

18.1 Concepts Review

- $r^2 + a_1r + a_2 = 0$; complex conjugate roots
- $C_1e^{-x} + C_2e^x$
- $(C_1 + C_2x)e^x$
- $C_1 \cos x + C_2 \sin x$

Problem Set 18.1

- Roots are 2 and 3. General solution is
 $y = C_1e^{2x} + C_2e^{3x}$.
- Roots are -6 and 1 . General solution is
 $y = C_1e^{-6x} + C_2e^x$.
- Auxiliary equation: $r^2 + 6r - 7 = 0$,
 $(r + 7)(r - 1) = 0$ has roots $-7, 1$.
General solution: $y = C_1e^{-7x} + C_2e^x$
 $y' = -7C_1e^{-7x} + C_2e^x$
If $x = 0, y = 0, y' = 4$, then $0 = C_1 + C_2$ and
 $4 = -7C_1 + C_2$, so $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$.
Therefore, $y = \frac{e^x - e^{-7x}}{2}$.
- Roots are -2 and 5 . General solution is
 $y = C_1e^{-2x} + C_2e^{5x}$. Particular solution is
 $y = \left(\frac{12}{7}\right)e^{5x} - \left(\frac{5}{7}\right)e^{-2x}$.
- Repeated root 2 . General solution is
 $y = (C_1 + C_2x)e^{2x}$.
- Auxiliary equation:
 $r^2 + 10r + 25 = 0, (r + 5)^2 = 0$ has one repeated
root -5 .

General solution: $y = C_1e^{-5x} + C_2xe^{-5x}$ or
 $y = (C_1 + C_2x)e^{-5x}$

- Roots are $2 \pm \sqrt{3}$. General solution is
 $y = e^{2x}(C_1e^{\sqrt{3}x} + C_2e^{-\sqrt{3}x})$.
- Roots are $-3 \pm \sqrt{11}$. General solution is
 $y = e^{-3x}(C_1e^{\sqrt{11}x} + C_2e^{-\sqrt{11}x})$.
- Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.
General solution: $y = C_1 \cos 2x + C_2 \sin 2x$
If $x = 0$ and $y = 2$, then $2 = C_1$; if $x = \frac{\pi}{4}$ and
 $y = 3$, then $3 = C_2$.
Therefore, $y = 2 \cos 2x + 3 \sin 2x$.
- Roots are $\pm 3i$. General solution is
 $y = (C_1 \cos 3x + C_2 \sin 3x)$. Particular solution is
 $y = -\sin 3x - 3 \cos 3x$.
- Roots are $-1 \pm i$. General solution is
 $y = e^{-x}(C_1 \cos x + C_2 \sin x)$.
- Auxiliary equation: $r^2 + r + 1 = 0$ has roots
 $\frac{-1 \pm \sqrt{3}}{2}i$.
General solution:
 $y = C_1e^{(-1/2)x} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2e^{(-1/2)x} \sin\left(\frac{\sqrt{3}}{2}x\right)$
 $y = e^{-x/2} \left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right]$
- Roots are $0, 0, -4, 1$.
General solution is
 $y = C_1 + C_2x + C_3e^{-4x} + C_4e^x$.
- Roots are $-1, 1, \pm i$. General solution is
 $y = C_1e^{-x} + C_2e^x + C_3 \cos x + C_4 \sin x$.

15. Auxiliary equation: $r^4 + 3r^2 - 4 = 0$,
 $(r+1)(r-1)(r^2+4) = 0$ has roots $-1, 1, \pm 2i$.
 General solution:
 $y = C_1 e^{-x} + C_2 e^x + C_3 \cos 2x + C_4 \sin 2x$

16. Roots are $-2, 3, \pm i$. General solution is
 $y = C_1 e^{-2x} + C_2 e^{3x} + C_3 \cos x + C_4 \sin x$.

18. $e^u = \cosh u + \sinh u$ and $e^{-u} = \cosh u - \sinh u$. (See Problems 1 and 2, Section 7.8.)

Auxiliary equation: $r^2 - 2br - c^2 = 0$

Roots of auxiliary equation: $\frac{2b \pm \sqrt{4b^2 + 4c^2}}{2} = b \pm \sqrt{b^2 + c^2}$

General solution: $y = C_1 e^{(b+\sqrt{b^2+c^2})x} + C_2 e^{(b-\sqrt{b^2+c^2})x}$

$$= e^{bx} \left[C_1 \left(\cosh(\sqrt{b^2+c^2}x) + \sinh(\sqrt{b^2+c^2}x) \right) + C_2 \left(\cosh(\sqrt{b^2+c^2}x) - \sinh(\sqrt{b^2+c^2}x) \right) \right]$$

$$= e^{bx} \left[(C_1 + C_2) \cosh(\sqrt{b^2+c^2}x) + (C_1 - C_2) \sinh(\sqrt{b^2+c^2}x) \right]$$

$$= e^{bx} \left[D_1 \cosh(\sqrt{b^2+c^2}x) + D_2 \sinh(\sqrt{b^2+c^2}x) \right]$$

19. Repeated roots $\left(-\frac{1}{2}\right) \pm \left(\frac{\sqrt{3}}{2}\right)i$.

General solution is $y = e^{-x/2} \left[(C_1 + C_2 x) \cos\left(\frac{\sqrt{3}}{2}x\right) + (C_3 + C_4 x) \sin\left(\frac{\sqrt{3}}{2}x\right) \right]$.

20. Roots $1 \pm i$. General solution is

$$y = e^x (C_1 \cos x + C_2 \sin x)$$

$$= e^x (c \sin \gamma \cos x + c \cos \gamma \sin x) = ce^x \sin(x + \gamma)$$

21. (*) $x^2 y'' + 5xy' + 4y = 0$

Let $x = e^z$. Then $z = \ln x$;

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx} \left(\frac{dy}{dz} \frac{1}{x} \right) = \frac{dy}{dz} \frac{-1}{x^2} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx}$$

$$= \frac{dy}{dz} \frac{-1}{x^2} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{1}{x}$$

$$\left(-\frac{dy}{dz} + \frac{d^2 y}{dz^2} \right) + \left(5 \frac{dy}{dz} \right) + 4y = 0$$

(Substituting y' and y'' into (*))

$$\frac{d^2 y}{dz^2} + 4 \frac{dy}{dz} + 4y = 0$$

17. Roots are $-2, 2$. General solution is

$$y = C_1 e^{-2x} + C_2 e^{2x}$$

$$y = C_1 (\cosh 2x - \sinh 2x) + C_2 (\sinh 2x + \cosh 2x)$$

$$= (-C_1 + C_2) \sinh 2x + (C_1 + C_2) \cosh 2x$$

$$= D_1 \sinh 2x + D_2 \cosh 2x$$

Auxiliary equation: $r^2 + 4r + 4 = 0$, $(r+2)^2 = 0$
 has roots $-2, -2$.

General solution: $y = (C_1 + C_2 z) e^{-2z}$,

$$y = (C_1 + C_2 \ln x) e^{-2 \ln x}$$

$$y = (C_1 + C_2 \ln x) x^{-2}$$

22. As done in Problem 21,

$$\left[-a \left(\frac{dy}{dz} \right) + a \left(\frac{d^2 y}{dz^2} \right) \right] + b \left(\frac{dy}{dz} \right) + cy = 0.$$

Therefore, $a \left(\frac{d^2 y}{dz^2} \right) + (b-a) \left(\frac{dy}{dz} \right) + cy = 0$.

23. We need to show that $y'' + a_1 y' + a_2 y = 0$ if r_1 and r_2 are distinct real roots of the auxiliary equation.

We have,

$$y' = C_1 r_1 e^{r_1 x} + C_2 r_2 e^{r_2 x}$$

$$y'' = C_1 r_1^2 e^{r_1 x} + C_2 r_2^2 e^{r_2 x}$$

When put into the differential equation, we

obtain

$$y'' + a_1y' + a_2y = C_1r_1^2e^{r_1x} + C_2r_2^2e^{r_2x} + a_1(C_1r_1e^{r_1x} + C_2r_2e^{r_2x}) + a_2(C_1e^{r_1x} + C_2e^{r_2x}) \quad (*)$$

The solutions to the auxiliary equation are given by

$$r_1 = \frac{1}{2}(-a_1 - \sqrt{a_1^2 - 4a_2}) \text{ and}$$

$$r_2 = \frac{1}{2}(-a_1 + \sqrt{a_1^2 - 4a_2}).$$

Putting these values into (*) and simplifying yields the desired result: $y'' + a_1y' + a_2y = 0$.

24. We need to show that $y'' + a_1y' + a_2y = 0$ if $\alpha \pm \beta i$ are complex conjugate roots of the auxiliary equation. We have,

$$y' = e^{\alpha x} ((\alpha C_1 + \beta C_2) \cos(\beta x) + (\alpha C_2 - \beta C_1) \sin(\beta x))$$

$$y'' = e^{\alpha x} ((\alpha^2 C_1 - \beta^2 C_1 + 2\alpha\beta C_2) \cos(\beta x) + (\alpha^2 C_2 - \beta^2 C_2 - 2\alpha\beta C_1) \sin(\beta x))'$$

When put into the differential equation, we obtain

$$y'' + a_1y' + a_2y = e^{\alpha x} ((\alpha^2 C_1 - \beta^2 C_1 + 2\alpha\beta C_2) \cos(\beta x) + (\alpha^2 C_2 - \beta^2 C_2 - 2\alpha\beta C_1) \sin(\beta x)) + a_1e^{\alpha x} ((\alpha C_1 + \beta C_2) \cos(\beta x) + (\alpha C_2 - \beta C_1) \sin(\beta x)) + a_2(C_1e^{\alpha x} \cos(\beta x) + C_2e^{\alpha x} \sin(\beta x)) \quad (*)$$

From the solutions to the auxiliary equation, we find that

$$\alpha = \frac{-a_1}{2} \text{ and } \beta = -\frac{1}{2}i\sqrt{a_1^2 - 4a_2}.$$

Putting these values into (*) and simplifying yields the desired result: $y'' + a_1y' + a_2y = 0$.

25. a. $e^{bi} = 1 + (bi) + \frac{(bi)^2}{2!} + \frac{(bi)^3}{3!} + \frac{(bi)^4}{4!} + \frac{(bi)^5}{5!} + \dots = \left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \dots\right) + i\left(b - \frac{b^3}{3!} + \frac{b^5}{5!} - \frac{b^7}{7!} + \dots\right)$

$$= \cos(b) + i \sin(b)$$

b. $e^{a+bi} = e^a e^{bi} = e^a [\cos(b) + i \sin(b)]$

c. $D_x [e^{(\alpha+\beta i)x}] = D_x [e^{\alpha x} (\cos \beta x + i \sin \beta x)] = \alpha e^{\alpha x} (\cos \beta x + i \sin \beta x) + e^{\alpha x} (-i \beta \sin \beta x + i \beta \cos \beta x)$

$$= e^{\alpha x} [(\alpha + \beta i) \cos \beta x + (i\alpha - \beta) \sin \beta x]$$

$$(\alpha + \beta i)e^{(\alpha+\beta i)x} = (\alpha + \beta i)[e^{\alpha x} (\cos \beta x + i \sin \beta x)] = e^{\alpha x} [(\alpha + \beta i) \cos \beta x + (i\alpha - \beta) \sin \beta x]$$

$$\text{Therefore, } D_x [e^{(\alpha+\beta i)x}] = (\alpha + i\beta)e^{(\alpha+\beta i)x}$$

26. $c_1e^{(\alpha+\beta i)x} + c_2e^{(\alpha+\beta i)x}$ [c_1 and c_2 are complex constants.]

$$= c_1e^{\alpha x} [\cos \beta x + i \sin \beta x] + c_2e^{\alpha x} [\cos(-\beta x) + i \sin(-\beta x)] = e^{\alpha x} [(c_1 + c_2) \cos \beta x + (c_1 - c_2)i \sin \beta x]$$

$$= e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x], \text{ where } C_1 = c_1 + c_2, \text{ and } C_2 = c_1 - c_2.$$

Note: If c_1 and c_2 are complex conjugates, then C_1 and C_2 are real.

27. $y = 0.5e^{5.16228x} + 0.5e^{-1.162278x}$

28. $y = 3.5xe^{-2.5x} + 2e^{-2.5x}$

29. $y = 1.29099e^{-0.25x} \sin(0.968246x)$

30. $y = e^{0.333333x} [2.5 \cos(0.471405x) - 4.94975 \sin(0.471405x)]$

18.2 Concepts Review

1. particular solution to the nonhomogeneous equation; homogeneous equation
2. $-6 + C_1e^{-2x} + C_2e^{3x}$
3. $y = Ax^2 + Bx + C$
4. $y = Bxe^{\frac{1}{3}x}$

Problem Set 18.2

1. $y_h = C_1e^{-3x} + C_2e^{3x}$
 $y_p = \left(-\frac{1}{9}\right)x + 0$
 $y = \left(-\frac{1}{9}\right)x + C_1e^{-3x} + C_2e^{3x}$
2. $y_h = C_1e^{-3x} + C_2e^{2x}$
 $y_p = \left(-\frac{1}{3}\right)x^2 + \left(-\frac{1}{9}\right)x + \left(-\frac{7}{54}\right)$
 $y = \left(-\frac{1}{3}\right)x^2 - \left(\frac{1}{9}\right)x - \left(\frac{7}{54}\right) + C_1e^{-3x} + C_2e^{2x}$
3. Auxiliary equation: $r^2 - 2r + 1 = 0$ has roots 1, 1.
 $y_h = (C_1 + C_2x)e^x$
Let $y_p = Ax^2 + Bx + C$; $y'_p = 2Ax + B$;
 $y''_p = 2A$.
Then $(2A) - 2(2Ax + B) + (Ax^2 + Bx + C) = x^2 + x$.
 $Ax^2 + (-4A + B)x + (2A - 2B + C) = x^2 + x$
Thus, $A = 1$, $-4A + B = 1$, $2A - 2B + C = 0$, so
 $A = 1$, $B = 5$, $C = 8$.
General solution: $y = x^2 + 5x + 8 + (C_1 + C_2x)e^x$
4. $y_h = C_1e^{-x} + C_2 \cdot y_p = 2x^2 + (-4)x$
 $y = 2x^2 - 4x + C_1e^{-x} + C_2$
5. $y_h = C_1e^{2x} + C_2e^{3x} \cdot y_p = \left(\frac{1}{2}\right)e^x \cdot y$
 $= \left(\frac{1}{2}\right)e^x + C_1e^{2x} + C_2e^{3x}$
6. Auxiliary equation: $r^2 + 6r + 9 = 0$, $(r + 3)^2 = 0$ has roots -3 , -3 .
 $y_h = (C_1 + C_2x)e^{-3x}$
Let $y_p = Be^{-x}$; $y'_p = -Be^{-x}$; $y''_p = Be^{-x}$.

Then $(Be^{-x}) + 6(-Be^{-x}) + 9(Be^{-x}) = 2e^{-x}$; $4Be^{-x} = 2e^{-x}$; $B = \frac{1}{2}$

General solution: $y = \left(\frac{1}{2}\right)e^{-x} + (C_1 + C_2x)e^{-3x}$

7. $y_h = C_1e^{-3x} + C_2e^{-x}$

$$y_p = \left(-\frac{1}{2}\right)xe^{-3x}$$

$$y = \left(-\frac{1}{2}\right)xe^{-3x} + C_1e^{-3x} + C_2e^{-x}$$

8. $y_h = e^{-x}(C_1 \cos x + C_2 \sin x)$

$$y_p = \left(\frac{3}{2}\right)e^{-2x}$$

$$y = \left(\frac{3}{2}\right)e^{-2x} + e^{-x}(C_1 \cos x + C_2 \sin x)$$

9. Auxiliary equation: $r^2 - r - 2 = 0$,

$(r + 1)(r - 2) = 0$ has roots $-1, 2$.

$$y_h = C_1e^{-x} + C_2e^{2x}$$

Let $y_p = B \cos x + C \sin x$; $y'_p = -B \sin x + C \cos x$; $y''_p = -B \cos x - C \sin x$.

Then $(-B \cos x - C \sin x) - (-B \sin x + C \cos x)$

$$-2(B \cos x + C \sin x) = 2 \sin x.$$

$(-3B - C) \cos x + (B - 3C) \sin x = 2 \sin x$, so $-3B - C = 0$ so $-3B - C = 0$ and $B - 3C = 2$; $B = \frac{1}{5}$; $C = \frac{-3}{5}$.

General solution: $\left(\frac{1}{5}\right)\cos x - \left(\frac{3}{5}\right)\sin x + C_1e^{2x} + C_2e^{-x}$

10. $y_h = C_1e^{-4x} + C_2$

$$y_p = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x$$

$$y = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x + C_1e^{-4x} + C_2$$

11. $y_h = C_1 \cos 2x + C_2 \sin 2x$

$$y_p = (0)x \cos 2x + \left(\frac{1}{2}\right)x \sin 2x$$

$$y = \left(\frac{1}{2}\right)x \sin 2x + C_1 \cos 2x + C_2 \sin 2x$$

12. Auxiliary equation: $r^2 + 9 = 0$ has roots $\pm 3i$, so $y_h = C_1 \cos 3x + C_2 \sin 3x$.

Let $y_p = Bx \cos 3x + Cx \sin 3x$; $y'_p = (-3bx + C) \sin 3x + (B + 3Cx) \cos 3x$;

$$y''_p = (-9Bx + 6C) \cos 3x + (-9Cx - 6B) \sin 3x.$$

Then substituting into the original equation and simplifying, obtain $6C \cos 3x - 6B \sin 3x = \sin 3x$, so $C = 0$ and

$$B = -\frac{1}{6}.$$

General solution: $y = \left(-\frac{1}{6}\right)x \cos 3x + C_1 \cos 3x + C_2 \sin 3x$

13. $y_h = C_1 \cos 3x + C_2 \sin 3x$

$$y_p = (0)\cos x + \left(\frac{1}{8}\right)\sin x + \left(\frac{1}{13}\right)e^{2x}$$

$$y = \left(\frac{1}{8}\right)\sin x + \left(\frac{1}{13}\right)e^{2x} + C_1 \cos 3x + C_2 \sin 3x$$

14. $y_h = C_1 e^{-x} + C_2$

$$y_p = \left(\frac{1}{2}\right)e^x + \left(\frac{3}{2}\right)x^2 + (-3)x$$

$$y = \left(\frac{1}{2}\right)e^x + \left(\frac{3}{2}\right)x^2 - 3x + C_1 e^{-x} + C_2$$

15. Auxiliary equation: $r^2 - 5r + 6 = 0$ has roots 2 and 3, so $y_h = C_1 e^{2x} + C_2 e^{3x}$.

Let $y_p = Be^x$; $y'_p = Be^x$; $y''_p = Be^x$.

Then $(Be^x) - 5(Be^x) + 6(Be^x) = 2e^x$; $2Be^x = 2e^x$; $B = 1$.

General solution: $y = e^x + C_1 e^{2x} + C_2 e^{3x}$

$$y' = e^x + 2C_1 e^{2x} + 3C_2 e^{3x}$$

If $x = 0, y = 1, y' = 0$, then $1 = 1 + C_1 + C_2$ and $0 = 1 + 2C_1 + 3C_2$; $C_1 = 1, C_2 = -1$.

Therefore, $y = e^x + e^{2x} - e^{3x}$.

16. $y_h = C_1 e^{-2x} + C_2 e^{2x}$

$$y_p = (0)\cos x + \left(-\frac{4}{5}\right)\sin x$$

$$y = \left(-\frac{4}{5}\right)\sin x + C_1 e^{-2x} + C_2 e^{2x}$$

$$y = \left(-\frac{4}{5}\right)\sin x + \left(\frac{9}{5}\right)e^{-2x} + \left(\frac{11}{5}\right)e^{2x} \text{ satisfies the conditions.}$$

17. $y_h = C_1 e^x + C_2 e^{2x}$

$$y_p = \left(\frac{1}{4}\right)(10x + 19)$$

$$y = \left(\frac{1}{4}\right)(10x + 19) + C_1 e^x + C_2 e^{2x}$$

18. Auxiliary equation: $r^2 - 4 = 0$ has roots 2, -2, so $y_h = C_1 e^{2x} + C_2 e^{-2x}$.

Let $y_p = v_1 e^{2x} + v_2 e^{-2x}$, subject to $v'_1 e^{2x} + v'_2 e^{-2x} = 0$, and $v'_1 (2e^{2x}) + v'_2 (-2e^{-2x}) = e^{2x}$.

Then $v'_1 (4e^{2x}) = e^{2x}$ and $v'_2 (-4e^{-2x}) = e^{2x}$; $v'_1 = \frac{1}{4}$ and $v'_2 = -e^{4x/4}$; $v_1 = \frac{x}{4}$ and $v_2 = -\frac{e^{4x}}{16}$.

General solution: $y = \frac{x e^{2x}}{4} - \frac{e^{2x}}{16} + C_1 e^{2x} + C_2 e^{-2x}$

19. $y_h = C_1 \cos x + C_2 \sin x$

$$y_p = -\cos x \ln|\sin x| - \cos x - x \sin x$$

$$y = -\cos x \ln|\sin x| - x \sin x + C_3 \cos x + C_2 \sin x \text{ (combined cos x terms)}$$

20. $y_h = C_1 \cos x + C_2 \sin x$

$$y_p = -\sin x \ln|\csc x + \cot x|$$

$$y = -\sin x \ln|\csc x + \cot x| + C_1 \cos x + C_2 \sin x$$

21. Auxiliary equation: $r^2 - 3r + 2 = 0$ has roots 1, 2, so $y_h = C_1 e^x + C_2 e^{2x}$.

Let $y_p = v_1 e^x + v_2 e^{2x}$ subject to $v_1' e^x + v_2' e^{2x} = 0$, and $v_1'(e^x) + v_2'(2e^{2x}) = e^x(e^x + 1)^{-1}$.

$$\begin{aligned} \text{Then } v_1' &= \frac{-e^x}{e^x(e^x + 1)} \text{ so } v_1 = \int \frac{-e^x}{e^x(e^x + 1)} dx = \int \frac{-1}{u(u+1)} du \\ &= \int \left(\frac{-1}{u} + \frac{1}{u+1} \right) du = -\ln u + \ln(u+1) = \ln \left(\frac{u+1}{u} \right) = \ln \frac{e^x + 1}{e^x} = \ln(1 + e^{-x}) \end{aligned}$$

$$v_2' = \frac{e^x}{e^{2x}(e^x + 1)} \text{ so } v_2 = -e^{-x} + \ln(1 + e^{-x})$$

(similar to finding v_1)

General solution: $y = e^x \ln(1 + e^{-x}) - e^{-x} + e^{2x} \ln(1 + e^{-x}) + C_1 e^x + C_2 e^{2x}$

$$y = (e^x + e^{2x}) \ln(1 + e^{-x}) + D_1 e^x + D_2 e^{2x}$$

22. $y_h = C_1 e^{2x} + C_2 e^{3x}$; $y_p = e^x$

$$y = e^x + C_1 e^{2x} + C_2 e^{3x}$$

23. $L(y_p) = (v_1 u_1 + v_2 u_2)'' + b(v_1 u_1 + v_2 u_2)' + c(v_1 u_1 + v_2 u_2)$

$$= (v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2') + b(v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2') + c(v_1 u_1 + v_2 u_2)$$

$$= (v_1'' u_1 + v_1' u_1' + v_1 u_1'' + v_2'' u_2 + v_2' u_2' + v_2 u_2'' + v_2 u_2') + b(v_1' u_1 + v_1 u_1' + v_2' u_2 + v_2 u_2') + c(v_1 u_1 + v_2 u_2)$$

$$= v_1(u_1'' + b u_1' + c u_1) + v_2(u_2'' + b u_2' + c u_2) + b(v_1' u_1 + v_2' u_2) + (v_1'' u_1 + v_1' u_1' + v_2'' u_2 + v_2' u_2') + (v_1 u_1' + v_2 u_2')$$

$$= v_1(u_1'' + b u_1' + c u_1) + v_2(u_2'' + b u_2' + c u_2) + b(v_1' u_1 + v_2' u_2) + (v_1 u_1' + v_2 u_2)'$$

$$= v_1(0) + v_2(0) + b(0) + (0) + k(x) = k(x)$$

24. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

Now write $\sin^3 x$ in a form involving $\sin \beta x$'s or $\cos \beta x$'s.

$$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

(C.R.C. Standard Mathematical Tables, or derive it using half-angle and product identities.)

Let $y_p = A \sin x + B \cos x + C \sin 3x + D \cos 3x$;

$$y_p' = A \cos x - B \sin x + 3C \cos 3x - 3D \sin 3x$$

$$y_p'' = -A \sin x - B \cos x - 9C \sin 3x - 9D \cos 3x$$

Then

$$y_p'' + 4y_p = 3A \sin x + 3B \cos x - 5C \sin 3x - 5D \cos 3x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x, \text{ so}$$

$$A = \frac{1}{4}, B = 0, C = \frac{1}{20}, D = 0.$$

General solution: $y = \frac{1}{4} \sin x + \frac{1}{20} \sin 3x + C_1 \cos 2x + C_2 \sin 2x$

18.3 Concepts Review

1. $3; \pi$
2. π ; decreases
3. 0
4. electric circuit

Problem Set 18.3

1. $k = 250, m = 10, B^2 = k/m = 250/10 = 25, B = 5$

(the problem gives the mass as $m = 10 \text{ kg}$)

Thus, $y'' = -25y$. The general solution is $y = C_1 \cos 5t + C_2 \sin 5t$. Apply the initial condition to get $y = 0.1 \cos 5t$.

The period is $\frac{2\pi}{5}$ seconds.

2. $k = 100 \text{ lb/ft}, w = 1 \text{ lb}, g = 32 \text{ ft/s}^2, y_0 = \frac{1}{12} \text{ ft},$

$$B = 40\sqrt{2}. \text{ Then } y = \left(\frac{1}{12}\right) \cos(40\sqrt{2})t.$$

Amplitude is $\frac{1}{12} \text{ ft} = 1 \text{ in.}$

Period is $\frac{2\pi}{40\sqrt{2}} \approx 0.1111 \text{ s.}$

3. $y = 0.1 \cos 5t = 0$ whenever $5t = \frac{\pi}{2} + \pi k$ or $t = \frac{\pi}{10} + \frac{\pi}{5}k$.

$$\begin{aligned} \left| y' \left(\frac{\pi}{10} + \frac{\pi}{5}k \right) \right| &= 0.5 \left| \sin 5 \left(\frac{\pi}{10} + \frac{\pi}{5}k \right) \right| \\ &= 0.5 \left| \sin \left(\frac{\pi}{2} + \pi k \right) \right| \\ &= 0.5 \text{ meters per second} \end{aligned}$$

4. $|10| = k \left(\frac{1}{3} \right)$, so $k = 30 \text{ lb/ft}, w = 20 \text{ lb},$

$$g = 32 \text{ ft/s}^2, y_0 = -1 \text{ ft}, v_0 = 2 \text{ ft/s}, B = 4\sqrt{3}$$

Then $y = C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)$.

$y = \cos(4\sqrt{3}t) + \left(\frac{\sqrt{3}t}{6}\right) \sin(4\sqrt{3}t)$ satisfies the initial conditions.

5. $k = 20 \text{ lb/ft}; w = 10 \text{ lb}; y_0 = 1 \text{ ft}; q = \frac{1}{10} \text{ s-lb/ft}, B = 8, E = 0.32$

$E^2 - 4B^2 < 0$, so there is damped motion. Roots of auxiliary equation are approximately $-0.16 \pm 8i$.

General solution is $y \approx e^{-0.16t} (C_1 \cos 8t + C_2 \sin 8t)$. $y \approx e^{-0.16t} (\cos 8t + 0.02 \sin 8t)$ satisfies the initial conditions.

6. $k = 20 \text{ lb/ft}; w = 10 \text{ lb}; y_0 = 1 \text{ ft}; q = 4 \text{ s-lb/ft}$

$$B = \sqrt{\frac{(20)(32)}{10}} = 8; E = \frac{(4)(32)}{10} = 12.8; E^2 - 4B^2 < 0, \text{ so damped motion.}$$

Roots of auxiliary equation are $\frac{-E \pm \sqrt{E^2 - 4B^2}}{2} = -6.4 \pm 4.8i$.

General solution is $y = e^{-6.4t} (C_1 \cos 4.8t + C_2 \sin 4.8t)$.

$$y' = e^{-6.4t} (-4.8C_1 \sin 4.8t + 4.8C_2 \cos 4.8t) - 6.4e^{-6.4t} (C_1 \cos 4.8t + C_2 \sin 4.8t)$$

If $t = 0, y = 1, y' = 0$, then $1 = C_1$ and $0 = 4.8C_2 - 6.4C_1$, so $C_1 = 1$ and $C_2 = \frac{4}{3}$.

$$\text{Therefore, } y = e^{-6.4t} \left[\cos 4.8t + \left(\frac{4}{3}\right) \sin 4.8t \right].$$

7. Original amplitude is 1 ft. Considering the contribution of the sine term to be negligible due to the 0.02 coefficient, the amplitude is approximately $e^{-0.16t}$.

$e^{-0.16t} \approx 0.1$ if $t \approx 14.39$, so amplitude will be about one-tenth of original in about 14.4 s.

8. $C_1 = 1$ and $C_2 = -0.105$, so $y = e^{-0.16t} (\cos 8t + 0.105 \sin 8t)$.

9. $LQ'' + RQ' + \frac{Q}{C} = E(t)$; $10^6 Q'' + 10^6 Q' = 1$; $Q'' + Q' = 10^{-6}$

Integrating factor: e^t

$$D[Qe^t] = 10^{-6} e^t; Qe^t = 10^{-6} e^t + C;$$

$$Q = 10^{-6} + Ce^{-t}$$

If $t = 0, Q = 0$, then $C = -10^{-6}$.

Therefore, $Q(t) = 10^{-6} - 10^{-6} e^{-t} = 10^{-6} (1 - e^{-t})$.

10. Same as Problem 9, except $C = 4 - 10^{-6}$, so $Q(t) = 10^{-6} + (4 - 10^{-6})e^{-t}$.

Then $I(t) = Q'(t) = -(4 - 10^{-6})e^{-t}$.

11. $\frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

a. $Q(t) = 0.00024 \sin 377t$

b. $I(t) = Q'(t) = 0.09048 \cos 377t$

12. $LQ'' + RQ' + \frac{Q}{C} = E$; $10^{-2} Q'' + \frac{Q}{10^{-7}} = 20$; $Q'' + 10^9 Q = 2000$

The auxiliary equation, $r^2 + 10^9 = 0$, has roots $\pm 10^{9/2}i$.

$$Q_h = C_1 \cos 10^{9/2}t + C_2 \sin 10^{9/2}t$$

$Q_p = 2000(10^{-9}) = 2(10^{-6})$ is a particular solution (by inspection).

$$\text{General solution: } Q(t) = 2(10^{-6}) + C_1 \cos 10^{9/2}t + C_2 \sin 10^{9/2}t$$

$$\text{Then } I(t) = Q'(t) = -10^{9/2} C_1 \sin 10^{9/2}t + 10^{9/2} C_2 \cos 10^{9/2}t.$$

If $t = 0, Q = 0, I = 0$, then $0 = 2(10^{-6}) + C_1$ and $0 = C_2$.

Therefore, $I(t) = -10^{9/2}(-2[10^{-6}]) \sin 10^{9/2}t = 2(10^{-3/2}) \sin 10^{9/2}t$.

13. $3.5Q'' + 1000Q' + \frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

(Values are approximated to 6 significant figures for the remainder of the problem.)

$$Q'' + 285.714Q' + 142857Q = 34.2857 \sin 377t$$

Roots of the auxiliary equation are

$$-142.857 \pm 349.927i.$$

$$Q_h = e^{-142.857t} (C_1 \cos 349.927t + C_2 \sin 349.927t)$$

$$Q_p = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t$$

Then, $Q = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t + Q_h$.

$$I = Q' = 0.119995 \sin 377t + 0.000810998 \cos 377t + Q'_h$$

$0.000888 \cos 377t$ is small and $Q'_h \rightarrow 0$ as $t \rightarrow \infty$, so the steady-state current is $I \approx 0.12 \sin 377t$.

14. a. Roots of the auxiliary equation are $\pm Bi$.

$$y_h = C_1 \cos Bt + C_2 \sin Bt.$$

$$y_p = \left[\frac{c}{(B^2 - A^2)} \right] \sin At$$

The desired result follows.

- b. $y_p = \left(-\frac{c}{2B} \right) t \cos Bt$, so

$$y = C_1 \cos Bt + C_2 \sin Bt - \left(\frac{c}{2B} \right) t \cos Bt.$$

- c. Due to the t factor in the last term, it increases without bound.

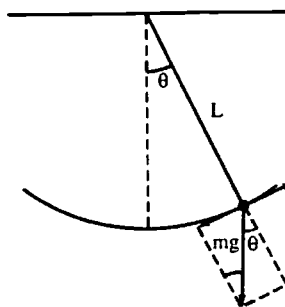
15. $A \sin(\beta t + \gamma) = A(\sin \beta t \cos \gamma + \cos \beta t \sin \gamma)$
 $= (A \cos \gamma) \sin \beta t + (A \sin \gamma) \cos \beta t$
 $= C_1 \sin \beta t + C_2 \cos \beta t$, where $C_1 = A \cos \gamma$ and
 $C_2 = A \sin \gamma$.

[Note that

$$C_1^2 + C_2^2 = A^2 \cos^2 \gamma + A^2 \sin^2 \gamma = A^2.)$$

16. The first two terms have period $\frac{2\pi}{B}$ and the last has period $\frac{2\pi}{A}$. Then the sum of the three terms is periodic if $m \left(\frac{2\pi}{B} \right) = n \left(\frac{2\pi}{A} \right)$ for some integers m, n ; equivalently, if $\frac{B}{A} = \frac{m}{n}$, a rational number.

17. The magnitudes of the tangential components of the forces acting on the pendulum bob must be equal.



Therefore, $-m \frac{d^2 s}{dt^2} = mg \sin \theta$.

$$s = L\theta, \text{ so } \frac{d^2 s}{dt^2} = L \frac{d^2 \theta}{dt^2}.$$

Therefore, $-mL \frac{d^2 \theta}{dt^2} = mg \sin \theta$.

$$\text{Hence, } \frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin \theta.$$

18. a. Since the roots of the auxiliary equation are $\pm \sqrt{\frac{g}{L}}i$, the solution of $\theta''(t) + \left(\frac{g}{L} \right) \theta = 0$ is

$$\theta = C_1 \cos \sqrt{\frac{g}{L}}t + C_2 \sin \sqrt{\frac{g}{L}}t, \text{ which can be}$$

$$\text{written as } \theta = C \left(\sqrt{\frac{g}{L}}t + \gamma \right)$$

(by Problem 15).

The period of this function is

$$\frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi \frac{L}{\sqrt{g}} = 2\pi \sqrt{\frac{LR^2}{GM}} = 2\pi R \sqrt{\frac{L}{GM}}.$$

$$\text{Therefore, } \frac{p_1}{p_2} = \frac{2\pi R_1 \sqrt{\frac{L_1}{GM}}}{2\pi R_2 \sqrt{\frac{L_2}{GM}}} = \frac{R_1 \sqrt{L_1}}{R_2 \sqrt{L_2}}.$$

- b. To keep perfect time at both places, require

$$p_1 = p_2. \text{ Then } 1 = \frac{R_2 \sqrt{80.85}}{3960 \sqrt{81}}, \text{ so}$$

$$R_2 \approx 3963.67.$$

The height of the mountain is about
 $3963.67 - 3960 = 3.67$ mi (about 19,387 ft).

18.4 Chapter Review

Concepts Test

- False: y^2 is not linear in y .
- True: y and y'' are linear in y and y'' , respectively.
- True: $y' = \sec^2 x + \sec x \tan x$
 $2y' - y^2 = (2\sec^2 x + 2\sec x \tan x) - (\tan^2 x + 2\sec x \tan x + \sec^2 x)$
 $= \sec^2 x - \tan^2 x = 1$
- False: It should involve 6.
- True: D^2 adheres to the conditions for linear operators.
 $D^2(kf) = kD^2(f)$
 $D^2(f+g) = D^2 f + D^2 g$
- False: Replacing y by $C_1 u_1(x) + C_2 u_2(x)$ would yield, on the left side,
 $C_1 f(x) + C_2 f(x) = (C_1 + C_2)f(x)$
 which is $f(x)$ only if $C_1 + C_2 = 1$ or $f(x) = 0$.
- True: -1 is a repeated root, with multiplicity 3, of the auxiliary equation.
- True: $L(u_1 - u_2) = L(u_1) - L(u_2)$
 $= f(x) - f(x) = 0$
- False: That is the form of y_h . y_p should have the form $Bx \cos 3x + Cx \sin 3x$.
- True: See Problem 15, Section 18.3.

Sample Test Problems

- $u' + 3u = e^x$. Integrating factor is e^{3x} .
 $D[ue^{3x}] = e^{4x}$

$$y = \left(\frac{1}{4}\right)e^x + C_1 e^{-3x}$$

$$y' = \left(\frac{1}{4}\right)e^x + C_1 e^{-3x}$$

$$y = \left(\frac{1}{4}\right)e^x + C_3 e^{-3x} + C_2$$

- Roots are $-1, 1$. $y = C_1 e^{-x} + C_2 e^x$
- (Second order homogeneous)
 The auxiliary equation, $r^2 - 3r + 2 = 0$, has roots 1, 2. The general solution is $y = C_1 e^x + C_2 e^{2x}$.
 $y' = C_1 e^x + 2C_2 e^{2x}$
 If $x = 0, y = 0, y' = 3$, then $0 = C_1 + C_2$ and $3 = C_1 + 2C_2$, so $C_1 = -3, C_2 = 3$.
 Therefore, $y = -3e^x + 3e^{2x}$.
- Repeated root $-\frac{3}{2}$. $y = (C_1 + C_2 x)e^{(-3/2)x}$
- $y_h = C_1 e^{-x} + C_2 e^x$ (Problem 2)
 $y_p = -1 + C_1 e^{-x} + C_2 e^x$
- (Second-order nonhomogeneous)
 The auxiliary equation, $r^2 + 4r + 4 = 0$, has roots $-2, -2$.
 $y_h = C_1 e^{-2x} + C_2 x e^{-2x} = (C_1 + C_2 x)e^{-2x}$
 Let $y_p = B e^x; y'_p = B e^x; y''_p = B e^x$.
 $(B e^x) + 4(B e^x) + 4(B e^x) = 3e^x$, so $B = \frac{1}{3}$.
 General solution: $y = \frac{e^x}{3} + (C_1 + C_2 x)e^{-2x}$
- $y_h = (C_1 + C_2 x)e^{-2x}$ (Problem 12)
 $y_p = \left(\frac{1}{2}\right)x^2 e^{-2x}$
 $y = \left[\left(\frac{1}{2}\right)x^2 + C_1 + C_2 x\right]e^{-2x}$

8. Roots are $\pm 2i$.
 $y = C_1 \cos 2x + C_2 \sin 2x$
 $y = \sin 2x$ satisfies the conditions.
9. (Second-order homogeneous)
The auxiliary equation, $r^2 + 6r + 25 = 0$, has roots $-3 \pm 4i$. General solution:
 $y = e^{-3x}(C_1 \cos 4x + C_2 \sin 4x)$
10. Roots are $\pm i$. $y_h = C_1 \cos x + C_2 \sin x$
 $y_p = x \cos x - \sin x + \sin x \ln |\cos x|$
 $y = x \cos x - \sin x \ln |\cos x| + C_1 \cos x + C_2 \sin x$
(combining the sine terms)
11. Roots are $-4, 0, 2$. $y = C_1 e^{-4x} + C_2 + C_3 e^{2x}$
12. (Fourth-order homogeneous)
The auxiliary equation, $r^4 - 3r^2 - 10 = 0$ or $(r^2 - 5)(r^2 + 2) = 0$, has roots $-\sqrt{5}, \sqrt{5}, \pm \sqrt{2}i$.
General solution:
 $y = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x} + C_3 \cos \sqrt{2}x + C_4 \sin \sqrt{2}x$
13. Repeated roots $\pm \sqrt{2}$
 $y = (C_1 + C_2 x)e^{-\sqrt{2}x} + (C_3 + C_4 x)e^{\sqrt{2}x}$
14. a. $Q'(t) = 3 - 0.02Q$
b. $Q'(t) + 0.02Q = 3$
Integrating factor is $e^{0.02t}$
 $D[Qe^{0.02t}] = 3e^{0.02t}$
- $Q(t) = 150 + Ce^{-0.02t}$
 $Q(t) = 150 - 30e^{-0.02t}$ goes through $(0, 120)$.
- c. $Q \rightarrow 150$ g, as $t \rightarrow \infty$.
15. (Simple harmonic motion)
 $k = 5$; $w = 10$; $y_0 = -1$
 $B = \sqrt{\frac{(5)(32)}{10}} = 4$
Then the equation of motion is $y = -\cos 4t$.
The amplitude is $|-1| = 1$; the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.
16. It is at equilibrium when $y = 0$ or $-\cos 4t = 0$, or
 $t = \frac{\pi}{8}, \frac{3\pi}{8}, \dots$
 $y'(t) = 4 \sin 4t$, so at equilibrium $|y'| = |\pm 4| = 4$.
17. $Q'' + 2Q' + 2Q = 1$
Roots are $-1 \pm i$.
 $Q_h = e^{-t}(C_1 \cos t + C_2 \sin t)$ and $Q_p = \frac{1}{2}$;
 $Q = e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{1}{2}$
 $I(t) = Q'(t) = -e^{-t}[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t]$
 $I(t) = e^{-t} \sin t$ satisfies the initial conditions.

Maple and Mathematica Code for some Technology Projects

Technology Project 12.1 Rotations in the Plane

Maple:

```
with(plots):  
f := (x,y) ->x^2 + 24*x*y + 8*y^2 - 136;  
implicitplot(f(x,y) = 0,x=-16..16,y=-16..16);
```

Mathematica:

```
<<Graphics`ImplicitPlot`  
f[x_,y_] = x^2 + 24 x y + 8 y^2 -136;  
ImplicitPlot[ f[x,y]==0 , {x,-16,16} , {y,-16,16} ]
```

Technology Project 12.1 Roses

Maple:

```
rect := (theta,r) ->[ r*cos(theta) , r*sin(theta) ] :  
n := 5:  
d := 49:  
degr := 2*Pi/360:  
rose := k ->evalf( rect( k*degr , sin(n*k*degr) ) ):  
path := [ seq ( rose(d*k) , k=1..360 ) ]:  
PLOT( CURVES(path) );
```

Mathematica:

```
rect[theta_,r_] := N[{r Cos[theta],r Sin[theta]}]  
Attributes[rose]={Listable}  
n = 5; d = 13;  
rose[k_] = rect[k Degree,Sin[n k Degree] ];  
path = Line[ rose[Range[0,360 d,d]] ];  
Show[Graphics[{Thickness[.0005],path}],AspectRatio->1]
```

Technology Project 13.2 Measuring Home Run Distance

Maple:

```
degr := 2*Pi/360:  
g := 3:  
d := 430:  
h := 80:
```

```

phi := 30*degr:
theta := arctan( tan(phi) + 2*(h-g)/d );
v0 := evalf( sqrt( 32*d / ((cos(theta))^2*(tan(phi)+tan(theta))) ) );
y := x ->-16*x^2/(v0^2 * (cos(theta))^2) + (tan(theta))*x + g;
plot(y(x), x=0..d, scaling=constrained);
HRdist := evalf(
( - tan(theta) - sqrt( (tan(theta))^2 + 64*g/(v0^2 * (cos(theta))^2) ) ) /
(-32/(v0^2 * (cos(theta))^2)) );

```

Technology Project 15.1 Newton's Method for Two Equations in Two Unknowns

Maple:

```

f := (x,y) ->x^2 + y^2 - 48:
g := (x,y) ->x + y - 6:
dfx := unapply(diff(f(x,y),x),(x,y)):
dfy := unapply(diff(f(x,y),y),(x,y)):
dgx := unapply(diff(g(x,y),x),(x,y)):
dgy := unapply(diff(g(x,y),y),(x,y)):
x1 := 2: y1 := 0.5:
for n from 1 to 5 do
  x2 := x1 - ( f(x1,y1)*dgy(x1,y1) - g(x1,y1)*dfy(x1,y1) ) /
  ( dfx(x1,y1)*dgy(x1,y1) - dgx(x1,y1)*dfy(x1,y1) ):1
  y2 := y1 - ( g(x1,y1)*dfx(x1,y1) - f(x1,y1)*dgx(x1,y1) ) /
  ( dfx(x1,y1)*dgy(x1,y1) - dgx(x1,y1)*dfy(x1,y1) ):
  x1 := x2:
  y1 := y2:
od;

```

Mathematica:

```

f[x_,y_] = x^2 + y^2 - 48;
g[x_,y_] = x + y - 6;
dfx[x_,y_] = D[f[x,y],x];
dfy[x_,y_] = D[f[x,y],y];
dgx[x_,y_] = D[g[x,y],x];
dgy[x_,y_] = D[g[x,y],y];
x1 = 2; y1 = 0.5;
Do[ x2 = x1 - ( f[x1,y1] dgy[x1,y1] - g[x1,y1] dfy[x1,y1] ) /
  ( dfx[x1,y1] dgy[x1,y1] - dgx[x1,y1] dfy[x1,y1] );
  y2 = y1 - ( g[x1,y1] dfx[x1,y1] - f[x1,y1] dgx[x1,y1] ) /
  ( dfx[x1,y1] dgy[x1,y1] - dgx[x1,y1] dfy[x1,y1] );
  x1 = x2;
  y1 = y2;

```

```
Print[x2, '' ', y2],
{n, 1, 7} ]
```

Technology Project 15.2 Visualizing the Directional Derivative

Maple

```
with(plots):
f := (x,y) ->3*x + 4*y + 4:
theta := -70: phi := 53:
x0 := 1: y0 := -1:
e := 0.2:
a := t ->(f(x0+e*cos(t),y0+e*sin(t))-f(x0,y0)) / e:
h := t ->f(x0+e*cos(t),y0+e*sin(t)):
p1 := plot3d( f(x,y) , x=-2..2 , y=-2..2 , axes=frame , color=grey ,
orientation=[theta,phi] ):
p2 := spacecurve( {[x0+e*cos(t),y0+e*sin(t) , h(t)]} , t=0..2*Pi ,
axes=FRAME , thickness=2 , color=black , orientation=[theta,phi]):
p3 := contourplot( f(x,y) , x=-2..2 , y=-2..2 , grid=[21,21] ,
color=black , scaling=constrained , contours=20 ):
p4 := plot( [ (x0+e*cos(t),y0+e*sin(t)) , t=0..2*Pi] ,
scaling=constrained , color=black , thickness=2 ):
display([p1,p2]);
display([p3,p4]);
plot( h(t) , t=0..2*Pi );
plot( a(t) , t=0..2*Pi );
evalf( int(a(t),t=0..2*Pi) );
```

Mathematica

```
f[x_,y_] = 3 x + 4 y + 4;
x0 = 1; y0 = -1;
e = 0.2;
a[t_] = ( f[x0+e*Cos[t],y0+e*Sin[t]] - f[x0,y0] ) / e;
h[t_] = f[x0+e*Cos[t],y0+e*Sin[t]];
p1 = Plot3D[ f[x,y] , {x,-2,2} , {y,-2,2} , AxesLabel->{'x','y','z'} ,
Boxed->True , ViewPoint ->{5, -15, 8} , DisplayFunction->Identity ];
p2 = ParametricPlot3D[ {x0+e*Cos[t] , y0+e*Sin[t] , h[t]} , {t,0,2Pi} ,
DisplayFunction->Identity ];
p3 = ContourPlot[ f[x,y] , {x,-2,2} , {y,-2,2} , PlotPoints->21 ,
ContourShading->False , DisplayFunction->Identity ];
p4 = ParametricPlot[ {x0+e*Cos[t] , y0+e*Sin[t]} , {t,0,2Pi} ,
DisplayFunction->Identity ];
Show[ {p1,p2} , DisplayFunction->$DisplayFunction ];
Show[ {p3,p4} , DisplayFunction->$DisplayFunction];
```

```

Plot[ h[t] , {t,0,2Pi} ];
Plot[ a[t] , {t,0,2Pi} ];
NIntegrate[ a[t] , {t,0,2Pi} ]

```

Technology Project 16.2 Monte Carlo Integration

Mathematica:

```

f[x_] := x (1-x) (Sin[100 x (1-x)])^2
xmin = 0; xmax = 1;
ymin = 0; ymax = 0.25;
numtrials = 1000;
p1 := Plot[f[x],{x,0,1},PlotStyle->Thickness[0.012],
  DisplayFunction->Identity]
x = Table[ Random[Real,{xmin,xmax}],{numtrials} ];
y = Table[ Random[Real,{ymin,ymax}],{numtrials} ];
data = Transpose[{x,y}];
p2 := ListPlot[data,PlotStyle->PointSize[0.012],
  DisplayFunction->Identity];
Show[p1,p2,DisplayFunction->$DisplayFunction]
numsucc = 0;
Do[If[ y[[k]] < f[x[[k]]],numsucc=numsucc+1],
  {k,1,numtrials}]
areaestimate = (xmax-xmin)(ymax-ymin)numsucc/numtrials;
Print[ areaestimate ]

```

Maple

```

with(plots):
with(stats):
with(linalg):
f := x ->x*(1-x)*sin(100*x*(1-x))^2:
xmin := 0: xmax := 1:
ymin := 0: ymax := 0.25:
numtrials := 1000:
p1 := plot( f(x) , x=xmin..xmax , y=ymin..ymax ):
randomize(Seed);
xx := [ stats[random, uniform](numtrials) ]:
yy := [ stats[random, uniform](numtrials) ]:
xx := [seq(xmin,i=1..numtrials)] + (xmax-xmin)*xx:
yy := [seq(ymin,i=1..numtrials)] + (ymax-ymin)*yy:
p2 := pointplot( transpose( [xx,yy] ) ):
display( [ p1,p2 ] );
numsucc := 0:
for i from 1 to numtrials do

```



```
if yy[i] <f(xx[i]) then numsucc := numsucc+1 fi:  
od;  
areaestimate := (xmax-xmin)*(ymax-ymin)*numsucc/numtrials;
```

Technology Project 17.2 Parametrized Surfaces

Maple

```
plot3d( [sin(s)*cos(t) , sin(s)*sin(t) , cos(s)] , s=0..Pi , t=0..2*Pi );
```

Mathematica

```
ParametricPlot3D[ { Sin[s]Cos[t] , Sin[s]Sin[t] , Cos[s] } ,  
{s,0,Pi} , {t,0,2 Pi} , PlotPoints->31 ]
```