Go Back

Appendixes

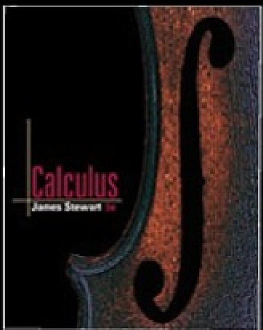
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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James Stewart
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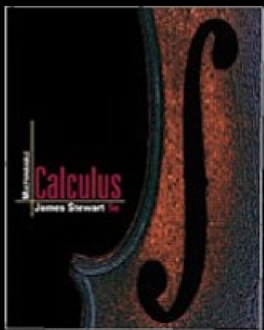

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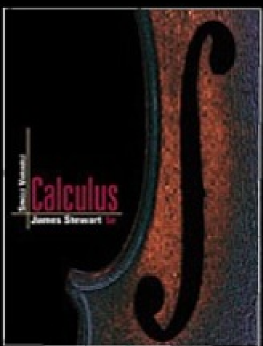

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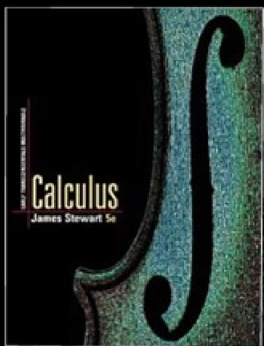

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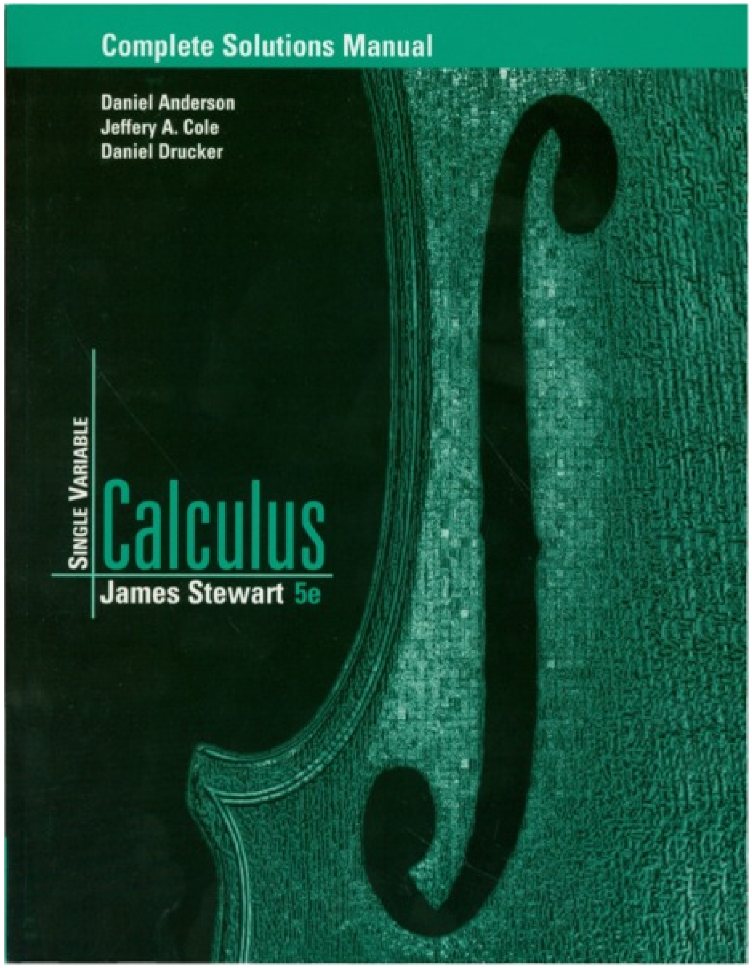


APPENDIXES

A Numbers, Inequalities, and Absolute Values

1. $|5 - 23| = |-18| = 18$

2. $|5| - |-23| = 5 - 23 = -18$

3. $|-\pi| = \pi$ because $\pi > 0$.

4. $|\pi - 2| = \pi - 2$ because $\pi - 2 > 0$.

5. $|\sqrt{5} - 5| = -(\sqrt{5} - 5) = 5 - \sqrt{5}$ because $\sqrt{5} - 5 < 0$.

6. $\big||-2| - |-3|\big| = |2 - 3| = |-1| = 1$

7. For $x < 2$, $x - 2 < 0$, so $|x - 2| = -(x - 2) = 2 - x$.

8. For $x > 2$, $x - 2 > 0$, so $|x - 2| = x - 2$.

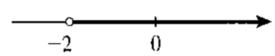
9. $|x + 1| = \begin{cases} x + 1 & \text{for } x + 1 \geq 0 \;\Leftrightarrow\; x \geq -1 \\ -(x + 1) & \text{for } x + 1 < 0 \;\Leftrightarrow\; x < -1 \end{cases}$

10. $|2x - 1| = \begin{cases} 2x - 1 & \text{for } 2x - 1 \geq 0 \;\Leftrightarrow\; x \geq \frac{1}{2} \\ 1 - 2x & \text{for } 2x - 1 < 0 \;\Leftrightarrow\; x < \frac{1}{2} \end{cases}$

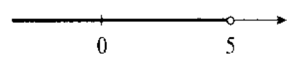
11. $|x^2 + 1| = x^2 + 1$ (since $x^2 + 1 \geq 0$ for all x).

12. Determine when $1 - 2x^2 < 0 \;\Leftrightarrow\; 1 < 2x^2 \;\Leftrightarrow\; x^2 > \frac{1}{2} \;\Leftrightarrow\; \sqrt{x^2} > \sqrt{\frac{1}{2}} \;\Leftrightarrow\; |x| > \sqrt{\frac{1}{2}} \;\Leftrightarrow$ $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$. Thus, $|1 - 2x^2| = \begin{cases} 1 - 2x^2 & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ 2x^2 - 1 & \text{if } x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}} \end{cases}$

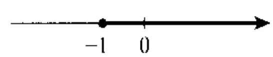
13. $2x + 7 > 3 \;\Leftrightarrow\; 2x > -4 \;\Leftrightarrow\; x > -2$, so $x \in (-2, \infty)$.



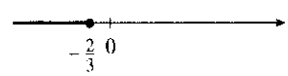
14. $3x - 11 < 4 \;\Leftrightarrow\; 3x < 15 \;\Leftrightarrow\; x < 5$, so $x \in (-\infty, 5)$.



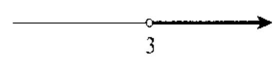
15. $1 - x \leq 2 \;\Leftrightarrow\; -x \leq 1 \;\Leftrightarrow\; x \geq -1$, so $x \in [-1, \infty)$.



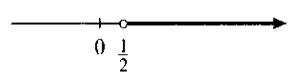
16. $4 - 3x \geq 6 \;\Leftrightarrow\; -3x \geq 2 \;\Leftrightarrow\; x \leq -\frac{2}{3}$, so $x \in \left(-\infty, -\frac{2}{3}\right]$.



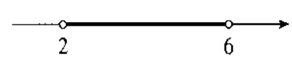
17. $2x + 1 < 5x - 8 \;\Leftrightarrow\; 9 < 3x \;\Leftrightarrow\; 3 < x$, so $x \in (3, \infty)$.



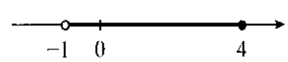
18. $1 + 5x > 5 - 3x \;\Leftrightarrow\; 8x > 4 \;\Leftrightarrow\; x > \frac{1}{2}$, so $x \in \left(\frac{1}{2}, \infty\right)$.



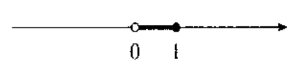
19. $-1 < 2x - 5 < 7 \;\Leftrightarrow\; 4 < 2x < 12 \;\Leftrightarrow\; 2 < x < 6$, so $x \in (2, 6)$.



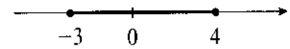
20. $1 < 3x + 4 \leq 16 \;\Leftrightarrow\; -3 < 3x \leq 12 \;\Leftrightarrow\; -1 < x \leq 4$, so $x \in (-1, 4]$.



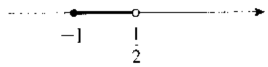
21. $0 \leq 1 - x < 1 \;\Leftrightarrow\; -1 \leq -x < 0 \;\Leftrightarrow\; 1 \geq x > 0$, so $x \in (0, 1]$.



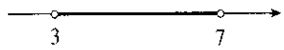
22. $-5 \le 3 - 2x \le 9 \quad \Leftrightarrow \quad -8 \le -2x \le 6 \quad \Leftrightarrow \quad 4 \ge x \ge -3$, so $x \in [-3, 4]$.



23. $4x < 2x + 1 \le 3x + 2$. So $4x < 2x + 1 \quad \Leftrightarrow \quad 2x < 1 \quad \Leftrightarrow \quad x < \frac{1}{2}$, and $2x + 1 \le 3x + 2 \quad \Leftrightarrow \quad -1 \le x$. Thus, $x \in \left[-1, \frac{1}{2}\right)$.



24. $2x - 3 < x + 4 < 3x - 2$. So $2x - 3 < x + 4 \quad \Leftrightarrow \quad x < 7$, and $x + 4 < 3x - 2 \quad \Leftrightarrow \quad 6 < 2x \quad \Leftrightarrow \quad 3 < x$, so $x \in (3, 7)$.



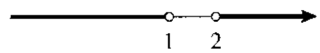
25. $(x - 1)(x - 2) > 0$. *Case 1:* (both factors are positive, so their product is positive)

$x - 1 > 0 \quad \Leftrightarrow \quad x > 1$, and $x - 2 > 0 \quad \Leftrightarrow \quad x > 2$, so $x \in (2, \infty)$.

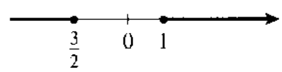
Case 2: (both factors are negative, so their product is positive)

$x - 1 < 0 \quad \Leftrightarrow \quad x < 1$, and $x - 2 < 0 \quad \Leftrightarrow \quad x < 2$, so $x \in (-\infty, 1)$.

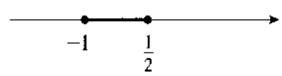
Thus, the solution set is $(-\infty, 1) \cup (2, \infty)$.



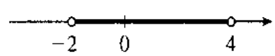
26. $(2x + 3)(x - 1) \ge 0$. *Case 1:* $2x + 3 \ge 0 \quad \Leftrightarrow \quad x \ge -\frac{3}{2}$, and $x - 1 \ge 0 \quad \Leftrightarrow \quad x \ge 1$, so $x \in [1, \infty)$. *Case 2:* $2x + 3 \le 0 \quad \Leftrightarrow \quad x \le -\frac{3}{2}$, and $x - 1 \le 0 \quad \Leftrightarrow \quad x \le 1$, so $x \in \left(-\infty, -\frac{3}{2}\right]$. Thus, the solution set is $\left(-\infty, -\frac{3}{2}\right] \cup [1, \infty)$.



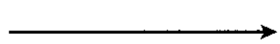
27. $2x^2 + x \le 1 \quad \Leftrightarrow \quad 2x^2 + x - 1 \le 0 \quad \Leftrightarrow \quad (2x - 1)(x + 1) \le 0$. *Case 1:* $2x - 1 \ge 0 \quad \Leftrightarrow \quad x \ge \frac{1}{2}$, and $x + 1 \le 0 \quad \Leftrightarrow \quad x \le -1$, which is an impossible combination. *Case 2:* $2x - 1 \le 0 \quad \Leftrightarrow \quad x \le \frac{1}{2}$, and $x + 1 \ge 0 \quad \Leftrightarrow \quad x \ge -1$, so $x \in \left[-1, \frac{1}{2}\right]$. Thus, the solution set is $\left[-1, \frac{1}{2}\right]$.



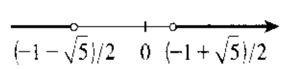
28. $x^2 < 2x + 8 \quad \Leftrightarrow \quad x^2 - 2x - 8 < 0 \quad \Leftrightarrow \quad (x - 4)(x + 2) < 0$. *Case 1:* $x > 4$ and $x < -2$, which is impossible. *Case 2:* $x < 4$ and $x > -2$. Thus, the solution set is $(-2, 4)$.



29. $x^2 + x + 1 > 0 \quad \Leftrightarrow \quad x^2 + x + \frac{1}{4} + \frac{3}{4} > 0 \quad \Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. But since $\left(x + \frac{1}{2}\right)^2 \ge 0$ for every real x, the original inequality will be true for all real x as well. Thus, the solution set is $(-\infty, \infty)$.

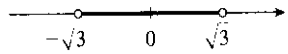


30. $x^2 + x > 1 \quad \Leftrightarrow \quad x^2 + x - 1 > 0$. Using the quadratic formula, we obtain $x^2 + x - 1 = \left(x - \frac{-1 - \sqrt{5}}{2}\right)\left(x - \frac{-1 + \sqrt{5}}{2}\right) > 0$. *Case 1:* $x - \frac{-1 - \sqrt{5}}{2} > 0$ and $x - \frac{-1 + \sqrt{5}}{2} > 0$, so that $x > \frac{-1 + \sqrt{5}}{2}$. *Case 2:* $x - \frac{-1 - \sqrt{5}}{2} < 0$ and $x - \frac{-1 + \sqrt{5}}{2} < 0$, so that $x < \frac{-1 - \sqrt{5}}{2}$. Thus, the solution set is $\left(-\infty, \frac{-1 - \sqrt{5}}{2}\right) \cup \left(\frac{-1 + \sqrt{5}}{2}, \infty\right)$.



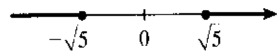
31. $x^2 < 3 \quad\Leftrightarrow\quad x^2 - 3 < 0 \quad\Leftrightarrow\quad (x - \sqrt{3})(x + \sqrt{3}) < 0$. *Case 1:* $x > \sqrt{3}$ and $x < -\sqrt{3}$, which is impossible. *Case 2:* $x < \sqrt{3}$ and $x > -\sqrt{3}$. Thus, the solution set is $(-\sqrt{3}, \sqrt{3})$.

Another method: $x^2 < 3 \quad\Leftrightarrow\quad |x| < \sqrt{3} \quad\Leftrightarrow\quad -\sqrt{3} < x < \sqrt{3}$.

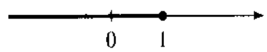


32. $x^2 \geq 5 \quad\Leftrightarrow\quad x^2 - 5 \geq 0 \quad\Leftrightarrow\quad (x - \sqrt{5})(x + \sqrt{5}) \geq 0$. *Case 1:* $x \geq \sqrt{5}$ and $x \geq -\sqrt{5}$, so $x \in [\sqrt{5}, \infty)$. *Case 2:* $x \leq \sqrt{5}$ and $x \leq -\sqrt{5}$, so $x \in (-\infty, -\sqrt{5}]$. Thus, the solution set is $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$.

Another method: $x^2 \geq 5 \quad\Leftrightarrow\quad |x| \geq \sqrt{5} \quad\Leftrightarrow\quad x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$.



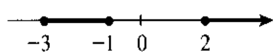
33. $x^3 - x^2 \leq 0 \quad\Leftrightarrow\quad x^2(x-1) \leq 0$. Since $x^2 \geq 0$ for all x, the inequality is satisfied when $x - 1 \leq 0 \quad\Leftrightarrow\quad x \leq 1$. Thus, the solution set is $(-\infty, 1]$.



34. $(x+1)(x-2)(x+3) = 0 \quad\Leftrightarrow\quad x = -1, 2$, or -3. Constructing a table:

Interval	$x+1$	$x-2$	$x+3$	$(x+1)(x-2)(x+3)$
$x < -3$	−	−	−	−
$-3 < x < -1$	−	−	+	+
$-1 < x < 2$	+	−	+	−
$x > 2$	+	+	+	+

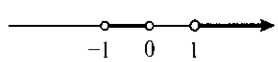
Thus, $(x+1)(x-2)(x+3) \geq 0$ on $[-3, -1]$ and $[2, \infty)$, and the solution set is $[-3, -1] \cup [2, \infty)$.



35. $x^3 > x \quad\Leftrightarrow\quad x^3 - x > 0 \quad\Leftrightarrow\quad x(x^2 - 1) > 0 \quad\Leftrightarrow\quad x(x-1)(x+1) > 0$. Constructing a table:

Interval	x	$x-1$	$x+1$	$x(x-1)(x+1)$
$x < -1$	−	−	−	−
$-1 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

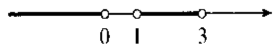
Since $x^3 > x$ when the last column is positive, the solution set is $(-1, 0) \cup (1, \infty)$.



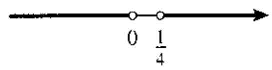
6. $x^3 + 3x < 4x^2 \quad \Leftrightarrow \quad x^3 - 4x^2 + 3x < 0 \quad \Leftrightarrow \quad x(x^2 - 4x + 3) < 0 \quad \Leftrightarrow \quad x(x-1)(x-3) < 0.$

Interval	x	$x-1$	$x-3$	$x(x-1)(x-3)$
$x < 0$	$-$	$-$	$-$	$-$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

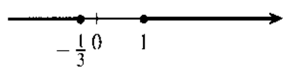
Thus, the solution set is $(-\infty, 0) \cup (1, 3)$.



17. $1/x < 4$. This is clearly true for $x < 0$. So suppose $x > 0$. then $1/x < 4 \quad \Leftrightarrow \quad 1 < 4x \quad \Leftrightarrow \quad \frac{1}{4} < x$. Thus, the solution set is $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$.



18. $-3 < 1/x \leq 1$. We solve the two inequalities separately and take the intersection of the solution sets. First, $-3 < 1/x$ is clearly true for $x > 0$. So suppose $x < 0$. Then $-3 < 1/x \quad \Leftrightarrow \quad -3x > 1 \quad \Leftrightarrow \quad x < -\frac{1}{3}$, so for this inequality, the solution set is $\left(-\infty, -\frac{1}{3}\right) \cup (0, \infty)$. Now $1/x \leq 1$ is clearly true if $x < 0$. So suppose $x > 0$. Then $1/x \leq 1 \quad \Leftrightarrow \quad 1 \leq x$, and the solution set here is $(-\infty, 0) \cup [1, \infty)$. Taking the intersection of the two solution sets gives the final solution set: $\left(-\infty, -\frac{1}{3}\right) \cup [1, \infty)$.



19. $C = \frac{5}{9}(F - 32) \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. So $50 \leq F \leq 95 \quad \Rightarrow \quad 50 \leq \frac{9}{5}C + 32 \leq 95 \quad \Rightarrow \quad 18 \leq \frac{9}{5}C \leq 63 \quad \Rightarrow$ $10 \leq C \leq 35$. So the interval is $[10, 35]$.

10. Since $20 \leq C \leq 30$ and $C = \frac{5}{9}(F - 32)$, we have $20 \leq \frac{5}{9}(F - 32) \leq 30 \quad \Rightarrow \quad 36 \leq F - 32 \leq 54 \quad \Rightarrow$ $68 \leq F \leq 86$. So the interval is $[68, 86]$.

11. (a) Let T represent the temperature in degrees Celsius and h the height in km. $T = 20$ when $h = 0$ and T decreases by 10 °C for every km (1 °C for each 100-m rise). Thus, $T = 20 - 10h$ when $0 \leq h \leq 12$.

(b) From part (a), $T = 20 - 10h \quad \Rightarrow \quad 10h = 20 - T \quad \Rightarrow \quad h = 2 - T/10$. So $0 \leq h \leq 5 \quad \Rightarrow$ $0 \leq 2 - T/10 \leq 5 \quad \Rightarrow \quad -2 \leq -T/10 \leq 3 \quad \Rightarrow \quad -20 \leq -T \leq 30 \quad \Rightarrow \quad 20 \geq T \geq -30 \quad \Rightarrow$ $-30 \leq T \leq 20$. Thus, the range of temperatures (in °C) to be expected is $[-30, 20]$.

12. The ball will be at least 32 ft above the ground if $h \geq 32 \quad \Leftrightarrow \quad 128 + 16t - 16t^2 \geq 32 \quad \Leftrightarrow$ $16t^2 - 16t - 96 \leq 0 \quad \Leftrightarrow \quad 16(t-3)(t+2) \leq 0$. $t = 3$ and $t = -2$ are endpoints of the interval we're looking for, and constructing a table gives $-2 \leq t \leq 3$. But $t \geq 0$, so the ball will be at least 32 ft above the ground in the time interval $[0, 3]$.

13. $|2x| = 3 \quad \Leftrightarrow \quad$ either $2x = 3$ or $2x = -3 \quad \Leftrightarrow \quad x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

14. $|3x + 5| = 1 \quad \Leftrightarrow \quad$ either $3x + 5 = 1$ or -1. In the first case, $3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$, and in the second case, $3x = -6 \quad \Leftrightarrow \quad x = -2$. So the solutions are -2 and $-\frac{4}{3}$.

15. $|x + 3| = |2x + 1| \quad \Leftrightarrow \quad$ either $x + 3 = 2x + 1$ or $x + 3 = -(2x + 1)$. In the first case, $x = 2$, and in the second case, $x + 3 = -2x - 1 \quad \Leftrightarrow \quad 3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$. So the solutions are $-\frac{4}{3}$ and 2.

46. $\left|\dfrac{2x-1}{x+1}\right| = 3 \quad\Leftrightarrow\quad$ either $\dfrac{2x-1}{x+1} = 3$ or $\dfrac{2x-1}{x+1} = -3$. In the first case, $2x - 1 = 3x + 3 \quad\Leftrightarrow\quad x = -4$, and in the second case, $2x - 1 = -3x - 3 \quad\Leftrightarrow\quad x = -\frac{2}{5}$.

47. By Property 5 of absolute values, $|x| < 3 \quad\Leftrightarrow\quad -3 < x < 3$, so $x \in (-3, 3)$.

48. By Properties 4 and 6 of absolute values, $|x| \geq 3 \quad\Leftrightarrow\quad x \leq -3$ or $x \geq 3$, so $x \in (-\infty, -3] \cup [3, \infty)$.

49. $|x - 4| < 1 \quad\Leftrightarrow\quad -1 < x - 4 < 1 \quad\Leftrightarrow\quad 3 < x < 5$, so $x \in (3, 5)$.

50. $|x - 6| < 0.1 \quad\Leftrightarrow\quad -0.1 < x - 6 < 0.1 \quad\Leftrightarrow\quad 5.9 < x < 6.1$, so $x \in (5.9, 6.1)$.

51. $|x + 5| \geq 2 \quad\Leftrightarrow\quad x + 5 \geq 2$ or $x + 5 \leq -2 \quad\Leftrightarrow\quad x \geq -3$ or $x \leq -7$, so $x \in (-\infty, -7] \cup [-3, \infty)$.

52. $|x + 1| \geq 3 \quad\Leftrightarrow\quad x + 1 \geq 3$ or $x + 1 \leq -3 \quad\Leftrightarrow\quad x \geq 2$ or $x \leq -4$, so $x \in (-\infty, -4] \cup [2, \infty)$.

53. $|2x - 3| \leq 0.4 \quad\Leftrightarrow\quad -0.4 \leq 2x - 3 \leq 0.4 \quad\Leftrightarrow\quad 2.6 \leq 2x \leq 3.4 \quad\Leftrightarrow\quad 1.3 \leq x \leq 1.7$, so $x \in [1.3, 1.7]$.

54. $|5x - 2| < 6 \quad\Leftrightarrow\quad -6 < 5x - 2 < 6 \quad\Leftrightarrow\quad -4 < 5x < 8 \quad\Leftrightarrow\quad -\frac{4}{5} < x < \frac{8}{5}$, so $x \in \left(-\frac{4}{5}, \frac{8}{5}\right)$.

55. $1 \leq |x| \leq 4$. So either $1 \leq x \leq 4$ or $1 \leq -x \leq 4 \quad\Leftrightarrow\quad -1 \geq x \geq -4$. Thus, $x \in [-4, -1] \cup [1, 4]$.

56. $0 < |x - 5| < \frac{1}{2}$. Clearly $0 < |x - 5|$ for $x \neq 5$. Now $|x - 5| < \frac{1}{2} \quad\Leftrightarrow\quad -\frac{1}{2} < x - 5 < \frac{1}{2} \quad\Leftrightarrow\quad 4.5 < x < 5.5$. So the solution set is $(4.5, 5) \cup (5, 5.5)$.

57. $a(bx - c) \geq bc \quad\Leftrightarrow\quad bx - c \geq \dfrac{bc}{a} \quad\Leftrightarrow\quad bx \geq \dfrac{bc}{a} + c = \dfrac{bc + ac}{a} \quad\Leftrightarrow\quad x \geq \dfrac{bc + ac}{ab}$

58. $a \leq bx + c < 2a \quad\Leftrightarrow\quad a - c \leq bx < 2a - c \quad\Leftrightarrow\quad \dfrac{a - c}{b} \leq x < \dfrac{2a - c}{b}$ (since $b > 0$)

59. $ax + b < c \quad\Leftrightarrow\quad ax < c - b \quad\Leftrightarrow\quad x > \dfrac{c - b}{a}$ (since $a < 0$)

60. $\dfrac{ax + b}{c} \leq b \quad\Leftrightarrow\quad ax + b \geq bc$ (since $c < 0$) $\quad\Leftrightarrow\quad ax \geq bc - b \quad\Leftrightarrow\quad x \leq \dfrac{b\,(c - 1)}{a}$ (since $a < 0$)

61. $|(x + y) - 5| = |(x - 2) + (y - 3)| \leq |x - 2| + |y - 3| < 0.01 + 0.04 = 0.05$

62. Use the Triangle Inequality: $|x + 3| < \frac{1}{2} \quad\Rightarrow$

$|4x + 13| = |4\,(x + 3) + 1| \leq |4\,(x + 3)| + |1| = 4\,|x + 3| + 1 < 4\left(\frac{1}{2}\right) + 1 = 3$

Another method: $|x + 3| < \frac{1}{2} \quad\Rightarrow\quad -\frac{1}{2} < x + 3 < \frac{1}{2} \quad\Rightarrow\quad -2 < 4x + 12 < 2 \quad\Rightarrow\quad -1 < 4x + 13 < 3 \quad\Rightarrow$
$|4x + 13| < 3$

63. If $a < b$ then $a + a < a + b$ and $a + b < b + b$. So $2a < a + b < 2b$. Dividing by 2, we get $a < \frac{1}{2}\,(a + b) < b$.

64. If $0 < a < b$, then $\dfrac{1}{ab} > 0$. So $a < b \quad\Rightarrow\quad \dfrac{1}{ab} \cdot a < \dfrac{1}{ab} \cdot b \quad\Leftrightarrow\quad \dfrac{1}{b} < \dfrac{1}{a}$.

65. $|ab| = \sqrt{(ab)^2} = \sqrt{a^2 b^2} = \sqrt{a^2}\,\sqrt{b^2} = |a|\,|b|$

66. $\left|\dfrac{a}{b}\right|\,|b| = \left|\dfrac{a}{b} \cdot b\right| = |a|$ (using the result of Exercise 65). Dividing the equation through by $|b|$ gives $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. If $0 < a < b$, then $a \cdot a < a \cdot b$ and $a \cdot b < b \cdot b$ [using Rule 3 of Inequalities]. So $a^2 < ab < b^2$ and hence $a^2 < b^2$.

68. Following the hint, the Triangle Inequality becomes $|(x - y) + y| \leq |x - y| + |y| \quad\Leftrightarrow\quad |x| \leq |x - y| + |y| \quad\Leftrightarrow\quad |x - y| \geq |x| - |y|$.

69. Observe that the sum, difference and product of two integers is always an integer. Let the rational numbers be represented by $r = m/n$ and $s = p/q$ (where m, n, p and q are integers with $n \neq 0$, $q \neq 0$). Now $r + s = \frac{m}{n} + \frac{p}{q} = \frac{mq + pn}{nq}$, but $mq + pn$ and nq are both integers, so $\frac{mq + pn}{nq} = r + s$ is a rational number by definition. Similarly, $r - s = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$ is a rational number. Finally, $r \cdot s = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$ but mp and nq are both integers, so $\frac{mp}{nq} = r \cdot s$ is a rational number by definition.

70. (a) No. Consider the case of $\sqrt{2}$ and $-\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} + \left(-\sqrt{2}\right) = 0$ and 0, being an integer, is not irrational.

(b) No. Consider the case of $\sqrt{2}$ and $\sqrt{2}$. Both are irrational numbers, yet $\sqrt{2} \cdot \sqrt{2} = 2$ is not irrational.

B Coordinate Geometry and Lines

1. From the Distance Formula with $x_1 = 1$, $x_2 = 4$, $y_1 = 1$, $y_2 = 5$, we find the distance from $(1, 1)$ to $(4, 5)$ to be $\sqrt{(4-1)^2 + (5-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

2. The distance from $(1, -3)$ to $(5, 7)$ is $\sqrt{(5-1)^2 + [7-(-3)]^2} = \sqrt{4^2 + 10^2} = \sqrt{116} = 2\sqrt{29}$.

3. $\sqrt{(-1-6)^2 + [3-(-2)]^2} = \sqrt{(-7)^2 + 5^2} = \sqrt{74}$

4. $\sqrt{(-1-1)^2 + [-3-(-6)]^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

5. $\sqrt{(4-2)^2 + (-7-5)^2} = \sqrt{2^2 + (-12)^2} = \sqrt{148} = 2\sqrt{37}$

6. $\sqrt{(b-a)^2 + (a-b)^2} = \sqrt{(a-b)^2 + (a-b)^2} = \sqrt{2\,(a-b)^2} = \sqrt{2}\,|a-b|$

7. From (2), the slope is $\frac{11-5}{4-1} = \frac{6}{3} = 2$.

8. $m = \frac{-3-6}{4-(-1)} = -\frac{9}{5}$

9. With $P(-3, 3)$ and $Q(-1, -6)$, the slope m of the line through P and Q is $m = \frac{-6-3}{-1-(-3)} = -\frac{9}{2}$.

10. $m = \frac{0-(-4)}{6-(-1)} = \frac{4}{7}$

11. Since $|AC| = \sqrt{(-4-0)^2 + (3-2)^2} = \sqrt{(-4)^2 + 1^2} = \sqrt{17}$ and $|BC| = \sqrt{[-4-(-3)]^2 + [3-(-1)]^2} = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$, the triangle has two sides of equal length, and so is isosceles.

12. (a) $|AB| = \sqrt{(11-6)^2 + [-3-(-7)]^2} = \sqrt{5^2 + 4^2} = \sqrt{41}$,

$|AC| = \sqrt{(2-6)^2 + [-2-(-7)]^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$, and

$|BC| = \sqrt{(2-11)^2 + [-2-(-3)]^2} = \sqrt{(-9)^2 + 1^2} = \sqrt{82}$, so

$|AB|^2 + |AC|^2 = 41 + 41 = 82 = |BC|^2$, and so $\triangle ABC$ is a right triangle.

(b) $m_{AB} = \dfrac{-3-(-7)}{11-6} = \dfrac{4}{5}$ and $m_{AC} = \dfrac{-2-(-7)}{2-6} = -\dfrac{5}{4}$. Thus $m_{AB} \cdot m_{AC} = -1$ and so AB is perpendicular to AC and $\triangle ABC$ must be a right triangle.

(c) Taking lengths from part (a), the base is $\sqrt{41}$ and the height is $\sqrt{41}$. Thus the area is $\frac{1}{2}bh = \frac{1}{2}\sqrt{41}\sqrt{41} = \frac{41}{2}$.

13. Using $A(-2,9)$, $B(4,6)$, $C(1,0)$, and $D(-5,3)$, we have

$|AB| = \sqrt{[4-(-2)]^2+(6-9)^2} = \sqrt{6^2+(-3)^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$,

$|BC| = \sqrt{(1-4)^2+(0-6)^2} = \sqrt{(-3)^2+(-6)^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$,

$|CD| = \sqrt{(-5-1)^2+(3-0)^2} = \sqrt{(-6)^2+3^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$, and

$|DA| = \sqrt{[-2-(-5)]^2+(9-3)^2} = \sqrt{3^2+6^2} = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$. So all sides are of equal length and we have a rhombus. Moreover, $m_{AB} = \dfrac{6-9}{4-(-2)} = -\dfrac{1}{2}$, $m_{BC} = \dfrac{0-6}{1-4} = 2$, $m_{CD} = \dfrac{3-0}{-5-1} = -\dfrac{1}{2}$, and $m_{DA} = \dfrac{9-3}{-2-(-5)} = 2$, so the sides are perpendicular. Thus, A, B, C, and D are vertices of a square.

14. (a) Using $A(-1,3)$, $B(3,11)$, and $C(5,15)$, we have

$|AB| = \sqrt{[3-(-1)]^2+(11-3)^2} = \sqrt{4^2+8^2} = \sqrt{80} = 4\sqrt{5}$,

$|BC| = \sqrt{(5-3)^2+(15-11)^2} = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$, and

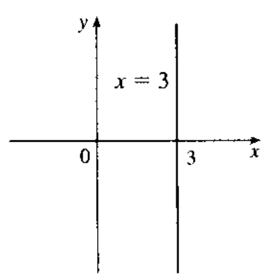
$|AC| = \sqrt{[5-(-1)]^2+(15-3)^2} = \sqrt{6^2+12^2} = \sqrt{180} = 6\sqrt{5}$. Thus, $|AC| = |AB| + |BC|$.

(b) $m_{AB} = \dfrac{11-3}{3-(-1)} = \dfrac{8}{4} = 2$ and $m_{AC} = \dfrac{15-3}{5-(-1)} = \dfrac{12}{6} = 2$. Since the segments AB and AC have the same slope, A, B and C must be collinear.

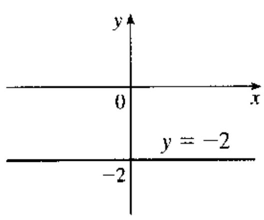
15. The slope of the line segment AB is $\dfrac{4-1}{7-1} = \dfrac{1}{2}$, the slope of CD is $\dfrac{7-10}{-1-5} = \dfrac{1}{2}$, the slope of BC is $\dfrac{10-4}{5-7} = -3$, and the slope of DA is $\dfrac{1-7}{1-(-1)} = -3$. So AB is parallel to CD and BC is parallel to DA. Hence $ABCD$ is a parallelogram.

16. The slopes of the four sides are $m_{AB} = \dfrac{3-1}{11-1} = \dfrac{1}{5}$, $m_{BC} = \dfrac{8-3}{10-11} = -5$, $m_{CD} = \dfrac{6-8}{0-10} = \dfrac{1}{5}$, and $m_{DA} = \dfrac{1-6}{1-0} = -5$. Hence $AB \parallel CD$, $BC \parallel DA$, $AB \perp BC$, $BC \perp CD$, $CD \perp DA$, and $DA \perp AB$, and so $ABCD$ is a rectangle.

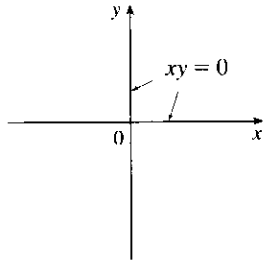
17. $x = 3$



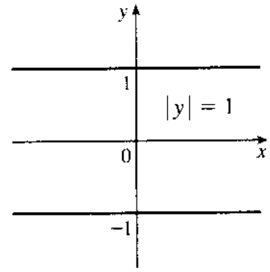
18. $y = -2$



19. $xy = 0 \quad \Leftrightarrow \quad x = 0$ or $y = 0$. The graph consists of the coordinate axes.



20. $|y| = 1 \quad \Leftrightarrow \quad y = 1$ or $y = -1$



21. By the point-slope form of the equation of a line, an equation of the line through $(2, -3)$ with slope 6 is $y - (-3) = 6(x - 2)$ or $y = 6x - 15$.

22. $y - 4 = -3\,[x - (-1)]$ or $y = -3x + 1$

23. $y - 7 = \frac{2}{3}\,(x - 1)$ or $y = \frac{2}{3}x + \frac{19}{3}$

24. $y - (-5) = -\frac{7}{2}\,[x - (-3)]$ or $y = -\frac{7}{2}x - \frac{31}{2}$

25. The slope of the line through $(2, 1)$ and $(1, 6)$ is $m = \dfrac{6 - 1}{1 - 2} = -5$, so an equation of the line is $y - 1 = -5(x - 2)$ or $y = -5x + 11$.

26. For $(-1, -2)$ and $(4, 3)$, $m = \dfrac{3 - (-2)}{4 - (-1)} = 1$. So $y - 3 = 1(x - 4)$ or $y = x - 1$.

27. By the slope-intercept form of the equation of a line, an equation of the line is $y = 3x - 2$.

28. By the slope-intercept form of the equation of a line, an equation of the line is $y = \frac{2}{5}x + 4$.

29. Since the line passes through $(1, 0)$ and $(0, -3)$, its slope is $m = \dfrac{-3 - 0}{0 - 1} = 3$, so an equation is $y = 3x - 3$.

Another method: From Exercise 61, $\dfrac{x}{1} + \dfrac{y}{-3} = 1 \quad \Rightarrow \quad -3x + y = -3 \quad \Rightarrow \quad y = 3x - 3$.

30. For $(-8, 0)$ and $(0, 6)$, $m = \dfrac{6 - 0}{0 - (-8)} = \dfrac{3}{4}$. So an equation is $y = \frac{3}{4}x + 6$.

Another method: From Exercise 61, $\dfrac{x}{-8} + \dfrac{y}{6} = 1 \quad \Rightarrow \quad -3x + 4y = 24 \quad \Rightarrow \quad y = \frac{3}{4}x + 6$.

31. Since $m = 0$, $y - 5 = 0(x - 4)$ or $y = 5$.

32. Since m is undefined, we have the vertical line $x = 4$.

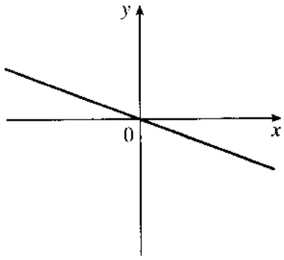
33. Putting the line $x + 2y = 6$ into its slope-intercept form gives us $y = -\frac{1}{2}x + 3$, so we see that this line has slope $-\frac{1}{2}$. Thus, we want the line of slope $-\frac{1}{2}$ that passes through the point $(1, -6)$: $y - (-6) = -\frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = -\frac{1}{2}x - \frac{11}{2}$.

34. $2x + 3y + 4 = 0 \quad \Leftrightarrow \quad y = -\frac{2}{3}x - \frac{4}{3}$, so $m = -\frac{2}{3}$ and the required line is $y = -\frac{2}{3}x + 6$.

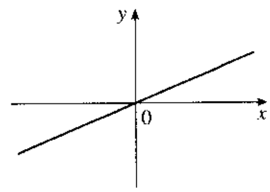
35. $2x + 5y + 8 = 0 \quad \Leftrightarrow \quad y = -\frac{2}{5}x - \frac{8}{5}$. Since this line has slope $-\frac{2}{5}$, a line perpendicular to it would have slope $\frac{5}{2}$, so the required line is $y - (-2) = \frac{5}{2}\,[x - (-1)] \quad \Leftrightarrow \quad y = \frac{5}{2}x + \frac{1}{2}$.

36. $4x - 8y = 1 \quad \Leftrightarrow \quad y = \frac{1}{2}x - \frac{1}{8}$. Since this line has slope $\frac{1}{2}$, a line perpendicular to it would have slope -2, so the required line is $y - \left(-\frac{2}{3}\right) = -2\left(x - \frac{1}{2}\right) \quad \Leftrightarrow \quad y = -2x + \frac{1}{3}$.

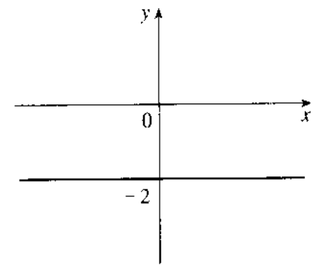
37. $x + 3y = 0 \quad \Leftrightarrow \quad y = -\frac{1}{3}x$, so the slope is $-\frac{1}{3}$ and the y-intercept is 0.



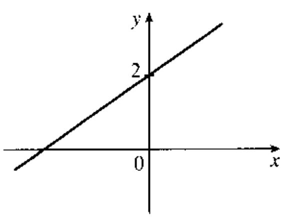
38. $2x - 5y = 0 \quad \Leftrightarrow \quad y = \frac{2}{5}x$, so the slope is $\frac{2}{5}$ and the y-intercept is 0.



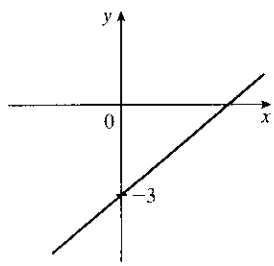
39. $y = -2$ is a horizontal line with slope 0 and y-intercept -2.



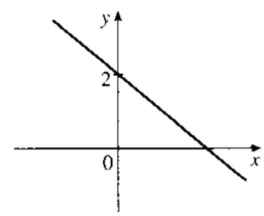
40. $2x - 3y + 6 = 0 \quad \Leftrightarrow$ $y = \frac{2}{3}x + 2$, so the slope is $\frac{2}{3}$ and the y-intercept is 2.



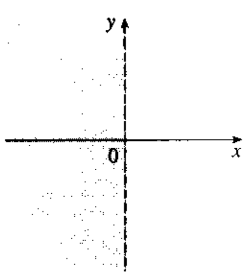
41. $3x - 4y = 12 \quad \Leftrightarrow$ $y = \frac{3}{4}x - 3$, so the slope is $\frac{3}{4}$ and the y-intercept is -3.



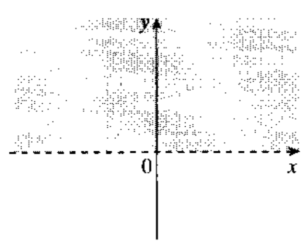
42. $4x + 5y = 10 \quad \Leftrightarrow$ $y = -\frac{4}{5}x + 2$, so the slope is $-\frac{4}{5}$ and the y-intercept is 2.



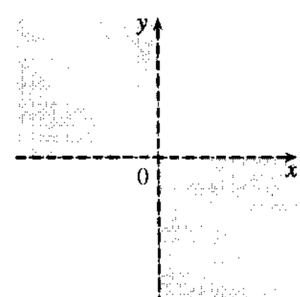
43. $\{(x, y) \mid x < 0\}$



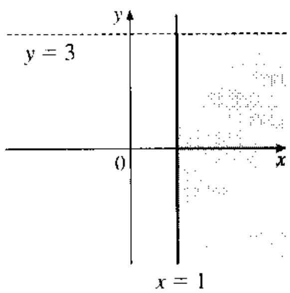
44. $\{(x, y) \mid y > 0\}$



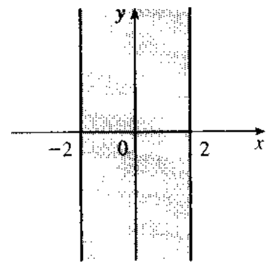
45. $\{(x, y) \mid xy < 0\} =$ $\{(x, y) \mid x < 0 \text{ and } y > 0\}$ $\cup \{(x, y) \mid x > 0 \text{ and } y < 0\}$



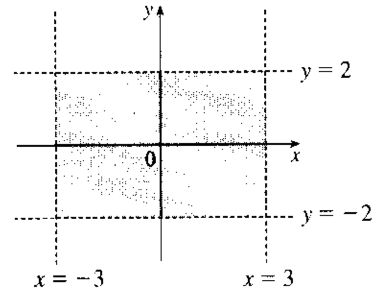
46. $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$



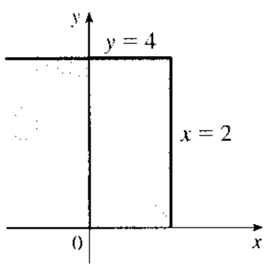
47. $\left\{(x, y) \,\middle|\, |x| \leq 2\right\} = \{(x, y) \mid -2 \leq x \leq 2\}$



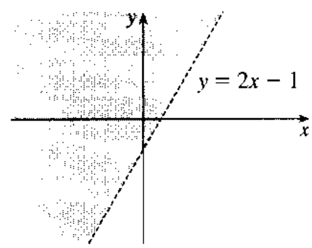
48. $\left\{(x, y) \,\middle|\, |x| < 3 \text{ and } |y| < 2\right\}$



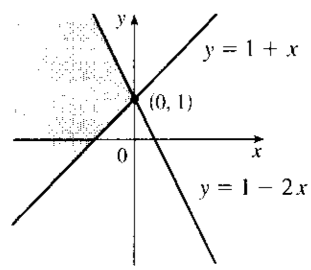
49. $\{(x, y) \mid 0 \leq y \leq 4, x \leq 2\}$



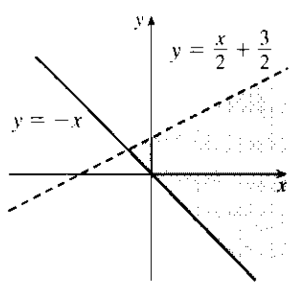
50. $\{(x, y) \mid y > 2x - 1\}$



51. $\{(x, y) \mid 1 + x \leq y \leq 1 - 2x\}$



52. $\left\{(x, y) \,\middle|\, -x \leq y < \dfrac{x + 3}{2}\right\}$



53. Let $P(0, y)$ be a point on the y-axis. The distance from P to $(5, -5)$ is $\sqrt{(5 - 0)^2 + (-5 - y)^2} = \sqrt{5^2 + (y + 5)^2}$. The distance from P to $(1, 1)$ is $\sqrt{(1 - 0)^2 + (1 - y)^2} = \sqrt{1^2 + (y - 1)^2}$. We want these distances to be equal: $\sqrt{5^2 + (y + 5)^2} = \sqrt{1^2 + (y - 1)^2} \Leftrightarrow$
$5^2 + (y + 5)^2 = 1^2 + (y - 1)^2 \Leftrightarrow$
$25 + (y^2 + 10y + 25) = 1 + (y^2 - 2y + 1) \Leftrightarrow 12y = -48 \Leftrightarrow$
$y = -4$. So the desired point is $(0, -4)$.

54. Let M be the point $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$. Then

$|MP_1|^2 = \left(x_1 - \dfrac{x_1 + x_2}{2}\right)^2 + \left(y_1 - \dfrac{y_1 + y_2}{2}\right)^2 = \left(\dfrac{x_1 - x_2}{2}\right)^2 + \left(\dfrac{y_1 - y_2}{2}\right)^2$ and

$|MP_2|^2 = \left(x_2 - \dfrac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \dfrac{y_1 + y_2}{2}\right)^2 = \left(\dfrac{x_2 - x_1}{2}\right)^2 + \left(\dfrac{y_2 - y_1}{2}\right)^2$. Hence, $|MP_1| = |MP_2|$; that is, M is equidistant from P_1 and P_2.

55. (a) Using the midpoint formula from Exercise 54 with $(1, 3)$ and $(7, 15)$, we get $\left(\frac{1+7}{2}, \frac{3+15}{2}\right) = (4, 9)$.

(b) Using the same formula, we get $\left(\frac{-1+8}{2}, \frac{6-12}{2}\right) = \left(\frac{7}{2}, -3\right)$.

56. The midpoint M_1 of AB is $\left(\frac{1+3}{2}, \frac{0+6}{2}\right) = (2,3)$, the midpoint M_2 of BC is $\left(\frac{3+8}{2}, \frac{6+2}{2}\right) = \left(\frac{11}{2}, 4\right)$, and the midpoint M_3 of CA is $\left(\frac{8+1}{2}, \frac{2+0}{2}\right) = \left(\frac{9}{2}, 1\right)$. The lengths of the medians are $|AM_2| = \sqrt{\left(\frac{11}{2} - 1\right)^2 + (4-0)^2} = \sqrt{\left(\frac{9}{2}\right)^2 + 4^2} = \sqrt{\frac{145}{4}} = \frac{\sqrt{145}}{2}$,
$|BM_3| = \sqrt{\left(\frac{9}{2} - 3\right)^2 + (1-6)^2} = \sqrt{\left(\frac{3}{2}\right)^2 + (-5)^2} = \sqrt{\frac{109}{4}} = \frac{\sqrt{109}}{2}$, and
$|CM_1| = \sqrt{(2-8)^2 + (3-2)^2} = \sqrt{(-6)^2 + 1^2} = \sqrt{37}$.

57. $2x - y = 4 \;\Leftrightarrow\; y = 2x - 4 \;\Rightarrow\; m_1 = 2$ and $6x - 2y = 10 \;\Leftrightarrow\; 2y = 6x - 10 \;\Leftrightarrow\; y = 3x - 5 \;\Rightarrow\; m_2 = 3$. Since $m_1 \neq m_2$, the two lines are not parallel. To find the point of intersection: $2x - 4 = 3x - 5 \;\Leftrightarrow\; x = 1 \;\Rightarrow\; y = -2$. Thus, the point of intersection is $(1, -2)$.

58. $3x - 5y + 19 = 0 \;\Leftrightarrow\; 5y = 3x + 19 \;\Leftrightarrow\; y = \frac{3}{5}x + \frac{19}{5} \;\Rightarrow\; m_1 = \frac{3}{5}$ and $10x + 6y - 50 = 0 \;\Leftrightarrow\; 6y = -10x + 50 \;\Leftrightarrow\; y = -\frac{5}{3}x + \frac{25}{3} \;\Rightarrow\; m_2 = -\frac{5}{3}$. Since $m_1 m_2 = \frac{3}{5}\left(-\frac{5}{3}\right) = -1$, the two lines are perpendicular. To find the point of intersection: $\frac{3}{5}x + \frac{19}{5} = -\frac{5}{3}x + \frac{25}{3} \;\Leftrightarrow\; 9x + 57 = -25x + 125 \;\Leftrightarrow\; 34x = 68 \;\Leftrightarrow\; x = 2 \;\Rightarrow\; y = \frac{3}{5} \cdot 2 + \frac{19}{5} = \frac{25}{5} = 5$. Thus, the point of intersection is $(2, 5)$.

59. With $A(1,4)$ and $B(7,-2)$, the slope of segment AB is $\frac{-2-4}{7-1} = -1$, so its perpendicular bisector has slope 1. The midpoint of AB is $\left(\frac{1+7}{2}, \frac{4+(-2)}{2}\right) = (4,1)$, so an equation of the perpendicular bisector is $y - 1 = 1(x - 4)$ or $y = x - 3$.

60. (a) Side PQ has slope $\frac{4-0}{3-1} = 2$, so its equation is $y - 0 = 2(x-1) \;\Leftrightarrow\; y = 2x - 2$. Side QR has slope $\frac{6-4}{-1-3} = -\frac{1}{2}$, so its equation is $y - 4 = -\frac{1}{2}(x-3) \;\Leftrightarrow\; y = -\frac{1}{2}x + \frac{11}{2}$. Side RP has slope $\frac{0-6}{1-(-1)} = -3$, so its equation is $y - 0 = -3(x-1) \;\Leftrightarrow\; y = -3x + 3$.

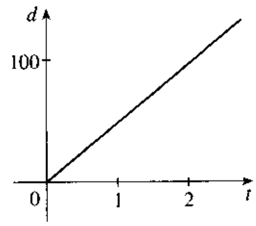
(b) M_1 (the midpoint of PQ) has coordinates $\left(\frac{1+3}{2}, \frac{0+4}{2}\right) = (2,2)$. M_2 (the midpoint of QR) has coordinates $\left(\frac{3-1}{2}, \frac{4+6}{2}\right) = (1,5)$. M_3 (the midpoint of RP) has coordinates $\left(\frac{1-1}{2}, \frac{0+6}{2}\right) = (0,3)$. RM_1 has slope $\frac{2-6}{2-(-1)} = -\frac{4}{3}$ and hence equation $y - 2 = -\frac{4}{3}(x-2) \;\Leftrightarrow\; y = -\frac{4}{3}x + \frac{14}{3}$. PM_2 is a vertical line with equation $x = 1$. QM_3 has slope $\frac{3-4}{0-3} = \frac{1}{3}$ and hence equation $y - 3 = \frac{1}{3}(x-0) \;\Leftrightarrow\; y = \frac{1}{3}x + 3$. PM_2 and RM_1 intersect where $x = 1$ and $y = -\frac{4}{3}(1) + \frac{14}{3} = \frac{10}{3}$, or at $\left(1, \frac{10}{3}\right)$. PM_2 and QM_3 intersect where $x = 1$ and $y = \frac{1}{3}(1) + 3 = \frac{10}{3}$, or at $\left(1, \frac{10}{3}\right)$, so this is the point where all three medians intersect.

61. (a) Since the x-intercept is a, the point $(a,0)$ is on the line, and similarly since the y-intercept is b, $(0,b)$ is on the line. Hence, the slope of the line is $m = \dfrac{b-0}{0-a} = -\dfrac{b}{a}$. Substituting into $y = mx + b$ gives $y = -\dfrac{b}{a}x + b \;\Leftrightarrow\; \dfrac{b}{a}x + y = b \;\Leftrightarrow\; \dfrac{x}{a} + \dfrac{y}{b} = 1$.

(b) Letting $a = 6$ and $b = -8$ gives $\dfrac{x}{6} + \dfrac{y}{-8} = 1 \;\Leftrightarrow\; -8x + 6y = -48$ [multiply by -48] $\Leftrightarrow$ $6y = 8x - 48 \;\Leftrightarrow\; 3y = 4x - 24 \;\Leftrightarrow\; y = \frac{4}{3}x - 8$.

62. (a) Let d = distance traveled (in miles) and t = time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus, we have two points: $(0,0)$ and $\left(\frac{5}{6}, 40\right)$, so $m = \dfrac{40-0}{5/6-0} = 48$ and $d = 48t$.

(b)



(c) The slope is 48 and represents the car's speed in mi/h.

C Graphs of Second-Degree Equations

1. An equation of the circle with center $(3,-1)$ and radius 5 is $(x-3)^2 + (y+1)^2 = 5^2 = 25$.

2. From (1), the equation is $(x+2)^2 + (y+8)^2 = 100$.

3. The equation has the form $x^2 + y^2 = r^2$. Since $(4,7)$ lies on the circle, we have $4^2 + 7^2 = r^2 \;\Rightarrow\; r^2 = 65$. So the required equation is $x^2 + y^2 = 65$.

4. The equation has the form $(x+1)^2 + (y-5)^2 = r^2$. Since $(-4,-6)$ lies on the circle, we have $r^2 = (-4+1)^2 + (-6-5)^2 = 130$. So an equation is $(x+1)^2 + (y-5)^2 = 130$.

5. $x^2 + y^2 - 4x + 10y + 13 = 0 \;\Leftrightarrow\; x^2 - 4x + y^2 + 10y = -13 \;\Leftrightarrow$ $(x^2 - 4x + 4) + (y^2 + 10y + 25) = -13 + 4 + 25 = 16 \;\Leftrightarrow\; (x-2)^2 + (y+5)^2 = 4^2$. Thus, we have a circle with center $(2,-5)$ and radius 4.

6. $x^2 + y^2 + 6y + 2 = 0 \;\Leftrightarrow\; x^2 + (y^2 + 6y + 9) = -2 + 9 \;\Leftrightarrow\; x^2 + (y+3)^2 = 7$. Thus, we have a circle with center $(0,-3)$ and radius $\sqrt{7}$.

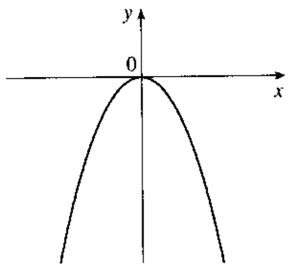
7. $x^2 + y^2 + x = 0 \;\Leftrightarrow\; \left(x^2 + x + \frac{1}{4}\right) + y^2 = \frac{1}{4} \;\Leftrightarrow\; \left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$. Thus, we have a circle with center $\left(-\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$.

8. $16x^2 + 16y^2 + 8x + 32y + 1 = 0 \;\Leftrightarrow\; 16\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + 16\left(y^2 + 2y + 1\right) = -1 + 1 + 16 \;\Leftrightarrow$ $16\left(x + \frac{1}{4}\right)^2 + 16(y+1)^2 = 16 \;\Leftrightarrow\; \left(x + \frac{1}{4}\right)^2 + (y+1)^2 = 1$. Thus, we have a circle with center $\left(-\frac{1}{4}, -1\right)$ and radius 1.

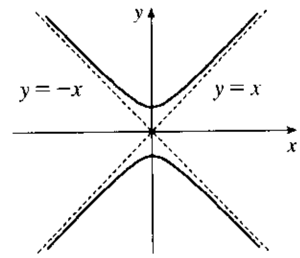
9. $2x^2 + 2y^2 - x + y = 1 \;\Leftrightarrow\; 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + 2\left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) = 1 + \frac{1}{8} + \frac{1}{8} \;\Leftrightarrow$ $2\left(x - \frac{1}{4}\right)^2 + 2\left(y + \frac{1}{4}\right)^2 = \frac{5}{4} \;\Leftrightarrow\; \left(x - \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{5}{8}$. Thus, we have a circle with center $\left(\frac{1}{4}, -\frac{1}{4}\right)$ and radius $\frac{\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{10}}{4}$.

10. $x^2 + y^2 + ax + by + c = 0 \;\Leftrightarrow\; \left(x^2 + ax + \frac{1}{4}a^2\right) + \left(y^2 + by + \frac{1}{4}b^2\right) = -c + \frac{1}{4}a^2 + \frac{1}{4}b^2 \;\Leftrightarrow$ $\left(x + \frac{1}{2}a\right)^2 + \left(y + \frac{1}{2}b\right)^2 = \frac{1}{4}\left(a^2 + b^2 - 4c\right)$. For this to represent a nondegenerate circle, $\frac{1}{4}\left(a^2 + b^2 - 4c\right) > 0$ or $a^2 + b^2 > 4c$. If this condition is satisfied, the circle has center $\left(-\frac{1}{2}a, -\frac{1}{2}b\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2 - 4c}$.

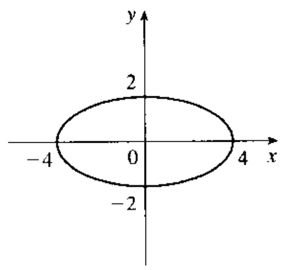
11. $y = -x^2$. Parabola



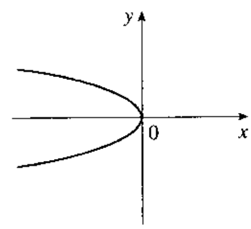
12. $y^2 - x^2 = 1$. Hyperbola



13. $x^2 + 4y^2 = 16 \quad \Leftrightarrow \quad \dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$. Ellipse

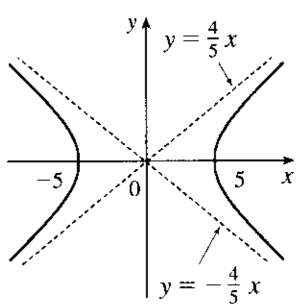


14. $x = -2y^2$. Parabola

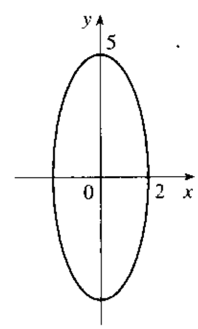


15. $16x^2 - 25y^2 = 400 \quad \Leftrightarrow \quad \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$.

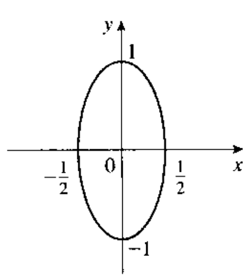
Hyperbola



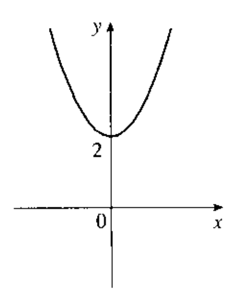
16. $25x^2 + 4y^2 = 100 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$. Ellipse



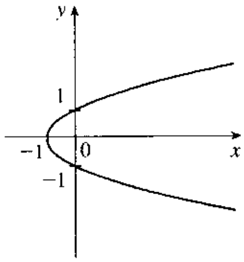
17. $4x^2 + y^2 = 1 \quad \Leftrightarrow \quad \dfrac{x^2}{1/4} + y^2 = 1$. Ellipse



18. $y = x^2 + 2$. Parabola with vertex at $(0, 2)$

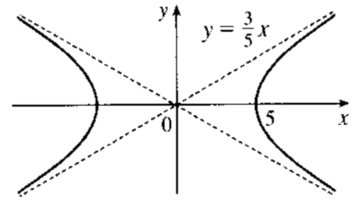


19. $x = y^2 - 1$. Parabola with vertex at $(-1, 0)$

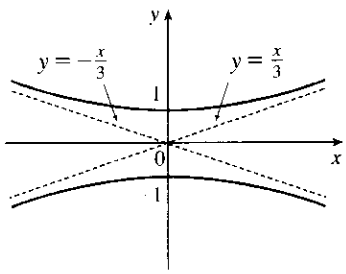


20. $9x^2 - 25y^2 = 225 \quad \Leftrightarrow \quad \dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$.

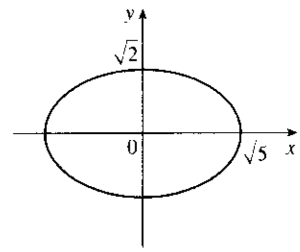
Hyperbola



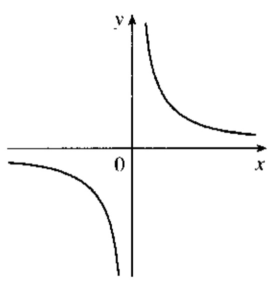
21. $9y^2 - x^2 = 9 \quad \Leftrightarrow \quad y^2 - \dfrac{x^2}{9} = 1$. Hyperbola



22. $2x^2 + 5y^2 = 10 \quad \Leftrightarrow \quad \dfrac{x^2}{5} + \dfrac{y^2}{2} = 1$. Ellipse

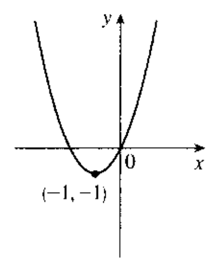


23. $xy = 4$. Hyperbola



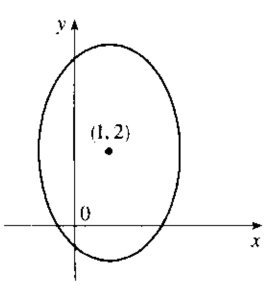
24. $y = x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$.

Parabola with vertex at $(-1, -1)$



25. $9(x - 1)^2 + 4(y - 2)^2 = 36 \quad \Leftrightarrow$

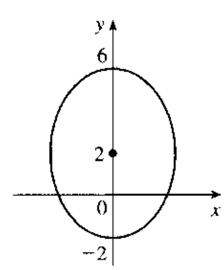
$\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$. Ellipse centered at $(1, 2)$



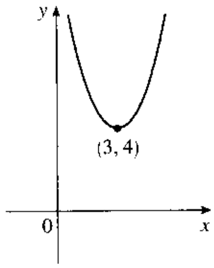
26. $16x^2 + 9y^2 - 36y = 108 \quad \Leftrightarrow$

$16x^2 + 9(y^2 - 4y + 4) = 108 + 36 = 144 \quad \Leftrightarrow$

$\dfrac{x^2}{9} + \dfrac{(y-2)^2}{16} = 1$. Ellipse centered at $(0, 2)$



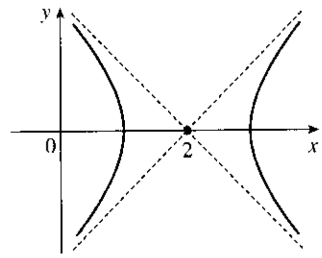
27. $y = x^2 - 6x + 13 = (x^2 - 6x + 9) + 4 = (x-3)^2 + 4$. Parabola with vertex at $(3, 4)$



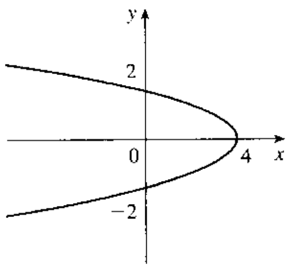
28. $x^2 - y^2 - 4x + 3 = 0 \quad \Leftrightarrow$

$(x^2 - 4x + 4) - y^2 = -3 + 4 = 1 \quad \Leftrightarrow$

$(x-2)^2 - y^2 = 1$. Hyperbola centered at $(2, 0)$



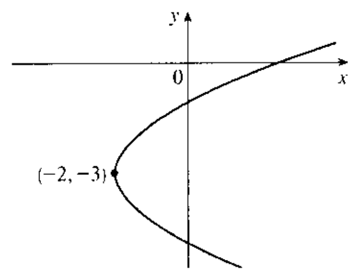
29. $x = -y^2 + 4$. Parabola with vertex at $(4, 0)$



30. $y^2 - 2x + 6y + 5 = 0 \quad \Leftrightarrow$

$y^2 + 6y + 9 = 2x + 4 \quad \Leftrightarrow$

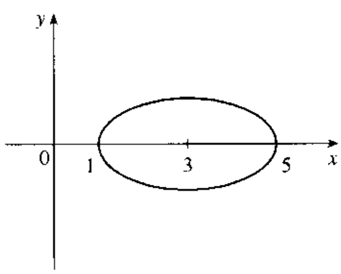
$(y+3)^2 = 2(x+2)$. Parabola with vertex $(-2, -3)$



31. $x^2 + 4y^2 - 6x + 5 = 0 \quad \Leftrightarrow$

$(x^2 - 6x + 9) + 4y^2 = -5 + 9 = 4 \quad \Leftrightarrow$

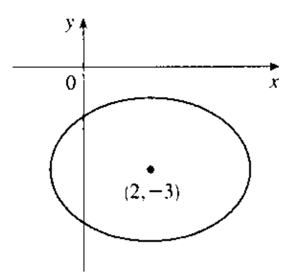
$\dfrac{(x-3)^2}{4} + y^2 = 1$. Ellipse centered at $(3, 0)$



32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0 \quad \Leftrightarrow$

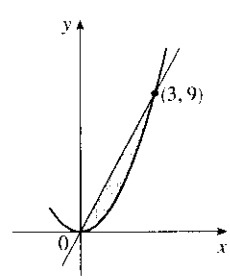
$4(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = -61 + 16 + 81 = 36 \quad \Leftrightarrow$

$\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{4} = 1$. Ellipse centered at $(2, -3)$

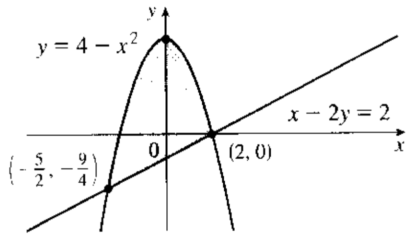


33. $y = 3x$ and $y = x^2$ intersect where $3x = x^2 \quad \Leftrightarrow$

$0 = x^2 - 3x = x(x-3)$, that is, at $(0, 0)$ and $(3, 9)$.



34.

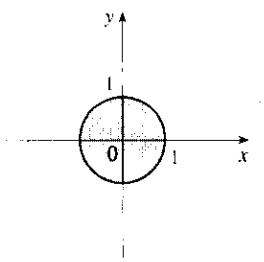


$y = 4 - x^2$, $x - 2y = 2$. Substitute y from the first equation into the second: $x - 2(4 - x^2) = 2 \;\Leftrightarrow\; 2x^2 + x - 10 = 0 \;\Leftrightarrow\; (2x+5)(x-2) = 0 \;\Leftrightarrow\; x = -\frac{5}{2}$ or 2. So the points of intersection are $\left(-\frac{5}{2}, -\frac{9}{4}\right)$ and $(2, 0)$.

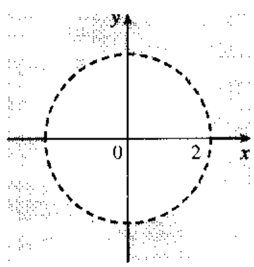
35. The parabola must have an equation of the form $y = a(x-1)^2 - 1$. Substituting $x = 3$ and $y = 3$ into the equation gives $3 = a(3-1)^2 - 1$, so $a = 1$, and the equation is $y = (x-1)^2 - 1 = x^2 - 2x$. Note that using the other point $(-1, 3)$ would have given the same value for a, and hence the same equation.

36. The ellipse has an equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Substituting $x = 1$ and $y = -\frac{10\sqrt{2}}{3}$ gives $\dfrac{1^2}{a^2} + \dfrac{\left(-10\sqrt{2}/3\right)^2}{b^2} = \dfrac{1}{a^2} + \dfrac{200}{9b^2} = 1$. Substituting $x = -2$ and $y = \dfrac{5\sqrt{5}}{3}$ gives $\dfrac{(-2)^2}{a^2} + \dfrac{\left(5\sqrt{5}/3\right)^2}{b^2} = \dfrac{4}{a^2} + \dfrac{125}{9b^2} = 1$. From the first equation, $\dfrac{1}{a^2} = 1 - \dfrac{200}{9b^2}$. Putting this into the second equation gives $4\left(1 - \dfrac{200}{9b^2}\right) + \dfrac{125}{9b^2} = 1 \;\Leftrightarrow\; 3 = \dfrac{675}{9b^2} \;\Leftrightarrow\; b^2 = \dfrac{675}{27} = 25$, so $b = 5$. Hence $\dfrac{1}{a^2} = 1 - \dfrac{200}{9(5)^2} = \dfrac{1}{9}$ and so $a = 3$. The equation of the ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$.

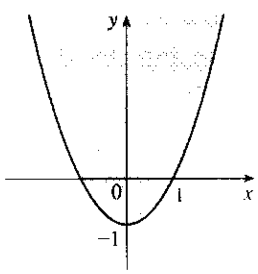
37. $\{(x, y) \mid x^2 + y^2 \le 1\}$



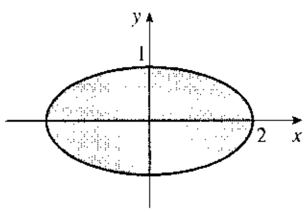
38. $\{(x, y) \mid x^2 + y^2 > 4\}$



39. $\{(x, y) \mid y \ge x^2 - 1\}$



40. $\{(x, y) \mid x^2 + 4y^2 \le 4\}$



D Trigonometry

1. $210° = 210\left(\frac{\pi}{180}\right) = \frac{7\pi}{6}$ rad

2. $300° = 300\left(\frac{\pi}{180}\right) = \frac{5\pi}{3}$ rad

3. $9° = 9\left(\frac{\pi}{180}\right) = \frac{\pi}{20}$ rad

4. $-315° = -315\left(\frac{\pi}{180}\right) = -\frac{7\pi}{4}$ rad

5. $900° = 900\left(\frac{\pi}{180}\right) = 5\pi$ rad

6. $36° = 36\left(\frac{\pi}{180}\right) = \frac{\pi}{5}$ rad

7. 4π rad $= 4\pi\left(\frac{180}{\pi}\right) = 720°$

8. $-\frac{7\pi}{2}$ rad $= -\frac{7\pi}{2}\left(\frac{180}{\pi}\right) = -630°$

9. $\frac{5\pi}{12}$ rad $= \frac{5\pi}{12}\left(\frac{180}{\pi}\right) = 75°$

10. $\frac{8\pi}{3}$ rad $= \frac{8\pi}{3}\left(\frac{180}{\pi}\right) = 480°$

11. $-\frac{3\pi}{8}$ rad $= -\frac{3\pi}{8}\left(\frac{180}{\pi}\right) = -67.5°$

12. 5 rad $= 5\left(\frac{180}{\pi}\right) = \left(\frac{900}{\pi}\right)°$

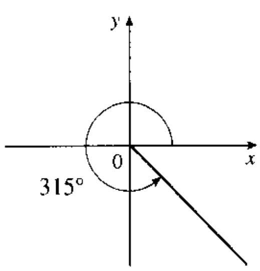
13. Using Formula 3, $a = r\theta = 36 \cdot \frac{\pi}{12} = 3\pi$ cm.

14. Using Formula 3, $a = r\theta = 10 \cdot 72\left(\frac{\pi}{180}\right) = 4\pi$ cm.

15. Using Formula 3, $\theta = a/r = \frac{1}{1.5} = \frac{2}{3}$ rad $= \frac{2}{3}\left(\frac{180}{\pi}\right) = \left(\frac{120}{\pi}\right)° \approx 38.2°$.

16. $a = r\theta \quad \Rightarrow \quad r = \frac{a}{\theta} = \frac{6}{3\pi/4} = \frac{8}{\pi}$ cm

17.



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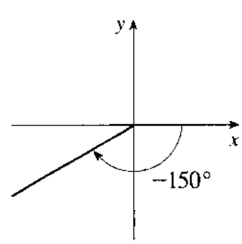

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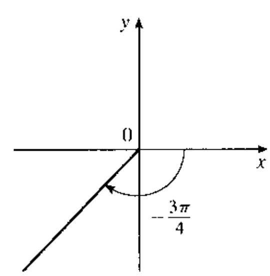

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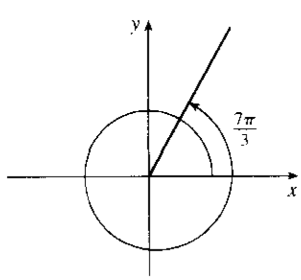

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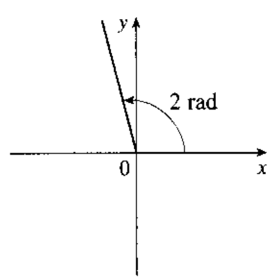

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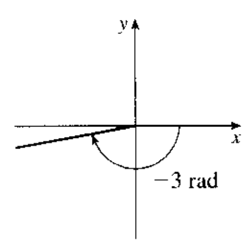

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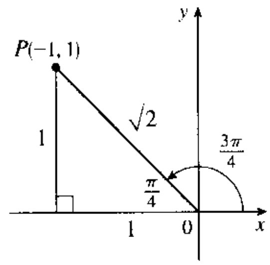
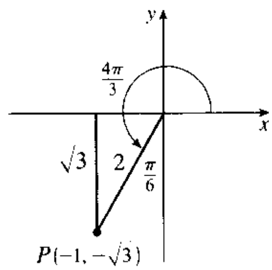


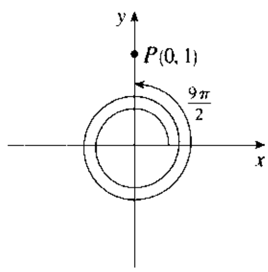
From the diagram we see that a point on the terminal side is $P(-1,1)$. Therefore, taking $x=-1$, $y=1$, $r=\sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}$, $\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}$, $\tan\frac{3\pi}{4}=-1$, $\csc\frac{3\pi}{4}=\sqrt{2}$, $\sec\frac{3\pi}{4}=-\sqrt{2}$, and $\cot\frac{3\pi}{4}=-1$.

24.



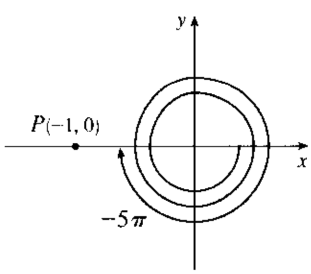
From the diagram and Figure 8, we see that a point on the terminal side is $P(-1,-\sqrt{3})$. Therefore, taking $x=-1$, $y=-\sqrt{3}$, $r=2$ in the definitions of the trigonometric ratios, we have $\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3}=-\frac{1}{2}$, $\tan\frac{4\pi}{3}=\sqrt{3}$, $\csc\frac{4\pi}{3}=-\frac{2}{\sqrt{3}}$, $\sec\frac{4\pi}{3}=-2$, and $\cot\frac{4\pi}{3}=\frac{1}{\sqrt{3}}$.

25.



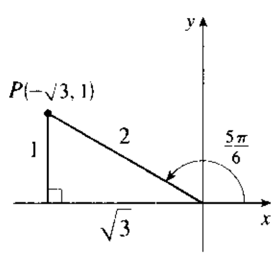
From the diagram we see that a point on the terminal line is $P(0,1)$. Therefore taking $x=0$, $y=1$, $r=1$ in the definitions of the trigonometric ratios, we have $\sin\frac{9\pi}{2}=1$, $\cos\frac{9\pi}{2}=0$, $\tan\frac{9\pi}{2}=y/x$ is undefined since $x=0$, $\csc\frac{9\pi}{2}=1$, $\sec\frac{9\pi}{2}=r/x$ is undefined since $x=0$, and $\cot\frac{9\pi}{2}=0$.

26.



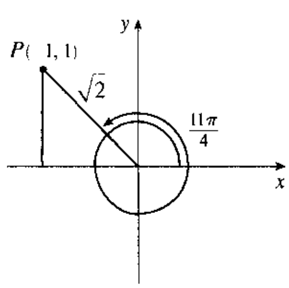
From the diagram, we see that a point on the terminal line is $P(-1,0)$. Therefore taking $x=-1$, $y=0$, $r=1$ in the definitions of the trigonometric ratios we have $\sin(-5\pi)=0$, $\cos(-5\pi)=-1$, $\tan(-5\pi)=0$, $\csc(-5\pi)$ is undefined, $\sec(-5\pi)=-1$, and $\cot(-5\pi)$ is undefined.

27.



Using Figure 8 we see that a point on the terminal line is $P(-\sqrt{3},1)$. Therefore taking $x=-\sqrt{3}$, $y=1$, $r=2$ in the definitions of the trigonometric ratios, we have $\sin\frac{5\pi}{6}=\frac{1}{2}$, $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$, $\tan\frac{5\pi}{6}=-\frac{1}{\sqrt{3}}$, $\csc\frac{5\pi}{6}=2$, $\sec\frac{5\pi}{6}=-\frac{2}{\sqrt{3}}$, and $\cot\frac{5\pi}{6}=-\sqrt{3}$.

28.



From the diagram, we see that a point on the terminal line is $P(-1,1)$. Therefore taking $x=-1$, $y=1$, $r=\sqrt{2}$ in the definitions of the trigonometric ratios we have $\sin\frac{11\pi}{4}=\frac{1}{\sqrt{2}}$, $\cos\frac{11\pi}{4}=-\frac{1}{\sqrt{2}}$, $\tan\frac{11\pi}{4}=-1$, $\csc\frac{11\pi}{4}=\sqrt{2}$, $\sec\frac{11\pi}{4}=-\sqrt{2}$, and $\cot\frac{11\pi}{4}=-1$.

29. $\sin\theta = y/r = \frac{3}{5} \Rightarrow y = 3, r = 5$, and $x = \sqrt{r^2 - y^2} = 4$ (since $0 < \theta < \frac{\pi}{2}$). Therefore taking $x = 4$, $y = 3$, $r = 5$ in the definitions of the trigonometric ratios, we have $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, and $\cot\theta = \frac{4}{3}$.

30. Since $0 < \alpha < \frac{\pi}{2}$, α is in the first quadrant where x and y are both positive. Therefore, $\tan\alpha = y/x = \frac{2}{1} \Rightarrow y = 2$, $x = 1$, and $r = \sqrt{x^2 + y^2} = \sqrt{5}$. Taking $x = 1$, $y = 2$, $r = \sqrt{5}$ in the definitions of the trigonometric ratios, we have $\sin\alpha = \frac{2}{\sqrt{5}}$, $\cos\alpha = \frac{1}{\sqrt{5}}$, $\csc\alpha = \frac{\sqrt{5}}{2}$, $\sec\alpha = \sqrt{5}$, and $\cot\alpha = \frac{1}{2}$.

31. $\frac{\pi}{2} < \phi < \pi \Rightarrow \phi$ is in the second quadrant, where x is negative and y is positive. Therefore $\sec\phi = r/x = -1.5 = -\frac{3}{2} \Rightarrow r = 3$, $x = -2$, and $y = \sqrt{r^2 - x^2} = \sqrt{5}$. Taking $x = -2$, $y = \sqrt{5}$, and $r = 3$ in the definitions of the trigonometric ratios, we have $\sin\phi = \frac{\sqrt{5}}{3}$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\frac{\sqrt{5}}{2}$, $\csc\phi = \frac{3}{\sqrt{5}}$, and $\cot\theta = -\frac{2}{\sqrt{5}}$.

32. Since $\pi < x < \frac{3\pi}{2}$, x is in the third quadrant where x and y are both negative. Therefore $\cos x = x/r = -\frac{1}{3} \Rightarrow x = -1$, $r = 3$, and $y = -\sqrt{r^2 - x^2} = -\sqrt{8} = -2\sqrt{2}$. Taking $x = -1$, $r = 3$, $y = -2\sqrt{2}$ in the definitions of the trigonometric ratios, we have $\sin x = -\frac{2\sqrt{2}}{3}$, $\tan x = 2\sqrt{2}$, $\csc x = -\frac{3}{2\sqrt{2}}$, $\sec x = -3$, and $\cot x = \frac{1}{2\sqrt{2}}$.

33. $\pi < \beta < 2\pi$ means that β is in the third or fourth quadrant where y is negative. Also since $\cot\beta = x/y = 3$ which is positive, x must also be negative. Therefore $\cot\beta = x/y = \frac{3}{1} \Rightarrow x = -3$, $y = -1$, and $r = \sqrt{x^2 + y^2} = \sqrt{10}$. Taking $x = -3$, $y = -1$ and $r = \sqrt{10}$ in the definitions of the trigonometric ratios, we have $\sin\beta = -\frac{1}{\sqrt{10}}$, $\cos\beta = -\frac{3}{\sqrt{10}}$, $\tan\beta = \frac{1}{3}$, $\csc\beta = -\sqrt{10}$, and $\sec\beta = -\frac{\sqrt{10}}{3}$.

34. Since $\frac{3\pi}{2} < \theta < 2\pi$, θ is in the fourth quadrant where x is positive and y is negative. Therefore $\csc\theta = r/y = -\frac{4}{3} \Rightarrow r = 4$, $y = -3$, and $x = \sqrt{r^2 - y^2} = \sqrt{7}$. Taking $x = \sqrt{7}$, $y = -3$, and $r = 4$ in the definitions of the trigonometric ratios, we have $\sin\theta = -\frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{3}{\sqrt{7}}$, $\sec\theta = \frac{4}{\sqrt{7}}$, and $\cot\theta = -\frac{\sqrt{7}}{3}$.

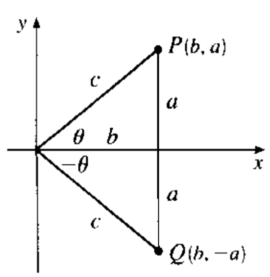
35. $\sin 35^\circ = \dfrac{x}{10} \Rightarrow x = 10\sin 35^\circ \approx 5.73576$ cm

36. $\cos 40^\circ = \dfrac{x}{25} \Rightarrow x = 25\cos 40^\circ \approx 19.15111$ cm

37. $\tan\frac{2\pi}{5} = \dfrac{x}{8} \Rightarrow x = 8\tan\frac{2\pi}{5} \approx 24.62147$ cm

38. $\cos\frac{3\pi}{8} = \dfrac{22}{x} \Rightarrow x = \dfrac{22}{\cos\frac{3\pi}{8}} \approx 57.48877$ cm

39.



(a) From the diagram we see that $\sin\theta = \dfrac{y}{r} = \dfrac{a}{c}$, and

$$\sin(-\theta) = \frac{-a}{c} = -\frac{a}{c} = -\sin\theta.$$

(b) Again from the diagram we see that $\cos\theta = \dfrac{x}{r} = \dfrac{b}{c} = \cos(-\theta)$.

40. (a) Using (12a) and (12b), we have

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\dfrac{\sin x\cos y}{\cos x\cos y} + \dfrac{\cos x \sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y} - \dfrac{\sin x\sin y}{\cos x\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

(b) From (10a) and (10b), we have $\tan(-\theta) = -\tan\theta$, so (14a) implies that

$$\tan(x-y) = \tan(x+(-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

41. (a) Using (12a) and (13a), we have

$$\begin{aligned} \tfrac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] &= \tfrac{1}{2}\left[\sin x\cos y + \cos x \sin y + \sin x \cos y - \cos x \sin y\right] \\ &= \tfrac{1}{2}(2\sin x \cos y) = \sin x \cos y \end{aligned}$$

(b) This time, using (12b) and (13b), we have

$$\begin{aligned} \tfrac{1}{2}\left[\cos(x+y) + \cos(x-y)\right] &= \tfrac{1}{2}\left[\cos x\cos y - \sin x \sin y + \cos x \cos y + \sin x \sin y\right] \\ &= \tfrac{1}{2}(2\cos x \cos y) = \cos x \cos y \end{aligned}$$

(c) Again using (12b) and (13b), we have

$$\begin{aligned} \tfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] &= \tfrac{1}{2}\left[\cos x\cos y + \sin x \sin y - \cos x \cos y + \sin x \sin y\right] \\ &= \tfrac{1}{2}(2\sin x \sin y) = \sin x \sin y \end{aligned}$$

42. Using (13b), $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = 0\cdot\cos x + 1\cdot\sin x = \sin x$.

43. Using (12a), we have $\sin\left(\frac{\pi}{2} + x\right) = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x = 1\cdot\cos x + 0\cdot\sin x = \cos x$.

44. Using (13a), we have $\sin(\pi - x) = \sin\pi\cos x - \cos\pi\,\sin x = 0\cdot\cos x - (-1)\sin x = \sin x$.

45. Using (6), we have $\sin\theta\cot\theta = \sin\theta\cdot\dfrac{\cos\theta}{\sin\theta} = \cos\theta$.

46. $$\begin{aligned}(\sin x + \cos x)^2 &= \sin^2 x + 2\sin x\cos x + \cos^2 x = \left(\sin^2 x + \cos^2 x\right) + \sin 2x \quad \text{[by (15a)]} \\ &= 1 + \sin 2x \quad \text{[by (7)]}\end{aligned}$$

47. $\sec y - \cos y = \dfrac{1}{\cos y} - \cos y$ [by (6)] $= \dfrac{1-\cos^2 y}{\cos y} = \dfrac{\sin^2 y}{\cos y}$ [by (7)] $= \dfrac{\sin y}{\cos y}\sin y = \tan y \sin y$ [by (6)]

48. $\tan^2\alpha - \sin^2\alpha = \dfrac{\sin^2\alpha}{\cos^2\alpha} - \sin^2\alpha = \dfrac{\sin^2\alpha - \sin^2\alpha\cos^2\alpha}{\cos^2\alpha} = \dfrac{\sin^2\alpha\left(1-\cos^2\alpha\right)}{\cos^2\alpha} = \tan^2\alpha\sin^2\alpha$ [by (6), (7)]

49. $$\begin{aligned}\cot^2\theta + \sec^2\theta &= \frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\cos^2\theta} \quad \text{[by (6)]} = \frac{\cos^2\theta\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} \\ &= \frac{\left(1-\sin^2\theta\right)\left(1-\sin^2\theta\right) + \sin^2\theta}{\sin^2\theta\cos^2\theta} \quad \text{[by (7)]} = \frac{1 - \sin^2\theta + \sin^4\theta}{\sin^2\theta\cos^2\theta} \\ &= \frac{\cos^2\theta + \sin^4\theta}{\sin^2\theta\cos^2\theta} \quad \text{[by (7)]} = \frac{1}{\sin^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \csc^2\theta + \tan^2\theta \quad \text{[by (6)]}\end{aligned}$$

50. $2\csc 2t = \dfrac{2}{\sin 2t} = \dfrac{2}{2\sin t\cos t}$ [by (15a)] $= \dfrac{1}{\sin t\cos t} = \sec t\csc t$

51. Using (14a), we have $\tan 2\theta = \tan(\theta + \theta) = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}$.

52. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = \dfrac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = \dfrac{2}{1-\sin^2\theta} = \dfrac{2}{\cos^2\theta}$ [by (7)] $= 2\sec^2\theta$

53. Using (15a) and (16a),

$$\begin{aligned}\sin x\sin 2x + \cos x\cos 2x &= \sin x\,(2\sin x\cos x) + \cos x\left(2\cos^2 x - 1\right) = 2\sin^2 x\cos x + 2\cos^3 x - \cos x\\ &= 2\left(1-\cos^2 x\right)\cos x + 2\cos^3 x - \cos x \quad \text{[by (7)]}\\ &= 2\cos x - 2\cos^3 x + 2\cos^3 x - \cos x = \cos x\end{aligned}$$

Or: $\sin x\sin 2x + \cos x\cos 2x = \cos(2x - x)$ [by 13(b)] $= \cos x$

54. Working backward, we start with equations (12a) and (13a):

$$\begin{aligned}\sin(x+y)\sin(x-y) &= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)\\ &= \sin^2 x\cos^2 y - \sin x\cos y\cos x\sin y + \cos x\sin y\sin x\cos y - \cos^2 x\sin^2 y\\ &= \sin^2 x\left(1-\sin^2 y\right) - \left(1-\sin^2 x\right)\sin^2 y \quad \text{[by (7)]}\\ &= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y = \sin^2 x - \sin^2 y\end{aligned}$$

55. $\dfrac{\sin\phi}{1-\cos\phi} = \dfrac{\sin\phi}{1-\cos\phi}\cdot\dfrac{1+\cos\phi}{1+\cos\phi} = \dfrac{\sin\phi\,(1+\cos\phi)}{1-\cos^2\phi} = \dfrac{\sin\phi\,(1+\cos\phi)}{\sin^2\phi}$ [by (7)]

$= \dfrac{1+\cos\phi}{\sin\phi} = \dfrac{1}{\sin\phi} + \dfrac{\cos\phi}{\sin\phi} = \csc\phi + \cot\phi$ [by (6)]

56. $\tan x + \tan y = \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y} = \dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y} = \dfrac{\sin(x+y)}{\cos x\cos y}$ [by (12a)]

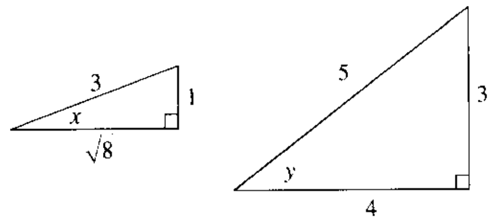
57. Using (12a),

$$\begin{aligned}\sin 3\theta + \sin\theta &= \sin(2\theta+\theta) + \sin\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta + \sin\theta\\ &= \sin 2\theta\cos\theta + \left(2\cos^2\theta - 1\right)\sin\theta + \sin\theta \quad \text{[by (16a)]}\\ &= \sin 2\theta\cos\theta + 2\cos^2\theta\sin\theta - \sin\theta + \sin\theta = \sin 2\theta\cos\theta + \sin 2\theta\cos\theta \quad \text{[by (15a)]}\\ &= 2\sin 2\theta\cos\theta\end{aligned}$$

58. We use (12b) with $x = 2\theta$, $y = \theta$ to get

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &= \left(2\cos^2\theta - 1\right)\cos\theta - 2\sin^2\theta\cos\theta \quad \text{[by (16a) and (15a)]}\\ &= \left(2\cos^2\theta - 1\right)\cos\theta - 2\left(1-\cos^2\theta\right)\cos\theta \quad \text{[by (7)]}\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = 4\cos^3\theta - 3\cos\theta\end{aligned}$$

59. Since $\sin x = \frac{1}{3}$ we can label the opposite side as having length 1, the hypotenuse as having length 3, and use the Pythagorean Theorem to get that the adjacent side has length $\sqrt{8}$. Then, from the diagram, $\cos x = \frac{\sqrt{8}}{3}$. Similarly we have that $\sin y = \frac{3}{5}$. Now use (12a):



$\sin(x+y) = \sin x \cos y + \cos x \sin y = \frac{1}{3} \cdot \frac{4}{5} + \frac{\sqrt{8}}{3} \cdot \frac{3}{5} = \frac{4}{15} + \frac{3\sqrt{8}}{15} = \frac{4+6\sqrt{2}}{15}$.

60. Use (12b) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59 to get

$$\cos(x+y) = \cos x \cos y - \sin x \sin y = \frac{\sqrt{8}}{3} \cdot \frac{4}{5} - \frac{1}{3} \cdot \frac{3}{5} = \frac{8\sqrt{2}-3}{15}$$

61. Using (13b) and the values for $\cos x$ and $\sin y$ obtained in Exercise 59, we have

$$\cos(x-y) = \cos x \cos y + \sin x \sin y = \frac{\sqrt{8}}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{3}{5} = \frac{8\sqrt{2}+3}{15}$$

62. Using (13a) and the values for $\sin y$ and $\cos x$ obtained in Exercise 59, we get

$$\sin(x-y) = \sin x \cos y - \cos x \sin y = \frac{1}{3} \cdot \frac{4}{5} - \frac{\sqrt{8}}{3} \cdot \frac{3}{5} = \frac{4-6\sqrt{2}}{15}$$

63. Using (15a) and the values for $\sin y$ and $\cos y$ obtained in Exercise 59, we have

$$\sin 2y = 2\sin y \cos y = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$$

64. Using (16a) with $\cos y = \frac{4}{5}$, we have $\cos 2y = 2\cos^2 y - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$.

65. $2\cos x - 1 = 0 \quad \Leftrightarrow \quad \cos x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$ for $x \in [0, 2\pi]$.

66. $3\cot^2 x = 1 \quad \Leftrightarrow \quad 3 = 1/\cot^2 x \quad \Leftrightarrow \quad \tan^2 x = 3 \quad \Leftrightarrow \quad \tan x = \pm\sqrt{3} \quad \Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

67. $2\sin^2 x = 1 \quad \Leftrightarrow \quad \sin^2 x = \frac{1}{2} \quad \Leftrightarrow \quad \sin x = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

68. $|\tan x| = 1 \quad \Leftrightarrow \quad \tan x = -1$ or $\tan x = 1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$ or $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

69. Using (15a), we have $\sin 2x = \cos x \quad \Leftrightarrow \quad 2\sin x \cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x(2\sin x - 1) = 0 \quad \Leftrightarrow$ $\cos x = 0$ or $2\sin x - 1 = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Therefore, the solutions are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

70. By (15a), $2\cos x + \sin 2x = 0 \quad \Leftrightarrow \quad 2\cos x + 2\sin x \cos x = 0 \quad \Leftrightarrow \quad 2\cos x\,(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $1 + \sin x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = -1 \quad \Rightarrow \quad x = \frac{3}{2}\pi$. So the solutions are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

71. $\sin x = \tan x \quad \Leftrightarrow \quad \sin x - \tan x = 0 \quad \Leftrightarrow \quad \sin x - \dfrac{\sin x}{\cos x} = 0 \quad \Leftrightarrow \quad \sin x\left(1 - \dfrac{1}{\cos x}\right) = 0 \quad \Leftrightarrow \quad \sin x = 0$ or $1 - \dfrac{1}{\cos x} = 0 \quad \Rightarrow \quad x = 0, \pi, 2\pi$ or $1 = \dfrac{1}{\cos x} \quad \Rightarrow \quad \cos x = 1 \quad \Rightarrow \quad x = 0, 2\pi$. Therefore the solutions are $x = 0, \pi, 2\pi$.

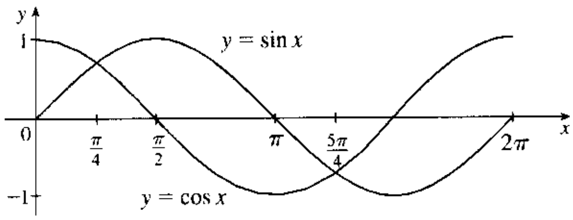
72. By (16a), $2 + \cos 2x = 3\cos x \quad \Leftrightarrow \quad 2 + 2\cos^2 x - 1 = 3\cos x \quad \Leftrightarrow \quad 2\cos^2 x - 3\cos x + 1 = 0 \quad \Leftrightarrow$ $(2\cos x - 1)(\cos x - 1) = 0 \quad \Leftrightarrow \quad \cos x = 1$ or $\cos x = \frac{1}{2} \quad \Rightarrow \quad x = 0, 2\pi$ or $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

73. We know that $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, and from Figure 13(a), we see that $\sin x \le \frac{1}{2} \Rightarrow 0 \le x \le \frac{\pi}{6}$ or $\frac{5\pi}{6} \le x \le 2\pi$ for $x \in [0, 2\pi]$.

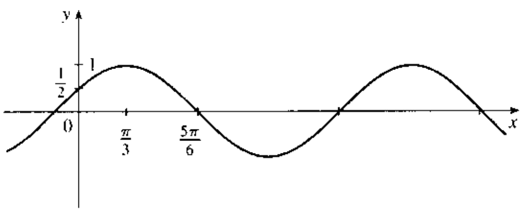
74. $2\cos x + 1 > 0 \Rightarrow 2\cos x > -1 \Rightarrow \cos x > -\frac{1}{2}$. $\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ and from Figure 13(b), we see that $\cos x > -\frac{1}{2}$ when $0 \le x < \frac{2\pi}{3}, \frac{4\pi}{3} < x \le 2\pi$.

75. $\tan x = -1$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, and $\tan x = 1$ when $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. From Figure 14 we see that $-1 < \tan x < 1 \Rightarrow 0 \le x < \frac{\pi}{4}, \frac{3\pi}{4} < x < \frac{5\pi}{4}$, and $\frac{7\pi}{4} < x \le 2\pi$.

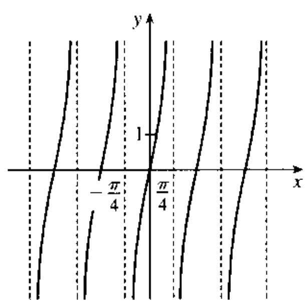
76. We know that $\sin x = \cos x$ when $x = \frac{\pi}{4}, \frac{5\pi}{4}$, and from the diagram we see that $\sin x > \cos x$ when $\frac{\pi}{4} < x < \frac{5\pi}{4}$.



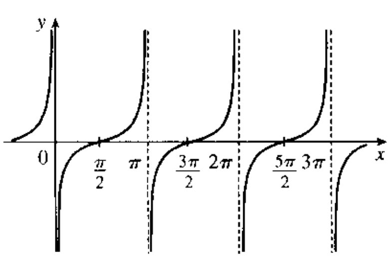
77. $y = \cos\left(x - \frac{\pi}{3}\right)$. We start with the graph of $y = \cos x$ and shift it $\frac{\pi}{3}$ units to the right.



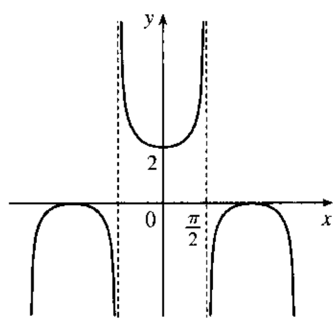
78. $y = \tan 2x$. Start with the graph of $y = \tan x$ with period π and compress it to a period of $\frac{\pi}{2}$.



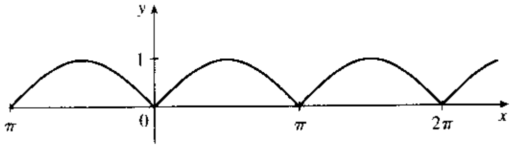
79. $y = \frac{1}{3}\tan\left(x - \frac{\pi}{2}\right)$. We start with the graph of $y = \tan x$, shift it $\frac{\pi}{2}$ units to the right and compress it to $\frac{1}{3}$ of its original vertical size.



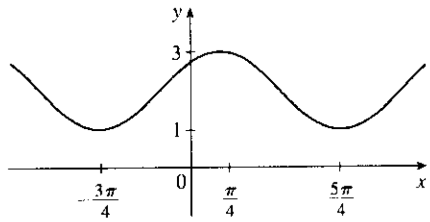
80. $y = 1 + \sec x$. Start with the graph of $y = \sec x$ and raise it by one unit.



81. $y = |\sin x|$. We start with the graph of $y = \sin x$ and reflect the parts below the x-axis about the x-axis.



82. $y = 2 + \sin\left(x + \frac{\pi}{4}\right)$. Start with the graph of $y = \sin x$, and shift it $\frac{\pi}{4}$ units to the left and 2 units up.



83. From the figure in the text, we see that $x = b\cos\theta$, $y = b\sin\theta$, and from the distance formula we have that the distance c from (x, y) to $(a, 0)$ is $c = \sqrt{(x-a)^2 + (y-0)^2} \quad \Rightarrow$

$$\begin{aligned} c^2 &= (b\cos\theta - a)^2 + (b\sin\theta)^2 = b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta \\ &= a^2 + b^2\left(\cos^2\theta + \sin^2\theta\right) - 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta \quad \text{[by (7)]} \end{aligned}$$

84. $|AB|^2 = |AC|^2 + |BC|^2 - 2\,|AC|\,|BC|\cos\angle C = (820)^2 + (910)^2 - 2(820)(910)\cos 103^\circ$

$\approx 1{,}836{,}217 \quad \Rightarrow \quad |AB| \approx 1355$ m

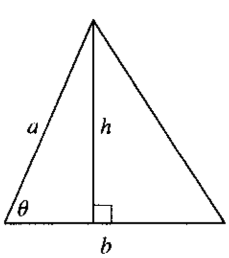
85. Using the Law of Cosines, we have $c^2 = 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta) = 2\,[1 - \cos(\alpha - \beta)]$. Now, using the distance formula, $c^2 = |AB|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$. Equating these two expressions for c^2, we get $2\,[1 - \cos(\alpha - \beta)] = \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \quad \Rightarrow$
$1 - \cos(\alpha - \beta) = 1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \Rightarrow \quad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

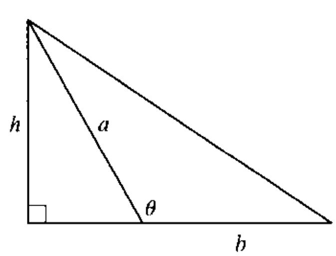
86. $\cos(x + y) = \cos(x - (-y)) = \cos x\cos(-y) + \sin x\sin(-y)$

$= \cos x\cos y - \sin x\sin y$ [using Equations 10a and 10b]

87. In Exercise 86 we used the subtraction formula for cosine to prove the addition formula for cosine. Using that formula with $x = \frac{\pi}{2} - \alpha$, $y = \beta$, we get $\cos\left[\left(\frac{\pi}{2} - \alpha\right) + \beta\right] = \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \quad \Rightarrow$
$\cos\left[\frac{\pi}{2} - (\alpha - \beta)\right] = \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$. Now we use the identities given in the problem, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, to get $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

88. If $0 < \theta < \frac{\pi}{2}$, we have the case depicted in the first diagram. In this case, we see that the height of the triangle is $h = a\sin\theta$. If $\frac{\pi}{2} \le \theta < \pi$, we have the case depicted in the second diagram. In this case, the height of the triangle is $h = a\sin(\pi - \theta) = a\sin\theta$ (by the identity proved in Exercise 78). So in either case, the area of the triangle is $\frac{1}{2}bh = \frac{1}{2}ab\sin\theta$.





89. Using the formula from part (a), the area of the triangle is $\frac{1}{2}(10)(3)\sin 107^\circ \approx 14.34457$ cm^2.

E Sigma Notation

1. $\sum_{i=1}^{5} \sqrt{i} = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$

2. $\sum_{i=1}^{6} \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

3. $\sum_{i=4}^{6} 3^i = 3^4 + 3^5 + 3^6$

4. $\sum_{i=4}^{6} i^3 = 4^3 + 5^3 + 6^3$

5. $\sum_{k=0}^{4} \frac{2k-1}{2k+1} = -1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$

6. $\sum_{k=5}^{8} x^k = x^5 + x^6 + x^7 + x^8$

7. $\sum_{i=1}^{n} i^{10} = 1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

8. $\sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$

9. $\sum_{j=0}^{n-1} (-1)^j = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$

10. $\sum_{i=1}^{n} f(x_i)\,\Delta x_i = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + f(x_3)\,\Delta x_3 + \cdots + f(x_n)\,\Delta x_n$

11. $1 + 2 + 3 + 4 + \cdots + 10 = \sum_{i=1}^{10} i$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} = \sum_{i=3}^{7} \sqrt{i}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20} = \sum_{i=1}^{19} \frac{i}{i+1}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27} = \sum_{i=3}^{23} \frac{i}{i+4}$

15. $2 + 4 + 6 + 8 + \cdots + 2n = \sum_{i=1}^{n} 2i$

16. $1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^{n} (2i-1)$

17. $1 + 2 + 4 + 8 + 16 + 32 = \sum_{i=0}^{5} 2^i$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} = \sum_{i=1}^{6} \frac{1}{i^2}$

19. $x + x^2 + x^3 + \cdots + x^n = \sum_{i=1}^{n} x^i$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{i=0}^{n} (-1)^i x^i$

21. $\sum_{i=4}^{8} (3i-2) = [3(4)-2] + [3(5)-2] + [3(6)-2] + [3(7)-2] + [3(8)-2] = 10 + 13 + 16 + 19 + 22 = 80$

22. $\sum_{i=3}^{6} i(i+2) = 3 \cdot 5 + 4 \cdot 6 + 5 \cdot 7 + 6 \cdot 8 = 15 + 24 + 35 + 48 = 122$

23. $\sum_{j=1}^{6} 3^{j+1} = 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 = 9 + 27 + 81 + 243 + 729 + 2187 = 3276$

(For a more general method, see Exercise 47.)

24. $\sum_{k=0}^{8} \cos k\pi = \cos 0 + \cos \pi + \cos 2\pi + \cos 3\pi + \cos 4\pi + \cos 5\pi + \cos 6\pi + \cos 7\pi + \cos 8\pi$

$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 1$

25. $\sum_{n=1}^{20} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 = 0$

26. $\displaystyle\sum_{i=1}^{100} 4 = \underbrace{4+4+4+\cdots+4}_{(100\text{ summands})} = 100\cdot 4 = 400$

27. $\displaystyle\sum_{i=0}^{4}\left(2^i+i^2\right) = (1+0)+(2+1)+(4+4)+(8+9)+(16+16) = 61$

28. $\displaystyle\sum_{i=-2}^{4} 2^{3-i} = 2^5+2^4+2^3+2^2+2^1+2^0+2^{-1} = 63.5$

29. $\displaystyle\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = 2\cdot\frac{n(n+1)}{2}$ [by Theorem 3(c)] $= n(n+1)$

30. $\displaystyle\sum_{i=1}^{n}(2-5i) = \sum_{i=1}^{n}2 - \sum_{i=1}^{n}5i = 2n - 5\sum_{i=1}^{n} i = 2n - \frac{5n(n+1)}{2} = \frac{4n}{2} - \frac{5n^2+5n}{2} = -\frac{n(5n+1)}{2}$

31. $\displaystyle\sum_{i=1}^{n}\left(i^2+3i+4\right) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 4 = \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 4n$

$$= \tfrac{1}{6}\left[\left(2n^3+3n^2+n\right)+\left(9n^2+9n\right)+24n\right] = \tfrac{1}{6}\left(2n^3+12n^2+34n\right)$$

$$= \tfrac{1}{3}n\left(n^2+6n+17\right)$$

32. $\displaystyle\sum_{i=1}^{n}(3+2i)^2 = \sum_{i=1}^{n}\left(9+12i+4i^2\right) = \sum_{i=1}^{n}9 + 12\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2$

$$= 9n + 6n(n+1) + \frac{2n(n+1)(2n+1)}{3} = \frac{27n+18n^2+18n+4n^3+6n^2+2n}{3}$$

$$= \tfrac{1}{3}\left(4n^3+24n^2+47n\right) = \tfrac{1}{3}n\left(4n^2+24n+47\right)$$

33. $\displaystyle\sum_{i=1}^{n}(i+1)(i+2) = \sum_{i=1}^{n}\left(i^2+3i+2\right) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n = \frac{n(n+1)}{6}\left[(2n+1)+9\right] + 2n$$

$$= \frac{n(n+1)}{3}(n+5) + 2n = \frac{n}{3}\left[(n+1)(n+5)+6\right] = \frac{n}{3}\left(n^2+6n+11\right)$$

34. $\displaystyle\sum_{i=1}^{n} i(i+1)(i+2) = \sum_{i=1}^{n}\left(i^3+3i^2+2i\right) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right] = \frac{n(n+1)}{4}\left(n^2+n+4n+2+4\right)$$

$$= \frac{n(n+1)}{4}\left(n^2+5n+6\right) = \frac{n(n+1)(n+2)(n+3)}{4}$$

35. $\displaystyle\sum_{i=1}^{n}\left(i^3-i-2\right) = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 = \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)}{2} - 2n$

$$= \tfrac{1}{4}n(n+1)\left[n(n+1)-2\right] - 2n = \tfrac{1}{4}n(n+1)(n+2)(n-1) - 2n$$

$$= \tfrac{1}{4}n\left[(n+1)(n-1)(n+2)-8\right] = \tfrac{1}{4}n\left[\left(n^2-1\right)(n+2)-8\right] = \tfrac{1}{4}n\left(n^3+2n^2-n-10\right)$$

36. By Theorem 3(c) we have that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 78 \quad \Leftrightarrow \quad n(n+1) = 156 \quad \Leftrightarrow \quad n^2 + n - 156 = 0 \quad \Leftrightarrow$ $(n+13)(n-12) = 0 \quad \Leftrightarrow \quad n = 12$ or -13. But $n = -13$ produces a negative answer for the sum, so $n = 12$.

37. By Theorem 2(a) and Example 3, $\sum_{i=1}^{n} c = c \sum_{i=1}^{n} 1 = cn$.

38. Let S_n be the statement that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

1. S_1 is true because $1^3 = \left(\frac{1 \cdot 2}{2}\right)^2$.

2. Assume S_k is true. Then $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$, so

$$\sum_{i=1}^{k+1} i^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 = \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}(k+2)^2$$
$$= \left(\frac{(k+1)\left[(k+1)+1\right]}{2}\right)^2$$

showing that S_{k+1} is true.

Therefore, S_n is true for all n by mathematical induction.

39. $$\sum_{i=1}^{n}\left[(i+1)^4 - i^4\right] = (2^4 - 1^4) + (3^4 - 2^4) + (4^4 - 3^4) + \cdots + \left[(n+1)^4 - n^4\right]$$
$$= (n+1)^4 - 1^4 = n^4 + 4n^3 + 6n^2 + 4n$$

On the other hand,

$$\sum_{i=1}^{n}\left[(i+1)^4 - i^4\right] = \sum_{i=1}^{n}(4i^3 + 6i^2 + 4i + 1) = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 4S + n(n+1)(2n+1) + 2n(n+1) + n \quad \left[\text{where } S = \sum_{i=1}^{n} i^3\right]$$
$$= 4S + 2n^3 + 3n^2 + n + 2n^2 + 2n + n = 4S + 2n^3 + 5n^2 + 4n$$

Thus, $n^4 + 4n^3 + 6n^2 + 4n = 4S + 2n^3 + 5n^2 + 4n$, from which it follows that $4S = n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$ and $S = \left[\frac{n(n+1)}{2}\right]^2$.

40. The area of G_i is

$$\left(\sum_{k=1}^{i} k\right)^2 - \left(\sum_{k=1}^{i-1} k\right)^2 = \left[\frac{i(i+1)}{2}\right]^2 - \left[\frac{(i-1)i}{2}\right]^2 = \frac{i^2}{4}\left[(i+1)^2 - (i-1)^2\right]$$
$$= \frac{i^2}{4}\left[(i^2 + 2i + 1) - (i^2 - 2i + 1)\right] = \frac{i^2}{4}(4i) = i^3$$

Thus, the area of $ABCD$ is $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

41. (a) $\sum_{i=1}^{n}\left[i^4-(i-1)^4\right]=(1^4-0^4)+(2^4-1^4)+(3^4-2^4)+\cdots+\left[n^4-(n-1)^4\right]=n^4-0=n^4$

(b) $\sum_{i=1}^{100}\left(5^i-5^{i-1}\right)=(5^1-5^0)+(5^2-5^1)+(5^3-5^2)+\cdots+(5^{100}-5^{99})=5^{100}-5^0=5^{100}-1$

(c) $\sum_{i=3}^{99}\left(\frac{1}{i}-\frac{1}{i+1}\right)=\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)=\frac{1}{3}-\frac{1}{100}=\frac{97}{300}$

(d) $\sum_{i=1}^{n}(a_i-a_{i-1})=(a_1-a_0)+(a_2-a_1)+(a_3-a_2)+\cdots+(a_n-a_{n-1})=a_n-a_0$

42. Summing the inequalities $-|a_i|\le a_i\le |a_i|$ for $i=1,2,\ldots,n$, we get $-\sum_{i=1}^{n}|a_i|\le\sum_{i=1}^{n}a_i\le\sum_{i=1}^{n}|a_i|$. Since $|x|\le c \;\Leftrightarrow\; -c\le x\le c$, we have $\left|\sum_{i=1}^{n}a_i\right|\le\sum_{i=1}^{n}|a_i|$. *Another method:* Use mathematical induction.

43. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2=\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}i^2=\lim_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}=\lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$
$=\frac{1}{6}(1)(2)=\frac{1}{3}$

44. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{i}{n}\right)^3+1\right]=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{i^3}{n^4}+\frac{1}{n}\right]=\lim_{n\to\infty}\left[\frac{1}{n^4}\sum_{i=1}^{n}i^3+\frac{1}{n}\sum_{i=1}^{n}1\right]$
$=\lim_{n\to\infty}\left[\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2+\frac{1}{n}(n)\right]=\lim_{n\to\infty}\frac{1}{4}\left(1+\frac{1}{n}\right)^2+1=\frac{1}{4}+1=\frac{5}{4}$

45. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left[\left(\frac{2i}{n}\right)^3+5\left(\frac{2i}{n}\right)\right]=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{16}{n^4}i^3+\frac{20}{n^2}i\right]=\lim_{n\to\infty}\left[\frac{16}{n^4}\sum_{i=1}^{n}i^3+\frac{20}{n^2}\sum_{i=1}^{n}i\right]$
$=\lim_{n\to\infty}\left[\frac{16}{n^4}\frac{n^2(n+1)^2}{4}+\frac{20}{n^2}\frac{n(n+1)}{2}\right]=\lim_{n\to\infty}\left[\frac{4(n+1)^2}{n^2}+\frac{10n(n+1)}{n^2}\right]$
$=\lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right)^2+10\left(1+\frac{1}{n}\right)\right]=4\cdot 1+10\cdot 1=14$

46. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(1+\frac{3i}{n}\right)^3-2\left(1+\frac{3i}{n}\right)\right]=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[1+\frac{9i}{n}+\frac{27i^2}{n^2}+\frac{27i^3}{n^3}-2-\frac{6i}{n}\right]$
$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{81}{n^4}i^3+\frac{81}{n^3}i^2+\frac{9}{n^2}i-\frac{3}{n}\right]$
$=\lim_{n\to\infty}\left[\frac{81}{n^4}\frac{n^2(n+1)^2}{4}+\frac{81}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{9}{n^2}\frac{n(n+1)}{2}-\frac{3}{n}n\right]$
$=\lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2+\frac{27}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{9}{2}\left(1+\frac{1}{n}\right)-3\right]=\frac{81}{4}+\frac{54}{2}+\frac{9}{2}-3=\frac{195}{4}$

47. Let $S = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplying both sides by r gives us $rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$. Subtracting the first equation from the second, we find $(r-1)S = ar^n - a = a(r^n - 1)$, so $S = \dfrac{a(r^n-1)}{r-1}$ (since $r \neq 1$).

48. $\sum_{i=1}^{n} \dfrac{3}{2^{i-1}} = 3\sum_{i=1}^{n}\left(\dfrac{1}{2}\right)^{i-1} = \dfrac{3\left[\left(\frac{1}{2}\right)^n - 1\right]}{\frac{1}{2} - 1}$ [using Exercise 47 with $a = 3$ and $r = \frac{1}{2}$] $= 6\left[1 - \left(\frac{1}{2}\right)^n\right]$

49. $\sum_{i=1}^{n}\left(2i + 2^i\right) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 \cdot 2^{i-1} = 2\dfrac{n(n+1)}{2} + \dfrac{2(2^n-1)}{2-1} = 2^{n+1} + n^2 + n - 2.$

For the first sum we have used Theorem 3(c), and for the second, Exercise 47 with $a = r = 2$.

50. $\sum_{i=1}^{m}\left[\sum_{j=1}^{n}(i+j)\right] = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} i + \sum_{j=1}^{n} j\right]$ [Theorem 2(b)] $= \sum_{i=1}^{m}\left[ni + \dfrac{n(n+1)}{2}\right]$ [Theorem 3(b) and (c)]

$$= \sum_{i=1}^{m} ni + \sum_{i=1}^{m} \frac{n(n+1)}{2} = \frac{nm(m+1)}{2} + \frac{nm(n+1)}{2} = \frac{nm}{2}(m+n+2)$$

G Complex Numbers

1. $(5 - 6i) + (3 + 2i) = (5 + 3) + (-6 + 2)i = 8 + (-4)i = 8 - 4i$

2. $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right) = (4 - 9) + \left(-\frac{1}{2} - \frac{5}{2}\right)i = -5 + (-3)i = -5 - 3i$

3. $(2 + 5i)(4 - i) = 2(4) + 2(-i) + (5i)(4) + (5i)(-i) = 8 - 2i + 20i - 5i^2$
$= 8 + 18i - 5(-1) = 8 + 18i + 5 = 13 + 18i$

4. $(1 - 2i)(8 - 3i) = 8 - 3i - 16i + 6(-1) = 2 - 19i$

5. $\overline{12 + 7i} = 12 - 7i$

6. $2i\left(\frac{1}{2} - i\right) = i - 2(-1) = 2 + i \quad \Rightarrow \quad \overline{2i\left(\frac{1}{2} - i\right)} = \overline{2 + i} = 2 - i$

7. $\dfrac{1+4i}{3+2i} = \dfrac{1+4i}{3+2i} \cdot \dfrac{3-2i}{3-2i} = \dfrac{3 - 2i + 12i - 8(-1)}{3^2 + 2^2} = \dfrac{11 + 10i}{13} = \dfrac{11}{13} + \dfrac{10}{13}i$

8. $\dfrac{3+2i}{1-4i} = \dfrac{3+2i}{1-4i} \cdot \dfrac{1+4i}{1+4i} = \dfrac{3 + 12i + 2i + 8(-1)}{1^2 + 4^2} = \dfrac{-5 + 14i}{17} = -\dfrac{5}{17} + \dfrac{14}{17}i$

9. $\dfrac{1}{1+i} = \dfrac{1}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{1-i}{1-(-1)} = \dfrac{1-i}{2} = \dfrac{1}{2} - \dfrac{1}{2}i$

10. $\dfrac{3}{4-3i} = \dfrac{3}{4-3i} \cdot \dfrac{4+3i}{4+3i} = \dfrac{12+9i}{16-9(-1)} = \dfrac{12}{25} + \dfrac{9}{25}i$

11. $i^3 = i^2 \cdot i = (-1)i = -i$

12. $i^{100} = \left(i^2\right)^{50} = (-1)^{50} = 1$

13. $\sqrt{-25} = \sqrt{25}\, i = 5i$

14. $\sqrt{-3}\sqrt{-12} = \sqrt{3}\, i\sqrt{12}\, i = \sqrt{3 \cdot 12}\, i^2 = \sqrt{36}\,(-1) = -6$

15. $\overline{12-5i} = 12+15i$ and $|12-15i| = \sqrt{12^2+(-5)^2} = \sqrt{144+25} = \sqrt{169} = 13$

16. $\overline{-1+2\sqrt{2}\,i} = -1-2\sqrt{2}\,i$ and $\left|-1+2\sqrt{2}\,i\right| = \sqrt{(-1)^2+\left(2\sqrt{2}\right)^2} = \sqrt{1+8} = \sqrt{9} = 3$

17. $\overline{-4i} = \overline{0-4i} = 0+4i = 4i$ and $|-4i| = \sqrt{0^2+(-4)^2} = \sqrt{16} = 4$

18. Let $z = a+bi$, $w = c+di$.

(a) $\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i}$

$= (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}$

(b) $\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i$.
On the other hand, $\overline{z}\,\overline{w} = (a-bi)(c-di) = (ac-bd)-(ad+bc)i = \overline{zw}$.

(c) Use mathematical induction and part (b): Let S_n be the statement that $\overline{z^n} = \overline{z}^{\,n}$.
S_1 is true because $\overline{z^1} = \overline{z} = \overline{z}^{\,1}$. Assume S_k is true, that is $\overline{z^k} = \overline{z}^{\,k}$. Then $\overline{z^{k+1}} = \overline{z^{1+k}} = \overline{zz^k} = \overline{z}\,\overline{z^k}$ [part (b) with $w = z^k$] $= \overline{z}^{\,1}\overline{z}^{\,k} = \overline{z}^{\,1+k} = \overline{z}^{\,k+1}$, which shows that S_{k+1} is true.
Therefore, by mathematical induction, $\overline{z^n} = \overline{z}^{\,n}$ for every positive integer n.
Another proof: Use part (b) with $w = z$, and mathematical induction.

19. $4x^2+9=0 \;\Leftrightarrow\; 4x^2=-9 \;\Leftrightarrow\; x^2=-\frac{9}{4} \;\Leftrightarrow\; x=\pm\sqrt{-\frac{9}{4}}=\pm\sqrt{\frac{9}{4}}\,i=\pm\frac{3}{2}i.$

20. $x^4=1 \;\Leftrightarrow\; x^4-1=0 \;\Leftrightarrow\; (x^2-1)(x^2+1)=0 \;\Leftrightarrow\; x^2-1=0$ or $x^2+1=0 \;\Leftrightarrow$
$x=\pm1$ or $x=\pm i$.

21. $x^2+2x+5=0 \;\Leftrightarrow\; x=\dfrac{-2\pm\sqrt{2^2-4\cdot1\cdot5}}{2\cdot1}=\dfrac{-2\pm\sqrt{-16}}{2}=\dfrac{-2\pm4i}{2}=-1\pm2i$

22. $2x^2-2x+1=0 \;\Leftrightarrow\; x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4\cdot2\cdot1}}{2\cdot2}=\dfrac{2\pm\sqrt{-4}}{4}=\dfrac{2\pm2i}{4}=\dfrac{1}{2}\pm\dfrac{1}{2}i$

23. By the quadratic formula, $z^2+z+2=0 \;\Leftrightarrow\; z=\dfrac{-1\pm\sqrt{1^2-4(1)(2)}}{2(1)}=\dfrac{-1\pm\sqrt{-7}}{2}=-\dfrac{1}{2}\pm\dfrac{\sqrt{7}}{2}i.$

24. $z^2+\frac{1}{2}z+\frac{1}{4}=0 \;\Leftrightarrow\; 4z^2+2z+1=0 \;\Leftrightarrow$

$$z=\frac{-2\pm\sqrt{2^2-4(4)(1)}}{2(4)}=\frac{-2\pm\sqrt{-12}}{8}=\frac{-2\pm2\sqrt{3}\,i}{8}=-\frac{1}{4}\pm\frac{\sqrt{3}}{4}i$$

25. For $z=-3+3i$, $r=\sqrt{(-3)^2+3^2}=3\sqrt{2}$ and $\tan\theta=\frac{3}{-3}=-1 \;\Rightarrow\; \theta=\frac{3\pi}{4}$ (since z lies in the second quadrant). Therefore, $-3+3i=3\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

26. For $z=1-\sqrt{3}\,i$, $r=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=2$ and $\tan\theta=\frac{-\sqrt{3}}{1}=-\sqrt{3} \;\Rightarrow\; \theta=\frac{5\pi}{3}$ (since z lies in the fourth quadrant). Therefore, $1-\sqrt{3}\,i=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$.

27. For $z=3+4i$, $r=\sqrt{3^2+4^2}=5$ and $\tan\theta=\frac{4}{3} \;\Rightarrow\; \theta=\tan^{-1}\left(\frac{4}{3}\right)$ (since z lies in the first quadrant).
Therefore, $3+4i=5\left[\cos\left(\tan^{-1}\frac{4}{3}\right)+i\sin\left(\tan^{-1}\frac{4}{3}\right)\right]$.

28. For $z=8i$, $r=\sqrt{0^2+8^2}=8$ and $\tan\theta=\frac{8}{0}$ is undefined, so $\theta=\frac{\pi}{2}$ (since z lies on the positive imaginary axis).
Therefore, $8i=8\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$.

29. For $z = \sqrt{3} + i$, $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$.

For $w = 1 + \sqrt{3}\,i$, $r = 2$ and $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow w = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.

Therefore, $zw = 2 \cdot 2\left[\cos(\frac{\pi}{6} + \frac{\pi}{3}) + i\sin(\frac{\pi}{6} + \frac{\pi}{3})\right] = 4(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$,

$z/w = \frac{2}{2}\left[\cos(\frac{\pi}{6} - \frac{\pi}{3}) + i\sin(\frac{\pi}{6} - \frac{\pi}{3})\right] = \cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})$, and $1 = 1 + 0i = 1(\cos 0 + i\sin 0) \Rightarrow$

$1/z = \frac{1}{2}\left[\cos(0 - \frac{\pi}{6}) + i\sin(0 - \frac{\pi}{6})\right] = \frac{1}{2}\left[\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right]$. For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{2}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$.

30. For $z = 4\sqrt{3} - 4i$, $r = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{64} = 8$ and $\tan\theta = \frac{-4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$

$z = 8(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$. For $w = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan\theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2} \Rightarrow$

$w = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$. Therefore, $zw = 8 \cdot 8\left[\cos(\frac{11\pi}{6} + \frac{\pi}{2}) + i\sin(\frac{11\pi}{6} + \frac{\pi}{2})\right] = 64(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$,

$z/w = \frac{8}{8}\left[\cos(\frac{11\pi}{6} - \frac{\pi}{2}) + i\sin(\frac{11\pi}{6} - \frac{\pi}{2})\right] = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$, and

$1 = 1 + 0i = 1(\cos 0 + i\sin 0) \Rightarrow 1/z = \frac{1}{8}\left[\cos(0 - \frac{11\pi}{6}) + i\sin(0 - \frac{11\pi}{6})\right] = \frac{1}{8}\left[\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right]$.

For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{8}(\cos\frac{11\pi}{6} - i\sin\frac{11\pi}{6})$.

31. For $z = 2\sqrt{3} - 2i$, $r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$ and $\tan\theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\Rightarrow \theta = -\frac{\pi}{6} \Rightarrow z = 4\left[\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right]$. For $w = -1 + i$, $r = \sqrt{2}$,

$\tan\theta = \frac{1}{-1} = -1 \Rightarrow \theta = \frac{3\pi}{4} \Rightarrow w = \sqrt{2}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. Therefore,

$zw = 4\sqrt{2}\left[\cos(-\frac{\pi}{6} + \frac{3\pi}{4}) + i\sin(-\frac{\pi}{6} + \frac{3\pi}{4})\right] = 4\sqrt{2}\,(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})$,

$z/w = \frac{4}{\sqrt{2}}\left[\cos(-\frac{\pi}{6} - \frac{3\pi}{4}) + i\sin(-\frac{\pi}{6} - \frac{3\pi}{4})\right] = \frac{4}{\sqrt{2}}\left[\cos(-\frac{11\pi}{12}) + i\sin(-\frac{11\pi}{12})\right]$

$= 2\sqrt{2}\,(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12})$, and

$1/z = \frac{1}{4}\left[\cos(-\frac{\pi}{6}) - i\sin(-\frac{\pi}{6})\right] = \frac{1}{4}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$.

32. For $z = 4(\sqrt{3} + i) = 4\sqrt{3} + 4i$, $r = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8$ and $\tan\theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow$

$z = 8(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. For $w = -3 - 3i$, $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$ and

$\tan\theta = \frac{-3}{-3} = 1 \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow w = 3\sqrt{2}\,(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$. Therefore,

$zw = 8 \cdot 3\sqrt{2}\left[\cos(\frac{\pi}{6} + \frac{5\pi}{4}) + i\sin(\frac{\pi}{6} + \frac{5\pi}{4})\right] = 24\sqrt{2}\,(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12})$,

$z/w = \frac{8}{3\sqrt{2}}\left[\cos(\frac{\pi}{6} - \frac{5\pi}{4}) + i\sin(\frac{\pi}{6} - \frac{5\pi}{4})\right] = \frac{4\sqrt{2}}{3}\left[\cos(-\frac{13\pi}{12}) + i\sin(-\frac{13\pi}{12})\right]$, and

$1/z = \frac{1}{8}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$.

33. For $z = 1 + i$, $r = \sqrt{2}$ and $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$. So by De Moivre's Theorem,

$$(1+i)^{20} = \left[\sqrt{2}\,(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4})\right]^{20} = \left(2^{1/2}\right)^{20}(\cos\tfrac{20\cdot\pi}{4} + i\sin\tfrac{20\cdot\pi}{4})$$

$$= 2^{10}(\cos 5\pi + i\sin 5\pi) = 2^{10}[-1 + i(0)] = -2^{10} = -1024$$

34. For $z = 1 - \sqrt{3}\,i$, $r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$ and $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \;\Rightarrow\; \theta = \frac{5\pi}{3} \;\Rightarrow$

$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$. So by De Moivre's Theorem,

$$\left(1 - \sqrt{3}\,i\right)^5 = \left[2\left(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3}\right)\right]^5 = 2^5\left(\cos\tfrac{5\cdot 5\pi}{3} + i\sin\tfrac{5\cdot 5\pi}{3}\right)$$
$$= 2^5\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) = 32\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\,i\right) = 16 + 16\sqrt{3}\,i$$

35. For $z = 2\sqrt{3} + 2i$, $r = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} = \sqrt{16} = 4$ and $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \;\Rightarrow\; \theta = \frac{\pi}{6} \;\Rightarrow$

$z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. So by De Moivre's Theorem,

$$\left(2\sqrt{3} + 2i\right)^5 = \left[4\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)\right]^5 = 4^5\left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right) = 1024\left[-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right] = -512\sqrt{3} + 512i$$

36. For $z = 1 - i$, $r = \sqrt{2}$ and $\tan\theta = \frac{-1}{1} = -1 \;\Rightarrow\; \theta = \frac{7\pi}{4} \;\Rightarrow\; z = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \;\Rightarrow$

$$(1 - i)^8 = \left[\sqrt{2}\left(\cos\tfrac{7\pi}{4} + i\sin\tfrac{7\pi}{4}\right)\right]^8 = 2^4\left(\cos\tfrac{8\cdot 7\pi}{4} + i\sin\tfrac{8\cdot 7\pi}{4}\right)$$
$$= 16(\cos 14\pi + i\sin 14\pi) = 16(1 + 0i) = 16$$

37. $1 = 1 + 0i = 1\,(\cos 0 + i\sin 0)$. Using Equation 3 with $r = 1$, $n = 8$, and $\theta = 0$, we have

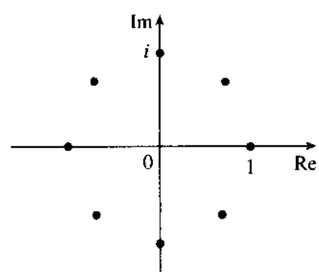
$$w_k = 1^{1/8}\left[\cos\left(\frac{0 + 2k\pi}{8}\right) + i\sin\left(\frac{0 + 2k\pi}{8}\right)\right] = \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}, \text{ where } k = 0, 1, 2, \ldots, 7.$$

$w_0 = 1(\cos 0 + i\sin 0) = 1$, $w_1 = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

$w_2 = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i$, $w_3 = 1\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

$w_4 = 1(\cos\pi + i\sin\pi) = -1$, $w_5 = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$,

$w_6 = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -i$, $w_7 = 1\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



38. $32 = 32 + 0i = 32\,(\cos 0 + i\sin 0)$. Using Equation 3 with $r = 32$, $n = 5$, and $\theta = 0$, we have

$$w_k = 32^{1/5}\left[\cos\left(\frac{0 + 2k\pi}{5}\right) + i\sin\left(\frac{0 + 2k\pi}{5}\right)\right] = 2\left(\cos\tfrac{2}{5}\pi k + i\sin\tfrac{2}{5}\pi k\right), \text{ where } k = 0, 1, 2, 3, 4.$$

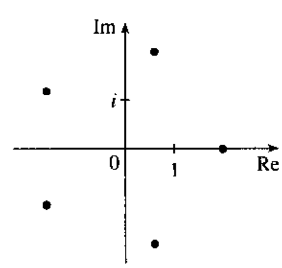
$w_0 = 2(\cos 0 + i\sin 0) = 2$

$w_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$w_2 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$w_3 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$w_4 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$



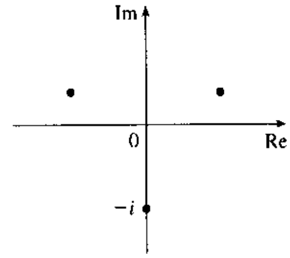
39. $i = 0 + i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Using Equation 3 with $r = 1$, $n = 3$, and $\theta = \frac{\pi}{2}$, we have

$$w_k = 1^{1/3}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right], \text{ where } k = 0, 1, 2.$$

$$w_0 = \left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$w_1 = \left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right) = -\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$

$$w_2 = \left(\cos\tfrac{9\pi}{6} + i\sin\tfrac{9\pi}{6}\right) = -i$$



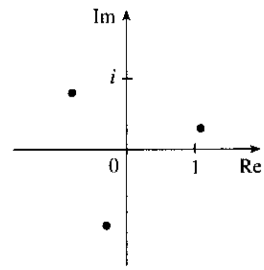
40. $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Using Equation 3 with $r = \sqrt{2}$, $n = 3$, and $\theta = \frac{\pi}{4}$, we have

$$w_k = \left(\sqrt{2}\right)^{1/3}\left[\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right], \text{ where } k = 0, 1, 2.$$

$$w_0 = 2^{1/6}\left(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}\right)$$

$$w_1 = 2^{1/6}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right) = 2^{1/6}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) = -2^{-1/3} + 2^{-1/3}i$$

$$w_2 = 2^{1/6}\left(\cos\tfrac{17\pi}{12} + i\sin\tfrac{17\pi}{12}\right)$$



41. Using Euler's formula (6) with $y = \frac{\pi}{2}$, we have $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + 1i = i$.

42. Using Euler's formula (6) with $y = 2\pi$, we have $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$.

43. Using Euler's formula (6) with $y = \dfrac{\pi}{3}$, we have $e^{i\pi/3} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

44. Using Euler's formula (6) with $y = -\pi$, we have $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1$.

45. Using Equation 7 with $x = 2$ and $y = \pi$, we have $e^{2+i\pi} = e^2e^{i\pi} = e^2(\cos\pi + i\sin\pi) = e^2(-1+0) = -e^2$.

46. Using Equation 7 with $x = \pi$ and $y = 1$, we have $e^{\pi+i} = e^\pi \cdot e^{1i} = e^\pi(\cos 1 + i\sin 1) = e^\pi\cos 1 + (e^\pi\sin 1)i$.

47. $F(x) = e^{rx} = e^{(a+bi)x} = e^{ax+bxi} = e^{ax}(\cos bx + i\sin bx) = e^{ax}\cos bx + i(e^{ax}\sin bx) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= (e^{ax}\cos bx)' + i(e^{ax}\sin bx)' = (ae^{ax}\cos bx - be^{ax}\sin bx) + i(ae^{ax}\sin bx + be^{ax}\cos bx) \\ &= a\left[e^{ax}(\cos bx + i\sin bx)\right] + b\left[e^{ax}(-\sin bx + i\cos bx)\right] = ae^{rx} + b\left[e^{ax}\left(i^2\sin bx + i\cos bx\right)\right] \\ &= ae^{rx} + bi\left[e^{ax}(\cos bx + i\sin bx)\right] = ae^{rx} + bie^{rx} = (a+bi)e^{rx} = re^{rx} \end{aligned}$$

48. (a) From Exercise 47, $F(x) = e^{(1+i)x} \quad \Rightarrow \quad F'(x) = (1+i)e^{(1+i)x}$. So

$$\int e^{(1+i)x}\,dx = \frac{1}{1+i}\int F'(x)\,dx = \frac{1}{1+i}F(x) + C = \frac{1-i}{2}F(x) + C = \frac{1-i}{2}e^{(1+i)x} + C$$

(b) $\int e^{(1+i)x}\,dx = \int e^x e^{ix}\,dx = \int e^x(\cos x + i\sin x)\,dx = \int e^x \cos x\,dx + i\int e^x \sin x\,dx$ **(1)**.

Also,

$$\begin{aligned}\frac{1-i}{2}e^{(1+i)x} &= \tfrac{1}{2}e^{(1+i)x} - \tfrac{1}{2}ie^{(1+i)x} = \tfrac{1}{2}e^{x+ix} - \tfrac{1}{2}ie^{x+ix} \\ &= \tfrac{1}{2}e^x(\cos x + i\sin x) - \tfrac{1}{2}ie^x(\cos x + i\sin x) \\ &= \tfrac{1}{2}e^x\cos x + \tfrac{1}{2}e^x\sin x + \tfrac{1}{2}ie^x\sin x - \tfrac{1}{2}ie^x\cos x \\ &= \tfrac{1}{2}e^x(\cos x + \sin x) + i\left[\tfrac{1}{2}e^x(\sin x - \cos x)\right] \qquad \textbf{(2)}\end{aligned}$$

Equating the real and imaginary parts in **(1)** and **(2)**, we see that $\int e^x \cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x) + C$ and $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.