14 Differentiation

Definition 14.1. Let f be a function on a neighborhood of x_0. f is differentiable at x_0 with value $f'(x)$ if

$$f'(x) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Define $D(f) = \{x : f'(x) \text{ exists}\}$.

The standard notations for the derivative will be used; e. g., $f'(x)$, $\frac{df(x)}{dx}$, $Df(x)$, etc.

Another way of stating this definition is to note that if $x_0 \in D(f)$, then

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

This can be interpreted in the standard way as the limiting slope of the secant line as the points of intersection approach each other.

Example 14.1. If $f(x) = c$ for some $c \in \mathbb{R}$, then

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

So, $f'(x) = 0$ everywhere.

Example 14.2. If $f(x) = x$, then

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{x_0+h-x_0}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

So, $f'(x) = 1$ everywhere.

Theorem 14.1. *For any function f, $D(f) \subset C(f)$.*

Proof. Suppose $x_0 \in D(f)$. Then

$$\begin{aligned} \lim_{x\to x_0} |f(x) - f(x_0)| &= \lim_{x\to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right| \\ &= f'(x_0)\ 0 = 0. \end{aligned}$$

This shows $\lim_{x\to x_0} f(x) = f(x_0)$, and $x_0 \in C(f)$. □

Example 14.3. The function $f(x) = |x|$ is continuous on $\mathbb{R}$, but

$$\lim_{h\downarrow 0} \frac{f(0+h) - f(0)}{h} = 1 = -\lim_{h\uparrow 0} \frac{f(0+h) - f(0)}{h},$$

so $f'(0)$ fails to exist.

Theorem 14.1 and Example 14.3 show that differentiability is a strictly stronger condition than continuity. For a long time most mathematicians thought that every continuous function must certainly be differentiable at some point. In 1887, Weierstrass constructed a function continuous on $\mathbb{R}$ which is differentiable nowhere. It has since been proved that the "typical" continuous function is nowhere differentiable.

Theorem 14.2. *Suppose f and g are functions such that $x_0 \in D(f) \cap D(g)$.*

(a) $x_0 \in D(f+g)$ and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

(b) If $a \in \mathbb{R}$, then $x_0 \in D(af)$ and $(af)'(x_0) = af'(x_0)$.

(c) $x_0 \in D(fg)$ and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(d) If $g(x_0) \neq 0$, then $x_0 \in D(f/g)$ and

$$\left(\frac{f}{g}\right)(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Proof. (a)

$$\lim_{h\to 0} \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h}$$
$$= \lim_{h\to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h}$$
$$= \lim_{h\to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h} \right) = f'(x_0) + g'(x_0)$$

(b)

$$\lim_{h\to 0} \frac{(af)(x_0+h) - (af)(x_0)}{h} = a \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = af'(x_0)$$

(c)

$$\lim_{h\to 0} \frac{(fg)(x_0+h) - (fg)(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}$$

Now, "slip a 0" into the numerator and factor the fraction.

$$= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h}$$
$$= \lim_{h\to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) + f(x_0) \frac{g(x_0+h) - g(x_0)}{h} \right)$$

Finally, use the definition of the derivative and the continuity of f and g at x_0.

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

(d) It will be proved that if $g(x_0) \neq 0$, then $(1/g)'(x_0) = -g'(x_0)/(g(x_0))^2$. This statement, combined with (c), yields (d).

$$\begin{aligned}
\lim_{h\to 0} \frac{(1/g)(x_0+h)-(1/g)(x_0)}{h} &= \lim_{h\to 0} \frac{\dfrac{1}{g(x_0+h)} - \dfrac{1}{g(x_0)}}{h} \\
&= \lim_{h\to 0} \frac{g(x_0)-g(x_0+h)}{h} \frac{1}{g(x_0+h)g(x_0)} \\
&= -\frac{g'(x_0)}{(g(x_0)^2}
\end{aligned}$$

Plug this into (c) to see

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \left(f\frac{1}{g}\right)'(x_0) \\
&= f'(x_0)\frac{1}{g(x_0)} + f(x_0)\frac{-g'(x_0)}{(g(x_0))^2} \\
&= \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}.
\end{aligned}$$

□

Combining Examples 14.1 and 14.2 with Theorem 14.2, the following theorem is immediate.

Theorem 14.3. *A rational function is differentiable at every point of its domain.*

Theorem 14.4 (Chain Rule). *If f and g are functions such that $x_0 \in D(f)$ and $f(x_0) \in D(g)$, then $x_0 \in D(g \circ f)$ and $(g \circ f)'(x_0) = g' \circ f(x_0) f'(x_0)$.*

Proof. Let $y_0 = f(x_0)$. By assumption, there is an open interval J containing $f(x_0)$ such that g is defined on J. Since J is open and $x_0 \in C(f)$, there is an open interval I containing x_0 such that $f(I) \subset J$.

Define $h : J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \dfrac{g(y)-g(y_0)}{y-y_0} - g'(y_0), & y \neq y_0 \\ 0, & y = y_0 \end{cases}.$$

Since $y_0 \in D(f)$, we see

$$\lim_{y\to y_0} h(y) = \lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0} - g'(y_0) = g'(y_0) - g'(y_0) = 0 = h(0),$$

so $y_0 \in C(h)$. Now, $x_0 \in C(f)$ and $f(x_0) = y_0 \in C(h)$, so Theorem 10.6 implies $x_0 \in C(h \circ f)$. In particular

$$\lim_{x\to x_0} h \circ f(x) = 0. \tag{5}$$

From the definition of $h \circ f$ for $x \in I$ with $f(x) \neq f(x_0)$, we can solve for

$$g \circ f(x) - g \circ f(x_0) = (h \circ f(x) + g' \circ f(x_0))(f(x) - f(x_0)). \qquad (6)$$

Notice that (6) is also true when $f(x) = f(x_0)$. Divide both sides of (6) by $x - x_0$, and use (5) to obtain

$$\begin{aligned}\lim_{x \to x_0} \frac{g \circ f(x) - g \circ f(x_0)}{x - x_0} &= \lim_{x \to x_0} (h \circ f(x) + g' \circ f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} \\ &= (0 + g' \circ f(x_0)) f'(x_0) \\ &= g' \circ f(x_0) f'(x_0).\end{aligned}$$

□

Theorem 14.5. *Suppose* $f : [a, b] \to \mathbb{R}$ *is continuous and invertible. If* $x_0 \in D(f)$ *and* $f'(x_0) \neq 0$ *for some* $x_0 \in (a, b)$*, then* $f(x_0) \in D(f^{-1})$ *and* $\left(f^{-1}\right)'(f(x_0)) = 1/f'(x_0)$.

Proof. Let $y_0 = f(x_0)$ and suppose y_n is any sequence in $f([a, b]) \setminus \{y_0\}$ converging to y_0 and $x_n = f^{-1}(y_n)$. By Theorem 12.5, f^{-1} is continuous, so

$$x_0 = f^{-1}(y_0) = \lim_{n \to \infty} f^{-1}(y_n) = \lim_{n \to \infty} x_n.$$

Therefore,

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y} = \lim_{n \to \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

□

Example 14.4. It follows easily from Theorem 14.2 that $f(x) = x^3$ is differentiable everywhere with $f'(x) = 3x^2$. Define $g(x) = \sqrt[3]{x}$. Then $g(x) = f^{-1}(x)$. Suppose $g(y_0) = x_0$ for some $y_0 \in \mathbb{R}$. According to Theorem 14.5,

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{3x_0^2} = \frac{1}{3(g(y_0))^2} = \frac{1}{3(\sqrt[3]{y_0})^2} = \frac{1}{3y_0^{2/3}}.$$

In the same manner as Example 14.4, the following corollary can be proved.

Corollary 14.6. *Suppose* $q \in \mathbb{Q}$, $f(x) = x^q$ *and* D *is the domain of* f*. Then* $f'(x) = qx^{q-1}$ *on the set*

$$\begin{cases} D, & \text{when } q \geq 1 \\ D \setminus \{0\}, & \text{when } q < 1 \end{cases}.$$

As is learned in calculus, the derivative is a powerful tool for determining the behavior of functions. The following theorems form the basis for much of differential calculus. First, we state a few familiar definitions.

Definition 14.2. Suppose $f : D \to \mathbb{R}$ and $x_0 \in D$. f is said to have a *relative maximum at* x_0 if there is a $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$. f has a *relative minimum at* x_0 if $-f$ has a relative maximum at x_0. If f has either a relative maximum or a relative minimum at x_0, then it is said that f *has a relative extreme value at* x_0.

The *absolute maximum* of f occurs at x_0 if $f(x_0) \geq f(x)$ for all $x \in D$. The definitions of *absolute minimum* and *absolute extreme* are analogous.

Examples like $f(x) = x$ on $(0, 1)$ show that even the nicest functions need not have relative extrema. Corollary 12.4 shows that if D is compact, then any continuous function defined on D assumes both an absolute maximum and an absolute minimum on D.

Theorem 14.7. *Suppose $f : (a, b) \to \mathbb{R}$ is differentiable. If f has a relative extreme value at x_0, then $f'(x_0) = 0$.*

Proof. Suppose $f(x_0)$ is a relative maximum value of f. Then there must be a $\delta > 0$ such that $f(x) \leq f(x_0)$ whenever $x \in (x_0 - \delta, x_0 + \delta)$. Since $f'(x_0)$ exists,

$$x \in (x_0 - \delta, x_0) \implies \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \implies f'(x_0) = \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \tag{7}$$

and

$$x \in (x_0, x_0 + \delta) \implies \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \implies f'(x_0) = \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0. \tag{8}$$

Combining (7) and (8) shows $f'(x_0) = 0$.

If $f(x_0)$ is a relative minimum value of f, apply the previous argument to $-f$. □

Theorem 14.7 is, of course, the basis for much of a beginning calculus course. If $f : [a, b] \to \mathbb{R}$, then the extreme values of f occur at points of the set

$$C = \{x \in (a, b) : f'(x) = 0\} \cup \{x \in [a, b] : f'(x) \text{ does not exist}\}.$$

The elements of C are often called the *critical points* of f on $[a, b]$. To find the maximum and minimum values of f on $[a, b]$, it suffices to find its maximum and minimum on the smaller set C.

Problem 26. If f is defined on an open set containing x_0, the *symmetric derivative* of f at x_0 is defined as

$$f^s(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

Prove that if $f'(x)$ exists, then so does $f^s(x)$. Is the converse true?