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# Chapter 18

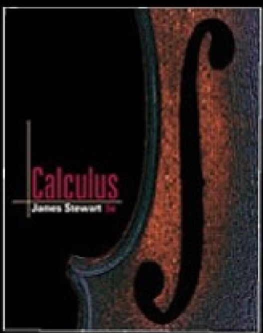
*Adapted from the*

# Complete Solutions Manual

*for*

## James Stewart's

## Calculus – 5th Edition

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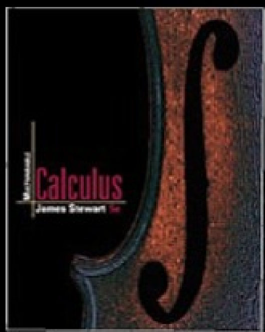


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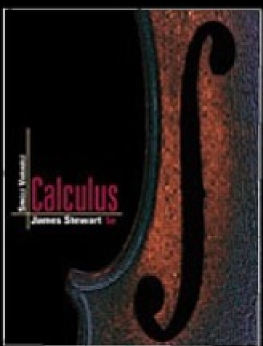


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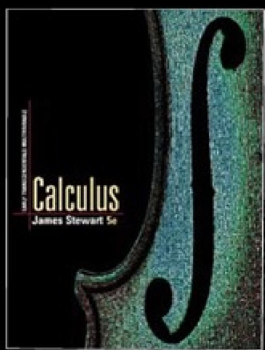


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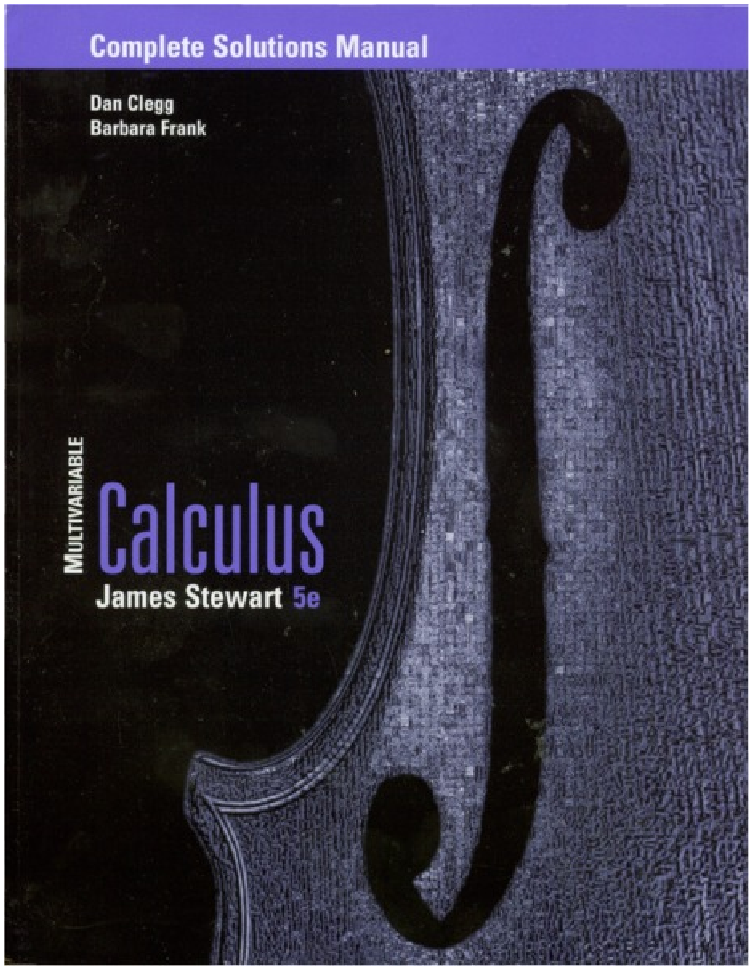


## 18.1 Second-Order Linear Equations ET 17.1

**1.** The auxiliary equation is $r^2 - 6r + 8 = 0 \Rightarrow (r-4)(r-2) = 0 \Rightarrow r = 4, r = 2$. Then by (8) the general solution is $y = c_1e^{4x} + c_2e^{2x}$.

**2.** The auxiliary equation is $r^2 - 4r + 8 = 0 \Rightarrow r = 2 \pm 2i$. Then by (11) the general solution is $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$.

**3.** The auxiliary equation is $r^2 + 8r + 41 = 0 \Rightarrow r = -4 \pm 5i$. Then by (11) the general solution is $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$.

**4.** The auxiliary equation is $2r^2 - r - 1 = (2r+1)(r-1) = 0 \Rightarrow r = 1, r = -\frac{1}{2}$. Then the general solution is $y = c_1e^x + c_2e^{-x/2}$.

**5.** The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$. Then by (10), the general solution is $y = c_1e^x + c_2xe^x$.

**6.** The auxiliary equation is $3r^2 - 5r = r(3r-5) = 0 \Rightarrow r = 0, r = \frac{5}{3}$, so $y = c_1 + c_2e^{5x/3}$.

**7.** The auxiliary equation is $4r^2 + 1 = 0 \Rightarrow r = \pm\frac{1}{2}i$, so $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$.

**8.** The auxiliary equation is $16r^2 + 24r + 9 = (4r+3)^2 = 0 \Rightarrow r = -\frac{3}{4}$, so $y = c_1e^{-3x/4} + c_2xe^{-3x/4}$.

**9.** The auxiliary equation is $4r^2 + r = r(4r+1) = 0 \Rightarrow r = 0, r = -\frac{1}{4}$, so $y = c_1 + c_2e^{-x/4}$.

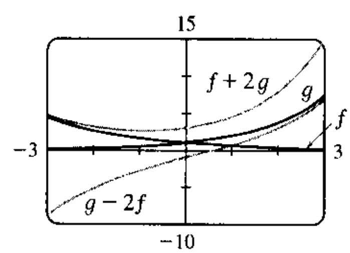
**10.** The auxiliary equation is $9r^2 + 4 = 0 \Rightarrow r = \pm\frac{2}{3}i$, so $y = c_1 \cos\left(\frac{2}{3}x\right) + c_2 \sin\left(\frac{2}{3}x\right)$.

**11.** The auxiliary equation is $r^2 - 2r - 1 = 0 \Rightarrow r = 1 \pm \sqrt{2}$, so $y = c_1e^{(1+\sqrt{2})t} + c_2e^{(1-\sqrt{2})t}$.

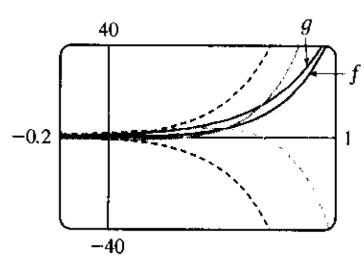
**12.** The auxiliary equation is $r^2 - 6r + 4 = 0 \Rightarrow r = 3 \pm \sqrt{5}$, so $y = c_1e^{(3+\sqrt{5})t} + c_2e^{(3-\sqrt{5})t}$.

**13.** The auxiliary equation is $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so $y = e^{-t/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$.

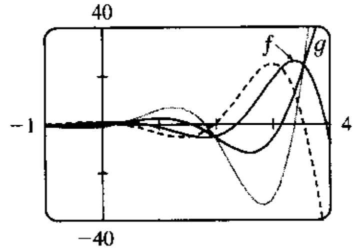
**14.** $6r^2 - r - 2 = (2r+1)(3r-2) = 0$ so $y = c_1e^{-x/2} + c_2e^{2x/3}$. The solutions $(c_1, c_2) = (0, 1)$, $(1, 0)$, $(1, 2)$, $(-2, 1)$ are shown. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x \to \pm\infty$.




**15.** $r^2 - 8r + 16 = (r-4)^2 = 0$ so $y = c_1e^{4x} + c_2xe^{4x}$. The graphs are all asymptotic to the $x$-axis as $x \to -\infty$, and as $x \to \infty$ the solutions tend to $\pm\infty$.



**16.** $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the solution is $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Graphs for $(c_1, c_2) = (1,0), (0,1), (1,-1), (-1,2)$ are shown. The solutions are all asymptotic to the $x$-axis as $x \to -\infty$ and they all oscillate. The amplitudes of the oscillations become arbitrarily large as $x \to \infty$ and arbitrarily small as $x \to -\infty$.




**17.** $2r^2 + 5r + 3 = (2r+3)(r+1) = 0$, so $r = -\frac{3}{2}$, $r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \Rightarrow c_1 + c_2 = 3$ and $y'(0) = -4 \Rightarrow -\frac{3}{2}c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.

**18.** $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}i$ and the general solution is $y = e^{0x}\left(c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)\right) = c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = 3 \Rightarrow c_2 = \sqrt{3}$, so the solution to the initial-value problem is $y = \cos(\sqrt{3}\,x) + \sqrt{3}\sin(\sqrt{3}\,x)$.

**19.** $4r^2 - 4r + 1 = (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = -1.5 \Rightarrow \frac{1}{2}c_1 + c_2 = -1.5$, so $c_2 = -2$ and the solution to the initial-value problem is $y = e^{x/2} - 2xe^{x/2}$.

**20.** $2r^2 + 5r - 3 = (2r-1)(r+3) = 0 \Rightarrow r = \frac{1}{2}$, $r = -3$ and the general solution is $y = c_1 e^{x/2} + c_2 e^{-3x}$. Then $1 = y(0) = c_1 + c_2$ and $4 = y'(0) = \frac{1}{2}c_1 - 3c_2$ so $c_1 = 2$, $c_2 = -1$ and the solution to the initial-value problem is $y = 2e^{x/2} - e^{-3x}$.

**21.** $r^2 + 16 = 0 \Rightarrow r = \pm 4i$ and the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$. Then $y\left(\frac{\pi}{4}\right) = -3 \Rightarrow -c_1 = -3 \Rightarrow c_1 = 3$ and $y'\left(\frac{\pi}{4}\right) = 4 \Rightarrow -4c_2 = 4 \Rightarrow c_2 = -1$, so the solution to the initial-value problem is $y = 3\cos 4x - \sin 4x$.

**22.** $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the general solution is $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Then $0 = y(\pi) = e^\pi(c_1 + 0) \Rightarrow c_1 = 0$ and $2 = y'(\pi) = (c_1 + 2c_2)e^\pi \Rightarrow c_2 = 1/e^\pi$ and the solution to the initial-value problem is $y = \dfrac{e^x}{e^\pi}\sin 2x = e^{x-\pi}\sin 2x$.

**23.** $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$ and the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $1 = y'(0) = c_2 - c_1 \Rightarrow c_2 = 3$ and the solution to the initial-value problem is $y = e^{-x}(2\cos x + 3\sin x)$.

**24.** $r^2 + 12r + 36 = (r+6)^2 = 0 \Rightarrow r = -6$ and the general solution is $y = c_1 e^{-6x} + c_2 x e^{-6x}$. Then $0 = y(1) = c_1 e^{-6} + c_2 e^{-6} \Rightarrow c_1 + c_2 = 0$ and $1 = y'(1) = -6c_1 e^{-6} - 5c_2 e^{-6} \Rightarrow 6c_1 + 5c_2 = -e^6$, so $c_1 = -e^6$ and $c_2 = e^6$. The solution to the initial-value problem is $y = -e^6 e^{-6x} + e^6 x e^{-6x} = (x-1)e^{6-6x}$.

**25.** $4r^2 + 1 = 0 \Rightarrow r = \pm\frac{1}{2}i$ and the general solution is $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$. Then $3 = y(0) = c_1$ and $-4 = y(\pi) = c_2$, so the solution of the boundary-value problem is $y = 3\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right)$.

**26.** $r^2 + 2r = r(2+r) = 0 \Rightarrow r = 0$, $r = -2$ and the general solution is $y = c_1 + c_2 e^{-2x}$. Then $1 = y(0) = c_1 + c_2$ and $2 = y(1) = c_1 + c_2 e^{-2}$ so $c_2 = \dfrac{e^2}{1-e^2}$, $c_1 = \dfrac{1-2e^2}{1-e^2}$. The solution of the boundary-value problem is $y = \dfrac{1-2e^2}{1-e^2} + \dfrac{e^2}{1-e^2} \cdot e^{-2x}$.

**27.** $r^2 - 3r + 2 = (r-2)(r-1) = 0 \quad \Rightarrow \quad r = 1, r = 2$ and the general solution is $y = c_1e^x + c_2e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(3) = c_1e^3 + c_2e^6$ so $c_2 = 1/(1-e^3)$ and $c_1 = e^3/(e^3-1)$. The solution of the boundary-value problem is $y = \dfrac{e^{x+3}}{e^3-1} + \dfrac{e^{2x}}{1-e^3}$.

**28.** $r^2 + 100 = 0 \quad \Rightarrow \quad r = \pm 10i$ and the general solution is $y = c_1 \cos 10x + c_2 \sin 10x$. But $2 = y(0) = c_1$ and $5 = y(\pi) = c_1$, so there is no solution.

**29.** $r^2 - 6r + 25 = 0 \quad \Rightarrow \quad r = 3 \pm 4i$ and the general solution is $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1e^{3\pi} \quad \Rightarrow \quad c_1 = 2/e^{3\pi}$, so there is no solution.

**30.** $r^2 - 6r + 9 = (r-3)^2 = 0 \quad \Rightarrow \quad r = 3$ and the general solution is $y = c_1e^{3x} + c_2xe^{3x}$. Then $1 = y(0) = c_1$ and $0 = y(1) = c_1e^3 + c_2e^3 \quad \Rightarrow \quad c_2 = -1$. The solution of the boundary-value problem is $y = e^{3x} - xe^{3x}$.

**31.** $r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r = -2 \pm 3i$ and the general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$. But $2 = y(0) = c_1$ and $1 = y\left(\frac{\pi}{2}\right) = e^{-\pi}(-c_2)$, so the solution to the boundary-value problem is $y = e^{-2x}(2\cos 3x - e^{\pi} \sin 3x)$.

**32.** $9r^2 - 18r + 10 = 0 \quad \Rightarrow \quad r = 1 \pm \frac{1}{3}i$ and the general solution is $y = e^x\left(c_1 \cos \frac{x}{3} + c_2 \sin \frac{x}{3}\right)$. Then $0 = y(0) = c_1$ and $1 = y(\pi) = e^{\pi}\left(\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2\right) \quad \Rightarrow \quad c_2 = \dfrac{2}{\sqrt{3}\,e^{\pi}}$. The solution of the boundary-value problem is $y = \dfrac{2e^x}{\sqrt{3}\,e^{\pi}} \sin\left(\dfrac{x}{3}\right) = \dfrac{2}{\sqrt{3}} e^{x-\pi} \sin\left(\dfrac{x}{3}\right)$.

**33.** (a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow$ $y = c_1 + c_2x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus, $y = 0$.

*Case 2* ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ (distinct and real since $\lambda < 0$) $\quad \Rightarrow \quad y = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1 + c_2$ (*) and $0 = y(L) = c_1e^{\sqrt{-\lambda}L} + c_2e^{-\sqrt{-\lambda}L}$ (†).

Multiplying (*) by $e^{\sqrt{-\lambda}L}$ and subtracting (†) gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from (*). Thus, $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}\,L = 0 \quad \Rightarrow \quad \sqrt{\lambda}\,L = n\pi$ where $n$ is an integer $\Rightarrow \quad \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where $n$ is an integer.

**34.** The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$ where $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. But $a$, $b$, and $c$ are all positive so both $r_1$ and $r_2$ are negative and $\lim_{x\to\infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1e^{rx} + c_2xe^{rx}$ where $r = -b/(2a) < 0$ since $a$, $b$ are positive. Hence $\lim_{x\to\infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form $y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ where $\alpha = -b/(2a) < 0$ since $a$ and $b$ are positive. Thus $\lim_{x\to\infty} y(x) = 0$.

## 18.2 Nonhomogeneous Linear Equations ET 17.2

**1.** The auxiliary equation is $r^2 + 3r + 2 = (r + 2)(r + 1) = 0$, so the complementary solution is $y_c(x) = c_1e^{-2x} + c_2e^{-x}$. We try the particular solution $y_p(x) = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substituting into the differential equation, we have $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ or $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$. Comparing coefficients gives $2A = 1$, $6A + 2B = 0$, and $2A + 3B + 2C = 0$, so $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{-2x} + c_2e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$.

**2.** The auxiliary equation is $r^2 + 9 = 0$ with roots $r = \pm 3i$, so the complementary solution is $y_c(x) = c_1\cos(3x) + c_2\sin(3x)$. Try the particular solution $y_p(x) = Ae^{3x}$, so $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substitution into the differential equation gives $9Ae^{3x} + 9(Ae^{3x}) = e^{3x}$ or $18Ae^{3x} = e^{3x}$. Thus $A = \frac{1}{18}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1\cos(3x) + c_2\sin(3x) + \frac{1}{18}e^{3x}$.

**3.** The auxiliary equation is $r^2 - 2r = r(r - 2) = 0$, so the complementary solution is $y_c(x) = c_1 + c_2e^{2x}$. Try the particular solution $y_p(x) = A\cos 4x + B\sin 4x$, so $y_p' = -4A\sin 4x + 4B\cos 4x$ and $y_p'' = -16A\cos 4x - 16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x - 16B\sin 4x) - 2(-4A\sin 4x + 4B\cos 4x) = \sin 4x \Rightarrow$ $(-16A - 8B)\cos 4x + (8A - 16B)\sin 4x = \sin 4x$. Then $-16A - 8B = 0$ and $8A - 16B = 1 \Rightarrow A = \frac{1}{40}$ and $B = -\frac{1}{20}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 + c_2e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$.

**4.** The auxiliary equation is $r^2 + 6r + 9 = (r + 3)^2 = 0$, so the complementary solution is $y_c(x) = c_1e^{-3x} + c_2xe^{-3x}$. Try the particular solution $y_p(x) = Ax + B$, so $y_p' = A$ and $y_p'' = 0$. Substitution into the differential equation gives $0 + 6A + 9(Ax + B) = 1 + x$ or $(9A)x + (6A + 9B) = 1 + x$. Comparing coefficients, we have $9A = 1$ and $6A + 9B = 1$, so $A = \frac{1}{9}$ and $B = \frac{1}{27}$. Thus the general solution is $y(x) = c_1e^{-3x} + c_2xe^{-3x} + \frac{1}{9}x + \frac{1}{27}$.

**5.** The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is $y_c(x) = e^{2x}(c_1\cos x + c_2\sin x)$. Try $y_p(x) = Ae^{-x}$, so $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \Rightarrow 10Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$.

**6.** $y_c(x) = e^{-x}(c_1x + c_2)$. Try $y_p(x) = x^2(Ax + B)e^{-x}$ so that no term in $y_p$ is a solution of the complementary equation. Then $y_p' = [-Ax^3 + (3A - B)x^2 + 2Bx]e^{-x}$, $y_p'' = [Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B]e^{-x}$ and substitution gives $[Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B] + 2[-Ax^3 + (3A - B)x^2 + 2Bx] + (Ax^3 + Bx^2) = x \Rightarrow$ $6Ax + 2B = x$. So $y_p(x) = x^2\left(\frac{1}{6}x\right)e^{-x}$ and the general solution is $y(x) = e^{-x}(c_1x + c_2) + \frac{1}{6}x^3e^{-x}$.

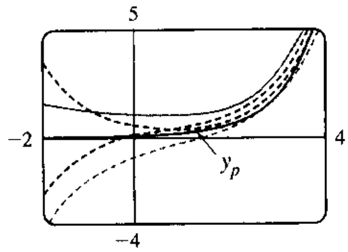
**7.** The auxiliary equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so the complementary solution is $y_c(x) = c_1 \cos x + c_2 \sin x$. For $y'' + y = e^x$ try $y_{p_1}(x) = Ae^x$. Then $y'_{p_1} = y''_{p_1} = Ae^x$ and substitution gives $Ae^x + Ae^x = e^x \Rightarrow A = \frac{1}{2}$, so $y_{p_1}(x) = \frac{1}{2}e^x$. For $y'' + y = x^3$ try $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$. Then $y'_{p_2} = 3Ax^2 + 2Bx + C$ and $y''_{p_2} = 6Ax + 2B$. Substituting, we have $6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$, so $A = 1$, $B = 0$, $6A + C = 0 \Rightarrow C = -6$, and $2B + D = 0 \Rightarrow D = 0$. Thus $y_{p_2}(x) = x^3 - 6x$ and the general solution is $y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x + x^3 - 6x$. But $2 = y(0) = c_1 + \frac{1}{2} \Rightarrow c_1 = \frac{3}{2}$ and $0 = y'(0) = c_2 + \frac{1}{2} - 6 \Rightarrow c_2 = \frac{11}{2}$. Thus the solution to the initial-value problem is $y(x) = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$.

**8.** The auxiliary equation is $r^2 - 4 = 0$ with roots $r = \pm 2$, so the complementary solution is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. Try $y_p(x) = e^x(A\cos x + B\sin x)$, so $y'_p = e^x(A\cos x + B\sin x + B\cos x - A\sin x)$ and $y''_p = e^x(2B\cos x - 2A\sin x)$. Substitution gives $e^x(2B\cos x - 2A\sin x) - 4e^x(A\cos x + B\sin x) = e^x \cos x \Rightarrow$ $(2B - 4A)e^x \cos x + (-2A - 4B)e^x \sin x = e^x \cos x \Rightarrow A = -\frac{1}{5}$, $B = \frac{1}{10}$. Thus the general solution is $y(x) = c_1 e^{2x} + c_2 e^{-2x} + e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$. But $1 = y(0) = c_1 + c_2 - \frac{1}{5}$ and $2 = y'(0) = 2c_1 - 2c_2 - \frac{1}{10}$. Then $c_1 = \frac{9}{8}$, $c_2 = \frac{3}{40}$, and the solution to the initial-value problem is $y(x) = \frac{9}{8}e^{2x} + \frac{3}{40}e^{-2x} + e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$.

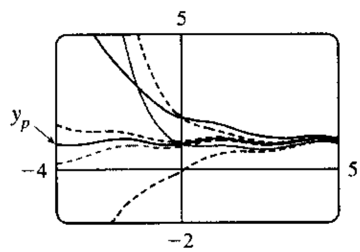
**9.** The auxiliary equation is $r^2 - r = 0$ with roots $r = 0$, $r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2 e^x$. Try $y_p(x) = x(Ax + B)e^x$ so that no term in $y_p$ is a solution of the complementary equation. Then $y'_p = (Ax^2 + (2A + B)x + B)e^x$ and $y''_p = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \Rightarrow$ $(2Ax + (2A + B))e^x = xe^x \Rightarrow A = \frac{1}{2}$, $B = -1$. Thus $y_p(x) = \left(\frac{1}{2}x^2 - x\right)e^x$ and the general solution is $y(x) = c_1 + c_2 e^x + \left(\frac{1}{2}x^2 - x\right)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The solution to the initial-value problem is $y(x) = 2e^x + \left(\frac{1}{2}x^2 - x\right)e^x = e^x\left(\frac{1}{2}x^2 - x + 2\right)$.

**10.** $y_c(x) = c_1 e^x + c_2 e^{-2x}$. For $y'' + y' - 2y = x$ try $y_{p_1}(x) = Ax + B$. Then $y'_{p_1} = A$, $y''_{p_1} = 0$, and substitution gives $0 + A - 2(Ax + B) = x \Rightarrow A = -\frac{1}{2}$, $B = -\frac{1}{4}$, so $y_{p_1}(x) = -\frac{1}{2}x - \frac{1}{4}$. For $y'' + y' - 2y = \sin 2x$ try $y_{p_2}(x) = A\cos 2x + B\sin 2x$. Then $y'_{p_2} = -2A\sin 2x + 2B\cos 2x$, $y''_{p_2} = -4A\cos 2x - 4B\sin 2x$, and substitution gives $(-4A\cos 2x - 4B\sin 2x) + (-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x) = \sin 2x$ $\Rightarrow A = -\frac{1}{20}$, $B = -\frac{3}{20}$. Thus $y_{p_2}(x) = -\frac{1}{20}\cos 2x + -\frac{3}{20}\sin 2x$ and the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$. But $1 = y(0) = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$ and $0 = y'(0) = c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} \Rightarrow c_1 = \frac{17}{15}$ and $c_2 = \frac{1}{6}$. Thus the solution to the initial-value problem is $y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$.

**11.** $y_c(x) = c_1e^{-x/4} + c_2e^{-x}$. Try $y_p(x) = Ae^x$. Then $10Ae^x = e^x$, so $A = \frac{1}{10}$ and the general solution is $y(x) = c_1e^{-x/4} + c_2e^{-x} + \frac{1}{10}e^x$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x \to -\infty$), they all approach either $\infty$ or $-\infty$ as $x \to -\infty$. As $x \to \infty$, all solutions are asymptotic to $y_p = \frac{1}{10}e^x$.




**12.** The auxiliary equation is $(2r+1)(r+1) = 0$, so $r = -1, -\frac{1}{2}$ and $y_c(x) = c_1e^{-x} + c_2e^{-x/2}$. For $2y'' + 3y' + y = 1$, try $y_{p_1}(x) = A$; substituting gives $y_{p_1}(x) = 1$. For $2y'' + 3y' + y = \cos 2x$ try $y_{p2} = A\cos 2x + B\sin 2x \quad \Rightarrow \quad y'_{p2} = -2A\sin 2x + 2B\cos 2x,\ y''_{p2} = -4A\cos 2x - 4B\sin 2x$. Substituting into the differential equation gives $\cos 2x = (6B - 7A)\cos 2x + (-7B - 6A)\sin 2x$. Then solving the equations $6B - 7A = 1$ and $-7B - 6A = 0$ gives $A = -\frac{7}{85}$, $B = \frac{6}{85}$. Thus, $y_{p2}(x) = -\frac{7}{85}\cos 2x + \frac{6}{85}\sin 2x$ and the general solution is $y(x) = c_1e^{-x} + c_2e^{-x/2} + 1 - \frac{7}{85}\cos 2x + \frac{6}{85}\sin 2x$.
The graph shows $y_p = y_{p1} + y_{p2}$ and several other solutions. Notice that all solutions are asymptotic to $y_p$ as $x \to \infty$.




**13.** Here $y_c(x) = c_1\cos 3x + c_2\sin 3x$. For $y'' + 9y = e^{2x}$ try $y_{p_1}(x) = Ae^{2x}$ and for $y'' + 9y = x^2\sin x$ try $y_{p_2}(x) = (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$. Thus a trial solution is
$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = Ae^{2x} + (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$.

**14.** Since $y_c(x) = c_1 + c_2e^{-9x}$, try $y_p(x) = (Ax + B)e^{-x}\cos \pi x + (Cx + D)e^{-x}\sin \pi x$.

**15.** Here $y_c(x) = c_1 + c_2e^{-9x}$. For $y'' + 9y' = 1$ try $y_{p_1}(x) = Ax$ (since $y = A$ is a solution to the complementary equation) and for $y'' + 9y' = xe^{9x}$ try $y_{p_2}(x) = (Bx + C)e^{9x}$.

**16.** Since $y_c(x) = c_1e^x + c_2e^{-4x}$ try $y_p(x) = x(Ax^3 + Bx^2 + Cx + D)e^x$ so that no term of $y_p(x)$ satisfies the complementary equation.

**17.** Since $y_c(x) = e^{-x}(c_1\cos 3x + c_2\sin 3x)$ we try
$y_p(x) = x(Ax^2 + Bx + C)e^{-x}\cos 3x + x(Dx^2 + Ex + F)e^{-x}\sin 3x$ (so that no term of $y_p$ is a solution of the complementary equation).

**18.** Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = e^{3x}$ try $y_{p_1}(x) = Ae^{3x}$ and for $y'' + 4y = x \sin 2x$ try $y_{p_2}(x) = x(Bx + C) \cos 2x + x(Dx + E) \sin 2x$ (so that no term of $y_{p_2}$ is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u_1' = -\frac{Gy_2}{a\,(y_1y_2' - y_2y_1')} \quad \text{and} \quad u_2' = \frac{Gy_1}{a\,(y_1y_2' - y_2y_1')}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

**19.** (a) The complementary solution is $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. A particular solution is of the form $y_p(x) = Ax + B$. Thus, $4Ax + 4B = x \;\Rightarrow\; A = \frac{1}{4}$ and $B = 0 \;\Rightarrow\; y_p(x) = \frac{1}{4}x$. Thus, the general solution is $y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

(b) In (a), $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, so set $y_1 = \cos 2x$, $y_2 = \sin 2x$. Then $y_1y_2' - y_2y_1' = 2\cos^2 2x + 2\sin^2 2x = 2$ so $u_1' = -\frac{1}{2}x \sin 2x \;\Rightarrow$ $u_1(x) = -\frac{1}{2}\int x \sin 2x\, dx = -\frac{1}{4}\left(-x \cos 2x + \frac{1}{2} \sin 2x\right)$ [by parts] and $u_2' = \frac{1}{2}x \cos 2x$ $\Rightarrow\; u_2(x) = \frac{1}{2}\int x \cos 2x dx = \frac{1}{4}\left(x \sin 2x + \frac{1}{2} \cos 2x\right)$ [by parts]. Hence $y_p(x) = -\frac{1}{4}\left(-x \cos 2x + \frac{1}{2} \sin 2x\right) \cos 2x + \frac{1}{4}\left(x \sin 2x + \frac{1}{2} \cos 2x\right) \sin 2x = \frac{1}{4}x$. Thus $y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

**20.** (a) Here $r^2 - 3r + 2 = 0 \;\Rightarrow\; r = 1$ or $2$ and $y_c(x) = c_1e^{2x} + c_2e^x$. We try a particular solution of the form $y_p(x) = A \cos x + B \sin x \;\Rightarrow\; y_p' = -A \sin x + B \cos x$ and $y_p'' = -A \cos x - B \sin x$. Then the equation $y'' - 3y' + 2y = \sin x$ becomes $(A - 3B) \cos x + (B + 3A) \sin x = \sin x \;\Rightarrow\; A - 3B = 0$ and $B + 3A = 1 \;\Rightarrow\; A = \frac{3}{10}$ and $B = \frac{1}{10}$. Thus, $y_p(x) = \frac{3}{10} \cos x + \frac{1}{10} \sin x$. Therefore, the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{2x} + c_2e^x + \frac{3}{10} \cos x + \frac{1}{10} \sin x$.

(b) From (a) we know that $y_c(x) = c_1e^{2x} + c_2e^x$. Setting $y_1 = e^{2x}$, $y_2 = e^x$, we have $y_1y_2' - y_2y_1' = e^{3x} - 2e^{3x} = -e^{3x}$. Thus $u_1' = -\dfrac{\sin xe^x}{-e^{3x}} = \sin xe^{-2x}$ and $u_2' = \dfrac{\sin xe^{2x}}{-e^{3x}} = -\sin xe^{-x}$. Then $u_1(x) = \int e^{-2x} \sin x\, dx = \frac{1}{5}e^{-2x}\,(-2 \sin x - \cos x)$ [by parts] and $u_2(x) = -\int e^{-x} \sin x\, dx = -\frac{1}{2}e^{-x}(-\sin x - \cos x)$. Thus $y_p(x) = \frac{1}{5}(-2 \sin x - \cos x) + \frac{1}{2}(\sin x + \cos x) = \frac{1}{10} \sin x + \frac{3}{10} \cos x$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{2x} + c_2e^x + \frac{1}{10} \sin x + \frac{3}{10} \cos x$.

**21.** (a) $r^2 - r = r(r - 1) = 0 \;\Rightarrow\; r = 0, 1$, so the complementary solution is $y_c(x) = c_1e^x + c_2xe^x$. A particular solution is of the form $y_p(x) = Ae^{2x}$. Thus $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \;\Rightarrow\; Ae^{2x} = e^{2x} \;\Rightarrow\; A = 1$ $\Rightarrow\; y_p(x) = e^{2x}$. So a general solution is $y(x) = y_c(x) + y_p(x) = c_1e^x + c_2xe^x + e^{2x}$.

(b) From (a), $y_c(x) = c_1e^x + c_2xe^x$, so set $y_1 = e^x$, $y_2 = xe^x$. Then, $y_1y_2' - y_2y_1' = e^{2x}(1 + x) - xe^{2x} = e^{2x}$ and so $u_1' = -xe^x \;\Rightarrow\; u_1(x) = -\int xe^x\, dx = -(x - 1)e^x$ [by parts] and $u_2' = e^x \;\Rightarrow$ $u_2(x) = \int e^x\, dx = e^x$. Hence $y_p(x) = (1 - x)e^{2x} + xe^{2x} = e^{2x}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^x + c_2xe^x + e^{2x}$.

**22.** (a) Here $r^2 - 2r + 1 = (r + 1)^2 = 0 \;\Rightarrow\; r = -1$ and $y_c(x) = c_1 + c_2e^x$ and so we try a particular solution of the form $y_p(x) = Axe^x$. Thus, after calculating the necessary derivatives, we get $y'' - y' = e^x \;\Rightarrow$ $Ae^x(2 + x) - Ae^x(1 + x) = e^x \;\Rightarrow\; A = 1$. Thus $y_p(x) = xe^x$ and the general solution is $y(x) = c_1 + c_2e^x + xe^x$.

(b) From (a) we know that $y_c(x) = c_1 + c_2 e^x$, so setting $y_1 = 1$, $y_2 = e^x$, then $y_1 y_2' - y_2 y_1' = e^x - 0 = e^x$. Thus $u_1' = -e^{2x}/e^x = -e^x$ and $u_2' = e^x/e^x = 1$. Then $u_1(x) = -\int e^x dx = -e^x$ and $u_2(x) = x$. Thus $y_p(x) = -e^x + xe^x$ and the general solution is $y(x) = c_1 + c_2 e^x - e^x + xe^x = c_1 + c_3 e^x + xe^x$.

**23.** As in Example 6, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$, so $u_1' = -\dfrac{\sec x \cos x}{-1} = 1 \quad\Rightarrow\quad u_1(x) = x$ and $u_2' = \dfrac{\sec x \sin x}{-1} = -\tan x \quad\Rightarrow\quad u_2(x) = -\int \tan x\, dx = \ln|\cos x| = \ln(\cos x)$ on $0 < x < \frac{\pi}{2}$. Hence $y_p(x) = x \sin x + \cos x \ln(\cos x)$ and the general solution is $y(x) = (c_1 + x)\sin x + [c_2 + \ln(\cos x)]\cos x$.

**24.** Setting $y_1 = \sin x$, $y_2 = \cos x$, then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$. Thus $u_1' = -\dfrac{\cot x \cos x}{-1} = \dfrac{\cos^2 x}{\sin x}$ and $u_2' = \dfrac{\cot x \sin x}{-1} = -\cos x$. Then $u_1(x) = \displaystyle\int \frac{\cos^2 x}{\sin x}\, dx = \int (\csc x - \sin x)\, dx = \ln(\csc x - \cot x) + \cos x$ and $u_2(x) = -\sin x$. Thus $y_p(x) = [\cos x + \ln(\csc x - \cot x)]\sin x + (-\sin x)(\cos x)$ and the general solution is $y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln(\csc x - \cot x)$.

**25.** $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y_2' - y_2 y_1' = e^{3x}$. So $u_1' = \dfrac{-e^{2x}}{(1 + e^{-x})e^{3x}} = -\dfrac{e^{-x}}{1 + e^{-x}}$ and

$$u_1(x) = \int -\frac{e^{-x}}{1 + e^{-x}}\, dx = \ln(1 + e^{-x}).\quad u_2' = \frac{e^x}{(1 + e^{-x})e^{3x}} = \frac{e^x}{e^{3x} + e^{2x}} \text{ so}$$

$$u_2(x) = \int \frac{e^x}{e^{3x} + e^{2x}}\, dx = \ln\left(\frac{e^x + 1}{e^x}\right) - e^{-x} = \ln(1 + e^{-x}) - e^{-x}.$$ Hence

$y_p(x) = e^x \ln(1 + e^{-x}) + e^{2x}[\ln(1 + e^{-x}) - e^{-x}]$ and the general solution is

$y(x) = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$.

**26.** $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and $y_1 y_2' - y_2 y_1' = -e^{-3x}$. So $u_1' = -\dfrac{(\sin e^x)e^{-2x}}{-e^{-3x}} = e^x \sin e^x$ and $u_2' = \dfrac{(\sin e^x)e^{-x}}{-e^{-3x}} = -e^{2x}\sin e^x$. Hence $u_1(x) = \int e^x \sin e^x dx = -\cos e^x$ and $u_2(x) = \int -e^{2x} \sin e^x dx = e^x \cos e^x - \sin e^x$. Then $y_p(x) = -e^{-x}\cos e^x - e^{-2x}[\sin e^x - e^x \cos e^x]$ and the general solution is $y(x) = (c_1 - \cos e^x)e^{-x} + [c_2 - \sin e^x + e^x \cos e^x]e^{-2x}$.

**27.** $y_1 = e^{-x}$, $y_2 = e^x$ and $y_1 y_2' - y_2 y_1' = 2$. So $u_1' = -\dfrac{e^x}{2x}$, $u_2' = \dfrac{e^{-x}}{2x}$ and

$y_p(x) = -e^{-x}\displaystyle\int \frac{e^x}{2x}\, dx + e^x \int \frac{e^{-x}}{2x}\, dx$. Hence the general solution is

$$y(x) = \left(c_1 - \int \frac{e^x}{2x}\, dx\right)e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x}\, dx\right)e^x.$$

**28.** $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$ and $y_1 y_2' - y_2 y_1' = e^{-4x}$. Then $u_1' = \dfrac{-e^{-2x}xe^{-2x}}{x^3 e^{-4x}} = -\dfrac{1}{x^2}$ so $u_1(x) = x^{-1}$ and $u_2' = \dfrac{e^{-2x}e^{-2x}}{x^3 e^{-4x}} = \dfrac{1}{x^3}$ so $u_2(x) = -\dfrac{1}{2x^2}$. Thus $y_p(x) = \dfrac{e^{-2x}}{x} - \dfrac{xe^{-2x}}{2x^2} = \dfrac{e^{-2x}}{2x}$ and the general solution is $y(x) = e^{-2x}[c_1 + c_2 x + 1/(2x)]$.

## 18.3 Applications of Second-Order Differential Equations ET 17.3

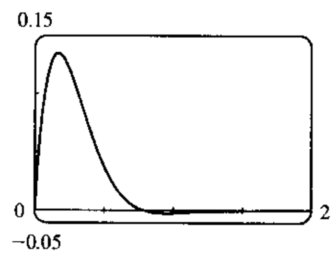
**1.** By Hooke's Law $k(0.6) = 20$ so $k = \frac{100}{3}$ is the spring constant and the differential equation is $3x'' + \frac{100}{3}x = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{10}{3}t\right) + c_2 \sin\left(\frac{10}{3}t\right)$. But $0 = x(0) = c_1$ and $1.2 = x'(0) = \frac{10}{3}c_2$, so the position of the mass after $t$ seconds is $x(t) = 0.36 \sin\left(\frac{10}{3}t\right)$.

**2.** $k(0.3) = 24.3$ or $k = 81$ is the spring constant and the resulting initial-value problem is $4x'' + 81x = 0$, $x(0) = -0.5$ (since compressed), $x'(0) = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{9}{2}t\right) + c_2 \sin\left(\frac{9}{2}t\right)$. But $-0.2 = x(0) = c_1$ and $0 = x'(0) = \frac{9}{2}c_2$. Thus the position is given by $x(t) = -0.2\cos(4.5t)$.

**3.** $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

**4.** (a) The differential equation is $3x'' + 30x' + 123x = 0$ with general solution $x(t) = e^{-5t}(c_1 \cos 4t + c_2 \sin 4t)$. Then $0 = x(0) = c_1$ and $2 = x'(0) = 4c_2$, so the position is given by $x(t) = \frac{1}{2}e^{-5t} \sin 4t$.

(b)




**5.** For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

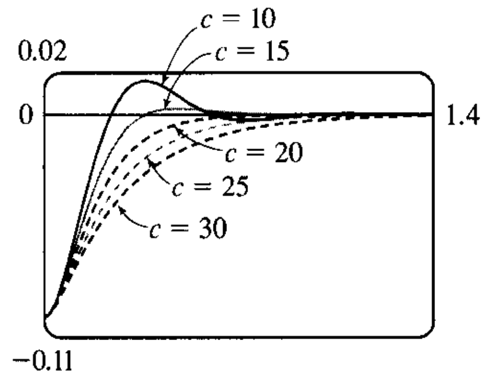
**6.** For critical damping we need $c^2 = 4mk$ or $c = 2\sqrt{mk} = 2\sqrt{3 \cdot 123} = 6\sqrt{41}$.

**7.** We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$. If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos(5\sqrt{3}t) + c_2 \sin(5\sqrt{3}t)\right]$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}c_2 - 5c_1 \quad\Rightarrow\quad c_2 = -\frac{1}{10\sqrt{3}}$, so $x = e^{-5t}\left[-0.1\cos(5\sqrt{3}t) - \frac{1}{10\sqrt{3}}\sin(5\sqrt{3}t)\right]$. If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}\left[c_1 \cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2 \sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1 = x(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}c_1 \quad\Rightarrow\quad c_2 = -\frac{3}{10\sqrt{7}}$. Thus $x = e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$. For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is $x = (c_1 + c_2t)e^{-10t}$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = -10c_1 + c_2 \quad\Rightarrow\quad c_2 = -1$, so $x = (-0.1 - t)e^{-10t}$. If $c = 25$ the auxiliary equation has roots $r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1e^{-5t} + c_2e^{-20t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \quad\Rightarrow\quad c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$, so $x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}$. If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x = c_1 e^{(-15+5\sqrt{5})t} + c_2 e^{(-15-5\sqrt{5})t}$. Then $-0.1 = x(0) = c_1 + c_2$ and

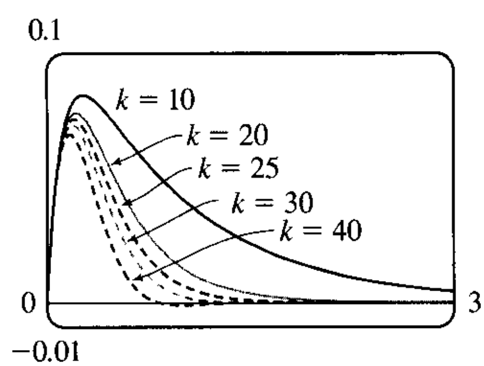
$0 = x'(0) = (-15+5\sqrt{5})\,c_1 + (-15-5\sqrt{5})\,c_2 \Rightarrow$

$c_1 = \frac{-5-3\sqrt{5}}{100}$ and $c_2 = \frac{-5+3\sqrt{5}}{100}$, so

$x = \left(\frac{-5-3\sqrt{5}}{100}\right) e^{(-15+5\sqrt{5})t} + \left(\frac{-5+3\sqrt{5}}{100}\right) e^{(-15-5\sqrt{5})t}$.




**8.** We are given $m = 1$, $c = 10$, $x(0) = 0$ and $x'(0) = 1$. The differential equation is $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + kx = 0$ with auxiliary equation $r^2 + 10r + k = 0$. $k = 10$: the auxiliary equation has roots $r = -5 \pm \sqrt{15}$ so we have overdamping and the solution is $x = c_1 e^{(-5+\sqrt{15})t} + c_2 e^{(-5-\sqrt{15})t}$. Entering the initial conditions gives $c_1 = \frac{1}{2\sqrt{15}}$ and $c_2 = -\frac{1}{2\sqrt{15}}$, so $x = \frac{1}{2\sqrt{15}}e^{(-5+\sqrt{15})t} - \frac{1}{2\sqrt{15}}e^{(-5-\sqrt{15})t}$. $k = 20$: $r = -5 \pm \sqrt{5}$ and the solution is $x = c_1 e^{(-5+\sqrt{5})t} + c_2 e^{(-5-\sqrt{5})t}$ so again the motion is overdamped. The initial conditions give $c_1 = \frac{1}{2\sqrt{5}}$ and $c_2 = -\frac{1}{2\sqrt{5}}$, so $x = \frac{1}{2\sqrt{5}}e^{(-5+\sqrt{5})t} - \frac{1}{2\sqrt{5}}e^{(-5-\sqrt{5})t}$. $k = 25$: we have equal roots $r_1 = r_2 = -5$, so the motion is critically damped and the solution is $x = (c_1 + c_2 t)e^{-5t}$. The initial conditions give $c_1 = 0$ and $c_2 = 1$, so $x = te^{-5t}$. $k = 30$: $r = -5 \pm \sqrt{5}\,i$ so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1\cos(\sqrt{5}\,t) + c_2\sin(\sqrt{5}\,t)\right]$. The initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{5}}$, so $x = \frac{1}{\sqrt{5}}e^{-5t}\sin(\sqrt{5}\,t)$. $k = 40$: $r = -5 \pm \sqrt{15}\,i$ so we again have underdamping. The solution is $x = e^{-5t}\left[c_1\cos(\sqrt{15}\,t) + c_2\sin(\sqrt{15}\,t)\right]$, and the initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{15}}$. Thus $x = \frac{1}{\sqrt{15}}e^{-5t}\sin(\sqrt{15}\,t)$.




**9.** The differential equation is $mx'' + kx = F_0 \cos\omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\,i = \pm\omega i$ so $x_c(t) = c_1\cos\omega t + c_2\sin\omega t$. Since $\omega_0 \neq \omega$, try

$x_p(t) = A\cos\omega_0 t + B\sin\omega_0 t$. Then we need

$(m)\left(-\omega_0^2\right)(A\cos\omega_0 t + B\sin\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0\cos\omega_0 t$ or $A\left(k - m\omega_0^2\right) = F_0$ and

$B\left(k - m\omega_0^2\right) = 0$. Hence $B = 0$ and $A = \dfrac{F_0}{k - m\omega_0^2} = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \dfrac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1\cos\omega t + c_2\sin\omega t + \dfrac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t$.

**10.** As in Exercise 9, $x_c(t) = c_1\cos\omega t + c_2\sin\omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t) = t(A\cos\omega t + B\sin\omega t)$. Then we need $m(2\omega B - \omega^2 At)\cos\omega t - m(2\omega A + \omega^2 Bt)\sin\omega t + kAt\cos\omega t + kBt\sin\omega t = F_0\cos\omega t$ or $2m\omega B = F_0$ and $-2m\omega A = 0$ (noting $-m\omega^2 A + kA = 0$ and $-m\omega^2 B + kB = 0$ since $\omega^2 = k/m$). Hence the general solution is $x(t) = c_1\cos\omega t + c_2\sin\omega t + [F_0 t/(2m\omega)]\sin\omega t$.

**11.** From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1\cos\omega t + c_2\sin\omega t$ and $g(t) = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t$. Then $f$ is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, $g$ is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say $\frac{\omega}{\omega_0} = \frac{a}{b} \;\Rightarrow\; a = \frac{b\omega}{\omega_0}$ where $a$ and $b$ are non-zero integers. Then

$$x\left(t + a\cdot\tfrac{2\pi}{\omega}\right) = f\left(t + a\cdot\tfrac{2\pi}{\omega}\right) + g\left(t + a\cdot\tfrac{2\pi}{\omega}\right) = f(t) + g\left(t + \tfrac{b\omega}{\omega_0}\cdot\tfrac{2\pi}{\omega}\right)$$
$$= f(t) + g\left(t + b\cdot\tfrac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t)$$

so $x(t)$ is periodic.

**12.** (a) The graph of $x = c_1e^{rt} + c_2te^{rt}$ has a $t$-intercept when $c_1e^{rt} + c_2te^{rt} = 0 \;\Leftrightarrow\; e^{rt}(c_1 + c_2t) = 0 \;\Leftrightarrow\; c_1 = -c_2t$. Since $t > 0$, $x$ has a $t$-intercept if and only if $c_1$ and $c_2$ have opposite signs.

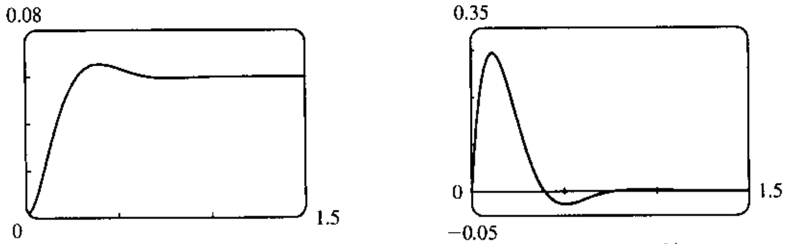
(b) For $t > 0$, the graph of $x$ crosses the $t$-axis when $c_1e^{r_1t} + c_2e^{r_2t} = 0 \;\Leftrightarrow\; c_2e^{r_2t} = -c_1e^{r_1t} \;\Leftrightarrow\; c_2 = -c_1\dfrac{e^{r_1t}}{e^{r_2t}} = -c_1e^{(r_1-r_2)t}$. But $r_1 > r_2 \;\Rightarrow\; r_1 - r_2 > 0$ and since $t > 0$, $e^{(r_1-r_2)t} > 1$. Thus $|c_2| = |c_1|\,e^{(r_1-r_2)t} > |c_1|$, and the graph of $x$ can cross the $t$-axis only if $|c_2| > |c_1|$.

**13.** Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t)$ and try $Q_p(t) = A \;\Rightarrow\; 500A = 12$ or $A = \frac{3}{125}$.
The general solution is $Q(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and $Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_2 - 20c_1)\sin 20t]$ but $0 = Q'(0) = -10c_1 + 20c_2$. Thus the charge is $Q(t) = -\frac{1}{250}e^{-10t}(6\cos 20t + 3\sin 20t) + \frac{3}{125}$ and the current is $I(t) = e^{-10t}\left(\frac{3}{5}\right)\sin 20t$.

**14.** (a) Here the initial-value problem for the charge is $2Q'' + 24Q' + 200Q = 12$ with $Q(0) = 0.001$ and $Q'(0) = 0$. Then $Q_c(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t)$ and try $Q_p(t) = A \;\Rightarrow\; A = \frac{3}{50}$ and the general solution is $Q(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t) + \frac{3}{50}$. But $0.001 = Q(0) = c + \frac{3}{50}$ so $c_1 = -0.059$. Also $Q'(t) = I(t) = e^{-6t}[(-6c_1 + 8c_2)\cos 8t + (-6c_2 - 8c_1)\sin 8t]$ and $0 = Q'(0) = -6c_1 + 8c_2$ so

$c_2 = -0.04425$. Hence the charge is $Q(t) = -e^{-6t}(0.059\cos 8t + 0.04425\sin 8t) + \frac{3}{50}$ and the current is $I(t) = e^{-6t}(0.7375)\sin 8t$.

(b)

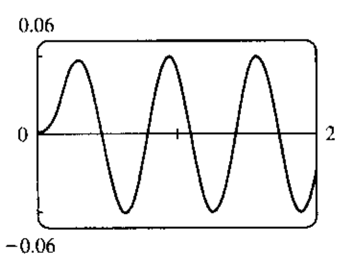



charge, $Q(t)$ current, $I(t) = Q'(t)$

**15.** As in Exercise 13, $Q_c(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t)$ but $E(t) = 12\sin 10t$ so try $Q_p(t) = A\cos 10t + B\sin 10t$. Substituting into the differential equation gives $(-100A + 200B + 500A)\cos 10t + (-100B - 200A + 500B)\sin 10t = 12\sin 10t \quad \Rightarrow \quad 400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is $Q(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t) - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$. But $0 = Q(0) = c_1 - \frac{3}{250}$ so $c_1 = \frac{3}{250}$. Also $Q'(t) = \frac{3}{25}\sin 10t + \frac{6}{25}\cos 10t + e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_2 - 20c_1)\sin 20t]$ and $0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by $Q(t) = e^{-10t}\left[\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right] - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$.

**16.** (a) As in Exercise 14, $Q_c(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t)$ but try $Q_p(t) = A\cos 10t + B\sin 10t$. Substituting into the differential equation gives $(-200A + 240B + 200A)\cos 10t + (-200B - 240A + 200B)\sin 10t = 12\sin 10t$, so $B = 0$ and $A = -\frac{1}{20}$. Hence, the general solution is $Q(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t) - \frac{1}{20}\cos 10t$. But $0.001 = Q(0) = c_1 - \frac{1}{20}$, $Q'(t) = e^{-6t}[(-6c_1 + 8c_2)\cos 8t + (-6c_2 - 8c_1)\sin 8t] - \frac{1}{2}\sin 10t$ and $0 = Q'(0) = -6c_1 + 8c_2$, so $c_1 = 0.051$ and $c_2 = 0.03825$. Thus the charge is given by $Q(t) = e^{-6t}(0.051\cos 8t + 0.03825\sin 8t) - \frac{1}{20}\cos 10t$.

(b)




**17.** $x(t) = A\cos(\omega t + \delta) \quad \Leftrightarrow \quad x(t) = A[\cos\omega t\cos\delta - \sin\omega t\sin\delta] \quad \Leftrightarrow \quad x(t) = A\left(\frac{c_1}{A}\cos\omega t + \frac{c_2}{A}\sin\omega t\right)$

where $\cos\delta = c_1/A$ and $\sin\delta = -c_2/A \quad \Leftrightarrow \quad x(t) = c_1\cos\omega t + c_2\sin\omega t$. (Note that $\cos^2\delta + \sin^2\delta = 1 \quad \Rightarrow$ $c_1^2 + c_2^2 = A^2$.)

**18.** (a) We approximate $\sin\theta$ by $\theta$ and, with $L = 1$ and $g = 9.8$, the differential equation becomes $\dfrac{d^2\theta}{dt^2} + 9.8\theta = 0$.

The auxiliary equation is $r^2 + 9.8 = 0 \Rightarrow r = \pm\sqrt{9.8}\,i$, so the general solution is

$\theta(t) = c_1\cos(\sqrt{9.8}\,t) + c_2\sin(\sqrt{9.8}\,t)$. Then $0.2 = \theta(0) = c_1$ and $1 = \theta'(0) = \sqrt{9.8}\,c_2 \Rightarrow c_2 = \frac{1}{\sqrt{9.8}}$,

so the equation is $\theta(t) = 0.2\cos(\sqrt{9.8}\,t) + \frac{1}{\sqrt{9.8}}\sin(\sqrt{9.8}\,t)$.

(b) $\theta'(t) = -0.2\sqrt{9.8}\sin(\sqrt{9.8}\,t) + \cos(\sqrt{9.8}\,t) = 0$ or $\tan(\sqrt{9.8}\,t) = \frac{5}{\sqrt{9.8}}$, so the critical numbers are

$t = \frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{n}{\sqrt{9.8}}\pi$ ($n$ any integer). The maximum angle from the vertical is

$\theta\left(\frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right) \approx 0.377$ radians (or about $21.7°$).

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\frac{2\pi}{\sqrt{9.8}} \approx 2.007$ seconds.

(d) $\theta(t) = 0 \Rightarrow 0.2\cos(\sqrt{9.8}\,t) + \frac{1}{\sqrt{9.8}}\sin(\sqrt{9.8}\,t) = 0 \Rightarrow \tan(\sqrt{9.8}\,t) = -0.2\sqrt{9.8} \Rightarrow$

$t = \frac{1}{\sqrt{9.8}}\left[\tan^{-1}(-0.2\sqrt{9.8}) + \pi\right] \approx 0.825$ seconds.

(e) $\theta'(0.825) \approx -1.180$ rad/s.

## 18.4 Series Solutions ET 17.4

**1.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes

$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$. Replacing $n$ by $n+1$ in the first sum gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=0}^{\infty} c_n x^n = 0$,

so $\sum_{n=0}^{\infty}[(n+1)c_{n+1} - c_n]x^n = 0$. Equating coefficients gives $(n+1)c_{n+1} - c_n = 0$, so the recursion relation is

$c_{n+1} = \dfrac{c_n}{n+1}$, $n = 0, 1, 2, \ldots$. Then $c_1 = c_0$, $c_2 = \dfrac{1}{2}c_1 = \dfrac{c_0}{2}$, $c_3 = \dfrac{1}{3}c_2 = \dfrac{1}{3}\cdot\dfrac{1}{2}c_0 = \dfrac{c_0}{3!}$, $c_4 = \dfrac{1}{4}c_3 = \dfrac{c_0}{4!}$, and

in general, $c_n = \dfrac{c_0}{n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}\frac{c_0}{n!}x^n = c_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = c_0 e^x$$

**2.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = xy \Rightarrow y' - xy = 0 \Rightarrow \sum_{n=1}^{\infty} nc_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$ or

$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$. Replacing $n$ with $n+1$ in the first sum and $n$ with $n-1$ in the second

gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_1 + \sum_{n=1}^{\infty}(n+1)\,c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$. Thus,

$c_1 + \sum_{n=1}^{\infty} [(n+1)c_{n+1} - c_{n-1}]\, x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)\, c_{n+1} - c_{n-1} = 0$. Thus, the recursion relation is $c_{n+1} = \dfrac{c_{n-1}}{n+1}$, $n = 1, 2, \ldots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also, $c_2 = \dfrac{c_0}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{4 \cdot 2} = \dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{6 \cdot 4 \cdot 2} = \dfrac{c_0}{2^3 \cdot 3!}$ and in general $c_{2n} = \dfrac{c_0}{2^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{c_0}{2^n \cdot n!}\, x^{2n} = c_0 \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}$$

**3.** Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ and $-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2} x^n$. Hence, the equation $y' = x^2 y$ becomes $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$ or $c_1 + 2c_2 x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}]\, x^n = 0$. Equating coefficients gives $c_1 = c_2 = 0$ and $c_{n+1} = \dfrac{c_{n-2}}{n+1}$ for $n = 2, 3, \ldots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general $c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally $c_3 = \dfrac{c_0}{3}$, $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6 \cdot 3} = \dfrac{c_0}{3^2 \cdot 2!}$, $c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9 \cdot 6 \cdot 3} = \dfrac{c_0}{3^3 \cdot 3!}$, ..., and $c_{3n} = \dfrac{c_0}{3^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}$$

**4.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \;\Rightarrow\; y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$. Then the differential equation becomes $(x-3) \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \;\Rightarrow$

$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} - 3 \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \;\Rightarrow$

$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0 \;\Rightarrow\; \sum_{n=0}^{\infty} [(n+2)c_n - 3(n+1)c_{n+1}]\, x^n = 0$

$\left(\text{since } \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n\right)$. Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \dfrac{(n+2)c_n}{3(n+1)}$, $n = 0, 1, 2, \ldots$. Then $c_1 = \dfrac{2c_0}{3}$, $c_2 = \dfrac{3c_1}{3(2)} = \dfrac{3c_0}{3^2}$, $c_3 = \dfrac{4c_2}{3(3)} = \dfrac{4c_0}{3^3}$, $c_4 = \dfrac{5c_3}{3(4)} = \dfrac{5c_0}{3^4}$, and in general, $c_n = \dfrac{(n+1)c_0}{3^n}$. Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n. \left[\text{Note that } c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n = \frac{9c_0}{(3-x)^2} \text{ for } |x| < 3.\right]$$

**5.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \;\Rightarrow\; y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + nc_n + c_n]x^n$ (since $\sum_{n=1}^{\infty} nc_n x^n = \sum_{n=0}^{\infty} nc_n x^n$). Equating coefficients gives $(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$, thus the recursion relation is $c_{n+2} = \dfrac{-(n+1)c_n}{(n+2)(n+1)} = -\dfrac{c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Then the even coefficients are given by $c_2 = -\dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2\cdot 4}$, $c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2\cdot 4\cdot 6}$, and in general, $c_{2n} = (-1)^n \dfrac{c_0}{2\cdot 4\cdot\cdots\cdot 2n} = \dfrac{(-1)^n c_0}{2^n\, n!}$. The odd coefficients are $c_3 = -\dfrac{c_1}{3}$, $c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3\cdot 5}$, $c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3\cdot 5\cdot 7}$, and in general, $c_{2n+1} = (-1)^n \dfrac{c_1}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)} = \dfrac{(-2)^n\, n!\, c_1}{(2n+1)!}$. The solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\, n!}x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n\, n!}{(2n+1)!}x^{2n+1}.$$

**6.** Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. Hence, the equation $y'' = y$ becomes $\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum\limits_{n=0}^{\infty} c_n x^n = 0$ or $\sum\limits_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - c_n]x^n = 0$. So the recursion relation is $c_{n+2} = \dfrac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \ldots$. Given $c_0$ and $c_1$, $c_2 = \dfrac{c_0}{2\cdot 1}$, $c_4 = \dfrac{c_2}{4\cdot 3} = \dfrac{c_0}{4!}$, $c_6 = \dfrac{c_4}{6\cdot 5} = \dfrac{c_0}{6!}, \ldots, c_{2n} = \dfrac{c_0}{(2n)!}$ and $c_3 = \dfrac{c_1}{3\cdot 2}$, $c_5 = \dfrac{c_3}{5\cdot 4} = \dfrac{c_1}{5\cdot 4\cdot 3\cdot 2} = \dfrac{c_1}{5!}$, $c_7 = \dfrac{c_5}{7\cdot 6} = \dfrac{c_1}{7!}, \ldots,$ $c_{2n+1} = \dfrac{c_1}{(2n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n}x^{2n} + \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1} = c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} + c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

The solution can be written as $y(x) = c_0 \cosh x + c_1 \sinh x$

$$\left[\text{or } y(x) = c_0\frac{e^x + e^{-x}}{2} + c_1\frac{e^x - e^{-x}}{2} = \frac{c_0 + c_1}{2}e^x + \frac{c_0 - c_1}{2}e^{-x}\right].$$

**7.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$, $xy' = \sum_{n=0}^{\infty} nc_n x^n$ and $(x^2+1)y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + [n(n-1) + n - 1]c_n]x^n = 0$. The recursion relation is $c_{n+2} = -\dfrac{(n-1)c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Given $c_0$ and $c_1$, $c_2 = \dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = -\dfrac{c_0}{2^2\cdot 2!}$, $c_6 = -\dfrac{3c_4}{6} = (-1)^2\dfrac{3c_0}{2^3\cdot 3!}, \ldots,$ $c_{2n} = (-1)^{n-1}\dfrac{1\cdot 3\cdot\cdots\cdot(2n-3)\,c_0}{2^n\, n!} = (-1)^{n-1}\dfrac{(2n-3)!\,c_0}{2^n 2^{n-2}\, n!\,(n-2)!} = (-1)^{n-1}\dfrac{(2n-3)!\,c_0}{2^{2n-2}\, n!\,(n-2)!}$ for $n = 2, 3, \ldots$. $c_3 = \dfrac{0\cdot c_1}{3} = 0 \;\Rightarrow\; c_{2n+1} = 0$ for $n = 1, 2, \ldots$. Thus the solution is

$$y(x) = c_0 + c_1 x + c_0\frac{x^2}{2} + c_0\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}\, n!\,(n-2)!}x^{2n}.$$

**8.** Assuming $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and

$-xy(x) = -\sum\limits_{n=0}^{\infty} c_n x^{n+1} = -\sum\limits_{n=1}^{\infty} c_{n-1}x^n$. The equation $y'' = xy$ becomes

$\sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum\limits_{n=1}^{\infty} c_{n-1}x^n = 0$ or $2c_2 + \sum\limits_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - c_{n-1}]x^n = 0$. Equating

coefficients gives $c_2 = 0$ and $c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \ldots$. Since $c_2 = 0$,

$c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Given $c_0$, $c_3 = \dfrac{c_0}{3 \cdot 2}$, $c_6 = \dfrac{c_3}{6 \cdot 5} = \dfrac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \ldots,$

$c_{3n} = \dfrac{c_0}{3n(3n-1)(3n-3)(3n-4) \cdot \cdots \cdot 6 \cdot 5 \cdot 3 \cdot 2}$. Given $c_1$, $c_4 = \dfrac{c_1}{4 \cdot 3}$, $c_7 = \dfrac{c_4}{7 \cdot 6} = \dfrac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}, \ldots,$

$c_{3n+1} = \dfrac{c_1}{(3n+1)3n(3n-2)(3n-3) \ldots 7 \cdot 6 \cdot 4 \cdot 3}$. The solution can be written as

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(3n-2)(3n-5) \cdot \cdots \cdot 7 \cdot 4 \cdot 1}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(3n-1)(3n-4) \cdot \cdots \cdot 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}$$

**9.** Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum\limits_{n=1}^{\infty} nc_n x^{n-1} = -\sum\limits_{n=1}^{\infty} nc_n x^n = -\sum\limits_{n=0}^{\infty} nc_n x^n$,

$y''(x) = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$, and the equation $y'' - xy' - y = 0$ becomes

$\sum\limits_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_n]x^n = 0$. Thus, the recursion relation is

$c_{n+2} = \dfrac{nc_n + c_n}{(n+2)(n+1)} = \dfrac{c_n(n+1)}{(n+2)(n+1)} = \dfrac{c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. One of the given conditions is

$y(0) = 1$. But $y(0) = \sum\limits_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \cdots = c_0$, so $c_0 = 1$. Hence, $c_2 = \dfrac{c_0}{2} = \dfrac{1}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{1}{2 \cdot 4}$,

$c_6 = \dfrac{c_4}{6} = \dfrac{1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n} = \dfrac{1}{2^n n!}$. The other given condition is $y'(0) = 0$. But

$y'(0) = \sum\limits_{n=1}^{\infty} nc_n(0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1$, so $c_1 = 0$. By the recursion relation, $c_3 = \dfrac{c_1}{3} = 0$, $c_5 = 0, \ldots,$

$c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}$$

**10.** Assuming that $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, we have $x^2 y = \sum\limits_{n=0}^{\infty} c_n x^{n+2}$ and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2}$$

$$= 2c_2 + 6c_3 x + \sum_{n=0}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2}$$

Thus, the equation $y'' + x^2y = 0$ becomes $2c_2 + 6c_3x + \sum_{n=0}^{\infty}[(n+4)(n+3)c_{n+4} + c_n]\,x^{n+2} = 0$. So

$c_2 = c_3 = 0$ and the recursion relation is $c_{n+4} = -\dfrac{c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, \ldots$.

But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation, $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \ldots$.

Also, $c_0 = y(0) = 1$, so

$$c_4 = -\frac{c_0}{4\cdot 3} = -\frac{1}{4\cdot 3},\ c_8 = -\frac{c_4}{8\cdot 7} = \frac{(-1)^2}{8\cdot 7\cdot 4\cdot 3},\ \ldots,\ c_{4n} = \frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}.$$

Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_nx^n = c_0 + \sum_{n=0}^{\infty} c_{4n}x^{4n} = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$$

**11.** Assuming that $y(x) = \sum_{n=0}^{\infty} c_nx^n$, we have $xy = x\sum_{n=0}^{\infty} c_nx^n = \sum_{n=0}^{\infty} c_nx^{n+1}$,

$x^2y' = x^2\sum_{n=1}^{\infty} nc_nx^{n-1} = \sum_{n=0}^{\infty} nc_nx^{n+1}$,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} = \sum_{n=-1}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1} \quad [\text{replace } n \text{ with } n+3]$$

$$= 2c_2 + \sum_{n=0}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1},$$

and the equation $y'' + x^2y' + xy = 0$ becomes $2c_2 + \sum_{n=0}^{\infty}[(n+3)(n+2)c_{n+3} + nc_n + c_n]\,x^{n+1} = 0$.

So $c_2 = 0$ and the recursion relation is $c_{n+3} = \dfrac{-nc_n - c_n}{(n+3)(n+2)} = -\dfrac{(n+1)c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \ldots$.

But $c_0 = y(0) = 0 = c_2$ and by the recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$ .

Also, $c_1 = y'(0) = 1$, so

$$c_4 = -\frac{2c_1}{4\cdot 3} = -\frac{2}{4\cdot 3},\ c_7 = -\frac{5c_4}{7\cdot 6} = (-1)^2\frac{2\cdot 5}{7\cdot 6\cdot 4\cdot 3} = (-1)^2\frac{2^2 5^2}{7!},\ \ldots,$$

$c_{3n+1} = (-1)^n\dfrac{2^2 5^2\cdot\cdots\cdot(3n-1)^2}{(3n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_nx^n = x + \sum_{n=1}^{\infty}\left[(-1)^n\frac{2^2 5^2\cdot\cdots\cdot(3n-1)^2x^{3n+1}}{(3n+1)!}\right]$$

**12.** (a) Let $y(x) = \sum_{n=0}^{\infty} c_nx^n$. Then $x^2y''(x) = \sum_{n=2}^{\infty} n(n-1)c_nx^n = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^{n+2}$,

$xy'(x) = \sum_{n=1}^{\infty} nc_nx^n = \sum_{n=-1}^{\infty}(n+2)c_{n+2}x^{n+2} = c_1x + \sum_{n=0}^{\infty}(n+2)c_{n+2}x^{n+2}$, and the equation

$x^2y'' + xy' + x^2y = 0$ becomes $c_1x + \sum_{n=0}^{\infty}\{[(n+2)(n+1) + (n+2)]c_{n+2} + c_n\}x^{n+2} = 0$. So $c_1 = 0$

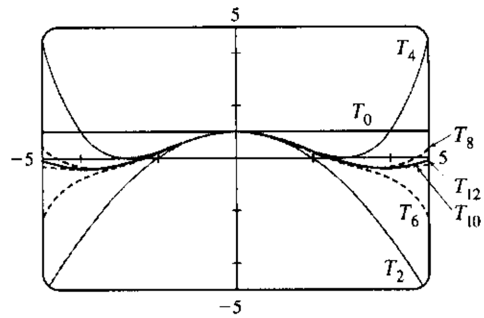
and the recursion relation is $c_{n+2} = -\dfrac{c_n}{(n+2)^2}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0$ so $c_{2n+1} = 0$ for

$n = 0, 1, 2, \ldots$. Also, $c_0 = y(0) = 1$, so $c_2 = -\frac{1}{2^2}$, $c_4 = -\frac{c_2}{4^2} = (-1)^2 \frac{1}{4^2 2^2} = (-1)^2 \frac{1}{2^4 (2!)^2}$,

$c_6 = -\frac{c_4}{6^2} = (-1)^3 \frac{1}{2^6 (3!)^2}, \ldots, c_{2n} = (-1)^n \frac{1}{2^{2n} (n!)^2}$. The solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} (n!)^2}$$

(b) The Taylor polynomials $T_0$ to $T_{12}$ are shown in the graph. Because $T_{10}$ and $T_{12}$ are close together throughout the interval $[-5, 5]$, it is reasonable to assume that $T_{12}$ is a good approximation to the Bessel function on that interval.




# 18 Review ET 17

## CONCEPT CHECK

**1.** (a) $ay'' + by' + cy = 0$ where $a$, $b$, and $c$ are constants.

(b) $ar^2 + br + c = 0$

(c) If the auxiliary equation has two distinct real roots $r_1$ and $r_2$, the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If the roots are real and equal, the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$ where $r$ is the common root. If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

**2.** (a) An initial-value problem consists of finding a solution $y$ of a second-order differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where $y_0$ and $y_1$ are constants.

(b) A boundary-value problem consists of finding a solution $y$ of a second-order differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

**3.** (a) $ay'' + by' + cy = G(x)$ where $a$, $b$, and $c$ are constants and $G$ is a continuous function.

(b) The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution $y_c$ of the complementary equation and $y_p$ is any particular solution of the original differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) See Examples 1–5 and the associated discussion in Section 18.2 [ET 17.2].

(d) See the discussion on pages 1188–1190 [ET 1152-1154].

**4.** Second-order linear differential equations can be used to describe the motion of a vibrating spring or to analyze an electric circuit; see the discussion in Section 18.3 [ET 17.3].

**5.** See Example 1 and the preceding discussion in Section 18.4 [ET 17.4].

## TRUE-FALSE QUIZ

**1.** True. See Theorem 18.1.3 [ ET 17.1.3].

**2.** False. The differential equation is not homogeneous.

**3.** True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.

**4.** False. $y = Ae^x$ is a solution of the complementary equation, so we have to take $y_p(x) = Axe^x$.

## EXERCISES

**1.** The auxiliary equation is $r^2 - 2r - 15 = 0 \;\Rightarrow\; (r-5)(r+3) = 0 \;\Rightarrow\; r = 5, r = -3$. Then the general solution is $y = c_1 e^{5x} + c_2 e^{-3x}$.

**2.** The auxiliary equation is $r^2 + 4r + 13 = 0 \;\Rightarrow\; r = -2 \pm 3i$, so $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

**3.** The auxiliary equation is $r^2 + 3 = 0 \;\Rightarrow\; r = \pm\sqrt{3}\, i$. Then the general solution is $y = c_1 \cos(\sqrt{3}\, x) + c_2 \sin(\sqrt{3}\, x)$.

**4.** The auxiliary equation is $4r^2 + 4r + 1 = 0 \;\Rightarrow\; (2r+1)^2 = 0 \;\Rightarrow\; r = -\frac{1}{2}$, so the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$.

**5.** $r^2 - 4r + 5 = 0 \;\Rightarrow\; r = 2 \pm i$, so $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{2x} \;\Rightarrow\; y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Substitution into the differential equation gives $4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \;\Rightarrow\; A = 1$ and the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}$.

**6.** $r^2 + r - 2 = 0 \;\Rightarrow\; r = 1, r = -2$ and $y_c(x) = c_1 e^x + c_2 e^{-2x}$. Try $y_p(x) = Ax^2 + Bx + C \;\Rightarrow\; y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitution gives $2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2 \;\Rightarrow\; A = B = -\frac{1}{2}, C = -\frac{3}{4}$ so the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$.

**7.** $r^2 - 2r + 1 = 0 \;\Rightarrow\; r = 1$ and $y_c(x) = c_1 e^x + c_2 x e^x$. Try $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x \;\Rightarrow\; y_p' = (C - Ax - B)\sin x + (A + Cx + D)\cos x$ and $y_p'' = (2C - B - Ax)\cos x + (-2A - D - Cx)\sin x$. Substitution gives $(-2Cx + 2C - 2A - 2D)\cos x + (2Ax - 2A + 2B - 2C)\sin x = x\cos x \;\Rightarrow\; A = 0$, $B = C = D = -\frac{1}{2}$. The general solution is $y(x) = c_1 e^x + c_2 x e^x - \frac{1}{2}\cos x - \frac{1}{2}(x+1)\sin x$.

**8.** $r^2 + 4 = 0 \;\Rightarrow\; r = \pm 2i$ and $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Try $y_p(x) = Ax\cos 2x + Bx\sin 2x$ so that no term of $y_p$ is a solution of the complementary equation. Then $y_p' = (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$ and $y_p'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$. Substitution gives $4B\cos 2x - 4A\sin 2x = \sin 2x \;\Rightarrow\; A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x\cos 2x$.

**9.** $r^2 - r - 6 = 0 \;\Rightarrow\; r = -2, r = 3$ and $y_c(x) = c_1 e^{-2x} + c_2 e^{3x}$. For $y'' - y' - 6y = 1$, try $y_{p_1}(x) = A$. Then $y_{p_1}'(x) = y_{p_1}''(x) = 0$ and substitution into the differential equation gives $A = -\frac{1}{6}$. For $y'' - y' - 6y = e^{-2x}$ try $y_{p_2}(x) = Bxe^{-2x}$ (since $y = Be^{-2x}$ satisfies the complementary equation). Then $y_{p_2}' = (B - 2Bx)e^{-2x}$ and $y_{p_2}'' = (4Bx - 4B)e^{-2x}$, and substitution gives $-5Be^{-2x} = e^{-2x} \;\Rightarrow\; B = -\frac{1}{5}$. The general solution then is $y(x) = c_1 e^{-2x} + c_2 e^{3x} + y_{p_1}(x) + y_{p_2}(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$.

**10.** Using variation of parameters, $y_c(x) = c_1 \cos x + c_2 \sin x$, $u_1'(x) = -\csc x \sin x = -1 \;\Rightarrow\; u_1(x) = -x$, and $u_2'(x) = \dfrac{\csc x \cos x}{x} = \cot x \;\Rightarrow\; u_2(x) = \ln|\sin x| \;\Rightarrow\; y_p = -x\cos x + \sin x \ln|\sin x|$. The solution is $y(x) = (c_1 - x)\cos x + (c_2 + \ln|\sin x|)\sin x$.

**11.** The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k^2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.

**12.** The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}\left(2\cos 4x - \frac{5}{4}\sin 4x\right)$.

**13.** The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.

**14.** $y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3)$. For $9y'' + y = 3x$, try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p_2}(x) = Ae^{-x}$. Then $9Ae^{-x} + Ae^{-x} = e^{-x}$ or $y_{p_2}(x) = \frac{1}{10}e^{-x}$. Thus the general solution is $y(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + 3x + \frac{1}{10}e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3}c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9\cos(x/3) - 27\sin(x/3)] + 3x + \frac{1}{10}e^{-x}$.

**15.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\dfrac{1}{3}$, $c_5 = \dfrac{(-1)^2}{3 \cdot 5}$, $c_7 = \dfrac{(-1)^3}{3 \cdot 5 \cdot 7} = \dfrac{(-1)^3 2^3 3!}{7!}, \ldots, c_{2n+1} = \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, n!}{(2n+1)!} x^{2n+1}$.

**16.** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (n+2)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = \dfrac{c_n}{n+1}$ for $n = 0, 1, 2, \ldots$. Given $c_0$ and $c_1$, we have $c_2 = \dfrac{c_0}{1}$, $c_4 = \dfrac{c_2}{3} = \dfrac{c_0}{1 \cdot 3}$, $c_6 = \dfrac{c_4}{5} = \dfrac{c_0}{1 \cdot 3 \cdot 5}, \ldots, c_{2n} = \dfrac{c_0}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = c_0 \dfrac{2^{n-1}(n-1)!}{(2n-1)!}$. Similarly $c_3 = \dfrac{c_1}{2}$, $c_5 = \dfrac{c_3}{4} = \dfrac{c_1}{2 \cdot 4}$, $c_7 = \dfrac{c_5}{6} = \dfrac{c_1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n+1} = \dfrac{c_1}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} = \dfrac{c_1}{2^n\, n!}$. Thus the general solution is
$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_0 \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)!\,x^{2n}}{(2n-1)!} + c \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n\, n!}.$$ But $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n\, n!} = x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}x^2\right)^n}{n!} = xe^{x^2/2}$,

so $y(x) = c_1 x e^{x^2/2} + c_0 + c_0 \displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)!\,x^{2n}}{(2n-1)!}$.

**17.** Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0!$. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A!$. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}!$. But $0.01 = Q'(0) = c_1 + 0.03$ and $0 = Q''(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2!$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

**18.** By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0$, $x(0) = 0$, $x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and $2.4 = x'(0) = -4c_1 + 4c_2 \quad\Rightarrow\quad c_1 = 0, c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

**19.** (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density $\rho$ as follows: $\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If $V_r$ is the volume of the portion of the earth which lies within a distance $r$ of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3} r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion, $m\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3} y$, so $y''(t) = -k^2 y(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so $g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the $r$ of Section 18.1 [ET 17.1], not the $r$ measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 18.1 [ET 17.1], the general solution of our differential equation for $t$ is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 18.3 [ET 17.3]) with amplitude $R$, frequency $k$, and phase angle 0. The period is $T = 2\pi/k$. $R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32$ ft/s$^2$, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3} s^{-1}$ and $T = 2\pi/k \approx 5079$ s $\approx 85$ min.

(d) $y(t) = 0 \quad\Leftrightarrow\quad \cos kt = 0 \quad\Leftrightarrow\quad kt = \frac{\pi}{2} + \pi n$ for some integer $n \quad\Rightarrow$ $y'(t) = -kR \sin\left(\frac{\pi}{2} + \pi n\right) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899$ mi/s $\approx 17{,}600$ mi/h.