CHAPTER

17 Vector Calculus

17.1 Concepts Review

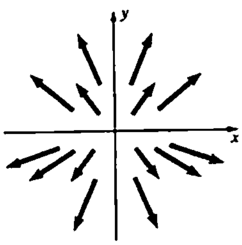
1. vector-valued function of three real variables or a vector field

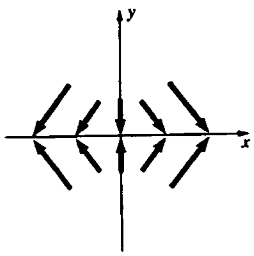
2. gradient field

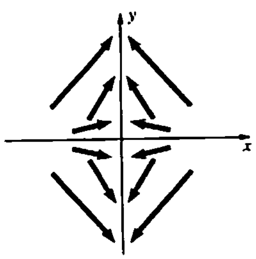
3. gravitational fields; electric fields

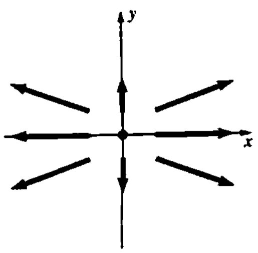
4. $\nabla \cdot \mathbf{F}, \nabla \times \mathbf{F}$

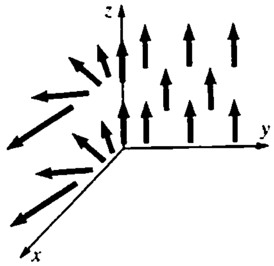
Problem Set 17.1

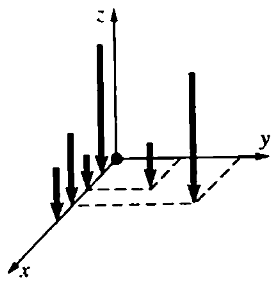
1. 

2. 

3. 

4. 

5. 

6. 

7. $\langle 2x-3y, -3x, 2\rangle$

8. $(\cos xyz)\langle yz, xz, xy\rangle$

9. $f(x, y, z) = \ln|x| + \ln|y| + \ln|z|$;
$\nabla f(x, y, z) = \langle x^{-1}, y^{-1}, z^{-1}\rangle$

10. $\langle x, y, z\rangle$

11. $e^y\langle \cos z, x\cos z, -x\sin z\rangle$

12. $\nabla f(x, y, z) = \langle 0, 2ye^{-2z}, -2y^2e^{-2z}\rangle$
$= 2ye^{-2z}\langle 0, 1, -y\rangle$

13. div $\mathbf{F} = 2x - 2x + 2yz = 2yz$
curl $\mathbf{F} = \langle z^2, 0, -2y\rangle$

14. div $\mathbf{F} = 2x + 2y + 2z$
curl $\mathbf{F} = \langle 0, 0, 0\rangle = \mathbf{0}$

15. div $\mathbf{F} = \nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0$
curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle x - x, y - y, z - z\rangle = 0$

16. div $\mathbf{F} = -\sin x + \cos y + 0$
curl $\mathbf{F} = \langle 0,0,0\rangle = \mathbf{0}$

17. div $\mathbf{F} = e^x \cos y + e^x \cos y + 1 = 2e^x \cos y + 1$
curl $\mathbf{F} = \left\langle 0,\ 0,\ 2e^x \sin y\right\rangle$

18. div $\mathbf{F} = \nabla \cdot \mathbf{F} = 0+0+0 = 0$
curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle 1-1, 1-1, 1-1\rangle = \mathbf{0}$

19. a. meaningless

b. vector field

c. vector field

d. scalar field

e. vector field

f. vector field

g. vector field

h. meaningless

i. meaningless

j. scalar field

k. meaningless

20. a. $\text{div}(\text{curl } \mathbf{F}) = \text{div}\cdot\left\langle P_y - N_z,\ M_z - P_x,\ N_x - M_y\right\rangle = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz}) = 0$

b. $\text{curl}(\text{grad}\, f) = \text{curl}\left\langle f_x,\ f_y,\ f_z\right\rangle = \left\langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy}\right\rangle = \mathbf{0}$

c. $\text{div}(f\mathbf{F}) = \text{div}\langle fM,\ fN,\ fP\rangle = (fM_x + f_xM) + (fN_y + f_yN) + (fP_z + f_zP)$
$= (f)(M_x + N_y + P_z) + (f_xM + f_yN + f_zP) = (f)(\text{div } \mathbf{F}) + (\text{grad}\, f)\cdot\mathbf{F}$

d. $\text{curl}(f\mathbf{F}) = \text{curl}\langle fM,\ fN,\ fP\rangle$
$= \left\langle (fP_y + f_yP) - (fN_z + f_zN),\ (fM_z + f_zM) - (fP_x + f_xP),\ (fN_x + f_xN) - (fM_y + f_yM)\right\rangle$
$= (f)\left\langle P_y - N_z,\ M_z - P_x,\ N_x - M_y\right\rangle + \left\langle f_x,\ f_y,\ f_z\right\rangle \times \langle M,\ N,\ P\rangle = (f)(\text{curl } \mathbf{F}) + (\text{grad}\, f) \times \mathbf{F}$

21. Let $f(x, y, z) = -c|\mathbf{r}|^{-3}$, so

$\text{grad}(f) = 3c|\mathbf{r}|^{-4}\dfrac{\mathbf{r}}{|\mathbf{r}|} = 3c|\mathbf{r}|^{-5}\mathbf{r}.$

Then $\text{curl } \mathbf{F} = \text{curl}\left[\left(-c|\mathbf{r}|^{-3}\right)\mathbf{r}\right]$

$= \left(-c|\mathbf{r}|^{-3}\right)(\text{curl } \mathbf{r}) + \left(3c|\mathbf{r}|^{-5} r\right)\times\mathbf{r}$ (by 20d)

$= \left(-c|\mathbf{r}|^{-3}\right)(0) + \left(3c|\mathbf{r}|^{-5}\right)(\mathbf{r}\times\mathbf{r}) = 0 + 0 = 0$

$\text{div } \mathbf{F} = \text{div}\left[\left(-c|\mathbf{r}|^{-3}\right)\mathbf{r}\right]$

$= \left(-c|\mathbf{r}|^{-3}\right)(\text{div } \mathbf{r}) + \left(3c|\mathbf{r}|^{-5}\mathbf{r}\right)\cdot\mathbf{r}$ (by 20c)

$= \left(-c|\mathbf{r}|^{-3}\right)(1+1+1) + \left(3c|\mathbf{r}|^{-5}\right)|\mathbf{r}|^2$

$= \left(-3c|\mathbf{r}|^{-3}\right) + 3c|\mathbf{r}|^3 = 0$

22. $\text{curl}\left[-c|\mathbf{r}|^{-m}\mathbf{r}\right] = \left(-c|\mathbf{r}|^{-m}\right)(0) + mc|\mathbf{r}|^{-m-2}(\mathbf{0})$
$= \mathbf{0}$
$\text{div}\left[-c|\mathbf{r}|^{-m}\mathbf{r}\right] = \left(-c|\mathbf{r}|^{-m}\right)(3) + mc|\mathbf{r}|^{-m-2}|\mathbf{r}|^2$
$= (m-3)c|\mathbf{r}|^{-m}$

23. $\text{grad}\, f = \left\langle f'(r)xr^{-1/2},\ f'(r)yr^{-1/2},\ f'(r)zr^{-1/2}\right\rangle$
(if $r \neq 0$)
$= f'(r)r^{-1/2}\langle x,\ y,\ z\rangle = f'(r)r^{-1/2}\mathbf{r}$
$\text{curl } \mathbf{F} = [f(r)][\text{curl } \mathbf{r}] + [f'(r)r^{-1/2}\mathbf{r}]\times\mathbf{r}$
$= [f(r)][\text{curl } \mathbf{r}] + [f'(r)r^{-1/2}\mathbf{r}]\times\mathbf{r}$
$= \mathbf{0} + \mathbf{0} = \mathbf{0}$

24. $\text{div } \mathbf{F} = \text{div}[f(r)\mathbf{r}] = [f(r)](\text{div } \mathbf{r}) + \text{grad}[f(r)]\cdot\mathbf{r}$
$= [f(r)](\text{div } \mathbf{r}) + [f'(r)r^{-1}\mathbf{r}]\cdot\mathbf{r}$
$= [f(r)](3) + [f'(r)r^{-1}](\mathbf{r}\cdot\mathbf{r})$
$= 3f(r) + [f'(r)r^{-1}](r^2) = 3f(r) + rf'(r)$

Now if div **F** = 0, and we let $y = f(r)$, we have the differential equation $3y + r\frac{dy}{dr} = 0$, which can be solved as follows:

$\frac{dy}{y} = -3\frac{dr}{r}$; $\ln|y| = -3\ln|r| + \ln|C| = \ln\left|Cr^{-3}\right|$,

for each $C \neq 0$. Then $y = Cr^{-3}$, or

$f(r) = Cr^{-3}$, is a solution (even for $C = 0$).

25. a. Let $P = (x_0, y_0)$.

div **F** = div **H** = 0 since there is no tendency toward P except along the line $x = x_0$, and along that line the tendencies toward and away from P are balanced; div **G** < 0 since there is no tendency toward P except along the line $x = x_0$, and along that line there is more tendency toward than away from P; div **L** > 0 since the tendency away from P is greater than the tendency toward P.

b. No rotation for **F**, **G**, **L**; clockwise rotation for **H** since the magnitudes of the forces to the right of P are less than those to the left.

c. div **F** = 0; curl **F** = **0**

div $\mathbf{G} = -2ye^{-y^2} < 0$ since $y > 0$ at P; curl **G** = **0**

div $\mathbf{L} = (x^2 + y^2)^{-1/2}$; curl **L** = **0**

div **H** = 0; curl $\mathbf{H} = \left\langle 0, 0, -2xe^{-x^2}\right\rangle$ which points downward at P, so the rotation is clockwise in a right-hand system.

26. div **v** = 0 + 0 + 0 = 0;

curl $\mathbf{v} = \langle 0, 0, w + w\rangle = 2\omega\mathbf{k}$

27. $\nabla f(x, y, z) = \frac{1}{2}m\omega^2\langle 2x, 2y, 2z\rangle = m\omega^2\langle x, y, z\rangle$

$= \mathbf{F}(x, y, z)$

28. $\nabla^2 f = \text{div}(\text{grad} f) = \text{div}\langle f_x, f_y, f_z\rangle$

$= f_{xx} + f_{yy} + f_{zz}$

a. $\nabla^2 f = 4 - 2 - 2 = 0$

b. $\nabla^2 f = 0 + 0 + 0 = 0$

c. $\nabla^2 f = 6x - 6x + 0 = 0$

d. $\nabla^2 f = \text{div}(\text{grad} f) = \text{div}\left(\text{grad } |\mathbf{r}|^{-1}\right)$

$= \text{div}\left(-|\mathbf{r}|^{-3}\mathbf{r}\right) = 0$ (by problem 21)

Hence, each is harmonic.

29. a. $\mathbf{F}\times\mathbf{G} = (f_y g_z - f_z g_y)\mathbf{i} - (f_x g_z - f_z g_x)\mathbf{j} + (f_x g_z - f_z g_x)\mathbf{k}$

Therefore,

$$div(\mathbf{F}\times\mathbf{G}) = \frac{\partial}{\partial x}(f_y g_z - f_z g_y) - \frac{\partial}{\partial y}(f_x g_z - f_z g_x) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x).$$

Using the product rule for partials and some algebra gives

$$div(\mathbf{F}\times\mathbf{G}) = g_x\left[\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right] + g_y\left[\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right] + g_z\left[\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right]$$

$$+ f_x\left[\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right] + f_y\left[\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right] + f_z\left[\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right]$$

$$= \mathbf{G}\cdot curl(\mathbf{F}) - \mathbf{F}\cdot curl(\mathbf{G})$$

b, $$\nabla f\times\nabla g = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\mathbf{k}$$

Therefore,

$$div(\nabla f\times\nabla g) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

Using the product rule for partials and some algebra will yield the result

$div(\nabla f\times\nabla g) = 0$

30. $\lim_{(x,y,z)\to(a,b,c)} F(x,y,z) = \mathbf{L}$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |\langle x, y, z\rangle - \langle a, b, c\rangle| < \delta$ implies that $|\mathbf{F}(x, y, z) - \mathbf{L}| < \varepsilon$.

F is continuous at (a, b, c) if and only if $\lim_{(x,y,z)\to(a,b,c)} = \mathbf{F}(a, b, c)$.

17.2 Concepts Review

1. Increasing values of t

2. $\sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta s_i$

3. $f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}$

4. $\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}$

Problem Set 17.2

1. $\int_0^1 (27t^3 + t^3)(9 + 9t^4)^{1/2} dt = 14\left(2\sqrt{2} - 1\right)$
≈ 25.5980

2. $\int_0^1 \left(\frac{t}{2}\right)(t)\left(\frac{1}{4} + \frac{25t^3}{4}\right)^{1/2} dt = \left(\frac{1}{450}\right)(26^{3/2} - 1)$
≈ 0.2924

3. Let $x = t$, $y = 2t$, t in $[0, \pi]$.
Then
$\int_C (\sin x + \cos y)ds = \int_0^{\pi} (\sin t + \cos 2t)\sqrt{1+4}dt$
$= 2\sqrt{5} \approx 4.4721$

4. Vector equation of the segment is
$\langle x, y\rangle = \langle -1, 2\rangle + t\langle 2, -1\rangle$, t in $[0, 1]$.
$\int_0^1 (-1 + 2t)e^{2-t}(4+1)^{1/2} dt = \sqrt{5}e^2(1 - 3e^{-1})$
≈ -1.7124

5. $\int_0^1 (2t + 9t^3)(1 + 4t^2 + 9t^4)^{1/2} dt = \left(\frac{1}{6}\right)(14^{3/2} - 1)$
≈ 8.5639

6. $\int_0^{2\pi} (16\cos^2 t + 16\sin^2 t + 9t^2)(16\sin t^2 + 16\cos^2 t + 9)^{1/2} dt = \int_0^{2\pi} (16 + 9t^2)(5)dt$
$= \left[80t + 15t^3\right]_0^{2\pi} = 160\pi + 120\pi^3 \approx 4878.11$

7. $\int_0^2 [(t^2 - 1)(2) + (4t^2)(2t)]dt = \frac{100}{3}$

8. $\int_0^4 (-1)dx + \int_{-1}^3 (4)^2 dy = 60$

9. $\int_C y^3 dx + x^3 dy = \int_{C_1} y^3 dx + x^3 dy$
$+ \int_{C_2} y^3 dx + x^3 dy$
$= \int_1^{-2} (-4)^3 dy + \int_{-4}^2 (-2)^3 dx = 192 + (-48) = 144$

10. $\int_{-2}^1 [(t^2 - 3)^3(2) + (2t)^3(2t)]dt = \frac{828}{35} \approx 23.6571$

11. $y = -x + 2$
$\int_1^3 ([x + 2(-x+2)](1) + [x - 2(-x+2)](-1))dx = 0$

12. $\int_0^1 [x^2 + (x)2x]dx = \int_0^1 3x^2 dx = 1$
(letting x be the parameter; i.e., $x = x$, $y = x^2$)

13. $\langle x,y,z\rangle = \langle 1,2,1\rangle + t\langle 1,-1,0\rangle$

$\int_0^1[(4-t)(1)+(1+t)(-1)-(2-3t+t^2)(-1)]dt = \frac{17}{6}$

≈ 2.8333

14 $\int_0^1[(e^{3t})(e^t)+(e^{-t}+e^{2t})(-3^{-t})+(e^t)(2e^{2t})]dt$

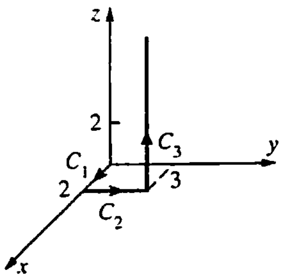
$=\left(\frac{1}{4}\right)e^4+\left(\frac{2}{3}\right)e^3-e+\left(\frac{1}{2}\right)e^{-2}-\frac{5}{12}$

≈ 23.9726

15. On $C_1: y=z=dy=dz=0$

On $C_2: x=2, z=dx=dz=0$

On $C_3: x=2, y=3, dx=dy=0$



$\int_0^2 x\,dx+\int_0^3(2-2y)dy+\int_0^4(4+3-z)dz$

$=\left[\frac{x^2}{2}\right]_0^2+[2y-y^2]_0^3+\left[7z-\frac{z^2}{2}\right]_0^4$

$=2+(-3)+20=19$

16. $\langle x,y,z\rangle = t\langle 2,3,4\rangle$, t in [0, 1].

$\int_0^1[(9t)(2)+(8t)(3)+(3t)(4)]dt = 27$

17. $m=\int_C k|x|ds=\int_{-2}^2 k|x|(1+4x^2)^{1/2}dx$

$=\left(\frac{k}{6}\right)(17^{3/2}-1)\approx 11.6821k$

18. Let $\delta(x, y, z) = k$ (a constant).

$m=k\int_C 1\,ds=k\int_0^{3\pi}1(a^2\sin^2 t+a^2\cos^2 t+b^2)^{1/2}dt$

$=3\pi k(a^2+b^2)^{1/2}$

$M_{xy}=k\int_C z\,ds=k(a^2+b^2)^{1/2}\int_0^{3\pi}bt\,dt$

$=\frac{9\pi^2bk(a^2+b^2)^{1/2}}{2}$

$M_{xz}=k\int_C y\,ds=k(a^2+b^2)^{1/2}\int_0^{3\pi}a\sin t\,dt$

$=ak(a^2+b^2)^{1/2}(2)=2ak(a^2+b^2)^{1/2}$

$M_{yz}=k\int_C x\,ds=k(a^2+b^2)^{1/2}\int_0^{3\pi}a\cos t\,dt$

$=ak(a^2+b^2)^{1/2}(0)=0$

Therefore, $\bar{x}=\frac{M_{yz}}{m}=0$; $\bar{y}=\frac{M_{xz}}{m}=\frac{2a}{3\pi}$;

$\bar{z}=\frac{M_{xy}}{m}=\frac{3\pi b}{2}$.

19 $\int_C(x^3-y^3)dx+xy^2dy$

$=\int_{-1}^0[(t^6-t^9)(2t)+(t^2)(t^6)(3t^2)]dt$

$=-\frac{7}{44}\approx -0.1591$

20. $\int_C e^x dx-e^{-y}dy=\int_1^5\left[(t^3)\left(\frac{3}{t}\right)-\left(\frac{1}{2t}\right)\left(\frac{1}{t}\right)\right]dt$

$=123.6$

21. $W=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C(x+y)dx+(x-y)dy=\int_0^{\pi/2}[(a\cos t+b\sin t)(-a\sin t)+(a\cos t-b\sin t)(b\cos t)]dt$

$=\int_0^{\pi/2}[-(a^2+b^2)\sin t\cos t+ab(\cos^2 t-\sin^2 t)]dt=\int_0^{\pi/2}\frac{-(a^2+b^2)\sin 2t}{2}+ab\cos 2t\,dt$

$=\left[\frac{(a^2+b^2)\cos 2t}{4}+\frac{ab\sin 2t}{2}\right]_0^{\pi/2}=\frac{a^2+b^2}{-2}$

22. $\langle x,y,z\rangle = t\langle 1,1,1\rangle$, t in [0, 1].

$\int_C(2x-y)dx+2z\,dy+(y-z)dz=\int_0^1(t+2t+0)dt=1.5$

23. $\int_0^\pi\left[\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)+\pi t\cos\left(\frac{\pi t}{2}\right)+\sin\left(\frac{\pi t}{2}\right)-t\right]dt=2-\frac{2}{\pi}\approx 1.3634$

24. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx + z\,dy + x\,dz = \int_0^2 [(t^2)(1) + (t^3)(2t) + (t)(3t^2)]dt$

$= \int_0^2 (2t^4 + 3t^3 + t^2)dt = \frac{64}{5} + 12 + \frac{8}{3} = \frac{412}{15} \approx 27.4667$

25. $\int_C \left(1 + \frac{y}{3}\right) ds = \int_0^2 (1 + 10\sin^3 t)[(-90\cos^2 t\sin t)^2 + (90\sin^2 t\cos t)^2]^{1/2} dt = 225$

Christy needs $\frac{450}{200} = 2.25$ gal of paint.

26. $\int_C \langle 0, 0, 1.2\rangle \cdot \langle dx, dy, dz\rangle = \int_C 1.2\,dz$

$= \int_0^{8\pi} 1.2(4)dt = 38.4\pi \approx 120.64$ ft-lb

Trivial way: The squirrel ends up 32π ft immediately above where it started.
(32π ft)(1.2 lb) ≈ 120.64 ft-lb

27. $C: x + y = a$

Let $x = t$, $y = a - t$, t in $[0, a]$.

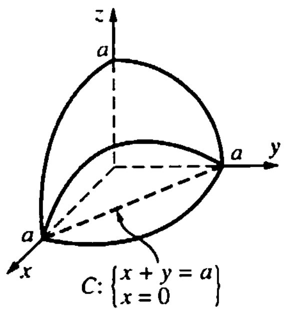
Cylinder: $x + y = a$; $(x + y)^2 = a^2$;

$x^2 + 2xy + y^2 = a^2$

Sphere: $x^2 + y^2 + z^2 = a^2$

The curve of intersection satisfies:

$z^2 = 2xy$; $z = \sqrt{2xy}$.



$\text{Area} = 8\int_C \sqrt{2xy}\,ds = 8\int_0^a \sqrt{2t(a-t)}\sqrt{(1)^2 + (-1)^2}\,dt$

$= 16\int_0^a \sqrt{at - t^2}\,dt$

$= 16\left[\frac{t - \frac{a}{2}}{2}\sqrt{at - t^2} + \frac{\left(\frac{a}{2}\right)^2}{2}\sin^{-1}\left(\frac{t - \frac{a}{2}}{\frac{a}{2}}\right)\right]_0^a$

$= 16\left(\left[0 + \left(\frac{a^2}{8}\right)\left(\frac{\pi}{2}\right)\right] - \left[0 + \left(\frac{a^2}{8}\right)\left(\frac{-\pi}{2}\right)\right]\right) = 2a^2\pi$

Trivial way: Each side of the cylinder is part of a plane that intersects the sphere in a circle. The radius of each circle is the value of z in $z = \sqrt{2xy}$ when $x = y = \frac{a}{2}$. That is, the radius is $\sqrt{2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)} = \frac{a\sqrt{2}}{2}$. Therefore, the total area of the part cut out is $r\left[\pi\left(\frac{a\sqrt{2}}{2}\right)^2\right] = 2a^2\pi$.

28. $I_y = \int_C kx^2 ds = 4k\int_0^a t^2\sqrt{2}dt = 4\sqrt{2}\frac{ka^3}{3}$

(using same parametric equations as in Problem 27)

$I_x = I_y$ (symmetry)

$I_z = I_x + I_y = 8\sqrt{2}\frac{ka^3}{3}$

29. Note that $r = a\cos\theta$ along C.

Then $(a^2 - x^2 - y^2)^{1/2} = (a^2 - r^2)^{1/2} = a\cos\theta$.

Let $\begin{Bmatrix} x = r\cos\theta = (a\sin\theta)\cos\theta \\ y = r\sin\theta = (a\sin\theta)\sin\theta \end{Bmatrix}$, θ in $\left[0, \frac{\pi}{2}\right]$.

Therefore, $x'(\theta) = a\cos 2\theta$; $y(\theta) = a\sin 2\theta$.

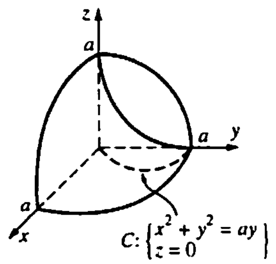
Then $\text{Area} = 4\int_C (a^2 - x^2 - y^2)^{1/2} ds$

$= 4\int_0^{\pi/2} (a\cos\theta)[(a\sin 2\theta)^2 + (a\cos 2\theta)^2]^{1/2} d\theta$

$= 4a^2$.

30. $C: x^2 + y^2 = a^2$

Let $x = a\cos\theta$, $y = a\sin\theta$, θ in $\left[0, \frac{\pi}{2}\right]$.



$\text{Area} = 8\int_C \sqrt{a^2 - x^2}\,ds$

$= 8\int_0^{\pi/2} (a\sin\theta)\sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}\,d\theta$

$= 8\int_0^{\pi/2}(a\sin\theta)\sqrt{a^2}\,d\theta = 8a^2[-\cos\theta]_0^{\pi/2}$

$= 8a^2$

(Note: The result of Problem 26, Section 12.8, could be used to arrive at this result more quickly.)

31. a. $\int_C x^2 y\,ds = \int_0^{\pi/2}(3\sin t)^2(3\cos t)[(3\cos t)^2 + (-3\sin t)^2]^{1/2}dt = 81\int_0^{\pi/2}\sin^2 t\cos t\,dt = 81\left[\left(\frac{1}{3}\right)\sin^3 t\right]_0^{\pi/2} = 27$

b. $\int_{C_4} xy^2dx + xy^2dy = \int_0^3(3-t)(5-t)^2(-1)dt + \int_0^3(3-t)(5-t)^2(-1)dt = 2\int_0^3(t^3 - 13t^2 + 55t - 75)dt = -148.5$

17.3 Concepts Review

1. $f(\mathbf{b}) - f(\mathbf{a})$

2. gradient; $\nabla f(\mathbf{r})$

3. 0; 0

4. **F** is conservative.

Problem Set 17.3

1. $M_y = -7 = N_x$, so **F** is conservative.

   $f(x, y) = 5x^2 - 7xy + y^2 + C$

2. $M_y = 6y + 5 = N_x$, so **F** is conservative.

   $f(x, y) = 4x^3 + 3xy^2 + 5xy - y^3 + C$

3. $M_y = 90x^4y - 36y^5 \neq N_x$ since

   $N_x = 90x^4y - 12y^5$, so **F** is not conservative.

4. $M_y = -12x^2y^3 + 9y^8 = N_x$, so **F** is conservative.

   $f(x, y) = 7x^5 - x^3y^4 + xy^9 + C$

5. $M_y = \left(-\frac{12}{5}\right)x^2y^{-3} = N_x$, so **F** is conservative.

   $f(x, y) = \left(\frac{2}{5}\right)x^3y^{-2} + C$

6. $M_y = (4y^2)(-2xy\sin xy^2) + (8y)(\cos xy^2) \neq N_x$ since $N_x = (8x)(-y^2\sin xy^2) + (8)(\cos xy^2)$, so **F** is not conservative.

7. $M_y = 2e^y - e^x = N_x$ so **F** is conservative.

   $f(x, y) = 2xe^y - ye^x + C$

8. $M_y = -e^{-x}y^{-1} = N_x$, so **F** is conservative.

   $f(x, y) = e^{-x}\ln y + C$

9. $M_y = 0 = N_x, M_z = 0 = P_x$, and $N_z = 0 = P_y$, so **F** is conservative. f satisfies

   $f_x(x, y, z) = 3x^2$, $f_y(x, y, z) = 6y^2$, and

   $f_z(x, y, z) = 9z^2$.

   Therefore, f satisfies

   1. $f(x, y, z) = x^3 + C_1(y, z)$,
   2. $f(x, y, z) = 2y^3 + C_2(x, z)$, and
   3. $f(x, y, z) = 3z^3 + C_3(x, y)$.

   A function with an arbitrary constant that satisfies 1, 2, and 3 is

   $f(x, y, z) = x^3 + 2y^3 + 3z^3 + C$.

10. $M_y = 2x = N_y, M_z = 2z = P_x$, and $N_z = 0 = P_y$, so **F** is conservative.

    $f(x, y, z) = x^2y + xz^2 + \sin\pi z + C$

11. $M_y = 2y + 2x = N_x$, so integral is path independent. $f(x, y) = xy^2 + x^2y$

    $\int_{(-1,2)}^{(3,1)}(y^2 + 2xy)dx + (x^2 + 2xy)dy = [xy^2 + x2y]_{(-1,2)}^{(3,1)}$

    = 14 (Or use paths.)

12. $M_y = e^x\cos y = N_x$, so the line integral is independent of the path.

    Let $f_x(x, y) = e^x\sin y$ and $f_y(x, y) = e^x\cos y$.

    Then $f(x, y) = e^x\sin y + C_1(y)$ and

$f(x, y) = e^x \sin y + C_2(x).$

Choose $f(x, y) = e^x \sin y.$

By Theorem A,

$\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\,dx + e^x \cos y\,dy = [e^x \sin y]_{(0,0)}^{(1,\pi/2)}$

$= e.$

(Or use line segments (0, 0) to (1, 0), then (1, 0) to $\left(1, \frac{\pi}{2}\right)$.)

13. $M_y = 18xy^2 = N_x, M_z = 4x = P_x, \quad N_z = 0 = P_y.$
By paths (0, 0, 0) to (1, 0, 0); (1, 0, 0) to (1, 1, 0); (1, 1, 0) to (1, 1, 1)

$\int_0^1 0\,dx + \int_0^1 9y^2 dy + \int_0^1 (4z+1)dz = 6$

(Or use $f(x, y) = 3x^2y^3 + 2xz^2 + z$.)

14. $M_y = z = N_x, M_z = y = P_x, \quad N_z = x = P_y.$

$f(x, y) = xyz + x + y + z$

Thus, the integral equals

$[xyz + x + y + z]_{(0,1,0)}^{(1,1,1)} = 3.$ (Or use paths.)

15. $M_y = 1 = N_x, M_z = 1 = P_x, \quad N_z = 1 = P_y$ (so path independent). From inspection observe that $f(x, y, z) = xy + xz + yz$ satisfies $f = \langle y+z, x+z, x+y \rangle$, so the integral equals

$[xy + xz + yz]_{(0,0,0)}^{(-1,0,\pi)} = -\pi.$ (Or use line segments $(0,1,0)$ to $(1,1,0)$, then $(1,1,0)$ to $(1,1,1)$.)

16. $M_y = 2z = N_x, M_z = 2y = P_x, \quad N_z = 2x = P_y$ by paths $(0,0,0)$ to $(\pi,0,0)$, $(\pi,0,0)$ to $(\pi,\pi,0)$.

$\int_0^\pi \cos x\,dx + \int_0^\pi \sin y\,dy = 2$

Or use $f(x, y, z) = \sin x + 2xyz - \cos y + \frac{z^2}{2}.$

17. $f_x = M, f_y = N, f_z = P$

$f_{xy} = M_y$, and $f_{yx} = N_x$, so $M_y = N_x$.

$f_{xz} = M_z$ and $f_{zx} = P_x$, so $M_z = P_x$.

$f_{yz} = N_z$ and $f_{zy} = P_y$, so $N_z = P_y$.

18. $f_x(x, y, z) = \frac{-kx}{x^2 + y^2 + z^2}$, so

$f(x, y, z) = \frac{-k}{2}\ln(x^2 + y^2 + z^2) + C_1(y, z).$

Similarly,

$f(x, y, z) = \frac{-k}{2}\ln(x^2 + y^2 + z^2) + C_2(y, z),$

using f_y; and

$f(x, y, z) = \frac{-k}{2}\ln(x^2 + y^2 + z^2) + C_3(y, z),$

using f_z.

Thus, one potential function for **F** is

$f(x, y, z) = \frac{-k}{2}\ln(x^2 + y^2 + z^2).$

19. $\mathbf{F}(x, y, z) = k|\mathbf{r}|\frac{\mathbf{r}}{|\mathbf{r}|} = k\mathbf{r} = k\langle x, y, z \rangle$

$f(x, y, z) = \left(\frac{k}{2}\right)(x^2 + y^2 + z^2)$ works.

20. Let $f = \left(\frac{1}{2}\right)h(u)$ where $u = x^2 + y^2 + z^2$.

Then $f_x = \left(\frac{1}{2}\right)h'(u)u_x = \left(\frac{1}{2}\right)g(u)(2x) = xg(u).$

Similarly, $f_y = yg(u)$ and $f_z = zg(u)$.

Therefore, $f(x, y, z) = g(u)\langle x, y, z \rangle$

$= g(x^2 + y^2 + z^2)\langle x, y, z \rangle = \mathbf{F}(x, y, z).$

21. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (m\mathbf{r}'' \cdot \mathbf{r}')dt$

$= m\int_a^b (x''x' + y''y' + z''z')dt$

$= m\left[\frac{(x')^2}{2} + \frac{(y')^2}{2} + \frac{(z')^2}{2}\right]_a^b$

$= \frac{m}{2}\left[|\mathbf{r}'(t)|^2\right]_a^b = \frac{m}{2}\left[|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2\right]$

22. The force exerted by Matt is not the only force acting on the object. There is also an equal but opposite force due to friction. The work done by the sum of the (equal but opposite) forces is zero since the sum of the forces is zero.

23. $f(x, y, z) = -gmz$ satisfies $\nabla f(x, y, z) = \langle 0, 0, -gm \rangle = \mathbf{F}$. Then, assuming the path is piecewise smooth,

Work $= \int_C \mathbf{F} \cdot d\mathbf{r} = [-gmz]_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)}$

$= -gm(z_2 - z_1) = gm(z_1 - z_2).$

24. a. Place the earth at the origin.

$GMm \approx 7.92(10^{44})$

$f(\mathbf{r}) = \frac{-GMm}{|\mathbf{r}|}$ is a potential function of $\mathbf{F}(\mathbf{r})$. (See Example 1.)

$$\text{Work} = \int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \left[\frac{-GMm}{|\mathbf{r}|}\right]_{|\mathbf{r}|=152.1(10^9)}^{147.1(10^9)}$$

$\approx -1.77(10^{32})$ joules

b. Zero

25. a. $M = \dfrac{y}{(x^2+y^2)}; M_y = \dfrac{(x^2-y^2)}{(x^2+y^2)^2}$

$N = -\dfrac{x}{(x^2+y^2)}; N_x = \dfrac{(x^2-y^2)}{(x^2+y^2)^2}$

b. $M = \dfrac{y}{(x^2+y^2)} = \dfrac{(\sin t)}{(\cos^2 t + \sin^2 t)} = \sin t$

$N = -\dfrac{x}{(x^2+y^2)} = \dfrac{(-\cos t)}{(\cos^2 t + \sin^2 t)} = -\cos t$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M dx + N dy$

$= \int_0^{2\pi} [(\sin t)(-\sin t) + (-\cos t)(\cos t)]dt$

$= -\int_0^{2\pi} 1\, dt = -2\pi \neq 0$

26. f is not continuously differentiable on C since f is undefined at two points of C (where x is 0).

17.4 Concepts Review

1. $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

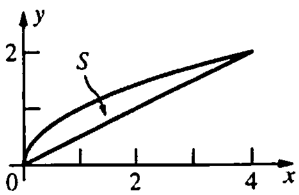
2. –2; –2

3. source; sink

4. rotate; irrotational

Problem Set 17.4

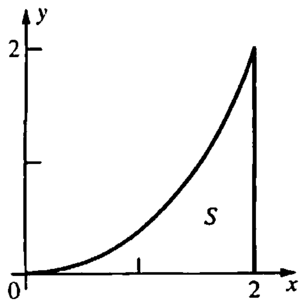
1.



$\oint_C 2xy\, dx + y^2 dy = \iint_S (0-2x)dA$

$= \int_0^2 \int_{y^2}^{2y} -2x\, dx\, dy \;= -\dfrac{64}{15} \approx -4.2667$

2.

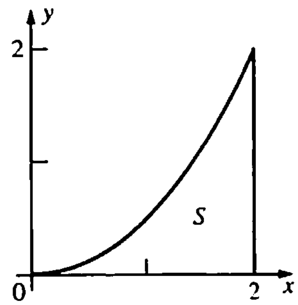


$\oint_C \sqrt{y}\, dx + \sqrt{x}\, dy = \iint_S \frac{1}{2}\left(x^{-1/2} - y^{-1/2}\right)dA$

$= \left(\dfrac{1}{2}\right)\int_0^2 \int_0^{x^2/2} (x^{-1/2} - y^{-1/2}) dy\, dx$

$= -\dfrac{3\sqrt{2}}{5} \approx -0.8485$

3.

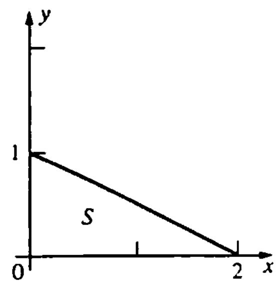


$\oint_C (2x+y^2)dx + (x^2+2y)dy = \iint_S (2x-2y)dA$

$= \int_0^2 \int_0^{x^3/4} (2x-2y)dy\, dx = \int_0^2 \left[\dfrac{x^4}{2} - \dfrac{x^6}{16}\right]dx$

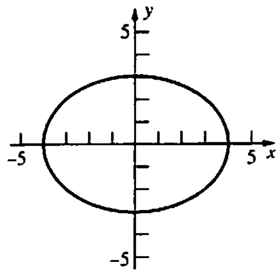
$= \dfrac{16}{5} - \dfrac{8}{7} = \dfrac{72}{35} \approx 2.0571$

4.

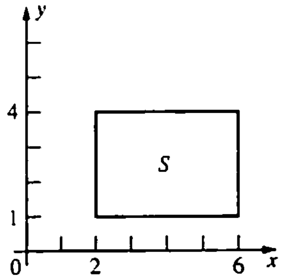


$\oint_C xy\, dx + (x+y)dy = \iint_S (1-x)dA$

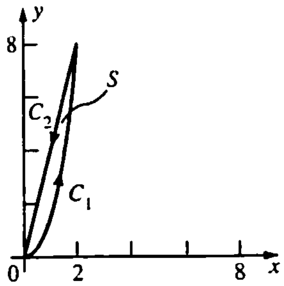
$= \int_0^1 \int_0^{-2y+2} (1-x)dx\, dy = \dfrac{1}{3}$

5. 

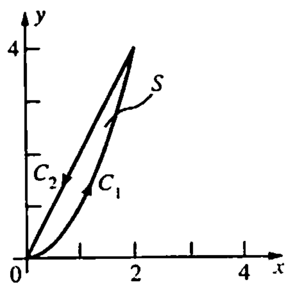
$$\oint_C (x^2+4xy)dx+(2x^2+3y)dy = \iint_S (4x-4x)dA$$
$$=0$$

6. 

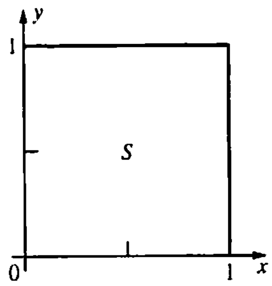
$$\oint_C (e^{3x}+2y)dx+(x^2+\sin y)dy = \iint_S (2x-2)dA$$
$$\int_1^4\int_2^6 (2x-2)dx\,dy = \int_1^4 24dy = 24(3) = 72$$

7. 

$$A(S) = \left(\frac{1}{2}\right)\oint_C x\,dy - y\,dx$$
$$= \left(\frac{1}{2}\right)\int_0^2 [4x^2-2x^2]dx + \left(\frac{1}{2}\right)\int_2^0 [4x-4x]dx = \frac{8}{3}$$

8. 

$$A(S) = \left(\frac{1}{2}\right)\oint_C x\,dy - y\,dx$$
$$= \left(\frac{1}{2}\right)\int_0^2 \left[\left(\frac{3}{2}\right)x^3 - \left(\frac{1}{2}\right)x^3\right]dx - \left(\frac{1}{2}\right)\int_2^0 [2x^2-x^2]dx$$
$$= \frac{2}{3}$$

9. 

a. $\iint_S \text{div}\,\mathbf{F}\,dA = \iint_S (M_x+N_y)dA$
$= \iint_S (0+0)dA = 0$

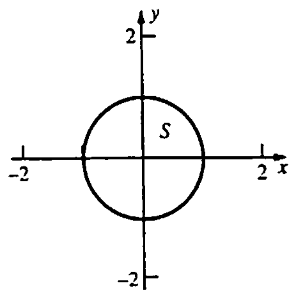
b. $\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dA = \iint_S (N_x-M_y)dA$
$= \iint_S (2x-2y)dA = \int_0^1\int_0^1 (2x-2y)dx\,dy$
$= \int_0^1 (1-2y)dy = 0$

10. a. $\iint_S (0+0)dA = 0$

b. $\iint_S (b-a)dA = \int_0^1\int_0^1 (b-a)dx\,dy = b-a$

11. a. $\iint_S (0+0)dA = 0$

b. $\iint_S (3x^2-3y^2)dA = 0$, since for the integrand, $f(y, x) = -f(x, y)$.

12. 

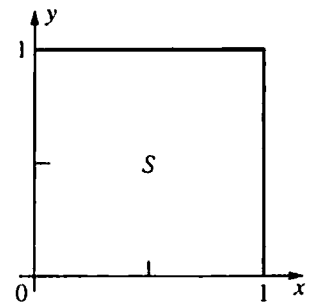
a. $\iint_S \text{div}\,\mathbf{F}\,dA = \iint_S (M_x+N_y)dA$
$= \iint_S (1+1)dA = 2[A(S)] = 2\pi$

b. $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_S (N_x - M_y) dA$
$= \iint_S (0-0) dA = 0$

13. $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA = \int_{C_1} \mathbf{F} \cdot \mathbf{T} ds - \int_{C_2} \mathbf{F} \cdot \mathbf{T} ds$
$= 30 - (-20) = 50$

14. $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S (2x + 2x) dA = \int_0^1 \int_0^1 4x\, dx\, dy = 2$

15.

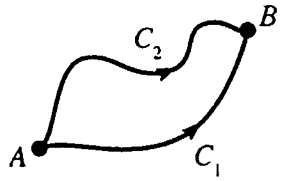


$W = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (N_x - M_y) dA$
$= \iint_S (-2y - 2y) dA = \int_0^1 \int_0^1 -4y dx dy$
$= \int_0^1 -4y\, dy = -2$

16. $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (2y - 2y) dA = 0$

17. $\mathbf{F}$ is a constant, so $N_x = M_y = 0$.
$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (N_x - M_y) dA = 0$

18. $\oint M\, dx + N\, dy = \iint_S (N_x - M_y) dA = 0$



Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path since
$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$
(Where C is the loop C_1 followed by $-C_2$.)
Therefore, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, so $\mathbf{F}$ is conservative.

19. a. Each equals $(x^2 - y^2)(x^2 + y^2)^{-2}$.

b. $\oint_C y(x^2 + y^2)^{-1} dx - x(x^2 + y^2)^{-1} dy = \int_0^{2\pi} (-\sin^2 t - \cos^2 t) dt$
$= \int_0^{2\pi} -1\, dt$

c. M and N are discontinuous at (0, 0).

20. a. Parameterization of the ellipse: $x = 3\cos t$, $y = 2\sin t$, t in $[0, 2\pi]$.
$$\int_0^{2\pi} \left[\frac{2\sin t}{9\cos^2 t + 4\sin^2 t}(-3\sin t) - \frac{3\cos t}{9\cos^2 t + 4\sin^2 t}(2\cos t) \right] dt = -2\pi$$

b. $\int_{-1}^{1} -(1 + y^2)^{-1} dy + \int_1^{-1} (x^2 + 1)^{-1} dx + \int_1^{-1} (1 + y^2)^{-1} dy + \int_{-1}^{1} -(x^2 + 1)^{-1} dx = -2\pi$

c. Green's Theorem applies here. The integral is 0 since $N_x - M_y$.

21. Use Green's Theorem with $M(x, y) = -y$ and $N(x, y) = 0$.
$\oint_C (-y) dx = \iint_S [0 - (-1)] dA = A(S)$
Now use Green's Theorem with $M(x, y) = 0$ and $N(x, y) = x$.
$\oint_C x\, dy = \iint_S (1 - 0) dA = A(S)$

22. $\oint_C \left(-\frac{1}{2}\right) y^2 dx = \iint_S (0 + y) dA = M_x \cdot \oint_C \left(\frac{1}{2}\right) x^2 dy$

$\iint_S (x - 0) dA = M_y$

23. $A(S) = \left(\frac{1}{2}\right) \oint_C x\,dy - y\,dx = \left(\frac{1}{2}\right) \int_0^{2\pi} [(a\cos^3 t)(3a\sin^2 t)(\cos t) - (a\sin^3 t)(3a\cos^2 t)(-\sin t)] dt = \left(\frac{3}{8}\right) a^2 \pi$

24. $W = \oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA = \iint_S (N_x - M_y) dA = \iint_S (-3 - 2) dA$

$= -5[A(S)] = -5\left(\frac{3a^2\pi}{8}\right) = -\frac{15a^2\pi}{15}$, using the result of Problem 23.

25. a. $\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \frac{1}{(x^2 + y^2)^{1/2}} = \frac{1}{a}$

Therefore, $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \frac{1}{a} \int_C 1\, ds = \frac{1}{a}(2\pi a) = 2\pi.$

b. $\text{div } \mathbf{F} = \frac{(x^2 + y^2)(1) - (x)(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)(1) - (y)(2y)}{(x^2 + y^2)^2} = 0$

c. $M = \frac{x}{(x^2 + y^2)}$ is not defined at (0, 0) which is inside C.

d. If origin is outside C, then $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S \text{div } \mathbf{F}\, dA = \iint_S 0\, dA = 0.$

If origin is inside C, let C' be a circle (centered at the origin) inside C and oriented clockwise. Let S be the region between C and C'.

Then $0 = \iint_S \text{div } \mathbf{F}\, dA$ (by "origin outside C" case)

$= \int_C \mathbf{F} \cdot \mathbf{n}\, ds - \int_{C'} \mathbf{F} \cdot \mathbf{n}\, ds$ (by Green's Theorem)

$= \int_C \mathbf{F} \cdot \mathbf{n}\, ds - 2\pi$ (by part a), so $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = 2\pi.$

26. a. Equation of C:

$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\langle x_1 - x_0, y_1 - y_0 \rangle$,

t in [0, 1].

Thus;

$\int_C x\,dy = \int_0^1 [x_0 + t(x_1 - x_0)](y_1 - y_0) dt$,

which equals the desired result.

b. $\text{Area}(P) = \int_C x\,dy$ where

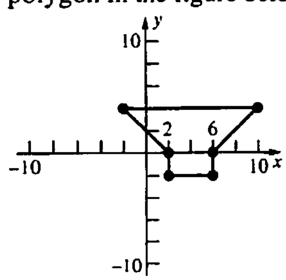
$C = C_1 \cup C_2 \cup \ldots \cup C_n$ and C_i is the ith edge. (by Problem 21)

$= \int_{C_1} x\,dy + \int_{C_2} x\,dy + \ldots + \int_{C_n} x\,dy$

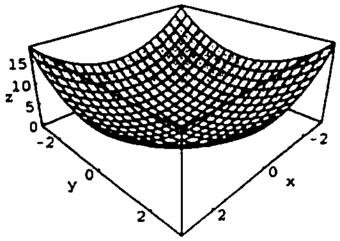
$= \sum_{i=1} \frac{(x_i - x_{i-1})(y_i - y_{i-1})}{2}$ (by part a)

c. Immediate result of part b if each x_i and each y_i is an integer.

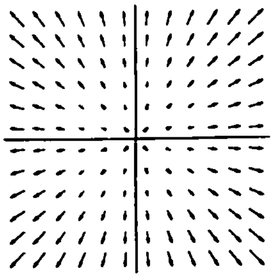
d. Formula gives 40 which is correct for the polygon in the figure below.



27. a. div $\mathbf{F} = 4$

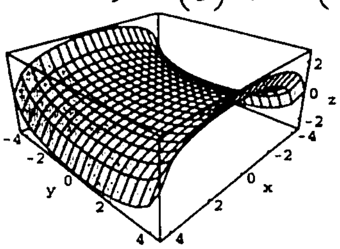


$x^2 + y^2$

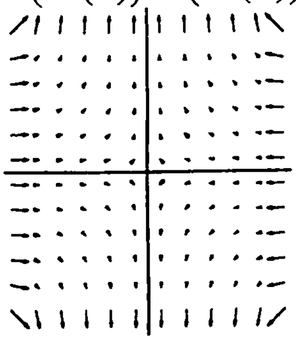


b. $4(36) = 144$

28. a. $\text{div}\ \mathbf{F} = -\frac{1}{9}\sec^2\left(\frac{x}{3}\right) + \frac{1}{9}\sec^2\left(\frac{y}{3}\right)$



$\ln\left(\cos\left(\frac{x}{3}\right)\right) - \ln\left(\cos\left(\frac{y}{3}\right)\right)$

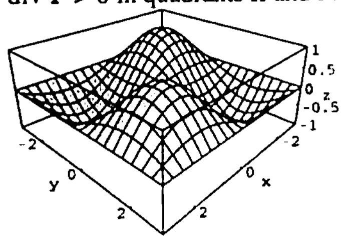


b. $\frac{1}{9}\int_{-3}^{3}\int_{-3}^{3}\left[-\sec^2\left(\frac{x}{3}\right) + \sec^2\left(\frac{y}{3}\right)\right]dy\,dx = 0$

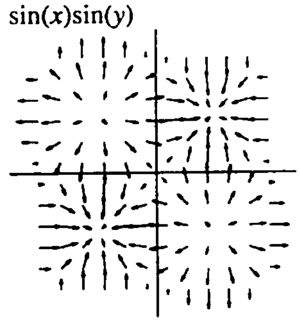
29. a. div $\mathbf{F} = -2 \sin x \sin y$

div $\mathbf{F} < 0$ in quadrants I and III

div $\mathbf{F} > 0$ in quadrants II and IV



sin(*x*)sin(*y*)

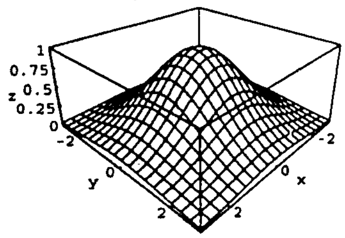


b. Flux across boundary of S is 0.

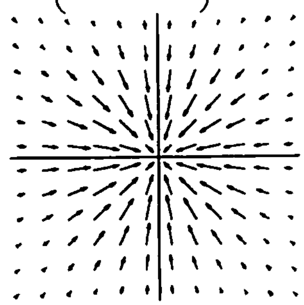
Flux across boundary T is $-2(1 - \cos 3)^2$.

30. $\text{div}\ \mathbf{F} = \frac{1}{4}e^{-(x^2+y^2)/4}(x^2 + y^2 - 4)$

so div $\mathbf{F} < 0$ when $x^2 + y^2 < 4$ and div $\mathbf{F} > 0$ when $x^2 + y^2 > 4$.



$\exp\left(-\frac{(x^2 + y^2)}{4}\right)$



17.5 Concepts Review

1. surface integral

2. $\sum_{i=1}^{n} g(\overline{x}_i, \overline{y}_i, \overline{z}_i)\Delta S_i$

3. $\sqrt{f_x^2 + f_y^2 + 1}$

4. 2; 18π

Problem Set 17.5

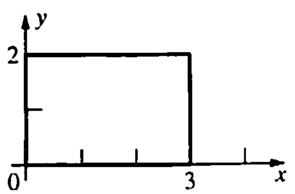
1. $\iint_R [x^2 + y^2 + (x + y + 1)](1 + 1 + 1)^{1/2} dA$
$= \int_0^1 \int_0^1 \sqrt{3}(x^2 + y^2 + x + y + 1)dx\,dy = \frac{8\sqrt{3}}{3}$

2. $\iint_R x\left(\frac{1}{4} + \frac{1}{4} + 1\right)^{1/2} dA = \int_0^1 \int_0^1 \left(\frac{\sqrt{6}}{2}\right) x\,dx\,dy$
$= \frac{\sqrt{6}}{4}$

3 $\iint_R (x + y)\sqrt{[-x(4 - x^2)^{-1/2}]^2 + 0 + 1}\,dA$
$= \int_0^{\sqrt{3}} \int_0^1 \frac{2(x + y)}{(4 - x^2)^{1/2}} dy\,dx$
$= \int_0^{\sqrt{3}} \frac{2x + 1}{(4 - x^2)^{1/2}} dx$
$= \left[-2(4 - x^2)^{1/2} + \sin^{-1}\left(\frac{x}{2}\right)\right]_0^{\sqrt{3}}$
$= \frac{\pi + 6}{3} = 2 + \frac{\pi}{3} \approx 3.0472$

4. $\int_0^{2\pi} \int_0^1 r^2 (4r^2 + 1)^{1/2} r\,dr\,d\theta = \left(\frac{\pi}{60}\right)(25\sqrt{5} + 1)$
$\approx$ 2.9794

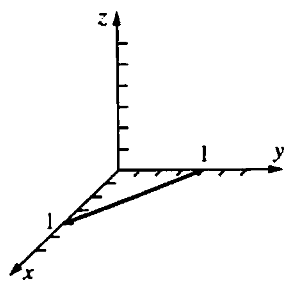
5. $\int_0^{\pi} \int_0^{\sin\theta} (4r^2 + 1) r\,dr\,d\theta = \left(\frac{5}{8}\right)\pi \approx 1.9635$

6. 

$\iint_R y(4y^2 + 1)^{1/2} dA = \int_0^3 \int_0^2 (4y^2 + 1)^{1/2} y\,dy\,dx$
$= \int_0^3 \frac{(17^{3/2} - 1)}{12} dx = \frac{17^{3/2} - 1}{4} \approx 17.2732$

7. $\iint_R (x + y)(0 + 0 + 1)^{1/2} dA$
Bottom $(z = 0)$: $\int_0^1 \int_0^1 (x + y)dx\,dy = 1$
Top $(z = 1)$: Same integral
Left side $(y = 0)$: $\int_0^1 \int_0^1 (x + 0)dx\,dz = \frac{1}{2}$
Right side $(y = 1)$: $\int_0^1 \int_0^1 (x + 1)dx\,dz = \frac{3}{2}$
Back $(x = 0)$: $\int_0^1 \int_0^1 (0 + y)dy\,dz = \frac{1}{2}$
Front $(x = 1)$: $\int_0^1 \int_0^1 (1 + y)dy\,dz = \frac{3}{2}$
Therefore, the integral equals
$1 + 1 + \frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{3}{2} = 6.$

8. Bottom $(z = 0)$: The integrand is 0 so the integral is 0.
Left face $(y = 0)$: $\int_0^4 \int_0^{8-2x} z\sqrt{1}dz\,dx = \frac{128}{3}$
Right face $(z = 8 - 2x - 4y)$:
$\int_0^2 \int_0^{4-2y} (8 - 2x - 4y)(4 + 16 + 1)^{1/2} dx\,dy$
$= \left(\frac{32}{3}\right)\sqrt{21}$
Back face $(x = 0)$: $\int_0^2 \int_0^{8-4y} z\sqrt{1}\,dz\,dy = \frac{64}{3}$
Therefore, integral $= 64 + \left(\frac{32}{3}\right)\sqrt{21} \approx 112.88.$

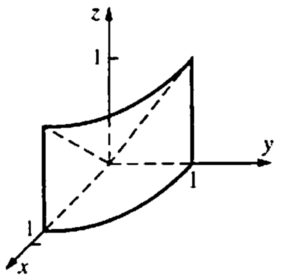
9. 

$\iint_G \mathbf{F} \cdot \mathbf{n}\,ds = \iint_R (-Mf_x - Nf_y + P)dA$
$= \int_0^1 \int_0^{1-y} (8y + 4x + 0)dx\,dy$
$= \int_0^1 [8(1 - y)y + 2(1 - y)^2]dy$
$= \int_0^1 (-6y^2 + 4y + 2)dy = 2$

10. $\int_0^3 \int_0^{(6-2x)/3} (x^2 - 9)\left(-\frac{1}{2}\right) dy\, dx = 11.25$

11. $\int_0^5 \int_{-1}^1 [-xy(1-y^2)^{-1/2} + 2] dy\, dx = 20$
(In the inside integral, note that the first term is odd in y.)

12.



$\iint_R [-Mf_x - Nf_y + P] dA$

$= \iint_R [-2x(x^2+y^2)^{-1/2} - 5y(x^2+y^2)^{-1/2} + 3] dA$

$= \int_0^{2\pi} \int_0^1 [(-2r\cos\theta - 5r\sin\theta)r^{-1} + 3] r\, dr\, d\theta$

$= \int_0^{2\pi} (-2\cos\theta - 5\sin\theta + 3) d\theta \int_0^1 r\, dr$

$= (6\pi)\left(\frac{1}{2}\right) = 3\pi \approx 9.4248$

13. $m = \iint_G kx^2 ds = \iint_R kx^2 \sqrt{3} dA$

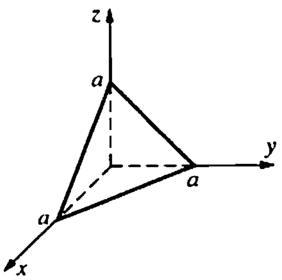
$= \sqrt{3}k \int_0^a \int_0^{a-x} x^2 dy\, dx = \left(\frac{\sqrt{3}k}{12}\right) a^4$

14. $m = \iint_G kxy\, ds = \iint_R kxy(x^2+y^2+1)^{1/2} dA$

$= \int_0^1 \int_0^1 kxy(x^2+y^2+1)^{1/2} dx\, dy$

$= \left(\frac{k}{15}\right)(9\sqrt{3} - 8\sqrt{2} + 1) \approx 0.3516k$

15.



Let $\delta = 1$.

$m = \iint_S 1\, ds = \iint_R (1+1+1)^{1/2} dA$

$= \sqrt{3} \int_0^a \int_0^{a-y} dx\, dy = \sqrt{3} \int_0^a (a-y) dy = \frac{a^2\sqrt{3}}{2}$

$M_{xy} = \iint_S z\, ds = \iint_R (a - x - y)\sqrt{3}\, dA$

$= \sqrt{3} \int_0^a \int_0^{a-y} (a-x-y) dx\, dy$

$= \sqrt{3} \int_0^a \left[a(a-y) - \frac{(a-y)^2}{2} - y(a-y) \right] dy$

$= \sqrt{3} \int_0^a \left(\frac{a^2}{2} - ay + \frac{y^2}{2} \right) dy = \frac{a^3\sqrt{3}}{6}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{a}{3}$; then $\bar{x} = \bar{y} = \frac{a}{3}$ (by symmetry).

16. $m = \iint_G z\, ds = \iint_R 3\, dA$
($= 3A(R) = 3\pi(3)^2 = 27\pi$, ignoring the subtlety)

$= \lim_{\varepsilon=0} \int_0^{2\pi} \int_0^{3-\varepsilon} 3r\, dr\, d\theta = \lim_{\varepsilon=0} 3(3-\varepsilon)^2 \pi = 27\pi$

17. a. 0 (By symmetry, since $g(x, y, -z) = -g(x, y, z)$.)

b. 0 (By symmetry, since $g(x, y, -z) = -g(x, y, z)$.)

c. $\iint_G (x^2+y^2+z^2) dS = \iint_G a^2 dS$

$= a^2 \text{Area}(G) = a^2(4\pi a^2) = 4\pi a^4$

d. Note:
$\iint_G (x^2+y^2+z^2) dS = \iint_G x^2 dS + \iint_G y^2 dS$
$= \iint_G z^2 dS = 3\iint_G x^2 dS$
(due to symmetry of the sphere with respect to the origin.)
Therefore,
$\iint_G x^2 dS = \left(\frac{1}{3}\right) \iint_G (x^2+y^2+z^2) dS$

$= \left(\frac{1}{3}\right) 4\pi a^4 = \frac{4\pi a^4}{3}$.

e. $\iint_G (x^2+y^2) dS = \left(\frac{2}{3}\right) 4\pi a^4 = \frac{8\pi a^4}{3}$

18. a. Let the diameter be along the z-axis.
$I_z = \iint_G k(x^2+y^2) dS$
1. $\iint_G x^2 dS = \iint_G y^2 dS = \iint_G z^2 dS$ (by symmetry of the sphere)
2. $\iint_G (x^2+y^2+z^2) dS = \iint_G a^2 dS$
$= a^2$(Area of sphere)$= a^2(4\pi a^2) = 4\pi a^4$
Thus,

$I_z = \iint_G k(x^2 + y^2)dS = \frac{2}{3}k(4\pi a^4) = \frac{8\pi a^4 k}{3}$.
(using 1 and 2)

b. Let the tangent line be parallel to the z-axis.

Then $I = I_z + ma^2 = \frac{8\pi a^4 k}{3} + [k(4\pi a^2)]a^2$

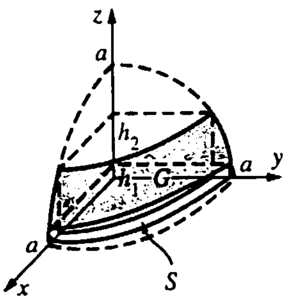
$= \frac{20\pi a^4 k}{3}$.

19. a. Place center of sphere at the origin.
$F = \iint_G k(a - z)dS = ka\iint_G 1\,dS - k\iint_G z\,dS$
$= ka(4\pi a^2) - 0 = 4\pi a^3 k$

b. Place hemisphere above xy-plane with center at origin and circular base in xy-plane.
F = Force on hemisphere + Force on circular base
$= \iint_G k(a - z)dS + ka(\pi a^2)$
$= ka\iint_G 1\,dS - k\iint_G z\,dS + \pi a^3 k$
$= ka(2\pi a^2) - k\iint_R z\sqrt{\frac{a^2}{a^2 - x^2 - y^2}}\,dA + \pi a^3 k$
$= 3\pi a^3 k - k\iint_R z\frac{a}{z}dA$
$= 3\pi a^3 k - ka(\pi a^2) = 2\pi a^3 k$

c. Place the cylinder above xy-plane with circular base in xy-plane with the center at the origin.
F = Force on top + Force on cylindrical side + Force on base
$= 0 + \iint_G k(h - z)dS + kh(\pi a^2)$
$= kh\iint_G 1\,dS - k\iint_G z\,dS + \pi a^2 hk$
$= kh(2\pi ah) - 4k\iint_R z\sqrt{\frac{a^2}{a^2 - y^2} + 0 + 1}\,dA + \pi a^2 hk$
(where R is a region in the yz-plane: $0 \le y = a, 0 \le z \le h$)
$= 2\pi ah^2 k + \pi a^2 hk - 4k\int_0^a \int_0^h \frac{az}{\sqrt{a^2 - y^2}}\,dz\,dy$
$= 2\pi ah^2 k + \pi a^2 hk - \pi kah^2$
$= \pi ah^2 k + \pi a^2 hk = \pi ahk(h + a)$

20. $\bar{x} = \bar{y} = 0$
Now let G' be the 1st octant part of G.
$M_{xy} = \iint_G k\,dS = 4\iint_{G'} kz\,dS = 4k\iint_{R'} z\left(\frac{a}{z}\right)dA$
(See Problem 19b.)
$= 4ak\ [\text{Area }(R')]$
$= 4ak\pi\left[\frac{(a^2 - h_1^2)}{4} - \frac{(a^2 - h_2^2)}{4}\right]$
$= ak\pi(h_2^2 - h_1^2)$



$m(G) = \iint_G k\,dS = k[\text{Area}(G)]$
$= k[2\pi a(h_2 - h_1)] = 2\pi ak(h_2 - h_1)$

Therefore, $\bar{z} = \frac{\pi ak(h_2^2 - h_1^2)}{2\pi ak(h_2 - h_1)} = \frac{h_1 + h_2}{2}$.

17.6 Concepts Review

1. boundary; ∂S

2. $\mathbf{F} \cdot \mathbf{n}$

3. div $\mathbf{F}$

4. flux; the shape

Problem Set 17.6

1. $\iiint_S (0 + 0 + 0)dV = 0$

2. $\iiint_S (1 + 2 + 3)dV = 6V(S) = 6$

3. $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S (M_x + N_y + P_z)dV$

$= \int_0^c \int_0^b \int_0^a (2xyz + 2xyz + 2xyz)dx\,dy\,dz$

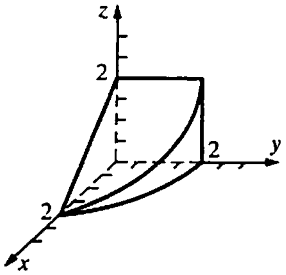
$= \int_0^c \int_0^b 3a^2 yz\,dy\,dz = \int_0^c \frac{3a^2b^2z}{2}dz$

$= \frac{3a^2b^2c^2}{4}$

4. $\iiint_S (3-2+4)dV = 5V(S) = 5\left[\left(\frac{4}{3}\right)\pi(3)^3\right]$

$= 180\pi = 565.49$

5. $2\iiint_S (x+y+z)dV \;= 2\int_0^{2\pi}\int_0^2\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)r\,dz\,dr\,d\theta \;= \frac{64\pi}{3} \approx 67.02$

6. 

$\iiint_S (M_x + N_y + P_z)dV = \iiint_S (2x+1+2z)dV \;= \int_0^{2\pi}\int_0^2\int_0^{2-r\cos\theta}(2r\cos\theta + 1 + 2z)r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 [(2r^2\cos\theta + r)(2-r\cos\theta) + r(2-r\cos\theta)^2]dr\,d\theta \;= \int_0^{2\pi}\int_0^2 (6r - r^3\cos^2\theta - r^2\cos\theta)dr\,d\theta$

$= \int_0^{2\pi}\left(12 - 4\cos^2\theta - \frac{8\cos\theta}{3}\right)d\theta = 20\pi$

7. $\iiint_S (1+1+0)dV = 2(\text{volume of cylinder}) \;= 2\pi(1)^2(2) = 4\pi \approx 12.5664$

8. $\iiint_S (2x+2y+2z)dV = \int_0^4\int_0^{4-x}\int_0^{4-x-y}(2x+2y+2z)dz\,dy\,dx \;= 64$

9. $\iiint_S (M_x + N_y + P_z)dV = \iiint_S (2+3+4)dV$

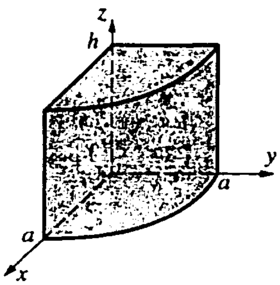
$= 9(\text{Volume of spherical shell})$

$= 9\left(\frac{4\pi}{3}\right)(5^3 - 3^3) = 1176\pi \approx 3694.51$

10. $\iiint_S (0+0+2z)dV = \int_0^{2\pi}\int_1^2\int_0^2 2zr\,dz\,dr\,d\theta$

$= 12\pi \approx 37.6991$

11. $\left(\frac{1}{3}\right)\iiint_S (1+1+1)dV = V(S)$

12. 

$V(S) = \frac{1}{3}\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F} = \langle x, y, z\rangle$

$= \frac{1}{3}\iiint_S 3\,dV = \int_0^{2\pi}\int_0^a\int_0^h r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^a rh\,dr\,d\theta = \int_0^{2\pi}\frac{a^2h}{2}d\theta = 2\pi\frac{a^2h}{2}$

$= \pi a^2 h$

13. Note:

1. $\iint_R (ax+by+cz)dS = \iint_R d\,dS = dD$ (R is the slanted face.)
2. $\mathbf{n} = \frac{\langle a, b, c\rangle}{(a^2+b^2+c^2)^{1/2}}$ (for slanted face)
3. $\mathbf{F}\cdot\mathbf{n} = 0$ on each coordinate-plane face.

Volume $= \left(\frac{1}{3}\right)\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ (where $\mathbf{F} = \langle x, y, z\rangle$).

$= \left(\frac{1}{3}\right)\iint_R \mathbf{F}\cdot\mathbf{n}\,dS$ (by Note 3)

$= \left(\frac{1}{3}\right)\iint_R \frac{(ax+by+cz)}{\sqrt{a^2+b^2+c^2}}dS$

$= \frac{dD}{3\sqrt{a^2+b^2+c^2}}$

14. $\iiint_S \operatorname{div}\mathbf{F}\,dV = \iiint_S 0\,dV = 0$ ("Nice" if there is an outer normal vector at each point of ∂S.)

15. a. $\operatorname{div}\mathbf{F} = 2 + 3 + 2z = 5 + 2z$

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S (5+2z)dV = \iiint_S 5\,dV + 2\iiint_S z\,dV = 5$ (Volume of S) $+ 2M_{xy}$

$= 5\left(\frac{4\pi}{3}\right) + 2\bar{z}$ (Volume of S) $= \frac{20\pi}{3} + 2(0)$(Volume) $= \frac{20\pi}{3}$

b. $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2+z^2)^{3/2}\langle x, y, z\rangle\cdot\frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^2 = 1$ on ∂S.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S 1\,dV = 4\pi(1)^2 = 4\pi$

c. $\operatorname{div}\mathbf{F} = 2x + 2y + 2z$

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S 2(x+y+z)dV$

$= 2\iiint_S x\,dV$ (Since $\bar{x} = \bar{z} = 0$ as in a.)

$= 2M_{yz} = 2(\bar{x})$(Volume of S) $= 2(2)\left(\frac{4\pi}{3}\right) = \frac{16\pi}{3}$

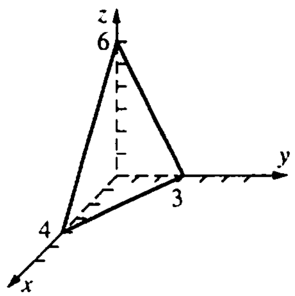
d. $\mathbf{F}\cdot\mathbf{n} = 0$ on each face except the face R in the plane $x = 1$.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \langle 1, 0, 0\rangle\cdot\langle 1, 0, 0\rangle dS = \iint_R 1\,dS = (1)^2 = 1$

e. $\operatorname{div}\mathbf{F} = 1 + 1 + 1 = 3$

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S 3\,dV = 3$(Volume of S) $= 3\left(\frac{1}{3}\left[\frac{1}{2}(4)(3)\right](6)\right) = 36$

f.



$\text{div}\,\mathbf{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3$ on ∂S.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = 3\iiint_S (x^2 + y^2 + z^2)\,dV = 3\left(\frac{3}{2}\frac{8\pi}{15}\right) = \frac{12\pi}{5}$

(That answer can be obtained by making use of symmetry and a change to spherical coordinates. Or you could go to the solution for Problem 24, Section 16.8, and realize that the value of the integral in this problem is $\frac{3}{2}$ (there are three terms instead of two) times the answer obtained there for I_z, letting $kabc = 1$.)

g. $\mathbf{F}\cdot\mathbf{n} = [\ln(x^2 + y^2)]\langle x, y, 0\rangle \cdot \langle 0, 0, 1\rangle = 0$ on top and bottom.

$\mathbf{F}\cdot\mathbf{n} = (\ln 4)\langle x, y, 0\rangle \cdot \frac{\langle x, y, 0\rangle}{\sqrt{x^2 + y^2}} = (\ln 4)\sqrt{x^2 + y^2} = (\ln 4)\sqrt{4} = 2\ln 4 = 4\ln 2$ on the side.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R 4\ln 2\,dS = (4\ln 2)[2\pi(2)(2)] = 32\pi\ln 2$

16. a. $\text{div}\,\mathbf{F} = 0$ (See Problem 21, Section 17.1.)

Therefore, $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S \text{div}\,\mathbf{F}\,dV = 0.$

b. $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi$ (by Gauss's law with $-cM = 1$ as in Example 5).

c. $\mathbf{F}\cdot\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{|\mathbf{r}|} = \frac{1}{a}$ on ∂S.

Thus, $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \left(\frac{1}{a}\right)\iint_{\partial S} 1\,dS$

$= \left(\frac{1}{a}\right)$(Surface area of sphere)

$= \left(\frac{1}{2}\right)(4\pi a^2) = 4\pi a$.

d. $\mathbf{F}\cdot\mathbf{n} = f(|r|)r\cdot\frac{r}{|r|} = |r|f(|r|) = af(a)$ on ∂S.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = af(a)\iint_{\partial S} 1\,dS = [af(a)](4\pi a^2) = 4\pi a^3 f(a)$

e. The ball is above the xy-plane, is tangent to the xy-plane at the origin, and has radius $\frac{a}{2}$.

$\text{div}\,\mathbf{F} = |\mathbf{r}|^n \,\text{div}\,\mathbf{r} + \left(\text{grad}\,|f|^n\right)\cdot\mathbf{r}$ (See Problem 20c, Section 17.1.)

$= |\mathbf{r}|^n (1+1+1) + n|f|^{n-1} \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{r} = 3|\mathbf{r}|^n + n|\mathbf{r}|^n = (3+n)|\mathbf{r}|^n$

$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\, dS = (3+n)\iiint_S |\mathbf{r}|^n \, dV = (3+n)\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a\cos\phi} \rho^n (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \frac{2\pi a^{n+3}}{n+4}$

17. $\iint_{\partial S} D_{\mathbf{n}} f\, dS = \iint_{\partial S} \nabla f \cdot \mathbf{n}\, dS = \iiint_S \operatorname{div}(\nabla f) dV = \iiint_S \nabla^2 f\, dV$ (See next problem.)

18. $\iint_{\partial S} f(\nabla f \cdot \mathbf{n}) dS = \iint_{\partial S} (f\nabla f) \cdot \mathbf{n}\, dS = \iiint_S \operatorname{div}(f\nabla f) dV$

$= \iiint_S \operatorname{div}(\nabla f) + (\nabla f)\cdot(\nabla f) dV$ (See Problem 20c, Section 17.1.)

$= \iiint_S \left[(f_{xx} + f_{yy} + f_{zz}) + |\nabla f|^2 \right] dV$

$= \iiint_S [(\nabla^2 f) + |\nabla f|^2] dV = \iiint_S |\nabla f|^2\, dV$ (Since it is given that $\nabla^2 f = 0$ on S.)

19. $\iint_{\partial S} f D_{\mathbf{n}} g\, dS = \iint_{\partial S} f(\nabla g \cdot \mathbf{n}) dS = \iint_{\partial S} (f\nabla g)\cdot \mathbf{n}\, dS = \iiint_S \operatorname{div}(f\nabla g) dV$ (Gauss)

$= \iiint_S [f(\operatorname{div} \nabla g) + (\nabla f)\cdot(\nabla g)] dV = \iiint_S (f\nabla^2 g + \nabla f \cdot \nabla g) dV$ (See Problem 20c, Section 17.1.)

20. $\iint_{\partial S} (f D_{\mathbf{n}} g - g D_{\mathbf{n}} f) dS = \iint_{\partial S} f D_{\mathbf{n}} g\, dS - \iint_{\partial S} g D_{\mathbf{n}} f\, dS$

$= \iiint_S (f\nabla^2 g + \nabla f \cdot \nabla g) dV - \iiint_S (g\nabla^2 f + \nabla g \cdot \nabla f) dV$ (by Green's 1st identity)

$= \iiint_S (f\nabla^2 g - g\nabla^2 f) dV$

17.7 Concepts Review

1. $(\text{curl } \mathbf{F})\cdot\mathbf{n}$

2. Möbius band

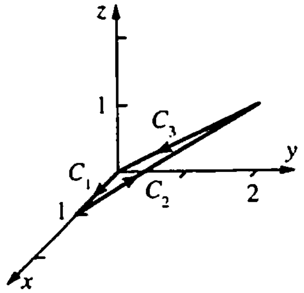
3. $\iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS$

4. curl **F**

Problem Set 17.7

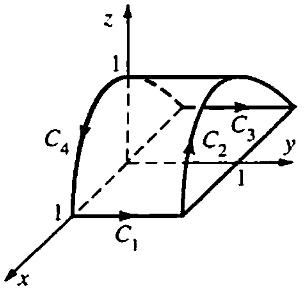
1. $\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\, dS = \iint_R (N_x - M_y) dA = \iint_R 0\, dA = 0$

2.



$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\, ds = \int_{C_1} 0\, dx + \int_{C_2} xy\, dx + yz\, dy + xz\, dz + \int_{C_3} yz\, dy = \int_0^1 (t^2 + 7t - 4) dt = -\frac{1}{6}$

3.



$\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds = \oint_{\partial S}(y+z)dx + (x^2+z^2)dy + y\,dz$

$= \int_0^1 1\,dt + \int_0^\pi [(1+\sin t)(-\sin t)+\cos t]dt + \int_0^1 -1\,dt + \int_0^\pi \sin^2 t\,dt$ (*)

$= \int_0^\pi (-\sin t + \cos t)dt = -2$

The result at (*) was obtained by integrating along S by doing so along C_1, C_2, C_3, C_4 in that order.

Along C_1 : $x = 1$, $y = t$, $z = 0$, $dx = dz = 0$, $dy = dt$, t in [0, 1]

Along C_2 : $x = \cos t$, $y = 1$, $z = \sin t$, $dx = -\sin dt$, $dy = 0$, $dz = \cos t\,dt$, t in $[0, \pi]$

Along C_3 : $x = -1$, $y = 1 - t$, $z = 0$, $dx = dz = 0$, $dy = dt$, t in [0, 1]

Along C_4 : $x = -\cos t$, $y = 0$, $z = \sin t$, $dx = \sin t\,dt$, $dy = 0$, $dz = \cos t\,dt$, t in $[0, \pi]$

4. ∂S is the circle $x^2 + y^2 = 1$, $z = 0$ (in the xy-plane).

$x = \cos t$
$y = \sin t$
$z = 0$
t in $[0, 2\pi]$

$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds = \oint_{\partial S} xy^2dx + x^3dy = (\cos xz)dz$

$= \oint_S x^3dy = \int_0^{2\pi}(\cos^3 t)(-\cos t)dt = \left(-\frac{3}{4}\right)\pi$

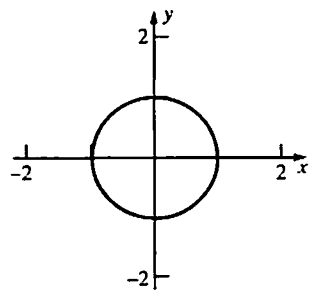
≈ -2.3562

5. ∂S is the circle $x^2 + y^2 = 12$, $z = 2$.
Parameterization of circle:
$x = \sqrt{12}\cos t$, $y = \sqrt{12}\sin t$, $z = 2$, t in $[0, 2\pi]$

$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds = \oint_{\partial S} yz\,dx + 3xz\,dy + z^2dz$

$= \int_0^{2\pi}(24\sin^2 t - 72\cos^2 t)dt = -48\pi \approx -150.80$

6. ∂S is the circle $x^2 + y^2 = 1$, $z = 0$



$\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{N}\,dS = \oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds$

$= \oint_S (z-y)dx + (z+x)dy + (-x-y)dz$

$= \int_0^{2\pi}[(-\sin t)(-\sin t) + (\cos t)(\cos t)]dt = 2\pi \approx 6.2832$

7. $(\text{curl } \mathbf{F})\cdot\mathbf{n} = \langle 3, 2, 1\rangle \cdot \left\langle \left(\frac{1}{\sqrt{2}}\right)1, 0, -1\right\rangle = \sqrt{2}$

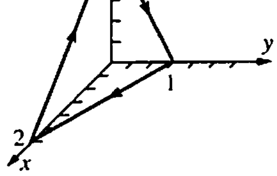
$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \sqrt{2}dS = \sqrt{2}A(S)$

$= \sqrt{2}[\sec(45°)]$(Area of a circle)$=8\pi \approx 25.1327$

8. $(\text{curl } \mathbf{F})\cdot\mathbf{n} = \langle -1, -1, -1\rangle \cdot \left[\left(\frac{1}{\sqrt{2}}\right)\langle 0, 1, -1\rangle\right] = 0$,

so the integral is 0.

9. $(\text{curl } \mathbf{F}) = \langle -1+1, 0-1, 1-1\rangle = \langle 0, -1, 0\rangle$
The unit normal vector that is needed to apply Stokes' Theorem points downward. It is
$n = \frac{\langle -1, -2, -1\rangle}{\sqrt{6}}$.



$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$

$= \iint_S \left(\frac{2}{\sqrt{6}}\right) dS = \iint_R \left(\frac{2}{\sqrt{6}}\right)(1+4+1)^{1/2} dA$

$= \iint_R 2\,dA = 2(\text{Area of triangle in } xy-\text{plane})$

$= 2(1) = 2$

10. $(\text{curl } \mathbf{F})\cdot\mathbf{n} = \langle 0, 0, -4x^2 - 4y^2\rangle \cdot \left[\left(\frac{1}{\sqrt{2}}\right)\langle -1, 0, 1\rangle\right]$

$= -2\sqrt{2}(x^2 + y^2)$

$\iint_S -\frac{2}{\sqrt{2}}(x^2 + y^2)\,dS$

$= -4\int_0^1\int_0^1 (x^2 + y^2)\,dx\,dy = -\frac{8}{3}$

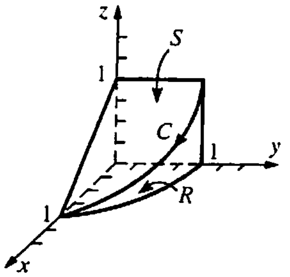
11. $(\text{curl } \mathbf{F})\cdot\mathbf{n} = \langle 0, 0, 1\rangle \cdot \langle x, y, z\rangle = z$

$\iint_S z\,dS = \iint_R 1\,dA = \text{Area of } R$

$\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \approx 0.7854$

12. $(\text{curl } \mathbf{F}) = \langle -1-1, -1-1, -1-1\rangle = -2\langle 1, 1, 1\rangle,$

$\mathbf{n} = \frac{\langle 1, 0, 1\rangle}{\sqrt{2}}$, so $(\text{curl } \mathbf{F})\cdot\mathbf{n} = -2\sqrt{2}.$



$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$

$= -2\sqrt{2}\iint_S 1\,dS = -2\sqrt{2}\iint_R (\sec 45°)\,dA$

$= -\frac{2}{\sqrt{2}\sqrt{2}}[A(R)] = -4\pi$

13. Let $H(x, y, z) = z - g(x, y) = 0.$

Then $\mathbf{n} = \frac{\nabla H}{|\nabla H|} = \frac{\langle -g_x, g_y, 1\rangle}{\sqrt{1 + g_x^2 + g_y^2}}$ points upward.

Thus,

$\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \iint_{S_{xy}} (\text{curl } \mathbf{F})\cdot\mathbf{n}\sec\gamma\,dA$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F})\frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}}\sqrt{g_x^2 + g_y^2 + 1}\,dA$

(Theorem A, Section 17.5)

$= \iint_{S_{xy}} (\text{curl } \mathbf{F})\cdot\langle -g_x, -g_y, 1\rangle\,dA$

14. $\text{curl } \mathbf{F} = \langle z^2, 0, -2y\rangle$

$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$ (Stoke's Theorem)

$= \iint_{S_{xy}} (\text{curl } \mathbf{F})\cdot\langle -g_x, -g_y, 1\rangle\,dA$ (Problem 13)

$= \iint_{S_{xy}} \langle z^2, 0, -2y\rangle\cdot\langle -y, -x, 1\rangle\,dA$

(where $z = xy$)

$= \int_0^1\int_0^1 (-x^2y^3 - 2y)\,dx\,dy = -\frac{13}{12}$

15. $\text{curl } F = \langle 0-x, 0-0, z-0\rangle = \langle -x, 0, z\rangle$

$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F})\cdot\langle -g_x, -g_y, 1\rangle\,dA$ where

$z = g(x, y) = xy^2.$ (Problem 13)

$= \iint_{S_{xy}} \langle -x, 0, z\rangle\cdot\langle -y^2, -2xy, 1\rangle\,dA$

$= \int_0^1\int_0^1 (xy^2 + 0 + xy^2)\,dx\,dy$

$= \int_0^1 \left([x^2y^2]_{x=0}^1\right)dy = \int_0^1 y^2\,dy = \frac{1}{3}$

16. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F})\cdot\langle -g_x, -g_y, 1\rangle\,dA$

$= \iint_{S_{xy}} \langle -x, 0, z\rangle\cdot\langle -2xy^2, -2x^2y, 1\rangle\,dA$

(where $z = x^2y^2$)

$= \iint_{S_{xy}} 3x^2y^2\,dA$

$= 12\int_0^{\pi/2}\int_0^a (r\cos\theta)^2(r\sin\theta)^2\,r\,dr\,d\theta = \frac{\pi a^6}{8}$

17. $\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F})\cdot\langle -g_x, -g_y, 1\rangle\,dA$

$= \iint_{S_{xy}} \langle 2, 2, 0\rangle\cdot\left[\frac{\langle x, y, (a^2 - x^2 - y^2)^{-1/2}\rangle}{(a^2 - x^2 - y^2)^{-1/2}}\right]dA$

$= 2\iint_{S_{xy}} (x + y)(a^2 - x^2 - y^2)^{-1/2}\,dA$

$= 2\iint_{S_{xy}} y(a^2 - x^2 - y^2)^{-1/2}\,dA$

$= 4\int_0^{\pi/2}\int_0^{a\sin\theta} (r\sin\theta)(a^2 - r^2)^{-1/2}\,dr\,d\theta$

$= \frac{4a^2}{3}$ joules

18. curl $\mathbf{F} = 0$ by Problem 23, Section 17.1. The result then follows from Stokes' Theorem since the left-hand side of the equation in the theorem is the work and the integrand of the right-hand side equals 0.

19. a. Let C be any piecewise smooth simple closed oriented curve C that separates the "nice" surface into two "nice" surfaces, S_1 and S_2.

$\iint_{\partial S} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= \oint_C \mathbf{F} \cdot \mathbf{T}\, ds + \oint_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = 0$ ($-C$ is C with opposite orientation.)

b. div(curl $\mathbf{F}$) = 0 (See Problem 20, Section 17.1.) Result follows.

20. $\oint_S (f\nabla g) \cdot \mathbf{T} ds = \iint_S \text{curl}(f\nabla g) \cdot \mathbf{n}\, dS$

$= \iint_S [f(\text{curl } \nabla g) + (\nabla f \times \nabla g)] \cdot \mathbf{n}\, dS$

$= \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, dS$, since curl $\nabla g = 0$.

(See 20b, Section 17.1.)

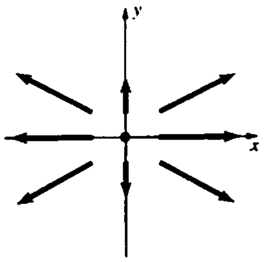
17.8 Chapter Review

Concepts Test

1. True: See Example 4, Section 17.1
2. False: It is a scalar field.
3. False: grad(curl $\mathbf{F}$) is not defined since curl $\mathbf{F}$ is not a scalar field.
4. True: See Problem 20b, Section 17.1.
5. True: See the three equivalent conditions in Section 17.3.
6. True: See the three equivalent conditions in Section 17.3.
7. False: $N_z = 0 \neq z^2 = P_y$
8. True: See discussion on text page 774.
9. True: It is the case in which the surface is in a plane.
10. False: See the Mobius band in Figure 7, Section 17.5.
11. True: See discussion on text page 777.
12. True: div $\mathbf{F} = 0$, so by Gauss's Divergence Theorem, the integral given equals $\iiint_D 0\, dV$ where D is the solid sphere for which $S = \partial D$.

Sample Test Problems

1.



2. div $\mathbf{F} = 2yz - 6y + 2y^2$

curl $\mathbf{F} = \langle 4yz, 2xy, -2xz \rangle$

grad(div $\mathbf{F}$) = $\langle 0, 2z - 6 + 4y, 2y \rangle$

div(curl $\mathbf{F}$) = 0

(See 20a, Section 17.1.)

3. $\text{curl}(f\nabla f) = (f)(\text{curl } \nabla f) + (\nabla f \times \nabla f)$

$= (f)(0) + 0 = 0$

4. a. $f(x, y) = x^2 y + xy + \sin y + C$

b. $f(x, y, z) = xyz + e^{-x} + e^y + C$

5. a. Parameterization is $x = \sin t$, $y = -\cos t$, t in $\left[0, \frac{\pi}{2}\right]$.

$\int_0^{\pi/2} (1 - \cos^2 t)(\sin^2 t + \cos^2 t)^{1/2} dt = \frac{\pi}{4}$

≈ 0.7854

b $\int_0^{\pi/2} [t \cos t - \sin^2 t \cos t + \sin t \cos t] dt$

$= \frac{(3\pi - 5)}{6} \approx 0.7375$

6. $M_x = 2y = N_y$ so the integral is independent of the path. Find any function $f(x, y)$ such that $f_x(x, y) = y^2$ and $f_y(x, y) = 2xy$.

$f(x, y) = xy^2 + C_1(y)$ and

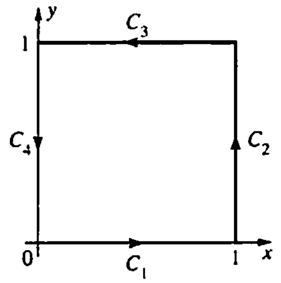
$f(x, y) = xy^2 + C_2(x)$, so let $f(x, y) = xy^2$.

Then the given integral equals $[xy^2]_{(0,\,0)}^{(1,\,2)} = 4$.

7. $[xy^2]_{(1,\,1)}^{(3,\,4)} = 47$

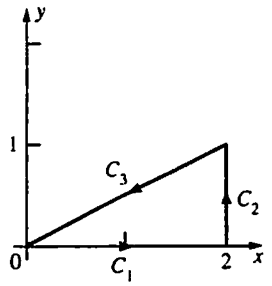
8. $[xyz + e^{-x} + e^y]_{(0,\,0,\,0)}^{(1,\,1,\,4)} = 2 + e^{-1} + e \approx 5.0862$

9. a.



$$\int_0^1 0\,dx + \int_0^1 (1+y^2)dy + \int_1^0 x\,dx + \int_1^0 y^2 dy = 0 + \frac{4}{3} - \frac{1}{2} - \frac{1}{3} = \frac{1}{2}$$

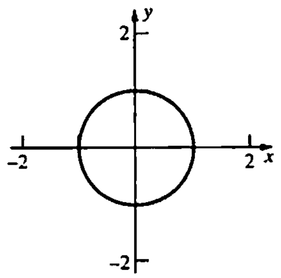
b.



A vector equation of C_3 is $\langle x, y\rangle = \langle 2,1\rangle + t\langle -2,-1\rangle$ for t in $[0, 1]$, so let $x = 2 - 2t$, $y = 1 - t$ for t in $[0, 1]$ be parametric equations of C_3.

$$\int_0^2 0\,dx + \int_0^1 (4+y^2)dy + \int_0^1 [2(1-t)^2(-2) + 5(1-t)^2(-1)]dt = 0 + \frac{13}{3} - 3 = \frac{4}{3}$$

c.



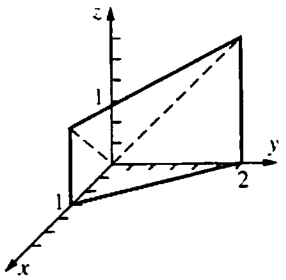
$x = \cos t$
$y = \sin t$
t in $[2, \pi]$

$$\int_0^{2\pi} [(\cos t)(\sin t)(-\sin t) + (\cos^2 t + \sin^2 t)(\cos t)]dt = \int_0^{2\pi} (1 - \sin^2 t)\cos t\,dt = \left[\sin t - \frac{\sin^3 t}{3}\right]_0^{2\pi} = 0$$

10. $\iint_S \text{div}\,\mathbf{F}\,dA = \iint_S 2\,dA = 2A(S) = 8$

11. Let $f(x, y) = (1 - x^2 - y^2)$ and $g(x, y) = -(1 - x^2 - y^2)$, the upper and lower hemispheres.
Then Flux $= \iint_G \mathbf{F}\cdot\mathbf{n}\,dS$
$= \iint_R [-Mf_x - Nf_y + P]dA + \iint_R [-Mg_x - Ng_y + P]dA = \iint_R 2P\,dA$ (since $f_x = -g_x$ and $f_y = -g_y$)
$= \iint_R 6\,dA = 6$ (Area of R, the circle $x^2 + y^2 = 1$, $z = 0$)
$= 6\pi \approx 18.8496$

12.



$\iint_G xyz\,dS = \iint_R xy(x+y)(\sec)dA = \sqrt{3}\int_0^1\int_0^{-2x+2}(x^2y + xy^2)dy\,dx = \sqrt{3}\int_0^1 \frac{4x^2(1-x)^2}{2} + \frac{8x(1-x)^3}{3}dx$
$= -\frac{2\sqrt{3}}{3}\int_0^1 (x^4 - 6x^3 + 9x^2 - 4x)dx = -\frac{2\sqrt{3}}{3}\frac{1}{5} - \frac{3}{2} + 3 - 2 = \frac{3}{5} \approx 0.3464$
$\cos\gamma = \frac{\langle -1, -1, 1\rangle \cdot \langle 0, 0, 1\rangle}{\sqrt{3}}$
Therefore, $\sec\gamma = \sqrt{3}$.

13. ∂S is the circle $x^2 + y^2 = 1$, $z = 1$.
A parameterization of the circle is $x = \cos t$,
$y = \sin t$, $z = 1$, t in $[0, 2\pi]$.
$\oint_{\partial S}\mathbf{F}\cdot\mathbf{T}\,ds = \oint_{\partial S}\left[x^3y\,dx + e^y dy + z\tan\left(\frac{xyz}{4}\right)dz\right] = \oint_{\partial S}(x^3y + e^y dy)$
$= \int_0^{2\pi}[(\cos t)^3(\sin t)(-\sin t) + (e^{\sin t})(\cos t)]dt = 0$

14. $\iiint_S \text{div}\,\mathbf{F}\,dv = \iiint_S [(\cos x) + (1 - \cos x) + (4)]dV = \iiint_S 5\,dV = 5V(S) = 5\left(\frac{1}{2}\right)\left[\left(\frac{4}{3}\right)\pi(3)^3\right]$
$= 90\pi \approx 282.7433$

15. curl $\mathbf{F} = \langle 3-0, 0-0, -1-1\rangle = \langle 3, 0, -2\rangle$
$\mathbf{n} = \frac{\langle a, b, 1\rangle}{\sqrt{a^2 + b^2 + 1}}$
$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS = \iint_S \frac{3a-2}{\sqrt{a^2+b^2+1}}dS = \frac{3a-2}{\sqrt{a^2+b^2+1}}[A(S)]$
$= \frac{3a-2}{\sqrt{a^2+b^2+1}}(9\pi)$ (S is a circle of radius 3.)
$= \frac{9\pi(3a-2)}{\sqrt{a^2+b^2+1}}$