12 Continuous Functions

Definition 12.1. Let $f : D \to \mathbb{R}$ and $A \subset D$. We say f *is continuous on* A if $A \subset C(f)$. If $D = C(f)$, then f is *continuous.*

Theorem 12.1. *$f : D \to \mathbb{R}$ is continuous iff whenever G is open in $\mathbb{R}$, then $f^{-1}(G)$ is relatively open in D.*

Proof. ($\Rightarrow$) Assume f is continuous on D and let G be open in $\mathbb{R}$. Let $x \in f^{-1}(G)$ and choose $\varepsilon > 0$ such that $(f(x)-\varepsilon, f(x)+\varepsilon) \subset G$. Using the continuity of f at x, we can find a $\delta > 0$ such that $f((x-\delta, x+\delta) \cap D) \subset G$. This implies at once that $(x-\delta, x+\delta) \cap D \subset f^{-1}(G)$. Because x was an arbitrary element of $f^{-1}(G)$, it follows that $f^{-1}(G)$ is open.

($\Leftarrow$) Choose $x \in D$ and let $\varepsilon > 0$. By assumption, the set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)$ is relatively open in D. This implies the existence of a $\delta > 0$ such that $(x-\delta, x+\delta) \cap D \subset f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)$. It follows at once from this that $f((x-\delta, x+\delta) \cap D) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, and $x \in C(f)$. ☐

Theorem 12.2. *If f is continuous on a compact set K, then $f(K)$ is compact.*

Proof. Let $\mathcal{O}$ be an open cover of $f(K)$ and $\mathcal{I} = \{f^{-1}(G) : G \in \mathcal{O}\}$. By Theorem 12.1, $\mathcal{I}$ is a collection of sets which are relatively open in K. Since $\mathcal{I}$ covers $f(K)$, its easy to see, $\mathcal{I}$ is an open cover of K. Using the fact that K is compact, we can choose a finite subcover of K from $\mathcal{I}$, say $\{G_1, G_2, \ldots, G_n\}$. There are $\{H_1, H_2, \ldots, H_n\} \subset \mathcal{O}$ such that $f^{-1}(H_k) = G_k$ for $1 \le k \le n$. Then

$$f(K) \subset f\left(\bigcup_{1\le k\le n} G_k\right) = \bigcup_{1\le k\le n} H_k.$$

Thus, $\{H_1, H_2, \ldots, H_3\}$ is a subcover of $f(K)$ from $\mathcal{O}$. ☐

Corollary 12.3. *If $f : K \to \mathbb{R}$ is continuous and K is compact, then f is bounded.*

Proof. By Theorem 12.2, $f(K)$ is compact. Now, use the Bolzano-Weierstrass theorem to conclude f is bounded. ☐

Corollary 12.4. *If $f : K \to \mathbb{R}$ is continuous and K is compact, then there are $m, M \in K$ such that $f(m) \le f(x) \le f(M)$ for all $x \in K$.*

Proof. According to Theorem 12.2 and the Bolzano-Weierstrass theorem, $f(K)$ is closed and bounded. Because of this, $\operatorname{glb} f(K) \in f(K)$ and $\operatorname{lub} f(K) \in f(K)$. It suffices to choose $m \in f^{-1}(\operatorname{glb} f(K))$ and $M \in f^{-1}(\operatorname{lub} f(K))$. ☐

Theorem 12.5. *If $f : K \to \mathbb{R}$ is continuous and invertible and K is compact, then $f^{-1} : f(K) \to K$ is continuous.*

Proof. Let G be open in K. According to Theorem 12.1, it suffices to show $f(G)$ is open in $f(K)$.

To do this, note that $K \setminus G$ is compact, so by Theorem 12.2, $f(K \setminus G)$ is compact, and therefore closed. Because f is injective, $f(G) = f(K) \setminus f(K \setminus G)$. This shows $f(G)$ is open in $f(K)$. □

Theorem 12.6. *If f is continuous on a connected set K, then $f(K)$ is connected.*

Proof. If $f(K)$ is not connected, there must exist two disjoint open sets, U and V, such that $f(K) \subset U \cup V$ and $f(K) \cap U \neq \emptyset \neq f(K) \cap V$. In this case, Theorem 12.1 implies $f^{-1}(U)$ and $f^{-1}(V)$ are both open. They are clearly disjoint and $f^{-1}(U) \cap K \neq \emptyset \neq f^{-1}(V) \cap K$. But, this implies $f^{-1}(U)$ and $f^{-1}(V)$ disconnect K, which is a contradiction. Therefore, $f(K)$ is connected. □

Corollary 12.7. *If $f : [a, b] \to \mathbb{R}$ is continuous and α is between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = \alpha$.*

Proof. This is an easy consequence of Theorem 12.6 and Theorem 7.1. □

Definition 12.2. A function $f : D \to \mathbb{R}$ has the *Darboux property* if whenever $a, b \in D$ and γ is between $f(a)$ and $f(b)$, then there is a c between a and b such that $f(c) = \gamma$.

The Darboux property is also often called the *intermediate value property.* Corollary 12.7 shows that a function continuous on an interval has the Darboux property. The next example shows continuity is not necessary for the Darboux property to hold.

Example 12.1. The function

$$f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not continuous, but does have the Darboux property. (See Figure 5.) It can be seen from Example 8.5 that $0 \notin C(f)$.

To see f has the Darboux property, choose two numbers $a < b$.

If $a > 0$ or $b < 0$, then f is continuous on $[a, b]$ and Corollary 12.7 suffices to finish the proof.

On the other hand, if $0 \in [a, b]$, then there must exist an $n \in \mathbb{Z}$ such that both $\frac{4}{(4n+1)\pi}, \frac{4}{(4n+3)\pi} \in [a, b]$. Since $f(\frac{4}{(4n+1)\pi}) = 1$, $f(\frac{4}{(4n+3)\pi}) = -1$ and f is continuous on the interval between them, we see $f([a, b]) = [-1, 1]$, which is the entire range of f. The claim now follows easily.

Problem 22. Let f and g be two functions which are continuous on a set $D \subset \mathbb{R}$. Prove or give a counter example: $\{x \in D : f(x) > g(x)\}$ is open.

Problem 23. If $f : [a, b] \to \mathbb{R}$ is continuous, not constant,

$$m = \operatorname{glb}\{f(x) : a \leq x \leq b\} \text{ and } M = \operatorname{lub}\{f(x) : a \leq x \leq b\},$$

then $f([a, b]) = [m, M]$.

Extra Credit 8. If $F \subset \mathbb{R}$ is closed, then there is an $f : \mathbb{R} \to \mathbb{R}$ such that $F = C(f)^c$.