CHAPTER 11

Numerical Methods, Approximations

11.1 Concepts Review

1. $f(1)$; $f'(1)$; $f''(1)$

2. $\frac{f^{(6)}(0)}{6!}$

3. error of the method; error of calculation

4. increase; decrease

Problem Set 11.1

1. $f(x) = e^{2x}$ $f(0) = 1$
$f'(x) = 2e^{2x}$ $f'(0) = 2$
$f''(x) = 4e^{2x}$ $f''(0) = 4$
$f^{(3)}(x) = 8e^{2x}$ $f^{(3)}(0) = 8$
$f^{(4)}(x) = 16e^{2x}$ $f^{(4)}(0) = 16$
$$f(x) \approx 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$
$$f(0.12) \approx 1 + 2(0.12) + 2(0.12)^2 + \frac{4}{3}(0.12)^3 + \frac{2}{3}(0.12)^4 \approx 1.2712$$

2. $f(x) = e^{-3x}$ $f(0) = 1$
$f'(x) = -3e^{-3x}$ $f'(0) = -3$
$f''(x) = 9e^{-3x}$ $f''(0) = 9$
$f^{(3)}(x) = -27e^{-3x}$ $f^{(3)}(0) = -27$
$f^{(4)}(x) = 81e^{-3x}$ $f^{(4)}(0) = 81$
$$f(x) \approx 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4$$
$$f(0.12) \approx 1 - 3(0.12) + \frac{9}{2}(0.12)^2 - \frac{9}{2}(0.12)^3 + \frac{27}{8}(0.12)^4 \approx 0.6977$$

3. $f(x) = \sin 2x$ $f(0) = 0$
$f'(x) = 2\cos 2x$ $f'(0) = 2$
$f''(x) = -4\sin 2x$ $f''(0) = 0$
$f^{(3)}(x) = -8\cos 2x$ $f^{(3)}(0) = -8$
$f^{(4)}(x) = 16\sin 2x$ $f^{(4)}(0) = 0$
$$f(x) \approx 2x - \frac{8}{3!}x^3 = 2x - \frac{4}{3}x^3$$
$$f(0.12) \approx 2(0.12) - \frac{4}{3}(0.12)^3 \approx 0.2377$$

4. $f(x) = \tan x \qquad f(0) = 0$

$f'(x) = \sec^2 x \qquad f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x \qquad f''(0) = 0$

$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f^{(3)}(0) = 2$

$f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$f^{(4)}(0) = 0$

$f(x) \approx x + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$

$f(0.12) \approx 0.12 + \frac{1}{3}(0.12)^3 \approx 0.1206$

5. $f(x) = \ln(1 + x) \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$

$f^{(3)}(x) = \frac{2}{(1+x)^3} \qquad f^{(3)}(0) = 2$

$f^{(4)}(x) = -\frac{6}{(1+x)^4} \qquad f^{(4)}(0) = -6$

$f(x) \approx x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

$f(0.12) \approx 0.12 - \frac{1}{2}(0.12)^2 + \frac{1}{3}(0.12)^3 - \frac{1}{4}(0.12)^4$

≈ 0.1133

6. $f(x) = \sqrt{1+x} \qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \qquad f''(0) = -\frac{1}{4}$

$f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2} \qquad f^{(3)}(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2} \qquad f^{(4)}(0) = -\frac{15}{16}$

$f(x) \approx 1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 - \frac{\frac{15}{16}}{4!}x^4$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$

$f(0.12) \approx 1 + \frac{1}{2}(0.12) - \frac{1}{8}(0.12)^2 + \frac{1}{16}(0.12)^3$

$- \frac{5}{128}(0.12)^4 \approx 1.0583$

7. $f(x) = \tan^{-1} x \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$

$f''(x) = -\frac{2x}{(1+x^2)^2} \qquad f''(0) = 0$

$f'''(x) = \frac{6x^2 - 2}{(1+x^2)^3} \qquad f'''(0) = -2$

$f^{(4)}(x) = \frac{-24x^3 + 24x}{(1+x^2)^4} \qquad f^{(4)}(0) = 0$

$f(x) \approx x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3$

$f(0.12) \approx 0.12 - \frac{1}{3}(0.12)^3 \approx 0.1194$

8. $f(x) = \sinh x \qquad f(0) = 0$

$f'(x) = \cosh x \qquad f'(0) = 1$

$f''(x) = \sinh x \qquad f''(0) = 0$

$f'''(x) = \cosh x \qquad f'''(0) = 1$

$f^{(4)}(x) = \sinh x \qquad f^{(4)}(0) = 0$

$f(x) \approx x + \frac{1}{3!}x^3 = x + \frac{1}{6}x^3$

$f(0.12) \approx 0.12 + \frac{1}{6}(0.12)^3 \approx 0.1203$

9. $f(x) = e^x \qquad f(1) = e$

$f'(x) = e^x \qquad f'(1) = e$

$f''(x) = e^x \qquad f''(1) = e$

$f'''(x) = e^x \qquad f'''(1) = e$

$P_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$

10. $f(x) = \sin x \qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$

$- \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$

11. $f(x) = \tan x$; $f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$f'(x) = \sec^2 x$; $f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$

$f''(x) = 2\sec^2 x \tan x$; $f''\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{9}$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$; $f'''\left(\frac{\pi}{6}\right) = \frac{16}{3}$

$$P_3(x) = \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3$$

12. $f(x) = \sec x$; $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$; $f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x$; $f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$f'''(x) = 5\sec^3 x \tan x + \sec x \tan^3 x$;

$f'''\left(\frac{\pi}{4}\right) = 11\sqrt{2}$

$$P_3(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3$$

13. $f(x) = \cot^{-1} x$; $f(1) = \frac{\pi}{4}$

$f'(x) = -\frac{1}{1+x^2}$; $f'(1) = -\frac{1}{2}$

$f''(x) = \frac{2x}{(1+x^2)^2}$; $f''(1) = \frac{1}{2}$

$f'''(x) = \frac{-6x^2 + 2}{(1+x^2)^3}$; $f'''(1) = -\frac{1}{2}$

$$P_3(x) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{12}(x-1)^3$$

14. $f(x) = \sqrt{x}$; $f(2) = \sqrt{2}$

$f'(x) = \frac{1}{2}x^{-1/2}$; $f'(2) = \frac{\sqrt{2}}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$; $f''(2) = -\frac{\sqrt{2}}{16}$

$f'''(x) = \frac{3}{8}x^{-5/2}$; $f'''(2) = \frac{3\sqrt{2}}{64}$

$$P_3(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3$$

15. $f(x) = x^3 - 2x^2 + 3x + 5$; $f(1) = 7$

$f'(x) = 3x^2 - 4x + 3$; $f'(1) = 2$

$f''(x) = 6x - 4$; $f''(1) = 2$

$f^{(3)}(x) = 6$; $f^{(3)}(1) = 6$

$P_3(x) = 7 + 2(x-1) + (x-1)^2 + (x-1)^3$

$= 5 + 3x - 2x^2 + x^3 = f(x)$

16. $f(x) = x^4$; $f(2) = 16$

$f'(x) = 4x^3$; $f'(2) = 32$

$f''(x) = 12x^2$; $f''(2) = 48$

$f^{(3)}(x) = 24x$; $f^{(3)}(2) = 48$

$f^{(4)}(x) = 24$; $f^{(4)}(2) = 24$

$P_4(x) = 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4$

$= x^4 = f(x)$

17. $f(x) = \frac{1}{1-x}$; $f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$; $f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3}$; $f''(0) = 2$

$f^{(3)}(x) = \frac{6}{(1-x)^4}$; $f^{(3)}(0) = 6$

$f^{(4)}(x) = \frac{24}{(1-x)^5}$; $f^{(4)}(0) = 24$

$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$; $f^{(n)}(0) = n!$

$f(x) \approx 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \ldots + \frac{n!}{n!}x^n$

$= 1 + x + x^2 + x^3 + \ldots + x^n$

Using $n = 4$, $f(x) \approx 1 + x + x^2 + x^3 + x^4$

a. $f(0.1) \approx 1.1111$

b. $f(0.5) \approx 1.9375$

c. $f(0.9) \approx 4.0951$

d. $f(2) \approx 31$

18. $f(x) = \sin x$; $f(0) = 0$
$f'(x) = \cos x$; $f'(0) = 1$
$f''(x) = -\sin x$; $f''(0) = 0$
$f^{(3)}(x) = -\cos x$; $f^{(3)}(0) = -1$
$f^{(4)}(x) = \sin x$; $f^{(4)}(0) = 0$

When n is odd,
$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{(n-1)/2} x^n}{n!}$$
Using $n = 5$, $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

a. $\sin(0.1) \approx 0.0998$

b. $\sin(0.5) \approx 0.4794$

c. $\sin(1) \approx 0.8417$

d. $\sin(10) \approx 676.67$

19. The area of the sector with angle t is $\frac{1}{2}tr^2$. The area of the triangle is
$$\frac{1}{2}\left(r\sin\frac{t}{2}\right)\left(2r\cos\frac{t}{2}\right) = r^2 \sin\frac{t}{2}\cos\frac{t}{2} = \frac{1}{2}r^2 \sin t$$
$$A = \frac{1}{2}tr^2 - \frac{1}{2}r^2 \sin t$$
Using $n = 3$, $\sin t \approx t - \frac{1}{6}t^3$.
$$A \approx \frac{1}{2}tr^2 - \frac{1}{2}r^2\left(t - \frac{1}{6}t^3\right) = \frac{1}{12}r^2 t^3$$

20. $m(v) = \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ $\quad m(0) = m_0$

$m'(v) = \dfrac{m_0 v}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$ $\quad m'(0) = 0$

$m''(v) = \dfrac{2m_0 v^2 + m_0 c^2}{c^4\left(1 - \frac{v^2}{c^2}\right)^{5/2}}$ $\quad m''(0) = \dfrac{m_0}{c^2}$

The Maclaurin polynomial of order 2 is:
$$m(v) \approx m_0 + \frac{1}{2}\frac{m_0}{c^2}v^2 = m_0 + \frac{m_0}{2}\left(\frac{v}{c}\right)^2.$$

21. a. $$\ln\left(1 + \frac{r}{12}\right)^{12n} = \ln 2$$
$$12n \ln\left(1 + \frac{r}{12}\right) = \ln 2$$
$$n = \frac{\ln 2}{12\ln\left(1 + \frac{r}{12}\right)}$$

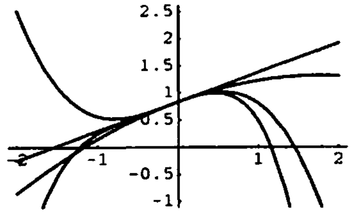
b. $f(x) = \ln(1 + x)$; $f(0) = 0$
$f'(x) = \dfrac{1}{1+x}$; $f'(0) = 1$
$f''(x) = -\dfrac{1}{(1+x)^2}$; $f''(0) = -1$
$$\ln(1 + x) \approx x - \frac{x^2}{2}$$
$$n \approx \frac{\ln 2}{r - \frac{r^2}{24}} = \left[\frac{24}{r(24 - r)}\right]\ln 2$$
$$= \frac{\ln 2}{r} + \frac{\ln 2}{24 - r}$$
$$\approx \frac{\ln 2}{r} + \frac{\ln 2}{24} \approx \frac{0.693}{r} + 0.029$$
We let $24 - r \approx 24$ since the interest rate r is going to be close to 0.

c.

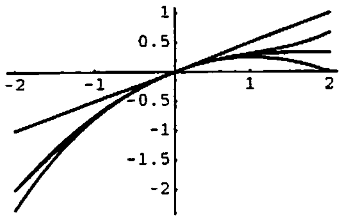
r	n (exact)	n (approx.)	n (rule 72)
0.05	13.8918	13.889	14.4
0.10	6.9603	6.959	7.2
0.15	4.6498	4.649	4.8
0.20	3.4945	3.494	3.6

22. $f(x) = 1 - e^{-(1+k)x}$; $f(0) = 0$
$f'(x) = (1 + k)e^{-(1+k)x}$; $f'(0) = (1 + k)$
$f''(x) = -(1 + k)^2 e^{-(1+k)x}$; $f''(0) = -(1 + k)^2$
$$1 - e^{-(1+k)x} \approx (1 + k)x - \frac{(1 + k)^2}{2}x^2$$
For $x = 2k$, the polynomial is
$2k - 4k^3 - 2k^4 \approx 2k$ when k is very small.
$1 - e^{-(1+0.01)(0.02)} \approx 0.019997 \approx 0.02$

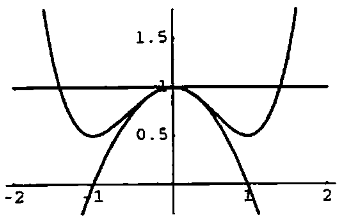
23. a.



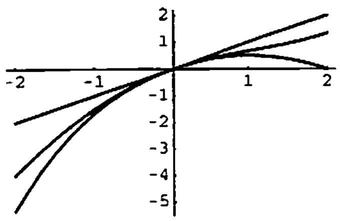
b.



24. a.



b.



25. $\left|e^{2c}+e^{-2c}\right| \le \left|e^{2c}\right| + \left|\frac{1}{e^{2c}}\right| \le e^6 + 1$

26. $|\tan c + \sec c| \le |\tan c| + |\sec c| \le 1 + 1 = 2$

27. $\left|\frac{4c}{\sin c}\right| = \frac{|4c|}{|\sin c|} \le \frac{2\pi}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}\pi$

28. $\left|\frac{4c}{c+4}\right| = \frac{|4c|}{|c+4|} \le \frac{4}{4} = 1$

29. $\left|\frac{e^c}{c+5}\right| = \frac{|e^c|}{|c+5|} \le \frac{e^4}{3}$

30. $\left|\frac{\cos c}{c+2}\right| = \frac{|\cos c|}{|c+2|} \le \frac{1}{2}$

31. $\left|\frac{c^2+\sin c}{10\ln c}\right| = \frac{|c^2+\sin c|}{|10\ln c|} \le \frac{|c^2| + |\sin c|}{|10\ln c|}$

$\le \frac{16+1}{10\ln 2} = \frac{17}{10\ln 2}$

32. $\left|\frac{c^2-c}{\cos c}\right| = \frac{|c^2-c|}{|\cos c|} \le \sqrt{2}\left|c^2-c\right| \le \sqrt{2}\left|\left(\frac{1}{2}\right)^2 - \frac{1}{2}\right|$

$= \frac{\sqrt{2}}{4}$ (Note that $|x^2-x|$ is maximum at $\frac{1}{2}$ in [0, 1].)

33. $f(x) = \ln(2+x);\ f'(x) = \frac{1}{2+x};$

$f''(x) = -\frac{1}{(2+x)^2};\ f^{(3)}(x) = \frac{2}{(2+x)^3};$

$f^{(4)}(x) = -\frac{6}{(2+x)^4};\ f^{(5)}(x) = \frac{24}{(2+x)^5};$

$f^{(6)}(x) = -\frac{120}{(2+x)^6};\ f^{(7)}(x) = \frac{720}{(2+x)^7}$

$R_6(x) = \frac{1}{7!}\cdot\frac{720}{(2+c)^7}x^7 = \frac{x^7}{7(2+c)^7}$

$|R_6(0.5)| \le \left|\frac{0.5^7}{7\cdot 2^7}\right| \approx 8.719\times 10^{-6}$

34. $f(x) = e^{-x};\ f'(x) = -e^{-x};$

$f^{(n)}(x) = \begin{cases} e^{-x} & \text{if } n \text{ is even} \\ -e^{-x} & \text{if } n \text{ is odd} \end{cases}$

$R_6(x) = \frac{-e^{-c}}{7!}(x-1)^7 = -\frac{(x-1)^7}{5040e^c}$

$|R_6(0.5)| \le \left|\frac{(-0.5)^7}{5040e^{0.5}}\right| \approx 9.402\times 10^{-7}$

35. $f(x) = \sin x;\ f^{(7)}(x) = -\cos x$

$R_6(x) = \frac{-\cos c}{7!}\left(x - \frac{\pi}{4}\right)^7 = \frac{-\cos c\left(x-\frac{\pi}{4}\right)^7}{5040}$

$|R_6(0.5)| \le \left|\frac{\cos 0.5\left(0.5 - \frac{\pi}{4}\right)^7}{5040}\right| \approx 2.685\times 10^{-8}$

36. $f(x)=\frac{1}{x-3}$; $f'(x)=-\frac{1}{(x-3)^2}$; $f''(x)=\frac{2}{(x-3)^3}$; $f^{(3)}(x)=-\frac{6}{(x-3)^4}$; $f^{(4)}(x)=\frac{24}{(x-3)^5}$;

$f^{(5)}(x)=-\frac{120}{(x-3)^6}$; $f^{(6)}(x)=\frac{720}{(x-3)^7}$; $f^{(7)}(x)=-\frac{5040}{(x-3)^8}$

$$R_6(x)=\frac{1}{7!}\cdot-\frac{5040}{(c-3)^8}(x-1)^7=-\frac{(x-1)^7}{(c-3)^8}$$

$$|R_6(0.5)|\le\left|\frac{(0.5-1)^7}{(1-3)^8}\right|=\left|\frac{0.5^7}{2^8}\right|\approx 3.052\times 10^{-5}$$

37. $f(x)=e^x$ $\quad f(0)=1$

$f'(x)=e^x$ $\quad f'(0)=1$

$f''(x)=e^x$ $\quad f''(0)=1$

$f^{(3)}(x)=e^x$ $\quad f^{(3)}(0)=1$

$f^{(4)}(x)=e^x$ $\quad f^{(4)}(c)=e^c$

$$e^x\approx 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$$

$$R_3(x)=\frac{e^c}{4!}x^4$$

$$R_3(-0.1)=\frac{e^c}{240,000}\le\frac{1}{240,000}\approx 4.17\times 10^{-6}$$

$$R_3(-0.1)\ge\frac{e^{-0.1}}{240,000}\approx 3.77\times 10^{-6}$$

$$e^{-0.1}-\left[1+(-0.1)+\frac{1}{2}(-0.1)^2+\frac{1}{6}(-0.1)^3\right]\approx 4.08\times 10^{-6}$$

38. $f(x)=\sin x$; $f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

$f'(x)=\cos x$; $f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

$f''(x)=-\sin x$; $f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$

$f^{(3)}(x)=-\cos x$; $f^{(3)}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$

$f^{(4)}(x)=\sin x$; $f^{(4)}(c)=\sin c$

$$\sin x\approx\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$

$$R_3(x)=\frac{\sin c}{4!}\left(x-\frac{\pi}{4}\right)^4$$

$$R_3\left(\frac{\pi}{8}\right)=\frac{\sin c}{24}\left(-\frac{\pi}{8}\right)^4\le\frac{\sin\left(\frac{\pi}{4}\right)}{24}\left(-\frac{\pi}{8}\right)^4\approx 7.01\times 10^{-4}$$

$$R_3\left(\frac{\pi}{8}\right) \ge \frac{\sin\left(\frac{\pi}{8}\right)}{24}\left(-\frac{\pi}{8}\right)^4 \approx 3.79\times 10^{-4}$$

$$\sin\left(\frac{\pi}{8}\right) - \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{\pi}{8}\right) - \frac{\sqrt{2}}{4}\left(-\frac{\pi}{8}\right)^2 - \frac{\sqrt{2}}{12}\left(-\frac{\pi}{8}\right)^3\right] \approx 6.42\times 10^{-4}$$

39. $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$

Note that $e^1 < 3$.

$|R_n(1)| < \frac{3}{(n+1)!}$

$\frac{3}{(n+1)!} < 0.000005$ or $600000 < (n+1)!$ when $n \ge 9$.

40. $f(x) = (1+x)^{3/2}$ $\quad f(0) = 1$

$f'(x) = \frac{3}{2}(1+x)^{1/2}$ $\quad f'(0) = \frac{3}{2}$

$f''(x) = \frac{3}{4}(1+x)^{-1/2}$ $\quad f''(0) = \frac{3}{4}$

$f^{(3)}(x) = -\frac{3}{8}(1+x)^{-3/2}$ $\quad f'''(0) = -\frac{3}{8}$

$f^{(4)}(x) = \frac{9}{16}(1+x)^{-5/2}$ $\quad f^{(4)}(c) = \frac{9}{16}(1+c)^{-5/2}$

$(1+x)^{3/2} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$

$R_3(x) = \frac{3}{128}(1+c)^{-5/2}x^4$

$|R_3(x)| \le \left|\frac{3}{128}(0.9)^{-5/2}(-0.1)^4\right| \approx 3.05\times 10^{-6}$

41. $f(x) = (1+x)^{-1/2}$ $\quad f(0) = 1$

$f'(x) = -\frac{1}{2}(1+x)^{-3/2}$ $\quad f'(0) = -\frac{1}{2}$

$f''(x) = \frac{3}{4}(1+x)^{-5/2}$ $\quad f''(0) = \frac{3}{4}$

$f^{(3)}(x) = -\frac{15}{8}(1+x)^{-7/2}$ $\quad f^{(3)}(0) = -\frac{15}{8}$

$f^{(4)}(x) = \frac{105}{16}(1+x)^{-9/2}$ $\quad f^{(4)}(c) = \frac{105}{16}(1+c)^{-9/2}$

$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$

$R_3(x) = \frac{35}{128}(1+c)^{-9/2}x^4$

$|R_3(x)| \le \left|\frac{35}{128}(0.95)^{-9/2}(0.05)^4\right| \approx 2.15\times 10^{-6}$

42. $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ $\quad f(0) = 0$

$f'(x) = \frac{2}{1-x^2}$ $\quad f'(0) = 2$

$f''(x) = \frac{4x}{(1-x^2)^2}$ $\quad f''(0) = 0$

$f^{(3)}(x) = \frac{4(1+3x^2)}{(1-x^2)^3}$ $\quad f^{(3)}(0) = 4$

$f^{(4)}(x) = \frac{48x(1+x^2)}{(1-x^2)^4}$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \frac{48(1+10x^2+5x^4)}{(1-x^2)^5}$

$f^{(5)}(c) = \frac{48(1+10c^2+5c^4)}{(1-c^2)^5}$

$\ln\left(\frac{1+x}{1-x}\right) \approx 2x + \frac{2}{3}x^3$

$R_4(x) = \frac{2}{5}\left[\frac{1+10c^2+5c^4}{(1-c^2)^5}\right]x^5$

$|R_4(x)| < \frac{2}{5}\left[\frac{1+10(0.5)^2+5(0.5)^4}{(1-(0.5)^2)^5}\right](0.5)^5$

≈ 0.201

43. $R_4(x) = \frac{\cos c}{5!}x^5$

$|R_4(x)| \le \frac{(0.5)^5}{5!} \approx 0.00026042 \le 0.0002605$

$\int_0^{0.5} \sin x\,dx \approx \int_0^{0.5}\left(x - \frac{1}{6}x^3\right)dx$

$= \left[\frac{1}{2}x^2 - \frac{1}{24}x^4\right]_0^{0.5} \approx 0.1224$

Error $\le 0.0002605(0.5 - 0) = 0.00013025$

44. $R_5(x) = -\frac{\cos c}{6!}x^6$

$|R_5(x)| \le \frac{1}{6!} \approx 0.001389$

$\int_0^1 \cos x\,dx \approx \int_0^1 \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) dx$

$= \left[x - \frac{x^3}{6} + \frac{x^5}{120}\right]_0^1 \approx 0.8417$

Error $\le 0.001389(1 - 0) = 0.001389$

45. $f(x) = x^4 - 3x^3 + 2x^2 + x - 2$; $f(1) = -1$

$f'(x) = 4x^3 - 9x^2 + 4x + 1$; $f'(1) = 0$

$f''(x) = 12x^2 - 18x + 4$; $f''(1) = -2$

$f^{(3)}(x) = 24x - 18$; $f^{(3)}(1) = 6$

$f^{(4)}(x) = 24$; $f^{(4)}(1) = 24$

$f^{(5)}(x) = 0$

Since $f^{(5)}(x) = 0$, $R_5(x) = 0$.

$x^4 - 3x^3 + 2x^2 + x - 2$

$= -1 - (x-1)^2 + (x-1)^3 + (x-1)^4$

46. $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$P_n'(x) = f'(a) + \frac{f''(a)}{2!}2(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}n(x-a)^{n-1}$

$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$

$P_n'(a) = f'(a) + 0 + 0 + \cdots + 0 = f'(a)$

$P_n'' = 0 + f''(a) + \frac{f'''(a)}{2!}2(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(n-1)(x-a)^{n-2}$

$= f''(a) + f'''(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}$

$P_n'' = f''(a) + 0 + 0 + \cdots + 0 = f''(a)$

$\vdots \quad \vdots \quad \vdots$

$P_n^{(n)}(x) = \frac{f^{(n)}(a)}{0!}(x-a)^0 = f^{(n)}(a)$

$P_n^{(n)}(a) = f^{(n)}(a)$

47. $f(x) = \sin x$; $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$; $f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$; $f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(3)}(x) = -\cos x$; $f^{(3)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$; $f^{(4)}(c) = \sin c$

$43° = \frac{\pi}{4} - \frac{\pi}{90}$ radians

$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$

$-\frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + R_3(x)$

$\sin\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{\pi}{90}\right) - \frac{\sqrt{2}}{4}\left(-\frac{\pi}{90}\right)^2$

$-\frac{\sqrt{2}}{12}\left(-\frac{\pi}{90}\right)^3 + R_3\left(\frac{\pi}{4} - \frac{\pi}{90}\right)$

$\approx 0.681998 + R_3$

$|R_3| = \left|\frac{\sin c}{4!}\left(-\frac{\pi}{90}\right)^4\right| < \frac{1}{24}\left(\frac{\pi}{90}\right)^4 \approx 6.19 \times 10^{-8}$

48. $63° = \frac{\pi}{3} + \frac{\pi}{60}$ radians

Since $f^{(n)}(x)$ is $\pm\sin x$ or $\pm\cos x$,

$$|R_n(x)| \le \frac{1}{(n+1)!}\left(x - \frac{\pi}{3}\right)^{n+1}$$

$$\left|R_n\left(\frac{\pi}{3} + \frac{\pi}{60}\right)\right| \le \frac{1}{(n+1)!}\left(\frac{\pi}{60}\right)^{n+1}$$

$$\frac{1}{(n+1)!}\left(\frac{\pi}{60}\right)^{n+1} \le 0.0005 \text{ when } n \ge 2.$$

$f(x) = \cos x$; $f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x$; $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$; $f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + R_3(x)$$

$$\cos 63° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{60}\right) - \frac{1}{4}\left(\frac{\pi}{60}\right)^2 \approx 0.45397$$

49. $|R_9(x)| \le \frac{1}{10!}x^{10} \le \frac{1}{10!}\left(\frac{\pi}{2}\right)^{10} \approx 2.5202 \times 10^{-5}$

50. a. $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{\sin c}{720}x^6$

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\left(\frac{1}{120} - \frac{\sin c}{720}x\right) = \frac{1}{120}$$

b. $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{\sin c}{5040}x^7$

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} = \lim_{x\to 0}\left(-\frac{1}{720} + \frac{\sin c}{5040}x\right) = -\frac{1}{720}$$

51. The kth derivative of $h(x)f(x)$ is

$\sum_{i=0}^{k}\binom{k}{i}h^{(i)}(x)f^{(k-i)}(x)$. If $h(x) = x^{n+1}$,

$$h^{(i)}(x) = \frac{(n+1)!}{(n+1-i)!}x^{n+1-i}.$$

Thus for $i \le n+1$, $h^{(i)}(0) = 0$. Let

$q(x) = x^{n+1}f(x)$. Then

$$q^{(k)}(0) = \sum_{i=0}^{k}\binom{k}{i}h^{(i)}(0)f^{(k-i)}(0) = 0$$

for $k \le n+1$.

$g^{(k)}(x) = p^{(k)}(x) + q^{(k)}(x)$, so

$g^{(k)}(0) = p^{(k)}(0) + q^{(k)}(0) = p^{(k)}(0)$

for $k \le n+1$.

The Maclaurin polynomial of order n for g is

$p(0) + p'(0)x + \frac{p''(0)}{2!}x^2 + \ldots + \frac{p^{(n)}(0)}{n!}x^n$ which is the Maclaurin polynomial of order n for $p(x)$. Since $p(x)$ is a polynomial of degree at most n, the remainder $R_n(x)$ of Maclaurin's Formula for $p(x)$ is 0, so the Maclaurin polynomial of order n for $g(x)$ is $p(x)$.

52. Using Taylor's formula,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

Since $f'(c) = f''(c) = f'''(c) = \ldots = f^{(n)}(c) = 0$,

$f(x) = f(c) + R_n(x)$

$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(a)(x-c)^{n+1}$ where a is between x and c.

(i) Since $f^{(n+1)}(x)$ is continuous near c, then $f^{(n+1)}(a) < 0$ when a is near c.
Thus $R_n(x) < 0$ when x is near c, so $f(x) < f(c)$ when x is near c. $f(c)$ is a local maximum.

(ii) Since $f^{(n+1)}(x)$ is continuous near c, then $f^{(n+1)}(a) > 0$ when a is near c.
Thus $R_n(x) > 0$ when x is near c, so $f(x) > f(c)$ when x is near c.
$f(c)$ is a local minimum.

Suppose $f(x) = x^4$. $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x > 0$. Thus $x = 0$ is a local minimum.

$f'(0) = f''(0) = f'''(0) = 0, f^{(4)}(0) = 24 > 0$

53. a. $L_{51}(x)$ is of degree 4 since it is the product of four linear factors.

$$L_{51}(x_1) = \frac{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)} = 1$$

$L_{51}(x_j) = 0$ for $j = 2, 3, 4, 5$ since $x - x_j$ is a linear factor of $L_{51}(x)$.

b. $$L_{52}(x) = \frac{(x - x_1)(x - x_3)(x - x_4)(x - x_5)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)}$$

$$L_{53}(x) = \frac{(x - x_1)(x - x_2)(x - x_4)(x - x_5)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)}$$

$$L_{54}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_5)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_4 - x_5)}$$

$$L_{55}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}$$

c. Since $L_{51}, L_{52}, L_{53}, L_{54}$, and L_{55} are of degree 4, L_5 is of degree less than or equal to 4. Since $L_{5j}(x_i) = 0$ for $i \neq j$ and $L_{5i}(x_i) = 1$, $L_5(x_i) = y_i$.

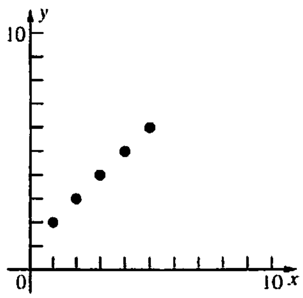
d. $$L_{31}(x) = \frac{(x-2)(x-0)}{(1-2)(1-0)} = -(x-2)(x) = -x^2 + 2x$$

$$L_{32}(x) = \frac{(x-1)(x-0)}{(2-1)(2-0)} = \frac{1}{2}(x-1)(x) = \frac{1}{2}x^2 - \frac{1}{2}x$$

$$L_{33}(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

$$L_3(x) = (-x^2 + 2x)(2) + \left(\frac{1}{2}x^2 - \frac{1}{2}x\right)(2.5) + \left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)(0) = -0.75x^2 + 2.75x$$

54.



$$L_{51}(x) = \frac{(x-2)(x-3)(x-4)(x-5)}{(1-2)(1-3)(1-4)(1-5)} = \frac{1}{24}(x-2)(x-3)(x-4)(x-5)$$

$$L_{52}(x) = \frac{(x-1)(x-3)(x-4)(x-5)}{(2-1)(2-3)(2-4)(2-5)} = -\frac{1}{6}(x-1)(x-3)(x-4)(x-5)$$

$$L_{53}(x) = \frac{(x-1)(x-2)(x-4)(x-5)}{(3-1)(3-2)(3-4)(3-5)} = \frac{1}{4}(x-1)(x-2)(x-4)(x-5)$$

$$L_{54}(x) = \frac{(x-1)(x-2)(x-3)(x-5)}{(4-1)(4-2)(4-3)(4-5)} = -\frac{1}{6}(x-1)(x-2)(x-3)(x-5)$$

$$L_{55}(x) = \frac{(x-1)(x-2)(x-3)(x-4)}{(5-1)(5-2)(5-3)(5-4)} = \frac{1}{24}(x-1)(x-2)(x-3)(x-4)$$

$$L_5(x) = L_{51}(x) \cdot 2 + L_{52}(x) \cdot 3 + L_{53}(x) \cdot 4 + L_{54}(x) \cdot 5 + L_{55}(x) \cdot 6$$

$$= \frac{1}{12}(x-2)(x-3)(x-4)(x-5) - \frac{1}{2}(x-1)(x-3)(x-4)(x-5) + (x-1)(x-2)(x-4)(x-5)$$
$$-\frac{5}{6}(x-1)(x-2)(x-3)(x-5) + \frac{1}{4}(x-1)(x-2)(x-3)(x-4) = x+1$$

55. $x_1 = 1;\ \ y_1 = 0$

$x_2 = 3;\ \ y_2 = 1.099$

$x_3 = 5;\ \ y_3 = 1.609$

$$L_{31}(x) = \frac{(x-3)(x-5)}{(1-3)(1-5)} = \tfrac{1}{8}(x-3)(x-5)$$
$$L_{32}(x) = \frac{(x-1)(x-5)}{(3-1)(3-5)} = -\tfrac{1}{4}(x-1)(x-5)$$
$$L_{33}(x) = \frac{(x-1)(x-3)}{(5-1)(5-3)} = \tfrac{1}{8}(x-1)(x-3)$$
$$L_3(x) = L_{31}(x)\cdot 0 + L_{32}(x)\cdot 1.099 + L_{33}(x)\cdot 1.609$$
$$= -\frac{1.099}{4}(x-1)(x-5) + \frac{1.609}{8}(x-1)(x-3)$$
$$= \tfrac{1}{4}(x-1)\left[-1.099(x-5) + 0.8045(x-3)\right]$$
$$= \tfrac{1}{4}(x-1)(-0.2945x + 3.0815)$$

Thus,
$$\ln 2 \approx \tfrac{1}{4}(2-1)(-0.2945x + 3.0815) \approx 0.623$$

The estimate for the maximum error is
$$R_2(x) = \frac{(x-1)(x-3)(x-5)}{3!} f^{(3)}(\alpha)$$
$$f(x) = \ln x; \qquad f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2}; \qquad f'''(x) = \frac{2}{x^3}$$
$$|R_2(2)| = \left|\frac{(2-1)(2-3)(2-5)}{3!}\right| \left|f^{(3)}(\alpha)\right|$$
$$= \left|\frac{1\cdot(-1)\cdot(-3)}{6}\frac{2}{\alpha^3}\right| = \frac{1}{\alpha^3} \le 1 \text{ for } \alpha \in [1,5].$$

A calculator gives $\ln 2 \approx 0.693$, so the actual error is approximately $0.693 - 0.623 \approx 0.07$.

56. $x_1 = 0;\ y_1 = 1$

$x_2 = 0.2;\ y_2 = 1.221$

$x_3 = 0.3;\ y_3 = 1.350$

$$L_{31}(x) = \frac{(x-0.2)(x-0.3)}{(0-0.2)(0-0.3)} = \tfrac{50}{3}(x-0.2)(x-0.3)$$
$$L_{32}(x) = \frac{(x-0)(x-0.3)}{(0.2-0)(0.2-0.3)} = -50x(x-0.3)$$
$$L_{33}(x) = \frac{(x-0)(x-0.2)}{(0.3-0)(0.3-0.2)} = \tfrac{100}{3}x(x-0.2)$$
$$L(x) = L_{31}(x)\cdot 1 + L_{32}(x)\cdot 1.221 + L_{33}(x)\cdot 1.350$$
$$= \tfrac{50}{3}(x-0.2)(x-0.3) - 61.05x(x-0.3) + 45x(x-0.2)$$
$$L(0.25) \approx \tfrac{50}{3}(0.25-0.2)(0.25-0.3) - 61.05\cdot 0.25(0.25-0.3) + 45\cdot 0.25(0.25-0.2) \approx 1.284$$

The error is
$$R_2(x) = \frac{x(x-0.2)(x-0.3)}{3!} f^{(3)}(\alpha)$$

The derivatives of $f(x)$ are

$f(x) = e^x \qquad f'(x) = e^x$

$f''(x) = e^x \qquad f'''(x) = e^x$

Thus,
$$R_2(0.25) = \frac{0.25(0.25-0.2)(0.25-0.3)}{3!}\cdot f^{(3)}(\alpha)$$
$$\approx -0.0001042e^{\alpha}$$

Thus $|R_2(0.25)| \le 0.0001042e^{0.3} \approx 0.0001407$ and $|R_2(0.25)| \ge 0.0001042e^{0} \approx 0.0001042$.

A calculator gives $e^{0.25} \approx 1.2840$ so the actual error is $1.284 - 1.284 = 0$, to three decimal places.

57. The second order Maclauring polynomial is
$$P_2(x) = 1 + x + \frac{x^2}{2}$$

From Problem 56, we know that the interpolating polynomial is $L(x) = \tfrac{50}{3}(x-0.2)(x-0.3) - 61.05x(x-0.3) + 45x(x-0.2)$

For the Maclaurin polynomial
$$R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \tfrac{1}{6}e^c x^3$$

Thus,

$|R_2(x)| = \frac{1}{6}e^c |x^3| \le \frac{1}{6}e^{0.3}0.3^3 \approx 0.0060744$

For the interpolating polynomial,

$|R_2(x)| = \left|\frac{x(x-0.2)(x-0.3)}{6}\right| e^c$

The expression in absolute values reaches a maximum of 0.0003521 when $x = 0.078475$. On the interval $[0, 0.3]$, the expression e^c reaches a maximum of $e^{0.3} \approx 1.350$. Thus, the largest possible error for the interpolating polynomial on $[0, 0.3]$ is

$R_2(x) \le 0.0003521 \cdot 1.350 \approx 0.0004753$

A calculator gives $e^{0.1} \approx 1.105$. When $x = 0.1$, the error for the Maclaurin polynomial is

$|1.105 - P_2(0.1)| = |1.105 - 1.105| = 0$

and the error for the interpolating polynomial is

$|1.105 - L_2(0.1)| = |1.105 - 1.104| = 0.001$

11.2 Concepts Review

1. 1, 2, 2, 2, ..., 2, 1

2. 1, 4, 2, 4, 2, ..., 4, 1

3. n^4

4. large

Problem Set 11.2

1. $f(x) = \frac{1}{x^2}; h = \frac{3-1}{8} = 0.25$

$x_0 = 1.00$	$f(x_0) = 1$	$x_5 = 2.25$	$f(x_5) \approx 0.1975$
$x_1 = 1.25$	$f(x_1) = 0.64$	$x_6 = 2.50$	$f(x_6) = 0.16$
$x_2 = 1.50$	$f(x_2) \approx 0.4444$	$x_7 = 2.75$	$f(x_7) \approx 0.1322$
$x_3 = 1.75$	$f(x_3) \approx 0.3265$	$x_8 = 3.00$	$f(x_8) \approx 0.1111$
$x_4 = 2.00$	$f(x_4) = 0.25$		

Left Riemann Sum: $\int_1^3 \frac{1}{x^2} dx \approx 0.25[f(x_0) + f(x_1) + \ldots + f(x_7)] \approx 0.7846$

Trapezoidal Rule: $\int_1^3 \frac{1}{x^2} dx \approx \frac{0.25}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_7) + f(x_8)] \approx 0.6766$

Parabolic Rule: $\int_1^3 \frac{1}{x^2} dx \approx \frac{0.25}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_7) + f(x_8)] \approx 0.6671$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^3 = -\frac{1}{3} + 1 = \frac{2}{3} \approx 0.6667$

2. $f(x) = \frac{1}{x}; h = \frac{3-1}{8} = 0.25$

$x_0 = 1.00$	$f(x_0) = 1$	$x_5 = 2.25$	$f(x_5) \approx 0.4444$
$x_1 = 1.25$	$f(x_1) = 0.8$	$x_6 = 2.50$	$f(x_6) = 0.4$
$x_2 = 1.50$	$f(x_2) \approx 0.6667$	$x_7 = 2.75$	$f(x_7) \approx 0.3636$
$x_3 = 1.75$	$f(x_3) \approx 0.5714$	$x_8 = 3.00$	$f(x_8) \approx 0.3333$
$x_4 = 2.00$	$f(x_4) = 0.5$		

Left Riemann Sum: $\int_1^3 \frac{1}{x} dx \approx 0.25[f(x_0) + f(x_1) + \ldots + f(x_7)] \approx 1.1865$

Trapezoidal Rule: $\int_1^3 \frac{1}{x} dx \approx \frac{0.25}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_7) + f(x_8)] \approx 1.1032$

Parabolic Rule: $\int_1^3 \frac{1}{x}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)] \approx 1.0987$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x}dx = \left[\ln|x|\right]_1^3 = \ln 3 \approx 1.0986$

3. $f(x) = \sqrt{x}; h = \frac{2-0}{8} = 0.25$

$x_0 = 0.00$	$f(x_0) = 0$	$x_5 = 1.25$	$f(x_5) \approx 1.1180$
$x_1 = 0.25$	$f(x_1) = 0.5$	$x_6 = 1.50$	$f(x_6) \approx 1.2247$
$x_2 = 0.50$	$f(x_2) \approx 0.7071$	$x_7 = 1.75$	$f(x_7) \approx 1.3229$
$x_3 = 0.75$	$f(x_3) \approx 0.8660$	$x_8 = 2.00$	$f(x_8) \approx 1.4142$
$x_4 = 1.00$	$f(x_4) = 1$		

Left Riemann Sum: $\int_0^2 \sqrt{x}\,dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)] \approx 1.6847$

Trapezoidal Rule: $\int_0^2 \sqrt{x}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)] \approx 1.8615$

Parabolic Rule: $\int_0^2 \sqrt{x}dx \approx \frac{0.25}{3}[f(x_1)+4f(x_2)+2f(x_3)+\ldots+4f(x_7)+f(x_8)] \approx 1.8755$

Fundamental Theorem of Calculus: $\int_0^2 \sqrt{x}dx = \left[\frac{2}{3}x^{3/2}\right]_0^2 = \frac{4\sqrt{2}}{3} \approx 1.8856$

4. $f(x) = x\sqrt{x^2+1}; h = \frac{3-1}{8} = 0.25$

$x_0 = 1.00$	$f(x_0) \approx 1.4142$	$x_5 = 2.25$	$f(x_5) \approx 5.5400$
$x_1 = 1.25$	$f(x_1) \approx 2.0010$	$x_6 = 2.50$	$f(x_6) \approx 6.7315$
$x_2 = 1.50$	$f(x_2) \approx 2.7042$	$x_7 = 2.75$	$f(x_7) \approx 8.0470$
$x_3 = 1.75$	$f(x_3) \approx 3.5272$	$x_8 = 3.00$	$f(x_8) \approx 9.4868$
$x_4 = 2.00$	$f(x_4) \approx 4.4721$		

Left Riemann Sum: $\int_1^3 x\sqrt{x^2+1}\,dx \approx 0.25[f(x_0)+f(x_1)+\cdots+f(x_7)] \approx 8.6903$

Trapezoidal Rule: $\int_1^3 x\sqrt{x^2+1}\,dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)] \approx 9.6184$

Parabolic Rule: $\int_1^3 x\sqrt{x^2+1}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)] \approx 9.5981$

Fundamental Theorem of Calculus: $\int_1^3 x\sqrt{x^2+1}dx = \left[\frac{1}{3}(x^2+1)^{3/2}\right]_1^3 = \frac{1}{3}\left(10\sqrt{10}-2\sqrt{2}\right) \approx 9.5981$

5. $f(x) = \sin x$

$n = 2$: $h = \frac{\pi}{2}$

$x_0 = 0$	$f(x_0) = 0$
$x_1 = \frac{\pi}{2}$	$f(x_1) = 1$
$x_2 = \pi$	$f(x_2) = 0$

$\int_0^\pi \sin x\,dx \approx \frac{\pi}{4}[f(x_0)+2f(x_1)+f(x_0)] = \frac{\pi}{2} \approx 1.5708$

$n = 6$: $h = \frac{\pi}{6}$

$x_0 = 0$ $\quad f(x_0) = 0$

$x_1 = \frac{\pi}{6}$ $\quad f(x_1) = \frac{1}{2}$

$x_2 = \frac{\pi}{3}$ $\quad f(x_2) = \frac{\sqrt{3}}{2}$

$x_3 = \frac{\pi}{2}$ $\quad f(x_3) = 1$

$x_4 = \frac{2\pi}{3}$ $\quad f(x_4) = \frac{\sqrt{3}}{2}$

$x_5 = \frac{5\pi}{6}$ $\quad f(x_5) = \frac{1}{2}$

$x_6 = \pi$ $\quad f(x_6) = 0$

$$\int_0^{\pi} \sin x\, dx \approx \frac{\pi}{12}[f(x_0) + 2f(x_1) + \ldots + 2f(x_5) + f(x_6)] = \frac{\pi}{12}\left(4 + 2\sqrt{3}\right) \approx 1.9541$$

$n = 12$: $h = \frac{\pi}{12}$

$x_0 = 0$ $\quad f(x_0) = 0$

$x_1 = \frac{\pi}{12}$ $\quad f(x_1) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$x_2 = \frac{\pi}{6}$ $\quad f(x_2) = \frac{1}{2}$

$x_3 = \frac{\pi}{4}$ $\quad f(x_3) = \frac{\sqrt{2}}{2}$

$x_4 = \frac{\pi}{3}$ $\quad f(x_4) = \frac{\sqrt{3}}{2}$

$x_5 = \frac{5\pi}{12}$ $\quad f(x_5) = \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$

$x_6 = \frac{\pi}{2}$ $\quad f(x_6) = 1$

$x_7 = \frac{7\pi}{12}$ $\quad f(x_7) = \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$

$x_8 = \frac{2\pi}{3}$ $\quad f(x_8) = \frac{\sqrt{3}}{2}$

$x_9 = \frac{3\pi}{4}$ $\quad f(x_9) = \frac{\sqrt{2}}{2}$

$x_{10} = \frac{5\pi}{6}$ $\quad f(x_{10}) = \frac{1}{2}$

$x_{11} = \frac{11\pi}{12}$ $\quad f(x_{11}) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$x_{12} = \pi$ $\quad f(x_{12}) = 0$

$$\int_0^{\pi} \sin x\, dx \approx \frac{\pi}{24}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{11}) + f(x_{12})] \approx \frac{\pi}{24}\left(4 + 2\sqrt{3} + 2\sqrt{2} + 2\sqrt{6}\right) \approx 1.9886$$

6. Use the calculations in Problem 5.

$n = 2$:

$$\int_0^{\pi} \sin x\, dx \approx \frac{\pi}{6}[f(x_0) + 4f(x_1) + f(x_2)] = \frac{2\pi}{3} \approx 2.0944$$

$n = 6$:

$$\int_0^{\pi} \sin x\, dx \approx \frac{\pi}{18}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_5) + f(x_6)] \approx 2.0009$$

$n = 12$:

$$\int_0^{\pi} \sin x\, dx \approx \frac{\pi}{36}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{11}) + f(x_{12})] \approx 2.0001$$

7. $f(x) = \frac{4}{1+x^2}$; $h = \frac{1}{10}$

$x_0 = 0.0$ $\quad f(x_0) = 4$

$x_1 = 0.1$ $\quad f(x_1) \approx 3.9604$

$x_2 = 0.2$ $\quad f(x_2) \approx 3.8462$

$x_3 = 0.3$ $\quad f(x_3) \approx 3.6697$

$x_4 = 0.4$ $\quad f(x_4) \approx 3.4483$

$x_5 = 0.5$ $\quad f(x_5) = 3.2$

$x_6 = 0.6$ $\quad f(x_6) \approx 2.9412$

$x_7 = 0.7$ $\quad f(x_7) \approx 2.6846$

$x_8 = 0.8$ $\quad f(x_8) \approx 2.4390$

$x_9 = 0.9$ $\quad f(x_9) \approx 2.2099$

$x_{10} = 1.0$ $\quad f(x_{10}) = 2$

$$\int_0^1 \frac{4}{1+x^2} dx \approx \frac{1}{30}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_9) + f(x_{10})] \approx 3.1416$$

8. $f(x)=\cos(\sin x);\ h=\frac{1}{10}$

$x_0=0.0$	$f(x_0)=1$	$x_6=0.6$	$f(x_6)\approx 0.8448$
$x_1=0.1$	$f(x_1)\approx 0.9950$	$x_7=0.7$	$f(x_7)\approx 0.7996$
$x_2=0.2$	$f(x_2)\approx 0.9803$	$x_8=0.8$	$f(x_8)\approx 0.7535$
$x_3=0.3$	$f(x_3)\approx 0.9567$	$x_9=0.9$	$f(x_9)\approx 0.7086$
$x_4=0.4$	$f(x_4)\approx 0.9251$	$x_{10}=1.0$	$f(x_{10})\approx 0.6664$
$x_5=0.5$	$f(x_5)\approx 0.8873$		

$$\int_0^1 \cos(\sin x)dx \approx \frac{1}{20}[f(x_0)+2f(x_1)+\ldots+2f(x_9)+f(x_{10})]\approx 0.8684$$

9. $f(x)=e^{-x^2}$

$f'(x)=-2xe^{-x^2}$

$f''(x)=-2e^{-x^2}+4x^2e^{-x^2}=e^{-x^2}(4x^2-2)$

$|E_n|=\frac{1}{12n^2}\left|e^{-c^2}(4c^2-2)\right|\le\frac{1}{12n^2}\cdot 2=\frac{1}{6n^2}$

$\frac{1}{6n^2}\le 0.01$ when $n\ge 5$.

$h=\frac{1}{5}=0.2$

$x_0=0.0$	$f(x_0)=1$	$x_3=0.6$	$f(x_3)\approx 0.6977$
$x_1=0.2$	$f(x_1)\approx 0.9608$	$x_4=0.8$	$f(x_4)\approx 0.5273$
$x_2=0.4$	$f(x_2)\approx 0.8521$	$x_5=1.0$	$f(x_5)\approx 0.3679$

$$\int_0^1 e^{-x^2}dx\approx\frac{0.2}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_4)+f(x_5)]\approx 0.74$$

10. $f(x)=e^{x^2}$

$f'(x)=2xe^{x^2}$

$f''(x)=2e^{x^2}+4x^2e^{x^2}=e^{x^2}(4x^2+2)$

$|E_n|=\frac{(0.6)^3}{12n^2}\left|e^{c^2}(4c^2+2)\right|$

$\le\frac{(0.6)^3}{12n^2}e^{0.36}[4(0.6)^2+2]\le\frac{1.06502}{12n^2}$

$\frac{1.06502}{12n^2}\le 0.01$ when $n\ge 3$.

$h=\frac{0.6}{3}=0.2$

$x_0=0.0$	$f(x_0)=1$
$x_1=0.2$	$f(x_1)\approx 1.0408$
$x_2=0.4$	$f(x_2)\approx 1.1735$
$x_3=0.6$	$f(x_3)\approx 1.4333$

$$\int_0^{0.6} e^{x^2}dx\approx\frac{0.2}{2}[f(x_0)+2f(x_1)+2f(x_2)+f(x_3)]$$
≈ 0.69

11. $f(x)=\sqrt{\cos x}$

$f'(x)=-\frac{\sin x}{2\sqrt{\cos x}}$

$f''(x)=-\frac{2\cos^2 x+\sin^2 x}{4\cos^{3/2} x}$

$|E_n|=\frac{(0.5)^3}{12n^2}\left|\frac{2\cos^2 x+\sin^2 x}{4\cos^{3/2} x}\right|$

$\le\frac{(0.5)^3}{12n^2}\left|\frac{2\cos^2 1+\sin^2 1.5}{4\cos^{3/2} 1.5}\right|\le\frac{2.6225}{12n^2}$

$\frac{2.6225}{12n^2}\le 0.01$ when $n\ge 5$.

$h=\frac{0.5}{5}=0.1$

$x_0 = 1.0 \quad f(x_0) \approx 0.7351$

$x_1 = 1.1 \quad f(x_1) \approx 0.6735$

$x_2 = 1.2 \quad f(x_2) \approx 0.6020$

$x_3 = 1.3 \quad f(x_3) \approx 0.5172$

$x_4 = 1.4 \quad f(x_4) \approx 0.4123$

$x_5 = 1.5 \quad f(x_5) \approx 0.2660$

$$\int_1^{1.5} \sqrt{\cos x}\, dx \approx \frac{0.1}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)] \approx 0.27$$

12. $f(x) = \cos\sqrt{x}$

$$f'(x) = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

$$f''(x) = -\frac{\cos\sqrt{x}}{4x} + \frac{\sin\sqrt{x}}{4x^{3/2}}$$

$$|E_n| = \frac{1}{12n^2}\left|-\frac{\cos\sqrt{c}}{4c} + \frac{\sin\sqrt{c}}{4c^{3/2}}\right|$$

$$\le \frac{1}{12n^2}\left(\left|\frac{\cos\sqrt{c}}{4c}\right| + \left|\frac{\sin\sqrt{c}}{4c^{3/2}}\right|\right) \le \frac{1}{12n^2}\left(\frac{0.6}{4} + \frac{1}{4}\right)$$

$$= \frac{0.4}{12n^2}$$

$\frac{0.4}{12n^2} \le 0.01$ when $n \ge 2$.

$$h = \frac{1}{2} = 0.5$$

$x_0 = 1.0 \quad f(x_0) \approx 0.5403$

$x_1 = 1.5 \quad f(x_1) \approx 0.3392$

$x_2 = 2.0 \quad f(x_2) \approx 0.1559$

$$\int_1^2 \cos\sqrt{x}\,dx \approx \frac{0.5}{2}[f(x_0) + 2f(x_1) + f(x_2)] \approx 0.34$$

13. $f(x) = \frac{1+x}{1-x}$

$$f'(x) = \frac{2}{(1-x)^2}$$

$$f''(x) = \frac{4}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{12}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{48}{(1-x)^5}$$

$$|E_n| = \frac{4^5}{180n^4}\left|\frac{48}{(1-c)^5}\right| \le \frac{4^5}{180n^4}\left(\frac{48}{1}\right) = \frac{4096}{15n^4}$$

$\frac{4096}{15n^4} \le 0.005$ when $n \ge 16$.

$$h = \frac{4}{16} = 0.25$$

$$\int_2^6 \frac{1+x}{1-x}dx \approx \frac{0.25}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{15}) + f(x_{16})] \approx -7.219$$

14. $f(x) = \ln x$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f^{(3)}(x) = \frac{2}{x^3}$$

$$f^{(4)}(x) = -\frac{6}{x^4}$$

$$|E_n| = \frac{2^5}{180n^4}\left|\frac{6}{c^4}\right| \le \frac{2^5}{180n^4}\left(\frac{6}{1}\right) = \frac{16}{15n^4}$$

$\frac{16}{15n^4} \le 0.005$ when $n \ge 4$.

$$h = \frac{2}{4} = 0.5$$

$$\int_1^3 \ln x\,dx \approx \frac{0.5}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \approx 1.295$$

15. $\int_{m-h}^{m+h}(ax^2+bx+c)dx=\left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_{m-h}^{m+h}$

$=\frac{a}{3}(m+h)^3+\frac{b}{2}(m+h)^2+c(m+h)-\frac{a}{3}(m-h)^3-\frac{b}{2}(m-h)^2-c(m-h)$

$=\frac{a}{3}(6m^2h+2h^3)+\frac{b}{2}(4mh)+c(2h)\ =\frac{h}{3}[a(6m^2+2h^2)+b(6m)+6]$

$=\frac{h}{3}[f(m-h)+4f(m)+f(m+h)]$

$=\frac{h}{3}[a(m-h)^2+b(m-h)+c+4am^2+4bm+4c+a(m+h)^2+b(m+h)+c]$

$=\frac{h}{3}[a(6m^2+2h^2)+b(6m)+6c]$

16. **a.** To show that the Parabolic Rule is exact, examine it on the interval $[m-h, m+h]$.
Let $f(x)=ax^3+bx^2+cx+d$, then

$\int_{m-h}^{m+h}f(x)\,dx$

$=\frac{a}{4}\left[(m+h)^4-(m-h)^4\right]+\frac{b}{3}\left[(m+h)^3-(m-h)^3\right]+\frac{c}{2}\left[(m+h)^2-(m-h)^2\right]+d[(m+h)-(m-h)]$

$=\frac{a}{4}(8m^3h+8h^3m)+\frac{b}{3}(6m^2h+2h^3)+\frac{c}{2}(4mh)+d(2h).$

The Parabolic Rule with $n = 2$ gives

$\int_{m-h}^{m+h}f(x)\,dx=\frac{h}{3}[f(m-h)+4f(m)+f(m+h)]=2am^3h+2amh^3+2bm^2h+\frac{2}{3}bh^3+2chm+2dh$

$=\frac{a}{4}(8m^3h+8mh^3)+\frac{b}{3}(6m^2h+2h^3)+\frac{c}{2}(4mh)+d(2h)$

which agrees with the direct computation. Thus, the Parabolic Rule is exact for any cubic polynomial.

b. The error in using the Parabolic Rule is given by $E_n=-\frac{(l-k)^5}{180n^4}f^{(4)}(m)$ for some m between l and k.

However, $f'(x)=3ax^2+2bx+c, f''(x)=6ax+2b, f^{(3)}(x)=6a$, and $f^{(4)}(x)=0$, so $E_n=0$.

17. $f(x)=\frac{1}{x}$

$f'(x)=-\frac{1}{x^2}$

$f''(x)=\frac{2}{x^3}$

$|E_n|=\frac{1}{12n^2}\left(\frac{2}{c^3}\right)\le\frac{1}{6n^2}$

$\frac{1}{6n^2}\le 10^{-10}$ when $n\ge 40825$.

18. $f^{(3)}(x)=-\frac{6}{x^4}$

$f^{(4)}(x)=\frac{24}{x^5}$

$|E_n|=\frac{1}{180n^4}\left(\frac{24}{c^5}\right)\le\frac{2}{15n^4}$

$\frac{2}{15n^4}\le 10^{-10}$ when $n\ge 192$.

19. Let $n = 2$.

$f(x) = x^k;\ h = a$

$x_0 = -a \qquad f(x_0) = -a^k$

$x_1 = 0 \qquad f(x_1) = 0$

$x_2 = a \qquad f(x_2) = a^k$

$$\int_{-a}^{a} x^k\,dx \approx \frac{a}{2}[-a^k + 2\cdot 0 + a^k] = 0$$

$$\int_{-a}^{a} x^k\,dx = \left[\frac{1}{k+1}x^{k+1}\right]_{-a}^{a} = \frac{1}{k+1}[a^{k+1} - (-a)^{k+1}] = \frac{1}{k+1}[a^{k+1} - a^{k+1}] = 0$$

20. **a.** From Example 1, $T \approx 48.9414$.

$f'(x) = 4x^3$

$$T - \frac{[4(3)^3 - 4(1)^3](0.25)^2}{12} \approx 48.9414 - 0.5417 = 48.3997$$

b. From Problem 5, $T \approx 1.9886$.

$f'(x) = \cos x$

$$T - \frac{[\cos\pi - \cos 0]\left(\frac{\pi}{12}\right)^2}{12} \approx 1.9886 + 0.0114 = 2.0000$$

21. $A \approx \frac{10}{2}[75 + 2\cdot 71 + 2\cdot 60 + 2\cdot 45 + 2\cdot 45 + 2\cdot 52 + 2\cdot 57 + 2\cdot 60 + 59] = 4570 \text{ ft}^2$

22. $A \approx \frac{3}{3}[23 + 4\cdot 24 + 2\cdot 23 + 4\cdot 21 + 2\cdot 18 + 4\cdot 15 + 2\cdot 12 + 4\cdot 11 + 2\cdot 10 + 4\cdot 8 + 0] = 465 \text{ ft}^2$

$V = A\cdot 6 \approx 2790 \text{ ft}^3$

23. $A \approx \frac{20}{3}[0 + 4\cdot 7 + 2\cdot 12 + 4\cdot 18 + 2\cdot 20 + 4\cdot 20 + 2\cdot 17 + 4\cdot 10 + 0] = 2120 \text{ ft}^2$

4 mi/h = 21,120 ft/h

(2120)(21,120)(24) = 1,074,585,600 ft^3

24. $h = \frac{210 - 0}{21} = 10 \text{ minutes} = \frac{1}{6} \text{ hour}$

Suppose t is time measured in hours and $v(t)$ is the velocity at time t. Then the distance traveled is

$$\int_0^{3.5} v(t)\,dt \approx \frac{1/6}{2}[0 + 2\cdot 55 + 2\cdot 57 + 2\cdot 60 + 2\cdot 70 + 2\cdot 70 + 2\cdot 70 + 2\cdot 70 + 2\cdot 19 + 2\cdot 0$$
$$+ 2\cdot 59 + 2\cdot 63 + 2\cdot 65 + 2\cdot 62 + 2\cdot 0 + 2\cdot 0 + 2\cdot 0 + 2\cdot 22 + 2\cdot 38 + 2\cdot 35 + 2\cdot 25 + 0]$$
$$= 140$$

25. $h = \frac{24 - 0}{8} = 3 \text{ minutes} = \frac{1}{20} \text{ hour}$

Suppose t is time measured in hours and $v(t)$ is the velocity at time t.

a. Using the Trapezoidal Rule, the distance traveled is

$$\int_0^{0.4} v(t)\,dt \approx \frac{1/20}{2}[0 + 2\cdot 31 + 2\cdot 54 + 2\cdot 53 + 2\cdot 52 + 2\cdot 35 + 2\cdot 31 + 2\cdot 28 + 0]$$
$$= \frac{464}{40} = 11.6 \text{ miles}$$

b. Using the Parabolic Rule, the distance traveled is

$$\int_0^{0.4} v(t)\,dt \approx \frac{1/20}{3}[1\cdot 0+4\cdot 31+2\cdot 54+4\cdot 53+2\cdot 52+4\cdot 35+2\cdot 31+4\cdot 28+1\cdot 0]$$

$$=\frac{568}{40}=14.2\text{ miles.}$$

26. Suppose t is time measured in hours and $r(t)$ is the rate of water usage at time t.

$$h=\frac{120}{10}\text{ minutes}=\frac{1}{5}\text{ hour}$$

a. Using the Trapezoidal Rule, the total water used is

$$\int_0^2 r(t)\,dt \approx \frac{1/5}{2}[65+2\cdot 71+2\cdot 68+2\cdot 78+2\cdot 105+2\cdot 111+2\cdot 108+2\cdot 144+2\cdot 160+2\cdot 152+148]$$

$$=\frac{1}{10}\cdot 2207=220.7\text{ gallons}$$

b. Using the Parabolic Rule, the total water used is

$$\int_0^2 r(t)\,dt \approx \frac{1/5}{2}[65+4\cdot 71+2\cdot 68+4\cdot 78+2\cdot 105+4\cdot 111+2\cdot 108+4\cdot 144+2\cdot 160+4\cdot 152+148]$$

$$=\frac{3319}{15}\approx 221.27\text{ gallons.}$$

27. a. Lay the part with the long flat side along the x-axis, with the upper left corner (as shown in Figure 22) at the origin. Let $h(x)$ denote the height of the lamina at the value of x. Then, using the Trapezoidal Rule,

$$m=\int_0^{40} h(x)\,dx \approx \frac{5}{2}\left[5+2(6.5+8+9+10+10.5+10.5+10)+8\right]=355$$

$$M_y=\int_0^{40} x\,h(x)\,dx \approx \frac{5}{2}\left[0\cdot 5+2(5\cdot 6.5+10\cdot 8+15\cdot 9+20\cdot 10+25\cdot 10.5+30\cdot 10.5+35\cdot 10)+40\cdot 8\right]=7675$$

$$\bar{x}=\frac{M_y}{m}\approx\frac{7675}{355}\approx 21.62$$

$$M_x=\frac{1}{2}\int_0^{40} h^2(x)\,dx \approx \frac{1}{2}\frac{5}{2}\left[5^2+2\left(6.5^2+8^2+9^2+10^2+10.5^2+10.5^2+10^2\right)+8^2\right]=1630.625$$

$$\bar{y}=\frac{M_x}{m}\approx\frac{1630.625}{355}=4.59$$

b. Using the Parabolic Rule,

$$m=\int_0^{40} h(x)\,dx \approx \frac{5}{3}[5+4\cdot 6.5+2\cdot 8+4\cdot 9+2\cdot 10+4\cdot 10.5+2\cdot 10.5+4\cdot 10+8]=\frac{5}{3}214\approx 356.67$$

$$M_y=\int_0^{40} x\,h(x)\,dx$$

$$\approx\frac{5}{3}\left[0\cdot 5+4(5\cdot 6.5)+2(10\cdot 8)+4(15\cdot 9)+2(20\cdot 10)+4(25\cdot 10.5)+2(30\cdot 10.5)+4(35\cdot 10)+40\cdot 8\right]$$

$$=\frac{5}{3}4630\approx 7716.67$$

$$\bar{x}=\frac{M_y}{m}\approx\frac{7716.67}{356.67}\approx 21.64$$

$$M_x=\frac{1}{2}\int_0^{40} h^2(x)\,dx$$

$$\approx\frac{1}{2}\frac{5}{3}\left[5^2+4\left(6.5^2\right)+2\left(8^2\right)+4\left(9^2\right)+2\left(10^2\right)+4\left(10.5^2\right)+2\left(10.5^2\right)+4\left(10^2\right)+8^2\right]$$

$$=\frac{5}{6}1971.5\approx 1642.\bar{9}$$

$$\bar{y}=\frac{M_x}{m}\approx\frac{1642.9}{356.67}=4.61$$

28. a. Place the lamina so that the origin is the center of the hole, and the long straight side is parallel to the x-axis. Let $h(x)$ denote the height of the lamina at the point x. Let R_1 be the lamina with the hole drilled in it, let R_2 be the lamina consisting of just the hole, and let R_3 be the lamina from Problem 27 without the hole. Using the Trapezoidal Rule,

$m(R_1) = m(R_3) - m(R_2) \approx 355 - \pi\left(2.5^2\right) \approx 335.37$

$M_y(R_3) = \int_{-30}^{10} x\,h(x)\,dx$

$\approx \frac{5}{2}\left[-30\cdot 5 + 2(-25\cdot 6.5 - 20\cdot 8 - 15\cdot 9 - 10\cdot 10 - 5\cdot 10.5 + 0\cdot 10.5 + 5\cdot 10) + 10\cdot 8\right] = -2975$

$M_x(R_3) = \frac{1}{2}\int_{-30}^{10}\left[f^2(x) - g^2(x)\right]dx = \frac{1}{2}\int_{-30}^{10}\left[(h(x)-4)^2 - (-4)^2\right]dx = \frac{1}{2}\int_{-30}^{10}\left[h^2(x) - 8h(x)\right]dx$

$\approx \frac{5}{4}\left[5^2 + 2\left(6.5^2 + 8^2 + 9^2 + 10^2 + 10.5^2 + 10.5^2 + 10^2\right) + 8^2\right] - 8\cdot\frac{5}{4}\left[5 + 2(6.5 + 8 + 10 + 10.5 + 10.5 + 10) + 8\right]$

$\approx 1630.625 - 4\cdot 355 = 210.625$

By symmetry, the centroid of R_2 is (0,0). Thus,

$M_y(R_2) = 0$, and $M_x(R_2) = 0$

$M_y(R_1) = M_y(R_3) - M_y(R_2) \approx -2975 - 0 = -2975$

$M_x(R_1) = M_x(R_3) - M_x(R_2) \approx 210.625 - 0 = 210.625$

$\bar{x}(R_3) = \frac{M_y}{m} \approx \frac{-2975}{335.37} \approx -8.87$

$\bar{y}(R_3) = \frac{M_x(R_3)}{m(R_3)} \approx \frac{210.625}{335.37} \approx 0.63$

b. Using the Parabolic Rule,

$m(R_1) = m(R_3) - m(R_2) \approx 356.67 - \pi\left(2.5^2\right) \approx 337.04$

$M_y(R_3) = \int_{-30}^{10} x\,h(x)\,dx$

$\approx \frac{5}{6}\left[-30\cdot 5 + 4(-25\cdot 6.5) + 2(-20\cdot 8) + 4(-15\cdot 9) + 2(-10\cdot 10) + 4(-5\cdot 10.5) + 2(0\cdot 10.5) + 4(5\cdot 10) + 10\cdot 8\right]$

$= \frac{5}{6}(-1790) \approx -1491.67$

$M_x(R_3) = \frac{1}{2}\int_{-30}^{10}\left[f^2(x) - g^2(x)\right]dx = \frac{1}{2}\int_{-30}^{10}\left[(h(x)-4)^2 - (-4)^2\right]dx = \frac{1}{2}\int_{-30}^{10}\left[h^2(x) - 8h(x)\right]dx$

$\approx \frac{5}{6}\left[5^2 + 4\left(6.5^2\right) + 2\left(8^2\right) + 4\left(9^2\right) + 2\left(10^2\right) + 4\left(10.5^2\right) + 2\left(10.5^2\right) + 4\left(10^2\right) + 8^2\right]$

$\quad - 8\cdot\frac{5}{6}\left[5 + 4\cdot 6.5 + 2\cdot 8 + 4\cdot 10 + 2\cdot 10.5 + 4\cdot 10.5 + 2\cdot 10 + 8\right]$

$\approx 1642.92 - 1426.67 = 216.25$

By symmetry, the centroid of R_2 is (0,0). Thus,

$M_y(R_2) = 0$, and $M_x(R_2) = 0$

$M_y(R_1) = M_y(R_3) - M_y(R_2) \approx -1491.67$

$M_x(R_1) = M_x(R_3) - M_x(R_2) \approx 216.25 - 0 = 216.25$

$\bar{x}(R_3) = \frac{M_y}{m} \approx \frac{-1491.67}{337.04} \approx -4.43$

$\bar{y}(R_3) = \frac{M_x(R_3)}{m(R_3)} \approx \frac{216.25}{337.04} \approx 0.64$

29. Rotate the map 90° clockwise and put the x-axis along the bottom, with the origin below the southernmost tip of Illinois. Let $f(x)$ denote the "upper" function, and $g(x)$ the "lower" function. Then the east/west dimensions given on the map are $f(x)-g(x)$, and $g(x)$ is equal to 85, 50, 30, 25, 15, and 10 at the southern end of the state, and 10 and 13 at the northern end.

$$m = \int_0^{380}[f(x)-g(x)]dx \approx \frac{20}{2}[0+2(58+79+\cdots+139+132)+140] = 57,000$$

$$M_y = \int_0^{380} x[f(x)-g(x)]dx \approx \frac{20}{2}[0\cdot 0+2(20\cdot 58+40\cdot 79+\cdots+340\cdot 139+360\cdot 132)+380\cdot 140] \approx 12,002,800$$

$$M_x = \frac{1}{2}\int_0^{380}\left[f^2(x)-g^2(x)\right]dx \approx \frac{1}{2}\frac{20}{2}\left\{\left(85^2-85^2\right)+2\left[\left(108^2-50^2\right)+\left(109^2-30^2\right)+\cdots+\left(142^2-10^2\right)\right]\right\}$$

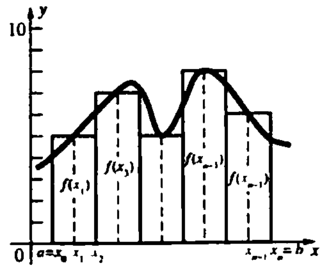
$\approx 4,994,480$

$$\bar{x} = \frac{M_y}{m} \approx \frac{12,002,800}{57,000} \approx 210.6$$

$$\bar{y} = \frac{M_x}{m} \approx \frac{4,994,480}{57,000} \approx 87.6$$

This is a point just southeast of Lincoln, IL and it is about 30 miles northeast of Springfield.

30. a.



b. $n = 16,\ h = \frac{3-1}{16} = 0.125$

$x_1 = 1.125$	$f(x_1) \approx 1.6018$	$x_9 = 2.125$	$f(x_9) \approx 20.3909$
$x_3 = 1.375$	$f(x_3) \approx 3.5745$	$x_{11} = 2.375$	$f(x_{11}) \approx 31.8167$
$x_5 = 1.625$	$f(x_5) \approx 6.9729$	$x_{13} = 2.625$	$f(x_{13}) \approx 47.4807$
$x_7 = 1.875$	$f(x_7) \approx 12.3596$	$x_{15} = 2.875$	$f(x_{15}) \approx 68.3206$

$$\int_1^3 x^4 dx \approx 0.25[f(x_1)+f(x_3)+\ldots+f(x_{15})] \approx 48.1294$$

31. a. Since $x > X > 0,\ x^2 > Xx$. Thus $e^{-x^2} < e^{-Xx}$.

$$\int_X^\infty e^{-x^2}dx < \int_X^\infty e^{-Xx}dx = \left[-\frac{1}{X}e^{-Xx}\right]_X^\infty = \frac{1}{X}e^{-X^2}$$

b. $$\int_5^\infty e^{-x^2}dx < \int_5^\infty e^{-5x}dx = \frac{1}{5}e^{-25} \approx 2.78\times 10^{-12} < 10^{-11}$$

c. $n = 10,\ h = \frac{5-0}{10} = 0.5$

$x_0 = 0.0$	$f(x_0) = 1$
$x_1 = 0.5$	$f(x_1) \approx 0.7788$
$x_2 = 1.0$	$f(x_2) \approx 0.3679$
$x_3 = 1.5$	$f(x_3) \approx 0.1054$
$x_4 = 2.0$	$f(x_4) \approx 0.0183$
$x_5 = 2.5$	$f(x_5) \approx 0.0019$
$x_6 = 3.0$	$f(x_6) \approx 1.234\times 10^{-4}$

$x_7 = 3.5 \qquad f(x_7) \approx 4.785\times10^{-6}$

$x_8 = 4.0 \qquad f(x_8) \approx 1.125\times10^{-7}$

$x_9 = 4.5 \qquad f(x_9) \approx 1.605\times10^{-9}$

$x_{10} = 5.0 \qquad f(x_{10}) \approx 1.389\times10^{-11}$

$$\int_0^5 e^{-x^2}\,dx \approx \frac{0.5}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_9)+f(x_{10})] \approx 0.8862$$

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx \approx \frac{2}{\sqrt{\pi}}(0.8862) \approx 0.99997$$

$f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$f''(x) = e^{-x^2}(4x^2-2)$

$f^{(3)}(x) = xe^{-x^2}(12-8x^2)$

$f^{(4)}(x) = e^{-x^2}(16x^4-48x^2+12)$

$$|E_{10}| = \frac{5^5}{180\times10^4}\left|\frac{16c^4-48c^2+12}{e^{c^2}}\right| \le \frac{5^5}{1,800,000}\cdot 12$$

A graphing utility or computer shows that

$$\left|\frac{16c^4-48c^2+12}{e^{c^2}}\right| \le 12 \text{ on } [0, 5].$$

$|E_{10}| \le 0.0209$

Thus, the error in computing

$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx$ is less than $\frac{2}{\sqrt{\pi}}(0.0209+10^{-11}) \approx 0.0236.$

32. a. First consider $t = \frac{x}{1+x}$. At $x = 0$, $t = 0$. $\frac{dt}{dx} = \frac{1}{(1+x)^2} > 0$ on $[0, \infty)$. $\lim_{x\to\infty}\frac{x}{1+x} = 1$. Since $t = \frac{x}{1+x}$ is increasing to 1 on $[0, \infty)$, the substitution transforms $[0, \infty)$ into $[0, 1)$.

Now consider $t = e^{-x}$. At $x = 0$, $t = 1$. $\frac{dt}{dx} = -e^{-x} < 0$ on $[0, \infty)$. $\lim_{x\to\infty} e^{-x} = 0$. Since $t = e^{-x}$ is decreasing to 0 on $[0, \infty)$, the substitution transforms $[0, \infty)$ into $(0, 1]$.

b. $\frac{dt}{dx} = \frac{2e^x}{(e^x+1)^2} > 0$ on $(-\infty, \infty)$. $\lim_{x\to-\infty}\frac{e^x-1}{e^x+1} = \frac{-1}{1} = -1$. $\lim_{x\to\infty}\frac{e^x-1}{e^x+1} = \lim_{x\to\infty}\frac{e^x}{e^x} = 1$. Since $t = \frac{e^x-1}{e^x+1}$ is increasing from -1 to 1 on $(-\infty, \infty)$, the substitution transforms $(-\infty, \infty)$ into $(-1, 1)$.

c. $t = e^{-x}$, $dt = -e^{-x}dx$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{1+e^{-2x}}}\,dx = \int_1^0 -\frac{1}{\sqrt{1+t^2}}\,dt = \int_0^1 \frac{1}{\sqrt{1+t^2}}\,dt$$

$f(t) = \frac{1}{\sqrt{1+t^2}}$, $h = 0.1$

i	t_i	$f(t_i)$
0	0.0	1.
1	0.1	0.99504
2	0.2	0.98058
3	0.3	0.95783
4	0.4	0.92848
5	0.5	0.89443
6	0.6	0.85749
7	0.7	0.81923
8	0.8	0.78087
9	0.9	0.74329
10	1.0	0.70711

$$\int_0^1 \frac{1}{\sqrt{1+t^2}}dt \approx \frac{0.1}{3}[f(t_0)+4f(t_1)+2f(t_2)+\ldots+4f(t_9)+f(t_{10})]$$

≈ 0.881374

$$f^{(4)}(t)=\frac{3(3-24t^2+8t^4)}{(1+t^2)^{9/2}}$$

$$|E_{10}| \le \frac{1}{180(10^4)}\frac{3(3+24+8)}{1} < 5.84\times10^{-5}$$

d. $\frac{1}{t}=x, -\frac{1}{t^2}dt=dx$

$$\int_1^\infty \frac{x}{1+x^3}dx=\int_1^0 \frac{\frac{1}{t}}{1+\left(\frac{1}{t}\right)^3}\left(-\frac{1}{t^2}\right)dt=\int_0^1 \frac{1}{t^3+1}dt \quad f(t)=\frac{1}{t^3+1},\ h=0.1$$

i	t_i	$f(t_i)$
0	0.0	1
1	0.1	0.99900
2	0.2	0.99206
3	0.3	0.97371
4	0.4	0.93985
5	0.5	0.88889
6	0.6	0.82237
7	0.7	0.74460
8	0.8	0.66138
9	0.9	0.57837
10	1.0	0.5

$$\int_0^1 \frac{1}{t^3+1} \approx \frac{0.1}{3}[f(t_0)+4f(t_1)+2f(t_2)+\cdots+4f(t_9)+f(t_{10})]$$

≈ 0.835653

33. a. $u = \frac{2}{\sqrt{4+x}}$ $\quad dv = \frac{1}{2\sqrt{x}}dx$

$du = -\frac{1}{(4+x)^{3/2}}dx \quad v = \sqrt{x}$

$$\int_0^2 \frac{dx}{\sqrt{4x+x^2}} = \left[\frac{2\sqrt{x}}{\sqrt{4+x}}\right]_0^2 + \int_0^2 \frac{\sqrt{x}}{(4+x)^{3/2}}dx = \frac{2}{\sqrt{3}} + \int_0^2 \frac{\sqrt{x}}{(4+x)^{3/2}}dx$$

b. $u = \frac{1}{x}$ $\quad dv = \sin x\, dx$

$du = -\frac{1}{x^2}dx \quad v = -\cos x$

$$\int_1^\infty \frac{\sin x}{x}dx = \left[-\frac{\cos x}{x}\right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2}dx = \cos 1 - \int_1^\infty \frac{\cos x}{x^2}dx$$

$|\cos x| \le 1$ for all x, so

$$\int_1^\infty \frac{\cos x}{x^2}dx \le \int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty = 1.$$

Therefore, $\int_1^\infty \frac{\sin x}{x}dx$ exists.

c. $u = \frac{1}{4+x}$ $\quad dv = \frac{1}{\sqrt{x}}dx$

$du = -\frac{1}{(4+x)^2} \quad v = 2\sqrt{x}$

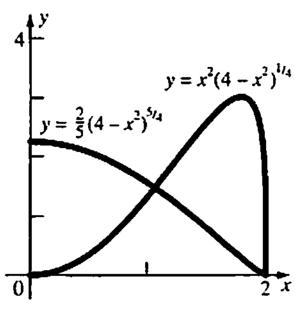
$$\int_0^1 \frac{1}{\sqrt{x}}\frac{1}{4+x}dx = \left[\frac{2\sqrt{x}}{4+x}\right]_0^1 + \int_0^1 \frac{2\sqrt{x}}{(4+x)^2}dx = \frac{2}{5} + \int_0^1 \frac{2\sqrt{x}}{(4+x)^2}dx$$

34. a. $u = x$ $\quad dv = x(4-x^2)^{1/4}dx$

$du = dx \quad v = -\frac{2}{5}(4-x^2)^{5/4}$

$$\int_0^2 x^2(4-x^2)^{1/4}dx = \left[-\frac{2}{5}x(4-x^2)^{5/4}\right]_0^2 + \int_0^2 \frac{2}{5}(4-x^2)^{5/4}dx = \int_0^2 \frac{2}{5}(4-x^2)^{5/4}dx$$

b.



c. First use the Trapezoidal Rule on the first part.

$f(x) = \frac{2}{5}(4-x^2)^{5/4}$

$x_0 = 0.00 \quad f(x_0) \approx 2.26274$

$x_1 = 0.25 \quad f(x_1) \approx 2.21863$

$x_2 = 0.50$ $\quad f(x_2) \approx 2.08737$

$x_3 = 0.75$ $\quad f(x_3) \approx 1.87225$

$x_4 = 1.00$ $\quad f(x_4) \approx 1.57929$

$$\int_0^1 \frac{2}{5}(4-x^2)^{5/4}\,dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)] \approx 2.02482$$

$$f''(x) = \frac{3x^2-8}{2(4-x^2)^{3/4}}$$

$$|E_4| = \frac{1}{12\cdot 4^2}|f''(c)| \le \frac{1}{192}\frac{8}{2(3)^{3/4}} < 0.00914$$

Next use integration by parts on the second part.

$u = \frac{1}{x}$ $\quad dv = x(4-x^2)^{5/4}$

$du = -\frac{1}{x^2}dx$ $\quad v = -\frac{2}{9}(4-x^2)^{9/4}$

$$= \frac{2}{5}\left(2\sqrt[4]{3} - \int_1^2 \frac{2}{9x^2}(4-x^2)^{9/4}\,dx\right) = \frac{4\sqrt[4]{3}}{5} - \int_1^2 \frac{4}{45x^2}(4-x^2)^{9/4}\,dx$$

Use the Trapezoidal Rule on the integral.

$$f(x) = \frac{4}{45x^2}(4-x^2)^{9/4}$$

$x_0 = 1.00$ $\quad f(x_0) \approx 1.05286$

$x_1 = 1.25$ $\quad f(x_1) \approx 0.42233$

$x_2 = 1.50$ $\quad f(x_2) \approx 0.13916$

$x_3 = 1.75$ $\quad f(x_3) \approx 0.02510$

$x_4 = 2.00$ $\quad f(x_4) = 0$

$$\int_1^2 \frac{4}{45x^2}(4-x^2)^{9/4}\,dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)] \approx 0.27825$$

$$f''(x) = \frac{(5x^4+8x^2+128)(4-x^2)^{1/4}}{15x^4}$$

$$|E_4| = \frac{1}{12\cdot 4^2}|f''(c)| \le \frac{1}{192}\frac{240(3)^{1/4}}{15} < 0.10968$$

Now estimate the value of the original integral.

$$\int_0^2 \frac{2}{5}(4-x^2)^{5/4}\,dx = \int_0^1 \frac{2}{5}(4-x^2)^{5/4}\,dx + \frac{4\sqrt[4]{3}}{5} - \int_1^2 \frac{4}{45x^2}(4-x^2)^{9/4}\,dx \approx 2.02482 + \frac{4\sqrt[4]{3}}{5} - 0.27825 \approx 2.7994$$

$|\text{Error}| \le 0.00914 + 0.10968 = 0.11882$

The error estimate for the original integral would fail since $f''(x)$ is not defined at 2, so the error estimate would be unbounded.

35. $\int_0^{\pi/2} \ln(\sin x)dx = \int_0^{\pi/2} \ln x\,dx + \int_0^{\pi/2} \ln\left(\frac{\sin x}{x}\right)dx$

$$\int_0^{\pi/2} \ln x\,dx = \lim_{a\to 0}\int_a^{\pi/2} \ln x\,dx = \lim_{a\to 0}[x\ln x - x]_a^{\pi/2} = \left(\frac{\pi}{2}\ln\frac{\pi}{2} - \frac{\pi}{2}\right) - \lim_{a\to 0} a\ln a = \frac{\pi}{2}\ln\frac{\pi}{2} - \frac{\pi}{2} \approx -0.8615$$

$f(x) = \ln\left(\frac{\sin x}{x}\right)$, $h = \frac{\pi}{8} \approx 0.3927$

Note that $\lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0.$

i	x_i	$f(x_i)$
0	0.0000	0
1	0.3927	–0.02584
2	0.7854	–0.10501
3	1.1781	–0.24307
4	1.5708	–0.45158

$$\int_0^{\pi/2} \ln\left(\frac{\sin x}{x}\right)dx \approx \frac{\pi}{24}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)] \approx -0.2274$$

$$\int_0^{\pi/2} \ln(\sin x)dx \approx -1.0889$$

36. Suppose $X = \frac{(2n+1)\pi}{2}$. Since the cosine function alternates in sign and $\left|\frac{\cos x}{1+x^4}\right|$ is decreasing,

$$\left|\int_X^\infty \frac{\cos x}{1+x^4}dx\right| < \left|\int_X^{X+\pi} \frac{\cos x}{1+x^4}dx\right| < (X+\pi-X)\frac{1}{1+\left[\left(n+\frac{1}{2}\right)\pi\right]^4} = \frac{\pi}{1+\pi^4\left(n+\frac{1}{2}\right)^4}$$

$\frac{\pi}{1+\pi^4\left(n+\frac{1}{2}\right)^4} < 10^{-5}$ when $n \ge 8$. Let $X = \frac{17\pi}{2}$

$$\int_0^\infty \frac{\cos x}{1+x^4}dx = \int_0^{17\pi/2} \frac{\cos x}{1+x^4}dx + \int_{17\pi/2}^\infty \frac{\cos x}{1+x^4}dx$$

Use the Parabolic Rule on the first part with $h = \frac{\pi}{8}$, so $n = 68$.

Calculating with a computer,

$$\int_0^{17\pi/2} \frac{\cos x}{1+x^4}dx \approx \frac{\pi}{24}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_{67})+f(x_{68})] \approx 0.77335$$

Using a computer to plot $\left|f^{(4)}(x)\right|$, we see that $\left|f^{(4)}(x)\right| < 65$ on $\left[0, \frac{17\pi}{2}\right]$.

$$|E_{68}| < \frac{\left(\frac{17\pi}{2}\right)^5}{180 \cdot 68^4}65 < 0.22933$$

Hence, $|\text{Error}| < 0.22933 + 10^{-5} = 0.22934$

11.3 Concepts Review

1. slowness of convergence

2. root; Intermediate Value

3. algorithms

4. $x_1, j; f'(r) = 0$

Problem Set 11.3

1. Let $f(x) = x^3 + 2x - 6$.
$f(1) = -3, f(2) = 6$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	0.375
2	0.25	1.25	−1.546875
3	0.125	1.375	−0.650391
4	0.0625	1.4375	−0.154541
5	0.03125	1.46875	0.105927
6	0.015625	1.45312	−0.0253716
7	0.0078125	1.46094	0.04001
8	0.00390625	1.45703	0.00725670
9	0.00195312	1.45508	−0.00907617

$r \approx 1.46$

2. Let $f(x) = x^4 + 5x^3 + 1$.
$f(-1) = -3, f(0) = 1$

n	h_n	m_n	$f(m_n)$
1	0.5	−0.5	0.4375
2	0.25	−0.75	−0.792969
3	0.125	−0.625	−0.0681152
4	0.0625	−0.5625	0.21022
5	0.03125	−0.59375	0.0776834
6	0.015625	−0.609375	0.00647169
7	0.0078125	−0.617187	−0.0303962
8	0.00390625	−0.613281	−0.011854
9	0.00195312	−0.611328	−0.00266589

$r \approx -0.61$

3. Let $f(x) = 2\cos x - e^{-x}$.
$f(1) \approx 0.712725, f(2) \approx -0.967629$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	−0.0816558
2	0.25	1.25	0.34414
3	0.125	1.375	0.136256
4	0.0625	1.4375	0.0282831
5	0.03125	1.46875	−0.0264745
6	0.015625	1.45313	0.000961516
7	0.0078125	1.46094	−0.0127427
8	0.00390625	1.45703	−0.00588708
9	0.00195312	1.45508	−0.0024619
10	0.000976562	1.4541	−0.000749968
11	0.000488281	1.45361	0.00010583

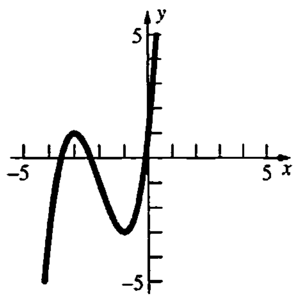
$r \approx 1.45$

4. Let $f(x) = x - 2 + 2\ln x$.
$f(1) = -1, f(2) \approx 1.38629$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	0.31093
2	0.25	1.25	−0.303713
3	0.125	1.375	0.0119075
4	0.0625	1.3125	−0.143633
5	0.03125	1.34375	−0.0653216
6	0.015625	1.35938	−0.0265749
7	0.0078125	1.36719	−0.00730108
8	0.00390625	1.37109	0.00231131
9	0.00195312	1.36914	−0.00249285

$r \approx 1.37$

5. Let $f(x) = x^3 + 6x^2 + 9x + 1 = 0$.

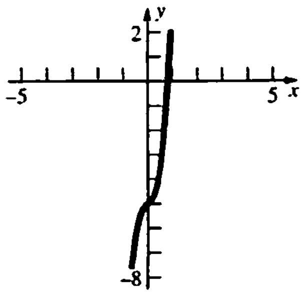


$f'(x) = 3x^2 + 12x + 9$

n	x_n
1	0
2	−0.1111111
3	−0.1205484
4	−0.1206148
5	−0.1206148

$r \approx -0.12061$

6. Let $f(x) = 7x^3 + x - 5$

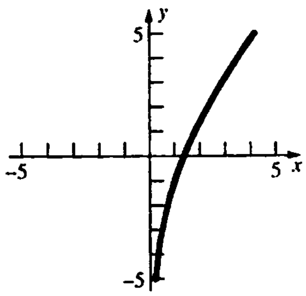


$f'(x) = 21x^2 + 1$

n	x_n
1	1
2	0.8636364
3	0.8412670
4	0.8406998
5	0.8406994
6	0.8406994

$r \approx 0.84070$

7. Let $f(x) = x - 2 + 2 \ln x$.

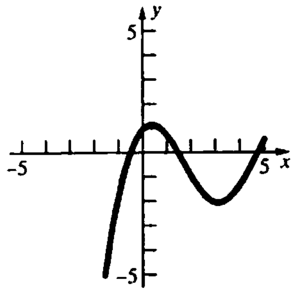


$f'(x) = 1 + \frac{2}{x}$

n	x_n
1	1.5
2	1.366744
3	1.370151
4	1.370154
5	1.370154

$r \approx 1.37015$

8. Let $f(x) = 2\cos x - e^{-x}$.

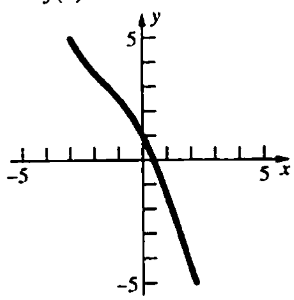


$f'(x) = -2\sin x + e^{-x}$

n	x_n
1	1.5
2	1.453915
3	1.453674
4	1.453674

$r \approx 1.45367$

9. Let $f(x) = \cos x - 2x$.

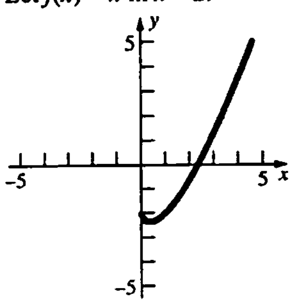


$f'(x) = -\sin x - 2$

n	x_n
1	0.5
2	0.4506267
3	0.4501836
4	0.4501836

$r \approx 0.45018$

10. Let $f(x) = x \ln x - 2$.

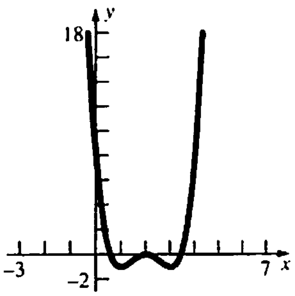


$f'(x) = \ln x + 1$

n	x_n
1	2.5
2	2.348287
3	2.345751
4	2.345751

$r \approx 2.34575$

11. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8.$



$f'(x) = 4x^3 - 24x^2 + 44x - 24$

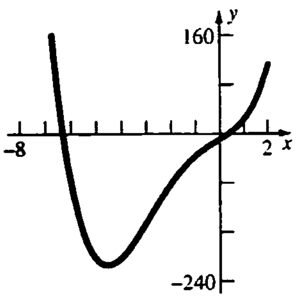
Note that $f(2) = 0$.

n	x_n
1	0.5
2	0.575
3	0.585586
4	0.585786

n	x_n
1	3.5
2	3.425
3	3.414414
4	3.414214
5	3.414214

$r = 2$, $r \approx 0.58579$, $r \approx 3.41421$

12. Let $f(x) = x^4 + 6x^3 + 2x^2 + 24x - 8.$



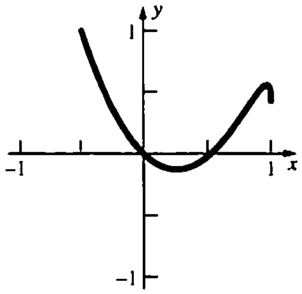
$f'(x) = 4x^3 + 18x^2 + 4x + 24$

n	x_n
1	–6.5
2	–6.3299632
3	–6.3167022
4	–6.3166248
5	–6.3166248

n	x_n
1	0.5
2	0.3286290
3	0.3166694
4	0.3166248
5	0.3166248

$r \approx -6.31662$, $r \approx 0.31662$

13. Let $f(x) = 2x^2 - \sin^{-1} x.$

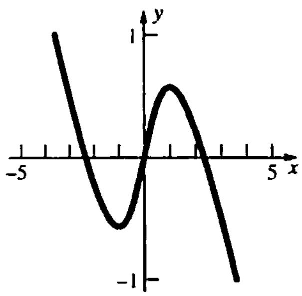


$f'(x) = 4x - \frac{1}{\sqrt{1-x^2}}$

n	x_n
1	0.5
2	0.527918
3	0.526583
4	0.526580
5	0.526580

$r \approx 0.52658$

14. Let $f(x) = 2\tan^{-1} x - x$.



$f'(x) = \frac{2}{1+x^2} - 1$

n	x_n
1	2.5
2	2.335087
3	2.331125
4	2.331122
5	2.331122

$r \approx 2.33112$

15. Let $f(x) = x^3 - 6$.

$f'(x) = 3x^2$

n	x_n
1	1.5
2	1.888889
3	1.819813
4	1.817125
5	1.817121
6	1.817121

$\sqrt[3]{6} \approx 1.81712$

16. Let $f(x) = x^4 - 47$.

$f'(x) = 4x^3$

n	x_n
1	2.5
2	2.627
3	2.618373
4	2.618330
5	2.618330

$\sqrt[4]{47} \approx 2.61833$

17. Let $g(x) = \frac{\sin x}{x}$.

$f(x) = g'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$

$f'(x) = -\frac{\sin x}{x} - \frac{2\cos x}{x^2} + \frac{2\sin x}{x^3}$

n	x_n
1	4.5
2	4.4934
3	4.49341
4	4.49341

Minimum at $x \approx 4.49341$
Minimum value is ≈ -0.21723.

18. Suppose $n = 1$, then $\frac{2m}{M}\left(\frac{M}{2m}|x_1 - r|\right)^{2^{n-1}} = |x_1 - r|$, so

$|x_1 - r| \le \frac{2m}{M}\left(\frac{M}{2m}|x_1 - r|\right)^{2^{1-1}}$.

Suppose the statement is true for $k \le n$.

$|x_{n+1} - r| \le \frac{M}{2m}(x_n - r)^2$

$\le \frac{M}{2m}\left[\frac{2m}{M}\left(\frac{M}{2m}|x_1 - r|\right)^{2^{n-1}}\right]^2$

$= \frac{2m}{M}\left(\frac{M}{2m}|x_1 - r|\right)^{2^n} = \frac{2m}{M}\left(\frac{M}{2m}|x_1 - r|\right)^{2^{n+1-1}}$

Thus, the statement is true for $n + 1$.

19. Let $f(x) = x^2 - 2$.

$f'(x) = 2x$, $f''(x) = 2$

$|f'(x)| \ge 2$ on $[1, 2]$, so $m = 2$.

$|f''(x)| \le 2$ on $[1, 2]$, so $M = 2$.

$|x_6 - \sqrt{2}| \le \frac{2(2)}{2}\left(\frac{2}{2(2)}|1.5 - r|\right)^{2^{6-1}}$

$\le 2(0.25)^{32} \approx 1.08 \times 10^{-19}$

20. From Problem 19, $m = 2$ and $M = 2$.

$|x_n - \sqrt{2}| \le \frac{2(2)}{2}\left(\frac{2}{2(2)}|1.5 - r|\right)^{2^{n-1}} \le 2(0.25)^{2^{n-1}}$

$2(0.25)^{2^{n-1}} \le 5 \times 10^{-41}$

$(0.25)^{2^{n-1}} \le 2.5 \times 10^{-41}$

$2^{n-1} \ge \frac{\log(2.5 \times 10^{-41})}{\log(0.25)}$

$n \ge \frac{\log\left[\frac{\log(2.5\times 10^{-41})}{\log(0.25)}\right]}{\log 2} + 1 \approx 7.08$

$n \ge 8$

21. Let $f(x) = \frac{1 + \ln x}{x}$.

$f'(x) = -\frac{\ln x}{x^2}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{x_n}{\ln x_n} + x_n$

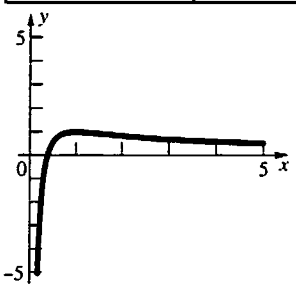
$= 2x_n + \frac{x_n}{\ln x_n}$

Suppose $x_1 = 1.2$.

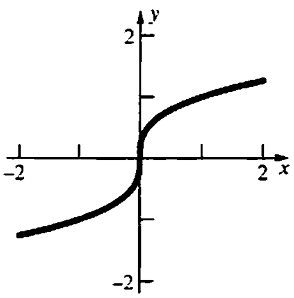
n	x_n
1	1.2
2	8.981778
3	22.05511
4	51.23963
5	115.4958
6	255.3103
7	556.6849
8	1201.425

Suppose $x_1 = 0.5$.

n	x_n
1	0.5
2	0.2786525
3	0.3392312
4	0.3646713
5	0.3678377
6	0.3678794
7	0.3678794



22.



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$$

$= x_n - 3x_n = -2x_n$

Thus, every iteration of Newton's Method gets further from zero. Note that $x_{n+1} = (-2)^{n+1}x_0$.

23. a. For Tom's car, $P = 2000$, $R = 100$, and $k = 24$, thus

$$2000 = \frac{100}{i}\left[1 - \frac{1}{(1+i)^{24}}\right] \text{ or } 20i = 1 - \frac{1}{(1+i)^{24}},$$

which is equivalent to

$20i(1+i)^{24} - (1+i)^{24} + 1 = 0$.

b. Let

$f(i) = 20i(1+i)^{24} - (1+i)^{24} + 1$

$= (1+i)^{24}(20i-1)+1$.

Then

$f'(i) = 20(1+i)^{24} + 480i(1+i)^{23} - 24(1+i)^{23}$

$= (1+i)^{23}(500i-4)$, so

$$i_{n+1} = i_n - \frac{f(i_n)}{f'(i_n)} = i_n - \frac{(1+i_n)^{24}(20i_n-1)+1}{(1+i_n)^{23}(500i_n-4)}$$

$$= i_n - \left[\frac{20i_n^2 + 19i_n - 1 + (1+i_n)^{-23}}{500i_n - 4}\right].$$

c.

n	i_n
1	0.012
2	0.0165297
3	0.0152651
4	0.0151323
5	0.0151308
6	0.0151308

$i = 0.0151308$

$r = 18.157\%$

24. From Newton's algorithm, $x_{n+1} - x_n = \frac{f(x_n)}{f'(x_n)}$.

$\lim_{x_n \to \bar{x}} (x_{n+1} - x_n) = \lim_{x_n \to \bar{x}} x_{n+1} - \lim_{x_n \to \bar{x}} x_n$

$= \bar{x} - \bar{x} = 0$

$\lim_{x_n \to \bar{x}} \frac{f(x_n)}{f'(x_n)}$ exists if f and f' are continuous at $\bar{x}$ and $f'(\bar{x}) \neq 0$.

Thus, $\lim_{x_n \to \bar{x}} \frac{f(x_n)}{f'(x_n)} = \frac{f(\bar{x})}{f'(\bar{x})} = 0$, so $f(\bar{x}) = 0$.

$\bar{x}$ is a solution of $f(x) = 0$.

25. a. The algorithm computes the root of $\frac{1}{x} - a = 0$ for x_1 close to $\frac{1}{a}$.

b. Let $f(x) = \frac{1}{x} - a$.

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(x)}{f'(x)} = -x + ax^2$$

The recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2x_n - ax_n^2.$$

26. $r \approx -1.25992, r \approx 1.25992$

27. $r \approx -1.87939, r \approx 0.34730, r \approx 1.53209$

28. $r \approx -0.28603, r \approx 1.03208, r \approx 1.08934,$ $r \approx 2.32816$

29. $r \approx -2.08204, r \approx 0.09251, r \approx 0.91314,$ $r \approx 1.62015, r \approx 1.85411$

30. $r = 1$

11.4 Concepts Review

1. fixed point

2. x_{n+1}

3. $|g'(x)| \le M < 1$

4. $|2x| > 1$ near $x = 2$

Problem Set 11.4

1. $x_{n+1} = \frac{1}{9}e^{-2x_n}$

n	x_n
1	1
2	0.015037
3	0.107819
4	0.089559
5	0.092890
6	0.092273
7	0.092387
8	0.092366
9	0.092370
10	0.092369
11	0.092369

$x \approx 0.09237$

2. $x_{n+1} = 2\tan^{-1} x_n$

n	x_n
1	2
2	2.214297
3	2.293208
4	2.319173
5	2.327392
6	2.329961
7	2.330761
8	2.331010
9	2.331087
10	2.331112
11	2.331119
12	2.331121
13	2.331122
14	2.331122

$x \approx 2.33112$

3. $x_{n+1} = \sqrt{2.7 + x_n}$

n	x_n
1	1
2	1.923538
3	2.150241
4	2.202326
5	2.214120
6	2.216781
7	2.217382
8	2.217517
9	2.217548
10	2.217554
11	2.217556
12	2.217556

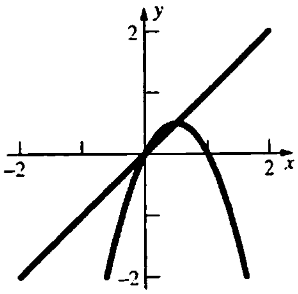
$x \approx 2.21756$

4. $x_{n+1} = \sqrt{3.2 + x_n}$

n	x_n
1	47
2	7.085196
3	3.207054
4	2.531216
5	2.393996
6	2.365163
7	2.359060
8	2.357766
9	2.357491
10	2.357433
11	2.357421
12	2.357418
13	2.357418

$x \approx 2.35742$

5. a.



$x \approx 0.5$

b. $x_{n+1} = 2(x_n - x_n^2)$

n	x_n
1	0.7
2	0.42
3	0.4872
4	0.4996723
5	0.4999998
6	0.5
7	0.5

c. $x = 2(x - x^2)$

$2x^2 - x = 0$

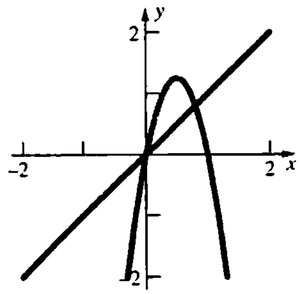
$x(2x - 1) = 0$

$x = 0, \; x = \frac{1}{2}$

d. $g'(x) = 2 - 4x$

$g'\left(\frac{1}{2}\right) = 0$

6. a.



$x \approx 0.8$

b. $x_{n+1} = 5(x_n - x_n^2)$

n	x_n
1	0.7
2	1.05
3	–0.2625
4	–1.657031
5	–22.01392
6	–2533.133

c. $x = 5(x - x^2)$

$5x^2 - 4x = 0$

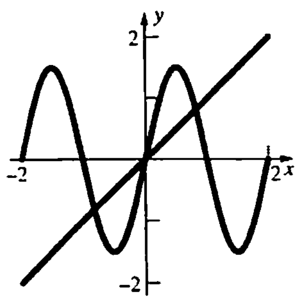
$x(5x - 4) = 0$

$x = 0, \; x = \frac{4}{5}$

d. $g'(x) = 5 - 10x$

$g'\left(\frac{4}{5}\right) = -3$

7. a.

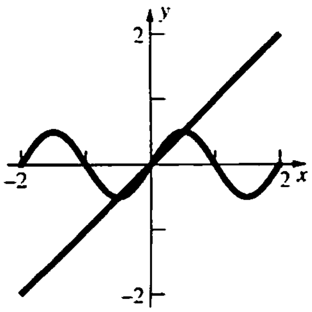


b. The algorithm does not yield a convergent sequence.

c. $g'(x) = \frac{3\pi}{2} \cos \pi x$

$|g'(x)| > 1$ in a neighborhood of the fixed points.

8. a.



b. The algorithm yields convergent sequences to 0.5 or –0.5.

c. $g'(x) = \frac{\pi}{2}\cos \pi x$

$|g'(x)| < 1$ in a neighborhood of the fixed points.

9. a. $x = \frac{3}{2}\sin \pi x$

$6x = 5x + \frac{3}{2}\sin \pi x$

$x = \frac{5}{6}x + \frac{1}{4}\sin \pi x$

b. Let $x_1 = 0.8$.

n	x_n
1	0.8
2	0.813613
3	0.816176
4	0.816629
5	0.816709
6	0.816723
7	0.816725
8	0.816726
9	0.816726

Let $x_1 = -0.8$.

n	x_n
1	-0.8
2	–0.813613
3	–0.816176
4	–0.816629
5	–0.816709
6	–0.816723
7	–0.816725
8	–0.816726
9	–0.816726

c. $g'(x) = \frac{5}{6} + \frac{\pi}{4}\cos \pi x$

$g'(0.81673) \approx 0.17456$

$g'(-0.81673) \approx 0.17456$

10. a. $x = 5(x - x^2)$

$6x = 10x - 5x^2$

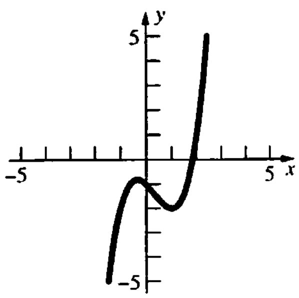
$x = \frac{5}{3}x - \frac{5}{6}x^2$

Let $g(x) = \frac{5}{3}x - \frac{5}{6}x^2$.

b. Let $x_1 = 0.75$.

n	x_n
1	0.75
2	0.78125
3	0.793457
4	0.797783
5	0.799257
6	0.799752
7	0.799917
8	0.799972
9	0.799991
10	0.799997
11	0.799999
12	0.8
13	0.8

11. Graph $y = x^3 - x^2 - x - 1$.



The positive root is near 2.

Rewrite the equation as $x = 1 + \frac{1}{x} + \frac{1}{x^2} = g(x)$.

Let $x_1 = 1.8$

n	x_n
1	1.8
2	1.8642
3	1.8242
4	1.8487
5	1.8335
6	1.8429
7	1.8371
8	1.8406
9	1.8384
10	1.8398
11	1.8390
12	1.8395
13	1.8392
14	1.8394
15	1.8392
16	1.8393
17	1.8393

$r \approx 1.839$

12. a. $x_1 = 0$

$x_2 = \sqrt{5} \approx 2.236068$

$x_3 = \sqrt{5+\sqrt{5}} \approx 2.689994$

$x_4 = \sqrt{5+\sqrt{5+\sqrt{5}}} \approx 2.7730839$

$x_5 = \sqrt{5+\sqrt{5+\sqrt{5+\sqrt{5}}}} \approx 2.7880251$

b. $x = \sqrt{5+x}$, and x must satisfy $x \ge 0$

$x^2 = 5 + x$

$x^2 - x - 5 = 0$

$$x = \frac{1 \pm \sqrt{1+4\cdot 1\cdot 5}}{2} = \frac{1 \pm \sqrt{21}}{2}$$

Taking the minus sign gives a negative solution for x, violating the requirement that $x \ge 0$. Hence, $x = \left(1+\sqrt{21}\right)/2 \approx 2.7912878$

c. Let $x = \sqrt{5+\sqrt{5+\sqrt{5+\ldots}}}$. Then x satisfies the equation $x = \sqrt{5+x}$. From part b we know that x must equal $\left(1+\sqrt{21}\right)/2 \approx 2.7912878$

13. a. $x_1 = 0$

$x_2 = \sqrt{1} = 1$

$x_3 = \sqrt{1+\sqrt{1}} = \sqrt{2} \approx 1.4142136$

$x_4 = \sqrt{1+\sqrt{1+\sqrt{1}}} \approx 1.553774$

$x_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} \approx 1.5980532$

b. $x = \sqrt{1+x}$

$x^2 = 1 + x$

$x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1+4\cdot 1\cdot 1}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Taking the minus sign gives a negative solution for x, violating the requirement that $x \ge 0$.

Hence, $x = \dfrac{1+\sqrt{5}}{2} \approx 1.618034$.

c. Let $x = \sqrt{5+\sqrt{5+\sqrt{5+\ldots}}}$. Then x satisfies the equation $x = \sqrt{5+x}$. From part b we know that x must equal $\left(1+\sqrt{5}\right)/2 \approx 1.618034$.

14. a. $x_1 = 1$

$x_2 = 1 + \frac{1}{1} = 2$

$x_3 = 1 + \frac{1}{1+\frac{1}{1}} = \frac{3}{2} = 1.5$

$x_4 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{5}{3} \approx 1.6666667$

$x_5 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = \frac{8}{5} = 1.6$

b. $x = 1 + \frac{1}{x}$

$x^2 = x + 1$

$x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1+4\cdot 1\cdot 1}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Taking the minus sign gives a negative solution for x, violating the requirement that $x \ge 0$.

Hence, $x = \dfrac{1+\sqrt{5}}{2} \approx 1.618034$.

c. Let

$$x = 1 + \frac{1}{1+\frac{1}{1+\cdots}}.$$

Then x satisfies the equation

$x = 1 + \frac{1}{x}$. From part b we know that

x must equal

$\left(1+\sqrt{5}\right)/2 \approx 1.618034$.

15. Let $a = \pi$ and $x_1 = 2$.

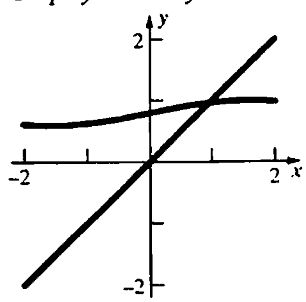
n	x_n
1	2
2	1.785398
3	1.772501
4	1.772454
5	1.772454

$\sqrt{\pi} \approx 1.77245$

$$g'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$$

$$g'(a) = \frac{1}{2}\left(1 - \frac{1}{a}\right) < 1 \text{ for } a > 0.$$

16. Graph $y = x$ and $y = 0.8 + 0.2 \sin x$.



$x_{n+1} = 0.8 + 0.2\sin x_n$

Let $x_1 = 1$.

n	x_n
1	1
2	0.96829
3	0.96478
4	0.96439
5	0.96434
6	0.96433
7	0.96433

$x \approx 0.9643$

17. a. $10,000 = \frac{R}{0.015}[1-(1.015)^{-48}]$

$$R = \frac{150}{1-(1.015)^{-48}} \approx 293.75$$

b. $10,000 = \frac{300}{i}[1-(1+i)^{-48}]$

$$i = \frac{3}{100}[1-(1+i)^{-48}]$$

Let $i_1 = 0.015$.

n	i_n
1	0.015
2	0.015319
3	0.015539
4	0.015689
5	0.015789
6	0.015857
7	0.015902
8	0.015932
9	0.015952
10	0.15965
11	0.015974
12	0.015980
13	0.015983
14	0.015986
15	0.015988
16	0.015989
17	0.015989

$i \approx 0.01599$

18. $500 = \frac{30}{i}[1-(1+i)^{-24}]$

$$i = \frac{3}{50}[1-(1+i)^{-24}]$$

Let $i_1 = 0.03$.

n	i_n
1	0.03
2	0.030484
3	0.030815
4	0.031039
5	0.031190
6	0.031290
7	0.031358
8	0.031403
9	0.031432
10	0.031452
11	0.031465
12	0.031474
13	0.031480
14	0.031484
15	0.031486

16	0.031488
17	0.031489
18	0.031490
19	0.031490

$i \approx 0.03149$

19. a. Suppose r is a root. Then

$$r = r - \frac{f(r)}{f'(r)}.$$

$$\frac{f(r)}{f'(r)} = 0, \text{ so } f(r) = 0.$$

Suppose $f(r) = 0$. Then

$$r - \frac{f(r)}{f'(r)} = r - 0 = r, \text{ so } r \text{ is a root of}$$

$$x = x - \frac{f(x)}{f'(x)}.$$

b. If we want to solve $f(x) = 0$ and $f'(x) \neq 0$ in $[a, b]$, then $\frac{f(x)}{f'(x)} = 0$ or

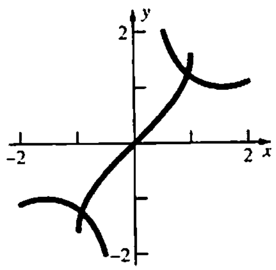
$$x = x - \frac{f(x)}{f'(x)} = g(x).$$

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)}{[f'(x)]^2} f''(x)$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

and $g'(r) = \frac{f(r)f''(r)}{[f'(r)]^2} = 0.$

20. a.



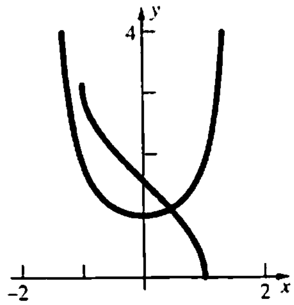
$$f(x) = \sin^{-1} x - \frac{1}{\sin x}$$

Use Newton's Method.

n	x_n
1	0.9
2	0.947423
3	0.944075
4	0.944039
5	0.944039

$x \approx 0.94404$

b.



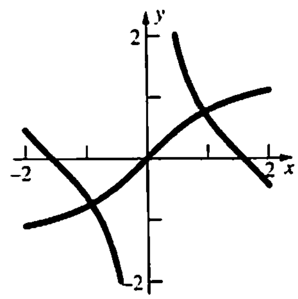
$$f(x) = \cos^{-1} x - \frac{1}{\cos x}$$

Use Newton's Method.

n	x_n
1	0.4
2	0.447464
3	0.446049
4	0.446048
5	0.446048

$x \approx 0.44605$

c.



$$f(x) = \tan^{-1} x - \frac{1}{\tan x}$$

Use Newton's Method.

n	x_n
1	0.9
2	0.9278324
3	0.9283946
4	0.9283949
5	0.9283949

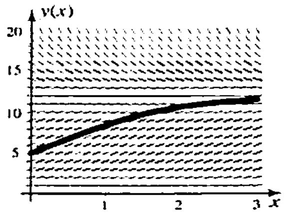
$x \approx 0.92839$

11.5 Concepts Review

1. slope field
2. tangent line, or Taylor polynomial of order 1 based at x_0.
3. $y_{n-1} + hf(x_{n-1}, y_{n-1})$
4. average

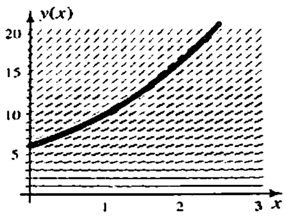
Problem Set 11.5

1.



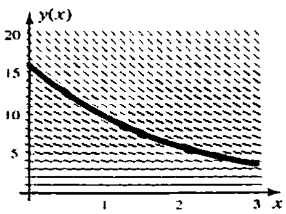
$\lim_{t\to\infty} y(t) = 12$ and $y(2) \approx 10.5$

2.



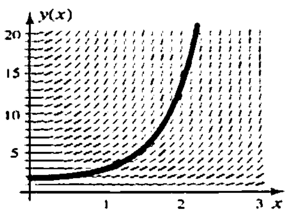
$\lim_{t\to\infty} y(t) = \infty$ and $y(2) \approx 16$

3.



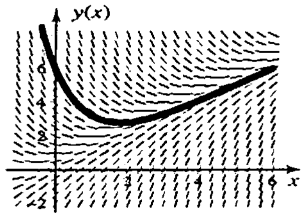
$\lim_{t\to\infty} y(t) = 0$ and $y(2) \approx 6$

4.



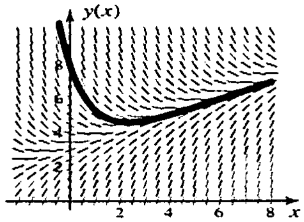
$\lim_{t\to\infty} y(t) = \infty$ and $y(2) \approx 13$

5.



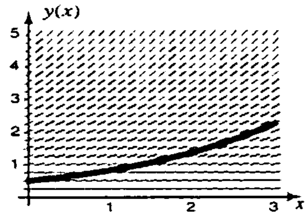
The oblique asymptote is $y = x$.

6.



The oblique asymptote is $y = 3 + x/2$.

7.



$\frac{dy}{dx} = \frac{1}{2}y; \quad y(0) = \frac{1}{2}$

$\frac{dy}{y} = \frac{1}{2}dx$

$\ln y = \frac{x}{2} + C$

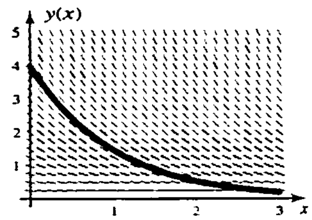
$y = C_1 e^{x/2}$

To find C_1, apply the initial condition:

$\frac{1}{2} = y(0) = C_1 e^0 = C_1$

$y = \frac{1}{2}e^{x/2}$

8.



$\frac{dy}{dx} = -y; \qquad y(0) = 4$

$\frac{dy}{y} = -dx$

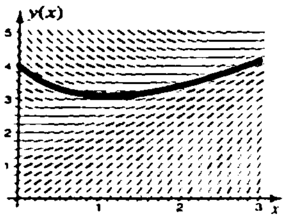
$\ln y = -x + C$

$y = C_1 e^{-x}$

To find C_1, apply the initial condition:

$4 = y(0) = C_1 e^{-0} = C_1$

$y = 4e^{-x}$

9. 

$y' + y = x + 2$

The integrating factor is $e^{\int 1 dx} = e^x$.

$e^x y' + ye^x = e^x(x+2)$

$\frac{d}{dx}\left(e^x y\right) = (x+2)e^x$

$e^x y = \int (x+2)e^x \, dx$

Integrate by parts: let $u = x+2,\ dv = e^x \, dx$.

Then $du = dx$ and $v = e^x$. Thus

$e^x y = (x+2)e^x - \int e^x \, dx$

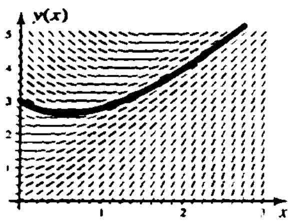
$e^x y = (x+2)e^x - e^x + C$

$y = x + 2 - 1 + Ce^{-x}$

To find C, apply the initial condition:

$4 = y(0) = 0 + 1 + Ce^{-0} = 1 + C$

Thus, $y = x + 1 + e^{-x}$.

10. 

$y' + y = 2x + \frac{3}{2}$

$e^x y' + ye^x = \left(2x + \frac{3}{2}\right)e^x$

$\frac{d}{dx}\left(e^x y\right) = \left(2x + \frac{3}{2}\right)e^x$

$e^x y = \int \left(2x + \frac{3}{2}\right)e^x \, dx$

Integrate by parts: let $u = 2x + \frac{3}{2}$,

$dv = e^x \, dx$. Then $du = 2dx$ and $v = e^x$.

Thus,

$e^x y = \left(2x + \frac{3}{2}\right)e^x - \int 2e^x \, dx$

$e^x y = \left(2x + \frac{3}{2}\right)e^x - 2e^x + C$

$y = 2x - \frac{1}{2} + Ce^{-x}$

To find C, apply the initial condition:

$3 = y(0) = 0 - \frac{1}{2} + Ce^{-0} = C - \frac{1}{2}$

Thus $C = \frac{7}{2}$, so the solution is

$y = 2x - \frac{1}{2} + \frac{7}{2}e^{-x}$

Note: Solutions to Problems 17-26 are given along with the corresponding solutions to 11-16.

11., 17.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	3.0	3.0
0.2	4.2	4.44
0.4	5.88	6.5712
0.6	8.232	9.72538
0.8	11.5248	14.39356
1.0	16.1347	21.30246

12., 18.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	2.0	2.0
0.2	1.6	1.64
0.4	1.28	1.3448
0.6	1.024	1.10274
0.8	0.8195	0.90424
1.0	0.65536	0.74148

13., 19.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	0.0	0.0
0.2	0.0	0.2
0.4	0.04	0.8
0.6	0.12	0.18
0.8	0.24	0.032
1.0	0.40	0.05

14., 20.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	0.0	0.0
0.2	0.0	0.004
0.4	0.008	0.024
0.6	0.040	0.076
0.8	0.112	0.176
1.0	0.240	0.340

15., 21.

x_n	Euler's Method y_n	Improved Euler Method y_n
1.0	1.0	1.0
1.2	1.2	1.244
1.4	1.488	1.60924
1.6	1.90464	2.16410
1.8	2.51412	3.02455
2.0	3.41921	4.391765

16., 22.

x_n	Euler's Method y_n	Improved Euler Method y_n
1.0	2.0	2.0
1.2	1.2	1.312
1.4	0.624	0.80609
1.6	0.27456	0.46689
1.8	0.09884	0.25698
2.0	0.02768	0.13568

23.

h	Error from Euler's Method	Error from Improved Euler Method
0.2	0.229962	0.015574
0.1	0.124539	0.004201
0.05	0.064984	0.001091
0.01	0.013468	0.000045
0.005	0.006765	0.000011

For Euler's method, the error is halved as the step size h is halved. Thus, the error is proportional to h. For the improved Euler method, when h is halved, the error decreases to approximately one-fourth of what is was. Hence, for the improved Euler method, the error is proportional to h^2.

24.

$h = 0.2$		
x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	1.0	1.0
0.2	1.0	1.0
0.4	0.95946	0.96028
0.6	0.87833	0.88251
0.8	0.75815	0.77002
1.0	0.60202	0.62778
1.2	0.41450	0.46269
1.4	0.20127	0.28589
$h = 0.1$		
0.0	1.0	1.0
0.1	1.0	1.0
0.2	0.98997	0.99002
0.3	0.96990	0.97015
0.4	0.93990	0.94061
0.5	0.90016	0.90168
0.6	0.85098	0.85376
0.7	0.79276	0.79735
0.8	0.72599	0.73302
0.9	0.65124	0.66143
1.0	0.56917	0.58333
1.1	0.48053	0.49956
1.2	0.38612	0.41105
1.3	0.28680	0.31892
1.4	0.18349	0.22473
1.5	0.07711	0.13221
$h = 0.05$		
0.0	1.0	1.0
0.05	1.0	1.0
0.10	0.99750	0.99750
0.15	0.99249	0.99251
0.20	0.98499	0.98504
0.25	0.97501	0.97510
0.30	0.96256	0.96273
0.35	0.94767	0.94796
0.40	0.93038	0.93082
0.45	0.91071	0.91135
0.50	0.88871	0.88960

0.55	0.86444	0.86563
0.60	0.83794	0.83950
0.65	0.80928	0.81128
0.70	0.77851	0.78103
0.75	0.74573	0.74883
0.80	0.71099	0.71476
0.85	0.67439	0.67891
0.90	0.63600	0.64137
0.95	0.59593	0.60223
1.00	0.55426	0.56159
1.05	0.51110	0.51957
1.10	0.46655	0.47625
1.15	0.42072	0.43176
1.20	0.37371	0.38622
1.25	0.32565	0.33974
1.30	0.276650	0.29247
1.35	0.22682	0.24453
1.40	0.17630	0.19613
1.45	0.12519	0.14751
1.50	0.07362	0.09927
1.55	0.02171	0.05395

For this example, Euler's method seems to be more accurate than the improved Euler method.

25. **a.** $y_0 = 1$

$y_1 = y_0 + hf(x_0, y_0)$
$= y_0 + hy_0 = (1+h)y_0$

$y_2 = y_1 + hf(x_1, y_1) = y_1 + hy_1$
$= (1+h)y_1 = (1+h)^2 y_0$

$y_3 = y_2 + hf(x_2, y_2) = y_2 + hy_2$
$= (1+h)y_2 = (1+h)^3 y_0$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) = y_{n-1} + hy_{n-1}$
$= (1+h)y_{n-1} = (1+h)^n y_0$

b. Let $N = 1/h$. Then y_N is an approximation to the solution at $x = Nh = (1/h)h = 1$. The exact solution is $y(1) = e$. Thus, $(1+1/N)^N \approx e$ for large N. From Chapter 7, we know that $\lim_{N\to\infty} (1+1/N)^N = e$.

26. $y_0 = y(x_0) = 0$

$y_1 = y_0 + hf(x_0) = 0 + hf(x_0) = hf(x_0)$

$y_2 = y_1 + hf(x_1) = hf(x_0) + hf(x_1)$
$= h(f(x_0) + f(x_1))$

$y_3 = y_2 + hf(x_2)$
$= h[f(x_0) + f(x_1)] + hf(x_2)$
$= h[f(x_0) + f(x_1) + f(x_2)] = h\sum_{i=0}^{3-1} f(x_i)$

At the nth step of Euler's method,

$$y_n = y_{n-1} + hf(x_{n-1}) = h\sum_{i=0}^{n-1} f(x_i)$$

27. **a.** $\int_{x_0}^{x_1} y'(x)dx = \int_{x_0}^{x_1} \sin x^2 dx$

$y(x_1) - y(x_0) \approx (x_1 - x_0)\sin x_0^2$
$y(x_1) - y(0) = h \sin x_0^2$
$y(x_1) - 0 \approx 0.1 \sin 0^2$
$y(x_1) \approx 0$

b. $\int_{x_0}^{x_2} y'(x)dx = \int_{x_0}^{x_2} \sin x^2 dx$

$y(x_2) - y(x_0) \approx (x_1 - x_0)\sin x_0^2 + (x_2 - x_1)\sin x_1^2$
$y(x_2) - y(0) = h \sin x_0^2 + h \sin x_1^2$
$y(x_2) - 0 \approx 0.1 \sin 0^2 + 0.1 \sin 0.1^2$
$y(x_2) \approx 0.00099998$

c. $\int_{x_0}^{x_3} y'(x)dx = \int_{x_0}^{x_3} \sin x^2 dx$

$y(x_3) - y(x_0) \approx (x_1 - x_0)\sin x_0^2 + (x_2 - x_1)\sin x_1^2 + (x_3 - x_2)\sin x_1^2$
$y(x_3) - y(0) = h \sin x_0^2 + h \sin x_1^2 + h \sin x_2^2$
$y(x_3) - 0 \approx 0.1 \sin 0^2 + 0.1 \sin 0.1^2 + 0.1 \sin 0.2^2$
$y(x_3) \approx 0.004999$

Continuing in this fashion, we have

$\int_{x_0}^{x_n} y'(x)dx = \int_{x_0}^{x_n} \sin x^2 dx$

$y(x_n) - y(x_0) \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i)\sin x_i^2$

$y(x_n) \approx h\sum_{i=0}^{n-1} f(x_{i-1})$

When $n = 10$, this becomes
$y(x_{10}) = y(1) \approx 0.269097$

d. The result $y(x_n) \approx h\sum_{i=0}^{n-1} f(x_{i-1})$ is the same as that given in Problem 26. Thus, when $f(x, y)$ depends only on x, then the two methods (1) Euler's method for approximating the solution to $y' = f(x)$ at x_n, and (2) the left-endpoint Riemann sum for approximating $\int_0^{x_n} f(x)\,dx$, are equivalent.

28. a. $\int_{x_0}^{x_1} y'(x)dx = \int_{x_0}^{x_1} \sqrt{x+1}\,dx$

$y(x_1) - y(x_0) \approx (x_1 - x_0)\sqrt{x_0+1}$

$y(x_1) - y(0) = h\sqrt{x_0+1}$

$y(x_1) - 0 \approx 0.1\sqrt{0+1}$

$y(x_1) \approx 0.1$

b. $\int_{x_0}^{x_2} y'(x)dx = \int_{x_0}^{x_2} \sqrt{x+1}\,dx$

$y(x_2) - y(x_0) \approx (x_1 - x_0)\sqrt{x_0+1} + (x_2 - x_1)\sqrt{x_1+1}$

$y(x_2) - y(0) = h\sqrt{x_0+1} + h\sqrt{x_1+1}$

$y(x_2) - 0 \approx 0.1\sqrt{0+1} + 0.1\sqrt{0.1+1}$

$y(x_2) \approx 0.204881$

c. $\int_{x_0}^{x_3} y'(x)dx = \int_{x_0}^{x_3} \sqrt{x+1}\,dx$

$y(x_3) - y(x_0) \approx (x_1 - x_0)\sqrt{x_0+1} + (x_2 - x_1)\sqrt{x_1+1} + (x_3 - x_2)\sqrt{x_2+1}$

$y(x_3) - y(0) = 0.1\sqrt{0+1} + 0.1\sqrt{0.1+1} + 0.1\sqrt{0.2+1}$

$y(x_3) \approx 0.314425$

Continuing in this fashion, we have

$\int_{x_0}^{x_n} y'(x)dx = \int_{x_0}^{x_n} \sqrt{x+1}\,dx$

$y(x_n) - y(x_0) \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i)\sqrt{x_{i-1}+1}$

$y(x_n) \approx h\sum_{i=0}^{n-1} \sqrt{x_{i-1}+1}$

When $n = 10$, this becomes

$y(x_{10}) = y(1) \approx 1.198119$

11.6 Chapter Review

Concepts Test

1. True: $P(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

2. True: If $p(x)$ and $q(x)$ are polynomials of degree less than or equal to n, satisfying $p(a) = q(a) = f(a)$ and $p^{(k)}(a) = q^{(k)}(a) = f^{(k)}(a)$ for $k \le n$, then $p(x) = q(x)$.

3. True: $f(0) = f'(0) = f''(0) = 0$, its second order Maclaurin polynomial is 0.

4. True: After simplifying, $P_3(x) = f(x)$.

5. True: Any Maclaurin polynomial for $\cos x$ involves only even powers of x.

6. True: The Maclaurin polynomial of an even function involves only even powers of x, so $f'(0) = 0$ if $f(x)$ is an even function.

7. True: Taylor's Formula with Remainder for $n = 0$ is $f(x) = f(a) + f'(c)(x-a)$ which is equivalent to the Mean Value Theorem.

8. False: A calculator can be limited to a certain number of significant digits.

9. False: For example $\int \frac{\sin x}{x}dx$ cannot be expressed in terms of elementary functions.

10. False: $E_{10} = -\frac{5^3}{12 \cdot 10^2}6c < 0$, so the Trapezoidal Rule will give a value greater than the true value.

11. True: $E_{10} = 0$ since the fourth derivative of x^3 is 0.

12. False: A computer can be limited.

13. True: $\left|e^{-x^2} + x^2 + \sin(x+1)\right| \le \left|e^{-x^2}\right| + \left|x^2\right| + \left|\sin(x+1)\right| \le 1 + 4 + 1 = 6$

14. True: This is the Parabolic Rule for $n = 2$. Since $f^{(4)}(x) = 0$, $E_2 = 0$, so the rule is exact.

15. True: Intermediate Value Theorem

16. False: See Example 1 of Section 10.4.

17. False: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$. (See Problem 22 of Problem Set 10.4.)

18. True: If $f'(x) > 1$, the Fixed Point Theorem fails.

19. False: $g(x) = 5(x - x^2) + 0.01$, $g'(x) = 5 - 10x$; $|g'(x)| > 1$ in a neighborhood of the fixed point.

20. True: Note that the fixed point is $\sqrt{a}$.

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right);$$

$$|g'(x)| = \left|\frac{1}{2}\left(1 - \frac{a}{x^2}\right)\right| < 1 \text{ when}$$

$$\sqrt{\frac{a}{3}} < x.$$

21. True: At (2,1) the slope is $y' = 2 \cdot 1 = 2$

22. False: $y' = 2y$, $y'' = 2 > 0$. Thus, the solution is concave up. The estimate from Euler's method will underestimate the solution.

Sample Test Problems

1. $f(0) = 0$

$f'(x) = \cos^2 x - 2x^2 \sin^2 x$

$f'(0) = 1$

$p(x) = x$; $p(0.2) = 0.2$; $f(0.2) = 0.1998$

2. a. $f(x) = xe^x$ $\quad f(0) = 0$

$f'(x) = e^x + xe^x$ $\quad f'(0) = 1$

$f''(x) = 2e^x + xe^x$ $\quad f''(0) = 2$

$f^{(3)}(x) = 3e^x + xe^x$ $\quad f^{(3)}(0) = 3$

$f^{(4)}(x) = 4e^x + xe^x$ $\quad f^{(4)}(0) = 4$

$$f(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

$f(0.1) \approx 0.11052$

b. $f(x) = \cosh x$ $\quad f(0) = 1$

$f'(x) = \sinh x$ $\quad f'(0) = 0$

$f''(x) = \cosh x$ $\quad f''(0) = 1$

$f^{(3)}(x) = \sinh x$ $\quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cosh x$ $\quad f^{(4)}(0) = 1$

$$f(x) \approx 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$f(0.1) \approx 1.0050042$

3. $g(x) = x^3 - 2x^2 + 5x - 7$ $\quad g(2) = 3$

$g'(x) = 3x^2 - 4x + 5$ $\quad g'(2) = 9$

$g''(x) = 6x - 4$ $\quad g''(2) = 8$

$g^{(3)}(x) = 6$ $\quad g^{(3)}(2) = 6$

Since $g^{(4)}(x) = 0$, $R_3(x) = 0$, so the Taylor polynomial of order 3 based at 2 is an exact representation.

$$g(x) = P_4(x) = 3 + 9(x-2) + 4(x-2)^2 + (x-2)^3$$

4. $g(2.1) = 3 + 9(0.1) + 4(0.1)^2 + (0.1)^3 = 3.941$

5. $f(x) = \frac{1}{x+1}$ $\quad f(1) = \frac{1}{2}$

$f'(x) = -\frac{1}{(x+1)^2}$ $\quad f'(1) = -\frac{1}{4}$

$f''(x) = \frac{2}{(x+1)^3}$ $\quad f''(1) = \frac{1}{4}$

$f^{(3)}(x) = -\frac{6}{(x+1)^4}$ $\quad f^{(3)}(1) = -\frac{3}{8}$

$f^{(4)}(x) = \frac{24}{(x+1)^5}$ $\quad f^{(4)}(1) = \frac{3}{4}$

$$f(x) \approx \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4$$

6. $f^{(5)}(x) = -\frac{120}{(x+1)^6}$, so $R_4(x) = -\frac{(x-1)^5}{(c+1)^6}$.

$$|R_4(1.2)| = \frac{(0.2)^5}{(c+1)^6} \le \frac{(0.2)^5}{(2)^6} = 0.000005$$

7. $f(x) = \frac{1}{2}(1 - \cos 2x)$ $\quad f(0) = 0$

$f'(x) = \sin 2x$ $\quad f'(0) = 0$

$f''(x) = 2\cos 2x$ $\quad f''(0) = 2$

$f^{(3)}(x) = -4\sin 2x$ $\quad f^{(3)}(0) = 0$

$f^{(4)}(x) = -8\cos 2x$ $\quad f^{(4)}(0) = -8$

$f^{(5)}(x) = 16\sin 2x$ $\quad f^{(5)}(0) = 0$

$f^{(6)}(x) = 32\cos 2x$ $\quad f^{(6)}(c) = 32\cos 2c$

$$\sin^2 x \approx \frac{2}{2!}x^2 - \frac{8}{4!}x^4 = x^2 - \frac{1}{3}x^4$$

$|R_4(x)| = |R_5(x)| = \left|\frac{32}{6!}(\cos 2c)x^6\right| \le \frac{2}{45}(0.2)^6$

$< 2.85\times 10^{-6}$

8. $f^{(n+1)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

$|R_n(x)| = \left|\frac{(-1)^n}{(n+1)c^{n+1}}(x-1)^{n+1}\right|$

$\le \frac{0.2^{n+1}}{(n+1)0.8^{n+1}} = \frac{(0.25)^{n+1}}{(n+1)}$

$\frac{(0.25)^{n+1}}{(n+1)} < 0.00005$ when $n \ge 5$.

9. From Problem 8,

$$\ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5.$$

$$|R_5(x)| = \left|\frac{1}{6c^6}(x-1)^6\right| \le \frac{0.2^6}{6\cdot 0.8^6} < 4.07\times 10^{-5}$$

$$\int_{0.8}^{1.2} \ln x\,dx \approx \int_{0.8}^{1.2}\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5\right]dx$$

$$= \left[\frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{30}(x-1)^6\right]_{0.8}^{1.2}$$

≈ -0.00269867

Error $\le (1.2-0.8)4.07\times 10^{-5} < 1.63\times 10^{-5}$

10. $f(x) = \ln x$, $h = 0.05$

n	x_n	$f(x_n)$
0	0.80	–0.22314
1	0.85	–0.16252
2	0.90	–0.10536
3	0.95	–0.051293
4	1.00	0
5	1.05	0.04879
6	1.10	0.09531
7	1.15	0.13976
8	1.20	0.18232

$$\int_{0.8}^{1.2} \ln x\,dx \approx \frac{0.05}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_7) + f(x_8)]$$

≈ -0.00278607

$$|E_8| = \left|\frac{0.4^3}{12\cdot 8^2}\frac{1}{c^2}\right| \le \frac{0.4^3}{12\cdot 8^2\cdot 0.8^2} < 1.31\times 10^{-4}$$

11. $f(x) = \ln x$, $h = 0.05$

n	x_n	$f(x_n)$
0	0.80	–0.22314
1	0.85	–0.16252
2	0.90	–0.10536
3	0.95	–0.051293
4	1.00	0
5	1.05	0.04879
6	1.10	0.09531
7	1.15	0.13976
8	1.20	0.18232

$$\int_{0.8}^{1.2} \ln x\,dx \approx \frac{0.05}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_7) + f(x_8)]$$

≈ -0.00269939

$$|E_8| = \left|\frac{0.4^5}{180\cdot 8^4}\frac{6}{c^4}\right| \le \frac{0.4^5\cdot 6}{180\cdot 8^4\cdot 0.8^4} < 2.04\times 10^{-7}$$

12. $\int_{0.8}^{1.2} \ln x\,dx = [x\ln x - x]_{0.8}^{1.2} \approx -0.00269929$

13. $f(x) = 3x - \cos 2x$, $f'(x) = 3 + 2\sin 2x$

Let $x_1 = 0.5$.

n	x_n
1	0.5
2	0.2950652
3	0.2818563
4	0.2817846
5	0.2817846

$x \approx 0.281785$

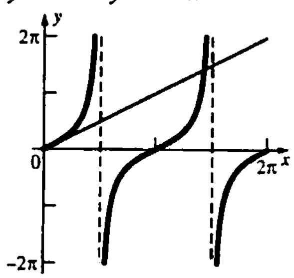
14. $x_{n+1} = \frac{\cos 2x_n}{3}$

n	x_n
1	0.5
2	0.18010
3	0.311942
4	0.270539
5	0.285718

6	0.280375
7	0.282285
8	0.281606
9	0.281848
10	0.281762
11	0.281793
12	0.281782
13	0.281786
14	0.281784
15	0.281785
16	0.281785

$x \approx 0.2818$

15. $y = x$ and $y = \tan x$



Let $x_1 = \frac{11\pi}{8}$.

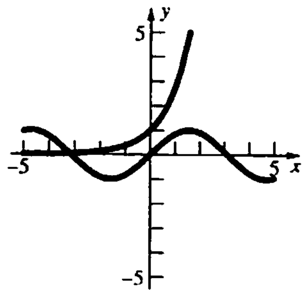
$f(x) = x - \tan x,\ f'(x) = 1 - \sec^2 x$.

n	x_n
1	$\frac{11\pi}{8}$
2	4.64661795
3	4.60091050
4	4.54662258
5	4.50658016
6	4.49422443
7	4.49341259
8	4.49340946

$x \approx 4.4934$

16. The Fixed-Point Method does not work because if $g(x) = \tan x,\ g'(x) = \sec^2 x$ and $|g'(x)| > 1$ in a neighborhood of the root.

17. Graph $y = e^x$ and $y = \sin x$



Let $x_1 = -3$.

$f(x) = e^x - \sin x, f'(x) = e^x - \cos x$

$$x_{n+1} = x_n - \frac{e^{x_n} - \sin x_n}{e^{x_n} - \cos x_n}$$

n	x_n
1	-3
2	-3.183603
3	-3.183063
4	-3.183063

$x \approx -3.18306$

18. Euler's Method:

x_n	y_n
1	2
1.2	2.4
1.4	2.976
1.6	3.80928
1.8	5.02825
2.0	6.83842

19. Improved Euler Method

x_n	y_n
0	2
0.2	3.56
0.4	6.3368
0.6	11.2795
0.8	20.07752
1.0	35.73798
1.2	63.61361
1.4	113.2322
1.6	201.5533
1.8	358.7650
2.0	638.6016

11.7 Additional Problem Set

1. Using a computer, the table is as follows:

i	(x_i, y_i)
0	$(\pi, 1)$
1	(3.4558, 0.9511)
2	(3.7699, 0.809)
3	(4.0841, 0.5878)
4	(4.3982, 0.309)
5	(4.7124, 0)
6	(5.0265, –0.309)
7	(5.3407, –0.5878)
8	(5.6549, –0.809)
9	(5.9690, –0.9511)
10	(6.2832, –1)

2. a. $y_{n+1} = y_n + 0.1(x_n + y_n) = 0.1x_n + 1.1y_n$;
$x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.1
2	0.2	1.22
3	0.3	1.362
4	0.4	1.5282

b. $y_{n+1} = y_n + 0.25(y_n) = 1.25y_n$;
$x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.25	1.25
2	0.5	1.5625
3	0.75	1.953125
4	1.00	2.44140625

c. $y_{n+1} = y_n + 0.1(y_n) = 1.1y_n$; $x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.1
2	0.2	1.21
3	0.3	1.331
4	0.4	1.4641
5	0.5	1.61051
6	0.6	1.771561
7	0.7	1.9487171
8	0.8	2.14358881
9	0.9	2.357947691
10	1.0	2.5937424601

3. a. $y_{n+1}^{\text{predicted}} = y_n + 0.25(y_n) = 1.25y_n$

$y_{n+1} = y_n + \frac{0.25}{2}(y_n + 1.25y_n) = 1.28125y_n$;

$x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.25	1.28125
2	0.50	1.641602
3	0.75	2.103302
4	1.00	2.694856

$y_{n+1}^{\text{predicted}} = y_n + 0.1(y_n) = 1.1y_n$

$y_{n+1} = y_n + \frac{0.1}{2}(y_n + 1.1y_n) = 1.105y_n$;

$x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.105
2	0.2	1.221025
3	0.3	1.349233
4	0.4	1.490902
5	0.5	1.647447
6	0.6	1.820429
7	0.7	2.011574
8	0.8	2.222789
9	0.9	2.456182
10	1.0	2.714081

b. $y_{n+1} = y_n + 0.05(x_n + y_n)$
$= 0.05x_n + 1.05y_n;$
$x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.05	1.05
2	0.10	1.105
3	0.15	1.16525
4	0.20	1.23101
5	0.25	1.30256
6	0.30	1.38019
7	0.35	1.46420
8	0.40	1.55491
9	0.45	1.65266
10	0.50	1.75779
11	0.55	1.87068
12	0.60	1.99171
13	0.65	2.12130
14	0.70	2.25986
15	0.75	2.40786
16	0.80	2.56575
17	0.85	2.73404
18	0.90	2.91324
19	0.95	3.10390
20	1.00	3.30660

$y_{n+1}^{\text{predicted}} = y_n + 0.1(x_n + y_n)$
$= 0.1x_n + 1.1y_n$
$y_{n+1} = y_n + \frac{0.1}{2}(x_n + y_n + x_{n+1} + 0.1x_n + 1.1y_n)$
$= y_n + 0.05(1.1x_n + 2.1y_n + x_n + 0.1)$
$= 0.105x_n + 1.105y_n + 0.005;$
$x_0 = 0, y_0 = 1$

n	x_n	y_n
1	0.1	1.11
2	0.2	1.24205
3	0.3	1.39847
4	0.4	1.58180
5	0.5	1.79489
6	0.6	2.04086
7	0.7	2.32315
8	0.8	2.64558
9	0.9	3.01236
10	1.0	3.42816

$y(1) = 2e - 1 - 1 = 3.4365$