CHAPTER

4 Applications of the Derivative

4.1 Concepts Review

1. continuous; closed
2. extreme
3. endpoints; stationary points; singular points
4. $f'(c) = 0$; $f'(c)$ does not exist

Problem Set 4.1

1. $f'(x) = 2x + 4$; $2x + 4 = 0$ when $x = -2$.
   Critical points: $-4, -2, 0$
   $f(-4) = 4, f(-2) = 0, f(0) = 4$
   Maximum value = 4, minimum value = 0

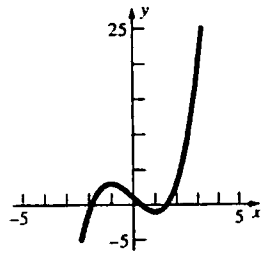
2. $h'(x) = 2x + 1$; $2x + 1 = 0$ when $x = -\frac{1}{2}$.
   Critical points: $-2, -\frac{1}{2}, 2$
   $h(-2) = 2, h\left(-\frac{1}{2}\right) = -\frac{1}{4}, h(2) = 6$
   Maximum value = 6, minimum value = $-\frac{1}{4}$

3. $\Psi'(x) = 2x + 3$; $2x + 3 = 0$ when $x = -\frac{3}{2}$.
   Critical points: $-2, -\frac{3}{2}, 1$
   $\Psi(-2) = -2, \Psi\left(-\frac{3}{2}\right) = -\frac{9}{4}, \Psi(1) = 4$
   Maximum value = 4, minimum value = $-\frac{9}{4}$

4. $G'(x) = \frac{1}{5}(6x^2 + 6x - 12) = \frac{6}{5}(x^2 + x - 2)$;
   $x^2 + x - 2 = 0$ when $x = -2, 1$
   Critical points: $-3, -2, 1, 3$
   $G(-3) = \frac{9}{5}, G(-2) = 4, G(1) = -\frac{7}{5}, G(3) = 9$
   Maximum value = 9,
   minimum value = $-\frac{7}{5}$

5. $f'(x) = 3x^2 - 3$; $3x^2 - 3 = 0$ when $x = -1, 1$.
   Critical points: $-1, 1$
   $f(-1) = 3, f(1) = -1$
   No maximum value, minimum value = -1

   (See graph.)



6. $f'(x) = 3x^2 - 3$; $3x^2 - 3 = 0$ when $x = -1, 1$.
   Critical points: $-\frac{3}{2}, -1, 1, 3$
   $f\left(-\frac{3}{2}\right) = \frac{17}{8}, f(-1) = 3, f(1) = -1, f(3) = 19$
   Maximum value = 19, minimum value = -1

7. $h'(r) = -\frac{1}{r^2}$; $h'(r)$ is never 0; $h'(r)$ is not defined when $r = 0$, but $r = 0$ is not in the domain on $[-1, 3]$ since $h(0)$ is not defined.
   Critical points: $-1, 3$
   Note that $\lim_{x \to 0^-} h(r) = -\infty$ and $\lim_{x \to 0^+} h(x) = \infty$.
   No maximum value, no minimum value.

8. $g'(x) = -\frac{2x}{(1+x^2)^2}$; $-\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$
   Critical points: $-3, 0, 1$
   $g(-3) = \frac{1}{10}, g(0) = 1, g(1) = \frac{1}{2}$
   Maximum value = 1, minimum value = $\frac{1}{10}$

9. $g'(x) = -\frac{2x}{(1+x^2)^2}$; $-\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$.

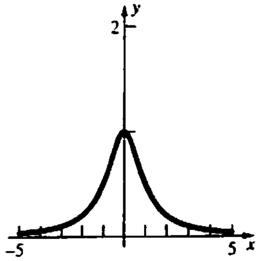
Critical point: 0

$g(0) = 1$

As $x \to \infty, g(x) \to 0^+$; as $x \to -\infty, g(x) \to 0^+$.

Maximum value = 1, no minimum value

(See graph.)



10. $f'(x) = \frac{1-x^2}{(1+x^2)^2}$;

$\frac{1-x^2}{(1+x^2)^2} = 0$ when $x = -1, 1$

Critical points: $-1, 1, 4$

$f(-1) = -\frac{1}{2}, f(1) = \frac{1}{2}, f(4) = \frac{4}{17}$

Maximum value = $\frac{1}{2}$,

minimum value $= -\frac{1}{2}$

11. $r'(\theta) = \cos\theta$; $\cos\theta = 0$ when $\theta = \frac{\pi}{2} + k\pi$

Critical points: $-\frac{\pi}{4}, \frac{\pi}{6}$

$r\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, r\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Maximum value $= \frac{\sqrt{3}}{2}$, minimum value $= -\frac{1}{\sqrt{2}}$

12. $s'(t) = \cos t + \sin t$; $\cos t + \sin t = 0$ when

$\tan t = -1$ or $t = -\frac{\pi}{4} + k\pi$.

Critical points: $0, \frac{3\pi}{4}, \pi$

$s(0) = -1, s\left(\frac{3\pi}{4}\right) = \sqrt{2}, s(\pi) = 1$.

Maximum value $= \sqrt{2}$,

minimum value $= -1$

13. $a'(x) = \frac{x-1}{|x-1|}$; $a'(x)$ does not exist when $x = 1$.

Critical points: 0, 1, 3

$a(0) = 1, a(1) = 0, a(3) = 2$

Maximum value = 2, minimum value = 0

14. $f'(s) = \frac{3(3s-2)}{|3s-2|}$; $f'(s)$ does not exist when

$s = \frac{2}{3}$.

Critical points: $-1, \frac{2}{3}, 4$

$f(-1) = 5, f\left(\frac{2}{3}\right) = 0, f(4) = 10$

Maximum value = 10, minimum value = 0

15. $g'(x) = \frac{1}{3x^{2/3}}$; $f'(x)$ does not exist when $x = 0$.

Critical points: $-1, 0, 27$

$g(-1) = -1, g(0) = 0, g(27) = 3$

Maximum value = 3, minimum value = -1

16. $s'(t) = \frac{2}{5t^{3/5}}$; $s'(t)$ does not exist when $t = 0$.

Critical points: $-1, 0, 32$

$s(-1) = 1, s(0) = 0, s(32) = 4$

Maximum value = 4, minimum value = 0

17. $f(x) = x(10-x) = 10x - x^2$; $f'(x) = 10 - 2x$;

$10 - 2x = 0; x = 5$

Critical points: 0, 5, 10

$f(0) = 0, f(5) = 25, f(10) = 0$; the numbers are $x = 5$ and $10 - x = 5$.

18. $x \ge x^2$ if $0 \le x \le 1$

$f(x) = x - x^2$; $f'(x) = 1 - 2x$;

$f'(x) = 0$ when $x = \frac{1}{2}$

Critical points: $0, \frac{1}{2}, 1$

$f(0) = 0, f(1) = 0, f\left(\frac{1}{2}\right) = \frac{1}{4}$; therefore, $\frac{1}{2}$ exceeds its square by the maximum amount.

19. Let x represent the width, then $100 - x$ represents the length.

$f(x) = x(100-x) = 100x - x^2$; $f'(x) = 100 - 2x$;

$100 - 2x = 0; x = 50$

Critical points: 0, 50, 100

$f(0) = f(100) = 0; f(50) = 50(50) = 2500$

The dimensions should be 50 ft by 50 ft.

20. For a rectangle with perimeter K and width x, the length is $\frac{K}{2} - x$. Then the area is

$A = x\left(\frac{K}{2} - x\right) = \frac{Kx}{2} - x^2$.

$\frac{dA}{dx} = \frac{K}{2} - 2x; \frac{dA}{dx} = 0$ when $x = \frac{K}{4}$

Critical points: $0, \frac{K}{4}, \frac{K}{2}$

At $x = 0$ or $\frac{K}{2}$, $A = 0$; at $x = \frac{K}{4}$, $A = \frac{K^2}{16}$.

The area is maximized when the width is one fourth of the perimeter, so the rectangle is a square.

21. Let x be the width of the square to be cut out and V the volume of the resulting open box.

$V = x(24 - 2x)^2 = 4x^3 - 96x^2 + 576x$

$\frac{dV}{dx} = 12x^2 - 192x + 576 = 12(x - 12)(x - 4)$;

$12(x - 12)(x - 4) = 0$; $x = 12$ or $x = 4$.

Critical points: 0, 4, 12

At $x = 0$ or 12, $V = 0$; at $x = 4$, $V = 1024$.

The volume of the largest box is 1024 in.^{3}

22. Let x be the length of the cut for the square, so $16 - x$ is the cut for the circle.

The area of the square is $\left(\frac{1}{4}x\right)\left(\frac{1}{4}x\right) = \frac{x^2}{16}$. The area of the circle is found by finding the radius first: $2\pi r = 16 - x$;

$r = \frac{16 - x}{2\pi}$; $A = \pi r^2 = \pi\left(\frac{16 - x}{2\pi}\right)^2 = \frac{(16 - x)^2}{4\pi}$.

Total area:

$A = \frac{x^2}{16} + \frac{(16 - x)^2}{4\pi}; \frac{dA}{dx} = \frac{x}{8} + \frac{2(16 - x)(-1)}{4\pi}$

$= \frac{x}{8} + \frac{x - 16}{2\pi}$;

$\frac{dA}{dx} = 0$ when $x = \frac{128}{2\pi + 8} \approx 8.96$

Critical numbers: 0, 8.96, 16

At $x = 0$, $A \approx 20.37$. At $x = 8.96$, $A \approx 8.96$.

At $x = 16$; $A = 16$.

Total area is minimum at $x = 8.96$; the total area is maximum with no cut and the wire bent to form a circle.

23. Let A be the area of the pen.

$A = x(80 - 2x) = 80x - 2x^2; \frac{dA}{dx} = 80 - 4x$;

$80 - 4x = 0$; $x = 20$

Critical points: 0, 20, 40.

At $x = 0$ or 40, $A = 0$; at $x = 20$, $A = 800$.

The dimensions are 20 ft by $80 - 2(20) = 40$ ft, with the length along the barn being 40 ft.

24. Let x be the width of each pen, then the length along the barn is $80 - 4x$.

$A = x(80 - 4x) = 80x - 4x^2; \frac{dA}{dx} = 80 - 8x$;

$\frac{dA}{dx} = 0$ when $x = 10$.

Critical points: 0, 10, 20

At $x = 0$ or 20, $A = 0$; at $x = 10$, $A = 400$.

The area is largest with width 10 ft and length 40 ft.

25. Let A be the area of the pen. The perimeter is $100 + 180 = 280$ ft.

$y + y - 100 + 2x = 180$; $y = 140 - x$

$A = x(140 - x) = 140x - x^2; \frac{dA}{dx} = 140 - 2x$;

$140 - 2x = 0$; $x = 70$

Since $0 \le x \le 40$, the critical points are 0 and 40. When $x = 0$, $A = 0$. When $x = 40$, $A = 4000$. The dimensions are 40 ft by 100 ft.

26. The perimeter will be $20 + 40 + 80 = 140$ ft, and x is limited by $20 \le x \le 30$.

$2x + 2y - 60 = 80$; $y = 70 - x$

$A = x(70 - x) = 70x - x^2; \frac{dA}{dx} = 70 - 2x$;

$70 - 2x = 0$; $x = 35$

Critical points: 20, 30

When $x = 20$, $A = 1000$. When $x = 30$, $A = 1200$. The pen has dimensions 30 ft by 40 ft.

27. x is limited by $0 \le x \le \sqrt{12}$.

$A = 2x(12 - x^2) = 24x - 2x^3; \frac{dA}{dx} = 24 - 6x^2$;

$24 - 6x^2 = 0$; $x = -2, 2$

Critical points: $0, 2, \sqrt{12}$.

When $x = 0$ or $\sqrt{12}$, $A = 0$.

When $x = 2$, $y = 12 - (2)^2 = 8$.

The dimensions are $2x = 2(2) = 4$ by 8.

28. Let the x-axis lie on the diameter of the semicircle and the y-axis pass through the middle.

Then the equation $y = \sqrt{r^2 - x^2}$ describes the semicircle. Let (x, y) be the upper-right corner of the rectangle. x is limited by $0 \le x \le r$.

$A = 2xy = 2x\sqrt{r^2 - x^2}$

$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x^2)$

$\frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x^2) = 0$; $x = \frac{r}{\sqrt{2}}$

Critical points: $0, \frac{r}{\sqrt{2}}, r$

When $x = 0$ or r, $A = 0$. When $x = \frac{r}{\sqrt{2}}, A = r^2$.

$$y = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \frac{r}{\sqrt{2}}$$

The dimensions are $\frac{r}{\sqrt{2}}$ by $\frac{2r}{\sqrt{2}}$.

29. The carrying capacity of the gutter is maximized when the area of the vertical end of the gutter is maximized. The height of the gutter is 3 sin θ. The area is

$$A = 3(3\sin\theta) + 2\left(\frac{1}{2}\right)(3\cos\theta)(3\sin\theta)$$

$$= 9\sin\theta + 9\cos\theta\sin\theta.$$

$$\frac{dA}{d\theta} = 9\cos\theta + 9(-\sin\theta)\sin\theta + 9\cos\theta\cos\theta$$

$$= 9(\cos\theta - \sin^2\theta + \cos^2\theta)$$

$$= 9(2\cos^2\theta + \cos\theta - 1)$$

$$2\cos^2\theta + \cos\theta - 1 = 0;\ \cos\theta = -1, \frac{1}{2}; \theta = \pi, \frac{\pi}{3}$$

Since $0 \le \theta \le \frac{\pi}{2}$, the critical points are

$0, \frac{\pi}{3}$, and $\frac{\pi}{2}$.

When $\theta = 0$, $A = 0$.

When $\theta = \frac{\pi}{3}, A = \frac{27\sqrt{3}}{4} \approx 11.7$.

When $\theta = \frac{\pi}{2}$, $A = 9$.

The carrying capacity is maximized when $\theta = \frac{\pi}{3}$.

30. The circumference of the top of the tank is the circumference of the circular sheet minus the arc length of the sector,
$20\pi - 10\theta$ meters. The radius of the top of the tank is $r = \frac{20\pi - 10\theta}{2\pi} = \frac{5}{\pi}(2\pi - \theta)$ meters. The slant height of the tank is 10 meters, so the height of the tank is

$$h = \sqrt{10^2 - \left(10 - \frac{5\theta}{\pi}\right)^2} = \frac{5}{\pi}\sqrt{4\pi\theta - \theta^2} \text{ meters.}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left[\frac{5}{\pi}(2\pi - \theta)\right]^2\left[\frac{5}{\pi}\sqrt{4\pi\theta - \theta^2}\right]$$

$$= \frac{125}{3\pi^2}(2\pi - \theta)^2\sqrt{4\pi\theta - \theta^2}$$

$$\frac{dV}{d\theta} = \frac{125}{3\pi^2}\left(2(2\pi - \theta)(-1)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2\left(\frac{1}{2}\right)(4\pi - 2\theta)}{\sqrt{4\pi\theta - \theta^2}}\right)$$

$$= \frac{125(2\pi - \theta)}{3\pi^2\sqrt{4\pi\theta - \theta^2}}(3\theta^2 - 12\pi\theta + 4\pi^2):$$

$$\frac{125(2\pi - \theta)}{3\pi^2\sqrt{4\pi\theta - \theta^2}}(3\theta^2 - 12\pi\theta + 4\pi^2) = 0$$

$2\pi - \theta = 0$ or $3\theta^2 - 12\pi\theta + 4\pi^2 = 0$

$$\theta = 2\pi, \theta = 2\pi - \frac{2\sqrt{6}}{3}\pi, \theta = 2\pi + \frac{2\sqrt{6}}{3}\pi$$

Since $0 < \theta < 2\pi$, the only critical point is $2\pi - \frac{2\sqrt{6}}{3}\pi$. A graph shows that this maximizes the volume.

31. Let c be the cost of driving the truck in cents. It takes $\frac{400}{x}$ hours to drive 400 miles.

$$c = \frac{400}{x}(1200) + 400\left(25 + \frac{x}{4}\right)$$

$$= 480{,}000x^{-1} + 10{,}000 + 100x$$

$$\frac{dc}{dx} = -480{,}000x^{-2} + 100;$$

$-480{,}000x^{-2} + 100 = 0; x \approx 69$

Since $40 \le x \le 55$, the critical points are 40 and 55.

At $x = 40$, $c = 26{,}000$.

At $x = 55$, $c \approx 24{,}227$.

The most economical allowed speed is at 55 mph.

32. Let c be the cost of driving the truck.

$$c = \frac{400}{x}(1200) + 400(40 + 0.05x^{3/2})$$

$$= 480{,}000x^{-1} + 16{,}000 + 20x^{3/2}$$

$\frac{dc}{dx} = -480{,}000x^{-2} + 30x^{1/2}; \frac{dc}{dx} = 0$ when $x \approx 48$

Critical points: 40, 48, 55

At $x = 40$, $c \approx \pi$ 33,060; At $x = 48$, $c \approx$ 32,651.

At $x = 55$, $c \approx$ 32,885.

The most economical speed is at approximately 48 mph.

33. Let D be the square of the distance.

$$D = (x - 0)^2 + (y - 4)^2 = x^2 + \left(\frac{x^2}{4} - 4\right)^2$$

$$= \frac{x^4}{16} - x^2 + 16$$

$\frac{dD}{dx}=\frac{x^3}{4}-2x; \frac{x^3}{4}-2x=0; x(x^2-8)=0$

$x=0, x=\pm 2\sqrt{2}$

Critical points: $0, 2\sqrt{2}, 2\sqrt{3}$

At $x=0, y=0$, and $D=16$. At $x=2\sqrt{2}, y=2$, and $D=12$. At $x=2\sqrt{3}, y=3$, and $D=13$.

$P(2\sqrt{2},2), Q(0,0)$

34. Note that $\cos t=\frac{h}{r}$, so $h=r\cos t$,

$\sin t=\frac{1}{r}\sqrt{r^2-h^2}$, and $\sqrt{r^2-h^2}=r\sin t$

Area of submerged region $=tr^2-h\sqrt{r^2-h^2}$

$=tr^2-(r\cos t)(r\sin t)=r^2(t-\cos t\sin t)$

A = area of exposed wetted region

$=\pi r^2-\pi h^2-r^2(t-\cos t\sin t)$

$=r^2(\pi-\pi\cos^2 t-t+\cos t\sin t)$

$\frac{dA}{dt}=r^2(2\pi\cos t\sin t-1+\cos^2 t-\sin^2 t)$

$=r^2(2\pi\cos t\sin t-2\sin^2 t)$

$=2r^2\sin t(\pi\cos t-\sin t)$

Since $0<t<\pi$, $\frac{dA}{dt}=0$ only when

$\pi\cos t=\sin t$ or $\tan t=\pi$. In terms of r and h,

this is $\frac{\frac{1}{r}\sqrt{r^2-h^2}}{\frac{h}{r}}=\pi$ or $h=\frac{r}{\sqrt{1+\pi^2}}$.

35. Let V be the volume. $y=4-x$ and $z=5-2x$.
x is limited by $\le x\le 2.5$.

$V=x(4-x)(5-2x)=20x-13x^2+2x^3$

$\frac{dV}{dx}=20-26x+6x^2; 2(3x^2-13x+10)=0;$

$2(3x-10)(x-1)=0;$

$x=1,\frac{10}{3}$

Critical points: 0, 1, 2.5
At $x=0$ or 2.5, $V=0$. At $x=1$, $V=9$.
Maximum volume when $x=1, y=4-1=3$, and $z=5-2(1)=3$.

36. a. $f'(x)=3x^2-12x+1; 3x^2-12x+1=0$

when $x=2-\frac{\sqrt{33}}{3}$ and $x=2+\frac{\sqrt{33}}{3}$.

Critical points: $-1, 2-\frac{\sqrt{33}}{3}, 2+\frac{\sqrt{33}}{3}, 5$

$f(-1)=-6$, $f\left(2-\frac{\sqrt{33}}{3}\right)\approx 2.04$,

$f\left(2+\frac{\sqrt{33}}{3}\right)\approx -26.04$, $f(5)=-18$

Maximum value ≈ 2.04,
minimum value ≈ -26.04

b. $g'(x)=\frac{(x^3-6x^2+x+2)(3x^2-12x+1)}{|x^3-6x^2+x+2|}$;

$g'(x)=0$ when $x=2-\frac{\sqrt{33}}{3}$ and

$x=2+\frac{\sqrt{33}}{3}$. $g'(x)$ does not exist when $f(x)=0$; on $[-1, 5]$, $f(x)=0$ when $x\approx -0.4836$ and $x\approx 0.7172$

Critical points: $-1, -0.4836, 2-\frac{\sqrt{33}}{3}$,

$0.7172, 2+\frac{\sqrt{33}}{3}, 5$

$g(-1)=6, g(-0.4836)=0$,

$g\left(2-\frac{\sqrt{33}}{3}\right)\approx 2.04$, $g(0.7172)=0$,

$g\left(2+\frac{\sqrt{33}}{3}\right)\approx 26.04$, $g(5)=18$

Maximum value ≈ 26.04,
minimum value = 0

37. a. $f'(x)=x\cos x$; on $[-1, 5]$, $x\cos x=0$ when

$x=0,\ x=\frac{\pi}{2}, x=\frac{3\pi}{2}$

Critical points: $-1, 0, \frac{\pi}{2}, \frac{3\pi}{2}, 5$

$f(-1)\approx 3.38, f(0)=3$, $f\left(\frac{\pi}{2}\right)\approx 3.57$,

$f\left(\frac{3\pi}{2}\right)\approx -2.71$, f(5) ≈ -2.51

Maximum value ≈ 3.57,
minimum value ≈ -2.71

b. $g'(x)=\frac{(\cos x+x\sin x+2)(x\cos x)}{|\cos x+x\sin x+2|}$;

$g'(x)=0$ when $x=0$, $x=\frac{\pi}{2}$, $x=\frac{3\pi}{2}$

$g'(x)$ does not exist when $f(x)=0$;
on $[-1, 5]$, $f(x)=0$ when $x\approx 3.45$

Critical points: $-1, 0, \frac{\pi}{2}, 3.45, \frac{3\pi}{2}, 5$

$g(-1) \approx 3.38, g(0) = 3,\ g\left(\frac{\pi}{2}\right) \approx 3.57,$

$g(3.45) = 0,\ g\left(\frac{3\pi}{2}\right) \approx 2.71,$

$g(5) \approx 2.51$
Maximum value ≈ 3.57;
minimum value = 0

4.2 Concepts Review

1. Increasing; concave up
2. $f'(x) > 0; f''(x) < 0$
3. An inflection point
4. $f''(c) = 0; f''(c)$ does not exist.

Problem Set 4.2

1. $f'(x) = 3$; $3 > 0$ for all x. $f(x)$ is increasing for all x.

2. $g'(x) = 2x - 1$; $2x - 1 > 0$ when $x > \frac{1}{2}$. $g(x)$ is increasing on $\left[\frac{1}{2}, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{2}\right]$.

3. $h'(t) = 2t + 2$; $2t + 2 > 0$ when $t > -1$. $h(t)$ is increasing on $[-1, \infty)$ and decreasing on $(-\infty, -1]$.

4. $f'(x) = 3x^2; 3x^2 > 0$ for $x \neq 0$.
$f(x)$ is increasing for all x.

5. $G'(x) = 6x^2 - 18x + 12 = 6(x-2)(x-1)$
Split the x-axis into the intervals $(-\infty, 1), (1, 2), (2, \infty)$.
Test points: $x = 0, \frac{3}{2}, 3; G'(0) = 12,\ G'\left(\frac{3}{2}\right) = -\frac{3}{2},$
$G'(3) = 12$
$G(x)$ is increasing on $(-\infty, 1] \cup [2, \infty)$ and decreasing on $[1, 2]$.

6. $f'(t) = 3t^2 + 6t = 3t(t+2)$
Split the x-axis into the intervals $(-\infty, -2)$, $(-2, 0), (0, \infty)$.
Test points: $t = -3, -1, 1$: $f'(-3) = 9$,
$f'(-1) = -3,\ f'(1) = 9$
$f(t)$ is increasing on $(-\infty, -2] \cup [0, \infty)$ and decreasing on $[-2, 0]$.

7. $h'(z) = z^3 - 2z^2 = z^2(z-2)$
Split the x-axis into the intervals $(-\infty, 0), (0, 2), (2, \infty)$.
Test points: $z = -1, 1, 3$; $h'(-1) = -3,\ h'(1) = -1,$
$h'(3) = 9$
$h(z)$ is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.

8. $f'(x) = \frac{2-x}{x^3}$
Split the x-axis into the intervals $(-\infty, 0), (0, 2), (2, \infty)$.
Test points: $-1, 1, 3$; $f'(-1) = -3,\ f'(1) = 1,$
$f'(3) = -\frac{1}{27}$
$f(x)$ is increasing on $(0, 2]$ and decreasing on $(-\infty, 0) \cup [2, \infty)$.

9. $H'(t) = \cos t; H'(t) > 0$ when $0 \le t < \frac{\pi}{2}$ and $\frac{3\pi}{2} < t \le 2\pi.$
$H(t)$ is increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

10. $R'(\theta) = -2\cos\theta\sin\theta; R'(\theta) > 0$ when $\frac{\pi}{2} < \theta < \pi$ and $\frac{3\pi}{2} < \theta < 2\pi.$
$R(\theta)$ is increasing on $\left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and decreasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$.

11. $f''(x) = 2$; $2 > 0$ for all x. $f(x)$ is concave up for all x; no inflection points.

12. $G''(w) = 2$; $2 > 0$ for all w. $G(w)$ is concave up for all w; no inflection points.

13. $T''(t) = 18t$; $18t > 0$ when $t > 0$. $T(t)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; $(0, 0)$ is the only inflection point.

14. $f''(z) = 2 - \frac{6}{z^4} = \frac{2}{z^4}(z^4 - 3)$; $z^4 - 3 > 0$ for $z < -\sqrt[4]{3}$ and $z > \sqrt[4]{3}$.
$f(z)$ is concave up on $(-\infty, -\sqrt[4]{3}) \cup (\sqrt[4]{3}, \infty)$ and concave down on $(-\sqrt[4]{3}, 0) \cup (0, \sqrt[4]{3})$: inflection points are $\left(-\sqrt[4]{3}, \sqrt{3} - \frac{1}{\sqrt{3}}\right)$ and $\left(\sqrt[4]{3}, \sqrt{3} - \frac{1}{\sqrt{3}}\right)$.

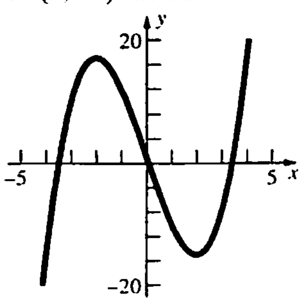
15. $q''(x) = 12x^2 - 36x - 48$; $q''(x) > 0$ when $x < -1$ and $x > 4$.
$q(x)$ is concave up on $(-\infty, -1) \cup (4, \infty)$ and concave down on $(-1, 4)$; inflection points are $(-1, -19)$ and $(4, -499)$.

16. $f''(x) = 12x^2 + 48x = 12x(x + 4)$; $f''(x) > 0$ when $x < -4$ and $x > 0$.
$f(x)$ is concave up on $(-\infty, -4) \cup (0, \infty)$ and concave down on $(-4, 0)$; inflection points are $(-4, -258)$ and $(0, -2)$.

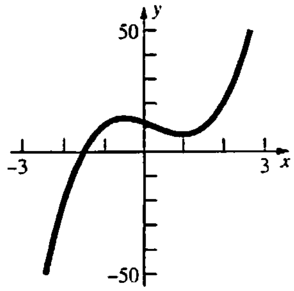
17. $F''(x) = 2\sin^2 x - 2\cos^2 x + 4 = 6 - 4\cos^2 x$;
$6 - 4\cos^2 x > 0$ for all x since $0 \le \cos^2 x \le 1$.
$F(x)$ is concave up for all x; no inflection points.

18. $G''(x) = 48 + 24\cos^2 x - 24\sin^2 x$
$= 24 + 48\cos^2 x$; $24 + 48\cos^2 x > 0$ for all x.
$G(x)$ is concave up for all x; no inflection points.

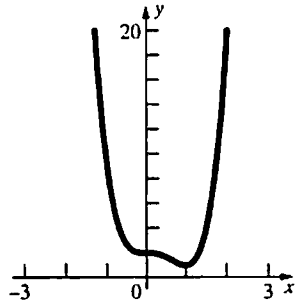
19. $f'(x) = 3x^2 - 12$; $3x^2 - 12 > 0$ when $x < -2$ or $x > 2$.
$f(x)$ is increasing on $(-\infty, -2] \cup [2, \infty)$ and decreasing on $[-2, 2]$.
$f''(x) = 6x$; $6x > 0$ when $x > 0$. $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.



20. $g'(x) = 12x^2 - 6x - 6 = 6(2x + 1)(x - 1)$; $g'(x) > 0$ when $x < -\frac{1}{2}$ or $x > 1$. $g(x)$ is increasing on $\left(-\infty, -\frac{1}{2}\right] \cup [1, \infty)$ and decreasing on $\left[-\frac{1}{2}, 1\right]$.
$g''(x) = 24x - 6 = 6(4x - 1)$; $g''(x) > 0$ when $x > \frac{1}{4}$.
$g(x)$ is concave up on $\left(\frac{1}{4}, \infty\right)$ and concave down on $\left(-\infty, \frac{1}{4}\right)$.



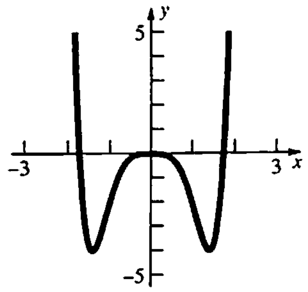
21. $g'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$; $g'(x) > 0$ when $x > 1$. $g(x)$ is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$.
$g''(x) = 36x^2 - 24x = 12x(3x - 2)$; $g''(x) > 0$ when $x < 0$ or $x > \frac{2}{3}$. $g(x)$ is concave up on $(-\infty, 0) \cup \left(\frac{2}{3}, \infty\right)$ and concave down on $\left(0, \frac{2}{3}\right)$.



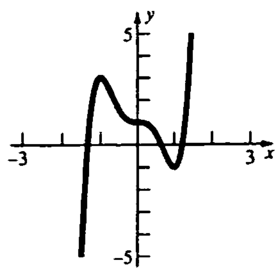
22. $F'(x) = 6x^5 - 12x^3 = 6x^3(x^2 - 2)$
Split the x-axis into the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, 0), (0, \sqrt{2}), (\sqrt{2}, \infty)$.
Test points: $x = -2, -1, 1, 2$: $F'(-2) = -96$, $F'(-1) = 6, F'(1) = -6, F'(2) = 96$
$F(x)$ is increasing on $[-\sqrt{2}, 0] \cup [\sqrt{2}, \infty)$ and decreasing on $(-\infty, -\sqrt{2}] \cup [0, \sqrt{2}]$
$F''(x) = 30x^4 - 36x^2 = 6x^2(5x^2 - 6)$; $5x^2 - 6 > 0$

when $x < -\sqrt{\frac{6}{5}}$ or $x > \sqrt{\frac{6}{5}}$.

$F(x)$ is concave up on $\left(-\infty, -\sqrt{\frac{6}{5}}\right) \cup \left(\sqrt{\frac{6}{5}}, \infty\right)$ and concave down on $\left(-\sqrt{\frac{6}{5}}, \sqrt{\frac{6}{5}}\right)$.

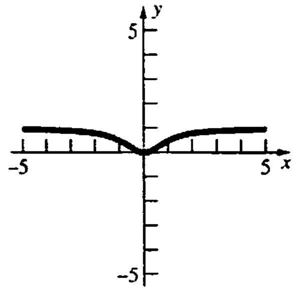


23. $G'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$; $G'(x) > 0$ when $x < -1$ or $x > 1$. $G(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $[-1, 1]$.
$G''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$;
Split the x-axis into the intervals $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0\right)$, $\left(0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \infty\right)$.
Test points: $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$; $G''(-1) = -30$, $G''\left(-\frac{1}{2}\right) = \frac{15}{2}$, $G''\left(\frac{1}{2}\right) = -\frac{15}{2}$, $G''(1) = 30$.
$G(x)$ is concave up on $\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$.



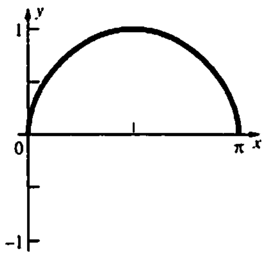
24. $H'(x) = \frac{2x}{(x^2 + 1)^2}$; $H'(x) > 0$ when $x > 0$.
$H(x)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.
$H''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}$; $H''(x) > 0$ when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

$H(x)$ is concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and concave down on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$.

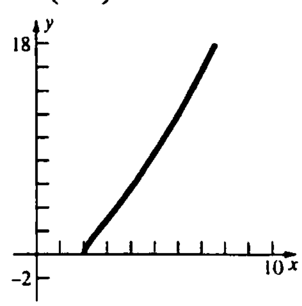


25. $f'(x) = \frac{\cos x}{2\sqrt{\sin x}}$; $f'(x) > 0$ when $0 < x < \frac{\pi}{2}$. $f(x)$ is increasing on $\left[0, \frac{\pi}{2}\right]$ and decreasing on $\left[\frac{\pi}{2}, \pi\right]$.

$f''(x) = \frac{-\cos^2 x - 2\sin^2 x}{4\sin^{3/2} x}$; $f''(x) < 0$ for all x in $(0, \infty)$. $f(x)$ is concave down on $(0, \pi)$.



26. $g'(x) = \frac{3x - 4}{2\sqrt{x - 2}}$; $3x - 4 > 0$ when $x > \frac{4}{3}$.
$g(x)$ is increasing on $[2, \infty)$.
$g''(x) = \frac{3x - 8}{4(x - 2)^{3/2}}$; $3x - 8 > 0$ when $x > \frac{8}{3}$.
$g(x)$ is concave up on $\left(\frac{8}{3}, \infty\right)$ and concave down on $\left(2, \frac{8}{3}\right)$.



27. $f'(x)=\frac{2-5x}{3x^{1/3}}$; $2-5x>0$ when $x<\frac{2}{5}$. $f'(x)$ does not exist at $x=0$.

Split the x-axis into the intervals $(-\infty,0)$, $\left(0,\frac{2}{5}\right),\left(\frac{2}{5},\infty\right)$.

Test points: $-1,\frac{1}{5},1$: $f'(-1)=-\frac{7}{3}$,

$f'\left(\frac{1}{5}\right)=\frac{\sqrt[3]{5}}{3}, f'(1)=-1.$

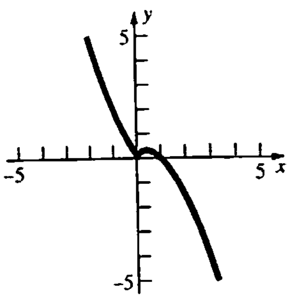
$f(x)$ is increasing on $\left[0,\frac{2}{5}\right]$ and decreasing on $(-\infty,0]\cup\left[\frac{2}{5},\infty\right)$.

$f''(x)=\frac{-2(5x+1)}{9x^{4/3}}$; $-2(5x+1)>0$ when $x<-\frac{1}{5}$. $f''(x)$ does not exist at $x=0$.

Test points: $-1,-\frac{1}{10},1$; $f''(-1)=\frac{8}{9}$,

$f''\left(-\frac{1}{10}\right)=-\frac{10^{4/3}}{9}, f(1)=-\frac{4}{3}.$

$f(x)$ is concave up on $\left(-\infty,-\frac{1}{5}\right)$ and concave down on $\left(-\frac{1}{5},0\right)\cup(0,\infty)$.



28. $g'(x)=\frac{4(x+2)}{3x^{2/3}}$; $x+2>0$ when $x>-2$, $g'(x)$ does not exist at $x=0$.

Split the x-axis into the intervals $(-\infty,-2)$, $(-2,0)$, $(0,\infty)$.

Test points: $-3,-1,1$; $g'(-3)=-\frac{4}{3^{5/3}}$,

$g'(-1)=\frac{4}{3}, g'(1)=4.$

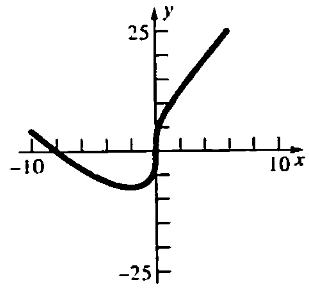
$g(x)$ is increasing on $[-2,\infty)$ and decreasing on $(-\infty,-2]$.

$g''(x)=\frac{4(x-4)}{9x^{5/3}}$; $x-4>0$ when $x>4$, $g''(x)$ does not exist at $x=0$.

Test points: $-1,1,5$; $g''(-1)=\frac{20}{9}$,

$g''(1)=-\frac{4}{3}, g''(5)=\frac{4}{9(5)^{5/3}}.$

$g(x)$ is concave up on $(-\infty,0)\cup(4,\infty)$ and concave down on $(0,4)$.



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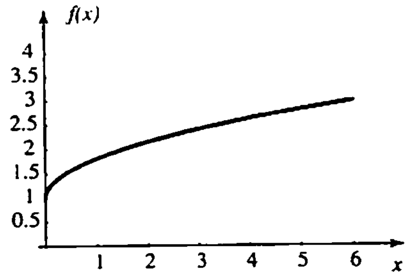

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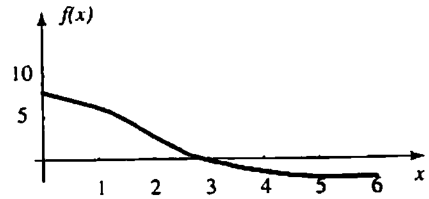

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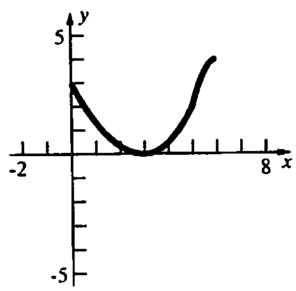

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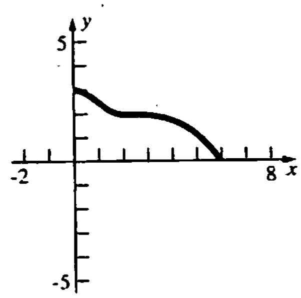

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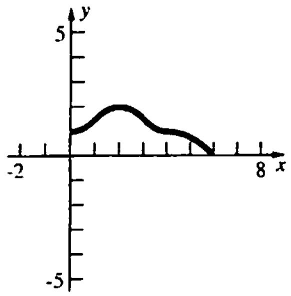
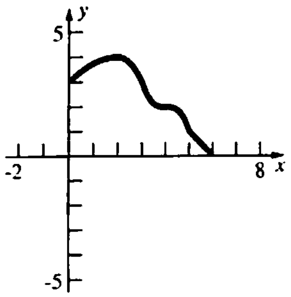


34.



35. $f(x) = ax^2 + bx + c; f'(x) = 2ax + b;$
$f''(x) = 2a$
An inflection point would occur where $f''(x) = 0$, or $2a = 0$. This would only occur when $a = 0$, but if $a = 0$, the equation is not quadratic. Thus, quadratic functions have no points of inflection.

36. $f(x) = ax^3 + bx^2 + cx + d;$
$f'(x) = 3ax^2 + 2bx + c; \ f''(x) = 6ax + 2b$
An inflection point occurs where $f''(x) = 0$, or $6ax + 2b = 0$.
The function will have an inflection point at
$x = -\frac{b}{3a}, \ a \neq 0.$

37. Suppose that there are points x_1 and x_2 in I where $f'(x_1) > 0$ and $f'(x_2) < 0$. Since f' is continuous on I, the Intermediate Value Theorem says that there is some number c between x_1 and x_2 such that $f'(c) = 0$, which is a contradiction. Thus, either $f'(x) > 0$ for all x in I and f is increasing throughout I or $f'(x) < 0$ for all x in I and f is decreasing throughout I.

38. Since $x^2 + 1 = 0$ has no real solutions, $f'(x)$ exists and is continuous everywhere.
$x^2 - x + 1 = 0$ has no real solutions. $x^2 - x + 1 > 0$ and $x^2 + 1 > 0$ for all x, so $f'(x) > 0$ for all x. Thus f is increasing everywhere.

39. a. Let $f(x) = x^2$ and let $I = [0, a], a > y$.
$f'(x) = 2x > 0$ on I. Therefore, $f(x)$ is increasing on I, so $f(x) < f(y)$ for $x < y$.

b. Let $f(x) = \sqrt{x}$ and let $I = [0, a], a > y$.
$f'(x) = \frac{1}{2\sqrt{x}} > 0$ on I. Therefore, $f(x)$ is increasing on I, so $f(x) < f(y)$ for $x < y$.

c. Let $f(x) = \frac{1}{x}$ and let $I = [0, a], a > y$.
$f'(x) = -\frac{1}{x^2} < 0$ on I. Therefore $f(x)$ is decreasing on I, so $f(x) > f(y)$ for $x < y$.

40. $f'(x) = 3ax^2 + 2bx + c$
In order for $f(x)$ to always be increasing, a, b, and c must meet the condition $3ax^2 + 2bx + c > 0$ for all x. More specifically, $a > 0$ and $b^2 - 3ac < 0$.

41. $f''(x) = \frac{3b - ax}{4x^{5/2}}$. If (4, 13) is an inflection point then $13 = 2a + \frac{b}{2}$ and $\frac{3b - 4a}{4 \cdot 32} = 0$. Solving these equations simultaneously, $a = \frac{39}{8}$ and $b = \frac{13}{2}$.

42. $f(x) = a(x - r_1)(x - r_2)(x - r_3)$
$f'(x) = a[(x - r_1)(2x - r_2 - r_3) + (x - r_2)(x - r_3)]$
$f'(x) = a[3x^2 - 2x(r_1 + r_2 + r_3) + r_1r_2 + r_2r_3 + r_1r_3]$
$f''(x) = a[6x - 2(r_1 + r_2 + r_3)]$
$a[6x - 2(r_1 + r_2 + r_3)] = 0$
$6x = 2(r_1 + r_2 + r_3); x = \frac{r_1 + r_2 + r_3}{3}$

43. a. $[f(x) + g(x)]' = f'(x) + g'(x)$.
Since $f'(x) > 0$ and $g'(x) > 0$ for all x, $f'(x) + g'(x) > 0$ for all x. No additional conditions are needed.

b. $[f(x) \cdot g(x)]' = f(x)g'(x) + f'(x)g(x)$.
$f(x)g'(x) + f'(x)g(x) > 0$ if
$f(x) > -\frac{f'(x)}{g'(x)} g(x)$ for all x.

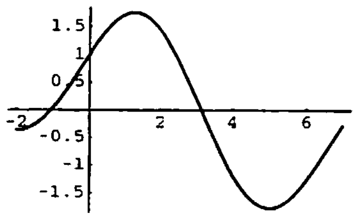
c. $[f(g(x))]' = f'(g(x))g'(x)$.
Since $f'(x) > 0$ and $g'(x) > 0$ for all x, $f'(g(x))g'(x) > 0$ for all x. No additional conditions are needed.

44. a. $[f(x)+g(x)]'' = f''(x)+g''(x)$.
Since $f''(x)>0$ and $g''>0$ for all x, $f''(x)+g''(x)>0$ for all x. No additional conditions are needed.

b. $[f(x)\cdot g(x)]'' = [f(x)g'(x)+f'(x)g(x)]'$
$= f(x)g''(x)+f''(x)g(x)+2f'(x)g'(x)$.
The additional condition is that $f(x)g''(x)+f''(x)g(x)+2f'(x)g'(x)>0$ for all x is needed.

c. $[f(g(x))]'' = [f'(g(x))g'(x)]'$
$= f'(g(x))g''(x)+f''(g(x))[g'(x)]^2$.
The additional condition is that
$$f'(g(x)) > -\frac{f''(g(x))[g'(x)]^2}{g''(x)} \text{ for all } x.$$

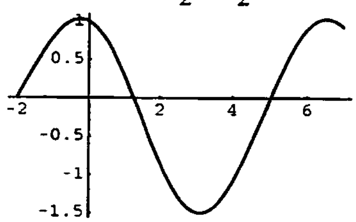
45. a.



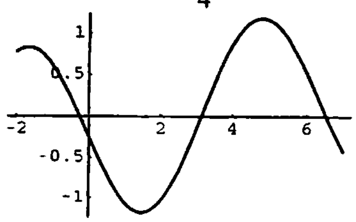
b. $f'(x)<0:(1.3, 5.0)$

c. $f''(x)<0:(-0.25, 3.1)\cup(6.5, 7]$

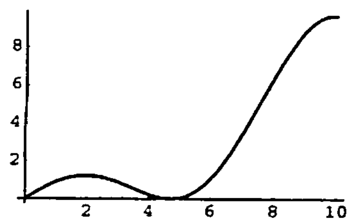
d. $f'(x)=\cos x-\frac{1}{2}\sin\frac{x}{2}$



e. $f''(x)=-\sin x-\frac{1}{4}\cos x$



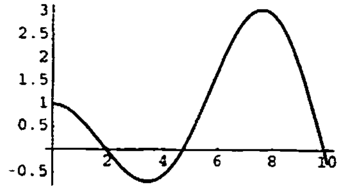
46. a.



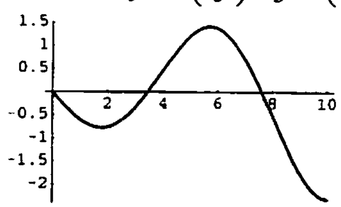
b. $f'(x)<0:(2.0, 4.7)\cup(9.9, 10]$

c. $f''(x)<0:[0, 3.4)\cup(7.6, 10]$

d. $f'(x)=x\left[-\frac{2}{3}\cos\left(\frac{x}{3}\right)\sin\left(\frac{x}{3}\right)\right]+\cos^2\left(\frac{x}{3}\right)$
$=\cos^2\left(\frac{x}{3}\right)-\frac{x}{3}\sin\left(\frac{2x}{3}\right)$



e. $f''(x)=-\frac{2x}{9}\cos\left(\frac{2x}{3}\right)-\frac{2}{3}\sin\left(\frac{2x}{3}\right)$

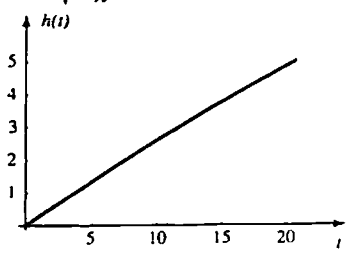


47. $f'(x)>0$ on $(-0.598, 0.680)$
f is increasing on $[-0.598, 0.680]$.

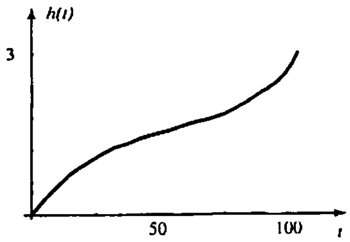
48. $f''(x)<0$ when $x>1.63$ in $[-2, 3]$
f is concave down on $(1.63, 3]$.

49. $\frac{dV}{dt}=2 \text{ in}^3/\text{sec}$
The cup is a portion of a cone with the bottom cut off. If we let x represent the height of the missing cone, we can use similar triangles to show that
$$\frac{x}{3}=\frac{x+5}{3.5}$$
$3.5x=3x+15$
$.5x=15$
$x=30$
Similar triangles can be used again to show that, at any given time, the radius of the cone at water level is
$$r=\frac{h+30}{20}$$
Therefore, the volume of water can be expressed as
$$V=\frac{\pi(h+30)^3}{1200}-\frac{45\pi}{2}.$$
We also know that $V=2t$ from above. Setting the two volume equations equal to each other and solving for h gives

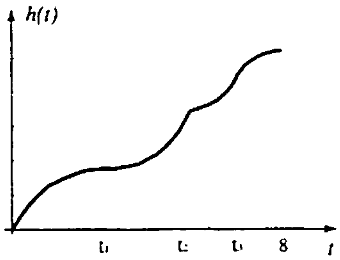
$h = \sqrt[3]{\frac{2400}{\pi}t + 27000} - 30$.



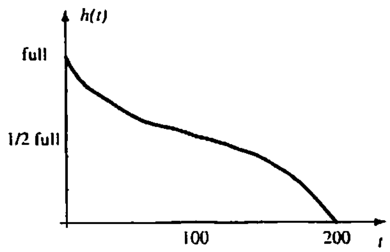
50. The height is always increasing so $h'(t) > 0$. The rate of change of the height decreases for the first 50 minutes and then increases over the next 50 minutes. Thus $h''(t) < 0$ for $0 \le t \le 50$ and $h''(t) > 0$ for $50 < t \le 100$.



51. $V = 3t$, $0 \le t \le 8$. The height is always increasing, so $h'(t) > 0$. The rate of change of the height decreases from time $t = 0$ until time t_1 when the water reaches the middle of the rounded bottom part. The rate of change then increases until time $t_2 = 2t_1$ when the water reaches the neck of the vase. Thus $h''(t) < 0$ for $0 < t < t_1$ and $h''(t) > 0$ for $t_1 < t < t_2$. For $t_2 < t < t_3$, the rate of change in the height increases until the water reaches the middle of the neck. Then the rate of change decreases until $t = 8$ and the vase is full. Thus, $h''(t) > 0$ for $t_2 < t < t_3$ and $h''(t) < 0$ for $t_3 < t < 8$.



52. $V = 20 - .1t$, $0 \le t \le 200$. The height of the water is always decreasing so $h'(t) < 0$. The rate of change in the height increases (the rate is negative, and its absolute value decreases) for the first 100 days and then decreases for the remaining time. Therefore we have $h''(t) > 0$ for $0 < t < 100$, and $h''(t) < 0$ for $100 < t < 200$.



4.3 Concepts Review

1. maximum

2. maximum; minimum

3. maximum

4. local maximum, local minimum, 0

Problem Set 4.3

1. $f'(x) = 3x^2 - 12x = 3x(x-4)$
   Critical points: 0, 4
   $f'(x) > 0$ on $(-\infty, 0)$, $f'(x) < 0$ on $(0, 4)$,
   $f'(x) > 0$ on $(4, \infty)$
   $f''(x) = 6x - 12$; $f''(0) = -12$, $f''(4) = 12$.
   Local minimum at $x = 4$; local maximum at $x = 0$

2. $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$
   Critical points: –2, 2
   $f'(x) > 0$ on $(-\infty, -2)$, $f'(x) < 0$ on $(-2, 2)$,
   $f'(x) > 0$ on $(2, \infty)$
   $f''(x) = 6x$; $f''(-2) = -12$, $f''(2) = 12$
   Local minimum at $x = 2$; local maximum at $x = -2$

3. $f'(\theta) = 2\cos 2\theta$; $2\cos 2\theta \ne 0$ on $\left(0, \frac{\pi}{4}\right)$
   No critical points; no local maxima or minima on $\left(0, \frac{\pi}{4}\right)$.

4. $f'(x)=\frac{1}{2}+\cos x; \frac{1}{2}+\cos x=0$ when $\cos x=-\frac{1}{2}$.

Critical points: $\frac{2\pi}{3}, \frac{4\pi}{3}$

$f'(x)>0$ on $\left(0,\frac{2\pi}{3}\right)$, $f'(x)<0$ on $\left(\frac{2\pi}{3},\frac{4\pi}{3}\right)$.

$f'(x)>0$ on $\left(\frac{4\pi}{3},2\pi\right)$

$f''(x)=-\sin x; f''\left(\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}, f''\left(\frac{4\pi}{3}\right)=\frac{\sqrt{3}}{2}$

Local minimum at $x=\frac{4\pi}{3}$; local maximum at $x=\frac{2\pi}{3}$.

5. $\Psi'(\theta)=2\sin\theta\cos\theta$

$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

Critical point: 0

$\Psi'(\theta)<0$ on $\left(-\frac{\pi}{2},0\right)$, $\Psi'(\theta)>0$ on $\left(0,\frac{\pi}{2}\right)$.

$\Psi''(\theta)=2\cos^2\theta-2\sin^2\theta;\ \Psi''(0)=2$

Local minima at $\theta=0$

6. $r'(z)=4z^3$

Critical point: 0

$r'(z)<0$ on $(-\infty,0)$;

$r'(z)>0$ on $(0,\infty)$

$r''(x)=12x^2; r''(0)=0$; the Second Derivative Test fails.

Local minimum at $z=0$; no local maxima

7. $f'(x)=3x^2-3=3(x^2-1)$

Critical points: $-1, 1$

$f''(x)=6x; f''(-1)=-6, f''(1)=6$

Local minimum value $f(1)=-2$;
local maximum value $f(-1)=2$

8. $g'(x)=4x^3+2x=2x(2x^2+1)$

Critical point: 0

$g''(x)=12x^2+2; g''(0)=2$

Local minimum value $g(0)=3$; no local maximum values

9. $H'(x)=4x^3-6x^2=2x^2(2x-3)$

Critical points: $0, \frac{3}{2}$

$H''(x)=12x^2-12x=12x(x-1);\ H''(0)=0$,

$H''\left(\frac{3}{2}\right)=9$

$H'(x)<0$ on $(-\infty,0)$, $H'(x)<0$ on $\left(0,\frac{3}{2}\right)$

Local minimum value $H\left(\frac{3}{2}\right)=-\frac{27}{16}$; no local maximum values ($x=0$ is neither a local minimum nor maximum)

10. $f'(x)=5(x-2)^4$

Critical point: 2

$f''(x)=20(x-2)^3; f''(2)=0$

$f'(x)>0$ on $(-\infty,2)$, $f'(x)>0$ on $(2,\infty)$

No local minimum or maximum values

11. $g'(t)=-\frac{2}{3(t-2)^{1/3}}; g'(t)$ does not exist at $t=2$.

Critical point: 2

$g'(1)=\frac{2}{3}, g'(3)=-\frac{2}{3}$

No local minimum values; local maximum value $g(2)=\pi$.

12. $r'(s)=3+\frac{2}{5s^{3/5}}=\frac{15s^{3/5}+2}{5s^{3/5}}; r'(s)=0$ when $s=-\left(\frac{2}{15}\right)^{5/3}$, $r'(s)$ does not exist at $s=0$.

Critical points: $-\left(\frac{2}{15}\right)^{5/3}, 0$

$r''(s)=-\frac{6}{25s^{8/5}}; r''\left(-\left(\frac{2}{15}\right)^{5/3}\right)=-\frac{6}{25}\left(\frac{15}{2}\right)^{8/3}$

$r'(s)<0$ on $\left(-\left(\frac{2}{15}\right)^{5/3},0\right)$, $r'(s)>0$ on $(0,\infty)$

Local minimum value $r(0)=0$; local maximum value

$r\left(-\left(\frac{2}{15}\right)^{5/3}\right)=-3\left(\frac{2}{15}\right)^{5/3}+\left(\frac{2}{15}\right)^{2/3}=\frac{3}{5}\left(\frac{2}{15}\right)^{2/3}$

13. $f'(t)=1+\frac{1}{t^2}$

No critical points

No local minimum or maximum values

14. $f'(x)=\frac{x(x^2+8)}{(x^2+4)^{3/2}}$

Critical point: 0

$f'(x)<0$ on $(-\infty,0)$, $f'(x)>0$ on $(0,\infty)$

Local minimum value $f(0) = 0$, no local maximum values

15. $\Lambda'(\theta) = -\frac{1}{1+\sin\theta}$; $\Lambda'(\theta)$ does not exist at $\theta = \frac{3\pi}{2}$, but $\Lambda(\theta)$ does not exist at that point either.
No critical points
No local minimum or maximum values

16. $g'(\theta) = \frac{\sin\theta\cos\theta}{|\sin\theta|}$; $g'(\theta) = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$;
$g'(\theta)$ does not exist at $\theta = \pi$.
Split the θ-axis into the intervals $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, $\left(\pi, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, 2\pi\right)$.
Test points: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; $g'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$,
$g'\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $g'\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $g'\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
Local minimum value $g(\pi) = 0$; local maximum values $g\left(\frac{\pi}{2}\right) = 1$ and $g\left(\frac{3\pi}{2}\right) = 1$

17. $F'(x) = \frac{3}{\sqrt{x}} - 4$; $\frac{3}{\sqrt{x}} - 4 = 0$ when $x = \frac{9}{16}$
Critical points: $0, \frac{9}{16}, 4$
$F(0) = 0$, $F\left(\frac{9}{16}\right) = \frac{9}{4}$, $F(4) = -4$
Minimum value $F(4) = -4$; maximum value $F\left(\frac{9}{16}\right) = \frac{9}{4}$

18. From Problem 17, the critical points are 0 and $\frac{9}{16}$.
$F'(x) > 0$ on $\left(0, \frac{9}{16}\right)$, $F'(x) < 0$ on $\left(\frac{9}{16}, \infty\right)$
F decreases without bound on $\left(\frac{9}{16}, \infty\right)$. No minimum values; maximum value $F\left(\frac{9}{16}\right) = \frac{9}{4}$

19. $f'(x) = 64(-1)(\sin x)^{-2}\cos x$
$+27(-1)(\cos x)^{-2}(-\sin x)$
$= -\frac{64\cos x}{\sin^2 x} + \frac{27\sin x}{\cos^2 x}$
$= \frac{(3\sin x - 4\cos x)(9\sin^2 x + 12\cos x\sin x + 16\cos^2 x)}{\sin^2 x\cos^2 x}$
On $\left(0, \frac{\pi}{2}\right)$, $f'(x) = 0$ only where $3\sin x = 4\cos x$;
$\tan x = \frac{4}{3}$;
$x = \tan^{-1}\frac{4}{3} \approx 0.9273$
Critical point: 0.9273
For $0 < x < 0.9273$, $f'(x) < 0$, while for $0.9273 < x < \frac{\pi}{2}$, $f'(x) > 0$
Minimum value $f\left(\tan^{-1}\frac{4}{3}\right) = \frac{64}{\frac{4}{5}} + \frac{27}{\frac{3}{5}} = 125$;
no maximum value

20. $f'(x) = 2x - \frac{2}{x^3}$; $f'(x) = 0$ when $x = -1, 1$ (only $x = 1$ in $(0, \infty)$)
$f'(x) < 0$ on $(0, 1)$, $f'(x) > 0$ on $(1, \infty)$
Minimum value $f(1) = 2$; no maximum value

21. $g'(x) = 2x + \frac{(8-x)^2(32x) - (16x^2)2(8-x)(-1)}{(8-x)^4}$
$= 2x + \frac{256x}{(8-x)^3} = \frac{2x[(8-x)^3 + 128]}{(8-x)^3}$
For $x > 8$, $g'(x) = 0$ when $(8-x)^3 + 128 = 0$;
$(8-x)^3 = -128$; $8 - x = -\sqrt[3]{128}$;
$x = 8 + 4\sqrt[3]{2} \approx 13.04$
$g'(x) < 0$ on $(8, 8 + 4\sqrt[3]{2})$,
$g'(x) > 0$ on $(8 + 4\sqrt[3]{2}, \infty)$
$g(13.04) \approx 277$ is the minimum value

22. $f'(x) = 2Ax + B$; $f'(x) = 0$ when $x = -\frac{B}{2A}$
$f''(x) = 2A > 0$ for all x, so $x = -\frac{B}{2A}$ is a minimum.
$f\left(-\frac{B}{2A}\right) = A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C$
$= \frac{B^2}{4A} - \frac{B^2}{2A} + C = -\frac{B^2}{4A} + C$

If $B^2 - 4AC \le 0$, then $\frac{B^2}{4A} \le C$ (recall $A > 0$), so

$f\left(-\frac{B}{2A}\right) \ge 0$.

Since the minimum value of f is nonnegative, $f(x) \ge 0$ for all x.

If $f\left(-\frac{B}{2A}\right) \ge 0$, then

$-\frac{B^2}{4A} + C \ge 0; C \ge \frac{B^2}{4A}; 4AC \ge B^2;$

$B^2 - 4AC \le 0$

23. $f'(x) = 0$ when $x = 0$ and $x = 1$. On the interval $(-\infty, 0)$ we get $f'(x) < 0$. On $(0, \infty)$, we get $f'(x) > 0$. Thus there is a local min at $x = 0$ but no local max.

24. $f'(x) = 0$ at $x = 1, 2, 3, 4$; $f'(x)$ is negative on $(-\infty, 1) \cup (2, 3) \cup (4, \infty)$ and positive on $(1, 2) \cup (3, 4)$. Thus, the function has a local minimum at $x = 1, 3$ and a local maximum at $x = 2, 4$.

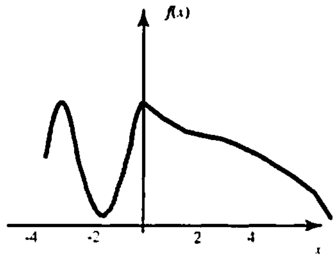
25. $f'(x) = 0$ at $x = 1, 2, 3, 4$; $f'(x)$ is negative on $(3, 4)$ and positive on $(-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (4, \infty)$ Thus, the function has a local minimum at $x = 4$ and a local maximum at $x = 3$.

26. The function is always increasing so there are no local extrema.

27. Critical points: $-2, -1, 2, 3$
Split the x-axis into the intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, 2)$, $(2, 3)$, $(3, \infty)$.
Use test values: -3, $-\frac{3}{2}$, 0, $\frac{5}{2}$, and 4 to find that $f'(x) < 0$ on $(-2, -1) \cup (-1, 2) \cup (2, 3)$ and $f'(x) > 0$ on $(-\infty, -2) \cup (3, \infty)$.
Local maximum at $x = -2$; local minimum at $x = 3$.

28. $f'''(c) > 0$ implies that f'' is increasing at c, so f is concave up to the right of c (since $f''(x) > 0$ to the right of c) and concave down to the left of c (since $f''(x) < 0$ to the left of c). Therefore f has a point of inflection at c.

29. Written response (graph)



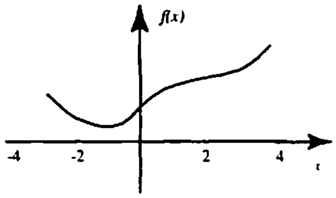
a. Increasing: $(-\infty, -3] \cup [-1, 0)$
Decreasing: $[-3, -1] \cup (0, \infty)$

b. Concave up: $(-2, 0) \cup (0, 2)$
Concave down: $(-\infty, -2) \cup (2, \infty)$

c. Local maximum at $x = -3$; local minimum at $x = -1$
$f'(0), f''(0)$ do not exist, but $f(x)$ is continuous. $x = 0$ is a critical point.
$f'(0) > 0$ on $[-1, 0]$, and $f'(0) < 0$ on $(0, \infty)$.
Thus, at $x = 0$ there is a local maximum.

d. Inflection points at $x = -2, 2$

30. Written response (graph)



a. Increasing: $[-1, \infty)$
Decreasing: $(-\infty, -1]$

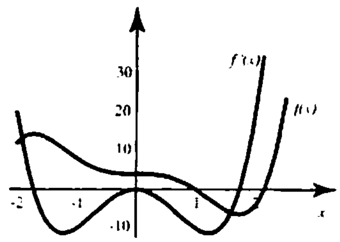
b. Concave up: $(-2, 0) \cup (2, \infty)$
Concave down: $(0, 2)$

c. No local maximum; local minimum at $x = -1$

d. Inflection points at $x = 0, 2$

31. $f'(x) = 5x^4 - 15x^2$
$f''(x) = -30x + 20x^3$
Solving $f'(x) = 0$ gives critical numbers of $x = 0, \pm\sqrt{3}$. Using the second derivative we get that $f''(0) = 0$, $f''(\sqrt{3}) > 0$, and $f''(-\sqrt{3}) < 0$. The second derivative fails at $x = 0$ (there is actually a point of inflection there) but indicates that we have a local maximum $f(-\sqrt{3}) \approx 14.392$ and a local minimum $f(\sqrt{3}) \approx -6.392$ (also the global minimum). The global maximum,

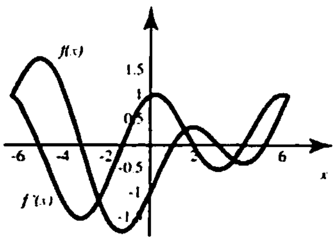
$f(2.5) \approx 23.531$, on $[-2, 2.5]$ occurs at the right endpoint.



32. $f'(x) = \cos x + \frac{1}{2}\sin\left(\frac{x}{2}\right)$

$f''(x) = \frac{1}{4}\cos\left(\frac{x}{2}\right) - \sin x$

Zooming in on the graph of the first derivative gives critical numbers of $x \approx -5.01, -1.27, 2.01, 4.28$.
Using the second derivative we get that
$f''(-5.01) < 0; f''(-1.27) > 0;$
$f''(2.01) < 0; f''(4.28) > 0.$
So we have a local maximum of
$f(-5.01) \approx 1.76$ (also the global max) and
$f(2.01) \approx 0.369$. We also have a local minimum of
$f(-1.27) \approx -1.76$ (the global minimum) and
$f(4.28) \approx -0.369$.

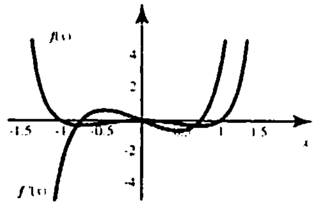


33. $f'(x) = (x^3 - x)\sec^2 x + (3x^2 - 1)\tan x$

$f''(x) = 6x\tan x$
$+ 2(\sec^2 x)(3x^2 - 1 + x(x^2 - 1)\tan x)$

Graphing the first derivative and zooming in gives critical numbers of $x = 0$ and $x \approx -0.745, 0.745$. Using the second derivative gives

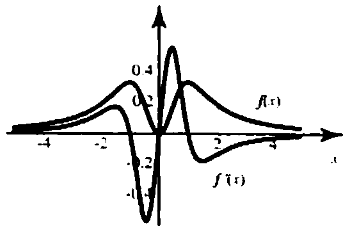
$f''(-0.745) > 0; f''(0) < 0; f''(0.745) > 0$. So have a local maximum of $f(0) = -2$ and local minimums of $f(-0.745) = f(0.745) \approx -0.306$ (also the global minimum). There is no global maximum since the graph blows up at the endpoints.



34. $f'(x) = \dfrac{-2x(x^4 - 1)}{(x^4 + x^2 + 1)^2}$

$f''(x) = \dfrac{6x^8 - 2x^6 - 24x^4 - 6x^2 + 2}{(x^4 + x^2 + 1)^3}$

Solving the equation $f'(x) = 0$ yields critical numbers of $x = -1, 0, 1$. Using the second derivative we see that
$f''(-1) < 0; f''(0) > 0; f''(1) < 0$. Therefore, we have a local minimum of $f(0) = 0$ (also the global min). We have two local maximums of $f(-1) = f(1) = \frac{1}{3}$ (also the global maximum).



4.4 Concepts Review

1. $0 < x < \infty$

2. $2x + \dfrac{200}{x}$

3. $3x + \dfrac{300}{x}$

4. minimum; maximum

Problem Set 4.4

1. Let x be one number, y be the other, and Q be the sum of the squares.
$xy = -16$
$y = -\frac{16}{x}$
The possible values for x are in $(-\infty, 0)$ or $(0, \infty)$.
$Q = x^2 + y^2 = x^2 + \frac{256}{x^2}$
$\frac{dQ}{dx} = 2x - \frac{512}{x^3}$
$2x - \frac{512}{x^3} = 0$
$x^4 = 256$
$x = \pm 4$
The critical points are –4, 4.
$\frac{dQ}{dx} < 0$ on $(-\infty, -4)$ and $(0, 4)$. $\frac{dQ}{dx} > 0$ on $(-4, 0)$ and $(4, \infty)$.
When $x = -4$, $y = 4$ and when $x = 4$, $y = -4$. The two numbers are –4 and 4.

2. Let x be the number.
$Q = \sqrt{x} - 8x$
x will be in the interval $(0, \infty)$.
$\frac{dQ}{dx} = \frac{1}{2}x^{-1/2} - 8$
$\frac{1}{2}x^{-1/2} - 8 = 0$
$x^{-1/2} = 16$
$x = \frac{1}{256}$
$\frac{dQ}{dx} > 0$ on $\left(0, \frac{1}{256}\right)$ and $\frac{dQ}{dx} < 0$ on $\left(\frac{1}{256}, \infty\right)$.
Q attains its maximum value at $x = \frac{1}{256}$.

3. Let x be the number.
$Q = \sqrt[4]{x} - 2x$
x will be in the interval $(0, \infty)$.
$\frac{dQ}{dx} = \frac{1}{4}x^{-3/4} - 2$
$\frac{1}{4}x^{-3/4} - 2 = 0$
$x^{-3/4} = 8$
$x = \frac{1}{16}$
$\frac{dQ}{dx} > 0$ on $\left(0, \frac{1}{16}\right)$ and $\frac{dQ}{dx} < 0$ on $\left(\frac{1}{16}, \infty\right)$
Q attains its maximum value at $x = \frac{1}{16}$.

4. Let x be one number, y be the other, and Q be the sum of the squares.
$xy = -12$
$y = -\frac{12}{x}$
The possible values for x are in $(-\infty, 0)$ or $(0, \infty)$.
$Q = x^2 + y^2 = x^2 + \frac{144}{x^2}$
$\frac{dQ}{dx} = 2x - \frac{288}{x^3}$
$2x - \frac{288}{x^3} = 0$
$x^4 = 144$
$x = \pm 2\sqrt{3}$
The critical points are $-2\sqrt{3}, 2\sqrt{3}$
$\frac{dQ}{dx} < 0$ on $(-\infty, -2\sqrt{3})$ and $(0, 2\sqrt{3})$.
$\frac{dQ}{dx} > 0$ on $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, \infty)$.
When $x = -2\sqrt{3}, y = 2\sqrt{3}$ and when $x = 2\sqrt{3}, y = -2\sqrt{3}$.
The two numbers are $-2\sqrt{3}$ and $2\sqrt{3}$.

5. Let Q be the square of the distance between (x, y) and $(0, 5)$.
$Q = (x-0)^2 + (y-5)^2 = x^2 + (x^2 - 5)^2$
$= x^4 - 9x^2 + 25$
$\frac{dQ}{dx} = 4x^3 - 18x$
$4x^3 - 18x = 0$
$2x(2x^2 - 9) = 0$
$x = 0, \pm\frac{3}{\sqrt{2}}$
$\frac{dQ}{dx} < 0$ on $\left(-\infty, -\frac{3}{\sqrt{2}}\right)$ and $\left(0, \frac{3}{\sqrt{2}}\right)$.
$\frac{dQ}{dx} > 0$ on $\left(-\frac{3}{\sqrt{2}}, 0\right)$ and $\left(\frac{3}{\sqrt{2}}, \infty\right)$.
When $x = -\frac{3}{\sqrt{2}}, y = \frac{9}{2}$ and when $x = \frac{3}{\sqrt{2}}$, $y = \frac{9}{2}$.
The points are $\left(-\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$.

6. Let Q be the square of the distance between (x, y) and $(10, 0)$.

$Q = (x-10)^2 + (y-0)^2 = (2y^2 - 10)^2 + y^2$

$= 4y^4 - 39y^2 + 100$

$\frac{dQ}{dy} = 16y^3 - 78y$

$16y^3 - 78y = 0$

$2y(8y^2 - 39) = 0$

$y = 0, \pm\frac{\sqrt{39}}{2\sqrt{2}}$

$\frac{dQ}{dy} < 0$ on $\left(-\infty, -\frac{\sqrt{39}}{2\sqrt{2}}\right)$ and $\left(0, \frac{\sqrt{39}}{2\sqrt{2}}\right)$.

$\frac{dQ}{dy} > 0$ on $\left(-\frac{\sqrt{39}}{2\sqrt{2}}, 0\right)$ and $\left(\frac{\sqrt{39}}{2\sqrt{2}}, \infty\right)$.

When $y = -\frac{\sqrt{39}}{2\sqrt{2}}, x = \frac{39}{4}$ and when $y = \frac{\sqrt{39}}{2\sqrt{2}}, x = \frac{39}{4}$.

The points are $\left(\frac{39}{4}, -\frac{\sqrt{39}}{2\sqrt{2}}\right)$ and $\left(\frac{39}{4}, \frac{\sqrt{39}}{2\sqrt{2}}\right)$.

7. $xy = 900; y = \frac{900}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 4x + 3y = 4x + 3\left(\frac{900}{x}\right) = 4x + \frac{2700}{x}$

$\frac{dQ}{dx} = 4 - \frac{2700}{x^2}$

$4 - \frac{2700}{x^2} = 0$

$x^2 = 675$

$x = \pm 15\sqrt{3}$

$x = 15\sqrt{3}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx} < 0$ on $(0, 15\sqrt{3})$ and

$\frac{dQ}{dx} > 0$ on $(15\sqrt{3}, \infty)$.

When $x = 15\sqrt{3}, y = \frac{900}{15\sqrt{3}} = 20\sqrt{3}$.

Q has a minimum when $x = 15\sqrt{3} \approx 25.98$ ft and $y = 20\sqrt{3} \approx 34.64$ ft.

8. $xy = 300;\ y = \frac{300}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 6x + 4y = 6x + \frac{1200}{x}$

$\frac{dQ}{dx} = 6 - \frac{1200}{x^2}$

$6 - \frac{1200}{x^2} = 0$

$x^2 = 200$

$x = \pm 10\sqrt{2}$

$x = 10\sqrt{2}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx} < 0$ on $(0, 10\sqrt{2})$ and $\frac{dQ}{dx} > 0$ on $(10\sqrt{2}, \infty)$

When $x = 10\sqrt{2}, y = \frac{300}{10\sqrt{2}} = 15\sqrt{2}$.

Q has a minimum when $x = 10\sqrt{2} \approx 14.14$ ft and $y = 15\sqrt{2} \approx 21.21$ ft.

9. $xy = 300;\ y = \frac{300}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 3(6x + 2y) + 2(2y) = 18x + 10y = 18x + \frac{3000}{x}$

$\frac{dQ}{dx} = 18 - \frac{3000}{x^2}$

$18 - \frac{3000}{x^2} = 0$

$x^2 = \frac{500}{3}$

$x = \pm\frac{10\sqrt{5}}{\sqrt{3}}$

$x = \frac{10\sqrt{5}}{\sqrt{3}}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx} < 0$ on $\left(0, \frac{10\sqrt{5}}{\sqrt{3}}\right)$ and

$\frac{dQ}{dx} > 0$ on $\left(\frac{10\sqrt{5}}{\sqrt{3}}, \infty\right)$.

When $x = \frac{10\sqrt{5}}{\sqrt{3}}, y = \frac{300}{\frac{10\sqrt{5}}{\sqrt{3}}} = 6\sqrt{15}$

Q has a minimum when $x = \frac{10\sqrt{5}}{\sqrt{3}} \approx 12.91$ ft and $y = 6\sqrt{15} \approx 23.24$ ft.

10. $xy = 900;\ y = \frac{900}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 6x + 4y = 6x + \frac{3600}{x}$

$\frac{dQ}{dx} = 6 - \frac{3600}{x^2}$

$6 - \frac{3600}{x^2} = 0$

$x^2 = 600$

$x = \pm 10\sqrt{6}$

$x = 10\sqrt{6}$ is the only critical point in (0, ∞).

$\frac{dQ}{dx} < 0$ on $(0, 10\sqrt{6})$ and $\frac{dQ}{dx} > 0$ on $(10\sqrt{6}, \infty)$.

When $x = 10\sqrt{6}, y = \frac{900}{10\sqrt{6}} = 15\sqrt{6}$

Q has a minimum when $x = 10\sqrt{6} \approx 24.49$ ft and $y = 15\sqrt{6} \approx 36.74$.

It appears that $\frac{x}{y} = \frac{2}{3}$.

Suppose that each pen has area A.

$xy = A;\ y = \frac{A}{x}$

The possible values for x are in (0, ∞).

$Q = 6x + 4y = 6x + \frac{4A}{x}$

$\frac{dQ}{dx} = 6 - \frac{4A}{x^2}$

$6 - \frac{4A}{x^2} = 0$

$x^2 = \frac{2A}{3}$

$x = \pm\sqrt{\frac{2A}{3}}$

$x = \sqrt{\frac{2A}{3}}$ is the only critical point on (0, ∞).

$\frac{dQ}{dx} < 0$ on $\left(0, \sqrt{\frac{2A}{3}}\right)$ and

$\frac{dQ}{dx} > 0$ on $\left(\sqrt{\frac{2A}{3}}, \infty\right)$.

When $x = \sqrt{\frac{2A}{3}}, y = \frac{A}{\sqrt{\frac{2A}{3}}} = \sqrt{\frac{3A}{2}}$

$\frac{x}{y} = \frac{\sqrt{\frac{2A}{3}}}{\sqrt{\frac{3A}{2}}} = \frac{2}{3}$

11. We are interested in the global extrema for the distance of the object from the observer. We obtain the same extrema by considering the squared distance

$D(x) = (x - 2.6656)^2 + (42 + x - .08x^2)^2$

The first derivative is given by

$D'(x) = \frac{16}{625}x^3 - \frac{12}{25}x^2 - \frac{236}{25}x - \frac{49168}{625}$.

This function can be plotted on a graphing calculator and then analyzed with the TRACE and ZOOM features. Doing so yields critical numbers of $x \approx 6.8267, 28.0$. The second derivative is

$D''(x) = \frac{48}{625}x^2 - \frac{24}{25}x - \frac{236}{25}$

which gives $D''(6.8267) < 0$ and $D''(28) > 0$. So we have a local maximum at $x \approx 6.827$ and a local minimum at $x \approx 28.0$. Thus, the object is at the point (28, 7.28) when it is closest to the observer and at the point (6.8267, 45.098) when it is farthest from the observer.

12. Let x be the distance from I_1.

$Q = \frac{kI_1}{x^2} + \frac{kI_2}{(s-x)^2}$

$\frac{dQ}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(s-x)^3}$

$-\frac{2kI_1}{x^3} + \frac{2kI_2}{(s-x)^3} = 0;\ \frac{x^3}{(s-x)^3} = \frac{I_1}{I_2};$

$x = \frac{s\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$

$\frac{d^2Q}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(s-x)^4} > 0$, so this point minimizes the sum.

13. Let x be the distance from P to where the woman lands the boat. She must row a distance of $\sqrt{x^2 + 4}$ miles and walk $10 - x$ miles. This will take her $T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{10-x}{4}$ hours;

$0 \le x \le 10$. $T'(x) = \frac{x}{3\sqrt{x^2+4}} - \frac{1}{4};\ T'(x) = 0$

when $x = \frac{6}{\sqrt{7}}$.

$T(0) = \frac{19}{6}$ hr = 3 hr, 10 min;

$T\left(\frac{6}{\sqrt{7}}\right) = \frac{15 + \sqrt{7}}{6} \approx 2.94$ hr,

$T(10) = \frac{\sqrt{104}}{3} \approx 3.40$ hr

She should land the boat $\frac{6}{\sqrt{7}} \approx 2.27$ mi down the shore from P.

14. $T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{10-x}{50}, 0 \le x \le 10.$

$T'(x) = \frac{x}{3\sqrt{x^2-4}} - \frac{1}{50}; T'(x) = 0$ when

$x = \frac{6}{\sqrt{2491}}$

$T(0) = \frac{13}{15} \approx 0.867$ hr; $T\left(\frac{6}{\sqrt{2491}}\right) \approx 0.865$ hr;

$T(10) \approx 3.399$ hr

She should land the boat $\frac{6}{\sqrt{2491}} \approx 0.12$ mi down the shore from P.

15. $T(x) = \frac{\sqrt{x^2+4}}{20} + \frac{10-x}{4}, 0 \le x \le 10.$

$T'(x) = \frac{x}{20\sqrt{x^2+4}} - \frac{1}{4}; T'(x) = 0$ has no solution.

$T(0) = \frac{2}{20} + \frac{10}{4} = \frac{13}{5}$ hr = 2 hr. 36 min

$T(10) = \frac{\sqrt{104}}{20} \approx 0.5$ hr

She should take the boat all the way to town.

16. Let x be the length of cable on land, $0 \le x \le L$. Let C be the cost.

$C = a\sqrt{(L-x)^2 + w^2} + bx$

$\frac{dC}{dx} = -\frac{a(L-x)}{\sqrt{(L-x)^2+w^2}} + b$

$-\frac{a(L-x)}{\sqrt{(L-x)^2+w^2}} + b = 0$ when

$b^2[(L-x)^2 + w^2] = a^2(L-x)^2$

$(a^2 - b^2)(L-x)^2 = b^2w^2$

$x = L - \frac{bw}{\sqrt{a^2-b^2}}$ ft on land;

$\frac{aw}{\sqrt{a^2-b^2}}$ ft under water

$\frac{d^2C}{dx^2} = \frac{aw^2}{[(L-x)^2+w^2]^{3/2}} > 0$ for all x, so this minimizes the cost.

17. Let the coordinates of the first ship at 7:00 a.m. be (0, 0). Thus, the coordinates of the second ship at 7:00 a.m. are (–60, 0). Let t be the time in hours since 7:00 a.m. The coordinates of the first and second ships at t are $(-20t, 0)$ and $\left(-60+15\sqrt{2}t, -15\sqrt{2}t\right)$ respectively. Let D be the square of the distances at t.

$D = \left(-20t + 60 - 15\sqrt{2}t\right)^2 + \left(0 + 15\sqrt{2}t\right)^2$

$= \left(1300 + 600\sqrt{2}\right)t^2 - \left(2400 + 1800\sqrt{2}\right)t + 3600$

$\frac{dD}{dt} = 2\left(1300+600\sqrt{2}\right)t - \left(2400+1800\sqrt{2}\right)$

$2\left(1300+600\sqrt{2}\right)t - \left(2400+1800\sqrt{2}\right) = 0$ when

$t = \frac{12+9\sqrt{2}}{13+6\sqrt{2}} \approx 1.15$ hrs or 1 hr, 9 min

D is the minimum at $t = \frac{12+9\sqrt{2}}{13+6\sqrt{2}}$ since $\frac{d^2D}{dt^2} > 0$ for all t.

The ships are closest at 8:09 A.M.

18. Write y in terms of x: $y = \frac{b}{a}\sqrt{a^2-x^2}$ (positive square root since the point is in the first quadrant). Compute the slope of the tangent line:

$y' = -\frac{bx}{a\sqrt{a^2-x^2}}.$

Find the y-intercept, y_0, of the tangent line through the point (x, y):

$\frac{y_0 - y}{0 - x} = -\frac{bx}{a\sqrt{a^2-x^2}}$

$y_0 = \frac{bx^2}{a\sqrt{a^2-x^2}} + y = \frac{bx^2}{a\sqrt{a^2-x^2}} + \frac{b}{a}\sqrt{a^2-x^2}$

$= \frac{ab}{\sqrt{a^2-x^2}}$

Find the x-intercept, x_0, of the tangent line through the point (x, y):

$\frac{y-0}{x-x_0} = -\frac{bx}{a\sqrt{a^2-x^2}}$

$x_0 = \frac{ay\sqrt{a^2-x^2}}{bx} + x = \frac{a^2-x^2}{x} + x = \frac{a^2}{x}$

Compute the Area A of the resulting triangle and maximize:

$A = \frac{1}{2}x_0y_0 = \frac{a^3b}{2x\sqrt{a^2-x^2}} = \frac{a^3b}{2}\left(x\sqrt{a^2-x^2}\right)^{-1}$

$\frac{dA}{dx} = -\frac{a^3b}{2}\left(x\sqrt{a^2-x^2}\right)^{-2}\left(\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}\right)$

$$= \frac{a^3 b}{2x^2(a^2 - x^2)^{3/2}}(2x^2 - a^2)$$

$$\frac{a^3 b}{2x^2(a^2 - x^2)^{3/2}}(2x^2 - a^2) = 0 \text{ when}$$

$$x = \frac{a}{\sqrt{2}}; y = \frac{b}{a}\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \frac{b}{\sqrt{2}}$$

$$y' = -\frac{b\left(\frac{a}{\sqrt{2}}\right)}{a\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}} = -\frac{b}{a}$$

Note that $\frac{dA}{dx} < 0$ on $\left(0, \frac{a}{\sqrt{2}}\right)$ and $\frac{dA}{dx} > 0$ on $\left(\frac{a}{\sqrt{2}}, a\right)$, so A is a minimum at $x = \frac{a}{\sqrt{2}}$. Then the equation of the tangent line is

$$y = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right) + \frac{b}{\sqrt{2}} \text{ or } bx + ay - ab\sqrt{2} = 0.$$

19. Let x be the radius of the base of the cylinder and h the height.

$$V = \pi x^2 h; r^2 = x^2 + \left(\frac{h}{2}\right)^2; x^2 = r^2 - \frac{h^2}{4}$$

$$V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi h r^2 - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}; V' = 0 \text{ when } h = \pm\frac{2\sqrt{3}r}{3}$$

Since $\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$, the volume is maximized when $h = \frac{2\sqrt{3}r}{3}$.

$$V = \pi\left(\frac{2\sqrt{3}}{3}r\right)r^2 - \frac{\pi\left(\frac{2\sqrt{3}}{3}r\right)^3}{4}$$

$$= \frac{2\pi\sqrt{3}}{3}r^3 - \frac{2\pi\sqrt{3}}{9}r^3 = \frac{4\pi\sqrt{3}}{9}r^3$$

20. Let r be the radius of the circle, x the length of the rectangle, and y the width of the rectangle.

$$P = 2x + 2y;\ r^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2; r^2 = \frac{x^2}{4} + \frac{y^2}{4};$$

$$y = \sqrt{4r^2 - x^2};\ P = 2x + 2\sqrt{4r^2 - x^2}$$

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4r^2 - x^2}};$$

$$2 - \frac{2x}{\sqrt{4r^2 - x^2}} = 0; 2\sqrt{4r^2 - x^2} = 2x;$$

$$16r^2 - 4x^2 = 4x^2; x = \pm\sqrt{2}r$$

$$\frac{d^2P}{dx^2} = -\frac{8r^2}{(4r^2 - x^2)^{3/2}} < 0 \text{ when } x = \sqrt{2}r;$$

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$$

The rectangle with maximum perimeter is a square with side length $\sqrt{2}r$.

21. Let x be the radius of the cylinder, r the radius of the sphere, and h the height of the cylinder.

$$A = 2\pi xh;\ r^2 = x^2 + \frac{h^2}{4}; x = \sqrt{r^2 - \frac{h^2}{4}}$$

$$A = 2\pi\sqrt{r^2 - \frac{h^2}{4}}h = 2\pi\sqrt{h^2r^2 - \frac{h^4}{4}}$$

$$\frac{dA}{dh} = \frac{\pi\left(2r^2h - h^3\right)}{\sqrt{h^2r^2 - \frac{h^4}{4}}}; A' = 0 \text{ when } h = 0, \pm\sqrt{2}r$$

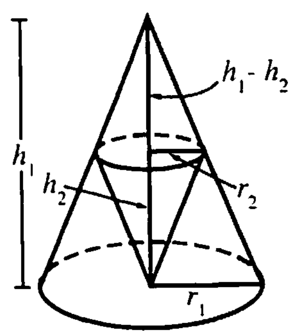
$\frac{dA}{dh} > 0$ on $(0, \sqrt{2}r)$ and $\frac{dA}{dh} < 0$ on $(\sqrt{2}r, 2r)$, so A is a maximum when $h = \sqrt{2}r$.

The dimensions are $h = \sqrt{2}r, x = \frac{r}{\sqrt{2}}$.

22. Let r_1 and h_1 be the radius and altitude of the outer cone; r_2 and h_2 the radius and altitude of the inner cone.

$V_1 = \frac{1}{3}\pi r_1^2 h_1$ is fixed. $r_1 = \sqrt{\frac{3V_1}{\pi h_1}}$

By similar triangles $\frac{h_1 - h_2}{h_1} = \frac{r_2}{r_1}$ (see figure).



$$r_2 = r_1\left(1 - \frac{h_2}{h_1}\right) = \sqrt{\frac{3V_1}{\pi h_1}}\left(1 - \frac{h_2}{h_1}\right)$$

$$V_2 = \frac{1}{3}\pi r_2^2 h_2 = \frac{1}{3}\pi\left[\sqrt{\frac{3V_1}{\pi h_1}}\left(1 - \frac{h_2}{h_1}\right)\right]^2 h_2$$

$$= \frac{\pi}{3} \cdot \frac{3V_1 h_2}{\pi h_1}\left(1 - \frac{h_2}{h_1}\right)^2 = V_1 \frac{h_2}{h_1}\left(1 - \frac{h_2}{h_1}\right)^2$$

Let $k = \frac{h_2}{h_1}$, the ratio of the altitudes of the cones,

then $V_2 = V_1 k(1-k)^2$.

$$\frac{dV_2}{dk} = V_1(1-k)^2 - 2kV_1(1-k) = V_1(1-k)(1-3k)$$

$0 < k < 1$ so $\frac{dV_2}{dk} = 0$ when $k = \frac{1}{3}$.

$$\frac{d^2V_2}{dk^2} = V_1(6k-4); \frac{d^2V_2}{dk^2} < 0 \text{ when } k = \frac{1}{3}$$

The altitude of the inner cone must be $\frac{1}{3}$ the altitude of the outer cone.

23. Let x be the length of a side of the square, so $\frac{100-4x}{3}$ is the side of the triangle, $0 \le x \le 25$

$$A = x^2 + \frac{1}{2}\left(\frac{100-4x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{100-4x}{3}\right)$$

$$= x^2 + \frac{\sqrt{3}}{4}\left(\frac{10,000 - 800x + 16x^2}{9}\right)$$

$$\frac{dA}{dx} = 2x - \frac{200\sqrt{3}}{9} + \frac{8\sqrt{3}}{9}x;\ A' = 0 \text{ when } x \approx 10.87$$

Critical points: $x = 0,\ 10.87,\ 25$
At $x = 0$, $A \approx 481$; at $x = 10.87$, $A \approx 272$; at $x = 25$, $A = 625$.

a. For minimum area, the cut should be approximately $4(10.87) = 43.48$ cm from one end and the shorter length should be bent to form the square.

b. For maximum area, the wire should not be cut, it should be bent to form a square.

24. Let x be the length of the sides of the base, y be the height of the box, and k be the cost per square inch of the material in the sides of the box.

$V = x^2 y$;

The cost is $C = 1.2kx^2 + 1.5kx^2 + 4kxy$

$$= 2.7kx^2 + 4kx\left(\frac{V}{x^2}\right) = 2.7kx^2 + \frac{4kV}{x}$$

$$\frac{dC}{dx} = 5.4kx - \frac{4kV}{x^2}; \frac{dC}{dx} = 0 \text{ when } x \approx 0.905\sqrt[3]{V}$$

$$y \approx \frac{V}{(0.905\sqrt[3]{V})^2} \approx 1.22\sqrt[3]{V}$$

25. Let r be the radius of the cylinder and h the height of the cylinder.

$$V = \pi r^2 h + \frac{2}{3}\pi r^3;\ h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{V}{\pi r^2} - \frac{2}{3}r$$

Let k be the cost per square foot of the cylindrical wall. The cost is

$$C = k(2\pi rh) + 2k(2\pi r^2)$$

$$= k\left(2\pi r\left(\frac{V}{\pi r^2} - \frac{2}{3}r\right) + 4\pi r^2\right) = k\left(\frac{2V}{r} + \frac{8\pi r^2}{3}\right)$$

$$\frac{dC}{dr} = k\left(-\frac{2V}{r^2} + \frac{16\pi r}{3}\right); k\left(-\frac{2V}{r^2} + \frac{16\pi r}{3}\right) = 0$$

when $r^3 = \frac{3V}{8\pi}$, $r = \frac{1}{2}\left(\frac{3V}{\pi}\right)^{1/3}$

$$h = \frac{4V}{\pi\left(\frac{3V}{\pi}\right)^{2/3}} - \frac{1}{3}\left(\frac{3V}{\pi}\right)^{1/3} = \left(\frac{3V}{\pi}\right)^{1/3}$$

For a given volume V, the height of the cylinder is $\left(\frac{3V}{\pi}\right)^{1/3}$ and the radius is $\frac{1}{2}\left(\frac{3V}{\pi}\right)^{1/3}$.

26. $\frac{dx}{dt} = 2\cos 2t - 2\sqrt{3}\sin 2t$;

$\frac{dx}{dt} = 0$ when $\tan 2t = \frac{1}{\sqrt{3}}$;

$2t = \frac{\pi}{6} + \pi n$ for any integer n

$$t = \frac{\pi}{12} + \frac{\pi}{2}n$$

When $t = \frac{\pi}{12} + \frac{\pi}{2}n$,

$$|x| = \left|\sin\left(\frac{\pi}{6} + \pi n\right) + \sqrt{3}\cos\left(\frac{\pi}{6} + \pi n\right)\right|$$

$$= \left|\sin\frac{\pi}{6}\cos \pi n + \cos\frac{\pi}{6}\sin \pi n + \sqrt{3}(\cos\frac{\pi}{6}\cos \pi n - \sin\frac{\pi}{6}\sin \pi n)\right|$$

$$= \left|(-1)^n \frac{1}{2} + (-1)^n \frac{3}{2}\right| = 2.$$

The farthest the weight gets from the origin is 2 units.

27. $A = \frac{r^2\theta}{2};\ \theta = \frac{2A}{r^2}$

The perimeter is

$$Q = 2r + r\theta = 2r + \frac{2Ar}{r^2} = 2r + \frac{2A}{r}$$

$$\frac{dQ}{dr} = 2 - \frac{2A}{r^2}; Q' = 0 \text{ when } r = \sqrt{A}$$

$$\theta = \frac{2A}{(\sqrt{A})^2} = 2$$

$$\frac{d^2Q}{dr^2} = \frac{4A}{r^3} > 0,$$ so this minimizes the perimeter.

28. The distance from the fence to the base of the ladder is $\frac{h}{\tan\theta}$.
The length of the ladder is x.

$$\cos\theta = \frac{\frac{h}{\tan\theta} + w}{x}; x\cos\theta = \frac{h}{\tan\theta} + w;$$

$$x = \frac{h}{\sin\theta} + \frac{w}{\cos\theta}$$

$$\frac{dx}{d\theta} = -\frac{h\cos\theta}{\sin^2\theta} + \frac{w\sin\theta}{\cos^2\theta}; \frac{w\sin^3\theta - h\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0$$

when $\tan^3\theta = \frac{h}{w}$

$$\theta = \tan^{-1}\sqrt[3]{\frac{h}{w}}$$

$$\tan\theta = \frac{\sqrt[3]{h}}{\sqrt[3]{w}}; \sin\theta = \frac{\sqrt[3]{h}}{\sqrt{h^{2/3} + w^{2/3}}},$$

$$\cos\theta = \frac{\sqrt[3]{w}}{\sqrt{h^{2/3} + w^{2/3}}}$$

$$x = h\left(\frac{\sqrt{h^{2/3} + w^{2/3}}}{\sqrt[3]{h}}\right) + w\left(\frac{\sqrt{h^{2/3} + w^{2/3}}}{\sqrt[3]{w}}\right)$$

$$= (h^{2/3} + w^{2/3})^{3/2}$$

29. Let d_1 be the distance that the light travels in medium 1 and let d_2 be the distance the light travels in medium 2.

$$d_1 = \sqrt{a^2 + x^2}, d_2 = \sqrt{b^2 + (d - x)^2}$$

The time that it takes the light to travel from A to B is

$$t = \frac{d_1}{c_1} + \frac{d_2}{c_2} = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d - x)^2}}{c_2}$$

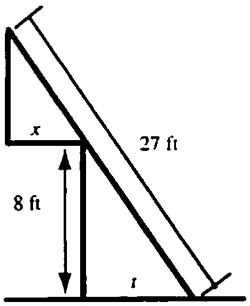
$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

$$= \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}$$

$\frac{dt}{dx} = 0$ implies $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$.

30. From Problem 29, since there is no change in the medium, $\sin\theta_1 = \sin\theta_2$ where θ_1 and θ_2 are measured from the perpendicular to the mirror. θ_1 and θ_2 are both less than 90° so $\sin\theta_1 = \sin\theta_2$ implies that $\theta_1 = \theta_2$. Thus $90° - \theta_1 = 90° - \theta_2$ or the angle of incidence equals the angle of reflection.

31. Consider the following sketch.



By similar triangles. $\frac{x}{27 - \sqrt{t^2 + 64}} = \frac{t}{\sqrt{t^2 + 64}}$.

$$x = \frac{27t}{\sqrt{t^2 + 64}} - t$$

$$\frac{dx}{dt} = \frac{27\sqrt{t^2 + 64} - \frac{27t^2}{\sqrt{t^2+64}}}{t^2 + 64} - 1 = \frac{1728}{(t^2 + 64)^{3/2}} - 1$$

$$\frac{1728}{(t^2 + 64)^{3/2}} - 1 = 0 \text{ when } t = 4\sqrt{5}$$

$$\frac{d^2x}{dt^2} = \frac{-5184t}{(t^2 + 64)^{5/2}}; \left.\frac{d^2x}{dt^2}\right|_{t=4\sqrt{5}} < 0$$

Therefore

$$x = \frac{27\left(4\sqrt{5}\right)}{\sqrt{\left(4\sqrt{5}\right)^2 + 64}} - 4\sqrt{5} = 5\sqrt{5} \approx 11.18$$ ft is the maximum horizontal overhang.

32. Let x be the length of the edges of the cube. The surface area of the cube is $6x^2$ so $0 \le x \le \frac{1}{\sqrt{6}}$.

The surface area of the sphere is $4\pi r^2$, so

$$6x^2 + 4\pi r^2 = 1, r = \sqrt{\frac{1 - 6x^2}{4\pi}}$$

$$V = x^3 + \frac{4}{3}\pi r^3 = x^3 + \frac{1}{6\sqrt{\pi}}(1 - 6x^2)^{3/2}$$

$$\frac{dV}{dx} = 3x^2 - \frac{3}{\sqrt{\pi}}x\sqrt{1 - 6x^2} = 3x\left(x - \sqrt{\frac{1 - 6x^2}{\pi}}\right)$$

$$\frac{dV}{dx} = 0 \text{ when } x = 0, \frac{1}{\sqrt{6+\pi}}$$

$$V(0) = \frac{1}{6\sqrt{\pi}} \approx 0.094 \text{ m}^3.$$

$$V\left(\frac{1}{\sqrt{6+\pi}}\right) = (6+\pi)^{-3/2} + \frac{1}{6\sqrt{\pi}}\left(1 - \frac{6}{6+\pi}\right)^{3/2}$$

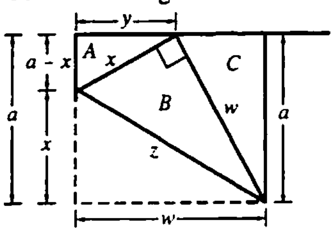
$$= \left(1+\frac{\pi}{6}\right)(6+\pi)^{-3/2} = \frac{1}{6\sqrt{6+\pi}} \approx 0.055 \text{ m}^3$$

For maximum volume: no cube, a sphere of radius $\frac{1}{2\sqrt{\pi}} \approx 0.282$ meters.

For minimum volume: cube with sides of length $\frac{1}{\sqrt{6+\pi}} \approx 0.331$ meters,

sphere of radius $\frac{1}{2\sqrt{6+\pi}} \approx 0.165$ meters

33. Consider the figure below.



a. $y = \sqrt{x^2 - (a-x)^2} = \sqrt{2ax - a^2}$

Area of $A = A = \frac{1}{2}(a-x)y$

$$= \frac{1}{2}(a-x)\sqrt{2ax - a^2}$$

$$\frac{dA}{dx} = -\frac{1}{2}\sqrt{2ax-a^2} + \frac{\frac{1}{2}(a-x)\left(\frac{1}{2}\right)(2a)}{\sqrt{2ax-a^2}}$$

$$= \frac{a^2 - \frac{3}{2}ax}{\sqrt{2ax-a^2}}$$

$$\frac{a^2 - \frac{3}{2}ax}{\sqrt{2ax-a^2}} = 0 \text{ when } x = \frac{2a}{3}.$$

$\frac{dA}{dx} > 0$ on $\left(\frac{a}{2}, \frac{2a}{3}\right)$ and $\frac{dA}{dx} < 0$ on $\left(\frac{2a}{3}, a\right)$,

so $x = \frac{2a}{3}$ maximizes the area of triangle A.

b. Triangle A is similar to triangle C, so

$$w = \frac{ax}{y} = \frac{ax}{\sqrt{2ax-a^2}}$$

Area of $B = B = \frac{1}{2}xw = \frac{ax^2}{2\sqrt{2ax-a^2}}$

$$\frac{dB}{dx} = \frac{a}{2}\left(\frac{2x\sqrt{2ax-a^2} - x^2\frac{a}{\sqrt{2ax-a^2}}}{2ax-a^2}\right)$$

$$= \frac{a}{2}\left(\frac{2x(2ax-a^2) - ax^2}{(2ax-a^2)^{3/2}}\right) = \frac{a}{2}\left(\frac{3ax^2 - 2xa^2}{(2ax-a^2)^{3/2}}\right)$$

$$\frac{a^2}{2}\left(\frac{3x^2 - 2xa}{(2ax-a^2)^{3/2}}\right) = 0 \text{ when } x = 0, \frac{2a}{3}$$

Since $x = 0$ is not possible, $x = \frac{2a}{3}$.

$\frac{dB}{dx} < 0$ on $\left(\frac{a}{2}, \frac{2a}{3}\right)$ and $\frac{dB}{dx} > 0$ on $\left(\frac{2a}{3}, a\right)$,

so $x = \frac{2a}{3}$ minimizes the area of triangle B.

c. $z = \sqrt{x^2 + w^2} = \sqrt{x^2 + \frac{a^2x^2}{2ax-a^2}}$

$$= \sqrt{\frac{2ax^3}{2ax-a^2}}$$

$$\frac{dz}{dx} = \frac{1}{2}\sqrt{\frac{2ax-a^2}{2ax^3}}\left(\frac{6ax^2(2ax-a^2) - 2ax^3(2a)}{(2ax-a^2)^2}\right)$$

$$= \frac{4a^2x^3 - 3a^3x^2}{\sqrt{2ax^3(2ax-a^2)^3}}$$

$$\frac{dz}{dx} = 0 \text{ when } x = 0, \frac{3a}{4}$$

$$x = \frac{3a}{4}$$

$\frac{dz}{dx} < 0$ on $\left(\frac{a}{2}, \frac{3a}{4}\right)$ and $\frac{dz}{dx} > 0$ on $\left(\frac{3a}{4}, a\right)$,

so $x = \frac{3a}{4}$ minimizes length z.

34. Let $2x$ be the length of a bar and $2y$ be the width of a bar.

$$x = a\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = a\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}+\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right) = \frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$$

$$y = a\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = a\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}-\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right) = \frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)$$

Compute the area A of the cross and maximize.

$$A = 2(2x)(2y)-(2y)^2$$

$$= 8\left[\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\right]\left[\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\right]-4\left[\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\right]^2$$

$$= 4a^2\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)-2a^2\left(1-2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right) = 4a^2\cos\theta-2a^2(1-\sin\theta)$$

$$\frac{dA}{d\theta} = -4a^2\sin\theta+2a^2\cos\theta$$

$-4a^2\sin\theta+2a^2\cos\theta = 0$ when $\tan\theta = \frac{1}{2}$

$$\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}$$

$\frac{d^2A}{d\theta^2} < 0$ when $\tan\theta = \frac{1}{2}$, so this maximizes the area.

$$A = 4a^2\left(\frac{2}{\sqrt{5}}\right)-2a^2\left(1-\frac{1}{\sqrt{5}}\right) = \frac{10a^2}{\sqrt{5}}-2a^2 = 2a^2(\sqrt{5}-1)$$

35. a. $L'(\theta) = 15(9+25-30\cos\theta)^{-1/2}\sin\theta = 15(34-30\cos\theta)^{-1/2}\sin\theta$

$$L''(\theta) = -\frac{15}{2}(34-30\cos\theta)^{-3/2}(30\sin\theta)\sin\theta+15(34-30\cos\theta)^{-1/2}\cos\theta$$

$$= -225(34-30\cos\theta)^{-3/2}\sin^2\theta+15(34-30\cos\theta)^{-1/2}\cos\theta$$

$$= 15(34-30\cos\theta)^{-3/2}[-15\sin^2\theta+(34-30\cos\theta)\cos\theta]$$

$$= 15(34-30\cos\theta)^{-3/2}[-15\sin^2\theta+34\cos\theta-30\cos^2\theta]$$

$$= 15(34-30\cos\theta)^{-3/2}[-15+34\cos\theta-15\cos^2\theta]$$

$$= -15(34-30\cos\theta)^{-3/2}[15\cos^2\theta-34\cos\theta+15]$$

$L'' = 0$ when $\cos\theta = \dfrac{34\pm\sqrt{(34)^2-4(15)(15)}}{2(15)} = \dfrac{5}{3}, \dfrac{3}{5}$

$$\theta = \cos^{-1}\left(\frac{3}{5}\right)$$

$$L'\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = 15\left(9+25-30\left(\frac{3}{5}\right)\right)^{-1/2}\left(\frac{4}{5}\right) = 3$$

$$L\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \left(9+25-30\left(\frac{3}{5}\right)\right)^{1/2} = 4$$

$\phi = 90°$ since the resulting triangle is a 3-4-5 right triangle.

b. $L'(\theta) = 65(25+169-130\cos\theta)^{-1/2}\sin\theta = 65(194-130\cos\theta)^{-1/2}\sin\theta$

$$L''(\theta) = -\frac{65}{2}(194-130\cos\theta)^{-3/2}(130\sin\theta)\sin\theta+65(194-130\cos\theta)^{-1/2}\cos\theta$$

$$= -4225(194-130\cos\theta)^{-3/2}\sin^2\theta+65(194-130\cos\theta)^{-1/2}\cos\theta$$

$= 65(194-130\cos\theta)^{-3/2}[-65\sin^2\theta+(194-130\cos\theta)\cos\theta]$

$= 65(194-130\cos\theta)^{-3/2}[-65\sin^2\theta+194\cos\theta-130\cos^2\theta]$

$= 65(194-130\cos\theta)^{-3/2}[-65\cos^2\theta+194\cos\theta-65]$

$= -65(194-130\cos\theta)^{-3/2}[65\cos^2\theta-194\cos\theta+65]$

$L'' = 0$ when $\cos\theta = \dfrac{194 \pm \sqrt{(194)^2-4(65)(65)}}{2(65)} = \dfrac{13}{5}, \dfrac{5}{13}$

$\theta = \cos^{-1}\left(\dfrac{5}{13}\right)$

$L'\left(\cos^{-1}\left(\dfrac{5}{13}\right)\right) = 65\left(25+169-130\left(\dfrac{5}{13}\right)\right)^{1/2}\left(\dfrac{12}{13}\right) = 5$

$L\left(\cos^{-1}\left(\dfrac{5}{13}\right)\right) = \left(25+169-130\left(\dfrac{5}{13}\right)\right)^{1/2} = 12$

$\phi = 90°$ since the resulting triangle is a 5-12-13 right triangle.

c. When the tips are separating most rapidly, $\phi = 90°$, $L = \sqrt{m^2-h^2}$, $L' = h$

d. $L'(\theta) = hm(h^2+m^2-2hm\cos\theta)^{-1/2}\sin\theta$

$L''(\theta) = -h^2m^2(h^2+m^2-2hm\cos\theta)^{-3/2}\sin^2\theta + hm(h^2+m^2-2hm\cos\theta)^{-1/2}\cos\theta$

$= hm(h^2+m^2-2hm\cos\theta)^{-3/2}[-hm\sin^2\theta+(h^2+m^2)\cos\theta-2hm\cos^2\theta]$

$= hm(h^2+m^2-2hm\cos\theta)^{-3/2}[-hm\cos^2\theta+(h^2+m^2)\cos\theta-hm]$

$= -hm(h^2+m^2-2hm\cos\theta)^{-3/2}[hm\cos^2\theta-(h^2+m^2)\cos\theta+hm]$

$L'' = 0$ when $hm\cos^2\theta-(h^2+m^2)\cos\theta+hm = 0$

$(h\cos\theta - m)(m\cos\theta - h) = 0$

$\cos\theta = \dfrac{m}{h}, \dfrac{h}{m}$

Since $h < m$, $\cos\theta = \dfrac{h}{m}$ so $\theta = \cos^{-1}\left(\dfrac{h}{m}\right)$.

$L'\left(\cos^{-1}\left(\dfrac{h}{m}\right)\right) = hm\left(h^2+m^2-2hm\left(\dfrac{h}{m}\right)\right)^{-1/2}\dfrac{\sqrt{m^2-h^2}}{m} = hm(m^2-h^2)^{-1/2}\dfrac{\sqrt{m^2-h^2}}{m} = h$

$L\left(\cos^{-1}\left(\dfrac{h}{m}\right)\right) = \left(h^2+m^2-2hm\left(\dfrac{h}{m}\right)\right)^{1/2} = \sqrt{m^2-h^2}$

Since $h^2 + L^2 = m^2$, $\phi = 90°$.

36. Following the same reasoning as in Problem 11, we are interested in finding the global extrema for the distance of the object from the observer. We will obtain the same results by considering the squared distance instead. The squared distance can be expressed as

$$D(x) = (x-2)^2 + \left(100 + x - \frac{1}{10}x^2\right)^2$$

The first and second derivatives are given by

$$D'(x) = \frac{1}{25}x^3 - \frac{3}{5}x^2 - 36x + 196 \text{ and}$$

$$D''(x) = \frac{3}{25}\left(x^2 - 10x - 300\right)$$

Using a computer package, we can solve the equation $D'(x) = 0$ to find the critical points. The critical points are $x \approx 5.1538, 36.148$. Using the second derivative we see that $D''(5.1538) \approx -38.9972$ (max) and

$D''(36.148) \approx 77.4237$ (min)

Therefore, the position of the object closest to the observer is $\approx (36.148, 5.48)$ while the position of the object farthest from the person is $\approx (5.1538, 102.5)$.

(Remember to go back to the original equation for the path of the object once you find the critical points.)

37. Here we are interested in minimizing the distance between the earth and the asteroid. Using the coordinates P and Q for the two bodies, we can use the distance formula to obtain a suitable equation. However, for simplicity, we will minimize the squared distance to find the critical points. The squared distance between the objects is given by

$$D(t) = (93\cos(2\pi t) - 60\cos[2\pi(1.51t-1)])^2 + (93\sin(2\pi t) - 120\sin[2\pi(1.51t-1)])^2$$

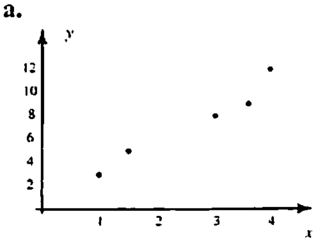
The first derivative is

$$D'(t) \approx -34359[\cos(2\pi t)][\sin(9.48761t)] + [\cos(9.48761t)][(204932\sin(9.48761t) - 141643\sin(2\pi t))]$$

Plotting the function and its derivative reveal a periodic relationship due to the orbiting of the objects. Careful examination of the graphs reveals that there is indeed a minimum squared distance (and hence a minimum distance) that occurs only once. The critical value for this occurrence is $t \approx 13.8279$. This value gives a distance between the objects of

≈ 0.047851 million miles $= 47{,}851$ miles

38. **a.**



b. There are only a few data points, but they do seem fairly linear.

c. The data values can be entered into most scientific calculators to utilize the Least Squares Regression feature. Alternately one could use the formulas for the slope and intercept provided in the text. The resulting line should be $y = 0.56852 + 2.6074x$

d. Using the result from c., the predicted number of surface imperfections on a sheet with area 2.0 square feet is

$y = 0.56852 + 2.6074(2.0) = 5.7833 \approx 6$

since we can't have partial imperfections

39. **a.**

$$\frac{dS}{db} = \frac{d}{db}\sum_{i=1}^{n}[y_i - (5 + bx_i)]^2$$
$$= \sum_{i=1}^{n}\frac{d}{db}[y_i - (5 + bx_i)]^2$$
$$= \sum_{i=1}^{n}2(y_i - 5 - bx_i)(-x_i)$$
$$= 2\left[\sum_{i=1}^{n}\left(-x_i y_i + 5x_i + bx_i^2\right)\right]$$
$$= -2\sum_{i=1}^{n}x_i y_i + 10\sum_{i=1}^{n}x_i + 2b\sum_{i=1}^{n}x_i^2$$

Setting $\frac{dS}{db} = 0$ gives

$$0 = -2\sum_{i=1}^{n}x_i y_i + 10\sum_{i=1}^{n}x_i + 2b\sum_{i=1}^{n}x_i^2$$
$$0 = -\sum_{i=1}^{n}x_i y_i + 5\sum_{i=1}^{n}x_i + b\sum_{i=1}^{n}x_i^2$$
$$b\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}x_i y_i - 5\sum_{i=1}^{n}x_i$$
$$b = \frac{\sum_{i=1}^{n}x_i y_i - 5\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}x_i^2}$$

You should check that this is indeed the value of b that minimizes the sum. Taking the second derivative yields

$$\frac{d^2S}{db^2} = 2\sum_{i=1}^{n}x_i^2$$

which is always positive (unless all the x values are zero). Therefore, the value for b above does minimize the sum as required.

b. Using the formula from a., we get that

$$b = \frac{(2037) - 5(52)}{590} \approx 3.0119$$

c. The Least Squares Regression line is

$y = 5 + 3.0119x$

Using this line, the predicted total number of labor hours to produce a lot of 15 brass bookcases is

$y = 5 + 3.0119(15)$

≈ 50.179 hours

4.5 Concepts Review

1. continuous

2. number of units sold; price per unit

3. fixed; variable

4. marginal revenue; marginal cost

Problem Set 4.5

1. Profit is $P(y) = ny - 10n$

$$= \frac{100y}{y-10} + 20y(100-y) - \frac{1000}{y-10} - 200(100-y)$$

$$= 20(-y^2 + 110y - 995)$$

$$\frac{dP}{dy} = 20(-2y + 110) = 40(55 - y)$$

$P'(y) = 0$ when $y = 55$

$P''(y) = -40$ so the maximum profit is when the items are sold at \$55 each.

2. $C(x) = 7000 + 100x$

3. $n = 100 + 10\frac{250 - p(n)}{5}$ so $p(n) = 300 - \frac{n}{2}$

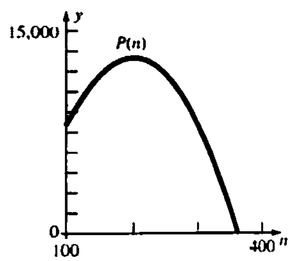
$$R(n) = np(n) = 300n - \frac{n^2}{2}$$

4. $P(n) = R(n) - C(n)$

$$= 300n - \frac{n^2}{2} - (7000 + 100n)$$

$$= -7000 + 200n - \frac{n^2}{2}$$

5.



Estimate $n \approx 200$

$P'(n) = 200 - n$; $200 - n = 0$ when $n = 200$.

$P''(n) = -1$, so profit is maximum at $n = 200$.

6. $\frac{C(x)}{x} = \frac{100}{x} + 3.002 - 0.0001x$

When $x = 1600$, $\frac{C(x)}{x} = 2.9045$ or \$2.90 per unit.

$$\frac{dC}{dx} = 3.002 - 0.0002x$$

$C'(1600) = 2.682$ or \$2.68

7. $\frac{C(n)}{n} = \frac{1000}{n} + \frac{n}{1200}$

When $n = 800$, $\frac{C(n)}{n} \approx 1.9167$ or \$1.92 per unit.

$$\frac{dC}{dn} = \frac{n}{600}$$

$C'(800) \approx 1.333$ or \$1.33

8. a. $\frac{dC}{dx} = 33 - 18x + 3x^2$

$\frac{d^2C}{dx^2} = -18 + 6x$; $\frac{d^2C}{dx^2} = 0$ when $x = 3$

$\frac{d^2C}{dx^2} < 0$ on $(0, 3)$, $\frac{d^2C}{dx^2} > 0$ on $(3, \infty)$

Thus, the marginal cost is a minimum when $x = 3$ or 300 units.

b. $33 - 18(3) + 3(3)^2 = 6$

9. a. $R(x) = xp(x) = 20x + 4x^2 - \frac{x^3}{3}$

$$\frac{dR}{dx} = 20 + 8x - x^2$$

b. Increasing when $\frac{dR}{dx} > 0$

$20 + 8x - x^2 > 0$ on $[0, 10)$

Total revenue is increasing if $0 \le x \le 10$.

c. $\frac{d^2R}{dx^2} = 8 - 2x$; $\frac{d^2R}{dx^2} = 0$ when $x = 4$

$\frac{d^3R}{dx^3} = -2$; $\frac{dR}{dx}$ is maximum at $x = 4$.

10. $R(x) = x\left(182 - \frac{x}{36}\right)^{1/2}$

$$\frac{dR}{dx} = x\frac{1}{2}\left(182 - \frac{x}{36}\right)^{-1/2}\left(-\frac{1}{36}\right) + \left(182 - \frac{x}{36}\right)^{1/2}$$

$$= \left(182 - \frac{x}{36}\right)^{-1/2}\left(182 - \frac{x}{24}\right)$$

$\frac{dR}{dx} = 0$ when $x = 4368$

$x_1 = 4368$; $R(4368) \approx 34{,}021.83$

At $x_1, \frac{dR}{dx} = 0$.

11. $R(x) = \frac{800x}{x+3} - 3x$

$\frac{dR}{dx} = \frac{(x+3)(800) - 800x}{(x+3)^2} - 3 = \frac{2400}{(x+3)^2} - 3;$

$\frac{dR}{dx} = 0$ when $x = 20\sqrt{2} - 3 \approx 25$

$x_1 = 25$; $R(25) \approx 639.29$

At $x_1, \frac{dR}{dx} = 0$.

12. $p(x) = 12 - (0.20)\frac{(x-400)}{10} = 20 - 0.02x$

$R(x) = 20x - 0.02x^2$

$\frac{dR}{dx} = 20 - 0.04x; \frac{dR}{dx} = 0$ when $x = 500$

Total revenue is maximized at $x_1 = 500$.

13. $x = 4000 + \frac{6 - p(x)}{0.15}(250);$

$p(x) = 6 - (0.15)\frac{(x-4000)}{250} = 8.4 - 0.0006x$

$R(x) = 8.4x - 0.0006x^2$

$\frac{dR}{dx} = 8.4 - 0.0012x; \frac{dR}{dx} = 0$ when $x = 7000$

Revenue is maximum when

$p(x) = 8.4 - 0.0006(7000) = \$4.20$ per yard.

14. a. $x = 500 + \frac{20 - p(x)}{0.50}(50);$

$p(x) = 20 - (0.50)\frac{(x-500)}{50} = 25 - 0.01x$

b. $R(x) = 25x - 0.01x^2;$

$P(x) = R(x) - C(x)$

$= (25x - 0.01x^2) - (4200 + 5.10x + 0.0001x^2)$

$= -4200 + 19.9x - 0.0101x^2$

$\frac{dP}{dx} = 19.9 - 0.0202x; \frac{dP}{dx} = 0$ when $x \approx 985$

$\frac{d^2P}{dx^2} = -0.0202$; profit is maximized at $x = 985$.

c. $p(985) = 25 - 0.01(985) = 15.15$

d. $\frac{dp}{dx} = -0.01$

15. a. $C(x) = \begin{cases} 6000 + 1.40x & \text{if } 0 \le x \le 4500 \\ 6000 + 1.60x & \text{if } 4500 < x \end{cases}$

b. $x = 4000 + \frac{7 - p(x)}{0.10}(100);$

$p(x) = 7 - (0.10)\frac{x - 4000}{100}$

$p(x) = 11 - 0.001x$

c. $R(x) = 11x - 0.001x^2$

For $0 \le x \le 4500$,

$P(x) = (11x - 0.001x^2) - (6000 + 1.40x)$

$= -6000 + 9.6x - 0.001x^2$

$\frac{dP}{dx} = 9.6 - 0.002x; \frac{dP}{dx} = 0$ when $x = 4800$;

this is not in the interval [0, 4500].

The critical numbers are 0 and 4500.

$P(0) = -6000$ and $P(4500) = 16{,}950$

For $4500 < x$,

$P(x) = (11x - 0.001x^2) - (6000 + 1.60x)$

$= -6000 + 9.4x - 0.001x^2$

$\frac{dP}{dx} = 9.4 - 0.002x; \frac{dP}{dx} = 0$ when $x = 4700$

$P(4700) = 16{,}090$

Therefore, the number of units for maximum profit is $x = 4500$.

16. $R(x) = 10x - 0.001x^2; 0 \le x \le 300$

$P(x) = (10x - 0.001x^2) - (200 + 4x - 0.01x^2)$

$= -200 + 6x + 0.009x^2$

$\frac{dP}{dx} = 6 + 0.018x; \frac{dP}{dx} = 0$ when $x \approx -333$

Critical numbers: $x = 0, 300$; $P(0) = -200$; $P(300) = 2410$; Maximum profit is \$2410 at $x = 300$.

17. $C(x) = \begin{cases} 200 + 4x - 0.01x^2 & \text{if } 0 \le x \le 300 \\ 800 + 3x - 0.01x^2 & \text{if } 300 < x \le 450 \end{cases}$

$P(x) = \begin{cases} -200 + 6x + 0.009x^2 & \text{if } 0 \le x \le 300 \\ -800 + 7x + 0.009x^2 & \text{if } 300 < x \le 450 \end{cases}$

There are no stationary points on the interval [0, 300]. On [300, 450]:

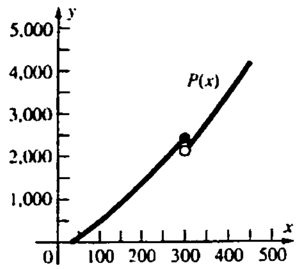
$\frac{dP}{dx} = 7 + 0.018x; \frac{dP}{dx} = 0$ when $x \approx -389$

The critical numbers are 0, 300, 450.

$P(0) = -200$, $P(300) = 2410$, $P(450) = 4172.5$

Monthly profit is maximized at $x = 450$.

$P(450) = 4172.50$



18. The number of lots ordered per year is $\frac{1000}{x}$. Let C represent the inventory cost.

$C = \frac{1000}{x}(200+3x) + 20\left(\frac{x}{2}\right)$

$= \frac{200,000}{x} + 3000 + 10x$

$\frac{dC}{dx} = -\frac{200,000}{x^2} + 10; -\frac{200,000}{x^2} + 10 = 0$ when

$x = \sqrt{20,000} \approx 141.42$

$\frac{d^2C}{dx^2} = \frac{400,000}{x^3}; \frac{d^2C}{dx^2} > 0$ at $x = \sqrt{20,000}$

Since the lot size must be a whole number

$x = 141$ or $x = 142$.

$C(141) \approx 5828.44$, $C(142) \approx 5828.45$

$x = 141$

19. The number of lots ordered per year is $\frac{N}{x}$. Let C represent the inventory cost.

$C = \frac{N}{x}(F + Bx) + \frac{x}{2}A = \frac{FN}{x} + BN + \frac{Ax}{2}$

$\frac{dC}{dx} = -\frac{FN}{x^2} + \frac{A}{2}; -\frac{FN}{x^2} + \frac{A}{2} = 0$ when

$x = \sqrt{\frac{2FN}{A}}$

20. If N is increased to $4N$,

$x = \sqrt{\frac{2F(4N)}{A}} = 2\sqrt{\frac{2FN}{A}}$.

Therefore, the ideal lot size will double.

4.6 Concepts Review

1. $f(x)$; $-f(x)$

2. decreasing; concave up

3. $x = -1, x = 2, x = 3; y = 1$

4. polynomial; rational.

Problem Set 4.6

1. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

Neither an even nor an odd function.

y-intercept: 5; x-intercept: ≈ -2.3

$f'(x) = 3x^2 - 3; 3x^2 - 3 = 0$ when $x = -1, 1$

Critical points: $-1, 1$

$f'(x) > 0$ when $x < -1$ or $x > 1$

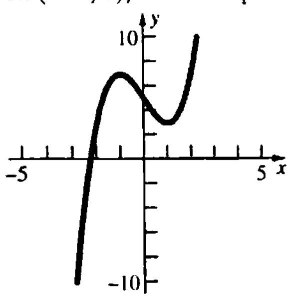
$f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $[-1, 1]$.

Local minimum $f(1) = 3$;

local maximum $f(-1) = 7$

$f''(x) = 6x; f''(x) > 0$ when $x > 0$.

$f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, 5)$.



2. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

Neither an even nor an odd function.

y-intercept: -10; x-intercept: 2

$f'(x) = 6x^2 - 3 = 3(2x^2 - 1); 2x^2 - 1 = 0$ when

$x = -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Critical points: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$f'(x) > 0$ when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$

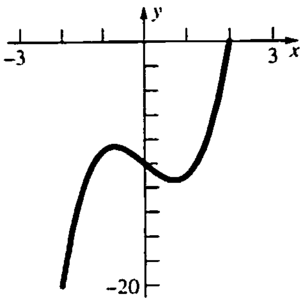
$f(x)$ is increasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and

decreasing on $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

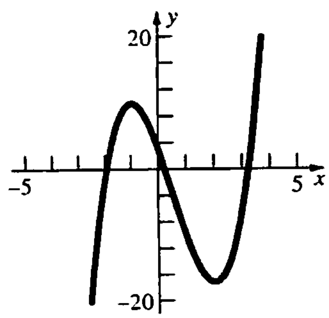
Local minimum $f\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2} - 10 \approx -11.4$

Local maximum $f\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2} - 10 \approx -8.6$

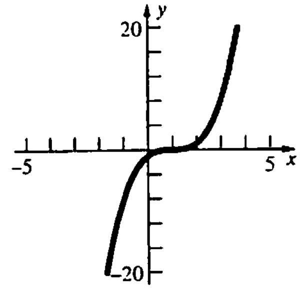
$f''(x) = 12x$; $f''(x) > 0$ when $x > 0$. $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, -10)$.



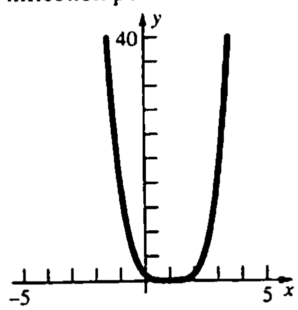
3. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
y-intercept: 3; x-intercepts: $\approx -2.0, 0.2, 3.2$
$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$;
$f'(x) = 0$ when $x = -1, 2$
Critical points: $-1, 2$
$f'(x) > 0$ when $x < -1$ or $x > 2$
$f(x)$ is increasing on $(-\infty, -1] \cup [2, \infty)$ and decreasing on $[-1, 2]$.
Local minimum $f(2) = -17$;
local maximum $f(-1) = 10$
$f''(x) = 12x - 6 = 6(2x-1)$;
$f''(x) > 0$ when $x > \frac{1}{2}$.
$f(x)$ is concave up on $\left(\frac{1}{2}, \infty\right)$ and concave down on $\left(-\infty, \frac{1}{2}\right)$; inflection point: $\left(\frac{1}{2}, -\frac{7}{2}\right)$



4. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function
y-intercept: -1; x-intercept: 1
$f'(x) = 3(x-1)^2$; $f'(x) = 0$ when $x = 1$
Critical point: 1
$f'(x) > 0$ for all $x \neq 0$
$f(x)$ is increasing on $(-\infty, \infty)$
No local minima or maxima
$f''(x) = 6(x-1)$; $f''(x) > 0$ when $x > 1$.
$f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; inflection point $(1, 0)$



5. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
Neither an even nor an odd function.
y-intercept: 1; x-intercept: 1
$G'(x) = 4(x-1)^3$; $G'(x) = 0$ when $x = 1$
Critical point: 1
$G'(x) > 0$ for $x > 1$
$G(x)$ is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$.
Global minimum $f(1) = 0$; no local maxima
$G''(x) = 12(x-1)^2$; $G''(x) > 0$ for all $x \neq 1$
$G(x)$ is concave up on $(-\infty, 1) \cup (1, \infty)$; no inflection points



6. Domain: $(-\infty, \infty)$; range: $\left[-\frac{1}{4}, \infty\right)$
$H(-t) = (-t)^2[(-t)^2 - 1] = t^2(t^2 - 1) = H(t)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; t-intercepts: $-1, 0, 1$
$H'(t) = 4t^3 - 2t = 2t(2t^2 - 1)$; $H'(t) = 0$ when $t = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

Critical points: $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

$H'(t) > 0$ for $-\frac{1}{\sqrt{2}} < t < 0$ or $\frac{1}{\sqrt{2}} < t$.

$H(t)$ is increasing on $\left[-\frac{1}{\sqrt{2}}, 0\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[0, \frac{1}{\sqrt{2}}\right]$

Global minima $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}, f\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$;

Local maximum $f(0) = 0$

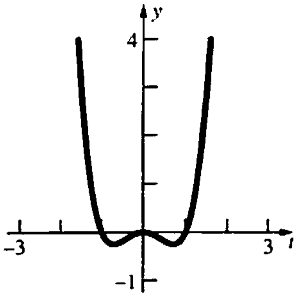
$H''(t) = 12t^2 - 2 = 2(6t^2 - 1)$; $H'' > 0$ when

$t < -\frac{1}{\sqrt{6}}$ or $t > \frac{1}{\sqrt{6}}$

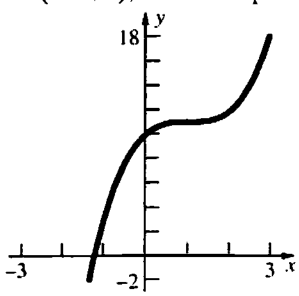
$H(t)$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{6}}\right) \cup \left(\frac{1}{\sqrt{6}}, \infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$; inflection

points $H\left(-\frac{1}{\sqrt{6}}, -\frac{5}{36}\right)$ and $H\left(\frac{1}{\sqrt{6}}, \frac{5}{36}\right)$



7. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
y-intercept: 10; x-intercept: $1 - 11^{1/3} \approx -1.2$
$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$; $f'(x) = 0$ when $x = 1$.
Critical point: 1
$f'(x) > 0$ for all $x \neq 1$.
$f(x)$ is increasing on $(-\infty, \infty)$ and decreasing nowhere.
No local maxima or minima
$f''(x) = 6x - 6 = 6(x-1)$; $f''(x) > 0$ when $x > 1$.
$f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; inflection point $(1, 11)$



8. Domain: $(-\infty, \infty)$; range: $\left[-\frac{16}{3}, \infty\right)$

$$F(-s) = \frac{4(-s)^4 - 8(-s)^2 - 12}{3} = \frac{4s^4 - 8s^2 - 12}{3}$$

$= F(s)$; even function; symmetric with respect to the y-axis

y-intercept: -4; s-intercepts: $-\sqrt{3}, \sqrt{3}$

$F'(s) = \frac{16}{3}s^3 - \frac{16}{3}s = \frac{16}{3}s(s^2 - 1)$; $F'(s) = 0$ when $s = -1, 0, 1$.

Critical points: $-1, 0, 1$

$F'(s) > 0$ when $-1 < x < 0$ or $x > 1$.

$F(s)$ is increasing on $[-1, 0] \cup [1, \infty)$ and decreasing on $(-\infty, -1] \cup [0, 1]$

Global minima $F(-1) = -\frac{16}{3}, F(1) = -\frac{16}{3}$; local maximum $F(0) = -4$

$F''(s) = 16s^2 - \frac{16}{3} = 16\left(s^2 - \frac{1}{3}\right)$; $F''(s) > 0$

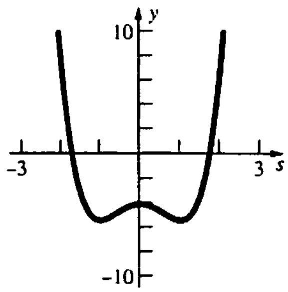
when $s < -\frac{1}{\sqrt{3}}$ or $s > \frac{1}{\sqrt{3}}$

$F(s)$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$;

inflection points

$F\left(-\frac{1}{\sqrt{3}}, -\frac{128}{27}\right), F\left(\frac{1}{\sqrt{3}}, -\frac{128}{27}\right)$



9. Domain: $(-\infty, -1) \cup (-1, \infty)$;
range: $(-\infty, 1) \cup (1, \infty)$
Neither an even nor an odd function
y-intercept: 0; x-intercept: 0

$g'(x) = \frac{1}{(x+1)^2}$; $g'(x)$ is never 0.

No critical points

$g'(x) > 0$ for all $x \neq -1$.

$g(x)$ is increasing on $(-\infty, -1) \cup (-1, \infty)$.

No local minima or maxima

$g''(x) = -\frac{2}{(x+1)^3}$; $g''(x) > 0$ when $x < -1$.

$g(x)$ is concave up on $(-\infty, -1)$ and concave down on $(-1, \infty)$; no inflection points (-1 is not in the domain of g).

$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$;

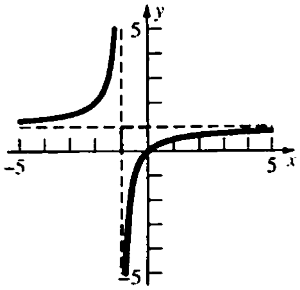
$$\lim_{x\to-\infty}\frac{x}{x+1} = \lim_{x\to-\infty}\frac{1}{1+\frac{1}{x}} = 1;$$

horizontal asymptote: $y = 1$

As $x \to -1^-, x+1 \to 0^-$ so $\lim_{x\to-1^-}\frac{x}{x+1} = \infty;$

as $x \to -1^+, x+1 \to 0^+$ so $\lim_{x\to-1^+}\frac{x}{x+1} = -\infty;$

vertical asymptote: $x = -1$



10. Domain: $(-\infty, 0) \cup (0, \infty)$;

range: $(-\infty, -4\pi] \cup [0, \infty)$

Neither an even nor an odd function

No y-intercept; s-intercept: π

$g'(s) = \frac{s^2 - \pi^2}{s^2}$; $g'(s) = 0$ when $s = -\pi, \pi$

Critical points: $-\pi, \pi$

$g'(s) > 0$ when $s < -\pi$ or $s > \pi$

$g(s)$ is increasing on $(-\infty, -\pi] \cup [\pi, \infty)$ and

decreasing on $[-\pi, 0) \cup (0, \pi]$.

Local minimum $g(\pi) = 0$;

local maximum $g(-\pi) = -4\pi$

$g''(s) = \frac{2\pi^2}{s^3}$; $g''(s) > 0$ when $s > 0$

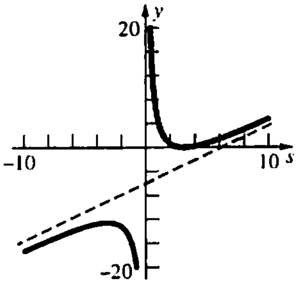
$g(s)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; no inflection points (0 is not in the domain of $g(s)$).

$g(s) = s - 2\pi + \frac{\pi^2}{s}$; $y = s - 2\pi$ is an oblique asymptote.

As $s \to 0^-, (s-\pi)^2 \to \pi^2$, so $\lim_{s\to 0^-} g(s) = -\infty;$

as $s \to 0^+, (s-\pi)^2 \to \pi^2$, so $\lim_{s\to 0^+} g(s) = \infty;$

$s = 0$ is a vertical asymptote.



11. Domain: $(-\infty, \infty)$; range: $\left[-\frac{1}{4}, \frac{1}{4}\right]$

$f(-x) = \frac{-x}{(-x)^2 + 4} = -\frac{x}{x^2 + 4} = -f(x)$; odd function; symmetric with respect to the origin.

y-intercept: 0; x-intercept: 0

$f'(x) = \frac{4 - x^2}{(x^2 + 4)^2}$; $f'(x) = 0$ when $x = -2, 2$

Critical points: $-2, 2$

$f'(x) > 0$ for $-2 < x < 2$

$f(x)$ is increasing on $[-2, 2]$ and decreasing on $(-\infty, -2] \cup [2, \infty)$.

Global minimum $f(-2) = -\frac{1}{4}$; global maximum

$f(2) = \frac{1}{4}$

$f''(x) = \frac{2x(x^2 - 12)}{(x^2 + 4)^3}$; $f''(x) > 0$ when

$-2\sqrt{3} < x < 0$ or $x > 2\sqrt{3}$

$f(x)$ is concave up on $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$ and

concave down on $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$;

inflection points $\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right)$, $(0, 0)$,

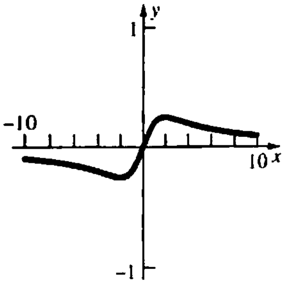
$\left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right)$

$$\lim_{x\to\infty}\frac{x}{x^2 + 4} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1 + \frac{4}{x^2}} = 0;$$

$$\lim_{x\to-\infty}\frac{x}{x^2 + 4} = \lim_{x\to-\infty}\frac{\frac{1}{x}}{1 + \frac{4}{x^2}} = 0;$$

$y = 0$ is a horizontal asymptote.

No vertical asymptotes



12. Domain: $(-\infty, \infty)$; range: $[0, 1)$

$\Lambda(-\theta) = \frac{(-\theta)^2}{(-\theta)^2+1} = \frac{\theta^2}{\theta^2+1} = \Lambda(\theta)$; even function; symmetric with respect to the y-axis.

y-intercept: 0; θ-intercept: 0

$\Lambda'(\theta) = \frac{2\theta}{(\theta^2+1)^2}$; $\Lambda'(\theta) = 0$ when $\theta = 0$

Critical point: 0

$\Lambda'(\theta) > 0$ when $\theta > 0$

$\Lambda(\theta)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

Global minimum $\Lambda(0) = 0$; no local maxima

$\Lambda''(\theta) = \frac{2(1-3\theta^2)}{(\theta^2+1)^3}$; $\Lambda''(\theta) > 0$ when

$-\frac{1}{\sqrt{3}} < \theta < \frac{1}{\sqrt{3}}$

$\Lambda(\theta)$ is concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and

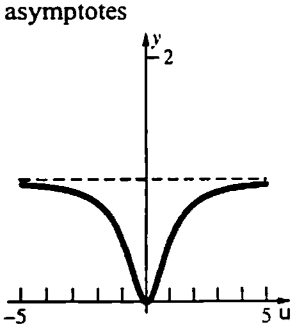
concave down on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$;

inflection points $\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$

$\lim\limits_{\theta\to\infty} \frac{\theta^2}{\theta^2+1} = \lim\limits_{\theta\to\infty} \frac{1}{1+\frac{1}{\theta^2}} = 1$;

$\lim\limits_{\theta\to-\infty} \frac{\theta^2}{\theta^2+1} = \lim\limits_{\theta\to-\infty} \frac{1}{1+\frac{1}{\theta^2}} = 1$;

$y = 1$ is a horizontal asymptote. No vertical asymptotes



13. Domain: $(-\infty, 1) \cup (1, \infty)$;
range $(-\infty, 1) \cup (1, \infty)$

Neither an even nor an odd function

y-intercept: 0; x-intercept: 0

$h(x) = -\frac{1}{(x-1)^2}$; $h'(x)$ is never 0.

No critical points

$h'(x) < 0$ for all $x \neq 1$.

$h(x)$ is increasing nowhere and decreasing on $(-\infty, 1) \cup (1, \infty)$.

No local maxima or minima

$h''(x) = \frac{2}{(x-1)^3}$; $h''(x) > 0$ when $x > 1$

$h'(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; no inflection points (1 is not in the domain of $h(x)$)

$\lim\limits_{x\to\infty} \frac{x}{x-1} = \lim\limits_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1$;

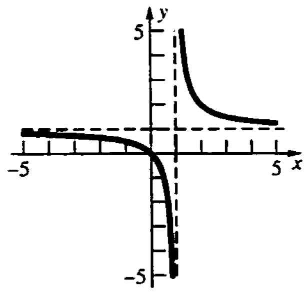
$\lim\limits_{x\to-\infty} \frac{x}{x-1} = \lim\limits_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = 1$;

$y = 1$ is a horizontal asymptote.

As $x \to 1^-, x-1 \to 0^-$ so $\lim\limits_{x\to1^-} \frac{x}{x-1} = -\infty$;

as $x \to 1^+, x-1 \to 0^+$ so $\lim\limits_{x\to1^+} \frac{x}{x-1} = \infty$;

$x = 1$ is a vertical asymptote.



14. Domain: $(-\infty, \infty)$

Range: $0 < P \le 1$

Even function since

$P(-x) = \frac{1}{(-x)^2+1} = \frac{1}{x^2+1} = P(x)$

so the function is symmetric with respect to the y-axis.

y-intercept: $y = 1$

x-intercept: none

$P'(x) = \frac{-2x}{(x^2+1)^2}$; $P'(x)$ is 0 when $x = 0$.

critical point: $x = 0$

$P'(x) > 0$ when $x < 0$ so $P(x)$ is increasing on

$(-\infty, 0]$ and decreasing on $[0, \infty)$. Global maximum $P(0) = 1$; no local minima.

$P''(x) = \dfrac{6x^2 - 2}{(x^2+1)^3}$

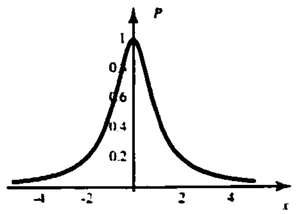
$P''(x) > 0$ on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$ (concave up) and $P''(x) < 0$ on $(-1/\sqrt{3}, 1/\sqrt{3})$ (concave down).

Inflection points: $\left(\pm\dfrac{1}{\sqrt{3}}, \dfrac{3}{4}\right)$

No vertical asymptotes.

$\lim\limits_{x\to\infty} P(x) = 0$; $\lim\limits_{x\to-\infty} P(x) = 0$

$y = 0$ is a horizontal asymptote.



15. Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$;
range: $(-\infty, \infty)$
Neither an even nor an odd function

y-intercept: $-\dfrac{3}{2}$; x-intercepts: 1, 3

$f'(x) = \dfrac{3x^2 - 10x + 11}{(x+1)^2(x-2)^2}$; $f'(x)$ is never 0.

No critical points

$f'(x) > 0$ for all $x \neq -1, 2$

$f(x)$ is increasing on

$(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.

No local minima or maxima

$f''(x) = \dfrac{-6x^3 + 30x^2 - 66x + 42}{(x+1)^3(x-2)^3}$; $f''(x) > 0$ when

$x < -1$ or $1 < x < 2$

$f(x)$ is concave up on $(-\infty, -1) \cup (1, 2)$ and concave down on $(-1, 1) \cup (2, \infty)$;
inflection point $f(1) = 0$

$\lim\limits_{x\to\infty} \dfrac{(x-1)(x-3)}{(x+1)(x-2)} = \lim\limits_{x\to\infty} \dfrac{x^2 - 4x + 3}{x^2 - x - 2}$

$= \lim\limits_{x\to\infty} \dfrac{1 - \frac{4}{x} + \frac{3}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 1$;

$\lim\limits_{x\to-\infty} \dfrac{(x-1)(x-3)}{(x+1)(x-2)} = \lim\limits_{x\to-\infty} \dfrac{1 - \frac{4}{x} + \frac{3}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 1$;

$y = 1$ is a horizontal asymptote.

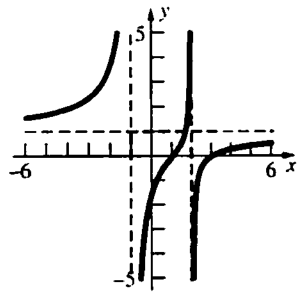
As $x \to -1^-$, $x - 1 \to -2$, $x - 3 \to -4$, $x - 2 \to -3$, and $x + 1 \to 0^-$ so $\lim\limits_{x\to-1^-} f(x) = \infty$;

as $x \to -1^+$, $x - 1 \to -2$, $x - 3 \to -4$, $x - 2 \to -3$, and $x + 1 \to 0^+$, so $\lim\limits_{x\to-1^+} f(x) = -\infty$

As $x \to 2^-$, $x - 1 \to 1$, $x - 3 \to -1$, $x + 1 \to 3$, and $x - 2 \to 0^-$, so $\lim\limits_{x\to 2^-} f(x) = \infty$; as

$x \to 2^+$, $x - 1 \to 1$, $x - 3 \to -1$, $x + 1 \to 3$, and $x - 2 \to 0^+$, so $\lim\limits_{x\to 2^+} f(x) = -\infty$

$x = -1$ and $x = 2$ are vertical asymptotes.



16. Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, -2] \cup [2, \infty)$

Odd function since

$w(-z) = \dfrac{(-z)^2 + 1}{-z} = -\dfrac{z^2 + 1}{z} = -w(z)$; symmetric with respect to the origin.

y-intercept: none
x-intercept: none

$w'(z) = 1 - \dfrac{1}{z^2}$; $w'(z) = 0$ when $z = \pm 1$.

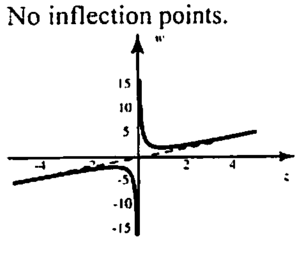
critical points: $z = \pm 1$. $w'(z) > 0$ on $(-\infty, -1) \cup (1, \infty)$ so the function is increasing on $(-\infty, -1] \cup [1, \infty)$. The function is decreasing on $[-1, 0) \cup (0, 1)$.

local maximum $w(1) = 2$ and local minimum $w(-1) = -2$. No global extrema.

$w''(z) = \dfrac{2}{z^3} > 0$ when $z > 0$. Concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

No horizontal asymptote; $x = 0$ is a vertical asymptote; the line $y = z$ is an oblique (or slant) asymptote.

No inflection points.



17. Domain: $(-\infty,1)\cup(1,\infty)$

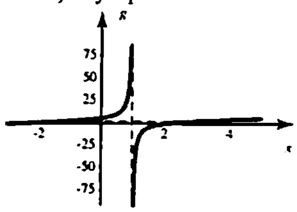
Range: $(-\infty,\infty)$

Neither even nor odd function.
y-intercept: $y=6$: x-intercept: $x=-3,2$

$g'(x)=\dfrac{x^2-2x+5}{(x-1)^2}$; $g'(x)$ is never zero. No critical points.

$g'(x)>0$ over the entire domain so the function is always increasing. No local extrema.

$f''(x)=\dfrac{-8}{(x-1)^3}$; $f''(x)>0$ when $x<1$ (concave up) and $f''(x)<0$ when $x>1$ (concave down); no inflection points.
No horizontal asymptote; $x=1$ is a vertical asymptote; the line $y=x+2$ is an oblique (or slant) asymptote.



18. Domain: $(-\infty,\infty)$; range $[0,\infty)$

$f(-x)=|-x|^3=|x|^3=f(x)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; x-intercept: 0

$f'(x)=3|x|^2\left(\dfrac{x}{|x|}\right)=3x|x|$; $f'(x)=0$ when $x=0$

Critical point: 0

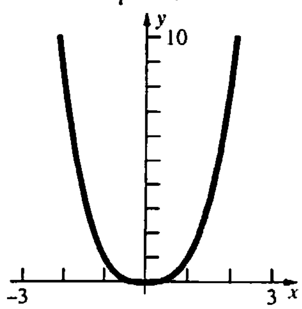
$f'(x)>0$ when $x>0$

$f(x)$ is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$.
Global minimum $f(0)=0$; no local maxima

$f''(x)=3|x|+\dfrac{3x^2}{|x|}=6|x|$ as $x^2=|x|^2$;

$f''(x)>0$ when $x\neq 0$

$f(x)$ is concave up on $(-\infty,0)\cup(0,\infty)$; no inflection points



19. Domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$

$R(-z)=-z|-z|=-z|z|=-R(z)$; odd function; symmetric with respect to the origin.
y-intercept: 0; z-intercept: 0

$R'(z)=|z|+\dfrac{z^2}{|z|}=2|z|$ since $z^2=|z|^2$ for all z;

$R'(z)=0$ when $z=0$

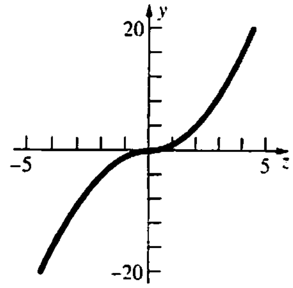
Critical point: 0

$R'(z)>0$ when $z\neq 0$

$R(z)$ is increasing on $(-\infty,\infty)$ and decreasing nowhere.
No local minima or maxima

$R''(z)=\dfrac{2z}{|z|}$; $R''(z)>0$ when $z>0$.

$R(z)$ is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$; inflection point $(0,0)$.



20. Domain: $(-\infty,\infty)$; range: $[0,\infty)$

$H(-q)=(-q)^2|-q|=q^2|q|=H(q)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; q-intercept: 0

$H'(q)=2q|q|+\dfrac{q^3}{|q|}=\dfrac{3q^3}{|q|}=3q|q|$ as $|q|^2=q^2$ for all q; $H'(q)=0$ when $q=0$

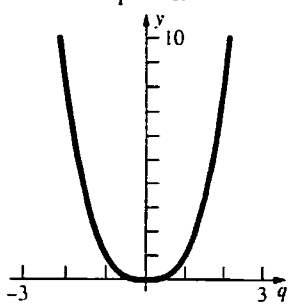
Critical point: 0

$H'(q)>0$ when $q>0$

$H(q)$ is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$.
Global minimum $H(0)=0$; no local maxima

$H''(q)=3|q|+\dfrac{3q^2}{|q|}=6|q|$; $H''(q)>0$ when $q\neq 0$.

$H(q)$ is concave up on $(-\infty,0)\cup(0,\infty)$; no inflection points.



21. Domain: $(-\infty,\infty)$; range: $[0,\infty)$
Neither an even nor an odd function.

Note that for $x \le 0$, $|x| = -x$ so $|x| + x = 0$, while for $x > 0$, $|x| = x$ so $\frac{|x|+x}{2} = x$.

$$g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 3x^2 + 2x & \text{if } x > 0 \end{cases}$$

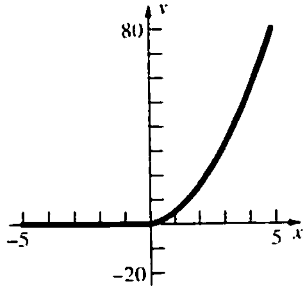
y-intercept: 0; x-intercepts: $(-\infty, 0]$

$$g'(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 6x + 2 & \text{if } x > 0 \end{cases}$$

No critical points for $x > 0$.
$g(x)$ is increasing on $[0, \infty)$ and decreasing nowhere.

$$g''(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 6 & \text{if } x > 0 \end{cases}$$

$g(x)$ is concave up on $(0, \infty)$; no inflection points



22. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
Neither an even nor an odd function. Note that for $x < 0$, $|x| = -x$ so $\frac{|x|-x}{2} = -x$, while for $x \ge 0$, $|x| = x$ so $\frac{|x|-x}{2} = 0$.

$$h(x) = \begin{cases} -x^3 + x^2 - 6x & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

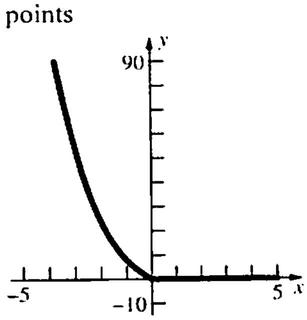
y-intercept: 0; x-intercepts: $[0, \infty)$

$$h'(x) = \begin{cases} -3x^2 + 2x - 6 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

No critical points for $x < 0$
$h(x)$ is increasing nowhere and decreasing on $(-\infty, 0]$.

$$h''(x) = \begin{cases} -6x + 2 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

$h(x)$ is concave up on $(-\infty, 0)$; no inflection points



23. Domain: $(-\infty, \infty)$; range: $[0, 1]$
$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; x-intercepts: $k\pi$ where k is any integer.

$f'(x) = \frac{\sin x}{|\sin x|}\cos x$; $f'(x) = 0$ when $x = \frac{\pi}{2} + k\pi$ and $f'(x)$ does not exist when $x = k\pi$, where k is any integer.

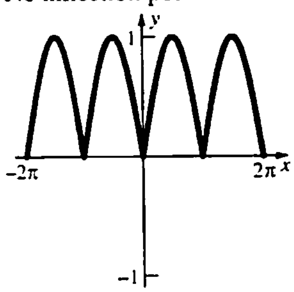
Critical points: $\frac{k\pi}{2}$, where k is any integer;
$f'(x) > 0$ when $\sin x$ and $\cos x$ are either both positive or both negative.
$f(x)$ is increasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right]$ and decreasing on $\left[k\pi + \frac{\pi}{2}, (k+1)\pi\right]$ where k is any integer.
Global minima $f(k\pi) = 0$; global maxima $f\left(k\pi + \frac{\pi}{2}\right) = 1$, where k is any integer.

$$f''(x) = \frac{\cos^2 x}{|\sin x|} - \frac{\sin^2 x}{|\sin x|} + \sin x \cos x\left(-\frac{1}{|\sin x|^2}\right)\left(\frac{\sin x}{|\sin x|}\right)(\cos x)$$

$$= \frac{\cos^2 x}{|\sin x|} - \frac{\sin^2 x}{|\sin x|} - \frac{\cos^2 x}{|\sin x|}$$

$$= -\frac{\sin^2 x}{|\sin x|} = -|\sin x|$$

$f''(x) < 0$ when $x \ne k\pi$, k any integer
$f(x)$ is never concave up and concave down on $(k\pi, (k+1)\pi)$ where k is any integer.
No inflection points



24. Domain: $[2k\pi, (2k+1)\pi]$ where k is any integer; range: $[0, 1]$
Neither an even nor an odd function
y-intercept: 0; x-intercepts: $k\pi$, where k is any integer.

$f'(x) = \frac{\cos x}{2\sqrt{\sin x}}$; $f'(x) = 0$ when $x = 2k\pi + \frac{\pi}{2}$

while $f'(x)$ does not exist when $x = k\pi$, k any integer.

Critical points: $k\pi, 2k\pi + \frac{\pi}{2}$ where k is any integer

$f'(x) > 0$ when $2k\pi < x < 2k\pi + \frac{\pi}{2}$

$f(x)$ is increasing on $\left[2k\pi, 2k\pi + \frac{\pi}{2}\right]$ and decreasing on $\left[2k\pi + \frac{\pi}{2}, (2k+1)\pi\right]$, k any integer.

Global minima $f(k\pi) = 0$; global maxima

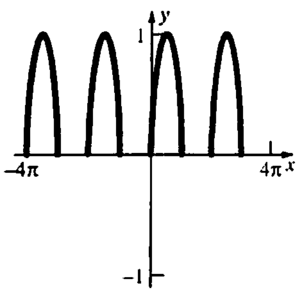
$f\left(2k\pi + \frac{\pi}{2}\right) = 1$, k any integer

$f''(x) = \frac{-\cos^2 x - 2\sin^2 x}{4\sin^{3/2} x} = \frac{-1 - \sin^2 x}{4\sin^{3/2} x}$

$= -\frac{1 + \sin^2 x}{4\sin^{3/2} x}$;

$f''(x) < 0$ for all x.

$f(x)$ is concave down on $(2k\pi, (2k+1)\pi)$; no inflection points



25. Domain: $(-\infty, \infty)$

Range: $[0, 1]$

Even function since

$h(-t) = \cos^2(-t) = \cos^2 t = h(t)$

so the function is symmetric with respect to the y-axis.

y-intercept: $y = 1$; t-intercepts: $x = \frac{\pi}{2} + k\pi$ where k is any integer.

$h'(t) = -2\cos t \sin t$; $h'(t) = 0$ at $t = \frac{k\pi}{2}$.

Critical points: $t = \frac{k\pi}{2}$

$h'(t) > 0$ when $k\pi + \frac{\pi}{2} < t < (k+1)\pi$. The function is increasing on the intervals $[k\pi + (\pi/2), (k+1)\pi]$ and decreasing on the intervals $[k\pi, k\pi + (\pi/2)]$.

Global maxima $h(k\pi) = 1$

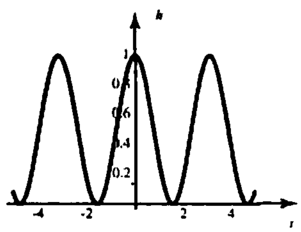
Global minima $h\left(\frac{\pi}{2} + k\pi\right) = 0$

$h''(t) = 2\sin^2 t - 2\cos^2 t = -2(\cos 2t)$

$h''(t) < 0$ on $\left(k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4}\right)$ so h is concave down, and $h''(t) > 0$ on $\left(k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}\right)$ so h is concave up.

Inflection points: $\left(\frac{k\pi}{2} + \frac{\pi}{4}, \frac{1}{2}\right)$

No vertical asymptotes; no horizontal asymptotes.



26. Domain: all reals except $t = \frac{\pi}{2} + k\pi$

Range: $[0, \infty)$

y-intercepts: $y = 0$; t-intercepts: $t = k\pi$ where k is any integer.

Even function since

$g(-t) = \tan^2(-t) = (-\tan t)^2 = \tan^2 t$

so the function is symmetric with respect to the y-axis.

$g'(t) = 2\sec^2 t \tan t = \frac{2\sin t}{\cos^3 t}$; $g'(t) = 0$ when $t = k\pi$.

Critical points: $k\pi$

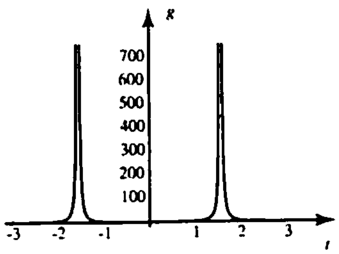
$g(t)$ is increasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(k\pi - \frac{\pi}{2}, k\pi\right]$.

Global minima $g(k\pi) = 0$; no local maxima

$$g''(t) = 2\frac{\cos^4 t + \sin t(3)\cos^2 t \sin t}{\cos^6 t} = 2\frac{\cos^2 t + 3\sin^2 t}{\cos^4 t}$$
$$= 2\frac{1 + 2\sin^2 t}{\cos^4 t} > 0$$

over the entire domain. Thus the function is concave up on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$; no inflection points.

No horizontal asymptotes; $t = \frac{\pi}{2} + k\pi$ are vertical asymptotes.



27. Domain: $\approx(-\infty, 0.44) \cup (0.44, \infty)$;

range: $(-\infty, \infty)$

Neither an even nor an odd function

y-intercept: 0; x-intercepts: 0, ≈ 0.24

$$f'(x) = \frac{74.6092x^3 - 58.2013x^2 + 7.82109x}{(7.126x - 3.141)^2};$$

$f'(x) = 0$ when $x = 0$, ≈ 0.17, ≈ 0.61

Critical points: 0, ≈ 0.17, ≈ 0.61

$f'(x) > 0$ when $0 < x < 0.17$ or $0.61 < x$

$f(x)$ is increasing on $\approx [0, 0.17] \cup [0.61, \infty)$ and decreasing on

$(-\infty, 0] \cup [0.17, 0.44) \cup (0.44, 0.61]$

Local minima $f(0) = 0$, $f(0.61) \approx 0.60$; local maximum $f(0.17) \approx 0.01$

$$f''(x) = \frac{531.665x^3 - 703.043x^2 + 309.887x - 24.566}{(7.126x - 3.141)^3};$$

$f''(x) > 0$ when $x < 0.10$ or $x > 0.44$

$f(x)$ is concave up on $(-\infty, 0.10) \cup (0.44, \infty)$ and concave down on $(0.10, 0.44)$;

inflection point $\approx (0.10, 0.003)$

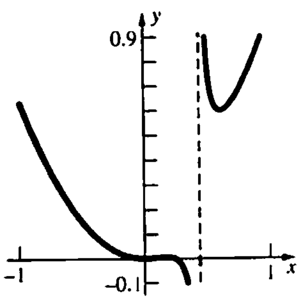
$$\lim_{x\to\infty} \frac{5.235x^3 - 1.245x^2}{7.126x - 3.141} = \lim_{x\to\infty} \frac{5.235x^2 - 1.245x}{7.126 - \frac{3.141}{x}} = \infty$$

so $f(x)$ does not have a horizontal asymptote.

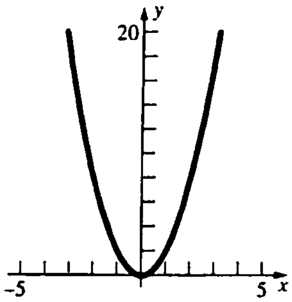
As $x \to 0.44^-$, $5.235x^3 - 1.245x^2 \to 0.20$ while $7.126x - 3.141 \to 0^-$, so $\lim_{x\to 0.44^-} f(x) = -\infty$;

as $x \to 0.44^+$, $5.235x^3 - 1.245x^2 \to 0.20$ while $7.126x - 3.141 \to 0^+$, so $\lim_{x\to 0.44^+} f(x) = \infty$;

$x \approx 0.44$ is a vertical asymptote of $f(x)$.



28.



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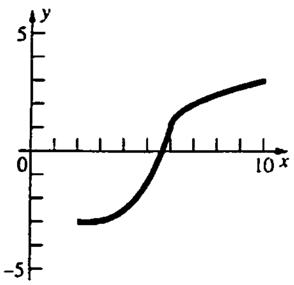

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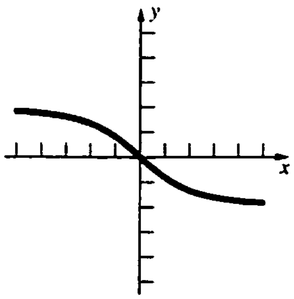

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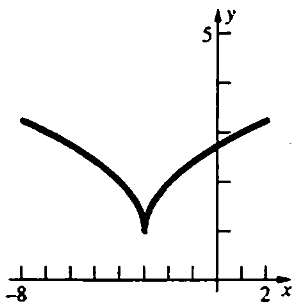

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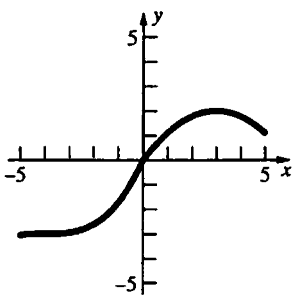

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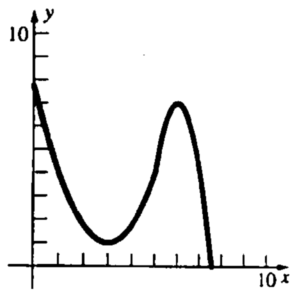

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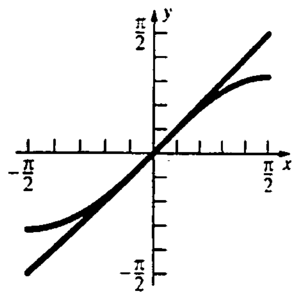

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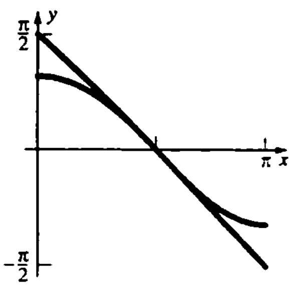
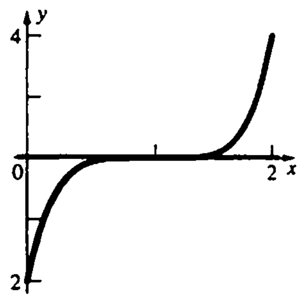


36. $y' = 5(x-1)^4$; $y'' = 20(x-1)^3$; $y''(x) > 0$
When $x > 1$; inflection point $(1, 3)$
At $x = 1$, $y' = 0$, so the linear approximation is a horizontal line.



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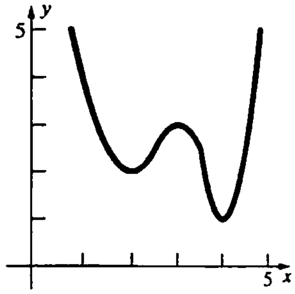

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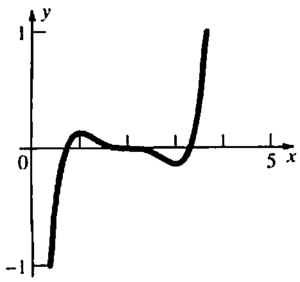

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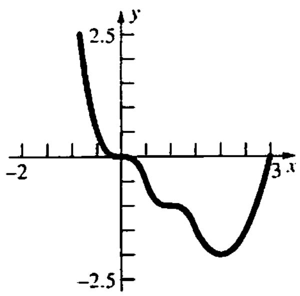
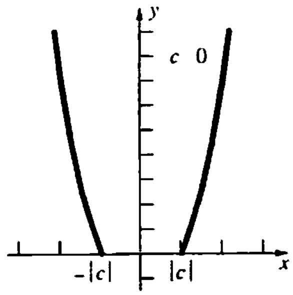


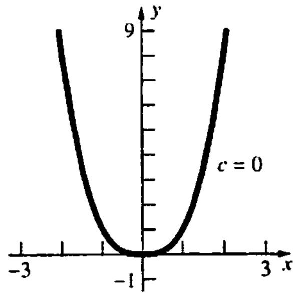
40. Let $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$ and $f''(x) = 2a$. An inflection point occurs where $f''(x)$ changes from positive to negative, but $2a$ is either always positive or always negative, so $f(x)$ does not have any inflection points.
($f''(x) = 0$ only when $a = 0$, but then $f(x)$ is not a quadratic curve.)

41. Let $f(x) = ax^3 + bx^2 + cx + d$, then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. As long as $a \neq 0$, $f''(x)$ will be positive on one side of $x = \frac{b}{3a}$ and negative on the other side.
$x = \frac{b}{3a}$ is the only inflection point.

42. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + c$, then
$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$ and
$f''(x) = 12ax^2 + 6bx + 2c = 2(6ax^2 + 3bx + c)$
Inflection points can only occur when $f''(x)$ changes sign from positive to negative and $f''(x) = 0$. $f''(x)$ has at most 2 zeros, thus $f(x)$ has at most 2 inflection points.

43. Since the c term is squared, the only difference occurs when $c = 0$. When $c = 0$,
$y = x^2\sqrt{x^2} = |x|^3$ which has domain $(-\infty, \infty)$ and range $[0, \infty)$. When $c \neq 0$, $y = x^2\sqrt{x^2 - c^2}$ has domain $(-\infty, -|c|] \cup [|c|, \infty)$ and range $[0, \infty)$.





44. Let $f(x) = \frac{cx}{4 + (cx)^2} = \frac{cx}{4 + c^2x^2}$

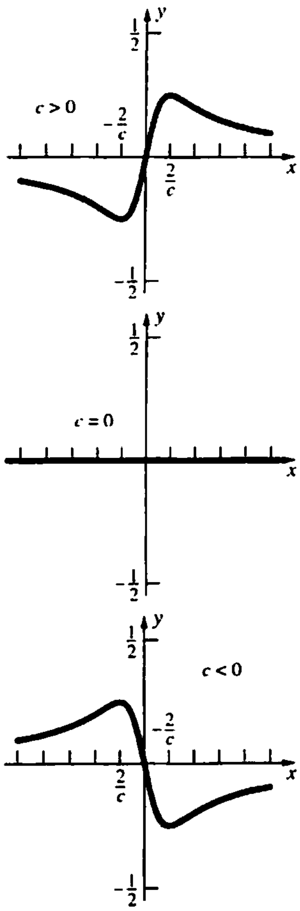
$f'(x) = \frac{c(4 - c^2x^2)}{(4 + c^2x^2)^2}$; $f'(x) = 0$ when $x = \pm\frac{2}{c}$

unless $c = 0$, in which case $f(x) = 0$ and $f'(x) = 0$.

If $c > 0$, $f(x)$ is increasing on $\left[-\frac{2}{c}, \frac{2}{c}\right]$ and decreasing on $\left(-\infty, -\frac{2}{c}\right] \cup \left[\frac{2}{c}, \infty\right)$, thus, $f(x)$ has a global minimum at $f\left(-\frac{2}{c}\right) = -\frac{1}{4}$ and a global maximum of $f\left(\frac{2}{c}\right) = \frac{1}{4}$.

If $c < 0$ $f(x)$ is increasing on $\left(-\infty, \frac{2}{c}\right] \cup \left[-\frac{2}{c}, \infty\right)$ and decreasing on $\left[\frac{2}{c}, -\frac{2}{c}\right]$. Thus, $f(x)$ has a global minimum at $f\left(-\frac{2}{c}\right) = -\frac{1}{4}$ and a global maximum at $f\left(\frac{2}{c}\right) = \frac{1}{4}$.

$f''(x) = \frac{2c^3x(c^2x^2 - 12)}{(4 + c^2x^2)^3}$, so $f(x)$ has inflection points at $x = 0, \pm\frac{2\sqrt{3}}{c}$, $c \neq 0$



45. Let $f(x) = \frac{1}{(cx^2 - 4)^2 + cx^2}$, then

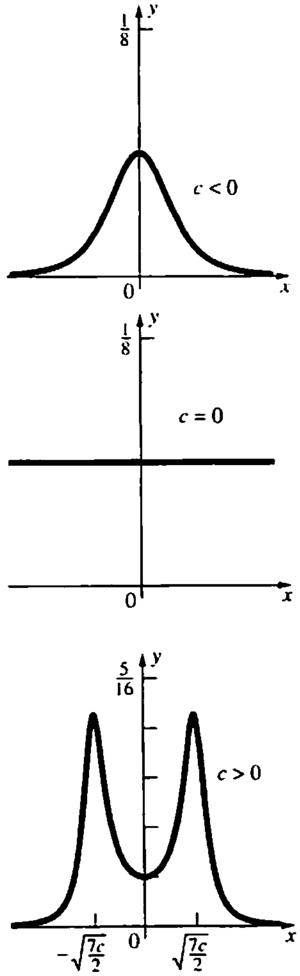
$$f'(x) = \frac{2cx(7 - 2cx^2)}{[(cx^2 - 4)^2 + cx^2]^2};$$

If $c > 0$, $f'(x) = 0$ when $x = 0, \pm\sqrt{\frac{7}{2c}}$.

If $c < 0$, $f'(x) = 0$ when $x = 0$.

Note that $f(x) = \frac{1}{16}$ (a horizontal line) if $c = 0$.

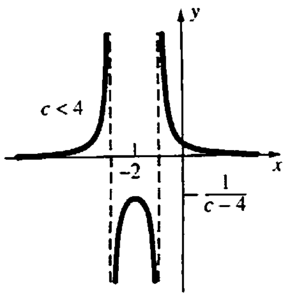
If $c > 0$, $f'(x) > 0$ when $x < -\sqrt{\frac{7}{2c}}$ and $0 < x < \sqrt{\frac{7}{2c}}$, so $f(x)$ is increasing on $\left(-\infty, -\sqrt{\frac{7}{2c}}\right] \cup \left[0, \sqrt{\frac{7}{2c}}\right]$ and decreasing on $\left[-\sqrt{\frac{7}{2c}}, 0\right] \cup \left[\sqrt{\frac{7}{2c}}, \infty\right)$. Thus, $f(x)$ has local maxima $f\left(-\sqrt{\frac{7}{2c}}\right) = \frac{4}{15}$, $f\left(\sqrt{\frac{7}{2c}}\right) = \frac{4}{15}$ and local minimum $f(0) = \frac{1}{16}$. If $c < 0$, $f'(x) > 0$ when $x < 0$, so $f(x)$ is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$. Thus, $f(x)$ has a local maximum $f(0) = \frac{1}{16}$. Note that $f(x) > 0$ and has horizontal asymptote $y = 0$.

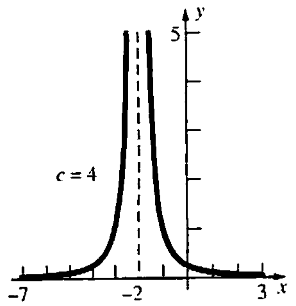


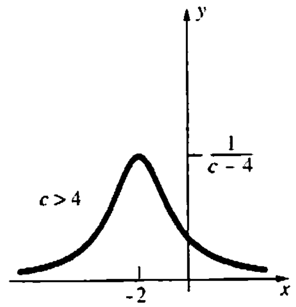
46. Let $f(x) = \frac{1}{x^2 + 4x + c}$. By the quadratic formula, $x^2 + 4x + c = 0$ when $x = -2 \pm \sqrt{4 - c}$. Thus $f(x)$ has vertical asymptote(s) at $x = -2 \pm \sqrt{4 - c}$ when $c \le 4$.

$f'(x) = \frac{-2x - 4}{(x^2 + 4x + c)^2}$; $f'(x) = 0$ when $x = -2$, unless $c = 4$ since then $x = -2$ is a vertical asymptote.

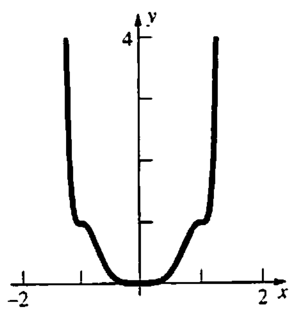
For $c \ne 4$, $f'(x) > 0$ when $x < -2$, so $f(x)$ is increasing on $(-\infty, -2]$ and decreasing on $[-2, \infty)$ (with the asymptotes excluded). Thus $f(x)$ has a local maximum at $f(-2) = \frac{1}{c - 4}$. For $c = 4$, $f'(x) = -\frac{2}{(x + 2)^3}$ so $f(x)$ is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.







47.



Justification:

$f(1) = g(1) = 1$

$f(-x) = g((-x)^4) = g(x^4) = f(x)$

f is an even function: symmetric with respect to the y-axis.

$f'(x) = g'(x^4)4x^3$

$f'(x) > 0$ for x on $(0,1) \cup (1, \infty)$

$f'(x) < 0$ for x on $(-\infty, -1) \cup (-1, 0)$

$f'(x) = 0$ for $x = -1, 0, 1$ since f' is continuous.

$f''(x) = g''(x^4)16x^6 + g'(x)12x^2$

$f''(x) = 0$ for $x = -1, 0, 1$

$f''(x) > 0$ for x on $(0, x_0) \cup (1, \infty)$

$f''(x) < 0$ for x on $(x_0, 1)$

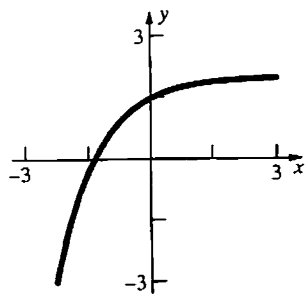
Where x_0 is a root of $f''(x) = 0$ (assume that there is only one root on $(0,1)$).

48. Suppose $H'''(1) < 0$, then $H''(x)$ is decreasing in a neighborhood around $x = 1$. Thus, $H''(x) > 0$ to the left of 1 and $H''(x) < 0$ to the right of 1, so $H(x)$ is concave up to the left of 1 and concave down to the right of 1. Suppose $H'''(1) > 0$, then $H''(x)$ is increasing in a neighborhood around $x = 1$. Thus, $H''(x) < 0$ to the left of 1 and $H''(x) > 0$ to the right of 1, so $H(x)$ is concave up to the right of 1 and concave down to the left of 1. In either case, $H(x)$ has a point of inflection at $x = 1$.

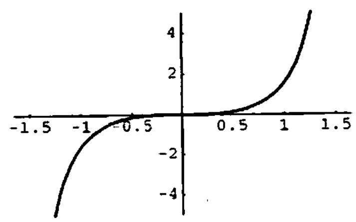
49. a. Not possible; $F'(x) > 0$ means that $F(x)$ is increasing. $F''(x) > 0$ means that the rate at which $F(x)$ is increasing never slows down. Thus the values of F must eventually become positive.

b. Not possible; If $F(x)$ is concave down for all x, then $F(x)$ cannot always be positive.

c.

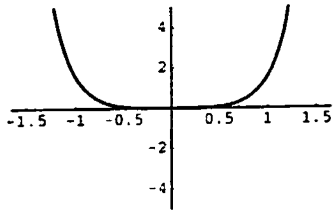


50. a.



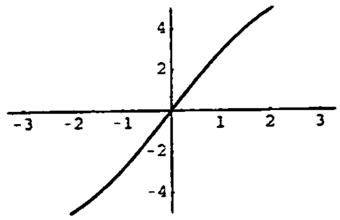
No global extrema; inflection point at (0, 0)

b.



No global maximum: global minimum at (0, 0); no inflection points

c.



Global minimum

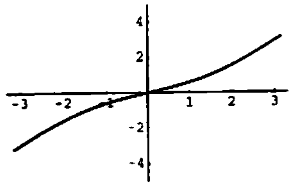
$f(-\pi) = -2\pi + \sin(-\pi) = -2\pi \approx -6.3$;

global maximum

$f(\pi) = 2\pi + \sin \pi = 2\pi \approx 6.3$;

inflection point at (0, 0)

d.



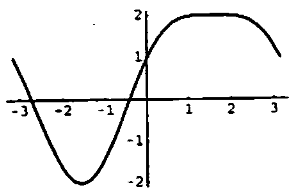
Global minimum

$f(-\pi) = -\pi - \dfrac{\sin(-\pi)}{2} = -\pi \approx 3.1$; global

maximum $f(\pi) = \pi + \dfrac{\sin \pi}{2} = \pi \approx 3.1$;

inflection point at (0, 0).

51. a.



$f'(x) = 2\cos x - 2\cos x \sin x$

$= 2\cos x(1 - \sin x)$;

$f'(x) = 0$ when $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

$f''(x) = -2\sin x - 2\cos^2 x + 2\sin^2 x$

$= 4\sin^2 x - 2\sin x - 2; f''(x) = 0$ when

$\sin x = -\dfrac{1}{2}$ or $\sin x = 1$ which occur when

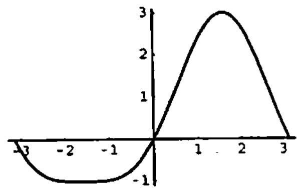
$x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{2}$

Global minimum $f\left(-\dfrac{\pi}{2}\right) = -2$; global

maximum $f\left(\dfrac{\pi}{2}\right) = 2$; inflection points

$f\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{4}, f\left(-\dfrac{5\pi}{6}\right) = -\dfrac{1}{4}$

b.



$f'(x) = 2\cos x + 2\sin x \cos x$

$= 2\cos x(1 + \sin x)$; $f'(x) = 0$ when

$x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

$f''(x) = -2\sin x + 2\cos^2 x - 2\sin^2 x$

$= -4\sin^2 x - 2\sin x + 2; f''(x) = 0$ when

$\sin x = -1$ or $\sin x = \dfrac{1}{2}$ which occur when

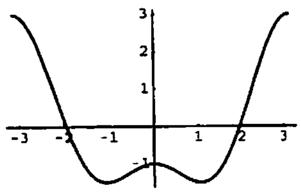
$x = -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Global minimum $f\left(-\dfrac{\pi}{2}\right) = -1$; global

maximum $f\left(\dfrac{\pi}{2}\right) = 3$; inflection points

$f\left(\dfrac{\pi}{6}\right) = \dfrac{5}{4}, f\left(\dfrac{5\pi}{6}\right) = \dfrac{5}{4}.$

c.



$f'(x) = -2\sin 2x + 2\sin x$

$= -4\sin x \cos x + 2\sin x = 2\sin x(1 - 2\cos x)$;

$f'(x) = 0$ when $x = -\pi, -\dfrac{\pi}{3}, 0, \dfrac{\pi}{3}, \pi$

$f''(x) = -4\cos 2x + 2\cos x$; $f''(x) = 0$ when

$x \approx -2.2, -0.6, 0.6, 2.2$

Global minimum $f\left(-\dfrac{\pi}{3}\right) = f\left(\dfrac{\pi}{3}\right) = -1.5$;

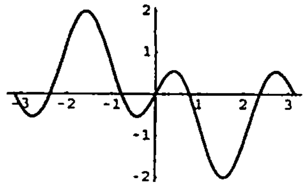
Global maximum $f(-\pi) = f(\pi) = 3$;

inflection points $f(-2.2) \approx 0.87$,

$f(-0.6) \approx -1.29, f(0.6) \approx -1.29$,

$f(2.2) \approx 0.87$

d.



$f'(x) = 3\cos 3x - \cos x$; $f'(x) = 0$ when $3\cos 3x = \cos x$ which occurs when

$x = -\frac{\pi}{2}, \frac{\pi}{2}$ and when

$x \approx -2.7, -0.4, 0.4, 2.7$

$f''(x) = -9\sin 3x + \sin x$ which occurs when

$x = -\pi, 0, \pi$ and when

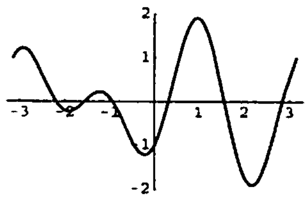
$x \approx -2.13, -1.02, 1.02, 2.13$

Global minimum $f\left(\frac{\pi}{2}\right) = -2$;

global maximum $f\left(-\frac{\pi}{2}\right) = 2$;

inflection points $f(-2.13) \approx 0.7$, $f(-1.02) \approx 0.8$; $f(0) = 0$, $f(1.02) \approx -0.8$, $f(2.13) \approx -0.7$

e.



$f'(x) = 2\cos 2x + 3\sin 3x$

Using the graphs, $f(x)$ has a global minimum at $f(2.17) \approx -1.9$ and a global maximum at $f(0.97) \approx 1.9$

$f''(x) = -4\sin 2x + 9\cos 3x$; $f''(x) = 0$ when

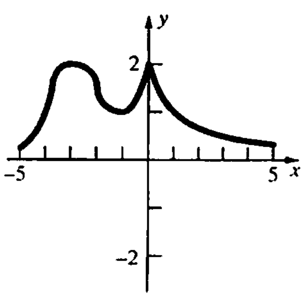
$x = -\frac{\pi}{2}, \frac{\pi}{2}$ and when

$x \approx -2.47, -0.67, 0.41, 2.73$.

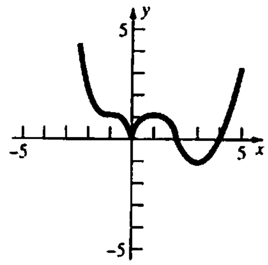
Inflection points $f(-2.47) \approx 0.54$, $f\left(-\frac{\pi}{2}\right) = 0$, $f(-0.67) \approx -0.55$,

$f(0.41) \approx 0.40$, $f\left(\frac{\pi}{2}\right) = 0$, $f(2.73) \approx -0.40$

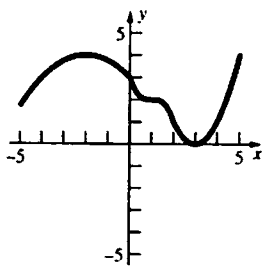
52.



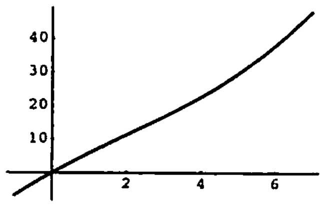
53.



54.



55. a.



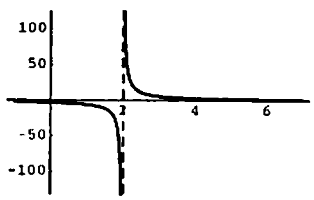
$f'(x) = \dfrac{2x^2 - 9x + 40}{\sqrt{x^2 - 6x + 40}}$; $f'(x)$ is never 0, and always positive, so $f(x)$ is increasing for all x. Thus, on $[-1, 7]$, the global minimum is $f(-1) \approx -6.9$ and the global maximum if $f(7) \approx 48.0$.

$f''(x) = \dfrac{2x^3 - 18x^2 + 147x - 240}{(x^2 - 6x + 40)^{3/2}}$; $f''(x) = 0$ when $x \approx 2.02$; inflection point $f(2.02) \approx 11.4$

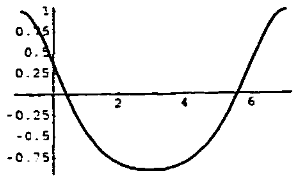
b.

Global minimum $f(0) = 0$; global maximum $f(7) \approx 124.4$; inflection point at $x \approx 2.34$, $f(2.34) \approx 48.09$

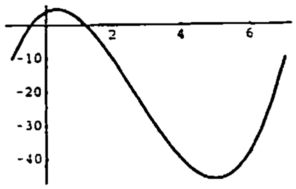
c.



No global minimum or maximum; no inflection points

d. 

Global minimum $f(3) \approx -0.9$;
global maximum $f(-1) \approx 1.0$ or $f(7) \approx 1.0$;
Inflection points at $x \approx 0.05$ and $x \approx 5.9$, $f(0.05) \approx 0.3$, $f(5.9) \approx 0.3$.

56. a. 

$f'(x) = 3x^2 - 16x + 5$; $f'(x) = 0$ when

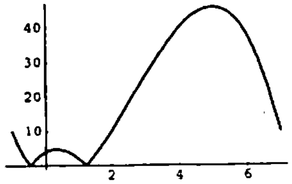
$x = \frac{1}{3}, 5.$

Global minimum $f(5) = -46$;

global maximum $f\left(\frac{1}{3}\right) \approx 4.8$

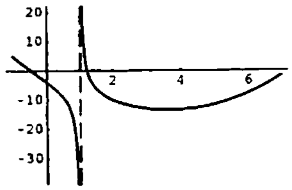
$f''(x) = 6x - 16$; $f''(x) = 0$ when $x = \frac{8}{3}$;

inflection point $f\left(\frac{8}{3}\right) \approx -20.6$

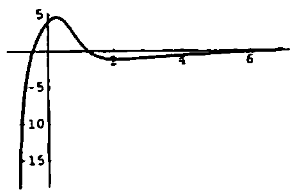
b. 

Global minimum when $x \approx -0.5$ and $x \approx 1.2$, $f(-0.5) \approx 0$, $f(1.2) \approx 0$;
global maximum $f(5) = 46$

Inflection point $f\left(\frac{8}{3}\right) \approx 20.6$

c. 

No global minimum or maximum;
inflection point at
$x \approx -0.26$, $f(-0.26) \approx -1.7$

d. 

No global minimum, global maximum when $x \approx 0.26$, $f(0.26) \approx 4.7$
Inflection points when $x \approx 0.75$ and $x \approx 3.15$, $f(0.75) \approx 2.6$, $f(3.15) \approx -0.88$

4.7 Concepts Review

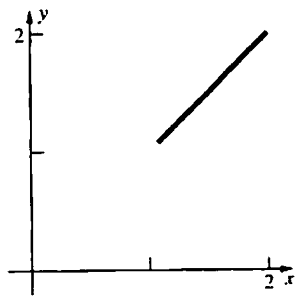
1. continuous; (a, b); $f(b) - f(a) = f'(c)(b - a)$
2. $f'(0)$ does not exist.
3. $F(x) = G(x) + C$
4. $x^4 + C$

Problem Set 4.7

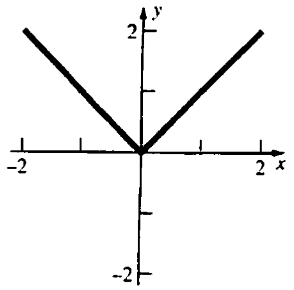
1. $f'(x) = \frac{x}{|x|}$

$\frac{f(2) - f(1)}{2 - 1} = \frac{2 - 1}{1} = 1$

$\frac{c}{|c|} = 1$ for all $c > 0$, hence for all c in $(1, 2)$



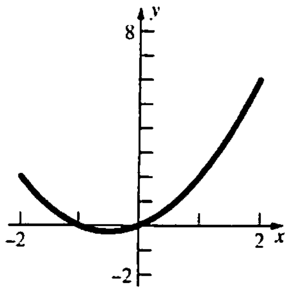
2. The Mean Value Theorem does not apply because $g'(0)$ does not exist.



3. $f'(x) = 2x + 1$

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 - 2}{4} = 1$$

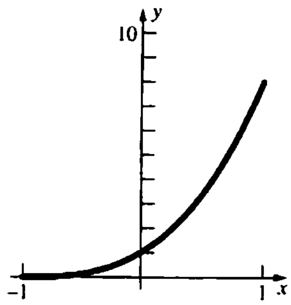
$2c + 1 = 1$ when $c = 0$



4. $g'(x) = 3(x+1)^2$

$$\frac{g(1) - g(-1)}{1 - (-1)} = \frac{8 - 0}{2} = 4$$

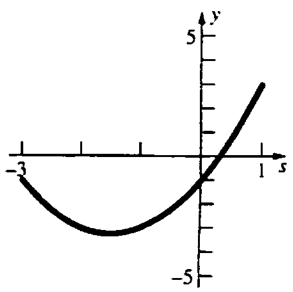
$3(c+1)^2 = 4$ when $c = -1 + \frac{2}{\sqrt{3}} \approx 0.15$



5. $H'(s) = 2s + 3$

$$\frac{H(1) - H(-3)}{1 - (-3)} = \frac{3 - (-1)}{1 - (-3)} = 1$$

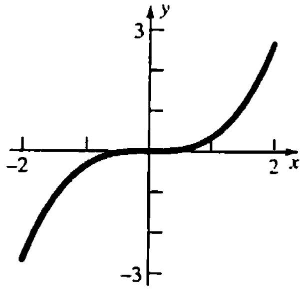
$2c + 3 = 1$ when $c = -1$



6. $F'(x) = x^2$

$$\frac{F(2) - F(-2)}{2 - (-2)} = \frac{\frac{8}{3} - \left(-\frac{8}{3}\right)}{4} = \frac{4}{3}$$

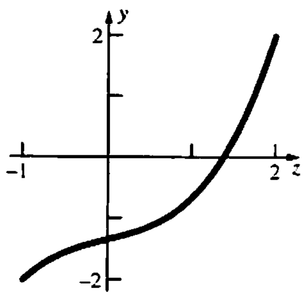
$c^2 = \frac{4}{3}$ when $c = \pm\frac{2}{\sqrt{3}} \approx \pm 1.15$



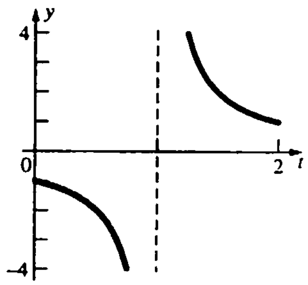
7. $f'(z) = \frac{1}{3}(3z^2 + 1) = z^2 + \frac{1}{3}$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{2 - (-2)}{3} = \frac{4}{3}$$

$c^2 + \frac{1}{3} = \frac{4}{3}$ when $c = -1, 1$, but -1 is not in $(-1, 2)$ so $c = 1$ is the only solution.



8. The Mean Value Theorem does not apply because $F(t)$ is not continuous at $t = 1$.

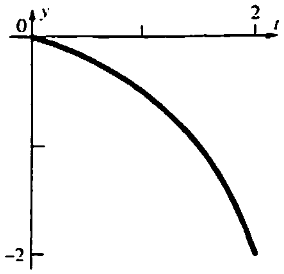


9. $h'(x) = -\dfrac{3}{(x-3)^2}$

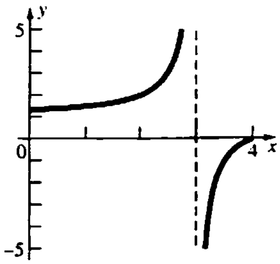
$\dfrac{h(2)-h(0)}{2-0} = \dfrac{-2-0}{2} = -1$

$-\dfrac{3}{(c-3)^2} = -1$ when $c = 3 \pm \sqrt{3}$,

$c = 3 - \sqrt{3} \approx 1.27$ $(3+\sqrt{3}$ is not in $(0, 2)$.)



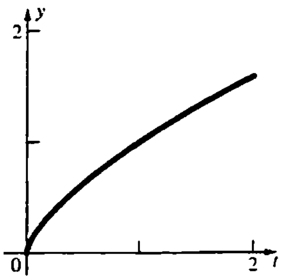
10. The Mean Value Theorem does not apply because$f(x)$ is not continuous at $x = 3$.



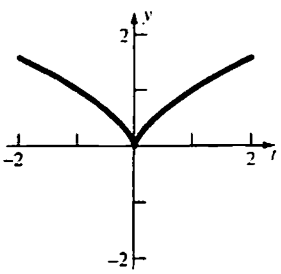
11. $h'(t) = \dfrac{2}{3t^{1/3}}$

$\dfrac{h(2)-h(0)}{2-0} = \dfrac{2^{2/3}-0}{2} = 2^{-1/3}$

$\dfrac{2}{3c^{1/3}} = 2^{-1/3}$ when $c = \dfrac{16}{27} \approx 0.59$



12. The Mean Value Theorem does not apply because $h'(0)$ does not exist.

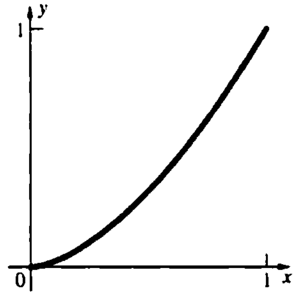


13. $g'(x) = \dfrac{5}{3}x^{2/3}$

$\dfrac{g(1)-g(0)}{1-0} = \dfrac{1-0}{1} = 1$

$\dfrac{5}{3}c^{2/3} = 1$ when $c = \pm\left(\dfrac{3}{5}\right)^{3/2}$,

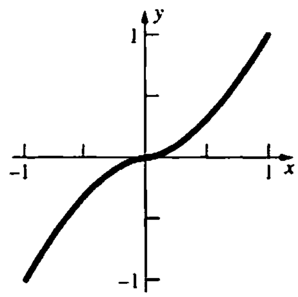
$c = \left(\dfrac{3}{5}\right)^{3/2} \approx 0.46$. $\left(-\left(\dfrac{3}{5}\right)^{3/2} \text{ is not in } (0, 1).\right)$



14. $g'(x) = \dfrac{5}{3}x^{2/3}$

$\dfrac{g(1)-g(-1)}{1-(-1)} = \dfrac{1-(-1)}{2} = 1$

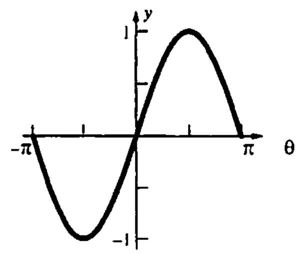
$\dfrac{5}{3}c^{2/3} = 1$ when $c = \pm\left(\dfrac{3}{5}\right)^{3/2} \approx \pm 0.46$



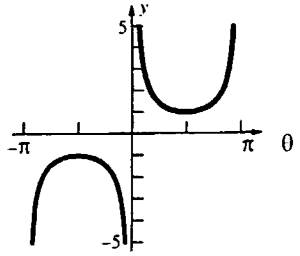
15. $S'(\theta) = \cos\theta$

$\dfrac{S(\pi)-S(-\pi)}{\pi-(-\pi)} = \dfrac{0-0}{2\pi} = 0$

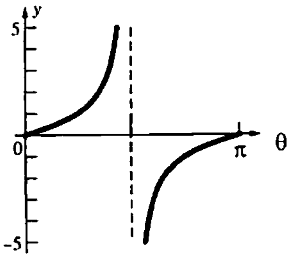
$\cos c = 0$ when $c = \pm\dfrac{\pi}{2}$.



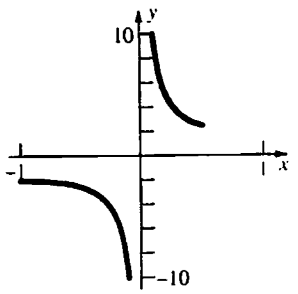
16. The Mean Value Theorem does not apply because $C(\theta)$ is not continuous at $\theta = -\pi, 0, \pi$.



17. The Mean Value Theorem does not apply because $T(\theta)$ is not continuous at $\theta = \frac{\pi}{2}$.



18. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 0$.

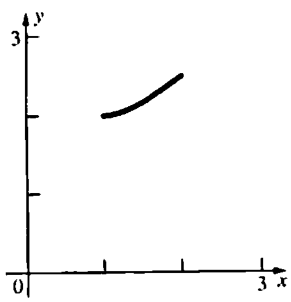


19. $f'(x) = 1 - \frac{1}{x^2}$

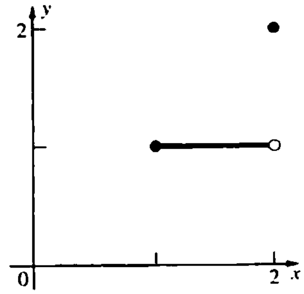
$$\frac{f(2) - f(1)}{2 - 1} = \frac{\frac{5}{2} - 2}{1} = \frac{1}{2}$$

$1 - \frac{1}{c^2} = \frac{1}{2}$ when $c = \pm\sqrt{2}$, $c = \sqrt{2} \approx 1.41$

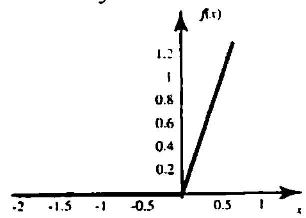
($c = -\sqrt{2}$ is not in (1, 2).)



20. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 2$.



21. The Mean Value Theorem does not apply because f is not differentiable at $x = 0$.



22. By the Mean Value Theorem

$\frac{f(b) - f(a)}{b - a} = f'(c)$ for some c in (a, b).

Since $f(b) = f(a)$, $\frac{0}{b - a} = f'(c)$; $f'(c) = 0$.

23. $\frac{f(8) - f(0)}{8 - 0} = -\frac{1}{4}$

There are three values for c such that

$f'(c) = -\frac{1}{4}$.

They are approximately 1.5, 3.75, and 7.

24. $f'(x) = 2\alpha x + \beta$

$$\frac{f(b) - f(a)}{b - a} = \frac{1}{b - a}[\alpha(b^2 - a^2) + \beta(b - a)]$$

$$= \alpha(a + b) + \beta$$

$2\alpha c + \beta = \alpha(a + b) + \beta$ when $c = \frac{a + b}{2}$ which is the midpoint of $[a, b]$.

25. By the Monotonicity Theorem, f is increasing on the intervals (a, x_0) and (x_0, b).

To show that $f(x_0) > f(x)$ for x in (a, x_0), consider f on the interval $(a, x_0]$.

f satisfies the conditions of the Mean Value Theorem on the interval $[x, x_0]$ for x in (a, x_0).

So for some c in (x, x_0),

$f(x_0) - f(x) = f'(c)(x_0 - x)$.

Because

$f'(c) > 0$ and $x_0 - x > 0$, $f(x_0) - f(x) > 0$,

so $f(x_0) > f(x)$.

Similar reasoning shows that
$f(x) > f(x_0)$ for x in (x_0, b).
Therefore, f is increasing on (a, b).

26. a. $f'(x) = 3x^2 > 0$ except at $x = 0$ in $(-\infty, \infty)$.
$f(x) = x^3$ is increasing on $(-\infty, \infty)$ by Problem 25.

b. $f'(x) = 5x^4 > 0$ except at $x = 0$ in $(-\infty, \infty)$.
$f(x) = x^5$ is increasing on $(-\infty, \infty)$ by Problem 25.

c. $f'(x) = \begin{cases} 3x^2 & x \le 0 \\ 1 & x > 0 \end{cases} > 0$ except at $x = 0$ in $(-\infty, \infty)$.
$f(x) = \begin{cases} x^3 & x \le 0 \\ x & x > 0 \end{cases}$ is increasing on $(-\infty, \infty)$ by Problem 25.

27. $s(t)$ is defined in any interval not containing $t = 0$.
$s'(c) = -\frac{1}{c^2} < 0$ for all $c \ne 0$. For any a, b with $a < b$ and both either positive or negative, the Mean Value Theorem says
$s(b) - s(a) = s'(c)(b - a)$ for some c in (a, b).
Since $a < b$, $b - a > 0$ while $s'(c) < 0$, hence $s(b) - s(a) < 0$, or $s(b) < s(a)$.
Thus, $s(t)$ is decreasing on any interval not containing $t = 0$.

28. $s'(c) = -\frac{2}{c^3} < 0$ for all $c > 0$. If $0 < a < b$, the Mean Value Theorem says
$s(b) - s(a) = s'(c)(b - a)$ for some c in (a, b).
Since $a < b$, $b - a > 0$ while $s'(c) < 0$, hence $s(b) - s(a) < 0$, or $s(b) < s(a)$. Thus, $s(t)$ is decreasing on any interval to the right of the origin.

29. $F'(x) = 0$ and $G(x) = 0$; $G'(x) = 0$.
By Theorem B,
$F(x) = G(x) + C$, so $F(x) = 0 + C = C$.

30. $F(x) = \cos^2 x + \sin^2 x$; $F(0) = 1^2 + 0^2 = 1$
$F'(x) = 2\cos x(-\sin x) + 2\sin x(\cos x) = 0$
By Problem 29, $F(x) = C$ for all x.
Since $F(0) = 1$, $C = 1$, so $\sin^2 x + \cos^2 x = 1$ for all x.

31. Let $G(x) = Dx$; $F'(x) = D$ and $G'(x) = D$.
By Theorem B, $F(x) = G(x) + C$; $F(x) = Dx + C$.

32. $F'(x) = 5$; $F(0) = 4$
$F(x) = 5x + C$ by Problem 31.
$F(0) = 4$ so $C = 4$.
$F(x) = 5x + 4$

33. Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. $f(x)$ is continuous on $[a, b]$, since it has a derivative. Thus, by the Intermediate Value Theorem, there is at least one point c,
$a < c < b$ with $f(c) = 0$.
Suppose there are two points, c and c', $c < c'$ in (a, b) with $f(c) = f(c') = 0$. Then by Rolle's Theorem, there is at least one number d in (c, c') with $f'(d) = 0$. This contradicts the given information that $f'(x) \ne 0$ for all x in $[a, b]$, thus there cannot be more than one x in $[a, b]$ where $f(x) = 0$.

34. $f'(x) = 6x^2 - 18x = 6x(x - 3)$; $f'(x) = 0$ when $x = 0$ or $x = 3$.
$f(-1) = -10$, $f(0) = 1$ so, by Problem 33, $f(x) = 0$ has exactly one solution on $(-1, 0)$.
$f(0) = 1$, $f(1) = -6$ so, by Problem 33, $f(x) = 0$ has exactly one solution on $(0, 1)$.
$f(4) = -15$, $f(5) = 26$ so, by Problem 33, $f(x) = 0$ has exactly one solution on $(4, 5)$.

35. Suppose there is more than one zero between successive distinct zeros of f'. That is, there are a and b such that $f(a) = f(b) = 0$ with a and b between successive distinct zeros of f'. Then by Rolle's Theorem, there is a c between a and b such that $f'(c) = 0$. This contradicts the supposition that a and b lie between successive distinct zeros.

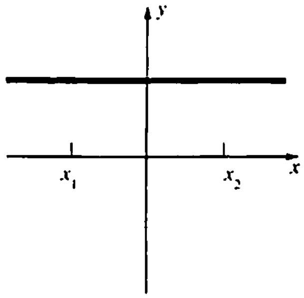
36. Let x_1, x_2, and x_3 be the three values such that $g(x_1) = g(x_2) = g(x_3) = 0$ and $a \le x_1 < x_2 < x_3 \le b$. By applying Rolle's Theorem (see Problem 22) there is at least one number x_4 in (x_1, x_2) and one number x_5 in (x_2, x_3) such that $g'(x_4) = g'(x_5) = 0$. Then by applying Rolle's Theorem to $g'(x)$, there is at least one number x_6 in (x_4, x_5) such that $g''(x_6) = 0$.

37. $f(x)$ is a polynomial function so it is continuous on $[0, 4]$ and $f''(x)$ exists for all x on $(0, 4)$.
$f(1) = f(2) = f(3) = 0$, so by Problem 36, there are at least two values of x in $[0, 4]$ where $f'(x) = 0$ and at least one value of x in $[0, 4]$ where $f''(x) = 0$.

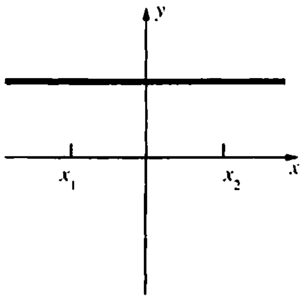
38. By applying the Mean Value Theorem and taking the absolute value of both sides,
$$\frac{|f(x_2)-f(x_1)|}{|x_2-x_1|} = |f'(c)|, \text{ for some } c \text{ in } (x_1, x_2).$$
Since $|f'(x)| \le M$ for all x in (a, b),
$$\frac{|f(x_2)-f(x_1)|}{|x_2-x_1|} \le M; |f(x_2)-f(x_1)| \le M|x_2-x_1|.$$

39. $f'(x) = 2\cos 2x; |f'(x)| \le 2$
$$\frac{|f(x_2)-f(x_1)|}{|x_2-x_1|} = |f'(x)|; \frac{|f(x_2)-f(x_1)|}{|x_2-x_1|} \le 2$$
$|f(x_2)-f(x_1)| \le 2|x_2-x_1|;$
$|\sin 2x_2 - \sin 2x_1| \le 2|x_2-x_1|$

40. a.



b.



41. Suppose $f'(x) \ge 0$. Let a and b lie in the interior of I such that $b > a$. By the Mean Value Theorem, there is a point c between a and b such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}; \frac{f(b)-f(a)}{b-a} \ge 0.$$
Since $a < b, f(b) \ge f(a)$, so f is nondecreasing.
Suppose $f'(x) \le 0$. Let a and b lie in the interior of I such that $b > a$. By the Mean Value Theorem, there is a point c between a and b such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}; \frac{f(b)-f(a)}{b-a} \le 0.$$ Since $a < b, f(a) \ge f(b)$, so f is nonincreasing.

42. $[f^2(x)]' = 2f(x)f'(x)$
Because $f(x) \ge 0$ and $f'(x) \ge 0$ on I, $[f^2(x)]' \ge 0$ on I.
As a consequence of the Mean Value Theorem,
$f^2(x_2) - f^2(x_1) \ge 0$ for all $x_2 > x_1$ on I.
Therefore f^2 is nondecreasing.

43. Let $f(x) = h(x) - g(x)$.
$f'(x) = h'(x) - g'(x); f'(x) \ge 0$ for all x in (a, b) since $g'(x) \le h'(x)$ for all x in (a, b), so f is nondecreasing on (a, b) by Problem 41. Thus
$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2);$
$h(x_1) - g(x_1) \le h(x_2) - g(x_2);$
$g(x_2) - g(x_1) \le h(x_2) - h(x_1)$ for all x_1 and x_2 in (a, b).

44. Let $f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$. Apply the Mean Value Theorem to f on the interval $[x, x+2]$ for $x > 0$.
Thus $\sqrt{x+2} - \sqrt{x} = \frac{1}{2\sqrt{c}}(2) = \frac{1}{\sqrt{c}}$ for some c in $(x, x+2)$.
Observe $\frac{1}{\sqrt{x+2}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{x}}$.
Thus as $x \to \infty, \frac{1}{\sqrt{c}} \to 0$.
Therefore $\lim_{x\to\infty}\left(\sqrt{x+2} - \sqrt{x}\right) = \lim_{x\to\infty}\frac{1}{\sqrt{c}} = 0$.

45. Let $f(x) = \sin x$. $f'(x) = \cos x$, so $|f'(x)| = |\cos x| \le 1$ for all x.
By the Mean Value Theorem,
$$\frac{f(x)-f(y)}{x-y} = f'(c) \text{ for some } c \text{ in } (x, y).$$
Thus, $\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)| \le 1;$
$|\sin x - \sin y| \le |x-y|$.

46. Let d be the difference in distance between horse A and horse B as a function of time t.
Then d' is the difference in speeds.
Let t_0 and t_1 and be the start and finish times of the race.
$d(t_0) = d(t_1) = 0$
By the Mean Value Theorem,
$$\frac{d(t_1)-d(t_0)}{t_1-t_0} = d'(c) \text{ for some } c \text{ in } (t_0, t_1).$$
Therefore $d'(c) = 0$ for some c in (t_0, t_1).

47. Let s be the difference in speeds between horse A and horse B as function of time t.
Then s' is the difference in accelerations.
Let t_2 be the time in Problem 46 at which the horses had the same speeds and let t_1 be the finish time of the race.
$s(t_2) = s(t_1) = 0$
By the Mean Value Theorem,
$\frac{s(t_1) - s(t_2)}{t_1 - t_2} = s'(c)$ for some c in (t_2, t_1).
Therefore $s'(c) = 0$ for some c in (t_2, t_1).

48. Suppose $x > c$. Then by the Mean Value Theorem,
$f(x) - f(c) = f'(a)(x - c)$ for some a in (c, x).
Since f is concave up, $f'' > 0$ and by the Monotonicity Theorem f' is increasing.
Therefore $f'(a) > f'(c)$ and
$f(x) - f(c) = f'(a)(x - c) > f'(c)(x - c)$
$f(x) > f(c) + f'(c)(x - c), x > c$
Suppose $x < c$. Then by the Mean Value Theorem,
$f(c) - f(x) = f'(a)(c - x)$ for some a in (x, c).
Since f is concave up, $f'' > 0$, and by the Monotonicity Theorem f' is increasing.
Therefore, $f'(c) > f'(a)$ and
$f(c) - f(x) = f'(a)(c - x) < f'(c)(c - x)$.
$-f(x) < -f(c) + f'(c)(c - x)$
$f(x) > f(c) - f'(c)(c - x)$
$f(x) > f(c) + f'(c)(x - c), x < c$
Therefore $f(x) > f(c) + f'(c)(x - c), x \neq c$.

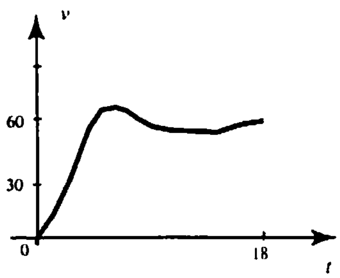
49. If $|f(y) - f(x)| \leq M(y - x)^2$, then
$\frac{|f(y) - f(x)|}{|y - x|} \leq M|y - x|$. By the Mean Value Theorem, there is some c between x and y such that $\frac{|f(y) - f(x)|}{|y - x|} = |f'(c)| \leq M|y - x|$. Since the given inequality is true for all x and y, we can choose x and y so close together that $M|y - x|$ is arbitrarily close to 0. Thus, in order for the last inequality to be true, $f'(c)$ must be 0, hence $f(x)$ is a constant function.

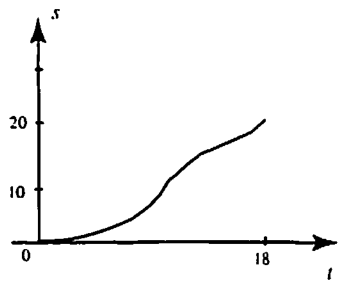
50. $f(x) = x^{1/3}$ on $[0, a]$ or $[-a, 0]$ where a is any positive number. $f'(0)$ does not exist, but $f(x)$ has a vertical tangent line at $x = 0$.

51. Let $f(t)$ be the distance traveled at time t.
$\frac{f(2) - f(0)}{2 - 0} = \frac{112 - 0}{2} = 56$
By the Mean Value Theorem, there is a time c such that $f'(c) = 56$.
At some time during the trip, Johnny must have gone 56 miles per hour.

52. s is differentiable with $s(0) = 0$ and $s(18) = 20$ so we can apply the Mean Value Theorem. There exists a c in the interval $(0, 18)$ such that

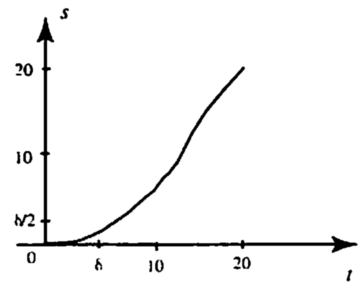
$$v(c) = s'(c) = \frac{(20 - 0)}{(18 - 0)}$$
≈ 1.11 miles per minute
≈ 66.67 miles per hour





53. Since the car is stationary at $t = 0$, and since v is continuous, there exists a δ such that $v(t) < \frac{1}{2}$ for all t in the interval $[0, \delta]$. $v(t)$ is therefore less than $\frac{1}{2}$ and $s(\delta) < \delta \cdot \frac{1}{2} = \frac{\delta}{2}$. By the Mean Value Theorem, there exists a c in the interval $(\delta, 20)$ such that

$$v(c) = s'(c) = \frac{\left(20 - \frac{\delta}{2}\right)}{(20-\delta)}$$
$$> \frac{20-\delta}{20-\delta}$$
$$= 1 \text{ mile per minute}$$
$$= 60 \text{ miles per hour}$$



4.8 Chapter Review

Concepts Test

1. True: Max-Min Existence Theorem

2. True: Since c is an interior point and f is differentiable ($f'(c)$ exists), by the Critical Point Theorem, c is a stationary point ($f'(c) = 0$).

3. True: For example, let $f(x) = \sin x$.

4. False: $f(x) = x^{1/3}$ is continuous and increasing for all x, but $f'(x)$ does not exist at $x = 0$.

5. True: $f'(x) = 18x^5 + 16x^3 + 4x$; $f''(x) = 90x^4 + 48x^2 + 4$, which is greater than zero for all x.

6. False: For example, $f(x) = x^3$ is increasing on [−1, 1] but $f'(0) = 0$.

7. True: When $f'(x) > 0$, $f(x)$ is increasing.

8. False: If $f''(c) = 0$, c is a candidate, but not necessarily an inflection point. For example, if $f(x) = x^4$, $f''(0) = 0$ but $x = 0$ is not an inflection point.

9. True: $f(x) = ax^2 + bx + c$; $f'(x) = 2ax + b$; $f''(x) = 2a$

10. True: If $f(x)$ is increasing for all x in [a, b], the maximum occurs at b.

11. False: $\tan^2 x$ has a minimum value of 0. This occurs whenever $x = k\pi$ where k is an integer.

12. True: $\lim_{x\to\frac{\pi}{2}^-} (2x^3 + x) = \infty$ while $\lim_{x\to-\frac{\pi}{2}^+} (2x^3 + x) = -\infty$

13. True: $\lim_{x\to\frac{\pi}{2}} (2x^3 + x + \tan x) = \infty$ while $\lim_{x\to-\frac{\pi}{2}} (2x^3 + x + \tan x) = -\infty$.

14. False: At $x = 3$ there is a removable discontinuity.

15. True: $\lim_{x\to\infty} \frac{x^2+1}{1-x^2} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{\frac{1}{x^2}-1} = \frac{1}{-1} = -1$ and $\lim_{x\to-\infty} \frac{x^2+1}{1-x^2} = \lim_{x\to-\infty} \frac{1+\frac{1}{x^2}}{\frac{1}{x^2}-1} = \frac{1}{-1} = -1$.

16. True: $\frac{3x^2+2x+\sin x}{x} - (3x+2) = \frac{\sin x}{x}$; $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ and $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$.

17. True: The function is differentiable on (0, 2).

18. False: $f'(x) = \frac{x}{|x|}$ so $f'(0)$ does not exist.

19. False: There are two points: $x = -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$.

20. True: Let $g(x) = D$ where D is any number. Then $g'(x) = 0$ and so, by Theorem B of Section 4.7, $f(x) = g(x) + C = D + C$, which is a constant, for all x in (a, b).

21. False: For example if $f(x) = x^4$, $f'(0) = f''(0) = 0$ but f has a minimum at $x = 0$.

22. True: $\frac{dy}{dx} = \cos x; \frac{d^2y}{dx^2} = -\sin x; -\sin x = 0$ has infinitely many solutions.

23. True: $A = xy = K; \ y = \frac{K}{x}$

$P = 2x + \frac{2K}{x}; \frac{dP}{dx} = 2 - \frac{2K}{x^2}; \frac{dP}{dx} = 0$ when $x = \sqrt{K}, y = \sqrt{K}$

24. True: By the Mean Value Theorem, the derivative must be zero between each pair of distinct x-intercepts.

25. True: If $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ for $x_1 < x_2$, $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $f + g$ is increasing.

26. False: Let $f(x) = g(x) = 2x$. $f'(x) > 0$ and $g'(x) > 0$ for all x, but $f(x)g(x) = 4x^2$ is decreasing on $(-\infty, 0)$.

27. True: Since $f''(x) > 0$, $f'(x)$ is increasing for $x \geq 0$. Therefore, $f'(x) > 0$ for x in $[0, \infty)$, so $f(x)$ is increasing.

28. False: If $f(3) = 4$, the Mean Value Theorem requires that at some point c in $[0, 3]$, $f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{4 - 1}{3 - 0} = 1$ which does not contradict that $f'(x) \leq 2$ for all x in $[0, 3]$.

29. True: If the function is nondecreasing, $f'(x)$ must be greater than or equal to zero, and if $f'(x) \geq 0$, f is nondecreasing. This can be seen using the Mean Value Theorem.

30. True: However, if the constant is 0, the functions are the same.

31. False: For example, let $f(x) = e^x$. $\lim_{x \to -\infty} e^x = 0$, so $y = 0$ is a horizontal asymptote.

32. True: If $f(c)$ is a global maximum then $f(c)$ is the maximum value of f on $(a, b) \leftrightarrow S$ where (a, b) is any interval containing c and S is the domain of f. Hence, $f(c)$ is a local maximum value.

33. True: $f'(x) = 3ax^2 + 2bx + c; \ f'(x) = 0$ when $x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$ by the Quadratic Formula. $f''(x) = 6ax + 2b$ so

$$f''\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) = \pm 2\sqrt{b^2 - 3ac}.$$

Thus, if $b^2 - 3ac > 0$, one critical point is a local maximum and the other is a local minimum. (If $b^2 - 3ac = 0$ the only critical point is an inflection point while if $b^2 - 3ac < 0$ there are no critical points.) On an open interval, no local maxima can come from endpoints, so there can be at most one local maximum in an open interval.

34. True: $f'(x) = a \neq 0$ so $f(x)$ has no local minima or maxima. On an open interval, no local minima or maxima can come from endpoints, so $f(x)$ has no local minima.

Sample Test Problems

1. $f'(x) = 2x - 2; \ 2x - 2 = 0$ when $x = 1$.
Critical points: 0, 1, 4
$f(0) = 0, f(1) = -1, f(4) = 8$
Global minimum $f(1) = -1$;
global maximum $f(4) = 8$

2. $f'(t) = -\frac{1}{t^2}; -\frac{1}{t^2}$ is never 0.
Critical points: 1, 4
$f(1) = 1, \ f(4) = \frac{1}{4}$
Global minimum $f(4) = \frac{1}{4}$;
global maximum $f(1) = 1$.

3. $f'(z) = -\frac{2}{z^3}; -\frac{2}{z^3}$ is never 0.

Critical points: $-2, -\frac{1}{2}$

$f(-2) = \frac{1}{4}, f\left(-\frac{1}{2}\right) = 4$

Global minimum $f(-2) = \frac{1}{4}$;

global maximum $f\left(-\frac{1}{2}\right) = 4$.

4. $f'(x) = -\frac{2}{x^3}; -\frac{2}{x^3}$ is never 0.

Critical point: –2

$f(-2) = \frac{1}{4}$

$f'(x) > 0$ for $x < 0$, so f is increasing.

Global minimum $f(-2) = \frac{1}{4}$; no global maximum.

5. $f'(x) = \frac{x}{|x|}; f'(x)$ does not exist at $x = 0$.

Critical points: $-\frac{1}{2}, 0, 1$

$f\left(-\frac{1}{2}\right) = \frac{1}{2}, f(0) = 0, f(1) = 1$

Global minimum $f(0) = 0$;
global maximum $f(1) = 1$

6. $f'(s) = 1 + \frac{s}{|s|}; f'(s)$ does not exist when $s = 0$.

For $s < 0$, $|s| = -s$ so $f(s) = s - s = 0$ and $f'(s) = 1 - 1 = 0$.

Critical points: 1 and all s in $[-1, 0]$
$f(1) = 2, f(s) = 0$ for s in $[-1, 0]$
Global minimum $f(s) = 0, -1 \le s \le 0$;
global maximum $f(1) = 2$.

7. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1); f'(x) = 0$ when $x = 0, 1$

Critical points: –2, 0, 1, 3
$f(-2) = 80, f(0) = 0, f(1) = -1, f(3) = 135$
Global minimum $f(1) = -1$;
global maximum $f(3) = 135$

8. $f'(u) = \frac{u(7u - 12)}{3(u - 2)^{2/3}}; f'(u) = 0$ when $u = 0, \frac{12}{7}$

$f'(2)$ does not exist.

Critical points: $-1, 0, \frac{12}{7}, 2, 3$

$f(-1) = \sqrt[3]{-3} \approx -1.44, f(0) = 0,$

$f\left(\frac{12}{7}\right) = \frac{144}{49}\sqrt[3]{-\frac{2}{7}} \approx -1.94, f(2) = 0, f(3) = 9$

Global minimum $f\left(\frac{12}{7}\right) \approx -1.94$;

global maximum $f(3) = 9$

9. $f'(x) = 10x^4 - 20x^3 = 10x^3(x - 2)$;
$f'(x) = 0$ when $x = 0, 2$
Critical points: –1, 0, 2, 3
$f(-1) = 0, f(0) = 7, f(2) = -9, f(3) = 88$
Global minimum $f(2) = -9$;
global maximum $f(3) = 88$

10. $f'(x) = 3(x-1)^2(x+2)^2 + 2(x-1)^3(x+2)$
$= (x-1)^2(x+2)(5x+4); f'(x) = 0$ when
$x = -2, -\frac{4}{5}, 1$

Critical points: $-2, -\frac{4}{5}, 1, 2$

$f(-2) = 0, f\left(-\frac{4}{5}\right) = -\frac{26,244}{3125} \approx -8.40,$
$f(1) = 0, f(2) = 16$

Global minimum $f\left(-\frac{4}{5}\right) \approx -8.40$;

global maximum $f(2) = 16$

11. $f'(\theta) = \cos\theta; f'(\theta) = 0$ when $\theta = \frac{\pi}{2}$ in $\left[\frac{\pi}{4}, \frac{4\pi}{3}\right]$

Critical points: $\frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{3}$

$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.71, f\left(\frac{\pi}{2}\right) = 1,$

$f\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \approx -0.87$

Global minimum $f\left(\frac{4\pi}{3}\right) \approx -0.87$;

global maximum $f\left(\frac{\pi}{2}\right) = 1$

12. $f'(\theta) = 2\sin\theta\cos\theta - \cos\theta = \cos\theta(2\sin\theta - 1)$;

$f'(\theta) = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ in $[0, \pi]$

Critical points: $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

$f(0) = 0, f\left(\frac{\pi}{6}\right) = -\frac{1}{4}, f\left(\frac{\pi}{2}\right) = 0.$

$f\left(\frac{5\pi}{6}\right) = -\frac{1}{4}$, $f(\pi) = 0$

Global minimum $f\left(\frac{\pi}{6}\right) = -\frac{1}{4}$ or $f\left(\frac{5\pi}{6}\right) = -\frac{1}{4}$;

global maximum $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 0$, or

$f(\pi) = 0$

13. $f'(x) = 3 - 2x$; $f'(x) > 0$ when $x < \frac{3}{2}$.
$f''(x) = -2$; $f''(x)$ is always negative.
$f(x)$ is increasing on $\left(-\infty, \frac{3}{2}\right]$ and concave down on $(-\infty, \infty)$.

14. $f'(x) = 9x^8$; $f'(x) > 0$ for all $x \neq 0$.
$f''(x) = 72x^7$; $f''(x) < 0$ when $x < 0$.
$f(x)$ is increasing on $(-\infty, \infty)$ and concave down on $(-\infty, 0)$.

15. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$; $f'(x) > 0$ when $x < -1$ or $x > 1$.
$f''(x) = 6x$; $f''(x) < 0$ when $x < 0$.
$f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and concave down on $(-\infty, 0)$.

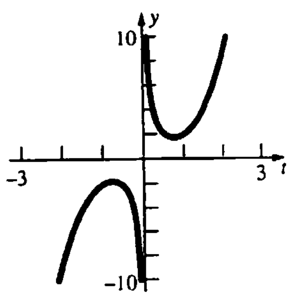
16. $f'(x) = -6x^2 - 6x + 12 = -6(x + 2)(x - 1)$;
$f'(x) > 0$ when $-2 < x < 1$.
$f''(x) = -12x - 6 = -6(2x + 1)$; $f''(x) < 0$ when $x > -\frac{1}{2}$.
$f(x)$ is increasing on $[-2, 1]$ and concave down on $\left(-\frac{1}{2}, \infty\right)$.

17. $f'(x) = 4x^3 - 20x^4 = 4x^3(1 - 5x)$; $f'(x) > 0$ when $0 < x < \frac{1}{5}$.
$f''(x) = 12x^2 - 80x^3 = 4x^2(3 - 20x)$; $f''(x) < 0$ when $x > \frac{3}{20}$.
$f(x)$ is increasing on $\left[0, \frac{1}{5}\right]$ and concave down on $\left(\frac{3}{20}, \infty\right)$.

18. $f'(x) = 3x^2 - 6x^4 = 3x^2(1 - 2x^2)$; $f'(x) > 0$ when $-\frac{1}{\sqrt{2}} < x < 0$ and $0 < x < \frac{1}{\sqrt{2}}$.
$f''(x) = 6x - 24x^3 = 6x(1 - 4x^2)$; $f''(x) < 0$ when $-\frac{1}{2} < x < 0$ or $x > \frac{1}{2}$.
$f(x)$ is increasing on $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ and concave down on $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$.

19. $f'(x) = 3x^2 - 4x^3 = x^2(3 - 4x)$; $f'(x) > 0$ when $x < \frac{3}{4}$.
$f''(x) = 6x - 12x^2 = 6x(1 - 2x)$; $f''(x) < 0$ when $x < 0$ or $x > \frac{1}{2}$.
$f(x)$ is increasing on $\left(-\infty, \frac{3}{4}\right]$ and concave down on $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$.

20. $g'(t) = 3t^2 - \frac{1}{t^2}$; $g'(t) > 0$ when $3t^2 > \frac{1}{t^2}$ or $t^4 > \frac{1}{3}$, so $t < -\frac{1}{3^{1/4}}$ or $t > \frac{1}{3^{1/4}}$.
$g'(t)$ is increasing on $\left(-\infty, -\frac{1}{3^{1/4}}\right] \cup \left[\frac{1}{3^{1/4}}, \infty\right)$ and decreasing on $\left[-\frac{1}{3^{1/4}}, 0\right) \cup \left(0, \frac{1}{3^{1/4}}\right]$.
Local minimum $g\left(\frac{1}{3^{1/4}}\right) = \frac{1}{3^{3/4}} + 3^{1/4} \approx 1.75$;
local maximum
$g\left(-\frac{1}{3^{1/4}}\right) = -\frac{1}{3^{3/4}} - 3^{1/4} \approx -1.75$
$g''(t) = 6t + \frac{2}{t^3}$; $g''(t) > 0$ when $t > 0$. $g(t)$ has no inflection point since $g(0)$ does not exist.



21. $f'(x) = 2x(x-4) + x^2 = 3x^2 - 8x = x(3x-8)$;

$f'(x) > 0$ when $x < 0$ or $x > \frac{8}{3}$

$f(x)$ is increasing on $(-\infty, 0] \cup \left[\frac{8}{3}, \infty\right)$ and

decreasing on $\left[0, \frac{8}{3}\right]$

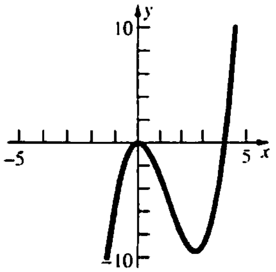
Local minimum $f\left(\frac{8}{3}\right) = -\frac{256}{27} \approx -9.48$;

local maximum $f(0) = 0$

$f''(x) = 6x - 8$; $f''(x) > 0$ when $x > \frac{4}{3}$.

$f(x)$ is concave up on $\left(\frac{4}{3}, \infty\right)$ and concave down

on $\left(-\infty, \frac{4}{3}\right)$; inflection point $\left(\frac{4}{3}, -\frac{128}{27}\right)$



22. $f'(x) = -\frac{8x}{(x^2+1)^2}$; $f'(x) = 0$ when $x = 0$.

$f''(x) = \frac{8(3x^2-1)}{(x^2+1)^3}$; $f''(0) = -8$, so $f(0) = 6$ is a

local maximum. $f'(x) > 0$ for $x < 0$ and

$f'(x) < 0$ for $x > 0$ so

$f(0) = 6$ is a global maximum value. $f(x)$ has no minimum value.

23. $f'(x) = 4x^3 - 2$; $f'(x) = 0$ when $x = \frac{1}{\sqrt[3]{2}}$.

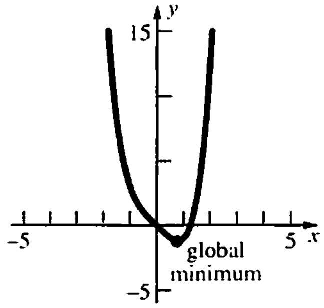
$f''(x) = 12x^2$; $f''(x) = 0$ when $x = 0$.

$f''\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{12}{2^{2/3}} > 0$, so

$f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{1}{2^{4/3}} - \frac{2}{2^{1/3}} = -\frac{3}{2^{4/3}}$ is a global minimum.

$f''(x) > 0$ for all $x \neq 0$; no inflection points

No horizontal or vertical asymptotes



24. $f'(x) = 2(x^2-1)(2x) = 4x(x^2-1) = 4x^3 - 4x$;

$f'(x) = 0$ when $x = -1, 0, 1$.

$f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$; $f''(x) = 0$ when

$x = \pm\frac{1}{\sqrt{3}}$.

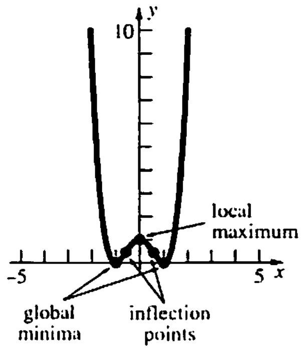
$f''(-1) = 8$, $f''(0) = -4$, $f''(1) = 8$

Global minima $f(-1) = 0$, $f(1) = 0$;

local maximum $f(0) = 1$

Inflection points $\left(\pm\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$

No horizontal or vertical asymptotes

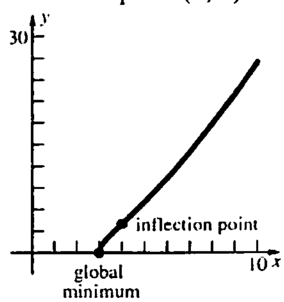


25. $f'(x) = \frac{3x-6}{2\sqrt{x-3}}$; $f'(x) = 0$ when $x = 2$, but $x = 2$ is not in the domain of $f(x)$. $f'(x)$ does not exist when $x = 3$.

$f''(x) = \frac{3(x-4)}{4(x-3)^{3/2}}$; $f''(x) = 0$ when $x = 4$.

Global minimum $f(3) = 0$; no local maxima

Inflection point (4, 4)



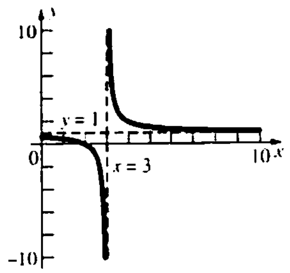
26. $f'(x) = -\frac{1}{(x-3)^2}$; $f'(x) < 0$ for all $x \neq 3$.

$f''(x) = \frac{2}{(x-3)^3}$; $f''(x) > 0$ when $x > 3$.

No local minima or maxima
No inflection points

$\lim_{x\to\infty} \frac{x-2}{x-3} = \lim_{x\to\infty} \frac{1-\frac{2}{x}}{1-\frac{3}{x}} = 1$

Horizontal asymptote $y = 1$
Vertical asymptote $x = 3$

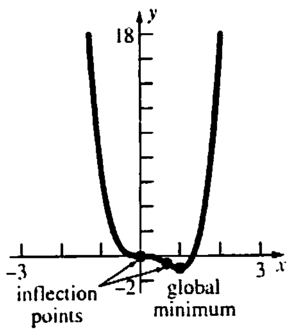


27. $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$; $f'(x) = 0$ when $x = 0, 1$.

$f''(x) = 36x^2 - 24x = 12x(3x-2)$; $f''(x) = 0$ when $x = 0, \frac{2}{3}$.

$f''(1) = 12$, so $f(1) = -1$ is a minimum.
Global minimum $f(1) = -1$; no local maxima
Inflection points $(0, 0)$, $\left(\frac{2}{3}, -\frac{16}{27}\right)$

No horizontal or vertical asymptotes.



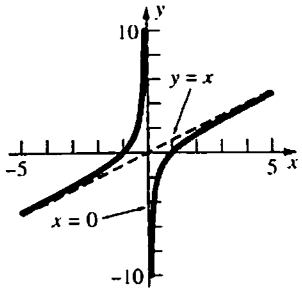
28. $f'(x) = 1 + \frac{1}{x^2}$; $f'(x) > 0$ for all $x \neq 0$.

$f''(x) = -\frac{2}{x^3}$; $f''(x) > 0$ when $x < 0$ and $f''(x) < 0$ when $x > 0$.

No local minima or maxima
No inflection points

$f(x) = x - \frac{1}{x}$, so

$\lim_{x\to\infty} [f(x) - x] = \lim_{x\to\infty} \left(-\frac{1}{x}\right) = 0$ and $y = x$ is an oblique asymptote.
Vertical asymptote $x = 0$



29. $f'(x) = 3 + \frac{1}{x^2}$; $f'(x) > 0$ for all $x \neq 0$.

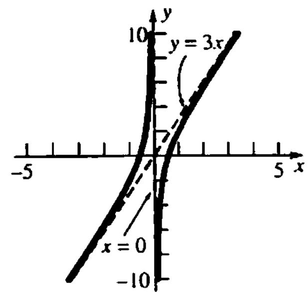
$f''(x) = -\frac{2}{x^3}$; $f''(x) > 0$ when $x < 0$ and

$f''(x) < 0$ when $x > 0$

No local minima or maxima
No inflection points

$f(x) = 3x - \frac{1}{x}$, so

$\lim_{x\to\infty} [f(x) - 3x] = \lim_{x\to\infty} \left(-\frac{1}{x}\right) = 0$ and $y = 3x$ is an oblique asymptote.
Vertical asymptote $x = 0$



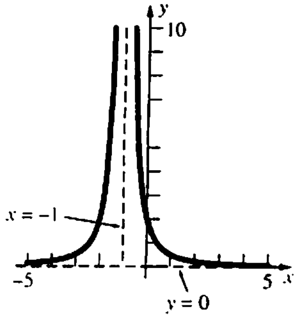
30. $f'(x) = -\frac{4}{(x+1)^3}$; $f'(x) > 0$ when $x < -1$ and

$f'(x) < 0$ when $x > -1$.

$f''(x) = \frac{12}{(x+1)^4}$; $f''(x) > 0$ for all $x \neq -1$.

No local minima or maxima
No inflection points
$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

Vertical asymptote $x = -1$



31. $f'(x) = -\sin x - \cos x$; $f'(x) = 0$ when $x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

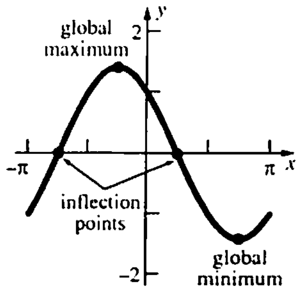
$f''(x) = -\cos x + \sin x$; $f''(x) = 0$ when $x = -\frac{3\pi}{4}, \frac{\pi}{4}$.

$f''\left(-\frac{\pi}{4}\right) = -\sqrt{2}, f''\left(\frac{3\pi}{4}\right) = \sqrt{2}$

Global minimum $f\left(\frac{3\pi}{4}\right) = -\sqrt{2}$;

global maximum $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$

Inflection points $\left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



32. $f'(x) = \cos x - \sec^2 x$; $f'(x) = 0$ when $x = 0$

$f''(x) = -\sin x - 2\sec^2 x \tan x$

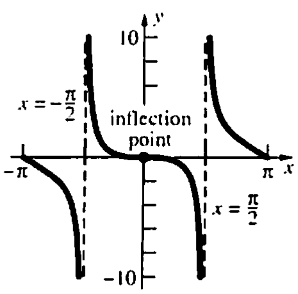
$= -\sin x(1 + 2\sec^3 x)$

$f''(x) = 0$ when $x = 0$

No local minima or maxima

Inflection point $f(0) = 0$

Vertical asymptotes $x = -\frac{\pi}{2}, \frac{\pi}{2}$

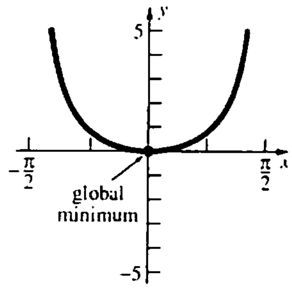


33. $f'(x) = x\sec^2 x + \tan x$; $f'(x) = 0$ when $x = 0$

$f''(x) = 2\sec^2 x(1 + x\tan x)$; $f''(x)$ is never 0 on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$f''(0) = 2$

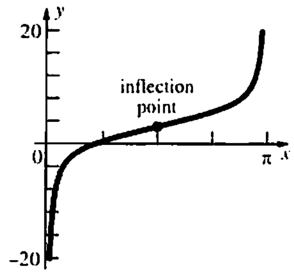
Global minimum $f(0) = 0$



34. $f'(x) = 2 + \csc^2 x$; $f'(x) > 0$ on $(0, \pi)$

$f''(x) = -2\cot x \csc^2 x$; $f''(x) = 0$ when $x = \frac{\pi}{2}$; $f''(x) > 0$ on $\left(\frac{\pi}{2}, \pi\right)$

Inflection point $\left(\frac{\pi}{2}, \pi\right)$



35. $f'(x) = \cos x - 2\cos x \sin x = \cos x(1 - 2\sin x)$;

$f'(x) = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

$f''(x) = -\sin x + 2\sin^2 x - 2\cos^2 x$; $f''(x) = 0$ when $x \approx -2.51, -0.63, 1.00, 2.14$

$f''\left(-\frac{\pi}{2}\right) = 3, f''\left(\frac{\pi}{6}\right) = -\frac{3}{2}, f''\left(\frac{\pi}{2}\right) = 1,$

$f''\left(\frac{5\pi}{6}\right) = -\frac{3}{2}$

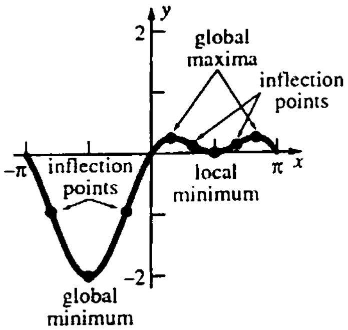
Global minimum $f\left(-\frac{\pi}{2}\right) = -2$,

local minimum $f\left(\frac{\pi}{2}\right) = 0$;

global maxima $f\left(\frac{\pi}{6}\right) = \frac{1}{4}, f\left(\frac{5\pi}{6}\right) = \frac{1}{4}$

Inflection points $(-2.51, -0.94)$,

(–0.63, –0.94), (1.00, 0.13), (2.14, 0.13)



36. $f'(x) = -2\sin x - 2\cos x$; $f'(x) = 0$ when

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

$f''(x) = -2\cos x + 2\sin x$; $f''(x) = 0$ when

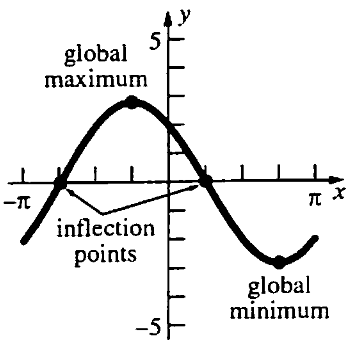
$x = -\frac{3\pi}{4}, \frac{\pi}{4}$.

$f''\left(-\frac{\pi}{4}\right) = -2\sqrt{2}, f''\left(\frac{3\pi}{4}\right) = 2\sqrt{2}$

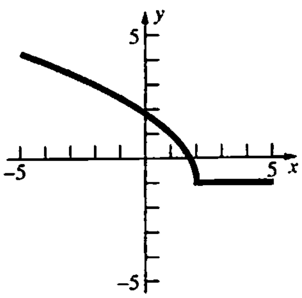
Global minimum $f\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$;

global maximum $f\left(-\frac{\pi}{4}\right) = 2\sqrt{2}$

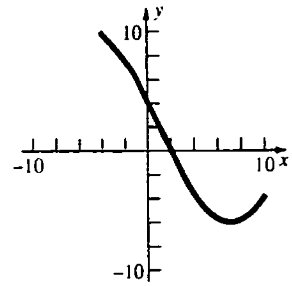
Inflection points $\left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



37.



38.



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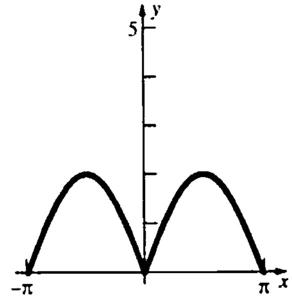


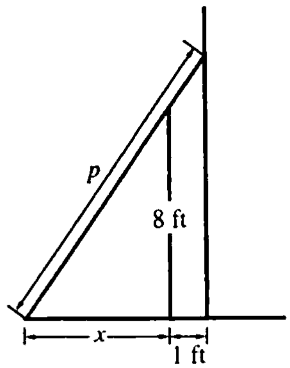
40. Let x be the length of a turned up side and let l be the (fixed) length of the sheet of metal.

$V = x(16 - 2x)l = 16xl - 2x^2l$

$\frac{dV}{dx} = 16l - 4xl$; $V' = 0$ when $x = 4$

$\frac{d^2V}{dx^2} = -4l$; 4 inches should be turned up for each side.

41. Let p be the length of the plank and let x be the distance from the fence to where the plank touches the ground.
See the figure below.



By properties of similar triangles,

$$\frac{p}{x+1} = \frac{\sqrt{x^2+64}}{x}$$

$$p = \left(1 + \frac{1}{x}\right)\sqrt{x^2+64}$$

Minimize p:

$$\frac{dp}{dx} = -\frac{1}{x^2}\sqrt{x^2+64} + \left(1 + \frac{1}{x}\right)\frac{x}{\sqrt{x^2+64}}$$

$= \frac{1}{x^2\sqrt{x^2+64}}\left(-(x^2+64)+\left(1+\frac{1}{x}\right)x^3\right)$

$= \frac{x^3-64}{x^2\sqrt{x^2+64}}$

$\frac{x^3-64}{x^2\sqrt{x^2+64}} = 0; x = 4$

$\frac{dp}{dx} < 0$ if $x < 4$, $\frac{dp}{dx} > 0$ if $x > 4$

When $x = 4$, $p = \left(1+\frac{1}{4}\right)\sqrt{16+64} \approx 11.18$ ft.

42. Let x be the width and y the height of a page. $A = xy$. Because of the margins,

$(y-4)(x-3) = 27$ or $y = \frac{27}{x-3}+4$

$A = \frac{27x}{x-3}+4x;$

$\frac{dA}{dx} = \frac{(x-3)(27)-27x}{(x-3)^2}+4 = -\frac{81}{(x-3)^2}+4$

$\frac{dA}{dx} = 0$ when $x = -\frac{3}{2}, \frac{15}{2}$

$\frac{d^2A}{dx^2} = \frac{162}{(x-3)^3}$; $\frac{d^2A}{dx^2} > 0$ when $x = \frac{15}{2}$

$x = \frac{15}{2}; y = 10$

43. $\frac{1}{2}\pi r^2 h = 128\pi$

$h = \frac{256}{r^2}$

Let S be the surface area of the trough.

$S = \pi r^2 + \pi r h = \pi r^2 + \frac{256\pi}{r}$

$\frac{dS}{dr} = 2\pi r - \frac{256\pi}{r^2}$

$2\pi r - \frac{256\pi}{r^2} = 0; r^3 = 128, r = 4\sqrt[3]{2}$

Since $\frac{d^2S}{dr^2} > 0$ when $r = 4\sqrt[3]{2}$, $r = 4\sqrt[3]{2}$ minimizes S.

$h = \frac{256}{\left(4\sqrt[3]{2}\right)^2} = 8\sqrt[3]{2}$

44. $f'(x) = \begin{cases} \frac{x}{2}+\frac{3}{2} & \text{if } -2 < x < 0 \\ -\frac{x+2}{3} & \text{if } 0 < x < 2 \end{cases}$

$\frac{x}{2}+\frac{3}{2} = 0; x = -3$, which is not in the domain.

$-\frac{x+2}{3} = 0; x = -2$, which is not in the domain.

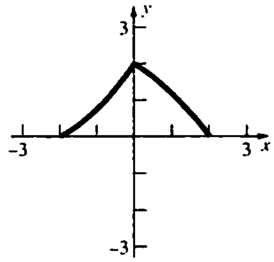
Critical points: $x = -2, 0, 2$

$f(-2) = 0, f(0) = 2, f(2) = 0$

Minima $f(-2) = 0, f(2) = 0$, maximum $f(0) = 2$.

$f''(x) = \begin{cases} \frac{1}{2} & \text{if } -2 < x < 0 \\ -\frac{1}{3} & \text{if } 0 < x < 2 \end{cases}$

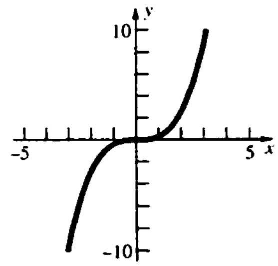
Concave up on $(-2, 0)$, concave down on $(0, 2)$



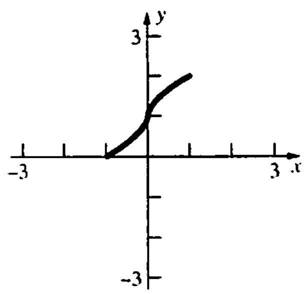
45. a. $f'(x) = x^2$

$\frac{f(3)-f(-3)}{3-(-3)} = \frac{9+9}{6} = 3$

$c^2 = 3; c = -\sqrt{3}, \sqrt{3}$



b. The Mean Value Theorem does not apply because $F'(0)$ does not exist.

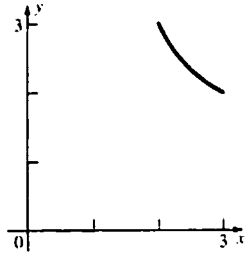


c. $g'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$\frac{g(3)-g(2)}{3-2} = \frac{2-3}{1} = -1$

$\frac{-2}{(c-1)^2} = -1; c = 1 \pm \sqrt{2}$

Only $c = 1+\sqrt{2}$ is in the interval (2, 3).



46. $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 3$

$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24; 12(x^2 - 3x + 2) = 0$ when $x = 1, 2$

Inflection points: $x = 1, y = 5$ and $x = 2, y = 11$

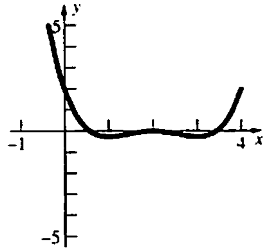
Slope at $x = 1$: $\left.\frac{dy}{dx}\right|_{x=1} = 7$

Tangent line: $y - 5 = 7(x - 1)$; $y = 7x - 2$

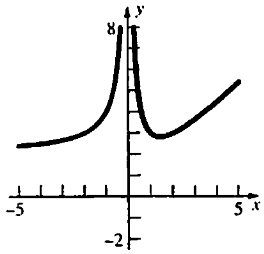
Slope at $x = 2$: $\left.\frac{dy}{dx}\right|_{x=2} = 5$

Tangent line: $y - 11 = 5(x - 2)$; $y = 5x + 1$

47.



48.



4.9 Additional Problem Set

1. a. $ab \le \left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4}$

$= \frac{a^2}{4} + \frac{1}{2}ab + \frac{b^2}{4}$

This is true if

$0 \le \frac{a^2}{4} - \frac{1}{2}ab + \frac{b^2}{4} = \left(\frac{a}{2} - \frac{b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2$

Since a square can never be negative, this is always true.

b. $F(b) = \frac{a^2+2ab+b^2}{4b}$

As $b \to 0^+$, $a^2 + 2ab + b^2 \to a^2$ while $4b \to 0^+$, thus $\lim_{b\to 0^+} F(b) = \infty$ which is not close to a.

$\lim_{b\to\infty} \frac{a^2+2ab+b^2}{4b} = \lim_{b\to\infty} \frac{\frac{a^2}{b}+2a+b}{4} = \infty$,

so when b is very large, $F(b)$ is not close to a.

$F'(b) = \frac{2(a+b)(4b) - 4(a+b)^2}{16b^2}$

$= \frac{4b^2 - 4a^2}{16b^2} = \frac{b^2 - a^2}{4b^2}$;

$F'(b) = 0$ when $b^2 = a^2$ or $b = a$ since a and b are both positive.

$F(a) = \frac{(a+a)^2}{4a} = \frac{4a^2}{4a} = a$

Thus $a \le \frac{(a+b)^2}{4b}$ for all $b > 0$ or

$ab \le \frac{(a+b)^2}{4}$ which leads to $\sqrt{ab} \le \frac{a+b}{2}$.

c. Let $F(b) = \frac{1}{b}\left(\frac{a+b+c}{3}\right)^3 = \frac{(a+b+c)^3}{27b}$

$F'(b) = \frac{3(a+b+c)^2(27b) - 27(a+b+c)^3}{27^2 b^2}$

$$= \frac{(a+b+c)^2[3b-(a+b+c)]}{27b^2}$$
$$= \frac{(a+b+c)^2(2b-a-c)}{27b^2};$$
$F'(b) = 0$ when $b = \frac{a+c}{2}$.
$$F\left(\frac{a+c}{2}\right) = \frac{2}{a+c}\cdot\left(\frac{a+c}{3}+\frac{a+c}{6}\right)^3$$
$$= \frac{2}{a+c}\left(\frac{3(a+c)}{6}\right)^3 = \frac{2}{a+c}\left(\frac{a+c}{2}\right)^3 = \left(\frac{a+c}{2}\right)^2$$
Thus $\left(\frac{a+c}{2}\right)^2 \le \frac{1}{b}\left(\frac{a+b+c}{3}\right)^3$ for all $b > 0$.

From Problem 1b, $ac \le \left(\frac{a+c}{2}\right)^2$, thus
$$ac \le \frac{1}{b}\left(\frac{a+b+c}{3}\right)^3 \text{ or } abc \le \left(\frac{a+b+c}{3}\right)^3$$
which gives the desired result
$$(abc)^{1/3} \le \frac{a+b+c}{3}.$$

2. Let $a = lw$, $b = lh$, and $c = hw$, then $S = 2(a + b + c)$ while $V^2 = abc$. By problem 1c, $(abc)^{1/3} \le \frac{a+b+c}{3}$ so $(V^2)^{1/3} \le \frac{2(a+b+c)}{2\cdot 3} = \frac{S}{6}$. In problem 1c, the minimum occurs, hence equality holds, when $b = \frac{a+c}{2}$. In the result used from Problem 1b, equality holds when $c = a$, thus $b = \frac{a+a}{2} = a$, so $a = b = c$. For the boxes, this means $l = w = h$, so the box is a cube.

3. (Proof by induction.)
Suppose that the result holds for all m with $1 \le m \le N$. Now let $n = N + 1$, and
$$F(x_1) = \frac{1}{x_1}\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n = \frac{(x_1+x_2+\ldots+x_n)^n}{n^n x_1}$$
$$F'(x_1) = \frac{n(x_1+x_2+\ldots+x_n)^{n-1}(n^n x_1) - n^n(x_1+x_2+\ldots+x_n)^n}{n^n x_1} = \frac{(x_1+x_2+\ldots+x_n)^{n-1}[nx_1-(x_1+x_2+\ldots+x_n)]}{x_1}$$
$$= \frac{(x_1+x_2+\ldots+x_n)^{n-1}[(n-1)x_1 - x_2 - \ldots - x_n]}{x_1}$$
$F'(x_1) = 0$ when $x_1 = \frac{x_2+x_3+\ldots+x_n}{n-1}$
$$F\left(\frac{x_2+x_3+\ldots+x_n}{n-1}\right) = \frac{n-1}{x_2+x_3+\ldots+x_n}\left(\frac{x_2+x_3+\ldots+x_n}{n(n-1)}+\frac{x_2+x_3+\ldots+x_n}{n}\right)^n$$
$$= \frac{n-1}{x_2+x_3+\ldots+x_n}\left(\frac{x_2+x_3+\ldots+x_n}{n-1}\right)^n = \left(\frac{x_2+x_3+\ldots x_n}{n-1}\right)^{n-1}$$
Thus, $\left(\frac{x_2+x_3+\ldots+x_n}{n-1}\right)^{n-1} \le \frac{1}{x_1}\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n$ for all $x_1 > 0$.

But, by assmption, $(x_2x_3\ldots x_n)^{\frac{1}{n-1}} \le \frac{x_2+x_3+\ldots+x_n}{n-1}$ with equality holding when $x_2 = x_3 = \ldots = x_n$.

Thus, $x_2x_3\ldots x_n \le \left(\frac{x_2+x_3+\ldots+x_3}{n-1}\right)^{n-1} \le \frac{1}{x_1}\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n$, and combining the first and last terms

$(x_1x_2\ldots x_n)^{\frac{1}{n}} \le \frac{x_1+x_2+\ldots+x_n}{n}$ with equality holding when $x_2 = x_3 = \ldots = x_n$ and

$x_1 = \frac{x_2+x_3+\ldots+x_n}{n-1} = \frac{(n-1)x_2}{n-1} = x_2$ or $x_1 = x_2 = \ldots = x_n$.

The case where $n = 1$ is clear, $n = 2$ and $n = 3$ were proved in Problems 1b and 1c, so the relation holds for any set of positive x_j.

4. a. Square both sides of the given inequality to get $\left(\frac{x+y}{2}\right)^2 \le \frac{x^2+y^2}{2}$

$\frac{x^2}{4}+\frac{xy}{2}+\frac{y^2}{4} \le \frac{x^2}{2}+\frac{y^2}{2}$

$0 \le \frac{x^2}{4}-\frac{xy}{2}+\frac{y^2}{4} = \left(\frac{x-y}{2}\right)^2$

which is always true since squares are never negative.

b. From Problem 4a, $\frac{x+y}{2} \le \sqrt{\frac{x^2+y^2}{2}}$ so it remains to show that $\frac{2xy}{x+y} \le \frac{x+y}{2}$. This is equivalent to $4xy \le (x+y)^2 = x^2+2xy+y^2$ or $0 \le x^2-2xy+y^2 = (x-y)^2$ which is true since squares are never negative.

5. $P_{ns}(p) = \frac{n!}{s!(n-s)!}p^s(1-p)^{n-s}$

$$P_{ns}'(p) = \frac{n!}{s!(n-s)!}[sp^{s-1}(1-p)^{n-s} - (n-s)p^s(1-p)^{n-s-1}]$$

$$= \frac{n!}{s!(n-s)!}p^{s-1}(1-p)^{n-s-1}[s(1-p)-(n-s)p]$$

$$= \frac{n!}{s!(n-s)!}p^{s-1}(1-p)^{n-s-1}(s-np)$$

$P_{ns}'(p) = 0$ when $p = 0, \frac{s}{n}, 1$

$P_{ns}(0) = P_{ns}(1) = 0$

$$P_{ns}\left(\frac{s}{n}\right) = \frac{n!}{s!(n-s)!}\left(\frac{s}{n}\right)^s\left(1-\frac{s}{n}\right)^{n-s} > 0,$$

thus P_{ns} is a maximum when $p = \frac{s}{n}$.

6. The ellipse has equation

$$y = \pm\sqrt{b^2 - \frac{b^2x^2}{a^2}} = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2-x^2}\right)$ be the upper right-hand corner of the rectangle (use a and b positive). Then the dimensions of the rectangle are $2x$ by $\frac{2b}{a}\sqrt{a^2-x^2}$ and the area is

$$A(x) = \frac{4bx}{a}\sqrt{a^2-x^2}.$$

$$A'(x) = \frac{4b}{a}\sqrt{a^2-x^2} - \frac{4bx^2}{a\sqrt{a^2-x^2}} = \frac{4b(a^2-2x^2)}{a\sqrt{a^2-x^2}};$$

$A'(x) = 0$ when $x = \frac{a}{\sqrt{2}}$, so the corner is at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. The corners of the rectangle are at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right), \left(\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right)$.

7. If the end of the cylinder has radius r and h is the height of the cylinder, the surface area is

$A = 2\pi r^2 + 2\pi rh$ so $h = \frac{A}{2\pi r} - r$.

The volume is

$$V = \pi r^2 h = \pi r^2\left(\frac{A}{2\pi r} - r\right) = \frac{Ar}{2} - \pi r^3.$$

$V'(r) = \frac{A}{2} - 3\pi r^2$; $V'(r) = 0$ when $r = \sqrt{\frac{A}{6\pi}}$,

$V''(r) = -6\pi r$, so the volume is maximum when $r = \sqrt{\frac{A}{6\pi}}$.

$$h = \frac{A}{2\pi r} - r = 2\sqrt{\frac{A}{6\pi}} = 2r$$

8. If the rectangle has length l and width w, the diagonal is $d = \sqrt{l^2+w^2}$, so $l = \sqrt{d^2-w^2}$. The area is $A = lw = w\sqrt{d^2-w^2}$.

$$A'(w) = \sqrt{d^2-w^2} - \frac{w^2}{\sqrt{d^2-w^2}} = \frac{d^2-2w^2}{\sqrt{d^2-w^2}};$$

$A'(w) = 0$ when $w = \frac{d}{\sqrt{2}}$ and so

$l = \sqrt{d^2 - \frac{d^2}{2}} = \frac{d}{\sqrt{2}}$. $A'(w) > 0$ on $\left(0, \frac{d}{\sqrt{2}}\right)$ and $A'(w) < 0$ on $\left(\frac{d}{\sqrt{2}}, d\right)$. Maximum area is for a square.