CHAPTER

14 Geometry in Space, Vectors

14.1 Concepts Review

1. coordinates

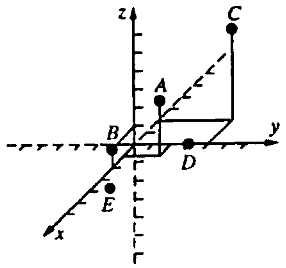
2. $\sqrt{(x+1)^2+(y-3)^2+(z-5)^2}$

3. (–1, 3, 5); 4

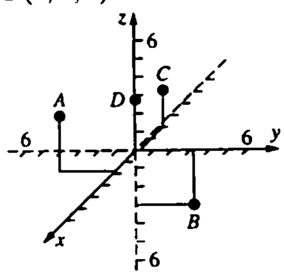
4. plane; 4; –6; 3

Problem Set 14.1

1. $A(1, 2, 3)$, $B(2, 0, 1)$, $C(-2, 4, 5)$, $D(0, 3, 0)$, $E(-1, -2, -3)$



2. $A\left(\sqrt{3}, -3, 3\right)$, $B(0, \pi, -3)$, $C\left(-2, \frac{1}{3}, 2\right)$, $D(0, 0, e)$



3. $x = 0$ in the yz-plane. $x = 0$ and $y = 0$ on the z-axis.

4. $y = 0$ in the xz-plane. $x = 0$ and $z = 0$ on the y-axis.

5. a. $\sqrt{(6-1)^2+(-1-2)^2+(0-3)^2}$
$= \sqrt{25+9+9} = \sqrt{43}$

b. $\sqrt{(-2-2)^2+(-2+2)^2+(0+3)^2}$
$= \sqrt{16+0+9} = 5$

c. $\sqrt{(e+\pi)^2+(\pi+4)^2+\left(0-\sqrt{3}\right)^2}$
$\sqrt{(e+\pi)^2+(\pi+4)^2+3} \approx 9.399$

6. $P(4, 5, 3)$, $Q(1, 7, 4)$, $R(2, 4, 6)$
$|PQ| = \sqrt{(4-1)^2+(5-7)^2+(3-4)^2}$
$= \sqrt{9+4+1} = \sqrt{14}$
$|PR| = \sqrt{(4-2)^2+(5-4)^2+(3-6)^2}$
$= \sqrt{4+1+9} = \sqrt{14}$
$|QR| = \sqrt{(1-2)^2+(7-4)^2+(4-6)^2}$
$= \sqrt{1+9+4} = \sqrt{14}$
Since the distances are equal, the triangle formed by joining P, Q, and R is equilateral.

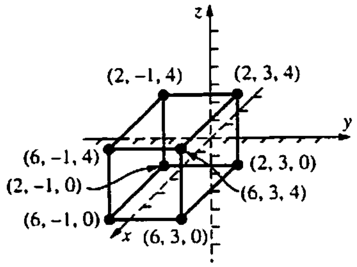
7. $P(2, 1, 6)$, $Q(4, 7, 9)$, $R(8, 5, -6)$
$|PQ| = \sqrt{(2-4)^2+(1-7)^2+(6-9)^2}$
$= \sqrt{4+36+9} = 7$
$|PR| = \sqrt{(2-8)^2+(1-5)^2+(6+6)^2}$
$= \sqrt{36+16+144} = 14$
$|QR| = \sqrt{(4-8)^2+(7-5)^2+(9+6)^2}$
$= \sqrt{16+4+225} = \sqrt{245}$
$|PQ|^2+|PR|^2 = 49+196 = 245 = |QR|^2$, so the triangle formed by joining P, Q, and R is a right triangle, since it satisfies the Pythagorean Theorem.

8. a. The distance to the xy-plane is 1 since the point is 1 unit below the plane.

b. The distance is $\sqrt{(2-0)^2+(3-3)^2+(-1-0)^2} = \sqrt{5}$ since the distance from a point to a line is the length of the shortest segment joining the point and the line. Using the point (0, 3, 0) on the y-axis clearly minimizes the length.

c. $\sqrt{(2-0)^2+(3-0)^2+(-1-0)^2}$
$=\sqrt{2^2+3^2+(-1)^2}=\sqrt{14}$

9. Since the faces are parallel to the coordinate planes, the sides of the box are in the planes $x=2, y=3, z=4, x=6, y=-1$, and $z=0$ and the vertices are at the points where 3 of these planes intersect. Thus, the vertices are (2, 3, 4), (2, 3, 0), (2, –1, 4), (2, –1, 0), (6, 3, 4), (6, 3, 0), (6, –1, 4), and (6, –1, 0)



10. It is parallel to the y-axis; $x=2$ and $z=3$. (If it were parallel to the x-axis, the y-coordinate could not change, similarly for the z-axis.)

11. a. $(x-1)^2+(y-2)^2+(z-3)^2=25$

b. $(x+2)^2+(y+3)^2+(z+6)^2=5$

c. $(x-\pi)^2+(y-e)^2+\left(z-\sqrt{2}\right)^2=\pi$

12. Since the sphere is tangent to the xy-plane, the point (2, 4, 0) is on the surface of the sphere. Hence, the radius of the sphere is 5 so the equation is $(x-2)^2+(y-4)^2+(z-5)^2=25$.

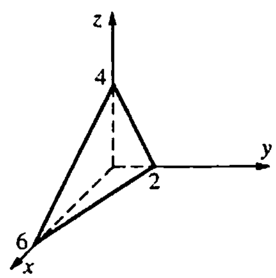
13. $(x^2-12x+36)+(y^2+14y+49)+(z^2-8z+16)=-1+36+49+16$
$(x-6)^2+(y+7)^2+(z-4)^2=100$
Center: (6, –7, 4); radius 10

14. $(x^2+2x+1)+(y^2-6y+9)+(z^2-10z+25)=-34+1+9+25$
$(x+1)^2+(y-3)^2+(z-5)^2=1$
Center: (–1, 3, 5); radius 1

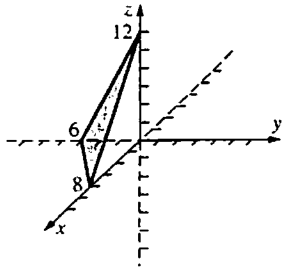
15. $x^2+y^2+z^2-x+2y+4z=\frac{13}{4}$
$\left(x^2-x+\frac{1}{4}\right)+(y^2+2y+1)+(z^2+4z+4)=\frac{13}{4}+\frac{1}{4}+1+4$
$\left(x-\frac{1}{2}\right)^2+(y+1)^2+(z+2)^2=\frac{17}{2}$
Center: $\left(\frac{1}{2},-1,-2\right)$; radius $\sqrt{\frac{17}{2}}\approx 2.92$

16. $(x^2+8x+16)+(y^2-4y+4)+(z^2-22z+121)=-77+16+4+121$
$(x+4)^2+(y-2)^2+(z-11)^2=64$
Center: (–4, 2, 11); radius 8

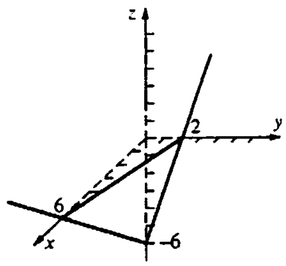
17. x-intercept: $y=z=0 \Rightarrow 2x=12, x=6$
y-intercept: $x=z=0 \Rightarrow 6y=12, y=2$
z-intercept: $x=y=0 \Rightarrow 3z=12, z=4$



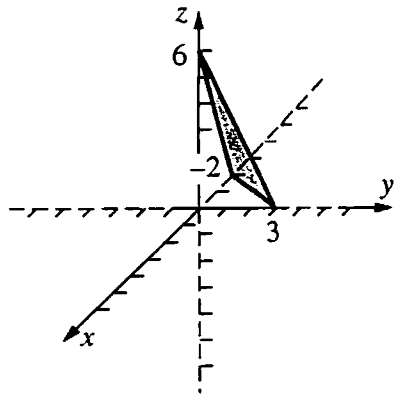
18. x-intercept: $y = z = 0 \Rightarrow 3x = 24, x = 8$
y-intercept: $y = z = 0 \Rightarrow -4y = 24, y = -6$
z-intercept: $x = y = 0 \Rightarrow 2z = 24, z = 12$



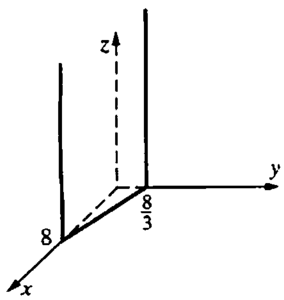
19. x-intercept: $y = z = 0 \Rightarrow x = 6$
y-intercept: $x = z = 0 \Rightarrow 3y = 6, y = 2$
z-intercept: $x = y = 0 \Rightarrow -z = 6, z = -6$



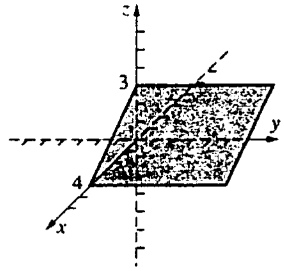
20. x-intercept: $y = z = 0 \Rightarrow -3x = 6, x = -2$
y-intercept: $x = z = 0 \Rightarrow 2y = 6, y = 3$
z-intercept: $x = y = 0 \Rightarrow z = 6$



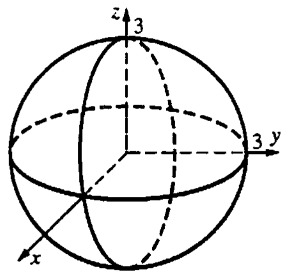
21. x and y cannot both be zero, so the plane is parallel to the z-axis.
x-intercept: $y = z = 0 \Rightarrow x = 8$
y-intercept: $x = z = 0 \Rightarrow 3y = 8,\ y = \frac{8}{3}$



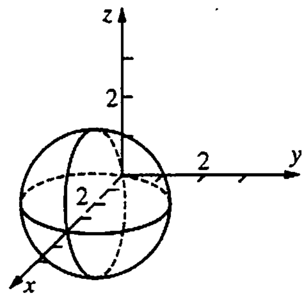
22. x and z cannot both be zero, so the plane is parallel to the y-axis.
x-intercept: $y = z = 0 \Rightarrow 3x = 12, x = 4$
z-intercept: $x = y = 0 \Rightarrow 4z = 12, z = 3$



23. This is a sphere with center (0, 0, 0) and radius 3.



24. This is a sphere with center (2, 0, 0) and radius 2.



25. The center of the sphere is the midpoint of the diameter, so it is $\left(\frac{-2+4}{2}, \frac{3-1}{2}, \frac{6+5}{2}\right) = \left(1, 1, \frac{11}{2}\right)$. The radius is $\frac{1}{2}\sqrt{(-2-4)^2 + (3+1)^2 + (6-5)^2} = \frac{\sqrt{53}}{2}$. The equation is $(x-1)^2 + (y-1)^2 + \left(z - \frac{11}{2}\right)^2 = \frac{53}{4}$.

26. Since the spheres are tangent and have equal radii, the radius of each sphere is $\frac{1}{2}$ of the distance between the centers.
$r = \frac{1}{2}\sqrt{(-3-5)^2 + (1+3)^2 + (2-6)^2} = 2\sqrt{6}$. The spheres are $(x+3)^2 + (y-1)^2 + (z-2)^2 = 24$ and $(x-5)^2 + (y+3)^2 + (z-6)^2 = 24$.

27. The center must be 6 units from each coordinate plane. Since it is in the first octant, the center is

(6, 6, 6). The equation is
$(x-6)^2+(y-6)^2+(z-6)^2=36.$

28. $x+y=12$ is parallel to the z-axis. The distance from (1, 1, 4) to the plane $x+y=12$ is the same as the distance in the xy-plane of (1, 1, 0) to the line $x+y-12=0$. That distance is
$\frac{|1+1-12|}{(1+1)^{1/2}}=5\sqrt{2}$. The equation of the sphere is
$(x-1)^2+(y-1)^2+(z-4)^2=50.$

29. **a.** Plane parallel to and two units above the xy-plane

b. Plane perpendicular to the xy-plane whose trace in the xy-plane is the line $x=y$.

c. Union of the yz-plane ($x=0$) and the xz-plane ($y=0$)

d. Union of the three coordinate planes

e. Cylinder of radius 2, parallel to the z-axis

f. Top half of the sphere with center (0, 0, 0) and radius 3

30. The points of the intersection satisfy both $(x-1)^2+(y+2)^2+(z+1)^2=10$ and $z=2$, so $(x-1)^2+(y+2)^2+(2+1)^2=10$. This simplifies to $(x-1)^2+(y+2)^2=1$, the equation of a circle of radius 1. The center is (1, –2, 2).

31. If $P(x, y, z)$ denotes the moving point,
$\sqrt{(x-1)^2+(y-2)^2+(z+3)^2}=2\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$,
which simplifies to $(x-1)^2+(y-2)^2+(z-5)^2=16$, is a sphere with radius 4 and center (1, 2, 5).

32. If $P(x, y, z)$ denotes the moving point,
$\sqrt{(x-1)^2+(y-2)^2+(z+3)^2}=\sqrt{(x-2)^2+(y-3)^2+(z-2)^2}$,
which simplifies to $x+y+5z=3/2$, a plane.

33. Note that the volume of a segment of height h in a hemisphere of radius r is $\pi h^2\left[r-\left(\frac{h}{3}\right)\right]$.

The resulting solid is the union of two segments, one for each ball. Since the two balls have the same radius, each segment will have the same value for h. h is the radius minus half the distance between the centers of the two balls.

$$h=2-\frac{1}{2}\sqrt{(2-1)^2+(4-2)^2+(3-1)^2}=2-\frac{3}{2}=\frac{1}{2}$$

$$V=2\left[\pi\left(\frac{1}{2}\right)^2\left(2-\frac{1}{6}\right)\right]=\frac{11\pi}{12}$$

34. As in Problem 33, the resulting solid is the union of two segments. Since the radii are not the same, the segments will have different heights. Let h_1 be the height of the segment from the first ball and let h_2 be the height from the second ball. $r_1=2$ is the radius of the first ball and $r_2=3$ is the radius of the second ball.

Solving for the equation of the plane containing the intersection of the spheres $(x-1)^2+(y-2)^2+(z-1)^2-4=0$ and $(x-2)^2+(y-4)^2+(z-3)^2-9=0$, we get $x+2y+2z-9=0$.

The distance from (1, 2, 1) to the plane is $\frac{2}{3}$, and the distance from (2, 4, 3) to the plane is $\frac{7}{3}$. (Use the formula in Example 6 of Section 14.2.)

$$h_1=2-\frac{2}{3}=\frac{4}{3};\ h_2=3-\frac{7}{3}=\frac{2}{3}$$

$$V=\pi\left(\frac{4}{3}\right)^2\left(2-\frac{4}{9}\right)+\pi\left(\frac{2}{3}\right)^2\left(3-\frac{2}{9}\right)=4\pi$$

14.2 Concepts Review

1. $\sqrt{2^2+(-3)^2+\left(\sqrt{3}\right)^2}=\sqrt{4+9+3}=4;$

   $2\cdot 3+(-3)\cdot 2+\sqrt{3}\cdot\left(-2\sqrt{3}\right)=6-6-6=-6$

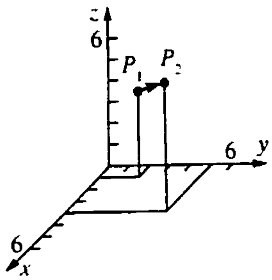
2. dot product; 0

3. $(2,-1,0)$; $\langle 3,-2,4\rangle$

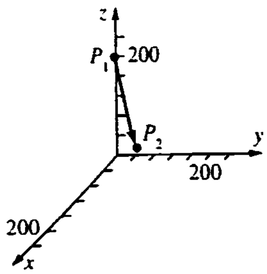
4. $|\mathbf{u}||\mathbf{v}|\cos\theta; \cos^{-1}\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

Problem Set 14.2

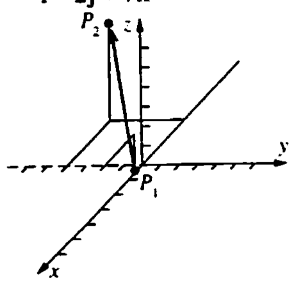
1. a. $(4-1)\mathbf{i}+(5-2)\mathbf{j}+(6-4)\mathbf{k}=3\mathbf{i}+3\mathbf{j}+2\mathbf{k}$



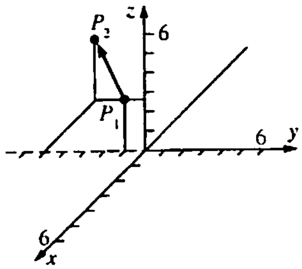
   b. $(-14+1)\mathbf{i}+(52+3)\mathbf{j}+(26-204)\mathbf{k}$
   $=-13\mathbf{i}+55\mathbf{j}-178\mathbf{k}$



2. a. $(-3+2)\mathbf{i}+(-4+2)\mathbf{j}+(5+2)\mathbf{k}$
   $=-\mathbf{i}-2\mathbf{j}+7\mathbf{k}$



   b. $\left(-\sqrt{14}-0\right)\mathbf{i}+(-5+1)\mathbf{j}+(\pi-e)\mathbf{k}$
   $=-\sqrt{14}\mathbf{i}-4\mathbf{j}+(\pi-e)\mathbf{k}$
   $\approx -3.74\mathbf{i}-4\mathbf{j}+0.42\mathbf{k}$



3. a. length $=\sqrt{4^2+1^2+2^2}=\sqrt{21}\approx 4.58$

   $\cos\alpha=\dfrac{4}{\sqrt{21}}\approx 0.87, \cos\beta=\dfrac{1}{\sqrt{21}}\approx 0.22,$

   $\cos\gamma=\dfrac{2}{\sqrt{21}}\approx 0.44$

   b. length $=\sqrt{(-2)^2+(-3)^2+7^2}=\sqrt{62}\approx 7.87$

   $\cos\alpha=-\dfrac{2}{\sqrt{62}}\approx -0.25,$

   $\cos\beta=-\dfrac{3}{\sqrt{62}}\approx -0.38,\ \cos\gamma=\dfrac{7}{\sqrt{62}}\approx 0.89$

4. a. length $=\sqrt{2^2+(-1)^2+(-2)^2}=3$

   $\cos\alpha=\dfrac{2}{3}, \cos\beta=-\dfrac{1}{3}, \cos\gamma=-\dfrac{2}{3}$

   b. length $=\sqrt{(-1)^2+2^2+(-2)^2}=3$

   $\cos\alpha=-\dfrac{1}{3}, \cos\beta=\dfrac{2}{3}, \cos\gamma=-\dfrac{2}{3}$

5. $|\langle 3,-4,5\rangle|=\sqrt{3^2+(-4)^2+5^2}=5\sqrt{2}$

   $\dfrac{1}{5\sqrt{2}}\langle 3,-4,5\rangle=\left\langle \dfrac{3}{5\sqrt{2}},-\dfrac{4}{5\sqrt{2}},\dfrac{1}{\sqrt{2}}\right\rangle$ is the unit vector with the same direction as $\langle 3,-4,5\rangle$.

   $-5\left\langle \dfrac{3}{5\sqrt{2}},-\dfrac{4}{5\sqrt{2}},\dfrac{1}{\sqrt{2}}\right\rangle=\left\langle -\dfrac{3}{\sqrt{2}},\dfrac{4}{\sqrt{2}},-\dfrac{5}{\sqrt{2}}\right\rangle$ is a vector of length 5 oriented in the opposite direction.

6. $|-4\mathbf{i}+3\mathbf{j}+-2\mathbf{k}|=\sqrt{(-4)^2+3^2+(-2)^2}=\sqrt{29}$

   $-\dfrac{10}{\sqrt{29}}(-4\mathbf{i}+3\mathbf{j}-2\mathbf{k})=\dfrac{40}{\sqrt{29}}\mathbf{i}-\dfrac{30}{\sqrt{29}}\mathbf{j}+\dfrac{20}{\sqrt{29}}\mathbf{k}$ is a vector of length of 10 with direction opposite $-4\mathbf{i}+3\mathbf{j}-2\mathbf{k}$.

7. $\cos\theta = \frac{\langle 4,-3,-1\rangle\cdot\langle -2,-3,5\rangle}{|\langle 4,-3,-1\rangle||\langle -2,-3,5\rangle|}$

$= \frac{(4)(-2)+(-3)(-3)+(-1)(5)}{\sqrt{16+9+1}\sqrt{4+9+25}} = \frac{-4}{\sqrt{26}\sqrt{38}}$

$= -\frac{2}{\sqrt{247}}$

$\theta = \cos^{-1}\left(-\frac{2}{\sqrt{247}}\right) \approx 97°$

8. $\cos\theta = \frac{(-4\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+5\mathbf{k})}{|-4\mathbf{i}+2\mathbf{j}+3\mathbf{k}||2\mathbf{i}+\mathbf{j}+5\mathbf{k}|}$

$= \frac{9}{\sqrt{29}\sqrt{30}} = \frac{9}{\sqrt{870}}$

$\theta = \cos^{-1}\frac{9}{\sqrt{870}} \approx 72°$

9. If $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is perpendicular to $-4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j}$, then $-4x + 5y + z = 0$ and $4x + y = 0$ since the dot product of perpendicular vectors is 0. Solving these equations yields $y = -4x$ and $z = 24x$. Hence, for any x, $x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}$ is perpendicular to the given vectors.

$|x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}| = \sqrt{x^2+16x^2+576x^2}$

$= |x|\sqrt{593}$

This length is 10 when $x = \pm\frac{10}{\sqrt{593}}$. The vectors are $\frac{10}{\sqrt{593}}\mathbf{i} - \frac{40}{\sqrt{593}}\mathbf{j} + \frac{240}{\sqrt{593}}\mathbf{k}$ and $-\frac{10}{\sqrt{593}}\mathbf{i} + \frac{40}{\sqrt{593}}\mathbf{j} - \frac{240}{\sqrt{593}}\mathbf{k}$.

10. If $\langle x, y, z\rangle$ is perpendicular to both $\langle 1, -2, -3\rangle$ and $\langle -3, 2, 0\rangle$, then $x - 2y - 3z = 0$ and $-3x + 2y = 0$. Solving these equations for y and z in terms of x yields $y = \frac{3}{2}x, z = -\frac{2}{3}x$. Thus, all the vectors have the form $\left\langle x, \frac{3}{2}x, -\frac{2}{3}x\right\rangle$ where x is a real number.

11. A vector equivalent to $\overline{BA}$ is $\mathbf{u} = \langle 1+4, 2-5, 3-6\rangle = \langle 5,-3,-3\rangle$.

A vector equivalent to $\overline{BC}$ is $\mathbf{v} = \langle 1+4, 0-5, 1-6\rangle = \langle 5,-5,-5\rangle$.

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{5\cdot5+(-3)(-5)+(-3)(-5)}{\sqrt{25+9+9}\sqrt{25+25+25}}$

$= \frac{55}{\sqrt{43}\sqrt{75}} = \frac{11}{\sqrt{129}}$, so $\theta = \cos^{-1}\frac{11}{\sqrt{129}} \approx 14.4°$.

12. A vector equivalent to $\overline{BA}$ is $\mathbf{u} = \langle 6-3, 3-1, 3+1\rangle = \langle 3, 2, 4\rangle$.

A vector equivalent to $\overline{BC}$ is $\mathbf{v} = \langle -1-3, 10-1, -2.5+1\rangle = \langle -4, 9, -1.5\rangle$.

$\mathbf{u}\cdot\mathbf{v} = 3(-4)+2\cdot9+4\cdot(-1.5) = 0$ so $\mathbf{u}$ is perpendicular to $\mathbf{v}$. Thus the angle at B is a right angle.

13. $\mathbf{u}\cdot\mathbf{v} = (-1)(-1)+5\cdot1+3(-1) = 3$

$|\mathbf{v}| = \sqrt{1+1+1} = \sqrt{3}$

$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{3}{\sqrt{3}} = \sqrt{3}$

14. $\mathbf{u}\cdot\mathbf{v} = 5(-\sqrt{5})+5(\sqrt{5})+2(1) = 2$

$|\mathbf{v}| = \sqrt{5+5+1} = \sqrt{11}$

$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{11}}$

15. $\mathbf{m} = \mathbf{pr_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$

$= \frac{(-3)(-3)+2\cdot5+1\cdot(-3)}{9+25+9}(-3\mathbf{i}+5\mathbf{j}-3\mathbf{k})$

$= \frac{16}{43}(-3\mathbf{i}+5\mathbf{j}-3\mathbf{k}) = -\frac{48}{43}\mathbf{i}+\frac{80}{43}\mathbf{j}-\frac{48}{43}\mathbf{k}$

$\mathbf{n} = \mathbf{u}-\mathbf{m} = \left(-3+\frac{48}{43}\right)\mathbf{i}+\left(2-\frac{80}{43}\right)\mathbf{j}+\left(1+\frac{48}{43}\right)\mathbf{k}$

$= -\frac{81}{43}\mathbf{i}+\frac{6}{43}\mathbf{j}+\frac{91}{43}\mathbf{k}$

16. $\mathbf{m} = \mathbf{pr_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$

$= \frac{e\cdot1+\pi\cdot1+1\cdot0}{1+1}(\mathbf{i}+\mathbf{j}) = \frac{e+\pi}{2}\mathbf{i}+\frac{e+\pi}{2}\mathbf{j}$

$\mathbf{n} = \mathbf{u}-\mathbf{m} = \left(e-\frac{e+\pi}{2}\right)\mathbf{i}+\left(\pi-\frac{e+\pi}{2}\right)\mathbf{j}+(1-0)\mathbf{k}$

$= \frac{e-\pi}{2}\mathbf{i}+\frac{\pi-e}{2}\mathbf{j}+\mathbf{k}$

17. **a.** $|\mathbf{u}| = \sqrt{9+4+1} = \sqrt{14}$

$\cos\alpha = -\frac{3}{\sqrt{14}}, \alpha = \cos^{-1}\left(-\frac{3}{\sqrt{14}}\right) \approx 143°$

$\cos\beta = \frac{2}{\sqrt{14}}, \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58°$

$\cos\gamma = \frac{1}{\sqrt{14}}, \gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74°$

b. $|\mathbf{u}| = \sqrt{9+36+1} = \sqrt{46}$

$\cos\alpha = \frac{3}{\sqrt{46}}, \alpha = \cos^{-1}\left(\frac{3}{\sqrt{46}}\right) \approx 64°$

$\cos\beta = \frac{6}{\sqrt{46}}, \beta = \cos^{-1}\left(\frac{6}{\sqrt{46}}\right) \approx 28°$

$\cos\gamma = -\frac{1}{\sqrt{46}}, \gamma = \cos^{-1}\left(-\frac{1}{\sqrt{46}}\right) \approx 98°$

18. $\cos^2(46°) + \cos^2(108°) + \cos^2\gamma = 1$
$\Rightarrow \cos\gamma \approx \pm 0.6496$
$\Rightarrow \gamma \approx 49.49°$ or $\gamma \approx 130.51°$

19. $|\mathbf{u}| = (4+9+z^2)^{1/2} = 5$ and $z > 0$, so
$z = 2\sqrt{3} \approx 3.4641$.

20. **u** and **v** perpendicular, so $\mathbf{u}\cdot\mathbf{v} = 0$, thus
$4 + 18 - 3z = 0,\ z = \frac{22}{3}$.

21. There are infinitely many such pairs. Note that
$\langle -4, 2, 5\rangle \cdot \langle 1, 2, 0\rangle = -4+4+0 = 0$, so
$\mathbf{u} = \langle 1, 2, 0\rangle$ is perpendicular to $\langle -4, 2, 5\rangle$. For any c, $\langle -2, 1, c\rangle \cdot \langle 1, 2, 0\rangle = -2+2+0 = 0$ so
$\mathbf{v} = \langle -2, 1, c\rangle$ is a candidate.
$\langle -4, 2, 5\rangle \cdot \langle -2, 1, c\rangle = 8+2+5c$
$8 + 2 + 5c = 0 \Rightarrow c = -2$, so one pair is
$\mathbf{u} = \langle 1, 2, 0\rangle$, $\mathbf{v} = \langle -2, 1, -2\rangle$.

22. The midpoint is
$\left(\frac{3+5}{2}, \frac{2-7}{2}, \frac{-1+2}{2}\right) = \left(4, -\frac{5}{2}, \frac{1}{2}\right)$, so the vector is $\left\langle 4, -\frac{5}{2}, \frac{1}{2}\right\rangle$.

23. The following do not make sense.

a. $\mathbf{v}\cdot\mathbf{w}$ is not a vector.

b. $\mathbf{u}\cdot\mathbf{w}$ is not a vector.

f. $|\mathbf{u}|$ is not a vector.

24. Pairs of opposite sides are parallel and of equal length in a parallelogram. Thus, side AB must be parallel to side CD and side BC must be parallel to side AD. In addition, the four points must not be colinear. Thus, $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$, $\mathbf{c} - \mathbf{b} = \mathbf{d} - \mathbf{a}$, and $\mathbf{b} - \mathbf{a} \neq k(\mathbf{c} - \mathbf{b})$ for any number k.

25. a. $2(x-1) - 4(y-2) + 3(z+3) = 0$ or
$2x - 4y + 3z = -15$.

b. $3(x+2) - 2(y+3) - 1(z-4) = 0$ or
$3x - 2y - z = -4$.

26. Normals to the planes are $\langle 3, -2, 5\rangle$ and $\langle 4, -2, -3\rangle$, so the cosine of the angle is
$\frac{12+4-15}{\sqrt{38}\sqrt{29}} = \frac{1}{\sqrt{1102}}$. The angle is $\approx 88.27°$.

27. The planes are $2x - 4y + 3z = -15$ and $3x - 2y - z = 4$. The normals to the planes are $\mathbf{u} = \langle 2, -4, 3\rangle$ and $\mathbf{v} = \langle 3, -2, -1\rangle$. If θ is the angle between the planes,
$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{6+8-3}{\sqrt{29}\sqrt{14}} = \frac{11}{\sqrt{406}}$, so
$\theta = \cos^{-1}\frac{11}{\sqrt{406}} \approx 56.91°$.

28. An equation of the plane has the form $2x + 4y - z = D$. And this equation must satisfy $2(-1) + 4(2) - (-3) = D$, so $D = 9$. Thus an equation of the plane is $2x + 4y - z = 9$.

29. a. Planes parallel to the xy-plane may be expressed as $z = D$, so $z = 2$ is an equation of the plane.

b. An equation of the plane is
$2(x+4) - 3(y+1) - 4(z-2) = 0$ or
$2x - 3y - 4z = -13$.

30. Distance $= \frac{|(1)+3(-1)+(2)-7|}{\sqrt{1+9+1}} = \frac{7}{\sqrt{11}} \approx 2.1106$

31. The distance is 0 since the point is in the plane.
$(|(-3)(2)+2(6)+(3)-9| = 0)$

32. (0, 0, 9) is on $-3x + 2y + z = 9$. The distance from (0, 0, 9) to $6x - 4y - 2z = 19$ is
$\frac{|6(0)-4(0)-2(9)-19|}{\sqrt{36+16+4}} = \frac{37}{\sqrt{56}} \approx 4.9443$ is the distance between the planes.

33. (1, 0, 0) is on $5x - 3y - 2z = 5$. The distance from (1, 0, 0) to $-5x + 3y + 2z = 7$ is
$\frac{|-5(1)+3(0)+2(0)-7|}{\sqrt{25+9+4}} = \frac{12}{\sqrt{38}} \approx 1.9467$.

34. The line segment between the points is perpendicular to the plane and its midpoint, (2, 1, 1), is in the plane. Then
$\langle 6-(-2), 1-1, -2-4\rangle = \langle 8, 0, -6\rangle$ is perpendicular to the plane. The equation of the plane is
$8(x-2) + 0(y-1) - 6(z-1) = 0$ or $8x - 6z = 10$.

35. $|\mathbf{u}+\mathbf{v}|^2+|\mathbf{u}-\mathbf{v}|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})+(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\ =[\mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]+[\mathbf{u}\cdot\mathbf{u}-2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]$

$=2(\mathbf{u}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v})=2|\mathbf{u}|^2+2|\mathbf{v}|^2$

36. $|\mathbf{u}+\mathbf{v}|^2-|\mathbf{u}-\mathbf{v}|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})-(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\ =[\mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]-[\mathbf{u}\cdot\mathbf{u}-2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]$

$=4(\mathbf{u}\cdot\mathbf{v})$ so $\mathbf{u}\cdot\mathbf{v}=\frac{1}{4}|\mathbf{u}+\mathbf{v}|^2-\frac{1}{4}|\mathbf{u}-\mathbf{v}|^2$.

37. Orient and scale the axes so that the cube has one end of a main diagonal at (0, 0, 0) and the other end at (1, 1, 1). Thus $\langle 1,1,1\rangle$ is along the diagonal. The angle between the main diagonal and one of its faces is the angle between $\langle 1,1,1\rangle$ and $\langle 1,1,0\rangle$. Let θ be the angle.

$\cos\theta=\frac{\langle 1,1,1\rangle\cdot\langle 1,1,0\rangle}{\sqrt{3}\cdot\sqrt{2}}=\frac{2}{\sqrt{6}}$,

$\theta=\cos^{-1}\frac{2}{\sqrt{6}}\approx 35.26°$

38. Let θ be the common direction angle. Then $3\cos^2\theta=1$, so $\cos\theta=\pm\frac{1}{\sqrt{3}}$. There are two unit vectors, $\pm\left\langle\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$.

39. Place the box so that its corners are at the points (0, 0, 0), (4, 0, 0), (0, 6, 0), (4, 6, 0), (0, 0, 10), (4, 0, 10), (0, 6, 10), and (4, 6, 10). The main diagonals are (0, 0, 0) to (4, 6, 10), (4, 0, 0) to (0, 6, 10), (0, 6, 0) to (4, 0, 10), and (0, 0, 10) to (4, 6, 0). The corresponding vectors are $\langle 4,6,10\rangle$, $\langle -4,6,10\rangle$, $\langle 4,-6,10\rangle$, and $\langle 4,6,-10\rangle$.

All of these vectors have length $\sqrt{16+36+100}=\sqrt{152}$. Thus, the smallest angle θ between any pair, $\mathbf{u}$ and $\mathbf{v}$ of the diagonals is found from the largest value of $\mathbf{u}\cdot\mathbf{v}$, since

$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}=\frac{\mathbf{u}\cdot\mathbf{v}}{152}$.

There are six ways of pairing the four vectors. The largest value of $\mathbf{u}\cdot\mathbf{v}$ is 120 which occurs with $\mathbf{u}=\langle 4,6,10\rangle$ and $\mathbf{v}=\langle -4,6,10\rangle$. Thus,

$\cos\theta=\frac{120}{152}=\frac{15}{19}$ so $\theta=\cos^{-1}\frac{15}{19}\approx 37.86°$.

40. Place the box so that its corners are at the points (0, 0, 0), (1, 0, 0), (0,1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0,1, 1), and (1, 1, 1). The main diagonals are (0, 0, 0) to (1, 1, 1), (1, 0, 0) to (0, 1, 1), (0, 1, 0) to (1, 0, 1), and (0, 0, 1) to (1, 1, 0). The corresponding vectors are $\langle 1,1,1\rangle$, $\langle -1,1,1\rangle$, $\langle 1,-1,1\rangle$, and $\langle 1,1,-1\rangle$.

All of these vectors have length $\sqrt{1+1+1}=\sqrt{3}$. Thus, the angle θ between any pair, u and v of the diagonals is found from $\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}=\frac{\mathbf{u}\cdot\mathbf{v}}{3}$.

There are six ways of pairing the four vectors, but due to symmetry, there are only two cases we need to consider. In these cases, $\mathbf{u}\cdot\mathbf{v}=1$ or $\mathbf{u}\cdot\mathbf{v}=-1$. Thus we get that $\cos\theta=\frac{1}{3}$ or $\cos\theta=-\frac{1}{3}$. Solving for θ gives $\theta\approx 70.53°$ or $\theta\approx 109.47°$.

41. $\mathbf{D}=(4-0)\mathbf{i}+(4-0)\mathbf{j}+(0-8)\mathbf{k}=4+4\mathbf{j}-8\mathbf{k}$
Thus, $W=\mathbf{F}\cdot\mathbf{D}=0(4)+0(4)-4(-8)=32$ joules.

42. $\mathbf{D}=(9-2)\mathbf{i}+(4-1)\mathbf{j}+(6-3)\mathbf{k}=7\mathbf{i}+3\mathbf{j}+3\mathbf{k}$
Thus, $W=\mathbf{F}\cdot\mathbf{D}=3(7)-6(3)+7(3)=24$ ft-lbs.

43. $\mathbf{D}=(3-0)\mathbf{i}+(5-1)\mathbf{j}+(7-2)\mathbf{k}=3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$

$\mathbf{F}=5\frac{(2\mathbf{i}+2\mathbf{j}-\mathbf{k})}{\sqrt{4+4+1}}=\frac{10}{3}\mathbf{i}+\frac{10}{3}\mathbf{j}-\frac{5}{3}\mathbf{k}$

$W=\mathbf{F}\cdot\mathbf{D}=\frac{10}{3}(3)+\frac{10}{3}(4)-\frac{5}{3}(5)=15$ joules.

44. The 3 wires must offset the weight of the object, thus
$(3\mathbf{i}+4\mathbf{j}+15\mathbf{k})+(-8\mathbf{i}-2\mathbf{j}+10\mathbf{k})+(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$
$=0\mathbf{i}+0\mathbf{j}+30\mathbf{k}$
Thus, $3-8+a=0$, so $a=5$;
$4-2+b=0$, so $b=-2$;
$15+10+c=30$, so $c=5$.

45. $\langle x,y,z\rangle=\langle 2,3,-1\rangle+\frac{1}{5}\langle 7-2,-2-3,9+1\rangle$
$=\langle 2+1,3-1,-1+2\rangle=\langle 3,2,1\rangle$,
so (3, 2, 1) is the point.

46. After reflecting from the xy-plane, the ray has direction $a\mathbf{i}+b\mathbf{j}-c\mathbf{k}$. After reflecting from the

xz-plane, the ray now has direction $a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$. After reflecting from the yz-plane, the ray now has direction $-a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$, the opposite of its original direction. ($a < 0, b < 0, c < 0$)

47. The equation of the sphere in standard form is $(x+1)^2 + (y+3)^2 + (z-4)^2 = 26$, so its center is $(-1, -3, 4)$ and radius is $\sqrt{26}$. The distance from the sphere to the plane is the distance from the center to the plane minus the radius of the sphere or

$$\frac{|3(-1)+4(-3)+1(4)-15|}{\sqrt{9+16+1}} - \sqrt{26} = \sqrt{26} - \sqrt{26} = 0,$$

so the sphere is tangent to the plane.

48. Let $P(x_0, y_0, z_0)$ be any point on $Ax + By + Cz = D$, so $Ax_0 + By_0 + Cz_0 = D$. The distance between the planes is the distance from $P(x_0, y_0, z_0)$ to $Ax + By + Cz = E$, which is

$$\frac{|Ax_0 + By_0 + Cz_0 - E|}{\sqrt{A^2+B^2+C^2}} = \frac{|D-E|}{\sqrt{A^2+B^2+C^2}}.$$

49. Let $\mathbf{x} = \langle x, y, z \rangle$, so

$$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b}) = \langle x-a_1, y-a_2, z-a_3\rangle \cdot \langle x-b_1, y-b_2, z-b_3\rangle$$
$$= x^2 - (a_1+b_1)x + a_1b_1 + y^2 - (a_2+b_2)y + a_2b_2 + z^2 - (a_3+b_3)z + a_3b_3$$

Setting this equal to 0 and completing the squares yields

$$\left(x - \frac{a_1+b_1}{2}\right)^2 + \left(y - \frac{a_2+b_2}{2}\right)^2 + \left(z - \frac{a_3+b_3}{2}\right)^2 = \frac{1}{4}\left[(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2\right].$$

A sphere with center $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$ and radius $\frac{1}{4}\left[(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2\right] = \frac{1}{4}|\mathbf{a}-\mathbf{b}|^2$.

50. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ represent the sides of the polygon connected tail to head in succession around the polygon. Then $\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n = \mathbf{0}$ since the polygon is closed, so $\mathbf{F}\cdot\mathbf{v}_1 + \mathbf{F}\cdot\mathbf{v}_2 + \ldots + \mathbf{F}\cdot\mathbf{v}_n = \mathbf{F}\cdot(\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n)$ $= \mathbf{F}\cdot\mathbf{0} = 0$.

51. If **a**, **b**, and **c** are the position vectors of the vertices labeled A, B, and C, respectively, then the side BC is represented by the vector $\mathbf{c}-\mathbf{b}$. The position vector of the midpoint of BC is $\mathbf{b} + \frac{1}{2}(\mathbf{c}-\mathbf{b}) = \frac{1}{2}(\mathbf{b}+\mathbf{c})$. The segment from A to the midpoint of BC is $\frac{1}{2}(\mathbf{b}+\mathbf{c}) - \mathbf{a}$. Thus, the position vector of P is $\mathbf{a} + \frac{2}{3}\left[\frac{1}{2}(\mathbf{b}+\mathbf{c}) - \mathbf{a}\right] = \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

If the vertices are (2, 6, 5), (4, –1, 2), and (6, 1, 2), the corresponding position vectors are $\langle 2, 6, 5\rangle$, $\langle 4, -1, 2\rangle$, and $\langle 6, 1, 2\rangle$. The position vector of P is $\frac{1}{3}\langle 2+4+6, 6-1+1, 5+2+2\rangle = \frac{1}{3}\langle 12, 6, 9\rangle = \langle 4, 2, 3\rangle$. Thus P is (4, 2, 3).

52. Let A, B, C, and D be the vertices of the tetrahedron with corresponding position vectors **a**, **b**, **c**, and **d**. The vector representing the segment from A to the centroid of the opposite face, triangle BCD, is $\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d}) - \mathbf{a}$ by Problem 49. Similarly, the segments from B, C, and D to the opposite faces are $\frac{1}{3}(\mathbf{a}+\mathbf{c}+\mathbf{d}) - \mathbf{b}$, $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{d}) - \mathbf{c}$, and $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) - \mathbf{d}$, respectively. If these segments meet in one point which has a nice formulation as some fraction of the way from a vertex to the centroid of the opposite face, then there is some number k, such that

$$\mathbf{a} + k\left[\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d}) - \mathbf{a}\right] = \mathbf{b} + k\left[\frac{1}{3}(\mathbf{a}+\mathbf{c}+\mathbf{d}) - \mathbf{b}\right] = \mathbf{c} + k\left[\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{d}) - \mathbf{c}\right] = \mathbf{d} + k\left[\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) - \mathbf{d}\right].$$

Thus, $\mathbf{a}(1-k) = \frac{k}{3}\mathbf{a}$, so $k = \frac{3}{4}$. Hence the segments joining the vertices to the centroids of the opposite faces meet in a common point which is $\frac{3}{4}$ of the way from a vertex to the corresponding centroid. With $k = \frac{3}{4}$, all of the above formulas simplify to $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$, the position vector of the point.

14.3 Concepts Review

1. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$
$= (-2-1)\mathbf{i} - (1-3)\mathbf{j} + (-1-6)\mathbf{k}$
$= -3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} = \langle -3, 2, -7 \rangle$

2. $|\mathbf{u}||\mathbf{v}|\sin\theta$

3. $-(\mathbf{v} \times \mathbf{u})$

4. parallel

Problem Set 14.3

1. a. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ -1 & 2 & -4 \end{vmatrix}$
$= \begin{vmatrix} 2 & -2 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -2 \\ -1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{k}$
$=(-8+4)\mathbf{i} - (12-2)\mathbf{j} + (-6+2)\mathbf{k}$
$= -4\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}$

b. $\mathbf{b} + \mathbf{c} = 6\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$, so
$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ 6 & 5 & -8 \end{vmatrix}$
$= \begin{vmatrix} 2 & -2 \\ 5 & -8 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -2 \\ 6 & -8 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 6 & 5 \end{vmatrix}\mathbf{k}$
$= (-16+10)\mathbf{i} - (24+12)\mathbf{j} + (-15-12)\mathbf{k}$
$= -6\mathbf{i} - 36\mathbf{j} - 27\mathbf{k}$

c. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = -3(6) + 2(5) - 2(-8) = 8$

d. $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -4 \\ 7 & 3 & -4 \end{vmatrix}$
$= \begin{vmatrix} 2 & -4 \\ 3 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -4 \\ 7 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ 7 & 3 \end{vmatrix}\mathbf{k}$
$= (-8+12)\mathbf{i} - (4+28)\mathbf{j} + (-3-14)\mathbf{k}$
$= 4\mathbf{i} - 32\mathbf{j} - 17\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ 4 & -32 & -17 \end{vmatrix}$
$= \begin{vmatrix} 2 & -2 \\ -32 & -17 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -2 \\ 4 & -17 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 4 & -32 \end{vmatrix}\mathbf{k}$
$= (-34-64)\mathbf{i} - (51+8)\mathbf{j} + (96-8)\mathbf{k}$
$= -98\mathbf{i} - 59\mathbf{j} + 88\mathbf{k}$

2. a. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ -2 & -1 & 0 \end{vmatrix}$
$= \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix}\mathbf{k}$
$= (0+1)\mathbf{i} - (0+2)\mathbf{j} + (-3+6)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
$= \langle 1, -2, 3 \rangle$

b. $\mathbf{b} + \mathbf{c} = \langle -2-2, -1-3, 0-1 \rangle$
$= \langle -4, -4, -1 \rangle$
$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ -4 & -4 & -1 \end{vmatrix}$
$= \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ -4 & -4 \end{vmatrix}\mathbf{k}$
$= (-3+4)\mathbf{i} - (-3+4)\mathbf{j} + (-12+12)\mathbf{k}$
$= \mathbf{i} - \mathbf{j} = \langle 1, -1, 0 \rangle$

c. $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 0 \\ -2 & -3 & -1 \end{vmatrix}$
$= \begin{vmatrix} -1 & 0 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & -1 \\ -2 & -3 \end{vmatrix}\mathbf{k}$
$= (1-0)\mathbf{i} - (2-0)\mathbf{j} + (6-2)\mathbf{k} = \langle 1, -2, 4 \rangle$
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 3(1) + 3(-2) + 1(4) = 1$

d. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix}$
$= \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{k}$

$= (12 + 2)\mathbf{i} - (12 - 1)\mathbf{j} + (-6 - 3)\mathbf{k}$
$= \langle 14, -11, -9\rangle$

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 2 & -4 \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$
$= (-8 - 6)\mathbf{i} - (-4 + 6)\mathbf{j} + (2 + 4)\mathbf{k} = -14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
is perpendicular to both **a** and **b**. All perpendicular vectors will have the form $c(-14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ where c is a real number.

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & -2 \\ 3 & -2 & 4 \end{vmatrix}$

$= \begin{vmatrix} 5 & -2 \\ -2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -2 \\ 3 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 5 \\ 3 & -2 \end{vmatrix}\mathbf{k}$
$= (20 - 4)\mathbf{i} - (-8 + 6)\mathbf{j} + (4 - 15)\mathbf{k}$
$= 16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$
All vectors perpendicular to both **a** and **b** will have the form $c(16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k})$ where c is a real number.

5. $\mathbf{u} = \langle 3-1, -1-3, 2-5\rangle = \langle 2, -4, -3\rangle$ and
$\mathbf{v} = \langle 4-1, 0-3, 1-5\rangle = \langle 3, -3, -4\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -3 & -4 \end{vmatrix}$

$= \begin{vmatrix} -4 & -3 \\ -3 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -4 \\ 3 & -3 \end{vmatrix}\mathbf{k}$
$= (16 - 9)\mathbf{i} - (-8 + 9)\mathbf{j} + (-6 + 12)\mathbf{k}$
$= \langle 7, -1, 6\rangle$ is perpendicular to the plane.

$\pm\frac{1}{\sqrt{49+1+36}}\langle 7, -1, 6\rangle = \pm\left\langle \frac{7}{\sqrt{86}}, -\frac{1}{\sqrt{86}}, \frac{6}{\sqrt{86}}\right\rangle$
are the vectors perpendicular to the plane.

6. $\mathbf{u} = \langle 5+1, 1-3, 2-0\rangle = \langle 6, -2, 2\rangle$ and
$\mathbf{v} = \langle 4+1, -3-3, -1-0\rangle = \langle 5, -6, -1\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 2 \\ 5 & -6 & -1 \end{vmatrix}$

$= \begin{vmatrix} -2 & 2 \\ -6 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 6 & 2 \\ 5 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & -2 \\ 5 & -6 \end{vmatrix}\mathbf{k}$
$= (2 + 12)\mathbf{i} - (-6 - 10)\mathbf{j} + (-36 + 10)\mathbf{k}$
$= \langle 14, 16, -26\rangle$
is perpendicular to the plane.

$\pm\frac{1}{\sqrt{196+256+676}}\langle 14, 16, -26\rangle$
$= \pm\left\langle \frac{7}{\sqrt{282}}, \frac{8}{\sqrt{282}}, -\frac{13}{\sqrt{282}}\right\rangle$
are the unit vectors perpendicular to the plane.

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -3 \\ 4 & 2 & -4 \end{vmatrix}$

$= \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -3 \\ 4 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{k}$
$= (-4 + 6)\mathbf{i} - (4 + 12)\mathbf{j} + (-2 - 4)\mathbf{k}$
$= 2\mathbf{i} - 16\mathbf{j} - 6\mathbf{k}$
Area of parallelogram $= |\mathbf{a}\times\mathbf{b}|$
$= \sqrt{4+256+36} = 2\sqrt{74}$

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix}$

$= \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{k}$
$= (-8 + 1)\mathbf{i} - (-8 - 1)\mathbf{j} + (2 + 2)\mathbf{k}$
$= -7\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$
Area of parallelogram $= |\mathbf{a}\times\mathbf{b}| = \sqrt{49+81+16}$
$= \sqrt{146}$

9. $\mathbf{a} = \langle 2-3, 4-2, 6-1\rangle = \langle -1, 2, 5\rangle$ and
$\mathbf{b} = \langle -1-3, 2-2, 5-1\rangle = \langle -4, 0, 4\rangle$ are adjacent sides of the triangle. The area of the triangle is half the area of the parallelogram with **a** and **b** as adjacent sides.

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ -4 & 0 & 4 \end{vmatrix}$

$= \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 5 \\ -4 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -4 & 0 \end{vmatrix}\mathbf{k}$
$= (8 - 0)\mathbf{i} - (-4 + 20)\mathbf{j} + (0 + 8)\mathbf{k} = \langle 8, -16, 8\rangle$

Area of triangle $= \frac{1}{2}\sqrt{64+256+64} = \frac{1}{2}(8\sqrt{6})$
$= 4\sqrt{6}$

10. $\mathbf{a} = \langle 3-1, 1-2, 5-3\rangle = \langle 2, -1, 2\rangle$ and
$\mathbf{b} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$ are adjacent sides of the triangle.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 2\\ 3 & 3 & 3\end{vmatrix}$$

$$=\begin{vmatrix}-1 & 2\\ 3 & 3\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & 2\\ 3 & 3\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & -1\\ 3 & 3\end{vmatrix}\mathbf{k}$$

$$=(-3-6)\mathbf{i}-(6-6)\mathbf{j}+(6+3)\mathbf{k}=\langle -9, 0, 9\rangle$$

$$\text{Area of triangle}=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|=\frac{1}{2}\sqrt{81+81}=\frac{9\sqrt{2}}{2}$$

11. $\mathbf{u}=\langle 0-1, 3-3, 0-2\rangle=\langle -1, 0, -2\rangle$ and $\mathbf{v}=\langle 2-1, 4-3, 3-2\rangle=\langle 1, 1, 1\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 0 & -2\\ 1 & 1 & 1\end{vmatrix}$$

$$=\begin{vmatrix}0 & -2\\ 1 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & -2\\ 1 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & 0\\ 1 & 1\end{vmatrix}\mathbf{k}$$

$$=(0+2)\mathbf{i}-(-1+2)\mathbf{j}+(-1-0)\mathbf{k}=\langle 2, -1, -1\rangle$$

The plane through (1, 3, 2) with normal $\langle 2, -1, -1\rangle$ has equation
$2(x-1)-1(y-3)-1(z-2)=0$ or
$2x-y-z=-3$.

12. $\mathbf{u}=\langle 0-1, 0-1, 1-2\rangle=\langle -1, -1, -1\rangle$ and $\mathbf{v}=\langle -2-1, -3-1, 0-2\rangle=\langle -3, -4, -2\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & -1\\ -3 & -4 & -2\end{vmatrix}$$

$$=\begin{vmatrix}-1 & -1\\ -4 & -2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & -1\\ -3 & -2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ -3 & -4\end{vmatrix}\mathbf{k}$$

$$=(2-4)\mathbf{i}-(2-3)\mathbf{j}+(4-3)\mathbf{k}=\langle -2, 1, 1\rangle$$

The plane through (0, 0, 1) with normal $\langle -2, 1, 1\rangle$ has equation $-2(x-0)+1(y-0)+1(z-1)=0$ or $-2x+y+z=1$.

13. The plane's normals will be perpendicular to the normals of the other two planes. Then a normal is $\langle 1, -3, 2\rangle\times\langle 2, -2, -1\rangle=\langle 7, 5, 4\rangle$. An equation of the plane is $7(x+1)+5(y+2)+4(z-3)=0$ or $7x+5y+4z=-5$.

14. $(4\mathbf{i}+3\mathbf{j}-\mathbf{k})\times(2\mathbf{i}-5\mathbf{j}+6\mathbf{k})=13\mathbf{i}-26\mathbf{j}-26\mathbf{k}$
$=13(\mathbf{i}-2\mathbf{j}-2\mathbf{k})$
is normal to the plane. An equation of the plane is
$1(x-2)-2(y+3)-2(z-2)=0$ or
$x-2y-2z=4$.

15. Each vector normal to the plane we seek is parallel to the line of intersection of the given planes. Also, the cross product of vectors normal to the given planes will be parallel to both planes, hence parallel to the line of intersection. Thus, $\langle 4, -3, 2\rangle\times\langle 3, 2, -1\rangle=\langle -1, 10, 17\rangle$ is normal to the plane we seek. An equation of the plane is $-1(x-6)+10(y-2)+17(z+1)=0$ or $-x+10y+17z=-3$.

16. **a** × **b** is perpendicular to the plane containing **a** and **b**, hence **a** × **b** is perpendicular to both **a** and **b**. (**a** × **b**) × **c** is perpendicular to **a** × **b** hence it is parallel to the plane containing **a** and **b**.

17. $\text{Volume}=|\langle 2, 3, 4\rangle\cdot(\langle 0, 4, -1\rangle\times\langle 5, 1, 3\rangle)|=|\langle 2, 3, 4\rangle\cdot\langle 13, -5, -20\rangle|=|-69|=69$

18. $\text{Volume}=|(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot[(-\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(3\mathbf{i}-2\mathbf{j}+5\mathbf{k})|=|(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot(12\mathbf{i}+8\mathbf{j}-4\mathbf{k})|=|-4|=4$

19. a. $\text{Volume}=|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|=|\langle 3, 2, 1\rangle\cdot\langle -3, -1, 2\rangle|=|-9|=9$

b. $\text{Area}=|\mathbf{u}\times\mathbf{v}|=|\langle 3, -5, 1\rangle|=\sqrt{9+25+1}=\sqrt{35}$

c. Let θ be the angle. Then θ is the complement of the smaller angle between **u** and **v** × **w**.

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{|\mathbf{u}||\mathbf{v}\times\mathbf{w}|}=\frac{9}{\sqrt{9+4+1}\sqrt{9+1+4}}=\frac{9}{14},\ \theta\approx 40.01°$$

20. From Theorem C, $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$, which leads to $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$. Again from Theorem C, $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = |(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}| = |-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}|$, which leads to $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = |\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|$. Therefore, we have that $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})| = |\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$.

21. Choice (c) does not make sense because $(\mathbf{a}\cdot\mathbf{b})$ is a scalar and can't be crossed with a vector. Choice (d) does not make sense because $(\mathbf{a}\times\mathbf{b})$ is a vector and can't be added to a constant.

22. $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ will both be perpendicular to the common plane. Hence $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are parallel so $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{0}$.

23. Let $\mathbf{b}$ and $\mathbf{c}$ determine the (triangular) base of the tetrahedron. Then the area of the base is $\frac{1}{2}|\mathbf{b}\times\mathbf{c}|$ which is half of the area of the parallelogram determined by $\mathbf{b}$ and $\mathbf{c}$. Thus,

$$\frac{1}{3}(\text{area of base})(\text{height}) = \frac{1}{3}\left[\frac{1}{2}(\text{area of corresponding parallelogram})(\text{height})\right]$$
$$= \frac{1}{6}(\text{area of corresponding parallelpiped}) = \frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

24. $\mathbf{a} = \langle 4+1, -1-2, 2-3\rangle = \langle 5, -3, -1\rangle$,
$\mathbf{b} = \langle 5+1, 6-2, 3-3\rangle = \langle 6, 4, 0\rangle$,
$\mathbf{c} = \langle 1+1, 1-2, -2-3\rangle = \langle 2, -1, -5\rangle$

$$\text{Volume} = \frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{1}{6}|\langle 5, -3, -1\rangle\cdot\langle -20, 30, -14\rangle| = \frac{1}{6}|-176| = \frac{88}{3}$$

25. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ then

$$\mathbf{u}\times\mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$$
$$|\mathbf{u}\times\mathbf{v}|^2 = u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2$$
$$= u_1^2(v_1^2 + v_2^2 + v_3^2) - u_1^2v_1^2 + u_2^2(v_1^2 + v_2^2 + v_3^2) - u_2^2v_2^2 + u_3^2(v_1^2 + v_2^2 + v_3^2)$$
$$- u_3^2v_3^2 - 2u_2u_3v_2v_3 - 2u_1u_3v_1v_3 - 2u_1u_2v_1v_2$$
$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_2u_3v_2v_3 + 2u_1u_3v_1v_3 + 2u_1u_2v_1v_2)$$
$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2$$

26. $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \langle (u_2v_3 - u_3v_2) + (u_2w_3 - u_3w_2), (u_3v_1 - u_1v_3) + (u_3w_1 - u_1w_3), (u_1v_2 - u_2v_1) + (u_1w_2 - u_2w_1)\rangle$$
$$= (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$$

27. $(\mathbf{v}+\mathbf{w})\times\mathbf{u} = -[\mathbf{u}\times(\mathbf{v}+\mathbf{w})] = -[(\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})] = -(\mathbf{u}\times\mathbf{v}) - (\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u}) + (\mathbf{w}\times\mathbf{u})$

28. $\mathbf{u}\times\mathbf{v} = \mathbf{0} \Rightarrow$ $\mathbf{u}$ and $\mathbf{v}$ are parallel. $\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow$ $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. Thus, either $\mathbf{u}$ or $\mathbf{v}$ is $\mathbf{0}$.

29. $\overline{PQ} = \langle -a, b, 0\rangle$, $\overline{PR} = \langle -a, 0, c\rangle$,

The area of the triangle is half the area of the parallelogram with $\overline{PQ}$ and $\overline{PR}$ as adjacent sides, so area

$$(\Delta PQR) = \frac{1}{2}|\langle -a, b, 0\rangle\times\langle -a, 0, c\rangle| = \frac{1}{2}|\langle bc, ac, ab\rangle| = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}.$$

30. The area of the triangle is

$$\frac{1}{2}|\langle x_2 - x_1, y_2 - y_1, 0\rangle\times\langle x_3 - x_1, y_3 - y_1, 0\rangle| = \frac{1}{2}|\langle 0, 0, (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\rangle|$$

$= \frac{1}{2}|(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)|$

which is half of the absolute value of the determinant given. (Expand the determinant along the third column to see the equality.)

31. From Problem 29, $D^2 = \frac{1}{4}(b^2c^2 + a^2c^2 + a^2b^2) = \left(\frac{1}{2}bc\right)^2 + \left(\frac{1}{2}ac\right)^2 + \left(\frac{1}{2}ab\right)^2 = A^2 + B^2 + C^2$.

32. Note that the area of the face determined by **a** and **b** is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

Label the tetrahedron so that $\mathbf{m} = \frac{1}{2}(\mathbf{a}\times\mathbf{b})$, $\mathbf{n} = \frac{1}{2}(\mathbf{b}\times\mathbf{c})$, and $\mathbf{p} = \frac{1}{2}(\mathbf{c}\times\mathbf{a})$ point outward.

The fourth face is determined by **a** – **c** and **b** – **c**, so

$\mathbf{q} = \frac{1}{2}[(\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{c})] = \frac{1}{2}[(\mathbf{b}\times\mathbf{a}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a}) + (\mathbf{c}\times\mathbf{c})] = \frac{1}{2}[-(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})]$.

$\mathbf{m}+\mathbf{n}+\mathbf{p}+\mathbf{q} = \frac{1}{2}[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) - (\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})] = \mathbf{0}$

33. The area of the triangle is $A = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. Thus,

$A^2 = \frac{1}{4}|\mathbf{a}\times\mathbf{b}|^2 = \frac{1}{4}\left(|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2\right) = \frac{1}{4}\left[|\mathbf{a}|^2|\mathbf{b}|^2 - \frac{1}{4}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\right)^2\right]$

$= \frac{1}{16}\left[4a^2b^2 - (a^2+b^2-c^2)^2\right] = \frac{1}{16}(2a^2b^2 - a^4 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4)$.

Note that $s - a = \frac{1}{2}(b+c-a)$,

$s - b = \frac{1}{2}(a+c-b)$, and $s - c = \frac{1}{2}(a+b-c)$.

$s(s-a)(s-b)(s-c) = \frac{1}{16}(a+b+c)(b+c-a)(a+c-b)(a+b-c) = \frac{1}{16}(2a^2b^2 - a^4 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4)$

which is the same as was obtained above.

34. $\mathbf{u}\times\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\times(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$

$= (u_1v_1)(\mathbf{i}\times\mathbf{i}) + (u_1v_2)(\mathbf{i}\times\mathbf{j}) + (u_1v_3)(\mathbf{i}\times\mathbf{k}) + (u_2v_1)(\mathbf{j}\times\mathbf{i}) + (u_2v_2)(\mathbf{j}\times\mathbf{j}) + (u_2v_3)(\mathbf{j}\times\mathbf{k}) + (u_3v_1)(\mathbf{k}\times\mathbf{i})$
$+ (u_3v_2)(\mathbf{k}\times\mathbf{j}) + (u_3v_3)(\mathbf{k}\times\mathbf{k})$

$= (u_1v_1)(0) + (u_1v_2)(\mathbf{k}) + (u_1v_3)(-\mathbf{j}) + (u_2v_1)(-\mathbf{k}) + (u_2v_2)(0) + (u_2v_3)(\mathbf{i}) + (u_3v_1)(\mathbf{j}) + (u_3v_2)(-\mathbf{i}) + (u_3v_3)(0)$

$= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

14.4 Concepts Review

1. $1 + 4t$; $-3 - 2t$; $2 - t$

2. $\frac{x-1}{4} = \frac{y+3}{-2} = \frac{z-2}{-1}$

3. $2t\mathbf{i} - 3\mathbf{j} + 3t^2\mathbf{k}$

4. $\langle 2, -3, 3\rangle$; $\frac{x-1}{2} = \frac{y+3}{-3} = \frac{z-1}{3}$

Problem Set 14.4

1. A parallel vector is
$\mathbf{v} = \langle 4 - 1, 5 + 2, 6 - 3\rangle = \langle 3, 7, 3\rangle$.
$x = 1 + 3t$, $y = -2 + 7t$, $z = 3 + 3t$

2. A parallel vector is
$\mathbf{v} = \langle 7 - 2, -2 + 1, 3 + 5\rangle = \langle 5, -1, 8\rangle$
$x = 2 + 5t$, $y = -1 - t$, $z = -5 + 8t$

3. A parallel vector is
$\mathbf{v} = \langle 6-4, 2-2, -1-3\rangle = \langle 2, 0, -4\rangle$ or $\langle 1, 0, -2\rangle$.
$x = 4 + t, y = 2, z = 3 - 2t$

4. A parallel vector is
$\mathbf{v} = \langle 5-5, 4+3, 2+3\rangle = \langle 0, 7, 5\rangle$
$x = 5, y = -3 + 7t, z = -3 + 5t$

5. $x = 4 + 3t, y = 5 + 2t, z = 6 + t$
$$\frac{x-4}{3} = \frac{y-5}{2} = \frac{z-6}{1}$$

6. $x = -1 - 2t, y = 3, z = -6 + 5t$
Since the second direction number is 0, the line does not have symmetric equations.

7. Another parallel vector is $\langle 1, 10, 100\rangle$.
$x = 1 + t, y = 1 + 10t, z = 1 + 100t$
$$\frac{x-1}{1} = \frac{y-1}{10} = \frac{z-1}{100}$$

8. $x = -2 + 7t, y = 2 - 6t, z = -2 + 3t$
$$\frac{x+2}{7} = \frac{y-2}{-6} = \frac{z+2}{3}$$

9. Set $z = 0$. Solving $4x + 3y = 1$ and $10x + 6y = 10$ yields $x = 4, y = -5$. Thus $P_1(4, -5, 0)$ is on the line. Set $y = 0$. Solving $4x - 7z = 1$ and $10x - 5z = 10$ yields $x = \frac{13}{10}, z = \frac{3}{5}$. Thus $P_2\left(\frac{13}{10}, 0, \frac{3}{5}\right)$ is on the line.
$\overline{P_1P_2} = \left\langle \frac{13}{10} - 4, 0 - (-5), \frac{3}{5} - 0\right\rangle = \left\langle -\frac{27}{10}, 5, \frac{3}{5}\right\rangle$ is a direction vector. Thus.
$\langle 27, -50, -6\rangle = -10\overline{P_1P_2}$ is also a direction vector. The symmetric equations are thus
$$\frac{x-4}{27} = \frac{y+5}{-50} = \frac{z}{-6}$$

10. With $x = 0, y - z = 2$ and $-2y + z = 3$ yield $(0, -5, -7)$.
With $y = 0, x - z = 2$ and $3x + z = 3$ yield $\left(\frac{5}{4}, 0, -\frac{3}{4}\right)$.
A vector parallel to the line is
$\left\langle \frac{5}{4}, 5, -\frac{3}{4} + 7\right\rangle = \left\langle \frac{5}{4}, 5, \frac{25}{4}\right\rangle$ or $\langle 1, 4, 5\rangle$.
$$\frac{x}{1} = \frac{y+5}{4} = \frac{z+7}{5}$$

11. $\mathbf{u} = \langle 1, 4, -2\rangle$ and $\mathbf{v} = \langle 2, -1, -2\rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -10, -2, -9\rangle$, and hence $\langle 10, 2, 9\rangle$ is parallel to the line.
With $y = 0, x - 2z = 13$ and $2x - 2z = 5$ yield $\left(-8, 0, -\frac{21}{2}\right)$. The symmetric equations are
$$\frac{x+8}{10} = \frac{y}{2} = \frac{z + \frac{21}{2}}{9}$$

12. $\mathbf{u} = \langle 1, -3, 1\rangle$ and $\mathbf{v} = \langle 6, -5, 4\rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -7, 2, 13\rangle$ is parallel to the line.
With $x = 0, -3y + z = -1$ and $-5y + 4z = 9$ yield $\left(0, \frac{13}{7}, \frac{32}{7}\right)$.
$$\frac{x}{-7} = \frac{y - \frac{13}{7}}{2} = \frac{z - \frac{32}{7}}{13}$$

13. $\langle 1, -5, 2\rangle$ is a vector in the direction of the line.
$$\frac{x-4}{1} = \frac{y}{-5} = \frac{z-6}{2}$$

14. $\langle 2, 1, -3\rangle \times \langle 5, 4, -1\rangle = \langle 11, -13, 3\rangle$ is in the direction of the line.
$$\frac{x+5}{11} = \frac{y-7}{-13} = \frac{z+2}{3}$$

15. The point of intersection on the z-axis is (0, 0, 4). A vector in the direction of the line is $\langle 5-0, -3-0, 4-4\rangle = \langle 5, -3, 0\rangle$. Parametric equations are $x = 5t, y = -3t, z = 4$.

16. $\langle 3, 1, -2\rangle \times \langle 2, 3, -1\rangle = \langle 5, -1, 7\rangle$ is in the direction of the line since the line is perpendicular to $\langle 3, 1, -2\rangle$ and $\langle 2, 3, -1\rangle$.
$$\frac{x-2}{5} = \frac{y+4}{-1} = \frac{z-5}{7}$$

17. Using $t = 0$ and $t = 1$, two points on the first line are (–2, 1, 2) and (0, 5, 1). Using $t = 0$, a point on the second line is (2, 3, 1). Thus, two nonparallel vectors in the plane are
$\langle 0+2, 5-1, 1-2\rangle = \langle 2, 4, 1\rangle$ and
$\langle 2+2, 3-1, 1-2\rangle = \langle 4, 2, -1\rangle$
Hence, $\langle 2, 4, -1\rangle \times \langle 4, 2, -1\rangle = \langle -2, -2, -12\rangle$ is a normal to the plane, and so is $\langle 1, 1, 6\rangle$. An equation of the plane is
$1(x + 2) + 1(y - 1) + 6(z - 2) = 0$ or
$x + y + 6z = 11$.

18. Solve $\frac{x-1}{-4}=\frac{y-2}{3}$ and $\frac{x-2}{-1}=\frac{y-1}{1}$ simultaneously to get $x = 1, y = 2$. From the first line $\frac{1-1}{-4}=\frac{z-4}{-2}$, so $z = 4$ and (1, 2, 4) is on the first line. This point also satisfies the equations of the second line, so the lines intersect.
$\langle -4,3,-2\rangle$ and $\langle -1,1,6\rangle$ are parallel to the plane determined by the lines, so
$\langle -4,3,-2\rangle \times \langle -1,1,6\rangle = \langle 20,26,-1\rangle$ is a normal to the plane. An equation of the plane is
$20(x-1)+26(y-2)-1(z-4)=0$ or
$20x+26y-z=68$.

19. Using $t = 0$, another point in the plane is (1, –1, 4) and $\langle 2,3,1\rangle$ is parallel to the plane. Another parallel vector is
$\langle 1-1,-1+1,5-4\rangle = \langle 0,0,1\rangle$. Thus,
$\langle 2,3,1\rangle \times \langle 0,0,1\rangle = \langle 3,-2,0\rangle$ is a normal to the plane. An equation of the plane is
$3(x-1)-2(y+1)+0(z-5)=0$ or $3x-2y=5$.

20. Using $t = 0$, one point of the plane is (0, 1, 0).
$\langle 2,-1,1\rangle \times \langle 0,1,1\rangle = \langle -2,-2,2\rangle = -2\langle 1,1,-1\rangle$ is perpendicular to the normals of both planes, hence parallel to their line of intersection.
$\langle 3,1,2\rangle$ is parallel to the line in the plane we seek, thus
$\langle 3,1,2\rangle \times \langle 1,1,-1\rangle = \langle -3,5,2\rangle$ is a normal to the plane. An equation of the plane is
$-3(x-0)+5(y-1)+2(z-0)=0$ or
$-3x+5y+2z=5$.

21. a. With $t = 0$ in the first line, $x = 2 - 0 = 2$, $y=3+4\cdot 0=3$, $z=2\cdot 0=0$, so (2, 3, 0) is on the first line.

b. $\langle -1,4,2\rangle$ is parallel to the first line, while $\langle 1,0,2\rangle$ is parallel to the second line, so
$\langle -1,4,2\rangle \times \langle 1,0,2\rangle = \langle 8,4,-4\rangle = 4\langle 2,1,-1\rangle$ is normal to both. Thus, π has equation
$2(x-2)+1(y-3)-1(z-0)=0$ or
$2x+y-z=7$, and contains the first line.

c. With $t = 0$ in the second line,
$x=-1+0=-1, y=2, z=-1+2\cdot 0=-1$, so $Q(-1, 2, -1)$ is on the second line.

d. From Example 6 of Section 14.2, the distance from Q to π is
$$\frac{|2(-1)+(2)-(-1)-7|}{\sqrt{4+1+1}}=\frac{6}{\sqrt{6}}=\sqrt{6}\approx 2.449.$$

22. With $t = 0$, (1, –3, –1) is on the first line.
$\langle 2,4,-1\rangle \times \langle -2,3,2\rangle = \langle 11,-2,14\rangle$ is perpendicular to both lines, so
$11(x-1)-2(y+3)+14(z+1)=0$ or
$11x-2y+14z=3$ is parallel to both lines and contains the first line.
With $t = 0$, (4, 1, 0) is on the second line. The distance from (4, 1, 0) to $11x-2y+14z=3$ is
$$\frac{|11(4)-2(1)+14(0)-3|}{\sqrt{121+4+196}}=\frac{39}{\sqrt{321}}\approx 2.1768.$$

23. $\mathbf{r}\left(\frac{\pi}{3}\right)=\mathbf{i}+3\sqrt{3}\mathbf{j}+\frac{\pi}{3}\mathbf{k}$, so $\left(1, 3\sqrt{3}, \frac{\pi}{3}\right)$ is on the tangent line.
$\mathbf{r}'(t)=-2\sin t\mathbf{i}+6\cos t\mathbf{j}+\mathbf{k}$, so
$\mathbf{r}'\left(\frac{\pi}{3}\right)=-\sqrt{3}\mathbf{i}+3\mathbf{j}+\mathbf{k}$ is parallel to the tangent line at $t=\frac{\pi}{3}$. The symmetric equations of the line are $\frac{x-1}{-\sqrt{3}}=\frac{y-3\sqrt{3}}{3}=\frac{z-\frac{\pi}{3}}{1}$.

24. The curve is given by $\mathbf{r}(t)=2t^2\mathbf{i}+4t\mathbf{j}+t^3\mathbf{k}$.
$\mathbf{r}(1)=2\mathbf{i}+4\mathbf{j}+\mathbf{k}$, so (2, 4, 1) is on the tangent line.
$\mathbf{r}'(t)=4t\mathbf{i}+4\mathbf{j}+3t^2\mathbf{k}$, so $\mathbf{r}'(1)=4\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ is parallel to the tangent line. The parametric equations of the line are $x = 2 + 4t$, $y = 4 + 4t$, $z = 1 + 3t$.

25. The curve is given by $\mathbf{r}(t)=3t\mathbf{i}+2t^2\mathbf{j}+t^5\mathbf{k}$.
$\mathbf{r}(-1)=-3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, so (–3, 2, –1) is on the plane.
$\mathbf{r}'(t)=3\mathbf{i}+4t\mathbf{j}+5t^4\mathbf{k}$, so $\mathbf{r}'(-1)=3\mathbf{i}-4\mathbf{j}+5\mathbf{k}$ is in the direction of the curve at $t=-1$, hence normal to the plane. An equation of the plane is
$3(x+3)-4(y-2)+5(z+1)=0$ or
$3x-4y+5z=-22$.

26. $\mathbf{r}\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\mathbf{i}+\frac{3\pi}{2}\mathbf{j}$, so $\left(\frac{\pi}{2}, \frac{3\pi}{2}, 0\right)$ is on the plane.
$\mathbf{r}'(t)=(t\cos t+\sin t)\mathbf{i}+3\mathbf{j}+(2\cos t-2t\sin t)\mathbf{k}$ so
$\mathbf{r}'\left(\frac{\pi}{2}\right)=\mathbf{i}+3\mathbf{j}-\pi\mathbf{k}$ is in the direction of the curve at $\frac{\pi}{2}$, hence normal to the plane. An equation of the plane is
$1\left(x-\frac{\pi}{2}\right)+3\left(y-\frac{3\pi}{2}\right)-\pi(z-0)=0$ or
$x+3y-\pi z=5\pi$.

27. a. $[x(t)]^2+[y(t)]^2+[z(t)]^2=(\sin t\cos t)^2+(\sin^2 t)^2+\cos^2 t \;=\sin^2 t\cos^2 t+\sin^4 t+\cos^2 t$

$=\sin^2 t(\cos^2 t+\sin^2 t)+\cos^2 t \;=\sin^2 t+\cos^2 t=1$

Thus, the curve lies on the sphere $x^2+y^2+z^2=1$ whose center is at the origin.

b. $\mathbf{r}\left(\frac{\pi}{6}\right)=\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\mathbf{i}+\frac{1}{4}\mathbf{j}+\frac{\sqrt{3}}{2}\mathbf{k} \;=\frac{\sqrt{3}}{4}\mathbf{i}+\frac{1}{4}\mathbf{j}+\frac{\sqrt{3}}{2}\mathbf{k}$, so $\left(\frac{\sqrt{3}}{4},\frac{1}{4},\frac{\sqrt{3}}{2}\right)$ is on the tangent line.

$\mathbf{r}'(t)=(\cos^2 t-\sin^2 t)\mathbf{i}+2\cos t\sin t\mathbf{j}-\sin t\mathbf{k}$ so $\mathbf{r}'\left(\frac{\pi}{6}\right)=\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}-\frac{1}{2}\mathbf{k}$ is parallel to the line.

The line has equations $x=\frac{\sqrt{3}}{4}+t,\ y=\frac{1}{4}+\sqrt{3}t,\ z=\frac{\sqrt{3}}{2}-t.$

The line intersects the xy-plane when $z=0$, so $t=\frac{\sqrt{3}}{2}$, hence $x=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{4},\ y=\frac{1}{4}+\frac{3}{2}=\frac{7}{4}.$

The point is $\left(\frac{3\sqrt{3}}{4},\frac{7}{4},0\right).$

28. In Figure 7, d is the magnitude of the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$. $\mathrm{pr}_{\mathbf{n}}\overrightarrow{PQ}=\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}$, so

$$\left|\mathrm{pr}_{\mathbf{n}}\overrightarrow{PQ}\right|=\left|\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}\right|=\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{|\mathbf{n}|^2}|\mathbf{n}|=\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{|\mathbf{n}|}$$

The point (0, 0, 1) is on the plane $4x-4y+2z=2$. With $P(0,0,1)$, $Q(4,-2,3)$, and $\mathbf{n}=\langle 2,-2,1\rangle=\frac{1}{2}\langle 4,-4,2\rangle$,

$\overrightarrow{PQ}=\langle 4-0,-2-0,3-1\rangle=\langle 4,-2,2\rangle$ and $d=\frac{|\langle 4,-2,2\rangle\cdot\langle 2,-2,1\rangle|}{\sqrt{4+4+1}}=\frac{14}{3}.$

From Example 6 of Section 14.2, $d=\frac{|4(4)-4(-2)+2(3)-2|}{\sqrt{16+16+4}}=\frac{28}{6}=\frac{14}{3}.$

29. Let $\overrightarrow{PR}$ be the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$. Then $\left|\overrightarrow{PQ}\right|^2=\left|\overrightarrow{PR}\right|^2+d^2$ so

$$d^2=\left|\overrightarrow{PQ}\right|^2-\left|\overrightarrow{PR}\right|^2=\left|\overrightarrow{PQ}\right|^2-\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|^2}{|\mathbf{n}|^2}$$

$$= \frac{\left|\overrightarrow{PQ}\right|^2 |\mathbf{n}|^2 - \left(\overrightarrow{PQ} \cdot \mathbf{n}\right)^2}{|\mathbf{n}|^2} = \frac{\left|\overrightarrow{PQ} \times \mathbf{n}\right|^2}{|\mathbf{n}|^2}$$

by Lagrange's Identity. Thus, $d = \frac{\left|\overrightarrow{PQ} \times \mathbf{n}\right|}{|\mathbf{n}|}$.

a. $P(3, -2, 1)$ is on the line, so $\overrightarrow{PQ} = \langle 1-3, 0+2, -4-1\rangle = \langle -2, 2, -5\rangle$ while $\mathbf{n} = \langle 2, -2, 1\rangle$, so

$$d = \frac{|\langle -2, 2, -5\rangle \times \langle 2, -2, 1\rangle|}{\sqrt{4+4+1}} = \frac{|\langle -8, -8, 0\rangle|}{3}$$

$$= \frac{8\sqrt{2}}{3} \approx 3.771$$

b. $P(1, -1, 0)$ is on the line, so $\overrightarrow{PQ} = \langle 2-1, -1+1, 3-0\rangle = \langle 1, 0, 3\rangle$ while $\mathbf{n} = \langle 2, 3, -6\rangle$.

$$d = \frac{|\langle 1, 0, 3\rangle \times \langle 2, 3, -6\rangle|}{\sqrt{4+9+36}} = \frac{|\langle -9, 12, 3\rangle|}{7}$$

$$= \frac{3\sqrt{26}}{7} \approx 2.185$$

30. d is the distance between the parallel planes containing the lines. Since $\mathbf{n}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, it is normal to the planes. Thus, d is the magnitude of the scalar projection of $\overline{PQ}$ on $\mathbf{n}$, which is $\frac{|\overline{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|}$.

a. $P(3, -2, 1)$ is on the first line, $Q(-4, -5, 0)$ is on the second line, $\mathbf{n}_1 = \langle 1, 1, 2\rangle$, and $\mathbf{n}_2 = \langle 3, 4, 5\rangle$.

$\overline{PQ} = \langle -4-3, -5+2, 0-1\rangle = \langle -7, -3, -1\rangle$

$\mathbf{n} = \langle 1, 1, 2\rangle \times \langle 3, 4, 5\rangle = \langle -3, 1, 1\rangle$

$$d = \frac{|\langle -7, -3, -1\rangle \cdot \langle -3, 1, 1\rangle|}{\sqrt{9+1+1}} = \frac{17}{\sqrt{11}} \approx 5.126$$

b. $P(1, -2, 0)$ is on the first line, $Q(0, 1, 0)$ is on the second line, $\mathbf{n}_1 = \langle 2, 3, -4\rangle$, and $\mathbf{n}_2 = \langle 3, 1, -5\rangle$.

$\overline{PQ} = \langle 0-1, 1+2, 0-0\rangle = \langle -1, 3, 0\rangle$

$\mathbf{n} = \langle 2, 3, -4\rangle \times \langle 3, 1, -5\rangle = \langle -11, -2, -7\rangle$

$$d = \frac{|\langle -1, 3, 0\rangle \cdot \langle -11, -2, -7\rangle|}{\sqrt{121+4+49}} = \frac{5}{\sqrt{174}} \approx 0.379$$

14.5 Concepts Review

1. $\mathbf{r}'(t); \mathbf{r}''(t)$

2. $\int_a^b |\mathbf{r}'(t)|\, dt$

3. parallel; concave

4. $\left|\frac{d\mathbf{T}}{ds}\right|$

Problem Set 14.5

1. $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 10t\mathbf{j} + 2\mathbf{k}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 10\mathbf{j}$

$\mathbf{v}(1) = 4\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}$; $\mathbf{a}(1) = 10\mathbf{j}$;

$s(1) = \sqrt{16+100+4} = 2\sqrt{30} \approx 10.954$

2. $\mathbf{v}(t) = \mathbf{i} + 2(t-1)\mathbf{j} + 3(t-3)^2\mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{j} + 6(t-3)\mathbf{k}$

$\mathbf{v}(0) = \mathbf{i} - 2\mathbf{j} + 27\mathbf{k}$; $\mathbf{a}(0) = 2\mathbf{j} - 18\mathbf{k}$;

$s(0) = \sqrt{1+4+729} = \sqrt{734} \approx 27.092$

3. $\mathbf{v}(t) = -\frac{1}{t^2}\mathbf{i} - \frac{2t}{(t^2-1)^2}\mathbf{j} + 5t^4\mathbf{k}$

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{i} + \frac{2+6t^2}{(t^2-1)^3}\mathbf{j} + 20t^3\mathbf{k}$

$\mathbf{v}(2) = -\frac{1}{4}\mathbf{i} - \frac{4}{9}\mathbf{j} + 80\mathbf{k};\ \mathbf{a}(2) = \frac{1}{4}\mathbf{i} + \frac{26}{27}\mathbf{j} + 160\mathbf{k};$

$s(2) = \sqrt{\frac{1}{16} + \frac{16}{81} + 6400} = \frac{\sqrt{8,294,737}}{36}$

≈ 80.002

4. $\mathbf{v}(t) = 6t^5\mathbf{i} + 72t(6t^2-5)^5\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 30t^4\mathbf{i} + 72(66t^2-5)(6t^2-5)^4\mathbf{j}$

$\mathbf{v}(1) = 6\mathbf{i} + 72\mathbf{j} + \mathbf{k};\ \mathbf{a}(1) = 30\mathbf{i} + 4392\mathbf{j};$

$s(1) = \sqrt{36+5184+1} = \sqrt{5221} \approx 72.256$

5. $\mathbf{v}(t) = t^2\mathbf{j} + \frac{2}{3}t^{-1/3}\mathbf{k}$; $\mathbf{a}(t) = 2t\mathbf{j} - \frac{2}{9}t^{-4/3}\mathbf{k}$

$\mathbf{v}(2) = 4\mathbf{j} + \frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2) = 4\mathbf{j} - \frac{2^{-1/3}}{9}\mathbf{k}$

$s(2) = \sqrt{16 + \frac{2^{4/3}}{9}} \approx 4.035$

$\mathbf{v}(2) = 4\mathbf{j} + \frac{2^{2/3}}{3}\mathbf{k};\ \mathbf{a}(2) = 4\mathbf{j} - \frac{1}{9\sqrt[3]{2}}\mathbf{k};$

$s(2) = \sqrt{16 + \frac{2^{4/3}}{9}} \approx 4.035$

6. $\mathbf{v}(t) = t^2\mathbf{i} + 5(t-1)^3\mathbf{j} + \sin \pi t\mathbf{k}$

$\mathbf{a}(t) = 2t\mathbf{i} + 15(t-1)^2\mathbf{j} + \pi\cos\pi t\mathbf{k}$

$\mathbf{v}(2) = 4\mathbf{i} + 5\mathbf{j};\ \mathbf{a}(2) = 4\mathbf{i} + 15\mathbf{j} + \pi\mathbf{k};$

$s(2) = \sqrt{16+25} = \sqrt{41} \approx 6.403$

7. $\mathbf{v}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{v}(\pi) = -\mathbf{j} + \mathbf{k};\ \mathbf{a}(\pi) = \mathbf{i};$

$s(\pi) = \sqrt{1+1} = \sqrt{2} \approx 1.414$

8. $\mathbf{v}(t) = 2\cos 2t\,\mathbf{i} - 3\sin 3t\,\mathbf{j} - 4\sin 4t\,\mathbf{k}$

$\mathbf{a}(t) = -4\sin 2t\,\mathbf{i} - 9\cos 3t\,\mathbf{j} - 16\cos 4t\,\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{2}\right) = -2\mathbf{i} + 3\mathbf{j};\ \mathbf{a}\left(\frac{\pi}{2}\right) = -16\mathbf{k};$

$s\left(\frac{\pi}{2}\right) = \sqrt{4+9} = \sqrt{13} \approx 3.606$

9. $\mathbf{v}(t) = \sec^2 t\mathbf{i} + 3e^t\mathbf{j} - 4\sin 4t\mathbf{k}$

$\mathbf{a}(t) = 2\sec^2 t\tan t\mathbf{i} + 3e^t\mathbf{j} - 16\cos 4t\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{4}\right) = 2\mathbf{i} + 3e^{\pi/4}\mathbf{j};\ \mathbf{a}\left(\frac{\pi}{4}\right) = 4\mathbf{i} + 3e^{\pi/4}\mathbf{j} + 16\mathbf{k};$

$s\left(\frac{\pi}{4}\right) = \sqrt{4+9e^{\pi/2}} \approx 6.877$

10. $\mathbf{v}(t) = -e^t\mathbf{i} - \sin\pi t\mathbf{j} + \frac{2}{3}t^{-1/3}\mathbf{k}$

$\mathbf{a}(t) = -e^t\mathbf{i} - \pi\cos\pi t\mathbf{j} - \frac{2}{9}t^{-4/3}\mathbf{k}$

$\mathbf{v}(2) = -e^2\mathbf{i} + \frac{2^{2/3}}{3}\mathbf{k};\ \mathbf{a}(2) = -e^2\mathbf{i} - \pi\mathbf{j} - \frac{1}{9\sqrt[3]{2}}\mathbf{k};$

$s(0) = \sqrt{e^4 + \frac{2^{4/3}}{9}} \approx 7.408$

11. $\mathbf{v}(t) = (\pi t\cos\pi t + \sin\pi t)\mathbf{i} + (\cos\pi t - \pi t\sin\pi t)\mathbf{j} - e^{-t}\mathbf{k}$

$\mathbf{a}(t) = (2\pi\cos\pi t - \pi^2 t\sin\pi t)\mathbf{i} + (-2\pi\sin\pi t - \pi^2 t\cos\pi t)\mathbf{j} + e^{-t}\mathbf{k}$

$\mathbf{v}(2) = 2\pi\mathbf{i} + \mathbf{j} - e^{-2}\mathbf{k};\ \mathbf{a}(2) = 2\pi\mathbf{i} - 2\pi^2\mathbf{j} + e^{-2}\mathbf{k};$

$s(2) = \sqrt{4\pi^2 + 1 + e^{-4}} \approx 6.364$

12. $\mathbf{v}(t) = \frac{1}{t}\mathbf{i} + \frac{2}{t}\mathbf{j} + \frac{3}{t}\mathbf{k}$

$\mathbf{a}(t) = -\frac{1}{t^2}\mathbf{i} - \frac{2}{t^2}\mathbf{j} - \frac{3}{t^2}\mathbf{k}$

$\mathbf{v}(2) = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k};\ \mathbf{a}(2) = -\frac{1}{4}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{3}{4}\mathbf{k};$

$s(2) = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2} \approx 1.871$

13. If $|\mathbf{v}| = C$, then $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = C^2$. Differentiate implicitly to get $D_t(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\mathbf{v}' = 0$. Thus, $\mathbf{v}\cdot\mathbf{v}' = \mathbf{v}\cdot\mathbf{a} = 0$, so **a** is perpendicular to **v**.

14. If $|\mathbf{r}(t)| = C$, similar to Problem 13, $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. Conversely, if $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, then $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. But since $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = D_t[\mathbf{r}(t)\cdot\mathbf{r}(t)]$, this means that $\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2$ is a constant, so $|\mathbf{r}(t)|$ is constant.

15. $s = \int_0^2 \sqrt{1^2+\cos^2 t+(-\sin t)^2}\,dt = \int_0^2 \sqrt{1+\cos^2 t+\sin^2 t}\,dt = \int_0^2 \sqrt{1+1}\,dt = \sqrt{2}\int_0^2 dt = \sqrt{2}(2-0) = 2\sqrt{2}$

16. $s = \int_0^2 \sqrt{(\cos t - t\sin t)^2+(\sin t + t\cos t)^2+2}\,dt = \int_0^2 \sqrt{t^2+3}\,dt = \left[\frac{t}{2}\sqrt{t^2+3}+\frac{3}{2}\ln\left|t+\sqrt{t^2+3}\right|\right]_0^2$

$= \sqrt{7}+\frac{3}{2}\ln\left(2+\sqrt{7}\right)-\frac{3}{2}\ln\sqrt{3} \approx 4.126$

Use Formula 44 with $u = t$ and $a = \sqrt{3}$ for $\int\sqrt{t^2+3}\,dt$.

17. $s = \int_3^6 \sqrt{24t^2+4t^4+36}\,dt = \int_3^6 2\sqrt{t^4+6t^2+9}\,dt$

$\int_3^6 2(t^2+3)\,dt = 2\left[\frac{t^3}{3}+3t\right]_3^6 = 2[72+18-(9+9)] = 144$

18. $s = \int_0^1 \sqrt{4t^2+36t^4+324t^4}\,dt = \int_0^1 2t\sqrt{1+90t^2}\,dt = \left[\frac{1}{135}(1+90t^2)^{3/2}\right]_0^1 = \frac{1}{135}(91^{3/2}-1) \approx 6.423$

19. $s = \int_0^1 \sqrt{9t^4+36t^4+324t^4}\,dt = \int_0^1 3\sqrt{41}t^2\,dt = \left[\sqrt{41}t^3\right]_0^1 = \sqrt{41} \approx 6.403$

20. $s = \int_0^1 \sqrt{343t^{12}+98t^{12}+1764t^{12}}\,dt = \int_0^1 21\sqrt{5}t^6\,dt = \left[3\sqrt{5}t^7\right]_0^1 = 3\sqrt{5} \approx 6.708$

21. $s = \int_0^{\pi/2} \sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt = \int_0^{\pi/2} 3\cos t\sin t\sqrt{\cos^2 t+\sin^2 t}\,dt$

$= \int_0^{\pi/2} 3\cos t\sin t\,dt = \left[\frac{3}{2}\sin^2 t\right]_0^{\pi/2} = \frac{3}{2}$

22. $s = \int_0^{\pi} \sqrt{e^{4t}(2\cos t-\sin t)^2+e^{4t}(2\sin t+\cos t)^2+4e^{4t}}\,dt = \int_0^{\pi} e^{2t}\sqrt{4\cos^2 t+\sin^2 t+4\sin^2 t+\cos^2 t+4}\,dt$

$= \int_0^{\pi} 3e^{2t}\,dt = \left[\frac{3}{2}e^{2t}\right]_0^{\pi} = \frac{3}{2}(e^{2\pi}-1) \approx 801.737$

23. $s = \int_0^{\pi} \sqrt{\cosh^2 t+\sinh^2 t+1}\,dt = \int_0^{\pi} \sqrt{2\cosh^2 t}\,dt = \int_0^{\pi} \sqrt{2}\cosh t\,dt = \left[\sqrt{2}\sinh t\right]_0^{\pi} = \sqrt{2}\sinh\pi \approx 16.332$

24. $s = \int_0^{\pi} \sqrt{9\cosh^2 3t+9\sinh^2 3t+9}\,dt = \int_0^{\pi} 3\sqrt{2\cosh^2 3t}\,dt = \int_0^{\pi} 3\sqrt{2}\cosh 3t\,dt = \left[\sqrt{2}\sinh 3t\right]_0^{\pi} = \sqrt{2}\sinh 3\pi \approx 8762$

25. $\mathbf{r}'(t) = 2t\mathbf{i}+2\mathbf{j}+(2t-4)\mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{i}+2\mathbf{k}$

$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{4\mathbf{i}+2\mathbf{j}}{\sqrt{16+4}} = \frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}$

$$a_T(2)=\frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{|\mathbf{r}'(2)|}=\frac{(4\mathbf{i}+2\mathbf{j})\cdot(2\mathbf{i}+2\mathbf{k})}{2\sqrt{5}}=\frac{8}{2\sqrt{5}}=\frac{4}{\sqrt{5}}$$

$$a_N(2)=\frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|}=\frac{|(4\mathbf{i}+2\mathbf{j})\times(2\mathbf{i}+2\mathbf{k})|}{2\sqrt{5}}=\frac{|4\mathbf{i}-8\mathbf{j}-4\mathbf{k}|}{2\sqrt{5}}=\frac{4\sqrt{6}}{2\sqrt{5}}=2\sqrt{\frac{6}{5}}$$

$$\mathbf{N}(2)=\frac{\mathbf{r}''(2)-a_T(2)\mathbf{T}(2)}{a_N(2)}=\frac{(2\mathbf{i}+2\mathbf{k})-\frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}\right)}{2\sqrt{\frac{6}{5}}}=\frac{\sqrt{5}}{2\sqrt{6}}\left(\frac{2}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}+2\mathbf{k}\right)=\frac{1}{\sqrt{30}}\mathbf{i}-\frac{2}{\sqrt{30}}\mathbf{j}+\frac{5}{\sqrt{30}}\mathbf{k}$$

$$\kappa(2)=\frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3}=\frac{4\sqrt{6}}{\left(2\sqrt{5}\right)^3}=\frac{\sqrt{6}}{10\sqrt{5}}$$

$$\mathbf{B}(2)=\mathbf{T}(2)\times\mathbf{N}(2)=\left(\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}\right)\times\left(\frac{1}{\sqrt{30}}\mathbf{i}-\frac{2}{\sqrt{30}}\mathbf{j}+\sqrt{\frac{5}{6}}\mathbf{k}\right)=\frac{1}{\sqrt{6}}\mathbf{i}-\frac{2}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k}$$

26. $\mathbf{r}'(t)=t^2\mathbf{i}+t\mathbf{k}$

$\mathbf{r}''(t)=2t\mathbf{i}+\mathbf{k}$

$$\mathbf{T}(1)=\frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|}=\frac{\mathbf{i}+\mathbf{k}}{\sqrt{1+1}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}$$

$$a_T(1)=\frac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{|\mathbf{r}'(1)|}=\frac{(\mathbf{i}+\mathbf{k})\cdot(2\mathbf{i}+\mathbf{k})}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

$$a_N(1)=\frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|}=\frac{|(\mathbf{i}+\mathbf{k})\times(2\mathbf{i}+\mathbf{k})|}{\sqrt{2}}=\frac{1}{\sqrt{2}}|\mathbf{j}|=\frac{1}{\sqrt{2}}$$

$$\mathbf{N}(1)=\frac{\mathbf{r}''(1)-a_T(1)\mathbf{T}(1)}{a_N(1)}=\frac{(2\mathbf{i}+\mathbf{k})-\frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}\right)}{\frac{1}{\sqrt{2}}}=\sqrt{2}\left(\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{k}\right)=\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k}$$

$$\kappa(1)=\frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3}=\frac{1}{\left(\sqrt{2}\right)^3}=2^{-3/2}$$

$$\mathbf{B}(1)=\mathbf{T}(1)\times\mathbf{N}(1)=\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{k}\right)\times\left(\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{k}\right)=\mathbf{j}$$

27. $\mathbf{r}'(t)=t\mathbf{i}+\mathbf{j}+t^2\mathbf{k}$

$\mathbf{r}''(t)=\mathbf{i}+2t\mathbf{k}$

$$\mathbf{T}(2)=\frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|}=\frac{2\mathbf{i}+\mathbf{j}+4\mathbf{k}}{\sqrt{4+1+16}}=\frac{2}{\sqrt{21}}\mathbf{i}+\frac{1}{\sqrt{21}}\mathbf{j}+\frac{4}{\sqrt{21}}\mathbf{k}$$

$$a_T(2)=\frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{|\mathbf{r}'(2)|}=\frac{(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+4\mathbf{k})}{\sqrt{21}}=\frac{18}{\sqrt{21}}$$

$$a_N(2)=\frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|}=\frac{|(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\times(\mathbf{i}+4\mathbf{k})|}{\sqrt{21}}=\frac{1}{\sqrt{21}}|4\mathbf{i}-4\mathbf{j}-\mathbf{k}|=\frac{\sqrt{33}}{\sqrt{21}}=\sqrt{\frac{11}{7}}$$

$$\mathbf{N}(2)=\frac{\mathbf{r}''(2)-a_T(2)\mathbf{T}(2)}{a_N(2)}=\frac{(\mathbf{i}+4\mathbf{k})-\frac{18}{\sqrt{21}}\left(\frac{2}{\sqrt{21}}\mathbf{i}+\frac{1}{\sqrt{21}}\mathbf{j}+\frac{4}{\sqrt{21}}\mathbf{k}\right)}{\sqrt{\frac{11}{7}}}=\sqrt{\frac{7}{11}}\left(-\frac{15}{21}\mathbf{i}-\frac{18}{21}\mathbf{j}+\frac{12}{21}\mathbf{k}\right)$$

$$=\sqrt{\frac{7}{11}}\left(-\frac{5}{7}\mathbf{i}-\frac{6}{7}\mathbf{j}+\frac{4}{7}\mathbf{k}\right)=-\frac{5}{\sqrt{77}}\mathbf{i}-\frac{6}{\sqrt{77}}\mathbf{j}+\frac{4}{\sqrt{77}}\mathbf{k}$$

$$\kappa(2)=\frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3}=\frac{\sqrt{33}}{\left(\sqrt{21}\right)^3}=\sqrt{\frac{33}{9261}}=\sqrt{\frac{11}{3087}}=\frac{\sqrt{11}}{21\sqrt{7}}$$

$$\mathbf{B}(2)=\mathbf{T}(2)\times\mathbf{N}(2)=\left(\frac{2}{\sqrt{21}}\mathbf{i}+\frac{1}{\sqrt{21}}\mathbf{j}+\frac{4}{\sqrt{21}}\mathbf{k}\right)\times\left(-\frac{5}{\sqrt{77}}\mathbf{i}-\frac{6}{\sqrt{77}}\mathbf{j}+\frac{4}{\sqrt{77}}\mathbf{k}\right)=\frac{4}{\sqrt{33}}\mathbf{i}-\frac{4}{\sqrt{33}}\mathbf{j}-\frac{1}{\sqrt{33}}\mathbf{k}$$

28. $\mathbf{r}(t)=\langle\sin 3t, \cos 3t, t\rangle$

$\mathbf{r}'(t)=\langle 3\cos 3t, -3\sin 3t, 1\rangle$

$\mathbf{r}''(t)=\langle -9\sin 3t, -9\cos 3t, 0\rangle$

$$\mathbf{T}\left(\frac{\pi}{9}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{9}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{9}\right)\right|}=\frac{\left\langle\frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1\right\rangle}{\sqrt{\frac{9}{4}+\frac{27}{4}+1}}=\left\langle\frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle$$

$$a_T\left(\frac{\pi}{9}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{9}\right)\cdot\mathbf{r}''\left(\frac{\pi}{9}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{9}\right)\right|}=\frac{\left\langle\frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1\right\rangle\cdot\left\langle-\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0\right\rangle}{\sqrt{10}}=0$$

$$a_N\left(\frac{\pi}{9}\right)=\frac{\left|\mathbf{r}'\left(\frac{\pi}{9}\right)\times\mathbf{r}''\left(\frac{\pi}{9}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{9}\right)\right|}=\frac{\left|\left\langle\frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1\right\rangle\times\left\langle-\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0\right\rangle\right|}{\sqrt{10}}=\frac{1}{\sqrt{10}}\left|\left\langle\frac{9}{2}, -\frac{9\sqrt{3}}{2}, -27\right\rangle\right|=\frac{1}{\sqrt{10}}\left(9\sqrt{10}\right)=9$$

$$\mathbf{N}\left(\frac{\pi}{9}\right)=\frac{\mathbf{r}''\left(\frac{\pi}{9}\right)-a_T\left(\frac{\pi}{9}\right)\mathbf{T}\left(\frac{\pi}{9}\right)}{a_N}=\frac{1}{9}\left\langle-\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0\right\rangle=\left\langle-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right\rangle$$

$$\kappa\left(\frac{\pi}{9}\right)=\frac{\left|\mathbf{r}'\left(\frac{\pi}{9}\right)\times\mathbf{r}''\left(\frac{\pi}{9}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{9}\right)\right|^3}=\frac{9\sqrt{10}}{\left(\sqrt{10}\right)^3}=\frac{9}{10}$$

$$\mathbf{B}\left(\frac{\pi}{9}\right)=\mathbf{T}\left(\frac{\pi}{9}\right)\times\mathbf{N}\left(\frac{\pi}{9}\right)=\left\langle\frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle\times\left\langle-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right\rangle=\left\langle\frac{1}{2\sqrt{10}}, -\frac{\sqrt{3}}{2\sqrt{10}}, -\frac{3}{\sqrt{10}}\right\rangle$$

29. $\mathbf{r}(t)=\langle 7\sin 3t, 7\cos 3t, 14t\rangle$

$\mathbf{r}'(t)=\langle 21\cos 3t, -21\sin 3t, 14\rangle$

$\mathbf{r}''(t)=\langle -63\sin 3t, -63\cos 3t, 0\rangle$

$$\mathbf{T}\left(\frac{\pi}{3}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|}=\frac{\langle -21, 0, 14\rangle}{\sqrt{441+196}}=\frac{1}{7\sqrt{13}}\langle -21, 0, 14\rangle=\left\langle-\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\right\rangle$$

$$a_T\left(\frac{\pi}{3}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{3}\right)\cdot\mathbf{r}''\left(\frac{\pi}{3}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|}=\frac{\langle -21, 0, 14\rangle\cdot\langle 0, -63, 0\rangle}{7\sqrt{13}}=0$$

$$a_N\left(\frac{\pi}{3}\right)=\frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|}=\frac{|\langle -21, 0, 14\rangle\times\langle 0, -63, 0\rangle|}{7\sqrt{13}}=\frac{|\langle 882, 0, -1323\rangle|}{7\sqrt{13}}=\frac{441\sqrt{13}}{7\sqrt{13}}=63$$

$$\mathbf{N}\left(\frac{\pi}{3}\right)=\frac{\mathbf{r}''\left(\frac{\pi}{3}\right)-a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N\left(\frac{\pi}{3}\right)}=\frac{1}{63}\langle 0, 63, 0\rangle=\langle 0, 1, 0\rangle$$

$$\kappa\left(\frac{\pi}{3}\right)=\frac{\left|\mathbf{r}\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|^3}=\frac{441\sqrt{13}}{\left(7\sqrt{13}\right)^3}=\frac{9}{91}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right)=\mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right)=\left\langle-\frac{3}{\sqrt{13}},\ 0,\ \frac{2}{\sqrt{13}}\right\rangle\times\langle 0,1,0\rangle=\left\langle-\frac{2}{\sqrt{13}},\ 0,-\frac{3}{\sqrt{13}}\right\rangle$$

30. $\mathbf{r}'(t)=-3\cos^2 t\sin t\mathbf{i}+3\sin^2 t\cos t\mathbf{k}$

$\mathbf{r}''(t)=(6\cos t\sin^2 t-3\cos^3 t)\mathbf{i}+(6\cos^2 t\sin t-3\sin^3 t)\mathbf{k}$

$\mathbf{r}'\left(\frac{\pi}{2}\right)=\mathbf{0}$ so the object is motionless at $t_1=\frac{\pi}{2}$.

κ, **T**, **N**, and **B** do not exist.

31. $\mathbf{r}'(t)=\sinh\frac{t}{3}\mathbf{i}+\mathbf{j}$

$\mathbf{r}''(t)=\frac{1}{3}\cosh\frac{t}{3}\mathbf{i}$

$$\mathbf{T}(1)=\frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|}=\frac{\sinh\frac{1}{3}\mathbf{i}+\mathbf{j}}{\sqrt{\sinh^2\frac{1}{3}+1}}=\frac{\sinh\frac{1}{3}\mathbf{i}+\mathbf{j}}{\cosh\frac{1}{3}}=\tanh\frac{1}{3}\mathbf{i}+\operatorname{sech}\frac{1}{3}\mathbf{j}$$

$$a_T(1)=\frac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{|\mathbf{r}'(1)|}=\frac{\left(\sinh\frac{1}{3}\mathbf{i}+\mathbf{j}\right)\cdot\left(\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}\right)}{\cosh\frac{1}{3}}=\frac{1}{3}\sinh\frac{1}{3}$$

$$a_N(1)=\frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|}=\frac{\left|\left(\sinh\frac{1}{3}\mathbf{i}+\mathbf{j}\right)\times\left(\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}\right)\right|}{\cosh\frac{1}{3}}=\frac{\left|-\frac{1}{3}\cos\frac{1}{3}\mathbf{k}\right|}{\cosh\frac{1}{3}}=\frac{1}{3}$$

$$\mathbf{N}(1)=\frac{\mathbf{r}''(1)-a_T(1)\mathbf{T}(1)}{a_N(1)}=3\left[\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}-\frac{1}{3}\sinh\frac{1}{3}\left(\tanh\frac{1}{3}\mathbf{i}+\operatorname{sech}\frac{1}{3}\mathbf{j}\right)\right]=\left(\cosh\frac{1}{3}-\frac{\sinh^2\frac{1}{3}}{\cosh\frac{1}{3}}\right)\mathbf{i}-\tanh\frac{1}{3}\mathbf{j}$$

$$=\operatorname{sech}\frac{1}{3}\mathbf{i}-\tanh\frac{1}{3}\mathbf{j}$$

$$\kappa(1)=\frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3}=\frac{\frac{1}{3}\cosh\frac{1}{3}}{\cosh^3\frac{1}{3}}=\frac{1}{3}\operatorname{sech}^2\frac{1}{3}$$

$$\mathbf{B}(1)=\mathbf{T}(1)\times\mathbf{N}(1)=\left(\tanh\frac{1}{3}\mathbf{i}+\operatorname{sech}\frac{1}{3}\mathbf{j}\right)\times\left(\operatorname{sech}\frac{1}{3}\mathbf{i}-\tanh\frac{1}{3}\mathbf{j}\right)=\left(-\operatorname{sech}^2\frac{1}{3}-\tanh^2\frac{1}{3}\right)\mathbf{k}=-\mathbf{k}$$

32. $\mathbf{r}'(t)=e^{7t}(7\cos 2t-2\sin 2t)\mathbf{i}+e^{7t}(7\sin 2t+2\cos 2t)\mathbf{j}+7e^{7t}\mathbf{k}$

$\mathbf{r}''(t)=e^{7t}(45\cos 2t-28\sin 2t)\mathbf{i}+e^{7t}(45\sin 2t+28\cos 2t)\mathbf{j}+49e^{7t}\mathbf{k}$

$$\mathbf{T}\left(\frac{\pi}{3}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|}=\frac{e^{7\pi/3}\left(-\frac{7}{2}-\sqrt{3}\right)\mathbf{i}+e^{7\pi/3}\left(\frac{7\sqrt{3}}{2}-1\right)\mathbf{j}+7e^{7\pi/3}\mathbf{k}}{e^{7\pi/3}\sqrt{\left(-\frac{7}{2}-\sqrt{3}\right)^2+\left(\frac{7\sqrt{3}}{2}-1\right)^2+49}}=\left(-\frac{7+2\sqrt{3}}{2\sqrt{102}}\right)\mathbf{i}+\left(\frac{7\sqrt{3}-2}{2\sqrt{102}}\right)\mathbf{j}+\frac{7}{\sqrt{102}}\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right)=\frac{\mathbf{r}'\left(\frac{\pi}{3}\right)\cdot\mathbf{r}''\left(\frac{\pi}{3}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|}=\frac{714e^{14\pi/3}}{e^{7\pi/3}\sqrt{102}}=7\sqrt{102}e^{7\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right)=\frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|}=\frac{\left|e^{14\pi/3}\left(49+14\sqrt{3}\right)\mathbf{i}+e^{14\pi/3}\left(14-49\sqrt{3}\right)\mathbf{j}+106e^{14\pi/3}\mathbf{k}\right|}{e^{7\pi/3}\sqrt{102}}=2\sqrt{53}e^{7\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right)=\frac{\mathbf{r}''\left(\frac{\pi}{3}\right)-a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N}=\frac{2-7\sqrt{3}}{2\sqrt{53}}\mathbf{i}-\frac{7+2\sqrt{3}}{2\sqrt{53}}\mathbf{j}$$

$$\kappa\left(\frac{\pi}{3}\right)=\frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|^3}=\frac{2\sqrt{5406}e^{14\pi/3}}{\sqrt{102^3}e^{7\pi}}=\frac{\sqrt{53}}{51}e^{-7\pi/3}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right)=\mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right)=\frac{42+49\sqrt{3}}{6\sqrt{1802}}\mathbf{i}+\frac{14\sqrt{3}-147}{6\sqrt{1802}}\mathbf{j}+\sqrt{\frac{53}{102}}\mathbf{k}$$

33. $\mathbf{r}'(t)=-2e^{-2t}\mathbf{i}+2e^{2t}\mathbf{j}+2\sqrt{2}\mathbf{k}$

$\mathbf{r}''(t)=4e^{-2t}\mathbf{i}+4e^{2t}\mathbf{j}$

$$\mathbf{T}(0)=\frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}=\frac{-2\mathbf{i}+2\mathbf{j}+2\sqrt{2}\mathbf{k}}{\sqrt{4+4+8}}=-\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{\sqrt{2}}{2}\mathbf{k}$$

$$a_T(0)=\frac{\mathbf{r}'(0)\cdot\mathbf{r}''(0)}{|\mathbf{r}'(0)|}=\frac{\left(-2\mathbf{i}+2\mathbf{j}+2\sqrt{2}\mathbf{k}\right)\cdot(4\mathbf{i}+4\mathbf{j})}{4}=0$$

$$a_N(0)=\frac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|}=\frac{\left|-8\sqrt{2}\mathbf{i}+8\sqrt{2}\mathbf{j}-16\mathbf{k}\right|}{4}=\frac{16\sqrt{2}}{4}=4\sqrt{2}$$

$$\mathbf{N}(0)=\frac{\mathbf{r}''(0)-a_T(0)\mathbf{T}(0)}{a_N(0)}=\frac{4\mathbf{i}+4\mathbf{j}}{4\sqrt{2}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$$\kappa(0)=\frac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3}=\frac{16\sqrt{2}}{64}=\frac{\sqrt{2}}{4}$$

$$\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)=\left(-\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{\sqrt{2}}{2}\mathbf{k}\right)\times\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\right)=-\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}-\frac{1}{\sqrt{2}}\mathbf{k}$$

34. $\mathbf{r}(t)=\left\langle \ln t, 3t, t^2\right\rangle$

$\mathbf{r}'(t)=\left\langle \frac{1}{t}, 3, 2t\right\rangle$

$\mathbf{r}''(t)=\left\langle -\frac{1}{t^2}, 0, 2\right\rangle$

$$T(2)=\frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|}=\frac{\left\langle \frac{1}{2}, 3, 4\right\rangle}{\sqrt{\frac{1}{4}+9+16}}=\frac{\left\langle \frac{1}{2}, 3, 4\right\rangle}{\frac{\sqrt{101}}{2}}=\left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle$$

$$a_T(2)=\frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{|\mathbf{r}'(2)|}=\frac{2}{\sqrt{101}}\left(\left\langle \frac{1}{2}, 3, 4\right\rangle\cdot\left\langle -\frac{1}{4}, 0, 2\right\rangle\right)=\frac{2}{\sqrt{101}}\left(\frac{63}{8}\right)=\frac{63}{4\sqrt{101}}$$

$$a_N(2)=\frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|}=\frac{2}{\sqrt{101}}\left|\left\langle 6, -2, \frac{3}{4}\right\rangle\right|=\frac{2}{\sqrt{101}}\left(\frac{\sqrt{649}}{4}\right)=\frac{\sqrt{649}}{2\sqrt{101}}$$

$$\mathbf{N}(2)=\frac{\mathbf{r}''(2)-a_T(2)\mathbf{T}(2)}{a_N(2)}=\frac{2\sqrt{101}}{\sqrt{649}}\left(\left\langle -\frac{1}{4}, 0, 2\right\rangle-\frac{63}{4\sqrt{101}}\left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle\right)$$

$$= \frac{2\sqrt{101}}{\sqrt{649}}\left\langle -\frac{41}{101}, -\frac{189}{202}, \frac{76}{101}\right\rangle = \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}}\right\rangle$$

$$\kappa(2) = \frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \left(\frac{\sqrt{649}}{4}\right)\left(\frac{2}{\sqrt{101}}\right)^3 = \frac{2\sqrt{649}}{101\sqrt{101}}$$

$$\mathbf{B}(2) = \mathbf{T}(2)\times\mathbf{N}(2) = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle \times \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}}\right\rangle = \left\langle \frac{24}{\sqrt{649}}, -\frac{8}{\sqrt{649}}, \frac{3}{\sqrt{649}}\right\rangle$$

35. $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{k}$

$$a_T(t) = \frac{(\mathbf{i}+3\mathbf{j}+2t\mathbf{k})\cdot(2\mathbf{k})}{\sqrt{1+9+4t^2}} = \frac{4t}{\sqrt{10+4t^2}}$$

$$a_N(t) = \frac{|(\mathbf{i}+3\mathbf{j}+2t\mathbf{k})\times(2\mathbf{k})|}{\sqrt{10+4t^2}} = \frac{|6\mathbf{i}-2\mathbf{j}|}{\sqrt{10+4t^2}} = \frac{\sqrt{36+4}}{\sqrt{10+4t^2}} = 2\sqrt{\frac{10}{10+4t^2}} = 2\sqrt{\frac{5}{5+2t^2}}$$

36. $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle$

$\mathbf{r}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$

$\mathbf{r}''(t) = \langle 0, 2, 6t\rangle$

$$a_T(t) = \frac{\left\langle 1, 2t, 3t^2\right\rangle\cdot\langle 0, 2, 6t\rangle}{\sqrt{1+4t^2+9t^4}} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$

$$a_N(t) = \frac{\left|\left\langle 1, 2t, 3t^2\right\rangle\times\langle 0, 2, 6t\rangle\right|}{\sqrt{1+4t^2+9t^4}} = \frac{\left|\left\langle 6t^2, -6t, 2\right\rangle\right|}{\sqrt{1+4t^2+9t^4}} = \sqrt{\frac{36t^4+36t^2+4}{1+4t^2+9t^4}} = 2\sqrt{\frac{9t^4+9t^2+1}{1+4t^2+9t^4}}$$

37. $\mathbf{r}(t) = \left\langle e^{-t}, 2t, e^t\right\rangle$

$\mathbf{r}'(t) = \left\langle -e^{-t}, 2, e^t\right\rangle$

$\mathbf{r}''(t) = \left\langle e^{-t}, 0, e^t\right\rangle$

$\mathbf{r}'(t)\cdot\mathbf{r}''(t) = -e^{-2t} + e^{2t}$

$|\mathbf{r}'(t)| = \sqrt{e^{-2t}+4+e^{2t}}$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \left|\left\langle 2e^t, 2, -2e^{-t}\right\rangle\right| = \sqrt{4e^{2t}+4+4e^{-2t}} = 2\sqrt{e^{2t}+1+e^{-2t}}$

$$a_T(t) = \frac{e^{2t}-e^{-2t}}{\sqrt{e^{2t}+4+e^{-2t}}}$$

$$a_N(t) = 2\sqrt{\frac{e^{2t}+1+e^{-2t}}{e^{2t}+4+e^{-2t}}}$$

38. $\mathbf{r}'(t) = 2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}$

$$a_T(t)=\frac{[2(t-2)\mathbf{i}-2t\mathbf{j}+\mathbf{k}]\cdot(2\mathbf{i}-2\mathbf{j})}{\sqrt{4(t-2)^2+4t^2+1}}=\frac{4(t-2)+4t}{\sqrt{8t^2-16t+17}}=\frac{8t-8}{\sqrt{8t^2-16t+17}}$$

$$a_N(t)=\frac{|[2(t-2)\mathbf{i}-2t\mathbf{j}+\mathbf{k}]\times(2\mathbf{i}-2\mathbf{j})|}{\sqrt{8t^2+-16t+17}}=\frac{|2\mathbf{i}+2\mathbf{j}+8\mathbf{k}|}{\sqrt{8t^2-16t+17}}=\frac{6\sqrt{2}}{\sqrt{8t^2-16t+17}}$$

39. $\mathbf{r}'(t)=(1-t^2)\mathbf{i}-(1+t^2)\mathbf{j}+\mathbf{k}$

$\mathbf{r}''(t)=-2t\mathbf{i}-2t\mathbf{j}$

$\mathbf{r}'(t)\cdot\mathbf{r}''(t)=-2t(1-t^2)+2t(1+t^2)=4t^3$

$|\mathbf{r}'(t)|=\sqrt{(1-t^2)^2+(1+t^2)^2+1}=\sqrt{2t^4+3}$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)|=|2t\mathbf{i}-2t\mathbf{j}-4t\mathbf{k}|=\sqrt{4t^2+4t^2+16t^2}=2\sqrt{6}|t|$

$$a_T(t)=\frac{4t^3}{\sqrt{2t^4+3}}$$

$$a_N(t)=\frac{2\sqrt{6}|t|}{\sqrt{2t^4+3}}$$

40. $\mathbf{r}'(t)=\mathbf{i}+t^2\mathbf{j}-\frac{1}{t^2}\mathbf{k},\ t>0$

$\mathbf{r}''(t)=2t\mathbf{j}+\frac{2}{t^3}\mathbf{k}$

$\mathbf{r}'(t)\cdot\mathbf{r}''(t)=2t^3-\frac{2}{t^5}=\frac{2}{t^5}(t^8-1)$

$|\mathbf{r}'(t)|=\sqrt{1+t^4+\frac{1}{t^4}}=\frac{1}{t^2}\sqrt{t^4+t^8+1}$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)|=\left|\frac{4}{t}\mathbf{i}-\frac{2}{t^3}\mathbf{j}+2t\mathbf{k}\right|=\sqrt{\frac{4}{t^6}+\frac{16}{t^2}+4t^2}=\frac{2}{t^3}\sqrt{1+4t^4+t^8}$

$$a_T(t)=\frac{\frac{2}{t^5}(t^8-1)}{\frac{1}{t^2}\sqrt{t^4+t^8+1}}=\frac{2(t^8-1)}{t^3\sqrt{t^8+t^4+1}}$$

$$a_N(t)=\frac{\frac{2}{t^3}\sqrt{1+4t^4+t^8}}{\frac{1}{t^2}\sqrt{t^4+t^8+1}}=\frac{2}{t}\sqrt{\frac{t^8+4t^4+1}{t^8+t^4+1}}$$

41. $\mathbf{r}'(t)=\cot t\mathbf{i}-\tan t\mathbf{j}+\mathbf{k}$

$\mathbf{r}''(t)=-\csc^2 t\mathbf{i}-\sec^2 t\mathbf{j}$

$\mathbf{r}'(t)\cdot\mathbf{r}''(t)=-\frac{\cos t}{\sin^3 t}+\frac{\sin t}{\cos^3 t}=\frac{-\cos^4 t+\sin^4 t}{\sin^3 t\cos^3 t}=\tan t\sec^2 t-\cot t\csc^2 t$

$|\mathbf{r}'(t)|=\sqrt{\cot^2 t+\tan^2 t+1}$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)|=\left|\sec^2 t\mathbf{i}-\csc^2 t\mathbf{j}-2\csc t\sec t\mathbf{k}\right|=\sqrt{\sec^4 t+\csc^4 t+4\csc^2 t\sec^2 t}$

$$a_T(t)=\frac{\tan t\sec^2 t-\cot t\csc^2 t}{\sqrt{\cot^2 t+\tan^2 t+1}}$$

$$a_N(t)=\frac{\sqrt{\sec^4 t+\csc^4 t+4\csc^2 t\sec^2 t}}{\sqrt{\cot^2 t+\tan^2 t+1}}$$

42. $\mathbf{r}'(t) = (t\cos t + \sin t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + 2t\mathbf{k}$

$\mathbf{r}''(t) = (2\cos t - t\sin t)\mathbf{i} + (-t\cos t - 2\sin t)\mathbf{j} + 2\mathbf{k}$

$\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t\cos t + \sin t)(2\cos t - t\sin t) + (\cos t - t\sin t)(-t\cos t - 2\sin t) + 4t = 5t$

$|\mathbf{r}'(t)| = \sqrt{(t\cos t + \sin t)^2 + (\cos t - t\sin t)^2 + 4t^2} = \sqrt{t^2 + 1 + 4t^2} = \sqrt{5t^2 + 1}$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \left|(2\cos t + 2t^2\cos t + 2t\sin t)\mathbf{i} + (2t\cos t - 2\sin t - 2t^2\sin t)\mathbf{j} + (-2 - t^2)\mathbf{k}\right| = \sqrt{5t^4 + 16t^2 + 8}$

$a_T(t) = \dfrac{5t}{\sqrt{5t^2 + 1}}$

$a_N(t) = \sqrt{\dfrac{5t^4 + 16t^2 + 8}{5t^2 + 1}}$

43. $\mathbf{r}(t) = \left\langle t,\ t^2,\ \frac{2}{3}t^3 \right\rangle$

$\mathbf{r}'(t) = \left\langle 1,\ 2t,\ 2t^2 \right\rangle$

$\mathbf{r}''(t) = \langle 0,\ 2,\ 4t \rangle$

$\mathbf{r}'(1) = \langle 1,\ 2,\ 2 \rangle$

$\mathbf{r}''(1) = \langle 0,\ 2,\ 4 \rangle$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{1+4+4}}\langle 1,\ 2,\ 2\rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

$a_T(1) = \frac{1}{3}(0 + 4 + 8) = 4$

$a_N(1) = \frac{1}{3}|\langle 4,\ -4,\ 2\rangle| = \frac{1}{3}\sqrt{16+16+4} = \frac{6}{3} = 2$

$\mathbf{N}(1) = \frac{1}{2}\left(\langle 0,\ 2,\ 4\rangle - 4\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle\right) = \frac{1}{2}\left\langle -\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right\rangle = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle$

$\mathbf{B}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle \times \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$

44. $\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{k}$

$\mathbf{r}''(t) = (6\cos t\sin^2 t - 3\cos^3 t)\mathbf{i} + (6\cos^2 t\sin t - 3\sin^3 t)\mathbf{k}$

$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{9}{8}\mathbf{i} + \frac{3\sqrt{3}}{8}\mathbf{k}$

$\mathbf{r}''\left(\frac{\pi}{6}\right) = -\frac{3\sqrt{3}}{8}\mathbf{i} + \frac{15}{8}\mathbf{k}$

$\mathbf{T}\left(\frac{\pi}{6}\right) = \dfrac{-\frac{9}{8}\mathbf{i} + \frac{3\sqrt{3}}{8}\mathbf{k}}{\sqrt{\frac{81}{64} + \frac{27}{64}}} = \frac{4}{3\sqrt{3}}\left(-\frac{9}{8}\mathbf{i} + \frac{3\sqrt{3}}{8}\mathbf{k}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{k}$

$a_T\left(\frac{\pi}{6}\right) = \frac{4}{3\sqrt{3}}\left(\frac{27\sqrt{3}}{64} + \frac{45\sqrt{3}}{64}\right) = \frac{3}{2}$

$a_N\left(\frac{\pi}{6}\right) = \frac{4}{3\sqrt{3}}\left|\frac{27}{16}\mathbf{j}\right| = \frac{9}{4\sqrt{3}} = \frac{3\sqrt{3}}{4}$

$$\mathbf{N}\left(\frac{\pi}{6}\right)=\frac{4}{3\sqrt{3}}\left[\left(-\frac{3\sqrt{3}}{8}\mathbf{i}+\frac{15}{8}\mathbf{k}\right)-\frac{3}{2}\left(-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{k}\right)\right]=\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{6}\right)=\left(-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{k}\right)\times\left(\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{k}\right)=\mathbf{j}$$

45. $\mathbf{r}'(t)=\cosh\frac{t}{c}\mathbf{i}+\mathbf{k}$

$$\mathbf{r}''(t)=\frac{1}{c}\sinh\frac{t}{c}\mathbf{i}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right)=\cosh\frac{\pi}{6c}\mathbf{i}+\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{6}\right)=\frac{1}{c}\sinh\frac{\pi}{6c}\mathbf{i}$$

$$\mathbf{T}\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{\cosh^2\frac{\pi}{6c}+1}}\left(\cosh\frac{\pi}{6c}\mathbf{i}+\mathbf{k}\right)$$

$$a_T\left(\frac{\pi}{6}\right)=\frac{\frac{1}{c}\cosh\frac{\pi}{6c}\sinh\frac{\pi}{6c}}{\sqrt{\cosh^2\frac{\pi}{6c}+1}}$$

$$a_N\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{\cosh^2\frac{\pi}{6c}+1}}\left|\frac{1}{c}\sinh\frac{\pi}{6c}\mathbf{j}\right|=\frac{\frac{1}{c}\sinh\frac{\pi}{6c}}{\sqrt{\cosh^2\frac{\pi}{6c}+1}}$$

$$\mathbf{N}\left(\frac{\pi}{6}\right)=\frac{\sqrt{\cosh^2\frac{\pi}{6c}+1}}{\frac{1}{c}\sinh\frac{\pi}{6c}}\left[\frac{1}{c}\sinh\frac{\pi}{6c}\mathbf{i}-\frac{\frac{1}{c}\cosh\frac{\pi}{6c}\sinh\frac{\pi}{6c}}{\cosh^2\frac{\pi}{6c}+1}\left(\cosh\frac{\pi}{6c}\mathbf{i}+\mathbf{k}\right)\right]=\frac{1}{\sqrt{\cosh^2\frac{\pi}{6c}+1}}\left(\mathbf{i}-\cosh\frac{\pi}{6c}\mathbf{k}\right)$$

$$\mathbf{B}\left(\frac{\pi}{6}\right)=\mathbf{T}\left(\frac{\pi}{6}\right)\times\mathbf{N}\left(\frac{\pi}{6}\right)=\mathbf{j}$$

46. $\mathbf{r}'(t)=(t\cos t+\sin t)\mathbf{i}+(\cos t-t\sin t)\mathbf{j}+2t\mathbf{k}$

$$\mathbf{r}''(t)=(2\cos t-t\sin t)\mathbf{i}+(-t\cos t-2\sin t)\mathbf{j}+2\mathbf{k}$$

$$\mathbf{r}'\left(\frac{\pi}{2}\right)=\mathbf{i}-\frac{\pi}{2}\mathbf{j}+\pi\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}\mathbf{i}-2\mathbf{j}+2\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right)=\frac{1}{\sqrt{1+\frac{\pi^2}{4}+\pi^2}}\left(\mathbf{i}-\frac{\pi}{2}\mathbf{j}+\pi\mathbf{k}\right)=\frac{1}{\sqrt{4+5\pi^2}}(2\mathbf{i}-\pi\mathbf{j}+2\pi\mathbf{k})$$

$$a_T\left(\frac{\pi}{2}\right)=\frac{2}{\sqrt{4+5\pi^2}}\left(-\frac{\pi}{2}+\pi+2\pi\right)=\frac{5\pi}{\sqrt{4+5\pi^2}}$$

$$a_N\left(\frac{\pi}{2}\right)=\frac{2}{\sqrt{4+5\pi^2}}\left|\pi\mathbf{i}+\left(-2-\frac{\pi^2}{2}\right)\mathbf{j}+\left(-2-\frac{\pi^2}{4}\right)\mathbf{k}\right|=\frac{2}{\sqrt{4+5\pi^2}}\cdot\frac{1}{4}\sqrt{128+64\pi^2+5\pi^4}=\frac{1}{2}\sqrt{\frac{128+64\pi^2+5\pi^4}{4+5\pi^2}}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right)=2\sqrt{\frac{4+5\pi^2}{128+64\pi^2+5\pi^4}}\left[\left(-\frac{\pi}{2}\mathbf{i}-2\mathbf{j}+2\mathbf{k}\right)-\frac{5\pi}{4+5\pi^2}(2\mathbf{i}-\pi\mathbf{j}+2\pi\mathbf{k})\right]$$

$$= \frac{1}{\sqrt{(128+64\pi^2+5\pi^4)(4+5\pi^2)}}\left[(-5\pi^3-24\pi)\mathbf{i}+(-16-10\pi^2)\mathbf{j}+16\mathbf{k}\right]$$

$$\mathbf{B}\left(\frac{\pi}{2}\right)=\mathbf{T}\left(\frac{\pi}{2}\right)\times\mathbf{N}\left(\frac{\pi}{2}\right)=\frac{1}{\sqrt{128+64\pi^2+5\pi^4}}\left(4\pi\mathbf{i}-(8+2\pi^2)\mathbf{j}-(8+\pi^2)\mathbf{k}\right)$$

47. $\mathbf{r}'(t)=(\cos t-t\sin t)\mathbf{i}+(t\cos t+\sin t)\mathbf{j}+2t\mathbf{k}$

$\mathbf{r}''(t)=(-t\cos t-2\sin t)\mathbf{i}+(2\cos t-t\sin t)\mathbf{j}+2\mathbf{k}$

$\mathbf{r}'(\pi)=-\mathbf{i}-\pi\mathbf{j}+2\pi\mathbf{k}$

$\mathbf{r}''(\pi)=\pi\mathbf{i}-2\mathbf{j}+2\mathbf{k}$

$$\mathbf{T}(\pi)=\frac{1}{\sqrt{1+5\pi^2}}(-\mathbf{i}-\pi\mathbf{j}+2\pi\mathbf{k})$$

$$a_T(\pi)=\frac{1}{\sqrt{1+5\pi^2}}(2\pi-\pi+4\pi)=\frac{5\pi}{\sqrt{1+5\pi^2}}$$

$$a_N(\pi)=\frac{1}{\sqrt{1+5\pi^2}}\left|2\pi\mathbf{i}+(2+2\pi^2)\mathbf{j}+(2+\pi^2)\mathbf{k}\right|=\sqrt{\frac{8+16\pi^2+5\pi^4}{1+5\pi^2}}$$

$$\mathbf{N}(\pi)=\sqrt{\frac{1+5\pi^2}{8+16\pi^2+5\pi^4}}\left[(\pi\mathbf{i}-2\mathbf{j}+2\mathbf{k})-\frac{5\pi}{1+5\pi^2}(-\mathbf{i}-\pi\mathbf{j}+2\pi\mathbf{k})\right]$$

$$=\frac{1}{\sqrt{(8+16\pi^2+5\pi^4)(1+5\pi^2)}}\left[(5\pi^3+6\pi)\mathbf{i}+(-2-5\pi^2)\mathbf{j}-2\mathbf{k}\right]$$

$$\mathbf{B}(\pi)=\frac{1}{\sqrt{8+16\pi^2+5\pi^4}}\left(2\pi\mathbf{i}+(2+2\pi^2)\mathbf{j}+(2+\pi^2)\mathbf{k}\right)$$

48. $\mathbf{r}'(t)=\mathbf{i}+t^2\mathbf{j}-\frac{1}{t^2}\mathbf{k}$

$\mathbf{r}''(t)=2t\mathbf{j}+\frac{2}{t^3}\mathbf{k}$

$\mathbf{r}'(1)=\mathbf{i}+\mathbf{j}-\mathbf{k}$

$\mathbf{r}''(1)=2\mathbf{j}+2\mathbf{k}$

$$\mathbf{T}(1)=\frac{1}{\sqrt{1+1+1}}(\mathbf{i}+\mathbf{j}-\mathbf{k})=\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$$

$$a_T(1)=\frac{1}{\sqrt{3}}(0+2-2)=0$$

$$a_N(1)=\frac{1}{\sqrt{3}}|4\mathbf{i}-2\mathbf{j}+2\mathbf{k}|=\frac{2\sqrt{6}}{\sqrt{3}}=2\sqrt{2}$$

$$\mathbf{N}(1)=\frac{1}{2\sqrt{2}}(2\mathbf{j}+2\mathbf{k})=\frac{1}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$$

$$\mathbf{B}(1)=\mathbf{T}(1)\times\mathbf{N}(1)=\sqrt{\frac{2}{3}}\mathbf{i}-\frac{1}{\sqrt{6}}\mathbf{j}+\frac{1}{\sqrt{6}}\mathbf{k}$$

49. $\mathbf{r}'(t)=e^t\mathbf{i}+e^t(\cos t-\sin t)\mathbf{j}+e^t(\cos t+\sin t)\mathbf{k}$

$\mathbf{r}''(t)=e^t\mathbf{i}-2e^t\sin t\mathbf{j}+2e^t\cos t\mathbf{k}$

$$\mathbf{r}'\left(\frac{\pi}{3}\right)=e^{\pi/3}\mathbf{i}+\frac{e^{\pi/3}}{2}\left(1-\sqrt{3}\right)\mathbf{j}+\frac{e^{\pi/3}}{2}\left(1+\sqrt{3}\right)\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{3}\right)=e^{\pi/3}\mathbf{i}-\sqrt{3}e^{\pi/3}\mathbf{j}+e^{\pi/3}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{3}e^{\pi/3}}\left[e^{\pi/3}\mathbf{i}+\frac{e^{\pi/3}}{2}\left(1-\sqrt{3}\right)\mathbf{j}+\frac{e^{\pi/3}}{2}\left(1+\sqrt{3}\right)\mathbf{k}\right]=\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}-1\right)\mathbf{j}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{3}e^{\pi/3}}\left[e^{2\pi/3}-\frac{\sqrt{3}}{2}e^{2\pi/3}\left(1-\sqrt{3}\right)+\frac{e^{2\pi/3}}{2}\left(1+\sqrt{3}\right)\right]=\frac{1}{\sqrt{3}e^{\pi/3}}(3e^{2\pi/3})=\sqrt{3}e^{\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{3}e^{\pi/3}}\left|2e^{2\pi/3}\mathbf{i}+\frac{e^{2\pi/3}}{2}\left(\sqrt{3}-1\right)\mathbf{j}+\frac{e^{2\pi/3}}{2}\left(-\sqrt{3}-1\right)\mathbf{k}\right|=\frac{1}{\sqrt{3}e^{\pi/3}}\left(\sqrt{6}e^{2\pi/3}\right)=\sqrt{2}e^{\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}e^{\pi/3}}\left[e^{\pi/3}\left(\mathbf{i}-\sqrt{3}\mathbf{j}+\mathbf{k}\right)-\sqrt{3}e^{\pi/3}\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}-1\right)\mathbf{j}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)\mathbf{k}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\left(-\frac{1+\sqrt{3}}{2}\mathbf{j}+\frac{1-\sqrt{3}}{2}\mathbf{k}\right)=-\frac{1+\sqrt{3}}{2\sqrt{2}}\mathbf{j}+\frac{1-\sqrt{3}}{2\sqrt{2}}\mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right)=\mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right)=\sqrt{\frac{2}{3}}\mathbf{i}+\frac{1}{12}\left(3\sqrt{2}-\sqrt{6}\right)\mathbf{j}-\frac{1}{12}\left(3\sqrt{2}+\sqrt{6}\right)\mathbf{k}$$

50. $\mathbf{r}'(t)=e^t\mathbf{i}+e^t(\cos t+\sin t)\mathbf{j}+e^t(\cos t-\sin t)\mathbf{j}$

$\mathbf{r}''(t)=e^t\mathbf{i}+2e^t\cos t\mathbf{j}-2e^t\sin t\mathbf{k}$

$$\mathbf{r}'\left(\frac{\pi}{3}\right)=e^{\pi/3}\mathbf{i}+\frac{e^{\pi/3}}{2}\left(1+\sqrt{3}\right)\mathbf{j}+\frac{e^{\pi/3}}{2}\left(1-\sqrt{3}\right)\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{3}\right)=e^{\pi/3}\mathbf{i}+e^{\pi/3}\mathbf{j}-\sqrt{3}e^{\pi/3}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)\mathbf{j}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}-1\right)\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right)=\sqrt{3}e^{\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right)=\sqrt{2}e^{\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right)=\frac{1-\sqrt{3}}{2\sqrt{2}}\mathbf{j}-\frac{1+\sqrt{3}}{2\sqrt{2}}\mathbf{k}b$$

$$\mathbf{B}\left(\frac{\pi}{3}\right)=-\sqrt{\frac{2}{3}}\mathbf{i}+\frac{1}{12}\left(3\sqrt{2}+\sqrt{6}\right)\mathbf{j}+\frac{1}{12}\left(\sqrt{6}-3\sqrt{2}\right)\mathbf{k}$$

Note that this is the $\mathbf{r}(t)$ from Problem 49 with the $\mathbf{j}$ and $\mathbf{k}$ components switched, thus the values computed in Problem 49 can be used.

51. $\mathbf{r}'(t)=\cosh t\mathbf{i}+\mathbf{j}+\sinh t\mathbf{k}$

$\mathbf{r}''(t)=\sinh t\mathbf{i}+\cosh t\mathbf{k}$

$$\mathbf{r}'\left(\frac{\pi}{3}\right)=\cosh\frac{\pi}{3}\mathbf{i}+\mathbf{j}+\sinh\frac{\pi}{3}\mathbf{k}$$

$$\mathbf{r}''\left(\frac{\pi}{3}\right)=\sinh\frac{\pi}{3}\mathbf{i}+\cosh\frac{\pi}{3}\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{\cosh^2\frac{\pi}{3}+1+\sinh^2\frac{\pi}{3}}}\left(\cosh\frac{\pi}{3}\mathbf{i}+\mathbf{j}+\sinh\frac{\pi}{3}\mathbf{k}\right)=\frac{1}{\sqrt{2}\cosh\frac{\pi}{3}}\left(\cosh\frac{\pi}{3}\mathbf{i}+\mathbf{j}+\sinh\frac{\pi}{3}\mathbf{k}\right)$$

$$=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{\operatorname{sech}\frac{\pi}{3}}{\sqrt{2}}\mathbf{j}+\frac{\tanh\frac{\pi}{3}}{\sqrt{2}}\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}\cosh\frac{\pi}{3}}\left(2\cosh\frac{\pi}{3}\sinh\frac{\pi}{3}\right)=\sqrt{2}\sinh\frac{\pi}{3}$$

$$a_N\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}\cosh\frac{\pi}{3}}\left|\cosh\frac{\pi}{3}\mathbf{i}-\mathbf{j}-\sinh\frac{\pi}{3}\mathbf{k}\right|=1$$

$$\mathbf{N}\left(\frac{\pi}{3}\right)=\left(\sinh\frac{\pi}{3}\mathbf{i}+\cosh\frac{\pi}{3}\mathbf{k}\right)-\sqrt{2}\sinh\frac{\pi}{3}\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{\operatorname{sech}\frac{\pi}{3}}{\sqrt{2}}\mathbf{j}+\frac{\tanh\frac{\pi}{3}}{\sqrt{2}}\mathbf{k}\right)=-\tanh\frac{\pi}{3}\mathbf{j}+\operatorname{sech}\frac{\pi}{3}\mathbf{k}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right)=\mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}\mathbf{i}-\frac{\operatorname{sech}\frac{\pi}{3}}{\sqrt{2}}\mathbf{j}-\frac{\tanh\frac{\pi}{3}}{\sqrt{2}}\mathbf{k}$$

52. a. $\mathbf{r}'(t)=\cos t\mathbf{i}-\sin t\mathbf{j}+(2t-3)\mathbf{k}$

$2t-3<0$ for $t<\frac{3}{2}$, so the particle moves downward for $0\le t<\frac{3}{2}$.

b. $|\mathbf{r}'(t)|=\sqrt{\cos^2 t+\sin^2 t+(2t-3)^2}=\sqrt{4t^2-12t+10}$

$4t^2-12t+10=0$ has no real-number solutions, so the particle never has speed 0, i.e., it never stops moving.

c. $t^2-3t+2=12$ when $t^2-3t-10=(t+2)(t-5)=0$, $t=-2, 5$. Since $t\ge 0$, the particle is 12 meters above the ground when $t=5$.

d. $\mathbf{v}(5)=\cos 5\mathbf{i}-\sin 5\mathbf{j}+7\mathbf{k}$

e. $\mathbf{v}(5)$ is tangent to the helix at the point where the particle is 12 meters above the ground. Its path is described by
$\mathbf{v}(5)=\cos 5\mathbf{i}-\sin 5\mathbf{j}+7\mathbf{k}$.

53. a. The motion of the planet with respect to the sun can be given by $x=R_p\cos t$, $y=R_p\sin t$.

Assume that when $t=0$, both the planet and the moon are on the x-axis.
Since the moon orbits the planet 10 times for every time the planet orbits the sun, the motion of the moon with respect to the planet can be given by $x=R_m\cos 10t$, $y=R_m\sin 10t$.
Combining these equations, the motion of the moon with respect to the sun is given by
$x=R_p\cos t+R_m\cos 10t$, $y=R_p\sin t+R_m\sin 10t$.

b. $x'(t)=-R_p\sin t-10R_m\sin 10t$
$y'(t)=R_p\cos t+10R_m\cos 10t$

The moon is motionless with respect to the sun when $x'(t)$ and $y'(t)$ are both 0.

Solve $x'(t)=0$ for $\sin t$ and $y'(t)=0$ for $\cos t$ to get $\sin t=-\frac{10R_m}{R_p}\sin 10t$, $\cos t=-\frac{10R_m}{R_p}\cos 10t$.

Since $\sin^2 t+\cos^2 t=1$

$$1=\frac{100R_m^2}{R_p^2}\sin^2 10t+\frac{100R_m^2}{R_p^2}\cos^2 10t$$

$=\frac{100R_m^2}{R_p^2}$. Thus, $R_p^2=100R_m^2$ or $R_p=10R_m$. Substitute this into $x'(t)=0$ and $y'(t)=0$ to get

$-R_p(\sin t+\sin 10t)=0$ and $R_p(\cos t+\cos 10t)=0$.

If $0\le t\le\frac{\pi}{2}$, then to have $\sin t+\sin 10t=0$ and $\cos t+\cos 10t=0$ it must be that

$10t=\pi+t$ or $t=\frac{\pi}{9}$.

Thus, when the radius of the planet's orbit around the sun is ten times the radius of the moon's orbit around the planet and $t = \frac{\pi}{9}$, the moon is motionless with respect to the sun.

54. a. Years

b. The sun orbits the earth once each year while the moon orbits the earth roughly 13 times each year.

c. $|\mathbf{r}(0)| = |93.24\mathbf{i}| = 93.24$ million mi, the sum of the orbital radii.

d. $93 - 0.24 = 92.76$ million mi

e. No; since the moon orbits the earth 13 times for each time the earth orbits the sun, the moon could not be stationary with respect to the sun unless the radius of its orbit around the earth were $\frac{1}{13}$ the the radius of the earth's orbit around the sun.

f. $\mathbf{v}(t) = [-186\pi \sin(2\pi t) - 6.24\pi \sin(26\pi t)]\mathbf{i} + [186\pi \cos(2\pi t) + 6.24\pi \cos(26\pi t)]\mathbf{j}$

$\mathbf{a}(t) = [-372\pi^2 \cos(2\pi t) - 162.24\pi^2 \cos(26\pi t)]\mathbf{i} + [-372\pi^2 \sin(2\pi t) - 162.24\pi^2 \sin(26\pi t)]\mathbf{j}$

$\mathbf{v}\left(\frac{1}{2}\right) = 0\mathbf{i} + (-186\pi - 6.24\pi)\mathbf{j} = -192.24\pi\mathbf{j}$

$s\left(\frac{1}{2}\right) = 192.24\pi$ million mi/yr

$\mathbf{a}\left(\frac{1}{2}\right) = (372\pi^2 + 162.24\pi^2)\mathbf{i} + 0\mathbf{j} = 534.24\pi^2\mathbf{i}$

55. a. Winding upward around the right circular cylinder $x = \sin t, y = \cos t$ as t increases.

b. Same as part a, but winding faster/slower by a factor of $3t^2$.

c. With standard orientation of the axes, the motion is winding to the right around the right circular cylinder $x = \sin t, z = \cos t$.

d. Spiraling upward, with increasing radius, along the spiral $x = t \sin t, y = t \cos t$.

e. Spiraling upward, with decreasing radius, along the spiral $x = \frac{1}{t^2}\sin t,\ y = \frac{1}{t^2}\cos t$.

f. Spiraling to the right, with increasing radius, along the spiral $x = t^2 \sin(\ln t)$, $z = t^2 \cos(\ln t)$.

56. Since $\lim_{x\to 0^-} y = \lim_{x\to 0^+} y = 0 = y(0)$, y is continuous.

$$y'(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 3x^2 & \text{if } x > 0 \end{cases}$$

is continuous since $\lim_{x\to 0^-} y' = \lim_{x\to 0^+} y' = 0 = y'(0)$.

$$y''(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 6x & \text{if } x > 0 \end{cases}$$

is continuous since $\lim_{x\to 0^-} y'' = \lim_{x\to 0^+} y'' = 0 = y''(0)$.

Thus, $\kappa = \frac{|y''|}{(1+y'^2)^{3/2}}$ is continuous also. If $x \ne 0$ then y' and κ are continuous as elementary functions.

57. Let

$P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$.

$P_5(0) = 0 \Rightarrow a_0 = 0$

$P_5'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4$, so

$P_5'(0) = 0 \Rightarrow a_1 = 0$.

$P''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3$, so

$P_5''(0) = 0 \Rightarrow a_2 = 0$.

Thus, $P_5(x) = a_3x^3 + a_4x^4 + a_5x^5$,

$P_5'(x) = 3a_3x^2 + 4a_4x^3 + 5a_5x^4$, and

$P_5''(x) = 6a_3x + 12a_4x^2 + 20a_5x^3$.

$P_5(1) = 1, P_5'(1) = 0,$ and $P_5''(1) = 0 \Rightarrow$
$a_3 + a_4 + a_5 = 1$
$3a_3 + 4a_4 + 5a_5 = 0$
$6a_3 + 12a_4 + 20a_5 = 0$
The simultaneous solution to these equations is
$a_3 = 10, a_4 = -15, a_5 = 6,$ so
$P_5(x) = 10x^3 - 15x^4 + 6x^5.$

58. $\mathbf{B}\cdot\mathbf{B} = 1$
$\frac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 2\mathbf{B}\cdot\frac{d\mathbf{B}}{ds} = 0$
Thus, $\frac{d\mathbf{B}}{ds}$ is perpendicular to **B**.

59. $\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds}$
Since $\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|}, \frac{d\mathbf{T}}{ds}\times\mathbf{N} = 0,$ so $\frac{d\mathbf{B}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}.$
Thus $\mathbf{T}\cdot\frac{d\mathbf{B}}{ds} = \mathbf{T}\cdot\left(\mathbf{T}\times\frac{d\mathbf{N}}{ds}\right) = (\mathbf{T}\times\mathbf{T})\cdot\frac{d\mathbf{N}}{ds} = 0,$ so
$\frac{d\mathbf{B}}{ds}$ is perpendicular to **T**.

60. **N** is perpendicular to **T**, and **B** = **T** × **N** is perpendicular to both **T** and **N**. Thus, since $\frac{d\mathbf{B}}{ds}$ is perpendicular to both **T** and **B**, it is parallel to **N**, and hence there is some number $\tau(s)$ such that $\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}.$

61. Let $ax + by + cz + d = 0$ be the equation of the plane containing the curve. Since **T** and **N** lie in the plane $\mathbf{B} = \pm\frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}}$. Thus, **B** is a constant vector and $\frac{d\mathbf{B}}{ds} = 0$, so $\tau(s) = 0$, since **N** will not necessarily be 0 everywhere.

62. $\mathbf{r}'(t) = a_0\mathbf{i} + b_0\mathbf{j} + c_0\mathbf{k}$
$\mathbf{r}''(t) = \mathbf{0}$
Thus, $\mathbf{r}'(t)\times\mathbf{r}''(t) = \mathbf{0}$ and since
$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}, \kappa = 0.$
To show that $\tau = 0$, note that the curve is confined to a plane. This means that the curve is two-dimensional and thus $\tau = 0$.

63. $\mathbf{r}(t) = 6\cos\pi t\mathbf{i} + 6\sin\pi t\mathbf{j}, + 2t\mathbf{k}, t > 0$
Let $(6\cos\pi t)^2 + (6\sin\pi t)^2 + (2t)^2 = 100.$
Then $36(\cos^2\pi t + \sin^2\pi t) + 4t^2 = 100;\ 4t^2 = 64;\ t = 4.$
$\mathbf{r}(4) = 6\mathbf{i} + 8\mathbf{k}$, so the fly will hit the sphere at the point (6, 0, 8).
$\mathbf{r}'(t) = -6\pi\sin\pi t\mathbf{i} + 6\pi\cos\pi t\mathbf{j} + 2\mathbf{k}$, so the fly will have traveled
$\int_0^4\sqrt{(-6\pi\sin\pi t)^2 + (6\pi\cos\pi t)^2 + (2)^2}\,dt = \int_0^4\sqrt{36\pi^2 + 4}\,dt = \sqrt{36\pi^2 + 4}(4 - 0)$
$= 8\sqrt{9\pi^2 + 1} \approx 75.8214$

64. $\mathbf{r}(t) = \left\langle 10\cos t, 10\sin t, \left(\frac{34}{2\pi}\right)t\right\rangle$
Using the result of Example 1 with $a = 10$ and $c = \frac{34}{2\pi}$, the length of one complete turn is
$2\pi\sqrt{(10)^2 + \left(\frac{34}{2\pi}\right)^2}$ angstroms $= 10^{-8}\sqrt{400\pi^2 + 34^2}$ cm. Therefore, the total length of the helix is
$(2.9)(10^8)(10^{-8})\sqrt{400\pi^2 + 34^2} \approx 207.1794$ cm.

65. a. $\mathbf{r}(t)$ = constant, so $\mathbf{r} = \mathbf{r}_0$.

b. $\mathbf{r}(t) = \mathbf{c}t + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{r}_0 \Rightarrow \mathbf{r}(t) = \mathbf{c}t + \mathbf{r}_0$

c. $\mathbf{r}'(t) = \mathbf{c}t + \mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{v}_0 \Rightarrow \mathbf{r}'(t) = \mathbf{c}t + \mathbf{v}_0$

$\mathbf{r}(t) = \frac{1}{2}\mathbf{c}t^2 + \mathbf{v}_0 t + \mathbf{C}_2$

$\mathbf{r}(0) = \mathbf{r}_0 \Rightarrow \mathbf{r}(t) = \frac{1}{2}\mathbf{c}t^2 + \mathbf{v}_0 t + \mathbf{r}_0$

d. Say $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

Then $\frac{dr}{dt} = cr$ is equivalent to $\frac{dx}{dt} = cx,\ \frac{dy}{dt} = cy,\ \frac{dz}{dt} = cz.$

These equations have the solutions $x(t) = x_0 e^{ct},\ y(t) = y_0 e^{ct},\ z(t) = z_0 e^{ct}$.

Hence, $\mathbf{r}(t) = \mathbf{r}_0 e^{ct}$.

66. $\mathbf{r}(t) = \langle \cos t, \sin t, 16t \rangle$

Death occurred at $\mathbf{r}(12) = \langle \cos 12, \sin 12, 192 \rangle$.

$\mathbf{r}'(t) = \langle -\sin t, \cos t, 16 \rangle$

$\mathbf{r}'(12) = \langle -\sin 12, \cos 12, 16 \rangle$

Now let $t = 0$ when the bee died, and let $\mathbf{s}(t)$ describe the path of the bee from the instant of death. Then $\mathbf{s}_0 = \mathbf{s}(0) = \langle \cos 12, \sin 12, 192 \rangle$ and $\mathbf{v}_0 = \mathbf{v}(0) = \langle -\sin 12, \cos 12, 16 \rangle$.

We know that $\frac{d^2\mathbf{s}}{dt^2} = \langle 0, 0, -32 \rangle$, so using the result of Problem 65 part c,

$\mathbf{s}(t) = \langle 0, 0, -32 \rangle \left[\left(\frac{1}{2} \right) t^2 \right] + \langle -\sin 12, \cos 12, 16 \rangle t + \langle \cos 12, \sin 12, 192 \rangle$.

It landed when $z = 0$: $-32\left[\left(\frac{1}{2} \right) t^2 \right] + 16t + 192 = 0;\ 0 = -16(t^2 - t - 12) = -16(t-4)(t+3)$ so $t = 4$.

$\mathbf{s}(4) = \langle -4\sin 12 + \cos 12, 4\cos 12 + \sin 12, 0 \rangle$, so it landed at approximately (2.99, 2.84, 0).

67. It left the path at the point $\mathbf{r}(4\pi) = \langle 4\pi, 0, 4\pi \rangle$; and $\mathbf{r}'(4\pi) = \langle 1, 4\pi, 1 \rangle$.

Hence, parametric equations of the tangent line are:

$x = 4\pi + t, y = 4\pi t, z = 4\pi + t$.

Then the point of intersection of the tangent line and the plane $x + y = 30$ occurs where $(4\pi + t) + (4\pi t) = 30;\ t = \frac{30 - 4\pi}{4\pi + 1}$.

The bee hits the plane at the point with coordinates:

$x = 4\pi + \frac{30 - 4\pi}{4\pi + 1} = \frac{2(8\pi^2 + 15)}{4\pi + 1} \approx 13.85,$

$y = 4\pi\left[\frac{30 - 4\pi}{4\pi + 1} \right] = \frac{8\pi(15 - 2\pi)}{4\pi + 1} \approx 16.15,\ z \approx 13.85$ (same as x).

68. $(\mathbf{F} \times \mathbf{G})' = \left[\langle f_2 g_3 - f_3 g_2, f_3 g_1 - f_1 g_3, f_1 g_2 - f_2 g_1 \rangle \right]'$

$= \langle (f_2 g_3' + f_2' g_3) - (f_3 g_2' + f_3' g_2), (f_3 g_1' + f_3' g_1) - (f_1 g_3' + f_1' g_3), (f_1 g_2' + f_1' g_2) - (f_2 g_1' + f_2' g_1) \rangle$

$\mathbf{F} \times \mathbf{G}' + \mathbf{F}' \times \mathbf{G} = \langle f_2 g_3' - f_3 g_2', f_3 g_1' - f_1 g_3', f_1 g_2' - f_2 g_1' \rangle + \langle f_2' g_3 - f_3' g_2, f_3' g_1 - f_1' g_3, f_1' g_2 - f_2' g_1 \rangle$

$= \langle (f_2 g_3' + f_2' g_3) - (f_3 g_2' + f_3' g_2), (f_3 g_1' + f_3' g_1) - (f_1 g_3' + f_1' g_3), (f_1 g_2' + f_1' g_2) - (f_2 g_1' + f_2' g_1) \rangle = (\mathbf{F} \times \mathbf{G})'$

Thus, $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{r}(t) \times \mathbf{r}''(t) + 0 = \mathbf{r}(t) \times \mathbf{r}''(t)$

69. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times c\mathbf{r}(t) = c[\mathbf{r}(t) \times \mathbf{r}(t)] = 0$

Hence, $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{c}$, a constant vector, with $\mathbf{r}'(t)$ perpendicular to **c** for all t. Thus, the motion is in a plane with **c** as a normal.

70. a. $\mathbf{L}'(t) = m\mathbf{r}(t) \times \mathbf{v}'(t) + m\mathbf{r}'(t) \times \mathbf{v}(t) = m\mathbf{r}(t) \times \mathbf{a}(t) + m[\mathbf{v}(t) \times \mathbf{v}(t)] = m\mathbf{r}(t) \times \mathbf{a}(t) = \tau(t)$

b. If $\tau(t) = \mathbf{0}$, then, by part a, $\mathbf{L}'(t) = \mathbf{0}$, so $\mathbf{L}(t)$ is a constant vector.

c. From Problem 69, a particle moving under a central force satisfies $\mathbf{a}(t) = \mathbf{r}''(t) = c\mathbf{r}(t)$.
Thus, for such a particle,
$\tau(t) = m\mathbf{r}(t) \times c\mathbf{r}(t) = mc[\mathbf{r}(t) \times \mathbf{r}(t)] = \mathbf{0}$
and angular momentum is conserved.

71. $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$ where $r = r(t)$ and $\theta = \theta(t)$ are functions of t.

$\mathbf{v}(t) = \mathbf{r}'(t) = (r'\cos\theta - r\sin\theta\theta')\mathbf{i} + (r'\sin\theta + r\cos\theta\theta')\mathbf{j}$

Thus $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ mr\cos\theta & mr\sin\theta & 0 \\ r'\cos\theta - r\sin\theta\theta' & r'\sin\theta + r\cos\theta\theta' & 0 \end{vmatrix}$

$= 0\mathbf{i} - 0\mathbf{j} + [mr\cos\theta(r'\sin\theta + r\cos\theta\theta') - mr\sin\theta(r'\cos\theta - r\sin\theta\theta')]\mathbf{k}$

$= (mr^2\cos^2\theta\theta' + mr^2\sin^2\theta\theta')\mathbf{k} = mr^2\theta'\mathbf{k} = mr^2\frac{d\theta}{dt}\mathbf{k}$

72. From Example 5, at $P\left(1, 1, \frac{1}{3}\right)$,

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$

$= \frac{1}{2\sqrt{3}}(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

So, $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is a normal to the osculating plane.

An equation of the plane is $1(x-1) - 1(y-1) + 1\left(z - \frac{1}{3}\right) = 0$ or $x - y + z = \frac{1}{3}$.

73. $\mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -[\mathbf{T} \times (\mathbf{T} \times \mathbf{N})] = -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}] = \mathbf{N}$

since **T** and **N** are perpendicular unit vectors, hence $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1^2 = 1$.

14.6 Concepts Review

1. traces; cross sections

2. cylinders; z-axis

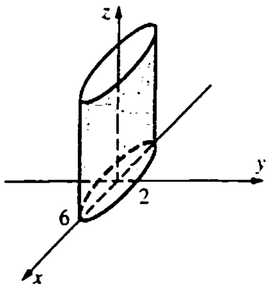
3. ellipsoid

4. elliptic paraboloid

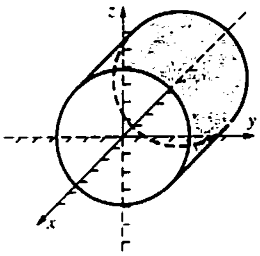
Problem Set 14.6

1. $\frac{x^2}{36}+\frac{y^2}{4}=1$

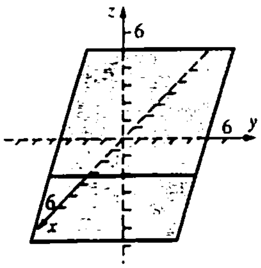
Elliptic cylinder



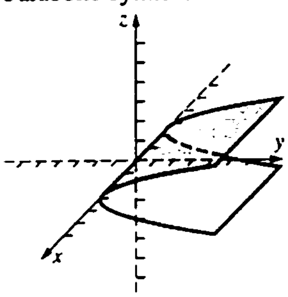
2. Circular cylinder



3. Plane

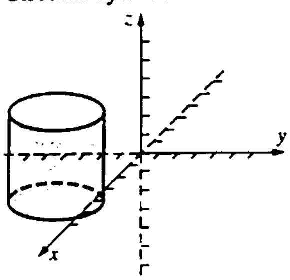


4. Parabolic cylinder

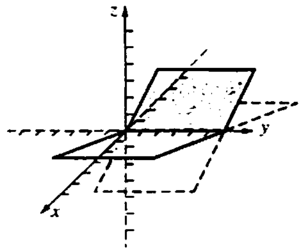


5. $(x-4)^2+(y+2)^2=7$

Circular cylinder

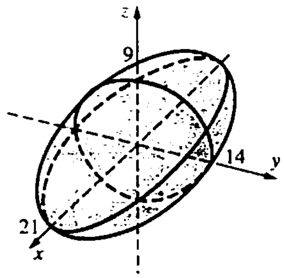


6. Two planes



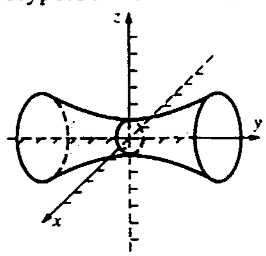
7. $\frac{x^2}{441}+\frac{y^2}{196}+\frac{z^2}{36}=1$

Ellipsoid



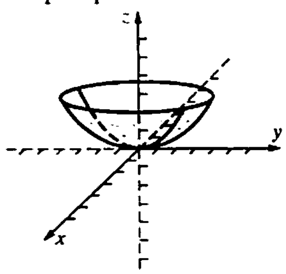
8. $\frac{x^2}{1}-\frac{y^2}{9}+\frac{z^2}{1}=1$

Hyperboloid of one sheet

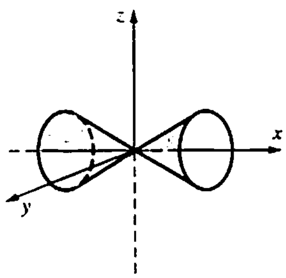


9. $z=\frac{x^2}{8}+\frac{y^2}{2}$

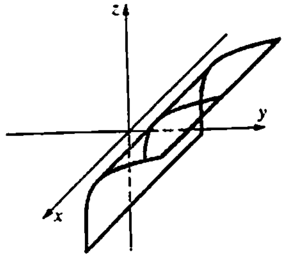
Elliptic paraboloid



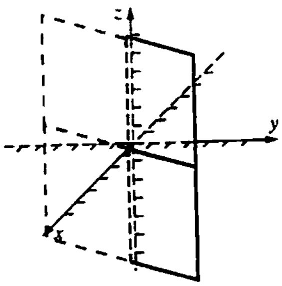
10. Circular cone



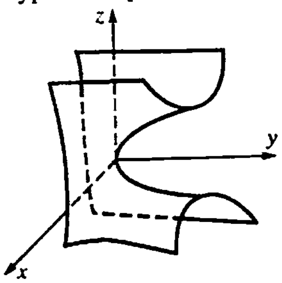
11. Cylinder



12. Plane

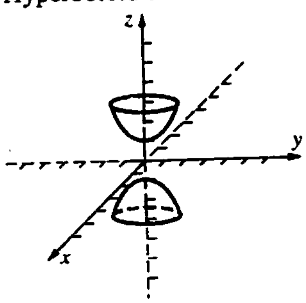


13. Hyperbolic paraboloid



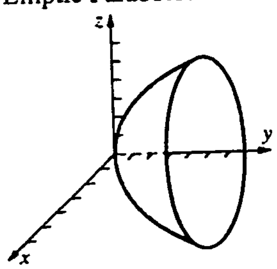
14. $-\frac{x^2}{4}-\frac{y^2}{4}+\frac{z^2}{1}=1$

Hyperboloid of two sheets



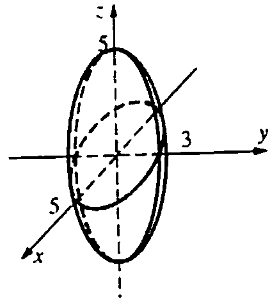
15. $y=\frac{x^2}{4}+\frac{z^2}{9}$

Elliptic Paraboloid

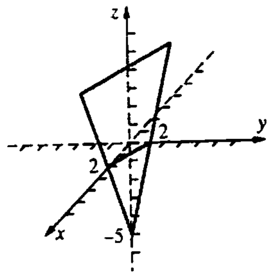


16. $\frac{x^2}{25}+\frac{y^2}{9}+\frac{z^2}{25}=1$

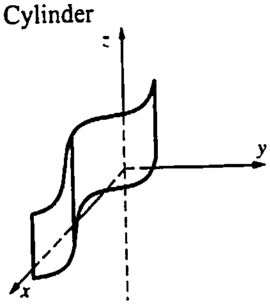
Ellipsoid



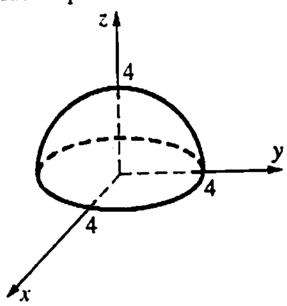
17. Plane



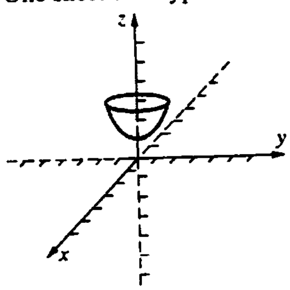
18. Cylinder



19. Hemisphere



20. One sheet of a hyperboloid of two sheets.



21. a. Replacing x by $-x$ results in an equivalent equation.

b. Replacing x by $-x$ and y by $-y$ results in an equivalent equation.

c. Replacing y by $-y$ and z by $-z$ results in an equivalent equation.

d. Replacing x by $-x$, y by $-y$, and z by $-z$, results in an equivalent equation.

22. a. 1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 18

b. 1, 2, 6, 7, 8, 9, 10, 14, 16, 19, 20

23. At $y = x$, the revolution generates a circle of radius $x = \sqrt{\frac{y}{2}} = \sqrt{\frac{k}{2}}$. Thus, the cross section in the plane $y = k$ is the circle $x^2 + z^2 = \frac{k}{2}$ or $2x^2 + 2z^2 = k$. The equation of the surface is $y = 2x^2 + 2z^2$.

24. At $z = k$, the revolution generates a circle of radius $y = \frac{z}{2} = \frac{k}{2}$. Thus, the cross section in the plane

$z = k$ is the circle $x^2 + y^2 = \frac{k^2}{4}$ or $4x^2 + 4y^2 = k^2$. The equation of the surface is $z^2 = 4x^2 + 4y^2$.

25. At $y = k$, the revolution generates a circle of radius $x = \sqrt{3 - \frac{3}{4}y^2} = \sqrt{3 - \frac{3}{4}k^2}$. Thus, the cross section in the plane $y = k$ is the circle $x^2 + z^2 = 3 - \frac{3}{4}k^2$ or $12 - 4x^2 - 4z^2 = 3k^2$. The equation of the surface is $4x^2 + 3y^2 + 4z^2 = 12$.

26. At $x = k$, the revolution generates a circle of radius $y = \sqrt{\frac{4}{3}x^2 - 4} = \sqrt{\frac{4}{3}k^2 - 4}$. Thus, the cross section in the plane $x = k$ is the circle $y^2 + z^2 = \frac{4}{3}k^2 - 4$ or $12 + 3y^2 + 3z^2 = 4k^2$. The equation of the surface is $4x^2 = 12 + 3y^2 + 3z^2$.

27. When $z = 4$ the equation is $4 = \frac{x^2}{4} + \frac{y^2}{9}$ or $1 = \frac{x^2}{16} + \frac{y^2}{36}$, so $a^2 = 36, b^2 = 16$, and $c^2 = a^2 - b^2 = 20$, hence $c = \pm 2\sqrt{5}$. The major axis of the ellipse is on the y-axis so the foci are at $\left(0, \pm 2\sqrt{5}, 4\right)$.

28. When $x = 4$, the equation is $z = \frac{16}{4} + \frac{y^2}{9}$ or $y^2 = 9(z-4) = 4 \cdot \frac{9}{4}(z-4)$, hence $p = \frac{9}{4}$. The vertex is at (4, 0, 4) so the focus is $\left(4, 0, 4 + \frac{9}{4}\right) = \left(4, 0, \frac{25}{4}\right)$.

29. When $z = h$, the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{h^2}{c^2} = 1$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^2 - h^2}{c^2}$ which is equivalent to $\frac{x^2}{\frac{a^2(c^2-h^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2-h^2)}{c^2}} = 1$, which is $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ with $A = \frac{a}{c}\sqrt{c^2 - h^2}$ and $B = \frac{b}{c}\sqrt{c^2 - h^2}$. Thus, the area is

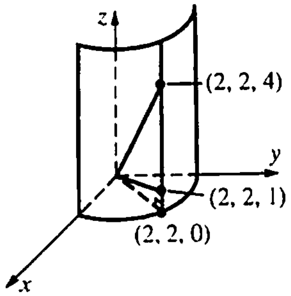
$$\pi\left(\frac{a}{c}\sqrt{c^2 - h^2}\right)\left(\frac{b}{c}\sqrt{c^2 - h^2}\right) = \frac{\pi ab(c^2 - h^2)}{c^2}$$

30. The equation of the elliptical cross section is $\frac{x^2}{a^2(h-z)} + \frac{y^2}{b^2(h-z)} = 1$, for each z in $[0, h)$.

Therefore, $\Delta V \approx \pi\left(a\sqrt{h-z}\right)\left(b\sqrt{h-z}\right)\Delta z$ $= \pi ab(h-z)\Delta z$, using the area formula mentioned in Problem 29.

Therefore, $V = \int_0^h \pi ab(h-z)dz = \pi ab\left[hz - \frac{z^2}{2}\right]_0^h$ $= \pi ab\left[\left(h - \frac{h^2}{2}\right) - 0\right] = \frac{\pi abh^2}{2}$, which is the height times one half the area of the base ($z = 0$), $\pi\left(a\sqrt{h}\right)\left(b\sqrt{h}\right) = \pi abh$.

31. Equating the expressions for y, $4 - x^2 = x^2 + z^2$ or $1 = \frac{x^2}{2} + \frac{z^2}{4}$ which is the equation of an ellipse in the xz-plane with major diameter of $2\sqrt{4} = 4$ and minor diameter $2\sqrt{2}$.

32. 

$y = x$ intersects the cylinder when $x = y = 2$. Thus, the vertices of the triangle are (0, 0, 0), (2, 2, 1), and (2, 2, 4). The area of the triangle with sides represented by $\langle 2, 2, 1\rangle$ and $\langle 2, 2, 4\rangle$ is

$$\frac{1}{2}|\langle 2, 2, 1\rangle \times \langle 2, 2, 4\rangle| = \frac{1}{2}|\langle 6, -6, 0\rangle|$$
$$= \frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}.$$

33. $(t\cos t)^2 + (t\sin t)^2 - t^2 = t^2(\cos^2 t + \sin^2 t) - t^2$
$= t^2 - t^2 = 0,$ hence every point on the spiral is on the cone.
For $\mathbf{r} = 3t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}$, every point satisfies $x^2 + 9y^2 - 9z^2 = 0$ so the spiral lies on the elliptical cone.

34. It is clear that $x = y$ at each point on the curve. Thus, the curve lies in the plane $x = y$. Since $z = t^2 = y^2$, the curve is the intersection of the plane $x = y$ with the parabolic cylinder $z = y^2$.
Let the line $y = x$ in the xy-plane be the u-axis, then the curve determined by $\mathbf{r}$ is in the uz-plane. The u-coordinate of a point on the curve, (y, y, z) is the signed distance of the point $(y, y, 0)$ from the origin, i.e., $u = \sqrt{2}y$. Thus, $u^2 = 2y^2 = 2z$ or $u^2 = 4\left(\frac{1}{2}\right)z$. This is a parabola in the uz-plane with vertex at $u = 0, z = 0$ and focus at $u = 0, z = \frac{1}{2}$. The focus is at $\left(0, 0, \frac{1}{2}\right)$.

14.7 Concepts Review

1. circular cylinder; sphere

2. plane; cone

3. $\rho^2 = r^2 + z^2$

4. $x^2 + y^2 + z^2 = 4z$, so $x^2 + y^2 + z^2 - 4z + 4 = 4$ or $x^2 + y^2 + (z-2)^2 = 4$.

Problem Set 14.7

1. Cylindrical to Spherical:
$\rho = \sqrt{r^2 + z^2}$
$\cos\phi = \frac{z}{\sqrt{r^2 + z^2}}$
$\theta = \theta$
Spherical to Cylindrical:
$r = \rho\sin\phi$
$z = \rho\cos\phi$
$\theta = \theta$

2. a. $(\rho, \theta, \phi) = \left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$

b. $(\rho, \theta, \phi) = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$

3. a. $x = 6\cos\left(\frac{\pi}{6}\right) = 3\sqrt{3}$
$y = 6\sin\left(\frac{\pi}{6}\right) = 3$
$z = -2$

b. $x = 4\cos\left(\frac{4\pi}{3}\right) = -2$
$y = 4\sin\left(\frac{4\pi}{3}\right) = -2\sqrt{3}$
$z = -8$

4. a. $x = 8\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}$
$y = 8\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$
$z = 8\cos\left(\frac{\pi}{6}\right) = 4\sqrt{3}$

b. $x = 4\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) = \sqrt{2}$

$y = 4\sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) = \sqrt{6}$

$z = 4\cos\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$

5. a. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+12+16} = 4\sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$ and (x, y) is in the 4th quadrant so $\theta = \frac{5\pi}{3}$.

$\cos\phi = \frac{z}{\rho} = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\phi = \frac{\pi}{4}$.

Spherical: $\left(4\sqrt{2}, \frac{5\pi}{3}, \frac{\pi}{4}\right)$

b. $\rho = \sqrt{2+2+12} = 4$

$\tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1$ and (x, y) is in the 2nd quadrant so $\theta = \frac{3\pi}{4}$.

$\cos\phi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ so $\frac{\pi}{6}$.

Spherical: $\left(4, \frac{3\pi}{4}, \frac{\pi}{6}\right)$

6. a. $r = \sqrt{4+4} = 2\sqrt{2}$

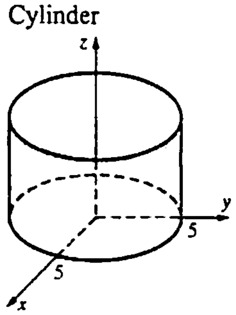
$\tan\theta = \frac{2}{2} = 1,\ x > 0, y > 0$, so $\theta = \frac{\pi}{4}$. $z = 3$

b. $r = \sqrt{48+16} = 8$

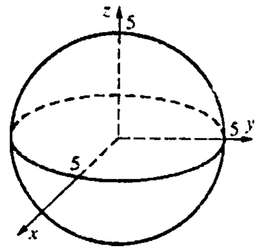
$\tan\theta = -\frac{4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}},\ x > 0, y < 0$, so

$\theta = \frac{11\pi}{6}$. $z = 6$

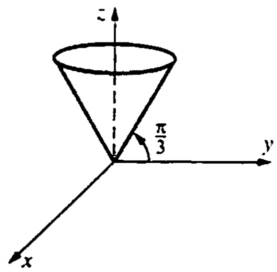
7. $r = 5$
Cylinder



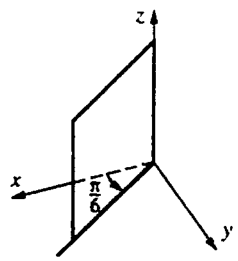
8. $\rho = 5$
Sphere



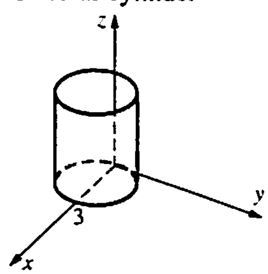
9. $\phi = \frac{\pi}{6}$
Cone



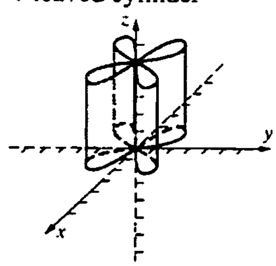
10. $\theta = \frac{\pi}{6}$
Plane



11. $r = 3\cos\theta$
Circular cylinder



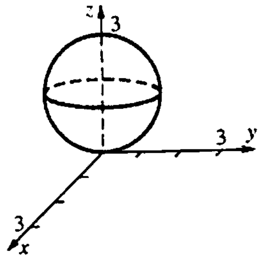
12. $r = 2\sin 2\theta$
4-leaved cylinder



13. $\rho = 3\cos\phi$

$x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = \frac{9}{4}$

Sphere

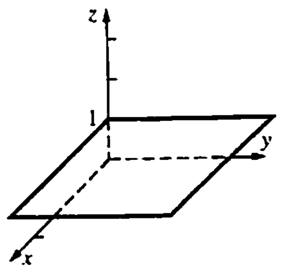


14. $\rho = \sec\phi$

$\rho\cos\phi = 1$

$z = 1$

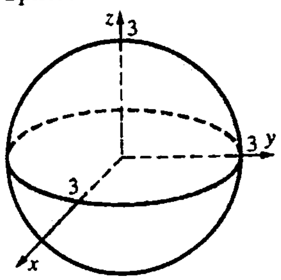
Plane



15. $r^2 + z^2 = 9$

$x^2 + y^2 + z^2 = 9$

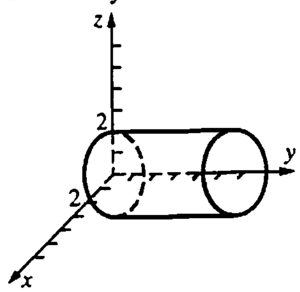
Sphere



16. $r^2\cos^2\theta + z^2 = 4$

$x^2 + z^2 = 4$

Circular cylinder



17. $x^2 + y^2 = 9; r^2 = 9;\ r = 3$

18. $r^2\cos^2\theta - r^2\sin^2\theta = 25; r^2\cos 2\theta = 25;$

$r^2 = 25\sec^2\theta; r = 5\sec\theta$

19. $r^2 + 4z^2 = 10$

20. $(x^2 + y^2 + z^2) + 3z^2 = 10; \rho^2 + 3\rho^2\cos^2\phi = 10;$

$\rho^2 = \frac{10}{1 + 3\cos^2\phi}$

21. $(x^2 + y^2 + z^2) - 3z^2 = 0; \rho^2 - 3\rho^2\cos^2\phi = 0;$

$\cos^2\phi = \frac{1}{3}$ (pole is not lost); $\cos^2\phi = \frac{1}{3}$ (or

$\sin^2\phi = \frac{2}{3}$ or $\tan^2\phi = 2$)

22. $\rho^2[\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - \cos^2\phi] = 1;$

$\rho^2[\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - 1 + \sin^2\phi] = 1;$

$\rho^2[\sin^2\phi\cos^2\theta - 1 + \sin^2\phi(1 - \sin^2\theta)] = 1;$

$\rho^2[\sin^2\phi\cos^2\theta - 1 + \sin^2\phi\cos^2\theta] = 1;$

$\rho^2[2\sin^2\phi\cos^2\theta - 1] = 1;$

$\rho^2 = \frac{1}{2\sin^2\phi\cos^2\theta - 1}$

23. $(r^2 + z^2) + z^2 = 4; \rho^2 + \rho^2\cos^2\phi = 4;$

$\rho^2 = \frac{4}{1 + \cos^2\phi}$

24. $\rho^2 = 2\rho\cos\phi; r^2 + z^2 = 2z; r^2 = 2z - z^2;$

$r = \sqrt{2z - z^2}$

25. $r\cos\theta + r\sin\theta = 4;\ r = \frac{4}{\sin\theta + \cos\theta}$

26. $\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi = 1;$

$\rho = \frac{1}{\sin\phi(\sin\theta + \cos\theta) + \cos\phi}$

27. $(x^2 + y^2 + z^2) - z^2 = 9;\ \rho^2 - \rho^2\cos^2\phi = 9;$

$\rho^2(1 - \cos^2\phi) = 9; \rho^2\sin^2\phi = 9$

28. $r^2 = 2r\sin\theta; x^2 + y^2 = 2y; x^2 + (y-1)^2 = 1$

29. $r^2\cos 2\theta = z; r^2(\cos^2\theta - \sin^2\theta) = z;$

$(r\cos\theta)^2 - (r\sin\theta)^2 = z;\ x^2 - y^2 = z$

30. $\rho\sin\phi = 1$ (spherical); $r = 1$ (cylindrical);

$x^2 + y^2 = 1$ (Cartesian)

31. $z = 2x^2 + 2y^2 = 2(x^2 + y^2)$ (Cartesian); $z = 2r^2$ (cylindrical)

32. $2x^2 + 2y^2 - z^2 = 2$ (Cartesian); $2r^2 - z^2 = 2$ (cylindrical)

33. For St. Paul:
$\rho = 3960,$
$\theta = 360° - 93.1° = 266.9° \approx 4.6583$ rad
$\phi = 90° - 45° = 45° = \frac{\pi}{4}$ rad
$x = 3960 \sin\frac{\pi}{4}\cos 4.6583 \approx -151.4$
$y = 3960 \sin\frac{\pi}{4}\sin 4.6583 \approx -2796.0$
$z = 3960 \cos\frac{\pi}{4} \approx 2800.1$
For Oslo:
$\rho = 3960$, $\theta = 10.5° \approx 0.1833$ rad.
$\phi = 90° - 59.6° = 30.4° \approx 0.5306$ rad
$x = 3960 \sin 0.5306 \cos 0.1833 \approx 1970.4$
$y = 3960 \sin 0.5306 \sin 0.1833 \approx 365.3$
$z = 3960 \cos 0.5306 \approx 3415.5$
As in Example 7,
$$\cos\gamma \approx \frac{(-151.4)(1970.4) + (-2796.0)(365.3) + (2800.1)(3415.5)}{3960^2} \approx 0.5257$$
so $\gamma \approx 1.0173$ and the great-circle distance is
$d \approx 3960(1.0173) \approx 4029$ mi

34. For New York:
$\rho = 3960$, $\theta = 360° - 74° = 286° \approx 4.9916$ rad
$\phi = 90° - 40.4° = 49.6° \approx 0.8657$ rad
$x = 3960 \sin 0.8657 \cos 4.9916 \approx 831.1$
$y = 3960 \sin 0.8657 \sin 4.9916 \approx -2898.9$
$z = 3960 \cos 0.8657 \approx 2566.5$
For Greenwich:
$\rho = 3960$, $\theta = 0$, $\phi = 90° - 51.3° = 38.7° \approx 0.6754$ rad
$x = 3960 \sin 0.6754 \cos 0 \approx 2475.8$
$y = 0$
$z = 3960 \cos 0.6754 \approx 3090.6$
$$\cos\gamma = \frac{(831.1)(2475.8) + (-2898.9)(0) + (2566.5)(3090.6)}{3960^2} \approx 0.6370$$
so $\gamma \approx 0.8802$ and the great-circle distance is
$d \approx 3960(0.8802) \approx 3485$ mi

35. From Problem 33, the coordinates of St. Paul are $P(-151.4, -2796.0, 2800.1)$.
For Turin:
$\rho = 3960$, $\theta = 7.4° \approx 0.1292$ rad, $\phi = \frac{\pi}{4}$ rad
$x = 3960 \sin\frac{\pi}{4}\cos 0.1292 \approx 2776.8$
$y = 3960 \sin\frac{\pi}{4}\sin 0.1292 \approx 360.8$
$z = 3960 \cos\frac{\pi}{4} \approx 2800.1$

$$\cos\gamma \approx \frac{(-151.4)(2776.8)+(-2796.0)(360.8)+(2800.1)(2800.1)}{3960^2} \approx 0.4088$$

so $\gamma \approx 1.1497$ and the great-circle distance is $d \approx 3960(1.1497) \approx 4553$ mi

36. The circle inscribed on the earth at 45° parallel ($\phi = 45°$) has radius $3960\cos\frac{\pi}{4}$. The longitudinal angle between St. Paul and Turin is 93.1° + 7.4° = 100.5° ≈ 1.7541 rad

Thus, the distance along the 45° parallel is $\left(3960\cos\frac{\pi}{4}\right)(1.7541) \approx 4912$ mi

37. Let St. Paul be at $P_1(-151.4, -2796.0, 2800.1)$ and Turin be at $P_2(2776.8, 360.8, 2800.1)$ and O be the center of the earth. Let β be the angle between the z-axis and the plane determined by O, P_1, and P_2. $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is normal to the plane. The angle between the z-axis and $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is complementary to β. Hence

$$\beta = \frac{\pi}{2} - \cos^{-1}\left(\frac{\left(\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right)\cdot \mathbf{k}}{\left|\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right||\mathbf{k}|}\right) \approx \frac{\pi}{2} - \cos^{-1}\left(\frac{7.709\times 10^6}{1.431\times 10^7}\right) \approx 0.5689.$$

The distance between the North Pole and the St. Paul-Turin great-circle is 3960(0.5689) ≈ 2253 mi

38. $x_i = \rho_i \sin\phi_i \cos\theta_i$, $y_i = \rho_i \sin\phi_i \sin\theta_i$, $z_i = \rho_i \cos\phi_i$ for $i = 1, 2$.

$d^2 = (\rho_2 \sin\phi_2 \cos\theta_2 - \rho_1 \sin\phi_1 \cos\theta_1)^2 + (\rho_2 \sin\phi_2 \sin\theta_2 - \rho_1 \sin\phi_1 \sin\theta_1)^2 + (\rho_2 \cos\phi_2 - \rho_1 \cos\phi_1)^2$

$= \rho_2^2 \sin^2\phi_2(\cos^2\theta_2 + \sin^2\theta_2) + \rho_1^2 \sin^2\phi_1(\cos^2\theta_1 + \sin^2\theta_1) + \rho_2^2\cos^2\phi_2 + \rho_1^2\cos^2\phi_1$
$\quad -2\rho_1\rho_2 \sin\phi_1 \sin\phi_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - 2\rho_1\rho_2\cos\phi_1\cos\phi_2$

$= \rho_2^2\sin^2\phi_2 + \rho_1^2\sin^2\phi_1 + \rho_2^2\cos^2\phi_2 + \rho_1^2\cos^2\phi_1 - 2\rho_1\rho_2\sin\phi_1\sin\phi_2[\cos(\theta_1-\theta_2)] - 2\rho_1\rho_2\cos\phi_1\cos\phi_2$

$= \rho_2^2(\sin^2\phi_2 + \cos^2\phi_2) + \rho_1^2(\sin^2\phi_1 + \cos^2\phi_1) + 2\rho_1\rho_2[-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$

$= \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2[-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$

$= \rho_1^2 - 2\rho_1\rho_2 + \rho_2^2 + 2\rho_1\rho_2[1 - \cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$

$= (\rho_1 - \rho_2)^2 + 2\rho_1\rho_2[1 - \cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$

Hence, $d = \{(\rho_1-\rho_2)^2 + 2\rho_1\rho_2[1 - \cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]\}^{1/2}$

39. Let P_1 be (a_1, θ_1, ϕ_1) and P_2 be (a_2, θ_2, ϕ_2). If γ is the angle between $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ then the great-circle distance between P_1 and P_2 is $a\gamma$. $|OP_1| = |OP_2| = a$ while the straight-line distance between P_1 and P_2 is (from Problem 38) $d^2 = (a-a)^2 + 2a^2[1 - \cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$

$= 2a^2\{1 - [\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2]\}$.

Using the Law Of Cosines on the triangle OP_1P_2,

$d^2 = a^2 + a^2 - 2a^2\cos\gamma = 2a^2(1-\cos\gamma)$.

Thus, γ is the central angle and $\cos\gamma = \cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2$.

40. The longitude/latitude system (α, β) is related to a spherical coordinate system (ρ, θ, ϕ) by the following relations: $\rho = 3960$; the trigonometric function values of α and θ are identical but

$-\pi \le \alpha \le \pi$ rather than $0 \le \theta \le 2\pi$, and $\beta = \frac{\pi}{2} - \phi$ so $\sin\beta = \cos\phi$ and $\cos\beta = \sin\phi$.

From Problem 39, the great-circle distance between $(3960, \theta_1, \phi_1)$ and $(3960, \theta_2, \phi_2)$ is 3960γ where $0 \le \gamma \le \pi$ and $\cos\gamma = \cos(\theta_1 - \theta_2)\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 = \cos(\alpha_1 - \alpha_2)\cos\beta_1 \cos\beta_2 + \sin\beta_1 \sin\beta_2$.

41. a. New York (–74°, 40.4°); Greenwich (0°, 51.3°)

$\cos\gamma = \cos(-74° - 0°)\cos(40.4°)\cos(51.3°) + \sin(40.4°)\sin(51.3°) \approx 0.637$

Then $\gamma \approx 0.880$ rad, so $d \approx 3960(0.8801) \approx 3485$ mi.

b. St. Paul (–93.1°, 45°); Turin (7.4°, 45°)

$\cos\gamma = \cos(-93.1° - 7.4°)\cos(45°)\cos(45°) + \sin(45°)\sin(45°) \approx 0.4089$

Then $\gamma \approx 1.495$ rad, so $d \approx 3960(1.1495) \approx 4552$ mi.

c. South Pole (7.4°, –90°); Turin (7.4°, 45°)

Note that any value of α can be used for the poles.

$\cos\gamma = \cos 0° \cos(-90°)\cos(45°) + \sin(-90°)\sin(45°) = -\frac{1}{\sqrt{2}}$

thus $\gamma = 135° = \frac{3\pi}{4}$ rad, so $d = 3960\left(\frac{3\pi}{4}\right) \approx 9331$ mi.

d. New York (–74°, 40.4°); Cape Town (18.4°, –33.9°)

$\cos\gamma = \cos(-74° - 18.4°)\cos(40.4°)\cos(-33.9°) + \sin(40.4°)\sin(-33.9°) \approx -0.3880$

Then $\gamma \approx 1.9693$ rad, so $d \approx 3960(1.9693) \approx 7798$ mi.

e. For these points $\alpha_1 = 100°$ and $\alpha_2 = -80°$ while $\beta_1 = \beta_2 = 0$, hence $\cos\gamma = \cos 180°$ and $\gamma = \pi$ rad, so $d = 3960\pi \approx 12{,}441$ mi.

42. $\rho = 2a \sin\phi$ is independent of θ so the cross section in each half-plane, $\theta = k$, is a circle tangent to the origin and with radius $2a$. Thus, the graph of $\rho = 2a\sin\phi$ is the surface of revolution generated by revolving about the z-axis a circle of radius $2a$ and tangent to the z-axis at the origin.

14.8 Chapter Review

Concepts Test

1. True: The coordinates are defined in terms of distances from the coordinate planes in such a way that they are unique.

2. False: The equation is $(x-2)^2 + y^2 + z^2 = -5$, so the solution set is the empty set.

3. True: See Section 14.2.

4. False: See previous problem. It represents a plane if A and B are not both zero.

5. False: The distance between (0, 0, 3) and (0, 0, –3) (a point from each plane) is 6, so the distance between the planes is less than or equal to 6 units.

6. False: It is normal to the plane.

7. True: Let $t = \frac{1}{2}$.

8. True: Direction cosines are $\frac{a}{|\mathbf{u}|}, \frac{b}{|\mathbf{u}|}, \frac{c}{|\mathbf{u}|}$ which are a, b, c.

9. True: $\left||\mathbf{u}|\mathbf{u}\right| = |\mathbf{u}||\mathbf{u}| = |\mathbf{u}|^2$

10. False: The dot product of a scalar and a vector is not defined.

11. True: $|\mathbf{u}\times\mathbf{v}| = |-\mathbf{v}\times\mathbf{u}| = |-1||\mathbf{v}\times\mathbf{u}| = |\mathbf{v}\times\mathbf{u}|$

12. True: $(k\mathbf{v})\times\mathbf{v} = k(\mathbf{v}\times\mathbf{v}) = k(\mathbf{0}) = \mathbf{0}$

13. False: Obviously not true if $\mathbf{u} = \mathbf{v}$. (More generally, it is only true when $\mathbf{u}$ and $\mathbf{v}$ are also perpendicular.)

14. False: It multiplies $\mathbf{v}$ by a; it multiplies the length of $\mathbf{v}$ by $|a|$.

15. True: $\frac{|\mathbf{u}\times\mathbf{v}|}{(\mathbf{u}\cdot\mathbf{v})} = \frac{|\mathbf{u}||\mathbf{v}|\sin\theta}{|\mathbf{u}||\mathbf{v}|\cos\theta} = \tan\theta$

16. True: The vectors are both parallel and perpendicular, so one or both must be **0**.

17. True: $|(2\mathbf{i}\times 2\mathbf{j})\cdot(\mathbf{j}\times\mathbf{i})| = 4|(\mathbf{k})\cdot(-\mathbf{k})|$
$= 4(\mathbf{k}\cdot\mathbf{k}) = 4$

18. False: Let $\mathbf{u} = \mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$.
Then $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{0}\times\mathbf{w} = \mathbf{0}$; but $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{i}\times\mathbf{k} = -\mathbf{j}$.

19. True: Since $\langle b_1, b_2, b_3\rangle$ is normal to the plane.

20. False: Each line can be represented by parametric equations, but lines with any zero direction number cannot be represented by symmetric equations.

21. True: $\kappa = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3} = 0 \Rightarrow |\mathbf{r}'\times\mathbf{r}''|$
$= |\mathbf{r}'||\mathbf{r}''|\sin\theta = 0$. Thus, either $\mathbf{r}'$ and $\mathbf{r}''$ are parallel or either $\mathbf{r}'$ or $\mathbf{r}''$ is **0**, which implies that the path is a straight line.

22. True: An ellipse bends the sharpest at points on the major axis.

23. False: κ depends only on the shape of the curve.

24. False: Suppose $\mathbf{v}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, then $|\mathbf{v}(t)| = 1$ but $\mathbf{a}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$ which is non-zero.

25. True: **T** depends only upon the shape of the curve, hence **N** and **B** also.

26. True: If **v** is perpendicular to **a**, then **T** is also perpendicular to **a**, so $\frac{d}{dt}\left(\frac{ds}{dt}\right) = a_T = \mathbf{T}\cdot\mathbf{a} = 0$. Thus speed $= \frac{ds}{dt}$ is a constant.

27. False: If $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$ then **v** is perpendicular to **a**, but the path of motion is a circular helix, not a circle.

28. False: The circular helix (see Problem 27) has constant curvature.

29. True: The curves are identical, although the motion of an object moving along the curves would be different.

30. False: At any time $0 < t < 1$, $r_1(t) \neq r_2(t)$

31. True: The parametrization affects only the rate at which the curve is traced out.

32. True: If a curve lies in a plane, then **T** and **N** will lie in the plane, so $\mathbf{T}\times\mathbf{N} = \mathbf{B}$ will be a unit vector normal to the plane.

33. False: For $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $|\mathbf{r}(t)| = 1$, but $\mathbf{r}'(t) \neq \mathbf{0}$.

34. True: The plane passes through the origin so its intersection with the sphere is a great circle. The radius of the circle is 1, so is curvature is $\frac{1}{1} = 1$.

35. False: It is the part of the z-axis with $z \geq 0$.

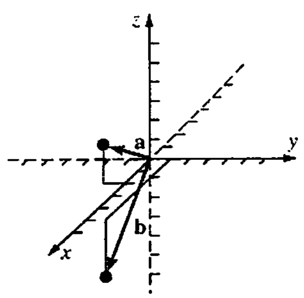
36. False: It is a parabolic cylinder.

37. False: The origin, $\rho = 0$, has infinitely many spherical coordinates, since any value of θ and ϕ can be used.

Sample Test Problems

1. The center of the sphere is the midpoint $\left(\frac{-2+4}{2}, \frac{3+1}{2}, \frac{3+5}{2}\right) = (1, 2, 4)$ of the diameter.
The radius is $r = \sqrt{(1+2)^2 + (2-3)^2 + (4-3)^2}$
$= \sqrt{9+1+1} = \sqrt{11}$. The equation of the sphere is
$(x-1)^2 + (y-2)^2 + (z-4)^2 = 11$

2. $(x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 8z + 16)$
$= 9 + 1 + 16$; $(x-3)^2 + (y+1)^2 + (z-4)^2 = 26$
Center: $(3, -1, 4)$; radius: $\sqrt{26}$

3.


a. $|\mathbf{a}| = \sqrt{4+1+4} = 3$; $|\mathbf{b}| = \sqrt{25+1+9} = \sqrt{35}$

b. $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, direction cosines $\frac{2}{3}, -\frac{1}{3}$, and $\frac{2}{3}$.

$\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{5}{\sqrt{35}}\mathbf{i} + \frac{1}{\sqrt{35}}\mathbf{j} - \frac{3}{\sqrt{35}}\mathbf{k}$, direction cosines $\frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}$

c. $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

d. $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{10-1-6}{3\sqrt{35}} = \frac{3}{3\sqrt{35}} = \frac{1}{\sqrt{35}}$

$\theta = \cos^{-1}\frac{1}{\sqrt{35}} \approx 1.4010 \approx 80.27°$

4. a. $\langle -5,-5,5\rangle = -5\langle 1,1,-1\rangle$

b. $\langle 2,-1,1\rangle \times \langle 0,5,1\rangle = \langle 6,-2,10\rangle$

c. $\langle 2,-1,1\rangle \cdot \langle -7,1,-5\rangle = -20$

d. $\langle 2,-1,1\rangle \times \langle -7,1,-5\rangle = \langle 4,3,-5\rangle$

5. $c\langle 3,3,-1\rangle \times \langle -1,-2,4\rangle = c\langle 10,-11,-3\rangle$ for any c in R.

6. Two vectors determined by the points are $\langle -1,7,-3\rangle$ and $\langle 3,-1,-3\rangle$. Then $\langle -1,7,-3\rangle \times \langle 3,-1,-3\rangle = -4\langle 6,3,5\rangle$ and $\langle 6,3,5\rangle$ are normal to the plane.

$\frac{\pm\langle 6,3,5\rangle}{\sqrt{36+9+25}} = \frac{\pm\langle 6,3,5\rangle}{\sqrt{70}}$ are the unit vectors normal to the plane.

7. a. $y = 7$, since y must be a constant.

b. $x = -5$, since it is parallel to the yz-plane.

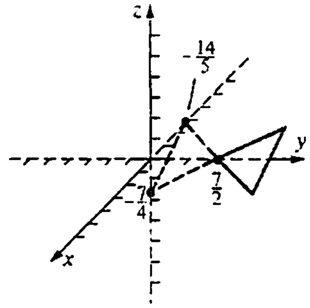
c. $z = -2$, since it is parallel to the xy-plane.

d. $3x - 4y + z = -45$, since it can be expressed as $3x - 4y + z = D$ and D must satisfy $3(-5) - 4(7) + (-2) = D$, so $D = -45$.

8. a. $\langle 4+1, 1-5, 1+7\rangle = \langle 5,-4,8\rangle$ is along the line, hence normal to the plane, which has equation $\langle x-2, y+4, z+5\rangle \cdot \langle 5,-4,8\rangle = 0$.

b. $5(x-2) - 4(y+4) + 8(z+5) = 0$ or
$5x - 4y + 8z = -14$

c.



9. If the planes are perpendicular, their normals will also be perpendicular. Thus
$0 = \langle 1,5,C\rangle \cdot \langle 4,-1,1\rangle = 4 - 5 + C$, so $C = 1$.

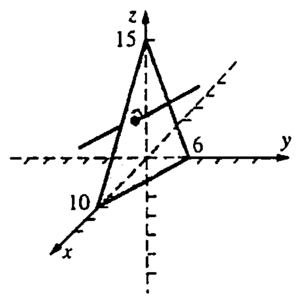
10. Two vectors in the same plane are $\langle 3,-2,-3\rangle$ and $\langle 3,7,0\rangle$. Their cross product, $3\langle 7,-3,9\rangle$, is normal to the plane. An equation of the plane is
$7(x-2) - 3(y-3) + 9(z+1) = 0$ or
$7x - 3y + 9z = -4$.

11. A vector in the direction of the line is $\langle 8,1,-8\rangle$. Parametric equations are $x = -2 + 8t$, $y = 1 + t$, $z = 5 - 8t$.

12. In the yz-plane, $x = 0$. Solve $-2y + 4z = 14$ and $2y - 5z = -30$, obtaining $y = 25$ and $z = 16$.
In the xz-plane, $y = 0$. Solve $x + 4z = 14$ and $-x - 5z = -30$, obtaining $x = -50$ and $z = 16$.
Therefore, the points are (0, 25, 16) and (–50, 0, 16).

13. (0, 25, 16) and (–50, 0, 16) are on the line, so $\langle 50,25,0\rangle = 25\langle 2,1,0\rangle$ is in the direction of the line. Parametric equations are $x = 0 + 2t$, $y = 25 + 1t$, $z = 16 + 0t$ or $x = 2t$, $y = 25 + t$, $z = 16$.

14. $\langle 3,5,2\rangle$ is normal to the plane, so is in the direction of the line. Symmetric equations of the line are $\frac{x-4}{3} = \frac{y-5}{5} = \frac{z-8}{2}$.

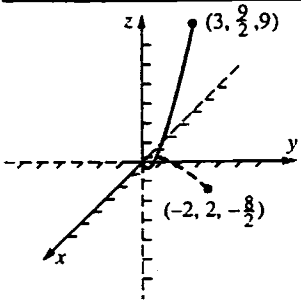


15. $\langle 5,-4,-3\rangle$ is a vector in the direction of the line, and $\langle 2,-2,1\rangle$ is a position vector to the line.

Then a vector equation of the line is
$\mathbf{r}(t) = \langle 2, -2, 1\rangle + t\langle 5, -4, -3\rangle$.

16.

t	x	y	z
–2	–2	2	–8/3
–1	–1	1/2	–1/3
0	0	0	0
1	1	1/2	1/3
2	2	2	8/3
3	3	9/2	9



17. $\mathbf{r}'(t) = \langle 1, t, t^2\rangle, \mathbf{r}'(2) = \langle 1, 2, 4\rangle$ and $\mathbf{r}(2) = \left\langle 2, 2, \frac{8}{3}\right\rangle$. Symmetric equations for the tangent line are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-\frac{8}{3}}{4}$. Normal plane is $1(x-2)+2(y-2)+4\left(z-\frac{8}{3}\right) = 0$ or $3x + 6y + 12z = 50$.

18. $\mathbf{r}(t) = \langle t\cos t, t\sin t, 2t\rangle$;

$\mathbf{r}'(t) = \langle -t\sin t + \cos t,\ t\cos t + \sin t,\ 2\rangle$;

$\mathbf{r}''(t) = \langle -t\cos t - 2\sin t,\ -t\sin t + 2\cos t,\ 0\rangle$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \left\langle -\frac{\pi}{2}, 1, 2\right\rangle; \mathbf{r}''\left(\frac{\pi}{2}\right) = \left\langle -2, -\frac{\pi}{2}, 0\right\rangle$

$\left|\mathbf{r}'\left(\frac{\pi}{2}\right)\right| = \frac{\sqrt{\pi^2+20}}{2}$;

$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{2}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{2}\right)\right|} = \frac{\langle -\pi, 2, 4\rangle}{\sqrt{\pi^2+20}}$

19. $\mathbf{r}'(t) = e^t\langle \cos t + \sin t, -\sin t + \cos t, 1\rangle$

$|\mathbf{r}'(t)| = \sqrt{3}e^t$

Length is

$\int_1^5 \sqrt{3}e^t\,dt = \left[\sqrt{3}e^t\right]_1^5 = \sqrt{3}(e^5 - e) \approx 252.3509.$

20. $\mathbf{v}(t) = \left\langle e^t, -e^{-t}, 2\right\rangle$

$\mathbf{a}(t) = \left\langle e^t, e^{-t}, 0\right\rangle$

$\mathbf{v}(\ln 2) = \left\langle 2, -\frac{1}{2}, 2\right\rangle$

$\mathbf{a}(\ln 2) = \left\langle 2, \frac{1}{2}, 0\right\rangle$

$\kappa(\ln 2) = \frac{|\mathbf{v}(\ln 2)\times\mathbf{a}(\ln 2)|}{|\mathbf{v}(\ln 2)|^3} = \frac{|\langle -1, 4, 2\rangle|}{\left(\sqrt{\frac{33}{4}}\right)^3}$

$= \frac{8}{33^{3/2}}\sqrt{21} = 8\sqrt{\frac{21}{35937}} = \frac{8\sqrt{7}}{\sqrt{11979}} \approx 0.1934$

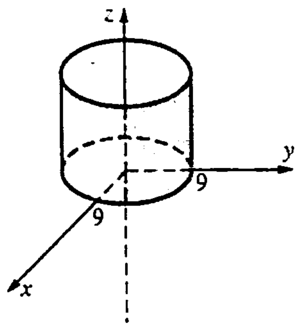
21. $\mathbf{v}(t) = \left\langle 1, 2t, 3t^2\right\rangle; \mathbf{a}(t) = \langle 0, 2, 6t\rangle$

$\mathbf{v}(1) = \langle 1, 2, 3\rangle; |\mathbf{v}(1)| = \sqrt{14}; \mathbf{a}(1) = \langle 0, 2, 6\rangle$

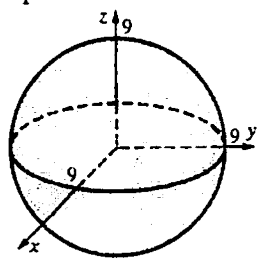
$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{0+4+18}{\sqrt{14}} = \frac{22}{\sqrt{14}} \approx 5.880;$

$a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{|\langle 6, -6, 2\rangle|}{\sqrt{14}} = \frac{2\sqrt{19}}{\sqrt{14}} \approx 2.330$

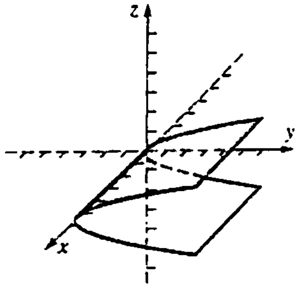
22. Circular cylinder



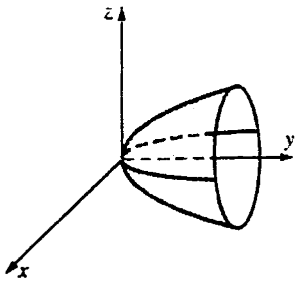
23. Sphere



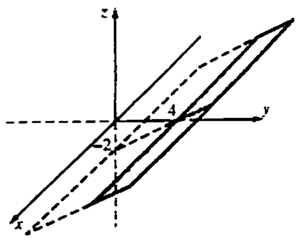
24. Parabolic cylinder



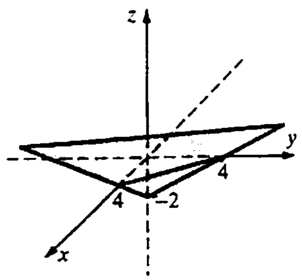
25. Circular paraboloid



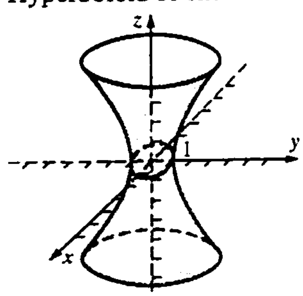
26. Plane



27. Plane

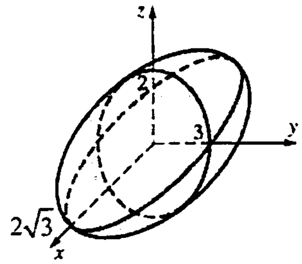


28. Hyperboloid of one sheet



29. $\frac{x^2}{12}+\frac{y^2}{9}+\frac{z^2}{4}=1$

Ellipsoid



30. The graph of $3x^2+4y^2+9z^2=-36$ is the empty set.

31. a. $r^2=9;\ r=3$

b. $(x^2+y^2)+3y^2=16$

$r^2+3r^2\sin^2\theta=16, r^2=\frac{16}{1+3\sin^2\theta}$

c. $r^2=9z$

d. $r^2+4z^2=10$

32. a. $x^2+y^2+z^2=9$

b. $x^2+z^2=4$

c. $r^2(\cos^2\theta-\sin^2\theta)+z^2=1;$

$x^2-y^2+z^2=1$

33. a. $\rho^2=4;\ \rho=2$

b. $x^2+y^2+z^2-2z^2=0;$

$\rho^2-2\rho^2\cos^2\phi=0;\ \ \rho^2(1-2\cos^2\phi)=0;$

$1-2\cos^2\phi=0;\ \ \cos^2\phi=\frac{1}{2};\ \ \phi=\frac{\pi}{4}$ or

$\phi=\frac{3\pi}{4}.$

Any of the following (as well as others) would be acceptable:

$\left(\phi-\frac{\pi}{4}\right)\left(\phi-\frac{3\pi}{4}\right)=0$

$\cos^2\phi=\frac{1}{2}$

$\sec^2\phi=2$

$\tan^2\phi=1$

c. $2x^2 - (x^2 + y^2 + z^2) = 1;$
$2\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 = 1;$
$\rho^2 = \dfrac{1}{2\sin^2 \phi \cos^2 \theta - 1}$

d. $x^2 + y^2 = z;$
$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = \rho \cos \phi$
$\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) = \rho \cos \phi;$
$\rho \sin^2 \phi = \cos \phi;\ \rho = \cot \phi \csc \phi$

(Note that when we divided through by ρ in part c and d we did not lose the pole since it is also a solution of the resulting equations.)

34. Cartesian coordinates are $\left(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}\right)$ and $\left(\sqrt{2}, \sqrt{6}, -2\sqrt{2}\right)$. Distance
$\left[2 + \left(2\sqrt{2} - \sqrt{6}\right)^2 + \left(4\sqrt{3} + 2\sqrt{2}\right)^2\right]^{1/2} \approx 9.8659.$

35. (2, 0, 0) is a point of the first plane. The distance between the planes is
$\dfrac{|2(2) - 3(0) + \sqrt{3}(0) - 9|}{\sqrt{4 + 9 + 3}} = \dfrac{5}{\sqrt{16}} = 1.25$

36. $\langle 2, -4, 1 \rangle$ and $\langle 3, 2, -5 \rangle$ are normal to the respective planes. The acute angle between the two planes is the same as the acute angle θ between the normal vectors.
$\cos \theta = \dfrac{|6 - 8 - 5|}{\sqrt{21}\sqrt{38}} = \dfrac{7}{\sqrt{798}}$, so $\theta \approx 1.3204$ rad $\approx 75.65°$

37. If speed $= \dfrac{ds}{dt} = c$, a constant, then
$\mathbf{a} = \dfrac{d^2 s}{dt^2}\mathbf{T} + \left(\dfrac{ds}{dt}\right)^2 \kappa \mathbf{N} = c^2 \kappa \mathbf{N}$ since $\dfrac{d^2 s}{dt^2} = 0$. **T** is in the direction of **v**, while **N** is perpendicular to **T** and hence to **v** also. Thus, $\mathbf{a} = c^2 \kappa \mathbf{N}$ is perpendicular to **v**.