CHAPTER

5 The Integral

5.1 Concepts Review

1. $rx^{r-1}; \frac{x^{r+1}}{r+1}+C, r \neq -1$

2. $r[f(x)]^{r-1} f'(x); [f(x)]^r f'(x)$

3. $u = x^4+3x^2+1, du = (4x^3+6x)dx$
$\int (x^4+3x^2+1)^8(4x^3+6x)dx = \int u^8 du$
$= \frac{u^9}{9}+C = \frac{(x^4+3x^2+1)^9}{9}+C$

4. $c_1 \int f(x)dx + c_2 \int g(x)dx$

Problem Set 5.1

1. $\int 5dx = 5x + C$

2. $\int (x-4)dx = \int xdx - 4\int 1dx$
$= \frac{x^2}{2} - 4x + C$

3. $\int (x^2+\pi)dx = \int x^2 dx + \pi \int 1dx = \frac{x^3}{3} + \pi x + C$

4. $\int \left(3x^2+\sqrt{3}\right)dx = 3\int x^2 dx + \sqrt{3}\int 1dx$
$= 3\frac{x^3}{3} + \sqrt{3}x + C = x^3 + \sqrt{3}x + C$

5. $\int x^{5/4} dx = \frac{x^{9/4}}{\frac{9}{4}} + C = \frac{4}{9}x^{9/4} + C$

6. $\int 3x^{2/3} dx = 3\int x^{2/3} dx = 3\left(\frac{x^{5/3}}{\frac{5}{3}} + C_1\right)$
$= \frac{9}{5}x^{5/3} + C$

7. $\int \frac{1}{\sqrt[3]{x^2}} dx = \int x^{-2/3} dx = 3x^{1/3} + C = 3\sqrt[3]{x} + C$

8. $\int 7x^{-3/4} dx = 7\int x^{-3/4} dx = 7(4x^{1/4} + C_1)$
$= 28x^{1/4} + C$

9. $\int (x^2 - x)dx = \int x^2 dx - \int x\,dx = \frac{x^3}{3} - \frac{x^2}{2} + C$

10. $\int (3x^2 - \pi x)dx = 3\int x^2 dx - \pi \int x\,dx$
$= 3\left(\frac{x^3}{3} + C_1\right) - \pi\left(\frac{x^2}{2} + C_2\right)$
$= x^3 - \frac{\pi x^2}{2} + C$

11. $\int (4x^5 - x^3)dx = 4\int x^5 dx - \int x^3 dx$
$= 4\left(\frac{x^6}{6} + C_1\right) - \left(\frac{x^4}{4} + C_2\right)$
$= \frac{2x^6}{3} - \frac{x^4}{4} + C$

12. $\int (x^{100} + x^{99})dx = \int x^{100} dx + \int x^{99} dx$
$= \frac{x^{101}}{101} + \frac{x^{100}}{100} + C$

13. $\int (27x^7 + 3x^5 - 45x^3 + \sqrt{2}x)dx$
$= 27\int x^7 dx + 3\int x^5 dx - 45\int x^3 dx + \sqrt{2}\int x\,dx$
$= \frac{27x^8}{8} + \frac{x^6}{2} - \frac{45x^4}{4} + \frac{\sqrt{2}x^2}{2} + C$

14. $\int \left[x^2\left(x^3 + 5x^2 - 3x + \sqrt{3}\right)\right]dx$
$= \int \left(x^5 + 5x^4 - 3x^3 + \sqrt{3}x^2\right)dx$
$= \int x^5 dx + 5\int x^4 dx - 3\int x^3 dx + \sqrt{3}\int x^2 dx$
$= \frac{x^6}{6} + x^5 - \frac{3x^4}{4} + \frac{\sqrt{3}x^3}{3} + C$

15. $\int\left(\frac{3}{x^2}-\frac{2}{x^3}\right)dx = \int(3x^{-2}-2x^{-3})\,dx$
$= 3\int x^{-2}\,dx - 2\int x^{-3}\,dx$
$= \frac{3x^{-1}}{-1}-\frac{2x^{-2}}{-2}+C$
$= -\frac{3}{x}+\frac{1}{x^2}+C$

16. $\int\left(\frac{\sqrt{2x}}{x}+\frac{3}{x^5}\right)dx = \int\left(\sqrt{2}\,x^{-1/2}+3x^{-5}\right)dx$
$= \frac{\sqrt{2}\,x^{1/2}}{\frac{1}{2}}+\frac{3x^{-4}}{-4}+C$
$= 2\sqrt{2x}-\frac{3}{4x^4}+C$

17. $\int\frac{4x^6+3x^4}{x^3}dx = \int(4x^3+3x)\,dx$
$= 4\int x^3\,dx + 3\int x\,dx$
$= x^4+\frac{3x^2}{2}+C$

18. $\int\frac{x^6-x}{x^3}dx = \int(x^3-x^{-2})\,dx$
$= \int x^3\,dx - \int x^{-2}\,dx = \frac{x^4}{4}-\frac{x^{-1}}{-1}+C$
$= \frac{x^4}{4}+\frac{1}{x}+C$

19. $\int(x^2+x)\,dx = \int x^2\,dx + \int x\,dx = \frac{x^3}{3}+\frac{x^2}{2}+C$

20. $\int\left(x^3+\sqrt{x}\right)dx = \int x^3\,dx + \int x^{1/2}\,dx$
$= \frac{x^4}{4}+\frac{x^{3/2}}{\frac{3}{2}}+C = \frac{x^4}{4}+\frac{2\sqrt{x^3}}{3}+C$

21. Let $u = x + 1$; then $du = dx$.
$\int(x+1)^2\,dx = \int u^2\,du = \frac{u^3}{3}+C = \frac{(x+1)^3}{3}+C$

22. $\int\left(z+\sqrt{2}\,z\right)^2 dz = \int\left[\left(1+\sqrt{2}\right)z\right]^2 dz$
$= \left(1+\sqrt{2}\right)^2\int z^2\,dz = \frac{\left(1+\sqrt{2}\right)^2 z^3}{3}+C$

23. $\int\frac{(z^2+1)^2}{\sqrt{z}}dz = \int\frac{z^4+2z^2+1}{\sqrt{z}}dz$
$= \int z^{7/2}\,dz + 2\int z^{3/2} + \int z^{-1/2}\,dz$
$= \frac{2}{9}z^{9/2}+\frac{4}{5}z^{5/2}+2z^{1/2}+C$

24. $\int\frac{s(s+1)^2}{\sqrt{s}}ds = \int\frac{s^3+2s^2+s}{\sqrt{s}}ds$
$= \int s^{5/2}ds + 2\int s^{3/2}ds + \int s^{1/2}ds$
$= \frac{2s^{7/2}}{7}+\frac{4s^{5/2}}{5}+\frac{2s^{3/2}}{3}+C$

25. $\int(\sin\theta-\cos\theta)d\theta = \int\sin\theta\,d\theta - \int\cos\theta\,d\theta$
$= -\cos\theta-\sin\theta+C$

26. $\int(t^2-2\cos t)dt = \int t^2\,dt - 2\int\cos t\,dt$
$= \frac{t^3}{3}-2\sin t+C$

27. Let $g(x)=\sqrt{2}\,x+1$; then $g'(x)=\sqrt{2}$.
$\int\left(\sqrt{2}\,x+1\right)^3\sqrt{2}\,dx = \int[g(x)]^3 g'(x)dx$
$= \frac{[g(x)]^4}{4}+C = \frac{\left(\sqrt{2}\,x+1\right)^4}{4}+C$

28. Let $g(x)=\pi x^3+1$; then $g'(x)=3\pi x^2$.
$\int(\pi x^3+1)^4 3\pi x^2\,dx = \int[g(x)]^4 g'(x)\,dx$
$= \frac{[g(x)]^5}{5}+C = \frac{(\pi x^3+1)^5}{5}+C$

29. Let $u = 5x^3+3x-8$; then $du = (15x^2+3)\,dx$.
$\int(5x^2+1)(5x^3+3x-8)^6\,dx$
$= \int\frac{1}{3}(15x^2+3)(5x^3+3x-8)^6\,dx$
$= \frac{1}{3}\int u^6\,du = \frac{1}{3}\left(\frac{u^7}{7}+C_1\right)$
$= \frac{(5x^3+3x-8)^7}{21}+C$

30. Let $u = 5x^3+3x-2$; then $du = (15x^2+3)dx$.
$\int(5x^2+1)\sqrt{5x^3+3x-2}\,dx$
$= \int\frac{1}{3}(15x^2+3)\sqrt{5x^3+3x-2}\,dx$
$= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\left(\frac{2}{3}u^{3/2}+C_1\right)$

$= \frac{2}{9}(5x^3 + 3x - 2)^{3/2} + C$

$= \frac{2}{9}\sqrt{(5x^3 + 3x - 2)^3} + C$

31. Let $u = 2t^2 - 11$; then $du = 4t\,dt$.

$\int 3t\sqrt[3]{2t^2 - 11}\,dt = \int \frac{3}{4}(4t)(2t^2 - 11)^{1/3}\,dt$

$= \frac{3}{4}\int u^{1/3}du = \frac{3}{4}\left(\frac{3}{4}u^{4/3} + C_1\right)$

$= \frac{9}{16}(2t^2 - 11)^{4/3} + C$

$= \frac{9}{16}\sqrt[3]{(2t^2 - 11)^4} + C$

32. Let $u = 2y^2 + 5$; then $du = 4y\,dy$

$\int \frac{3y}{\sqrt{2y^2 + 5y}}dy = \int \frac{3}{4}(4y)(2y^2 + 5)^{-1/2}\,dy$

$= \frac{3}{4}\int u^{-1/2}du = \frac{3}{4}(2u^{1/2} + C_1)$

$= \frac{3}{2}\sqrt{2y^2 + 5} + C$

33. $f'(x) = \int (3x + 1)dx = \frac{3}{2}x^2 + x + C_1$

$f(x) = \int\left(\frac{3}{2}x^2 + x + C_1\right)dx$

$= \frac{1}{2}x^3 + \frac{1}{2}x^2 + C_1x + C_2$

34. $f'(x) = \int (-2x + 3)\,dx = -x^2 + 3x + C_1$

$f(x) = \int (-x^2 + 3x + C_1)dx$

$= -\frac{1}{3}x^3 + \frac{3}{2}x^2 + C_1x + C_2$

35. $f'(x) = \int x^{1/2}dx = \frac{2}{3}x^{3/2} + C_1$

$f(x) = \int\left(\frac{2}{3}x^{3/2} + C_1\right)dx$

$= \frac{4}{15}x^{5/2} + C_1x + C_2$

36. $f'(x) = \int x^{4/3}\,dx = \frac{3}{7}x^{7/3} + C_1$

$f(x) = \int\left(\frac{3}{7}x^{7/3} + C_1\right)dx = \frac{9}{70}x^{10/3} + C_1x + C_2$

37. $f''(x) = x + x^{-3}$

$f'(x) = \int (x + x^{-3})dx = \frac{x^2}{2} - \frac{x^{-2}}{2} + C_1$

$f(x) = \int\left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2} + C_1\right)dx$

$= \frac{1}{6}x^3 + \frac{1}{2}x^{-1} + C_1x + C_2$

$= \frac{1}{6}x^3 + \frac{1}{2x} + C_1x + C_2$

38. $f'(x) = 2\int (x+1)^{1/3}\,dx = \frac{3}{2}(x+1)^{4/3} + C_1$

$f(x) = \int\left[\frac{3}{2}(x+1)^{4/3} + C_1\right]dx$

$= \frac{9}{14}(x+1)^{7/3} + C_1x + C_2$

39. The Product Rule for derivatives says

$\frac{d}{dx}[f(x)g(x) + C] = f(x)g'(x) + f'(x)g(x)$.

Thus,

$\int [f(x)g'(x) + f'(x)g(x)]dx = f(x)g(x) + C$.

40. The Quotient Rule for derviatives says

$\frac{d}{dx}\left[\frac{f(x)}{g(x)} + C\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$.

Thus, $\int \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}dx = \frac{f(x)}{g(x)} + C$.

41. Let $f(x) = x^2$, $g(x) = \sqrt{x-1}$.

$f'(x) = 2x$, $g'(x) = \frac{1}{2\sqrt{x-1}}$

$\int\left[\frac{x^2}{2\sqrt{x-1}} + 2x\sqrt{x-1}\right]dx$

$= \int [f(x)g'(x) + f'(x)g(x)]\,dx = f(x)g(x) + C$

$= x^2\sqrt{x-1} + C$

42. Let $f(x) = x^3$, $g(x) = (2x+5)^{-1/2}$.

$f'(x) = 3x^2$, $g'(x) = -(2x+5)^{-3/2}$

$= -\frac{1}{(2x+5)^{3/2}}$

$\int\left[\frac{-x^3}{(2x+5)^{3/2}} + \frac{3x^2}{\sqrt{2x+5}}\right]dx$

$= \int [f(x)g'(x) + g(x)f'(x)]\,dx$

$$= f(x)g(x) + C = x^3(2x+5)^{-1/2} + C$$
$$= \frac{x^3}{\sqrt{2x+5}} + C$$

43. $\int f''(x)dx = \int \frac{d}{dx} f'(x)dx = f'(x) + C$

$f'(x) = \sqrt{x^3+1} + \frac{3x^3}{2\sqrt{x^3+1}} = \frac{5x^3+2}{2\sqrt{x^3+1}}$ so

$\int f''(x)dx = \frac{5x^3+2}{2\sqrt{x^3+1}} + C.$

44. $\frac{d}{dx}\left(\frac{f(x)}{\sqrt{g(x)}} + C\right)$

$$= \frac{\sqrt{g(x)}f'(x) - f(x)\frac{1}{2}[g(x)]^{-1/2}g'(x)}{g(x)}$$
$$= \frac{2g(x)f'(x) - f(x)g'(x)}{2[g(x)]^{3/2}}$$

Thus,

$$\int \frac{2g(x)f'(x) - f(x)g'(x)}{2[g(x)]^{3/2}} = \frac{f(x)}{\sqrt{g(x)}} + C$$

45. The Product Rule for derivatives says that

$\frac{d}{dx}[f^m(x)g^n(x) + C]$

$= f^m(x)[g^n(x)]' + [f^m(x)]'g^n(x)$

$= f^m(x)[ng^{n-1}(x)g'(x)] + [mf^{m-1}(x)f'(x)]g^n(x)$

$= f^{m-1}(x)g^{n-1}(x)[nf(x)g'(x) + mg(x)f'(x)].$

Thus,

$\int f^{m-1}(x)g^{n-1}(x)[nf(x)g'(x) + mg(x)f'(x)]dx$

$= f^m(x)g^n(x) + C.$

46. Let $u = \sin[(x^2+1)^4]$;

then $du = \cos\left[(x^2+1)^4\right]4(x^2+1)^3(2x)dx.$

$du = 8x\cos\left[(x^2+1)^4\right](x^2+1)^3\,dx$

$\int \sin^3\left[(x^2+1)^4\right]\cos\left[(x^2+1)^4\right](x^2+1)^3 x\,dx$

$= \int u^3 \cdot \frac{1}{8}du = \frac{1}{8}\int u^3\,du = \frac{1}{8}\left(\frac{u^4}{4} + C_1\right)$

$= \frac{\sin^4\left[(x^2+1)^4\right]}{32} + C$

47. If $x \ge 0$, then $|x| = x$ and $\int |x|dx = \frac{1}{2}x^2 + C$.

If $x < 0$, then $|x| = -x$ and $\int |x|dx = -\frac{1}{2}x^2 + C$.

$$\int |x|\,dx = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \ge 0 \\ -\frac{1}{2}x^2 + C & \text{if } x < 0 \end{cases}$$

48. Using $\sin^2\frac{u}{2} = \frac{1-\cos u}{2}$.

$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

49. Different software may produce different, but equivalent answers. These answers were produced by Mathematica.

a. $\int 6\sin(3(x-2))dx = -2\cos(3(x-2)) + C$

b. $\int \sin^3\left(\frac{x}{6}\right)dx = \frac{1}{2}\cos\left(\frac{x}{2}\right) - \frac{9}{2}\cos\left(\frac{x}{6}\right) + C$

c. $\int (x^2\cos 2x + x\sin 2x)dx = \frac{x^2\sin 2x}{2} + C$

50. a. $F_1(x) = \int (x\sin x)dx = \sin x - x\cos x + C_1$

$F_2(x) = \int (\sin x - x\cos x + C_1)dx$

$= -2\cos x - x\sin x + C_1x + C_2$

$F_3(x) = \int (-2\cos x - x\sin x + C_1x + C_2)dx$

$= x\cos x - 3\sin x + \frac{1}{2}C_1x^2 + C_2x + C_3$

$F_4(x) = \int (x\cos x - 3\sin x + \frac{1}{2}C_1x^2 + C_2x + C_3)dx$

$= x\sin x + 4\cos x + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$

b. $F_{16}(x) = x\sin x + 16\cos x + \sum_{n=1}^{16}\frac{C_n x^{16-n}}{(16-n)!}$

5.2 Concepts Review

1. differential equation

2. function

3. separate variables

4. $-32t + v_0; -16t^2 + v_0t + s_0$

Problem Set 5.2

1. $\frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$

$\frac{dy}{dx} + \frac{x}{y} = \frac{-x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = 0$

2. $\frac{dy}{dx} = C$

$-x\frac{dy}{dx} + y = -Cx + Cx = 0$

3. $\frac{dy}{dx} = C_1 \cos x - C_2 \sin x;$

$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$

$\frac{d^2y}{dx^2} + y$

$= (-C_1 \sin x - C_2 \cos x) + (C_1 \sin x + C_2 \cos x) = 0$

4. For $y = \sin(x + C)$, $\frac{dy}{dx} = \cos(x + C)$

$\left(\frac{dy}{dx}\right)^2 + y^2 = \cos^2(x+C) + \sin^2(x+C) = 1$

For $y = \pm 1$, $\frac{dy}{dx} = 0$.

$\left(\frac{dy}{dx}\right)^2 + y^2 = 0^2 + (\pm 1)^2 = 1$

5. $\frac{dy}{dx} = x^2 + 1$

$dy = (x^2+1)\,dx$

$\int dy = \int (x^2+1)\,dx$

$y + C_1 = \frac{x^3}{3} + x + C_2$

$y = \frac{x^3}{3} + x + C$

At $x = 1, y = 1$:

$1 = \frac{1}{3} + 1 + C; C = -\frac{1}{3}$

$y = \frac{x^3}{3} + x - \frac{1}{3}$

6. $\frac{dy}{dx} = x^{-3} + 2$

$dy = (x^{-3} + 2)\,dx$

$\int dy = \int (x^{-3} + 2)\,dx$

$y + C_1 = -\frac{x^{-2}}{2} + 2x + C_2$

$y = -\frac{1}{2x^2} + 2x + C$

At $x = 1, y = 3$:

$3 = -\frac{1}{2} + 2 + C; C = \frac{3}{2}$

$y = -\frac{1}{2x^2} + 2x + \frac{3}{2}$

7. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} + C_1 = \frac{x^2}{2} + C_2$

$y^2 = x^2 + C$

$y = \pm\sqrt{x^2 + C}$

At $x = 1, y = 1$:

$1 = \pm\sqrt{1+C}; C = 0$ and the square root is positive.

$y = \sqrt{x^2}$ or $y = |x|$

8. $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$

$\int \sqrt{y}\,dy = \int \sqrt{x}\,dx$

$\frac{2}{3}y^{3/2} + C_1 = \frac{2}{3}x^{3/2} + C_2$

$y^{3/2} = x^{3/2} + C$

$y = (x^{3/2} + C)^{2/3}$

At $x = 1, y = 4$:

$4 = (1+C)^{2/3}; C = 7$

$y = (x^{3/2} + 7)^{2/3}$

9. $\frac{dz}{dt} = t^2 z^2$

$\int z^{-2}dz = \int t^2\,dt$

$-z^{-1} + C_1 = \frac{t^3}{3} + C_2$

$\frac{1}{z} = -\frac{t^3}{3} + C_3 = \frac{C - t^3}{3}$

$z = \frac{3}{C - t^3}$

At $t = 1$, $z = \frac{1}{3}$:

$\frac{1}{3}=\frac{3}{C-1}; C-1=9; C=10$

$z=\frac{3}{10-t^3}$

10. $\frac{dy}{dt}=y^4$

$\int y^{-4}dy=\int dt$

$-\frac{1}{3y^3}+C_1=t+C_2$

$y=-\frac{1}{\sqrt[3]{3t+C}}$

At $t=0, y=1$:

$C=-1$

$y=-\frac{1}{\sqrt[3]{3t-1}}$

11. $\frac{ds}{dt}=16t^2+4t-1$

$\int ds=\int(16t^2+4t-1)\,dt$

$s+C_1=\frac{16}{3}t^3+2t^2-t+C_2$

$s=\frac{16}{3}t^3+2t^2-t+C$

At $t=0, s=100$:

$C=100$

$s=\frac{16}{3}t^3+2t^2-t+100$

12. $\frac{du}{dt}=u^3(t^3-t)$

$\int u^{-3}\,du=\int(t^3-t)\,dt$

$-\frac{1}{2u^2}+C_1=\frac{t^4}{4}-\frac{t^2}{2}+C_2$

$u^{-2}=t^2-\frac{t^4}{2}+C$

$u=\left(t^2-\frac{t^4}{2}+C\right)^{-1/2}$

At $t=0, u=4$:

$4=C^{-1/2}; C=\frac{1}{16}$

$u=\left(t^2-\frac{t^4}{2}+\frac{1}{16}\right)^{-1/2}$

13. $\frac{dy}{dx}=(2x+1)^4$

$y=\int(2x+1)^4dx=\frac{1}{2}\int(2x+1)^4 2\,dx$

$=\frac{1}{2}\frac{(2x+1)^5}{5}+C=\frac{(2x+1)^5}{10}+C$

At $x=0, y=6$:

$6=\frac{1}{10}+C; C=\frac{59}{10}$

$y=\frac{(2x+1)^5}{10}+\frac{59}{10}=\frac{(2x+1)^5+59}{10}$

14. $\frac{dy}{dx}=-y^2x(x^2+2)^4$

$-\int y^{-2}dy=\frac{1}{2}\int 2x(x^2+2)^4\,dx$

$\frac{1}{y}+C_1=\frac{1}{2}\frac{(x^2+2)^5}{5}+C_2$

$\frac{1}{y}=\frac{(x^2+2)^5+C}{10}$

$y=\frac{10}{(x^2+2)^5+C}$

At $x=0, y=1$:

$1=\frac{10}{32+C}; C=10-32=-22$

$y=\frac{10}{(x^2+2)^5-22}$

15. $\frac{dy}{dx}=3x$

$y=\int 3x\,dx=\frac{3}{2}x^2+C$

At (1, 2):

$2=\frac{3}{2}+C$

$C=\frac{1}{2}$

$y=\frac{3}{2}x^2+\frac{1}{2}=\frac{3x^2+1}{2}$

16. $\frac{dy}{dx}=3y^2$

$\int y^{-2}\,dy=3\int dx$

$-\frac{1}{y}+C_1=3x+C_2$

$\frac{1}{y}=-3x+C$

$$y=\frac{1}{C-3x}$$

At (1, 2):

$$2=\frac{1}{C-3}$$

$$C=\frac{7}{2}$$

$$y=\frac{1}{\frac{7}{2}-3x}=\frac{2}{7-6x}$$

17. $v=\int t\,dt=\frac{t^2}{2}+v_0$

$$v=\frac{t^2}{2}+3$$

$$s=\int\left(\frac{t^2}{2}+3\right)dt=\frac{t^3}{6}+3t+s_0$$

$$s=\frac{t^3}{6}+3t+0=\frac{t^3}{6}+3t$$

At $t=2$:

$v=5$ cm/s

$s=\frac{22}{3}$ cm

18. $v=\int(1+t)^{-4}dt=-\frac{1}{3(1+t)^3}+C$

$$v_0=0:0=-\frac{1}{3(1+0)^3}+C;C=\frac{1}{3}$$

$$v=-\frac{1}{3(1+t)^3}+\frac{1}{3}$$

$$s=\int\left(-\frac{1}{3(1+t)^3}+\frac{1}{3}\right)dt=\frac{1}{6(1+t)^2}+\frac{1}{3}t+C$$

$$s_0=10:10=\frac{1}{6(1+0)^2}+\frac{1}{3}(0)+C;C=\frac{59}{6}$$

$$s=\frac{1}{6(1+t)^2}+\frac{1}{3}t+\frac{59}{6}$$

At $t=2$:

$v=-\frac{1}{81}+\frac{1}{3}=\frac{26}{81}$ cm/s

$s=\frac{1}{54}+\frac{2}{3}+\frac{59}{6}=\frac{284}{27}$ cm

19. $v=\int(2t+1)^{1/3}dt=\frac{1}{2}\int(2t+1)^{1/3}2dt$

$$=\frac{3}{8}(2t+1)^{4/3}+C_1$$

$$v_0=0:0=\frac{3}{8}+C_1;C_1=-\frac{3}{8}$$

$$v=\frac{3}{8}(2t+1)^{4/3}-\frac{3}{8}$$

$$s=\frac{3}{8}\int(2t+1)^{4/3}dt-\frac{3}{8}\int 1dt$$

$$=\frac{3}{16}\int(2t+1)^{4/3}2dt-\frac{3}{8}\int 1dt$$

$$=\frac{9}{112}(2t+1)^{7/3}-\frac{3}{8}t+C_2$$

$$s_0=10:10=\frac{9}{112}+C_2;C_2=\frac{1111}{112}$$

$$s=\frac{9}{112}(2t+1)^{7/3}-\frac{3}{8}t+\frac{1111}{112}$$

At $t=2$: $v=\frac{3}{8}(5)^{4/3}-\frac{3}{8}\approx 2.83$

$$s=\frac{9}{112}(5)^{7/3}-\frac{6}{8}+\frac{1111}{112}\approx 12.6$$

20. $v=\int(3t+1)^{-3}dt=\frac{1}{3}\int(3t+1)^{-3}3dt$

$$=-\frac{1}{6}(3t+1)^{-2}+C_1$$

$$v_0=4:4=-\frac{1}{6}+C_1;C_1=\frac{25}{6}$$

$$v=-\frac{1}{6}(3t+1)^{-2}+\frac{25}{6}$$

$$s=-\frac{1}{6}\int(3t+1)^{-2}dt+\int\frac{25}{6}dt$$

$$=-\frac{1}{18}\int(3t+1)^{-2}3dt+\frac{25}{6}\int dt$$

$$=\frac{1}{18}(3t+1)^{-1}+\frac{25}{6}t+C_2$$

$$s_0=0:0=\frac{1}{18}+C_2;C_2=-\frac{1}{18}$$

$$s=\frac{1}{18}(3t+1)^{-1}+\frac{25}{6}t-\frac{1}{18}$$

At $t=2$: $v=-\frac{1}{6}(7)^{-2}+\frac{25}{6}\approx 4.16$

$$s=\frac{1}{18}(7)^{-1}+\frac{25}{3}-\frac{1}{18}\approx 8.29$$

21. $v=-32t+96$,

$s=-16t^2+96t+s_0=-16t^2+96t$

$v=0$ at $t=3$

At $t=3$, $s=-16(3^2)+96(3)=144$ ft

22. $a = \frac{dv}{dt} = k$

$v = \int k\,dt = kt + v_0 = \frac{ds}{dt}$;

$s = \int (kt + v_0)dt = \frac{k}{2}t^2 + v_0 t + s_0 = \frac{k}{2}t^2 + v_0 t$

$v = 0$ when $t = -\frac{v_0}{k}$. Then

$s = \frac{k}{2}\left(-\frac{v_0}{k}\right)^2 + \left(-\frac{v_0^2}{k}\right) = -\frac{v_0^2}{2k}$.

23. $\frac{dv}{dt} = -5.28$

$\int dv = -\int 5.28 dt$

$v = \frac{ds}{dt} = -5.28t + v_0 = -5.28t + 56$

$\int ds = \int (-5.28t + 56)dt$

$s = -2.64t^2 + 56t + s_0 = -2.64t^2 + 56t + 1000$

When $t = 4.5$, $v = 32.24$ ft/s and $s = 1198.54$ ft

24. $v = 0$ when $t = \frac{-56}{-5.28} \approx 10.6061$. Then

$s \approx -2.64(10.6061)^2 + 56(10.6061) + 1000$

$\approx$ 1296.97 ft

25. $\frac{dV}{dt} = -kS$

Since $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$,

$4\pi r^2 \frac{dr}{dt} = -k4\pi r^2$ so $\frac{dr}{dt} = -k$.

$\int dr = -\int k\,dt$

$r = -kt + C$

$2 = -k(0) + C$ and $0.5 = -k(10) + C$, so

$C = 2$ and $k = \frac{3}{20}$. Then, $r = -\frac{3}{20}t + 2$.

26. Solving $v = -136 = -32t$ yields $t = \frac{17}{4}$.

Then $s = 0 = -16\left(\frac{17}{4}\right)^2 + (0)\left(\frac{17}{4}\right) + s_0$, so

$s_0 = 289$ ft.

27. $v_{esc} = \sqrt{2gR}$

For the Moon, $v_{esc} \approx \sqrt{2(0.165)(32)(1080 \cdot 5280)}$

$\approx$ 7760 ft/s $\approx$ 1.470 mi/s.

For Venus, $v_{esc} \approx \sqrt{2(0.85)(32)(3800 \cdot 5280)}$

$\approx$ 33,038 ft/s $\approx$ 6.257 mi/s.

For Jupiter, $v_{esc} \approx$ 194,369 ft/s $\approx$ 36.812 mi/s.

For the Sun, $v_{esc} \approx$ 2,021,752 ft/s

$\approx$ 382.908 mi/s.

28. $v_0 = 60$ mi/h = 88 ft/s

$v = 0 = -11t + 88$; $t = 8$ sec

29. $a = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{60-45}{10} = 1.5$ mi/h/s = 2.2 ft/s^2

30. $75 = \frac{8}{2}(3.75)^2 + v_0(3.75) + 0; v_0 = 5$ ft/s

31. For the first 10 s, $a = \frac{dv}{dt} = 6t, v = 3t^2$, and

$s = t^3$. So $v(10) = 300$ and $s(10) = 1000$. After

10 s, $a = \frac{dv}{dt} = -10$, $v = -10(t - 10) + 300$, and

$s = -5(t-10)^2 + 300(t-10) + 1000$. $v = 0$ at

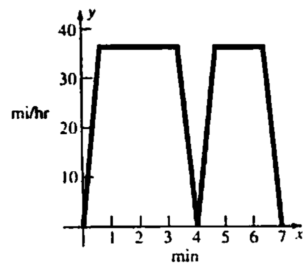
$t = 40$, at which time $s = 5500$ m.

32. a. After accelerating for 8 seconds, the velocity is $8 \cdot 3 = 24$ m/s.

b. Since acceleration and deceleration are constant, the average velocity during those times is $\frac{24}{2} = 12$ m/s. Solve $0 = -4t + 24$ to get the time spent decelerating. $t = \frac{24}{4} = 6$ s;

$d = (12)(8) + (24)(100) + (12)(6) = 2568$ m.

33. a.



b. Since the trip that involves 1 min more travel time at speed v_m is 0.6 mi longer,

$v_m = 0.6$ mi/min

$= 36$ mi/h.

c. From part b, $v_m = 0.6$ mi/min. Note that the average speed during acceleration and deceleration is $\frac{v_m}{2} = 0.3$ mi/min. Let t be the time spent between stop C and stop D at the constant speed v_m, so

$0.6t + 0.3(4 - t) = 2$ miles. Therefore,

$t = 2\frac{2}{3}$ min and the time spent accelerating is $\frac{4-2\frac{2}{3}}{2} = \frac{2}{3}$ min.

$a = \frac{0.6-0}{\frac{2}{3}} = 0.9$ mi/min^2.

34. For the balloon, $\frac{dh}{dt} = 4$, so $h(t) = 4t + C_1$. Set $t = 0$ at the time when Victoria threw the ball, and height 0 at the ground, then $h(t) = 4t + 64$. The height of the ball is given by $s(t) = -16t^2 + v_0 t$, since $s_0 = 0$. The maximum height of the ball is when $t = \frac{v_0}{32}$, since then $s'(t) = 0$. At this time

$h(t) = s(t)$ or $4\left(\frac{v_0}{32}\right) + 64 = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right)$.

Solve this for v_0 to get $v_0 \approx 68.125$ feet per second.

35. a. $\frac{dV}{dt} = C_1\sqrt{h}$ where h is the depth of the water. Here, $V = \pi r^2 h = 100h$, so $h = \frac{V}{100}$.

Hence $\frac{dV}{dt} = C_1\frac{\sqrt{V}}{10}$, $V(0) = 1600$, $V(40) = 0$.

b. $\int 10V^{-1/2}dV = \int C_1 dt; 20\sqrt{V} = C_1 t + C_2;$

$V(0) = 1600$: $C_2 = 20 \cdot 40 = 800;$

$V(40) = 0$: $C_1 = -\frac{800}{40} = -20$

$V(t) = \frac{1}{400}(-20t + 800)^2$

c. $V(10) = \frac{1}{400}(-200 + 800)^2 = 900 \text{ cm}^3$

36. a. $\frac{dP}{dt} = C_1\sqrt[3]{P}$, $P(0) = 1000$, $P(10) = 1700$ where t is the number of years since 1970.

b. $\int P^{-1/3}dP = \int C_1 dt; \frac{3}{2}P^{2/3} = C_1 t + C_2$

$P(0) = 1000$: $C_2 = \frac{3}{2}\cdot 1000^{2/3} = 150$

$P(10) = 1700$: $C_1 = \frac{\frac{3}{2}\cdot 1700^{2/3} - 150}{10} \approx 6.3660$

$P = (4.2440t + 100)^{3/2}$

c. $4000 = (4.2440t + 100)^{3/2}$

$t = \frac{4000^{2/3} - 100}{4.2440} \approx 35.812$

$t \approx 36$ years, so the population will reach 4000 by 2016.

37. Initially, $v = -32t$ and $s = -16t^2 + 16$. $s = 0$ when $t = 1$. Later, the ball falls 9 ft in a time given by $0 = -16t^2 + 9$, or $\frac{3}{4}$ s, and on impact has a velocity of $-32\left(\frac{3}{4}\right) = -24$ ft/s. By symmetry, 24 ft/s must be the velocity right after the first bounce. So

a. $v(t) = \begin{cases} -32t & \text{for } 0 \le t \le 1 \\ -32(t-1) + 24 & \text{for } 1 < t \le 2.5 \end{cases}$

b. $9 = -16t^2 + 16 \Rightarrow t \approx 0.66$ sec; s also equals 9 at the apex of the first rebound at $t = 1.75$ sec.

5.3 Concepts Review

1. $2\cdot\frac{5(6)}{2} = 30; 2(5) = 10$

2. $3(9) - 2(7) = 13; 9 + 4(10) = 49$

3. $1 - \frac{1}{10} = 0.9$

4. $2\cdot\frac{6(7)}{2} - \frac{6(7)(13)}{6} = -49$

Problem Set 5.3

1. $\sum_{k=1}^{6}(k-1) = \sum_{k=1}^{6}k - \sum_{k=1}^{6}1 = \frac{6(7)}{2} - 6(1) = 15$

2. $\sum_{i=1}^{6}i^2 = \frac{6(7)(13)}{6} = 91$

3. $\displaystyle\sum_{k=1}^{7}\frac{1}{k+1}=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}$

$+\frac{1}{4+1}+\frac{1}{5+1}+\frac{1}{6+1}+\frac{1}{7+1}$

$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=\frac{1443}{840}=\frac{481}{280}$

4. $\displaystyle\sum_{l=3}^{8}(l+1)^2=4^2+5^2+6^2+7^2+8^2+9^2=271$

5. $\displaystyle\sum_{m=1}^{8}(-1)^m 2^{m-2}=(-1)^1 2^{-1}+(-1)^2 2^0+(-1)^3 2^1$

$+(-1)^4 2^2+(-1)^5 2^3+(-1)^6 2^4$

$+(-1)^7 2^5+(-1)^8 2^6$

$=-\frac{1}{2}+1-2+4-8+16-32+64=\frac{85}{2}$

6. $\displaystyle\sum_{k=3}^{7}\frac{(-1)^k 2^k}{(k+1)}$

$=\frac{(-1)^3 2^3}{4}+\frac{(-1)^4 2^4}{5}$

$+\frac{(-1)^5 2^5}{6}+\frac{(-1)^6 2^6}{7}+\frac{(-1)^7 2^7}{8}=-\frac{1154}{105}$

7. $\displaystyle\sum_{n=1}^{6}n\cos(n\pi)=\cos\pi+2\cos 2\pi+3\cos 3\pi$

$+4\cos 4\pi+5\cos 5\pi+6\cos 6\pi$

$=-1+2-3+4-5+6=3$

8. $\displaystyle\sum_{k=-1}^{6}k\sin\left(\frac{k\pi}{2}\right)$

$=-\sin\left(-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)+2\sin(\pi)$

$+3\sin\left(\frac{3\pi}{2}\right)+4\sin(2\pi)+5\sin\left(\frac{5\pi}{2}\right)+6\sin(3\pi)$

$=1+1+0-3+0+5+0=4$

9. $1+2+3+\cdots+41=\displaystyle\sum_{i=1}^{41}i$

10. $2+4+6+8+\cdots+50=\displaystyle\sum_{i=1}^{25}2i$

11. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}=\displaystyle\sum_{i=1}^{100}\frac{1}{i}$

12. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{100}=\displaystyle\sum_{i=1}^{100}\frac{(-1)^{i+1}}{i}$

13. $a_1+a_3+a_5+a_7+\cdots+a_{99}=\displaystyle\sum_{i=1}^{50}a_{2i-1}$

14. $b_{-1}+b_1+b_3+b_5+\cdots+b_{1001}=\displaystyle\sum_{i=1}^{502}b_{2i-3}$

15. $f(c_1)+f(c_2)+\cdots+f(c_n)=\displaystyle\sum_{i=1}^{n}f(c_i)$

16. $f(w_1)\Delta x+f(w_2)\Delta x+\cdots+f(w_n)\Delta x$

$=\displaystyle\sum_{i=1}^{n}f(w_i)\Delta x$

17. $\displaystyle\sum_{i=1}^{10}(a_i+b_i)=\sum_{i=1}^{10}a_i+\sum_{i=1}^{10}b_i=40+50=90$

18. $\displaystyle\sum_{n=1}^{10}(3a_n+2b_n)=3\sum_{n=1}^{10}a_n+2\sum_{n=1}^{10}b_n$

$=3(40)+2(50)=220$

19. $\displaystyle\sum_{p=0}^{9}(a_{p+1}-b_{p+1})=\sum_{p=1}^{10}a_p-\sum_{p=1}^{10}b_p=40-50$

$=-10$

20. $\displaystyle\sum_{q=1}^{10}(a_q-b_q-q)=\sum_{q=1}^{10}a_q-\sum_{q=1}^{10}b_q-\sum_{q=1}^{10}q$

$=40-50-\frac{10(11)}{2}=-65$

21. $\displaystyle\sum_{k=1}^{40}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{40}-\frac{1}{41}\right)$

$=1-\frac{1}{41}=\frac{40}{41}$

22. $\displaystyle\sum_{k=1}^{10}(2^k-2^{k-1})$

$=(2-1)+(4-2)+(8-4)+\cdots+(1024-512)$

$=-1+1024=1023$

23. $\sum_{k=3}^{20}\left(\frac{1}{(k+1)^2}-\frac{1}{k^2}\right)$
$=\left(\frac{1}{4^2}-\frac{1}{3^2}\right)+\left(\frac{1}{5^2}-\frac{1}{4^2}\right)+\cdots\cdots+\left(\frac{1}{21^2}-\frac{1}{20^2}\right)$
$=-\frac{1}{3^2}+\frac{1}{21^2}=-\frac{49}{441}+\frac{1}{441}=-\frac{48}{441}=-\frac{16}{147}$

24. $\sum_{k=3}^{m+1}(a_k-a_{k-1})$
$=(a_3-a_2)+(a_4-a_3)+(a_5-a_4)+\cdots$
$+(a_m-a_{m-1})+(a_{m+1}-a_m) \;=-a_2+a_{m+1}$

25. $\sum_{i=1}^{100}(3i-2)=3\sum_{i=1}^{100}i-\sum_{i=1}^{100}2=3(5050)-2(100)$
$=14{,}950$

26. $\sum_{i=1}^{10}[(i-1)(4i+3)]=\sum_{i=1}^{10}(4i^2-i-3)$
$=4\sum_{i=1}^{10}i^2-\sum_{i=1}^{10}i-\sum_{i=1}^{10}3=4(385)-55-3(10)$
$=1455$

27. $\sum_{k=1}^{10}(k^3-k^2)=\sum_{k=1}^{10}k^3-\sum_{k=1}^{10}k^2=3025-385$
$=2640$

28. $\sum_{k=1}^{10}5k^2(k+4)=\sum_{k=1}^{10}(5k^3+20k^2)$
$=5\sum_{k=1}^{10}k^3+20\sum_{k=1}^{10}k^2=5(3025)+20(385)$
$=22{,}825$

29. $\sum_{i=1}^{n}(2i^2-3i+1)=2\sum_{i=1}^{n}i^2-3\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$
$=\frac{2n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{2}+n$
$=\frac{2n^3+3n^2+n}{3}-\frac{3n^2+3n}{2}+n\;=\frac{4n^3-3n^2-n}{6}$

30. $\sum_{i=1}^{n}(2i-3)^2=\sum_{i=1}^{n}(4i^2-12i+9)$
$=4\sum_{i=1}^{n}i^2-12\sum_{i=1}^{n}i+\sum_{i=1}^{n}9$
$=\frac{4n(n+1)(2n+1)}{6}-\frac{12n(n+1)}{2}+9n$
$=\frac{4n^3-12n^2+11n}{3}$

31. $\sum_{i=3}^{19}i(i-2)=\sum_{k=1}^{17}(k+2)k$

32. $\sum_{k=5}^{14}k2^{k-4}=\sum_{i=1}^{10}(i+4)2^i$

33. $\sum_{k=0}^{10}\frac{k}{k+1}=\sum_{i=1}^{11}\frac{i-1}{i}$

34. $\sum_{k=4}^{13}(k-3)\sin\left(\frac{\pi}{k-3}\right)=\sum_{i=1}^{10}i\sin\left(\frac{\pi}{i}\right)$

35. $\sum_{i=1}^{10}\frac{3i}{5}\cdot\frac{1}{5}=\frac{3}{25}\sum_{i=1}^{10}i=\frac{33}{5}$

36. $S-rS=a+ar+ar^2+\cdots+ar^n$
$\quad-\;(ar+ar^2+\cdots+ar^n+ar^{n+1})$
$=a-ar^{n+1}$
$=S(1-r);\ S=\frac{a-ar^{n+1}}{1-r}$

37. a. $\sum_{k=0}^{10}\left(\frac{1}{2}\right)^k=\frac{1-\left(\frac{1}{2}\right)^{11}}{\frac{1}{2}}=2-\left(\frac{1}{2}\right)^{10}$, so
$\sum_{k=1}^{10}\left(\frac{1}{2}\right)^k=1-\left(\frac{1}{2}\right)^{10}$.

b. $\sum_{k=0}^{10}2^k=\frac{1-2^{11}}{-1}=2^{11}-1$, so
$\sum_{k=1}^{10}2^k=2^{11}-2$.

38. $S=1+2+3+\cdots+(n-2)+(n-1)+n$
$+S=n+(n-1)+(n-2)+\cdots+3+2+1$
$2S=(n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+$

$2S=n(n+1)$
$S=\frac{n(n+1)}{2}$

39. $$\begin{aligned} S &= a+(a+d)+(a+2d)+\cdots+[a+(n-2)d]+[a+(n-1)d]+(a+nd) \\ +S &= (a+nd)+[a+(n-1)d]+[a+(n-2)d]+\cdots+(a+2d)+(a+d)+a \end{aligned}$$
$$2S=(2a+nd)+(2a+nd)+(2a+nd)+\cdots+(2a+nd)+(2a+nd)+(2a+nd)$$
$$2S=(n+1)(2a+nd)$$
$$S=\frac{(n+1)(2a+nd)}{2}$$

40. $\sum_{k=1}^{1}\frac{1}{\sqrt{k}}=\frac{1}{\sqrt{1}}=1\ge 1;\ \sum_{k=1}^{2}\frac{1}{\sqrt{k}}=1+\frac{1}{\sqrt{2}}\approx 1.707\ge\sqrt{2}\approx 1.414$

Assume that $\sum_{k=1}^{N}\frac{1}{\sqrt{k}}\ge\sqrt{N}$ for all $N\le n$.

Then $\sum_{k=1}^{n+1}\frac{1}{\sqrt{k}}=\sum_{k=1}^{n}\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{n+1}}\ge\sqrt{n}+\frac{1}{\sqrt{n+1}}=\frac{\sqrt{n^2+n}+1}{\sqrt{n+1}}\ge\frac{n+1}{\sqrt{n+1}}=\sqrt{n+1}$

41. $\bar{x}=\frac{1}{7}(2+5+7+8+9+10+14)=\frac{55}{7}\approx 7.86$

$$s^2=\frac{1}{7}\left[\left(2-\frac{55}{7}\right)^2+\left(5-\frac{55}{7}\right)^2+\left(7-\frac{55}{7}\right)^2+\left(8-\frac{55}{7}\right)^2+\left(9-\frac{55}{7}\right)^2+\left(10-\frac{55}{7}\right)^2+\left(14-\frac{55}{7}\right)^2\right]\approx 12.4$$

42. a. $\bar{x}=1,\ s^2=0$

b. $\bar{x}=1001,\ s^2=0$

c. $\bar{x}=2$

$$s^2=\tfrac{1}{3}\left[(1-2)^2+(2-2)^2+(3-2)^2\right]=\tfrac{1}{3}\left[(-1)^2+0^2+1^2\right]=\tfrac{1}{3}(2)=\tfrac{2}{3}$$

d. $\bar{x}=1{,}000{,}002$

$$s^2=\tfrac{1}{3}\left[(-1)^2+0^2+1^2\right]=\tfrac{2}{3}$$

43. a. $\sum_{i=1}^{n}(x_i-\bar{x})=\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\bar{x}=n\bar{x}-n\bar{x}=0$

b. $$s^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2=\frac{1}{n}\sum_{i=1}^{n}(x_i^2-2\bar{x}x_i+\bar{x}^2)$$
$$=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{2\bar{x}}{n}\sum_{i=1}^{n}x_i+\frac{1}{n}\sum_{i=1}^{n}\bar{x}^2$$
$$=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{2\bar{x}}{n}(n\bar{x})+\frac{1}{n}(n\bar{x}^2)$$
$$=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-2\bar{x}^2+\bar{x}^2=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-\bar{x}^2$$

44. The variance of n identical numbers is 0. Let c be the constant. Then

$$\bar{x}=\tfrac{1}{n}\left[(c-c)^2+(c-c)^2+\cdots+(c-c)^2\right]=0$$

45. Let $S(c)=\sum_{i=1}^{n}(x_i-c)^2$. Then

$$S'(c)=\frac{d}{dc}\sum_{i=1}^{n}(x_i-c)^2=\sum_{i=1}^{n}\frac{d}{dc}(x_i-c)^2=\sum_{i=1}^{n}2(x_i-c)(-1)=-2\sum_{i=1}^{n}x_i+2nc$$

$S''(c)=2n$

Set $S'(c)=0$ and solve for c:

$$-2\sum_{i=1}^{n}x_i+2nc=0$$

$$c=\tfrac{1}{n}\sum_{i=1}^{n}x_i=\bar{x}$$

Since $S''(\bar{x})=2n>0$ we know that $\bar{x}$ minimizes $S(c)$.

46. $$\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})=\sum_{i=1}^{n}(x_iy_i-x_i\overline{y}-y_i\overline{x}+\overline{x}\,\overline{y})$$
$$=\sum_{i=1}^{n}x_iy_i-\overline{y}\sum_{i=1}^{n}x_i-\overline{x}\sum_{i=1}^{n}y_i+n\,\overline{x}\,\overline{y}$$
$$=\sum_{i=1}^{n}x_iy_i-\overline{y}n\overline{x}-\overline{x}n\overline{y}+n\,\overline{x}\,\overline{y}$$
$$=\sum_{i=1}^{n}x_iy_i-n\,\overline{x}\,\overline{y}$$

47. $$(i+1)^4-i^4=4i^3+6i^2+4i+1$$
$$\sum_{i=1}^{n}\left[(i+1)^4-i^4\right]=\sum_{i=1}^{n}\left(4i^3+6i^2+4i+1\right)$$
$$(n+1)^4-1^4=4\sum_{i=1}^{n}i^3+6\sum_{i=1}^{n}i^2+4\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$$
$$n^4+4n^3+6n^2+4n=4\sum_{i=1}^{n}i^3+6\frac{n(n+1)(2n+1)}{6}$$
$$+4\frac{n(n+1)}{2}+n$$

Solving for $\sum_{i=1}^{n}i^3$ gives

$$4\sum_{i=1}^{n}i^3=n^4+4n^3+6n^2+4n-\left(2n^3+3n^2+n\right)$$
$$-\left(2n^2+2n\right)-n$$
$$4\sum_{i=1}^{n}i^3=n^4+2n^3+n^2$$
$$\sum_{i=1}^{n}i^3=\frac{n^4+2n^3+n^2}{4}=\left[\frac{n(n+1)}{2}\right]^2$$

48. $$(i+1)^5-i^5=5i^4+10i^3+10i^2+5i+1$$
$$\sum_{i=1}^{n}\left[(i+1)^5-i^5\right]=5\sum_{i=1}^{n}i^4+10\sum_{i=1}^{n}i^3+10\sum_{i=1}^{n}i^2+5\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$$
$$(n+1)^5-1^5=5\sum_{i=1}^{n}i^4+10\frac{n^2(n+1)^2}{4}$$
$$+10\frac{n(n+1)(2n+1)}{6}+5\frac{n(n+1)}{2}+n$$
$$n^5+5n^4+10n^3+10n^2+5n=5\sum_{i=1}^{n}i^4+\tfrac{5}{2}n^2(n+1)^2$$
$$+\tfrac{10}{3}n(n+1)(2n+1)+\frac{5}{2}n(n+1)+n$$

Solving for $\sum_{i=1}^{n} i^4$ yields

$$\sum_{i=1}^{n} i^4 = \frac{1}{5}\left[n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n\right]$$

$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

49. $$\sum_{i=1}^{n}(a_i t + b_i)^2 = \sum_{i=1}^{n}(a_i^2 t^2 + 2a_i b_i t + b_i^2)$$

$$= t^2\sum_{i=1}^{n} a_i^2 + 2t\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2$$

$$= At^2 + 2Bt + C$$

Because $\sum_{i=1}^{n}(a_i t + b_i)^2 \ge 0$, $At^2 + 2Bt + C \ge 0$.

50. Since $At^2 + 2Bt + C \ge 0$ for all t

$$(2B)^2 - 4AC \le 0$$

$$4B^2 - 4AC \le 0$$

$$B^2 - AC \le 0$$

$$B^2 \le AC$$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

51. $$\sum_{i=1}^{n}(a_i - a_{i-1})b_{i-1} = \sum_{i=1}^{n} a_i b_{i-1} - \sum_{i=1}^{n} a_{i-1}b_{i-1}$$

$$= \sum_{i=1}^{n} a_i b_{i-1} - \sum_{i=0}^{n-1} a_i b_i$$

$$= \sum_{i=1}^{n} a_i b_{i-1} - \sum_{i=1}^{n} a_i b_i + a_n b_n - a_0 b_0$$

$$= a_n b_n - a_0 b_0 + \sum_{i=1}^{n} a_i(b_{i-1} - b_i)$$

$$= a_n b_n - a_0 b_0 - \sum_{i=1}^{n} a_i(b_i - b_{i-1})$$

52. $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

53. Suppose we have a $(n+1)\times n$ grid. Shade in $n + 1 - k$ boxes in the kth column. There are n columns, and the shaded area is $1+2+\cdots+n$. The shaded area is also half the area of the grid or $\frac{n(n+1)}{2}$. Thus, $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Suppose we have a square grid with sides of length $1+2+\cdots+n = \frac{n(n+1)}{2}$. From the diagram the area is $1^3 + 2^3 + \cdots + n^3$ or $\left[\frac{n(n+1)}{2}\right]^2$. Thus, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

54. **a.** The number of gifts given on the nth day is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

The total number of gifts is $\sum_{i=1}^{12} \frac{i(i+1)}{2} = 364$.

b. For n days, the total number of gifts is $\sum_{i=1}^{n} \frac{i(i+1)}{2}$.

$$\sum_{i=1}^{n} \frac{i(i+1)}{2} = \sum_{i=1}^{n} \frac{i^2}{2} + \sum_{i=1}^{n} \frac{i}{2} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{1}{4}n(n+1)\left(\frac{2n+1}{3}+1\right) = \frac{1}{12}n(n+1)(2n+4)$$
$$= \frac{1}{6}n(n+1)(n+2)$$

55. The bottom layer contains $10 \cdot 16 = 160$ oranges, the next layer contains $9 \cdot 15 = 135$ oranges, the third layer contains $8 \cdot 14 = 112$ oranges, and so on, up to the top layer, which contains $1 \cdot 7 = 7$ oranges. The stack contains
$$1 \cdot 7 + 2 \cdot 8 + \cdots + 9 \cdot 15 + 10 \cdot 16$$
$$= \sum_{i=1}^{10} i(6+i)$$
$$= \sum_{i=1}^{10} i(16-10+i) = 715 \text{ oranges.}$$

If the bottom layer is 50 oranges by 60 oranges, the stack contains
$$\sum_{i=1}^{50} i(60-50+i) = \sum_{i=1}^{50} i(10+i) = 55{,}675.$$
For a general stack whose base is m rows of n oranges with $m \le n$, the stack contains
$$\sum_{i=1}^{m} i(n-m+i) = (n-m)\sum_{i=1}^{m} i + \sum_{i=1}^{m} i^2$$
$$= (n-m)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6}$$
$$= \frac{m(m+1)(3n-m+1)}{6}$$

5.4 Concepts Review

1. 16

2. inscribed; circumscribed

3. $\frac{1}{2}(4)(4) = 8; 8 + \frac{1}{2}(2)(2) = 10$

4. 6

Problem Set 5.4

1. $A = \frac{1}{2}\left[1 + \frac{3}{2} + 2 + \frac{5}{2}\right] = \frac{7}{2}$

2. $A = \frac{1}{4}\left[1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 + \frac{9}{4} + \frac{5}{2} + \frac{11}{4}\right] = \frac{15}{4}$

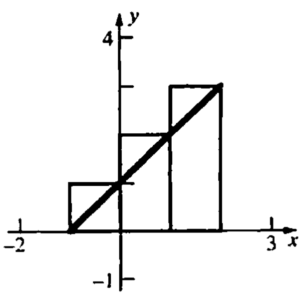
3. $A = \frac{1}{2}\left[\frac{3}{2} + 2 + \frac{5}{2} + 3\right] = \frac{9}{2}$

4. $A = \frac{1}{4}\left[\frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 + \frac{9}{4} + \frac{5}{2} + \frac{11}{4} + 3\right] = \frac{17}{4}$

5. $A = \frac{1}{2}\left[\left(\frac{1}{2}\cdot 0^2 + 1\right) + \left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 + 1\right) + \left(\frac{1}{2}\cdot 1^2 + 1\right) + \left(\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2 + 1\right)\right] = \frac{1}{2}\left(1 + \frac{9}{8} + \frac{3}{2} + \frac{17}{8}\right) = \frac{23}{8}$

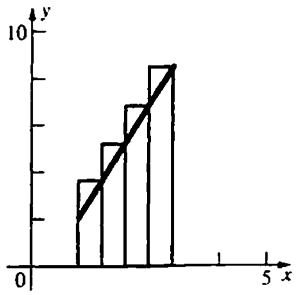
6. $A = \frac{1}{2}\left[\left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 + 1\right) + \left(\frac{1}{2}\cdot 1^2 + 1\right) + \left(\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2 + 1\right) + \left(\frac{1}{2}\cdot 2^2 + 1\right)\right] = \frac{1}{2}\left(\frac{9}{8} + \frac{3}{2} + \frac{17}{8} + 3\right) = \frac{31}{8}$

7.



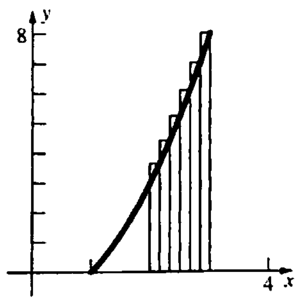
$A = 1(1 + 2 + 3) = 6$

8.



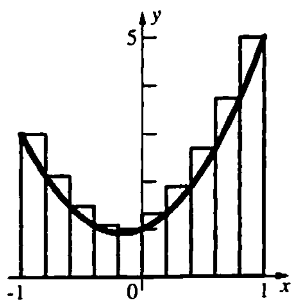
$$A=\frac{1}{2}\left[\left(3\cdot\frac{3}{2}-1\right)+(3\cdot 2-1)+\left(3\cdot\frac{5}{2}-1\right)+(3\cdot 3-1)\right]$$
$$=\frac{1}{2}\left(\frac{7}{2}+5+\frac{13}{2}+8\right)=\frac{23}{2}$$

9.



$$A=\frac{1}{6}\left[\left(\left(\frac{13}{6}\right)^2-1\right)+\left(\left(\frac{7}{3}\right)^2-1\right)+\left(\left(\frac{5}{2}\right)^2-1\right)+\left(\left(\frac{8}{3}\right)^2-1\right)+\left(\left(\frac{17}{6}\right)^2-1\right)+(3^2-1)\right]$$
$$=\frac{1}{6}\left(\frac{133}{36}+\frac{40}{9}+\frac{21}{4}+\frac{55}{9}+\frac{253}{36}+8\right)=\frac{1243}{216}$$

10.



$$A=\frac{1}{5}\left[(3(-1)^2+(-1)+1)+\left(3\left(-\frac{4}{5}\right)^2+\left(-\frac{4}{5}\right)+1\right)+\left(3\left(-\frac{3}{5}\right)^2+\left(-\frac{3}{5}\right)+1\right)+\left(3\left(-\frac{2}{5}\right)^2+\left(-\frac{2}{5}\right)+1\right)+(3(0)^2+0+1)\right.$$
$$\left.+\left(3\left(\frac{1}{5}\right)^2+\frac{1}{5}+1\right)+\left(3\left(\frac{2}{5}\right)^2+\frac{2}{5}+1\right)+\left(3\left(\frac{3}{5}\right)^2+\frac{3}{5}+1\right)\left(3\left(\frac{4}{5}\right)^2+\frac{4}{5}+1\right)+(3(1)^2+1+1)\right]$$
$$=\frac{1}{5}[3+2.12+1.48+1.08+1+1.32+1.88+2.68+3.72+5]=4.656$$

11. $\Delta x=\frac{1}{n}, x_i=\frac{i}{n}$

$$f(x_i)\Delta x=\left(\frac{i}{n}+2\right)\left(\frac{1}{n}\right)=\frac{i}{n^2}+\frac{2}{n}$$

$$A(S_n)=\left[\left(\frac{1}{n^2}+\frac{2}{n}\right)+\left(\frac{2}{n^2}+\frac{2}{n}\right)+\cdots+\left(\frac{n}{n^2}+\frac{2}{n}\right)\right]=\frac{1}{n^2}(1+2+3+\cdots+n)+2=\frac{n(n+1)}{2n^2}+2=\frac{1}{2n}+\frac{5}{2}$$

$$\lim_{n\to\infty} A(S_n)=\lim_{n\to\infty}\left(\frac{1}{2n}+\frac{5}{2}\right)=\frac{5}{2}$$

12. $\Delta x=\frac{1}{n},\ x_i=\frac{i}{n}$

$$f(x_i)\Delta x=\left[\frac{1}{2}\cdot\left(\frac{i}{n}\right)^2+1\right]\left(\frac{1}{n}\right)=\frac{i^2}{2n^3}+\frac{1}{n}$$

$$A(S_n)=\left[\left(\frac{1^2}{2n^3}+\frac{1}{n}\right)+\left(\frac{2^2}{2n^3}+\frac{1}{n}\right)+\cdots+\left(\frac{n^2}{2n^3}+\frac{1}{n}\right)\right]=\frac{1}{2n^3}(1^2+2^2+3^2+\cdots+n^2)+1$$

$$=\frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+1=\frac{1}{12}\left[\frac{2n^3+3n^2+n}{n^3}\right]+1=\frac{1}{12}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]+1$$

$$\lim_{n\to\infty} A(S_n)=\lim_{n\to\infty}\left[\frac{1}{12}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+1\right]=\frac{7}{6}$$

13. $\Delta x=\frac{2}{n},\ x_i=-1+\frac{2i}{n}$

$$f(x_i)\Delta x=\left[2\left(-1+\frac{2i}{n}\right)+2\right]\left(\frac{2}{n}\right)=\frac{8i}{n^2}$$

$$A(S_n)=\left[\left(\frac{8}{n^2}\right)+\left(\frac{16}{n^2}\right)+\cdots+\left(\frac{8n}{n^2}\right)\right]$$

$$=\frac{8}{n^2}(1+2+3+\cdots+n)=\frac{8}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$=4\left[\frac{n^2+n}{n^2}\right]=4+\frac{4}{n}$$

$$\lim_{n\to\infty} A(S_n)=\lim_{n\to\infty}\left(4+\frac{4}{n}\right)=4$$

14. First, consider $a=0$ and $b=2$.

$\Delta x=\frac{2}{n},\ x_i=\frac{2i}{n}$

$$f(x_i)\Delta x=\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)=\frac{8i^2}{n^3}$$

$$A(S_n)=\left[\left(\frac{8}{n^3}\right)+\left(\frac{8(2^2)}{n^3}\right)+\cdots+\left(\frac{8n^2}{n^3}\right)\right]$$

$$=\frac{8}{n^3}(1^2+2^2+\cdots+n^2)=\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\frac{4}{3}\left[\frac{2n^3+3n^2+n}{n^3}\right]=\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}$$

$$\lim_{n\to\infty} A(S_n)=\lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right)=\frac{8}{3}.$$

By symmetry, $A=2\left(\frac{8}{3}\right)=\frac{16}{3}$.

15. $\Delta x=\frac{1}{n},\ x_i=\frac{i}{n}$

$$f(x_i)\Delta x=\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right)=\frac{i^3}{n^4}$$

$$A(S_n)=\left[\frac{1}{n^4}(1^3)+\frac{1}{n^4}(2^3)+\cdots+\frac{1}{n^4}(n^3)\right]$$

$$=\frac{1}{n^4}(1^3+2^3+\cdots+n^3)=\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

$$=\frac{1}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right]=\frac{1}{4}\left[1+\frac{2}{n}+\frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} A(S_n)=\lim_{n\to\infty}\frac{1}{4}\left[1+\frac{2}{n}+\frac{1}{n^2}\right]=\frac{1}{4}$$

16. $\Delta x=\frac{1}{n},\ x_i=\frac{i}{n}$

$$f(x_i)\Delta x=\left[\left(\frac{i}{n}\right)^3+\frac{i}{n}\right]\left(\frac{1}{n}\right)=\frac{i^3}{n^4}+\frac{i}{n^2}$$

$$A(S_n)=\frac{1}{n^4}(1^3+2^3+\cdots+n^3)+\frac{1}{n^2}(1+2+\cdots+n)$$

$$=\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2+\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{n^2+2n+1}{4n^2} + \frac{n^2+n}{2n^2} = \frac{3n^2+4n+1}{4n^2} = \frac{3}{4} + \frac{1}{n} + \frac{1}{4n^2}$$

$$\lim_{n\to\infty} A(S_n) = \frac{3}{4}$$

17. $f(t_i)\Delta t = \left[\frac{i}{n}+2\right]\frac{1}{n} = \frac{i}{n^2} + \frac{2}{n}$

$$A(S_n) = \sum_{i=1}^{n}\left(\frac{i}{n^2}+\frac{2}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} i + \sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] + 2$$

$$= \left[\frac{n^2+n}{2n^2}\right] + 2$$

$$= \left(\frac{1}{2}+\frac{1}{2n}\right) + 2$$

$$\lim_{n\to\infty} A(S_n) = \frac{1}{2} + 2 = \frac{5}{2}$$

The object traveled $2\frac{1}{2}$ ft.

18. $f(t_i)\Delta t = \left[\frac{1}{2}\left(\frac{i}{n}\right)^2 + 1\right]\frac{1}{n} = \frac{i^2}{2n^3} + \frac{1}{n}$

$$A(S_n) = \sum_{i=1}^{n}\left(\frac{1i^2}{2n^3}+\frac{1}{n}\right) = \frac{1}{2n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\frac{1}{n}$$

$$= \frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + 1 = \frac{1}{12}\left[2+\frac{3}{n}+\frac{1}{n^2}\right] + 1$$

$$\lim_{n\to\infty} A(S_n) = \frac{1}{12}(2) + 1 = \frac{7}{6} \approx 1.17$$

The subject traveled about 1.17 feet.

19. **a.** $f(x_i)\Delta x = \left(\frac{ib}{n}\right)^2\left(\frac{b}{n}\right) = \frac{b^3 i^2}{n^3}$

$$A_0^b = \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 = \frac{b^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{b^3}{6}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} A_0^b = \frac{2b^3}{6} = \frac{b^3}{3}$$

b. Since $a \ge 0$, $A_0^b = A_0^a + A_a^b$, or

$$A_a^b = A_0^b - A_0^a = \frac{b^3}{3} - \frac{a^3}{3}.$$

20. $A_3^5 = \frac{5^3}{3} - \frac{3^3}{3} = \frac{98}{3} \approx 32.7$

The object traveled about 32.7 m.

21. **a.** $A_0^5 = \frac{5^3}{3} = \frac{125}{3}$

b. $A_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$

c. $A_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{117}{3} = 39$

22. **a.** $\Delta x = \frac{b}{n}, x_i = \frac{bi}{n}$

$$f(x_i)\Delta x = \left(\frac{bi}{n}\right)^m\left(\frac{b}{n}\right) = \frac{b^{m+1}i^m}{n^{m+1}}$$

$$A(S_n) = \frac{b^{m+1}}{n^{m+1}}\sum_{i=1}^{n} i^m$$

$$= \frac{b^{m+1}}{n^{m+1}}\left[\frac{n^{m+1}}{m+1} + C_n\right]$$

$$= \frac{b^{m+1}}{m+1} + \frac{b^{m+1}C_n}{n^{m+1}}$$

$$A_0^b(x^m) = \lim_{n\to\infty} A(S_n) = \frac{b^{m+1}}{m+1}$$

$\lim_{n\to\infty}\frac{C_n}{n^{m+1}} = 0$ since C_n is a polynomial in n of degree m.

b. Notice that $A_a^b(x^m) = A_0^b(x^m) - A_0^a(x^m)$.

Thus, using part a, $A_a^b(x^m) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}$.

23. **a.** $A_0^2(x^3) = \frac{2^{3+1}}{3+1} = 4$

b. $A_1^2(x^3) = \frac{2^{3+1}}{3+1} - \frac{1^{3+1}}{3+1} = 4 - \frac{1}{4} = \frac{15}{4}$

c. $A_1^2(x^5) = \frac{2^{5+1}}{5+1} - \frac{1^{5+1}}{5+1} = \frac{32}{3} - \frac{1}{6} = \frac{63}{6}$

$$= \frac{21}{2} = 10.5$$

d. $A_0^2(x^9) = \frac{2^{9+1}}{9+1} = \frac{1024}{10} = 102.4$

24. Inscribed:
Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The angle at the center is $\frac{2\pi}{n}$. The length of the base

is $2r\sin\frac{\pi}{n}$. The height is $r\cos\frac{\pi}{n}$. Thus the area of the triangle is $r^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}=\frac{1}{2}r^2\sin\frac{2\pi}{n}$.

$$A_n=n\left(\frac{1}{2}r^2\sin\frac{2\pi}{n}\right)=\frac{1}{2}nr^2\sin\frac{2\pi}{n}$$

Circumscribed:
Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The angle at the center is $\frac{2\pi}{n}$. The length of the base is $2r\tan\frac{\pi}{n}$. The height is r. Thus the area of the triangle is $r^2\tan\frac{\pi}{n}$.

$$B_n=n\left(r^2\tan\frac{\pi}{n}\right)=nr^2\tan\frac{\pi}{n}$$

$$\lim_{n\to\infty}A_n=\lim_{n\to\infty}\frac{1}{2}nr^2\sin\frac{2\pi}{n}=\lim_{n\to\infty}\pi r^2\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$$

$$=\pi r^2$$

$$\lim_{n\to\infty}B_n=\lim_{n\to\infty}nr^2\tan\frac{\pi}{n}=\lim_{n\to\infty}\frac{\pi r^2}{\cos\frac{\pi}{n}}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$$

$$=\pi r^2$$

5.5 Concepts Review

1. Riemann sum

2. definite integral: $\int_a^b f(x)dx$

3. $A_{up}-A_{down}$

4. $8-\frac{1}{2}=\frac{15}{2}$

Problem Set 5.5

1. $R_P=f(2)(2.5-1)+f(3)(3.5-2.5)+f(4.5)(5-3.5)\ =4(1.5)+3(1)+(-2.25)(1.5)=5.625$

2. $R_P=f(0.5)(0.7-0)+f(1.5)(1.7-0.7)+f(2)(2.7-1.7)+f(3.5)(4-2.7)$
$=1.25(0.7)+(-0.75)(1)+(-1)(1)+1.25(1.3)=0.75$

3. $R_P=\sum_{i=1}^{5}f(\overline{x}_i)\Delta x_i\ =f(3)(3.75-3)+f(4)(4.25-3.75)+f(4.75)(5.5-4.25)+f(6)(6-5.5)+f(6.5)(7-6)$
$=2(0.75)+3(0.5)+3.75(1.25)+5(0.5)+5.5(1)=15.6875$

4. $R_P=\sum_{i=1}^{4}f(\overline{x}_i)\Delta x_i\ =f(-2)(-1.3+3)+f(-0.5)(0+1.3)+f(0)(0.9-0)+f(2)(2-0.9)$
$=4(1.7)+3.25(1.3)+3(0.9)+2(1.1)=15.925$

5. $R_P=\sum_{i=1}^{8}f(\overline{x}_i)\Delta x_i\ =[f(-1.75)+f(-1.25)+f(-0.75)+f(-0.25)+f(0.25)+f(0.75)+f(1.25)+f(1.75)](0.5)$
$=[-0.21875-0.46875-0.46875-0.21875+0.28125+1.03125+2.03125+3.28125](0.5)=2.625$

6. $R_P=\sum_{i=1}^{6}f(\overline{x}_i)\Delta x_i\ =[f(0.5)+f(1)+f(1.5)+f(2)+f(2.5)+f(3)](0.5)$
$=[1.5+5+14.5+33+63.5+109](0.5)=113.25$

7. $\int_1^3 x^3\,dx$

8. $\int_0^2 (x+1)^3\,dx$

9. $\int_{-1}^{1} \frac{x^2}{1+x} dx$

10. $\int_0^{\pi} (\sin x)^2 \, dx$

11. $\Delta x = \frac{2}{n}, \overline{x}_i = \frac{2i}{n}$

$$f(\overline{x}_i) = \overline{x}_i + 1 = \frac{2i}{n} + 1$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n} i \quad = \frac{2}{n}(n) + \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 2 + 2\left(1 + \frac{1}{n}\right)$$

$$\int_0^2 (x+1)dx = \lim_{n\to\infty}\left[2 + 2\left(1 + \frac{1}{n}\right)\right] = 4$$

12. $\Delta x = \frac{2}{n}, \overline{x}_i = \frac{2i}{n}$

$$f(\overline{x}_i) = \left(\frac{2i}{n}\right)^2 + 1 = \frac{4i^2}{n^2} + 1$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i^2\left(\frac{4}{n^2}\right)\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^3}\sum_{i=1}^{n} i^2 \quad = \frac{2}{n}(n) + \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 2 + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty}\left[2 + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = \frac{14}{3}$$

13. $\Delta x = \frac{3}{n}, \overline{x}_i = -2 + \frac{3i}{n}$

$$f(\overline{x}_i) = 2\left(-2 + \frac{3i}{n}\right) + \pi = \pi - 4 + \frac{6i}{n}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[\pi - 4 + \frac{6i}{n}\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n}(\pi - 4) + \frac{18}{n^2}\sum_{i=1}^{n} i = 3(\pi - 4) + \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 3\pi - 12 + 9\left(1 + \frac{1}{n}\right)$$

$$\int_{-2}^{1} (2x+\pi)\,dx = \lim_{n\to\infty}\left[3\pi - 12 + 9\left(1 + \frac{1}{n}\right)\right] = 3\pi - 3$$

14. $\Delta x = \frac{3}{n}, \overline{x}_i = -2 + \frac{3i}{n}$

$$f(\overline{x}_i) = 3\left(-2 + \frac{3i}{n}\right)^2 + 2 = 14 - \frac{36i}{n} + \frac{27i^2}{n^2}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[14 - \left(\frac{36}{n}\right)i + \left(\frac{27}{n^2}\right)i^2\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n} 14 - \frac{108}{n^2}\sum_{i=1}^{n} i + \frac{81}{n^3}\sum_{i=1}^{n} i^2$$

$$= 42 - \frac{108}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{81}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 42 - 54\left(1 + \frac{1}{n}\right) + \frac{27}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_{-2}^{1} (3x^2 + 2)\,dx$$

$$= \lim_{n\to\infty}\left[42 - 54\left(1 + \frac{1}{n}\right) + \frac{27}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 15$$

15. $\Delta x = \frac{5}{n}, \overline{x}_i = \frac{5i}{n}$

$$f(\overline{x}_i) = 1 + \frac{5i}{n}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{5}{n}\right)\right]\frac{5}{n}$$

$$= \frac{5}{n}\sum_{i=1}^{n} 1 + \frac{25}{n^2}\sum_{i=1}^{n} i \quad = 5 + \frac{25}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 5 + \frac{25}{2}\left(1 + \frac{1}{n}\right)$$

$$\int_0^5 (x+1)\,dx = \lim_{n\to\infty}\left[5 + \frac{25}{2}\left(1 + \frac{1}{n}\right)\right] = \frac{35}{2}$$

16. $\Delta x = \frac{20}{n}, \overline{x}_i = -10 + \frac{20i}{n}$

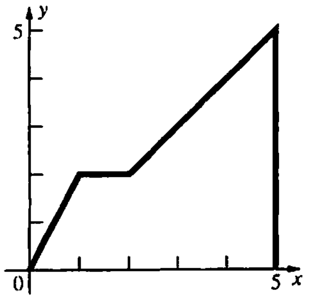
$$f(\overline{x}_i) = \left(-10 + \frac{20i}{n}\right)^2 + \left(-10 + \frac{20i}{n}\right) = 90 - \frac{380i}{n} + \frac{400i^2}{n^2}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[90 - i\left(\frac{380}{n}\right) + i^2\left(\frac{400}{n^2}\right)\right]\frac{20}{n} \quad = \frac{20}{n}\sum_{i=1}^{n} 90 - \frac{7600}{n^2}\sum_{i=1}^{n} i + \frac{8000}{n^3}\sum_{i=1}^{n} i^2$$

$$= 1800 - \frac{7600}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{8000}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = 1800 - 3800\left(1+\frac{1}{n}\right) + \frac{4000}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

$$\int_{-10}^{10}(x^2+x)\,dx = \lim_{n\to\infty}\left[1800-3800\left(1+\frac{1}{n}\right)+\frac{4000}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right] = \frac{2000}{3}$$

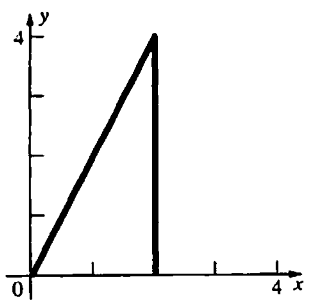
17.



$\int_0^5 f(x)\,dx$

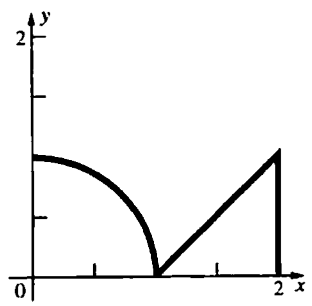
$= \frac{1}{2}(1)(2) + 1(2) + 3(2) + \frac{1}{2}(3)(3) = \frac{27}{2}$

18.



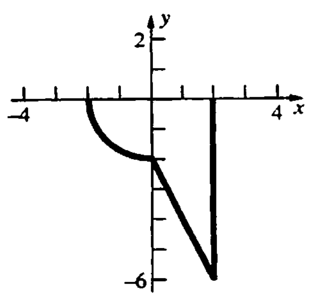
$\int_0^2 f(x)\,dx = \frac{1}{2}(2)(4) = 4$

19.



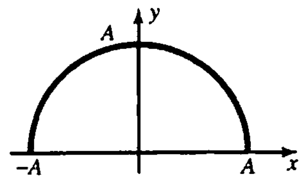
$\int_0^2 f(x)\,dx = \frac{1}{4}(\pi \cdot 1^2) + \frac{1}{2}(1)(1) = \frac{1}{2} + \frac{\pi}{4}$

20.



$\int_{-2}^{2} f(x)\,dx = -\frac{1}{4}(\pi \cdot 2^2) - (2)(2) - \frac{1}{2}(2)(4)$
$= -\pi - 8$

21. The area under the curve is equal to the area of a semi-circle: $\int_{-A}^{A}\sqrt{A^2 - x^2}\,dx = \frac{1}{2}\pi A^2$.



22. Partition [0, 1] into n regular intervals, so $|P| = \frac{1}{n}$.

If $\bar{x}_i = \frac{i}{n} + \frac{1}{2n}$, $f(\bar{x}_i) = 1$.

$\lim_{|P|\to 0}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n} = 1$

If $\bar{x}_i = \frac{i}{n} + \frac{1}{\pi n}$, $f(\bar{x}_i) = 0$.

$\lim_{|P|\to 0}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} 0 = 0$

Thus f is not integrable on [0, 1].

23. a. $\int_{-3}^{3} [\![x]\!]\,dx = (-3-2-1+0+1+2)(1) = -3$

b. $\int_{-3}^{3} [\![x]\!]^2\,dx = [(-3)^2 + (-2)^2$
$+(-1)^2 + 0 + 1 + 4](1) = 19$

c. $\int_{-3}^{3}\left(x - [\![x]\!]\right)dx = 6\left[\frac{1}{2}(1)(1)\right] = 3$

d. $\int_{-3}^{3}(x-[\![x]\!])^2 = \int_{-3}^{3}(x^2 - 2x[\![x]\!] + [\![x]\!]^2)\,dx$

$= \int_{-3}^{3} x^2 dx - 2\int_{-3}^{3} x[\![x]\!]\,dx + \int_{-3}^{3}[\![x]\!]^2\,dx$

$= 2\frac{3^3}{3} - 2\left[-3\int_{-3}^{-2} x\,dx - 2\int_{-2}^{-1} x\,dx - \int_{-1}^{0} x\,dx\right.$

$\left.+0\int_0^1 x\,dx + \int_1^2 x\,dx + 2\int_2^3 x\,dx\right] + 19$

$= 18 - 2\left[(-3)\left(1(-2) - \frac{1}{2}(1)(1)\right)\right.$

$-2\left(1(-1) - \frac{1}{2}(1)(1)\right) - \left(-\frac{1}{2}(1)(1)\right) + 0$

$\left.+\left(1\cdot 1 + \frac{1}{2}(1)(1)\right) + 2\left(1\cdot 2 + \frac{1}{2}(1)(1)\right)\right] + 19$

$= 37 - 2\left[(-3)\left(-\frac{5}{2}\right) - 2\left(-\frac{3}{2}\right) + \frac{1}{2} + \frac{3}{2} + 2\left(\frac{5}{2}\right)\right]$

$= 37 - 35 = 2$

e. $\int_{-3}^{3} |x|\, dx = \frac{1}{2}(3)(3) + \frac{1}{2}(3)(3) = 9$

f. $\int_{-3}^{3} x|x|\, dx = \frac{(-3)^3}{3} + \frac{(3)^3}{3} = 0$

g. $\int_{-1}^{2} |x|[\![x]\!]\, dx = -\int_{-1}^{0} |x|\, dx + 0\int_{0}^{1} |x|\, dx + \int_{1}^{2} |x|\, dx$

$= -\frac{1}{2}(1)(1) + 1(1) + \frac{1}{2}(1)(1) = 1$

h. $\int_{-1}^{2} x^2 [\![x]\!]\, dx = -\int_{-1}^{0} x^2\, dx + 0\int_{0}^{1} x^2\, dx + \int_{1}^{2} x^2\, dx$

$= -\frac{1^3}{3} + \left(\frac{2^3}{3} - \frac{1^3}{3}\right) = 2$

24. a. $\int_{-1}^{1} f(x)\, dx = -3 + 3 = 0$

b. $\int_{-1}^{1} g(x)\, dx = 3 + 3 = 6$

c. $\int_{-1}^{1} |f(x)|\, dx = 3 + 3 = 6$

d. $\int_{-1}^{1} [-g(x)]\, dx = -3 + (-3) = -6$

e. $\int_{-1}^{1} xg(x)\, dx = 0$ because $xg(x)$ is an odd function.

f. $\int_{-1}^{1} f^3(x)g(x)\, dx = 0$ because $f^3(x)g(x)$ is an odd function.

25. $R_P = \frac{1}{2}\sum_{i=1}^{n} (x_i + x_{i-1})(x_i - x_{i-1})$

$= \frac{1}{2}\sum_{i=1}^{n} \left(x_i^2 - x_{i-1}^2\right)$

$= \frac{1}{2}\left[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + (x_3^2 - x_2^2) + \cdots + (x_n^2 - x_{n-1}^2)\right]$

$= \frac{1}{2}(x_n^2 - x_0^2)$

$= \frac{1}{2}(b^2 - a^2)$

$\lim_{n\to\infty} \frac{1}{2}(b^2 - a^2) = \frac{1}{2}(b^2 - a^2)$

26. Note that $\bar{x}_i = \left[\frac{1}{3}\left(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right)\right]^{1/2}$

$\geq \left[\frac{1}{3}(x_{i-1}^2 + x_{i-1}^2 + x_{i-1}^2)\right]^{1/2} = x_{i-1}$ and

$\bar{x}_i = \left[\frac{1}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2)\right]^{1/2}$

$\leq \left[\frac{1}{3}(x_i^2 + x_i^2 + x_i^2)\right]^{1/2} = x_i$.

$R_P = \sum_{i=1}^{n} \bar{x}_i^2 \Delta x_i$

$= \sum_{i=1}^{n} \frac{1}{3}(x_i^2 + x_{i-1}x_i + x_{i-1}^2)(x_i - x_{i-1})$

$= \frac{1}{3}\sum_{i=1}^{n} (x_i^3 - x_{i-1}^3)$

$= \frac{1}{3}\left[(x_1^3 - x_0^3) + (x_2^3 - x_1^3) + (x_3^3 - x_2^3) + \cdots + (x_n^3 - x_{n-1}^3)\right]$

$= \frac{1}{3}(x_n^3 - x_0^3) = \frac{1}{3}(b^3 - a^3)$

27. Left: $\int_{0}^{2} (x^3 + 1)\, dx \approx 5.24$

Right: $\int_{0}^{2} (x^3 + 1)\, dx \approx 6.84$

Midpoint: $\int_{0}^{2} (x^3 + 1)\, dx \approx 5.98$

28. Left: $\int_{0}^{1} \tan x\, dx \approx 0.5398$

Right: $\int_{0}^{1} \tan x\, dx \approx 0.6955$

Midpoint: $\int_{0}^{1} \tan x\, dx \approx 0.6146$

29. Left: $\int_0^1 \cos x\,dx \approx 0.8638$

Right: $\int_0^1 \cos x\,dx \approx 0.8178$

Midpoint: $\int_0^1 \cos x\,dx \approx 0.8418$

30. Left: $\int_1^3 \left(\frac{1}{x}\right)dx \approx 1.1682$

Right: $\int_1^3 \left(\frac{1}{x}\right)dx \approx 1.0349$

Midpoint: $\int_1^3 \left(\frac{1}{x}\right)dx \approx 1.0971$

31. $\int_{-2}^{4} (-1+|x|)dx = 4$

32. $\int_0^6 \sin x\,dx \approx 0.0398$

33. $\int_{-1}^{2} (x^4 - 3x^2 + 1)\,dx = 0.6$

34. $\int_{-2}^{2} \left(\frac{x-1}{x^2+1}\right)dx \approx -2.2143$

35. $\int_0^1 \left(\frac{1}{x}\right)dx$ is undefined; $\frac{1}{x}$ is not bounded near $x = 0$.

36. $\int_0^2 \tan x\,dx$ is undefined; $\tan x$ is not bounded near $x = \frac{\pi}{2}$.

5.6 Concepts Review

1. $4(4-2) = 8;\ 16(4-2) = 32$

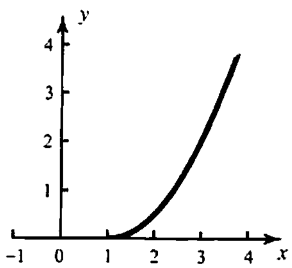
2. $\sin^3 x$

3. $\int_1^4 f(x)\,dx;\ \int_2^5 \sqrt{x}\,dx$

4. 5

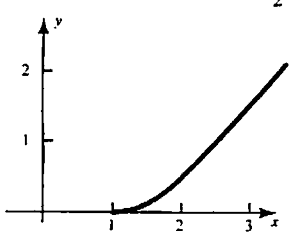
Problem Set 5.6

1. $A(x) = \frac{1}{2}(x-1)(-1+x),\quad x > 1$



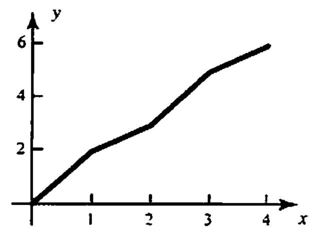
2. If $1 \le x \le 2$, then $A(x) = \frac{1}{2}(x-1)^2$.

If $2 \le x$, then $A(x) = x - \frac{3}{2}$



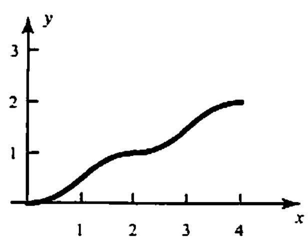
3.

$$A(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 2+(x-1) & 1 < x \le 2 \\ 3+2(x-2) & 2 < x \le 3 \\ 5+(x-3) & 3 < x \le 4 \\ \text{etc.} & \end{cases}$$



4.

$$A(x) = \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{2}+\frac{1}{2}(3-x)(x-1) & 1 < x \le 2 \\ 1+\frac{1}{2}(x-2)^2 & 2 < x \le 3 \\ \frac{3}{2}+\frac{1}{2}(5-x)(x-3) & 3 < x \le 4 \\ 2+\frac{1}{2}(x-4)^2 & 4 < x \le 5 \\ \text{etc.} & \end{cases}$$



5. $\int_1^2 2f(x)dx = 2\int_1^2 f(x)dx = 2(3) = 6$

6. $\int_0^2 2f(x)dx = 2\int_0^2 f(x)dx$

$= 2\left[\int_0^1 f(x)dx + \int_1^2 f(x)dx\right] = 2(2+3) = 10$

7. $\int_0^2 [2f(x)+g(x)]dx = 2\int_0^2 f(x)dx + \int_0^2 g(x)dx$

$= 2\left[\int_0^1 f(x)dx + \int_1^2 f(x)dx\right] + \int_0^2 g(x)dx$

$= 2(2+3) + 4 = 14$

8. $\int_0^1 [2f(s)+g(s)]ds = 2\int_0^1 f(s)ds + \int_0^1 g(s)ds$

$= 2(2) + (-1) = 3$

9. $\int_2^1 [2f(s)+5g(s)]ds = -2\int_1^2 f(s)ds - 5\int_1^2 g(s)ds$

$= -2(3) - 5\left[\int_0^2 g(s)ds - \int_0^1 g(s)ds\right]$

$= -6 - 5[4+1] = -31$

10. $\int_1^1 [3f(x)+2g(x)]dx = 0$

11. $\int_0^2 [3f(t)+2g(t)]dt$

$= 3\left[\int_0^1 f(t)dt + \int_1^2 f(t)dt\right] + 2\int_0^2 g(t)dt$

$= 3(2+3) + 2(4) = 23$

12. $\int_0^2 \left[\sqrt{3}f(t)+\sqrt{2}g(t)+\pi\right]dt$

$= \sqrt{3}\left[\int_0^1 f(t)dt + \int_1^2 f(t)dt\right] + \sqrt{2}\int_0^2 g(t)dt$

$+\pi\int_0^2 dt$

$= \sqrt{3}(2+3) + \sqrt{2}(4) + 2\pi = 5\sqrt{3} + 4\sqrt{2} + 2\pi$

13. $G'(x) = D_x\left[\int_1^x 2t\,dt\right] = 2x$

14. $G'(x) = D_x\left[\int_x^1 2t\,dt\right] = D_x\left[-\int_1^x 2t\,dt\right] = -2x$

15. $G'(x) = D_x\left[\int_0^x \left(2t^2+\sqrt{t}\right)dt\right] = 2x^2 + \sqrt{x}$

16. $G'(x) = D_x\left[\int_1^x \cos^3(2t)\tan(t)\,dt\right]$

$= \cos^3(2x)\tan(x)$

17. $G'(x) = D_x\left[\int_x^{\pi/2} (s-2)\cot(2s)ds\right]$

$= D_x\left[-\int_{\pi/2}^x (s-2)\cot(2s)ds\right] = -(x-2)\cot(2x)$

18. $G'(x) = D_x\left[\int_1^x xt\,dt\right] = D_x\left[x\int_1^x t\,dt\right]$

$= D_x\left[x\left[\frac{t^2}{2}\right]_1^x\right] = D_x\left[x\left(\frac{x^2-1}{2}\right)\right]$

$= D_x\left(\frac{x^3}{2} - \frac{x}{2}\right) = \frac{3}{2}x^2 - \frac{1}{2}$

19. $G'(x) = D_x\left[\int_1^{x^2} \sin t\,dt\right] = 2x\sin(x^2)$

20. $G'(x) = D_x\left[\int_1^{x^2+x} \sqrt{2z+\sin z}\,dz\right]$

$= (2x+1)\sqrt{2(x^2+x)+\sin(x^2+x)}$

$= (2x+1)\sqrt{2x^2+2x+\sin(x^2+x)}$

21. $G(x) = \int_{-x^2}^x \frac{t^2}{1+t^2}dt = \int_{-x^2}^0 \frac{t^2}{1+t^2}dt + \int_0^x \frac{t^2}{1+t^2}dt$

$= -\int_0^{-x^2} \frac{t^2}{1+t^2}dt + \int_0^x \frac{t^2}{1+t^2}dt$

$G'(x) = -\frac{\left(-x^2\right)^2}{1+\left(-x^2\right)^2}(-2x) + \frac{x^2}{1+x^2}$

$= \frac{2x^5}{1+x^4} + \frac{x^2}{1+x^2}$

22. $G(x) = D_x\left[\int_{\cos x}^{\sin x} t^5 dt\right]$

$= D_x\left[\int_0^{\sin x} t^5 dt + \int_{\cos x}^0 t^5 dt\right]$

$= D_x\left[\int_0^{\sin x} t^5 dt - \int_0^{\cos x} t^5 dt\right]$

$= \sin^5 x\cos x + \cos^5 x\sin x$

23. $f'(x)=\dfrac{x}{\sqrt{a^2+x^2}}$

$f''(x)=\dfrac{1}{\sqrt{a^2+x^2}}-\dfrac{x^2}{\sqrt{(a^2+x^2)^3}}$

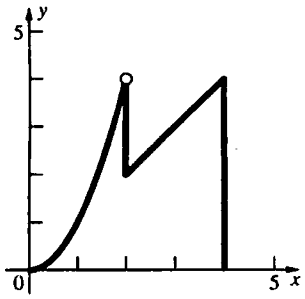
$=\dfrac{a^2}{\sqrt{(a^2+x^2)^3}}$, which is always positive.

24. The graph is concave up when $f''(x)$ is positive:

$f'(x)=\dfrac{1+x}{1+x^2}$, $f''(x)=\dfrac{1-2x-x^2}{(1+x^2)^2}$, $f''(x)>0$

when $x^2+2x-1<0$. By the quadratic formula the left side is zero for $x=-1\pm\sqrt{2}$, and testing the point $x=0$ indicates that $f''(x)>0$ on $(-1-\sqrt{2},-1+\sqrt{2})$.

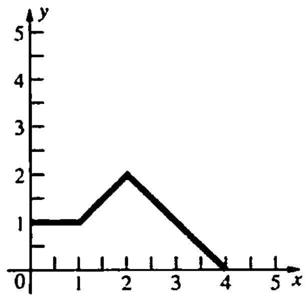
25.



$\int_0^4 f(x)\,dx=\int_0^2 x^2dx+\int_2^4 x\,dx$

$=\left[\dfrac{x^3}{3}\right]_0^2+\left[\dfrac{x^2}{2}\right]_2^4=\left(\dfrac{8}{3}-0\right)+(8-2)=\dfrac{26}{3}$

26.



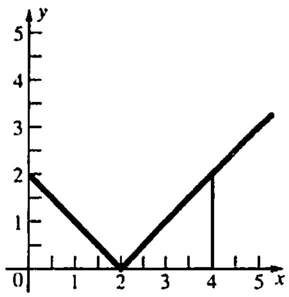
$\int_0^4 f(x)\,dx=\int_0^1 dx+\int_1^2 x\,dx+\int_2^4(4-x)\,dx$

$=\int_0^1 dx+\int_1^2 x\,dx+4\int_2^4 dx-\int_2^4 x\,dx$

$=[x]_0^1+\left[\dfrac{x^2}{2}\right]_1^2+4[x]_2^4-\left[\dfrac{x^2}{2}\right]_2^4$

$=(1-0)+\left(2-\dfrac{1}{2}\right)+4(4-2)-(8-2)\ =\dfrac{9}{2}$

27.



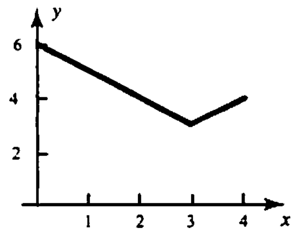
$\int_0^4 f(x)\,dx=\int_0^2(2-x)dx+\int_2^4(x-2)\,dx$

$=2\int_0^2 dx-\int_0^2 x\,dx+\int_2^4 x\,dx-2\int_2^4 dx$

$=2[x]_0^2-\left[\dfrac{x^2}{2}\right]_0^2+\left[\dfrac{x^2}{2}\right]_2^4-2[x]_2^4$

$=2(2-0)-(2-0)+(8-2)-2(4-2)=4$

28.



$\int_0^4(3+|x-3|)\,dx$

$=\int_0^3(3+|x-3|)\,dx+\int_3^4(3+|x-3|)\,dx$

$=3\cdot\frac{9}{2}+1\cdot\frac{7}{2}=17$

29. a. $F(0)=\int_0^0\left(t^4+1\right)dt=0$

b. $y=F(x)$

$\dfrac{dy}{dx}=F'(x)=x^4+1$

$dy=\left(x^4+1\right)dx$

$y=\frac{1}{5}x^5+x+C$

c. Now apply the initial condition $y(0)=0$:

$0=\frac{1}{5}0^5+0+C$

$C=0$

Thus $y=F(x)=\frac{1}{5}x^5+x$

d. $\int_0^1\left(x^4+1\right)dx=F(1)=\dfrac{1}{5}1^5+1=\dfrac{6}{5}$.

30. a. $G(x) = \int_0^x \sin t\, dt$

$G(0) = \int_0^0 \sin t\, dt = 0$

$G(2\pi) = \int_0^{2\pi} \sin t\, dt = 0$

b. Let $y = G(x)$. Then

$\frac{dy}{dx} = G'(x) = \sin x$.

c. $dy = \sin x\, dx$

$y = -\cos x + C$

Apply the initial condition

$0 = y(0) = -\cos 0 + C$. Thus, $C = 1$,

and hence $y = G(x) = 1 - \cos x$.

d. $\int_0^{\pi} \sin x\, dx = G(\pi) = 1 - \cos \pi = 2$

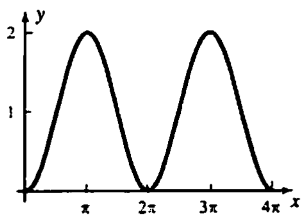
e. G attains the maximum of 2 when $x = 0, 2\pi, 4\pi$.

G attains the minimum of 0 when $x = \pi, 3\pi$

Inflection points of G occur at

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

f.



31. $1 \le \sqrt{1+x^4} \le 1 + x^4$, so

$$\int_0^1 dx \le \int_0^1 \sqrt{1+x^4}\, dx \le \int_0^1 (1+x^4)\, dx$$

$$\int_0^1 dx \le \int_0^1 \sqrt{1+x^4}\, dx \le \int_0^1 dx + \int_0^1 x^4 dx$$

$$[x]_0^1 \le \int_0^1 \sqrt{1+x^4}\, dx \le [x]_0^1 + \left[\frac{x^5}{5}\right]_0^1$$

$$(1-0) \le \int_0^1 \sqrt{1+x^4}\, dx \le (1-0) + \left(\frac{1}{5} - 0\right)$$

$$1 \le \int_0^1 \sqrt{1+x^4}\, dx \le \frac{6}{5}$$

32. On the interval [0,1], $2 \le \sqrt{4+x^4} \le 4 + x^4$. Thus

$$\int_0^1 2\, dx \le \int_0^1 \sqrt{4+x^2}\, dx \le \int_0^1 \left(4+x^2\right) dx$$

$$2 \le \int_0^1 \sqrt{4+x^2}\, dx \le \frac{21}{5}$$

Here, we have used the result from problem 29:

$$\int_0^1 \left(4+x^4\right) dx = \int_0^1 \left(3+1+x^4\right) dx$$

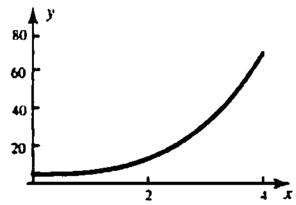
$$= \int_0^1 3\, dx + \int_0^1 \left(1+x^4\right) dx$$

$$= 3 + \tfrac{6}{5} = \tfrac{21}{5}$$

33. $5 \le f(x) \le 69$ so

$$4 \cdot 5 \le \int_0^4 \left(5+x^3\right) dx \le 4 \cdot 69$$

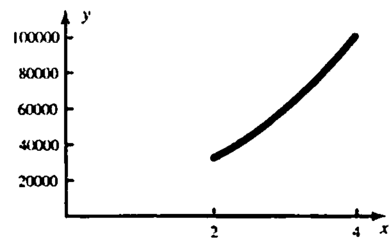
$$20 \le \int_0^4 \left(5+x^3\right) dx \le 276$$



34. On [2,4], $8^5 \le (x+6)^5 \le 10^5$. Thus,

$$2 \cdot 8^5 \le \int_2^4 (x+6)^5\, dx \le 2 \cdot 10^5$$

$$65{,}536 \le \int_2^4 (x+6)^5\, dx \le 200{,}000$$

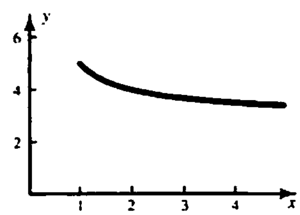


35. On [1,5],

$$3 + \frac{2}{5} \le 3 + \frac{2}{x} \le 3 + \frac{2}{1}$$

$$4\left(\frac{17}{5}\right) \le \int_1^5 \left(3 + \frac{2}{x}\right) dx \le 4 \cdot 5$$

$$\frac{68}{5} \le \int_1^5 \left(3 + \frac{2}{x}\right) dx \le 20$$



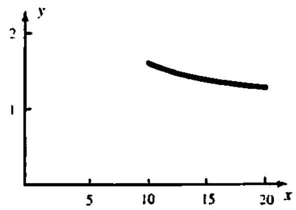
36. On [10,20],

$$\left(1+\frac{1}{20}\right)^5 \le \left(1+\frac{1}{x}\right)^5 \le \left(1+\frac{1}{10}\right)$$

$$10\left(\frac{21}{20}\right)^5 \le \int_{10}^{20}\left(1+\frac{1}{x}\right)^5 dx \le 10\left(\frac{11}{10}\right)^5$$

$$\frac{4,084,101}{320,000} \le \int_{10}^{20}\left(1+\frac{1}{x}\right)^5 dx \le \frac{161,051}{10,000}$$

$$12.7628 \le \int_{10}^{20}\left(1+\frac{1}{x}\right)^5 dx \le 16.1051$$

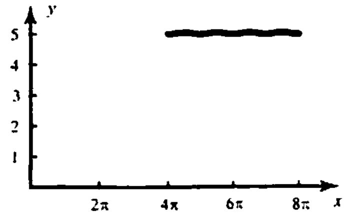


37. On $[4\pi, 8\pi]$

$$5 \le 5+\tfrac{1}{20}\sin^2 x \le 5+\tfrac{1}{20}xdx$$

$$(4\pi)(5) \le \int_{4\pi}^{8\pi}\left(5+\tfrac{1}{20}\sin^2 x\right)dx \le (4\pi)\left(5+\tfrac{1}{20}\right)$$

$$20\pi \le \int_{4\pi}^{8\pi}\left(5+\tfrac{1}{20}\sin^2 x\right)dx \le \frac{101}{5}\pi$$



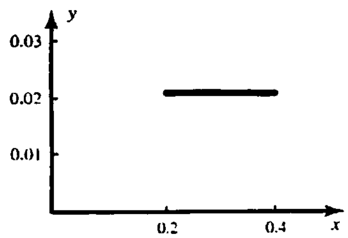
38. On [0.2,0.4],

$$0.002+0.0001\cos^2 0.4 \le 0.002+0.0001\cos^2 x$$
$$\le 0.002+0.0001\cos^2 0.2$$

$$0.2\left(0.002+0.0001\cos^2 0.4\right)$$
$$\le \int_{0.2}^{0.4}\left(0.002+0.0001\cos^2 x\right)dx$$
$$\le 0.2\left(0.002+0.0001\cos^2 0.2\right)$$

Thus,

$$0.000417 \le \int_{0.2}^{0.4}\left(0.002+0.0001\cos^2 x\right)dx$$
$$\le 0.000419$$



39. Let $F(x)=\int_0^x \frac{1+t}{2+t}dt$. Then

$$\lim_{x\to 0}\frac{1}{x}\int_0^x \frac{1+t}{2+t}dt = \lim_{x\to 0}\frac{F(x)-F(0)}{x-0}$$
$$= F'(0) = \frac{1+0}{2+0} = \frac{1}{2}$$

40.

$$\lim_{x\to 1}\frac{1}{x-1}\int_0^x \frac{1+t}{2+t}dt$$
$$= \lim_{x\to 1}\frac{1}{x-1}\left[\int_0^x \frac{1+t}{2+t}dt - \int_0^1 \frac{1+t}{2+t}dt\right]$$
$$= \lim_{x\to 1}\frac{F(x)-F(1)}{x-1}$$
$$= F'(1) = \frac{1+1}{2+1} = \frac{2}{3}$$

41. $\int_1^x f(t)\,dt = 2x-2$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_1^x f(t)\,dt = \frac{d}{dx}(2x-2)$$
$$f(x)=2$$

If such a function exists, it must satisfy $f(x)=2$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When $x=1$ the left side is $\int_1^1 f(t)\,dt=0$ and the right side is $2\cdot 1-2=0$. Thus the function $f(x)=2$ satisfies $\int_1^x f(t)\,dt = 2x-2$.

42. $\int_0^x f(t)\,dt = x^2$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}x^2$$
$$f(x)=2x$$

If such a function exists, it must satisfy $f(x)=2x$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When $x=0$ the left side is $\int_0^0 f(t)\,dt=0$ and the right side is $0^2=0$. Thus the function $f(x)=2x$ satisfies $\int_0^x f(t)\,dt = x^2$.

43. $\int_0^{x^2} f(t)\,dt = \frac{1}{3}x^3$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{dx}\left(\tfrac{1}{3}x^3\right)$$
$$f\left(x^2\right)(2x) = x^2$$
$$f\left(x^2\right) = \frac{x}{2}$$
$$f(x) = \frac{\sqrt{x}}{2}$$

If such a function exists, it must satisfy $f(x) = \frac{\sqrt{x}}{2}$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When $x = 0$ the left side is $\int_0^{0^2} f(t)\,dt = 0$ and the right side is $\frac{\sqrt{0}}{2} = 0$.

Thus the function $f(x) = \frac{\sqrt{x}}{2}$ satisfies

$$\int_0^{x^2} f(t)\,dt = \tfrac{1}{3}x^3.$$

44. No such function exists. When $x = 0$ the left side is 0, whereas the right side is 1

45. True. $f(x)$ will have a minimum, m on $[a, b]$, $m \ge 0$, so $\int_a^b f(x)\,dx \ge \int_a^b m\,dx = m(b-a) \ge 0$.

46. False. $a = -1$, $b = 2$, $f(x) = x$ is a counterexample.

47. False. $a = -1$, $b = 1$, $f(x) = x$ is a counterexample.

48. True. The integral can be zero only when the area above the x-axis equals the area below, but since $f(x) \ge 0$, the area below is zero. Hence $f(x)$ never rises above 0.

49. True. $\int_a^b f(x)dx - \int_a^b g(x)dx$
$= \int_a^b [f(x) - g(x)]dx$

50. False. $a = 0$, $b = 1$, $f(x) = 0$, $g(x) = -1$ is a counterexample.

51. $-|f(x)| \le f(x) \le |f(x)|$, so
$$\int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \Rightarrow$$
$$\int_a^b |f(x)|\,dx \ge -\int_a^b f(x)\,dx$$
and combining this with
$$\int_a^b |f(x)|\,dx \ge \int_a^b f(x)\,dx,$$
we can conclude that
$$\left|\int_a^b f(x)\,dx\right| \ge \int_a^b |f(x)|\,dx$$

52. If $x > a$, $\int_a^x |f'(x)|\,dx \le M(x-a)$ by the Boundedness Property. If $x < a$, $\int_x^a |f(x)|\,dx = -\int_a^x |f'(x)|\,dx \ge -M(x-a)$ by the Boundedness Property. Thus
$$\int_a^x |f'(x)|\,dx \le M|x-a|.$$
From Problem 51, $\int_a^x |f'(x)|\,dx \ge \left|\int_a^x f'(x)\,dx\right|$.
$$\left|\int_a^x f'(x)\,dx\right| = |f(x) - f(a)| \ge |f(x)| - |f(a)|$$
Therefore, $|f(x)| - |f(a)| \le M|x-a|$ or $|f(x)| \le |f(a)| + M|x-a|$.

5.7 Concepts Review

1. antiderivative; $F(b) - F(a)$

2. $F(b) - F(a)$

3. $\frac{(3)^3}{3} - \frac{(1)^3}{3} = \frac{26}{3}$

4. $-\frac{1}{x}; 1 - \frac{1}{2} = \frac{1}{2}$

Problem Set 5.7

1. $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4 - 0 = 4$

2. $\int_{-1}^2 x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^2 = \frac{32}{5} + \frac{1}{5} = \frac{33}{5}$

3. $\int_{-1}^2 (3x^2 - 2x + 3)\,dx = \left[x^3 - x^2 + 3x\right]_{-1}^2$
$= (8 - 4 + 6) - (-1 - 1 - 3) = 15$

4. $\int_1^2 (4x^3+7)\,dx = \left[x^4+7x\right]_1^2$
$= (16+14)-(1+7) = 22$

5. $\int_1^4 \frac{1}{w^2}\,dw = \left[-\frac{1}{w}\right]_1^4 = \left(-\frac{1}{4}\right)-(-1) = \frac{3}{4}$

6. $\int_1^3 \frac{2}{t^3}\,dt = \left[-\frac{1}{t^2}\right]_1^3 = \left(-\frac{1}{9}\right)-(-1) = \frac{8}{9}$

7. $\int_0^4 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^4 = \left(\frac{2}{3}\cdot 8\right)-0 = \frac{16}{3}$

8. $\int_1^8 \sqrt[3]{w}\,dw = \left[\frac{3}{4}w^{4/3}\right]_1^8 = \left(\frac{3}{4}\cdot 16\right)-\left(\frac{3}{4}\cdot 1\right) = \frac{45}{4}$

9. $\int_{-4}^{-2}\left(y^2+\frac{1}{y^3}\right)dy = \left[\frac{y^3}{3}-\frac{1}{2y^2}\right]_{-4}^{-2}$
$= \left(-\frac{8}{3}-\frac{1}{8}\right)-\left(-\frac{64}{3}-\frac{1}{32}\right) = \frac{1783}{96}$

10. $\int_1^4 \frac{s^4-8}{s^2}\,ds = \int_1^4 (s^2-8s^{-2})\,ds = \left[\frac{s^3}{3}+\frac{8}{s}\right]_1^4$
$= \left(\frac{64}{3}+2\right)-\left(\frac{1}{3}+8\right) = 15$

11. $\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1-0 = 1$

12. $\int_{\pi/6}^{\pi/2} 2\sin t\,dt = [-2\cos t]_{\pi/6}^{\pi/2} = 0+\sqrt{3} = \sqrt{3}$

13. $\int_0^1 (2x^4-3x^2+5)\,dx = \left[\frac{2}{5}x^5-x^3+5x\right]_0^1$
$= \left(\frac{2}{5}-1+5\right)-0 = \frac{22}{5}$

14. $\int_0^1 (x^{4/3}-2x^{1/3})\,dx = \left[\frac{3}{7}x^{7/3}-\frac{3}{2}x^{4/3}\right]_0^1$
$= \left(\frac{3}{7}-\frac{3}{2}\right)-0 = -\frac{15}{14}$

15. $u = x^2+1, du = 2x\,dx$
$\int (x^2+1)^{10}(2x)dx = \int u^{10}\,du = \frac{u^{11}}{11}+C$
$= \frac{1}{11}(x^2+1)^{11}+C$
$\int_0^1 (x^2+1)^{10}(2x)dx = \left[\frac{1}{11}(x^2+1)^{11}\right]_0^1$
$= \left[\frac{1}{11}(2)^{11}\right]-\left[\frac{1}{11}(1)^{11}\right] = \frac{2047}{11}$

16. $u = x^3+1, du = 3x^2\,dx$
$\int \sqrt{x^3+1}(3x^2)\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+C$
$= \frac{2}{3}(x^3+1)^{3/2}+C$
$\int_{-1}^0 \sqrt{x^3+1}(3x^2)\,dx = \left[\frac{2}{3}(x^3+1)^{3/2}\right]_{-1}^0$
$= \left(\frac{2}{3}\cdot 1^{3/2}\right)-\left(\frac{2}{3}\cdot 0\right) = \frac{2}{3}$

17. $u = t+2, du = dt$
$\int \frac{1}{(t+2)^2}\,dt = \int u^{-2}\,du = -\frac{1}{u}+C = -\frac{1}{t+2}+C$
$\int_{-1}^3 \frac{1}{(t+2)^2}\,dt = \left[-\frac{1}{t+2}\right]_{-1}^3 = \left[-\frac{1}{5}\right]-[-1] = \frac{4}{5}$

18. $u = y-1, du = dy$
$\int \sqrt{y-1}\,dy = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+C$
$= \frac{2}{3}(y-1)^{3/2}+C$
$\int_2^{10} \sqrt{y-1}\,dy = \left[\frac{2}{3}(y-1)^{3/2}\right]_2^{10}$
$= \left[\frac{2}{3}(27)\right]-\left[\frac{2}{3}(1)\right] = \frac{52}{3}$

19. $u = 3x+1, du = 3\,dx$
$\int \sqrt{3x+1}\,dx = \frac{1}{3}\int \sqrt{3x+1}\,3dx = \frac{1}{3}\int \sqrt{u}\,du$
$= \frac{2}{9}u^{3/2}+C = \frac{2}{9}(3x+1)^{3/2}+C$
$\int_5^8 \sqrt{3x+1}\,dx = \left[\frac{2}{9}(3x+1)^{3/2}\right]_5^8$
$= \left[\frac{2}{9}(125)\right]-\left[\frac{2}{9}(64)\right] = \frac{122}{9}$

20. $u = 2x+2, du = 2\,dx$
$\int \frac{1}{\sqrt{2x+2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{2x+2}}\,dx = \frac{1}{2}\int u^{-1/2}\,du$
$= \sqrt{u}+C = \sqrt{2x+2}+C$
$\int_1^7 \frac{1}{\sqrt{2x+2}}\,dx = \left[\sqrt{2x+2}\right]_1^7 = 4-2 = 2$

21. $u = 7 + 2t^2, du = 4t\,dt$

$$\int \sqrt{7+2t^2}(8t)dt = 2\int \sqrt{7+2t^2}(4t)dt$$

$$= 2\int \sqrt{2}\,du = \frac{4}{3}u^{3/2} + C = \frac{4}{3}(7+2t^2)^{3/2} + C$$

$$\int_{-3}^{3} \sqrt{7+2t^2}(8t)dt = \left[\frac{4}{3}(7+2t^2)^{3/2}\right]_{-3}^{3}$$

$$= \left[\frac{4}{3}(125)\right] - \left[\frac{4}{3}(125)\right] = 0$$

22. $u = x^3 + 3x, du = (3x^2+3)dx$

$$\int \frac{x^2+1}{\sqrt{x^3+3x}}dx = \frac{1}{3}\int \frac{3x^2+3}{\sqrt{x^3+3x}}dx$$

$$= \frac{1}{3}\int u^{-1/2}du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}\sqrt{x^3+3x} + C$$

$$\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}dx = \left[\frac{2}{3}\sqrt{x^3+3x}\right]_1^3$$

$$= \left(\frac{2}{3}\cdot 6\right) - \left(\frac{2}{3}\cdot 2\right) = \frac{8}{3}$$

23. $u = \cos x, du = -\sin x\,dx$

$$\int \cos^2 x \sin x\,dx = -\int \cos^2 x(-\sin x)dx$$

$$= -\int u^2\,du = -\frac{u^3}{3} + C = -\frac{1}{3}\cos^3 x + C$$

$$\int_0^{\pi/2} \cos^2 x \sin x\,dx = \left[-\frac{1}{3}\cos^3 x\right]_0^{\pi/2}$$

$$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$$

24. $u = \sin 3x, du = 3\cos 3x\,dx$

$$\int \sin^2 3x \cos 3x\,dx = \frac{1}{3}\int \sin^2 3x(3\cos 3x)dx$$

$$= \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}\sin^3 3x + C$$

$$\int_0^{\pi/2} \sin^2 3x \cos 3x\,dx = \left[\frac{1}{9}\sin^3 3x\right]_0^{\pi/2}$$

$$= \left(-\frac{1}{9}\right) - 0 = -\frac{1}{9}$$

25. $\int_0^{\pi/2} (2x + \sin x)dx$

$$= 2\int_0^{\pi/2} x\,dx + \int_0^{\pi/2} \sin x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^{\pi/2} + [-\cos x]_0^{\pi/2}$$

$$= 2\left(\frac{\pi^2}{8} - 0\right) + (0+1) = \frac{\pi^2}{4} + 1$$

26. $\int_0^{\pi/2} [4x + \cos x]dx$

$$= 4\int_0^{\pi/2} x\,dx + \int_0^{\pi/2} \cos x\,dx$$

$$= 4\left[\frac{x^2}{2}\right]_0^{\pi/2} + [\sin x]_0^{\pi/2}$$

$$= 4\left(\frac{\pi^2}{8} - 0\right) + (1-0)$$

$$= \frac{\pi^2}{2} + 1 = \frac{\pi^2+2}{2}$$

27. $\int_0^4 \left[\sqrt{x} + \sqrt{2x+1}\right]dx = \int_0^4 \sqrt{x}\,dx + \int_0^4 \sqrt{2x+1}\,dx$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^4 + \frac{1}{2}\left[\frac{2}{3}(2x+1)^{3/2}\right]_0^4$$

$$= \left(\frac{16}{3} - 0\right) + \frac{1}{2}\left(18 - \frac{2}{3}\right) = 14$$

28. $\int_{-4}^{-1} \frac{1-s^4}{2s^2}ds = \int_{-4}^{-1} \frac{1}{2s^2}ds - \int_{-4}^{-1} \frac{s^2}{2}ds$

$$= \frac{1}{2}\left[-\frac{1}{s}\right]_{-4}^{-1} - \frac{1}{2}\left[\frac{s^3}{3}\right]_{-4}^{-1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{4}\right) - \frac{1}{2}\left(-\frac{1}{3} + \frac{64}{3}\right) = -\frac{81}{8}$$

29. $\int_0^1 (x^2+2x)^2\,dx = \int_0^1 (x^4 + 4x^3 + 4x^2)dx$

$$= \int_0^1 x^4\,dx + 4\int_0^1 x^3 dx + 4\int_0^1 x^2\,dx$$

$$= \left[\frac{x^5}{5}\right]_0^1 + 4\left[\frac{x^4}{4}\right]_0^1 + 4\left[\frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{5} - 0\right) + 4\left(\frac{1}{4} - 0\right) + 4\left(\frac{1}{3} - 0\right) = \frac{38}{15}$$

30. $\int_a^{8a} (a^{1/3} - x^{1/3})^3\,dx$

$$= \int_a^{8a} (a - 3a^{2/3}x^{1/3} + 3a^{1/3}x^{2/3} - x)dx$$

$$= a\int_a^{8a} dx - 3a^{2/3}\int_a^{8a} x^{1/3}dx + 3a^{1/3}\int_a^{8a} x^{2/3}dx - \int_a^{8a} x\,dx$$

$$= a[x]_a^{8a} - 3a^{2/3}\left[\frac{3}{4}x^{4/3}\right]_a^{8a} + 3a^{1/3}\left[\frac{3}{5}x^{5/3}\right]_a^{8a} - \left[\frac{x^2}{2}\right]_a^{8a}$$

$$= a(8a-a) - 3a^{2/3}\left(12a^{4/3} - \frac{3}{4}a^{4/3}\right) + 3a^{1/3}\left(\frac{96}{5}a^{5/3} - \frac{3}{5}a^{5/3}\right) - \left(32a^2 - \frac{a^2}{2}\right)$$

$$= 7a^2 - \frac{135}{4}a^2 + \frac{279}{5}a^2 - \frac{63}{2}a^2$$

$$= -\frac{49}{20}a^2$$

31. $\int_1^x t^3\,dt = \left[\frac{t^4}{4}\right]_1^x = \frac{x^4}{4} - \frac{1}{4}$

32. $\int_{-1}^x (t+|t|)\,dt = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases}$

33. $\frac{1}{3-1}\int_1^3 4x^3dx = \frac{1}{2}\left[x^4\right]_1^3 = 40$

34. $\frac{1}{3-0}\int_0^3 \frac{x}{\sqrt{x^2+16}}dx = \frac{1}{3}\left[\sqrt{x^2+16}\right]_0^3 = \frac{1}{3}$

35. $\frac{1}{1+2}\int_{-2}^1 (2+|x|)dx$

$$= \frac{1}{3}\left[\int_{-2}^0 (2-x)dx + \int_0^1 (2+x)dx\right]$$

$$= \frac{1}{3}\left\{\left[2x - \tfrac{1}{2}x^2\right]_{-2}^0 + \left[2x + \tfrac{1}{2}x^2\right]_0^1\right\}$$

$$= \frac{1}{3}\left(-2(-2) + \tfrac{1}{2}(-2)^2 + 2 + \tfrac{1}{2}\right) = \frac{17}{6}$$

36. $\frac{1}{2+3}\int_{-3}^2 (x+|x|)dx$

$$= \frac{1}{5}\left(\int_{-3}^0 (-x+x)dx + \int_0^2 2x\,dx\right)$$

$$= \frac{1}{5}\left[x^2\right]_0^2 = \frac{4}{5}$$

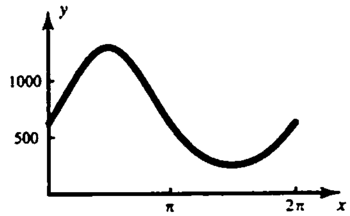
37. $\frac{1}{\pi}\int_0^{\pi} \cos x\,dx = \frac{1}{\pi}[\sin x]_0^{\pi}$

$$= \frac{1}{\pi}[\sin\pi - \sin 0] = 0$$

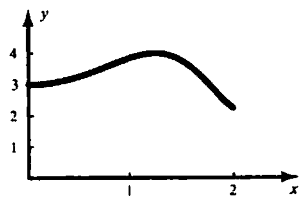
38. $\frac{1}{\pi/2-0}\int_0^{\pi/2} \sin^2 x\cos x\,dx$

$$= \frac{2}{\pi}\left[\frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{2}{3\pi}$$

39. Using $c = \pi$ yields $2\pi(5)^4 = 1250\pi \approx 3927$

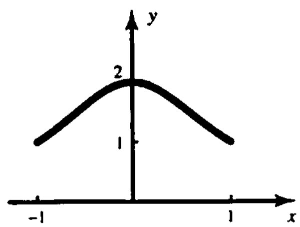


40. Using $c = 0.8$ yields
$2\left(3 + \sin 0.8^2\right) \approx 7.19$

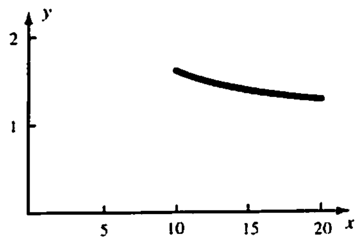


41. Using $c = 0.5$ yields

$$2\frac{2}{1+0.5^2} = 3.2$$



42. Using $c = 15$ yields $\left(\frac{16}{15}\right)^5(20-10) \approx 13.8$.



43. $f(c)(3-0)=\int_0^3 \sqrt{x+1}\,dx$

$3\sqrt{c+1}=\frac{2}{3}\left[(x+1)^{3/2}\right]_0^3$

$\sqrt{c+1}=\frac{2}{9}(8-1)$

$c=\left(\frac{14}{9}\right)^2-1=\frac{115}{81}$

44. $f(c)(1-(-1))=\int_{-1}^1 x^2\,dx$

$2f(c)=\left[\frac{x^3}{3}\right]_{-1}^1$

$2c^2=\frac{2}{3}$

$c=\pm\frac{\sqrt{3}}{3}$

Both $\pm\sqrt{3}/3$ are in the interval $(-1,1)$.

45. $2f(c)=\int_0^2 |x|\,dx$

$2|c|=\int_0^2 x\,dx$

$2|c|=\left[\frac{x^2}{2}\right]_0^2$

$2|c|=2$

$c=\pm1$

Only $c=1$ is in the interval $[0,2]$.

46. $4f(c)=\int_{-2}^2 |x|\,dx$

$4|c|=2\int_0^2 x\,dx=2\left[\frac{x^2}{2}\right]_0^2$

$4|c|=4$

$c=\pm1$

Both ±1 are in the interval $[-2,2]$.

47. $\int_0^3 x^2dx=\left[\frac{x^3}{3}\right]_0^3=9-0=9$

48. $\int_0^2 x^3\,dx=\left[\frac{x^4}{4}\right]_0^2=4-0=4$

49. $\int_0^\pi \sin x\,dx=[-\cos x]_0^\pi=1+1=2$

50. $\int_0^2 (1+x+x^2)dx=\int_0^2 dx+\int_0^2 x\,dx+\int_0^2 x^2\,d$

$=[x]_0^2+\left[\frac{x^2}{2}\right]_0^2+\left[\frac{x^3}{3}\right]_0^2$

$=(2-0)+(2-0)+\left(\frac{8}{3}-0\right)=\frac{20}{3}$

51. The right-endpoint Riemann sum is

$\sum_{i=1}^{n}\left(0+\frac{1-0}{n}i\right)^2\left(\frac{1}{n}\right)=\frac{1}{n^3}\sum_{i=1}^{n}i^2$, which for

$n=10$ equals $\frac{77}{200}=0.385$.

$\int_0^1 x^2dx=\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{3}=0.\overline{333}$

52. $\int_{-2}^4 (2[\![x]\!]-3|x|)dx=2\int_{-2}^4 [\![x]\!]dx-3\int_{-2}^4 |x|dx$

$=2[(-2-1+0+1+2+3)(1)]$

$-3\left[\frac{1}{2}(2)(2)+\frac{1}{2}(4)(4)\right]$

$=-24$

53. $\frac{d}{dx}\left(\frac{1}{2}x|x|\right)=\frac{1}{2}x\left(\frac{|x|}{x}\right)+\frac{|x|}{2}=|x|$

$\int_a^b |x|\,dx=\left[\frac{1}{2}x|x|\right]_a^b=\frac{1}{2}(b|b|-a|a|)$

54. For $b>0$, if b is an integer,

$\int_0^b [\![x]\!]dx=0+1+2+\cdots+(b-1)$

$=\sum_{i=1}^{b-1} i=\frac{(b-1)b}{2}.$

If b is not an integer, let $n = [\![b]\!]$. Then

$$\int_0^b [\![x]\!]dx = 0+1+2+\cdots+(n-1)+n(b-n)$$

$$= \frac{(n-1)n}{2} + n(b-n)$$

$$= \frac{([\![b]\!]-1)[\![b]\!]}{2} + [\![b]\!](b-[\![b]\!]).$$

55. a. $\int_a^b x^n dx = B_n;\ \int_{a^n}^{b^n} \sqrt[n]{y}\,dy = A_n$

Using Figure 3 of the text,
$(a)(a^n) + A_n + B_n = (b)(b^n)$ or

$B_n + A_n = b^{n+1} - a^{n+1}$. Thus

$$\int_a^b x^n dx + \int_{a^n}^{b^n} \sqrt[n]{y}\,dy = b^{n+1} - a^{n+1}$$

b. $\int_a^b x^n\,dx + \int_{a^n}^{b^n} \sqrt[n]{y}\,dy$

$$= \left[\frac{x^{n+1}}{n+1}\right]_a^b + \left[\frac{n}{n+1}y^{(n+1)/n}\right]_{a^n}^{b^n}$$

$$= \left(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}\right) + \left(\frac{n}{n+1}b^{n+1} - \frac{n}{n+1}a^{n+1}\right)$$

$$= \frac{(n+1)b^{n+1} - (n+1)a^{n+1}}{n+1} = b^{n+1} - a^{n+1}$$

c. $B_n = \int_a^b x^n\,dx = \frac{1}{n+1}\left[x^{n+1}\right]_a^b$

$$= \frac{1}{n+1}(b^{n+1} - a^{n+1})$$

$$A_n = \int_{a^n}^{b^n} \sqrt[n]{y}\,dy = \left[\frac{n}{n+1}y^{(n+1)/n}\right]_{a^n}^{b^n}$$

$$= \frac{n}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

$$nB_n = \frac{n}{n+1}(b^{n+1} - a^{n+1}) = A_n$$

56. Since f is continuous on a closed interval $[a,b]$ there exist (by the Min-Max Existence Theorem) an m and M in $[a,b]$ such that $f(m) \le f(x) \le f(M)$ for all x in $[a,b]$. Thus

$$\int_a^b f(m)dx \le \int_a^b f(x)dx \le \int_a^b f(M)dx$$

$$(b-a)f(m) \le \int_a^b f(x)dx \le (b-a)f(M)$$

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)dx \le f(M)$$

Since f is continuous, we can apply the Intermediate Value Theorem and say that f takes on every value between $f(m)$ and $f(M)$. Since $\frac{1}{b-a}\int_a^b f(x)dx$ is between $f(m)$ and $f(M)$, there exists a c in $[a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

57. a. Let c be in $[a,b]$. Then $G'(c) = f(c)$ by the First Fundamental Theorem of Calculus. Since G is differentiable at c, G is continuous there. Now suppose $c = a$. Then

$\lim_{x\to c} G(x) = \lim_{x\to a} \int_a^x f(t)\,dt$. Since f is continuous on $[a,b]$, there exist (by the Min-Max Existence Theorem) m and M such that $f(m) \le f(x) \le f(M)$ for all x in $[a,b]$. Then

$$\int_a^x f(m)\,dt \le \int_a^x f(t)\,dt \le \int_a^x f(M)\,dt$$

$$(x-a)f(m) \le G(x) \le (x-a)f(M)$$

By the Squeeze Theorem

$$\lim_{x\to a^+}(x-a)f(m) \le \lim_{x\to a^+} G(x) \le \lim_{x\to a^+}(x-a)f(M)$$

Thus

$$\lim_{x\to a^+} G(x) = 0 = \int_a^a f(t)\,dt = G(a)$$

Therefore G is right-continuous at $x = a$.
Now, suppose $c = b$. Then

$$\lim_{x\to b^-} G(x) = \lim_{x\to b^-} \int_x^b f(t)\,dt$$

As before,

$$(b-x)f(m) \le G(x) \le (b-x)f(M)$$

so we can apply the Squeeze Theorem again to obtain

$$\lim_{x\to b^-}(b-x)f(m) \le \lim_{x\to b^-} G(x) \le \lim_{x\to b^-}(b-x)f(M)$$

Thus

$$\lim_{x\to b^-} G(x) = 0 = \int_b^b f(t)\,dt = G(b)$$

Therefore, G is left-continuous at $x = b$.

b. Let F be any antiderivative of f. Note that G is also an antiderivative of f. Thus, $F(x) = G(x) + C$. We know from part (a) that $G(x)$ is continuous on $[a,b]$. Thus $F(x)$, being equal to $G(x)$ plus a constant, is also continuous on $[a,b]$.

58. Let

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \ge 0 \end{cases}$$

and $F(x) = \int_{-1}^{x} f(t)\,dt$. If $x < 0$, then $F(x) = 0$. If $x \ge 0$, then

$$F(x) = \int_{-1}^{x} f(t)\,dt$$
$$= \int_{-1}^{0} 0\,dt + \int_{0}^{x} 1\,dt$$
$$= 0 + x = x$$

Thus,

$$F(x) = \begin{cases} x, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

which is continuous everywhere even though $f(x)$ is not continuous everywhere.

59. **a.** Between 0 and 3, $f(x) > 0$. Thus,

$$\int_0^3 f(x)\,dx > 0.$$

b. Since f is an antiderivative of f',

$$\int_0^3 f'(x)\,dx = f(3) - f(0)$$
$$= 0 - 2 = -2 < 0$$

c. $$\int_0^3 f''(x)\,dx = f'(3) - f'(0)$$
$$= -1 - 0 = -1 < 0$$

d. $$\int_0^3 f'''(x)\,dx = f''(3) - f''(0)$$
$$= 0 - (\text{negative number}) > 0$$

Since f is concave down at 0, $f''(0) < 0$.

60. **a.** On $[0,4]$, $f(x) > 0$. Thus,

$$\int_0^4 f(x)\,dx > 0.$$

b. Since f is an antiderivative of f',

$$\int_0^4 f'(x)\,dx = f(4) - f(0)$$
$$= 1 - 2 = -1 < 0$$

c. $$\int_0^4 f''(x)\,dx = f'(4) - f'(0)$$
$$= \frac{1}{4} - (-2) = \frac{9}{4} > 0$$

d. $$\int_0^4 f'''(x)\,dx = f''(4) - f''(0)$$
$$= (\text{positive}) - (\text{negative}) > 0$$

61. Let $y = G(x) = \int_a^x f(t)\,dt$. Then

$$\frac{dy}{dx} = G'(x) = f(x)$$
$$dy = f(x)\,dx$$

Let F be any antiderivative of f. Then $y = F(x) + C$. When $x = a$, we must have $y = 0$. Thus, $C = -F(a)$ and $y = F(x) - F(a)$. Now choose $x = b$ to obtain

$$\int_a^b f(t)\,dt = y = F(b) - F(a)$$

62. **a.** A rectangle with height 30 and width 24 has approximately the same area as that under the curve. Thus,

$$\frac{1}{24-0}\int_0^{24} T(t)\,dt \approx 30$$

b. Yes. The Mean Value Theorem for Integrals guarantees the existence of a c such that

$$\frac{1}{24-0}\int_0^{24} T(t)\,dt = T(c)$$

The figure indicates that there are actually two such values of c, roughly, $c = 11$ and $c = 16$.

63. A rectangle with height 25 and width 7 has approximately the same area as that under the curve. Thus

$$\frac{1}{7}\int_0^7 H(t)\,dt \approx 25$$

5.8 Concepts Review

1. $\frac{1}{3}\int_1^2 u^4\, du$

2. 0; $2\int_0^2 f(x)\,dx$

3. $\int_0^2 (2+x^2)\,dx$; 0

4. $f(x+p)=f(x)$; period

Problem Set 5.8

1. $u=3x+2,\ du=3\,dx$

$\int \sqrt{u}\cdot\frac{1}{3}du=\frac{2}{9}u^{3/2}+C=\frac{2}{9}(3x+2)^{3/2}+C$

2. $u=2x-4,\ du=2\,dx$

$\int u^{1/3}\cdot\frac{1}{2}du=\frac{3}{8}u^{4/3}+C=\frac{3}{8}(2x-4)^{4/3}+C$

3. $u=6x-7,du=6dx$

$\int u^{1/8}\cdot\frac{1}{6}du=\frac{4}{27}u^{9/8}+C=\frac{4}{27}(6x-7)^{9/8}+C$

4. $x=5u-\pi,dx=5du$

$\int x^{21/8}\cdot\frac{1}{5}dx=\frac{8}{145}x^{29/8}+C$

$=\frac{8}{145}(5u-\pi)^{29/8}+C$

5. $u=3x+2,\ du=3\,dx$

$\int \cos(u)\cdot\frac{1}{3}du=\frac{1}{3}\sin u+C=\frac{1}{3}\sin(3x+2)+C$

6. $u=2x-4,\ du=2\,dx$

$\int \sin u\cdot\frac{1}{2}du=-\frac{1}{2}\cos u+C$

$=-\frac{1}{2}\cos(2x-4)+C$

7. $u=6x-7,\ du=6dx$

$\int \sin u\cdot\frac{1}{6}du=-\frac{1}{6}\cos u+C$

$=-\frac{1}{6}\cos(6x-7)+C$

8. $u=\pi v-\sqrt{7},\ du=\pi\,dv$

$\int \cos u\cdot\frac{1}{\pi}du=\frac{1}{\pi}\sin u+C=\frac{1}{\pi}\sin(\pi v-\sqrt{7})+C$

9. $u=x^2+4,du=2x\,dx$

$\int \sqrt{u}\cdot\frac{1}{2}du=\frac{1}{3}u^{3/2}+C=\frac{1}{3}(x^2+4)^{3/2}+C$

10. $u=x^3+5,du=3x^2\,dx$

$\int u^9\cdot\frac{1}{3}du=\frac{1}{30}u^{10}+C=\frac{1}{30}(x^3+5)^{10}+C$

11. $u=x^2+3,du=2x\,dx$

$\int u^{-12/7}\cdot\frac{1}{2}du=-\frac{7}{10}u^{-5/7}+C$

$=-\frac{7}{10}(x^2+3)^{-5/7}+C$

12. $u=\sqrt{3}\,v^2+\pi,du=2\sqrt{3}v\,dv$

$\int u^{7/8}\cdot\frac{1}{2\sqrt{3}}du=\frac{4}{15\sqrt{3}}u^{15/8}+C$

$=\frac{4}{15\sqrt{3}}\left(\sqrt{3}\,v^2+\pi\right)^{15/8}+C$

13. $u=x^2+4,du=2x\,dx$

$\int \sin(u)\cdot\frac{1}{2}du=-\frac{1}{2}\cos u+C$

$=-\frac{1}{2}\cos(x^2+4)+C$

14. $u=x^3+5,du=3x^2\,dx$

$\int \cos u\cdot\frac{1}{3}du=\frac{1}{3}\sin u+C=\frac{1}{3}\sin(x^3+5)+C$

15. $y=6x^3-7,du=18x^2\,dx$

$\int \sin u\cdot\frac{1}{18}du=-\frac{1}{18}\cos u+C$

$=-\frac{1}{18}\cos(6x^3-7)+C$

16. $u=\pi v^5-\sqrt{7},du=5\pi v^4dv$

$\int \cos u\cdot\frac{1}{5\pi}du=\frac{1}{5\pi}\sin u+C$

$=\frac{1}{5\pi}\sin(\pi v^5-\sqrt{7})+C$

17. $u = \sqrt{x^2+4}, du = \frac{x}{\sqrt{x^2+4}}dx$

$\int \sin u\, du = -\cos u + C = -\cos\sqrt{x^2+4} + C$

18. $u = \sqrt[3]{z^2+3}, du = \frac{2z}{3\left(\sqrt[3]{z^2+3}\right)^2}dz$

$\int \cos u \cdot \frac{3}{2}du = \frac{3}{2}\sin u + C = \frac{3}{2}\sin\sqrt[3]{z^2+3} + C$

19. $u = (x^3+5)^9$,

$du = 9(x^3+5)^8(3x^2)dx = 27x^2(x^3+5)^8 dx$

$\int \cos u \cdot \frac{1}{27}du = \frac{1}{27}\sin u + C$

$= \frac{1}{27}\sin\left[(x^3+5)^9\right] + C$

20. $u = (7x^7+\pi)^9, du = 441x^6(7x^7+\pi)^8 dx$

$\int \sin u \cdot \frac{1}{441}du = -\frac{1}{441}\cos u + C$

$= -\frac{1}{441}\cos(7x^7+\pi)^9 + C$

21. $u = \sin(x^2+4), du = 2x\cos(x^2+4)dx$

$\int \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{3}u^{3/2} + C$

$= \frac{1}{3}\left[\sin(x^2+4)\right]^{3/2} + C$

22. $u = \cos(3x^7+9)$

$du = -21x^6\sin(3x^7+9)dx$

$\int \sqrt[3]{u}\cdot\left(-\frac{1}{21}\right)du = -\frac{1}{28}u^{4/3} + C$

$= -\frac{1}{28}\left[\cos(3x^7+9)\right]^{4/3} + C$

23. $u = \cos(x^3+5), du = -3x^2\sin(x^3+5)dx$

$\int u^9\cdot\left(-\frac{1}{3}\right)du = -\frac{1}{30}u^{10} + C$

$= -\frac{1}{30}\cos^{10}(x^3+5) + C$

24. $u = \tan(x^{-3}+1)$

$du = -3x^{-4}\sec^2(x^{-3}+1)dx$

$\int \sqrt[5]{u}\cdot\left(-\frac{1}{3}\right)du = -\frac{5}{18}u^{6/5} + C$

$= -\frac{5}{18}\left[\tan(x^{-3}+1)\right]^{6/5} + C$

25. $\int (x+1)\sec^{1/2}(x^2+2x)\tan(x^2+2x)dx$

$= \int (x+1)\frac{\sin(x^2+2x)}{\cos^{3/2}(x^2+2x)}dx$

$u = \cos(x^2+2x), du = -(2x+2)\sin(x^2+2x)dx$

$\int u^{-3/2}\cdot -\frac{1}{2}du = u^{-1/2} + C$

$= \frac{1}{\cos^{1/2}(x^2+2x)} + C = \sec^{1/2}(x^2+2x) + C$

26. $u = \sqrt{x}+4, du = \frac{1}{2\sqrt{x}}dx$

$\int u^2\cdot 2du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(\sqrt{x}+4\right)^3 + C$

27. $u = 3x+1, du = 3dx$

$\frac{1}{3}\int_1^4 u^3\,du = \left[\frac{1}{12}\cdot u^4\right]_1^4 = \left(\frac{64}{3}-\frac{1}{12}\right) = \frac{85}{4}$

28. $u = 2t+1, du = 2dt$

$\frac{1}{2}\int_1^9 \sqrt{u}\,du = \frac{1}{3}\left[u^{3/2}\right]_1^9 = \frac{1}{3}(27-1) = \frac{26}{3}$

29. $u = t^2+9, du = 2t\,dt$

$\frac{1}{2}\int_9^{13}\frac{1}{u^2}du = -\frac{1}{2}\left[\frac{1}{u}\right]_9^{13} = -\frac{1}{2}\left(\frac{1}{13}-\frac{1}{9}\right) = \frac{2}{117}$

30. $u = 9-x^2, du = -2x\,dx$

$-\frac{1}{2}\int_9^4 \sqrt{u}\,du = -\frac{1}{3}\left[u^{3/2}\right]_9^4 = -\frac{1}{3}(8-27) = \frac{19}{3}$

31. $u = x^2+4x+1, du = (2x+4)dx$

$\frac{1}{2}\int_1^6 u^{-2}du = -\frac{1}{2}\left[\frac{1}{u}\right]_1^6 = -\frac{1}{2}\left(\frac{1}{6}-1\right) = \frac{5}{12}$

32. $u = 9-x^3, du = -3x^2\,dx$

$-\frac{1}{3}\int_9^1 u^{-3/2}du = \frac{2}{3}\left[u^{-1/2}\right]_9^1 = \frac{2}{3}\left(1-\frac{1}{3}\right) = \frac{4}{9}$

33. $u = \sin\theta, du = \cos\theta\,d\theta$

$\int_0^{1/2} u^3 du = \left[\frac{u^4}{4}\right]_0^{1/2} = \frac{1}{64} - 0 = \frac{1}{64}$

34. $u = \cos\theta, du = -\sin\theta\, d\theta$

$$-\int_1^{\sqrt{3}/2} u^{-3} du = \frac{1}{2}\left[u^{-2}\right]_1^{\sqrt{3}/2} = \frac{1}{2}\left(\frac{4}{3}-1\right) = \frac{1}{6}$$

35. $u = 3x - 3, du = 3dx$

$$\frac{1}{3}\int_{-3}^{0} \cos u\, du = \frac{1}{3}[\sin u]_{-3}^{0} = \frac{1}{3}(0 - \sin(-3))$$

$$= \frac{\sin 3}{3}$$

36. $u = 2\pi x, du = 2\pi dx$

$$\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = -\frac{1}{2\pi}[\cos u]_0^{\pi} = -\frac{1}{2\pi}(-1-1)$$

$$= \frac{1}{\pi}$$

37. $u = \pi x^2, du = 2\pi x\, dx$

$$\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = -\frac{1}{2\pi}[\cos u]_0^{\pi} = -\frac{1}{2\pi}(-1-1)$$

$$= \frac{1}{\pi}$$

38. $u = 2x^5, du = 10x^4 dx$

$$\frac{1}{10}\int_0^{2\pi^5} \cos u\, du = \frac{1}{10}[\sin u]_0^{2\pi^5}$$

$$= \frac{1}{10}(\sin(2\pi^5) - 0) = \frac{1}{10}\sin(2\pi^5)$$

39. $u = 2x, du = 2dx$

$$\frac{1}{2}\int_0^{\pi/2} \cos u\, du + \frac{1}{2}\int_0^{\pi/2} \sin u\, du$$

$$= \frac{1}{2}[\sin u]_0^{\pi/2} - \frac{1}{2}[\cos u]_0^{\pi/2}$$

$$= \frac{1}{2}(1-0) - \frac{1}{2}(0-1) = 1$$

40. $u = 3x, du = 3dx; v = 5x, dv = 5dx$

$$\frac{1}{3}\int_{-3\pi/2}^{3\pi/2} \cos u\, du + \frac{1}{5}\int_{-5\pi/2}^{5\pi/2} \sin v\, dv$$

$$= \frac{1}{3}[\sin u]_{-3\pi/2}^{3\pi/2} - \frac{1}{5}[\cos v]_{-5\pi/2}^{5\pi/2}$$

$$= \frac{1}{3}[(-1)-1] - \frac{1}{5}[0-0] = -\frac{2}{3}$$

41. $u = \cos x, du = -\sin x\, dx$

$$-\int_1^0 \sin u\, du = [\cos u]_1^0 = 1 - \cos 1$$

42. $u = \pi\sin\theta, du = \pi\cos\theta\, d\theta$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos u\, du = \frac{1}{\pi}[\sin u]_{-\pi}^{\pi} = 0$$

43. $u = \cos(x^2),\ du = -2x\sin(x^2)dx$

$$-\frac{1}{2}\int_1^{\cos 1} u^3 du = -\frac{1}{2}\left[\frac{u^4}{4}\right]_1^{\cos 1} = -\frac{\cos^4 1}{8} + \frac{1}{8}$$

$$= \frac{1 - \cos^4 1}{8}$$

44. $u = \sin(x^3), du = 3x^2\cos(x^3)dx$

$$\frac{1}{3}\int_{-\sin(\pi^3/8)}^{\sin(\pi^3/8)} u^2\, du = \frac{1}{9}\left[u^3\right]_{-\sin(\pi^3/8)}^{\sin(\pi^3/8)} = \frac{2\sin^3\left(\frac{\pi^3}{8}\right)}{9}$$

45. $u = \sqrt{t} + 1, du = \frac{1}{2\sqrt{t}}dt$

$$2\int_2^3 u^{-3} du = \left[-u^{-2}\right]_2^3 = \frac{5}{36}$$

46. $u = 1 + \frac{1}{t}, du = -\frac{1}{t^2}dt$

$$-\int_2^{3/2} u^2\, du = -\frac{1}{3}\left[u^3\right]_2^{3/2} = \frac{37}{24}$$

47. $$\int_{-\pi}^{\pi} (\sin x + \cos x)\, dx = \int_{-\pi}^{\pi} \sin x\, dx + 2\int_0^{\pi} \cos x\, dx$$

$$= 0 + 2[\sin x]_0^{\pi} = 0$$

48. $\int_{-1}^{1} \frac{x^3}{(1+x^2)^4} dx = 0$, since

$$\frac{(-x)^3}{(1+(-x)^2)^4} = -\frac{x^3}{(1+x^2)^4}$$

49. $\int_{-\pi/2}^{\pi/2} \frac{\sin x}{1+\cos x} dx = 0$, since

$$\frac{\sin(-x)}{1+\cos(-x)} = -\frac{\sin x}{1+\cos x}$$

50. $$\int_{-\sqrt{3}\,\pi}^{\sqrt{3}\,\pi} x^2\cos(x^3)\, dx = 2\int_0^{\sqrt{3}\,\pi} x^2\cos(x^3)\, dx$$

$$= \frac{2}{3}\left[\sin(x^3)\right]_0^{\sqrt{3}\,\pi} = \frac{2}{3}\sin\left(3\sqrt{3}\,\pi^3\right)$$

51. $\int_{-\pi}^{\pi} (\sin x + \cos x)^2\, dx$

$= \int_{-\pi}^{\pi} (\sin^2 x + 2\sin x \cos x + \cos^2 x)\, dx$

$= \int_{-\pi}^{\pi} (1 + 2\sin x \cos x)\, dx = \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sin 2x\, dx$

$= 2\int_0^{\pi} dx + 0 = 2[x]_0^{\pi} = 2\pi$

52. $\int_{-\pi/2}^{\pi/2} z\sin^2(z^3)\cos(z^3)dz = 0$, since

$(-z)\sin^2[(-z)^3]\cos[(-z)^3]$

$= -z\sin^2(-z^3)\cos(-z^3)$

$= -z[-\sin(z^3)]^2 \cos(z^3)$

$= -z\sin^2(z^3)\cos(z^3)$

53. $\int_{-1}^{1} (1 + x + x^2 + x^3)\, dx$

$= \int_{-1}^{1} dx + \int_{-1}^{1} x\, dx + \int_{-1}^{1} x^2 dx + \int_{-1}^{1} x^3 dx$

$= 2[x]_0^1 + 0 + 2\left[\frac{x^3}{3}\right]_0^1 + 0 = \frac{8}{3}$

54. $\int_{-100}^{100} (v + \sin v + v\cos v + \sin^3 v)^5\, dv = 0$

since $(-v + \sin(-v) - v\cos(-v) + \sin^3(-v))^5$

$= (-v - \sin v - v\cos v - \sin^3 v)^5$

$= -(v + \sin v + v\cos v + \sin^3 v)^5$

55. $\int_{-1}^{1} \left(\left|x^3\right| + x^3\right) dx = 2\int_0^1 \left|x^3\right| dx + \int_{-1}^{1} x^3 dx$

$= 2\left[\frac{x^4}{4}\right]_0^1 + 0 = \frac{1}{2}$

56. $\int_{-\pi/4}^{\pi/4} \left(|x|\sin^5 x + |x|^2 \tan x\right) dx = 0$

since $|-x|\sin^5(-x) + |-x|^2 \tan(-x)$

$= -|x|\sin^5 x - |x|^2 \tan x$

57. $\int_{-b}^{-a} f(x)\, dx = \int_a^b f(x)\, dx$ when f is even.

$\int_{-b}^{-a} f(x)\, dx = -\int_a^b f(x)\, dx$ when f is odd.

58. $u = -x,\ du = -dx$

$-\int_{-a}^{-b} f(u)\, du = \int_{-b}^{-a} f(u)\, du = \int_{-b}^{-a} f(x)\, dx$

since the variable used in the integration is not important.

59. $\int_0^{4\pi} |\cos x|\, dx = \int_0^{\pi} |\cos x|\, dx + \int_{\pi}^{2\pi} |\cos x|\, dx$

$+ \int_{2\pi}^{3\pi} |\cos x|\, dx + \int_{3\pi}^{4\pi} |\cos x|\, dx$

$= \int_0^{\pi} |\cos x|\, dx + \int_0^{\pi} |\cos x|\, dx + \int_0^{\pi} |\cos x|\, dx$

$+ \int_0^{\pi} |\cos x|\, dx$

$= 4\int_0^{\pi} |\cos x|\, dx = 8\int_0^{\pi/2} |\cos x|\, dx$

$= 8[\sin x]_0^{\pi/2} = 8$

60. Since $\sin x$ is periodic with period 2π, $\sin 2x$ is periodic with period π.

$\int_0^{4\pi} |\sin 2x|\, dx$

$= \int_0^{\pi} |\sin 2x|\, dx + \int_{\pi}^{2\pi} |\sin 2x|\, dx + \int_{2\pi}^{3\pi} |\sin 2x|\, dx$

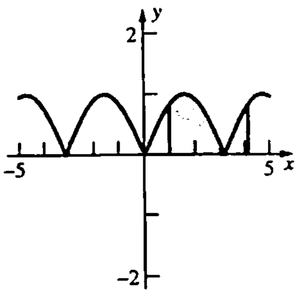
$+ \int_{3\pi}^{4\pi} |\sin 2x|\, dx$

$= 4\int_0^{\pi} |\sin 2x|\, dx = 8\int_0^{\pi/2} \sin 2x\, dx$

$= 8\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = -4(-1 - 1) = 8$

61. $\int_1^{1+\pi} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx$

$= [-\cos x]_0^{\pi} = 2$



62. $\int_2^{2+\pi/2} |\sin 2x|\, dx = \int_0^{\pi/2} |\sin 2x|\, dx$

$= \frac{1}{2}[-\cos 2x]_0^{\pi/2} = 1$

63. $\frac{1}{12}\int_6^{18} T(t)\, dt = \left[70t - \frac{96}{\pi}\cos\left[\frac{\pi}{12}(t-9)\right]\right]_6^{18}$

$= 70(18-6) - \frac{96}{\pi}\left(\cos\frac{3\pi}{4} - \cos\frac{\pi}{4}\right)$

$= 840 + \frac{96\sqrt{2}}{\pi} \approx 883.2$

64. a. Written response.

b. $A = \int_0^a g(x)\,dx = \int_0^a \frac{a}{c} f\left(\frac{c}{a}x\right)dx$

$= \int_0^c \frac{a}{c} f(x)\frac{a}{c}\,dx = \frac{a^2}{c^2}\int_0^c f(x)\,dx$

$B = \int_0^b h(x)\,dx = \int_0^b \frac{b}{c} f\left(\frac{c}{b}x\right)dx$

$= \int_0^c \frac{b}{c} f(x)\frac{b}{c}\,dx = \frac{b^2}{c^2}\int_0^c f(x)\,dx$

Thus, $\int_0^a g(x)\,dx + \int_0^b h(x)\,dx$

$= \frac{a^2}{c^2}\int_0^c f(x)\,dx + \frac{b^2}{c^2}\int_0^c f(x)\,dx$

$= \frac{a^2+b^2}{c^2}\int_0^c f(x)\,dx = \int_0^c f(x)\,dx$ since

$a^2+b^2=c^2$ from the triangle.

65. $y' = -k^2 \sin kx$

At $x = \frac{\pi}{2k}$, $y' = -k^2$. The equation of the tangent line at $x = \frac{\pi}{2k}$ is $y = -k^2\left(x - \frac{\pi}{2k}\right)$.

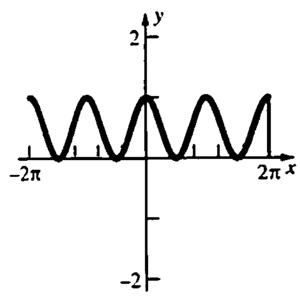
When $x = 0$, $y = \frac{k\pi}{2}$.

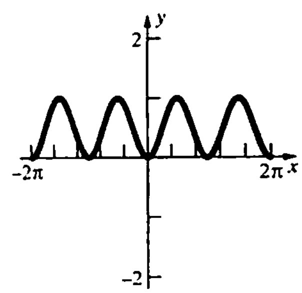
$A = \frac{1}{2}\left(\frac{\pi}{k}\right)\left(\frac{k\pi}{2}\right) - \int_{-\pi/2k}^{\pi/2k} k\cos kx\,dx$

$= \frac{\pi^2}{4} - [\sin kx]_{-\pi/2k}^{\pi/2k} = \frac{\pi^2}{4} - 2$

66. a. $\int_0^{2\pi}(\sin^2 x + \cos^2 x)\,dx = \int_0^{2\pi} dx = [x]_0^{2\pi} =$ 2π

b.





c. $2\pi = \int_0^{2\pi}\cos^2 x\,dx + \int_0^{2\pi}\sin^2 x\,dx$

$= 2\int_0^{2\pi}\cos^2 x\,dx$, thus $\int_0^{2\pi}\cos^2 x\,dx$

$= \int_0^{2\pi}\sin^2 x\,dx = \pi$

67. a. Even

b. 2π

c.

Interval	Value of Integral
$\left[0, \frac{\pi}{2}\right]$	0.46
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	0.92
$\left[0, \frac{3\pi}{2}\right]$	–0.46
$\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$	–0.92
$[0, 2\pi]$	0
$\left[\frac{\pi}{6}, \frac{13\pi}{6}\right]$	0
$\left[\frac{\pi}{6}, \frac{4\pi}{3}\right]$	–0.44
$\left[\frac{13\pi}{6}, \frac{10\pi}{3}\right]$	–0.44

68. a. Odd

b. 2π

c.

Interval	Value of Integral
$\left[0, \frac{\pi}{2}\right]$	0.69
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	0
$\left[0, \frac{3\pi}{2}\right]$	0.69
$\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$	0
$[0, 2\pi]$	0
$\left[\frac{\pi}{6}, \frac{13\pi}{6}\right]$	0
$\left[\frac{\pi}{6}, \frac{4\pi}{3}\right]$	1.06
$\left[\frac{13\pi}{6}, \frac{10\pi}{3}\right]$	1.06

69. Using a right-Riemann sum,

$$\text{Distance} = \int_0^{24} v(t)\,dt$$

$$\approx \sum_{i=1}^{8} v(t_i)\Delta t$$

$$= (31+54+53+52+35+31+28)\frac{3}{60}$$

$$= \frac{852}{60} = 14.2 \text{ miles}$$

70. Using a right-Riemann sum,

$$\text{Water Usage} = \int_0^{120} F(t)\,dt$$

$$\approx \sum_{i=1}^{10} F(t_i)\Delta t$$

$$= (71+68+78+105+111+108 +144+160+152+148)12$$

$$= 1145 \cdot 12 = 13,740 \text{ gallons}$$

71. If f is odd, then $f(-x) = -f(x)$ and we can write

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0}[-f(-x)]\,dx$$

$$= \int_a^0 f(u)\,du$$

$$= -\int_0^a f(u)\,du$$

$$= -\int_0^a f(x)\,dx$$

On the second line, we have made the substitution $u = -x$.

5.9 Chapter Review

Concepts Test

1. True: Theorem 5.1.C

2. True: Obtained by integrating both sides of the Product Rule

3. True: $(-\sin x)^2 = \sin^2 x = 1 - \cos^2 x$

4. True: Theorem 5.6A

5. True: If $F(x) = \int f(x)\,dx$, $f(x)$ is a derivative of $F(x)$.

6. False: $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 7x - 5$ are a counterexample.

7. False: The two sides will in general differ by a constant term.

8. True: At any given height, speed on the downward trip is the negative of speed on the upward.

9. True: $a_1 + a_0 + a_2 + a_1 + a_3 + a_2 + \cdots + a_{n-1} + a_{n-2} + a_n + a_{n-1}$
$= a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n$

10. True: $\sum_{i=1}^{100}(2i-1) = 2\sum_{i=1}^{100} i - \sum_{i=1}^{100} 1$
$= \frac{2(100)(100+1)}{2} - 100 = 10,000$

11. True: $\sum_{i=1}^{10}(a_i+1)^2 = \sum_{i=1}^{10} a_i^2 + 2\sum_{i=1}^{10} a_i + \sum_{i=1}^{100} 1$
$= 100 + 2(20) + 10 = 150$

12. False: f must also be continuous except at a finite number of points on $[a, b]$.

13. True: The area of a vertical line segment is 0.

14. False: $\int_{-1}^{1} x\,dx$ is a counterexample.

15. True: $[f(x)]^2 \ge 0$, and if $[f(x)]^2$ is greater than 0 on $[a, b]$, the integral will be also.

16. False: $D_x\left[\int_a^x f(z)\,dz\right] = f(x)$

17. True: $\sin x + \cos x$ has period 2π, so $\int_x^{x+2\pi}(\sin x + \cos x)\,dx$ is independent of x.

18. True: $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$ and $\lim_{x\to a}[f(x)+g(x)]$
$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ when all the limits exist.

19. True: $\sin^{13} x$ is an odd function.

20. True: Theorem 5.7.A

21. False: The statement is not true if $c > d$.

22. False: $D_x\left[\int_0^{x^2} \frac{1}{1+t^2}dt\right] = \frac{2x}{1+x^2}$

23. True: Both sides equal 4.

24. True: Both sides equal 4.

25. True: The derivatives of even functions are odd.

26. False: $f(x) = x^2$ is a counterexample.

27. False: $f(x) = x^2$ is a counterexample.

28. False: $f(x) = x^2$ is a counterexample.

29. False: $f(x) = x^2$, $v(x) = 2x + 1$ is a counterexample.

30. False: $f(x) = x^3$ is a counterexample.

31. False: $f(x) = \sqrt{x}$ is a counterexample.

32. True: All rectangles have height 4, regardless of $\overline{x}_i$.

33. True: $F(b) - F(a) = \int_a^b F'(x)\,dx$
$= \int_a^b G'(x)\,dx = G(b) - G(a)$

34. False: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ because f is even.

35. False: $z(t) = t^2$ is a counterexample.

36. False: $\int_0^b f(x)\,dx = F(b) - F(0)$

37. True: Odd-exponent terms cancel themselves out over the interval, since they are odd.

38. False: $a = 0, b = 1, f(x) = -1, g(x) = 0$ is a counterexample.

39. False: $a = 0, b = 1, f(x) = -1, g(x) = 0$ is a counterexample.

40. True: $|a_1 + a_2 + a_3 + \cdots + a_n|$
$\le |a_1| + |a_2| + |a_3| + \cdots + |a_n|$ because any negative values of a_i make the left side smaller than the right side.

41. True: See Problem 28 of Section 5.7.

42. True: Definition of Definite Integral

43. True: Definition of Definite Integral

Sample Test Problems

1. $\left[\frac{1}{4}x^4 - x^3 + 2x^{3/2}\right]_0^1 = \frac{5}{4}$

2. $\frac{2}{3}x^3 - 3x - \frac{1}{x} + C$

3. $\frac{1}{3}y^3 + 9\cos y - \frac{26}{y} + C$

4. $\frac{1}{3}(y^2 - 4)^{3/2} + C$

5. $\frac{1}{4}\cdot\frac{3}{4}(2z^2 - 3)^{4/3} + C$
$= \frac{3}{16}(2z^2 - 3)^{4/3} + C$

6. $\left[-\frac{1}{5}\cos^5 x\right]_0^{\pi/2} = \frac{1}{5}$

7. $u = \tan(3x^2 + 6x), du = (6x + 6)\sec^2(3x^2 + 6x)$
$\frac{1}{6}\int u^2 du = \frac{1}{18}u^3 + C$
$\frac{1}{18}\left[\tan^3(3x^2 + 6x)\right]_0^{\pi} = \frac{1}{18}\tan^3(3\pi^2)$

8. $u = t^4 + 9, du = 4t^3 dt$
$\frac{1}{4}\int_9^{25} u^{-1/2}du = \frac{1}{2}\left[u^{1/2}\right]_9^{25} = 1$

9. $\frac{1}{5}\left[\frac{3}{5}(t^5 + 5)^{5/3}\right]_1^2 \approx 46.9$

10. $-\frac{1}{3(y^3 - 3y)} + C$

11. $u = 2y^3 + 3y^2 + 6y, du = (6y^2 + 6y + 6)dy$
$\frac{1}{6}\int u^{-1/5}du = \frac{5}{24}(2y^3 + 3y^2 + 6y)^{4/5} + C$

12. $\int dy = \int \sin x\,dx$
$y = -\cos x + C$
$y = -\cos x + 3$

13. $\int dy = \int \frac{1}{\sqrt{x+1}}dx$

$y = 2\sqrt{x+1} + C$

$y = 2\sqrt{x+1} + 14$

14. $\int \sin y\, dy = \int dx$

$-\cos y = x + C$

$x = -1 - \cos y$

15. $\int dy = \int \sqrt{2t-1}\, dt$

$y = \frac{1}{3}(2t-1)^{3/2} + C$

$y = \frac{1}{3}(2t-1)^{3/2} - 1$

16. $\int y^{-4} dy = \int t^2 dt$

$-\frac{1}{3y^3} = \frac{t^3}{3} + C$

$-\frac{1}{3y^3} = \frac{t^3}{3} - \frac{2}{3}$

$y = \sqrt[3]{\frac{1}{2-t^3}}$

17. $\int 2y\, dy = \int (6x - x^3)dx$

$y^2 = 3x^2 - \frac{1}{4}x^4 + C$

$y^2 = 3x^2 - \frac{1}{4}x^4 + 9$

$y = \sqrt{3x^2 - \frac{1}{4}x^4 + 9}$

18. $\int \cos y\, dy = \int x\, dx$

$\sin y = \frac{x^2}{2} + C$

$y = \sin^{-1}\left(\frac{x^2}{2}\right)$

19. On $xy = 2$, $\frac{dy}{dx} = -\frac{2}{x^2}$, so the curve has slope

$\frac{dy}{dx} = \frac{x^2}{2}$.

$\int dy = \int \frac{x^2}{2} dx$

$y = \frac{1}{6}x^3 + C$

$-\frac{1}{3} = -\frac{4}{3} + C$

$y = \frac{1}{6}x^3 + 1$

20. $a = 15\sqrt{t} + 8 = \frac{dv}{dt}$

$v = \int (15\sqrt{t} + 8)dt = 10t^{3/2} + 8t + v_0$

$v = 10t^{3/2} + 8t - 6 = \frac{dx}{dt}$

$x = \int (10t^{3/2} + 8t - 6)\, dt = 4t^{5/2} + 4t^2 - 6t + x_0$

$x = 4t^{5/2} + 4t^2 - 6t - 44$

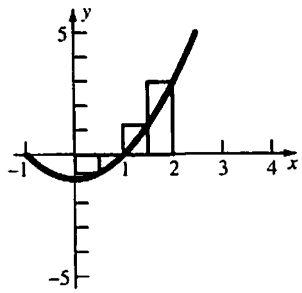
When $t = 4$ sec, $x = 124$ ft.

21. $s = -16t^2 + 48t + 448$; $s = 0$ at $t = 7$;
when $t = 7$, $v = -32(7) + 48 = -176$ ft/s

22. 45 miles per hour = 66 feet per second;
60 miles per hour = 88 feet per second.

$a = \frac{88-66}{10} = 2.2$ ft/s^2.

23. $\sum_{i=1}^{4}\left[\left(\frac{i}{2}\right)^2 - 1\right]\left(\frac{1}{2}\right) = \frac{7}{4}$



24. $f'(x) = \frac{1}{x+3}, f'(7) = \frac{1}{10}$

25. $\int_0^3 (2 - \sqrt{x+1})^2 dx$

$= \int_0^3 \left(x + 5 - 4\sqrt{x+1}\right)dx$

$= \left[\frac{1}{2}x^2 + 5x - \frac{8}{3}(x+1)^{3/2}\right]_0^3 = \frac{5}{6}$

26. $\frac{1}{5-2}\int_2^5 3x^2\sqrt{x^3-4}\, dx = \frac{1}{3}\left[\frac{2}{3}(x^3-4)^{3/2}\right]_2^5$

$= 294$

27. $\int_2^4 \left(5 - \frac{1}{x^2}\right)dx = \left[5x + \frac{1}{x}\right]_2^4 = \frac{39}{4}$

28. $\sum_{i=1}^{n}(3^i - 3^{i-1})$

$= (3-1)+(3^2-3)+(3^3-3^2)+\cdots+(3^n-3^{n-1})$

$= 3^n - 1$

29. $\sum_{i=1}^{10}(6i^2-8i) = 6\sum_{i=1}^{10} i^2 - 8\sum_{i=1}^{10} i$

$= 6\left[\frac{10(11)(21)}{6}\right] - 8\left[\frac{10(11)}{2}\right] = 1870$

30. **a.** $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$

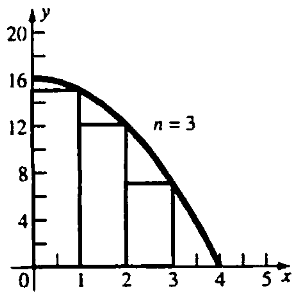
b. $1+0+(-1)+(-2)+(-3)+(-4)=-9$

c. $1+\frac{\sqrt{2}}{2}+0-\frac{\sqrt{2}}{2}-1=0$

31. **a.** $\sum_{n=2}^{78}\frac{1}{n}$

b. $\sum_{n=1}^{50} nx^{2n}$

32.



$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left[16-\left(\frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[\frac{48}{n}-\frac{27}{n^3}i^2\right]\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{48}{n}\sum_{i=1}^{n}1-\frac{27}{n^3}\sum_{i=1}^{n}i^2\right\}$$

$$= \lim_{n\to\infty}\left\{48-\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right\}$$

$$= \lim_{n\to\infty}\left\{48-\frac{9}{2}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]\right\}$$

$= 48 - 9 = 39$

33. **a.** $\int_1^2 f(x)\,dx = \int_1^0 f(x)\,dx + \int_0^2 f(x)\,dx$

$= -4 + 2 = -2$

b. $\int_1^0 f(x)\,dx = -\int_0^1 f(x)\,dx = -4$

c. $\int_0^2 3f(u)\,du = 3\int_0^2 f(u)\,du = 3(2) = 6$

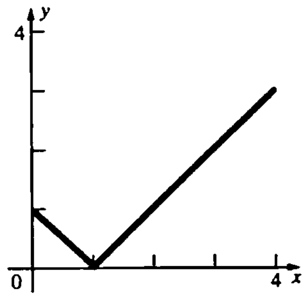
d. $\int_0^2 [2g(x)-3f(x)]\,dx$

$= 2\int_0^2 g(x) - 3\int_0^2 f(x)\,dx$

$= 2(-3) - 3(2) = -12$

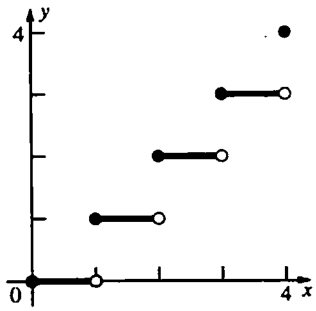
e. $\int_0^{-2} f(-x)\,dx = \int_0^2 f(x)\,dx = 2$

34. **a.**



$\int_0^4 |x-1|\,dx = \frac{1}{2}(1)(1)+\frac{1}{2}(3)(3)=5$

b.



$\int_0^4 [\![x]\!]\,dx = 1+2+3=6$

c. $\int_0^4 (x-[\![x]\!])\,dx = \int_0^4 x\,dx - \int_0^4 [\![x]\!]\,dx$

$\left[\frac{1}{2}x^2\right]_0^4 - 6 = 8 - 6 = 2$

35. **a.** $\int_{-2}^{2} f(x)\,dx = 2\int_0^2 f(x)\,dx = 2(-4) = -8$

b. $\int_{-2}^{2} |f(x)|\,dx = 2\int_0^2 |f(x)|\,dx = 2|-4| = 8$

c. $\int_{-2}^{2} g(x)\,dx = 0$

d. $\int_{-2}^{2} [f(x)+f(-x)]dx$
$= 2\int_0^2 f(x)dx + 2\int_0^2 f(x)dx$
$= 4(-4) = -16$

e. $\int_0^2 [2g(x)+3f(x)]dx$
$= 2\int_0^2 g(x)dx + 3\int_0^2 f(x)dx$
$= 2(5) + 3(-4) = -2$

f. $\int_{-2}^{0} g(x)dx = \int_2^0 g(-x)\cdot -dx = \int_2^0 g(x)dx$
$= -\int_0^2 g(x)dx = -5$

36. $\int_{-100}^{100} (x^3 + \sin^5 x)dx = 0$

37. $\int_{-4}^{-1} 3x^2 dx = 3c^2(-1+4)$
$\left[x^3\right]_{-4}^{-1} = 9c^2$
$c^2 = 7$
$c = -\sqrt{7} \approx -2.65$

38. a. $G'(x) = \dfrac{1}{x^2+1}$

b. $G'(x) = \dfrac{2x}{x^4+1}$

c. $G'(x) = \dfrac{3x^2}{x^6+1} - \dfrac{1}{x^2+1}$

39. a. $G'(x) = \sin^2 x$

b. $G'(x) = f(x+1) - f(x)$

c. $G'(x) = -\dfrac{1}{x^2}\int_0^x f(z)dz + \dfrac{1}{x}f(x)$

d. $G'(x) = \int_0^x f(t)dt$

e. $G(x) = \int_0^{g(x)} \dfrac{dg(u)}{du}du = [g(u)]_0^{g(x)}$
$= g(g(x)) - g(0)$
$G'(x) = g'(g(x))g'(x)$

f. $G(x) = \int_0^{-x} f(-t)dt = \int_0^x f(u)(-du)$
$= -\int_0^x f(u)du$
$G'(x) = -f(x)$

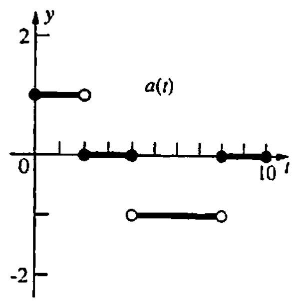
40. a. $\int_0^4 \sqrt{x}\,dx = \dfrac{2}{3}\left[x^{3/2}\right]_0^4 = \dfrac{16}{3}$

b. $\int_1^3 x^2 dx = \dfrac{1}{3}\left[x^3\right]_1^3 = \dfrac{26}{3}$

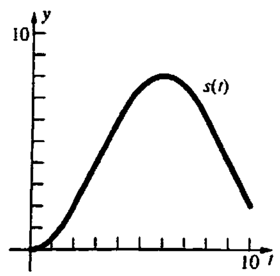
41. $f(x) = \int_{2x}^{5x} \dfrac{1}{t}dt = \int_0^{5x} \dfrac{1}{t}dt - \int_0^{2x} \dfrac{1}{t}dt$
$f'(x) = \dfrac{1}{5x}\cdot 5 - \dfrac{1}{2x}\cdot 2 = 0$

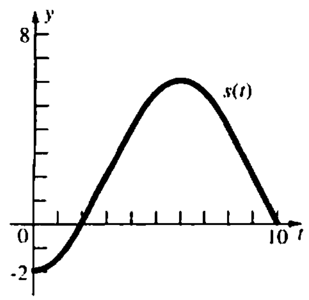
5.10 Additional Problem Set

1. Recall that by implicitly differentiating the equation of the circle $x^2 + y^2 = r^2$, we get $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Thus, the solution must be a graph of a circle. Since the point (0, 1) must lie on the graph, the answer is Figure 1.

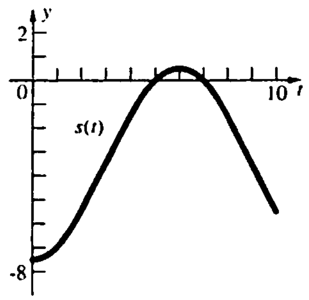


2. a.



b. 

The graph of $a(t)$ is the same as in part a.

c. 

The graph of $a(t)$ is the same as in part a.

3. a. $30[88 + 80 + 66 + 51 + 37 + 26 + 14 + 5]$
$= 11{,}010$ feet

b. $30[80 + 66 + 51 + 37 + 26 + 14 + 5 + 0]$
$= 8370$ feet

c. $30\left[84+73+\frac{117}{2}+44+\frac{63}{2}+20+\frac{19}{2}+\frac{5}{2}\right]$
$= 9690$ feet

d. Part **a.** uses the largest value of each pair and **b.** uses the smallest. Part **c.** averages each pair, so the total value is the average of **a.** and **b.**

4. Recall that $\bar{f}=\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\int_a^b \bar{f}dx=\bar{f}\int_a^b dx=\frac{1}{b-a}\int_a^b f(x)dx\cdot\int_a^b dx$$
$$=\frac{1}{b-a}\int_a^b f(x)dx\cdot(b-a)=\int_a^b f(x)\,dx$$

5. a. $$\bar{u}+\bar{v}=\frac{1}{b-a}\int_a^b u\,dx+\frac{1}{b-a}\int_a^b v\,dx$$
$$=\frac{1}{b-a}\int_a^b (u+v)\,dx=\overline{u+v}$$

b. $$k\bar{u}=\frac{k}{b-a}\int_a^b u\,dx=\frac{1}{b-a}\int_a^b ku\,dx=\overline{ku}$$

c. $$\bar{u}=\frac{1}{b-a}\int_a^b u\,dx\le\frac{1}{b-a}\int_a^b v\,dx=\bar{v}$$

6. a. $$\int_{-\phi/120\pi}^{-\phi/120\pi+1}\hat{V}\,dt=\int_{-\phi/120\pi}^{-\phi/120\pi+1}\hat{V}\sin(120\pi t+\phi)\,dt=\left[-\frac{1}{120\pi}\hat{V}\cos(120\pi t+\phi)\right]_{-\phi/120\pi}^{-\phi/120\pi+1}=0$$

b. $$\frac{1}{\frac{1}{120}}\int_{-\phi/120\pi}^{-\phi/120\pi+1/120}\hat{V}\sin(120\pi t+\phi)\,dt=120\left[-\frac{1}{120\pi}\hat{V}\cos(120\pi t+\phi)\right]_{-\phi/120\pi}^{-\phi/120\pi+1/120}=-\frac{\hat{V}}{\pi}(-1-1)=\frac{2\hat{V}}{\pi}$$

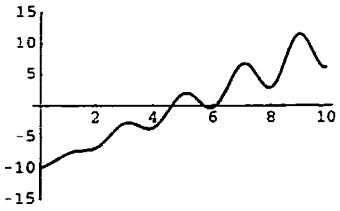
c. $$V_{rms}=\sqrt{\int_\phi^{\phi+1}\hat{V}^2\sin^2(120\pi t+\phi)\,dt}=\sqrt{\hat{V}^2\left[-\frac{1}{240}\sin(120\pi t+\phi)\cos(120\pi t+\phi)+\frac{1}{2}t\right]_\phi^{\phi+1}}$$
$$=\hat{V}\sqrt{-\frac{1}{240\pi}\sin(120\pi\phi+120\pi+\phi)\cos(120\pi\phi+120\pi+\phi)+\frac{1}{2}\phi+\frac{1}{2}-\left[-\frac{1}{240\pi}\sin(120\pi\phi+\phi)\cos(120\pi\phi+\phi)+\frac{1}{2}\phi\right]}$$
$$=\hat{V}\sqrt{\frac{1}{2}}=\frac{\hat{V}\sqrt{2}}{2}$$

d. $$120=\frac{\hat{V}\sqrt{2}}{2}$$
$\hat{V}=120\sqrt{2}\approx 169.71$ Volts

7. a. Local minima at 0, ≈ 3.8, ≈ 5.8, ≈ 7.9, ≈ 9.9, ≈ 10;
local maxima at ≈ 3.1, ≈ 5, ≈ 7.1, ≈ 9

b. Absolute minimum at 0, absolute maximum at ≈ 9

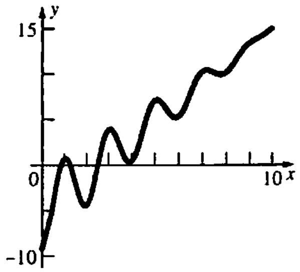
c. $\approx (0.7, 1.5), (2.5, 3.5), (4.5, 5.5), (6.5, 7.5), (8.5, 9.5)$

d. 

8. a. Local minima at 0, ≈ 1.8, ≈ 3.8, ≈ 5.8, ≈ 7.8;
local maxima at ≈ 1, ≈ 3.1, ≈ 5.2, ≈ 7.3

b. Absolute minimum at 0, absolute maximum at 10

c. $(0.5, 1.5), (2.5, 3.5), (4.5, 5.5), (6.5, 7.5), (8.5, 9.5)$

d. 

9. a. $f(x) > 0$ so $\int_{-1}^{3} f(x)\,dx > 0$

b. $\int_{-1}^{3} f'(x)\,dx = f(3) - f(-1) = 1 - 2 = -1 < 0$

c. $\int_{-1}^{3} f''(x)\,dx = f'(3) - f'(1) = -1 - 1 = -2 < 0$

d. $\int_{-1}^{3} f'''(x)\,dx = f''(3) - f''(1)$

$= \text{positive} - \text{negative} > 0$