DISTRIBUTION THEORY
(GENERALIZED FUNCTIONS)
NOTES

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Chapter 1

Introduction

The so-called Dirac delta function (on $\mathbb{R}$) obeys $\delta(x) = 0$ for all $x \neq 0$ but is supposed to satisfy $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. (The δ function on $\mathbb{R}^d$ is similarly described.) Consequently,

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} (f(x) - f(0))\,\delta(x)\,dx + f(0) \int_{-\infty}^{\infty} \delta(x)\,dx = f(0)$$

because $(f(x) - f(0))\,\delta(x) \equiv 0$ on $\mathbb{R}$. Moreover, if $H(x)$ denotes the Heaviside step-function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0\,, \end{cases}$$

then we see that $H' = \delta$, in the following sense. If f vanishes at infinity, then integration by parts gives

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,H'(x)\,dx &= \big[f(x)\,H(x)\big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,H(x)\,dx \\ &= -\int_{-\infty}^{\infty} f'(x)\,H(x)\,dx \\ &= -\int_{0}^{\infty} f'(x)\,dx \\ &= -\big[f(x)\big]_{0}^{\infty} \\ &= f(0) \\ &= \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx\,. \end{aligned}$$

Of course, there is no such function δ with these properties and we cannot interpret $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$ as an integral in the usual sense. The δ function is thought of as a *generalized function.*

However, what does make sense is the assignment $f \mapsto f(0) = \langle \delta, f \rangle$, say. Clearly $\langle \delta, \alpha f + \beta g \rangle = \alpha \langle \delta, f \rangle + \beta \langle \delta, g \rangle$ for functions f, g and constants α and β. In other words, the Dirac delta-function can be defined not as a function but as a functional on a suitable linear space of functions. The development of this is the theory of distributions of Laurent Schwartz.

One might think of $\delta(x)$ as a kind of limit of some sequence of functions whose graphs become very tall and thin, as indicated in the figure.

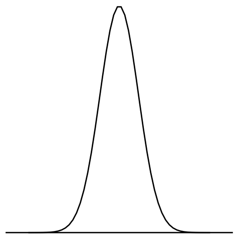

Figure 1.1: Approximation to the δ-function.

The Dirac δ function can be thought of as a kind of continuous version of the discrete Kronecker δ and is used in quantum mechanics to express the orthogonality properties of non square-integrable wave functions.

Distributions play a crucial rôle in the study of (partial) differential equations. As an introductory remark, consider the equations

$$\frac{\partial^2 u}{\partial_x \partial_y} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial_y \partial_x} = 0\,.$$

These "ought" to be equivalent. However, the first holds for any function u independent of y, whereas the second may not make any sense. By (formally) integrating by parts twice and discarding the surface terms, we get

$$\int \varphi \, \frac{\partial^2 u}{\partial_x \partial_y} \, dx\, dy = \int u \, \frac{\partial^2 \varphi}{\partial_x \partial_y} \, dx\, dy\,.$$

So we might interpret $\dfrac{\partial^2 u}{\partial_x \partial_y} = 0$ as

$$\int u \, \frac{\partial^2 \varphi}{\partial_x \partial_y} \, dx\, dy = 0$$

for all φ in some suitably chosen set of smooth functions. The point is that this makes sense for non-differentiable u and, since φ is supposed smooth,

$$\int u \, \frac{\partial^2 \varphi}{\partial_x \partial_y} \, dx\, dy = \int u \, \frac{\partial^2 \varphi}{\partial_y \partial_x} \, dx\, dy\,,$$

that is, $\dfrac{\partial^2 u}{\partial_x \partial_y} = \dfrac{\partial^2 u}{\partial_y \partial_x}$ in a certain weak sense. These then are weak or distributional derivatives.

Finally, we note that distributions also play a central rôle in quantum field theory, where quantum fields are defined as operator-valued distributions.

Bibliography

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M. Reed and B. Simon, *Methods of Mathematical Physics, Volume II*, Academic Press, Inc., 1975.

W. Rudin, *Functional Analysis*, McGraw-Hill, Inc,. 1973.

L. Schwartz, *Théorie des distributions*, Hermann & Cie, Paris, 1966.

The proper approach to the theory is via topological vector spaces—see Rudin's excellent book for the development along these lines, as well as much background material. The approach via approximating sequences of functions is to be found in Lighthill's book.

For the preparation of these lecture notes, extensive use was made of the books of Rudin and Reed and Simon.

Chapter 2

The spaces $\mathscr{S}$ and $\mathscr{S}'$

Let $\mathbb{Z}_+^n$ denote the set of n-tuples $(\alpha_1, \dots, \alpha_n)$ where each α_i is a non-negative integer and write $\mathbb{Z}_+$ for $\mathbb{Z}_+^1$. For $\alpha \in I_+^n$, let $|\alpha| = \sum_{i=1}^n \alpha_i$ and let D^α denote the partial differential operator $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

Finally, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let x^α denote the product $x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Definition 2.1. The complex linear space of bounded continuous complex-valued functions on $\mathbb{R}^n$ is denoted $C_b(\mathbb{R}^n)$. It is equipped with the norm

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)| \ .$$

Theorem 2.2. *For any $d \in \mathbb{N}$, $C_b(\mathbb{R}^d)$ is a complete normed space with respect to the norm $\|\cdot\|_\infty$.*

Proof. Suppose (f_n) is a Cauchy sequence in $C_b(\mathbb{R}^d)$, that is, $\|f_n - f_m\|_\infty \to 0$ as $m, n \to \infty$. We must show that there is some $f \in C_b(\mathbb{R}^d)$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. To see where such an f comes from, we note that the inequality $|g(x)| \le \|g\|_\infty$ implies that for each $x \in \mathbb{R}^d$, the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{C}$ and therefore converges. Let $f(x) = \lim_n f_n(x)$.

We claim that $f \in C_b(\mathbb{R}^d)$ and that $\|f_n - f\|_\infty \to 0$. Let $\varepsilon > 0$ be given. Since (f_n) is a Cauchy sequence in $C_b(\mathbb{R}^d)$ there is $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_\infty < \tfrac{1}{2}\,\varepsilon \qquad (*)$$

for all $n, m \ge N$. But then, for any $x \in \mathbb{R}^d$,

$$|f_{N+k}(x)| \le |f_{N+k}(x) - f_N(x)| + |f_N(x)| < \tfrac{1}{2}\,\varepsilon + \|f_N\|_\infty$$

by $(*)$. Letting $k \to \infty$ gives

$$|f(x)| \le \tfrac{1}{2}\,\varepsilon + \|f_N\|_\infty$$

which shows that f is bounded on $\mathbb{R}^d$.

Next, we note that for any $x \in \mathbb{R}^d$, $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ and so the inequality $(*)$ implies that

$$|f_n(x) - f_m(x)| < \tfrac{1}{2}\,\varepsilon$$

provided $n, m \ge N$. Letting $m \to \infty$, we see that

$$|f_n(x) - f(x)| \le \tfrac{1}{2}\,\varepsilon \qquad (**)$$

for any $x \in \mathbb{R}^d$, provided $n \ge N$. In other words, $f_n(x) \to f(x)$ uniformly on $\mathbb{R}^d$. However, each f_n is continuous and so the same is true of f. But then this means that $f \in C_b(\mathbb{R}^d)$. The inequality $(**)$ gives

$$\|f_n - f\|_\infty \le \tfrac{1}{2}\,\varepsilon < \varepsilon$$

whenever $n \ge N$ and so $f_n \to f$ with respect to $\|\cdot\|_\infty$ and the proof is complete. ∎

Definition 2.3. The linear space of infinitely-differentiable bounded functions on $\mathbb{R}^n$ is denoted by $C_b^\infty(\mathbb{R}^n)$. Evidently $C_b^\infty(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$.

The space $\mathscr{S}(\mathbb{R}^n)$ is the linear subspace of $C_b^\infty(\mathbb{R}^n)$ formed by the set of functions f on $\mathbb{R}^n$ such that $x^\alpha\, D^\beta f(x)$ is bounded on $\mathbb{R}^n$ for each $\alpha, \beta \in \mathbb{Z}_+^n$.

$\mathscr{S}(\mathbb{R}^n)$ is equipped with the family of norms

$$|||f|||_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |\, x^\alpha\, D^\beta f(x)\,|$$

for $\alpha, \beta \in \mathbb{Z}_+^n$. The elements of $\mathscr{S}(\mathbb{R}^n)$ are said to be rapidly decreasing functions.

Example 2.4. Evidently, the function $f(x) = x^m\, e^{-x^2}$ belongs to $\mathscr{S}(\mathbb{R})$ for any $m \in \mathbb{Z}_+$. Indeed, $\mathscr{S}(\mathbb{R})$ contains all the Hermite functions.
For any polynomial $p(x_1, \dots, x_n)$ on $\mathbb{R}^n$, the function $p(x_1, \dots, x_n) e^{-(x_1^2 + \cdots + x_n^2)}$ belongs to $\mathscr{S}(\mathbb{R}^n)$.

Definition 2.5. We say that a sequence (f_n) in $\mathscr{S}(\mathbb{R}^d)$ converges to f in $\mathscr{S}(\mathbb{R}^d)$ if, for each $\alpha, \beta \in \mathbb{Z}_+^d$, $|||f_n - f|||_{\alpha,\beta} \to 0$ as $n \to \infty$.

The sequence (f_n) in $\mathscr{S}(\mathbb{R}^d)$ is said to be a Cauchy sequence in $\mathscr{S}(\mathbb{R}^d)$ if $|||f_n - f_m|||_{\alpha,\beta} \to 0$ as $n, m \to \infty$, for each $\alpha, \beta \in \mathbb{Z}_+^d$.

Theorem 2.6. *$\mathscr{S}(\mathbb{R}^d)$ is complete, that is, every Cauchy sequence in $\mathscr{S}(\mathbb{R}^d)$ converges in $\mathscr{S}(\mathbb{R}^d)$.*

Proof. First consider the case $d = 1$. So suppose that (f_n) is a Cauchy sequence in $\mathscr{S}(\mathbb{R})$. Fix $\alpha, \beta \in \mathbb{Z}_+$. Then we know that

$$|||f_n - f_m|||_{\alpha,\beta} \to 0 \text{ as } n, m \to \infty.$$

In other words, the sequence $x^\alpha D^\beta f_n(x)$ is a Cauchy sequence with respect to the norm $\|\cdot\|_\infty$ and so converges to some function $g_{\alpha,\beta}$, say.

We shall show that $x^\alpha D^\beta g = g_{\alpha,\beta}$. This follows from the equality

$$f_n(x) = f_n(0) + \int_0^x f_n'(t)\, dt\,.$$

Indeed, $f_n' = D^1 f_n \to g_{0,1}$ on $\mathbb{R}$ and so, letting $n \to \infty$, we may say that

$$g_{0,0}(x) = g_{0,0}(0) + \int_0^x g_{0,1}(t)\, dt\,.$$

Hence $g_{0,0}$ is continuously differentiable and $g_{0,0}' = g_{0,1}$. Repeating this argument, we see that $g_{0,0}$ is infinitely-differentiable and that $D^\beta g_{0,0} = g_{0,\beta}$.

Now, $D^\beta f_n \to g_{0,\beta} = D^\beta g_{0,0}$ uniformly and so $x^\alpha D^\beta f_n(x) \to x^\alpha D^\beta g_{0,0}(x)$ pointwise. But we also know that $x^\alpha D^\beta f_n \to g_{\alpha,\beta}$ uniformly and so it follows that $g_{\alpha,\beta} = x^\alpha D^\beta g_{0,0}$. We note that $g_{\alpha,\beta}$ is bounded and so $g_{0,0} \in \mathscr{S}(\mathbb{R})$. Hence $f_n \to g_{0,0}$ in $\mathscr{S}(\mathbb{R})$ and we conclude that $\mathscr{S}(\mathbb{R})$ is complete.

For the general d-dimensional case, suppose that (f_n) is a Cauchy sequence in $\mathscr{S}(\mathbb{R}^d)$. Then for each $\alpha, \beta \in \mathbb{Z}_+^d$ the sequence $x^\alpha D^\beta f_n$ is a Cauchy sequence in $C_b(\mathbb{R}^d)$ and so converges; $x^\alpha D^\beta f_n \to g_{\alpha,\beta}$ uniformly on $\mathbb{R}^d$, for some $g_{\alpha,\beta} \in C_b(\mathbb{R}^d)$.

Fix $x_2, x_3, \dots, x_d$. Then as in the 1-dimensional argument above, we know that for any $\alpha_1, \beta_1 \in \mathbb{Z}_+$, (all relevant partial derivatives exist and)

$$\begin{aligned} x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} f_n(x_1, x_2, \dots, x_d) &\to g_{(\alpha_1,0,\dots,0),(\beta_1,0,\dots,0)}(x_1, x_2, \dots, x_d) \\ &= x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} g_{0,0}(x_1, x_2, \dots, x_d)\,. \end{aligned}$$

Considering now the function $x_2 \mapsto x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} f_n(x_1, x_2, \dots, x_d)$, we similarly see that

$$\begin{aligned} x_2^{\alpha_2}\, \partial_{x_2}^{\beta_2}\, x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} f_n(x_1, \dots, x_d) &\to g_{(\alpha_1,\alpha_2,0,\dots,0),(\beta_1,\beta_2,0,\dots,0)}(x_1, \dots, x_d) \\ &= x_2^{\alpha_2}\, \partial_{x_2}^{\beta_2}\, x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} g_{0,0}(x_1, \dots, x_d) \end{aligned}$$

for any $\alpha_2, \beta_2 \in \mathbb{Z}_+$.

Repeating this for the function $x_3 \mapsto x_2^{\alpha_2}\, \partial_{x_2}^{\beta_2}\, x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} f_n(x_1, x_2, \dots, x_d)$, we find that

$$\begin{aligned} &g_{(\alpha_1,\alpha_2,\alpha_3,0,\dots,0),(\beta_1,\beta_2,\beta_3,0,\dots,0)}(x_1, \dots, x_d) \\ &\qquad = x_3^{\alpha_3}\, \partial_{x_3}^{\beta_3}\, x_2^{\alpha_2}\, \partial_{x_2}^{\beta_2}\, x_1^{\alpha_1}\, \partial_{x_1}^{\beta_1} g_{0,0}(x_1, \dots, x_d) \end{aligned}$$

for $\alpha_3, \beta_3 \in \mathbb{Z}_+$. Continuing this way, we conclude that

$$g_{\alpha,\beta} = x^\alpha D^\beta g_{0,0}$$

for any $\alpha, \beta \in \mathbb{Z}_+^d$. It follows that $f_n \to g_{0,0}$ in $\mathscr{S}(\mathbb{R}^d)$. ∎

Definition 2.7. Continuous linear functionals on $\mathscr{S}(\mathbb{R}^d)$ are called tempered distributions. The linear space of tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^d)$. Thus $T \in \mathscr{S}'(\mathbb{R}^d)$ if and only if $T : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C}$ is linear and $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d)$ implies that $T(f_n) \to T(f)$ in $\mathbb{C}$.

Example 2.8. For fixed $a \in \mathbb{R}^d$, let δ_a be the map on $\mathscr{S}(\mathbb{R}^d)$ given by the prescription $\delta_a : f \mapsto f(a)$. Evidently $\delta_a \in \mathscr{S}'(\mathbb{R}^d)$. δ_a is called the Dirac delta function (at $a \in \mathbb{R}^d$).

Remark 2.9. Since $T(f_n) - T(f) = T(f_n - f)$ and $f_n \to f$ in $\mathscr{S}$ if and only if $(f_n - f) \to 0$ in $\mathscr{S}$, we see that a linear map on $\mathscr{S}$ is continuous if and only if it is continuous at $0 \in \mathscr{S}$.

Proposition 2.10. *Suppose that $T : \mathscr{S} \to \mathscr{S}$ is linear and that there are $\alpha, \beta \in \mathbb{Z}_+^d$ such that*

$$|T(f)| \le |||f|||_{\alpha,\beta}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$. Then $T \in \mathscr{S}'(\mathbb{R}^d)$.

Proof. According to the remark above, we need only verify the continuity of T at 0. But if $f_n \to 0$ in $\mathscr{S}$, it follows, in particular, that $|||f_n|||_{\alpha,\beta} \to 0$ and so

$$|T(f_n)| \le |||f_n|||_{\alpha,\beta} \to 0$$

as $n \to \infty$. This means that T is continuous at 0, as required. ■

To establish a converse, we shall introduce another family of norms on $\mathscr{S}$.

Definition 2.11. For each $k, m \in \mathbb{Z}_+$ and $f \in \mathscr{S}(\mathbb{R}^d)$, set

$$\|f\|_{k,m} = \sum_{\substack{|\alpha| \le k \\ |\beta| \le m}} |||f|||_{\alpha,\beta} \,.$$

These norms on $\mathscr{S}$ have the property of being directed, that is, for any (k', m') and (k'', m'') there is (k, m) such that

$$\max\{\, \|f\|_{k',m'}, \|f\|_{k'',m''} \,\} \le \|f\|_{k,m}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$. (Any (k,m) with $k \ge \max\{\, k', k'' \,\}$ and $m \ge \max\{\, m', m'' \,\}$ will do).

Remark 2.12. It is clear that $|||f_n - f|||_{\alpha,\beta} \to 0$ for each $\alpha, \beta \in \mathbb{Z}_+^d$ if and only if $\|f_n - f\|_{k,m} \to 0$ for each $k, m \in \mathbb{Z}_+$. It follows that a linear functional T on $\mathscr{S}(\mathbb{R}^d)$ is a tempered distribution if and only if $T(f_n) \to 0$ whenever $\|f_n\|_{k,m} \to 0$ for all $k, m \in \mathbb{Z}_+$.

Theorem 2.13. *A linear functional T on $\mathscr{S}(\mathbb{R}^d)$ is a tempered distribution if and only if there is $C > 0$ and some $k, m \in \mathbb{Z}_+$ such that*

$$|T(f)| \le C\|f\|_{k,m}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$.

Proof. If such a bound exists, it is clear that $T \in \mathscr{S}'(\mathbb{R}^d)$. For the converse, suppose that $T \in \mathscr{S}'(\mathbb{R}^d)$ but that there exists no such bound. Then for any $n \in \mathbb{N}$, it is false that

$$|T(f)| \le n\,\|f\|_{n,n}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$. In other words, there is a sequence (g_n) in $\mathscr{S}$ such that

$$|T(g_n)| > n\,\|g_n\|_{n,n}\,.$$

Set $f_n = g_n/n\,\|g_n\|_{n,n}$ so that $|T(f_n)| > 1$. However,

$$\|f_n\|_{k,m} = \frac{\|g_n\|_{k,m}}{n\,\|g_n\|_{n,n}} \le \frac{1}{n}$$

whenever $n \ge \max\{\,k, m\,\}$. It follows that $f_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. This is a contradiction because it is false that $T(f_n) \to 0$. The result follows. ∎

Proposition 2.14. *Let $g \in L^2(\mathbb{R}^d)$. Then the linear map*

$$T_g : f \mapsto \int g(x)\,f(x)\,dx$$

on $\mathscr{S}(\mathbb{R}^d)$ defines a tempered distribution.

Proof. For $f \in \mathscr{S}(\mathbb{R}^d)$, we have $|T_g(f)| = \left|\int g(x)\,f(x)\,dx\right| \le \|g\|_{L^2}\,\|f\|_{L^2}$. But

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int |f(x)|\,|f(x)|\,dx \\
&\le \|f\|_{0,0} \int |f(x)|\,dx \\
&= \|f\|_{0,0} \int \Big(\prod_{j=1}^{d}(1+x_j^2)\Big)\,|f(x)|\,\frac{1}{\prod_{k=1}^{d}(1+x_k^2)}\,dx \\
&\le \|f\|_{0,0}\,\|f\|_{2d,0} \int \frac{1}{\prod_{k=1}^{d}(1+x_k^2)}\,dx_1\,dx_2\ldots dx_d \\
&= \pi^d\,\|f\|_{0,0}\,\|f\|_{2d,0} \\
&\le \pi^d\,\|f\|_{2d,0}^2\,.
\end{aligned}$$

This leads to the estimate

$$|T_g(f)| \le \|g\|_{L^2}\,\pi^{d/2}\,\|f\|_{2d,0}$$

which shows that $T \in \mathscr{S}'(\mathbb{R}^d)$, as claimed. ∎

The next result indicates that polynomially bounded functions determine tempered distributions, via integration.

Theorem 2.15. *Suppose that $g(x)$ (is measurable and) is such that for some $m \in \mathbb{N}$, $\prod_{j=1}^{d}(1+x_j^2)^{-m} g(x)$ is bounded on $\mathbb{R}^d$. Then the map*

$$T_g : f \mapsto \int g(x)\, f(x)\, dx$$

is a tempered distribution.

Proof. Let $p(x) = \prod_{j=1}^{d}(1+x_j^2)$. Then the hypotheses mean that for any $f \in \mathscr{S}(\mathbb{R}^d)$

$$\begin{aligned} |g(x)\, f(x)| &= p(x)^{-m}\ |g(x)|\ p(x)^m\ |f(x)| \\ &< M p(x)^m\ |f(x)| \end{aligned}$$

for some $M > 0$. It follows that $g(x)f(x)$ is integrable and so T_g is well-defined on $\mathscr{S}(\mathbb{R}^d)$.

To show that $T \in \mathscr{S}'(\mathbb{R}^d)$, we estimate

$$\begin{aligned} |T_g(f)| &< M \int p(x)^m\, |f(x)|\ dx \\ &= M \int p(x)^{m+1}\, |f(x)|\ \frac{1}{p(x)}\ dx \\ &\le M\, \|f\|_{2d(m+1),0} \int \frac{1}{p(x)}\ dx \\ &= M\, \|f\|_{2d(m+1),0}\ \pi^d\,. \end{aligned}$$

It follows that $T_g \in \mathscr{S}'(\mathbb{R}^d)$. ∎

Theorem 2.16 (Cauchy Principal Part Integral).
The functional

$$P(\tfrac{1}{x}) : f \mapsto \lim_{\varepsilon \downarrow 0} \int_{|x| \ge \varepsilon} \frac{1}{x}\ f(x)\ dx$$

belongs to $\mathscr{S}'(\mathbb{R})$.

Proof. We first show that $P(\frac{1}{x})$ is well-defined on $\mathscr{S}(\mathbb{R})$. For $f \in \mathscr{S}(\mathbb{R})$

$$\int_{|x| \ge \varepsilon} \frac{1}{x}\ f(x)\ dx = \int_{\varepsilon}^{\infty} \frac{f(x) - f(-x)}{x}\ dx\,.$$

However, $\dfrac{f(x) - f(-x)}{x} \to 2f'(0)$ as $x \to 0$ and therefore $\dfrac{f(x) - f(-x)}{x}$ is integrable on $[0, \infty)$ and $P(\frac{1}{x})$ is indeed well-defined.

Clearly $P(\frac{1}{x})$ is linear and so we need only verify its continuity on $\mathscr{S}(\mathbb{R})$. To do this, we observe that for $x > 0$

$$\begin{aligned}\Big|\,\frac{f(x)-f(-x)}{x}\,\Big| &= \Big|\,\frac{1}{x}\int_{-x}^{x} f'(t)\,dt\,\Big| \\ &\le \frac{1}{x}\int_{-x}^{x} \big|f'(t)\big|\,dt \\ &\le 2\,\|f'\|_\infty\,.\end{aligned}$$

Therefore

$$\begin{aligned}\big|P(\tfrac{1}{x})(f)\big| &= \Big|\int_0^1 \frac{f(x)-f(-x)}{x}\,dx + \int_1^\infty \frac{f(x)-f(-x)}{x}\,dx\,\Big| \\ &\le \int_0^1 2\,\|f'\|_\infty\,dx + \int_1^\infty \{\,|f(x)|+|f(-x)|\,\}\,x\,\frac{dx}{x^2} \\ &\le 2\,\|f'\|_\infty + 2\,\|xf(x)\|_\infty \int_1^\infty \frac{dx}{x^2} \\ &= 2\,\|\!|f|\!\|_{0,1} + 2\,\|\!|f|\!\|_{1,0}\,.\end{aligned}$$

The result follows. ■

Definition 2.17. A sequence (T_n) in $\mathscr{S}'(\mathbb{R}^d)$ is said to converge in $\mathscr{S}'(\mathbb{R}^d)$ if $T_n(f) \to T(f)$ for each $f \in \mathscr{S}(\mathbb{R}^d)$. One also says that T_n converges to T in the sense of distributions.

Example 2.18. Lebesgue's Dominated Convergence Theorem implies that (if each g_n is measurable and) if $g_n(x) \to g(x)$ pointwise and if $|g_n(x)| \le \varphi(x)$ for some integrable function φ, then $\int g_n(x)f(x)\,dx \to \int g(x)f(x)\,dx$ for each $f \in \mathscr{S}$. In other words, the sequence T_{g_n} of tempered distributions converges to T_g in $\mathscr{S}'(\mathbb{R}^d)$.

We wish to discuss the well-known formula

$$\lim_{\varepsilon\downarrow 0} \frac{1}{x-x_0+i\varepsilon} = P(\tfrac{1}{x-x_0}) - i\pi\,\delta(x-x_0)\,.$$

We will see that this holds with convergence in the sense of distributions.

Theorem 2.19. *Let* $g_\varepsilon(x) = \dfrac{x}{x^2+\varepsilon^2}$. *Then* $T_{g_\varepsilon} \to P(\frac{1}{x})$ *in* $\mathscr{S}'(\mathbb{R})$ *as* $\varepsilon \downarrow 0$.

Proof. Let $f \in \mathscr{S}(\mathbb{R})$ and let $\delta > 0$. Then

$$\begin{aligned}\Big|\;P(\tfrac{1}{x})(f) - T_{g_\varepsilon}(f)\;\Big| &= \Big|\;P(\tfrac{1}{x})(f) - \int_{-\infty}^\infty \frac{x\,f(x)}{x^2+\varepsilon^2}\,dx\;\Big| \\ &= \Big|\int_0^\infty \frac{f(x)-f(-x)}{x}\,dx - \int_0^\infty \frac{xf(x)-xf(-x)}{x^2+\varepsilon^2}\,dx\;\Big|\end{aligned}$$

$$
\begin{aligned}
&= \Big| \int_0^\infty \frac{\varepsilon^2}{(x^2+\varepsilon^2)} \, \frac{\big(f(x)-f(-x)\big)}{x} \; dx \, \Big| \\
&\le \Big| \int_0^\delta \frac{\varepsilon^2}{(x^2+\varepsilon^2)} \, \frac{\big(f(x)-f(-x)\big)}{x} \; dx \, \Big| \\
&\qquad + \Big| \int_\delta^\infty \frac{\varepsilon^2}{(x^2+\varepsilon^2)} \, \frac{\big(f(x)-f(-x)\big)}{x} \; dx \, \Big| \\
&\le \int_0^\delta \Big| \frac{f(x)-f(-x)}{x} \Big| \; dx \\
&\qquad + \int_\delta^\infty \frac{\varepsilon^2}{\delta^2} \, \Big| \frac{f(x)-f(-x)}{x} \Big| \; dx \,.
\end{aligned}
$$

Now, $\frac{f(x)-f(-x)}{x} \to 2f'(0)$ as $x \downarrow 0$ and so the first term on the right hand side can be made arbitrarily small by choosing δ sufficiently small. But for fixed $\delta > 0$, the fact that $\frac{f(x)-f(-x)}{x}$ is integrable means that the second term approaches 0 as $\varepsilon \to 0$. [Alternatively, one can set $\delta = \sqrt{\varepsilon}$ in the discussion above. Another proof is to use Lebesgue's Monotone Convergence Theorem together with the fact that $\frac{\varepsilon^2}{x^2+\varepsilon^2} \big| \frac{f(x)-f(-x)}{x} \big| \downarrow 0$ on $(0,\infty)$ as $\varepsilon \downarrow 0$.] ∎

The next theorem tells us that the Dirac delta function is the limit, in the sense of distributions, of a sequence of functions whose graphs become thin, tall peaks around $x = 0$.

Theorem 2.20. *Let* (φ_n) *be a sequence of functions on* $\mathbb{R}$ *such that*

(i) $\varphi_n(x) \ge 0$ *for all* $x \in \mathbb{R}$.

(ii) $\int \varphi_n(x)\, dx = 1$ *for all* n.

(iii) *For any* $a > 0$, $\int_{|x|\ge a} \varphi_n(x)\, dx \to 0$ *as* $n \to \infty$.

Then $\varphi_n \to \delta$ *in* $\mathscr{S}'(\mathbb{R})$ *as* $n \to \infty$ *(that is,* $T_{\varphi_n} \to \delta$ *in* $\mathscr{S}'(\mathbb{R})$*).*

Proof. Fix $f \in \mathscr{S}(\mathbb{R})$ (with $f \not\equiv 0$). To show that $\int \varphi_n(x)\, f(x)\, dx \to f(0)$ as $n \to \infty$, let $\varepsilon > 0$ be given. Then, for any $\eta > 0$,

$$
\begin{aligned}
\Big| \int \varphi_n(x)\, f(x)\, dx - f(0) \Big| &= \Big| \int \varphi_n(x)\, (f(x)-f(0))\, dx \Big| \\
&\le \int_{-\eta}^{\eta} \varphi_n(x)\, |f(x)-f(0)| \; dx \\
&\qquad + \int_{x \ge \eta} \varphi_n(x)\, |f(x)-f(0)| \; dx \,.
\end{aligned}
$$

Fix $\eta > 0$ such that $|f(x) - f(0)| < \frac{1}{2}\varepsilon$ for all $|x| \le \eta$. Then we can estimate the first term on the right hand side by

$$\begin{aligned}\int_{-\eta}^{\eta} \varphi_n(x)\, |f(x) - f(0)|\, dx &\le \tfrac{1}{2}\varepsilon \int_{-\eta}^{\eta} \varphi_n(x)\, dx \\ &\le \tfrac{1}{2}\varepsilon \int_{-\infty}^{\infty} \varphi_n(x)\, dx \\ &= \tfrac{1}{2}\varepsilon\end{aligned}$$

for all n. Furthermore, by hypothesis, there is $N \in \mathbb{N}$ such that if $n > N$ then

$$\int_{|x|\ge\eta} \varphi_n(x)\, dx < \frac{\varepsilon}{4\|f\|_\infty}\,.$$

So for $n > N$, the second term on the right hand side above is estimated according to

$$\begin{aligned}\int_{x\ge\eta} \varphi_n(x)\, |f(x) - f(0)|\, dx &\le 2\|f\|_\infty \int_{x\ge\eta} \varphi_n(x)\, dx \\ &\le \tfrac{1}{2}\varepsilon\,.\end{aligned}$$

Hence, for all $n > N$, we find that

$$\Big| \int \varphi_n(x)\, f(x)\, dx - f(0) \Big| < \varepsilon$$

as required. ∎

Remark 2.21. If we replace (iii) by the requirement that $\int_{|x-x_0|\ge a} \varphi_n(x)\, dx \to 0$ as $n \to \infty$, then one sees that $\varphi_n \to \delta_{x_0}$ in $\mathscr{S}'(\mathbb{R})$.

Corollary 2.22. *For $\varepsilon > 0$, let $g_\varepsilon = \dfrac{\varepsilon}{(x-x_0)^2 + \varepsilon^2}$. Then $g_\varepsilon \to \pi\,\delta_{x_0}$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \to 0$.*

Proof. Clearly $g_\varepsilon(x) \ge 0$ for all $x \in \mathbb{R}$ and $\int g_\varepsilon(x)\, dx = \pi$. Also, for any $a > 0$,

$$\begin{aligned}\int_{|x-x_0|\ge a} g_\varepsilon(x)\, dx &= 2\int_a^\infty \frac{\varepsilon}{x^2+\varepsilon^2}\, dx \\ &= 2\Big[\tan^{-1}\tfrac{x}{\varepsilon}\Big]_a^\infty \\ &= 2\Big(\tfrac{\pi}{2} - \tan^{-1}\tfrac{a}{\varepsilon}\Big) \\ &\to 0\end{aligned}$$

as $\varepsilon \to 0$. The result now follows from the theorem (applied to $\frac{1}{\pi}\, g_\varepsilon$). ∎

Theorem 2.23. *For $\varepsilon > 0$, let $h_\varepsilon = \dfrac{1}{x - x_0 + i\varepsilon}$. Then*

$$h_\varepsilon \to P(\tfrac{1}{x-x_0}) - i\pi\,\delta_{x_0}$$

in $\mathscr{S}'(\mathbb{R})$, as $\varepsilon \to 0$.

Proof. We have

$$\begin{aligned} h_\varepsilon(x) &= \frac{1}{x - x_0 + i\varepsilon} = \frac{x - x_0 - i\varepsilon}{(x-x_0)^2 + \varepsilon^2} \\ &= \frac{(x-x_0)}{(x-x_0)^2+\varepsilon^2} - \frac{i\,\varepsilon}{(x-x_0)^2+\varepsilon^2} \\ &\to P(\tfrac{1}{x-x_0}) - i\pi\,\delta_{x_0} \end{aligned}$$

in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \to 0$, by the previous theorems. ∎

To motivate the next definition, consider the integral $\int g'(x)\,f(x)\;dx$ where $f, g \in \mathscr{S}(\mathbb{R})$. Integrating by parts, we find that

$$\int g'(x)\,f(x)\;dx = -\int g(x)\,f'(x)\;dx\,.$$

Using our notation introduced earlier, identifying a function g with the distribution T_g, this equality becomes

$$T_{g'}(f) = -T(g)(f')\,.$$

If we think of $T_{g'}$ as the derivative of T_g, then the following definition is quite natural.

Definition 2.24. Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{Z}_+^d$. The weak derivative $D^\alpha T$ (or the derivative in the sense of distributions) is defined by

$$(D^\alpha T)(f) = (-1)^{|\alpha|}\,T(D^\alpha f)$$

for $f \in \mathscr{S}(\mathbb{R}^d)$.

This corresponds to $D^\alpha T_g = T_{D^\alpha g}$. Note that a distribution *always* has a weak derivative. Of course, we should verify that the weak derivative of a tempered distribution is also a tempered distribution. We do this next.

Theorem 2.25. *For any $\alpha \in \mathbb{Z}_+^d$, $D^\alpha : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous. In particular, for any $T \in \mathscr{S}'(\mathbb{R}^d)$, $D^\alpha T \in \mathscr{S}'(\mathbb{R}^d)$.*

Proof. Suppose that $f_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. Let $\gamma, \delta \in \mathbb{Z}_+^d$. Then

$$\begin{aligned} |||D^\alpha f_n|||_{\gamma,\delta} &= \|x^\gamma D^\delta D^\alpha f_n\|_\infty \\ &= \|x^\gamma D^{\delta+\alpha} f_n\|_\infty \end{aligned}$$

$$= \|f_n\|_{\gamma,\alpha+\delta} \to 0$$

as $n \to \infty$. Hence $D^\alpha : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous.

Now let $T \in \mathscr{S}'(\mathbb{R}^d)$. Evidently $D^\alpha T$ is well-defined and is a linear map on $\mathscr{S}(\mathbb{R}^d)$. If $f_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$, then $D^\alpha f_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$, by the first part. Hence

$$(D^\alpha T)(f_n) = (-1)^{|\alpha|}\, T(D^\alpha f_n) \to 0$$

and so $D^\alpha T \in \mathscr{S}'(\mathbb{R}^d)$, as required. ■

Examples 2.26.

1. We find that $\delta_a'(f) = -\delta_a(f) = -f'(a)$.

2. Let $g(x) = \begin{cases} x, & x > 0 \\ 0, & x \le 0. \end{cases}$

   Then we know that $T_g \in \mathscr{S}'(\mathbb{R})$ and

$$\begin{aligned} T_g'(f) &= -T_g(f') \\ &= -\int_0^\infty x\, f'(x)\, dx \\ &= -[xf(x)]_0^\infty + \int_0^\infty f(x)\, dx \\ &= \int_0^\infty f(x)\, dx \\ &= \int_{-\infty}^\infty H(x)\, f(x)\, dx \end{aligned}$$

   where $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside step-function.

   So $T_g' = T_H$. Moreover,

$$\begin{aligned} T_H'(f) &= -T_H(f') \\ &= -\int_0^\infty f'(x)\, dx \\ &= -[f(x)]_0^\infty \\ &= f(0) \\ &= \delta(f) \end{aligned}$$

   and so $T_H' = \delta$. Therefore $T_g' = T_H$ and $T_g'' = T_H = \delta$. We say $g' = H$ and $g'' = H' = \delta$, in the sense of distributions.

Remark 2.27. We notice that although δ is not a function, it is the second distributional derivative of a continuous function, namely g. We will see that every tempered distribution is the weak derivative (of a suitable order) of some continuous function.

Definition 2.28. The support of a function f on $\mathbb{R}^d$, denoted by $\operatorname{supp} f$, is the closure of the set where f does not vanish;

$$\operatorname{supp} f = \overline{\{ x \in \mathbb{R}^d : f(x) \neq 0 \}}\,.$$

Let $C_0^\infty(\mathbb{R}^d)$ denote the linear subspace of $C^\infty(\mathbb{R}^d)$ of those functions with compact support. Clearly $C_0^\infty(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$.

Example 2.29. For $x \in \mathbb{R}$, let

$$h(x) = \begin{cases} e^{-1/(1-x^2)}, & |x| \le 1 \\ 0, & |x| > 1\,. \end{cases}$$

Then h is infinitely-differentiable and one finds that its n^{th} derivative has the form $h^{(n)}(x) = p_n(x, 1/(1-x^2))\,h(x)$ for some polynomial $p_n(s,t)$ and therefore $h \in C_0^\infty(\mathbb{R})$.

Let $g(x) = \int_{-\infty}^x h(t)\ dt$. Then g is infinitely-differentiable, $g(x) = 0$ for $x < -1$ and g is constant for $x > 1$. Evidently $g \notin \mathscr{S}(\mathbb{R})$.

Let $g_\lambda(x) = g(\lambda x)$, where $\lambda > 0$. Then g_λ is zero for $x < -1/\lambda$ and constant for $x > 1/\lambda$. Now let $g_{\lambda,a}(x) = g(\lambda(x-a))$. Then $g_{\lambda,a}$ vanishes for $x < a - \frac{1}{\lambda}$ and is constant when $x > a + \frac{1}{\lambda}$.

Let $a < b$ and suppose that λ, μ are such that $a < a + \frac{1}{\lambda} < b - \frac{1}{\mu} < b$. Let $f(x) = g_{\lambda,a}(x)\,g_{\mu,-b}(-x)$. Then $f \in C^\infty(\mathbb{R})$ and we see that $f(x) = 0$ for $x < a - \frac{1}{\lambda}$, $f(x) = 0$ for $x > b + \frac{1}{\mu}$ and f is constant for $a + \frac{1}{\lambda} < x < b - \frac{1}{\mu}$. Evidently $f \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} f \subset [a - \frac{1}{\lambda}, b + \frac{1}{\mu}]$.

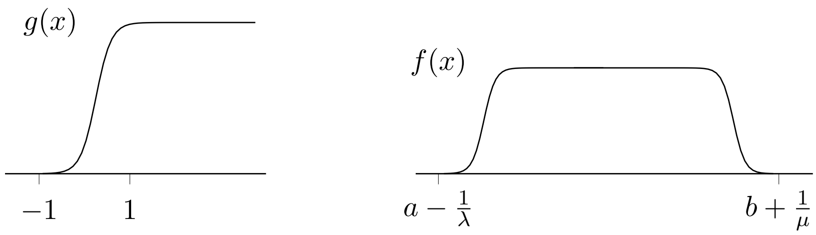

Figure 2.1: The functions $g(x)$ and $f(x)$.

In d-dimensions, set $f(x_1, \dots, x_d) = g(|x|^2)$ where $g \in C^\infty(\mathbb{R})$ is such that $g(t) = 1$ for $0 \le t \le a$ and $g(t) = 0$ for $x \ge b$, where $0 < a < b$. Then $f \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp} f \subset \{ x \in \mathbb{R}^d : |x|^2 \le b^2 \}$ and $f(x) = 1$ for $|x|^2 \le a^2$.

In particular, if $N_r(x_0)$ denotes the open ball of radius r, centred at x_0 in $\mathbb{R}^d$, then there exist functions $f \in C_0^\infty(\mathbb{R}^d)$ such that $f = 1$ on $N_{r/2}(x_0)$ and f vanishes outside $N_r(x_0)$.

Theorem 2.30. *Let K be compact and let A be an open set in $\mathbb{R}^d$ with $K subset A$. Then there exists a C^∞-function φ such that $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^d$, $\varphi(x) = 1$ for all $x \in K$ and $\varphi(x) = 0$ for $x \notin A$.*

Proof. For each $x \in K$, there is $r(x) > 0$ such that $N_{r(x)}(x) \subset A$. The collection $\{ N_{r(x)/2}(x) : x \in K \}$ is an open cover of the compact set K and so has a finite subcover, that is, there is $x_1, \dots, x_m \in K$ such that

$$K \subset N_{r_1/2}(x_1) \cup \dots \cup N_{r_m/2}(x_m)$$

where $r_i = r(x_i)$.

Let $\varphi_i \in C^\infty(\mathbb{R}^d)$ be such that $0 \le \varphi_i(x) \le 1$, $\varphi_i(x) = 1$ for $x \in N_{r_i/2}(x_i)$ and $\varphi_i(x) = 0$ if $x \notin N_{r_i}(x)$. (Such functions can be constructed as in the previous example.)

Set

$$\varphi(x) = 1 - (1 - \varphi_1(x))(1 - \varphi_2(x)) \dots (1 - \varphi_m(x)) .$$

Then $\varphi \in C^\infty(\mathbb{R}^d)$ and obeys $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^d$. Furthermore, for any $x \in K$, there is some $1 \le i \le m$ such that $x \in N_{r_i/2}(x_i)$ and so $\varphi_i(x) = 1$ and therefore $\varphi(x) = 1$.

On the other hand, for any $x \notin A$, it is true that $x \notin N_{r_i}(x_i)$ for all $1 \le i \le m$ (since $N_{r_i}(x_i) \subset A$). Hence $\varphi_i(x) = 0$ for all $1 \le i \le m$ and so $\varphi(x) = 1 - 1 = 0$. Therefore φ satisfies the requirements and the proof is complete. ■

Theorem 2.31. *$C_0^\infty(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$.*

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and let $f_n \in C_0^\infty(\mathbb{R}^d)$ be a sequence of functions such that $\operatorname{supp} f_n \subset \{ x \in \mathbb{R}^d : |x| < n + 1 \}$, $f_n(x) = 1$ for $|x| \le n - 1$ and such that the shape of the graph of f_n for $|x|$ between $n - 1$ and $n + 1$ is independent of n. This means that for any given multi-index $\gamma \in \mathbb{Z}_+^d$, $D^\gamma f_n(x)$ is bounded uniformly in n. (Such functions can be constructed as in example 2.29).

Let $\varphi_n = \varphi f_n$. Then $\varphi_n \in C_0^\infty(\mathbb{R}^d)$ and

$$\begin{aligned} \|\varphi_n - \varphi\|_{\alpha,\beta} &= \|\varphi(f_n - 1)\|_{\alpha,\beta} \\ &= \sup_{x \ge n-1} | x^\alpha D^\beta(\varphi(x)(f_n(x) - 1)) | . \end{aligned}$$

But for each $\gamma \in \mathbb{Z}_+^d$, $\|D^\gamma(f_n(x) - 1)\|_\infty$ is bounded uniformly in n and so it follows by Leibnitz' formula and the fact that $\varphi \in \mathscr{S}$ that $\|\varphi_n - \varphi\|_{\alpha,\beta} \to 0$ as $n \to \infty$ for each $\alpha, \beta \in \mathbb{Z}_+^d$. ■

Definition 2.32. Let G be an open set in $\mathbb{R}^d$. We say that a distribution T vanishes on G if $T(\varphi) = 0$ for each $\varphi \in \mathscr{S}(\mathbb{R}^d)$ with $\operatorname{supp} \varphi \subset G$.

If $g \in \mathscr{S}$ vanishes on G (as a function) then evidently $\int g(x)\, \varphi(x)\, dx = 0$ for all $\varphi \in \mathscr{S}$ with $\operatorname{supp} \varphi \subset G$, that is, T_g vanishes on G as a distribution.

Remark 2.33. Suppose that $T \in \mathscr{S}'(\mathbb{R}^d)$ vanishes on the open sets G_1 and G_2 where $G_1 \cap G_2 = \varnothing$. Then T also vanishes on $G_1 \cup G_2$. To see this, let $\varphi \in \mathscr{S}$ with $\operatorname{supp} \varphi \subset G_1 \cup G_2$.

Now $\operatorname{supp} \varphi \cap G_1$ is a closed set in G_1 and so there is an infinitely-differentiable function f_1 such that $f_1 = 1$ on $\operatorname{supp} \varphi \cap G_1$ and $f_1 = 0$ outside some closed set F_1 containing $\operatorname{supp} \varphi \cap G_1$. Hence $f_1 \varphi$ has support in G_1. Similarly, there is some f_2 such that $f_2 \varphi$ has support in G_2.

But $\varphi = f_1 \varphi + f_2 \varphi$ and therefore $T(\varphi) = T(f_1 \varphi) + T(f_2 \varphi) = 0$ since the distribution T vanishes on both G_1 and G_2.

This result has a satisfactory generalization, as follows.

Theorem 2.34. *Suppose that $T \in \mathscr{S}'(\mathbb{R}^d)$ vanishes on each member of a collection $\{ G_\alpha \}$ of open sets. Then T vanishes on $\bigcup_\alpha G_\alpha$.*

Proof. A proof of this result may be found in Rudin's book[1]. ∎

Thanks to this theorem, the following (desirable) definition makes sense.

Definition 2.35. For any $T \in \mathscr{S}'(\mathbb{R}^d)$, let W denote the union of all open sets on which T vanishes. The support of T is defined to be $\operatorname{supp} T = W^c$, the complement of W in $\mathbb{R}^d$.

Examples 2.36.

1. One sees that $\operatorname{supp} \delta_a = \{ a \}$, for any $a \in \mathbb{R}^d$.
2. If H is the Heaviside function, then we see that $\operatorname{supp} H = [0, \infty)$.

[1] [*Functional Analysis*, by Walter Rudin, Tata McGraw-Hill, 1973]

Chapter 3

The spaces $\mathscr{D}$ and $\mathscr{D}'$

In this section, we consider another space of functions and the associated collection of continuous linear functionals.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$. $C_0^\infty(\Omega)$ denotes the linear subset consisting of those functions in $C_0^\infty(\mathbb{R}^d)$ which have support in Ω. Suppose that (φ_n) is a sequence in $C_0^\infty(\Omega)$ and let $\varphi \in C_0^\infty(\mathbb{R}^d)$. We say that $\varphi_n \to \varphi$ in $C_0^\infty(\Omega)$ if

(i) there is some compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for all n, and

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly as $n \to \infty$, for each $\alpha \in \mathbb{Z}_+^d$.

Note that it follows immediately that $\operatorname{supp} \varphi \subset K$.

$\mathscr{D}(\Omega)$ is the space $C_0^\infty(\Omega)$ equipped with this notion of convergence and we say that $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$.

Example 3.2. Let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(x) = 0$ for $|x| > 1$. For each $n \in \mathbb{N}$, let $\psi_n(x) = \psi(x-1) + \frac{1}{2}\,\psi(x-2) + \cdots + \frac{1}{n}\,\psi(x-n)$ (so that ψ_n comprises n smaller and smaller smooth "bumps").

Evidently, $\psi_n \in C_0^\infty(\mathbb{R})$ and (ψ_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_\infty$. Indeed, $\|\psi_n - \varphi\|_\infty \to 0$ where $\varphi(x) = \sum_{k=1}^\infty \frac{1}{k}\,\psi(x-k)$. Clearly $\varphi \in C^\infty(\mathbb{R})$ but ψ does not converge to φ in $\mathscr{D}(\mathbb{R})$ because the supports of the ψ_n are not all contained in a compact set (and $\varphi \notin C_0^\infty(\mathbb{R})$, anyway).

Continuing with this notation, let $h_n(x) = \frac{1}{n}\,\psi(x-n)$. Then $h_n \in C_0^\infty(\mathbb{R})$ and $\|h_n\|_\infty \to 0$ but (h_n) does not converge to 0 in $\mathscr{D}(\mathbb{R})$ (because there is no compact set K such that $\operatorname{supp} h_n \subset K$ for all n).

The notion of convergence in $\mathscr{D}$ ensures its completeness, as we show next.

Definition 3.3. We say that (φ_n) is a Cauchy sequence in $\mathscr{D}(\Omega)$ if there is some compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subseteq K$ for all n and such that $\|D^\alpha(\varphi_n - \varphi_m)\|_\infty \to 0$ as $n, m \to \infty$ for every $\alpha \in \mathbb{Z}_+^d$.

Theorem 3.4. *$\mathscr{D}(\Omega)$ is complete.*

Proof. Exactly as in the proof of the completeness of $\mathscr{S}$, we see that if (φ_n) is a Cauchy sequence in $\mathscr{D}(\Omega)$, then there is some $f \in C^\infty(\mathbb{R}^d)$ such that $\|D^\alpha(\varphi_n - f)\|_\infty \to 0$ for all $\alpha \in \mathbb{Z}_+^d$. But if $\operatorname{supp} \varphi_n \subset K$ for all n, then it follows that $\operatorname{supp} f \subset K$ also. Hence $f \in C_0^\infty(\Omega)$ and $\varphi_n \to f$ in $\mathscr{D}(\Omega)$. ∎

Definition 3.5. A linear functional $u : \mathscr{D}(\Omega) \to \mathbb{C}$ is said to be continuous if $u(\varphi_n) \to u(\varphi)$ whenever $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$ as $n \to \infty$. Such a continuous linear functional is called a distribution. The linear space of distributions is denoted $\mathscr{D}'(\Omega)$.

The derivatives of a distribution are defined as for tempered distributions, namely by the formula

$$D^\alpha u(\varphi) = (-1)^{|\alpha|}\, u(D^\alpha \varphi)$$

for $\varphi \in \mathscr{D}(\Omega)$ and $\alpha \in \mathbb{Z}_+^d$.

Example 3.6. Clearly the map $\delta_a : \varphi \mapsto \varphi(a)$ (Dirac delta "function") is a distribution (i.e., belongs to $\mathscr{D}'(\mathbb{R}^d)$) for any $a \in \mathbb{R}^d$.

Example 3.7. Suppose u is a locally integrable function (that is, $u \in L^1(K)$ for each compact set $K \subset \mathbb{R}^d$). Then the map $T_u : \varphi \mapsto \int_{\mathbb{R}^d} u(x)\, \varphi(x)\, dx$ is a distribution. (In particular, $u(x) = e^{x^2}$ defines a distribution, $T_u \in \mathscr{D}'(\mathbb{R})$ but $T_u \notin \mathscr{S}'(\mathbb{R})$. Indeed, T_u is not defined on every element of $\mathscr{S}(\mathbb{R})$.)

To see this, we first note that T_u is well-defined because φ has compact support if it belongs to $\mathscr{D}(\mathbb{R}^d)$. Furthermore,

$$\begin{aligned} |T_u(\varphi)| &= \Big| \int u(x)\, \varphi(x)\, dx \Big| \\ &\le \int_K |u(x)|\, |\varphi(x)|\, dx\,, \quad \text{where } K = \operatorname{supp} \varphi, \\ &\le \|\varphi\|_\infty \int_K |u(x)|\, dx\,. \end{aligned}$$

From this, we see that if $\varphi_n \to \varphi$ in $\mathscr{D}(\mathbb{R}^d)$, then certainly $T_u(\varphi_n) \to T_u(\varphi)$ so that $T_u \in \mathscr{D}'(\mathbb{R}^d)$, as claimed.

Remark 3.8. It is sometimes convenient to identify u with T_u and to consider the function u as being a distribution (namely that given by T_u).

The next result tells us that every tempered distribution is a distribution. (If this were not true then the terminology would be most inappropriate.)

Theorem 3.9. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$. Then $T \restriction C_0^\infty(\mathbb{R}^d) \in \mathscr{D}'(\mathbb{R}^d)$.*

Proof. Suppose that $\varphi_n \to \varphi$ in $\mathscr{D}(\mathbb{R}^d)$. Then there is some compact set $K \subset \mathbb{R}^d$ such that $\operatorname{supp} \varphi_n \subset K$ for all n. So for any $\alpha, \beta \in \mathbb{Z}_+^d$

$$\begin{aligned}
|||\varphi_n - \varphi|||_{\alpha,\beta} &= \|x^\alpha D^\beta(\varphi_n - \varphi)\|_\infty \\
&= \sup_{x \in K} |\, x^\alpha D^\beta(\varphi_n - \varphi)\,| \\
&\le C_\alpha \sup_x |\, D^\beta(\varphi_n - \varphi)\,| \\
&\to 0\,, \quad \text{as } n \to \infty,
\end{aligned}$$

where C_α is some constant such that $|x^\alpha| < C_\alpha$ for all $x \in K$. It follows that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R}^d)$ and so $T(\varphi_n) \to T(\varphi)$. Hence $\varphi \mapsto T(\varphi)$ is continuous on $\mathscr{D}(\mathbb{R}^d)$. ∎

Theorem 3.10. *A linear functional u on $C_0^\infty(\Omega)$ is a distribution if and only if for each compact subset $K \subset \Omega$ there is a constant C and an integer N such that*

$$|u(\varphi)| \le C\, |||\varphi|||_N\,, \quad \textit{for all } \varphi \in C_0^\infty(K),$$

where $|||\varphi|||_N \equiv \sum_{|\beta| \le N} \|D^\beta \varphi\|_\infty$.

Proof. Clearly u is continuous on $\mathscr{D}(\Omega)$ if such bounds hold. Conversely, suppose that $u \in \mathscr{D}'(\Omega)$ but that no such bounds exist. Then there is some compact set $K_0 \subset \Omega$ and a sequence (φ_n) in $C_0^\infty(K_0)$ such that

$$|u(\varphi_j)| > j\, |||\varphi_j|||_j\,, \quad \text{for all } j \in \mathbb{N}.$$

Set $f_j = \varphi_j / u(\varphi_j)$ so that $u(f_j) = 1$ for all j. However, for any $\beta \in \mathbb{Z}_+^d$,

$$\begin{aligned}
\|D^\beta f_j\|_\infty &= \frac{\|D^\beta \varphi_j\|_\infty}{|u(\varphi_j)|} \\
&\le \frac{|||\varphi|||_j}{|u(\varphi_j)|}\,, \quad \text{for all } j > |\beta|, \\
&< \tfrac{1}{j} \to 0\,, \quad \text{as } j \to \infty.
\end{aligned}$$

Hence $f_j \to 0$ in $\mathscr{D}(\Omega)$ which forces $u(f_j) \to 0$. This contradicts the fact that, by construction, $u(f_j) = 1$ for all j. The result follows. ∎

Remark 3.11. Note that the same proof works if the norm $|||\varphi|||_N$ above is redefined to be $\max_{|\beta| \le N} \|D^\beta \varphi\|_\infty$.

Definition 3.12. If the integer N in theorem 3.10 can be chosen independently of K, then the distribution u is said to of finite order. The smallest such N is called the order of u.

Example 3.13. Let $u \in \mathscr{D}'(\mathbb{R})$ be given by $u = \delta'$, that is, $u : \varphi \mapsto \varphi'(0)$. Then we see that u has order 1. On the other hand, if $u = \sum_{n=0}^{\infty} \delta_n^{(n)}$ (where $\delta_n^{(n)}(\varphi) = (-1)^n \varphi^{(n)}(n)$), then u is an element of $\mathscr{D}'(\mathbb{R})$ but its order is infinite.

Note that in this latter case, $\operatorname{supp} u = \{ 0, 1, 2, \dots \}$. We show next that distributions with compact support must have finite order.

Theorem 3.14. *Suppose that $u \in \mathscr{D}'(\Omega)$ and that $\operatorname{supp} u$ is compact. Then u has finite order. In fact, there is $C > 0$ and $N \in \mathbb{Z}_+$ such that*

$$|u(\varphi)| \le C \, \|\!|\varphi|\!\|_N$$

for all $\varphi \in C_0^\infty(\Omega)$ (i.e., C does not depend on φ nor on $\operatorname{supp} \varphi$).

Proof. Suppose that $u \in \mathscr{D}'(\Omega)$ and that $\operatorname{supp} u$ is compact. Let W be an open set with $\operatorname{supp} u \subset W$ and let $\psi \in C_0^\infty(\Omega)$ be such that $\psi = 1$ on W. For any $\varphi \in C_0^\infty(\Omega)$, we have

$$u(\varphi) = u(\psi\,\varphi + (1-\psi)\,\varphi) = u(\psi\,\varphi) + u((1-\psi)\,\varphi)\,.$$

But $(1-\psi)\,\varphi = 0$ on W and so $u((1-\psi)\,\varphi) = 0$ and therefore $u(\varphi) = u(\psi\,\varphi)$ for all $\varphi \in C_0^\infty(\Omega)$. It follows by theorem 3.10 that there is some $C' > 0$ and $N \in \mathbb{Z}_+$ such that

$$|u(\varphi)| = |u(\varphi\,\psi)| \le C' \, \|\!|\varphi\,\psi|\!\|_N$$

for all $\varphi \in C_0^\infty(\Omega)$ (since $\operatorname{supp} \varphi\,\psi \subseteq \operatorname{supp} \psi$). Note that C' does not depend on φ nor $\operatorname{supp} \varphi$ but may depend on $\operatorname{supp} \psi$. An application of Leibnitz formula implies that

$$\begin{aligned} |u(\varphi)| &= |u(\varphi\,\psi)| \\ &\le C' \, \|\!|\varphi\,\psi|\!\|_N \\ &\le C \, \|\!|\varphi|\!\|_N \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$ for some $C > 0$ which may depend on ψ but does not depend on φ nor on $\operatorname{supp} \varphi$. The result follows. ∎

Remark 3.15. The converse is false. Indeed, suppose that $u(x)$ is a non-zero constant, say $u(x) = c \ne 0$, for all $x \in \mathbb{R}$. Then $\operatorname{supp} u = \mathbb{R}$ which is not compact. However, for any compact set K, and any $\varphi \in C_0^\infty(K)$, we have

$$\begin{aligned} |u(\varphi)| &= \Big| \int c\,\varphi(x)\,dx \Big| \\ &\le |c| \operatorname{diam} K \, \|\varphi\|_\infty\,. \end{aligned}$$

So u has order zero but its support is not compact.

Theorem 3.16. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$. Then $T \restriction \mathscr{D}(\mathbb{R}^d)$ is a distribution of finite order.*

Proof. We have already seen that $T \restriction \mathscr{D}(\mathbb{R}^d)$ is a distribution, so we need only show that it has finite order. Since $T \in \mathscr{S}'(\mathbb{R}^d)$, there is $C_0 > 0$ and integers $n, k \in \mathbb{Z}_+$ such that

$$|T(f)| \le C_0 \, \|f\|_{n,k} \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^d).$$

But if $K \subset \mathbb{R}^d$ is compact, there is $M > 0$ such that $|x^\alpha| \le M$ for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \le n$ and all $x \in K$. Hence there is $C' > 0$ such that

$$|T(f)| \le C' \, \|f\|_{n,k} \,, \quad \text{for all } f \in C_0^\infty(K).$$

It follows that

$$|T(\varphi)| \le C' \, |\!|\!|\varphi|\!|\!|_k$$

for all $\varphi \in C_0^\infty(K)$, where k does not depend on K and so we see that the order of T on $\mathscr{D}(\mathbb{R}^d)$ is finite (no larger than k). ∎

Remark 3.17. Again, the converse is false. For example, the linear map $\varphi \mapsto \int e^{x^2} \varphi(x)\, dx$ defines an element of $\mathscr{D}'(\mathbb{R})$ which does not extend to a continuous functional on $\mathscr{S}(\mathbb{R})$. However, for any compact set $K \subset \mathbb{R}$,

$$|u(\varphi)| \le C \, |\!|\!|\varphi|\!|\!|_0$$

for all $\varphi \in C_0^\infty(K)$ so that u has order 0. There is no $T \in \mathscr{S}'(\mathbb{R})$ such that $T \restriction \mathscr{D}(\mathbb{R}) = u$.

Theorem 3.18. *Let $u \in \mathscr{D}'(\mathbb{R}^d)$ and suppose that* $\operatorname{supp} u$ *is compact. Then $u \in \mathscr{S}'(\mathbb{R}^d)$, that is, there is a unique $T \in \mathscr{S}'(\mathbb{R}^d)$ such that $T \restriction \mathscr{D}(\mathbb{R}^d) = u$.*

Proof. Let W be an open ball in $\mathbb{R}^d$ with $\operatorname{supp} u \subset W$ and let $\psi \in C_0^\infty(\mathbb{R}^d)$ be such that $W \subset \operatorname{supp} \psi$ and $\psi = 1$ on W. For any $f \in \mathscr{S}(\mathbb{R}^d)$, $f\psi \in C_0^\infty(\mathbb{R}^d)$ and so we may define the linear map T on $\mathscr{S}(\mathbb{R}^d)$ by

$$T(f) = u(f\psi) \quad \text{for } f \in \mathscr{S}(\mathbb{R}^d).$$

Now, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi\,\psi = \varphi$ on W and so $\operatorname{supp}(\varphi\,\psi - \varphi) \subseteq W^c$ and therefore $u(\varphi\,\psi) = u(\varphi)$. It follows that $T(\varphi) = u(\varphi)$ for any $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Suppose that $f_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. Then (using Leibnitz' formula) it follows that $f_n\psi \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. But $\operatorname{supp} f_n\psi \subseteq \operatorname{supp}\psi$ for all n and so $f_n\psi \to 0$ in $\mathscr{D}(\mathbb{R}^d)$. This means that $T(f_n) = u(f_n\psi) \to 0$ as $n \to \infty$ and so T is continuous on $\mathscr{S}(\mathbb{R}^d)$, that is, $T \in \mathscr{S}'(\mathbb{R}^d)$.

To see that T is unique, suppose that $S \in \mathscr{S}'(\mathbb{R}^d)$ and $S \restriction \mathscr{D}(\mathbb{R}^d) = u$. Then $S - T \in \mathscr{S}'(\mathbb{R}^d)$ and vanishes on $C_0^\infty(\mathbb{R}^d)$ which is dense in $\mathscr{S}(\mathbb{R}^d)$. By continuity, it follows that $S = T$ on $\mathscr{S}(\mathbb{R}^d)$. ∎

We know that δ and all its derivatives are distributions whose support is the singleton set $\{\,0\,\}$. The next theorem gives a precise converse.

Theorem 3.19. *Let* $u \in \mathscr{D}'(\mathbb{R}^d)$ *be such that* $\operatorname{supp} u = \{\,0\,\}$. *Then there is* $N \in \mathbb{Z}_+$ *and constants* a_j *such that*

$$u = \sum_{|j| \le N} a_j \, D^j \delta \,.$$

Proof. We will only give the proof for $d = 1$. The general case is similar.

So suppose that $u \in \mathscr{D}'(\mathbb{R})$ with $\operatorname{supp} u = \{\,0\,\}$. Since $\{\,0\,\}$ is compact, it follows that u has finite order, N, say. Then there is $C > 0$ such that

$$|u(\varphi)| \le C \, |\!|\!|\varphi|\!|\!|_N \,, \quad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}).$$

Claim: if the derivatives $\varphi^{(j)}(0) = 0$ for all $0 \le j \le N$, then $u(\varphi) = 0$.

Proof of Claim. Let $h \in C_0^\infty(\mathbb{R})$ be such that $h(x) = 1$ for $|x| \le 1$ and such that $h(x) = 0$ if $|x| \ge 2$. Set $h_n(x) = h(nx)$ so that $h_n(x) = 1$ if $|x| \le \frac{1}{n}$ but $h_n(x) = 0$ if $|x| \ge \frac{2}{n}$. Let $\varphi_n(x) = h_n(x)\,\varphi(x)$. Evidently, $\operatorname{supp} \varphi_n \subseteq \{\, x : |x| \le 2 \,\}$.

We will show that $|\!|\!|\varphi_n|\!|\!|_N \to 0$ as $n \to \infty$. Let $k \in \mathbb{Z}_+$ be fixed such that $0 \le k \le N$ and let $\varepsilon > 0$ be given. Since $\varphi^{(k)}(0) = 0$, there is $\rho > 0$ such that $|\varphi^{(k)}(x)| < \varepsilon$ for all $|x| < \rho$. But then

$$|\, \varphi^{(k-1)}(x) \,| = \Big| \int_0^x \varphi^{(k)}(t) \; dt \Big| \le |x| \; \varepsilon$$

for all $|x| < \rho$. Continuing in this way, we obtain

$$|\varphi^{(k-2)}(x)| \le \frac{|x|^2}{2!} \; \varepsilon$$

$$\vdots$$

$$|\varphi^{(k-r)}(x)| \le \frac{|x|^r}{r!} \; \varepsilon$$

for all $|x| < \rho$. Using Leibnitz' formula, we find that

$$\begin{aligned} |\, D^k \varphi_n(x) \,| &= |\, D^k (h_n(x) \varphi(x)) \,| \\ &= \Big| \sum_{r=0}^{k} \binom{k}{r} \; D^r h_n(x) \, D^{k-r} \varphi(x) \Big| \\ &= \Big| \sum_{r=0}^{k} \binom{k}{r} \; n^r \, h^{(r)}(nx) \, \varphi^{(k-r)}(x) \; \Big| \\ &\le \sum_{r=0}^{k} \binom{k}{r} \; \frac{|nx|^r}{r!} \; |h^{(r)}(nx)| \, \varepsilon \,. \end{aligned}$$

Hence

$$\sup_x |\, D^k \varphi_n(x)\,| = \sup_{|x|\le 2/n} |\, D^k \varphi_n(x)\,| \le C'\,\varepsilon$$

for some constant $C' > 0$ (which may depend on k but not on φ_n) provided $2/n < \rho$, that is, $n > 2/\rho$. It follows that $\sup_x |\, D^k\varphi_n(x)\,| \to 0$ as $n \to \infty$ for each $0 \le k \le N$ and so $|\!|\!| \varphi_n |\!|\!|_N \to 0$, as required.

To complete the proof of the claim, we note that the bound for u implies that $u(\varphi_n) \to 0$. However, $\varphi(x) - \varphi_n(x) = 0$ if $|x| < 1/n$, and therefore $\operatorname{supp}(\varphi - \varphi_n) \subset \{\, x : |x| \ge 1/n \,\}$. Since $\operatorname{supp} u = \{\, 0 \,\}$, it follows that $u(\varphi - \varphi_n) = 0$ so $u(\varphi) = u(\varphi_n)$ for all n which forces $u(\varphi) = 0$ and the claim is proved.

To continue with the proof of the theorem, let $\varphi \in C_0^\infty(\mathbb{R})$ be given and define ψ by

$$\varphi(x) = \underbrace{\varphi(0) + x\,\varphi'(0) + \tfrac{x^2}{2!}\,\varphi^{(2)}(0) + \cdots + \tfrac{x^N}{N!}\,\varphi^{(N)}(0)}_{=p(x)} + \psi(x)\,.$$

Then ψ is infinitely-differentiable and $\psi^{(k)}(0) = 0$ for all $0 \le k \le N$. Note, however, that $\psi \notin C_0^\infty(\mathbb{R})$ — indeed, ψ is not even bounded.

Let $g \in C_0^\infty(\mathbb{R})$ be such that $g(x) = 1$ if $|x| \le 1$. Then $u(\varphi) = u(\varphi\, g)$ and

$$\varphi\, g = p\, g + \psi\, g\,.$$

Now, $\psi\, g \in C_0^\infty(\mathbb{R})$ and (by Leibnitz' formula) we see that $(\psi\, g)^{(k)}(0) = 0$ for all $0 \le k \le N$ and so $u(\psi\, g) = 0$, according to the claim above. Hence

$$\begin{aligned} u(\varphi) &= u(\varphi\, g) \\ &= u(\,p\, g\,) + u(\psi\, g) \\ &= u(\,p\, g\,) \\ &= \varphi(0)\, u(g) + \varphi'(0)\, u(xg) + \cdots + \varphi^{(N)}(0)\, u(x^N g/N!) \\ &= \sum_{k=0}^N a_k\, D^k \delta(\varphi) \end{aligned}$$

where $a_k = (-1)^k\, u(x^k\, g)/k!$. ∎

Definition 3.20. For any $\psi \in C_0^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$, the product $\psi\, u$ is the linear functional

$$\psi\, u : \varphi \mapsto u(\psi\, \varphi)\,, \quad \text{for } \varphi \in \mathscr{D}(\Omega).$$

Theorem 3.21.

(i) *For any* $\psi \in C_0^\infty(\Omega)$*, the product* $\psi\, u \in \mathscr{D}'(\Omega)$*.*

(ii) *Suppose that* ψ *and its derivatives are polynomially bounded, that is, for each multi-index* $\alpha \in \mathbb{Z}_+^d$*, there is some integer* $N_\alpha \in \mathbb{N}$ *and constant* $C_\alpha > 0$ *such that* $|\, D^\alpha \psi(x) \,| \le C_\alpha\, (1 + |x|^2)^{N_\alpha}$ *for all* $x \in \mathbb{R}^d$*. Then for any* $f \in \mathscr{S}(\mathbb{R}^d)$*, the function* $\psi\, f \in \mathscr{S}(\mathbb{R}^d)$ *and the map* $\psi\, T : f \mapsto T(\psi\, f)$ *defines a tempered distribution.*

Proof. (i) For any $\varphi \in C_0^\infty(\Omega)$, the product $\psi\, \varphi \in C_0^\infty(\Omega)$ also and so $\psi\, u$ is a well-defined linear functional on $\mathscr{D}(\Omega)$. Now if $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$, it follows from Leibnitz' formula that $\psi\, \varphi_n \to \psi\, \varphi$ in $\mathscr{D}(\Omega)$ and so $\psi\, u \in \mathscr{D}'(\Omega)$, as claimed.

(ii) If ψ and its derivatives are polynomially bounded, then $\psi\, f \in \mathscr{S}(\mathbb{R}^d)$ for any $f \in \mathscr{S}(\mathbb{R}^d)$. Furthermore, again by Leibnitz' formula, we see that if $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d)$, then also $\psi\, f_n \to \psi\, f$ in $\mathscr{S}(\mathbb{R}^d)$ and so $\psi T \in \mathscr{S}'(\mathbb{R}^d)$. ∎

Chapter 4

The Fourier transform

We begin with the definition of the Fourier transform and the inverse Fourier transform for smooth functions.

Definition 4.1. The Fourier transform of the function $f \in \mathscr{S}(\mathbb{R}^d)$ is the function $\mathscr{F}f$ given by

$$\mathscr{F}f(\lambda) = \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\lambda\, x}\, f(x)\, dx$$

where $\lambda \in \mathbb{R}^d$ and $\lambda\, x = \sum_{j=1}^d \lambda_j\, x_j$ for $x \in \mathbb{R}^d$.

The inverse Fourier transform of $f \in \mathscr{S}(\mathbb{R}^d)$ is the function $\mathscr{F}^{-1}f$ given by

$$\mathscr{F}^{-1}f(\lambda) = \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\lambda\, x}\, f(x)\, dx\,.$$

It is often convenient to also use the notation $\widehat{f}$ for $\mathscr{F}f$.

Of course, the terminology must be justified, that is, we must show that these transforms really are inverses of each other.

Proposition 4.2. *Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then $\widehat{f} \in \mathbb{C}^\infty(\mathbb{R}^d)$ and for any $\alpha, \beta \in \mathbb{Z}_+^d$*

$$\big(\,(i\lambda)^\alpha\, D^\beta \widehat{f}\,\big)(\lambda) = \big(\,D^\alpha((-ix)^\beta\, f(x))\,\big)\widehat{\ }\,(\lambda)\,.$$

In particular, $i\lambda_j\, \widehat{f}\,(\lambda) = (D_j f)\widehat{\ }\,(\lambda)$ *and* $(D_j \widehat{f}\,)(\lambda) = (-ix_j\, f(x))\widehat{\ }\,(\lambda)$.

Proof. Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_d = (0, 0, \ldots, 0, 1)$ denote the standard basis vectors for $\mathbb{R}^d$. For $f \in \mathscr{S}(\mathbb{R}^d)$ and $h \neq 0$,

$$\begin{aligned}
&\Big|\, \frac{\widehat{f}\,(\lambda + h\, e_j) - \widehat{f}\,(\lambda)}{h} \;+\; \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} i\, x_j\, e^{-i\lambda\, x}\, f(x)\, dx\,\Big| \\
&\qquad = \Big|\, \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \Big\{\, \frac{(e^{-i(\lambda + h e_j)\, x} - e^{-i\lambda\, x})}{h} \;+\; i\, x_j\, e^{-i\lambda\, x}\,\Big\}\, f(x)\, dx \Big| \\
&\qquad \to 0 \quad \text{as } h \to 0
\end{aligned}$$

since $f \in \mathscr{S}(\mathbb{R}^d)$. In other words, differentiation under the integral sign is justified. Repeated differentiation (since $x_j f(x) \in \mathscr{S}(\mathbb{R}^d)$) shows that

$$(D^\beta \widehat{f}\,)(\lambda) = \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-ix)^\beta \, e^{-i\lambda\, x} \, f(x)\, dx = \big((-ix)^\beta \, f(x) \big)\widehat{\ }\,(\lambda)\,.$$

Furthermore,

$$\begin{aligned}
(\,\lambda^\alpha \, D^\beta \widehat{f}\,)(\lambda) &= \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \lambda^\alpha \, (-ix)^\beta \, e^{-i\lambda\, x} \, f(x)\, dx \\
&= \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-i)^{-\alpha} \, (\, D_x^\alpha \, e^{-i\lambda\, x}\,)\, (-ix)^\beta \, f(x)\, dx \\
&= \tfrac{(-1)^\alpha}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-i)^{-\alpha} \, e^{-i\lambda\, x} \, D_x^\alpha \big\{ (-ix)^\beta \, f(x) \big\}\, dx \\
&\qquad\qquad \text{(integrating by parts)} \\
&= \tfrac{(-i)^\alpha}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\lambda\, x} \, D_x^\alpha \big\{ (-ix)^\beta \, f(x) \big\}\, dx
\end{aligned}$$

so that

$$(\,(i\lambda)^\alpha \, D^\beta \widehat{f}\,\,)(\lambda) = (\, D^\alpha (\,(-ix)^\beta \, f(x)\,)\,)\widehat{\ }\,(\lambda)$$

and the proof is complete. ∎

Remark 4.3. Clearly, similar formulae also hold for the inverse Fourier transform $\mathscr{F}^{-1} f$ (replacing i by $-i$).

Theorem 4.4. *Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are continuous maps on $\mathscr{S}(\mathbb{R}^d)$.*

Proof. We first show that if $f \in \mathscr{S}(\mathbb{R}^d)$, then so are $\mathscr{F} f$ and $\mathscr{F}^{-1} f$. For any $\alpha, \beta \in \mathbb{Z}_+^d$, we have

$$\begin{aligned}
|\,(\lambda^\alpha \, D^\beta \widehat{f}\,)(\lambda)\,| &= \Big| \tfrac{(-i)^\alpha}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\lambda x} \, D^\alpha \big((-ix)^\beta \, f(x) \big)\, dx \Big| \\
&\le \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\, D^\alpha \big(x^\beta \, f(x) \big)\,|\, dx\,.
\end{aligned}$$

It follows that $\| \widehat{f} \|_{\alpha,\beta} = \sup_{\lambda \in \mathbb{R}^d} |\,(\lambda^\alpha \, D^\beta \widehat{f}\,)(\lambda)\,|$ is finite for each pair of multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$ and therefore $\widehat{f} \in \mathscr{S}(\mathbb{R}^d)$. A similar proof holds for $\mathscr{F}^{-1} f$ (or one can simply note that $(\mathscr{F}^{-1} f)(\lambda) = (\mathscr{F} f)(-\lambda)$).

To show that $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous, we use the estimate obtained above. We have

$$\begin{aligned}
\| \widehat{f} \|_{\alpha,\beta} &\le \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\, D^\alpha \big(x^\beta \, f(x) \big)\,|\, dx \\
&\le \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{x_1^2 \, x_2^2 \cdots \, x_d^2}{(1 + x_1^2)(1 + x_2^2) \cdots (1 + x_d^2)} \,|\, D^\alpha \big(x^\beta \, f(x) \big)\,|\, dx
\end{aligned}$$

$$\le \sup_x |\, x_1^2 \cdots x_d^2 \, D^\alpha\big(x^\beta f(x)\big) \,| \, \tfrac{1}{(2\pi)^{d/2}} \, \Big(\int_{\mathbb{R}} \frac{1}{(1+t^2)} \, dt \,\Big)^d$$
$$\le C \, \|f\|_{m,n}$$

by Leibnitz' formula, for some constant $C > 0$ and integers $m, n \in \mathbb{Z}_+$ depending on α and β. (In fact, we can take $m = |\beta| + 2d$ and $n = |\alpha|$.) From this, it follows that if $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d)$, then $\widehat{f_n} \to \widehat{f}$ in $\mathscr{S}(\mathbb{R}^d)$, that is, the map $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous.

Similarly, one sees that $\mathscr{F}^{-1}$ is continuous on $\mathscr{S}(\mathbb{R}^d)$. ∎

The next theorem justifies the terminology.

Theorem 4.5 (Fourier Inversion Theorem). *For any $f \in \mathscr{S}(\mathbb{R}^d)$,*

$$\mathscr{F}^{-1}(\mathscr{F} f) = f = \mathscr{F}(\mathscr{F}^{-1} f)$$

(so that the Fourier transform $\mathscr{F}$ is a linear bicontinuous bijection of $\mathscr{S}(\mathbb{R}^d)$ onto $\mathscr{S}(\mathbb{R}^d)$ with inverse $\mathscr{F}^{-1}$).

Proof. For any $f, g \in \mathscr{S}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} g(\lambda) \, \widehat{f}\,(\lambda) \, e^{i\lambda y} \, d\lambda = \int_{\mathbb{R}^d} g(\lambda) \, \Big\{ \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\lambda x} f(x) \, dx \Big\} \, e^{i\lambda y} \, d\lambda$$
$$= \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \Big\{ \int_{\mathbb{R}^d} g(\lambda) \, e^{-i\lambda(x-y)} \, d\lambda \Big\} f(x) \, dx$$
$$= \int_{\mathbb{R}^d} \widehat{g}\,(x-y) \, f(x) \, dx$$
$$= \int_{\mathbb{R}^d} \widehat{g}\,(x) \, f(y+x) \, dx \,.$$

Let $g_\varepsilon(\lambda) = g(\varepsilon\lambda)$, so that

$$\widehat{g_\varepsilon}\,(x) = \tfrac{1}{(2\pi)^{d/2)}} \int_{\mathbb{R}^d} e^{-ix\lambda} \, g(\varepsilon\lambda) \, d\lambda$$
$$= \tfrac{1}{(2\pi)^{d/2)}} \, \varepsilon^{-d} \int_{\mathbb{R}^d} e^{-ixu/\varepsilon} \, g(u) \, du$$
$$= \varepsilon^{-d} \, \widehat{g}\,(x/\varepsilon) \,.$$

Therefore

$$\int_{\mathbb{R}^d} g(\varepsilon\lambda) \, \widehat{f}\,(\lambda) \, e^{i\lambda y} \, d\lambda = \int_{\mathbb{R}^d} \widehat{g_\varepsilon}\,(x) \, f(y+x) \, dx$$
$$= \int_{\mathbb{R}^d} \widehat{g}\,(x/\varepsilon) \, f(y+x) \, dx/\varepsilon^d$$
$$= \int_{\mathbb{R}^d} \widehat{g}\,(x) \, f(y+\varepsilon x) \, dx \,.$$

Letting $\varepsilon \downarrow 0$, we obtain

$$g(0) \int_{\mathbb{R}^d} \widehat{f}\,(\lambda)\, e^{iy\lambda}\, d\lambda = f(y) \int_{\mathbb{R}^d} \widehat{g}\,(x)\, dx\,.$$

Now set $g(x) = e^{-x^2/2}$. Then $g(0) = 1$ and one knows that $\widehat{g}\,(u) = e^{-u^2/2}$ and $\int_{\mathbb{R}^d} \widehat{g}\,(u)\, du = (2\pi)^{d/2}$. Substituting this into the equation above gives

$$(\mathscr{F}^{-1}\widehat{f}\,)(y) = f(y)\,,$$

that is, $\mathscr{F}^{-1}(\mathscr{F}f) = f$. Similarly, one shows that $\mathscr{F}(\mathscr{F}^{-1}f) = f$ and the result follows. ∎

Remark 4.6. We see that

$$(\mathscr{F}^2 f)(x) = (\mathscr{F}\widehat{f}\,)(x) = (\mathscr{F}^{-1}\widehat{f}\,)(-x) = f(-x)\,.$$

It follows that $\mathscr{F}^4 f = f$ so that $\mathscr{F}$ and $\mathscr{F}^{-1}$ have period 4.

Furthermore, writing the identity $i\lambda_j\, \widehat{f}\,(\lambda) = (D_j f)\widehat{\ }\,(\lambda)$ obtained earlier as $i\lambda_j\,(\mathscr{F}f)(\lambda) = (\mathscr{F}D_j f)(\lambda)$ and replacing f by $\mathscr{F}^{-1}f$, we get the formula $i\lambda_j\, f(\lambda) = (\mathscr{F}D_j\mathscr{F}^{-1}f)(\lambda)$. This gives the identity

$$i\lambda_j = \mathscr{F}D_j\mathscr{F}^{-1}$$

as operators on $\mathscr{S}(\mathbb{R}^d)$.

We also find that $\mathscr{F}^{-1}(i\lambda_j)\mathscr{F} = D_j$ and $\mathscr{F}(i\lambda_j)\mathscr{F}^{-1} = \mathscr{F}^2 D_j\mathscr{F}^{-2} = -D_j$ on $\mathscr{S}(\mathbb{R}^d)$.

Corollary 4.7 (Parseval's formula). *For any* $f, g \in \mathscr{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \overline{\widehat{f}\,(x)}\, \widehat{g}\,(x)\, dx = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx\,.$$

In particular, $\|\widehat{f}\,\|_{L^2} = \|f\|_{L^2}$ *(Plancheral's formula).*

Proof. We have seen that

$$\int_{\mathbb{R}^d} g(\lambda)\, \widehat{f}\,(\lambda)\, e^{iy\lambda}\, d\lambda = \int_{\mathbb{R}^d} \widehat{g}\,(x)\, f(y+x)\, dx\,.$$

Setting $y = 0$, we get

$$\int_{\mathbb{R}^d} g(\lambda)\, \widehat{f}\,(\lambda)\, d\lambda = \int_{\mathbb{R}^d} \widehat{g}\,(x)\, f(x)\, dx\,.$$

Replacing f by $\mathscr{F}^{-1}f$ and using the identity $\mathscr{F}^{-1}f(x) = \widehat{f}\,(-x)$, we obtain

$$\int_{\mathbb{R}^d} g(x)\, f(x) = \int_{\mathbb{R}^d} \widehat{g}\,(x)\, \widehat{f}\,(-x)\, dx\,.$$

However, $\overline{\widehat{f}(x)} = (\overline{f})^{\widehat{}}(-x)$, so putting $h = \overline{f}$, we see that $\widehat{f}(-x) = (\overline{h})^{\widehat{}}(-x) = \overline{\widehat{h}(x)}$. Hence

$$\int_{\mathbb{R}^d} g(x)\,\overline{h(x)}\,dx = \int_{\mathbb{R}^d} \widehat{g}(x)\,\overline{\widehat{h}(x)}\,dx\,,$$

as required. ■

It is now easy to see that the Fourier transform is a unitary operator on the Hilbert space $L^2(\mathbb{R}^d)$.

Theorem 4.8 (Plancherel). *The Fourier transform $\mathscr{F}$ extends from $\mathscr{S}(\mathbb{R}^d)$ to a unitary operator on $L^2(\mathbb{R}^d)$.*

Proof. We have seen above (Plancherel's formula) that the Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is isometric with respect to $\|\cdot\|_2$ (and maps $\mathscr{S}(\mathbb{R}^d)$ onto $\mathscr{S}(\mathbb{R}^d)$). However, $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ and so the result follows by standard density arguments.

[The details are as follows. Let $h \in L^2(\mathbb{R}^d)$. Then there is a sequence (f_n) in $\mathscr{S}(\mathbb{R}^d)$ such that $\|f_n - h\|_2 \to 0$. In particular, (f_n) is L^2-Cauchy. But $\|\widehat{\varphi}\|_2 = \|\varphi\|_2$ for $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and so $\widehat{f_n}$ is also L^2-Cauchy and therefore converges to some element, F, say, in $L^2(\mathbb{R}^d)$. We define $\mathscr{F}h = F$. Then

$$\|\mathscr{F}h\|_2 = \lim_n \|\widehat{f_n}\|_2 = \lim_n \|f_n\|_2 = \|h\|_2\,.$$

To see that that F is independent of the particular sequence (f_n), suppose that (g_n) is any sequence in $\mathscr{S}(\mathbb{R}^d)$ such that $\|g_n - h\|_2 \to 0$. Define a new sequence (φ_n) in $\mathscr{S}(\mathbb{R}^d)$ by setting

$$\varphi_n = \begin{cases} f_n, & n \text{ odd} \\ g_n, & n \text{ even.} \end{cases}$$

Arguing as above (but with φ_n rather than f_n), we see that the sequence $(\widehat{\varphi_n})$ converges in $L^2(\mathbb{R}^d)$. But then

$$\mathscr{F}h = L^2\text{-}\lim_n \widehat{f_n} = L^2\text{-}\lim_n \widehat{\varphi_n} = L^2\text{-}\lim_n \widehat{g_n}$$

so that $\mathscr{F}h$ is well-defined and $\|\mathscr{F}h\|_2 = \|h\|_2$.

A similar argument holds for the inverse Fourier transform $\mathscr{F}^{-1}$. Moreover, $\mathscr{F}\mathscr{F}^{-1}$ and $\mathscr{F}^{-1}\mathscr{F}$ are both equal to the identity operator on $\mathscr{S}(\mathbb{R}^d)$ which is dense in $L^2(\mathbb{R}^d)$ and so $\mathscr{F}\mathscr{F}^{-1} = \mathscr{F}^{-1}\mathscr{F} = \mathbb{1}$ on $L^2(\mathbb{R}^d)$.] ■

Corollary 4.9. *For any $f, g \in L^2(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} \widehat{f}(x)\,g(x)\,dx = \int_{\mathbb{R}^d} f(x)\,\widehat{g}(x)\,dx\,.$$

Proof. The fact that $\mathscr{F}$ is unitary on $L^2(\mathbb{R}^d)$ means that

$$\int_{\mathbb{R}^d} \overline{\widehat{f}}\, \widehat{g}\ dx = \int_{\mathbb{R}^d} \overline{f}\, g\ dx\,.$$

Replacing f by $\overline{f}$ and g by $\widehat{g}$ and using the facts that $\overline{(\mathscr{F}\overline{f})(x)} = (\mathscr{F}f)(-x)$ and $(\mathscr{F}\mathscr{F}g)(x) = g(-x)$ we see that

$$\int_{\mathbb{R}^d} \widehat{f}\,(-x)\, g(-x)\ dx = \int_{\mathbb{R}^d} f(x)\,\widehat{g}\,(x)\ dx$$

and the result follows. ∎

Remark 4.10. The unbounded self-adjoint operator $-iD_j$ on the Hilbert space $L^2(\mathbb{R}^d)$ is unitarily equivalent to the operator of multiplication by x_j. This follows because $\mathscr{F}(-i)D_j\mathscr{F}^{-1} = x_j$ on $\mathscr{S}(\mathbb{R}^d)$ which is a core for the multiplication operator x_j.

Definition 4.11. The Fourier transform $\mathscr{F}T$ of the tempered distribution $T \in \mathscr{S}\,'(\mathbb{R}^d)$ is given by

$$\mathscr{F}T(f) = T(\mathscr{F}f) \quad \text{for } f \in \mathscr{S}(\mathbb{R}^d).$$

We often write $\widehat{T}$ for $\mathscr{F}T$. Similarly, the inverse Fourier transform $\mathscr{F}^{-1}T$ is given by

$$\mathscr{F}^{-1}T(f) = T(\mathscr{F}^{-1}f) \quad \text{for } f \in \mathscr{S}(\mathbb{R}^d).$$

Remark 4.12. Note that $\mathscr{F}: \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous and so the Fourier transform $\mathscr{F}$ maps $\mathscr{S}\,'(\mathbb{R}^d)$ into $\mathscr{S}\,'(\mathbb{R}^d)$. Similarly, $\mathscr{F}^{-1}T \in \mathscr{S}\,'(\mathbb{R}^d)$ for every $T \in \mathscr{S}\,'(\mathbb{R}^d)$. Evidently, $\mathscr{F}^{-1}\mathscr{F}T = T = \mathscr{F}\mathscr{F}^{-1}T$.

If T is given by some element g of $L^2(\mathbb{R}^d)$, so $T = T_g$, then

$$\mathscr{F}T_g(f) = T_g(\widehat{f}\,) = \int_{\mathbb{R}^d} g(x)\widehat{f}\,(x)\ dx = \int_{\mathbb{R}^d} \widehat{g}\,(x)\, f(x)\ dx = T_{\widehat{g}}\,(f)\,.$$

This means that we can think of the Fourier transform on $\mathscr{S}\,'(\mathbb{R}^d)$ as an extension of that on $L^2(\mathbb{R}^d)$.

Examples 4.13.

1. We compute $\widehat{\delta_b}$ for $b \in \mathbb{R}^d$. For any $\varphi \in \mathscr{S}(\mathbb{R}^d)$

$$\begin{aligned}\widehat{\delta_b}\,(\varphi) &= \delta_b(\widehat{\varphi}\,) \\ &= \widehat{\varphi}\,(b) \\ &= \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ibx}\ \varphi(x)\ dx \\ &= \int_{\mathbb{R}^d} \frac{e^{-ibx}}{(2\pi)^{d/2}}\ \varphi(x)\ dx\end{aligned}$$

so that $\widehat{\delta_b} = T_\psi$ where $\psi(x) = e^{-ibx}/(2\pi)^{d/2}$. In particular, with $b = 0$, we find that $\widehat{\delta} = (2\pi)^{-d/2}$.

2. We shall determine the Fourier transform $\mathscr{F}(\delta_b')$. As above,

$$\begin{aligned}\widehat{(\delta_b')}\,(\varphi) &= \delta_b'(\widehat{\varphi}\,) \\ &= \delta_b(-\widehat{\varphi}\,') \\ &= -\widehat{\varphi}\,'(b) \\ &= \int_{\mathbb{R}^d} \frac{ix\, e^{-ibx}}{(2\pi)^{d/2}}\, \varphi(x)\, dx\end{aligned}$$

and therefore $\widehat{\delta_b'} = ix\, e^{-ibx}/(2\pi)^{d/2}$.

Theorem 4.14. *$\mathscr{F}$ and $\mathscr{F}^{-1}$ are continuous on $\mathscr{S}\,'(\mathbb{R}^d)$.*

Proof. Suppose that $T_n \to T$ in $\mathscr{S}\,'(\mathbb{R}^d)$. Then $T_n(\varphi) \to T(\varphi)$ for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Hence

$$\widehat{T_n}\,(\varphi) = T_n(\widehat{\varphi}\,) \to T(\widehat{\varphi}\,) = \widehat{T}\,(\varphi)$$

so $\widehat{T_n} \to \widehat{T}$ in $\mathscr{S}\,'(\mathbb{R}^d)$. A similar argument holds for the inverse Fourier transform. ∎

Theorem 4.15. *For any $T \in \mathscr{S}\,'(\mathbb{R}^d)$ and multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$,*

$$(ix)^\alpha\, \widehat{T} = (D^\alpha T)\widehat{\ } \quad \textit{and} \quad D^\beta \widehat{T} = (\,(-ix)^\beta T\,)\widehat{\ }\,.$$

In general, $(ix)^\alpha\, D^\beta \widehat{T} = \{D^\alpha(\,(-ix)^\beta T\,)\}\widehat{\ }\,.$

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\begin{aligned}(ix)^\alpha\, \widehat{T}\,(\varphi) &= \widehat{T}\,(\,(ix)^\alpha\, \varphi\,) \\ &= T\big(\,((ix^\alpha)\,\varphi)\widehat{\ }\,\big) \\ &= T\big(\,(-D)^\alpha \widehat{\varphi}\,\big) \\ &= D^\alpha T(\widehat{\varphi}\,) \\ &= (D^\alpha T)\widehat{\ }\,(\varphi)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}D^\beta \widehat{T}\,(\varphi) &= \widehat{T}\,(\,(-D)^\beta \varphi\,) \\ &= T(\,((-D)^\beta\, \varphi)\widehat{\ }\,) \\ &= T(\,(-ix)^\beta\, \widehat{\varphi}\,) \\ &= (-ix)^\beta\, T(\widehat{\varphi}\,) \\ &= (\,(-ix)^\beta\, T\,)\widehat{\ }\,(\varphi)\,.\end{aligned}$$

The general case is proved in the same way. ∎

Chapter 5

Convolution

Definition 5.1. Let $f, g \in \mathscr{S}(\mathbb{R}^d)$. The convolution of f and g, denoted by $f * g$, is the function

$$(f * g)(y) = \int_{\mathbb{R}^d} f(y-x)\, g(x)\, dx\,.$$

Theorem 5.2. *$f * g \in \mathscr{S}(\mathbb{R}^d)$ for any $f, g \in \mathscr{S}(\mathbb{R}^d)$. Moreover,*

$$(2\pi)^{d/2}\, \widehat{f\, g} = \widehat{f} * \widehat{g} \quad \textit{and} \quad (2\pi)^{d/2}\, \widehat{f}\, \widehat{g} = \widehat{f * g}\,.$$

Furthermore, $f * g = g * f$ and $f * (g * h) = (f * g) * h$ for any $f, g, h \in \mathscr{S}(\mathbb{R}^d)$.

Proof. We have

$$(2\pi)^{d/2}\, \widehat{fg}\,(y) = \int_{\mathbb{R}^d} e^{-iyx}\, f(x)\, g(x)\, dx\,.$$

But we know that $\int \widehat{\varphi}\, \psi\, dx = \int \varphi\, \widehat{\psi}\, dx$, for any $\varphi\,, \psi \in \mathscr{S}(\mathbb{R}^d)$. If we let $\check{\varphi}$ denote $\mathscr{F}^{-1}\varphi$, then, replacing φ by $\check{\varphi}$, we get $\int \varphi\, \psi\, dx = \int \check{\varphi}\, \widehat{\psi}\, dx$.

Now, for fixed $y \in \mathbb{R}^d$, let $\varphi(x) = e^{-iyx} f(x)$ and set $\psi(x) = g(x)$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(x)\, \psi(x)\, dx &= \int_{\mathbb{R}^d} \check{\varphi}(x)\, \widehat{\psi}\,(x)\, dx \\
&= \int_{\mathbb{R}^d} \Big\{ (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ixt}\, e^{-iyt}\, f(t)\, dt \Big\}\, \widehat{g}\,(x)\, dx \\
&= \int_{\mathbb{R}^d} \widehat{f}\,(y-x)\, \widehat{g}\,(x)\, dx \\
&= (\widehat{f} * \widehat{g}\,)(y)
\end{aligned}$$

giving

$$(2\pi)^{d/2}\, \widehat{fg}\,(y) = (\widehat{f} * \widehat{g}\,)(y)\,,$$

as required.

Replacing f by $\mathscr{F}^{-1}f$ and g by $\mathscr{F}^{-1}g$ in this identity, we find that

$$(2\pi)^{d/2}\,\mathscr{F}(\mathscr{F}^{-1}f\,\mathscr{F}^{-1}g) = f * g$$

which shows that $f * g \in \mathscr{S}(\mathbb{R}^d)$ (because this is true of the left hand side). The left hand side is unaltered if we interchange f and g and therefore

$$f * g = g * f\,.$$

Next, taking the Fourier transform once again gives

$$\begin{aligned}\mathscr{F}(f * g)(y) &= (2\pi)^{d/2}\,\mathscr{F}\mathscr{F}((\mathscr{F}^{-1}f)(\mathscr{F}^{-1}g))(y)\\ &= (2\pi)^{d/2}\,((\mathscr{F}^{-1}f)(\mathscr{F}^{-1}g))(-y)\\ &= (2\pi)^{d/2}\,\mathscr{F}^{-1}f(-y)\,\mathscr{F}^{-1}g(-y)\\ &= (2\pi)^{d/2}\,\mathscr{F}f(y)\,\mathscr{F}g(y)\end{aligned}$$

and so $\mathscr{F}(f * g) = (2\pi)^{d/2}\,(\mathscr{F}f)(\mathscr{F}g)$, as claimed.

Finally, we have

$$\begin{aligned}\mathscr{F}(f * (g * h)) &= (2\pi)^{d/2}\,\widehat{f}\;\widehat{g * h}\\ &= (2\pi)^{d}\,\widehat{f}\;\widehat{g}\;\widehat{h}\\ &= \mathscr{F}((f * g) * h)\end{aligned}$$

and therefore (taking the inverse Fourier transform) $f*(g*h) = (f*g)*h$. ∎

Corollary 5.3. *If $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$, then $\varphi * \psi \in C_0^\infty(\mathbb{R}^d)$. Moreover,*

$$\operatorname{supp} \varphi * \psi \subseteq \operatorname{supp} \varphi + \operatorname{supp} \psi\,.$$

Proof. It follows from the theorem that $\varphi * \psi \in \mathscr{S}(\mathbb{R}^d)$ for $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$. Now

$$(\varphi * \psi)(y) = \int_{\mathbb{R}^d} \varphi(y - x)\,\psi(x)\,dx$$

which certainly vanishes if it is false that $y - x \in \operatorname{supp} \varphi$ for some $x \in \operatorname{supp} \psi$, that is, if it is false that $y = x_1 + x_2$ for some $x_1 \in \operatorname{supp} \varphi$ and $x_2 \in \operatorname{supp} \psi$. Hence $\operatorname{supp} \varphi * \psi \subseteq \operatorname{supp} \varphi + \operatorname{supp} \psi$, as required.

Moreover, each of $\operatorname{supp} \varphi$ and $\operatorname{supp} \psi$ is compact and so therefore is $\operatorname{supp} \varphi * \psi$. We conclude that $\varphi * \psi \in C_0^\infty(\mathbb{R}^d)$. ∎

Corollary 5.4. *For fixed $f \in \mathscr{S}(\mathbb{R}^d)$, the mapping $g \mapsto f * g$ is continuous from $\mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}(\mathbb{R}^d)$.*

Proof. Fix $f \in \mathscr{S}(\mathbb{R}^d)$ and suppose $g_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. Then also $\widehat{g_n} \to 0$ in $\mathscr{S}(\mathbb{R}^d)$ and so (by Leibnitz' formula) $\widehat{f}\,\widehat{g_n} \to 0$ in $\mathscr{S}(\mathbb{R}^d)$. But then $f * g_n = (2\pi)^{d/2}\,\mathscr{F}^{-1}(\widehat{f}\,\widehat{g_n}\,) \to 0$ in $\mathscr{S}(\mathbb{R}^d)$ and the result follows. ∎

Definition 5.5. For any function u on $\mathbb{R}^d$, we define the translation $\tau_x u$ and the inversion $\widetilde{u}$ by the formulae

$$(\tau_x u)(y) = u(y-x) \quad \text{and} \quad \widetilde{u}\,(y) = u(-y)\,.$$

Then $(\tau_x \widetilde{u}\,)(y) = \widetilde{u}\,(y-x) = u(x-y)$ and for $u, v \in \mathscr{S}(\mathbb{R}^d)$, we have

$$(u * v)(y) = \int_{\mathbb{R}^d} u(x)\, v(y-x)\, dx = \int_{\mathbb{R}^d} u(x)\, (\tau_y \widetilde{v}\,)(x)\, dx\,.$$

One readily checks that for fixed $x \in \mathbb{R}^d$, τ_x and $\widetilde{\ }$ are continuous maps from $\mathscr{D}(\mathbb{R}^d)$ onto $\mathscr{D}(\mathbb{R}^d)$ and from $\mathscr{S}(\mathbb{R}^d)$ onto $\mathscr{S}(\mathbb{R}^d)$.

Definition 5.6. For $u \in \mathscr{D}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the convolution $u * \varphi$ is the function

$$(u * \varphi)(x) = u(\tau_x \widetilde{\varphi}\,) = u(\varphi(x - \cdot\,))\,.$$

For $T \in \mathscr{S}\,'(\mathbb{R}^d)$ and $f \in \mathscr{S}(\mathbb{R}^d)$, the convolution $T * f$ is the function

$$(T * f)(x) = T(\tau_x \widetilde{f}\,) = T(f(x - \cdot\,))\,.$$

Note that $u(\varphi)$ can be expressed as a convolution. Indeed, we see that $\varphi = (\widetilde{\varphi}\,)\widetilde{\ } = \tau_0((\widetilde{\varphi}\,)\widetilde{\ })$. Hence $u(\varphi) = u(\tau_0((\widetilde{\varphi}\,)\widetilde{\ })) = (u * \widetilde{\varphi}\,)(0)$. Similarly, $T(f) = (T * \widetilde{f}\,)(0)$.

Lemma 5.7. *For $f \in \mathscr{S}(\mathbb{R}^d)$ and $\kappa \neq 0$, set $f_\kappa(x) = \dfrac{f(x+\kappa e_j) - f(x)}{\kappa}$ where $e_1 = (1, 0, \dots, 0)$, $\dots$, $e_d = (0, 0, \dots, 0, 1)$ are the standard basis vectors of $\mathbb{R}^d$. Then $f_\kappa \to \partial_j f$ in $\mathscr{S}(\mathbb{R}^d)$ as $\kappa \to 0$.*

Proof. We shall show that $|\!|\!| f_\kappa - \partial_j f |\!|\!|_{\alpha,\beta} \to 0$ for each $\alpha, \beta \in \mathbb{Z}_+^d$. Since $D^\beta(f_\kappa - \partial_j f) = g_\kappa - \partial_j g$ where $g = D^\beta f$, we may assume that $\beta = 0$.

For notational simplicity, let $h = \partial_j f \in \mathscr{S}(\mathbb{R}^d)$ and let $\|x\|_1 \equiv \sum_{i=1}^d |x_i|$ for $x \in \mathbb{R}^d$. Let $|\kappa| < 1$. By the Mean Value Theorem, for each $x \in \mathbb{R}^d$ there is some $\theta \in \mathbb{R}$ (depending on x) with $|\theta| < 1$ such that $f_\kappa(x) = h(x + \theta\kappa e_j)$ and so

$$\begin{aligned}
\sup_{x \in \mathbb{R}^d} |\, x^\alpha (f_\kappa(x) - \partial_j f(x))\,| &= \sup_{x \in \mathbb{R}^d} |\, x^\alpha (h(x+\theta\kappa e_j) - h(x))\,| \\
&\le \sup_{\|x\|_1 \le M} |\, x^\alpha (h(x+\theta\kappa e_j) - h(x))\,| \\
&\qquad + \sup_{\|x\|_1 > M} |\, x^\alpha h(x+\theta\kappa e_j)\,| \\
&\qquad\qquad + \sup_{\|x\|_1 > M} |\, x^\alpha h(x)\,|\,. \qquad (*)
\end{aligned}$$

We shall estimate each of the three terms on the right hand side. Let $\varepsilon > 0$ be given. Since $h \in \mathscr{S}(\mathbb{R}^d)$, the third term on the right hand side is smaller than $\frac{1}{3}\,\varepsilon$ for all sufficiently large M.

To estimate the middle term, note that $|x_j| \le |x_j + \theta\kappa| + |\theta\kappa| \le |x_j + \theta\kappa| + 1$ and so

$$| x^\alpha | \le \Big(\prod_{i \ne j} |x_i|^{\alpha_i} \Big) \, (1 + |x_j + \theta\kappa| \,)^{\alpha_j} \,.$$

Furthermore, if $\|x\|_1 > M$, then $\|x + \theta\kappa e_j\|_1 > M - 1$. Combining these remarks, it follows that

$$\begin{aligned}
\sup_{\|x\|_1 > M} & | \, x^\alpha \, h(x + \theta\kappa e_j) \, | \\
& \le \sup_{\|x\|_1 > M} \Big(\prod_{i \ne j} |x_i|^{\alpha_i} \Big) \, (1 + |x_j + \theta\kappa| \,)^{\alpha_j} \, | \, h(x + \theta\kappa e_j) \, | \\
& \le \sup_{\|x\|_1 > M - 1} \Big(\prod_{i \ne j} |x_i|^{\alpha_i} \Big) \, (1 + |x_j| \,)^{\alpha_j} \, | \, h(x) \, | \\
& < \tfrac{1}{3} \, \varepsilon
\end{aligned}$$

provided M is sufficiently large (again because $h \in \mathscr{S}(\mathbb{R}^d)$).

Fix M sufficiently large (according to the discussion above) so that each of the second and third terms on the right hand side of the inequality $(*)$ is smaller than $\frac{1}{3}\,\varepsilon$. The function h is uniformly continuous on the compact set $\{ \, x \in \mathbb{R}^d : \|x\|_1 \le M \, \}$ and so the first term on the right hand side of $(*)$ is smaller than $\frac{1}{3}\,\varepsilon$ for all $|\kappa|$ sufficiently small. The result follows. ∎

Lemma 5.8. *For any $f \in \mathscr{S}(\mathbb{R}^d)$, $\tau_a f \to f$ in $\mathscr{S}(\mathbb{R}^d)$ as $a \to 0$ in $\mathbb{R}^d$.*

Proof. Fix $\alpha \in \mathbb{Z}_+^d$ and suppose that $\|a\|_1 < 1$. Then

$$\begin{aligned}
\sup_x | \, x^\alpha \, (f(x-a) - f(x)) \, | \le \sup_{\|x\|_1 \le M} & | \, x^\alpha \, (f(x-a) - f(x)) \, | \\
& + \sup_{\|x\|_1 > M} | \, x^\alpha \, f(x-a) \, | \\
& \quad + \sup_{\|x\|_1 > M} | \, x^\alpha \, f(x) \, |
\end{aligned}$$

Since $f \in \mathscr{S}(\mathbb{R}^d)$, for any given $\varepsilon > 0$, we may fix M sufficiently large that the second and third terms on the right hand side are each smaller than $\frac{1}{3}\,\varepsilon$. But then for all sufficiently small a, the first term is also smaller than $\frac{1}{3}\,\varepsilon$ because f is uniformly continuous on $\{ \, x \in \mathbb{R}^d : \|x\|_1 \le M \, \}$. It follows that $\sup_x | \, x^\alpha \, (f(x-a) - f(x)) \, | \to 0$ as $a \to 0$. Replacing f by $D^\beta f$, we see that $\|\tau_a f - f\|_{\alpha,\beta} \to 0$ as $a \to 0$ for any $\alpha, \beta \in \mathbb{Z}_+^d$, that is, $\tau_a f \to f$ in $\mathscr{S}(\mathbb{R}^d)$ as $a \to 0$. ∎

Corollary 5.9. *Suppose that* $\varphi \in \mathscr{D}(\mathbb{R}^d)$. *Then*

(i) $\varphi_\kappa \to \partial_j \varphi$ *in* $\mathscr{D}(\mathbb{R}^d)$ *as* $\kappa \to 0$, *and*

(ii) $\tau_a \varphi \to \varphi$ *in* $\mathscr{D}(\mathbb{R}^d)$ *as* $a \to 0$.

Proof. We have seen that $\varphi_\kappa \to \partial_j \varphi$ in $\mathscr{S}(\mathbb{R}^d)$ as $\kappa \to 0$ and that $\tau_a \varphi \to \varphi$ in $\mathscr{S}(\mathbb{R}^d)$ as $a \to 0$. However, for all $|\kappa| < 1$, say, there is some fixed compact set K such that each $\operatorname{supp} \varphi_\kappa \subset K$ (and $\operatorname{supp} \partial_j \varphi \subset K$) which means that $\varphi_\kappa \to \partial_j \varphi$ in $\mathscr{D}(\mathbb{R}^d)$ as $\kappa \to 0$.

Similarly, for all $\|a\|_1 < 1$, say, the supports of φ and $\tau_a \varphi$ all lie within some fixed compact set and so $\tau_a \varphi \to \varphi$ in $\mathscr{D}(\mathbb{R}^d)$ as $a \to 0$. ∎

Theorem 5.10. *Let* $u \in \mathscr{D}'(\mathbb{R}^d)$ *and let* $\varphi \in \mathscr{D}(\mathbb{R}^d)$. *Then* $u * \varphi \in C^\infty(\mathbb{R}^d)$ *and*

$$D^\alpha (u * \varphi) = (D^\alpha u) * \varphi = u * (D^\alpha \varphi)$$

for any $\alpha \in \mathbb{Z}_+^d$. *Furthermore,* $\operatorname{supp}(u * \varphi) \subseteq \operatorname{supp} u + \operatorname{supp} \varphi$.

Proof. By the corollary, it follows that $\tau_y \widetilde{\varphi} \to \tau_x \widetilde{\varphi}$ in $\mathscr{D}(\mathbb{R}^d)$ if $y \to x$ in $\mathbb{R}^d$. Hence $u(\tau_y \widetilde{\varphi}) \to u(\tau_x \widetilde{\varphi})$, that is, $(u * \varphi)(y) \to (u * \varphi)(x)$ if $y \to x$ which shows that $u * \varphi$ is continuous on $\mathbb{R}^d$.

Again, using the corollary, we see that for fixed $x \in \mathbb{R}^d$,

$$\begin{aligned}
\frac{(u * \varphi)(x + \kappa e_j) - (u * \varphi)(x)}{\kappa} &= \frac{u(\,\tau_{x+\kappa e_j} \widetilde{\varphi} - \tau_x \widetilde{\varphi}\,)}{\kappa} \\
&= u\Big(\frac{\tau_x\,(\tau_{\kappa e_j} \widetilde{\varphi} - \widetilde{\varphi}\,)}{\kappa} \Big) \\
&\to u(\,\tau_x(-D_j \widetilde{\varphi}\,)\,)\,, \quad \text{as } \kappa \to 0, \\
&= u(\,\tau_x(\widetilde{D_j \varphi}\,)\,) \\
&= (u * D_j \varphi)(x)\,.
\end{aligned}$$

Hence the partial derivative $D_j(u * \varphi)(x)$ exists at each $x \in \mathbb{R}^d$ and it is equal to $(u * D_j \varphi)(x)$. Furthermore, for fixed x,

$$\begin{aligned}
u(\,\tau_x(-D_j \widetilde{\varphi}\,)\,) &= -u(D_j \tau_x \widetilde{\varphi}\,) \\
&= (D_j u)(\tau_x \widetilde{\varphi}\,) \\
&= \big(\,(D_j u) * \varphi\,\big)(x)
\end{aligned}$$

and therefore

$$D_j(u * \varphi) = u * D_j \varphi = (D_j u) * \varphi\,,$$

as required. The general case follows by induction.

For the last part, we note that $(u*\varphi)(x) = 0$ if $\operatorname{supp} u \cap \operatorname{supp} \tau_x \widetilde{\varphi} = \varnothing$, that is to say, if $\operatorname{supp} u \cap \operatorname{supp} \varphi(x - \cdot\,) = \varnothing$. Hence

$$\begin{aligned}\operatorname{supp}(u*\varphi) &\subseteq \{\, x \in \mathbb{R}^d : \operatorname{supp} u \cap \operatorname{supp} \varphi(x - \cdot\,) \neq \varnothing \,\} \\ &= \{\, x : \text{there is } y \in \operatorname{supp} u \text{ such that } x - y \in \operatorname{supp} \varphi \,\} \\ &= \{\, x : x \in \operatorname{supp} u + \operatorname{supp} \varphi \,\}\end{aligned}$$

and the proof is complete. ∎

Corollary 5.11. *For $u \in \mathscr{D}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$, $u * \varphi \in \mathscr{D}'(\mathbb{R}^d)$.*

Proof. The function $u*\varphi$ belongs to C^∞ and so is bounded on each compact subset of $\mathbb{R}^d$. Hence $\psi \mapsto \int_{\mathbb{R}^d} (u*\varphi)(x)\,\psi(x)\,dx$ is a continuous linear map on $\mathscr{D}(\mathbb{R}^d)$, that is, it is a distribution. ∎

There is an analogous result for $\mathscr{S}$ and $\mathscr{S}'$.

Theorem 5.12. *Let $u \in \mathscr{S}'(\mathbb{R}^d)$ and $f \in \mathscr{S}(\mathbb{R}^d)$. Then $u * f \in C^\infty$ and*

$$D^\alpha(u*f) = (D^\alpha u) * f = u * (D^\alpha f)$$

for any $\alpha \in \mathbb{Z}^d_+$. Furthermore, $u * f$ is polynomially bounded and hence determines a tempered distribution.

Proof. The first part is just as for $\mathscr{D}(\mathbb{R}^d)$. We need to show that $u * f$ is polynomially bounded. Since $u \in \mathscr{S}'(\mathbb{R}^d)$, it follows that there is some constant $C > 0$ and integers k, n such that

$$|u(g)| \le C\, \|g\|_{k,n}$$

for all $g \in \mathscr{S}(\mathbb{R}^d)$. Hence, for $x \in \mathbb{R}^d$,

$$\begin{aligned}|(u*f)(x)| &= |\, u(\tau_x \widetilde{f}\,)\,| \\ &\le C\, \|\tau_x \widetilde{f}\,\|_{k,n} \\ &= C \sum_{\substack{|\alpha|\le k\\ |\beta|\le n}} \sup_y |y^\alpha|\,|\, D^\beta_y f(x-y)\,| \\ &= C \sum_{\substack{|\alpha|\le k\\ |\beta|\le n}} \sup_y |(-y+x)^\alpha|\,|\, D^\beta f(y)\,|\end{aligned}$$

which is polynomially bounded in x. ∎

Proposition 5.13. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and suppose that $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d)$ as $n \to \infty$. Then $(T * f_n)(x) \to (T * f)(x)$ uniformly on compact sets in $\mathbb{R}^d$.*

Proof. By replacing f_n by $f_n - f$, we may assume that $f = 0$. Now, since $T \in \mathscr{S}'(\mathbb{R}^d)$, there is $C > 0$ and $k, n \in \mathbb{Z}_+$ such that

$$|\,T(g)\,| \le C \sum_{\substack{|\alpha|\le k\\ |\beta|\le n}} \|g\|_{\alpha,\beta}$$

for all $g \in \mathscr{S}(\mathbb{R}^d)$. Therefore

$$|\,(T * f_n)(x)\,| = |\,T(\tau_x \widetilde{f_n}\,)\,| \le C \sum_{\substack{|\alpha|\le k\\ |\beta|\le n}} \|\tau_x \widetilde{f_n}\,\|_{\alpha,\beta}\,.$$

However, if $\|x\|_1 \le M$, say, then

$$\begin{aligned}
\|\tau_x \widetilde{f_n}\,\|_{\alpha,\beta} &= \sup_y |\,y^\alpha\, D_y^\beta f_n(x-y)\,| \\
&\le \sup_y |(-y+x)^\alpha|\,|D^\beta f_n(y)| \\
&\le \sup_y \Big(\prod_{i=1}^d (M + |y_i|)^{\alpha_i} \Big)\, |D^\beta f_n(y)| \\
&\to 0
\end{aligned}$$

as $n \to \infty$ (because $f_n \to 0$ in $\mathscr{S}(\mathbb{R}^d)$). It follows that $(T * f_n)(x) \to 0$ uniformly on $\{\, x : \|x\|_1 \le M \,\}$ for any fixed $M > 0$ which establishes the result. ∎

Theorem 5.14. *Let $u \in \mathscr{D}'(\mathbb{R}^d)$ and let $\varphi, \psi \in \mathscr{D}(\mathbb{R}^d)$. Then*

$$(u * \varphi) * \psi = u * (\varphi * \psi)\,.$$

Proof. We first observe that the statement of the theorem makes sense because $u * \varphi \in C^\infty$ and $\varphi * \psi \in C_0^\infty(\mathbb{R}^d)$.

Let $\varepsilon > 0$ and consider the Riemann sum

$$f_\varepsilon(x) = \varepsilon^d \sum_{\kappa \in \mathbb{Z}^d} \varphi(x - \kappa\varepsilon)\, \psi(\kappa\varepsilon)\,.$$

This is always a finite sum because the functions φ and ψ have compact support. Furthermore, $\operatorname{supp} f_\varepsilon \subseteq \operatorname{supp}\varphi + \operatorname{supp}\psi$ and $f_\varepsilon \in \mathscr{D}(\mathbb{R}^d)$. Now, using the uniform continuity of $D^\alpha\varphi$, we see that

$$D^\alpha f_\varepsilon(x) = \varepsilon^d \sum_{\kappa \in \mathbb{Z}^d} D^\alpha\varphi(x - \kappa\varepsilon)\, \psi(\kappa\varepsilon) \to (\,(D^\alpha\varphi) * \psi\,)(x) = D^\alpha(\varphi * \psi)(x)$$

uniformly as $\varepsilon \to 0$. Hence $f_\varepsilon \to \varphi * \psi$ in $\mathscr{D}(\mathbb{R}^d)$ as $\varepsilon \to 0$ and therefore $\tau_x \widetilde{f_\varepsilon} \to \tau_x(\widetilde{\varphi * \psi})$ in $\mathscr{D}(\mathbb{R}^d)$ as $\varepsilon \to 0$. It follows that

$$\begin{aligned} u * (\varphi * \psi)(x) &= u(\,\tau_x(\widetilde{\varphi * \psi}\,)\,) \\ &= \lim_{\varepsilon \to 0} u(\tau_x \widetilde{f_\varepsilon}\,) \\ &= \lim_{\varepsilon \to 0} \varepsilon^d \sum_{\kappa \in \mathbb{Z}^d} u(\,\varphi(x - \,\cdot\, - \kappa\varepsilon)\,\psi(\kappa\varepsilon)\,) \\ &= \lim_{\varepsilon \to 0} \varepsilon^d \sum_{\kappa \in \mathbb{Z}^d} (\,u * \varphi\,)(x - \kappa\varepsilon)\,\psi(\kappa\varepsilon) \\ &= (\,(u * \varphi) * \psi\,)(x) \end{aligned}$$

and the proof is complete. ■

Corollary 5.15. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and let $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then*

$$(\,T * f\,) * g = T * (\,f * g\,)\,.$$

Proof. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$, there are sequences (φ_n) and (ψ_n) in $C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \to f$ and $\psi_n \to g$ in $\mathscr{S}(\mathbb{R}^d)$. Furthermore, since $T \restriction C_0^\infty(\mathbb{R}^d) \in \mathscr{D}'(\mathbb{R}^d)$, it follows from the theorem that

$$(\,T * \varphi_n\,) * \psi_k = T * (\,\varphi_n * \psi_k\,)\,.$$

However, we know that $(T * \varphi_n)(x) \to (T * f)(x)$ uniformly on compact sets in $\mathbb{R}^d$ and so, for fixed $y \in \mathbb{R}^d$,

$$\begin{aligned} (\,(T * \varphi_n) * \psi_k\,)(y) &= (T * \varphi_n)(\tau_y \widetilde{\psi_k}\,) \\ &= \int_{\mathbb{R}^d} (T * \varphi_n)(x)\,\psi_k(y - x)\,dx\,, \\ &\qquad \text{since } T * \varphi_n \in C^\infty(\mathbb{R}^d) \text{ (and is polynomially bounded),} \\ &\to \int_{\mathbb{R}^d} (T * f)(x)\,\psi_k(y - x)\,dx\,, \\ &\qquad \text{since } \operatorname{supp} \psi_k \text{ is compact,} \\ &= (T * f) * \psi_k(y)\,. \end{aligned}$$

On the other hand, $\varphi_n * \psi_k \to f * \psi_k$ as $n \to \infty$ in $\mathscr{S}(\mathbb{R}^d)$ and so

$$(T * \varphi_n) * \psi_k = T * (\varphi_n * \psi_k) \to T * (f * \psi_k)\,.$$

It follows that $(T * f) * \psi_k = T * (f * \psi_k)$ for each k. But $\psi_k \to g$ in $\mathscr{S}(\mathbb{R}^d)$, so

$$(T * f) * \psi_k(y) = (T * f)(\tau_y \widetilde{\psi_k}\,) \to (T * f)(\tau_y \widetilde{g}\,) = (T * f) * g(y)$$

and

$$T * (f * \psi_k)(y) = T(\,\tau_y(\widetilde{f * \psi_k}\,)\,) \to T(\,\tau_y(\widetilde{f * g}\,)\,) = T * (f * g)(y)\,.$$

Hence $(T * f) * g = T * (f * g)$, as required. ■

Theorem 5.16. *Let $g \in \mathscr{S}(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} g(x)\,dx = 1$ and for $\varepsilon \neq 0$ set $g_\varepsilon(x) = \varepsilon^{-d} g(x/\varepsilon)$. Then for any $f \in \mathscr{S}(\mathbb{R}^d)$, $g_\varepsilon * f \to f$ in $\mathscr{S}(\mathbb{R}^d)$, as $\varepsilon \to 0$.*

Proof. It is enough to show that $(g_\varepsilon * f)\widehat{\ } \to \widehat{f}$ in $\mathscr{S}(\mathbb{R}^d)$ (the result then follows by taking the inverse Fourier transform). However, we know that $(g_\varepsilon * f)\widehat{\ } = (2\pi)^{d/2}\,\widehat{g_\varepsilon}\,\widehat{f}$ so we must show that $((2\pi)^{d/2}\,\widehat{g_\varepsilon} - 1)\,\widehat{f} \to 0$ in $\mathscr{S}(\mathbb{R}^d)$, as $\varepsilon \to 0$.

Using Leibnitz' formula together with the fact that $\lambda^\alpha D^\beta \widehat{f}\,(\lambda) \in \mathscr{S}(\mathbb{R}^d)$ for any $\alpha, \beta \in \mathbb{Z}_+^d$, it is enough to show that

$$\big(D^\alpha(\,(2\pi)^{d/2}\,\widehat{g_\varepsilon} - 1\,)\big)\,\varphi \to 0 \quad \text{uniformly on } \mathbb{R}^d, \text{ as } \varepsilon \to 0,$$

for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{Z}_+^d$. Note that

$$\widehat{g_\varepsilon}\,(\lambda) = \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\lambda x}\, g(\tfrac{x}{\varepsilon})\,\tfrac{1}{\varepsilon^d}\,dx = \widehat{g}\,(\varepsilon\lambda)\,.$$

(So we see that $\widehat{g_\varepsilon}\,(\lambda) \to \widehat{g}\,(0) = (2\pi)^{-d/2}$ as $\varepsilon \to 0$.)

We consider two cases.

(i) Suppose $|\alpha| = 0$. Fix $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\begin{aligned}
|\,(\,(2\pi)^{d/2}\,\widehat{g_\varepsilon}\,(\lambda) - 1\,)\,\varphi(\lambda)\,| &= |\,(\,(2\pi)^{d/2}\,\widehat{g}\,(\varepsilon\lambda) - 1\,)\,\varphi(\lambda)\,| \\
&= \Big|\int_{\mathbb{R}^d} g(x)\,(\,e^{-i\varepsilon\lambda x} - 1\,)\,dx\,\varphi(\lambda)\Big| \\
&\le \int_{\mathbb{R}^d} |g(x)|\,|\varepsilon\lambda x|\,dx\,|\varphi(\lambda)| \\
&= \varepsilon \int_{\mathbb{R}^d} |xg(x)|\,dx\,|\lambda\,\varphi(\lambda)| \\
&< \varepsilon\, M
\end{aligned}$$

for some constant $M > 0$ independent of λ. So $(\,(2\pi)^{d/2}\,\widehat{g_\varepsilon}\,(\lambda) - 1\,)\,\varphi(\lambda) \to 0$ uniformly in λ, as $\varepsilon \to 0$.

(ii) Suppose that $|\alpha| > 0$. For fixed $\varphi \in \mathscr{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\Big|\big(D^\alpha(\,(2\pi)^{d/2}\,\widehat{g_\varepsilon} - 1\,)\big)(\lambda)\,\varphi(\lambda)\Big| &= |\,(2\pi)^{d/2}\,\varepsilon^{|\alpha|}\,(D^\alpha \widehat{g}\,)(\varepsilon\lambda)\,\varphi(\lambda)\,| \\
&< \varepsilon^{|\alpha|}\, M'
\end{aligned}$$

for some constant $M' > 0$, since both $D^\alpha \widehat{g}$ and φ are bounded on $\mathbb{R}^d$.

The result follows. ■

Theorem 5.17. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and let $g \in \mathscr{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)\,dx = 1$. Then $T * g_\varepsilon \to T$ in $\mathscr{S}'(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$, where $g_\varepsilon(x) = \varepsilon^{-d}\, g(x/\varepsilon)$.*

If $u \in \mathscr{D}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$, then $u * \varphi_\varepsilon \to u$ in $\mathscr{D}'(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$, where $\varphi_\varepsilon(x) = \varepsilon^{-d}\, \varphi(x/\varepsilon)$.

Proof. Fix $f \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\begin{aligned}(T * g_\varepsilon)(f) &= (T * g_\varepsilon) * \widetilde{f}\,(0)\\ &= T * (g_\varepsilon * \widetilde{f}\,)(0)\\ &\underset{\varepsilon\to 0}{\to} T * \widetilde{f}\,(0)\,, \quad \text{by the previous theorem,}\\ &= T(f)\end{aligned}$$

which proves the first part.

Next, we note that for given $\psi \in C_0^\infty(\mathbb{R}^d)$ and $0 < \varepsilon < 1$, say, the supports of $\varphi_\varepsilon * \psi$ and ψ all lie in some fixed compact set (independently of ε). Hence $\varphi_\varepsilon * \psi \to \psi$ in $\mathscr{D}(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$. Arguing now as above, we deduce that for any $u \in \mathscr{D}'(\mathbb{R}^d)$ and $\psi \in \mathscr{D}(\mathbb{R}^d)$, $(u * \varphi_\varepsilon)(\psi) \to u(\psi)$ as $\varepsilon \downarrow 0$, that is, $u * \varphi_\varepsilon \to u$ in $\mathscr{D}'(\mathbb{R}^d)$. ■

Remark 5.18. The infinitely-differentiable function $T * g_\varepsilon$ is called the regularization of T. This is easier to deal with than T itself, but of course, one must eventually take the limit $\varepsilon \downarrow 0$ in order to recover the distribution T.

Theorem 5.19.

(i) *$C_0^\infty(\mathbb{R}^d)$ is dense in $\mathscr{D}'(\mathbb{R}^d)$, that is, for any $u \in \mathscr{D}'(\mathbb{R}^d)$ there is some sequence (f_n) in $\mathscr{D}(\mathbb{R}^d)$ such that $f_n \to u$ in $\mathscr{D}'(\mathbb{R}^d)$.*

(ii) *$C_0^\infty(\mathbb{R}^d)$ is dense in $\mathscr{S}'(\mathbb{R}^d)$, that is, for any $T \in \mathscr{S}'(\mathbb{R}^d)$ there is some sequence (f_n) in $\mathscr{S}(\mathbb{R}^d)$ such that $f_n \to T$ in $\mathscr{S}'(\mathbb{R}^d)$.*

Proof. (i) Fix $u \in \mathscr{D}'(\mathbb{R}^d)$ and let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\int \varphi(x)\,dx = 1$ and set $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. For $n \in \mathbb{N}$ and $t \geq 0$, let the function $\lambda_n(t)$ be as shown in the diagram.

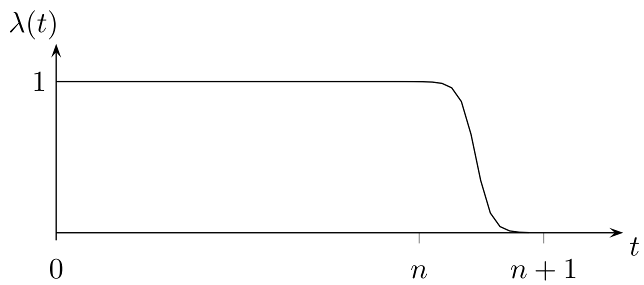

Figure 5.1: The function $\lambda(t)$.

For $x \in \mathbb{R}^d$, let $\gamma_n(x) = \lambda_n(|x|)$. Evidently, $\gamma_n \in C_0^\infty(\mathbb{R}^d)$ and $\gamma_n(x) = 1$ for $|x| \le n$ and $\gamma_n(x) = 0$ when $|x| \ge n+1$. Now, $\gamma_n u$ has compact support and so $\gamma_n u * \varphi_{1/n} \in C_0^\infty(\mathbb{R}^d)$. We claim that $\gamma_n u * \varphi_{1/n} \to u$ in $\mathscr{D}'(\mathbb{R}^d)$ as $n \to \infty$.

Indeed, for $\psi \in \mathscr{D}(\mathbb{R}^d)$, we have

$$\begin{aligned}
(\gamma_n u * \varphi_{1/n})(\psi) &= (\gamma_n u * \varphi_{1/n}) * \widetilde{\psi}\,(0) \\
&= \gamma_n u * (\varphi_{1/n} * \widetilde{\psi}\,)(0) \\
&= \gamma_n u(\widetilde{\varphi}_{1/n} * \psi)) \\
&= u(\gamma_n(\widetilde{\varphi}_{1/n} * \psi)) \\
&= u(\widetilde{\varphi}_{1/n} * \psi)\,, \quad \text{for all sufficiently large } n, \\
&= u(\,(\varphi_{1/n} * \widetilde{\psi}\,)^{\sim}\,) \\
&\to u((\widetilde{\psi}\,)^{\sim}\,) \\
&= u(\psi)\,,
\end{aligned}$$

as required.

(ii) The functions λ_n are supposed to be smooth and obey the requirements that $\lambda_n(t) = 1$ when $0 \le t \le n$, $\lambda_n(t) = 0$ for $t \ge n+1$ and $\lambda_{n+1}(t) = \lambda_n(t-1)$ for $n+1 \le t \le n+2$. As n increases, so the graph of λ_n extends out but maintains its general shape as it decreases from 1 to 0. The point is that the λ_ns and any derivatives are bounded independently of n. That is, for any $k = 0, 1, 2, \dots$, there is $M_k > 0$ such that $\sup_n \sup_{t \ge 0} |\lambda_n^{(k)}(t)| < M_k$.

Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and let $f \in \mathscr{S}(\mathbb{R}^d)$. Then, with notation as in part (i), we note that $\gamma_n T * \varphi_{1/n} \in C_0^\infty(\mathbb{R}^d)$. We claim that $\gamma_n T * \varphi_{1/n} \to T$ in $\mathscr{S}'(\mathbb{R}^d)$ as $n \to \infty$.

To see this, first we observe that $\gamma_n(\varphi_{1/n} * f) \to f$ in $\mathscr{S}(\mathbb{R}^d)$. This follows from the inequalities

$$\|\gamma_n(\varphi_{1/n} * f) - f\|_{\alpha,\beta} \le \|\gamma_n(\varphi_{1/n} * f - f)\|_{\alpha,\beta} + \|\gamma_n f - f\|_{\alpha,\beta}$$

for $\alpha, \beta \in \mathbb{Z}_+^d$, together with (Leibnitz' formula and) the bounds on $\|D^\tau \gamma_n\|_\infty$ for each fixed $\tau \in \mathbb{Z}_+^d$ uniformly in n.

Similarly, $\gamma_n(\varphi_{1/n} * \widetilde{f}\,)^{\sim} \to f$ in $\mathscr{S}(\mathbb{R}^d)$ and therefore

$$\begin{aligned}
(\gamma_n T * \varphi_{1/n})(f) &= T(\gamma_n(\varphi_{1/n} * \widetilde{f}\,)^{\sim}\,)\,, \quad \text{as in part (i)}, \\
&\to T(f)\,,
\end{aligned}$$

as $n \to \infty$, that is, $\gamma_n T * \varphi_{1/n} \to T$ in $\mathscr{S}'(\mathbb{R}^d)$ and the proof is complete. ■

Remark 5.20. This result ties up the approach to distributions taken here with the "generalized function" approach in which distributions are defined via *sequences* of functions in $\mathscr{S}(\mathbb{R}^d)$. If $T \in \mathscr{S}'(\mathbb{R}^d)$ and $f_n \in \mathscr{S}(\mathbb{R}^d)$ is such that $f_n \to T$ in $\mathscr{S}'(\mathbb{R}^d)$, then the generalized function approach would be to consider T to *be* the sequence (f_n) (or strictly speaking, equivalence classes of such sequences so as to allow for different sequences in $\mathscr{S}(\mathbb{R}^d)$ which converge in $\mathscr{S}'(\mathbb{R}^d)$ to the same distribution). This is in the same spirit as defining real numbers via Cauchy sequences of rational numbers.

The next result tells us that, under the Fourier transform, convolution becomes essentially multiplication.

Theorem 5.21. *For any* $T \in \mathscr{S}'(\mathbb{R}^d)$ *and* $f \in \mathscr{S}(\mathbb{R}^d)$,

(i) $\mathscr{F}(T * f) = (2\pi)^{d/2}\, \mathscr{F}f\, \mathscr{F}T$ *and*

(ii) $\mathscr{F}T * \mathscr{F}f = (2\pi)^{d/2}\, \mathscr{F}(f\,T)$.

Proof. (i) We know that there is a sequence (φ_n) in $C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \to T$ in $\mathscr{S}'(\mathbb{R}^d)$. So for given $g \in \mathscr{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}
(T*f)\widehat{\ }\,(g) &= (T*f)(\,\widehat{g}\,) = (T*f)*\widetilde{\widehat{g}}\,(0)\\
&= T*(\,f*\widetilde{\widehat{g}}\,)(0) = T(\,(\,f*\widetilde{\widehat{g}}\,)\widetilde{\ }\,)\\
&= T(\,\widetilde{f}*\widehat{g}\,)\\
&= \lim_n \varphi_n(\,\widetilde{f}*\widehat{g}\,) = \lim_n \varphi_n*(\,f*\widetilde{\widehat{g}}\,)(0)\\
&= \lim_n (\,\varphi_n*f\,)(\,\widehat{g}\,) = \lim_n (\,\varphi_n*f\,)\widehat{\ }\,(g)\\
&= (2\pi)^{d/2}\,\lim_n (\widehat{\varphi}_n\,\widehat{f}\,)(g) = \lim_n \int_{\mathbb{R}^d} \widehat{\varphi}_n(x)\,\widehat{f}\,(x)\,g(x)\,dx\\
&= \lim_n \int_{\mathbb{R}^d} \varphi_n(x)\,(\widehat{f}\,g)\widehat{\ }\,(x)\,dx\\
&= (2\pi)^{d/2}\,T(\,(\widehat{f}\,g)\widehat{\ }\,) = (2\pi)^{d/2}\,\widehat{T}\,(\,\widehat{f}\,g\,)\\
&= (2\pi)^{d/2}\,(\widehat{f}\,\widehat{T}\,)(g)\,,
\end{aligned}$$

that is, $(T*f)\widehat{\ } = (2\pi)^{d/2}\,\widehat{f}\,\widehat{T}$, as required.

The second part can be established in a similar way. ∎

Chapter 6

Fourier-Laplace Transform

We have seen that any tempered distribution has a Fourier transform which is also a tempered distribution. We will see here, however, that the Fourier transform of a distribution with compact support is actually given by a *function*. By way of motivation, we note that the Fourier transform of an integrable function u, say, is given by

$$\widehat{u}\,(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e_y(x)\, u(x)\; dx$$

where e_y denotes the function $x \mapsto e^{-iyx}$ for $x \in \mathbb{R}^d$. We cannot write this in the distributional sense as $T_u(e_y)$ because the function e_y does not belong to $\mathscr{D}(\mathbb{R}^d)$ (or $\mathscr{S}(\mathbb{R}^d)$). However, if u has compact support, then we can write

$$\widehat{u}\,(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e_y(x)\, u(x)\; dx = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e_y(x)\, \psi(x)\, u(x)\; dx$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$ is chosen such that $\psi = 1$ on some open set containing $\operatorname{supp} u$. In this case, we see that

$$\widehat{u}\,(y) = (2\pi)^{-d/2}\, T_u(e_y\, \psi)\,.$$

This formula makes sense if $y \in \mathbb{R}^d$ is replaced by any $z \in \mathbb{C}^d$. The following definition seems appropriate.

Definition 6.1. Suppose that $u \in \mathscr{D}'(\mathbb{R}^d)$ and that u has compact support. The Fourier-Laplace transform of u is the function $\widehat{u}\,(\cdot)$ on $\mathbb{C}^d$ given by

$$\widehat{u}\,(z) = (2\pi)^{-d/2}\, u(e_z\, \psi)$$

where $e_z(x) = e^{-izx}$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ is such that $\psi = 1$ on some open set W with $\operatorname{supp} u \subset W$.

We note straightaway that $e_z\psi \in \mathscr{D}(\mathbb{R}^d)$ and that if W_1 and W_2 are open sets with $\operatorname{supp} u \subset W_1$ and $\operatorname{supp} u \subset W_2$ and if $\psi_1 \in C_0^\infty(\mathbb{R}^d)$ and $\psi_2 \in C_0^\infty(\mathbb{R}^d)$ are such that $\psi_1 = 1$ on W_1 and $\psi_2 = 1$ on W_2, then it follows that $u(e_z\,\psi_1) = u(e_z\,\psi_2)$ so that $\widehat{u}$ is a well-defined function on $\mathbb{C}^d$. Furthermore, if u is given by an integrable function, then, as discussed above, $\widehat{u}\,(y)$ is the Fourier transform of u.

Now, any element of $\mathscr{D}'(\mathbb{R}^d)$ with compact support determines a tempered distribution and for any $u \in \mathscr{S}'(\mathbb{R}^d)$ we have already defined its Fourier transform as the tempered distribution $\widehat{u} : f \mapsto u(\widehat{f}\,)$. We can then ask whether there is any relationship between the tempered distribution $\widehat{u}$ and the function $\widehat{u}\,(x)$, $x \in \mathbb{R}^d$. If u is given by a square-integrable function, then by Plancherel's Theorem, Corollary 4.9, we find that for any $f \in \mathscr{S}(\mathbb{R}^d)$,

$$u(\widehat{f}\,) = \int u(x)\,\widehat{f}\,(x)\,dx = \int \widehat{u}\,(x)\,f(x)\,dx\,.$$

This shows that in this case, the tempered distribution $\widehat{u}$ is indeed given by the function $\widehat{u}\,(x)$. This is true in general, as we now show.

Theorem 6.2. *Let $u \in \mathscr{D}'(\mathbb{R}^d)$ and suppose that u has compact support. Then the Fourier-Laplace transform $\widehat{u}\,(z)$ is an entire function whose restriction to $\mathbb{R}^d$ determines the tempered distribution $\widehat{u}$, that is, $\widehat{u} \in \mathscr{S}'(\mathbb{R}^d)$ is given by the function $\widehat{u}\,(\cdot) \restriction \mathbb{R}^d$.*

Proof. Let $\psi \in C_0^\infty$ be such that $\psi = 1$ on some open set $W \subset \mathbb{R}^d$ with $\operatorname{supp} u \subset W$. Then $\widehat{u}\,(z) = (2\pi)^{-d/2}\,u(e_z\,\psi)$ for $z \in \mathbb{C}^d$ and

$$e_z(y)\,\psi(y) = e^{-izy}\,\psi(y) = \sum_{\alpha \in \mathbb{Z}_+^d} \tfrac{(-i)^\alpha}{\alpha!}\,z^\alpha\,y^\alpha\,\psi(y)$$

for any $y \in \mathbb{R}^d$. Since $\operatorname{supp}\psi$ is compact, it follows that for fixed $z \in \mathbb{C}^d$ the partial sums converge uniformly in y and the same is true of any partial derivatives (with respect to y). In other words, the partial sums converge in $\mathscr{D}(\mathbb{R}^d)$ and so

$$u(e_z\,\psi) = \sum_{\alpha \in \mathbb{Z}_+^d} \tfrac{(-i)^\alpha}{\alpha!}\,z^\alpha\,u(g_\alpha)$$

where $g_\alpha \in \mathscr{D}(\mathbb{R}^d)$ is the function $g_\alpha(y) = y^\alpha\,\psi(y)$.

Moreover, u has finite order by Theorem 3.14, so that $|u(\varphi)| \le C\,|\!|\!|\varphi|\!|\!|_N$ for some constants $C > 0$ and $N \in \mathbb{Z}_+$ and any $\varphi \in \mathscr{D}(\mathbb{R}^d)$. It follows that $|u(g_\alpha)| \le C' R^{|\alpha|}$ for constants C' and R (depending, of course, on u and $\operatorname{supp} u$). The series above for $u(e_z\,\psi)$ therefore converges absolutely for all $z \in \mathbb{C}^d$ and so $u(e_z\,\psi)$ is entire.

We must now show that $\widehat{u}$, the tempered distribution, is given by the function $\widehat{u}(x)$, $x \in \mathbb{R}^d$.

Now, $u = u\,\psi$ as distributions, that is,

$$u(\varphi) = (\psi\, u)(\varphi) = u(\psi\, \varphi)$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and so

$$\widehat{u} = (\psi\, u)\widehat{\ } = (2\pi)^{-d/2}\, \widehat{u} * \widehat{\psi}\ .$$

It follows that

$$\widehat{u}(\varphi) = \int (2\pi)^{-d/2}\, (\widehat{u} * \widehat{\psi})(y)\, \varphi(y)\, dy\,,$$

that is, $\widehat{u}$ is determined by the C^∞-function $(2\pi)^{-d/2}\, (\widehat{u} * \widehat{\psi})(x)$. Our aim is to show that $(2\pi)^{-d/2}\, (\widehat{u} * \widehat{\psi})(x) = \widehat{u}(x)$. To see this, let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\widehat{\varphi} = \psi$. Then

$$\begin{aligned}
(\widehat{u} * \widehat{\psi})(x) &= (\widehat{u} * \widehat{\widehat{\varphi}})(x) \\
&= (\widehat{u} * \widetilde{\varphi})(x) \\
&= \widehat{u}(\tau_x \varphi) \\
&= u(\, (\tau_x \varphi)\widehat{\ }\,) \\
&= u(e_x \widehat{\varphi}\,) \\
&= u(e_x \psi) \\
&= (2\pi)^{d/2}\, \widehat{u}(x)
\end{aligned}$$

and the result follows. ∎

Next we consider the relationship between the support of u and growth properties of the function $\widehat{u}(z)$. We need a preliminary result.

Lemma 6.3. *Suppose that f is analytic on $\mathbb{C}^d$ and vanishes on $\mathbb{R}^d$. Then $f = 0$ everywhere.*

Proof. Let $z_1, z_2, \dots, z_d \in \mathbb{C}^d$. Fix $a_2 \in \mathbb{R}, \dots,\ a_d \in \mathbb{R}$ and consider the map $z \mapsto f(z, a_2, \dots, a_d)$. This is entire and vanishes on $\mathbb{R}$ and so vanishes everywhere on $\mathbb{C}$, by the Identity Theorem. Since $a_2 \in \mathbb{R}$ is arbitrary, we may say that $f(z_1, a_2, \dots, a_d) = 0$ for all $a_2 \in \mathbb{R}$. But then the map $z \mapsto f(z_1, z, a_3, \dots, a_d)$ is entire and vanishes on $\mathbb{R}$ and so vanishes on $\mathbb{C}$. In particular, $f(z_1, z_2, a_3, \dots, a_d) = 0$ for any $a_3 \in \mathbb{R}$.

Continuing in this way, we see that $f(z_1, z_2, \dots, z_d) = 0$ and the result follows. ∎

For the following, let K_r denote the closed ball $K_r = \{\, x \in \mathbb{R}^d : |x| \le r \,\}$.

Theorem 6.4.

(a) *Suppose that* $\varphi \in C_0^\infty(\mathbb{R}^d)$ *and that* $\operatorname{supp}\varphi \subseteq K_r$.

Then

$$(1) \qquad f(z) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-izt}\,\varphi(t)\,dt, \text{ for } z \in \mathbb{C}^d, \text{ is entire}$$

and there are constants γ_N *such that*

$$(2) \qquad |f(z)| \le \gamma_N\,(1+|z|)^{-N}\,e^{r\,|\operatorname{Im} z|} \text{ for } z \in \mathbb{C}^d \text{ and } N = 0,1,2,\dots.$$

(b) *Conversely, if an entire function* f *satisfies* (2)*, then there is some* $\varphi \in C_0^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}\varphi \subseteq K_r$ *such that* (1) *holds.*

Proof. (a) First note that if $z = x + iy \in \mathbb{C}^d$, then

$$|e^{-izt}| = e^{yt} \le e^{|y|\,|t|} \le e^{r|\operatorname{Im} z|}$$

for all $t \in \mathbb{R}^d$ with $|t| \le r$ (and where $|\operatorname{Im} z| = (y_1^2 + \cdots + y_d^2)^{1/2}$). It follows that $e^{-izt}\,\varphi(t)$ is integrable for each $z \in \mathbb{C}^d$. Moreover, for each fixed $z \in \mathbb{C}^d$, the power series expansion for e^{-izt} converges uniformly for $t \in K_r$ and therefore

$$\begin{aligned} f(z) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \varphi(t) \sum_{\alpha \in \mathbb{Z}_+^d} \frac{(-iz)^\alpha t^\alpha}{\alpha!}\,dt \\ &= \sum_{\alpha \in \mathbb{Z}_+^d} (2\pi)^{-d/2} \int_{\mathbb{R}^d} \varphi(t)\,\frac{(-iz)^\alpha t^\alpha}{\alpha!}\,dt\,. \end{aligned}$$

Now

$$\begin{aligned} \Big| \int \varphi(t)\,t^\alpha\,dt \Big| &\le \int |\varphi(t)|\,r^{\alpha_1} \dots r^{\alpha_d}\,dt \\ &= r^{|\alpha|}\,\|\varphi\|_{L^1} \end{aligned}$$

and so we see that the series expression for $f(z)$ converges absolutely for all $z \in \mathbb{C}^d$ and so f is analytic on the whole of $\mathbb{C}^d$.

Next, integrating by parts, we find that

$$\begin{aligned} i^\alpha\,z^\alpha\,f(z) &= (2\pi)^{-d/2} \int \varphi(t)\,(iz)^\alpha\,e^{-izt}\,dt \\ &= (2\pi)^{-d/2} \int (D^\alpha\varphi)(t)\,e^{-izt}\,dt \end{aligned}$$

and so

$$|z^\alpha|\,|f(z)| \le (2\pi)^{-d/2}\,\|D^\alpha\varphi\|_{L^1}\,e^{r|\operatorname{Im} z|}\,.$$

This, together with the inequality $(1+|z|)^N \le (1+|z_1|+\cdots+|z_d|)^N$ implies that

$$(1+|z|)^N\,|f(z)| \le \gamma_N\; e^{r|\operatorname{Im} z|}$$

for a suitable constant γ_N, which is (2).

(b) Suppose that f is entire and satisfies the inequalities (2). For $t \in \mathbb{R}^d$, let

$$\varphi(t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, e^{itx}\; dx\,.$$

Since $(1+|x|)^N f(x)$ is bounded on $\mathbb{R}^d$ for any N (by (2)), it follows that φ is a well-defined function and that $\varphi \in \mathbb{C}^\infty(\mathbb{R}^d)$.

We wish to show that it is possible to replace x by $x+iy$ in this formula for φ without any other changes. To see this, let

$$I(\eta) = \int_{-\infty}^{\infty} f(x+i\eta, z_2, \ldots, z_d)\; e^{(t_1(x+i\eta)+t_2z_2+\cdots+t_dz_d)}\; dx$$

where $t_1, \ldots, t_d \in \mathbb{R}$, $z_2, \ldots z_d \in \mathbb{C}$ and $\eta \in \mathbb{R}$.

We shall show that $I(\eta) = I(0)$ which shows that, in fact, I does not depend on η. Let Γ be the (closed, simple) rectangular contour in $\mathbb{C}$ with vertices at the points $\pm X$ and $\pm X + i\eta$, where $X > 0$. Since the function $\zeta \mapsto f(\zeta, z_2, \ldots, z_d)\; e^{(t_1\zeta+t_2z_2+\cdots+t_dz_d)}$ is analytic, it follows that

$$\int_\Gamma f(\zeta, z_2, \ldots, z_d)\; e^{(t_1\zeta+t_2z_2+\cdots+t_dz_d)}\; d\zeta = 0\,, \qquad (*)$$

by Cauchy's Theorem. Now we use (2) to estimate the integrand along the vertical sides of the rectangular contour Γ.

We have, with $z = (\pm X + iy, z_2, \ldots, z_d)$,

$$\begin{aligned}
&\big|\, f(\pm X + iy, z_2, \ldots, z_d)\, e^{i(t_1(\pm X+iy)+t_2z_2+\cdots+t_dz_d)}\,\big| \\
&\qquad \le \frac{\gamma_N\; e^{r|\operatorname{Im} z|}\; e^{-t_1 y}\; e^{-t_2|\operatorname{Im} z_2|} \ldots e^{-t_d|\operatorname{Im} z_d|}}{\big(1 + (\,|\pm X + iy|^2 + |z_2|^2 + \cdots + |z_d|^2\,)^{1/2}\big)^N} \\
&\qquad \le \frac{\gamma_N}{(\,1 + |\pm X + iy|\,)^N}\; e^{r|\operatorname{Im} z|}\; e^{-t_1 y}\; e^{-t_2|\operatorname{Im} z_2|} \ldots e^{-t_d|\operatorname{Im} z_d|} \\
&\qquad \le \frac{\gamma_N}{(1+X)^N}\; e^{r|\operatorname{Im} z|}\; e^{-t_1 y}\; e^{-t_2|\operatorname{Im} z_2|} \ldots e^{-t_d|\operatorname{Im} z_d|} \\
&\qquad \le \frac{\gamma_N}{(1+X)^N}\; e^{r|\operatorname{Im} z|}\; e^{-t_1 y}\; e^{-t_2|\operatorname{Im} z_2|} \ldots e^{-t_d|\operatorname{Im} z_d|} \\
&\qquad \to 0
\end{aligned}$$

as $X \to \infty$ for all $|y| \le |\eta|$. It follows that the part of the contour integral along the vertical sides of Γ converges to zero, as $X \to \infty$ and so from $(*)$ we conclude that $I(0) - I(\eta) = 0$, as required.

Repeating this argument coordinate by coordinate, we deduce that

$$\begin{aligned}\varphi(t) &= (2\pi)^{-d/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x)\, e^{itx}\, dx_1\, dx_2 \dots dx_d \\ &= (2\pi)^{-d/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x+iy)\, e^{it(x+iy)}\, dx_1\, dx_2 \dots dx_d\end{aligned}$$

for any $y \in \mathbb{R}^d$.

Now, let $N \in \mathbb{N}$ be such that $(1+|x|)^{-N} \in L^1(\mathbb{R}^d)$. Then, using (2), together with the inequality $(1+|x+iy|)^{-N} \le (1+|x|)^{-N}$, we find that

$$\begin{aligned}|\varphi(t)| &\le (2\pi)^{-d/2} \gamma_N \int_{\mathbb{R}^d} (1+|x|)^{-N} e^{r|y|} e^{-ty}\, dx \\ &\le (2\pi)^{-d/2} \gamma_N\, e^{(r|y|-ty)} \int_{\mathbb{R}^d} (1+|x|)^{-N}\, dx\end{aligned}$$

for all $y \in \mathbb{R}^d$. That is, there is some constant $C > 0$ such that

$$|\varphi(t)| \le C\, e^{r|y|-ty}$$

for any $y \in \mathbb{R}^d$. We shall show that this implies that $\operatorname{supp} \varphi \subseteq K_r$. Indeed, let $t \in \mathbb{R}^d$ be fixed such that $|t| > r$. Setting $y = \lambda t$ with $\lambda > 0$, it follows that $ty = \lambda\, |t|^2$ and we see that

$$|\varphi(t)| \le C\, e^{\lambda |t|(r-|t|)}$$

for any $\lambda > 0$. Letting $\lambda \to \infty$, it follows that $\varphi(t) = 0$ and so we conclude that $\operatorname{supp} \varphi \subseteq K_r$, as claimed.

By the Fourier inversion theorem, we have

$$f(x) = (2\pi)^{-d/2} \int \varphi(t)\, e^{-ixt}\, dt\,.$$

It follows that the entire function $z \mapsto g(z) \equiv \int \varphi(t)\, e^{-izt}\, dt$ agrees with f on $\mathbb{R}^d$ and so, by the Lemma, $f = g$ on $\mathbb{C}^d$, that is, f is given as in (1) and the proof is complete. ∎

There is a version of this result for distributions, as follows.

Theorem 6.5.

(a) *Suppose that* $u \in \mathscr{D}'(\mathbb{R}^d)$ *and that* $\operatorname{supp} u \subseteq K_r$. *Then the Fourier-Laplace transform* $f(z) = \widehat{u}\,(z)$ *is entire, its restriction to* $\mathbb{R}^d$ *is the Fourier transform of* u*, and there is a constant* $\gamma > 0$ *such that*

$$|f(z)| \le \gamma\, (1+|z|)^N\, e^{r|\operatorname{Im} z|} \qquad (*)$$

where N *is the order of* u.

(b) *Conversely, if f is entire and satisfies $(*)$ for some $N \in \mathbb{N}$, some $\gamma > 0$ and $r > 0$, then $f(z) = \widehat{u}\,(z)$ for some $u \in \mathscr{D}'(\mathbb{R}^d)$ with* $\operatorname{supp} u \subseteq K_r$.

Proof. (a) We have already proved everything except for the estimate $(*)$. Let $h \in C^\infty(\mathbb{R})$ be such that $h(s) = 1$ when $s \leq 1$ and $h(s) = 0$ for all $s \geq 2$. For given $z \in \mathbb{C}^d$ (with $z \neq 0$), let

$$\chi(x) = h(\,|x|\,|z| - r\,|z|\,)\,.$$

Then $\chi \in C^\infty(\mathbb{R})$ and $\chi(x) = 0$ whenever $|x|\,|z| - r\,|z| > 2$, that is χ vanishes whenever $|x| > \frac{2}{|z|} + r$. So $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subseteq K_{r+2/|z|}$. Furthermore, if $|x| < r + \frac{1}{|z|}$, then $\chi(x) = 1$ and so $\widehat{u}\,(z) = u(e_z\chi)$ (since $\chi = 1$ on some open set W with $\sup u \subset W$).

By hypothesis, $\operatorname{supp} u$ is compact and so u has finite order, say N. Then there is $C > 0$ such that

$$|\,u(e_z\chi)\,| \leq C \sum_{|\alpha| \leq N} \|D^\alpha(e^{-izx}\chi(x))\|_\infty\,.$$

Now $\partial h/\partial x_j = |z|\,\dfrac{x_j}{|x|}\,h'(|x|\,|z| - r\,|z|)$ and so for any $\alpha, \beta \in \mathbb{Z}_+^d$, we have

$$\begin{aligned}
\big|\,(D^\alpha e^{-izx})(D^\beta\chi(x))\,\big| &= \big|\,(-iz)^\alpha e^{-izx}\,|z|^{|\beta|}\,\frac{x^\beta}{|x|^{|\beta|}}\,D^{|\beta|}\,\big| \\
&\leq |z|^{|\alpha|+|\beta|}\;\|\,D^{|\beta|}h\,\|_\infty\,e^{|\operatorname{Im} z|(r+2/|z|)} \\
&\text{(since } |z_j^{\alpha_j}| \leq |z|^{\alpha_j} \text{ and if } x \in \operatorname{supp}\chi \text{ then } |x| \leq r + 2/\,|z|) \\
&\leq |z|^{|\alpha|+|\beta|}\;\|\,D^{|\beta|}h\,\|_\infty\,e^{r|\operatorname{Im} z|+2} \\
&\text{(since } |\operatorname{Im} z| \leq |z|)\,.
\end{aligned}$$

Hence, by Leibnitz' formula, we conclude that

$$|\widehat{u}\,(z)| \leq \gamma\,(1 + |z|\,)^N\,e^{r|\operatorname{Im} z|}$$

for a suitable constant γ.

(b) Suppose that f satisfies $|f(x)| \leq \gamma(1 + |x|\,)^N e^{r\operatorname{Im} z}$ for some constants γ, $r > 0$ and some $N \in \mathbb{N}$. Then the map from $\mathscr{S} \to \mathbb{C}$ given by

$$\varphi \mapsto f(\varphi) = \int f(x)\,\varphi(x)\,dx$$

is a tempered distribution. Hence there is some $T \in \mathscr{S}'(\mathbb{R}^d)$ such that $\widehat{T} = f$ (namely, $T = \mathscr{F}^{-1}f$). We will show that $\operatorname{supp} T \subseteq K_r$ and that $f(z) = \widehat{T}\,(z)$. The idea is to regularize T and apply the Paley-Wiener theorem for functions.

Let $g \in C_0^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp} g \subset K_1$ and $\int g(x)\,dx = 1$. For $\varepsilon > 0$, set $g_\varepsilon(x) = e^{-d}\, g(x/\varepsilon)$. Then $\operatorname{supp} g_\varepsilon \subset K_\varepsilon$ and we have seen that for any tempered distribution $S \in \mathscr{S}'(\mathbb{R}^d)$, it is true that $S * g_\varepsilon \to S$ in $\mathscr{S}'(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$. Now

$$T * g_\varepsilon = (\mathscr{F}^{-1} f) * g_\varepsilon = \mathscr{F}^{-1}((2\pi)^{d/2} \widehat{g_\varepsilon}\ f)$$

since $(S * \varphi)\widehat{\ } = (2\pi)^{d/2} \widehat{\varphi}\, \widehat{S}$, for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $S \in \mathscr{S}'(\mathbb{R}^d)$. Since $\operatorname{supp} g_\varepsilon \subseteq K_\varepsilon$, the Paley-Wiener Theorem implies that $\widehat{g_\varepsilon}$ is entire and that for any integer k there is a constant γ_k such that

$$\begin{aligned} |\, \widehat{g_\varepsilon}\,(z)\,| &= |\,(2\pi)^{-d/2} \int g_\varepsilon(y)\, e^{-izy}\, dy\,| \\ &\le \gamma_k\, (1 + |z|\,)^{-k}\, e^{\varepsilon\, |\mathrm{Im}\, z|} \end{aligned}$$

for any $z \in \mathbb{C}^d$. But, by hypothesis,

$$|f(z)| \le \gamma\, (1 + |z|\,)^N\, e^{r\, |\mathrm{Im}\, z|}$$

and so, for any integer m,

$$\begin{aligned} |\, \widehat{g_\varepsilon}\,(z)\, f(z)\,| &\le \gamma_{N+m} \gamma\, (1 + |z|\,)^{-N-m}\, (1 + |z|\,)^N\, e^{(r+\varepsilon)\, |\mathrm{Im}\, z|} \\ &= \gamma_{N+m} \gamma\, (1 + |z|\,)^{-m}\, e^{(r+\varepsilon)\, |\mathrm{Im}\, z|} \end{aligned}$$

for $z \in \mathbb{C}^d$.

Again, by the Paley-Wiener Theorem, it follows that there is $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \varphi_\varepsilon \subseteq K_{r+\varepsilon}$ such that

$$\widehat{g_\varepsilon}\,(z)\, f(z) = (2\pi)^{-d/2} \int e^{-izt}\, \varphi_\varepsilon(t)\, dt\,.$$

In particular, $\widehat{g_\varepsilon}\,(x)\, f(x) \in \mathscr{S}(\mathbb{R}^d)$ and is the Fourier transform of φ_ε. That is, $\mathscr{F}^{-1}(\widehat{g_\varepsilon}\ f) = \varphi_\varepsilon$ as functions and so therefore as tempered distributions.

Now, let $\psi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\operatorname{supp} \psi \cap K_r = \varnothing$. Then, for all $\varepsilon > 0$ sufficiently small, $\operatorname{supp} \psi \cap K_{r+\varepsilon} = \varnothing$ and therefore $\int \varphi_\varepsilon(x)\, \psi(x)\, dx = 0$ for small ε. It follows that

$$\begin{aligned} T(\psi) &= \lim_{\varepsilon \downarrow 0} T * g_\varepsilon(\psi) \\ &= \lim_{\varepsilon \downarrow 0} (2\pi)^{d/2} \int \varphi_\varepsilon(x)\, \psi(x)\, dx \\ &= 0 \end{aligned}$$

and so $\operatorname{supp} T \subseteq K_r$, as required.

It remains to verify that $\widehat{T}\,(z) = f(z)$ on $\mathbb{C}^d$. However, $\widehat{T}$ is entire and $\widehat{T}\,(x)$ is the Fourier transform of T, $\widehat{T}\,(x) = f(x)$ for $x \in \mathbb{R}^d$. But, by hypothesis, f is entire and so $\widehat{T} = f$ on $\mathbb{R}^d$ means that $\widehat{T} = f$ on $\mathbb{C}^d$ and the proof is complete. ■

Chapter 7

Structure Theorem for Distributions

Any polynomially bounded continuous function certainly determines a tempered distribution. It turns out that such functions, together with their distributional derivatives exhaust $\mathscr{S}'(\mathbb{R}^d)$, as we discuss next.

Theorem 7.1. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$. Then there is a continuous polynomially bounded function $F(x)$ and a multi-index $\alpha \in \mathbb{Z}_+^d$ such that $T = D^\alpha F$.*

Proof. For notational convenience, we shall only consider the one-dimensional case, $d = 1$. So let $T \in \mathscr{S}(\mathbb{R})$ be given. Then we know that there exist $k, m \in \mathbb{N}$ and $C > 0$ such that

$$|T(f)| \le C\|f\|_{k,m} = C \sum_{\substack{\alpha \le k \\ \beta \le m}} \|\!|f|\!\|_{\alpha,\beta}$$

for all $f \in \mathscr{S}(\mathbb{R})$. It follows that there is $C' > 0$ such that

$$|T(f)| \le C' \sum_{\beta \le m} \sup_x |\,(1+x^2)^k \, D^\beta f(x)\,|$$

for all $f \in \mathscr{S}(\mathbb{R})$. Now, if $\varphi \in C_0^\infty(\mathbb{R})$, then $\varphi(x) = \int_{-\infty}^x \varphi'(t)\,dt$ and so

$$|\varphi(x)| \le \int_{-\infty}^x \left|\varphi'(t)\right| \,dt \le \int_{-\infty}^\infty \left|\varphi'(t)\right| \,dt = \|\varphi'\|_{L^1} \,.$$

Hence, for any $\varphi \in C_0^\infty(\mathbb{R})$, $\sup_x |\varphi(x)| \le \|\varphi'\|_{L^1}$ and so

$$|T(\varphi)| \le C' \sum_{\beta \le m} \|\, D\big((1+x^2)^k D^\beta \varphi(x)\big)\,\|_{L^1}$$

since $(1+x^2)^k D^\beta \varphi \in C_0^\infty(\mathbb{R})$. Using the inequality

$$D(1+x^2)^k = k\,2x(1+x^2)^{k-1} \le k(1+x^2)^k$$

we get

$$|\,D\big((1+x^2)^k D^\beta \varphi(x)\big)\,| \le k\,|\,(1+x^2)^k D^\beta \varphi(x)\,| + |\,(1+x^2)^k D^{\beta+1}\varphi(x)\,|$$

and so there is $C'' > 0$ such that

$$|T(\varphi)| \le C'' \sum_{j \le m+1} \| (1+x^2)^k D^j \varphi(x) \| \tag{$*$}$$

for $\varphi \in C_0^\infty(\mathbb{R})$.

Let $J : C_0^\infty(\mathbb{R}) \to L^1(\mathbb{R}) \oplus \cdots \oplus L^1(\mathbb{R})$ (with $(m+2)$ terms) be the map given by

$$J(\varphi) = (1+x^2)^k \varphi(x) \oplus (1+x^2)^k D\varphi(x) \oplus \cdots \oplus (1+x^2)^k D^{m+1}\varphi(x)\,.$$

Note that J is one-one, because if $J(\varphi) = J(\psi)$, then, in particular, the first components agree and so $(1+x^2)^k\varphi(x) = (1+x^2)^k\psi(x)$ which means that $\varphi = \psi$. It is also clear that J is a linear map. Now we define the map $\Lambda : J(C_0^\infty(\mathbb{R})) \to \mathbb{C}$ by setting

$$\Lambda(J(\varphi)) = T(\varphi)\,.$$

Then Λ is well-defined, since $J(\varphi)$ is uniquely determined by φ, and is linear. Moreover, the bound $(*)$ implies that Λ is a bounded linear functional on $J(C_0^\infty(\mathbb{R}))$ when considered as a subspace of $(L^1(\mathbb{R}))^{m+2}$.

By the Hahn-Banach theorem, Λ has an extension to a bounded linear functional on the whole of $(L^1(\mathbb{R}))^{m+2}$. But then Λ must have the form

$$\Lambda(f_0 \oplus f_1 \oplus \cdots \oplus f_{m+1}) = \sum_{j=0}^{m+1} \int_{\mathbb{R}} g_j(x)\, f_j(x)\, dx$$

for suitable $g_0, \dots, g_{m+1} \in L^\infty(\mathbb{R})$. Hence

$$T(\varphi) = \Lambda(J\varphi) = \sum_{j=0}^{m+1} \int_{\mathbb{R}} g_j(x)\, (1+x^2)^k D^j \varphi(x)\, dx\,.$$

and so, as distributions,

$$T = \sum_{j=0}^{m+1} (-1)^j D^j \big((1+x^2)^k g_j(x) \big)$$

on $C_0^\infty(\mathbb{R})$. But $C_0^\infty(R)$ is dense in $\mathscr{S}(\mathbb{R})$ and therefore T has this same form on $\mathscr{S}(\mathbb{R})$.

For each j, set

$$h_j(x) = \int_0^x (1+t^2)^k\, g_j(t)\, dt\,.$$

Evidently h_j is continuous and polynomially bounded (because g_j is bounded) and

$$T = \sum_{j=0}^{m+1} (-1)^j D^{j+1} h_j\,.$$

To obtain the stated form for T, define f_j by

$$f_j(x) = \int_0^x dt_{m+1-j} \dots \int_0^{t_2} dt_1\, h_j(t_1)$$

i.e., by integrating h_j $(m+1-j)$ times. Then f_j is continuous, polynomially bounded and $D^{m+2} f_j(x) = D^{j+1} h_j$ and so

$$T = \sum_{j \le m+1} (-1)^j D^{m+2} f_j \,.$$

Set $F(x) = \sum_{j \le m+1} (-1)^j f_j(x)$. Then F is continuous, polynomially bounded and $T = D^{m+2} F$. ∎

Theorem 7.2. *Let $u \in \mathscr{D}'(\Omega)$. Then for any compact set $K \subset \Omega$, there exists a continuous function F and a multi-index α such that $u = D^\alpha F$ on $C_0^\infty(K)$.*

Proof. Let $u \in \mathscr{D}'(\Omega)$ and let $K \subset \Omega$ with K compact. Let $\chi \in C_0^\infty(\Omega)$ be such that $K \subset W \subset \operatorname{supp} \chi$ for some open set W with $\chi = 1$ on W. Then $u = \chi u$ on $C_0^\infty(K)$. However, χu has compact support and so defines a tempered distribution and therefore has the form $\chi u = D^\alpha F$ for some continuous function F and $\alpha \in \mathbb{Z}_+^d$. Hence, for all $\varphi \in C_0^\infty(K)$,

$$u(\varphi) = \chi u(\varphi) = D^\alpha F(\varphi)\,,$$

as required. ∎

Chapter 8

Partial Differential Equations

We will make a few sketchy remarks in this last chapter. We have seen that every distribution is differentiable (in the distributional sense) and so we can consider (partial) differential equations satisfied by distributions. Indeed, we might expect a differential equation to have a distribution as solution rather than a function.

Example 8.1. Consider the differential equation $u' = H$ where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is the Heaviside step-function. For $x < 0$, $u'(x) = 0$, so $u(x) = a$ and for $x > 0$, $u'(x) = 1$ giving $u(x) = x + b$, for suitable constants a and b. However, continuity of u at $x = 0$ would require $u(0) = a = b$ and so

$$u(x) = \begin{cases} a, & x < 0 \\ 1, & x + a \geq 0. \end{cases}$$

But such u is not differentiable at $x = 0$ and so cannot satisfy the original differential equation at this point. However, for any $\varphi \in \mathscr{S}(\mathbb{R})$, we see that $u'(\varphi) = H(\varphi)$, that is, u is a distributional solution to the differential equation. Indeed, integrating by parts, we get

$$\begin{aligned} u'(\varphi) = -u(\varphi') &= -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx \\ &= -\int_{-\infty}^{0} a\varphi'(x)\,dx - \int_{0}^{\infty} (x+a)\varphi'(x)\,dx \\ &= -a\varphi(0) + a\varphi(0) - \int_{0}^{\infty} x\varphi'(x)\,dx \\ &= \int_{0}^{\infty} \varphi(x)\,dx \\ &= \int_{-\infty}^{\infty} H(x)\varphi(x)\,dx \\ &= H(\varphi)\,. \end{aligned}$$

Notation. Suppose that $P(x_1, \dots, x_d)$ is a polynomial in d-variables. Then the symbol $P(D)$ denotes the partial differential operator obtained after the substitution $D_j = \frac{\partial}{\partial x_j}$ for x_j in the polynomial expression $P(x_, \dots, x_d)$. For example, if $P(x_1, \dots, x_d) = x_1^3 + x_3 x_4$, then $P(D) = \frac{\partial^3}{\partial x_1^3} + \frac{\partial^2}{\partial x_3 \partial x_4}$.

For a given function g on $\mathbb{R}^d$ and polynomial P, a distributional solution $u \in \mathscr{D}'(\mathbb{R}^d)$ to the partial differential equation $P(D)u = g$ is called a *weak solution*. We have seen above that a partial differential equation may possess a weak solution but no solution in the classical sense. Notice that a partial differential equation such as $P(D)u = g$ is meaningful even if $g \in \mathscr{D}'(\mathbb{R}^d)$. Of particular interest is the case for which $g = \delta$.

Definition 8.2. A distribution $E \in \mathscr{D}'(\mathbb{R}^d)$ satisfying the partial differential equation $P(D)E = \delta$ is said to be a fundamental solution for the partial differential operator $P(D)$.

The importance of fundamental solutions is their part in the solution of inhomogeneous partial differential equations of the form $P(D)u = g$, with $g \in C_0^\infty(\mathbb{R}^d)$. Indeed, if we set $u = E * g$, where E is a fundamental solution for $P(D)$, then $u \in C_0^\infty(\mathbb{R}^d)$ and we find that

$$\begin{aligned} P(D)u &= P(D)(E * g) \\ &= (P(D)E\,) * g \\ &= \delta * g \\ &= g\,, \end{aligned}$$

that is, $P(D)u = g$. So $E * g$ is a solution in the *classical sense*.

Example 8.3 (Poisson's Equation). Consider $\Delta u = g$ (in $\mathbb{R}^3$).

We claim that a fundamental solution for Δ is $E = -\frac{1}{4\pi|x|}$. To see this, let $\varphi \in C_0^\infty(\mathbb{R}^3)$. Then

$$\begin{aligned} (\Delta E)(\varphi) &= E(\Delta\varphi) \\ &= -\int_{\mathbb{R}^3} \frac{1}{4\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}\ \Delta\varphi\, dx_1\, dx_2\, dx_3 \\ &= -\int_{\mathbb{R}^3} \frac{1}{4\pi r}\ \Delta\varphi\, r^2 \cos\theta \sin\phi\, dr\, d\theta\, d\phi \\ &= \lim_{\varepsilon\downarrow 0} -\int_{r\ge\varepsilon} \frac{1}{4\pi r}\ \Delta\varphi\, r^2\, dr\, dS \end{aligned}$$

where $dS = \cos\theta \sin\phi\, d\theta\, d\phi$. Integrating by parts with respect to r, one finds (after some manipulation) that

$$\int_{r\ge\varepsilon} E(r)\, r^2\, \Delta\varphi\, dr = \int_{r\ge\varepsilon} \Delta E(r)\, \varphi\, r^2\, dr - r^2 E(r)\, \partial_r\varphi\Big|_{r=\varepsilon} + r^2\varphi\, \partial_r E(r)\Big|_{r=\varepsilon}.$$

But $\Delta E = 0$ for $x \neq 0$, so the first term on the right hand side above vanishes and we have

$$(\Delta E)(\varphi) = \lim_{\varepsilon \downarrow 0} \Big\{ \iint_{r=\varepsilon} E(r)\, \partial_r \varphi \; r^2 \, dS - \iint_{r \geq \varepsilon} \varphi \, \partial_r E(r) \; r^2 \, dS \Big\} .$$

Now, the first term in brackets gives zero in the limit $\varepsilon \downarrow 0$ ($\partial_r \varphi$ is bounded) and the second can be written as

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \iint_{r=\varepsilon} \varphi(r, \theta, \phi) \, \frac{1}{4\pi r^2} \; r^2 \, d\theta \, d\phi \\ &\quad = \iint \varphi(0) \, \tfrac{1}{4\pi} \, d\theta \, d\phi + \lim_{\varepsilon \downarrow 0} \iint \big(\varphi(\varepsilon, \theta, \phi) - \varphi(0) \big) \, \tfrac{1}{4\pi} \, d\theta \, d\phi \\ &\quad = \varphi(0) + 0 \end{aligned}$$

since φ is continuous at 0. We have shown that $(\Delta E)(\varphi) = \varphi(0)$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, so that $\Delta E = \delta$ as required.

Example 8.4. The heat operator (or diffusion operator) is $P(D) = \partial_t - \Delta$ where Δ is the Laplacian in $\mathbb{R}^d$ (so we are working with $(x, t) \in \mathbb{R}^d \times \mathbb{R}$). One checks that

$$E(x, t) = H(t) \, \frac{1}{2^d} \, \frac{1}{(\pi t)^{d/2}} \; e^{-|x|^2/4t}$$

satisfies $P(D)E = 0$ on $\mathbb{R}^d \times (\mathbb{R} \setminus \{ 0 \})$. We claim that E is a fundamental solution for $P(D)$.

To see this, let $\varphi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R})$. Then

$$\begin{aligned} (P(D)E)(\varphi) &= (\, (\partial_t - \Delta) E \,)(\varphi) \\ &= -E((\partial_t + \Delta)\varphi) \\ &= - \int E(x, t) \, (\partial_t \varphi + \Delta \varphi) \; dx \, dt \\ &\qquad \text{noting that } E \text{ is locally integrable,} \\ &= - \lim_{\varepsilon \downarrow 0} \int_{t \geq \varepsilon} E(x, t) \, (\partial_t \varphi + \Delta \varphi) \; dx \, dt \\ &= \lim_{\varepsilon \downarrow 0} \int_{t \geq \varepsilon} E(x, \varepsilon) \, \varphi(x, \varepsilon) \; dx \\ &\qquad \text{integrating by parts and using } P(D)E = 0 \\ &\qquad \text{on } \mathbb{R}^d \times (\mathbb{R} \setminus \{ 0 \}), \\ &= \frac{1}{\pi^{d/2}} \, \lim_{\varepsilon \downarrow 0} \int \varphi(2\varepsilon^{1/2} y, \varepsilon) \; e^{-|y|^2} \; dy \\ &\qquad \text{changing variable, } x = 2\varepsilon^{1/2} y, \end{aligned}$$

$$= \frac{1}{\pi^{d/2}} \int \varphi(0,0)\, e^{-|y|^2}\, dy$$
$$= \varphi(0)\,,$$

so $P(D)E = \delta$, as claimed.

Of course, one might enquire as to the existence of fundamental solutions. We state the following theorem, without proof.

Theorem 8.5 (Malgrange-Ehrenpreis). *For every constant coefficient partial differential operator $P(D)$ on $\mathbb{R}^d$, there is a distribution $E \in \mathscr{D}'(\mathbb{R}^d)$ such that $P(D)E = \delta$.*

Suppose now that we can show that a particular partial differential equation has a weak solution. Is it possible to show that under certain circumstances this solution is actually a solution in the classical sense? A result in this vein is the following.

Theorem 8.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set and suppose that u and f are continuous on Ω and that $D_j u = f$ as distributions. Then $D_j u = f$ in the classical sense, that is, $D_j u$ exists (as a function) and is equal to f on Ω.*

Proof. Let W be any open ball in Ω and let $\chi \in C_0^\infty(\Omega)$ be such that $\chi = 1$ on W. Then χu is continuous and has compact support. Also, for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}(D_j(\chi u))(\varphi) &= -(\chi u)(D_j \varphi) \\ &= -u(\chi D_j \varphi) \\ &= -u(D_j(\chi\varphi)) + u(D_j\chi\, \varphi) \\ &= (D_j u)(\chi\varphi) + u(D_j\chi\,\varphi) \\ &= (\chi D_j u)(\varphi) + ((D_j\chi)u)(\varphi)\,,\end{aligned}$$

that is, $D_j(\chi u) = (D_j\chi)u + \chi(D_j u)$, as distributions.

Let $v = \chi u$ and $g = D_j v = (D_j\chi)u + \chi(D_j u)$. Then $g = (D_j\chi)u + \chi f$ since $D_j u = f$. So g is continuous, as is v and both v and g have compact support. We have shifted the problem to the case of compact support.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\varphi \geq 0$ and $\int \varphi(y)\, dy = 1$. For $\varepsilon > 0$ let

$$\begin{aligned}v_\varepsilon(x) &= \int v(x - \varepsilon y)\, \varphi(y)\, dy \\ &= \tfrac{1}{\varepsilon^d} \int v(y)\, \varphi\big(\tfrac{x-y}{\varepsilon}\big)\, dy\,.\end{aligned}$$

Then $v_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $v_\varepsilon \to v$ uniformly as $\varepsilon \downarrow 0$. Also,

$$D_j v_\varepsilon(x) = \tfrac{1}{\varepsilon^d} \int v(y)\, D_{x_j}\varphi\big(\tfrac{x-y}{\varepsilon}\big)\, dy$$

$$
\begin{aligned}
&= -\tfrac{1}{\varepsilon^d} \int v(y)\, D_{y_j}\varphi(\tfrac{x-y}{\varepsilon})\, dy \\
&= \tfrac{1}{\varepsilon^d} \int (D_j v)(y)\, \varphi(\tfrac{x-y}{\varepsilon})\, dy \\
&\qquad\qquad \text{integrating by parts,} \\
&= \tfrac{1}{\varepsilon^d} \int g(y)\, \varphi(\tfrac{x-y}{\varepsilon})\, dy \\
&\equiv g_\varepsilon\,.
\end{aligned}
$$

It follows that $g_\varepsilon = D_j v_\varepsilon \to g$ uniformly as $\varepsilon \downarrow 0$. Let $e_1, \dots, e_d$ denote the usual orthonormal basis vectors for $\mathbb{R}^d$ (so that $e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$, with the 1 in the j^{th} coordinate position). Then

$$
\begin{aligned}
v_\varepsilon(x + \lambda e_j) - v_\varepsilon(x) &= \int_{x_j}^{x_j+\lambda} D_j v_\varepsilon(y)\, dy_j \\
&= \int_{x_j}^{x_j+\lambda} g_\varepsilon(y)\, dy_j\,.
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$, we deduce that

$$
v(x + \lambda e_j) - v(x) = \int_{x_j}^{x_j+\lambda} g(y)\, dy_j
$$

and hence $D_j v$ exists and $D_j v(x) = g(x)$. But for any $x \in W$, we have $v(x) = \chi(x)u(x) = u(x)$ and $D_j v(x) = D_j u(x)$ and $g(x) = f(x)$. It follows that $D_j u$ exists on W and $D_j u = f$ on W. Since W is arbitrary, the result follows. ∎

Definition 8.7. Let $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ be a linear differential operator of order m, with constant coefficients, defined on $\mathbb{R}^d$. Then the polynomial $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$ is called the symbol of P. The sum of those terms of order m in $P(\xi)$ is called the principal symbol of P, denoted σ_P, that is,

$$
\sigma_P(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha\,.
$$

Note that σ_P is homogeneous of degree m. The differential operator $P(D)$ is said to be elliptic if $\sigma_P(\xi) \ne 0$ for all $0 \ne \xi \in \mathbb{R}^d$.

Examples 8.8.

1. Δ is elliptic on $\mathbb{R}^d$.
2. $\partial_1 + i\partial_2$ is elliptic on $\mathbb{R}^2$.

Theorem 8.9 (Elliptic Regularity). *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $P(D)$ be an elliptic differential operator (constant coefficients). If $v \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$ is a weak solution to $P(D)u = v$, then $u \in C^\infty(\Omega)$.*

In particular, every weak solution to the homogeneous partial differential equation $P(D)u = 0$ belongs to $C^\infty(\Omega)$.

We will not prove this here.

Example 8.10. Suppose that $P(D)$ is elliptic (constant coefficients) and that E is a fundamental solution: $P(D) = \delta$. Then $P(D)E = 0$ on $\mathbb{R}^d \setminus \{0\}$, that is $(P(D)E)(\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. So $E \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$.

Example 8.11. Let Ω be an open set in $\mathbb{R}^2$ and suppose that $u \in \mathscr{D}'(\Omega)$ satisfies $(\partial_1 + i\partial_2)u = 0$ on Ω. Since $\partial_1 + i\partial_2$ is elliptic, the theorem tells us that $u \in C_0^\infty(\Omega)$. But then $(\partial_1 + i\partial_2)u = 0$ is just the Cauchy-Riemann equations and so we conclude that u is analytic in Ω. (Note that $\partial_1 u$ and $\partial_2 u$ are continuous because $u \in C^\infty$.) In other words, an analytic distribution is an analytic function.

Definition 8.12. For $s \in \mathbb{R}$, the Sobolev space $\mathscr{H}_s(\mathbb{R}^d)$ is defined to be the set of tempered distributions $T \in \mathscr{S}'(\mathbb{R}^d)$ such that $\widehat{T}$ is a function with the property that $\int |\widehat{T}(\lambda)|^2 (1 + |\lambda|^2)^s \, d\lambda < \infty$. Evidently, $\mathscr{H}_s \supseteq \mathscr{H}_t$ if $s \le t$.

$\mathscr{H}_s$ is a Hilbert space with respect to the inner product

$$(T_1, T_2) = \int \widehat{T_1}(\lambda)\, \overline{\widehat{T_2}(\lambda)}\, (1 + |\lambda|^2)^s \, d\lambda \,.$$

Definition 8.13. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. The local Sobolev space $\mathscr{H}_s(\Omega)$, for $s \in \mathbb{R}$, is the set of distributions $u \in \mathscr{D}'(\Omega)$ such that $\varphi u \in \mathscr{H}_s(\mathbb{R}^d)$ for all $\varphi \in C_0^\infty(\Omega)$.

The Sobolev spaces are used in the proof of the elliptic regularity theorem. In fact, one can prove the following stronger version.

Theorem 8.14 (Elliptic Regularity). *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $P(D)$ be an elliptic operator of order N. Suppose that $P(D)u = v$ where $v \in \mathscr{H}_s(\Omega)$ for some $s \in \mathbb{R}$. Then $u \in \mathscr{H}_{s+N}(\Omega)$.*

The theorem says that u is "better behaved" than v (by order N).

If $v \in C^\infty(\Omega)$, then $\varphi v \in C_0^\infty(\Omega)$ for all $\varphi \in C_0^\infty(\Omega)$. Hence $\varphi v \in \mathscr{S}(\mathbb{R}^d)$ and so $(\varphi v)\widehat{\ } \in \mathscr{S}(\mathbb{R}^d)$ and therefore $\varphi v \in \mathscr{H}_s(\Omega)$ for all s. Hence $u \in \mathscr{H}_s(\Omega)$ for all s. One then shows that this implies that $u \in C^\infty(\Omega)$ (Sobolev's Lemma).