left(2-\frac{y}{2}\right)^2 dy = \pi\left[-\frac{2}{3}\left(2-\frac{y}{2}\right)^3\right]_0^4$

$= \frac{16\pi}{3} \approx 16.76$

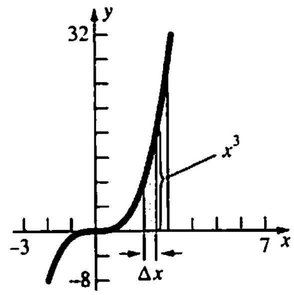
5.




$\Delta V \approx \pi\left(\frac{x^2}{\pi}\right)^2 \Delta x = \frac{x^4}{\pi}\Delta x$

$V = \int_0^4 \frac{x^4}{\pi}dx = \frac{1}{\pi}\left[\frac{1}{5}x^5\right]_0^4 = \frac{1024}{5\pi} \approx 65.19$

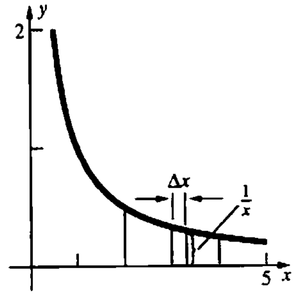
6.




$\Delta V \approx \pi(x^3)^2 \Delta x = \pi x^6 \Delta x$

$V = \pi \int_0^3 x^6 dx = \pi\left[\frac{1}{7}x^7\right]_0^3 = \frac{2187\pi}{7} \approx 981.52$

7.

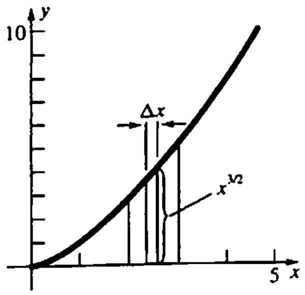



$\Delta V \approx \pi\left(\frac{1}{x}\right)^2 \Delta x = \pi\left(\frac{1}{x^2}\right)\Delta x$

$V = \pi \int_2^4 \frac{1}{x^2}dx = \pi\left[-\frac{1}{x}\right]_2^4 = \pi\left(-\frac{1}{4}+\frac{1}{2}\right) = \frac{\pi}{4}$

$\approx 0.79$

**8.**

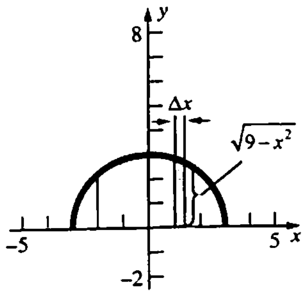



$$\Delta V \approx \pi(x^{3/2})^2 \Delta x = \pi x^3 \Delta x$$

$$V = \pi \int_2^3 x^3 dx = \pi \left[ \frac{1}{4} x^4 \right]_2^3 = \pi \left( \frac{81}{4} - 4 \right)$$

$$= \frac{65\pi}{4} \approx 51.05$$

**9.**

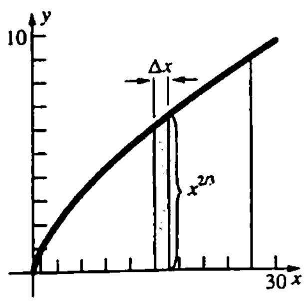



$$\Delta V \approx \pi \left( \sqrt{9-x^2} \right)^2 \Delta x = \pi(9 - x^2)\Delta x$$

$$V = \pi \int_{-2}^3 (9 - x^2) dx = \pi \left[ 9x - \frac{1}{3} x^3 \right]_{-2}^3$$

$$= \pi \left[ (27 - 9) - \left( -18 + \frac{8}{3} \right) \right] = \frac{100\pi}{3} \approx 104.72$$

**10.**

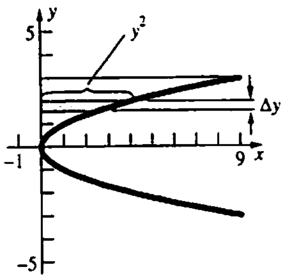



$$\Delta V \approx \pi(x^{2/3})^2 \Delta x = \pi x^{4/3} \Delta x$$

$$V = \pi \int_1^{27} x^{4/3} dx = \pi \left[ \frac{3}{7} x^{7/3} \right]_1^{27} = \pi \left( \frac{6561}{7} - \frac{3}{7} \right)$$

$$= \frac{6558\pi}{7} \approx 2943.22$$

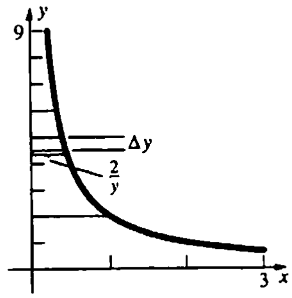
**11.**




$$\Delta V \approx \pi(y^2)^2 \Delta y = \pi y^4 \Delta y$$

$$V = \pi \int_0^3 y^4 dy = \pi \left[ \frac{1}{5} y^5 \right]_0^3 = \frac{243\pi}{5} \approx 152.68$$

**12.**

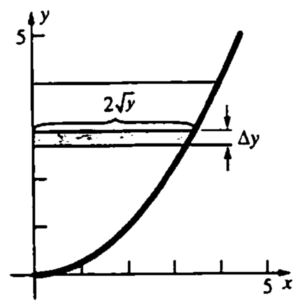



$$\Delta V \approx \pi \left( \frac{2}{y} \right)^2 \Delta y = 4\pi \left( \frac{1}{y^2} \right) \Delta y$$

$$V = 4\pi \int_2^6 \frac{1}{y^2} dy = 4\pi \left[ -\frac{1}{y} \right]_2^6 = 4\pi \left( -\frac{1}{6} + \frac{1}{2} \right)$$

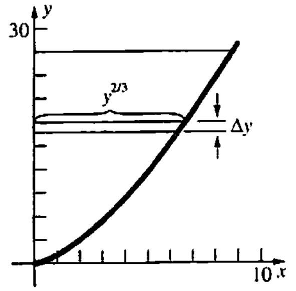
$$= \frac{4\pi}{3} \approx 4.19$$

**13.**




$$\Delta V \approx \pi \left( 2\sqrt{y} \right)^2 \Delta y = 4\pi y \Delta y$$

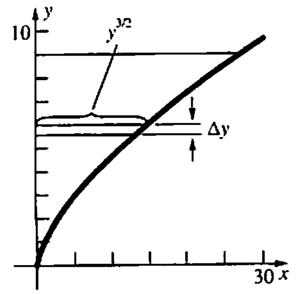
$$V = 4\pi \int_0^4 y\, dy = 4\pi \left[ \frac{1}{2} y^2 \right]_0^4 = 32\pi \approx 100.53$$

14. 


$\Delta V \approx \pi(y^{2/3})^2 \Delta y = \pi y^{4/3} \Delta y$

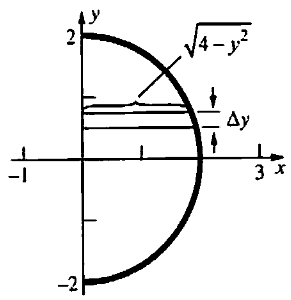
$V = \pi \int_0^{27} y^{4/3}\,dy = \pi\left[\frac{3}{7} y^{7/3}\right]_0^{27} = \frac{6561\pi}{7}$

$\approx 2944.57$

15. 


$\Delta V \approx \pi(y^{3/2})^2 \Delta y = \pi y^3 \Delta y$

$V = \pi \int_0^9 y^3 dy = \pi\left[\frac{1}{4} y^4\right]_0^9 = \frac{6561\pi}{4} \approx 5153.00$

16. 


$\Delta V \approx \pi\left(\sqrt{4-y^2}\right)^2 \Delta y = \pi(4-y^2)\Delta y$

$V = \pi \int_{-2}^{2} (4-y^2)dy = \pi\left[4y - \frac{1}{3}y^3\right]_{-2}^{2}$

$= \pi\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right] = \frac{32\pi}{3} \approx 33.51$

17. The equation of the upper half of the ellipse is

$y = b\sqrt{1-\frac{x^2}{a^2}}$ or $y = \frac{b}{a}\sqrt{a^2-x^2}$.

$V = \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2-x^2)dx$

$= \frac{b^2\pi}{a^2}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$

$= \frac{b^2\pi}{a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right] = \frac{4}{3}ab^2\pi$

18. To find the intersection points, solve $6x = 6x^2$.

$6(x^2 - x) = 0$

$6x(x-1) = 0$

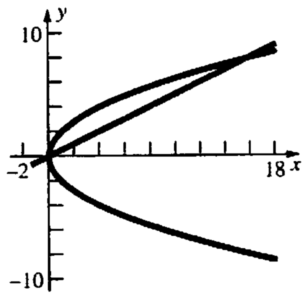
$x = 0, 1$

$\Delta V \approx \pi\left[(6x)^2 - (6x^2)^2\right]\Delta x = 36\pi(x^2 - x^4)\Delta x$

$V = 36\pi \int_0^1 (x^2 - x^4)dx = 36\pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$

$= 36\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{24\pi}{5} \approx 15.08$

19. Sketch the region.




To find the intersection points, solve $\frac{x}{2} = 2\sqrt{x}$.

$\frac{x^2}{4} = 4x$

$x^2 - 16x = 0$

$x(x-16) = 0$

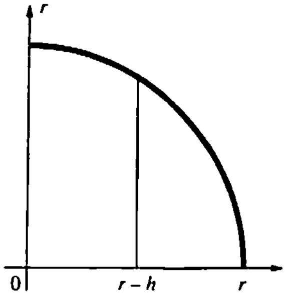
$x = 0, 16$

$\Delta V \approx \pi\left[\left(2\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2\right]\Delta x = \pi\left(4x - \frac{x^2}{4}\right)\Delta x$

$V = \pi \int_0^{16}\left(4x - \frac{x^2}{4}\right)dx = \pi\left[2x^2 - \frac{x^3}{12}\right]_0^{16}$

$= \pi\left(512 - \frac{1024}{3}\right) = \frac{512\pi}{3} \approx 536.17$

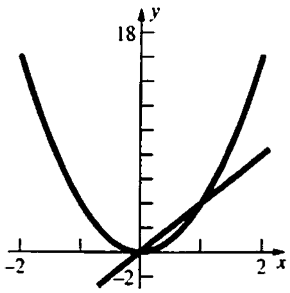
**20.** Sketch the region.




$$V = \pi \int_{r-h}^{r} (r^2 - x^2)dx = \pi \left[ r^2 x - \frac{1}{3}x^3 \right]_{r-h}^{r}$$

$$= \frac{1}{3}\pi h^2 (3r - h)$$

**21.** Sketch the region.




To find the intersection points, solve $\frac{y}{4} = \frac{\sqrt{y}}{2}$.

$$\frac{y^2}{16} = \frac{y}{4}$$

$$y^2 - 4y = 0$$

$$y(y-4) = 0$$

$$y = 0, 4$$

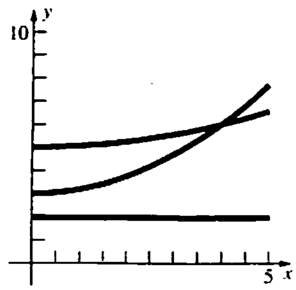
$$\Delta V \approx \pi \left[ \left( \frac{\sqrt{y}}{2} \right)^2 - \left( \frac{y}{4} \right)^2 \right] \Delta y = \pi \left( \frac{y}{4} - \frac{y^2}{16} \right) \Delta y$$

$$V = \pi \int_0^4 \left( \frac{y}{4} - \frac{y^2}{16} \right) dy = \pi \left[ \frac{y^2}{8} - \frac{y^3}{48} \right]_0^4$$

$$= \frac{2\pi}{3} \approx 2.0944$$

**22.** $y = \frac{3}{16}x^2 + 3,\ y = \frac{1}{16}x^2 + 5$

Sketch the region.




To find the intersection point, solve

$$\frac{3}{16}x^2 + 3 = \frac{1}{16}x^2 + 5.$$

$$\frac{1}{8}x^2 - 2 = 0$$

$$x^2 - 16 = 0$$

$$(x+4)(x-4) = 0$$

$$x = -4, 4$$

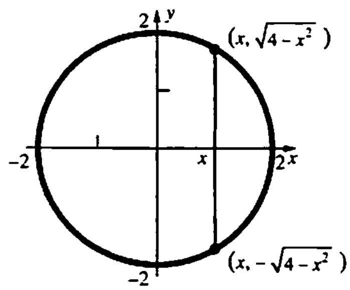
$$V = \pi \int_0^4 \left[ \left( \frac{1}{16}x^2 + 5 - 2 \right)^2 - \left( \frac{3}{16}x^2 + 3 - 2 \right)^2 \right] dx$$

$$= \pi \int_0^4 \left[ \left( \frac{1}{256}x^4 - \frac{3}{8}x^2 + 9 \right) - \left( \frac{9}{256}x^4 - \frac{3}{8}x^2 + 1 \right) \right] dx$$

$$= \pi \int_0^4 \left( 8 - \frac{1}{32}x^4 \right) dx = \pi \left[ 8x - \frac{1}{160}x^5 \right]_0^4$$

$$= \pi \left( 32 - \frac{32}{5} \right) = \frac{128\pi}{5} \approx 80.42$$

**23.**




The square at $x$ has sides of length $2\sqrt{4-x^2}$, as shown.

$$V = \int_{-2}^{2} \left( 2\sqrt{4-x^2} \right)^2 dx = \int_{-2}^{2} 4(4-x^2)dx$$

$$= 4\left[ 4x - \frac{x^3}{3} \right]_{-2}^{2} = 4\left[ \left( 8 - \frac{8}{3} \right) - \left( -8 + \frac{8}{3} \right) \right] = \frac{128}{3}$$

$$\approx 42.67$$

**24.** The area of each cross section perpendicular to the $x$-axis is $\frac{1}{2}(4)\left(2\sqrt{4-x^2}\right) = 4\sqrt{4-x^2}$.

The area of a semicircle with radius 2 is

$$\int_{-2}^{2} \sqrt{4-x^2}\, dx = 2\pi.$$

$$V = \int_{-2}^{2} 4\sqrt{4-x^2}\, dx = 4(2\pi) = 8\pi \approx 25.13$$

**25.** The square at $x$ has sides of length $\sqrt{\cos x}$.

$$V = \int_{-\pi/2}^{\pi/2} \cos x\, dx = [\sin x]_{-\pi/2}^{\pi/2} = 2$$

26. The area of each cross section perpendicular to the $x$-axis is $[(1-x^2)-(1-x^4)]^2 = x^8 - 2x^6 + x^4$.

$$V = \int_{-1}^{1}(x^8 - 2x^6 + x^4)dx$$

$$= \left[\frac{1}{9}x^9 - \frac{2}{7}x^7 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16}{315} \approx 0.051$$

27. The square at $x$ has sides of length $\sqrt{1-x^2}$.

$$V = \int_0^1 (1-x^2)dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} \approx 0.67$$

28. From Problem 27 we see that horizontal cross sections of one octant of the common region are squares. The length of a side at height $y$ is $\sqrt{r^2 - y^2}$ where $r$ is the common radius of the cylinders. The volume of the "+" can be found by adding the volumes of each cylinder and subtracting off the volume of the common region (which is counted twice). The volume of one octant of the common region is

$$\int_0^r (r^2 - y^2)dy = r^2 y - \frac{1}{3}y^2 \Big|_0^r$$

$$= r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$$

Thus, the volume of the "+" is

$V =$ vol. of cylinders - vol. of common region

$$=2(\pi r^2 l) - 8\left(\frac{2}{3}r^3\right)$$

$$= 2\pi(2^2)(12) - 8\left(\frac{2}{3}(2)^3\right) = 96\pi - \frac{128}{3}$$

$$\approx 258.93 \text{ in}^2$$

29. Using the result from Problem 28, the volume of one octant of the common region in the "+" is

$$\int_0^r (r^2 - y^2)dy = r^2 y - \frac{1}{3}y^2 \Big|_0^r$$

$$= r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$$

Thus, the volume inside the "+" for two cylinders of radius $r$ and length $L$ is

$V =$ vol. of cylinders - vol. of common region

$$= 2(\pi r^2 L) - 8\left(\frac{2}{3}r^3\right)$$

$$= 2\pi r^2 L - \frac{16}{3}r^3$$

30. From Problem 28, the volume of one octant of the common region is $\frac{2}{3}r^3$. We can find the volume of the "T" similarly. Since the "T" has one-half the common region of the "+" in Problem 28, the volume of the "T" is given by

$V =$ vol. of cylinders - vol. of common region

$$=(\pi r^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$$

With $r = 2, L_1 = 12$, and $L_2 = 8$ (inches), the volume of the "T" is

$V =$ vol. of cylinders - vol. of common region

$$=(\pi r^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$$

$$= (\pi 2^2)(12+8) - 4\left(\frac{2}{3}2^3\right)$$

$$= 80\pi - \frac{64}{3} \text{ in}^3$$

$$\approx 229.99 \text{ in}^3$$

31. From Problem 30, the general form for the volume of a "T" formed by two cylinders with the same radius is

$V =$ vol. of cylinders - vol. of common region

$$=(\pi r^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$$

$$= \pi r^2(L_1 + L_2) - \frac{8}{3}r^3$$

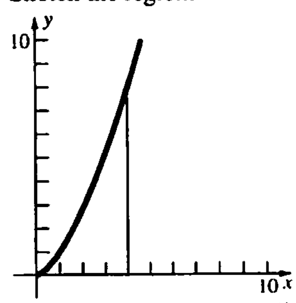
32. The area of each cross section perpendicular to the $x$-axis is $\frac{1}{2}\pi\left[\frac{1}{2}\left(\sqrt{x} - x^2\right)\right]^2$

$$= \frac{\pi}{8}(x^4 - 2x^{5/2} + x).$$

$$V = \frac{\pi}{8}\int_0^1 (x^4 - 2x^{5/2} + x)dx$$

$$= \frac{\pi}{8}\left[\frac{1}{5}x^5 - \frac{4}{7}x^{7/2} + \frac{1}{2}x^2\right]_0^1 = \frac{9\pi}{560} \approx 0.050$$

33. Sketch the region.




a. Revolving about the line $x = 4$, the radius of the disk at $y$ is $4 - \sqrt[3]{y^2} = 4 - y^{2/3}$.

$$V = \pi\int_0^8 (4 - y^{2/3})^2\, dy$$

$$= \pi\int_0^8 (16 - 8y^{2/3} + y^{4/3})dy$$

$$= \pi\left[16y - \frac{24}{5}y^{5/3} + \frac{3}{7}y^{7/3}\right]_0^8$$

$$= \pi\left(128 - \frac{768}{5} + \frac{384}{7}\right)$$

$$= \frac{1024\pi}{35} \approx 91.91$$

**b.** Revolving about the line $y = 8$, the inner radius of the disk at $x$ is $8 - \sqrt{x^3} = 8 - x^{3/2}$.

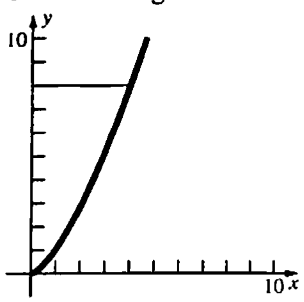
$$V = \pi\int_0^4 \left[8^2 - (8 - x^{3/2})^2\right]dx$$

$$= \pi\int_0^4 (16x^{3/2} - x^3)dx$$

$$= \pi\left[\frac{32}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^4 = \pi\left(\frac{1024}{5} - 64\right)$$

$$= \frac{704\pi}{5} \approx 442.34$$

**34.** Sketch the region.




**a.** Revolving about the line $x = 4$, the inner radius of the disk at $y$ is $4 - \sqrt[3]{y^2} = 4 - y^{2/3}$.

$$V = \pi\int_0^8 \left[4^2 - \left(4 - y^{2/3}\right)^2\right]dy$$

$$= \pi\int_0^8 (8y^{2/3} - y^{4/3})dy$$

$$= \pi\left[\frac{24}{5}y^{5/3} - \frac{3}{7}y^{7/3}\right]_0^8$$

$$= \pi\left(\frac{768}{5} - \frac{384}{7}\right) = \frac{3456\pi}{35} \approx 310.21$$

**b.** Revolving about the line $y = 8$, the radius of the disk at $x$ is $8 - \sqrt{x^3} = 8 - x^{3/2}$.

$$V = \pi\int_0^4 (8 - x^{3/2})^2 dx$$

$$= \pi\int_0^4 (64 - 16x^{3/2} + x^3)dx$$

$$= \pi\left[64x - \frac{32}{5}x^{5/2} + \frac{1}{4}x^4\right]_0^4$$

$$= \pi\left[256 - \frac{1024}{5} + 64\right] = \frac{576\pi}{5} \approx 361.91$$

**35.** The area of a quarter circle with radius 2 is $\int_0^2 \sqrt{4 - y^2}\, dy = \pi$.

$$\int_0^2 \left[2\sqrt{4 - y^2} + 4 - y^2\right]dy$$

$$= 2\int_0^2 \sqrt{4 - y^2}\, dy + \int_0^2 (4 - y^2)dy$$

$$= 2\pi + \left[4y - \frac{1}{3}y^3\right]_0^2 = 2\pi + \left(8 - \frac{8}{3}\right)$$

$$= 2\pi + \frac{16}{3} \approx 11.62$$

**36.** Let the $x$-axis lie along the diameter at the base perpendicular to the water level and slice perpendicular to the $x$-axis. Let $x = 0$ be at the center. The slice has base length $2\sqrt{r^2 - x^2}$ and height $\frac{hx}{r}$.

$$V = \frac{2h}{r}\int_0^r x\sqrt{r^2 - x^2}\, dx$$

$$= \frac{2h}{r}\left[-\frac{1}{3}\left(r^2 - x^2\right)^{3/2}\right]_0^r = \frac{2h}{r}\left(\frac{1}{3}r^3\right) = \frac{2}{3}r^2 h$$

**37.** Let the $x$-axis lie on the base perpendicular to the diameter through the center of the base. The slice at $x$ is a rectangle with base of length $2\sqrt{r^2 - x^2}$ and height $x\tan\theta$.

$$V = \int_0^r 2x\tan\theta\sqrt{r^2 - x^2}\, dx$$

$$= \left[-\frac{2}{3}\tan\theta(r^2 - x^2)^{3/2}\right]_0^r$$

$$= \frac{2}{3}r^3\tan\theta$$

**38. a.** $x = \sqrt[4]{\frac{y}{k}}$

Slice horizontally.

$$\Delta V \approx \pi\left(\sqrt[4]{\frac{y}{k}}\right)^2 \Delta y = \pi\left(\sqrt{\frac{y}{k}}\right)\Delta y$$

If the depth of the tank is $h$, then

$$V = \pi\int_0^h \sqrt{\frac{y}{k}}dy = \frac{\pi}{\sqrt{k}}\left[\frac{2}{3}y^{3/2}\right]_0^h$$

$$= \frac{2\pi}{3\sqrt{k}}h^{3/2}.$$

The volume as a function of the depth of the tank is $V(y) = \frac{2\pi}{3\sqrt{k}}y^{3/2}$.

b. It is given that $\frac{dV}{dt} = -m\sqrt{y}$.

From part a, $\frac{dV}{dt} = \frac{\pi}{\sqrt{k}} y^{1/2} \frac{dy}{dt}$.

Thus, $\frac{\pi}{\sqrt{k}} \sqrt{y} \frac{dy}{dt} = -m\sqrt{y}$ and $\frac{dy}{dt} = \frac{-m\sqrt{k}}{\pi}$ which is constant.

39. Let $A$ lie on the $xy$-plane. Suppose $\Delta A = f(x)\Delta x$ where $f(x)$ is the length at $x$, so $A = \int f(x)dx$.
Slice the general cone at height $z$ parallel to $A$. The slice of the resulting region is $A_z$ and $\Delta A_z$ is a region related to $f(x)$ and $\Delta x$ by similar triangles:

$$\Delta A_z = \left(1 - \frac{z}{h}\right) f(x) \cdot \left(1 - \frac{z}{h}\right) \Delta x$$

$$= \left(1 - \frac{z}{h}\right)^2 f(x)\Delta x$$

Therefore, $A_z = \left(1 - \frac{z}{h}\right)^2 \int f(x)dx = \left(1 - \frac{z}{h}\right)^2 A.$

$$\Delta V \approx A_z \Delta z = A\left(1 - \frac{z}{h}\right)^2 \Delta z \quad V = A\int_0^h \left(1 - \frac{z}{h}\right)^2 dz$$

$$= A\left[-\frac{h}{3}\left(1 - \frac{z}{h}\right)^3\right]_0^h = \frac{1}{3}Ah.$$

a. $A = \pi r^2$

$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$

b. A face of a regular tetrahedron is an equilateral triangle. If the side of an equilateral triangle has length $r$, then the area is $A = \frac{1}{2} r \cdot \frac{\sqrt{3}}{2} r = \frac{\sqrt{3}}{4} r^2$.

The center of an equilateral triangle is $\frac{2}{3} \cdot \frac{\sqrt{3}}{2} r = \frac{1}{\sqrt{3}} r$ from a vertex. Then the height of a regular tetrahedron is

$$h = \sqrt{r^2 - \left(\frac{1}{\sqrt{3}} r\right)^2} = \sqrt{\frac{2}{3} r^2} = \frac{\sqrt{2}}{\sqrt{3}} r.$$

$$V = \frac{1}{3}Ah = \frac{\sqrt{2}}{12} r^3$$

40. If two solids have the same cross sectional area at every $x$ in $[a, b]$, then they have the same volume.

41. First we examine the cross-sectional areas of each shape.
*Hemisphere:* cross-sectional shape is a circle. The radius of the circle at height $y$ is $\sqrt{r^2 - y^2}$. Therefore, the cross-sectional area for the hemisphere is

$$A_h = \pi(\sqrt{r^2 - y^2})^2 = \pi(r^2 - y^2)$$

*Cylinder w/o cone:* cross-sectional shape is a washer. The outer radius is a constant, $r$. The inner radius at height $y$ is equal to $y$. Therefore, the cross-sectional area is

$$A_2 = \pi r^2 - \pi y^2 = \pi(r^2 - y^2).$$

Since both cross-sectional areas are the same, we can apply Cavaleri's Principle. The volume of the hemisphere of radius $r$ is

$V =$ vol. of cylinder - vol. of cone

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$

$$= \frac{2}{3}\pi r^2 h$$

With the height of the cylinder and cone equal to $r$, the volume of the hemisphere is

$$V = \frac{2}{3}\pi r^2 (r) = \frac{2}{3}\pi r^3$$

## 6.3 Concepts Review

1. $2\pi x f(x)\Delta x$

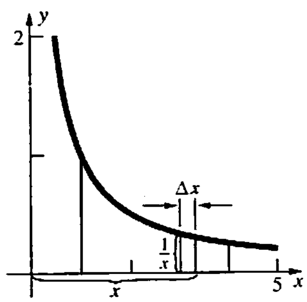
2. $2\pi\int_0^2 x^2 dx; \pi\int_0^2 (4 - y^2)dy$

3. $2\pi\int_0^2 (1 + x)x\,dx$

4. $2\pi\int_0^2 (1 + y)(2 - y)dy$

## Problem Set 6.3

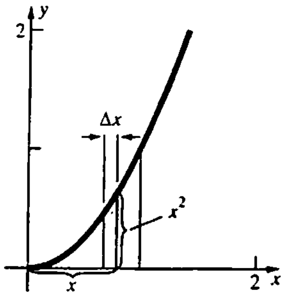
1. a,b.




c. $\Delta V \approx 2\pi x\left(\frac{1}{x}\right)\Delta x = 2\pi\Delta x$

**d,e.** $V = 2\pi\int_1^4 dx = 2\pi[x]_1^4 = 6\pi \approx 18.85$

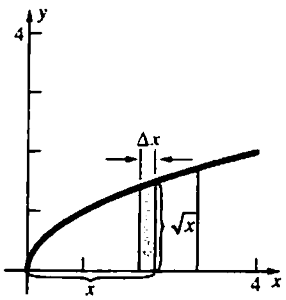
**2. a,b.**




**c.** $\Delta V \approx 2\pi x(x^2)\Delta x = 2\pi x^3 \Delta x$

**d,e.** $V = 2\pi\int_0^1 x^3 dx = 2\pi\left[\frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2} \approx 1.57$

**3. a,b.**

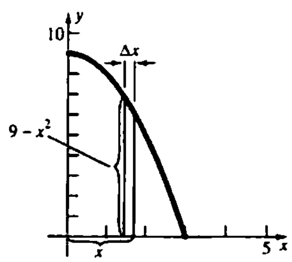



**c.** $\Delta V \approx 2\pi x\sqrt{x}\,\Delta x = 2\pi x^{3/2}\Delta x$

**d,e.** $V = 2\pi\int_0^3 x^{3/2}dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^3$

$= \frac{36\sqrt{3}}{5}\pi \approx 39.18$

**4. a,b.**

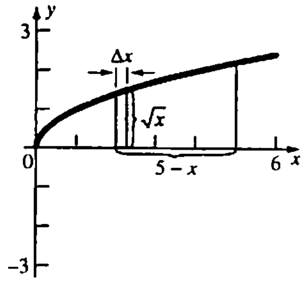



**c.** $\Delta V \approx 2\pi x(9 - x^2)\Delta x = 2\pi(9x - x^3)\Delta x$

**d,e.** $V = 2\pi\int_0^3 (9x - x^3)dx = 2\pi\left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81\pi}{2} \approx 127.23$

**5. a,b.**




**c.** $\Delta V \approx 2\pi(5 - x)\sqrt{x}\,\Delta x$

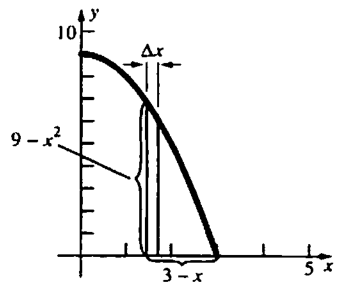
$= 2\pi(5x^{1/2} - x^{3/2})\Delta x$

**d,e.** $V = 2\pi\int_0^5 (5x^{1/2} - x^{3/2})dx$

$= 2\pi\left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^5$

$= 2\pi\left(\frac{50\sqrt{5}}{3} - 10\sqrt{5}\right) = \frac{40\sqrt{5}}{3}\pi \approx 93.66$

**6. a,b.**




**c.** $\Delta V \approx 2\pi(3 - x)(9 - x^2)\Delta x$

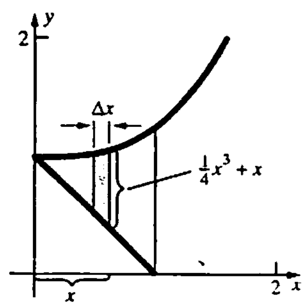
$= 2\pi(27 - 9x - 3x^2 + x^3)\Delta x$

**d,e.** $V = 2\pi\int_0^3 (27 - 9x - 3x^2 + x^3)dx$

$= 2\pi\left[27x - \frac{9}{2}x^2 - x^3 + \frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(81 - \frac{81}{2} - 27 + \frac{81}{4}\right) = \frac{135\pi}{2} \approx 212.06$

**7. a,b.**



c. $\Delta V \approx 2\pi x\left[\left(\frac{1}{4}x^3+1\right)-(1-x)\right]\Delta x$

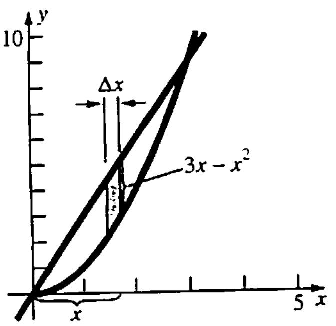
$= 2\pi\left(\frac{1}{4}x^4+x^2\right)\Delta x$

d,e. $V = 2\pi\int_0^1 \left(\frac{1}{4}x^4+x^2\right)dx$

$= 2\pi\left[\frac{1}{20}x^5+\frac{1}{3}x^3\right]_0^1 = 2\pi\left(\frac{1}{20}+\frac{1}{3}\right)$

$= \frac{23\pi}{30} \approx 2.41$

8. a,b.

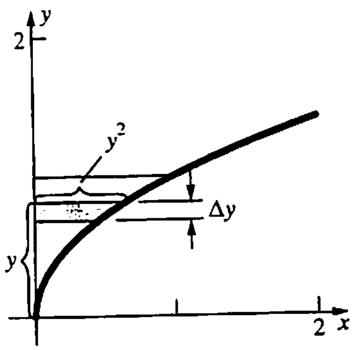



c. $\Delta V \approx 2\pi x(3x-x^2)\Delta x = 2\pi(3x^2-x^3)\Delta x$

d,e. $V = 2\pi\int_0^3 (3x^2-x^3)dx = 2\pi\left[x^3-\frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(27-\frac{81}{4}\right) = \frac{27\pi}{2} \approx 42.41$

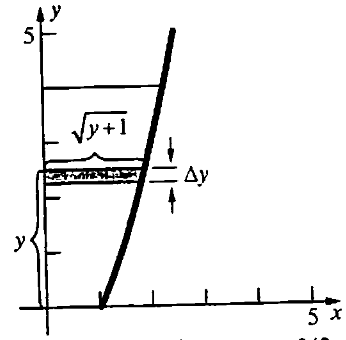
9. a,b.




c. $\Delta V \approx 2\pi y(y^2)\Delta y = 2\pi y^3\Delta y$

d,e. $V = 2\pi\int_0^1 y^3 dy = 2\pi\left[\frac{1}{4}y^4\right]_0^1 = \frac{\pi}{2} \approx 1.57$

10. a,b.



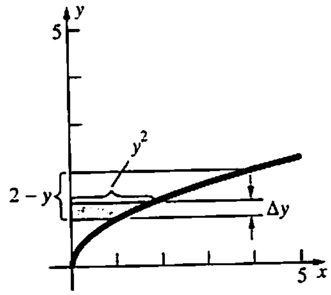

c. $\Delta V \approx 2\pi y\left(\sqrt{y}+1\right)\Delta y = 2\pi(y^{3/2}+y)\Delta y$

d,e. $V = 2\pi\int_0^4 (y^{3/2}+y)dy$

$= 2\pi\left[\frac{2}{5}y^{5/2}+\frac{1}{2}y^2\right]_0^4 = 2\pi\left(\frac{64}{5}+8\right)$

$= \frac{208\pi}{5} \approx 130.69$

11. a,b.

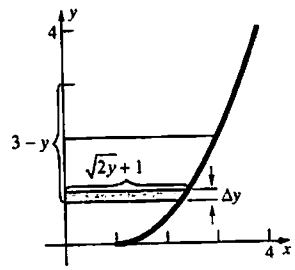



c. $\Delta V \approx 2\pi(2-y)y^2\Delta y = 2\pi(2y^2-y^3)\Delta y$

d,e. $V = 2\pi\int_0^2 (2y^2-y^3)dy = 2\pi\left[\frac{2}{3}y^3-\frac{1}{4}y^4\right]_0^2$

$= 2\pi\left(\frac{16}{3}-4\right) = \frac{8\pi}{3} \approx 8.38$

12. a,b.




c. $\Delta V \approx 2\pi(3-y)\left(\sqrt{2y}+1\right)\Delta y$

$= 2\pi\left(3+3\sqrt{2}y^{1/2}-y-\sqrt{2}y^{3/2}\right)\Delta y$

**d,e.** $V = 2\pi\int_0^2 \left(3+3\sqrt{2}y^{1/2} - y - \sqrt{2}\,y^{3/2}\right)dy$

$= 2\pi\left[3y + 2\sqrt{2}y^{3/2} - \frac{1}{2}y^2 - \frac{2\sqrt{2}}{5}y^{5/2}\right]_0^2$

$= 2\pi\left(6+8-2-\frac{16}{5}\right) = \frac{88\pi}{5} \approx 55.29$

**13. a.** $\pi\int_a^b \left[f(x)^2 - g(x)^2\right]dx$

**b.** $2\pi\int_a^b x\left[f(x) - g(x)\right]dx$

**c.** $2\pi\int_a^b (x-a)\left[f(x)-g(x)\right]dx$

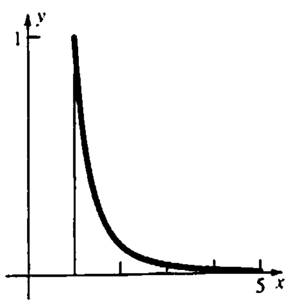
**d.** $2\pi\int_a^b (b-x)\left[f(x)-g(x)\right]dx$

**14. a.** $\pi\int_c^d \left[f(y)^2 - g(y)^2\right]dy$

**b.** $2\pi\int_c^d y\left[f(y)-g(y)\right]dy$

**c.** $2\pi\int_c^d (3-y)\left[f(y)-g(y)\right]dy$

**15.**




**a.** $A = \int_1^3 \frac{1}{x^3}dx$

**b.** $V = 2\pi\int_1^3 x\left(\frac{1}{x^3}\right)dx = 2\pi\int_1^3 \frac{1}{x^2}dx$

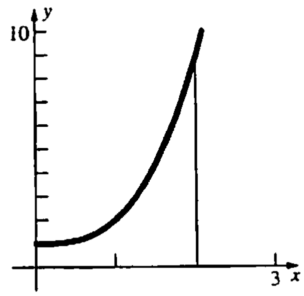
**c.** $V = \pi\int_1^3 \left[\left(\frac{1}{x^3}+1\right)^2 - (-1)^2\right]dx$

$= \pi\int_1^3 \left(\frac{1}{x^6} + \frac{2}{x^3}\right)dx$

**d.** $V = 2\pi\int_1^3 (4-x)\left(\frac{1}{x^3}\right)dx$

$= 2\pi\int_1^3 \left(\frac{4}{x^3} - \frac{1}{x^2}\right)dx$

**16.**




**a.** $A = \int_0^2 (x^3+1)\,dx$

**b.** $V = 2\pi\int_0^2 x(x^3+1)dx = 2\pi\int_0^2 (x^4+x)dx$

**c.** $V = \pi\int_0^2 \left[(x^3+2)^2 - (-1)^2\right]$

$= \pi\int_0^2 (x^6+4x^3+3)dx$

**d.** $V = 2\pi\int_0^2 (4-x)(x^3+1)dx$

$= 2\pi\int_0^2 (-x^4+4x^3-x+4)dx$

**17.** To find the intersection point, solve $\sqrt{y} = \frac{y^3}{32}$.

$y = \frac{y^6}{1024}$

$y^6 - 1024y = 0$

$y(y^5 - 1024) = 0$

$y = 0, 4$

$V = 2\pi\int_0^4 y\left(\sqrt{y} - \frac{y^3}{32}\right)dy$

$= 2\pi\int_0^4 \left(y^{3/2} - \frac{y^4}{32}\right)dy$

$= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{y^5}{160}\right]_0^4 = 2\pi\left(\frac{64}{5} - \frac{32}{5}\right) = \frac{64\pi}{5}$

$\approx 40.21$

18. $V = 2\pi \int_0^4 (4-y)\left(\sqrt{y} - \frac{y^3}{32}\right)dy$

$= 2\pi \int_0^4 \left(4y^{1/2} - y^{3/2} - \frac{y^3}{8} + \frac{y^4}{32}\right)dy$

$= 2\pi \left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2} - \frac{y^4}{32} + \frac{y^5}{160}\right]_0^4$

$= 2\pi\left(\frac{64}{3} - \frac{64}{5} - 8 + \frac{32}{5}\right) = \frac{208\pi}{15} \approx 43.56$

19. Let $R$ be the region bounded by $y = \sqrt{b^2 - x^2}$, $y = -\sqrt{b^2 - x^2}$, and $x = a$. When $R$ is revolved about the $y$-axis, it produces the desired solid.

$V = 2\pi \int_a^b x\left(\sqrt{b^2 - x^2} + \sqrt{b^2 - x^2}\right)dx$

$= 4\pi \int_a^b x\sqrt{b^2 - x^2}\, dx = 4\pi\left[-\frac{1}{3}(b^2 - x^2)^{3/2}\right]_a^b$

$= 4\pi\left[\frac{1}{3}(b^2 - a^2)^{3/2}\right] = \frac{4\pi}{3}(b^2 - a^2)^{3/2}$

20. $y = \pm\sqrt{a^2 - x^2}, -a \le x \le a$

$V = 2\pi \int_{-a}^{a} (b-x)\left(2\sqrt{a^2 - x^2}\right)dx$

$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx - 4\pi \int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx$

$= 4\pi b\left(\frac{1}{2}\pi a^2\right) - 4\pi\left[-\frac{1}{3}(a^2 - x^2)^{3/2}\right]_{-a}^{a}$

$= 2\pi^2 a^2 b$

(Note that the area of a semicircle with radius $a$ is $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\pi a^2$.)

21. To find the intersection point, solve $\sin(x^2) = \cos(x^2)$.

$\tan(x^2) = 1$

$x^2 = \frac{\pi}{4}$

$x = \frac{\sqrt{\pi}}{2}$

$V = 2\pi \int_0^{\sqrt{\pi}/2} x\left[\cos(x^2) - \sin(x^2)\right]dx$

$= 2\pi \int_0^{\sqrt{\pi}/2} \left[x\cos(x^2) - x\sin(x^2)\right]dx$

$= 2\pi\left[\frac{1}{2}\sin(x^2) + \frac{1}{2}\cos(x^2)\right]_0^{\sqrt{\pi}/2}$

$= 2\pi\left[\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) - \frac{1}{2}\right] = \pi\left(\sqrt{2} - 1\right) \approx 1.30$

22. $V = 2\pi \int_0^{2\pi} x(2 + \sin x)dx$

$= 2\pi \int_0^{2\pi} (2x + x\sin x)dx$

$= 2\pi \int_0^{2\pi} 2x\, dx + 2\pi \int_0^{2\pi} x\sin x\, dx$

$= 2\pi\left[x^2\right]_0^{2\pi} + 2\pi\left[\sin x - x\cos x\right]_0^{2\pi}$

$= 2\pi(4\pi^2) + 2\pi(-2\pi) = 4\pi^2(2\pi - 1) \approx 208.57$

23. a. The curves intersect when $x = 0$ and $x = 1$.

$V = \pi \int_0^1 [x^2 - (x^2)^2]dx = \pi \int_0^1 (x^2 - x^4)dx$

$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15} \approx 0.42$

b. $V = 2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$

$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$

$\approx 0.52$

c. Slice perpendicular to the line $y = x$. At $(a, a)$, the perpendicular line has equation $y = -(x - a) + a = -x + 2a$. Substitute $y = -x + 2a$ into $y = x^2$ and solve for $x \ge 0$.

$x^2 + x - 2a = 0$

$x = \frac{-1 \pm \sqrt{1 + 8a}}{2}$

$x = \frac{-1 + \sqrt{1 + 8a}}{2}$

Substitute into $y = -x + 2a$, so $y = \frac{1 + 4a - \sqrt{1 + 8a}}{2}$. Find an expression for $r^2$, the square of the distance from $(a, a)$ to $\left(\frac{-1 + \sqrt{1 + 8a}}{2}, \frac{1 + 4a - \sqrt{1 + 8a}}{2}\right)$.

$r^2 = \left[a - \frac{-1 + \sqrt{1 + 8a}}{2}\right]^2$

$+ \left[a - \frac{1 + 4a - \sqrt{1 + 8a}}{2}\right]^2$

$$= \left[\frac{2a+1-\sqrt{1+8a}}{2}\right]^2 + \left[-\frac{2a+1-\sqrt{1+8a}}{2}\right]^2$$

$$= 2\left[\frac{2a+1-\sqrt{1+8a}}{2}\right]^2$$

$$= 2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a}$$

$$\Delta V \approx \pi r^2 \Delta a$$

$$V = \pi \int_0^1 (2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a})\, da$$

$$= \pi\left[\frac{2}{3}a^3 + 3a^2 + a - \frac{1}{12}(1+8a)^{3/2}\right]_0^1 - \pi \int_0^1 2a\sqrt{1+8a}\, da$$

$$= \pi\left[\left(\frac{2}{3}+3+1-\frac{9}{4}\right)-\left(-\frac{1}{12}\right)\right] - \pi \int_0^1 2a\sqrt{1+8a}\, da$$

$$= \frac{5\pi}{2} - \pi \int_0^1 2a\sqrt{1+8a}\, da$$

To integrate $\int_0^1 2a\sqrt{1+8a}\, da$, use the substitution $u = 1 + 8a$.

$$\int_0^1 2a\sqrt{1+8a}\, da = \int_1^9 \frac{1}{4}(u-1)\sqrt{u}\,\frac{1}{8}du$$

$$= \frac{1}{32}\int_1^9 (u^{3/2} - u^{1/2})du$$

$$= \frac{1}{32}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9$$

$$= \frac{1}{32}\left[\left(\frac{486}{5} - 18\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{149}{60}$$

$$V = \frac{5\pi}{2} - \frac{149\pi}{60} = \frac{\pi}{60} \approx 0.052$$

24. $\Delta V \approx 4\pi x^2 \Delta x$

$$V = 4\pi \int_0^r x^2\, dx = 4\pi\left[\frac{1}{3}x^3\right]_0^r = \frac{4}{3}\pi r^3$$

25. $\Delta V \approx \frac{x^2}{r^2} S \Delta x$

$$V = \frac{S}{r^2}\int_0^r x^2 dx = \frac{S}{r^2}\left[\frac{1}{3}x^3\right]_0^r = \frac{1}{3}rS$$

## 6.4 Concepts Review

1. Circle

   $x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$

2. $x^2 + 1; x$

3. $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

4. Mean Value Theorem (for derivatives)

## Problem Set 6.4

1. $f(x) = x^2; a = -1; b = 3$

   **a.** $n = 2; x = -1, 1, 3; y = 1, 1, 9$

$$\sum_{i=1}^{n} \Delta w_i = \sqrt{(1+1)^2 + (1-1)^2} + \sqrt{(3-1)^2 + (9-1)^2}$$

$$= \sqrt{4} + \sqrt{68} = 2 + 2\sqrt{17} \approx 10.2462$$

   **b.** $n = 4; x = -1, 0, 1, 2, 3; y = 1, 0, 1, 4, 9$

$$\sum_{i=1}^{n} \Delta w_i = \sqrt{(0+1)^2 + (0-1)^2} + \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (4-1)^2} + \sqrt{(3-2)^2 + (9-4)^2}$$

$$= \sqrt{2} + \sqrt{2} + \sqrt{10} + \sqrt{26}$$

$$\approx 11.0897$$

2. $f(x) = \sqrt{x}; a = 0; b = 4$

**a.** $n = 2; x = 0, 2, 4; y = 0, \sqrt{2}, 2$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(2-0)^2 + (\sqrt{2}-0)^2} + \sqrt{(4-2)^2 + (2-\sqrt{2})^2} \approx 4.5335$$

**b.** $n = 4; x = 0, 1, 2, 3, 4; y = 0, 1, \sqrt{2}, \sqrt{3}, 2$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (\sqrt{2}-1)^2} + \sqrt{(3-2)^2 + (\sqrt{3}-\sqrt{2})^2} + \sqrt{(4-3)^2 + (2-\sqrt{3})^2} \approx 4.5812$$

3. $f(x) = \sin x; a = 0; b = 2\pi$

**a.** $n = 2; x = 0, \pi, 2\pi; y = 0, 0, 0$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(\pi-0)^2 + (0-0)^2} + \sqrt{(2\pi-\pi)^2 + (0-0)^2} = 2\pi \approx 6.2832$$

**b.** $n = 4; x = 0, \pi/2, \pi, 3\pi/2, 2\pi;$
$y = 0, 1, 0, -1, 0$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(\pi/2-0)^2 + (1-0)^2} + \sqrt{(\pi-\pi/2)^2 + (0-1)^2} + \sqrt{(3\pi/2-\pi)^2 + (-1-0)^2} + \sqrt{(2\pi-3\pi/2)^2 + (0+1)^2} \approx 7.4484$$

**c.** $n = 8; x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi;$
$y = 0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}, -1, \frac{-\sqrt{2}}{2}, 0$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(\pi/4-0)^2 + (\sqrt{2}/2-0)^2} + \sqrt{(\pi/2-\pi/4)^2 + (1-\sqrt{2}/2)^2} + \sqrt{(3\pi/4-\pi/2)^2 + (\sqrt{2}/2-1)^2} + \sqrt{(\pi-3\pi/4)^2 + (0-\sqrt{2}/2)^2} + \sqrt{(5\pi/4-\pi)^2 + (-\sqrt{2}/2-0)^2} + \sqrt{(3\pi/2-5\pi/4)^2 + (-1+\sqrt{2}/2)^2} + \sqrt{(7\pi/4-3\pi/2)^2 + (-\sqrt{2}/2+1)^2} + \sqrt{(2\pi-7\pi/4)^2 + (0+\sqrt{2}/2)^2} \approx 7.5802$$

4. $f(x) = \sin^2 x; a = 0; b = 2\pi$

**a.** $n = 2; x = 0, \pi, 2\pi; y = 0, 0, 0$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(\pi-0)^2 + (0-0)^2} + \sqrt{(2\pi-\pi)^2 + (0-0)^2} = 2\pi \approx 6.2832$$

**b.** $n = 4; x = 0, \pi/2, \pi, 3\pi/2, 2\pi;$
$y = 0, 1, 0, 1, 0$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(\pi/2-0)^2 + (1-0)^2} + \sqrt{(\pi-\pi/2)^2 + (0-1)^2} + \sqrt{(3\pi/2-\pi)^2 + (1-0)^2} + \sqrt{(2\pi-3\pi/2)^2 + (0-1)^2} \approx 7.4484$$

**c.** $n = 8; x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi;$
$y = 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \frac{1}{2}, 1, \frac{1}{2}, 0$

$$\sum_{i=1}^{n} \Delta w = \sqrt{(\pi/4-0)^2 + (1/2-0)^2} + \sqrt{(\pi/2-\pi/4)^2 + (1-1/2)^2} + \sqrt{(3\pi/4-\pi/2)^2 + (1/2-1)^2}$$

$$+\sqrt{(\pi-3\pi/4)^2+(0-1/2)^2}$$
$$+\sqrt{(5\pi/4-\pi)^2+(1/2-0)^2}$$
$$+\sqrt{(3\pi/2-5\pi/4)^2+(1-1/2)^2}$$
$$+\sqrt{(7\pi/4-3\pi/2)^2+(1/2-1)^2}$$
$$+\sqrt{(2\pi-7\pi/4)^2+(0-1/2)^2}$$
$$\approx 7.4484$$

5. $f(x)=2x+3, f'(x)=2$

$$L=\int_1^3\sqrt{1+(2)^2}\,dx=\sqrt{5}\int_1^3 dx=2\sqrt{5}$$

At $x=1, y=2(1)+3=5.$
At $x=3, y=2(3)+3=9.$

$$d=\sqrt{(3-1)^2+(9-5)^2}=\sqrt{20}=2\sqrt{5}$$

6. $x=y+\frac{3}{2}$

$$g(y)=y+\frac{3}{2}, g'(y)=1$$

$$L=\int_1^3\sqrt{1+(1)^2}=\sqrt{2}\int_1^3 dy=2\sqrt{2}$$

At $y=1,\ x=1+\frac{3}{2}=\frac{5}{2}.$

At $y=3,\ x=3+\frac{3}{2}=\frac{9}{2}.$

$$d=\sqrt{\left(\frac{9}{2}-\frac{5}{2}\right)^2+(3-1)^2}=\sqrt{8}=2\sqrt{2}$$

7. $f(x)=4x^{3/2}, f'(x)=6x^{1/2}$

$$L=\int_{1/3}^5\sqrt{1+(6x^{1/2})^2}\,dx=\int_{1/3}^5\sqrt{1+36x}\,dx$$
$$=\left[\frac{1}{36}\cdot\frac{2}{3}(1+36x)^{3/2}\right]_{1/3}^5$$
$$=\frac{1}{54}\left(181\sqrt{181}-13\sqrt{13}\right)\approx 44.23$$

8. $f(x)=\frac{2}{3}(x^2+1)^{3/2}, f'(x)=2x(x^2+1)^{1/2}$

$$L=\int_1^2\sqrt{1+\left[2x(x^2+1)^{1/2}\right]^2}\,dx$$
$$=\int_1^2\sqrt{4x^4+4x^2+1}\,dx=\int_1^2(2x^2+1)dx$$
$$=\left[\frac{2}{3}x^3+x\right]_1^2=\left(\frac{16}{3}+2\right)-\left(\frac{2}{3}+1\right)=\frac{17}{3}\approx 5.67$$

9. $f(x)=(4-x^{2/3})^{3/2},$

$$f'(x)=\frac{3}{2}(4-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$$
$$=-x^{-1/3}(4-x^{2/3})^{1/2}$$
$$L=\int_1^8\sqrt{1+\left[-x^{-1/3}(4-x^{2/3})^{1/2}\right]^2}\,dx$$
$$=\int_1^8\sqrt{4x^{-2/3}}\,dx=\int_1^8 2x^{-1/3}dx$$
$$=2\left[\frac{3}{2}x^{2/3}\right]_1^8=3(4-1)=9$$

10. $f(x)=\frac{x^4+3}{6x}=\frac{x^3}{6}+\frac{1}{2x}$

$$f'(x)=\frac{x^2}{2}-\frac{1}{2x^2}$$
$$L=\int_1^3\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$
$$=\int_1^3\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}\,dx=\int_1^3\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\,dx$$
$$=\int_1^3\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^3$$
$$=\left(\frac{9}{2}-\frac{1}{6}\right)-\left(\frac{1}{6}-\frac{1}{2}\right)=\frac{14}{3}\approx 4.67$$

11. $g(y)=\frac{y^4}{16}+\frac{1}{2y^2}, g'(y)=\frac{y^3}{4}-\frac{1}{y^3}$

$$L=\int_{-3}^{-2}\sqrt{1+\left(\frac{y^3}{4}-\frac{1}{y^3}\right)^2}\,dy$$
$$=\int_{-3}^{-2}\sqrt{\frac{y^6}{16}+\frac{1}{2}+\frac{1}{y^6}}\,dy=\int_{-3}^{-2}\sqrt{\left(\frac{y^3}{4}+\frac{1}{y^3}\right)^2}\,dy$$
$$=\int_{-3}^{-2}-\left(\frac{y^3}{4}+\frac{1}{y^3}\right)dy=-\left[\frac{y^4}{16}-\frac{1}{2y^2}\right]_{-3}^{-2}$$
$$=-\left[\left(1-\frac{1}{8}\right)-\left(\frac{81}{16}-\frac{1}{18}\right)\right]=\frac{595}{144}\approx 4.13$$

12. $x=\frac{y^5}{30}+\frac{1}{2y^3}$

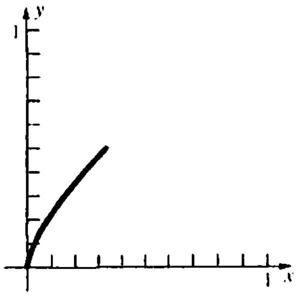
$$g(y)=\frac{y^5}{30}+\frac{1}{2y^3}, g'(y)=\frac{y^4}{6}-\frac{3}{2y^4}$$
$$L=\int_1^3\sqrt{1+\left(\frac{y^4}{6}-\frac{3}{2y^4}\right)^2}\,dy$$
$$=\int_1^3\sqrt{\frac{y^8}{36}+\frac{1}{2}+\frac{9}{4y^8}}\,dy$$

$= \int_1^3 \sqrt{\left(\frac{y^4}{6}+\frac{3}{2y^4}\right)^2}\, dy = \int_1^3 \left(\frac{y^4}{6}+\frac{3}{2y^4}\right) dy$

$= \left[\frac{y^5}{30}-\frac{1}{2y^3}\right]_1^3 = \left(\frac{81}{10}-\frac{1}{54}\right)-\left(\frac{1}{30}-\frac{1}{2}\right) = \frac{1154}{135}$

$\approx 8.55$

13.



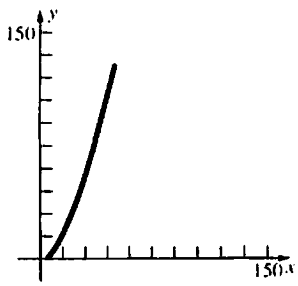
$\frac{dx}{dt} = t^2, \frac{dy}{dt} = t$

$L = \int_0^1 \sqrt{(t^2)^2+(t)^2}\, dt = \int_0^1 \sqrt{t^4+t^2}\, dt$

$= \int_0^1 t\sqrt{t^2+1}\, dt = \left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^1$

$= \frac{1}{3}\left(2\sqrt{2}-1\right) \approx 0.61$

14.



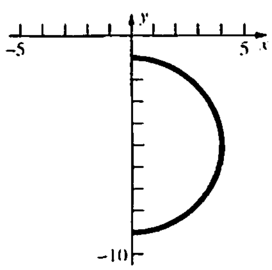
$\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$

$L = \int_1^4 \sqrt{(6t)^2+(6t^2)^2}\, dt = \int_1^4 \sqrt{36t^2+36t^4}\, dt$

$= \int_1^4 6t\sqrt{1+t^2}\, dt = \left[2(1+t^2)^{3/2}\right]_1^4$

$= 2\left(17\sqrt{17}-2\sqrt{2}\right) \approx 134.53$

15.



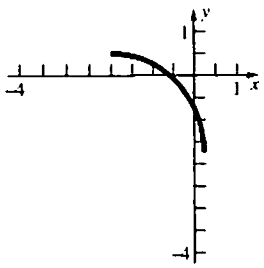
$\frac{dx}{dt} = 4\cos t, \frac{dy}{dt} = -4\sin t$

$L = \int_0^\pi \sqrt{(4\cos t)^2+(-4\sin t)^2}\, dt$

$= \int_0^\pi \sqrt{16\cos^2 t+16\sin^2 t}\, dt$

$= \int_0^\pi 4\, dt = 4\pi \approx 12.57$

16.



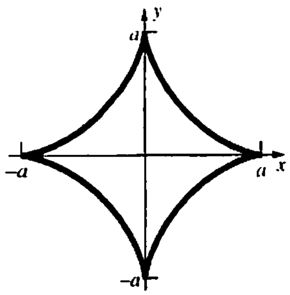
$\frac{dx}{dt} = 2\sqrt{5}\cos 2t, \frac{dy}{dt} = -2\sqrt{5}\sin 2t$

$L = \int_0^{\pi/4} \sqrt{\left(2\sqrt{5}\cos 2t\right)^2+\left(-2\sqrt{5}\sin 2t\right)^2}\, dt$

$= \int_0^{\pi/4} \sqrt{20\cos^2 2t+20\sin^2 2t}\, dt = \int_0^{\pi/4} 2\sqrt{5}\, dt$

$= \frac{\sqrt{5}\,\pi}{2} \approx 3.51$

17.



$\frac{dx}{dt} = 3a\cos t\sin^2 t, \frac{dy}{dt} = -3a\sin t\cos^2 t$

The first quadrant length is $L$

$= \int_0^{\pi/2} \sqrt{(3a\cos t\sin^2 t)^2+(-3a\sin t\cos^2 t)^2}\, dt$

$= \int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^4 t+9a^2\sin^2 t\cos^4 t}\, dt$

$= \int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^2 t(\sin^2 t+\cos^2 t)}\, dt$

$= \int_0^{\pi/2} 3a\cos t\sin t\, dt = 3a\left[-\frac{1}{2}\cos^2 t\right]_0^{\pi/2}$

$= \frac{3a}{2}$

(The integral can also be evaluated as

$3a\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2}$ with the same result.)

The total length is $6a$.

18. a. $\overline{OT} = \text{length}\left(\widehat{PT}\right) = a\theta$

b. From Figure 18 of the text,

$\sin\theta = \frac{\overline{PQ}}{\overline{PC}} = \frac{\overline{PQ}}{a}$ and $\cos\theta = \frac{\overline{QC}}{\overline{PC}} = \frac{\overline{QC}}{a}$.

Therefore $\overline{PQ} = a\sin\theta$ and $\overline{QC} = a\cos\theta$.

c. $x = \overline{OT} - \overline{PQ} = a\theta - a\sin\theta = a(\theta - \sin\theta)$

$y = \overline{CT} - \overline{CQ} = a - a\cos\theta = a(1-\cos\theta)$

19. From Problem 18,

$x = a(\theta - \sin\theta), y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1-\cos\theta), \frac{dy}{d\theta} = a\sin\theta$ so

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [a(1-\cos\theta)]^2 + [a\sin\theta]^2$

$= a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta$

$= 2a^2 - 2a^2\cos\theta = 2a^2(1-\cos\theta)$

$= 4a^2\frac{1-\cos\theta}{2} = 4a^2\sin^2\left(\frac{\theta}{2}\right)$.

The length of one arch of the cycloid is

$\int_0^{2\pi}\sqrt{4a^2\sin^2\left(\frac{\theta}{2}\right)}\,d\theta = \int_0^{2\pi} 2a\sin\left(\frac{\theta}{2}\right)d\theta$

$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 2a(2+2) = 8a$

20. a. Using $\theta = \omega t$, the point $P$ is at $x = a\omega t - a\sin(\omega t)$, $y = a - a\cos(\omega t)$ at time $t$.

$\frac{dx}{dt} = a\omega - a\omega\cos(\omega t) = a\omega(1-\cos(\omega t))$

$\frac{dy}{dt} = a\omega\sin(\omega t)$

$\frac{ds}{dt} = \sqrt{\left[\frac{dy}{dt}\right]^2 + \left[\frac{dx}{dt}\right]^2}$

$= \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2 - 2a^2\omega^2\cos(\omega t) + a^2\omega^2\cos^2(\omega t)} = \sqrt{2a^2\omega^2 - 2a^2\omega^2\cos(\omega t)}$

$= 2a\omega\sqrt{\frac{1}{2}(1-\cos(\omega t))}$

$= 2a\omega\sqrt{\sin^2\frac{\omega t}{2}} = 2a\omega\left|\sin\frac{\omega t}{2}\right|$

b. The speed is a maximum when $\left|\sin\frac{\omega t}{2}\right| = 1$, which occurs when $t = \frac{\pi}{\omega}(2k+1)$. The speed is a minimum when $\left|\sin\frac{\omega t}{2}\right| = 0$, which occurs when $t = \frac{2k\pi}{\omega}$.

c. From Problem 18a, the distance traveled by the wheel is $a\theta$, so at time $t$, the wheel has gone $a\theta = a\omega t$ miles. Since the car is going 60 miles per hour, the wheel has gone $60t$ miles at time $t$. Thus, $a\omega = 60$ and the maximum speed of the bug on the wheel is $2a\omega = 2(60) = 120$ miles per hour.

21. a. $\frac{dy}{dx} = \sqrt{x^3 - 1}$

$L = \int_1^2\sqrt{1 + x^3 - 1}\,dx = \int_1^2 x^{3/2}dx$

$= \left[\frac{2}{5}x^{5/2}\right]_1^2 = \frac{2}{5}\left(4\sqrt{2} - 1\right) \approx 1.86$

b. $f'(t) = 1 - \cos t, g'(t) = \sin t$

$L = \int_0^{4\pi}\sqrt{2 - 2\cos t}\,dt = \int_0^{4\pi} 2\left|\sin\left(\frac{t}{2}\right)\right|dt$

$\sin\left(\frac{t}{2}\right)$ is positive for $0 < t < 2\pi$, and by symmetry, we can double the integral from 0 to $2\pi$.

$$L = 4\int_0^{2\pi} \sin\left(\frac{t}{2}\right)dt = \left[-8\cos\frac{t}{2}\right]_0^{2\pi}$$

$$= 8 + 8 = 16$$

**22.** **a.** $\frac{dy}{dx} = \sqrt{64\sin^2 x\cos^4 x - 1}$

$$L = \int_{\pi/6}^{\pi/3} \sqrt{1 + 64\sin^2 \cos^4 x - 1}\, dx$$

$$= \int_{\pi/6}^{\pi/3} 8\sin x\cos^2 x dx$$

$$= \left[-\frac{8}{3}\cos^3 x\right]_{\pi/6}^{\pi/3} = -\frac{1}{3} + \sqrt{3} \approx 1.40$$

**b.** $\frac{dx}{dt} = -a\sin t + a\sin t + at\cos t = at\cos t$

$$\frac{dy}{dt} = a\cos t - a\cos t + at\sin t = at\sin t$$

$$L = \int_{-1}^{1} \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}\, dt$$

$$= \int_{-1}^{1} |at|\, dt = \int_0^1 at\, dt - \int_{-1}^{0} at\, dt$$

$$= \left[\frac{a}{2}t^2\right]_0^1 - \left[\frac{a}{2}t^2\right]_{-1}^{0} = \frac{a}{2} + \frac{a}{2}$$

$$= a$$

**23.** $f(x) = 6x,\ f'(x) = 6$

$$A = 2\pi\int_0^1 6x\sqrt{1+36}\, dx = 12\sqrt{37}\pi\int_0^1 x\, dx$$

$$= 12\sqrt{37}\pi\left[\frac{1}{2}x^2\right]_0^1 = 6\sqrt{37}\pi \approx 114.66$$

**24.** $f(x) = \sqrt{25 - x^2},\ f'(x) = -\frac{x}{\sqrt{25-x^2}}$

$$A = 2\pi\int_{-2}^{3} \sqrt{25-x^2}\sqrt{1 + \frac{x^2}{25-x^2}}\, dx$$

$$= 2\pi\int_{-2}^{3} \sqrt{25 - x^2 + x^2}\, dx$$

$$= 2\pi\int_{-2}^{3} 5dx = 10\pi[x]_{-2}^{3} = 50\pi \approx 157.08$$

**25.** $f(x) = \frac{x^3}{3},\ f'(x) = x^2$

$$A = 2\pi\int_1^{\sqrt{7}} \frac{x^3}{3}\sqrt{1+x^4}\, dx$$

$$= 2\pi\left[\frac{1}{18}(1+x^4)^{3/2}\right]_1^{\sqrt{7}} = \frac{\pi}{9}\left(250\sqrt{2} + 2\sqrt{2}\right)$$

$$= 28\sqrt{2}\pi \approx 124.40$$

**26.** $f(x) = \frac{x^6+2}{8x^2} = \frac{x^4}{8} + \frac{1}{4x^2},\ f'(x) = \frac{x^3}{2} - \frac{1}{2x^3}$

$$A = 2\pi\int_1^3 \left(\frac{x^4}{8} + \frac{1}{4x^2}\right)\sqrt{1 + \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2}$$

$$= 2\pi\int_1^3 \left(\frac{x^4}{8} + \frac{1}{4x^2}\right)\sqrt{\frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}}\, dx$$

$$= 2\pi\int_1^3 \left(\frac{x^4}{8} + \frac{1}{4x^2}\right)\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)dx$$

$$= 2\pi\int_1^3 \left(\frac{x^7}{16} + \frac{3x}{16} + \frac{1}{8x^5}\right)dx$$

$$= 2\pi\left[\frac{x^8}{128} + \frac{3x^2}{32} - \frac{1}{32x^4}\right]_1^3$$

$$= 2\pi\left[\left(\frac{6561}{128} + \frac{27}{32} - \frac{1}{2592}\right) - \left(\frac{1}{128} + \frac{3}{32} - \frac{1}{32}\right)\right]$$

$$= \frac{8429\pi}{81} \approx 326.92$$

**27.** $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 3t^2$

$$A = 2\pi\int_0^1 t^3\sqrt{1+9t^4}\, dt$$

$$= 2\pi\left[\frac{1}{54}(1+9t^4)^{3/2}\right]_0^1 = \frac{\pi}{27}\left(10\sqrt{10} - 1\right)$$

$$\approx 3.56$$

**28.** $\frac{dx}{dt} = -2t,\ \frac{dy}{dt} = 2$

$$A = 2\pi\int_0^1 2t\sqrt{4t^2+4}\, dt = 8\pi\int_0^1 t\sqrt{t^2+1}\, dt$$

$$= 8\pi\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^1 = \frac{8\pi}{3}(2\sqrt{2} - 1)$$

$$\approx 15.32$$

**29.** $y = f(x) = \sqrt{r^2 - x^2}$

$$f'(x) = -x(r^2 - x^2)^{-1/2}$$

$$A = 2\pi\int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{1 + \left[-x(r^2-x^2)^{-1/2}\right]^2}\, dx$$

$$= 2\pi\int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{1 + x^2(r^2-x^2)^{-1}}\, dx$$

$$= 2\pi\int_{-r}^{r} \sqrt{\left(r^2-x^2\right)\left(1 + x^2(r^2-x^2)^{-1}\right)}dx$$

$$= 2\pi\int_{-r}^{r} \sqrt{r^2 - x^2 + x^2}\, dx$$

$$= 2\pi\int_{-r}^{r} \sqrt{r^2}\, dx = 2\pi\int_{-r}^{r} r dx$$

$$= 2\pi rx\,|_{-r}^{r} = 4\pi r^2$$

30. $x = f(t) = r\cos t$

$y = g(t) = r\sin t$

$f'(t) = -r\sin t$

$g'(t) = r\cos t$

$$A = 2\pi \int_0^{\pi} r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt$$

$$= 2\pi \int_0^{\pi} r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$$

$$= 2\pi \int_0^{\pi} r\sin t\sqrt{r^2}\,dt$$

$$= 2\pi \int_0^{\pi} r^2 \sin t\,dt = -2\pi r^2 \cos t \,|_0^{\pi}$$

$$= -2r^2(-1-1) = 4\pi r^2$$

31. a. The base circumference is equal to the arc length of the sector, so $2\pi r = \theta l$. Therefore, $\theta = \frac{2\pi r}{l}$.

b. The area of the sector is equal to the lateral surface area. Therefore, the lateral surface area is $\frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi rl$.

c. Assume $r_2 > r_1$. Let $l_1$ and $l_2$ be the slant heights for $r_1$ and $r_2$, respectively. Then

$A = \pi r_2 l_2 - \pi r_1 l_1 = \pi r_2(l_1 + l) - \pi r_1 l_1$.

From part a, $\theta = \frac{2\pi r_2}{l_2} = \frac{2\pi r_2}{l_1 + l} = \frac{2\pi r_1}{l_1}$.

Solve for $l_1 : l_1 r_2 = l_1 r_1 + l r_1$

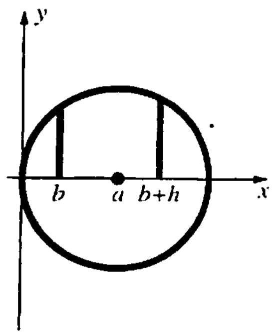
$l_1(r_2 - r_1) = l r_1$

$$l_1 = \frac{l r_1}{r_2 - r_1}$$

$$A = \pi r_2\left(\frac{l r_1}{r_2 - r_1} + l\right) - \pi r_1\left(\frac{l r_1}{r_2 - r_1}\right)$$

$$= \pi(l r_1 + l r_2) = 2\pi\left[\frac{r_1 + r_2}{2}\right]l$$

32. Put the center of a circle of radius $a$ at $(a, 0)$. Revolving the portion of the circle from $x = b$ to $x = b + h$ about the $x$-axis results in the surface in question. (See figure.)




The equation of the top half of the circle is $y = \sqrt{a^2 - (x-a)^2}$.

$$\frac{dy}{dx} = \frac{-(x-a)}{\sqrt{a^2 - (x-a)^2}}$$

$$A = 2\pi \int_b^{b+h} \sqrt{a^2 - (x-a)^2}\sqrt{1 + \frac{(x-a)^2}{a^2 - (x-a)^2}}\,dx$$

$$= 2\pi \int_b^{b+h} \sqrt{a^2 - (x-a)^2 + (x-a)^2}\,dx$$

$$= 2\pi \int_b^{b+h} a\,dx = 2\pi a[x]_b^{b+h} = 2\pi ah$$

A right circular cylinder of radius $a$ and height $h$ has surface area $2\pi ah$.

33. a. $\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = a\sin t$

$$A = 2\pi \int_0^{2\pi} a(1-\cos t)\cdot \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,dt$$

$$= 2\pi a \int_0^{2\pi} (1-\cos t)\sqrt{2a^2 - 2a^2\cos t}\,dt$$

$$= 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1-\cos t)^{3/2}\,dt$$

b. $1 - \cos t = 2\sin^2\left(\frac{t}{2}\right)$, so

$$A = 2\sqrt{2}\pi a^2 \int_0^{2\pi} 2^{3/2}\sin^3\left(\frac{t}{2}\right)dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin\left(\frac{t}{2}\right)\left[1 - \cos^2\left(\frac{t}{2}\right)\right]dt$$

$$= 8\pi a^2\left[-2\cos\left(\frac{t}{2}\right) + \frac{2}{3}\cos^3\left(\frac{t}{2}\right)\right]_0^{2\pi}$$

$$= 8\pi a^2\left[\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right] = \frac{64}{3}\pi a^2$$

34. $\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$

Since the circle is being revolved about the line $x = b$, the surface area is

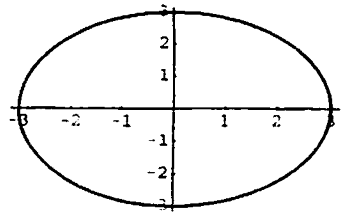
$$A = 2\pi \int_0^{2\pi} (b - a\cos t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$$

$$= 2\pi a \int_0^{2\pi} (b - a\cos t)\,dt$$

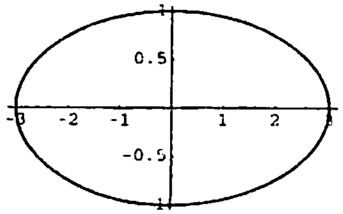
$$= 2\pi a[bt - a\sin t]_0^{2\pi}$$

$$= 4\pi^2 ab$$

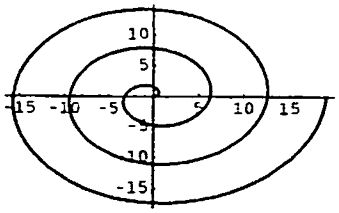
**35. a.**



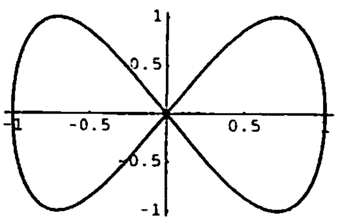

**b.**



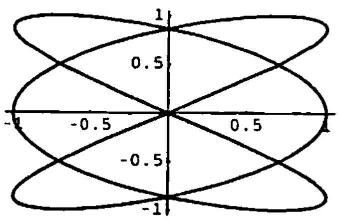

**c.**



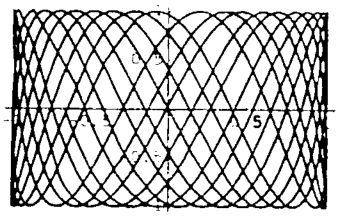

**d.**




**e.**




**f.**



**36. a.** $f'(t) = -3\sin t, g'(t) = 3\cos t$

$$L = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\,dt$$

$$= \int_0^{2\pi} 3\,dt = 3[t]_0^{2\pi} = 6\pi \approx 18.850$$

**b.** $f'(t) = -3\sin t, g'(t) = \cos t$

$$L = \int_0^{2\pi} \sqrt{9\sin^2 t + \cos^2 t}\,dt \approx 13.365$$

**c.** $f'(t) = \cos t - t\sin t, g'(t) = t\cos t + \sin t$

$$L = \int_0^{6\pi} \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}\,dt$$

$$= \int_0^{6\pi} \sqrt{1 + t^2}\,dt \approx 179.718$$

**d.** $f'(t) = -\sin t, g'(t) = 2\cos 2t$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t}\,dt \approx 9.429$$

**e.** $f'(t) = -3\sin 3t, g'(t) = 2\cos 2t$

$$L = \int_0^{2\pi} \sqrt{9\sin^2 3t + 4\cos^2 2t}\,dt$$

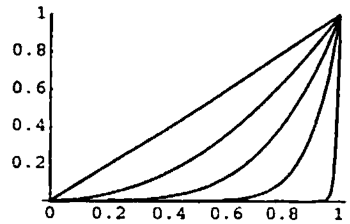
$\approx 15.289$

**f.** $f'(t) = -\sin t, g'(t) = \pi\cos \pi t$

$$L = \int_0^{40} \sqrt{\sin^2 t + \pi^2 \cos^2 \pi t}\,dt$$

$\approx 86.58$

**37.**




$y = x, y' = 1$,

$L = \int_0^1 \sqrt{2}\,dx = \left[\sqrt{2}\,x\right]_0^1 = \sqrt{2} \approx 1.41421$

$y = x^2, y' = 2x$, $L = \int_0^1 \sqrt{1+4x^2}\,dx \approx 1.47894$

$y = x^4, y' = 4x^3$, $L = \int_0^1 \sqrt{1+16x^6}\,dx \approx 1.60023$

$y = x^{10}, y' = 10x^9$,

$L = \int_0^1 \sqrt{1+100^{18}}\,dx \approx 1.75441$

$y = x^{100}, y' = 100x^{99}$,

$L = \int_0^1 \sqrt{1+10,000x^{198}}\,dx \approx 1.95167$

When $n$ = 10,000 the length will be close to 2.

## 6.5 Concepts Review

1. $F \cdot (b-a)$; $\int_a^b F(x)\,dx$

2. $kx$

3. $30 \cdot 10 = 300$

4. $\delta\pi 5^2 \Delta y(12-y) = 62.4(12-y)\pi(25)\Delta y$;
   $62.4\pi(25)\int_0^{12} (12-y)dy$

## Problem Set 6.5

1. $F\left(\frac{1}{2}\right) = 6; k \cdot \frac{1}{2} = 6, k = 12$
   $F(x) = 12x$
   $W = \int_0^{1/2} 12x\,dx = \left[6x^2\right]_0^{1/2} = \frac{3}{2} = 1.5$ ft-lb

2. From Problem 1, $F(x) = 12x$.
   $W = \int_0^2 12x\,dx = \left[6x^2\right]_0^2 = 24$ ft-lb

3. $F(0.01) = 0.6; k = 60$
   $F(x) = 60x$
   $W = \int_0^{0.02} 60x\,dx = \left[30x^2\right]_0^{0.02} = 0.012$ Joules

4. $F(x) = kx$ and let $l$ be the natural length of the spring.
   $W = \int_{8-l}^{9-l} kx\,dx = \left[\frac{1}{2}kx^2\right]_{8-l}^{9-l}$
   $= \frac{1}{2}k\left[(81-18l+l^2)-(64-16l+l^2)\right]$
   $= \frac{1}{2}k(17-2l) = 0.05$
   Thus, $k = \frac{0.1}{17-2l}$.
   $W = \int_{9-l}^{10-l} kx\,dx = \left[\frac{1}{2}kx^2\right]_{9-l}^{10-l}$
   $= \frac{1}{2}k\left[(100-20l+l^2)-(81-18l+l^2)\right]$
   $= \frac{1}{2}k(19-2l) = 0.1$
   Thus, $k = \frac{0.2}{19-2l}$.
   Solving $\frac{0.1}{17-2l} = \frac{0.2}{19-2l}, l = \frac{15}{2}$.
   Thus $k = 0.05$, and the natural length is 7.5 cm.

5. $W = \int_0^d kx\,dx = \left[\frac{1}{2}kx^2\right]_0^d$
   $= \frac{1}{2}k(d^2-0) = \frac{1}{2}kd^2$

6. $F(8) = 2; k16 = 2, k = \frac{1}{8}$
   $W = \int_0^{27} \frac{1}{8}s^{4/3}\,ds = \frac{1}{8}\left[\frac{3}{7}s^{7/3}\right]_0^{27} = \frac{6561}{56}$
   $\approx 117.16$ inch-pounds

7. $W = \int_0^2 9s\,ds = 9\left[\frac{1}{2}s^2\right]_0^2 = 18$ ft-lb

8. One spring will move from 2 feet beyond its natural length to 3 feet beyond its natural length. The other will move from 2 feet beyond its natural length to 1 foot beyond its natural length.
   $W = \int_2^3 6s\,ds + \int_2^1 6s\,ds = \left[3s^2\right]_2^3 + \left[3s^2\right]_2^1$
   $= 3(9-4) + 3(1-4) = 6$ ft-lb

9. A slab of thickness $\Delta y$ at height $y$ has width $4 - \frac{4}{5}y$ and length 10. The slab will be lifted a distance $10 - y$.
   $\Delta W \approx \delta \cdot 10 \cdot \left(4 - \frac{4}{5}y\right)\Delta y(10-y)$
   $= 8\delta(y^2 - 15y + 50)\Delta y$
   $W = \int_0^5 8\delta(y^2 - 15y + 50)\,dy$
   $= 8(62.4)\left[\frac{1}{3}y^3 - \frac{15}{2}y^2 + 50y\right]_0^5$
   $= 8(62.4)\left(\frac{125}{3} - \frac{375}{2} + 250\right)$
   $= 52{,}000$ ft-lb

10. A slab of thickness $\Delta y$ at height $y$ has width $4 - \frac{4}{3}y$ and length 10. The slab will be lifted a distance $8 - y$.
    $\Delta W \approx \delta \cdot 10 \cdot \left(4 - \frac{4}{3}y\right)\Delta y(8-y)$
    $= \frac{40}{3}\delta(24 - 11y + y^2)\Delta y$
    $W = \int_0^3 \frac{40}{3}\delta(24 - 11y + y^2)\,dy$
    $= \frac{40}{3}(62.4)\left[24y - \frac{11}{2}y^2 + \frac{1}{3}y^3\right]_0^3$

$= \frac{40}{3}(62.4)\left(72 - \frac{99}{2} + 9\right)$

$= 26{,}208$ ft-lb

11. A slab of thickness $\Delta y$ at height $y$ has width $\frac{3}{4}y + 3$ and length 10. The slab will be lifted a distance $9 - y$. $\Delta W \approx \delta \cdot 10 \cdot \left(\frac{3}{4}y + 3\right)\Delta y(9 - y)$

$= \frac{15}{2}\delta(36 + 5y - y^2)\Delta y$

$W = \int_0^4 \frac{15}{2}\delta(36 + 5y - y^2)dy$

$= \frac{15}{2}(62.4)\left[36y + \frac{5}{2}y^2 - \frac{1}{3}y^3\right]_0^4$

$= \frac{15}{2}(62.4)\left(144 + 40 - \frac{64}{3}\right)$

$= 76{,}128$ ft-lb

12. A slab of thickness $\Delta y$ at height $y$ has width $2\sqrt{6y - y^2}$ and length 10. The slab will be lifted a distance $8 - y$.

$\Delta W \approx \delta \cdot 10 \cdot 2\sqrt{6y - y^2}\,\Delta y(8 - y)$

$= 20\delta\sqrt{6y - y^2}\,(8 - y)\Delta y$

$W = \int_0^3 20\delta\sqrt{6y - y^2}\,(8 - y)dy$

$= 20\delta \int_0^3 \sqrt{6y - y^2}\,(3 - y)\,dy$

$+20\delta \int_0^3 \sqrt{6y - y^2}\,(5)dy$

$= 20\delta\left[\frac{1}{3}(6y - y^2)^{3/2}\right]_0^3$

$+100\delta \int_0^3 \sqrt{6y - y^2}\,dy$

Notice that $\int_0^3 \sqrt{6y - y^2}\,dy$ is the area of a quarter of a circle with radius 3.

$W = 20\delta(9) + 100\delta\left(\frac{1}{4}\pi 9\right)$

$= (62.4)(180 + 225\pi) \approx 55{,}340$ ft-lb

13. The volume of a disk with thickness $\Delta y$ is $16\pi\Delta y$. If it is at height $y$, it will be lifted a distance $10 - y$.

$\Delta W \approx \delta 16\pi\Delta y(10 - y) = 16\pi\delta(10 - y)\Delta y$

$W = \int_0^{10} 16\pi\delta(10 - y)dy = 16\pi(50)\left[10y - \frac{1}{2}y^2\right]_0^{10}$

$= 16\pi(50)(100 - 50) \approx 125{,}664$ ft-lb

14. The volume of a disk with thickness $\Delta x$ at height $x$ is $\pi(4 + x)^2\Delta x$. It will be lifted a distance of $10 - x$.

$\Delta W \approx \delta\pi(4 + x)^2\Delta x(10 - x)$

$= \pi\delta(160 + 64x + 2x^2 - x^3)\Delta x$

$W = \int_0^{10} \pi\delta(160 + 64x + 2x^2 - x^3)dx$

$= \pi(50)\left[160x + 32x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^{10}$

$= \pi(50)\left(1600 + 3200 + \frac{2000}{3} - 2500\right)$

$\approx 466{,}003$ ft-lb

15. The total force on the face of the piston is $A \cdot f(x)$ if the piston is $x$ inches from the cylinder head. The work done by moving the piston from $x_1$ to $x_2$ is $W = \int_{x_1}^{x_2} A \cdot f(x)dx = A\int_{x_1}^{x_2} f(x)dx$. This is the work done by the gas in moving the piston. The work done by the piston to compress the gas is the opposite of this or $A\int_{x_2}^{x_1} f(x)dx$.

16. $c = 40(16)^{1.4}$

$A = 1;\ p(v) = cv^{-1.4}$

$f(x) = cx^{-1.4}$

$x_1 = \frac{16}{1} = 16,\ x_2 = \frac{2}{1} = 2$

$W = \int_2^{16} cx^{-1.4}dx = c\left[-2.5x^{-0.4}\right]_2^{16}$

$= 40(16)^{1.4}(-2.5)(16^{-0.4} - 2^{-0.4})$

$\approx 2075.83$ in.-lb

17. $c = 40(16)^{1.4}$

$A = 2;\ p(v) = cv^{-1.4}$

$f(x) = c(2x)^{-1.4}$

$x_1 = \frac{16}{2} = 8,\ x_2 = \frac{2}{2} = 1$

$W = 2\int_1^8 c(2x)^{-1.4}dx = 2c\left[-1.25(2x)^{-0.4}\right]_1^8$

$= 80(16)^{1.4}(-1.25)(16^{-0.4} - 2^{-0.4})$

$\approx 2075.83$ in.-lb

18. 80 lb/in.$^2$ = 11,520 lb/ft$^2$

$c = 11{,}520(1)^{1.4} = 11{,}520$

$\Delta W \approx p(v)\Delta v = 11{,}520v^{-1.4}\Delta v$

$W = \int_1^4 11{,}520v^{-1.4}dv = \left[-28{,}800v^{-0.4}\right]_1^4$

$= -28{,}800(4^{-0.4} - 1^{-0.4}) \approx 12{,}259$ ft-lb

**19.** The total work is equal to the work $W_1$ to haul the load by itself and the work $W_2$ to haul the rope by itself.

$W_1 = 200 \cdot 500 = 100{,}000$ ft-lb

Let $y = 0$ be the bottom of the shaft. When the rope is at $y$, $\Delta W_2 \approx 2\Delta y(500 - y)$.

$W_2 = \int_0^{500} 2(500 - y)dy = 2\left[500y - \frac{1}{2}y^2\right]_0^{500}$

$= 2(250{,}000 - 125{,}000) = 250{,}000$ ft-lb

$W = W_1 + W_2 = 100{,}000 + 250{,}000$

$= 350{,}000$ ft-lb

**20.** The total work is equal to the work $W_1$ to lift the monkey plus the work $W_2$ to lift the chain.

$W_1 = 10 \cdot 20 = 200$ ft-lb

Let $y = 20$ represent the top. As the monkey climbs the chain, the piece of chain at height $y$ $(0 \le y \le 10)$ will be lifted $20 - 2y$ ft.

$\Delta W_2 \approx \frac{1}{2}\Delta y(20 - 2y) = (10 - y)\Delta y$

$W_2 = \int_0^{10}(10 - y)dy = \left[10y - \frac{1}{2}y^2\right]_0^{10}$

$= 100 - 50 = 50$ ft-lb

$W = W_1 + W_2 = 250$ ft-lb

**21.** $f(x) = \frac{k}{x^2}$; $f(4000) = 5000$

$\frac{k}{4000^2} = 5000$, $k = 80{,}000{,}000{,}000$

$W = \int_{4000}^{4200} \frac{80{,}000{,}000{,}000}{x^2}dx$

$= 80{,}000{,}000{,}000\left[-\frac{1}{x}\right]_{4000}^{4200}$

$= \frac{20{,}000{,}000}{21} \approx 952.381$ mi-lb

**22.** $F(x) = \frac{k}{x^2}$ where $x$ is the distance between the charges. $F(2) = 10; \frac{k}{4} = 10, k = 40$

$W = \int_1^5 \frac{40}{x^2}dx = \left[-\frac{40}{x}\right]_1^5 = 32$ ergs

**23.** The relationship between the height of the bucket and time is $y = 2t$, so $t = \frac{1}{2}y$. When the bucket is a height $y$, the sand has been leaking out of the bucket for $\frac{1}{2}y$ seconds. The weight of the bucket and sand is $100 + 500 - 3\left(\frac{1}{2}y\right) = 600 - \frac{3}{2}y$.

$\Delta W \approx \left(600 - \frac{3}{2}y\right)\Delta y$

$W = \int_0^{80}\left(600 - \frac{3}{2}y\right)dy = \left[600y - \frac{3}{4}y^2\right]_0^{80}$

$= 48{,}000 - 4800 = 43{,}200$ ft-lb

**24.** The total work is equal to the work $W_1$ needed to fill the pipe plus the work $W_2$ needed to fill the tank.

$\Delta W_1 = \delta\pi\left(\frac{1}{2}\right)^2 \Delta y(y) = \frac{\delta\pi y}{4}\Delta y$

$W_1 = \int_0^{30} \frac{\delta\pi y}{4}dy = \frac{(62.4)\pi}{4}\left[\frac{1}{2}y^2\right]_0^{30}$

$\approx 22{,}054$ ft-lb

The cross sectional area at height $y$ feet $(30 \le y \le 50)$ is $\pi r^2$ where

$r = \sqrt{10^2 - (40 - y)^2} = \sqrt{-y^2 + 80y - 1500}$.

$\Delta W_2 = \delta\pi r^2 \Delta y\, y = \delta\pi(-y^3 + 80y^2 - 1500y)\Delta y$

$W_2 = \int_{30}^{50} \delta\pi(-y^3 + 80y^2 - 1500y)dy$

$= (62.4)\pi\left[-\frac{1}{4}y^4 + \frac{80}{3}y^3 - 750y^2\right]_{30}^{50}$

$= (62.4)\pi\left[\left(-1{,}562{,}500 + \frac{10{,}000{,}000}{3} - 1{,}875{,}000\right)\right.$
$\left. - (-202{,}500 + 720{,}000 - 675{,}000)\right]$

$\approx 10{,}455{,}220$ ft-lb

$W = W_1 + W_2 \approx 10{,}477{,}274$ ft-lb

**25.** Let $W_1$ be the work to lift $V$ to the surface and $W_2$ be the work to lift $V$ from the surface to 15 feet above the surface. The volume displaced by the buoy $y$ feet above its original position is

$\frac{1}{3}\pi\left(a - \frac{a}{h}y\right)^2 (h - y) = \frac{1}{3}\pi a^2 h\left(1 - \frac{y}{h}\right)^3$.

The weight displaced is $\frac{\delta}{3}\pi a^2 h\left(1 - \frac{y}{h}\right)^3$.

Note by Archimede's Principle $m = \frac{\delta}{3}\pi a^2 h$ or

$a^2 h = \frac{3m}{\delta\pi}$, so the displaced weight is

$m\left(1-\frac{y}{h}\right)^3$.

$$\Delta W_1 \approx \left(m - m\left(1-\frac{y}{h}\right)^3\right)\Delta y = m\left(1-\left(1-\frac{y}{h}\right)^3\right)\Delta y$$

$$W_1 = m\int_0^h \left(1-\left(1-\frac{y}{h}\right)^3\right)dy$$

$$= m\left[y + \frac{h}{4}\left(1-\frac{y}{h}\right)^4\right]_0^h = \frac{3mh}{4}$$

$W_2 = m\cdot 15 = 15m$

$$W = W_1 + W_2 = \frac{3mh}{4} + 15m$$

26. Since $\delta\left(\frac{1}{3}\pi a^2\right)(8) = 300,\ a = \sqrt{\frac{225}{2\pi\delta}}$.

When the buoy is at $z$ feet ($0 \le z \le 2$) below floating position, the radius $r$ at the water level is

$$r = \left(\frac{8+z}{8}\right)a = \sqrt{\frac{225}{2\pi\delta}}\left(\frac{8+z}{8}\right).$$

$$F = \delta\left(\frac{1}{3}\pi r^2\right)(8+z) - 300$$

$$= \frac{75}{128}(8+z)^3 - 300$$

$$W = \int_0^2 \left[\frac{75}{128}(8+z)^3 - 300\right]dz$$

$$= \left[\frac{75}{512}(8+z)^4 - 300z\right]_0^2$$

$$= \left(\frac{46,875}{32} - 600\right) - (600 - 0)$$

$$= \frac{8475}{32} \approx 264.84 \text{ ft-lb}$$

27. First calculate the work $W_1$ needed to lift the contents of the bottom tank to 10 feet.

$\Delta W_1 \approx \delta 40 \Delta y(10-y)$

$$W_1 = \int_0^4 \delta 40(10-y)dy$$

$$= (62.4)(40)\left[-\frac{1}{2}(10-y)^2\right]_0^4$$

$= (62.4)(40)(-18 + 50)$

$= 79,872$ ft-lb

Next calculate the work $W_2$ needed to fill the top tank. Let $y$ be the distance from the bottom of the top tank.

$\Delta W_2 \approx \delta(36\pi)\Delta y\, y$

Solve for the height of the top tank:

$$36\pi h = 160;\ h = \frac{160}{36\pi} = \frac{40}{9\pi}$$

$$W_2 = \int_0^{40/9\pi} \delta 36\pi y\, dy$$

$$= (62.4)(36\pi)\left[\frac{1}{2}y^2\right]_0^{40/9\pi}$$

$$= (62.4)(36\pi)\left(\frac{800}{81\pi^2}\right)$$

$\approx 7062$ ft-lbs

$W = W_1 + W_2 \approx 86,934$ ft-lbs

## 6.6 Concepts Review

1. right; $\frac{4\cdot 1 + 6\cdot 3}{4+6} = 2.2$

2. 2.5; right; $x(1+x)$; $1+x$

3. 1; 3

4. $\frac{24}{16}; \frac{40}{16}$

The second lamina balances at $\bar{x} = 3,\ \bar{y} = 1$.
The first lamina has area 12 and the second lamina has area 4.

$$\bar{x} = \frac{12\cdot 1 + 4\cdot 3}{12+4} = \frac{24}{16},\ \bar{y} = \frac{12\cdot 3 + 4\cdot 1}{12+4} = \frac{40}{16}$$

## Problem Set 6.6

1. $$\bar{x} = \frac{2\cdot 5 + (-2)\cdot 7 + 1\cdot 9}{5+7+9} = \frac{5}{21}$$

2. Let $x$ measure the distance from the end where John sits.

$$\frac{180\cdot 0 + 80\cdot x + 110\cdot 12}{180+80+110} = 6$$

$80x + 1320 = 6 \cdot 370$

$80x = 900$

$x = 11.25$

Tom should be 11.25 feet from John, or, equivalently, 0.75 feet from Mary.

3. $\bar{x}=\dfrac{\int_0^7 x\sqrt{x}\,dx}{\int_0^7 \sqrt{x}\,dx}=\dfrac{\left[\frac{2}{5}x^{5/2}\right]_0^7}{\left[\frac{2}{3}x^{3/2}\right]_0^7}=\dfrac{\frac{2}{5}\left(49\sqrt{7}\right)}{\frac{2}{3}\left(7\sqrt{7}\right)}=\dfrac{21}{5}$

4. $\bar{x}=\dfrac{\int_0^7 x(1+x^3)dx}{\int_0^7 (1+x^3)dx}=\dfrac{\left[\frac{1}{2}x^2+\frac{1}{5}x^5\right]_0^7}{\left[x+\frac{1}{4}x^4\right]_0^7}$

$=\dfrac{\left(\frac{49}{2}+\frac{16,807}{5}\right)}{\left(7+\frac{2401}{4}\right)}=\dfrac{\frac{33,859}{10}}{\frac{2429}{4}}=\dfrac{9674}{1735}\approx 5.58$

5. $M_y=1\cdot 2+7\cdot 3+(-2)\cdot 4+(-1)\cdot 6+4\cdot 2=17$

$M_x=1\cdot 2+1\cdot 3+(-5)\cdot 4+0\cdot 6+6\cdot 2=-3$

$m=2+3+4+6+2=17$

$\bar{x}=\dfrac{M_y}{m}=1, \bar{y}=\dfrac{M_x}{m}=-\dfrac{3}{17}$

6. $M_y=(-3)\cdot 5+(-2)\cdot 6+3\cdot 2+4\cdot 7+7\cdot 1=14$

$M_x=2\cdot 5+(-2)\cdot 6+5\cdot 2+3\cdot 7+(-1)\cdot 1=28$

$m=5+6+2+7+1=21$

$\bar{x}=\dfrac{M_y}{m}=\dfrac{2}{3}, \bar{y}=\dfrac{M_x}{m}=\dfrac{4}{3}$

7. Consider two regions $R_1$ and $R_2$ such that $R_1$ is bounded by *f(x)* and the *x*-axis, and $R_2$ is bounded by *g(x)* and the *x*-axis. Let $R_3$ be the region formed by $R_1-R_2$. Make a regular partition of the homogeneous region $R_3$ such that each sub-region is of width , $\Delta x$ and let $x$ be the distance from the *y*-axis to the center of mass of a sub-region. The heights of $R_1$ and $R_2$ at $x$ are approximately *f(x)* and *g(x)* respectively. The mass of $R_3$ is approximately

$\Delta m=\Delta m_1-\Delta m_2$

$\approx \delta f(x)\Delta x-\delta g(x)\Delta x$

$=\delta[f(x)-g(x)]\Delta x$

where $\delta$ is the density. The moments for $R_3$ are approximately

$M_x=M_x(R_1)-M_x(R_2)$

$\approx \frac{\delta}{2}[f(x)]^2\Delta x-\frac{\delta}{2}[g(x)]^2\Delta x$

$=\frac{\delta}{2}\left[(f(x))^2-(g(x))^2\right]\Delta x$

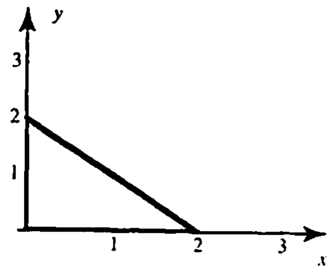
$M_y=M_y(R_1)-M_y(R_2)$

$\approx x\delta f(x)\Delta x-x\delta g(x)\Delta x$

$=x\delta[f(x)-g(x)]\Delta x$

Taking the limit of the regular partition as $\Delta x\to 0$ yields the resulting integrals in Figure 10.

8.




$f(x)=2-x; g(x)=0$

$\bar{x}=\dfrac{\int_0^2 x[(2-x)-0]dx}{\int_0^2[(2-x)-0]dx}$

$=\dfrac{\int_0^2[2x-x^2]dx}{\int_0^2[2-x]dx}$

$=\dfrac{(x^2-\frac{1}{3}x^3)_0^2}{(2x-\frac{1}{2}x^2)_0^2}=\dfrac{4-\frac{8}{3}}{4-2}$

$=\dfrac{2}{3}$

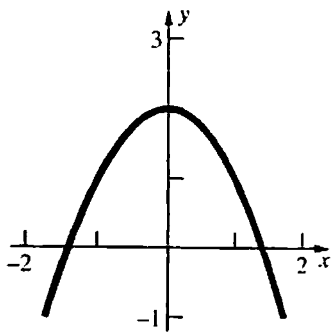
$\bar{y}=\dfrac{\frac{1}{2}\int_0^2[(2-x)^2-0^2]dx}{\int_0^2[(2-x)-0]dx}$

$=\dfrac{\int_0^2[4-4x+x^2]dx}{4}$

$=\dfrac{(4x-2x^2+\frac{1}{3}x^3)_0^2}{4}=\dfrac{8-8+\frac{8}{3}}{4}$

$=\dfrac{2}{3}$

9.



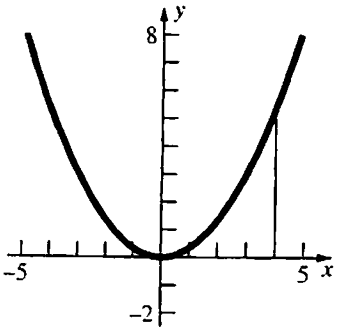

$\bar{x}=0$ (by symmetry)

$\bar{y}=\dfrac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}(2-x^2)^2dx}{\int_{-\sqrt{2}}^{\sqrt{2}}(2-x^2)dx}$

$$= \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (4-4x^2+x^4)dx}{\left[2x-\frac{1}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$= \frac{\frac{1}{2}\left[4x-\frac{4}{3}x^3+\frac{1}{5}x^5\right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{32\sqrt{2}}{15}}{\frac{8\sqrt{2}}{3}} = \frac{4}{5}$$

10.



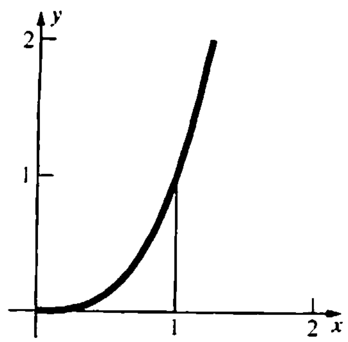

$$\bar{x} = \frac{\int_0^4 x\left(\frac{1}{3}x^2\right)dx}{\int_0^4 \frac{1}{3}x^2 dx} = \frac{\frac{1}{3}\int_0^4 x^3 dx}{\frac{1}{3}\int_0^4 x^2 dx}$$

$$= \frac{\frac{1}{3}\left[\frac{1}{4}x^4\right]_0^4}{\frac{1}{3}\left[\frac{1}{3}x^3\right]_0^4} = \frac{\frac{64}{3}}{\frac{64}{9}} = 3$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^4 \left(\frac{1}{3}x^2\right)^2 dx}{\int_0^4 \frac{1}{3}x^2 dx} = \frac{\frac{1}{18}\int_0^4 x^4 dx}{\frac{64}{9}} = \frac{\frac{1}{18}\left[\frac{1}{5}x^5\right]_0^4}{\frac{64}{9}}$$

$$= \frac{\frac{512}{45}}{\frac{64}{9}} = \frac{8}{5}$$

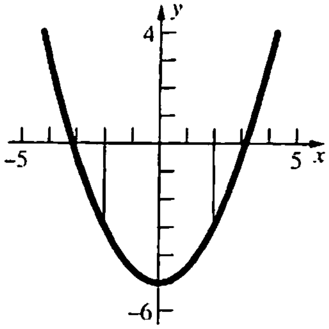
11.




$$\bar{x} = \frac{\int_0^1 x(x^3)dx}{\int_0^1 x^3 dx} = \frac{\int_0^1 x^4 dx}{\left[\frac{1}{4}x^4\right]_0^1} = \frac{\left[\frac{1}{5}x^5\right]_0^1}{\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^1 (x^3)^2 dx}{\int_0^1 x^3 dx} = \frac{\frac{1}{2}\int_0^1 x^6 dx}{\frac{1}{4}} = \frac{\left[\frac{1}{14}x^7\right]_0^1}{\frac{1}{4}} = \frac{\frac{1}{14}}{\frac{1}{4}} = \frac{2}{7}$$

12.



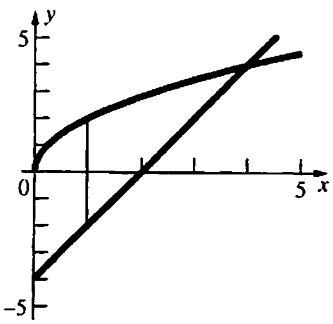

$\bar{x} = 0$ (by symmetry)

$$\bar{y} = \frac{\frac{1}{2}\int_{-2}^{2}\left[-\left(\frac{1}{2}(x^2-10)\right)^2\right]dx}{\int_{-2}^{2}\left[-\frac{1}{2}(x^2-10)\right]dx}$$

$$= \frac{-\frac{1}{8}\int_{-2}^{2}(x^4-20x^2+100)dx}{-\frac{1}{2}\int_{-2}^{2}(x^2-10)dx}$$

$$= \frac{-\frac{1}{8}\left[\frac{1}{5}x^5-\frac{20}{3}x^3+100x\right]_{-2}^{2}}{-\frac{1}{2}\left[\frac{1}{3}x^3-10x\right]_{-2}^{2}} = \frac{-\frac{574}{15}}{\frac{52}{3}} = -\frac{287}{130}$$

13.




To find the intersection point, solve

$2x-4 = 2\sqrt{x}$.

$x-2 = \sqrt{x}$

$x^2-4x+4 = x$

$x^2-5x+4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ ($x = 1$ is extraneous.)

$$\bar{x} = \frac{\int_1^4 x\left[2\sqrt{x}-(2x-4)\right]dx}{\int_1^4 \left[2\sqrt{x}-(2x-4)\right]dx}$$

$$= \frac{2\int_1^4 (x^{3/2}-x^2+2x)dx}{2\int_1^4 (x^{1/2}-x+2)dx}$$

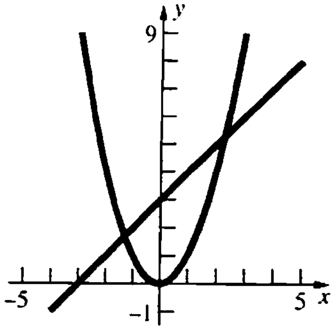
$$= \frac{2\left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^3+x^2\right]_1^4}{2\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_1^4} = \frac{\frac{64}{5}}{\frac{19}{3}} = \frac{192}{95}$$

$$\bar{y}=\frac{\frac{1}{2}\int_1^4\left[\left(2\sqrt{x}\right)^2-(2x-4)^2\right]dx}{\int_1^4\left[2\sqrt{x}-(2x-4)\right]dx}$$

$$=\frac{2\int_1^4(-x^2+5x-4)dx}{\frac{19}{3}}$$

$$=\frac{2\left[-\frac{1}{3}x^3+\frac{5}{2}x^2-4x\right]_1^4}{\frac{19}{3}}=\frac{9}{\frac{19}{3}}=\frac{27}{19}$$

14.




To find the intersection points, $x^2=x+3$.

$x^2-x-3=0$

$x=\frac{1\pm\sqrt{13}}{2}$

$$\bar{x}=\frac{\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}x(x+3-x^2)dx}{\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}(x+3-x^2)dx}$$

$$=\frac{\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}(x^2+3x-x^3)dx}{\left[\frac{1}{2}x^2+3x-\frac{1}{3}x^3\right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}}$$

$$=\frac{\left[\frac{1}{3}x^3+\frac{3}{2}x^2-\frac{1}{4}x^4\right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}}{\frac{13\sqrt{13}}{6}}=\frac{\frac{13\sqrt{3}}{12}}{\frac{13\sqrt{13}}{6}}=\frac{1}{2}$$

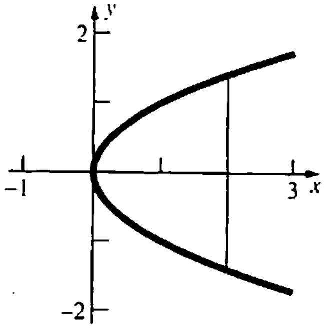
$$\bar{y}=\frac{\frac{1}{2}\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}\left[(x+3)^2-(x^2)^2\right]dx}{\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}(x+3-x^2)dx}$$

$$=\frac{\frac{1}{2}\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}(x^2+6x+9-x^4)}{\frac{13\sqrt{13}}{6}}$$

$$=\frac{\frac{1}{2}\left[\frac{1}{3}x^3+3x^2+9x-\frac{1}{5}x^5\right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}}}{\frac{13\sqrt{13}}{6}}$$

$$=\frac{\frac{143\sqrt{13}}{30}}{\frac{13\sqrt{13}}{6}}=\frac{11}{5}$$

15.




To find the intersection points, solve $y^2=2$.

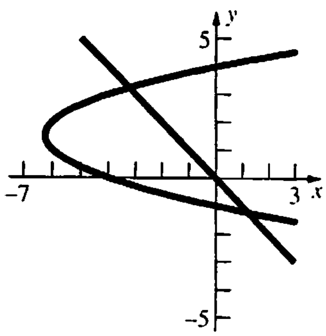
$y=\pm\sqrt{2}$

$$\bar{x}=\frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\left[2^2-(y^2)^2\right]dy}{\int_{-\sqrt{2}}^{\sqrt{2}}(2-y^2)dy}=\frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}(4-y^4)dy}{\left[2y-\frac{1}{3}y^3\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$=\frac{\frac{1}{2}\left[4y-\frac{1}{5}y^5\right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}}=\frac{\frac{16\sqrt{2}}{5}}{\frac{8\sqrt{2}}{3}}=\frac{6}{5}$$

$\bar{y}=0$ (by symmetry)

16.




To find the intersection points, solve

$y^2-3y-4=-y$.

$y^2-2y-4=0$

$y=\frac{2\pm\sqrt{20}}{2}$

$y = 1 \pm \sqrt{5}$

$$\bar{x} = \frac{\frac{1}{2}\int_{1-\sqrt{5}}^{1+\sqrt{5}}\left[(-y)^2 - (y^2 - 3y - 4)^2\right]dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}}\left[(-y) - (y^2 - 3y - 4)\right]dy}$$

$$= \frac{\frac{1}{2}\int_{1-\sqrt{5}}^{1+\sqrt{5}}(-y^4 + 6y^3 - 24y - 16)dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}}(-y^2 + 2y + 4)dy}$$

$$= \frac{\frac{1}{2}\left[-\frac{1}{5}y^5 + \frac{3}{2}y^4 - 12y^2 - 16y\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\left[-\frac{1}{3}y^3 + y^2 + 4y\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}$$

$$= \frac{-20\sqrt{5}}{\frac{20\sqrt{5}}{3}} = -3$$

$$\bar{y} = \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}} y\left[(-y) - (y^2 - 3y - 4)\right]dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}}\left[(-y) - (y^2 - 3y - 4)\right]dy}$$

$$= \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}}(-y^3 + 2y^2 + 4y)dy}{\frac{20\sqrt{5}}{3}}$$

$$= \frac{\left[-\frac{1}{4}y^4 + \frac{2}{3}y^3 + 2y^2\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\frac{20\sqrt{5}}{3}} = \frac{\frac{20\sqrt{5}}{3}}{\frac{20\sqrt{5}}{3}} = 1$$

**17.** We let $\delta$ be the density of the regions and $A_i$ be the area of region $i$.

Region $R_1$:

$$m(R_1) = \delta A_1 = \delta(1/2)(1)(1) = \frac{1}{2}\delta$$

$$\bar{x}_1 = \frac{\int_0^1 x(x)dx}{\int_0^1 xdx} = \frac{\frac{1}{3}x^3\Big|_0^1}{\frac{1}{2}x^2\Big|_0^1} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Since $R_1$ is symmetric about the line $y = 1 - x$, the centroid must lie on this line. Therefore,

$\bar{y}_1 = 1 - \bar{x}_1 = 1 - \frac{2}{3} = \frac{1}{3}$; and we have

$$M_y(R_1) = \bar{x}_2 \cdot m(R_1) = \frac{1}{3}\delta$$

$$M_x(R_1) = \bar{y}_2 \cdot m(R_1) = \frac{1}{6}\delta$$

Region $R_2$:

$m(R_2) = \delta A_2 = \delta(2)(1) = 2\delta$

By symmetry we get

$\bar{x}_2 = 2$ and $\bar{y}_2 = \frac{1}{2}$.

Thus,

$M_y(R_2) = \bar{x}_2 \cdot m(R_2) = 4\delta$

$M_x(R_2) = \bar{y}_2 \cdot m(R_2) = \delta$

**18.** We can obtain the mass and moments for the whole region by adding the individual regions. Using the results from Problem 17 we get that

$$m = m(R_1) + m(R_2) = \frac{1}{2}\delta + 2\delta = \frac{5}{2}\delta$$

$$M_y = M_y(R_1) + M_y(R_2) = \frac{1}{3}\delta + 4\delta = \frac{13}{3}\delta$$

$$M_x = M_x(R_1) + M_x(R_2) = \frac{1}{6}\delta + \delta = \frac{7}{6}\delta$$

Therefore, the centroid is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{13}{3}\delta}{\frac{5}{2}\delta} = \frac{26}{15}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{7}{6}\delta}{\frac{5}{2}\delta} = \frac{7}{15}$$

**19.** $m(R_1) = \delta\int_a^b (g(x) - f(x))dx$

$m(R_2) = \delta\int_b^c (g(x) - f(x))dx$

$M_x(R_1) = \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx$

$M_x(R_2) = \frac{\delta}{2}\int_b^c ((g(x))^2 - (f(x))^2)dx$

$M_y(R_1) = \delta\int_a^b x(g(x) - f(x))dx$

$M_y(R_2) = \delta\int_b^c x(g(x) - f(x))dx$

Now,

$$m(R_3) = \delta\int_a^c (g(x) - f(x))dx$$
$$= \delta\int_a^b (g(x) - f(x))dx + \delta\int_b^c (g(x) - f(x))dx$$
$$= m(R_1) + m(R_2)$$

$$M_x(R_3) = \frac{\delta}{2}\int_a^c ((g(x))^2 - (f(x))^2)dx$$
$$= \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx$$
$$+ \frac{\delta}{2}\int_b^c ((g(x))^2 - (f(x))^2)dx$$
$$= M_x(R_1) + M_x(R_2)$$

$$M_y(R_3) = \delta \int_a^c x(g(x) - f(x))dx$$
$$= \delta \int_a^b x(g(x) - f(x))dx$$
$$+\delta \int_b^c x(g(x) - f(x))dx$$
$$= M_y(R_1) + M_y(R_2)$$

**20.** $m(R_1) = \delta \int_a^b (h(x) - g(x))dx$

$$m(R_2) = \delta \int_a^b (g(x) - f(x))dx$$
$$M_x(R_1) = \frac{\delta}{2} \int_a^b ((h(x))^2 - (g(x))^2)dx$$
$$M_x(R_2) = \frac{\delta}{2} \int_a^b ((g(x))^2 - (f(x))^2)dx$$
$$M_y(R_1) = \delta \int_a^b x(h(x) - g(x))dx$$
$$M_y(R_2) = \delta \int_a^b x(g(x) - f(x))dx$$

Now,

$$m(R_3) = \delta \int_a^b (h(x) - f(x))dx$$
$$= \delta \int_a^b (h(x) - g(x) + g(x) - f(x))dx$$
$$= \delta \int_a^b (h(x) - g(x))dx + \delta \int_a^b (g(x) - f(x))dx$$
$$= m(R_1) + m(R_2)$$
$$M_x(R_3) = \frac{\delta}{2} \int_a^b ((h(x))^2 - (f(x))^2)dx$$
$$= \frac{\delta}{2} \int_a^b ((h(x))^2 - (g(x))^2 + (g(x))^2 - (f(x))^2)dx$$
$$= \frac{\delta}{2} \int_a^b ((h(x))^2 - (g(x))^2)dx$$
$$+\frac{\delta}{2} \int_a^b ((g(x))^2 - (f(x))^2)dx$$
$$= M_x(R_1) + M_x(R_2)$$
$$M_y(R_3) = \delta \int_a^b x(h(x) - f(x))dx$$
$$= \delta \int_a^b x(h(x) - g(x) + g(x) - f(x))dx$$
$$= \delta \int_a^b x(h(x) - g(x))dx + \delta \int_a^b x(g(x) - f(x))dx$$
$$= M_y(R_1) + M_y(R_2)$$

**21.** Let region 1 be the region bounded by $x = -2$, $x = 2, y = 0$, and $y = 1$, so $m_1 = 4 \cdot 1 = 4$.

By symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = \frac{1}{2}$. Therefore $M_{1y} = \bar{x}_1 m_1 = 0$ and $M_{1x} = \bar{y}_1 m_1 = 2$.

Let region 2 be the region bounded by $x = -2$, $x = 1, y = -1$, and $y = 0$, so $m_2 = 3 \cdot 1 = 3$.

By symmetry, $\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -\frac{1}{2}$. Therefore $M_{2y} = \bar{x}_2 m_2 = -\frac{3}{2}$ and $M_{2x} = \bar{y}_2 m_2 = -\frac{3}{2}$.

$$\bar{x} = \frac{M_{1y} + M_{2y}}{m_1 + m_2} = \frac{-\frac{3}{2}}{7} = -\frac{3}{14}$$
$$\bar{y} = \frac{M_{1x} + M_{2x}}{m_1 + m_2} = \frac{\frac{1}{2}}{7} = \frac{1}{14}$$

**22.** Let region 1 be the region bounded by $x = -3$, $x = 1, y = -1$, and $y = 4$, so $m_1 = 20$. By symmetry, $\bar{x} = -1$ and $\bar{y}_1 = \frac{3}{2}$. Therefore, $M_{1y} = \bar{x}_1 m_1 = -20$ and $M_{1x} = \bar{y}_1 m_1 = 30$. Let region 2 be the region bounded by $x = -3, x = -2$, $y = -3$, and $y = -1$, so $m_2 = 2$. By symmetry, $\bar{x}_2 = -\frac{5}{2}$ and $\bar{y}_2 = -2$. Therefore, $M_{2y} = \bar{x}_2 m_2 = -5$ and $M_{2x} = \bar{y}_2 m_2 = -4$. Let region 3 be the region bounded by $x = 0, x = 1$, $y = -2$, and $y = -1$, so $m_3 = 1$. By symmetry, $\bar{x}_3 = \frac{1}{2}$ and $\bar{y}_3 = -\frac{3}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = \frac{1}{2}$ and $M_{3x} = \bar{y}_3 m_3 = -\frac{3}{2}$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{-\frac{49}{2}}{23} = -\frac{49}{46}$$
$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{\frac{49}{2}}{23} = \frac{49}{46}$$

**23.** Let region 1 be the region bounded by $x = -2$, $x = 2, y = 2$, and $y = 4$, so $m_1 = 4 \cdot 2 = 8$. By symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = 3$. Therefore, $M_{1y} = \bar{x}_1 m_1 = 0$ and $M_{1x} = \bar{y}_1 m_1 = 24$. Let region 2 be the region bounded by $x = -1$, $x = 2, y = 0$, and $y = 2$, so $m_2 = 3 \cdot 2 = 6$. By symmetry, $\bar{x}_2 = \frac{1}{2}$ and $\bar{y}_2 = 1$. Therefore, $M_{2y} = \bar{x}_2 m_2 = 3$ and $M_{2x} = \bar{y}_2 m_2 = 6$. Let region 3 be the region bounded by $x = 2, x = 4$, $y = 0$, and $y = 1$, so $m_3 = 2 \cdot 1 = 2$. By symmetry, $\bar{x}_3 = 3$ and $\bar{y}_2 = \frac{1}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = 6$ and $M_{3x} = \bar{y}_3 m_3 = 1$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{9}{16}$$
$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{31}{16}$$

24. Let region 1 be the region bounded by $x = -3$, $x = -1$, $y = -2$, and $y = 1$, so $m_1 = 6$. By symmetry, $\bar{x}_1 = -2$ and $\bar{y}_1 = -\frac{1}{2}$. Therefore, $M_{1y} = \bar{x}_1 m_1 = -12$ and $M_{1x} = \bar{y}_1 m_1 = -3$. Let region 2 be the region bounded by $x = -1$, $x = 0$, $y = -2$, and $y = 0$, so $m_2 = 2$. By symmetry, $\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -1$. Therefore, $M_{2y} = \bar{x}_2 m_2 = -1$ and $M_{2x} = \bar{y}_2 m_2 = -2$. Let region 3 be the remaining region, so $m_3 = 22$. By symmetry, $\bar{x}_3 = 2$ and $\bar{y}_3 = -\frac{1}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = 44$ and $M_{3x} = \bar{y}_3 m_3 = -11$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{31}{30}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = -\frac{16}{30} = -\frac{8}{15}$$

25. $A = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$

From Problem 11, $\bar{x} = \frac{4}{5}$.

$V = A(2\pi\bar{x}) = \frac{1}{4}\left(2\pi \cdot \frac{4}{5}\right) = \frac{2\pi}{5}$

Using cylindrical shells:

$$V = 2\pi \int_0^1 x \cdot x^3 dx = 2\pi \int_0^1 x^4 dx = 2\pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{5}$$

26. The area of the region is $\pi a^2$. The centroid is the center (0, 0) of the circle. It travels a distance of $2\pi(2a) = 4\pi a$. $V = 4\pi^2 a^3$

27. The volume of a sphere of radius $a$ is $\frac{4}{3}\pi a^3$. If the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about the $x$-axis the result is a sphere of radius $a$. The centroid of the region travels a distance of $2\pi\bar{y}$.

The area of the region is $\frac{1}{2}\pi a^2$. Pappus's Theorem says that

$(2\pi\bar{y})\left(\frac{1}{2}\pi a^2\right) = \pi^2 a^2 \bar{y} = \frac{4}{3}\pi a^3$.

$\bar{y} = \frac{4a}{3\pi}$, $\bar{x} = 0$ (by symmetry)

28. Consider a slice at $x$ rotated about the $y$-axis.

$\Delta V = 2\pi x h(x)\Delta x$, so $V = 2\pi \int_a^b xh(x)dx$.

$\Delta m \approx h(x)\Delta x$, so $m = \int_a^b h(x)dx = A$.

$\Delta M_y \approx xh(x)\Delta x$ so $M_y = \int_a^b xh(x)dx$.

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b xh(x)dx}{A}$$

The distance traveled by the centroid is $2\pi\bar{x}$.

$(2\pi\bar{x})A = 2\pi \int_a^b xh(x)dx$

Therefore, $V = 2\pi\bar{x}A$.

29. a. $\Delta V \approx 2\pi(K - y)w(y)\Delta y$

$V = 2\pi \int_c^d (K - y)w(y)dy$

b. $\Delta m \approx w(y)\Delta y$, so $m = \int_c^d w(y)dy = A$.

$\Delta M_x \approx yw(y)\Delta y$, so $M_x = \int_c^d yw(y)dy$.

$$\bar{y} = \frac{\int_c^d yw(y)dy}{A}$$

The distance traveled by the centroid is $2\pi(K - \bar{y})$.

$2\pi(K - \bar{y})A = 2\pi(KA - M_x)$

$= 2\pi\left(\int_c^d Kw(y)dy - \int_c^d yw(y)dy\right)$

$= 2\pi \int_c^d (K - y)w(y)dy$

Therefore, $V = 2\pi(K - \bar{y})A$.

30. a. $m = \frac{1}{2}bh$

The length of a segment at $y$ is $b - \frac{b}{h}y$.

$\Delta M_x \approx y\left(b - \frac{b}{h}y\right)\Delta y = \left(by - \frac{b}{h}y^2\right)\Delta y$

$M_x = \int_0^h \left(by - \frac{b}{h}y^2\right)dy$

$= \left[\frac{1}{2}by^2 - \frac{b}{3h}y^3\right]_0^h = \frac{1}{6}bh^2$

$\bar{y} = \frac{M_x}{m} = \frac{h}{3}$

b. $A = \frac{1}{2}bh$; the distance traveled by the centroid is $2\pi\left(k - \frac{h}{3}\right)$.

$V = 2\pi\left(k - \frac{h}{3}\right)\left(\frac{1}{2}bh\right) = \frac{\pi bh}{3}(3k - h)$

31. a. The area of a regular polygon $P$ of $2n$ sides is $2r^2 n \sin\frac{\pi}{2n}\cos\frac{\pi}{2n}$. (To find this consider the isosceles triangles with one vertex at the center of the polygon and the other vertices on adjacent corners of the polygon. Each such triangle has base of length $2r\sin\frac{\pi}{2n}$ and height $r\cos\frac{\pi}{2n}$.) Since $P$ is a regular polygon the centroid is at its center. The distance from the centroid to any side is $r\cos\frac{\pi}{2n}$, so the centroid travels a distance of $2\pi r\cos\frac{\pi}{2n}$.

Thus, by Pappus's Theorem, the volume of the resulting solid is

$$\left(2\pi r\cos\frac{\pi}{2n}\right)\left(2r^2 n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}\right)$$

$$= 4\pi r^3 n\sin\frac{\pi}{2n}\cos^2\frac{\pi}{2n}.$$

b. $\lim_{n\to\infty} 4\pi r^3 n\sin\frac{\pi}{2n}\cos^2\frac{\pi}{2n}$

$$\lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}2\pi^2 r^3\cos^2\frac{\pi}{2n} = 2\pi^2 r^3$$

As $n\to\infty$, the regular polygon approaches a circle. Using Pappus's Theorem on the circle of area $\pi r^2$ whose centroid (= center) travels a distance of $2\pi r$, the volume of the solid is $(\pi r^2)(2\pi r) = 2\pi^2 r^3$ which agrees with the results from the polygon.

32. a. The graph of $f(\sin x)$ on $[0, \pi]$ is symmetric about the line $x = \frac{\pi}{2}$ since $f(\sin x) = f(\sin(\pi - x))$. Thus $\bar{x} = \frac{\pi}{2}$.

$$\bar{x} = \frac{\int_0^\pi x f(\sin x)\,dx}{\int_0^\pi f(\sin x)dx} = \frac{\pi}{2}$$

Therefore

$$\int_0^\pi x f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$$

b. $\sin x\cos^4 x = \sin x(1-\sin^2 x)^2$, so $f(x) = x(1-x^2)^2$.

$$\int_0^\pi x\sin x\cos^4 x\,dx = \frac{\pi}{2}\int_0^\pi \sin x\cos^4 x\,dx$$

$$= \frac{\pi}{2}\left[-\frac{1}{5}\cos^5 x\right]_0^\pi = \frac{\pi}{5}$$

33. Consider the region $S - R$.

$$\bar{y}_{S-R} = \frac{\frac{1}{2}\int_0^1\left[g^2(x) - f^2(x)\right]dx}{S-R} \ge \bar{y}_R$$

$$= \frac{\frac{1}{2}\int_0^1 f^2(x)dx}{R}$$

$$\frac{1}{2}R\int_0^1\left[g^2(x)-f^2(x)\right]dx \ge \frac{1}{2}(S-R)\int_0^1 f^2(x)dx$$

$$\frac{1}{2}R\int_0^1\left[g^2(x)-f^2(x)\right]dx + \frac{1}{2}R\int_0^1 f^2(x)dx$$

$$\ge \frac{1}{2}(S-R)\int_0^1 f^2(x)dx + \frac{1}{2}R\int_0^1 f^2(x)dx$$

$$\frac{1}{2}R\int_0^1 g^2(x)dx \ge \frac{1}{2}S\int_0^1 f^2(x)dx$$

$$\frac{\frac{1}{2}\int_0^1 g^2(x)\,dx}{S} \ge \frac{\frac{1}{2}\int_0^1 f^2(x)dx}{R}$$

$$\bar{y}_S \ge \bar{y}_R$$

34. To approximate the centroid, we can lay the figure on the x-axis (flat side down) and put the shortest side against the y-axis. Next we can use the eight regions between measurements to approximate the centroid. We will let $h_i$, the height of the $i$th region, be approximated by the height at the right end of the interval. Each interval is of width $\Delta x = 5$ cm. The centroid can be approximated as

$$\bar{x} \approx \frac{\sum_{i=1}^8 x_i h_i}{\sum_{i=1}^8 h_i} = \frac{(5)(6.5)+(10)(8)+\cdots+(35)(10)+(40)(8)}{6.5+8+\cdots+10+8}$$

$$= \frac{1695}{72.5} \approx 23.38$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^8 (h_i)^2}{\sum_{i=1}^8 h_i} = \frac{(1/2)(6.5^2+8^2+\cdots+10^2+8^2)}{(6.5+8+\cdots+10+8)}$$

$$= \frac{335.875}{72.5} \approx 4.63$$

35. First we place the lamina so that the origin is centered inside the hole. We then recompute the centroid of Problem 34 (in this position) as

$$\bar{x} \approx \frac{\sum_{i=1}^{8} x_i h_i}{\sum_{i=1}^{8} h_i} = \frac{(-25)(6.5)+(-15)(8)+\cdots+(5)(10)+(10)(8)}{6.5+8+\cdots+10+8}$$

$$= \frac{-480}{72.5} \approx -6.62$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{8}((h_i-4)^2-(-4)^2)}{\sum_{i=1}^{8} h_i}$$

$$= \frac{(1/2)((2.5^2-(-4)^2)+\cdots+(4^2-(-4)^2))}{6.5+8+\cdots+10+8}$$

$$= \frac{45.875}{72.5} \approx 0.633$$

A quick computation will show that these values agree with those in Problem 34 (using a different reference point).

Now consider the whole lamina as $R_3$, the circular hole as $R_2$, and the remaining lamina as $R_1$. We can find the centroid of $R_1$ by noting that

$M_x(R_1) = M_x(R_3) - M_x(R_2)$

and similarly for $M_y(R_1)$.

From symmetry, we know that the centroid of a circle is at the center. Therefore, both $M_x(R_2)$ and $M_y(R_2)$ must be zero in our case. This leads to the following equations

$$\bar{x} = \frac{M_y(R_3)-M_y(R_2)}{m(R_3)-m(R_2)}$$

$$= \frac{\delta\Delta x(-480)}{\delta\Delta x(72.5)-\delta\pi(2.5)^2}$$

$$= \frac{-2400}{342.87} \approx -7$$

$$\bar{y} = \frac{M_x(R_3)-M_x(R_2)}{m(R_3)-m(R_2)}$$

$$= \frac{\delta\Delta x(45.875)}{\delta\Delta x(72.5)-\delta\pi(2.5)^2}$$

$$= \frac{229.375}{342.87} \approx 0.669$$

36. This problem is much like Problem 34 except we don't have one side that is completely flat. In this problem, it will be necessary, in some regions, to find the value of $g(x)$ instead of just $f(x) - g(x)$. We will use the 19 regions in the figure to approximate the centroid. Again we choose the height of a region to be approximately the value at the right end of that region. Each region has a width of 20 miles. We will place the north-east corner of the state at the origin.
The centroid is approximately

$$\bar{x} \approx \frac{\sum_{i=1}^{19} x_i(f(x_i)-g(x_i))}{\sum_{i=1}^{19}(f(x_i)-g(x_i))}$$

$$= \frac{(20)(145-13)+(40)(149-10)+\cdots(380)(85-85)}{(145-13)+(149-19)+\cdots(85-85)}$$

$$= \frac{482,860}{2780} \approx 173.69$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{19}[(f(x_i))^2-(g(x_i))^2]}{\sum_{i=1}^{19}(f(x_i)-g(x_i))}$$

$$= \frac{\frac{1}{2}\left[(145^2-13^2)+(149^2-10^2)+\cdots+(85^2-85^2)\right]}{(145-13)+(149-19)+\cdots+(85-85)}$$

$$= \frac{230,805}{2780} \approx 83.02$$

This would put the geographic center of Illinois just south-east of Lincoln, IL.

## 6.7 Chapter Review

### Concepts Test

1. False: $\int_0^{\pi} \cos x \, dx = 0$ because half of the area lies above the $x$-axis and half below the $x$-axis.

2. True: The integral represents the area of the region in the first quadrant if the center of the circle is at the origin.

3. False: The statement would be true if either $f(x) \geq g(x)$ or $g(x) \geq f(x)$ for $a \leq x \leq b$. Consider Problem 1 with $f(x) = \cos x$ and $g(x) = 0$.

4. True: The area of a cross section of a cylinder will be the same in any plane parallel to the base.

5. True: Since the cross sections in all planes parallel to the bases have the same area, the integrals used to compute the volumes will be equal.

6. False: The volume of a right circular cone of radius $r$ and height $h$ is $\frac{1}{3}\pi r^2 h$. If the radius is doubled and the height halved the volume is $\frac{2}{3}\pi r^2 h$..

7. False: Using the method of shells, $V = 2\pi \int_0^1 x(-x^2 + x)dx$. To use the method of washers we need to solve $y = -x^2 + x$ for $x$ in terms of $y$.

8. True: The bounded region is symmetric about the line $x = \frac{1}{2}$. Thus the solids obtained by revolving about the lines $x = 0$ and $x = 1$ have the same volume.

9. False: Consider the curve given by $x = \frac{\cos t}{t}$, $y = \frac{\sin t}{t}, 2 \le t < \infty$.

10. False: The work required to stretch a spring 2 inches beyond its natural length is $\int_0^2 kx\,dx = 2k$, while the work required to stretch it 1 inch beyond its natural length is $\int_0^1 kx\,dx = \frac{1}{2}k$.

11. False: If the cone-shaped tank is placed with the point downward, then the amount of water that needs to be pumped from near the bottom of the tank is much less than the amount that needs to be pumped from near the bottom of the cylindrical tank.

12. True: $(100)(10) + (100)(15) + (200)(-12.5) = 0$

13. True: This is the definition of the center of mass.

14. True: The region is symmetric about the point ($\pi$, 0).

15. True: By symmetry, the centroid is on the line $x = \frac{\pi}{2}$, so the centroid travels a distance of $2\pi\left(\frac{\pi}{2}\right) = \pi^2$.

16. True: At slice $y$, $\Delta A \approx (9 - y^2)\Delta y$.

17. True: Since the density is proportional to the square of the distance from the midpoint, equal masses are on either side of the midpoint.

18. True: See Problem 30 in Section 6.6.

## Sample Test Problems

1. $A = \int_0^1 (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$

2. $V = \pi \int_0^1 (x - x^2)^2 dx$
$= \pi \int_0^1 (x^2 - 2x^3 + x^4)dx$
$= \pi\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{\pi}{30}$

3. $V = 2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$
$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{6}$

4. $V = \pi \int_0^1 \left[(x - x^2 + 2)^2 - (2)^2\right]dx$
$= \pi \int_0^1 (x^4 - 2x^3 - 3x^2 + 4x)\,dx$
$= \pi\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 2x^2\right]_0^1 = \frac{7\pi}{10}$

5. $V = 2\pi \int_0^1 (3 - x)(x - x^2)dx$
$= 2\pi \int_0^1 (x^3 - 4x^2 + 3x)\,dx$
$= 2\pi\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1 = \frac{5\pi}{6}$

6. $\bar{x} = \dfrac{\int_0^1 x(x - x^2)dx}{\int_0^1 (x - x^2)dx} = \dfrac{\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1}{\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1} = \dfrac{1}{2}$

$\bar{y} = \dfrac{\frac{1}{2}\int_0^1 (x - x^2)^2 dx}{\int_0^1 (x - x^2)dx} = \dfrac{\frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1}{\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1}$
$= \dfrac{1}{10}$

7. From Problem 1, $A = \frac{1}{6}$.

From Problem 6, $\bar{x} = \frac{1}{2}$ and $\bar{y} = \frac{1}{10}$.

$$V(S_1) = 2\pi\left(\frac{1}{10}\right)\left(\frac{1}{6}\right) = \frac{\pi}{30}$$

$$V(S_2) = 2\pi\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{\pi}{6}$$

$$V(S_3) = 2\pi\left(\frac{1}{10}+2\right)\left(\frac{1}{6}\right) = \frac{7\pi}{10}$$

$$V(S_4) = 2\pi\left(3-\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{5\pi}{6}$$

8. $8 = F(8) = 8k, k = 1$

a. $W = \int_2^8 x\,dx = \left[\frac{1}{2}x^2\right]_2^8 = \frac{1}{2}(64-4)$

$= 30$ in.-lb

b. $W = \int_0^{12} x\,dx = \left[\frac{1}{2}x^2\right]_0^{12} = 72$ in.-lb

9. $W = \int_0^6 (62.4)(5^2)\pi(10-y)dy$

$= 1560\pi \int_0^6 (10-y)dy$

$= 1560\pi\left[10y - \frac{1}{2}y^2\right]_0^6 = 65{,}520\pi \approx 205{,}837$ ft-lb

10. The total work is equal to the work $W_1$ to pull up the object to the top without the cable and the work $W_2$ to pull up the cable.

$W_1 = 200\cdot 100 = 20{,}000$ ft-lb

The cable weighs $\frac{120}{100} = \frac{6}{5}$ lb/ft.

$$\Delta W_2 = \frac{6}{5}\Delta y \cdot y = \frac{6}{5}y\Delta y$$

$$W_2 = \int_0^{100} \frac{6}{5}y\,dy = \frac{6}{5}\left[\frac{1}{2}y^2\right]_0^{100}$$

$= 6000$ ft-lb

$W = W_1 + W_2 = 26{,}000$ ft-lb

11. a. To find the intersection points, solve

$4x = x^2$.

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

$$A = \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4$$

$$= \left(32 - \frac{64}{3}\right) = \frac{32}{3}$$

b. To find the intersection points, solve

$\frac{y}{4} = \sqrt{y}$.

$\frac{y^2}{16} = y$

$y^2 - 16y = 0$

$y(y-16) = 0$

$y = 0, 16$

$$A = \int_0^{16}\left(\sqrt{y} - \frac{y}{4}\right)dy = \left[\frac{2}{3}y^{3/2} - \frac{1}{8}y^2\right]_0^{16}$$

$$= \left(\frac{128}{3} - 32\right) = \frac{32}{3}$$

12. $$\bar{x} = \frac{\int_0^4 x(4x-x^2)dx}{\int_0^4 (4x-x^2)dx} = \frac{\int_0^4 (4x^2 - x^3)dx}{\frac{32}{3}}$$

$$= \frac{\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4}{\frac{32}{3}} = \frac{\frac{64}{3}}{\frac{32}{3}} = 2$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^4 \left[(4x)^2 - (x^2)^2\right]dx}{\int_0^4 (4x-x^2)dx}$$

$$= \frac{\frac{1}{2}\int_0^4 (16x^2 - x^4)dx}{\frac{32}{3}}$$

$$= \frac{\frac{1}{2}\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4}{\frac{32}{3}} = \frac{\frac{1024}{15}}{\frac{32}{3}} = \frac{32}{5}$$

13. $$V = \pi\int_0^4 \left[(4x)^2 - (x^2)^2\right]dx$$

$$= \pi\int_0^4 (16x^2 - x^4)dx$$

$$= \pi\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4 = \frac{2048\pi}{15}$$

Using Pappus's Theorem:

From Problem 11, $A = \frac{32}{3}$.

From Problem 12, $\bar{y} = \frac{32}{5}$.

$$V = 2\pi\bar{y}\cdot A = 2\pi\left(\frac{32}{5}\right)\left(\frac{32}{3}\right) = \frac{2048\pi}{15}$$

14. $V = 2\pi\int_0^4 x(4x-x^2)dx$

$= 2\pi\int_0^4 (4x^2-x^3)dx$

$= 2\pi\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]_0^4 = \frac{128\pi}{3}$

Using Pappus's Theorem:

From Problem 11, $A = \frac{32}{3}$.

From Problem 12, $\bar{x} = 2$.

$V = 2\pi\bar{x}\cdot A = 2\pi(2)\left(\frac{32}{3}\right) = \frac{128\pi}{3}$

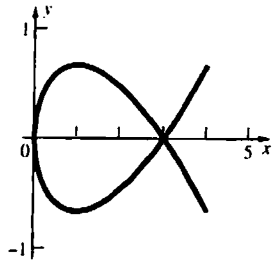
15. $\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$

$L = \int_1^3 \sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}\,dx$

$= \int_1^3 \sqrt{x^4+\frac{1}{2}+\frac{1}{16x^4}}\,dx = \int_1^3\left(x^2+\frac{1}{4x^2}\right)dx$

$= \left[\frac{1}{3}x^3-\frac{1}{4x}\right]_1^3 = \left(9-\frac{1}{12}\right)-\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{53}{6}$

16.




The loop is $-\sqrt{3} \le t \le \sqrt{3}$. By symmetry, we can double the length of the loop from $t = 0$ to $t = \sqrt{3}$. $\frac{dx}{dt} = 2t; \frac{dy}{dt} = t^2 - 1$

$L = 2\int_0^{\sqrt{3}} \sqrt{t^4+2t^2+1}\,dt = 2\int_0^{\sqrt{3}}(t^2+1)dt$

$= 2\left[\frac{1}{3}t^3+t\right]_0^{\sqrt{3}} = 4\sqrt{3}k$

17. $V = \int_{-3}^{3}\left(\sqrt{9-x^2}\right)^2 dx = \int_{-3}^{3}(9-x^2)dx$

$= \left[9x-\frac{1}{3}x^3\right]_{-3}^{3} = (27-9)-(-27+9) = 36$

18. $A = \int_a^b [f(x)-g(x)]dx$

19. $V = \pi\int_a^b\left[f^2(x)-g^2(x)\right]dx$

20. $V = 2\pi\int_a^b (x-a)[f(x)-g(x)]dx$

21. $M_y = \delta\int_a^b x[f(x)-g(x)]dx$

$M_x = \frac{\delta}{2}\int_a^b\left[f^2(x)-g^2(x)\right]dx$

22. $L_1 = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

$L_2 = \int_a^b \sqrt{1+[g'(x)]^2}\,dx$

$L_3 = f(a)-g(a)$

$L_4 = f(b)-g(b)$

Total length $= L_1+L_2+L_3+L_4$

23. $A_1 = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

$A_2 = 2\pi\int_a^b g(x)\sqrt{1+[g'(x)]^2}\,dx$

$A_3 = \pi\left[f^2(a)-g^2(a)\right]$

$A_4 = \pi\left[f^2(b)-g^2(b)\right]$

Total surface area $= A_1+A_2+A_3+A_4$.

## 6.8 Additional Problem Set

1. a. feet/second
   $F(T)$ is the change in velocity from 6 seconds to $T$ seconds.

   b. foot-pounds
   $F(s)$ is the change in work from 3 feet to $s$ feet.

   c. inches
   $F(r)$ is the center of mass of an object $r$ inches long whose mass at $x$ is $f(x)$.

2. This follows from the Fundamental Theorem of Calculus.

3. $F(t)=\int_0^t f(u)du$

a. $\lim_{t\to\infty} F(t)=\int_0^3 A\,du=\left[Au\right]_0^3=3A=1$

$A=\frac{1}{3}$

b. $\lim_{t\to\infty} F(t)=\int_0^7 A(u^2-7u)du$

$=A\left[\frac{1}{3}u^3-\frac{7}{2}u^2\right]_0^7=\frac{343}{6}A=1$

$A=\frac{6}{343}$

c. $\lim_{t\to\infty} F(t)=\int_0^3 Au^2du+\int_3^9 A(9-u)du$

$=A\left[\frac{1}{3}u^3\right]_0^3+A\left[9u-\frac{1}{2}u^2\right]_3^9$

$=9A+18A=27A=1$

$A=\frac{1}{27}$

4. a. $\lim_{t\to\infty} F(t)=\int_0^4 15Au^2(4-u)^2du$

$=15A\int_0^4(u^4-8u^3+16u^2)du$

$=15A\left[\frac{1}{5}u^5-2u^4+\frac{16}{3}u^3\right]_0^4$

$=15A\left(\frac{1024}{5}-512+\frac{1024}{3}\right)=512A=1$

$A=\frac{1}{512}$

b. $F(3)=\int_0^3\frac{15}{512}u^2(4-u)^2du$

$=\frac{15}{512}\left[\frac{1}{5}u^5-2u^4+\frac{16}{3}u^3\right]_0^3$

$=\frac{15}{512}\left(\frac{153}{5}\right)=\frac{459}{512}$

Pr(fail after 3 years) $=1-F(3)$

$=1-(\frac{459}{512})\approx 0.1035$

c. $F(t)=15A\int_0^t\left[x^2(16-8x+x^2)\right]dx+F(0)$

$=15A\int_0^t(16x^2-8x^3+x^4)dx+F(0)$

$=15A\left[\frac{16}{3}t^3-2t^4+\frac{1}{5}t^5\right]+F(0)$

From the initial conditions, we know that

$F(0)=0$ and that

$F(4)=1=15A\left[\frac{1024}{3}-512+\frac{1024}{5}\right]$

$=A[512]=1$

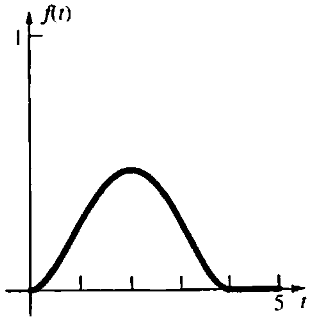
Therefore,

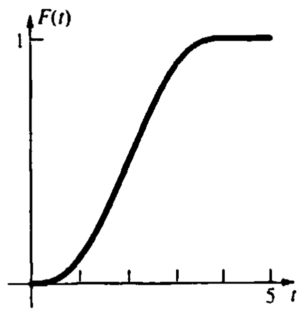
$A=\frac{1}{512}$

and we get:

$$F(t)=\begin{cases}\frac{15}{512}\left(\frac{1}{5}t^5-2t^4+\frac{16}{3}t^3\right) & 0\le t\le 4\\ 1 & t>4\end{cases}$$

d.




5. a. We can consider the expression

$\int_0^B tf(t)dt$

as a moment if we consider that the total mass of the region is one unit. The total area under the curve must equal one as well, and so we can consider the region to have a density of $\delta=1$. A moment is defined to be the product of a region's mass times it's directed distance from the rotation point. *f(t)* represents the height of small region and *dt* represents a very small change in t. *f(t)dt* can be considered as the area of a very small sliver of the region. Since the density is equal to 1, the quantity *f(t)dt* also represents the mass of the sliver. Multiplying by *t*, the direct distance from the y-axis, we obtain the moment for that sliver about the y-axis. The integral represents an infinite sum over all the slivers that make up the region, and represents the moment of the region about the y-axis.

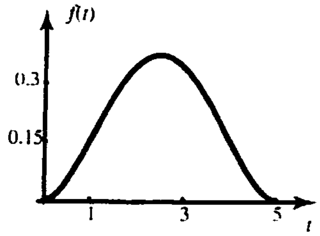
**b.** $E = \int_0^4 \frac{15}{512} t \left[ t^2 (4-t)^2 \right] dt$

$= \frac{15}{512} \int_0^4 (16t^3 - 8t^4 + t^5) dt$

$= \frac{15}{512} \left[ \frac{16}{4} t^4 - \frac{8}{5} t^5 + \frac{1}{6} t^6 \right]_0^4$

$= \frac{15}{512} \left[ 1024 - \frac{8192}{5} + \frac{4096}{6} \right] = 2$

**6. a.**




**b.** $F(t) = \int_0^t f(u) du$

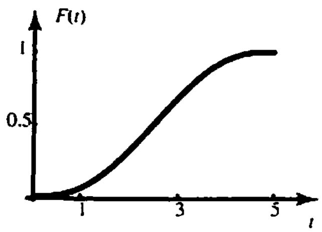
For $0 \le t \le 5$:

$F(t) = \frac{6}{625} \int_0^t u^2 (5-u)^2 du$

$= \frac{6}{625} \left( \frac{1}{5} t^5 - \frac{5}{2} t^4 + \frac{25}{3} t^3 \right)$

$= \frac{6}{3125} t^5 - \frac{3}{125} t^4 + \frac{2}{25} t^3$

$$F(t) = \begin{cases} \frac{6}{3125} t^5 - \frac{3}{125} t^4 + \frac{2}{25} t^3 & \text{if } 0 \le t \le 5 \\ 1 & t > 5 \end{cases}$$




**c.** $F(t \ge 3) = 1 - F(3)$

$= 1 - \frac{6}{3125} 3^5 + \frac{3}{125} 3^4 - \frac{2}{25} 3^3 = 0.31744$

**d.** We would expect
$100{,}000 \cdot 0.31744 = 31{,}744$
batteries to be working.

**7.** $F = (62.4)(10)(0.0004) = 0.2496$ lb
$p = (62.4)(10) = 624$ lb/ft$^2$

**8. a.** The total force is the sum of the forces at each depth. For example, the force will be approximately equal to the sum of the forces at every $\frac{1}{100}$ foot. We can approximate the force on the wall by using $n = 100$, $\Delta h = \frac{1}{100}$, and depth $= 10 - h_i$, where $h_i = \frac{i}{10}$ is the $i$th height. At a particular depth, the force on the wall will be approximately $F \approx 62.4 \cdot 75 \cdot (10 - h)(\Delta h)$

**b.** Given the coordinate system, the depth at $h$ is $(10 - h)$. On the interval [0, 10],

$\lim_{n \to \infty} 62.4 \cdot 75 \sum_{i=1}^{n} (10 - h_i) \Delta h_i$

$= 62.4 \cdot 75 \int_0^{10} (10 - h) dh$.

**9. a.** At height $h$, a strip $\Delta h$ has $F \approx 62.4(10 - h) \Delta h \cdot W(h)$.

Thus $F = 62.4 \int_0^{10} (10 - h) W(h) dh$.

**b.** $W(h) = \frac{2}{\sqrt{3}} h$ follows from the geometry of an equilateral triangle. (Similar triangles, 30-60-90 triangle, etc.)

**c.** $F = 62.4 \int_0^{10} (10 - h)(2h / \sqrt{3}) dh$

$= 62.4 \int_0^{10} \left( \frac{20}{\sqrt{3}} h - \frac{2}{\sqrt{3}} h^2 \right) dh$

$= 62.4 \left[ \frac{10}{\sqrt{3}} h^2 - \frac{2}{3\sqrt{3}} h^3 \right]_0^{10} \approx 12009$

**10.** Let $h$ measure the distance from the top of the plate, where $0 \le h \le 3\sqrt{3}$. This also gives the depth at that distance. The width of the plate at $h$ is given by $W(h) = -\frac{2}{\sqrt{3}}(h - 3\sqrt{3})$. At distance $h$ from the top, a strip $\Delta h$ has

$F \approx 62.4 h \Delta h \left[ -\frac{2}{\sqrt{3}} (h - 3\sqrt{3}) \right]$.

$F = -\frac{124.8}{\sqrt{3}} \int_0^{3\sqrt{3}} h(h - 3\sqrt{3}) dh$

$= -\frac{124.8}{\sqrt{3}} \left[ \frac{1}{3} h^3 - \frac{3\sqrt{3}}{2} h^2 \right]_0^{3\sqrt{3}}$

$= -\frac{124.8}{\sqrt{3}} \left( -\frac{27\sqrt{3}}{2} \right) = 1684.8$ lb

**11.** Let $h$ measure the distance from the top of the plate, where $0 \le h \le 3\sqrt{3}$. The depth at that distance is $h + 3$. The width of the plate at $h$ is given by $W(h) = -\frac{2}{\sqrt{3}}(h - 3\sqrt{3})$. At distance $h$ from the top a strip $\Delta h$ has

$$F \approx 62.4(h+3)\Delta h\left[-\frac{2}{\sqrt{3}}(h-3\sqrt{3})\right].$$

$$F = -\frac{124.8}{\sqrt{3}}\int_0^{3\sqrt{3}}(h+3)(h-3\sqrt{3})dh$$

$$= -\frac{124.8}{\sqrt{3}}\int_0^{3\sqrt{3}}[h^2 + 3(1-\sqrt{3})h - 9\sqrt{3}]dh$$

$$= -\frac{124.8}{\sqrt{3}}\left[\frac{1}{3}h^3 + \frac{3(1-\sqrt{3})}{2}h^2 - 9\sqrt{3}h\right]_0^{3\sqrt{3}}$$

$$= -\frac{124.8}{\sqrt{3}}\left[-\frac{27(3+\sqrt{3})}{2}\right]$$

$$= 1684.8(1+\sqrt{3}) \approx 4602.96 \text{ lb}$$

**12.** Let $y$ measure the distance from the bottom of the patch, where $0 \le y \le 2$. Let $l$ be the length of a side of the cube. The depth at $y$ is $l - y$. The width of the patch at $y$ is $2\sqrt{y}$. At $y$, a strip $\Delta y$ has

$$F \approx 62.4(l-y)\Delta y \cdot 2\sqrt{y}.$$

$$F = 124.8\int_0^2 \sqrt{y}(l-y)dy$$

$$= 124.8\int_0^2 (l\sqrt{y} - y^{3/2})dy$$

$$= 124.8\left[\frac{2l}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^2$$

$$= 33.28\sqrt{2}(5l-6)$$

$33.28\sqrt{2}(5l-6) = 200$, so $l \approx 2.05$ ft.

**13.** Let $y$ measure the distance from the bottom of the window, where $0 \le y \le 2$. Let $h$ be the height of the tank. The depth at $y$ is $h - y$. The width of the window at $y$ is $y$. At $y$, the strip $\Delta y$ has

$$F \approx 62.4(h-y)\Delta y \cdot y.$$

$$F = 62.4\int_0^2 y(h-y)dy$$

$$= 62.4\int_0^2 (hy - y^2)dy$$

$$= 62.4\left[\frac{h}{2}y^2 - \frac{1}{3}y^3\right]_0^2$$

$$= 62.4\left(2h - \frac{8}{3}\right)$$

$62.4\left(2h - \frac{8}{3}\right) = 400$, so $h \approx 4.54$ ft.

**14. a.** The mass of the car is

$\frac{3200 \text{ lb}}{32 \text{ ft/s}^2} = 100 \text{ lb-s}^2/\text{ft}.$

Since acceleration is uniform,

$a = \frac{66-0}{6} = 11 \text{ ft/s}^2$. Thus,

$F = ma = (100 \text{ lb-s}^2/\text{ft})(11 \text{ ft/s}^2) = 1100$ lb.

$v(t) = 11t$, so

$P(t) = F \cdot v(t) = (1100)(11t) = 12{,}100t$. The most power needed is at 6 seconds, so

$P(6) = 12{,}100 \cdot 6 = 72{,}600$ ft-lb/s.

$\frac{72{,}600 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 132$ hp is needed.

**b.** Since the amount of horsepower needed depends on the acceleration, we cannot necessarily determine the horsepower needed, but we can determine the minimum horsepower needed. To calculate the minimum horsepower needed, assume that power is constant. Then

$P = 100a(t)v(t)$. Since $a(t) = \frac{dv(t)}{dt}$,

$P\,dt = 100v(t)\,dv(t)$. Integrate both sides to get $Pt = 50v(t)^2 + C$. Since $v(0) = 0$, $C = 0$,

so $P = \frac{50v(t)^2}{t}$. At $t = 6$,

$P = \frac{50(66)^2}{6} = 36{,}300$ ft-lb/s.

$\frac{36{,}300 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 66$ hp is the minimum horsepower needed.

**15. a.** The additional force on the car is the component of the force due to gravity along the incline.

$F = 100 \cdot 32\sin(\tan^{-1} 0.05) \approx 159.8$ lb.

$P \approx 159.8 \cdot 66 = 10{,}546.8$ ft-lb/s

$\frac{10{,}546.8 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} \approx 19$ hp additional is required.

**b.** Since acceleration is not assumed to be uniform, we will calculate the minimum horsepower required, so we will assume that the additional power is constant while accelerating.

$P = 100a(t)v(t)$. Since $a(t) = \frac{dv(t)}{dt}$,

$P\,dt = 100v(t)\,dv(t)$.

Integrate both sides to get $Pt = 50v(t)^2 + C$.

60 mi/h = 88 ft/s

At $t = 0$, $v = 66$ and at $t = 4$, $v = 88$.

Therefore, $C = -217{,}800$ and

$P = \dfrac{50(88)^2 - 217{,}800}{4} = 42{,}350$ ft-lb/s.

$\dfrac{42{,}350 \text{ ft-lb/s}}{550 \text{ (ft-lb/s)/hp}} = 77$ hp additional is required. Note that since the car is already on the incline, we do not need to consider the power needed to overcome the force due to gravity.

If we assume uniform acceleration, then $a = \dfrac{88-66}{4} = 5.5 \text{ ft/s}^2$, so the additional force is $F = 100 \cdot 5.5 = 550$ lb. The most power is needed when $v = 88$ ft/s, so $P = 550 \cdot 88 = 48{,}400$ ft-lb/s.

$\dfrac{48{,}400 \text{ ft-lb/s}}{550 \text{ (ft-ls/2)/hp}} = 88$ hp additional is required.

c. The force due to gravity along the incline is $F = 100 \cdot 32 \sin(\tan^{-1} 0.05) \approx 159.8$ lb. The distance traveled is (66 ft/s)(180 s) = 11,880 ft. Thus the work is $W \approx 159.8 \cdot 11{,}800 = 1{,}898{,}424$ ft-lb