7 Connectedness

Definition 7.1. A set $S \subset \mathbb{R}$ is *disconnected* if there are two open sets U and V such that $U \cap V = \emptyset$, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$ and $S \subset U \cup V$. Otherwise, it is *connected*. Such sets U and V are said to be a disconnection of S.

Example 7.1. Let $S = \{x\}$ be a set containing a single point. S is connected because there cannot exist nonempty disjoint open sets U and V such that $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. The same argument shows that $\emptyset$ is connected.

Example 7.2. If $S = [-1, 0) \cup (0, 1]$, then $U = (-2, 0)$ and $V = (0, 2)$ are open sets such that $U \cap V = \emptyset$, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$ and $S \subset U \cup V$. This shows S is disconnected.

Example 7.3. The sets $U = (-\infty, \sqrt{2})$ and $V = (\sqrt{2}, \infty)$ are open sets such that $U \cap V = \emptyset$, $U \cap \mathbb{Q} \neq \emptyset$, $V \cap \mathbb{Q} \neq \emptyset$ and $\mathbb{Q} \subset U \cup V = \mathbb{R} \setminus \{\sqrt{2}\}$. This shows $\mathbb{Q}$ is disconnected.

Theorem 7.1. *A nonempty set $S \subset \mathbb{R}$ is connected iff it is either a single point or an interval.*

Proof. ($\Rightarrow$) If S is not a single point or an interval, there must be numbers $r < s < t$ such that $r, t \in S$ and $s \notin S$. In this case, the sets $U = (-\infty, s)$ and $V = (s, \infty)$ are a disconnection of S.

($\Leftarrow$) It was shown in Example 7.1 that a set containing a single point is connected. So, assume S is an interval.

Suppose S is not connected with U and V forming a disconnection of S. Choose $u \in U \cap S$ and $v \in V \cap S$. There is no generality lost by assuming $u < v$, so that $[u, v] \subset S$.

Let $A = \{x : [u, x) \subset U\}$.

We claim $A \neq \emptyset$. To see this, use the fact that U is open to find $\varepsilon \in (0, v-u)$ such that $(u - \varepsilon, u + \varepsilon) \subset U$. Then $u < u + \varepsilon/2 < v$, so $u + \varepsilon/2 \in A$.

Define $w = \operatorname{lub} A$.

Since $v \in V$ it is evident $u < w \leq v$ and $w \in S$.

If $w \in U$, then $u < w < v$ and there is $\varepsilon \in (0, v-w)$ such that $(w-\varepsilon, w+\varepsilon) \subset U$ and $[u, w + \varepsilon) \subset S$ because $w + \varepsilon < v$. This clearly contradicts the definition of w, so $w \notin U$.

If $w \in V$, then there is an $\varepsilon > 0$ such that $(w - \varepsilon, w] \subset V$. In particular, this shows $w = \operatorname{lub} A \leq w - \varepsilon < w$. This contradiction forces the conclusion that $w \notin V$.

Now, putting all this together, we see $w \in S \subset U \cup V$ and $w \notin U \cup V$. This is a clear contradiction, so we're forced to conclude there is no separation of S. ☐

Problem 19. (a) Give an example of a set S such that S is disconnected, but $S \cup \{1\}$ is connected.

(b) Prove that 1 must be a limit point of S.