16 Applications of the Mean Value Theorem

For the following sections, we require the standard idea of higher order derivatives. If $n \in \mathbb{N}$, then the n'th order derivative of f at x_0 is written $f^{(n)}(x_0)$. We also use the convention that $f^{(0)} = f$.

16.1 Taylor's Theorem

The motivation behind Taylor's theorem is the attempt to approximate a function f near a number a by a polynomial. The polynomial of degree 0 which does the best job is clearly $p_0(x) = f(a)$. The best polynomial of degree 1 is the tangent line to the graph of the function $p_1(x) = f(a) + f'(a)(x-a)$. Continuing in this way, we approximate f near a by the polynomial p_n of degree n such that $f^{(k)}(a) = p_n^{(k)}(a)$ for $k = 0, 1, \ldots, n$. A simple induction argument shows that

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{9}$$

This is the well-known Taylor polynomial of f at a.

The fact which makes the Taylor polynomial important is that in many cases it is possible to determine how large n must be to achieve a desired accuracy in the approximation of f by p_n. This is accomplished by using Taylor's Theorem, which is also known as the Extended Mean Value Theorem.

Theorem 16.1 (Taylor's Theorem). *If f is a function such that $f, f', \ldots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b), then there is a $c \in (a, b)$ such that*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Proof. Let the constant α be defined by

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{\alpha}{(n+1)!}(b-a)^{n+1} \tag{10}$$

and define

$$F(x) = f(b) - \left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{\alpha}{(n+1)!}(b-x)^{n+1} \right).$$

From (10) we see that $F(a) = 0$. Direct substitution in the definition of F shows that $F(b) = 0$. From the assumptions in the statement of the theorem, it is easy to see that F is continuous on $[a, b]$ and differentiable on (a, b). An application of Rolle's Theorem yields a $c \in (a, b)$ such that

$$0 = F'(c) = -\left(\frac{f^{(n+1)}(c)}{n!}(b-c)^n - \frac{\alpha}{n!}(b-c)^n \right) \implies \alpha = f^{(n+1)}(c),$$

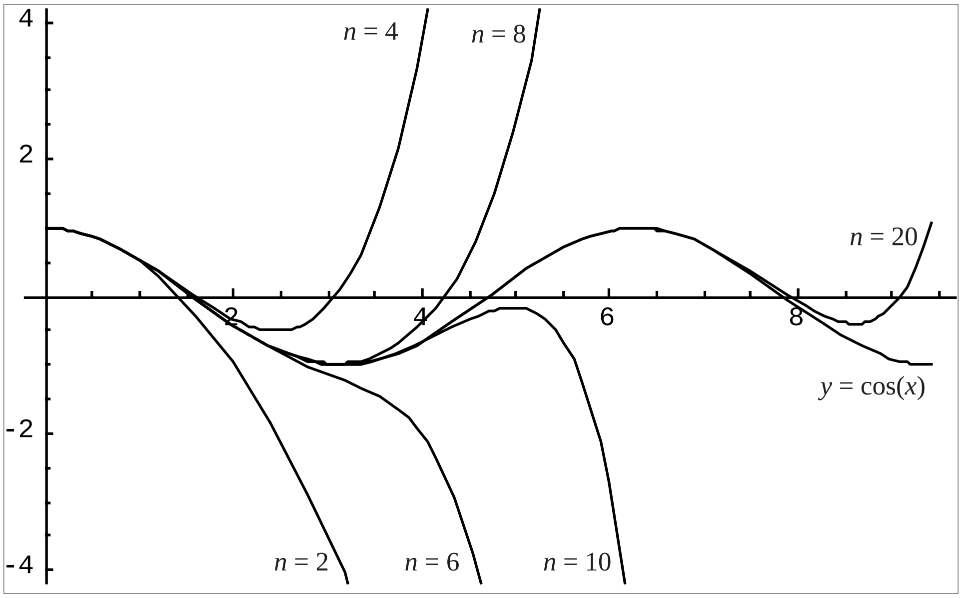

Figure 11: Here are several of the Taylor polynomials for the function $f(x) = \cos(x)$ graphed along with f.

as desired. □

Now, suppose f is defined on an open interval I with $a, x \in I$. If f is $n+1$ times differentiable on I, then Theorem 16.1 implies there is a c between a and x such that

$$f(x) = p_n(x) + R_f(n, x, a),$$

where $R_n(c, x, a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ is the error in the approximation.

Example 16.1. Let $f(x) = \cos(x)$. Suppose we want to approximate $f(2)$ to 5 decimal places of accuracy. Since it's an easy point to work with, we'll choose $a = 0$. Then, for some $c \in (0, 2)$,

$$|R_f(n, 2, 0)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} 2^{n+1} \leq \frac{2^{n+1}}{(n+1)!}. \tag{11}$$

A bit of experimentation with a calculator shows that $n = 12$ is the smallest n such that the right-hand side of (11) is less than 5×10^{-6}. After doing some arithmetic, it follows that

$$p_{12}(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \frac{2^{10}}{10!} + \frac{2^{12}}{12!} = -\frac{27809}{66825} \approx -0.41614.$$

is a 5 decimal place approximation to $\cos(2)$.

16.2 L'Hôspital's Rules and Indeterminate Forms

According to Theorem 8.3,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$$

whenever $\lim_{x\to a}f(x)$ and $\lim_{x\to a}g(x)$ both exist and $\lim_{x\to a}g(x)\neq 0$. But, it is easy to find examples where both $\lim_{x\to a}f(x)=0$ and $\lim_{x\to a}g(x)=0$ and $\lim_{x\to a}f(x)/g(x)$ exists, as well as similar examples where $\lim_{x\to a}f(x)/g(x)$ fails to exist. Because of this, such a limit problem is said to be in the *indeterminate form* 0/0. The following theorem allows us to determine many such limits.

Theorem 16.2 (Easy L'Hôspital's Rule). *Suppose f and g are each continuous on $[a,b]$, differentiable on (a,b) and $f(b)=g(b)=0$. If $g'(x)\neq 0$ on (a,b) and $\lim_{x\uparrow b}f'(x)/g'(x)=L$, where L could be infinite, then $\lim_{x\uparrow b}f(x)/g(x)=L$.*

Proof. Let $x\in[a,b)$, so f and g are continuous on $[x,b]$ and differentiable on (x,b). Cauchy's Mean Value Theorem, Theorem 15.2, implies there is a $c(x)\in(x,b)$ such

$$f'(c(x))g(x)=g'(c(x))f(x)\implies\frac{f(x)}{g(x)}=\frac{f'(c(x))}{g'(c(x))}.$$

Since $x<c(x)<b$, it follows that $\lim_{x\uparrow b}c(x)=c$. This shows that

$$L=\lim_{x\uparrow b}\frac{f'(x)}{g'(x)}=\lim_{x\uparrow b}\frac{f'(c(x))}{g'(c(x))}=\lim_{x\uparrow b}\frac{f(x)}{g(x)}.$$

□

Several things should be noted about this proof. First, there is nothing special about the left-hand limit used in the statement of the theorem. It could just as easily be written in terms of the right-hand limit. Second, if $\lim_{x\to a}f(x)/g(x)$ is not of the indeterminate form 0/0, then applying L'Hôspital's rule will give a wrong answer. To see this, consider

$$\lim_{x\to 0}\frac{x}{x+1}=0\neq 1=\lim_{x\to 0}\frac{1}{1}.$$

Corollary 16.3. *Suppose f and g are differentiable on (a,∞) and $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=0$. If $g'(x)\neq 0$ on (a,∞) and $\lim_{x\to\infty}f'(x)/g'(x)=L$, where L could be infinite, then $\lim_{x\to\infty}f(x)/g(x)=L$.*

Proof. There is no generality lost by assuming $a>0$. Let

$$F(x)=\begin{cases}f(1/x), & x\in[a,\infty)\\ 0, & x=0\end{cases}\quad\text{and}\quad G(x)=\begin{cases}g(1/x), & x\in[a,\infty)\\ 0, & x=0\end{cases}.$$

Then

$$\lim_{x\downarrow 0} F(x) = \lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) \lim_{x\downarrow 0} G(x),$$

so both F and G are continuous at 0. It follows that both F and G are continuous on $[0, 1/a]$ and differentiable on $(0, 1/a)$ with $G'(x) = -g'(x)/x^2 \neq 0$ on $(0, 1/a)$ and $\lim_{x\downarrow 0} F'(x)/G'(x) = \lim_{x\to\infty} f'(x)/g'(x) = L$. The rest follows from Theorem 16.2. □

The other standard indeterminate form is when $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$. This is called an ∞/∞ indeterminate form. This is handled by the following theorem.

Theorem 16.4 (Hard L'Hôspital's Rule). *Suppose that f and g are differentiable on (a, ∞) and $g'(x) \neq 0$ on (a, ∞). If*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \textit{and} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = L \in \mathbb{R} \cup \{-\infty, \infty\},$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Proof. First, suppose $L \in \mathbb{R}$ and let $\varepsilon > 0$. Choose $a_1 > a$ large enough so that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon, \ \forall x > a_1. \tag{12}$$

Since $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$, we can assume there is an $a_2 > a_1$ such that both $f(x) > 0$ and $g(x) > 0$ when $x > a_2$. Finally, choose $a_3 > a_2$ such that whenever $x > a_3$, then $f(x) > f(a_2)$ and $g(x) > g(a_2)$.

Let $x > a_3$ and apply Cauchy's Mean Value Theorem, Theorem 15.2, to f and g on $[a_2, x]$ to find a $c(x) \in (a_2, x)$ such that

$$\frac{f'(c(x))}{g'(c(x))} = \frac{f(x) - f(a_2)}{g(x) - g(a_2)} = \frac{f(x)\left(1 - \frac{f(a_2)}{f(x)}\right)}{g(x)\left(1 - \frac{g(a_2)}{g(x)}\right)}. \tag{13}$$

If

$$h(x) = \frac{1 - \frac{g(a_2)}{g(x)}}{1 - \frac{f(a_2)}{f(x)}},$$

then (13) implies

$$\frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))} h(x).$$

Since $\lim_{x\to\infty} h(x) = 1$, there is an $a_4 > a_3$ such that whenever $x > a_4$, then $|h(x) - 1| < \varepsilon$. If $x > a_4$, then

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &= \left|\frac{f'(c(x))}{g'(c(x))}h(x) - L\right| \\
&= \left|\frac{f'(c(x))}{g'(c(x))}h(x) - Lh(x) + Lh(x) - L\right| \\
&\leq \left|\frac{f'(c(x))}{g'(c(x))} - L\right| |h(x)| + |L||h(x) - 1| \\
&< \varepsilon(1+\varepsilon) + |L|\varepsilon = (1 + |L| + \varepsilon)\varepsilon.
\end{aligned}$$

Therefore $\lim_{x\to\infty} f(x)/g(x) = L$.

The case when $L = \infty$ is done similarly by first choosing a $B > 0$ and adjusting (13) so that $f'(x)/g'(x) > B$ when $x > a_1$. A similar adjustment is necessary when $L = -\infty$. □

There is a companion corollary to Theorem 16.4 which is proved in the same way as Corollary 16.3.

Corollary 16.5. *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0$ on (a, b). If*

$$\lim_{x\downarrow a} f(x) = \lim_{x\downarrow a} g(x) = \infty \quad \text{and} \quad \lim_{x\downarrow a} \frac{f(x)}{g(x)} = L \in \mathbb{R} \cup \{-\infty, \infty\},$$

then

$$\lim_{x\downarrow a} \frac{f(x)}{g(x)} = L.$$