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Chapter 17

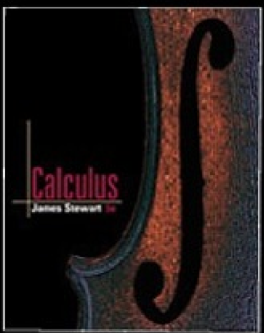
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



Calculus 5e
James Stewart
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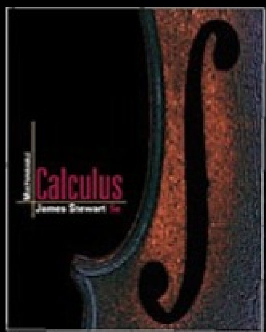

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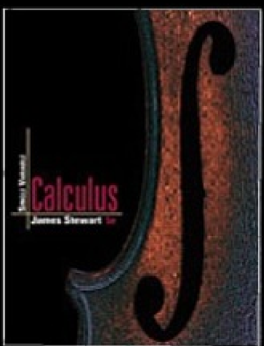

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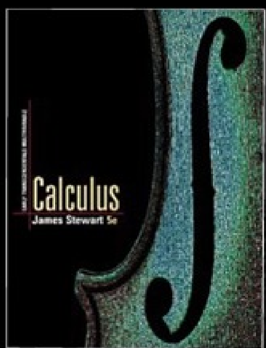

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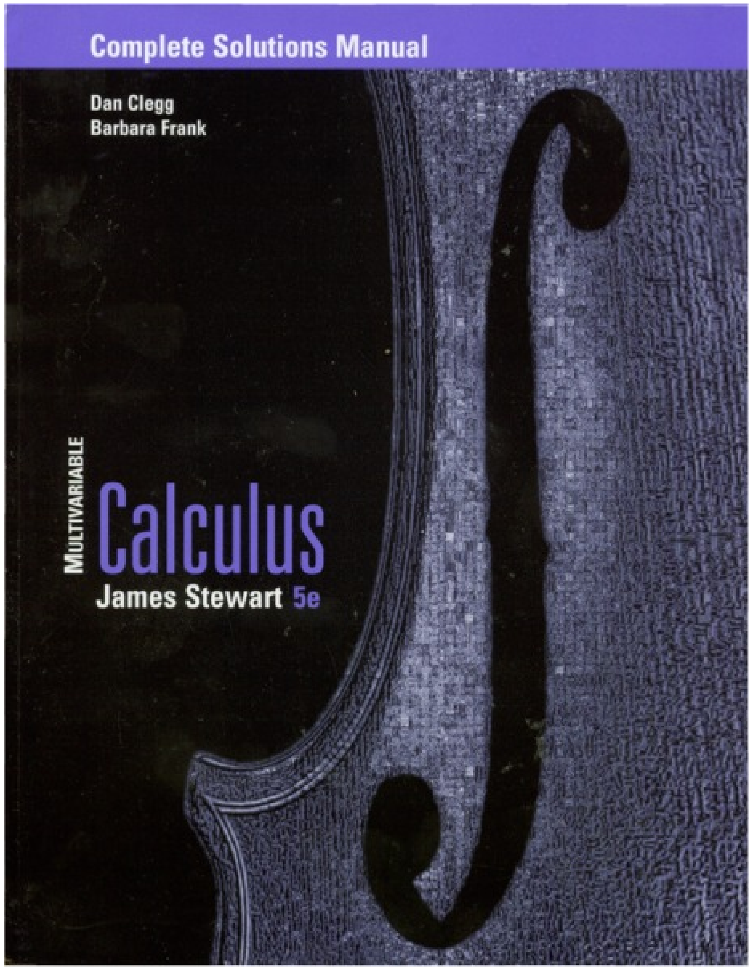
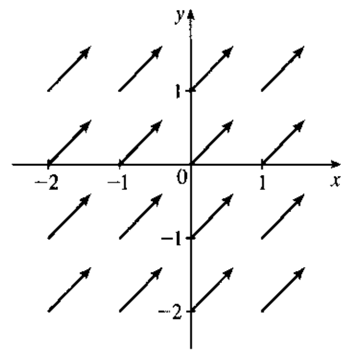


17 ☐ VECTOR CALCULUS ☐ ET 16

17.1 Vector Fields ET 16.1

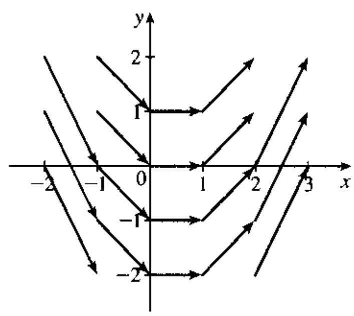
1. $\mathbf{F}(x,y) = \frac{1}{2}(\mathbf{i}+\mathbf{j})$

All vectors in this field are identical, with length $\frac{1}{\sqrt{2}}$ and direction parallel to the line $y = x$.



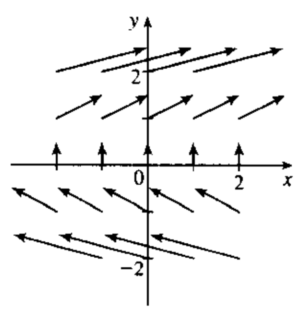
2. $\mathbf{F}(x,y) = \mathbf{i} + x\,\mathbf{j}$

The length of the vector $\mathbf{i} + x\,\mathbf{j}$ is $\sqrt{1+x^2}$. Vectors are tangent to parabolas opening about the y-axis.



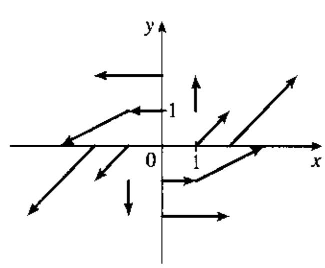
3. $\mathbf{F}(x,y) = y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$.

The length of the vector $y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$ is $\sqrt{y^2 + \frac{1}{4}}$. Vectors are tangent to parabolas opening about the x-axis.



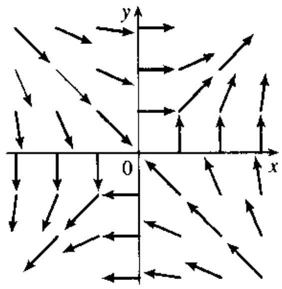
4. $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + x\,\mathbf{j}$

The length of the vector $(x-y)\,\mathbf{i} + x\,\mathbf{j}$ is $\sqrt{(x-y)^2 + x^2}$. Vectors along the line $y = x$ are vertical.



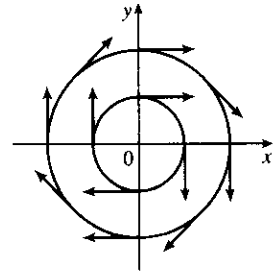
5. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2+y^2}}$ is 1.



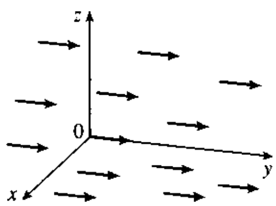
6. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

All the vectors $\mathbf{F}(x,y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2+y^2}$.



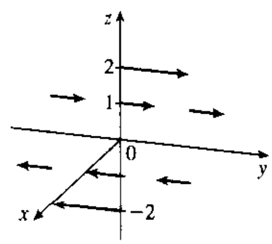
7. $\mathbf{F}(x,y,z) = \mathbf{j}$

All vectors in this field are parallel to the y-axis and have length 1.



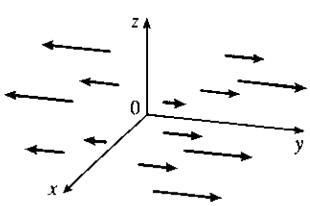
8. $\mathbf{F}(x,y,z) = z\,\mathbf{j}$

At each point (x,y,z), $\mathbf{F}(x,y,z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive y-axis while for $z < 0$, all are in the direction of the negative y-axis.



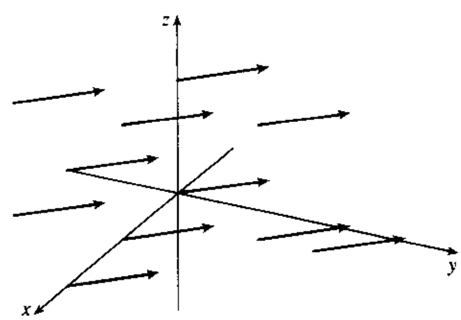
9. $\mathbf{F}(x,y,z) = y\,\mathbf{j}$

The length of $\mathbf{F}(x,y,z)$ is $|y|$. No vectors emanate from the xz-plane since $y = 0$ there. In each plane $y = b$, all the vectors are identical.



10. $\mathbf{F}(x,y,z) = \mathbf{j} - \mathbf{i}$

All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xy-plane.



11. $\mathbf{F}(x,y) = \langle y, x\rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. In addition, the vectors get shorter as we approach the origin.

12. $\mathbf{F}(x, y) = \langle 1, \sin y \rangle$ corresponds to graph IV since the x-component of each vector is constant, the vectors are independent of x (vectors along horizontal lines are identical), and the vector field appears to repeat the same pattern vertically.

13. $\mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x- and the y-components get larger.

14. $\mathbf{F}(x, y) = \langle y, 1/x \rangle$ corresponds to graph III. As in Exercise 11, all the vectors in the first quadrant have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. Also, the vectors become longer as we approach the y-axis.

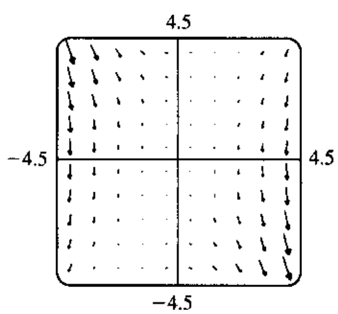
15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy-plane point generally upward while the vectors below the xy-plane point generally downward.

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

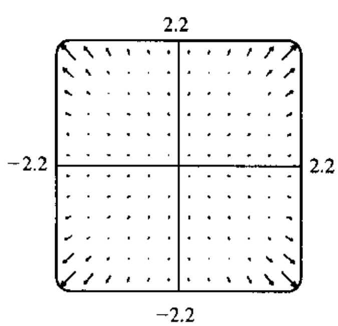
18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z), and therefore the vectors all point directly away from the origin.

19.



The vector field seems to have very short vectors near the line $y = 2x$. For $\mathbf{F}(x, y) = \langle 0, 0 \rangle$ we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$. The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x, y) = \mathbf{0}$] along the line $y = 2x$.

20.



From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\mathbf{x}| = 2$ and near the origin. Note that $\mathbf{F}(\mathbf{x}) = \mathbf{0} \quad \Leftrightarrow \quad r(r-2) = 0 \quad \Leftrightarrow \quad r = 0$ or 2, so as we suspected, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ for $|\mathbf{x}| = 2$ and for $|\mathbf{x}| = 0$. Note that where $r^2 - r < 0$, the vectors point towards the origin, and where $r^2 - r > 0$, they point away from the origin.

21. $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\,\mathbf{j} = \dfrac{1}{x + 2y}\,\mathbf{i} + \dfrac{2}{x + 2y}\,\mathbf{j}$

22. $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \left[x^\alpha\left(-\beta e^{-\beta x}\right) + \alpha x^{\alpha-1} e^{-\beta x}\right]\mathbf{i} + 0\,\mathbf{j} = (\alpha - \beta x)x^{\alpha-1}e^{-\beta x}\,\mathbf{i}$

23. $\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$$

24. $\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$

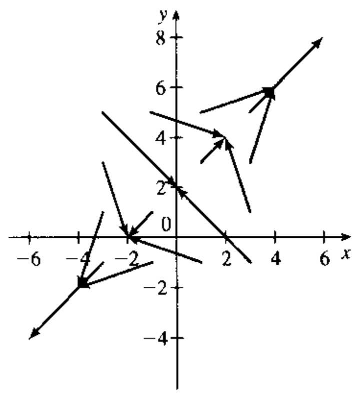
$$= \left(\cos\frac{y}{z}\right)\mathbf{i} - x\left(\sin\frac{y}{z}\right)\left(\frac{1}{z}\right)\mathbf{j} - x\left(\sin\frac{y}{z}\right)\left(-\frac{y}{z^2}\right)\mathbf{k}$$

$$= \left(\cos\frac{y}{z}\right)\mathbf{i} - \frac{x}{z}\left(\sin\frac{y}{z}\right)\mathbf{j} + \frac{xy}{z^2}\left(\sin\frac{y}{z}\right)\mathbf{k}$$

25. $f(x,y) = xy - 2x \quad\Rightarrow$

$\nabla f(x,y) = (y-2)\,\mathbf{i} + x\,\mathbf{j}$.

The length of $\nabla f(x,y)$ is $\sqrt{(y-2)^2 + x^2}$ and $\nabla f(x,y)$ terminates on the line $y = x + 2$ at the point $(x + y - 2, x + y)$.

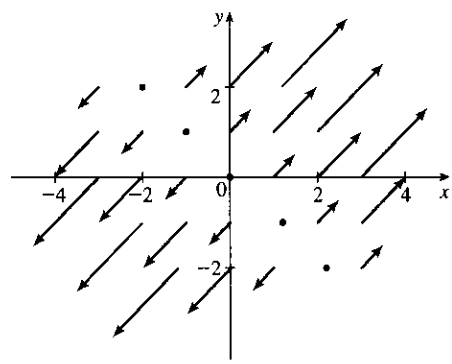


26. $f(x,y) = \frac{1}{4}(x+y)^2 \quad\Rightarrow$

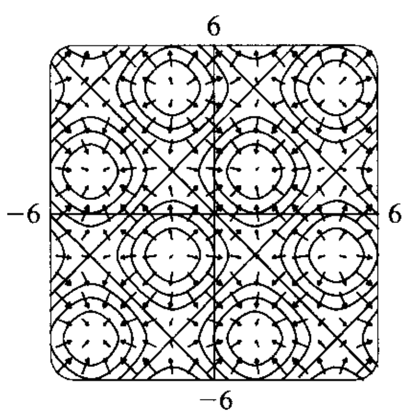
$\nabla f(x,y) = \frac{1}{2}(x+y)\,\mathbf{i} + \frac{1}{2}(x+y)\,\mathbf{j}$.

The length of $\nabla f(x,y)$ is

$\sqrt{\frac{1}{2}(x+y)^2} = \frac{1}{\sqrt{2}}\,|x+y|$. The vectors are perpendicular to the line $y = -x$ and point away from the line, with length that increases as the distance from the line $y = -x$ increases.

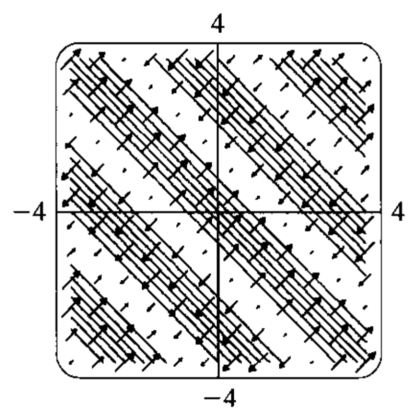


27. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

28. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

29. $f(x,y) = xy \;\Rightarrow\; \nabla f(x,y) = y\,\mathbf{i} + x\,\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x-component is positive while its y-component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.

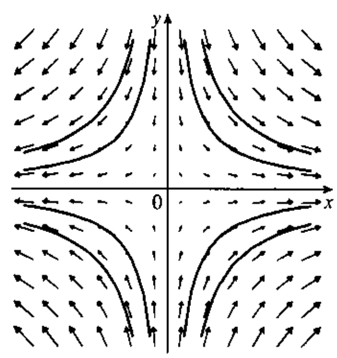
30. $f(x,y) = x^2 - y^2 \;\Rightarrow\; \nabla f(x,y) = 2x\,\mathbf{i} - 2y\,\mathbf{j}$. In the first quadrant, the x-component of each vector is positive while the y-component is negative. The other three quadrants are similar, where the x-component of each vector has the same sign as the x-value of its initial point, and the y-component has sign opposite that of the y-value of the initial point. Thus, ∇f is graph III.

31. $f(x,y) = x^2 + y^2 \;\Rightarrow\; \nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

32. $f(x,y) = \sqrt{x^2+y^2} \;\Rightarrow\; \nabla f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$. Then

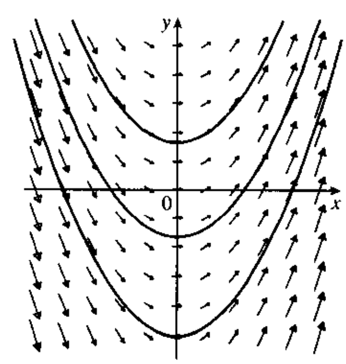
$|\nabla f(x,y)| = \dfrac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = 1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x,y)$ has the same direction as the position vector of the point (x,y), so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. (a) We sketch the vector field $\mathbf{F}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x,y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\,\mathbf{i} - y\,\mathbf{j} \;\Rightarrow\; dx/dt = x,\ dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \;\Rightarrow\; dx/x = dt \;\Rightarrow\; \ln|x| = t + C \;\Rightarrow\; x = \pm e^{t+C} = Ae^t$ for some constant A, and $dy/dt = -y \;\Rightarrow\; dy/y = -dt \;\Rightarrow\; \ln|y| = -t + K \;\Rightarrow\; y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^tBe^{-t} = AB =$ constant. If the flow line passes through $(1,1)$ then $(1)(1) =$ constant $= 1 \;\Rightarrow\; xy = 1 \;\Rightarrow\; y = 1/x,\ x > 0$.

34. (a) We sketch the vector field $\mathbf{F}(x,y) = \mathbf{i} + x\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be parabolas.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = \mathbf{i} + x\,\mathbf{j} \quad\Rightarrow\quad \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = x$. Thus $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x}{1} = x$.

(c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0,0)$ is on the curve, so $0 = 0 + c \quad\Rightarrow\quad c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

17.2 Line Integrals ET 16.2

1. $x = t^2$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y\,ds = \int_0^2 t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 t\,\sqrt{(2t)^2 + (1)^2}\,dt$$
$$= \int_0^2 t\,\sqrt{4t^2+1}\,dt = \tfrac{1}{12}\left(4t^2+1\right)^{3/2}\Big]_0^2 = \tfrac{1}{12}\left(17\sqrt{17}-1\right)$$

2. $\int_C \frac{y}{x}\,ds = \int_{1/2}^1 \frac{t^3}{t^4}\sqrt{(4t^3)^2 + (3t^2)^2}\,dt = \int_{1/2}^1 \frac{1}{t}\sqrt{16t^6 + 9t^4}\,dt = \int_{1/2}^1 t\,\sqrt{16t^2+9}\,dt$

$$= \tfrac{1}{48}(16t^2+9)^{3/2}\Big]_{1/2}^1 = \tfrac{1}{48}(25^{3/2} - 13^{3/2}) = \tfrac{1}{48}(125 - 13\sqrt{13})$$

3. Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}dt$$
$$= \int_{-\pi/2}^{\pi/2} 4^5\cos t\sin^4 t\sqrt{16\left(\sin^2 t + \cos^2 t\right)}dt$$
$$= 4^5\int_{-\pi/2}^{\pi/2}\left(\sin^4 t\cos t\right)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \tfrac{2\cdot 4^6}{5} = 1638.4$$

4. Parametric equations for C are $x = 1 + 3t$, $y = 2 + 5t$, $0 \le t \le 1$. Then

$$\int_C ye^x\,ds = \int_0^1 (2+5t)\,e^{1+3t}\sqrt{3^2+5^2}\,dt = \sqrt{34}\int_0^1 (2+5t)\,e^{1+3t}\,dt$$

Integrating by parts with $u = 2 + 5t \quad\Rightarrow\quad du = 5\,dt$, $dv = e^{1+3t} \quad\Rightarrow\quad v = \frac{1}{3}e^{1+3t}\,dt$ gives

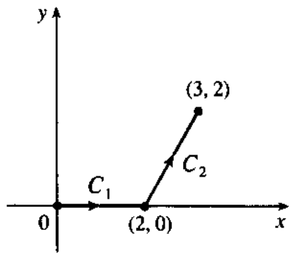
$$\int_C ye^x\,ds = \sqrt{34}\left[\tfrac{1}{3}(2+5t)\,e^{1+3t} - \tfrac{5}{9}e^{1+3t}\right]_0^1$$
$$= \sqrt{34}\left[\left(\tfrac{7}{3} - \tfrac{5}{9}\right)e^4 - \left(\tfrac{2}{3} - \tfrac{5}{9}\right)e\right] = \tfrac{\sqrt{34}}{9}\left(16e^4 - e\right)$$

5. If we choose x as the parameter, parametric equations for C are $x = x$, $y = x^2$ for $1 \le x \le 3$ and

$$\int_C (xy + \ln x)\;dy = \int_1^3 \left(x\cdot x^2 + \ln x\right)2x\,dx = \int_1^3 2\left(x^4 + x\ln x\right)dx$$
$$= 2\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^3 \quad \text{(by integrating by parts in the second term)}$$
$$= 2\left(\tfrac{243}{5} + \tfrac{9}{2}\ln 3 - \tfrac{9}{4} - \tfrac{1}{5} + \tfrac{1}{4}\right) = \tfrac{464}{5} + 9\ln 3$$

6. Choosing y as the parameter, we have $x = e^y$, $y = y$, $0 \le y \le 1$. Then $\int_c xe^y\,dx = \int_0^1 e^y(e^y)e^y\,dy = \int_0^1 e^{3y}\,dy = \frac{1}{3}e^{3y}\Big]_0^1 = \frac{1}{3}(e^3 - 1)$.

7.



$C = C_1 + C_2$

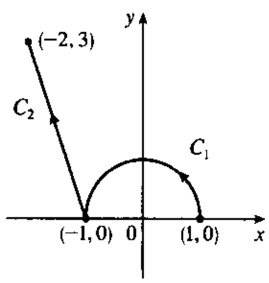
On C_1: $x = x, y = 0 \quad \Rightarrow \quad dy = 0\,dx, 0 \le x \le 2.$

On C_2: $x = x, y = 2x - 4 \quad \Rightarrow \quad dy = 2\,dx, 2 \le x \le 3.$

Then

$$\int_C xy\,dx + (x-y)\,dy = \int_{C_1} xy\,dx + (x-y)\,dy + \int_{C_2} xy\,dx + (x-y)\,dy$$
$$= \int_0^2 (0+0)\,dx + \int_2^3 [(2x^2 - 4x) + (-x+4)(2)]\,dx$$
$$= \int_2^3 (2x^2 - 6x + 8)\,dx = \tfrac{17}{3}$$

8.



$C = C_1 + C_2$

On C_1: $x = \cos t \quad \Rightarrow \quad dx = -\sin t\,dt, y = \sin t \quad \Rightarrow$

$dy = \cos t\;dt, 0 \le t \le \pi.$

On C_2: $x = -1 - t \quad \Rightarrow \quad dx = -dt, y = 3t \quad \Rightarrow$

$dy = 3\,dt, 0 \le t \le 1.$

Then
$$\int_C \sin x\,dx + \cos y\,dy = \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy$$
$$= \int_0^\pi \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)\cos t\,dt$$
$$+ \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt)$$
$$= \left[-\cos(\cos t) + \sin(\sin t)\right]_0^\pi + \left[-\cos(-1-t) + \sin(3t)\right]_0^1$$
$$= -\cos(\cos\pi) + \sin(\sin\pi) + \cos(\cos 0) - \sin(\sin 0)$$
$$- \cos(-2) + \sin(3) + \cos(-1) - \sin(0)$$
$$= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1)$$
$$= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3$$

where we have used the identity $\cos(-\theta) = \cos\theta$.

9. $x = 4\sin t, y = 4\cos t, z = 3t, 0 \le t \le \frac{\pi}{2}$. Then by Formula 9,

$$\int_C xy^3\,ds = \int_0^{\pi/2} (4\sin t)(4\cos t)^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
$$= \int_0^{\pi/2} 4^4 \cos^3 t \sin t\,\sqrt{(4\cos t)^2 + (-4\sin t)^2 + (3)^2}\,dt$$
$$= \int_0^{\pi/2} 256\cos^3 t \sin t\,\sqrt{16(\cos^2 t + \sin^2 t) + 9}\,dt$$
$$= 1280 \int_0^{\pi/2} \cos^3 t \sin t\,dt = -320\cos^4 t\Big]_0^{\pi/2} = 320$$

10. Parametric equations for C are $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$, $0 \le t \le 1$. Then

$$\int_C x^2 z\,ds = \int_0^1 (4t)^2(6t-1)\sqrt{4^2+(-5)^2+6^2}\,dt = \sqrt{77}\int_0^1 (96t^3 - 16t^2)\,dt$$
$$= \sqrt{77}\left[96\cdot\frac{t^4}{4} - 16\cdot\frac{t^3}{3}\right]_0^1 = \tfrac{56}{3}\sqrt{77}$$

11. Parametric equations for C are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt$$
$$= \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}(e^6 - 1)$$

12. $\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2} = \sqrt{1^2+(2t)^2+(3t^2)^2} = \sqrt{1+4t^2+9t^4}$. Then

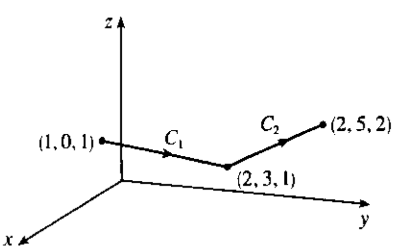
$$\int_C (2x+9z)\,ds = \int_0^1 (2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt \quad [\text{let } u = 1+4t^2+9t^4 \;\Rightarrow\; \tfrac14\,du = (2t+9t^3)\,dt]$$
$$= \int_1^{14}\tfrac14\sqrt{u}\,du = \tfrac16 u^{3/2}\Big]_1^{14} = \tfrac16(14^{3/2}-1)$$

13. $\int_C x^2y\sqrt{z}\,dz = \int_0^1 (t^3)^2(t)\sqrt{t^2}\cdot 2t\,dt = \int_0^1 2t^9\,dt = \tfrac15 t^{10}\Big]_0^1 = \tfrac15$

14. $\int_C z\,dx + x\,dy + y\,dz = \int_0^1 t^2\cdot 2t\,dt + t^2\cdot 3t^2\,dt + t^3\cdot 2t\,dt = \int_0^1 (2t^3+5t^4)\,dt$

$$= \left[\tfrac12 t^4 + t^5\right]_0^1 = \tfrac12 + 1 = \tfrac32$$

15.

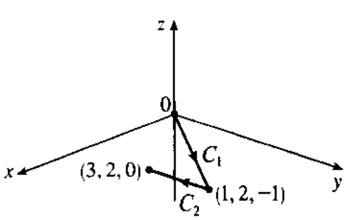


On C_1: $x = 1 + t \;\Rightarrow\; dx = dt$, $y = 3t \;\Rightarrow\; dy = 3\,dt$, $z = 1 \;\Rightarrow\; dz = 0\,dt$, $0 \le t \le 1$.
On C_2: $x = 2 \;\Rightarrow\; dx = 0\,dt$, $y = 3 + 2t \;\Rightarrow\; dy = 2\,dt$, $z = 1 + t \;\Rightarrow\; dz = dt$, $0 \le t \le 1$.

Then $\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$

$$= \int_{C_1}(x+yz)\,dx + 2x\,dy + xyz\,dz + \int_{C_2}(x+yz)\,dx + 2x\,dy + xyz\,dz$$
$$= \int_0^1 (1+t+(3t)(1))\,dt + 2(1+t)\cdot 3\,dt + (1+t)(3t)(1)\cdot 0\,dt$$
$$+ \int_0^1 (2+(3+2t)(1+t))\cdot 0\,dt + 2(2)\cdot 2\,dt + (2)(3+2t)(1+t)dt$$
$$= \int_0^1 (10t+7)\,dt + \int_0^1 (4t^2+10t+14)\,dt$$
$$= \left[5t^2+7t\right]_0^1 + \left[\tfrac43 t^3 + 5t^2 + 14t\right]_0^1 = 12 + \tfrac{61}{3} = \tfrac{97}{3}$$

16.



On C_1: $x = t \;\Rightarrow\; dx = dt$, $y = 2t \;\Rightarrow\; dy = 2\,dt$, $z = -t \;\Rightarrow\; dz = -dt$, $0 \le t \le 1$.
On C_2: $x = 1 + 2t \;\Rightarrow\; dx = 2\,dt$, $y = 2 \;\Rightarrow\; dy = 0\,dt$, $z = -1 + t \;\Rightarrow\; dz = dt$, $0 \le t \le 1$.

Then $\int_C x^2\,dx + y^2\,dy + z^2\,dz$

$$= \int_{C_1} x^2\,dx + y^2\,dy + z^2\,dz + \int_{C_2} x^2\,dx + y^2\,dy + z^2\,dz$$

$$= \int_0^1 t^2\,dt + (2t)^2 \cdot 2\,dt + (-t)^2(-dt) + \int_0^1 (1+2t)^2 \cdot 2\,dt + 2^2 \cdot 0\,dt + (-1+t)^2\,dt$$

$$= \int_0^1 8t^2\,dt + \int_0^1 (9t^2 + 6t + 3)\,dt = \left[\tfrac{8}{3}t^3\right]_0^1 + \left[3t^3 + 3t^2 + 3t\right]_0^1 = \tfrac{35}{3}$$

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F} \cdot \mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F} \cdot \mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\,ds$ is negative.

18. Vectors starting on C_1 point in roughly the same direction as C_1, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive. Then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\,ds$ to be negative.

19. $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = (t^2)^2(-t^3)^3\,\mathbf{i} - (-t^3)\sqrt{t^2}\,\mathbf{j} = -t^{13}\,\mathbf{i} + t^4\,\mathbf{j}$ and $\mathbf{r}'(t) = 2t\,\mathbf{i} - 3t^2\,\mathbf{j}$.
Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^1 \left(-2t^{14} - 3t^6\right) dt = \left[-\tfrac{2}{15}t^{15} - \tfrac{3}{7}t^7\right]_0^1 = -\tfrac{59}{105}$.

20. $\mathbf{F}(\mathbf{r}(t)) = (t^2)(t^3)\,\mathbf{i} + (t)(t^3)\,\mathbf{j} + (t)(t^2)\,\mathbf{k} = t^5\,\mathbf{i} + t^4\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$.
Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^2 \left(t^5 + 2t^5 + 3t^5\right) dt = \left.t^6\right]_0^2 = 64$.

21. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left\langle \sin t^3, \cos\left(-t^2\right), t^4 \right\rangle \cdot \left\langle 3t^2, -2t, 1 \right\rangle dt$

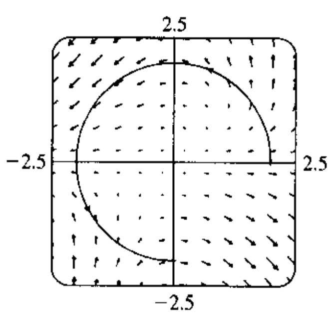
$$= \int_0^1 \left(3t^2 \sin t^3 - 2t\cos t^2 + t^4\right) dt = \left[-\cos t^3 - \sin t^2 + \tfrac{1}{5}t^5\right]_0^1 = \tfrac{6}{5} - \cos 1 - \sin 1$$

22. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \langle \cos t, \sin t, -t \rangle \cdot \langle 1, \cos t, -\sin t \rangle\,dt = \int_0^\pi (\cos t + \sin t \cos t + t \sin t)\,dt$

$$= \left[\sin t + \tfrac{1}{2}\sin^2 t + (\sin t - t\cos t)\right]_0^\pi = \pi$$

23. We graph $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$ to be positive.

To verify, we evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2\cos t - 2\sin t)\,\mathbf{i} + 4\cos t \sin t\,\mathbf{j}$ and $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Then



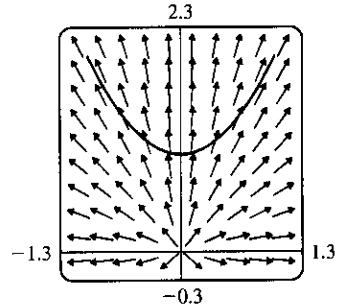
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{3\pi/2} \left[-2\sin t(2\cos t - 2\sin t) + 2\cos t(4\cos t \sin t)\right] dt$$

$$= 4\int_0^{3\pi/2} (\sin^2 t - \sin t \cos t + 2\sin t \cos^2 t)\,dt$$

$$= 3\pi + \tfrac{2}{3} \quad \text{[using a CAS]}$$

24. We graph $\mathbf{F}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$ and the curve C.



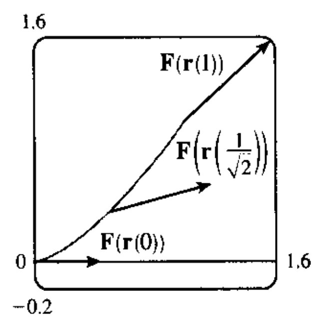
In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\mathbf{F}\cdot\mathbf{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants counteract each other, so it seems reasonable to guess that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is zero. To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = t\,\mathbf{i} + (1+t^2)\,\mathbf{j}$, $-1 \le t \le 1$, so

$\mathbf{F}(\mathbf{r}(t)) = \dfrac{t}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{i} + \dfrac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_{-1}^{1}\left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}}\right)dt \\ &= \int_{-1}^{1}\frac{t(3+2t^2)}{\sqrt{t^4+3t^2+1}}\,dt = 0 \quad \text{[since the integrand is an odd function]}\end{aligned}$$

25. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, t^5\right\rangle \cdot \langle 2t, 3t^2\rangle\,dt = \int_0^1 \left(2te^{t^2-1} + 3t^7\right)dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0\rangle$;



$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}}\right\rangle$;

$\mathbf{r}(1) = \langle 1, 1\rangle$, $\mathbf{F}(\mathbf{r}(1)) = \langle 1, 1\rangle$.

In order to generate the graph with Maple, we use the `PLOT` command (not to be confused with the `plot` command) to define each of the vectors. For example,

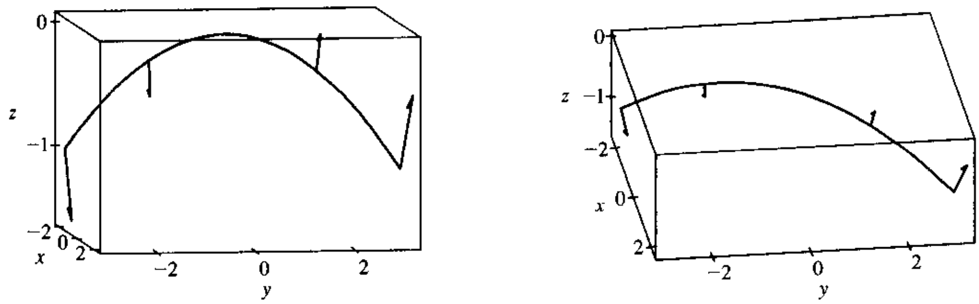
```
v1:=PLOT(CURVES([[0,0],[evalf(1/exp(1)),0]]));
```

generates the vector from the vector field at the point $(0,0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command. In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

26. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \langle 2t, t^2, 3t\rangle\cdot\langle 2, 3, -2t\rangle\,dt = \int_{-1}^{1}\left(4t + 3t^2 - 6t^2\right)dt$

$= \left[2t^2 - t^3\right]_{-1}^{1} = -2$

(b) Now $\mathbf{F}(\mathbf{r}(t)) = \langle 2t, t^2, 3t \rangle$, so $\mathbf{F}(\mathbf{r}(-1)) = \langle -2, 1, -3 \rangle$, $\mathbf{F}(\mathbf{r}(-\frac{1}{2})) = \langle -1, \frac{1}{4}, -\frac{3}{2} \rangle$, $\mathbf{F}(\mathbf{r}(\frac{1}{2})) = \langle 1, \frac{1}{4}, \frac{3}{2} \rangle$, and $\mathbf{F}(\mathbf{r}(1)) = \langle 2, 1, 3 \rangle$.



27. The part of the astroid that lies in the quadrant is parametrized by $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$.

Now $\dfrac{dx}{dt} = 3\cos^2 t\,(-\sin t)$ and $\dfrac{dy}{dt} = 3\sin^2 t \cos t$, so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\cos t \sin t\,\sqrt{\cos^2 t + \sin^2 t} = 3\cos t \sin t.$$

Therefore $\int_C x^3 y^5\,ds = \int_0^{\pi/2} \cos^9 t \sin^{15} t\,(3\cos t \sin t)\,dt = \frac{945}{16{,}777{,}216}\pi$.

28. We parametrize the line as $\mathbf{r}(t) = \langle 1, 2, 1 \rangle + t\,(\langle 6, 4, 5 \rangle - \langle 1, 2, 1 \rangle) = (1 + 5t)\,\mathbf{i} + (2 + 2t)\,\mathbf{j} + (1 + 4t)\,\mathbf{k}$, $0 \le t \le 1$. Using a CAS, we calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left\langle (1 + 5t)^4 e^{2+2t}, \ln(1 + 4t), \sqrt{(2 + 2t)^2 + (1 + 4t)^2} \right\rangle \cdot \langle 5, 2, 4 \rangle\,dt$$

$$= \frac{5235e^4}{4} - \frac{6285e^2}{4} + \frac{9\sqrt{5}\sinh^{-1}\left(\frac{14}{3}\right)}{25} - \frac{9\sqrt{5}\sinh^{-1}\left(\frac{4}{3}\right)}{25} + \frac{5\ln 5}{2} + \frac{14\sqrt{41}}{5} - \frac{4\sqrt{5}}{5} - 2$$

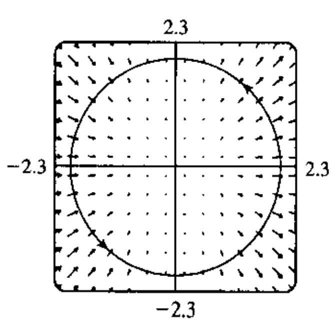
$$= \frac{5235e^4}{4} - \frac{6285e^2}{4} - \frac{18\sqrt{5}\ln 3}{25} + \frac{9\sqrt{5}\ln\left(14 + \sqrt{205}\right)}{25} + \frac{5\ln 5}{2} + \frac{14\sqrt{41} - 4\sqrt{5}}{5} - 2$$

The first answer is the one given by Maple. The two answers are equivalent by Equation 7.6.3 [ET 3.9.3].

29. A calculator or CAS gives $\int_C x \sin y\,ds = \int_1^2 \ln t \sin\left(e^{-t}\right) \sqrt{(1/t)^2 + (-e^{-t})^2}\,dt \approx 0.052$.

30. (a) We parametrize the circle C as $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. So $\mathbf{F}(\mathbf{r}(t)) = \langle 4\cos^2 t, 4\cos t \sin t \rangle$, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t \rangle$, and $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-8\cos^2 t \sin t + 8\cos^2 t \sin t)\,dt = 0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C, $\mathbf{F} \cdot \mathbf{T} = 0$, and so certainly $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

31. We use the parametrization $x = 2\cos t$, $y = 2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

Then $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = 2\,dt$, so

$m = \int_C k\,ds = 2k\int_{-\pi/2}^{\pi/2} dt = 2k(\pi)$, $\overline{x} = \frac{1}{2\pi k}\int_C xk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt = \frac{1}{2\pi}\left[4\sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$,

$\overline{y} = \frac{1}{2\pi k}\int_C yk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt = 0$. Hence $(\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, 0\right)$.

32. We use the parametrization $x = r\cos t$, $y = r\sin t$, $0 \le t \le \frac{\pi}{2}$.

Then $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt = r\,dt$, so

$m = \int_C (x+y)\,ds = \int_0^{\pi/2}(r\cos t + r\sin t)\,r\,dt = r^2\left[\sin t - \cos t\right]_0^{\pi/2} = 2r^2$,

$$\overline{x} = \frac{1}{2r^2}\int_C x(x+y)\,ds = \frac{1}{2r^2}\int_0^{\pi/2}(r^2\cos^2 t + r^2\cos t\sin t)r\,dt = \frac{r}{2}\left[\frac{t}{2} + \frac{\sin 2t}{4} - \frac{\cos 2t}{4}\right]_0^{\pi/2}$$
$$= \frac{r(\pi+2)}{8}, \text{ and}$$

$$\overline{y} = \frac{1}{2r^2}\int_C y(x+y)\,ds = \frac{1}{2r^2}\int_0^{\pi/2}(r^2\sin t\cos t + r^2\sin^2 t)r\,dt$$
$$= \frac{r}{2}\left[-\frac{\cos 2t}{4} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi/2} = \frac{r(\pi+2)}{8}.$$

Therefore $(\overline{x}, \overline{y}) = \left(\frac{r(\pi+2)}{8}, \frac{r(\pi+2)}{8}\right)$.

33. (a) $\overline{x} = \frac{1}{m}\int_C x\rho(x,y,z)\,ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x,y,z)\,ds$, $\overline{z} = \frac{1}{m}\int_C z\rho(x,y,z)\,ds$ where $m = \int_C \rho(x,y,z)\,ds$.

(b) $m = \int_C k\,ds = k\int_0^{2\pi}\sqrt{4\sin^2 t + 4\cos^2 t + 9}\,dt = k\sqrt{13}\int_0^{2\pi} dt = 2\pi k\sqrt{13}$,

$$\overline{x} = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi} k2\sqrt{13}\sin t\,dt = 0,\ \overline{y} = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi} k2\sqrt{13}\cos t\,dt = 0,$$
$$\overline{z} = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}\left(k\sqrt{13}\right)(3t)\,dt = \frac{3}{2\pi}\left(2\pi^2\right) = 3\pi.$$ Hence $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 3\pi)$.

34. $m = \int_C (x^2 + y^2 + z^2)\,ds = \int_0^{2\pi}(t^2+1)\sqrt{(1)^2 + (-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi}(t^2+1)\sqrt{2}\,dt$
$= \sqrt{2}\left(\frac{8}{3}\pi^3 + 2\pi\right)$,

$$\overline{x} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3 + 2\pi\right)}\int_0^{2\pi}\sqrt{2}\,(t^3 + t)\,dt = \frac{4\pi^4 + 2\pi^2}{\frac{8}{3}\pi^3 + 2\pi} = \frac{3\pi(2\pi^2+1)}{4\pi^2+3},$$

$$\overline{y} = \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi}\left(\sqrt{2}\cos t\right)(t^2+1)\,dt = 0, \text{ and}$$

$$\overline{z} = \frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi}\left(\sqrt{2}\sin t\right)(t^2+1)\,dt = 0.$$ Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{3\pi(2\pi^2+1)}{4\pi^2+3}, 0, 0\right)$.

35. From Example 3, $\rho(x,y) = k(1-y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi \;\Rightarrow$

$$I_x = \int_C y^2 \rho(x,y)\,ds = \int_0^\pi \sin^2 t\,[k(1-\sin t)]\,dt = k\int_0^\pi (\sin^2 t - \sin^3 t)\,dt$$

$$= \tfrac{1}{2}k\int_0^\pi (1-\cos 2t)\,dt - k\int_0^\pi (1-\cos^2 t)\sin t\,dt \qquad \begin{bmatrix}\text{Let } u = \cos t,\ du = -\sin t\,dt \\ \text{in the second integral}\end{bmatrix}$$

$$= k\left[\tfrac{\pi}{2} + \int_1^{-1}(1-u^2)\,du\right] = k\left(\tfrac{\pi}{2} - \tfrac{4}{3}\right)$$

$$I_y = \int_C x^2\rho(x,y)\,ds = k\int_0^\pi \cos^2 t\,(1-\sin t)\,dt = \tfrac{k}{2}\int_0^\pi (1+\cos 2t)\,dt - k\int_0^\pi \cos^2 t\sin t\,dt$$

$$= k\left(\tfrac{\pi}{2} - \tfrac{2}{3}\right), \text{ using the same substitution as above.}$$

36. The wire is given as $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \le t \le 2\pi$ with $\rho(x,y,z) = k$. Then

$$ds = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + 3^2} = \sqrt{4(\cos^2 t + \sin^2 t) + 9} = \sqrt{13} \text{ and}$$

$$I_x = \int_C (y^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\cos^2 t + 9t^2)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$$

$$= \sqrt{13}\,k(4\pi + 24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$$

$$I_y = \int_C (x^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\sin^2 t + 9t^2)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$$

$$= \sqrt{13}\,k(4\pi + 24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$$

$$I_z = \int_C (x^2+y^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\sin^2 t + 4\cos^2 t)(k)\sqrt{13}\,dt = 4\sqrt{13}\,k\int_0^{2\pi} dt = 8\pi\sqrt{13}\,k$$

37. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle t - \sin t, 3 - \cos t\rangle \cdot \langle 1 - \cos t, \sin t\rangle\,dt$

$$= \int_0^{2\pi}(t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\,dt$$

$$= \int_0^{2\pi}(t - t\cos t + 2\sin t)\,dt = \left[\tfrac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi} \qquad \begin{bmatrix}\text{by integrating by parts} \\ \text{in the second term}\end{bmatrix}$$

$$= 2\pi^2$$

38. $x = x$, $y = x^2$, $-1 \le x \le 2$,

$$W = \int_{-1}^2 \langle x\sin x^2, x^2\rangle \cdot \langle 1, 2x\rangle\,dx = \int_{-1}^2 (x\sin x^2 + 2x^3)\,dx = \left[-\tfrac{1}{2}\cos x^2 + \tfrac{1}{2}x^4\right]_{-1}^2$$

$$= \tfrac{1}{2}(15 + \cos 1 - \cos 4)$$

39. $\mathbf{r}(t) = \langle 1 + 2t, 4t, 2t\rangle$, $0 \le t \le 1$,

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 6t, 1+4t, 1+6t\rangle \cdot \langle 2,4,2\rangle\,dt = \int_0^1 (12t + 4(1+4t) + 2(1+6t))\,dt$$

$$= \int_0^1 (40t + 6)\,dt = \left[20t^2 + 6t\right]_0^1 = 26$$

40. $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{j} + 5t\,\mathbf{k}$, $0 \le t \le 1$. Therefore

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \frac{K\langle 2, t, 5t\rangle}{(4+26t^2)^{3/2}} \cdot \langle 0,1,5\rangle\,dt = K\int_0^1 \frac{26t}{(4+26t^2)^{3/2}}\,dt$$

$$= K\left[-(4+26t^2)^{-1/2}\right]_0^1 = K\left(\tfrac{1}{2} - \tfrac{1}{\sqrt{30}}\right)$$

41. Let $\mathbf{F} = 185\,\mathbf{k}$. To parametrize the staircase, let

$x = 20\cos t,\ y = 20\sin t,\ z = \frac{90}{6\pi}t = \frac{15}{\pi}t,\ 0 \le t \le 6\pi \quad\Rightarrow$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{6\pi} \langle 0,0,185\rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi}\right\rangle dt = (185)\tfrac{15}{\pi}\int_0^{6\pi} dt = (185)(90)$$

$$\approx 1.67\times 10^4 \text{ ft-lb}$$

42. This time m is a function of t: $m = 185 - \frac{9}{6\pi}t = 185 - \frac{3}{2\pi}t$. So let $\mathbf{F} = \left(185 - \frac{3}{2\pi}t\right)\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t$, $y = 20\sin t$, $z = \frac{90}{6\pi}t = \frac{15}{\pi}t$, $0 \le t \le 6\pi$. Therefore

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{6\pi} \left\langle 0,0,185 - \tfrac{3}{2\pi}t\right\rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi}\right\rangle dt = \tfrac{15}{\pi}\int_0^{6\pi}\left(185 - \tfrac{3}{2\pi}t\right) dt$$

$$= \tfrac{15}{\pi}\left[185t - \tfrac{3}{4\pi}t^2\right]_0^{6\pi} = 90\left(185 - \tfrac{9}{2}\right) \approx 1.62\times 10^4 \text{ ft-lb}$$

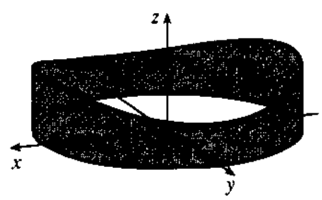
43. (a) $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, and let $\mathbf{F} = \langle a, b\rangle$. Then

$$\begin{aligned} W = \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \langle a,b\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} (-a\sin t + b\cos t)\, dt = \left[a\cos t + b\sin t\right]_0^{2\pi} \\ &= a + 0 - a + 0 = 0 \end{aligned}$$

(b) Yes. $\mathbf{F}(x,y) = k\,\mathbf{x} = \langle kx, ky\rangle$ and

$$\begin{aligned} W = \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \langle k\cos t, k\sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^{2\pi} (-k\sin t\cos t + k\sin t\cos t)\, dt = \int_0^{2\pi} 0\, dt = 0 \end{aligned}$$

44. Consider the base of the fence in the xy-plane, centered at the origin, with the height given by $z = h(x,y)$. The fence can be graphed using the parametric equations



$x = 10\cos u$, $y = 10\sin u$,

$$\begin{aligned} z &= v\left[4 + 0.01((10\cos u)^2 - (10\sin u)^2)\right] \\ &= v(4 + \cos^2 u - \sin^2 u) \\ &= v(4 + \cos 2u),\ \ 0 \le u \le 2\pi,\ \ 0 \le v \le 1. \end{aligned}$$

The area of the fence is $\int_C h(x,y)\, ds$ where C, the base of the fence, is given by $x = 10\cos t$, $y = 10\sin t$, $0 \le t \le 2\pi$. Then

$$\begin{aligned} \int_C h(x,y)\, ds &= \int_0^{2\pi} \left[4 + 0.01((10\cos t)^2 - (10\sin t)^2)\right] \sqrt{(-10\sin t)^2 + (10\cos t)^2}\, dt \\ &= \int_0^{2\pi} (4 + \cos 2t)\sqrt{100}\, dt = 10\left[4t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} \\ &= 10(8\pi) = 80\pi \text{ m}^2 \end{aligned}$$

If we paint both sides of the fence, the total surface area to cover is 160π m^2, and since 1 L of paint covers 100 m^2, we require $\frac{160\pi}{100} = 1.6\pi \approx 5.03$ L of paint.

45. The work done in moving the object is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\, ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F}\cdot\mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F} \cdot \mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds \approx \sum_{i=1}^{7} [\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)]\, \Delta s = [2+2+2+2+1+1+1](2) = 22$. Thus, we estimate the work done to be approximately 22 J.

46. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\, \mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta$, $y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}| \langle -\sin\theta, \cos\theta\rangle$. Then

$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}| \langle -\sin\theta, \cos\theta\rangle \cdot \langle -r\sin\theta, r\cos\theta\rangle\, d\theta = \int_0^{2\pi} |\mathbf{B}|\, r\, d\theta = 2\pi r\, |\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

17.3 The Fundamental Theorem for Line Integrals ET 16.3

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

2. C is represented by the vector function $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + (t^3+t)\,\mathbf{j}$, $0 \le t \le 1$, so $\mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2+1)\,\mathbf{j}$. Since $3t^2 + 1 \ne 0$, we have $\mathbf{r}'(t) \ne \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2,2) - f(1,0) = 9 - 3 = 6$.

3. $\partial(6x+5y)/\partial y = 5 = \partial(5x+4y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = 6x + 5y$ and $f_y(x,y) = 5x + 4y$. But $f_x(x,y) = 6x + 5y$ implies $f(x,y) = 3x^2 + 5xy + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = 5x + g'(y)$. Thus $5x + 4y = 5x + g'(y)$ so $g'(y) = 4y$ and $g(y) = 2y^2 + K$ where K is a constant. Hence $f(x,y) = 3x^2 + 5xy + 2y^2 + K$ is a potential function for $\mathbf{F}$.

4. $\partial(x^3 + 4xy)/\partial y = 4x$, $\partial(4xy - y^3)/\partial x = 4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

5. $\partial(xe^y)/\partial y = xe^y$, $\partial(ye^x)/\partial x = ye^x$. Since these are not equal, $\mathbf{F}$ is not conservative.

6. $\partial(e^y)/\partial y = e^y = \partial(xe^y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = e^y$ implies $f(x,y) = xe^y + g(y)$ and $f_y(x,y) = xe^y + g'(y)$. But $f_y(x,y) = xe^y$ so $g'(y) = 0 \;\Rightarrow\; g(y) = K$. Then $f(x,y) = xe^y + K$ is a potential function for $\mathbf{F}$.

7. $\partial(2x\cos y - y\cos x)/\partial y = -2x\sin y - \cos x = \partial(-x^2\sin y - \sin x)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 2x\cos y - y\cos x$ implies $f(x,y) = x^2\cos y - y\sin x + g(y)$ and $f_y(x,y) = -x^2\sin y - \sin x + g'(y)$. But $f_y(x,y) = -x^2\sin y - \sin x$ so $g'(y) = 0 \;\Rightarrow\; g(y) = K$. Then $f(x,y) = x^2\cos y - y\sin x + K$ is a potential function for $\mathbf{F}$.

8. $\partial(1 + 2xy + \ln x)/\partial y = 2x = \partial(x^2)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid x > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 1 + 2xy + \ln x$ implies $f(x,y) = x + x^2y + x\ln x - x + g(y)$ and $f_y(x,y) = x^2 + g'(y)$. But $f_y(x,y) = x^2$ so $g'(y) = 0 \;\Rightarrow\; g(y) = K$. Then $f(x,y) = x^2y + x\ln x + K$ is a potential function for $\mathbf{F}$.

9. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

10. $\dfrac{\partial(xy\cosh xy + \sinh xy)}{\partial y} = x^2y\sinh xy + x\cosh xy + x\cosh xy = x^2y\sinh xy + 2x\cosh xy = \dfrac{\partial(x^2\cosh xy)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = xy\cosh xy + \sinh xy$ implies $f(x,y) = x\sinh xy + g(y) \Rightarrow f_y(x,y) = x^2\cosh xy + g'(y)$. But $f_y(x,y) = x^2\cosh xy$ so $g(y) = K$ and $f(x,y) = x\sinh xy + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}2xy = 2x = \dfrac{\partial}{\partial x}(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F}\cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, a constant. Thus $f(x,y) = x^2y + K$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

12. (a) $f_x(x,y) = y$ implies $f(x,y) = xy + g(y)$ and $f_y(x,y) = x + g'(y)$. But $f_y(x,y) = x + 2y$ so $g'(y) = 2y \Rightarrow g(y) = y^2 + K$. We can take $K = 0$, so $f(x,y) = xy + y^2$.

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,1) - f(0,1) = 3 - 1 = 2$.

13. (a) $f_x(x,y) = x^3y^4$ implies $f(x,y) = \frac{1}{4}x^4y^4 + g(y)$ and $f_y(x,y) = x^4y^3 + g'(y)$. But $f_y(x,y) = x^4y^3$ so $g'(y) = 0 \Rightarrow g(y) = K$, a constant. We can take $K = 0$, so $f(x,y) = \frac{1}{4}x^4y^4$.

(b) The initial point of C is $\mathbf{r}(0) = (0,1)$ and the terminal point is $\mathbf{r}(1) = (1,2)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2) - f(0,1) = 4 - 0 = 4$.

14. (a) $f_x(x,y) = y^2/(1+x^2)$ implies $f(x,y) = y^2\arctan x + g(y) \Rightarrow f_y(x,y) = 2y\arctan x + g'(y)$. But $f_y(x,y) = 2y\arctan x$ so $g'(y) = 0 \Rightarrow g(y) = K$. We can take $K = 0$, so $f(x,y) = y^2\arctan x$.

(b) The initial point of C is $\mathbf{r}(0) = (0,0)$ and the terminal point is $\mathbf{r}(1) = (1,2)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2) - f(0,0) = 4\arctan 1 - 0 = 4\cdot\frac{\pi}{4} = \pi$.

15. (a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$ so $g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$ and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z \Rightarrow h(z) = z^2 + K$. Hence $f(x,y,z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

16. (a) $f_x(x,y,z) = 2xz + y^2$ implies $f(x,y,z) = x^2z + xy^2 + g(y,z)$ and so $f_y(x,y,z) = 2xy + g_y(y,z)$. But $f_y(x,y,z) = 2xy$ so $g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$. Thus $f(x,y,z) = x^2z + xy^2 + h(z)$ and $f_z(x,y,z) = x^2 + h'(z)$. But $f_z(x,y,z) = x^2 + 3z^2$, so $h'(z) = 3z^2 \Rightarrow h(z) = z^3 + K$. Hence $f(x,y,z) = x^2z + xy^2 + z^3$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,1,-1)$ and $t = 1$ corresponds to $(1,2,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2,1) - f(0,1,-1) = 6 - (-1) = 7$.

17. (a) $f_x(x,y,z) = y^2 \cos z$ implies $f(x,y,z) = xy^2 \cos z + g(y,z)$ and so $f_y(x,y,z) = 2xy \cos z + g_y(y,z)$. But $f_y(x,y,z) = 2xy \cos z$ so $g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$. Thus $f(x,y,z) = xy^2 \cos z + h(z)$ and $f_z(x,y,z) = -xy^2 \sin z + h'(z)$. But $f_z(x,y,z) = -xy^2 \sin z$, so $h'(z) = 0 \Rightarrow h(z) = K$. Hence $f(x,y,z) = xy^2 \cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0,0,0 \rangle$, $\mathbf{r}(\pi) = \langle \pi^2, 0, \pi \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi^2, 0, \pi) - f(0,0,0) = 0 - 0 = 0$.

18. (a) $f_x(x,y,z) = e^y$ implies $f(x,y,z) = xe^y + g(y,z)$ and so $f_y(x,y,z) = xe^y + g_y(y,z)$. But $f_y(x,y,z) = xe^y$ so $g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$. Thus $f(x,y,z) = xe^y + h(z)$ and $f_z(x,y,z) = 0 + h'(z)$. But $f_z(x,y,z) = (z+1)e^z$, so $h'(z) = (z+1)e^z \Rightarrow h(z) = ze^z + K$ (using integration by parts). Hence $f(x,y,z) = xe^y + ze^z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0,0,0 \rangle$, $\mathbf{r}(1) = \langle 1,1,1 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 2e - 0 = 2e$.

19. Here $\mathbf{F}(x,y) = \tan y\,\mathbf{i} + x \sec^2 y\,\mathbf{j}$. Then $f(x,y) = x \tan y$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C \tan y\,dx + x \sec^2 y\,dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2, \frac{\pi}{4}) - f(1,0) = 2 \tan \frac{\pi}{4} - \tan 0 = 2$.

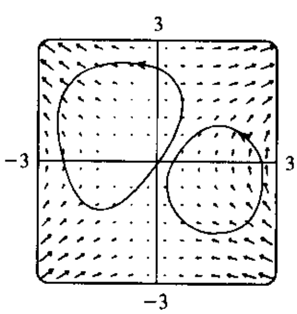
20. Here $\mathbf{F}(x,y) = (1 - ye^{-x})\,\mathbf{i} + e^{-x}\,\mathbf{j}$. Then $f(x,y) = x + ye^{-x}$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C (1 - ye^{-x})\,dx + e^{-x}\,dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2) - f(0,1) = (1 + 2e^{-1}) - 1 = 2/e$.

21. $\mathbf{F}(x,y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(2y^{3/2})/\partial y = 3\sqrt{y} = \partial(3x\sqrt{y})/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x,y) = 2y^{3/2} \Rightarrow f(x,y) = 2xy^{3/2} + g(y) \Rightarrow f_y(x,y) = 3xy^{1/2} + g'(y)$. But $f_y(x,y) = 3x\sqrt{y}$ so $g'(y) = 0$ or $g(y) = K$. We can take $K = 0 \Rightarrow f(x,y) = 2xy^{3/2}$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2,4) - f(1,1) = 2(2)(8) - 2(1) = 30$.

22. $\mathbf{F}(x,y) = \dfrac{y^2}{x^2}\,\mathbf{i} - \dfrac{2y}{x}\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\dfrac{\partial}{\partial y}\left(\dfrac{y^2}{x^2}\right) = \dfrac{2y}{x^2} = \dfrac{\partial}{\partial x}\left(-\dfrac{2y}{x}\right)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = y^2/x^2 \Rightarrow f(x,y) = -y^2/x + g(y) \Rightarrow f_y = -2y/x + g'(y) \Rightarrow g'(y) = 0$, so we can take $f(x,y) = -y^2/x$ as a potential function for $\mathbf{F}$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(4,-2) - f(1,1) = -\left[(-2)^2/4\right] + (1/1) = 0$.

23. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be some circle centered at the origin, oriented counterclockwise. All of the field vectors along C oppose motion along C, so the integral around C will be negative. Therefore the field is not conservative.

24.

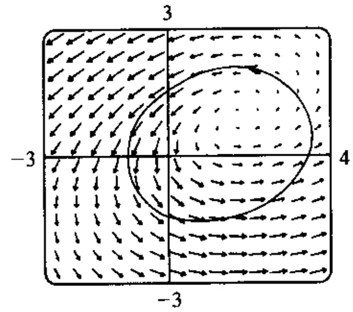


From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate

$$\frac{\partial}{\partial y}(2xy + \sin y) = 2x + \cos y, \quad \frac{\partial}{\partial x}(x^2 + x\cos y) = 2x + \cos y$$

Thus $\mathbf{F}$ is conservative, by Theorem 6.

25. From the graph, it appears that $\mathbf{F}$ is not conservative. For example, any closed curve containing the point $(2, 1)$ seems to have many field vectors pointing counterclockwise along it, and none pointing clockwise. So along this path the integral $\int \mathbf{F} \cdot d\mathbf{r} \neq 0$. To confirm our guess, we calculate



$$\frac{\partial}{\partial y}\left(\frac{x-2y}{\sqrt{1+x^2+y^2}}\right) = (x-2y)\left[\frac{-y}{(1+x^2+y^2)^{3/2}}\right] - \frac{2}{\sqrt{1+x^2+y^2}} = \frac{-2-2x^2-xy}{(1+x^2+y^2)^{3/2}},$$

$$\frac{\partial}{\partial x}\left(\frac{x-2}{\sqrt{1+x^2+y^2}}\right) = (x-2)\left[\frac{-x}{(1+x^2+y^2)^{3/2}}\right] + \frac{1}{\sqrt{1+x^2+y^2}} = \frac{1+y^2+2x}{(1+x^2+y^2)^{3/2}}.$$

These are not equal, so the field is not conservative, by Theorem 5.

26. $\nabla f(x, y) = \cos(x-2y)\,\mathbf{i} - 2\cos(x-2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0,0)$ to (π,π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j}$, $0 \leq t \leq 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2} t\,\mathbf{i}$, $0 \leq t \leq 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$.

27. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

28. Here $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 27, $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

29. $D = \{(x,y) \mid x > 0, y > 0\}$ = the first quadrant (excluding the axes).

(a) D is open because around every point in D we can put a disk that lies in D.

(b) D is connected because the straight line segment joining any two points in D lies in D.

(c) D is simply-connected because it's connected and has no holes.

30. $D = \{(x,y) \mid x \neq 0\}$ consists of all points in the xy-plane except for those on the y-axis.

(a) D is open.

(b) Points on opposite sides of the y-axis cannot be joined by a path that lies in D, so D is not connected.

(c) D is not simply-connected because it is not connected.

31. $D = \{(x,y) \mid 1 < x^2 + y^2 < 4\}$ = the annular region between the circles with center $(0,0)$ and radii 1 and 2.

(a) D is open.

(b) D is connected.

(c) D is not simply-connected. For example, $x^2 + y^2 = (1.5)^2$ is simple and closed and lies within D but encloses points that are not in D. (Or we can say, D has a hole, so is not simply-connected.)

32. $D = \{(x,y) \mid x^2+y^2 \le 1 \text{ or } 4 \le x^2+y^2 \le 9\}$ = the points on or inside the circle $x^2+y^2=1$, together with the points on or between the circles $x^2+y^2=4$ and $x^2+y^2=9$.

(a) D is not open because, for instance, no disk with center $(0,2)$ lies entirely within D.

(b) D is not connected because, for example, $(0,0)$ and $(0,2.5)$ lie in D but cannot be joined by a path that lies entirely in D.

(c) D is not simply-connected because, for example, $x^2+y^2=9$ is a simple closed curve in D but encloses points that are not in D.

33. (a) $P = -\dfrac{y}{x^2+y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ and $Q = \dfrac{x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x=\cos t$, $y=\sin t$, $0 \le t \le \pi$, C_2: $x=\cos t$, $y=\sin t$, $t=2\pi$ to $t=\pi$. Then

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t)+(\cos t)(\cos t)}{\cos^2 t+\sin^2 t}\,dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{2\pi}^\pi dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2+y^2=1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

34. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 17.1 [ET 16.1].) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$.

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(P_2)-f(P_1) = -\frac{c}{(x_2^2+y_2^2+z_2^2)^{1/2}} + \frac{c}{(x_1^2+y_1^2+z_1^2)^{1/2}} = c\left(\frac{1}{d_1}-\frac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \;\Rightarrow$

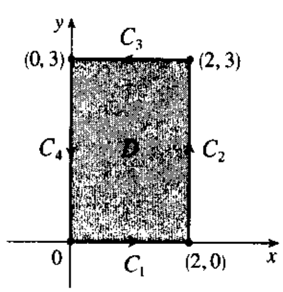
$$W = -mMG\left(\frac{1}{1.52\times10^8}-\frac{1}{1.47\times10^8}\right)$$
$$= -(5.97\times10^{24})(1.99\times10^{30})(6.67\times10^{-11})(-2.2377\times10^{-10}) \approx 1.77\times10^{35}\text{ J}$$

(c) In this case, $c = \epsilon qQ \;\Rightarrow$

$$W = \epsilon qQ\left(\tfrac{1}{10^{-12}}-\tfrac{1}{5\times10^{-13}}\right) = (8.985\times10^{10})(1)(-1.6\times10^{-19})(-10^{12}) \approx 1.4\times10^4\text{ J}.$$

17.4 Green's Theorem ET 16.4

1. (a)



C_1: $x=t \;\Rightarrow\; dx=dt,\ y=0 \;\Rightarrow\; dy=0\,dt,\ 0\le t\le 2$.

C_2: $x=2 \;\Rightarrow\; dx=0\,dt,\ y=t \;\Rightarrow\; dy=dt,\ 0\le t\le 3$.

C_3: $x=2-t \;\Rightarrow\; dx=-dt,\ y=3 \;\Rightarrow\; dy=0\,dt,\ 0\le t\le 2$.

C_4: $x=0 \;\Rightarrow\; dx=0\,dt,\ y=3-t \;\Rightarrow\; dy=-dt,\ 0\le t\le 3$.

Thus $\oint_C xy^2\,dx+x^3\,dy = \oint_{C_1+C_2+C_3+C_4} xy^2\,dx+x^3\,dy$

$$= \int_0^2 0\,dt + \int_0^3 8\,dt + \int_0^2 -9(2-t)\,dt + \int_0^3 0\,dt$$
$$= 0+24-18+0 = 6$$

(b) $\oint_C xy^2\,dx + x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(xy^2)\right] dA = \int_0^2 \int_0^3 (3x^2 - 2xy)\,dy\,dx$

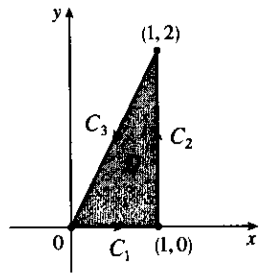
$= \int_0^2 (9x^2 - 9x)\,dx = 24 - 18 = 6$

2. (a) $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$. Then

$\oint_C y\,dx - x\,dy = \int_0^{2\pi} [\sin t(-\sin t) - \cos t(\cos t)]\,dt = -\int_0^{2\pi} dt = -2\pi$.

(b) $\oint_C y\,dx - x\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right] dA = -2\iint_D dA = -2A(D) = -2\pi(1)^2 = -2\pi$

3. (a)



C_1: $x = t \Rightarrow dx = dt,\ y = 0 \Rightarrow dy = 0\,dt,\ 0 \le t \le 1$.

C_2: $x = 1 \Rightarrow dx = 0\,dt,\ y = t \Rightarrow dy = dt,\ 0 \le t \le 2$.

C_3: $x = 1 - t \Rightarrow dx = -dt,\ y = 2 - 2t \Rightarrow dy = -2\,dt,\ 0 \le t \le 1$.

Thus $\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1 + C_2 + C_3} xy\,dx + x^2y^3\,dy$

$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$

$= 0 + \left[\frac{1}{4}t^4\right]_0^2 + \left[\frac{2}{3}(1-t)^3 + \frac{8}{3}(1-t)^6\right]_0^1 = 4 - \frac{10}{3} = \frac{2}{3}$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$= \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

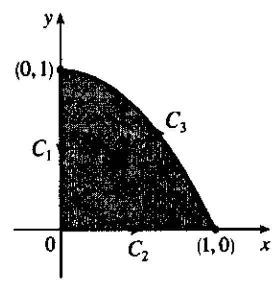
4. (a) $C_1 : x = 0 \Rightarrow dx = 0\,dt,\ y = 1 - t \Rightarrow$

$dy = -dt,\ 0 \le t \le 1$

$C_2 : x = t \Rightarrow dx = dt,\ y = 0 \Rightarrow dy = 0\,dt,\ 0 \le t \le 1$

$C_3 : x = 1 - t \Rightarrow dx = -dt,\ y = 1 - (1-t)^2 = 2t - t^2 \Rightarrow$

$dy = (2 - 2t)\,dt,\ 0 \le t \le 1$



Thus

$\oint_C x\,dx + y\,dy = \oint_{C_1 + C_2 + C_3} x\,dx + y\,dy$

$= \int_0^1 (0\,dt + (1-t)(-dt)) + \int_0^1 (t\,dt + 0\,dt) + \int_0^1 ((1-t)(-dt) + (2t - t^2)(2 - 2t)\,dt)$

$= \left[\frac{1}{2}t^2 - t\right]_0^1 + \left[\frac{1}{2}t^2\right]_0^1 + \left[\frac{1}{2}t^4 - 2t^3 + \frac{5}{2}t^2 - t\right]_0^1$

$= -\frac{1}{2} + \frac{1}{2} + \left(\frac{1}{2} - 2 + \frac{5}{2} - 1\right) = 0$

(b) $\oint_C x\,dx + y\,dy = \iint_D \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right] dA = \iint_D 0\,dA = 0$

5. We can parametrize C as $x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$. Then the line integral is $\oint_C P\,dx + Q\,dy = \int_0^{2\pi} \cos^4\theta \sin^5\theta\,(-\sin\theta)\,d\theta + \int_0^{2\pi} (-\cos^7\theta \sin^6\theta)\cos\theta\,d\theta = -\frac{29\pi}{1024}$, according to a CAS. The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-7x^6y^6 - 5x^4y^4)\,dy\,dx = -\frac{29\pi}{1024},$$

verifying Green's Theorem in this case.

6. Since $y = x^2$ along the first part of C and $y = x$ along the second part, the line integral is

$$\oint_C P\,dx + Q\,dy = \int_0^1 \left[x^4 \sin x + x^2 \sin(x^2)(2x)\right] dx + \int_1^0 (x^2 \sin x + x^2 \sin x)\,dx$$
$$= -16\cos 1 - 23\sin 1 + 28$$

according to a CAS. The double integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_{x^2}^x (2x \sin y - 2y \sin x)\,dy\,dx = -16\cos 1 - 23\sin 1 + 28$$

7. The region D enclosed by C is $[0,1] \times [0,1]$, so

$$\int_C e^y\,dx + 2xe^y\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xe^y) - \frac{\partial}{\partial y}(e^y)\right] dA = \int_0^1 \int_0^1 (2e^y - e^y)\,dy\,dx$$
$$= \int_0^1 dx \int_0^1 e^y\,dy = (1)(e^1 - e^0) = e - 1$$

8. The region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, 3x \le y \le 3\}$, so

$$\int_C x^2y^2\,dx + 4xy^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(4xy^3) - \frac{\partial}{\partial y}(x^2y^2)\right] dA = \int_0^1 \int_{3x}^3 (4y^3 - 2x^2y)\,dy\,dx$$
$$= \int_0^1 \left[y^4 - x^2y^2\right]_{y=3x}^{y=3} dx = \int_0^1 (81 - 9x^2 - 72x^4)\,dx = 81 - 3 - \tfrac{72}{5} = \tfrac{318}{5}$$

9. $$\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$$
$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy = \tfrac{1}{3}$$

10. $$\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^4 + 2x^2y^2) - \frac{\partial}{\partial y}(xe^{-2x})\right] dA = \iint_D (4x^3 + 4xy^2 - 0)\,dA$$
$$= 4\iint_D x(x^2 + y^2)\,dA = 4\int_0^{2\pi} \int_1^2 (r\cos\theta)(r^2)\,r\,dr\,d\theta$$
$$= 4\int_0^{2\pi} \cos\theta\,d\theta \int_1^2 r^4\,dr = 4\left[\sin\theta\right]_0^{2\pi} \left[\tfrac{1}{5}r^5\right]_1^2 = 0$$

11. $$\int_C y^3\,dx - x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 (-3r^2)\,r\,dr\,d\theta$$
$$= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3(2\pi)(4) = -24\pi$$

12. $$\int_C \sin y\,dx + x\cos y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(\sin y)\right] dA = \iint_D (\cos y - \cos y)\,dA = \iint_D 0\,dA = 0$$

13. $\mathbf{F}(x,y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y} \right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \sin x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \left(\sqrt{x} + y^3\right) dx + \left(x^2 + \sqrt{y}\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \frac{\partial}{\partial y}\left(\sqrt{x} + y^3\right)\right] dA$$
$$= -\int_0^\pi \int_0^{\sin x} (2x - 3y^2)\,dy\,dx = -\int_0^\pi \left[2xy - y^3\right]_{y=0}^{y=\sin x} dx$$
$$= -\int_0^\pi (2x\sin x - \sin^3 x)\,dx = -\int_0^\pi (2x\sin x - (1 - \cos^2 x)\sin x)\,dx$$
$$= -\left[2\sin x - 2x\cos x + \cos x - \tfrac{1}{3}\cos^3 x\right]_0^\pi \quad \text{[integrate by parts in the first term]}$$
$$= -\left(2\pi - 2 + \tfrac{2}{3}\right) = \tfrac{4}{3} - 2\pi$$

14. $\mathbf{F}(x,y) = \langle y^2\cos x, x^2 + 2y\sin x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(y^2\cos x)\,dx + (x^2 + 2y\sin x)\,dy \\ &= -\iint_D \left[\tfrac{\partial}{\partial x}(x^2 + 2y\sin x) - \tfrac{\partial}{\partial y}(y^2\cos x)\right] dA \\ &= -\iint_D (2x + 2y\cos x - 2y\cos x)\,dA = -\int_0^2\int_0^{3x} 2x\,dy\,dx \\ &= -\int_0^2 2x\,[y]_{y=0}^{y=3x}\,dx = -\int_0^2 6x^2\,dx = -2x^3\Big]_0^2 = -16\end{aligned}$$

15. $\mathbf{F}(x,y) = \langle e^x + x^2y, e^y - xy^2\rangle$ and the region D enclosed by C is the disk $x^2 + y^2 \le 25$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(e^x + x^2y)\,dx + (e^y - xy^2)\,dy \\ &= -\iint_D \left[\tfrac{\partial}{\partial x}(e^y - xy^2) - \tfrac{\partial}{\partial y}(e^x + x^2y)\right] dA = -\iint_D(-y^2 - x^2)\,dA \\ &= \iint_D (x^2 + y^2)\,dA = \int_0^{2\pi}\int_0^5 (r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^5 r^3\,dr = 2\pi\left[\tfrac14 r^4\right]_0^5 = \tfrac{625}{2}\pi\end{aligned}$$

16. $\mathbf{F}(x,y) = \left\langle y - \ln(x^2 + y^2), 2\tan^{-1}\left(\frac{y}{x}\right)\right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at $(2,3)$. C is oriented positively, so

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (y - \ln(x^2+y^2))\,dx + \left(2\tan^{-1}\left(\frac{y}{x}\right)\right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(y - \ln(x^2+y^2)\right)\right] dA \\ &= \iint_D \left[2\left(\frac{-yx^{-2}}{1 + (y/x)^2}\right) - \left(1 - \frac{2y}{x^2+y^2}\right)\right] dA = \iint_D \left[-\frac{2y}{x^2+y^2} - 1 + \frac{2y}{x^2+y^2}\right] dA \\ &= -\iint_D dA = -(\text{area of } D) = -\pi\end{aligned}$$

17. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dy\,dx$ where C is the path described in the question and D is the triangle bounded by C. So

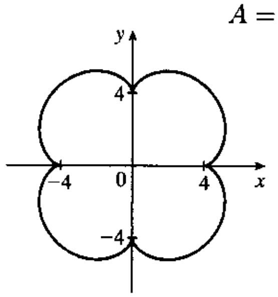
$$\begin{aligned}W &= \int_0^1\int_0^{1-x}(y^2 - x)\,dy\,dx = \int_0^1 \left[\tfrac13 y^3 - xy\right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\tfrac13(1-x)^3 - x(1-x)\right) dx \\ &= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac12 x^2 + \tfrac13 x^3\right]_0^1 = \left(-\tfrac12 + \tfrac13\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}\end{aligned}$$

18. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2\int_0^\pi r^2\cdot r\,d\theta\,dr = 3\pi\left[\frac14 r^4\right]_0^2 = 12\pi$.

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt)$$
$$= \int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$$

20.



$$A = \oint_C x\,dy = \int_0^{2\pi}(5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$
$$= \int_0^{2\pi}(25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$
$$= \left[25\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - 30\left(\tfrac{1}{8}\sin 4t + \tfrac{1}{12}\sin 6t\right) + 5\left(\tfrac{1}{2}t + \tfrac{1}{20}\sin 10t\right)\right]_0^{2\pi}$$

[Use Formula 80 in the Table of Integrals]

$$= 30\pi$$

21. (a) Using Equation 17.2.8 [ET 16.2.8], we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt$$
$$= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt$$
$$= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt = x_1 y_2 - x_2 y_1$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon} = A(D) = \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx$$
$$= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)$$

To evaluate these integrals we use the formula from (a) to get
$A(D) = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$.

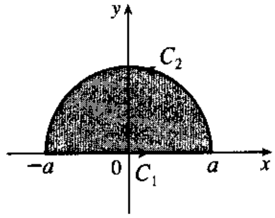
(c) $A = \frac{1}{2}[(0\cdot 1 - 2\cdot 0) + (2\cdot 3 - 1\cdot 1) + (1\cdot 2 - 0\cdot 3) + (0\cdot 1 - (-1)\cdot 2) + (-1\cdot 0 - 0\cdot 1)]$
$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and
$-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D(-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. Here $A = \frac{1}{2}(1)(1) = \frac{1}{2}$ and $C = C_1 + C_2 + C_3$, where C_1: $x = x$, $y = 0$, $0 \le x \le 1$;
C_2: $x = x$, $y = 1 - x$, $x = 1$ to $x = 0$; and C_3: $x = 0$, $y = 1$ to $y = 0$. Then
$\overline{x} = \frac{1}{2A}\int_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = 0 + \int_1^0 (x^2)(-dx) + 0 = \frac{1}{3}$. Similarly,
$\overline{y} = -\frac{1}{2A}\int_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = 0 + \int_1^0 (1-x)^2(-dx) + 0 = \frac{1}{3}$.
Therefore $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$.

24. $A = \frac{\pi a^2}{2}$ so $\overline{x} = \frac{1}{\pi a^2} \oint_C x^2\,dy$ and $\overline{y} = -\frac{1}{\pi a^2} \oint_C y^2\,dx$.

Orienting the semicircular region as in the figure,



$\overline{x} = \frac{1}{\pi a^2} \oint_{C_1+C_2} x^2\,dy$

$= \frac{1}{\pi a^2}\left[0 + \int_0^\pi (a^2\cos^2 t)(a\cos t)\,dt\right] = 0$

and

$\overline{y} = -\frac{1}{\pi a^2}\left[\int_{-a}^a 0\,dx + \int_0^\pi (a^2\sin^2 t)(-a\sin t)\,dt\right] = \frac{a}{\pi}\int_0^\pi \sin^3 t\,dt = \frac{a}{\pi}\left[-\cos t + \frac{1}{3}(\cos^3 t)\right]_0^\pi = \frac{4a}{3\pi}$.

Thus $(\overline{x}, \overline{y}) = \left(0, \frac{4a}{3\pi}\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho \oint_C y^3\,dx = -\frac{1}{3}\rho \iint_D (-3y^2)\,dA = \iint_D y^2\rho\,dA = I_x$ and

$\frac{1}{3}\rho \oint_C x^3\,dy = \frac{1}{3}\rho \iint_D (3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$.

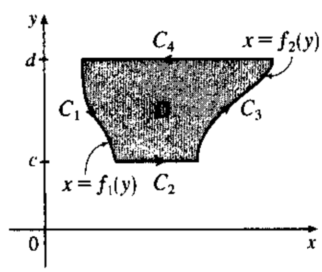
26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \tfrac{1}{3}\rho \oint_C x^3\,dy = \tfrac{1}{3}\rho \int_0^{2\pi} (a^4\cos^4 t)\,dt = \tfrac{1}{3}a^4\rho \int_0^{2\pi}\left[\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\right]dt$$

$$= \tfrac{1}{3}a^4\rho \cdot \tfrac{3(2\pi)}{8} = \tfrac{1}{4}\pi a^4\rho$$

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) [ET 16.3.33(a)], $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D 0\,dA = 0$.

28. We express D as a type II region: $D = \{(x,y) \mid f_1(y) \le x \le f_2(y), c \le y \le d\}$ where f_1 and f_2 are continuous functions. Then $\displaystyle\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(f_2(y),y) - Q(f_1(y),y)]\,dy$ by the Fundamental Theorem of Calculus. But referring to the figure,



$\oint_C Q\,dy = \oint_{C_1+C_2+C_3+C_4} Q\,dy$. Then $\int_{C_1} Q\,dy = \int_d^c Q(f_1(y),y)\,dy$,

$\int_{C_2} Q\,dy = \int_{C_4} Q\,dy = 0$, and $\int_{C_3} Q\,dy = \int_c^d Q(f_2(y),y)\,dy$. Hence

$\oint_C Q\,dy = \int_c^d [Q(f_2(y),y) - Q(f_1(y),y)]\,dy = \iint_D (\partial Q/\partial x)\,dA$.

29. Using the first part of (5), we have that $\iint_R dx\,dy = A(R) = \int_{\partial R} x\,dy$. But $x = g(u,v)$, and $dy = \dfrac{\partial h}{\partial u}du + \dfrac{\partial h}{\partial v}dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R, so

$$\int_{\partial R} x\,dy = \int_{\partial S} g(u,v)\left(\frac{\partial h}{\partial u}\,du + \frac{\partial h}{\partial v}\,dv\right) = \int_{\partial S} g(u,v)\,\frac{\partial h}{\partial u}\,du + g(u,v)\,\frac{\partial h}{\partial v}\,dv$$

$$= \pm\iint_S \left[\frac{\partial}{\partial u}\left(g(u,v)\,\frac{\partial h}{\partial v}\right) - \frac{\partial}{\partial v}\left(g(u,v)\,\frac{\partial h}{\partial u}\right)\right] dA \quad \text{[using Green's Theorem in the } uv\text{-plane]}$$

$$= \pm\iint_S \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + g(u,v)\,\frac{\partial^2 h}{\partial u\,\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} - g(u,v)\,\frac{\partial^2 h}{\partial v\,\partial u}\right) dA \quad \text{[using the Chain Rule]}$$

$$= \pm\iint_S \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) dA \ \text{[by the equality of mixed partials]} \ = \pm\iint_S \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\frac{\partial(x,y)}{\partial(u,v)}$. Therefore $A(R) = \iint_R dx\,dy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

17.5 Curl and Divergence ET 16.5

1. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 0 & -x^2y \end{vmatrix} = (-x^2-0)\,\mathbf{i} - (-2xy - xy)\,\mathbf{j} + (0 - xz)\,\mathbf{k}$

$= -x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-x^2y) = yz + 0 + 0 = yz$

2. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = (xz^2 - xy^2)\,\mathbf{i} - (yz^2 - x^2y)\,\mathbf{j} + (y^2z - x^2z)\,\mathbf{k}$

$= x(z^2 - y^2)\,\mathbf{i} + y(x^2 - z^2)\,\mathbf{j} + z(y^2 - x^2)\,\mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$

3. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \sqrt{z} \end{vmatrix} = (x - y)\,\mathbf{i} - (y - 0)\,\mathbf{j} + (1 - 0)\,\mathbf{k}$

$= (x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(x + yz) + \frac{\partial}{\partial z}(xy - \sqrt{z}) = z - \frac{1}{2\sqrt{z}}$

4. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \cos xz & -\sin xy \end{vmatrix}$

$$= (-x\cos xy + x\sin xz)\,\mathbf{i} - (-y\cos xy - 0)\,\mathbf{j} + (-z\sin xz - 0)\,\mathbf{k}$$
$$= x(\sin xz - \cos xy)\,\mathbf{i} + y\cos xy\,\mathbf{j} - z\sin xz\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(0) + \dfrac{\partial}{\partial y}(\cos xz) + \dfrac{\partial}{\partial z}(-\sin xy) = 0+0+0 = 0$

5. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin y & e^x\cos y & z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (e^x\cos y - e^x\cos y)\,\mathbf{k} = \mathbf{0}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(e^x\sin y) + \dfrac{\partial}{\partial y}(e^x\cos y) + \dfrac{\partial}{\partial z}(z) = e^x\sin y - e^x\sin y + 1 = 1$

6. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{x^2+y^2+z^2} & \dfrac{y}{x^2+y^2+z^2} & \dfrac{z}{x^2+y^2+z^2} \end{vmatrix}$

$$= \frac{1}{(x^2+y^2+z^2)^2}\left[(-2yz+2yz)\,\mathbf{i} - (-2xz+2xz)\,\mathbf{j} + (-2xy+2xy)\,\mathbf{k}\right] = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2+z^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2+z^2}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{x^2+y^2+z^2}\right)$

$$= \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$$
$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x & \ln(xy) & \ln(xyz) \end{vmatrix} = \left(\dfrac{xz}{xyz} - 0\right)\mathbf{i} - \left(\dfrac{yz}{xyz} - 0\right)\mathbf{j} + \left(\dfrac{y}{xy} - 0\right)\mathbf{k}$

$$= \left\langle \frac{1}{y}, -\frac{1}{x}, \frac{1}{x} \right\rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(\ln x) + \dfrac{\partial}{\partial y}(\ln(xy)) + \dfrac{\partial}{\partial z}(\ln(xyz)) = \dfrac{1}{x} + \dfrac{x}{xy} + \dfrac{xy}{xyz} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$

8. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{-y} & xz & ze^y \end{vmatrix} = (ze^y - x)\,\mathbf{i} - (0-0)\,\mathbf{j} + (z - xe^{-y}(-1))\,\mathbf{k}$

$$= \left\langle ze^y - x, 0, z + xe^{-y} \right\rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(xe^{-y}) + \dfrac{\partial}{\partial y}(xz) + \dfrac{\partial}{\partial z}(ze^y) = e^{-y} + 0 + e^y = e^y + e^{-y}$

9. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the x-component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\dfrac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x- or z-directions, so $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial z} = 0$.

(a) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + \dfrac{\partial Q}{\partial y} + 0 < 0$

(b) $\operatorname{curl} \mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

10. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, P and Q don't vary in the z-direction, so $\dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial z} = \dfrac{\partial Q}{\partial z} = 0$. As x increases, the x-component of each vector of $\mathbf{F}$ increases while the y-component remains constant, so $\dfrac{\partial P}{\partial x} > 0$ and $\dfrac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y-component of each vector increases while the x-component remains constant, so $\dfrac{\partial Q}{\partial y} > 0$ and $\dfrac{\partial P}{\partial y} = 0$.

(a) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + 0 > 0$

(b) $\operatorname{curl} \mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

11. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the y-component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial y} = \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. P increases as y increases, so $\dfrac{\partial P}{\partial y} > 0$, but P doesn't change in the x- or z-directions, so $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + 0 + 0 = 0$

(b)
$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(0 - \frac{\partial P}{\partial y}\right)\mathbf{k} = -\frac{\partial P}{\partial y}\mathbf{k}\end{aligned}$$

Since $\dfrac{\partial P}{\partial y} > 0$, $-\dfrac{\partial P}{\partial y}\mathbf{k}$ is a vector pointing in the negative z-direction.

12. (a) $\operatorname{curl} f = \nabla \times f$ is meaningless because f is a scalar field.

(b) $\operatorname{grad} f$ is a vector field.

(c) $\operatorname{div} \mathbf{F}$ is a scalar field.

(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.

(e) $\operatorname{grad} \mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) $\operatorname{grad}(\operatorname{div} \mathbf{F})$ is a vector field.

(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.

(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.

(i) curl (curl $\mathbf{F}$) is a vector field.

(j) div(div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(k) (grad f) $\times$ (div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(l) div(curl (grad f)) is a scalar field.

13. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x - x)\,\mathbf{i} - (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = \mathbf{0}$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$, so $g(y,z) = h(z)$ and $f(x,y,z) = xyz + h(z)$. Thus $f_z(x,y,z) = xy + h'(z)$ but $f_z(x,y,z) = xy$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xyz + K$.

14. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z^2 & \cos y & 2xz \end{vmatrix} = (0 - 0)\,\mathbf{i} - (2z - 6z)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = 4z\,\mathbf{j} \neq \mathbf{0}$,

so $\mathbf{F}$ is not conservative.

15. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 + 2yz & y^2 \end{vmatrix} = (2y - 2y)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (2x - 2x)\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = 2xy$ implies $f(x,y,z) = x^2y + g(y,z)$ and $f_y(x,y,z) = x^2 + g_y(y,z)$. But $f_y(x,y,z) = x^2 + 2yz$, so $g(y,z) = y^2z + h(z)$ and $f(x,y,z) = x^2y + y^2z + h(z)$. Thus $f_z(x,y,z) = y^2 + h'(z)$ but $f_z(x,y,z) = y^2$ so $h(z) = K$ and $f(x,y,z) = x^2y + y^2z + K$.

16. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & 1 & xe^z \end{vmatrix} = (0 - 0)\,\mathbf{i} - (e^z - e^z)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = \mathbf{0}$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions that have continuous partial deriatives, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^z$ implies $f(x,y,z) = xe^z + g(y,z) \Rightarrow f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = 1$, so $g(y,z) = y + h(z)$ and $f(x,y,z) = xe^z + y + h(z)$. Thus $f_z(x,y,z) = xe^z + h'(z)$ but $f_z(x,y,z) = xe^z$, so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xe^z + y + K$.

17. $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{-x} & e^{-x} & 2z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (-e^{-x} - e^{-x})\,\mathbf{k} = -2e^{-x}\,\mathbf{k} \neq \mathbf{0}$,

so $\mathbf{F}$ is not conservative.

18. $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos xy & x\cos xy & -\sin z \end{vmatrix}$

$$= (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + [(-xy\sin xy + \cos xy) - (-xy\sin xy + \cos xy)]\,\mathbf{k} = \mathbf{0}$$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = y\cos xy$ implies $f(x,y,z) = \sin xy + g(y,z) \;\Rightarrow\; f_y(x,y,z) = x\cos xy + g_y(y,z)$. But $f_y(x,y,z) = x\cos xy$, so $g(y,z) = h(z)$ and $f(x,y,z) = \sin xy + h(z)$. Thus $f_z(x,y,z) = h'(z)$ but $f_z(x,y,z) = -\sin z$ so $h(z) = \cos z + K$ and a potential function for $\mathbf{F}$ is $f(x,y,z) = \sin xy + \cos z + K$.

19. No. Assume there is such a $\mathbf{G}$. Then $\text{div}(\text{curl}\,\mathbf{G}) = y^2 + z^2 + x^2 \neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\mathbf{G}$. Then $\text{div}(\text{curl}\,\mathbf{G}) = xz \neq 0$ which contradicts Theorem 11.

21. $\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$.

Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$ is irrotational.

22. $\text{div}\,\mathbf{F} = \dfrac{\partial(f(y,z))}{\partial x} + \dfrac{\partial(g(x,z))}{\partial y} + \dfrac{\partial(h(x,y))}{\partial z} = 0$ so $\mathbf{F}$ is incompressible.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

23. $\text{div}\,(\mathbf{F}+\mathbf{G}) = \dfrac{\partial(P_1+P_2)}{\partial x} + \dfrac{\partial(Q_1+Q_2)}{\partial y} + \dfrac{\partial(R_1+R_2)}{\partial z}$

$$= \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_3}{\partial z}\right) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$$

24. $\text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G} = \left[\left(\dfrac{\partial R_1}{\partial y} - \dfrac{\partial Q_1}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P_1}{\partial z} - \dfrac{\partial R_1}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q_1}{\partial x} - \dfrac{\partial P_1}{\partial y}\right)\mathbf{k}\right]$

$$+ \left[\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\mathbf{k}\right]$$

$$= \left[\frac{\partial(R_1+R_2)}{\partial y} - \frac{\partial(Q_1+Q_2)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(P_1+P_2)}{\partial z} - \frac{\partial(R_1+R_2)}{\partial x}\right]\mathbf{j}$$

$$+ \left[\frac{\partial(Q_1+Q_2)}{\partial x} - \frac{\partial(P_1+P_2)}{\partial y}\right]\mathbf{k} = \text{curl}\,(\mathbf{F}+\mathbf{G})$$

25. $\operatorname{div}(f\mathbf{F}) = \dfrac{\partial(fP_1)}{\partial x} + \dfrac{\partial(fQ_1)}{\partial y} + \dfrac{\partial(fR_1)}{\partial z}$

$$= \left(f\,\frac{\partial P_1}{\partial x} + P_1\,\frac{\partial f}{\partial x}\right) + \left(f\,\frac{\partial Q_1}{\partial y} + Q_1\,\frac{\partial f}{\partial y}\right) + \left(f\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$$

26. $\operatorname{curl}(f\mathbf{F}) = \left[\dfrac{\partial(fR_1)}{\partial y} - \dfrac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\dfrac{\partial(fP_1)}{\partial z} - \dfrac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\dfrac{\partial(fQ_1)}{\partial x} - \dfrac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$

$$= \left[f\,\frac{\partial R_1}{\partial y} + R_1\,\frac{\partial f}{\partial y} - f\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\,\frac{\partial P_1}{\partial z} + P_1\,\frac{\partial f}{\partial z} - f\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial f}{\partial x}\right]\mathbf{j}$$

$$+ \left[f\,\frac{\partial Q_1}{\partial x} + Q_1\,\frac{\partial f}{\partial x} - f\,\frac{\partial P_1}{\partial y} - P_1\,\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k}$$

$$+ \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$$

27. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \dfrac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \dfrac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \dfrac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$

$$= \left[Q_1\,\frac{\partial R_2}{\partial x} + R_2\,\frac{\partial Q_1}{\partial x} - Q_2\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial Q_2}{\partial x}\right]$$

$$- \left[P_1\,\frac{\partial R_2}{\partial y} + R_2\,\frac{\partial P_1}{\partial y} - P_2\,\frac{\partial R_1}{\partial y} - R_1\,\frac{\partial P_2}{\partial y}\right]$$

$$+ \left[P_1\,\frac{\partial Q_2}{\partial z} + Q_2\,\frac{\partial P_1}{\partial z} - P_2\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial P_2}{\partial z}\right]$$

$$= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right]$$

$$- \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right]$$

$$= \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

28. $\operatorname{div}(\nabla f\times\nabla g) = \nabla g\cdot\operatorname{curl}(\nabla f) - \nabla f\cdot\operatorname{curl}(\nabla g)$ [by Exercise 27] $= 0$ (by Theorem 3)

29. $\text{curl}\,\text{curl}\,\mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$

$$= \left(\frac{\partial^2 Q_1}{\partial y \partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z \partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z \partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x \partial y}\right)\mathbf{j}$$

$$+ \left(\frac{\partial^2 P_1}{\partial x \partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y \partial z}\right)\mathbf{k}$$

Now let's consider grad div $\mathbf{F} - \nabla^2\mathbf{F}$ and compare with the above.
(Note that $\nabla^2\mathbf{F}$ is defined on page 1130 [ET 1094].)

$$\text{grad}\,\text{div}\,\mathbf{F} - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$

$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$

$$= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}$$

$$+ \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have curl curl $\mathbf{F} =$ grad div $\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

30. (a) $\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 1 + 1 + 1 = 3$

(b) $\nabla \cdot (r\mathbf{r}) = \nabla \cdot \sqrt{x^2 + y^2 + z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

$$= \left(\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{z^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(4x^2 + 4y^2 + 4z^2\right) = 4\sqrt{x^2+y^2+z^2} = 4r$$

Another method:
By Exercise 25, $\nabla \cdot (r\mathbf{r}) = \text{div}(r\mathbf{r}) = r\,\text{div}\,\mathbf{r} + \mathbf{r} \cdot \nabla r = 3r + \mathbf{r} \cdot \frac{\mathbf{r}}{r}$ [see Exercise 31(a) below] $= 4r$.

(c) $\nabla^2 r^3 = \nabla^2 \left(x^2+y^2+z^2\right)^{3/2}$

$$= \frac{\partial}{\partial x}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\right] + \frac{\partial}{\partial y}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\right] + \frac{\partial}{\partial z}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\right]$$

$$= 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(x) + (x^2+y^2+z^2)^{1/2}\right] + 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(y) + (x^2+y^2+z^2)^{1/2}\right] + 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(z) + (x^2+y^2+z^2)^{1/2}\right]$$

$$= 3(x^2+y^2+z^2)^{-1/2}(4x^2+4y^2+4z^2) = 12(x^2+y^2+z^2)^{1/2}$$

$$= 12r$$

Another method: $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2} = 3x\sqrt{x^2+y^2+z^2} \;\Rightarrow\; \nabla r^3 = 3r(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}) = 3r\,\mathbf{r}$, so $\nabla^2 r^3 = \nabla\cdot\nabla r^3 = \nabla\cdot(3r\,\mathbf{r}) = 3(4r) = 12r$ by part (b).

31. (a) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$

$$= \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r}$$

(b) $\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$$

(c) $\nabla\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{-\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\frac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\frac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla\ln r = \nabla\ln\left(x^2+y^2+z^2\right)^{1/2} = \tfrac{1}{2}\nabla\ln\left(x^2+y^2+z^2\right)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

32. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad\Rightarrow\quad r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\dfrac{\partial}{\partial x}\dfrac{x}{(x^2+y^2+z^2)^{p/2}} = \dfrac{(x^2+y^2+z^2) - px^2}{(x^2+y^2+z^2)^{1+p/2}} = \dfrac{r^2 - px^2}{r^{p+2}}$. Similarly,

$\dfrac{\partial}{\partial y}\dfrac{y}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2 - py^2}{r^{p+2}}$ and $\dfrac{\partial}{\partial z}\dfrac{z}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2 - pz^2}{r^{p+2}}$. Thus

$$\begin{aligned}\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{r^2-px^2}{r^{p+2}} + \frac{r^2-py^2}{r^{p+2}} + \frac{r^2-pz^2}{r^{p+2}} = \frac{3r^2 - px^2 - py^2 - pz^2}{r^{p+2}} \\ &= \frac{3r^2 - p(x^2+y^2+z^2)}{r^{p+2}} = \frac{3r^2 - pr^2}{r^{p+2}} = \frac{3-p}{r^p}\end{aligned}$$

Consequently, if $p = 3$ we have $\operatorname{div}\mathbf{F} = 0$.

33. By (13), $\oint_C f(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

34. By Exercise 33, $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$ and $\iint_D g\nabla^2 f\,dA = \oint_C g(\nabla f)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$. Hence

$$\begin{aligned}\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA &= \oint_C [f(\nabla g)\cdot\mathbf{n} - g(\nabla f)\cdot\mathbf{n}]\,ds + \iint_D (\nabla f\cdot\nabla g - \nabla g\cdot\nabla f)\,dA \\ &= \oint_C [f\nabla g - g\nabla f]\cdot\mathbf{n}\,ds\end{aligned}$$

35. (a) We know that $\omega = v/d$, and from the diagram $\sin\theta = d/r \quad\Rightarrow\quad v = d\omega = (\sin\theta)r\omega = |\mathbf{w}\times\mathbf{r}|$. But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) From (a), $\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & \omega \\ x & y & z\end{vmatrix} = (0\cdot z - \omega y)\,\mathbf{i} + (\omega x - 0\cdot z)\,\mathbf{j} + (0\cdot y - x\cdot 0)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$

(c) $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0\end{vmatrix}$

$$\begin{aligned}&= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k} \\ &= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w}\end{aligned}$$

36. Let $\mathbf{H} = \langle h_1, h_2, h_3 \rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$.

(a) $$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (\operatorname{curl} \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{1}{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$$

$$= -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\,\partial t} - \frac{\partial^2 h_2}{\partial z\,\partial t} \right)\mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z\,\partial t} - \frac{\partial^2 h_3}{\partial x\,\partial t} \right)\mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x\,\partial t} - \frac{\partial^2 h_1}{\partial y\,\partial t} \right)\mathbf{k} \right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right)\mathbf{k} \right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(b) $$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times (\operatorname{curl} \mathbf{H}) = \nabla \times \left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \right) = \frac{1}{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix}$$

$$= \frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y\,\partial t} - \frac{\partial^2 E_2}{\partial z\,\partial t} \right)\mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z\,\partial t} - \frac{\partial^2 E_3}{\partial x\,\partial t} \right)\mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x\,\partial t} - \frac{\partial^2 E_1}{\partial y\,\partial t} \right)\mathbf{k} \right]$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \right)\mathbf{k} \right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= \frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that $\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} \Rightarrow$

$$\nabla^2\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad} 0 + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \text{ [from part (a)] } = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2\mathbf{H} = \operatorname{grad}\operatorname{div}\mathbf{H} - \operatorname{curl}\operatorname{curl}\mathbf{H} = \operatorname{grad} 0 + \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$ [using part (b)] $= \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$.

37. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$ where $g(x, y, z) = \int_0^x f(t, y, z)\,dt$. Then

$\operatorname{div}\mathbf{G} = \frac{\partial}{\partial x}(g(x, y, z)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = \frac{\partial}{\partial x}\int_0^x f(t, y, z)\,dt = f(x, y, z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

17.6 Parametric Surfaces and Their Areas ET 16.6

1. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = u^2$. For any point (x,y,z) on the surface, we have $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z$. Since no restrictions are placed on the parameters, the surface is $z = x^2 + y^2$, which we recognize as a circular paraboloid opening upward whose axis is the z-axis.

2. $\mathbf{r}(u,v) = (1+2u)\,\mathbf{i} + (-u+3v)\,\mathbf{j} + (2+4u+5v)\,\mathbf{k} = \langle 1,0,2\rangle + u\langle 2,-1,4\rangle + v\langle 0,3,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(1,0,2)$ and containing vectors $\mathbf{a} = \langle 2,-1,4\rangle$ and $\mathbf{b} = \langle 0,3,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix} = -17\,\mathbf{i} - 10\,\mathbf{j} + 6\,\mathbf{k}$$

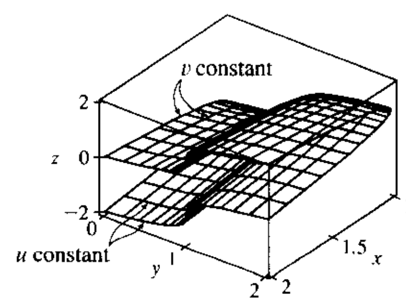
and an equation of the plane is $-17(x-1) - 10(y-0) + 6(z-2) = 0$ or $-17x - 10y + 6z = -5$.

3. $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x$, $y = \cos\theta$, $z = \sin\theta$. For any point (x,y,z) on the surface, we have $y^2 + z^2 = \cos^2\theta + \sin^2\theta = 1$, so any vertical trace in $x = k$ is the circle $y^2 + z^2 = 1$, $x = k$. Since $x = x$ with no restriction, the surface is a circular cylinder with radius 1 whose axis is the x-axis.

4. $\mathbf{r}(x,\theta) = \langle x, x\cos\theta, x\sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x$, $y = x\cos\theta$, $z = x\sin\theta$. For any point (x,y,z) on the surface, we have $y^2 + z^2 = x^2\cos^2\theta + x^2\sin^2\theta = x^2$. With $x = x$ and no restrictions on the parameters, the surface is $x^2 = y^2 + z^2$, which we recognize as a circular cone whose axis is the x-axis.

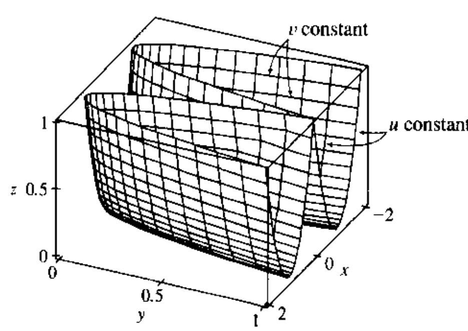
5. $\mathbf{r}(u,v) = \langle u^2+1, v^3+1, u+v\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u^2+1$, $y = v^3+1$, $z = u+v$, $-1 \le u \le 1$, $-1 \le v \le 1$. If we keep u constant at u_0, $x = u_0^2 + 1$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^3 + 1$, a constant, so these grid curves are the curves parallel to the xz-plane.



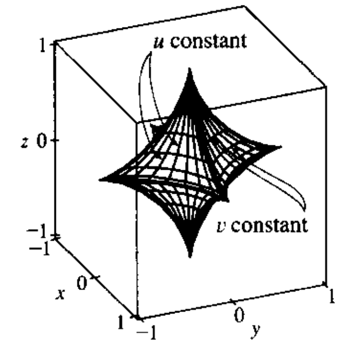
6. $\mathbf{r}(u,v) = \langle u+v, u^2, v^2\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u+v$, $y = u^2$, $z = v^2$, $-1 \le u \le 1$, $-1 \le v \le 1$. If $u = u_0$ is constant, $y = u_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane. If $v = v_0$ is constant, $z = v_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



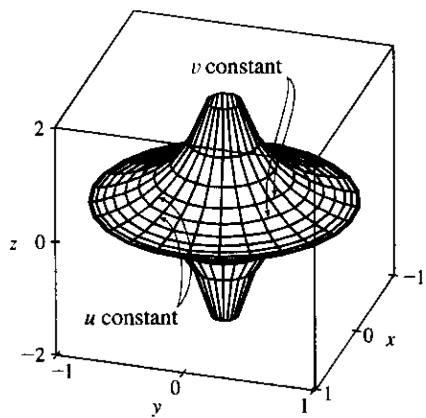
7. $\mathbf{r}(u,v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$.

The surface has parametric equations $x = \cos^3 u \cos^3 v$, $y = \sin^3 u \cos^3 v$, $z = \sin^3 v$, $0 \le u \le \pi$, $0 \le v \le 2\pi$. Note that if $v = v_0$ is constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curves must be the curves parallel to the xy-plane. The vertically oriented grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos^3 u_0 \cos^3 v$, $y = \sin^3 u_0 \cos^3 v$, $z = \sin^3 v$. These curves lie in vertical planes that contain the z-axis.



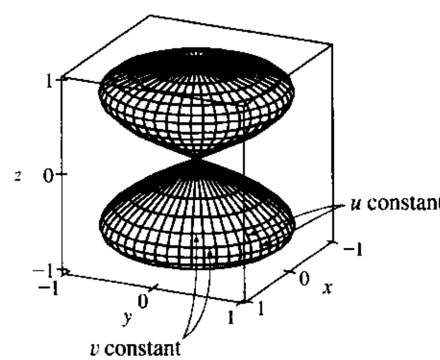
8. $\mathbf{r}(u,v) = \langle \cos u \sin v, \sin u \sin v, \cos v + \ln\tan(v/2) \rangle$.

The surface has parametric equations $x = \cos u \sin v$, $y = \sin u \sin v$, $z = \cos v + \ln\tan(v/2)$, $0 \le u \le 2\pi$, $0.1 \le v \le 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin v_0$, $y = \sin u \sin v_0$, $z = \cos v_0 + \ln\tan(v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln\tan(v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin v$, $y = \sin u_0 \sin v$, $z = \cos v + \ln\tan(v/2)$. These curves lie in vertical planes that contain the z-axis.

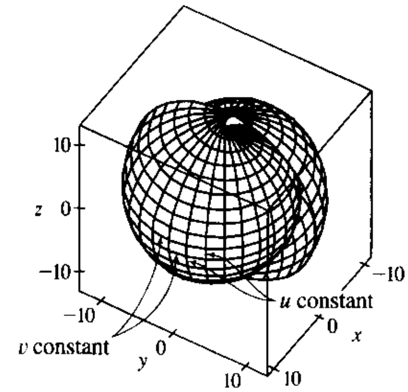


9. $x = \cos u \sin 2v$, $y = \sin u \sin 2v$, $z = \sin v$.

The complete graph of the surface is given by the parametric domain $0 \le u \le \pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin 2v_0$, $y = \sin u \sin 2v_0$, $z = \sin v_0$ which represent a circle of radius $\sin 2v_0$ in the plane $z = \sin v_0$. So the circular grid curves we see lying horizontally are the grid curves which have v constant. The vertical grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin 2v$ and $y = \sin u_0 \sin 2v$ with $z = \sin v$ which has a "figure-eight" shape.



10. $x = u\sin u\cos v$, $y = u\cos u\cos v$, $z = u\sin v$.
We graph the portion of the surface with parametric domain $0 \le u \le 4\pi$, $0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = u\sin u\cos v_0$, $y = u\cos u\cos v_0$, $z = u\sin v_0$. The equations for x and y show that the projections onto the xy-plane give a spiral shape, so the corresponding grid curves are the almost-horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0\sin u_0\cos v$, $y = u_0\cos u_0\cos v$, $z = u_0\sin v$.



11. $\mathbf{r}(u,v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + u\,\mathbf{k}$. The parametric equations for the surface are $x = \cos v$, $y = \sin v$, $z = u$. Then $x^2 + y^2 = \cos^2 v + \sin^2 v = 1$ and $z = u$ with no restriction on u, so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z = u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z-axis.

12. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = u$. Then $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z^2$, which represents the equation of a cone with axis the z-axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2 + y^2 = u^2$ in the plane $z = u$. If v is constant, x, y, z are each scalar multiples of u, corresponding to the straight line grid curves through the origin.

13. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x = u^3$, $y = u\sin v$, $z = u\cos v$. Then $y^2 + z^2 = u^2\sin v^2 + u^2\cos v^2 = u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x = u^3$. The graph then must be graph III. If v is held constant, so $v = v_0$, we have $y = u\sin v_0$ and $z = u\cos v_0$. Then $y = (\tan v_0)z$, so the grid curves we see running lengthwise along the surface in the planes $y = kz$ correspond to keeping v constant.

15. $x = (u - \sin u)\cos v$, $y = (1 - \cos u)\sin v$, $z = u$. If u is held constant, x and y give an equation of an ellipse in the plane $z = u$, thus the grid curves are horizontally oriented ellipses. Note that when $u = 0$, the "ellipse" is the single point $(0,0,0)$, and when $u = \pi$, we have $y = 0$ while x ranges from $-\pi$ to π, a line segment parallel to the x-axis in the plane $z = \pi$. This is the upper "seam" we see in graph II. When v is held constant, $z = u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x = (1-u)(3+\cos v)\cos 4\pi u$, $y = (1-u)(3+\cos v)\sin 4\pi u$, $z = 3u + (1-u)\sin v$. These equations correspond to graph VI: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3,0,0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2}\cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. From Example 3, parametric equations for the plane through the point $(1, 2, -3)$ that contains the vectors $\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle 1, -1, 1 \rangle$ are $x = 1 + u(1) + v(1) = 1 + u + v$, $y = 2 + u(1) + v(-1) = 2 + u - v$, $z = -3 + u(-1) + v(1) = -3 - u + v$.

18. Solving the equation for z gives $z^2 = 1 - 2x^2 - 4y^2 \quad \Rightarrow \quad z = -\sqrt{1 - 2x^2 - 4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x = x$, $y = y$, $z = -\sqrt{1 - 2x^2 - 4y^2}$.

Alternate solution: The equation can be rewritten as $\dfrac{x^2}{\left(1/\sqrt{2}\right)^2} + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, and if we let $x = \dfrac{1}{\sqrt{2}} u \cos v$ and $y = \frac{1}{2} u \sin v$, then $z = -\sqrt{1 - 2x^2 - 4y^2} = -\sqrt{1 - u^2 \cos^2 v - u^2 \sin^2 v} = -\sqrt{1 - u^2}$, where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

19. Solving the equation for y gives $y^2 = 1 - x^2 + z^2 \quad \Rightarrow \quad y = \sqrt{1 - x^2 + z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y \geq 0$.) If we let x and z be the parameters, parametric equations are $x = x$, $z = z$, $y = \sqrt{1 - x^2 + z^2}$.

20. $x = 4 - y^2 - 2z^2$, $y = y$, $z = z$ where $y^2 + 2z^2 \leq 4$ since $x \geq 0$. Then the associated vector equation is $\mathbf{r}(y, z) = (4 - y^2 - 2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

21. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 2$.

Alternate solution: Using spherical coordinates, $x = 2 \sin\phi \cos\theta$, $y = 2 \sin\phi \sin\theta$, $z = 2 \cos\phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$.

22. In spherical coordinates, parametric equations are $x = 4 \sin\phi \cos\theta$, $y = 4 \sin\phi \sin\theta$, $z = 4 \cos\phi$. The intersection of the sphere with the plane $z = 2$ corresponds to $z = 4 \cos\phi = 2 \quad \Rightarrow \quad \cos\phi = \frac{1}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z = -2$ corresponds to $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Thus the surface is described by $0 \leq \theta \leq 2\pi$, $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$.

23. Parametric equations are $x = x$, $y = 4 \cos\theta$, $z = 4 \sin\theta$, $0 \leq x \leq 5$, $0 \leq \theta \leq 2\pi$.

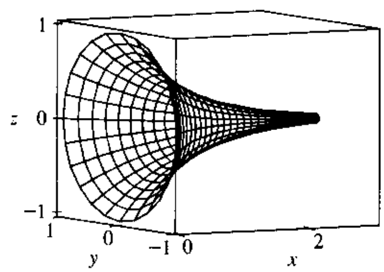
24. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \leq x^2 + y^2 \leq 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \leq s \leq 1$ and $0 \leq \theta \leq 2\pi$.

25. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \leq x \leq 5$, $y \leq 0$ to obtain the portion shown.
To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter

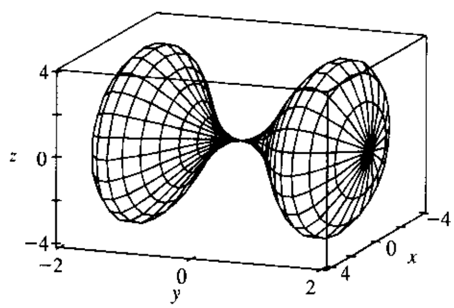
domain $0 \le u \le 5$, $\frac{\pi}{2} \le v \le \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \le x \le 5$ and $-3 \le z \le 3$.

26. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$.

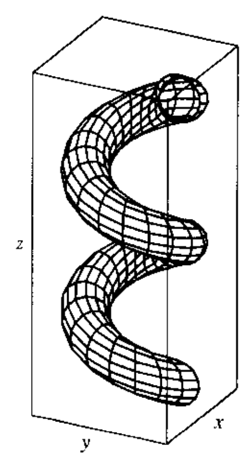
27. Using Equations 3, we have the parametrization $x = x$, $y = e^{-x}\cos\theta$, $z = e^{-x}\sin\theta$, $0 \le x \le 3$, $0 \le \theta \le 2\pi$.



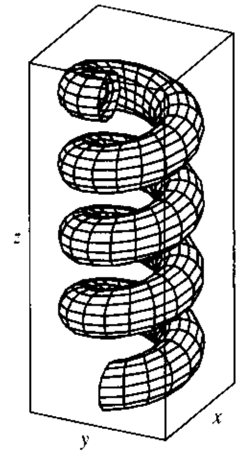
28. Letting θ be the angle of rotation about the y-axis, we have the parametrization $x = (4y^2 - y^4)\cos\theta$, $y = y$, $z = (4y^2 - y^4)\sin\theta$, $-2 \le y \le 2$, $0 \le \theta \le 2\pi$.



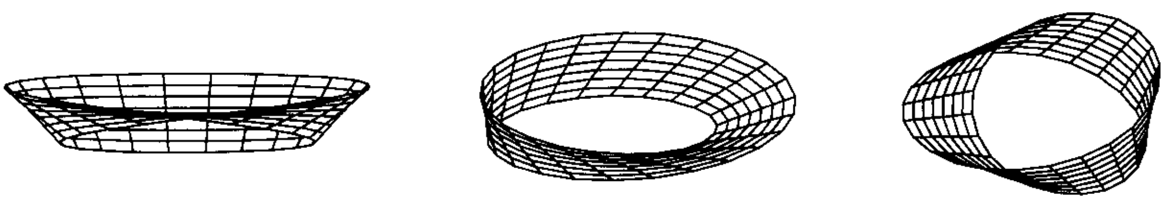
29. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.

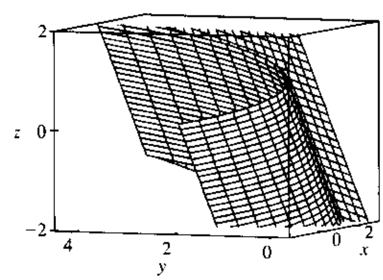


30. First we graph the surface as viewed from the front, then from two additional viewpoints.

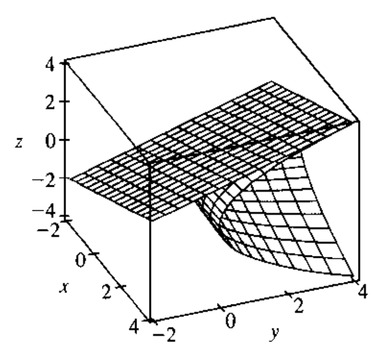


The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 17.7 [ET 16.7].)

31. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.
$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.



32. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + v^2\,\mathbf{j} + uv\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}(1, 1) = (1, 1, 1)$.
$\mathbf{r}_u = 2u\,\mathbf{i} + v\,\mathbf{k}$ and $\mathbf{r}_v = 2v\,\mathbf{j} + u\,\mathbf{k}$, so a normal vector to the surface at the point $(1, 1, 1)$ is
$\mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j} + \mathbf{k}) = -2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}$.
Thus an equation of the tangent plane at the point $(1, 1, 1)$ is
$-2(x - 1) - 2(y - 1) + 4(z - 1) = 0$ or $x + y - 2z = 0$.



33. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}(1,0) = (1,0,1)$.

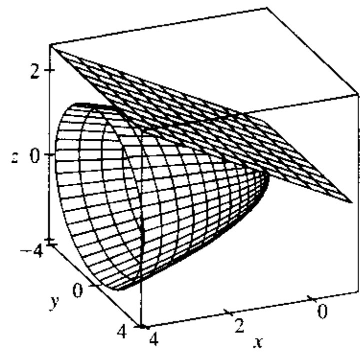
$\mathbf{r}_u = 2u\,\mathbf{i} + 2\sin v\,\mathbf{j} + \cos v\,\mathbf{k}$ and $\mathbf{r}_v = 2u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$,

so a normal vector to the surface at the point $(1,0,1)$ is

$\mathbf{r}_u(1,0) \times \mathbf{r}_v(1,0) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j}) = -2\,\mathbf{i} + 4\,\mathbf{k}$.

Thus an equation of the tangent plane at $(1,0,1)$ is

$-2(x-1) + 0(y-0) + 4(z-1) = 0$ or $-x + 2z = 1$.



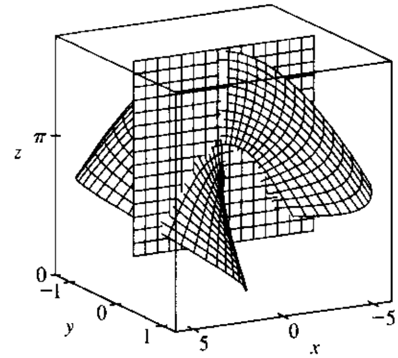
34. $\mathbf{r}(u,v) = uv\,\mathbf{i} + u\sin v\,\mathbf{j} + v\cos u\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}(0,\pi) = (0,0,\pi)$.

$\mathbf{r}_u = v\,\mathbf{i} + \sin v\,\mathbf{j} - v\sin u\,\mathbf{k}$ and $\mathbf{r}_v = u\,\mathbf{i} + u\cos v\,\mathbf{j} + \cos u\;\mathbf{k}$, so

a normal vector to the surface at the point $(0,0,\pi)$ is

$\mathbf{r}_u(0,\pi) \times \mathbf{r}_v(0,\pi) = (\pi\,\mathbf{i}) \times (\mathbf{k}) = -\pi\,\mathbf{j}$. Thus an equation of the

tangent plane is $-\pi(y-0) = 0$ or $y = 0$.



35. Here $z = f(x,y) = 4 - x - 2y$ and D is the disk $x^2 + y^2 \le 4$. Thus, by Formula 9,

$$A(S) = \iint_D \sqrt{1 + (-1)^2 + (-2)^2}\,dA = \sqrt{6}\iint_D dA = \sqrt{6}\,A(D) = 4\sqrt{6}\,\pi$$

36. $\mathbf{r}_u = \langle 0,1,-5\rangle$, $\mathbf{r}_v = \langle 1,-2,1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle -9,-5,-1\rangle$. Then by Definition 6,

$$A(S) = \iint_D |\,\mathbf{r}_u \times \mathbf{r}_v\,|\,dA = \int_0^1\int_0^1 |\,\langle -9,-5,-1\rangle\,|\,du\,dv = \sqrt{107}\int_0^1 du\int_0^1 dv = \sqrt{107}$$

37. $z = f(x,y) = xy$ with $0 \le x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$A(S) = \iint_D \sqrt{1 + y^2 + x^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{3}\left(r^2+1\right)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\frac{1}{3}\left(2\sqrt{2}-1\right)d\theta = \frac{2\pi}{3}\left(2\sqrt{2}-1\right)$$

38. $z = f(x,y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus, by Formula 9,

$$A(S) = \iint_D \sqrt{1 + 3^2 + (4y)^2}\,dA = \int_0^1\int_0^{2y} \sqrt{10 + 16y^2}\,dx\,dy = \int_0^1 2y\sqrt{10 + 16y^2}\,dy$$

$$= \tfrac{1}{16}\cdot\tfrac{2}{3}\left(10 + 16y^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{24}\left(26^{3/2} - 10^{3/2}\right)$$

39. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{1 + 4r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_1^2 \sqrt{1+4r^2}\,r\,dr$$

$$= \big[\,\theta\,\big]_0^{2\pi}\left[\tfrac{1}{12}\left(1 + 4r^2\right)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

40. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \le y^2 + z^2 \le 9$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$.

Note: In general, if $x = f(y,z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$, and

$A(S) = \iint_D \sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \iint_{0 \le y^2 + z^2 \le 9} \sqrt{1 + 4y^2 + 4z^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\,dr = 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^3 = \tfrac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

41. A parametric representation of the surface is $x = x$, $y = 4x + z^2$, $z = z$ with $0 \le x \le 1$, $0 \le z \le 1$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 4\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 4\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x,z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \dfrac{\partial f}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$ and

$A(S) = \iint_D \sqrt{1 + \left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \int_0^1\int_0^1 \sqrt{17 + 4z^2}\,dx\,dz = \int_0^1 \sqrt{17 + 4z^2}\,dz$$

$$= \tfrac{1}{2}\left(z\sqrt{17 + 4z^2} + \tfrac{17}{2}\ln\left|2z + \sqrt{4z^2 + 17}\right|\right)\Big]_0^1 = \tfrac{\sqrt{21}}{2} + \tfrac{17}{4}\left[\ln\left(2 + \sqrt{21}\right) - \ln\sqrt{17}\right]$$

42. Let S_1 be that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$ by symmetry.

On S_1, $z = \sqrt{a^2 - x^2}$ so $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2}} = \dfrac{a}{\sqrt{a^2 - x^2}}$. Hence

$$A(S_1) = \iint_{0 \le x^2 + y^2 \le a^2} \frac{a}{\sqrt{a^2 - x^2}}\,dA = \int_{-a}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}}\,dy\,dx = \int_{-a}^{a} 2a\,dx = 4a^2.$$

Thus $A(S) = 8a^2$.

Alternate solution: If $A(S_2)$ is the surface area in the first octant, then $A(S) = 8A(S_2)$. A parametric representation of the surface in the first octant is $x = a\sin\theta$, $y = y$, $z = a\cos\theta$ (θ being the angle in the xz-plane measured from the positive z-axis), where $0 \le \theta \le \frac{\pi}{2}$ and $0 \le y \le a\cos\theta$. The restrictions on y follow from: $x^2 + y^2 \le a^2$ or $a^2\sin^2\theta + y^2 \le a^2$ so $y^2 \le a^2(1 - \sin^2\theta)$; thus in the first octant $0 \le y \le a\cos\theta$. Then $\mathbf{r}_y \times \mathbf{r}_\theta = \langle -a\sin\theta, 0, -a\cos\theta\rangle$ and $A(S_2) = \int_0^{\pi/2}\int_0^{a\cos\theta} a\,dy\,d\theta = \int_0^{\pi/2} a^2\cos\theta\,d\theta = a^2$.

Hence $A(S) = 8a^2$.

43. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$,

$z = a\cos\phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$. For D, $0 \le \phi \le \frac{\pi}{2}$ and for each fixed ϕ, $\left(x - \frac{1}{2}a\right)^2 + y^2 \le \left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta - \frac{1}{2}a\right]^2 + a^2\sin^2\phi\sin^2\theta \le (a/2)^2$ implies $a^2\sin^2\phi - a^2\sin\phi\cos\theta \le 0$ or $\sin\phi\,(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \frac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\sin\left(\frac{\pi}{2} + \theta\right) \ge \sin\phi$ or $\phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi$. Hence $D = \left\{(\phi,\theta) \mid 0 \le \phi \le \frac{\pi}{2}, \phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi\right\}$. Then

$$A(S_1) = \int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi} a^2\sin\phi\,d\theta\,d\phi = a^2\int_0^{\pi/2}(\pi - 2\phi)\sin\phi\,d\phi$$

$$= a^2\left[(-\pi\cos\phi) - 2(-\phi\cos\phi + \sin\phi)\right]_0^{\pi/2} = a^2(\pi - 2)$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$, $z = \sqrt{a^2 - x^2 - y^2}$. Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$A(S_1) = \iint_{0 \le (x-(a/2))^2 + y^2 \le (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -a(a^2 - r^2)^{1/2}\Big]_{r=0}^{r=a\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} a^2\left[1 - (1 - \cos^2\theta)^{1/2}\right]d\theta$$

$$= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin\theta|)\,d\theta = 2a^2\int_0^{\pi/2}(1 - \sin\theta)\,d\theta = 2a^2\left(\frac{\pi}{2} - 1\right)$$

Thus $A(S) = 4a^2\left(\frac{\pi}{2} - 1\right) = 2a^2(\pi - 2)$.

Notes:

(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

44. $\mathbf{r}_u = \langle\cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle\sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^{\pi}\int_0^1 \sqrt{1 + u^2}\,du\,dv = \int_0^{\pi} dv \int_0^1 \sqrt{1 + u^2}\,du$$

$$= \pi\left[\tfrac{u}{2}\sqrt{u^2 + 1} + \tfrac{1}{2}\ln\left|u + \sqrt{u^2 + 1}\right|\right]_0^1 = \tfrac{\pi}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

45. $\mathbf{r}_u = \langle v, 1, 1\rangle$, $\mathbf{r}_v = \langle u, 1, -1\rangle$ and $\mathbf{r}_u \times \mathbf{r}_v = \langle -2, u + v, v - u\rangle$. Then

$$A(S) = \iint_{u^2 + v^2 \le 1} \sqrt{4 + 2u^2 + 2v^2}\,dA = \int_0^{2\pi}\int_0^1 r\sqrt{4 + 2r^2}\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{4 + 2r^2}\,dr$$

$$= 2\pi\left[\tfrac{1}{6}\left(4 + 2r^2\right)^{3/2}\right]_0^1 = \tfrac{\pi}{3}\left(6\sqrt{6} - 8\right) = \pi\left(2\sqrt{6} - \tfrac{8}{3}\right)$$

46. $z = f(x,y) = \cos(x^2+y^2)$ with $x^2+y^2 \le 1$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1+(-2x\sin(x^2+y^2))^2+(-2y\sin(x^2+y^2))^2}\,dA \\ &= \iint_D \sqrt{1+4x^2\sin^2(x^2+y^2)+4y^2\sin^2(x^2+y^2)}\,dA \\ &= \iint_D \sqrt{1+4(x^2+y^2)\sin^2(x^2+y^2)}\,dA \\ &= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2\sin^2(r^2)}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{1+4r^2\sin^2(r^2)}\,dr \\ &= 2\pi \int_0^1 r\sqrt{1+4r^2\sin^2(r^2)}\,dr \approx 4.1073 \end{aligned}$$

47. $z = f(x,y) = e^{-x^2-y^2}$ with $x^2+y^2 \le 4$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1+\left(-2xe^{-x^2-y^2}\right)^2+\left(-2ye^{-x^2-y^2}\right)^2}\,dA \\ &= \iint_D \sqrt{1+4(x^2+y^2)e^{-2(x^2+y^2)}}\,dA \\ &= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2e^{-2r^2}}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr \\ &= 2\pi\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr \approx 13.9783 \end{aligned}$$

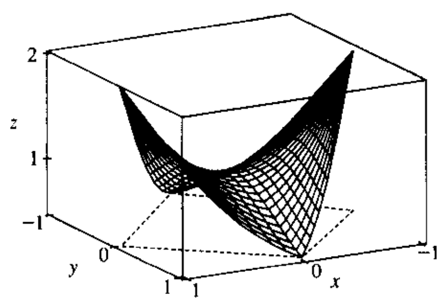
48. Let $f(x,y) = \dfrac{1+x^2}{1+y^2}$. Then $f_x = \dfrac{2x}{1+y^2}$,

$f_y = (1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right] = -\dfrac{2y(1+x^2)}{(1+y^2)^2}$.

We use a CAS to estimate

$\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|} \sqrt{1+f_x^2+f_y^2}\,dy\,dx \approx 2.6959$.

In order to graph only the part of the surface above the square, we use $-(1-|x|) \le y \le 1-|x|$ as the y-range in our plot command.



49. (a) The midpoints of the four squares are $\left(\frac14,\frac14\right)$, $\left(\frac14,\frac34\right)$, $\left(\frac34,\frac14\right)$, and $\left(\frac34,\frac34\right)$; the derivatives of the function $f(x,y) = x^2+y^2$ are $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$, so the Midpoint Rule gives

$$\begin{aligned} A(S) &= \int_0^1\int_0^1 \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dy\,dx \\ &\approx \tfrac14\left(\sqrt{\left[2\left(\tfrac14\right)\right]^2+\left[2\left(\tfrac14\right)\right]^2+1}+\sqrt{\left[2\left(\tfrac14\right)\right]^2+\left[2\left(\tfrac34\right)\right]^2+1}\right. \\ &\qquad \left.+\sqrt{\left[2\left(\tfrac34\right)\right]^2+\left[2\left(\tfrac14\right)\right]^2+1}+\sqrt{\left[2\left(\tfrac34\right)\right]^2+\left[2\left(\tfrac34\right)\right]^2+1}\right) \\ &= \tfrac14\left(\sqrt{\tfrac32}+2\sqrt{\tfrac72}+\sqrt{\tfrac{11}{2}}\right) \approx 1.8279 \end{aligned}$$

(b) A CAS estimates the integral to be

$A(S) = \int_0^1\int_0^1 \sqrt{f_x^2+f_y^2+1}\,dy\,dx = \int_0^1\int_0^1 \sqrt{4x^2+4y^2+1}\,dy\,dx \approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

50. $\mathbf{r}(u,v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, so $\mathbf{r}_u = \langle -3\cos^2 u \sin u \cos^3 v, 3\sin^2 u \cos u \cos^3 v, 0 \rangle$,

$\mathbf{r}_v = \langle -3\cos^3 u \cos^2 v \sin v, -3\sin^3 u \cos^2 v \sin v, 3\sin^2 v \cos v \rangle$, and

$\mathbf{r}_u \times \mathbf{r}_v = \langle 9\cos u \sin^2 u \cos^4 v \sin^2 v, 9\cos^2 u \sin u \cos^4 v \sin^2 v, 9\cos^2 u \sin^2 u \cos^5 v \sin v \rangle$. Then

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= 9\sqrt{\cos^2 u \sin^4 u \cos^8 v \sin^4 v + \cos^4 u \sin^2 u \cos^8 v \sin^4 v + \cos^4 u \sin^4 u \cos^{10} v \sin^2 v} \\ &= 9\sqrt{\cos^2 u \sin^2 u \cos^8 v \sin^2 v\,(\sin^2 v + \cos^2 u \sin^2 u \cos^2 v)} \\ &= 9\cos^4 v\,|\cos u \sin u \sin v|\,\sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v} \end{aligned}$$

Using a CAS, we have

$A(S) = \int_0^{\pi} \int_0^{2\pi} 9\cos^4 v\,|\cos u \sin u \sin v|\,\sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v}\,dv\,du \approx 4.4506$.

51. $z = 1 + 2x + 3y + 4y^2$, so

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3+8y)^2}\,dy\,dx \\ &= \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\,dy\,dx. \end{aligned}$$

Using a CAS, we have

$\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\,dy\,dx = \frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\!\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \frac{15}{16}\ln\!\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$

or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}$.

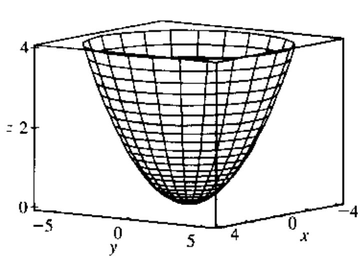
52. (a) $\mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and

$\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\,\mathbf{k}$.

$A(S) = \int_0^{2\pi}\int_0^2 |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2 \;\Rightarrow\; x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D, notice that $0 \le u \le 2 \;\Rightarrow\; 0 \le z \le 4 \;\Rightarrow\; 0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using Formula 9, we have $A(S) = \displaystyle\int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\,dy\,dx$.

(c)



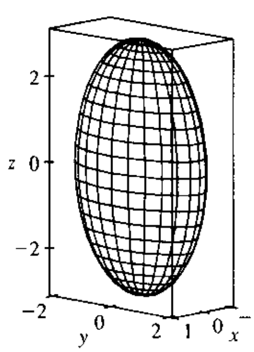
(d) We substitute $a = 2$, $b = 3$ in the integral in part (a) to get $A(S) = \int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v + 4u^2\sin^2 v + 9}\,du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set `Digits:=7;` (in Maple) or use the approximation command `N` (in Mathematica). We find that $A(S) \approx 115.6596$.

53. (a) $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u$ $\Rightarrow$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2$$
$$= \sin^2 u + \cos^2 u = 1$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

(b)



(c) From the parametric equations (with $a=1$, $b=2$, and $c=3$), we calculate $\mathbf{r}_u = \cos u\cos v\,\mathbf{i} + 2\cos u\sin v\,\mathbf{j} - 3\sin u\,\mathbf{k}$ and $\mathbf{r}_v = -\sin u\sin v\,\mathbf{i} + 2\sin u\cos v\,\mathbf{j}$. So $\mathbf{r}_u\times\mathbf{r}_v = 6\sin^2 u\cos v\,\mathbf{i} + 3\sin^2 u\sin v\,\mathbf{j} + 2\sin u\cos u\,\mathbf{k}$, and the surface area is given by

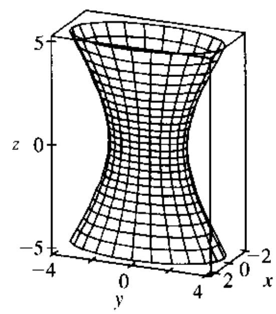
$$A(S) = \int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv$$
$$= \int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv$$

54. (a) $x = a\cosh u\cos v$, $y = b\cosh u\sin v$, $z = c\sinh u$ $\Rightarrow$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = \cosh^2 u\cos^2 v + \cosh^2 u\sin^2 v - \sinh^2 u$$
$$= \cosh^2 u - \sinh^2 u = 1$$

and the parametric equations represent a hyperboloid of one sheet.

(b)



(c) $\mathbf{r}_u = \sinh u\cos v\,\mathbf{i} + 2\sinh u\sin v\,\mathbf{j} + 3\cosh u\,\mathbf{k}$ and $\mathbf{r}_v = -\cosh u\sin v\,\mathbf{i} + 2\cosh u\cos v\,\mathbf{j}$, so $\mathbf{r}_u\times\mathbf{r}_v = -6\cosh^2 u\cos v\,\mathbf{i} - 3\cosh^2 u\sin v\,\mathbf{j} + 2\cosh u\sinh u\,\mathbf{k}$. We integrate between $u = \sinh^{-1}(-1) = -\ln(1+\sqrt{2})$ and $u = \sinh^{-1}1 = \ln(1+\sqrt{2})$, since then z varies between -3 and 3, as desired. So the surface area is

$$A(S) = \int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv$$
$$= \int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})}\sqrt{36\cosh^4 u\cos^2 v + 9\cosh^4 u\sin^2 v + 4\cosh^2 u\sinh^2 u}\,du\,dv$$

55. $\mathbf{r}(u,v) = \langle\cos^3 u\cos^3 v, \sin^3 u\cos^3 v, \sin^3 v\rangle$, so $\mathbf{r}_u = \langle -3\cos^2 u\sin u\cos^3 v, 3\sin^2 u\cos u\cos^3 v, 0\rangle$, $\mathbf{r}_v = \langle -3\cos^3 u\cos^2 v\sin v, -3\sin^3 u\cos^2 v\sin v, 3\sin^2 v\cos v\rangle$, and $\mathbf{r}_u\times\mathbf{r}_v = \langle 9\cos u\sin^2 u\cos^4 v\sin^2 v, 9\cos^2 u\sin u\cos^4 v\sin^2 v, 9\cos^2 u\sin^2 u\cos^5 v\sin v\rangle$. Then

$$|\mathbf{r}_u\times\mathbf{r}_v| = 9\sqrt{\cos^2 u\sin^4 u\cos^8 v\sin^4 v + \cos^4 u\sin^2 u\cos^8 v\sin^4 v + \cos^4 u\sin^4 u\cos^{10} v\sin^2 v}$$
$$= 9\sqrt{\cos^2 u\sin^2 u\cos^8 v\sin^2 v\,(\sin^2 v + \cos^2 u\sin^2 u\cos^2 v)}$$
$$= 9\cos^4 v\,|\cos u\sin u\sin v|\sqrt{\sin^2 v + \cos^2 u\sin^2 u\cos^2 v}$$

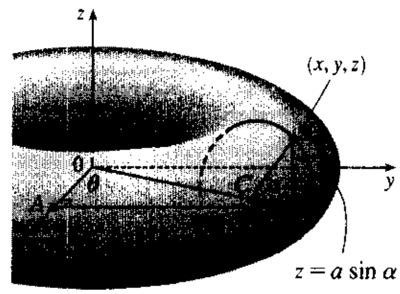
Using a CAS, we have $A(S) = \int_0^{\pi}\int_0^{2\pi} 9\cos^4 v\,|\cos u\sin u\sin v|\sqrt{\sin^2 v + \cos^2 u\sin^2 u\cos^2 v}\,dv\,du \approx 4.4506$.

56. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But

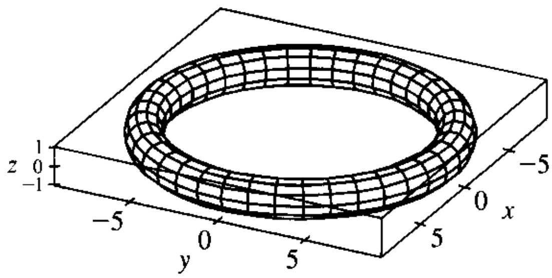
$|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that

$y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so

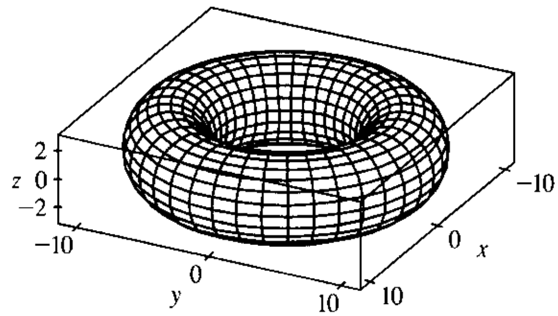
$x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\cos\theta$, $y = b\sin\theta + a\cos\alpha\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



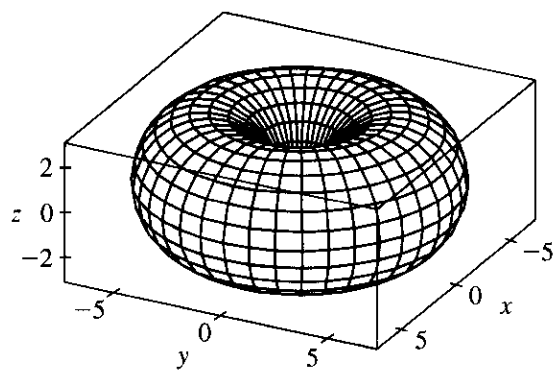
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\cos\theta$, $y = b\sin\theta + a\cos\alpha\sin\theta$, $z = a\sin\alpha$, so

$\mathbf{r}_\alpha = \langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\cos\theta - a^2\cos\alpha\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\cos\theta - a^2\sin\alpha\cos^2\theta\right)\mathbf{j} \\ &\quad + \left(-ab\cos^2\alpha\sin\theta - a^2\cos^2\alpha\sin\theta\cos\theta - ab\sin^2\alpha\sin\theta - a^2\sin^2\alpha\sin\theta\cos\theta\right)\mathbf{k} \\ &= -a(b + a\cos\alpha)\left[(\cos\theta\cos\alpha)\,\mathbf{i} + (\sin\theta\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}\right]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\cos^2\alpha + \sin^2\theta\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.

17.7 Surface Integrals ET 16.7

1. Each face of the cube has surface area $2^2 = 4$, and the points P_{ij}^* are the points where the cube intersects the coordinate axes. Here, $f(x,y,z) = \sqrt{x^2 + 2y^2 + 3z^2}$, so by Definition 1,

$$\begin{aligned}\iint_S f(x,y,z)\,dS &\approx [f(1,0,0)](4) + [f(-1,0,0)](4) + [f(0,1,0)](4) + [f(0,-1,0)](4)\\ &\quad + [f(0,0,1)](4) + [f(0,0,-1)](4)\\ &= 4\left(1 + 1 + 2\sqrt{2} + 2\sqrt{3}\right) = 8\left(1 + \sqrt{2} + \sqrt{3}\right) \approx 33.170\end{aligned}$$

2. Each quarter-cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can take $(0,0,1)$ as a sample point in the top disk, $(0,0,-1)$ in the bottom disk, and $(\pm1,0,0)$, $(0,\pm1,0)$ in the four quarter-cylinders. Then $\iint_S f(x,y,z)\,dS$ can be approximated by the Riemann sum

$$\begin{aligned}&f(1,0,0)(\pi) + f(-1,0,0)(\pi) + f(0,1,0)\,(\pi) + f(0,-1,0)(\pi) + f(0,0,1)(\pi) + f(0,0,-1)(\pi)\\ &\qquad = (2+2+3+3+4+4)\pi = 18\pi \approx 56.5.\end{aligned}$$

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}(4)\pi\left(\sqrt{50}\right)^2 = 25\pi$. Then $(\pm3, \pm4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\begin{aligned}\iint_H f(x,y,z)\,dS &\approx f(3,4,5)\,\Delta S + f(3,-4,5)\,\Delta S + f(-3,4,5)\,\Delta S + f(-3,-4,5)\,\Delta S\\ &= (7+8+9+12)(25\pi) = 900\pi \approx 2827\end{aligned}$$

4. On the surface, $f(x,y,z) = g\left(\sqrt{x^2+y^2+z^2}\right) = g(2) = -5$. So since the area of a sphere is $4\pi r^2$, $\iint_S f(x,y,z)\,dS = \iint_S g(2)\,dS = -5\iint_S dS = -5[4\pi(2)^2] = -80\pi$.

5. $z = 1 + 2x + 3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 2,

$$\begin{aligned}\iint_S x^2yz\,dS &= \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA\\ &= \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\,dy\,dx\\ &= \sqrt{14}\int_0^3\int_0^2 (x^2y + 2x^3y + 3x^2y^2)\,dy\,dx\\ &= \sqrt{14}\int_0^3 \left[\tfrac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2} dx\\ &= \sqrt{14}\int_0^3 (10x^2 + 4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14}\end{aligned}$$

6. S is the region in the plane $2x + y + z = 2$ or $z = 2 - 2x - y$ over $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\}$. Thus

$$\begin{aligned}\iint_S xy\,dS &= \iint_D xy\sqrt{(-2)^2 + (-1)^2 + 1}\,dA\\ &= \sqrt{6}\int_0^1\int_0^{2-2x} xy\,dy\,dx = \sqrt{6}\int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=2-2x} dx\\ &= \tfrac{\sqrt{6}}{2}\int_0^1 (4x - 8x^2 + 4x^3)\,dx = \tfrac{\sqrt{6}}{2}\left(2 - \tfrac{8}{3} + 1\right) = \tfrac{\sqrt{6}}{6}\end{aligned}$$

7. S is the part of the plane $z = 1 - x - y$ over the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$. Thus

$$\begin{aligned}\iint_S yz\,dS &= \iint_D y(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\,dA \\ &= \sqrt{3}\int_0^1\int_0^{1-x}(y - xy - y^2)\,dy\,dx = \sqrt{3}\int_0^1\left[\tfrac{1}{2}y^2 - \tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3\right]_{y=0}^{y=1-x}dx \\ &= \sqrt{3}\int_0^1 \tfrac{1}{6}(1-x)^3\,dx = -\tfrac{\sqrt{3}}{24}(1-x)^4\Big]_0^1 = \tfrac{\sqrt{3}}{24}\end{aligned}$$

8. $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ and

$$\begin{aligned}\iint_S y\,dS &= \iint_D y\sqrt{(\sqrt{x}\,)^2 + (\sqrt{y}\,)^2 + 1}\,dA = \int_0^1\int_0^1 y\sqrt{x+y+1}\,dx\,dy \\ &= \int_0^1 y\left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{x=0}^{x=1}dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2} - (y+1)^{3/2}\right]dy\end{aligned}$$

Substituting $u = y + 2$ in the first term and $t = y + 1$ in the second, we have

$$\begin{aligned}\iint_S y\,dS &= \tfrac{2}{3}\int_2^3(u-2)u^{3/2}\,du - \tfrac{2}{3}\int_1^2(t-1)t^{3/2}\,dt \\ &= \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2} - \tfrac{4}{5}u^{5/2}\right]_2^3 - \tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2} - \tfrac{2}{5}t^{5/2}\right]_1^2 \\ &= \tfrac{2}{3}\left[\tfrac{2}{7}(3^{7/2} - 2^{7/2}) - \tfrac{4}{5}(3^{5/2} - 2^{5/2}) - \tfrac{2}{7}(2^{7/2} - 1) + \tfrac{2}{5}(2^{5/2} - 1)\right] \\ &= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3} + \tfrac{8}{35}\sqrt{2} - \tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3} + 4\sqrt{2} - 2\right)\end{aligned}$$

9. S is the portion of the cone $z^2 = x^2 + y^2$ for $1 \le z \le 3$, or equivalently, S is the part of the surface $z = \sqrt{x^2 + y^2}$ over the region $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 9\}$. Thus

$$\begin{aligned}\iint_S x^2z^2\,dS &= \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\,dA \\ &= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2} + 1}\,dA = \iint_D \sqrt{2}\,x^2(x^2+y^2)\,dA \\ &= \sqrt{2}\int_0^{2\pi}\int_1^3(r\cos\theta)^2(r^2)\,r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta\int_1^3 r^5\,dr \\ &= \sqrt{2}\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}r^6\right]_1^3 = \sqrt{2}\,(\pi)\cdot\tfrac{1}{6}(3^6 - 1) = \frac{364\sqrt{2}}{3}\,\pi\end{aligned}$$

10. Using y and z as parameters, we have $\mathbf{r}(y, z) = (y + 2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $0 \le y \le 1, 0 \le z \le 1$. Then $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i} + \mathbf{j}) \times (4z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - \mathbf{j} - 4z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{2 + 16z^2}$. Thus

$$\begin{aligned}\iint_S z\,dS &= \int_0^1\int_0^1 z\sqrt{2+16z^2}\,dy\,dz = \int_0^1 z\sqrt{2+16z^2}\,dz = \left[\tfrac{1}{32}\cdot\tfrac{2}{3}(2+16z^2)^{3/2}\right]_0^1 \\ &= \tfrac{1}{48}(18^{3/2} - 2^{3/2}) = \tfrac{13}{12}\sqrt{2}\end{aligned}$$

11. Using x and z as parameters, we have $\mathbf{r}(x, z) = x\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + z\,\mathbf{k}$, $x^2 + z^2 \le 4$. Then $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2 + 1 + 4z^2} = \sqrt{1 + 4(x^2 + z^2)}$.

Thus

$$\iint_S y\,dS = \iint_{x^2+z^2\le 4} (x^2+z^2)\sqrt{1+4(x^2+z^2)}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^2\sqrt{1+4r^2}\,r\,dr = 2\pi\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr$$

$$\left[\text{let } u = 1+4r^2 \quad\Rightarrow\quad r^2 = \tfrac{1}{4}(u-1) \text{ and } \tfrac{1}{8}du = r\,dr\right]$$

$$= 2\pi\int_1^{17} \tfrac{1}{4}(u-1)\sqrt{u}\cdot\tfrac{1}{8}du = \tfrac{1}{16}\pi\int_1^{17}(u^{3/2}-u^{1/2})\,du$$

$$= \tfrac{1}{16}\pi\left[\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}\right]_1^{17} = \tfrac{1}{16}\pi\left[\tfrac{2}{5}(17)^{5/2}-\tfrac{2}{3}(17)^{3/2}-\tfrac{2}{5}+\tfrac{2}{3}\right] = \frac{\pi}{60}(391\sqrt{17}+1)$$

12. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$. On S_1: using cylindrical coordinates, $\mathbf{r}(\theta,y) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + \cos\theta\,\mathbf{k}$, $0\le\theta\le 2\pi$, $0\le y\le 2-\sin\theta$, $|\mathbf{r}_\theta\times\mathbf{r}_y| = 1$ and

$\iint_{S_1} xy\,dS = \int_0^{2\pi}\int_0^{2-\sin\theta}(\sin\theta)\,y\,dy\,d\theta = \int_0^{2\pi}\left[2\sin\theta - 2\sin^2\theta + \tfrac{1}{2}\sin^3\theta\right]d\theta = -2\pi.$

On S_2: $\mathbf{r}(x,z) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_x\times\mathbf{r}_z| = |-\mathbf{i}-\mathbf{j}| = \sqrt{2}$, where $x^2+z^2\le 1$ and

$$\iint_{S_2} xy\,dS = \iint_{x^2+z^2\le 1} x(2-x)\sqrt{2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{2}\,(2r\sin\theta - r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\tfrac{2}{3}\sin\theta - \tfrac{1}{4}\sin^2\theta\right]d\theta = -\tfrac{\sqrt{2}}{4}\pi$$

On S_3: $y=0$ so $\iint_{S_3} xy\,dS = 0$. Hence $\iint_S xy\,dS = -2\pi - \frac{\sqrt{2}}{4}\pi = -\frac{1}{4}(8+\sqrt{2})\pi$.

13. Using spherical coordinates and Example 17.6.10 [ET 16.6.10] we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$ and $|\mathbf{r}_\phi\times\mathbf{r}_\theta| = 4\sin\phi$. Then

$\iint_S (x^2z+y^2z)\,dS = \int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta = 16\pi\sin^4\phi\big]_0^{\pi/2} = 16\pi.$

14. Using spherical coordinates, $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0\le\phi\le\frac{\pi}{4}$, $0\le\theta\le 2\pi$, and $|\mathbf{r}_\phi\times\mathbf{r}_\theta| = \sin\phi$ (see Example 17.6.10 [ET 16.6.10]). Then

$\iint_S xyz\,dS = \int_0^{2\pi}\int_0^{\pi/4}(\sin^3\phi\cos\phi\cos\theta\sin\theta)\,d\phi\,d\theta = 0$ since $\int_0^{2\pi}\cos\theta\sin\theta\,d\theta = 0$.

15. Using cylindrical coordinates, we have $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0\le\theta\le 2\pi$, $0\le z\le 2$, and $|\mathbf{r}_\theta\times\mathbf{r}_z| = 3$.

$$\iint_S (x^2y+z^2)\,dS = \int_0^{2\pi}\int_0^2 (27\cos^2\theta\sin\theta + z^2)\,3\,dz\,d\theta = \int_0^{2\pi}(162\cos^2\theta\sin\theta + 8)\,d\theta = 16\pi$$

16. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1: $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0\le\theta\le 2\pi$, $0\le z\le 2$, $|\mathbf{r}_\theta\times\mathbf{r}_z| = 3$,

$\iint_{S_1}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2 (9+z^2)\,3\,dz\,d\theta = 2\pi(54+8) = 124\pi.$

On S_2: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0\le r\le 3$, $0\le\theta\le 2\pi$, $|\mathbf{r}_\theta\times\mathbf{r}_r| = r$,

$\iint_{S_2}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2+4)\,r\,dr\,d\theta = 2\pi\left(\tfrac{81}{4}+18\right) = \tfrac{153}{2}\pi.$

On S_3: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0\le r\le 3$, $0\le\theta\le 2\pi$, $|\mathbf{r}_\theta\times\mathbf{r}_r| = r$,

$\iint_{S_3}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2+0)\,r\,dr\,d\theta = 2\pi\left(\tfrac{81}{4}\right) = \tfrac{81}{2}\pi.$

Hence $\iint_S (x^2+y^2+z^2)\,dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

17. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$, $0\le u\le 1$, $0\le v\le \pi/2$ and

$\mathbf{r}_u\times\mathbf{r}_v = (2u\,\mathbf{i} + \sin v\,\mathbf{j} + \cos v\,\mathbf{k})\times(u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}) = -u\,\mathbf{i} + 2u^2\sin v\,\mathbf{j} + 2u^2\cos v\,\mathbf{k}$ and

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2 + 4u^4 \sin^2 v + 4u^4 \cos^2 v} = \sqrt{u^2 + 4u^4(\sin^2 v + \cos^2 v)} = u\sqrt{1+4u^2}$ (since $u \geq 0$). Then

$$\iint_S yz\,dS = \int_0^{\pi/2}\int_0^1 (u\sin v)(u\cos v)\cdot u\sqrt{1+4u^2}\,du\,dv = \int_0^1 u^3\sqrt{1+4u^2}\,du\int_0^{\pi/2}\sin v\cos v\,dv$$

$$[\text{let } t = 1+4u^2 \;\Rightarrow\; u^2 = \tfrac{1}{4}(t-1) \text{ and } \tfrac{1}{8}\,dt = u\,du]$$

$$= \int_1^5 \tfrac{1}{8}\cdot\tfrac{1}{4}(t-1)\sqrt{t}\,dt\int_0^{\pi/2}\sin v\cos v\,dv = \tfrac{1}{32}\int_1^5\left(t^{3/2}-\sqrt{t}\right)dt\int_0^{\pi/2}\sin v\cos v\,dv$$

$$= \tfrac{1}{32}\left[\tfrac{2}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}\right]_1^5\left[\tfrac{1}{2}\sin^2 v\right]_0^{\pi/2} = \tfrac{1}{32}\left(\tfrac{2}{5}(5)^{5/2} - \tfrac{2}{3}(5)^{3/2} - \tfrac{2}{5} + \tfrac{2}{3}\right)\cdot\tfrac{1}{2}(1-0)$$

$$= \tfrac{5}{48}\sqrt{5} + \tfrac{1}{240}$$

18. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k} \;\Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+u^2}$, so $\iint_S \sqrt{1+x^2+y^2}\,dS = \int_0^\pi\int_0^1 \sqrt{1+u^2}\,\sqrt{1+u^2}\,du\,dv = \frac{4}{3}\pi$.

19. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 8

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D[-xy(-2x) - yz(-2y) + zx]\,dA$$

$$= \int_0^1\int_0^1[2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx$$

$$= \int_0^1\left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right)dx = \tfrac{713}{180}$$

20. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + 4x^2\,\mathbf{j} + yz\,\mathbf{k}$, $z = g(x,y) = xe^y$, and D is the square $[0,1]\times[0,1]$, so by Equation 8

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D[-xy(e^y) - 4x^2(xe^y) + yz]dA = \int_0^1\int_0^1(-xye^y - 4x^3e^y + xye^y)\,dy\,dx$$

$$= \int_0^1\left[-4x^3e^y\right]_{y=0}^{y=1}dx = (e-1)\int_0^1(-4x^3)\,dx = 1-e$$

21. $\mathbf{F}(x,y,z) = xze^y\,\mathbf{i} - xze^y\,\mathbf{j} + z\,\mathbf{k}$, $z = g(x,y) = 1-x-y$, and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1-x\}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D[-xze^y(-1) - (-xze^y)(-1) + z]\,dA = -\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$= -\int_0^1\left(\tfrac{1}{2}x^2 - x + \tfrac{1}{2}\right)dx = -\tfrac{1}{6}$$

22. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z^4\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the disk $\{(x,y) \mid x^2+y^2 \le 1\}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D\left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - y\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^4\right]dA$$

$$= -\iint_D\left[\frac{-x^2-y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^4\right]dA = -\int_0^{2\pi}\int_0^1\left(\frac{-r^2}{r} + r^4\right)r\,dr\,d\theta$$

$$= -\int_0^{2\pi}d\theta\int_0^1(r^5 - r^2)\,dr = -2\pi\left(\tfrac{1}{6} - \tfrac{1}{3}\right) = \frac{\pi}{3}$$

23. $\mathbf{F}(x,y,z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk $\{(x,y)\,|\,0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\}$. S has downward orientation, so by Formula 8,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-x\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y\right] dA$$

$$= -\iint_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y\right) dA$$

$$= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta$$

$$= -\int_0^{\pi/2}\cos^2\theta\,d\theta \int_0^2 r^3(4-r^2)^{-1/2}\,dr$$

$$[\text{let } u = 4-r^2 \;\Rightarrow\; r^2 = 4-u \text{ and } -\tfrac{1}{2}\,du = r\,dr]$$

$$= -\int_0^{\pi/2}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta \int_4^0 -\tfrac{1}{2}(4-u)(u)^{-1/2}\,du$$

$$= -\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}\left(-\tfrac{1}{2}\right)\left[8\sqrt{u} - \tfrac{2}{3}u^{3/2}\right]_4^0$$

$$= -\tfrac{\pi}{4}\left(-\tfrac{1}{2}\right)\left(-16+\tfrac{16}{3}\right) = -\frac{4}{3}\pi$$

24. $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$

Using spherical coordinates, S is given by $x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$, $z = 5\cos\phi$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = (5\sin\phi\cos\theta)(5\cos\phi)\,\mathbf{i} + (5\sin\phi\cos\theta)\,\mathbf{j} + (5\sin\phi\sin\theta)\,\mathbf{k}$ and $\mathbf{r}_\phi \times \mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\cos\phi\sin\phi\,\mathbf{k}$, so

$$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta$$

Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)]\,dA$$

$$= \int_0^\pi\int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\theta\,d\phi$$

$$= 125\int_0^\pi \left[5\sin^3\phi\cos\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + \sin^3\phi\left(\tfrac{1}{2}\sin^2\theta\right) + \sin^2\phi\cos\phi\,(-\cos\theta)\right]_{\theta=0}^{\theta=\pi} d\phi$$

$$= 125\int_0^\pi \left(\tfrac{5}{2}\pi\sin^3\phi\cos\phi + 2\sin^2\phi\cos\phi\right)d\phi$$

$$= 125\left[\tfrac{5}{2}\pi\cdot\tfrac{1}{4}\sin^4\phi + 2\cdot\tfrac{1}{3}\sin^3\phi\right]_0^\pi = 0$$

25. Let S_1 be the paraboloid $y = x^2+z^2$, $0 \le y \le 1$ and S_2 the disk $x^2+z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation. On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}[-(x^2+z^2) - 2z^2]\,dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\tfrac{1}{4}(1+2\cos^2\theta)\,d\theta = -\left(\tfrac{\pi}{2}+\tfrac{\pi}{2}\right) = -\pi$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}(1)\,dA = \pi$.

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi + \pi = 0$.

26. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 2$; and the back, S_3, in the plane $y = 0$.

On S_1: $\mathbf{F}(\mathbf{r}(\theta, y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \Rightarrow$

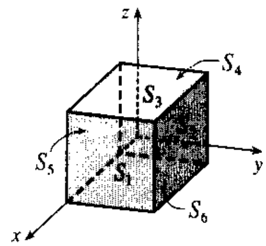
$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{2-\sin\theta} (\sin^2\theta + 5\cos\theta)\,dy\,d\theta$$
$$= \int_0^{2\pi} (2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\cos\theta)\,d\theta = 2\pi$$

On S_2: $\mathbf{F}(\mathbf{r}(x, z)) = x\,\mathbf{i} + (2 - x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2 + z^2 \le 1} [x + (2 - x)]\,dA = 2\pi$$

On S_3: $\mathbf{F}(\mathbf{r}(x, z)) = x\,\mathbf{i} + 5\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ so $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = 0$. Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi$.

27. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$;

S_2: $\mathbf{F} = x\,\mathbf{i} + 2\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 2\,dx\,dz = 8$;

S_3: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\,\mathbf{i} - 2\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 3\,dx\,dy = 12$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^{6} \iint_{S_i} \mathbf{F} \cdot d\mathbf{S} = 48$.

28. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(u, v)) = u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + v^2\,\mathbf{k}$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi}\int_0^{1} (u\sin^2 v - u\cos^2 v + uv^2)\,du\,dv = \int_0^{\pi}\int_0^{1} (-u\cos 2v + uv^2)\,du\,dv$$
$$= \int_0^{\pi} \left[-\tfrac{1}{2}\cos 2v + \tfrac{1}{2}v^2\right] dv = \tfrac{1}{6}\pi^3.$$

29. $z = xy \Rightarrow \partial z/\partial x = y$, $\partial z/\partial y = x$, so by Formula 2, a CAS gives

$\iint_S xyz\,dS = \int_0^1\int_0^1 xy\,(xy)\,\sqrt{y^2 + x^2 + 1}\,dx\,dy \approx 0.1642$.

30. As in Exercise 29, we use a CAS to calculate

$$\iint_S x^2yz\,dS = \int_0^1\int_0^1 x^2y\,(xy)\,\sqrt{y^2 + x^2 + 1}\,dx\,dy$$
$$= \tfrac{1}{60}\sqrt{3} - \tfrac{1}{12}\ln\left(1 + \sqrt{3}\right) - \tfrac{1}{192}\ln\left(\sqrt{2} + 1\right) + \tfrac{317}{2880}\sqrt{2} + \tfrac{1}{24}\ln 2$$

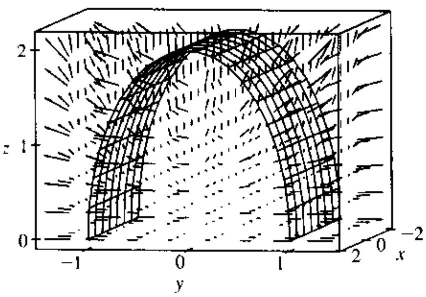
31. We use Formula 2 with $z = 3 - 2x^2 - y^2 \Rightarrow \partial z/\partial x = -4x$, $\partial z/\partial y = -2y$. The boundaries of the region $3 - 2x^2 - y^2 \ge 0$ are $-\sqrt{\frac{3}{2}} \le x \le \sqrt{\frac{3}{2}}$ and $-\sqrt{3 - 2x^2} \le y \le \sqrt{3 - 2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation takes a very long time) to calculate

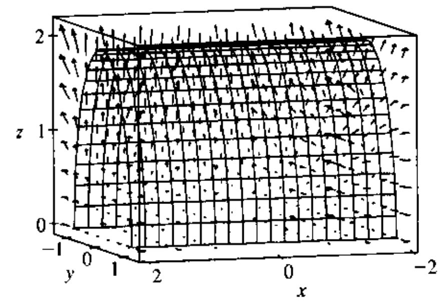
$$\iint_S x^2y^2z^2\,dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}}\int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2(3 - 2x^2 - y^2)^2\sqrt{16x^2 + 4y^2 + 1}\,dy\,dx \approx 3.4895$$

32. The flux of $\mathbf{F}$ across S is given by $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$. Now on S, $z = g(x,y) = 2\sqrt{1-y^2}$, so $\partial g/\partial x = 0$ and $\partial g/\partial y = -2y(1-y^2)^{-1/2}$. Therefore, by (8),

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_{-2}^{2}\int_{-1}^{1}\left(-x^2y\left[-2y(1-y^2)^{-1/2}\right] + \left[2\sqrt{1-y^2}\right]^2 e^{x/5}\right)dy\,dx$$

$$= \tfrac{1}{3}(16\pi + 80e^{2/5} - 80e^{-2/5})$$





33. If S is given by $y = h(x,z)$, then S is also the level surface $f(x,y,z) = y - h(x,z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \dfrac{-h_x\,\mathbf{i} + \mathbf{j} - h_z\,\mathbf{k}}{\sqrt{h_x^2+1+h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (8), using Formula 2 to evaluate

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

$$= \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\,\frac{\dfrac{\partial h}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial h}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + 1 + \left(\dfrac{\partial h}{\partial z}\right)^2}}\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}\,dA$$

where D is the projection of $f(x,y,z)$ onto the xz-plane. Therefore

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(P\frac{\partial h}{\partial x} - Q + R\frac{\partial h}{\partial z}\right)dA$$

34. If S is given by $x = k(y,z)$, then S is also the level surface $f(x,y,z) = x - k(y,z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \dfrac{\mathbf{i} - k_y\,\mathbf{j} - k_z\,\mathbf{k}}{\sqrt{1+k_y^2+k_z^2}}$, and since the x-component is positive this is the unit normal that points forward. Now we proceed as in the derivation of (8), using Formula 2 for

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

$$= \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\,\frac{\mathbf{i} - \dfrac{\partial k}{\partial y}\mathbf{j} - \dfrac{\partial k}{\partial z}\mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial k}{\partial y}\right)^2 + \left(\dfrac{\partial k}{\partial z}\right)^2}}\sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2}\,dA$$

where D is the projection of $f(x,y,z)$ onto the yz-plane. Therefore

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z}\right)dA$$

35. $m = \iint_S K\,dS = K \cdot 4\pi\left(\frac{1}{2}a^2\right) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \iint_S zK\,dS = K\int_0^{2\pi}\int_0^{\pi/2}(a\cos\phi)(a^2\sin\phi)\,d\phi\,d\theta = 2\pi Ka^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \pi Ka^3$.

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{2}a\right)$.

36. S is given by $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2+y^2}\,\mathbf{k}$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2+y^2}{x^2+y^2}} = \sqrt{2}$ so

$$m = \iint_S \left(10 - \sqrt{x^2+y^2}\right)dS = \iint_{1 \le x^2+y^2 \le 16}\left(10 - \sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$

$$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10-r)\,r\,dr\,d\theta = 2\pi\sqrt{2}\left[5r^2 - \tfrac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\,\pi$$

37. (a) $I_z = \iint_S (x^2+y^2)\rho(x,y,z)\,dS$

(b) $I_z = \iint_S (x^2+y^2)\left(10 - \sqrt{x^2+y^2}\right)dS = \iint_{1 \le x^2+y^2 \le 16}(x^2+y^2)\left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA$

$$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10r^3 - r^4)\,dr\,d\theta = 2\sqrt{2}\,\pi\left(\tfrac{4329}{10}\right) = \tfrac{4329}{5}\sqrt{2}\,\pi$$

38. S is given by $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2+y^2}\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$.

(a) $m = \iint_S k\,dS = k\iint_{0 \le x^2+y^2 \le a^2}\sqrt{2}\,dS = \sqrt{2}\,a^2 k\pi$; by symmetry $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \iint_S zk\,dS = k\int_0^{2\pi}\int_0^a \sqrt{2}\,r^2\,dr\,d\theta = \frac{2}{3}\sqrt{2}\,a^3 k\pi$. Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{3}a\right)$.

(b) $I_z = \iint_S (x^2+y^2)k\,dS = \int_0^{2\pi}\int_0^a \sqrt{2}\,kr^3\,dr\,d\theta = 2\pi\sqrt{2}\,k\left(\frac{1}{4}a^4\right) = \frac{\sqrt{2}}{2}\pi ka^4$.

39. $\rho(x,y,z) = 1200$, $\mathbf{V} = y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{F} = \rho\mathbf{V} = (1200)(y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k})$. S is given by

$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \left[9 - \frac{1}{4}(x^2+y^2)\right]\mathbf{k}$, $0 \le x^2+y^2 \le 36$ and $\mathbf{r}_x \times \mathbf{r}_y = \frac{1}{2}x\,\mathbf{i} + \frac{1}{2}y\,\mathbf{j} + \mathbf{k}$.

Thus the rate of flow is given by

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{0 \le x^2+y^2 \le 36}(1200)\left(\tfrac{1}{2}xy + \tfrac{1}{2}y + \left[9 - \tfrac{1}{4}(x^2+y^2)\right]\right)dA$$

$$= 1200\int_0^6\int_0^{2\pi}\left[\tfrac{1}{2}r^2\sin\theta\cos\theta + \tfrac{1}{2}r\sin\theta + 9 - \tfrac{1}{4}r^2\right]r\,d\theta\,dr$$

$$= 1200\int_0^6 2\pi\left(9r - \tfrac{1}{4}r^3\right)dr = (1200)(2\pi)(81) = 194{,}400\pi$$

40. $\rho(x,y,z) = 1500$, $\mathbf{F} = \rho\mathbf{V} = (1500)(-y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k})$. S is given by

$\mathbf{r}(\phi,\theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$, and

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\sin\phi\cos\phi\,\mathbf{k}$. Thus the rate of outward flow is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 1500\int_0^{2\pi}\int_0^{\pi}\left(-125\sin^3\phi\sin\theta\cos\theta + 125\sin^3\phi\sin\theta\cos\theta + 250\sin\phi\cos^2\phi\right)d\phi\,d\theta$$

$$= (3000\pi)(250)\left(-\tfrac{1}{3}\cos^3\phi\right)\Big]_0^{\pi} = 500{,}000\pi.$$

41. S consists of the hemisphere S_1 given by $z = \sqrt{a^2 - x^2 - y^2}$ and the disk S_2 given by $0 \le x^2 + y^2 \le a^2$, $z = 0$. On S_1: $\mathbf{E} = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + 2a\cos\phi\,\mathbf{k}$, $\mathbf{T}_\phi \times \mathbf{T}_\theta = a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$. Thus

$$\iint_{S_1} \mathbf{E}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi + 2a^3\sin\phi\cos^2\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi + a^3\sin\phi\cos^2\phi)\,d\phi\,d\theta = (2\pi)a^3\left(1+\tfrac{1}{3}\right) = \tfrac{8}{3}\pi a^3$$

On S_2: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2} \mathbf{E}\cdot d\mathbf{S} = 0$.

Hence the total charge is $q = \varepsilon_0 \iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{8}{3}\pi a^3\varepsilon_0$.

42. Referring to the figure in Exercise 27, on

S_1: $\mathbf{E} = \mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{E}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$;

S_2: $\mathbf{E} = x\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{E}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\,dz = 4$;

S_3: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{E}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\,dy = 4$;

S_4: $\mathbf{E} = -\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{E}\cdot d\mathbf{S} = 4$.

Similarly $\iint_{S_5} \mathbf{E}\cdot d\mathbf{S} = \iint_{S_6} \mathbf{E}\cdot d\mathbf{S} = 4$. Hence $q = \varepsilon_0 \iint_S \mathbf{E}\cdot d\mathbf{S} = \varepsilon_0 \sum_{i=1}^{6} \iint_{S_i} \mathbf{E}\cdot d\mathbf{S} = 24\varepsilon_0$.

43. $K\nabla u = 6.5(4y\,\mathbf{j} + 4z\,\mathbf{k})$. S is given by $\mathbf{r}(x,\theta) = x\,\mathbf{i} + \sqrt{6}\cos\theta\,\mathbf{j} + \sqrt{6}\sin\theta\,\mathbf{k}$ and since we want the inward heat flow, we use $\mathbf{r}_x \times \mathbf{r}_\theta = -\sqrt{6}\cos\theta\,\mathbf{j} - \sqrt{6}\sin\theta\,\mathbf{k}$. Then the rate of heat flow inward is given by $\iint_S (-K\nabla u)\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^4 -(6.5)(-24)\,dx\,d\theta = (2\pi)(156)(4) = 1248\pi$.

44. $u(x,y,z) = c/\sqrt{x^2+y^2+z^2}$,

$$\mathbf{F} = -K\nabla u = -K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} - \frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} - \frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right]$$

$$= \frac{cK}{(x^2+y^2+z^2)^{3/2}}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and the outward unit normal is $\mathbf{n} = \dfrac{1}{a}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.

Thus $\mathbf{F}\cdot\mathbf{n} = \dfrac{cK}{a(x^2+y^2+z^2)^{3/2}}\,(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2 = a^2$ so $\mathbf{F}\cdot\mathbf{n} = \dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{cK}{a^2}\iint_S dS = \frac{cK}{a^2}\,(4\pi a^2) = 4\pi Kc$.

17.8 Stokes' Theorem ET 16.8

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2 + y^2 = 4$, $z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ (where C is the boundary curve).

2. The plane $z = 5$ intersects the paraboloid $z = 9 - x^2 - y^2$ in the circle $x^2 + y^2 = 4$, $z = 5$. This boundary curve C is oriented in the counterclockwise direction, so the vector equation is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j} + 4\cos t\sin t\,\mathbf{k}$, and by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-20\sin^2 t + 20\cos^2 t)\,dt \\ &= 20\int_0^{2\pi} \cos 2t\,dt = 0\end{aligned}$$

3. The boundary curve C is the circle $x^2 + y^2 = 4$, $z = 0$ oriented in the counterclockwise direction. The vector equation is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = (2\cos t)^2 e^{(2\sin t)(0)}\,\mathbf{i} + (2\sin t)^2 e^{(2\cos t)(0)}\,\mathbf{j} + (0)^2 e^{(2\cos t)(2\sin t)}\,\mathbf{k} = 4\cos^2 t\,\mathbf{i} + 4\sin^2 t\,\mathbf{j}$. Then, by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-8\cos^2 t\sin t + 8\sin^2 t\cos t)\,dt \\ &= 8\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = 0\end{aligned}$$

4. The boundary curve C is the circle $x^2 + z^2 = 9$, $y = 3$ with vector equation $\mathbf{r}(t) = 3\sin t\,\mathbf{i} + 3\,\mathbf{j} + 3\cos t\,\mathbf{k}$, $0 \le t \le 2\pi$ which gives the positive orientation. Then $\mathbf{F}(\mathbf{r}(t)) = 729\sin^2 t\cos t\,\mathbf{i} + \sin(27\sin t\cos t)\,\mathbf{j} + 27\sin t\cos t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 2187\sin^2 t\cos^2 t - 81\sin^2 t\cos t$. Thus

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_0^{2\pi} (2187\sin^2 t\cos^2 t - 81\sin^2 t\cos t)\,dt = \int_0^{2\pi} \left(2187\left(\tfrac{1}{2}\sin 2t\right)^2 - 81\sin^2 t\cos t\right) dt \\ &= \left[\tfrac{2187}{4}\left(\tfrac{1}{2}t - \tfrac{1}{8}\sin 4t\right) - 81 \cdot \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = \tfrac{2187}{4}(\pi) - 0 = \tfrac{2187}{4}\pi\end{aligned}$$

5. C is the square in the plane $z = -1$. By (3), $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl} \mathbf{F} = x^2 z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus

$\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} (x + y)\,dx\,dy = 0$ so $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

6. The boundary curve C is the unit circle in the yz-plane. By Equation 3,

$\iint_{S_1} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ where S_1 is the original hemisphere and S_2 is the disk $y^2+z^2\le 1$, $x=0$. $\text{curl}\,\mathbf{F} = (x-x^2)\,\mathbf{i} - (y+e^{xy}\sin z)\,\mathbf{j} + (2xz - xe^{xy}\cos z)\,\mathbf{k}$, and for S_2 we choose $\mathbf{n}=\mathbf{i}$ so that C has the same orientation for both surfaces. Then $\text{curl}\,\mathbf{F}\cdot\mathbf{n} = x - x^2$ on S_2, where $x=0$. Thus

$$\iint_{S_2} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{y^2+z^2\le 1} (x-x^2)\,dA = \iint_{y^2+z^2\le 1} 0\,dA = 0.$$

Alternatively, we can evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$: C with positive orientation is given by $\mathbf{r}(t) = \langle 0, \cos t, \sin t\rangle$, $0\le t\le 2\pi$, and

$$\begin{aligned}\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} &= \oint_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^{2\pi} \left\langle e^{0(\cos t)}\cos(\sin t), (0)^2(\sin t), (0)(\cos t)\right\rangle \cdot \langle 0, -\sin t, \cos t\rangle\,dt\\ &= \int_0^{2\pi} 0\,dt = 0\end{aligned}$$

7. $\text{curl}\,\mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x+y+z=1$ over $D = \{(x,y)\mid 0\le x\le 1, 0\le y\le 1-x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 17.7.8 [ET 16.7.8], we have $z = g(x,y) = 1-x-y$, $P=-2z$, $Q=-2x$, $R=-2y$, and

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} &= \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)]\,dA\\ &= \int_0^1\int_0^{1-x}(-2)\,dy\,dx = -2\int_0^1(1-x)\,dx = -1\end{aligned}$$

8. $\text{curl}\,\mathbf{F} = e^x\,\mathbf{k}$ and S is the portion of the plane $2x+y+2z=2$ over $D = \{(x,y)\mid 0\le x\le 1, 0\le y\le 2-2x\}$. We orient S upward and use Equation 17.7.8 [ET 16.7.8] with $z = g(x,y) = 1 - x - \frac{1}{2}y$:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} &= \iint_D (0+0+e^x)\,dA = \int_0^1\int_0^{2-2x} e^x\,dy\,dx\\ &= \int_0^1 (2-2x)e^x\,dx = \left[(2-2x)e^x + 2e^x\right]_0^1 \qquad \text{[by integrating by parts]}\\ &= 2e - 4\end{aligned}$$

9. $\text{curl}\,\mathbf{F} = (xe^{xy} - 2x)\,\mathbf{i} - (ye^{xy} - y)\,\mathbf{j} + (2z - z)\,\mathbf{k}$ and we take S to be the disk $x^2+y^2\le 16$, $z=5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\mathbf{n}=\mathbf{k}$ and $\text{curl}\,\mathbf{F}\cdot\mathbf{n} = 2z - z$ on S, where $z=5$. Thus

$$\oint \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (2z-z)\,dS = \iint_S (10-5)\,dS = 5(\text{area of } S) = 5(\pi\cdot 4^2) = 80\pi$$

10. S is the part of the surface $z = 1 - x^2 - y^2$ in the first octant. $\text{curl}\,\mathbf{F} = 2y\,\mathbf{i} - 2x\,\mathbf{j}$.

Using Equation 17.7.8 [ET 16.7.8] with $g(x,y) = 1-x^2-y^2$, $P=2y$, $Q=-2x$, we have

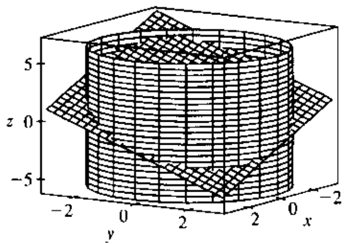
$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_D [-2y(-2x) + (2x)(-2y)]\,dA = \iint_D 0\,dA = 0.$

11. (a) The curve of intersection is an ellipse in the plane $x+y+z=1$ with unit normal $\mathbf{n}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\operatorname{curl}\mathbf{F}=x^2\,\mathbf{j}+y^2\,\mathbf{k}$ and $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=\frac{1}{\sqrt{3}}(x^2+y^2)$. Then

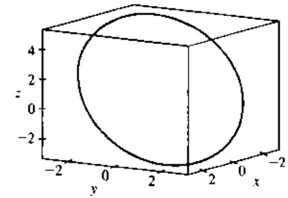
$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \tfrac{1}{\sqrt{3}}(x^2+y^2)\,dS=\iint_{x^2+y^2\le 9}(x^2+y^2)\,dx\,dy$$

$$=\int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta=2\pi\left(\tfrac{81}{4}\right)=\tfrac{81\pi}{2}$$

(b)



(c) One possible parametrization is $x=3\cos t$, $y=3\sin t$, $z=1-3\cos t-3\sin t$, $0\le t\le 2\pi$.

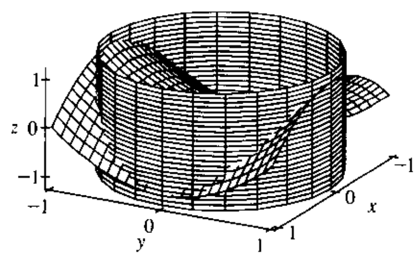


12. (a) S is the part of the surface $z=y^2-x^2$ that lies above the unit disk D. $\operatorname{curl}\mathbf{F}=x\,\mathbf{i}-y\,\mathbf{j}+(x^2-x^2)\,\mathbf{k}=x\,\mathbf{i}-y\,\mathbf{j}$. Using Equation 17.7.8 [ET 16.7.8] with $g(x,y)=y^2-x^2$, $P=x$, $Q=-y$, we have

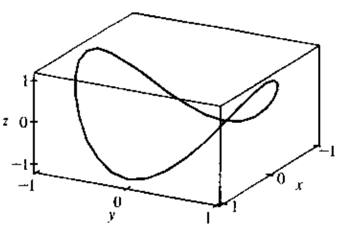
$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_D[-x(-2x)-(-y)(2y)]\,dA=2\iint_D(x^2+y^2)\,dA$$

$$=2\int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta=2(2\pi)\left[\tfrac14 r^4\right]_0^1=\pi$$

(b)



(c) One possible set of parametric equations is $x=\cos t$, $y=\sin t$, $z=\sin^2 t-\cos^2 t$, $0\le t\le 2\pi$.



13. The boundary curve C is the circle $x^2+y^2=1$, $z=1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+\mathbf{k}$, $0\le t\le 2\pi$, and then $\mathbf{r}'(t)=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t))=\sin^2 t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=\cos^2 t-\sin^3 t$, and

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(\cos^2 t-\sin^3 t)\,dt=\int_0^{2\pi}\tfrac12(1+\cos 2t)\,dt-\int_0^{2\pi}(1-\cos^2 t)\sin t\,dt$$

$$=\tfrac12\left[t+\tfrac12\sin 2t\right]_0^{2\pi}-\left[-\cos t+\tfrac13\cos^3 t\right]_0^{2\pi}=\pi$$

Now $\operatorname{curl}\mathbf{F}=(1-2y)\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2+y^2\le 1$, so by Equation 17.7.8 [ET 16.7.8] with $z=g(x,y)=x^2+y^2$ we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_D(1-2y)\,dA=\int_0^{2\pi}\int_0^1(1-2r\sin\theta)\,r\,dr\,d\theta=\int_0^{2\pi}\left(\tfrac12-\tfrac23\sin\theta\right)d\theta=\pi$$

14. The plane intersects the coordinate axes at $x = 1$, $y = z = 2$ so the boundary curve C consists of the three line segments C_1: $\mathbf{r}_1(t) = (1-t)\,\mathbf{i} + 2t\,\mathbf{j}$, $0 \le t \le 1$, C_2: $\mathbf{r}_2(t) = (2-2t)\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \le t \le 1$, C_3: $\mathbf{r}_3(t) = t\,\mathbf{i} + (2-2t)\,\mathbf{k}$, $0 \le t \le 1$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [(1-t)\,\mathbf{i} + 2t\,\mathbf{j}]\cdot(-\mathbf{i}+2\,\mathbf{j})\,dt + \int_0^1 [(2-2t)\,\mathbf{j}]\cdot(-2\,\mathbf{j}+2\,\mathbf{k})\,dt + \int_0^1 (t\,\mathbf{i})\cdot(\mathbf{i}-2\,\mathbf{k})\,dt$$
$$= \int_0^1 (5t-1)\,dt + \int_0^1 (4t-4)\,dt + \int_0^1 t\,dt = \tfrac{3}{2} - 2 + \tfrac{1}{2} = 0$$

Now curl $\mathbf{F} = xz\,\mathbf{i} - yz\,\mathbf{j}$, so by Equation 17.7.8 [ET 16.7.8] with $z = g(x,y) = 2 - 2x - y$ we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_D [-x(2-2x-y)(-2) + y(2-2x-y)(-1)]\,dA$$
$$= \int_0^1 \int_0^{2-2x} (4x - 4x^2 - 2y + y^2)\,dy\,dx$$
$$= \int_0^1 \left[4x(2-2x) - 4x^2(2-2x) - (2-2x)^2 + \tfrac{1}{3}(2-2x)^3\right] dx$$
$$= \int_0^1 \left(\tfrac{16}{3}x^3 - 12x^2 + 8x - \tfrac{4}{3}\right) dx = \left[\tfrac{4}{3}x^4 - 4x^3 + 4x^2 - \tfrac{4}{3}x\right]_0^1 = 0$$

15. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ oriented in the counterclockwise direction as viewed from the positive y-axis. Then C can be described by $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -\cos^2 t$, and

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} -\cos^2 t\,dt = -\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\Big]_0^{2\pi} = -\pi$$

Now curl $\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, and S can be parametrized (see Example 17.6.10 [ET 16.6.10]) by $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$ and

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} \text{curl}\,\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA$$
$$= \int_0^\pi \int_0^\pi (-\sin^2\phi\cos\theta - \sin^2\phi\sin\theta - \sin\phi\cos\phi)\,d\theta\,d\phi$$
$$= \int_0^\pi (-2\sin^2\phi - \pi\sin\phi\cos\phi)\,d\phi = \left[\tfrac{1}{2}\sin 2\phi - \phi - \tfrac{\pi}{2}\sin^2\phi\right]_0^\pi = -\pi$$

16. The components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$, and both the curve C and the surface S meet the requirements of Stokes' Theorem. If there is a vector field $\mathbf{G}$ where $\mathbf{F} = \text{curl}\,\mathbf{G}$, then Stokes' Theorem says $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \text{curl}\,\mathbf{G}\cdot d\mathbf{S}$ depends only on the values of $\mathbf{G}$ on C, and hence is independent of the choice of S. By Theorem 17.5.11 [ET 16.5.11], div curl $\mathbf{G} = 0$, so div $\mathbf{F} = 0$ $\Leftrightarrow$ $(3ax^2 - 3z^2) + (x^2 + 3by^2) + (3cz^2) = 0$ $\Leftrightarrow$ $(3a+1)x^2 + 3by^2 + (3c-3)z^2 = 0$ $\Leftrightarrow$ $a = -\tfrac{1}{3}$, $b = 0$, $c = 1$.

17. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^x + z^2 & y^y + x^2 & z^z + y^2 \end{vmatrix} = 2y\,\mathbf{i} + 2z\,\mathbf{j} + 2x\,\mathbf{k}$ and $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$.

To parametrize the surface, let $x = 2\cos\theta\sin\phi$, $y = 2\sin\theta\sin\phi$, $z = 2\cos\phi$, so that $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$, and $\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\sin\phi\cos\phi\,\mathbf{k}$. Then $\operatorname{curl}\mathbf{F}(\mathbf{r}(\phi,\theta)) = 4\sin\phi\sin\theta\,\mathbf{i} + 4\cos\phi\,\mathbf{j} + 4\sin\phi\cos\theta\,\mathbf{k}$, and $\operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 16\sin^3\phi\sin\theta\cos\theta + 16\cos\phi\sin^2\phi\sin\theta + 16\sin^2\phi\cos\phi\cos\theta$. Therefore

$$\begin{aligned}
\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA \\
&= 16\left[\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^3\phi\,d\phi\right] + 16\left[\int_0^{\pi/2}\sin\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\right] \\
&\qquad + 16\left[\int_0^{\pi/2}\cos\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\right] \\
&= 8\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2} + 16(1)\left[\tfrac{1}{3}\sin^3\phi\right]_0^{\pi/2} + 16(1)\left[\tfrac{1}{3}\sin^3\phi\right]_0^{\pi/2} \\
&= 8\left[0 + 1 + 0 - \tfrac{1}{3}\right] + 16\left(\tfrac{1}{3}\right) + 16\left(\tfrac{1}{3}\right) = \tfrac{16}{3} + \tfrac{16}{3} + \tfrac{16}{3} = 16
\end{aligned}$$

18. $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y+\sin x)\,\mathbf{i} + (z^2+\cos y)\,\mathbf{j} + x^3\,\mathbf{k}$ $\Rightarrow$ $\operatorname{curl}\mathbf{F} = -2z\,\mathbf{i} - 3x^2\,\mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2\sin t\cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk D $(x^2 + y^2 \le 1)$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 17.7.8 [ET 16.7.8] with $g(x,y) = 2xy$, $P = -2(2xy) = -4xy$, $Q = -3x^2$, $R = -1$, we have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= -\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) - 1\right] dA \\
&= -\iint_D (8xy^2 + 6x^3 - 1)\,dA = -\int_0^{2\pi}\int_0^1 (8r^3\cos\theta\sin^2\theta + 6r^3\cos^3\theta - 1)\,r\,dr\,d\theta \\
&= -\int_0^{2\pi}\left(\tfrac{8}{5}\cos\theta\sin^2\theta + \tfrac{6}{5}\cos^3\theta - \tfrac{1}{2}\right) r\,dr\,d\theta \\
&= -\left[\tfrac{8}{15}\sin^3\theta + \tfrac{6}{5}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) - \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi
\end{aligned}$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{H_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} + \iint_{H_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_{C_1}\mathbf{F}\cdot d\mathbf{r} + \oint_{C_2}\mathbf{F}\cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2}\mathbf{F}\cdot d\mathbf{r} = -\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}$ so $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$ as desired.

20. (a) By Exercise 17.5.26 [ET 16.5.26], $\operatorname{curl}(f\nabla g) = f\operatorname{curl}(\nabla g) + \nabla f\times\nabla g = \nabla f\times\nabla g$ since $\operatorname{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f\nabla g)\cdot d\mathbf{r} = \iint_S (\nabla f\times\nabla g)\cdot d\mathbf{S}$.

(b) As in (a), $\operatorname{curl}(f\nabla f) = \nabla f\times\nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f\nabla f)\cdot d\mathbf{r} = \iint_S [\operatorname{curl}(f\nabla f)]\cdot d\mathbf{S} = 0$.

(c) As in part (a),

$$\begin{aligned}\operatorname{curl}(f\nabla g + g\nabla f) &= \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) \qquad \text{(by Exercise 17.5.24 [ET 16.5.24])}\\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} \qquad [\text{since } \mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})]\end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f)\cdot d\mathbf{S} = 0$.

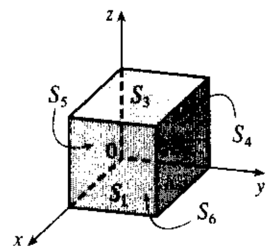
17.9 The Divergence Theorem — ET 16.9

1. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

2. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

   (b) $\mathbf{F}(x,y) = \langle x, y^2\rangle \;\Rightarrow\; \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1+2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1+2y$ is negative, and P_2 is a sink.

3. $\operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$, so



$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, on S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and

$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

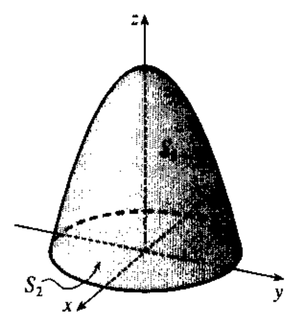
S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} x\,dS = \frac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_6} 0\,dS = 0$. Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{9}{2}$.

4. $\operatorname{div}\mathbf{F} = 2x + x + 1 = 3x + 1$ so



$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E (3x+1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r\cos\theta + 1)\,r\,dz\,dr\,d\theta\\ &= \int_0^2\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)\,d\theta\,dr\\ &= \int_0^{2\pi} r(4-r^2)\big[3r\sin\theta + \theta\big]_{\theta=0}^{\theta=2\pi}\,dr\\ &= 2\pi\int_0^2(4r - r^3)\,dr = 2\pi\big[2r^2 - \tfrac{1}{4}r^4\big]_0^2\\ &= 2\pi(8-4) = 8\pi\end{aligned}$$

On S_1: The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \le 4$, with upward orientation, and $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}$. Then

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D[-(x^2)(-2x) - (xy)(-2y) + (4 - x^2 - y^2)]\,dA\\ &= \iint_D\left[2x(x^2+y^2) + 4 - (x^2+y^2)\right]\,dA = \int_0^{2\pi}\int_0^2(2r\cos\theta\cdot r^2 + 4 - r^2)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos\theta + 2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2}\,d\theta = \int_0^{2\pi}\left(\tfrac{64}{5}\cos\theta + 4\right)\,d\theta\\ &= \left[\tfrac{64}{5}\sin\theta + 4\theta\right]_0^{2\pi} = 8\pi\end{aligned}$$

On S_2: The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} 0\,dS = 0$.

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = 8\pi$.

5. $\operatorname{div}\mathbf{F} = x + y + z$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^1 (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^2\cos\theta + r^2\sin\theta + \tfrac{1}{2}r\right) dr\,d\theta$$
$$= \int_0^{2\pi} \left(\tfrac{1}{3}\cos\theta + \tfrac{1}{3}\sin\theta + \tfrac{1}{4}\right) d\theta = \tfrac{1}{4}(2\pi) = \tfrac{\pi}{2}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1, $z = 1$, $\mathbf{n} = \mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, so

$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} x\,dS = \int_0^{2\pi}\int_0^1 (r\cos\theta)\,r\,dr\,d\theta = \left[\sin\theta\right]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^1 = 0$. On S_2, $z = 0$, $\mathbf{n} = -\mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i}$ so $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} 0\,dS = 0$. S_3 is given by $\mathbf{r}(\theta, z) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$. Then $\mathbf{r}_\theta \times \mathbf{r}_z = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and

$$\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_\theta \times \mathbf{r}_z)\,dA = \int_0^{2\pi}\int_0^1 (\cos^2\theta\sin\theta + z\sin^2\theta)\,dz\,d\theta$$
$$= \int_0^{2\pi} \left(\cos^2\theta\sin\theta + \tfrac{1}{2}\sin^2\theta\right) d\theta = \left[-\tfrac{1}{3}\cos^3\theta + \tfrac{1}{4}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \tfrac{\pi}{2}$$

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0 + 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

6. $\operatorname{div}\mathbf{F} = 1 + 1 + 1 = 3$, so $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 3\,dV = 3\,(\text{volume of ball}) = 3\left(\frac{4}{3}\pi\right) = 4\pi$. To find $\iint_S \mathbf{F}\cdot d\mathbf{S}$ we use spherical coordinates. S is the unit sphere, represented by $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$ (see Example 17.6.10 [ET 16.6.10]) and $\mathbf{F}(\mathbf{r}(\phi,\theta)) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$. Thus

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi} \left(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right) d\phi\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\,d\phi = (2\pi)(2) = 4\pi$$

7. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(e^x\sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(yz^2) = e^x\sin y - e^x\sin y + 2yz = 2yz$, so by the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^2 2yz\,dz\,dy\,dx = 2\int_0^1 dx \int_0^1 y\,dy \int_0^1 z\,dz$$
$$= 2\left[x\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^2 = 2$$

8. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$, so by the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_{-1}^1\int_{-2}^2\int_{-3}^3 8xz^3\,dz\,dy\,dx = 8\int_{-1}^1 x\,dx \int_{-2}^2 dy \int_{-3}^3 z^3\,dz$$
$$= 8\left[\tfrac{1}{2}x^2\right]_{-1}^1 \left[y\right]_{-2}^2 \left[\tfrac{1}{4}z^4\right]_{-3}^3 = 0$$

9. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (3y^2 + 3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta$$
$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_{-1}^2 dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}$$

10. $\operatorname{div}\mathbf{F} = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

11. $\operatorname{div}\mathbf{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

12. $\operatorname{div}\mathbf{F} = 2xy + 2xy + 2xy = 6xy$, so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 6xy\,dV \\ &= \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 6xy\,dz\,dx\,dy = \int_0^1 \int_0^{2-2y} 6xy(2-x-2y)\,dx\,dy \\ &= \int_0^1 \int_0^{2-2y} (12xy - 6x^2y - 12xy^2)\,dx\,dy = \int_0^1 \left[6x^2y - 2x^3y - 6x^2y^2\right]_{x=0}^{x=2-2y} dy \\ &= \int_0^1 y(2-2y)^3\,dy = \left[-\tfrac{8}{5}y^5 + 6y^4 - 8y^3 + 4y^2\right]_0^1 = \tfrac{2}{5}\end{aligned}$$

13. $\operatorname{div}\mathbf{F} = y^2 + 0 + x^2 = x^2 + y^2$ so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 (4r^3 - r^5)\,dr = 2\pi\left[r^4 - \tfrac{1}{6}r^6\right]_0^2 = \tfrac{32}{3}\pi\end{aligned}$$

14. $\operatorname{div}\mathbf{F} = 4x^3 + 4xy^2$ so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 4x(x^2+y^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2} (4r^3\cos\theta)\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (4r^5\cos^2\theta + 8r^4\cos\theta)\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{2}{3}\cos^2\theta + \tfrac{8}{5}\cos\theta\right)d\theta = \tfrac{2}{3}\pi\end{aligned}$$

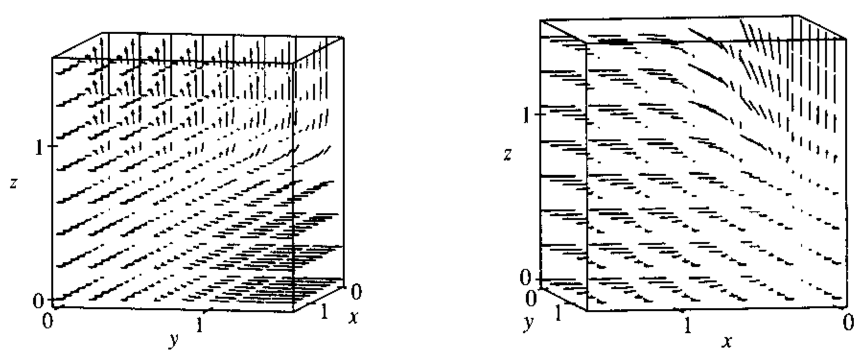
15. $\operatorname{div}\mathbf{F} = 12x^2z + 12y^2z + 12z^3$ so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 12z(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R 12(\rho\cos\phi)(\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 12\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\cos\phi\,d\phi \int_0^R \rho^5\,d\rho = 12(2\pi)\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^R = 0\end{aligned}$$

16. $$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3(x^2+y^2+1)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 3(\rho^2\sin^2\phi + 1)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi\int_0^{\pi/2}\left[\tfrac{93}{5}\sin^3\phi + 7\sin\phi\right]d\phi = 2\pi\left[\tfrac{93}{5}\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) - 7\cos\phi\right]_0^{\pi/2} = \tfrac{194}{5}\pi\end{aligned}$$

17. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \sqrt{3-x^2}\,dV = \int_{-1}^1\int_{-1}^1\int_0^{2-x^4-y^4}\sqrt{3-x^2}\,dz\,dy\,dx = \frac{341}{60}\sqrt{2} + \frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

18.



By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\cos^2 y + 3\sin^2 y\cos y\cos^4 z + 5\sin^4 z\cos z\cos^6 x\right]dz\,dy\,dx = \tfrac{19}{64}\pi^2.$$

19. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D(-y^2)\,dA = -\int_0^{2\pi}\int_0^1 r^2(\sin^2\theta)\,r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \tan z\right) + \frac{\partial}{\partial z}(x^2z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get

$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi.$

20. As in the hint to Exercise 19, we create a closed surface $S_2 = S \cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F} \cdot (-\mathbf{k}) = -z = -1$ on S_1. So $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} (-1)\, dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2. Then

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 1\, dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r\, dz\, d\theta\, dr = \int_0^1 \int_0^{2\pi} (r - r^3)\, d\theta\, dr$$
$$= (2\pi)\tfrac{1}{4} = \tfrac{\pi}{2}$$

Thus the flux of $\mathbf{F}$ across S is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) = \dfrac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2 + y^2 + z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}\right)$, we have $\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0$, except at $(0, 0, 0)$ where it is undefined.

22. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (2x + 2y + z^2)\, dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$. But for S, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and $\operatorname{div} \mathbf{F} = 1$. If $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$, then $\iint_S (2x + 2y + z^2)\, dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

23. $\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{a}\, dV = 0$ since $\operatorname{div} \mathbf{a} = 0$.

24. $\frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \iiint_E \operatorname{div} \mathbf{F}\, dV = \frac{1}{3} \iiint_E 3\, dV = V(E)$

25. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

26. $\iint_S D_{\mathbf{n}} f\, dS = \iint_S (\nabla f \cdot \mathbf{n})\, dS = \iiint_E \operatorname{div}(\nabla f)\, dV = \iiint_E \nabla^2 f\, dV$

27. $\iint_S (f \nabla g) \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div}(f \nabla g)\, dV = \iiint_E (f \nabla^2 g + \nabla g \cdot \nabla f)\, dV$ by Exercise 17.5.25 [ET 16.5.25].

28. $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E \left[(f \nabla^2 g + \nabla g \cdot \nabla f) - (g \nabla^2 f + \nabla g \cdot \nabla f)\right] dV$ [by Exercise 27]. But $\nabla g \cdot \nabla f = \nabla f \cdot \nabla g$, so that $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f \nabla^2 g - g \nabla^2 f)\, dV$.

29. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then $\operatorname{div} \mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x} c_1 + \dfrac{\partial f}{\partial y} c_2 + \dfrac{\partial f}{\partial z} c_3 = \nabla f \cdot \mathbf{c}$ and the Divergence Theorem says $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV \;\Rightarrow\; \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla f \cdot \mathbf{c}\, dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then $\iint_S f\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \nabla f \cdot \mathbf{i}\, dV \;\Rightarrow\; \displaystyle\iint_S f n_1\, dS = \iiint_E \frac{\partial f}{\partial x}\, dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$). Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S f n_2\, dS = \iiint_E \frac{\partial f}{\partial y}\, dV$, and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S f n_3\, dS = \iiint_E \frac{\partial f}{\partial z}\, dV$. Then

$$\iint_S f\mathbf{n}\, dS = \left(\iint_S f n_1\, dS\right)\mathbf{i} + \left(\iint_S f n_2\, dS\right)\mathbf{j} + \left(\iint_S f n_3\, dS\right)\mathbf{k}$$
$$= \left(\iiint_E \frac{\partial f}{\partial x}\, dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\, dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\, dV\right)\mathbf{k}$$
$$= \iiint_E \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) dV = \iiint_E \nabla f\, dV$$

as desired.

30. By Exercise 29, $\iint_S p\mathbf{n}\,dS = \iiint_E \nabla p\,dV$, so

$$\mathbf{F} = -\iint_S p\mathbf{n}\,dS = -\iiint_E \nabla p\,dV = -\iiint_E \nabla(\rho g z)\,dV = -\iiint_E (\rho g\,\mathbf{k})\,dV$$
$$= -\rho g\left(\iiint_E dV\right)\mathbf{k} = -\rho g V(E)\,\mathbf{k}$$

But the weight of the displaced liquid is volume × density × $g = \rho g V(E)$, thus $\mathbf{F} = -W\mathbf{k}$ as desired.

17 Review ET 16

CONCEPT CHECK

1. See Definitions 1 and 2 in Section 17.1 [ET 16.1]. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

(b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 17.2.2 [ET 16.2.2].

(b) We normally evaluate the line integral using Formula 17.2.3 [ET 16.2.3].

(c) The mass is $m = \int_C \rho(x,y)\,ds$, and the center of mass is $(\overline{x},\overline{y})$ where $\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$.

(d) See (5) and (6) in Section 17.2 [ET 16.2] for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) in Section 17.2 [ET 16.2]).

(e) For plane curves, see Equations 17.2.7 [ET 16.2.7]. We have similar results for space curves (see the equation preceding (10) in Section 17.2 [ET 16.2]).

4. (a) See Definition 17.2.13 [ET 16.2.13].

(b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F}\cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

(c) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 17.3.2 [ET 16.3.2].

6. (a) $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

(b) See Theorem 17.3.4 [ET 16.3.4].

7. See the statement of Green's Theorem on page 1119 [ET 1083].

8. See Equations 17.4.5 [ET 16.4.5].

9. (a) $\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \nabla\times\mathbf{F}$

(b) $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla\cdot\mathbf{F}$

(c) For curl $\mathbf{F}$, see the discussion accompanying Figure 1 on page 1129 [ET 1093] as well as Figure 6 and the accompanying discussion on page 1160 [ET 1124]. For div $\mathbf{F}$, see the discussion following Example 5 on page 1130 [ET 1094] as well as the discussion preceding (8) on page 1167 [ET 1131].

10. See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

11. (a) See (1) and (2) and the accompanying discussion in Section 17.6 [ET 16.6]; See Figure 4 and the accompanying discussion on page 1135 [ET 1099].

(b) See Definition 17.6.6 [ET 16.6.6].

(c) See Equation 17.6.9 [ET 16.6.9].

12. (a) See (1) in Section 17.7 [ET 16.7].

(b) We normally evaluate the surface integral using Formula 17.7.3 [ET 16.7.3].

(c) See Formula 17.7.2 [ET 16.7.2].

(d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 7 and 8 and the accompanying discussion in Section 17.7 [ET 16.7]. A Möbius strip is a nonorientable surface; see Figures 5 and 6 and the accompanying discussion on page 1149 [ET 1113].

(b) See Definition 17.7.7 [ET 16.7.7].

(c) See Formula 17.7.9 [ET 16.7.9].

(d) See Formula 17.7.8 [ET 16.7.8].

14. See the statement of Stokes' Theorem on page 1157 [ET 1121].

15. See the statement of the Divergence Theorem on page 1163 [ET 1127].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; div $\mathbf{F}$ is a scalar field.

2. True. (See Definition 17.5.1 [ET 16.5.1].)

3. True, by Theorem 17.5.3 [ET 16.5.3] and the fact that div $\mathbf{0} = 0$.

4. True, by Theorem 17.3.2 [ET 16.3.2].

5. False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if D is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)

6. False. See the discussion accompanying Figure 8 on page 1103 [ET 1067].

7. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

8. False by Theorem 17.5.11 [ET 16.5.11], because if it were true, then div curl $\mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

(b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and div $\mathbf{F}\,(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \leq x \leq 1$ so

$$\int_C x\, ds = \int_0^1 x\, \sqrt{1 + (2x)^2}\, dx = \tfrac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}\big(5\sqrt{5} - 1\big).$$

3. $\int_C x^3 z\,ds = \int_0^{\pi/2}(2\sin t)^3(2\cos t)\sqrt{(2\cos t)^2+(1)^2+(-2\sin t)^2}\,dt = \int_0^{\pi/2}(16\sin^3 t\cos t)\sqrt{5}\,dt$

$= 4\sqrt{5}\sin^4 t\Big]_0^{\pi/2} = 4\sqrt{5}$

4. $\int_C xy\,dx + y\,dy = \int_0^{\pi/2}(x\sin x + \sin x\cos x)\,dx = -x\cos x + \sin x - \frac{1}{4}\cos 2x\Big]_0^{\pi/2} = \frac{3}{2}$

5. $x = \cos t \Rightarrow dx = -\sin t\,dt$, $y = \sin t \Rightarrow dy = \cos t\,dt$, $0 \le t \le 2\pi$ and

$\int_C x^3 y\,dx - x\,dy = \int_0^{2\pi}(-\cos^3 t\sin^2 t - \cos^2 t)\,dt = \int_0^{2\pi}(-\cos^3 t\sin^2 t - \cos^2 t)\,dt = -\pi$

Or: Since C is a simple closed curve, apply Green's Theorem giving

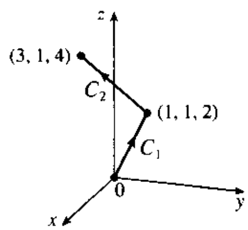
$\iint_{x^2+y^2\le 1}(-1-x^3)\,dA = \int_0^1\int_0^{2\pi}(-r - r^4\cos^3\theta)\,d\theta = -\pi.$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1\left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right)dt$

$= \int_0^1\left(4t^6 + 2te^{t^2} + 3t^9\right)dt = \left[\frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10}\right]_0^1$

$= e - \frac{9}{70}$

7.



C_1: $x = t$, $y = t$, $z = 2t$, $0 \le t \le 1$;

C_2: $x = 1 + 2t$, $y = 1$, $z = 2 + 2t$, $0 \le t \le 1$.

Then

$\int_C y\,dx + z\,dy + x\,dz = \int_0^1 5t\,dt + \int_0^1(4+4t)\,dt = \frac{17}{2}$

8. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi((1+t)\sin t\cos t + \sin^2 t)\,dt = \int_0^\pi\left(\frac{1}{2}(1+t)\sin 2t + \sin^2 t\right)dt$

$= \left[\frac{1}{2}\left((1+t)(-\frac{1}{2}\cos 2t) + \frac{1}{4}\sin 2t\right) + \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^\pi = \frac{\pi}{4}.$

9. $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2+t^3)\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1(2te^{-t} - 3t^5 - (t^2+t^3))\,dt = \left[-2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right]_0^1 = \frac{11}{12} - \frac{4}{e}.$

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \le t \le 1$. Then

$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\left[3t\,\mathbf{i} + (3-3t)\,\mathbf{j} + \frac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \frac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right]dt = \int_0^1\left[-9t + \frac{3\pi}{2}\right]dt$

$= \frac{1}{2}(3\pi - 9)$

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}(3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$= \int_0^{\pi/2}(-9\sin^2 t + 3\cos t + 3t\cos t)\,dt$

$= \left[-\frac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2} = -\frac{9\pi}{4} + 3 + \frac{3\pi}{2} - 3 = -\frac{3\pi}{4}$

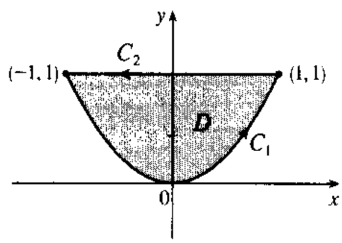
11. $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x,y) = e^y + x^2e^{xy}$ implies $f(x,y) = e^y + xe^{xy} + g(x)$ and then $f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$, so $g'(x) = 0 \Rightarrow g(x) = K$. Thus $f(x,y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 17.5.4 [ET 16.5.4]. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and then $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$, so $g_y(y,z) = 0 \;\Rightarrow\; g(y,z) = h(z)$. Then $f(x,y,z) = x\sin y + h(z)$ implies $f_z(x,y,z) = h'(z)$. But $f_z(x,y,z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x,y,z) = x\sin y + \cos z + K$.

13. Since $\frac{\partial}{\partial y}\left(4x^3y^2 - 2xy^3\right) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}\left(2x^4y - 3x^2y^2 + 4y^3\right)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x,y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0,1)$ and $t = 1$ corresponds to $(1,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(0,1) = 1 - 1 = 0$.

14. Here $\operatorname{curl}\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x,y,z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,0,3) - f(0,2,0) = 4 - 2 = 2$.

15.



C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;

C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.

Then

$$\begin{aligned}\int_C xy^2\,dx - x^2y\,dy &= \int_{-1}^{1}(t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt \\ &= \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0\end{aligned}$$

Using Green's Theorem, we have

$$\begin{aligned}\int_C xy^2\,dx - x^2y\,dy &= \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA = \iint_D (-2xy - 2xy)\,dA \\ &= \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx = \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1} dx \\ &= \int_{-1}^{1}(2x^5 - 2x)\,dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0\end{aligned}$$

16. $$\begin{aligned}\int_C \sqrt{1+x^3}\,dx + 2xy\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(2xy) - \tfrac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right] dA = \int_0^1\int_0^{3x}(2y - 0)\,dy\,dx \\ &= \int_0^1 9x^2\,dx = \left.3x^3\right]_0^1 = 3\end{aligned}$$

17. $$\begin{aligned}\int_C x^2y\,dx - xy^2\,dy &= \iint_{x^2+y^2\le 4}\left[\tfrac{\partial}{\partial x}(-xy^2) - \tfrac{\partial}{\partial y}(x^2y)\right] dA \\ &= \iint_{x^2+y^2\le 4}(-y^2 - x^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi\end{aligned}$$

18. $\operatorname{curl}\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,
$\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}
&\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} \\
&= \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right] \\
&\quad - \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j} + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} = &\left[\left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\frac{\partial Q_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right. \\
&\left. + \left(P_2\frac{\partial R_1}{\partial x} + Q_2\frac{\partial R_1}{\partial y} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right] \\
&- \left[\left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right. \\
&\left. + \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} \\
&= \left[\left(P_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial P_1}{\partial x}\right) - \left(P_2\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial P_2}{\partial y}\right)\right. \\
&\qquad \left. - \left(P_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial P_2}{\partial z}\right) + \left(P_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i} \\
&\quad + \left[\left(Q_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial Q_2}{\partial z}\right)\right. \\
&\qquad \left. - \left(P_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial P_1}{\partial x}\right) + \left(P_2\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j} \\
&\quad + \left[\left(P_2\frac{\partial R_1}{\partial x} + R_1\frac{\partial P_2}{\partial x}\right) - \left(P_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial P_1}{\partial x}\right)\right. \\
&\qquad \left. - \left(Q_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\frac{\partial R_1}{\partial y} + R_1\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k} \\
&= \left[\frac{\partial}{\partial y}(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}(P_2R_1 - P_1R_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}(P_1Q_2 - P_2Q_1)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}(Q_1R_2 - Q_2R_1)\right]\mathbf{k} \\
&= \operatorname{curl}(\mathbf{F}\times\mathbf{G})
\end{aligned}$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx + g(y)\,dy = \iint_D \left[\frac{\partial}{\partial x}\,g(y) - \frac{\partial}{\partial y}\,f(x)\right] dA = \iint_D 0\,dA = 0$.

22. $$\nabla^2(fg) = \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\,g + f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\,g + f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\,g + f\,\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}\,g + 2\,\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\,\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\,g + 2\,\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\,\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,g + 2\,\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

Another method: Using the rules in Exercises 15.6.37(b) [ET 14.6.37(b)] and 17.5.25 [ET 16.5.25], we have

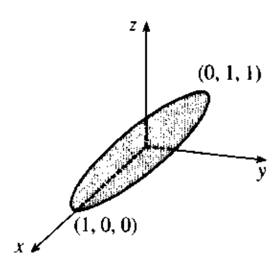
$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (g\,\nabla f + f\,\nabla g) = \nabla g \cdot \nabla f + g\nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\nabla \cdot \nabla g$$
$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f \cdot \nabla g$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA = -\iint_D (f_{xx} + f_{yy})\,dA$$
$$= -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



(b) Apply Stokes' Theorem, $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix}$$

$$= \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

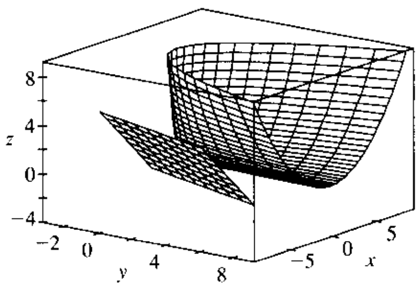
Therefore $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0$.

25. $z = f(x, y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\, dA = \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2}\, dy\, dx = \int_0^1 2x\sqrt{5 + 4x^2}\, dx$$
$$= \tfrac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{6}\left(27 - 5\sqrt{5}\right).$$

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4, -2, 1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 17.6.6 [ET 16.6.6], the area of S is given by
$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\, dv\, du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\, dv\, du.$

(d) By Equation 17.7.9 [ET 16.7.9], the surface integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1 + (v^2)^2}, \frac{(v^2)^2}{1 + (-uv)^2}, \frac{(-uv)^2}{1 + (u^2)^2} \right\rangle \cdot \left\langle 2u^2, 4uv, 2v^2 \right\rangle dv\, du$$
$$= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1 + v^4} + \frac{4uv^5}{1 + u^2v^2} + \frac{2u^2v^4}{1 + u^4} \right) dv\, du \approx 1524.0190$$

27. $z = f(x, y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\, dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)\sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_0^2 r^3 s\sqrt{1 + 4r^2}\, dr\, d\theta$$
$$= \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

28. $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2z + y^2z)\, dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\, dA$$
$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\, r^3(4 + r\cos\theta + r\sin\theta)\, d\theta\, dr = \int_0^2 8\pi\sqrt{3}\, r^3\, dr = 32\pi\sqrt{3}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and
$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (z - 2)\, dV = \iiint_E z\, dV - 2\iiint_E dV = m\bar{z} - 2\left(\frac{4}{3}\pi 2^3\right) = -\frac{64}{3}\pi.$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi, \theta)) = 4\sin\phi\cos\theta\cos\phi\,\mathbf{i} - 4\sin\phi\sin\theta\,\mathbf{j} + 6\sin\phi\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\sin\phi\cos\phi\,\mathbf{k}$, and

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\cos^2\theta\cos\phi - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta$. Then

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_0^\pi (16\sin^3\phi\cos\phi\cos^2\theta - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta)\, d\phi\, d\theta$$
$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\, d\theta = -\tfrac{64}{3}\pi$$

30. $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2\le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA$$

$$= \int_0^1 \int_0^{2\pi} (-2r^3 \cos^3\theta - 2r^3 \cos\theta \sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3(2\pi)\,dr = \tfrac{\pi}{2}$$

31. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t)\,dt = \frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0$.

32. $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t \sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t \sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t + 4\sin t \cos t$. Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-16\cos^2 t \sin^2 t + 4\sin t \cos t)\,dt$$

$$= \left[-16\left(-\tfrac{1}{4}\sin t \cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi$$

33. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D (-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dA$$

$$= \iint_D (-1)\,dA = -(\text{area of } D) = -\tfrac{1}{2}$$

34. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi} \int_0^1 \int_0^2 (3r^2 + 3z^2)\,r\,dz\,dr\,d\theta = 2\pi \int_0^1 (6r^3 + 8r)\,dr = 11\pi$

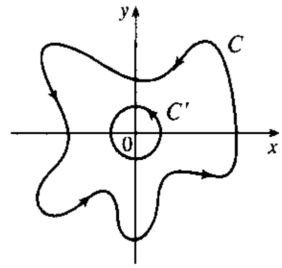
35. $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_{x^2+y^2+z^2\le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then

$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi \cos^2\theta + \sin^3\phi \sin^2\theta + \sin\phi \cos^2\phi = \sin\phi$ and $\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi$.

36. Here we must use Equation 17.9.6 [ET 16.9.6] since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\,dS + \iiint_E \operatorname{div}\mathbf{F}\,dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so $\operatorname{div}\mathbf{F} = \nabla \cdot (|\mathbf{r}|^{-3}\,\mathbf{r}) = |\mathbf{r}|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot (\nabla |\mathbf{r}|^{-3}) = |\mathbf{r}|^{-3}(3) + \mathbf{r} \cdot (-3|\mathbf{r}|^{-4})(\mathbf{r}\,|\mathbf{r}|^{-1}) = 0$. (Here we have used Exercises 17.5.30(a) [ET 16.5.30(a)] and 17.5.31(a) [ET 16.5.31(a)].) And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1. Thus $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = \iint_{S_1} dS + \iiint_E 0\,dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi$.

37. Because $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x, y, z) = x^3yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, 3, 0) - f(0, 0, 2) = 0 - 4 = -4$.

38. Let C' be the circle with center at the origin and radius a as in the figure.



Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3 \cos^3 t + 2a^3 \cos t \sin^2 t - 2a \sin t}{a^2} (-a \sin t) + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \sin t + 2a \cos t}{a^2} (a \cos t) \right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi$$

39. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = 3(\text{volume of } E) = 3(8 - 1) = 21.$

40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

PROBLEMS PLUS

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S, and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have $\iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV$. But

$$\nabla\cdot\frac{\mathbf{r}}{r^3} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle\cdot\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$$
$$= \frac{(x^2+y^2+z^2-3x^2)+(x^2+y^2+z^2-3y^2)+(x^2+y^2+z^2-3z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\Rightarrow \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} 0\,dV = 0$. On the other hand, notice that for the surfaces of ∂S_1 other than $S(a)$ and S, $\mathbf{r}\cdot\mathbf{n} = 0 \;\Rightarrow$

$$0 = \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S_L} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$$
$$= \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$$

$\Rightarrow \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = -\iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$. Notice that on $S(a)$, $r = a \;\Rightarrow\; \mathbf{n} = -\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{a}$ and $\mathbf{r}\cdot\mathbf{r} = r^2 = a^2$, so that $-\iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{r}}{a^4}\,dS = \iint_{S(a)} \frac{a^2}{a^4}\,dS = \frac{1}{a^2}\iint_{S(a)} dS = \frac{\text{area of } S(a)}{a^2} = |\Omega(S)|$.

Therefore $|\Omega(S)| = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$.

2. By Green's Theorem

$$\int_C (y^3 - y)\,dx - 2x^3\,dy = \iint_D \left[\frac{\partial(-2x^3)}{\partial x} - \frac{\partial(y^3-y)}{\partial y}\right] dA = \iint_D (1 - 6x^2 - 3y^2)\,dA$$

Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ can be expressed as $\int_C \mathbf{F}\cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x,y,z) = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \frac{1}{2}(bz - cy)\,\mathbf{i} + \frac{1}{2}(cx - az)\,\mathbf{j} + \frac{1}{2}(ay - bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$. Now

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \left(\tfrac{1}{2}a + \tfrac{1}{2}a\right)\mathbf{i} + \left(\tfrac{1}{2}b + \tfrac{1}{2}b\right)\mathbf{j} + \left(\tfrac{1}{2}c + \tfrac{1}{2}c\right)\mathbf{k} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = \mathbf{n}$$

so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S dS$ which is simply the surface area of S. Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ is the plane area enclosed by C.

4. (a) First we place the piston on coordinate axes so the top of the cylinder is at the origin and $x(t) \geq 0$ is the distance from the top of the cylinder to the piston at time t. Let C_1 be the curve traced out by the piston during one four-stroke cycle, so C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leq t \leq b$. (Thus, the curve lies on the positive x-axis and reverses direction several times.) The force on the piston is $AP(t)\,\mathbf{i}$, where A is the area of the top of the piston and $P(t)$ is the pressure in the cylinder at time t. As in Section 17.2 [ET 16.2], the work done on the piston is $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b AP(t)\,\mathbf{i} \cdot x'(t)\,\mathbf{i}\,dt = \int_a^b AP(t)\,x'(t)\,dt$. Here, the volume of the cylinder at time t is $V(t) = Ax(t) \Rightarrow V'(t) = Ax'(t) \Rightarrow \int_a^b AP(t)\,x'(t)\,dt = \int_a^b P(t)\,V'(t)\,dt$. Since the curve C in the PV-plane corresponds to the values of P and V at time t, $a \leq t \leq b$, we have

$$W = \int_a^b AP(t)\,x'(t)\,dt = \int_a^b P(t)\,V'(t)\,dt = \int_C P\,dV$$

Another method: If we divide the time interval $[a, b]$ into n subintervals of equal length Δt, the amount of work done on the piston in the ith time interval is approximately $AP(t_i)[x(t_i) - x(t_{i-1})]$. Thus we estimate the total work done during one cycle to be $\sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})]$. If we allow $n \to \infty$, we have

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})] = \lim_{n \to \infty} \sum_{i=1}^{n} P(t_i)[Ax(t_i) - Ax(t_{i-1})]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} P(t_i)[V(t_i) - V(t_{i-1})] = \int_C P\,dV$$

(b) Let C_L be the lower loop of the curve C and C_U the upper loop. Then $C = C_L \cup C_U$. C_L is positively oriented, so from Formula 17.4.5 [ET 16.4.5] we know the area of the lower loop in the PV-plane is given by $-\oint_{C_L} P\,dV$. C_U is negatively oriented, so the area of the upper loop is given by $-\left(-\oint_{C_U} P\,dV\right) = \oint_{C_U} P\,dV$. From part (a),

$$W = \int_C P\,dV = \int_{C_L \cup C_U} P\,dV = \oint_{C_L} P\,dV + \oint_{C_U} P\,dV = \oint_{C_U} P\,dV - \left(-\oint_{C_L} P\,dV\right),$$

the difference of the areas enclosed by the two loops of C.