4 The Topology of $\mathbb{R}$

Definition 4.1. A set $G \subset \mathbb{R}$ is *open* if for every $x \in G$ there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset G$. A set $F \subset \mathbb{R}$ is *closed* if F^c is open.

Example 4.1. Any open interval (a, b) is open. To see this, let $x \in (a, b)$ and $\varepsilon = \min\{x - a, b - x\}$. Then $(x - \varepsilon, x + \varepsilon) \subset (a, b)$.

Open half-lines are also open sets. For example, let $x \in (a, \infty)$ and $\varepsilon = x - a$. Then $(x - \varepsilon, x + \varepsilon) \subset (a, \infty)$.

A singleton set $\{a\}$ is closed. To see this, suppose $x \neq a$ and $\varepsilon = |x - a|$. Then $a \notin (x - \varepsilon, x + \varepsilon)$, and $\{a\}^c$ must be open. The definition of a closed set then implies $\{a\}$ is closed.

There are sets which are neither open nor closed. For example, consider the half-open interval $[0, 1)$. To see it isn't open or closed, let $\varepsilon > 0$. Then $(0 - \varepsilon, 0 + \varepsilon) \not\subset [0, 1)$ shows it cannot be open. Since $(1 - \varepsilon, 1 + \varepsilon) \not\subset [0, 1)^c$, we see $[0, 1)^c$ is not open, so $[0, 1)$ cannot be closed.

Theorem 4.1. *(a) If* $\{G_\lambda : \lambda \in \Lambda\}$ *is a collection of open sets, then* $\bigcup_{\lambda \in \Lambda} G_\lambda$ *is open.*

(b) If $\{G_k : 1 \leq k \leq n\}$ *is a finite collection of open sets, then* $\bigcap_{k=1}^n G_k$ *is open.*

(c) Both $\emptyset$ *and* $\mathbb{R}$ *are open.*

Proof. (a) If $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$, then there is a $\lambda_x \in \Lambda$ such that $x \in G_{\lambda_x}$. Since G_{λ_x} is open, there is an $\varepsilon > 0$ such that $x \in (x - \varepsilon, x + \varepsilon) \subset G_{\lambda_x} \subset \bigcup_{\lambda \in \Lambda} G_\lambda$. This shows $\bigcup_{\lambda \in \Lambda} G_\lambda$ is open.

(b) If $x \in \bigcap_{k=1}^n G_k$, then $x \in G_k$ for $1 \leq k \leq n$. For each G_k there is an ε_k such that $(x - \varepsilon_k, x + \varepsilon_k) \subset G_k$. Let $\varepsilon = \min\{\varepsilon_k : 1 \leq k \leq n\}$. Then $(x - \varepsilon, x + \varepsilon) \subset G_k$ for $1 \leq k \leq n$, so $(x - \varepsilon, x + \varepsilon) \subset \bigcap_{k=1}^n G_k$. Therefore $\bigcap_{k=1}^n G_k$ is open.

(c) $\emptyset$ is open vacuously. $\mathbb{R}$ is obviously open.

□

Applying DeMorgan's laws to the parts of Theorem 4.1 immediately yields the following.

Corollary 4.2. *(a) If* $\{F_\lambda : \lambda \in \Lambda\}$ *is a collection of closed sets, then* $\bigcap_{\lambda \in \Lambda} G_\lambda$ *is closed.*

(b) If $\{F_k : 1 \leq k \leq n\}$ *is a finite collection of closed sets, then* $\bigcup_{k=1}^n G_k$ *is closed.*

(c) Both $\emptyset$ *and* $\mathbb{R}$ *are closed.*

Notice that $\emptyset$ and $\mathbb{R}$ are both open and closed. They are the only subsets of $\mathbb{R}$ with this dual personality.

Definition 4.2. x_0 is a limit point of $S \subset \mathbb{R}$ if for every $\varepsilon > 0$, $(x_0 - \varepsilon, x_0 + \varepsilon) \cap S \setminus \{x_0\} \neq \emptyset$. The *derived set* of S is

$$S' = \{x : x \text{ is a limit point of } S\}.$$

A point $x_0 \in S \setminus S'$ is an *isolated point* of S.

Notice that limit points of S need not be elements of S, but isolated points of S must be elements of S. In a sense, limit points and isolated points are at opposite extremes. The definitions can be restated as follows:

$$x_0 \text{ is a limit point of } S \text{ iff } \forall \varepsilon > 0, S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \neq \emptyset$$
$$x_0 \in S \text{ is an isolated point of } S \text{ iff } \exists \varepsilon > 0, S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} = \emptyset$$

Example 4.2. If $S = (0, 1]$, then $S' = [0, 1]$ and S has no isolated points.

Example 4.3. If $T = \{1/n : n \in \mathbb{Z} \setminus \{0\}\}$, then $T' = \{0\}$ and all points of T are isolated points of T.

Theorem 4.3. *x_0 is a limit point of S iff there is a sequence $x_n \in S \setminus \{x_0\}$ such that $x_n \to x_0$.*

Proof. ($\Rightarrow$) For each $n \in \mathbb{N}$ choose $x_n \in S \cap (x_0 - 1/n, x_0 + 1/n) \setminus \{x_0\}$. Then $|x_n - x_0| < 1/n$ for all $n \in \mathbb{N}$, so $x_n \to x_0$.

($\Leftarrow$) Suppose x_n is a sequence from $x_n \in S \setminus \{x_0\}$ converging to x_0. If $\varepsilon > 0$, the definition of convergence for a sequence yields an $N \in \mathbb{N}$ such that whenever $n \geq N$, then $x_n \in S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}$. This shows $S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \neq \emptyset$, and x_0 must be a limit point of S. □

Theorem 4.4. *A set $S \subset \mathbb{R}$ is closed iff it contains all its limit points.*

Proof. ($\Rightarrow$) Suppose S is closed and x_0 is a limit point of S. If $x_0 \notin S$, then S^c open implies the existence of $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap S = \emptyset$. This contradicts the fact that x_0 is a limit point of S. Therefore, $x_0 \in S$, and S contains all its limit points.

($\Leftarrow$) Since S contains all its limit points, if $x_0 \notin S$, there must exist an $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap S \neq \emptyset$. It follows from this that S^c is open. Therefore S is closed. □

Definition 4.3. The *closure* of a set S is the set $\overline{S} = S \cup S'$.

For the set S of Example 4.2, $\overline{S} = [0, 1]$. In Example 4.3, $\overline{T} = \{1/n : n \in \mathbb{Z} \setminus \{0\}\} \cup \{0\}$. According to Theorem 4.4, the closure of any set is a closed set.

Problem 16. If $S \subset \mathbb{R}$, then $\overline{S}$ is the smallest closed set containing S. (In this case "smallest" means that if T is any closed set with $S \subset T$, then $\overline{S} \subset T$.)

Theorem 4.5 (Bolzano-Weierstrass Theorem). *A set which is both bounded and infinite has a limit point.*

Proof. For the purposes of this proof, if $I = [a, b]$ is a closed interval, let $I^L = [a, (a+b)/2]$ be the closed left half of I and $I^R = [(a+b)/2, b]$ be the closed right half of I.

Suppose S is a bounded and infinite set. The assumption that S is bounded implies the existence of an interval $I_1 = [-B, B]$ containing S. Since S is infinite, at least one of the two sets $I_1^L \cap S$ or $I_1^R \cap S$ is infinite. Let I_2 be either I_1^L or I_1^R such that $I_2 \cap S$ is infinite.

If I_n is such that $I_n \cap S$ is infinite, let I_{n+1} be either I_n^L or I_n^R, where $I_{n+1} \cap S$ is infinite.

In this way, a nested sequence of intervals, I_n for $n \in \mathbb{N}$, is defined such that $I_n \cap S$ is infinite for all $n \in \mathbb{N}$ and the length of I_n is $B/2^{n-2} \to 0$. According to the Nested Interval Theorem, there is an $x_0 \in \mathbb{R}$ such that $\bigcap_{n \in \mathbb{N}} I_n = \{x_0\}$.

To see that x_0 is a limit point of S, let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ so that $B/2^{n-2} < \varepsilon$. Then $x_0 \in I_n \subset (x_0 - \varepsilon, x_0 + \varepsilon)$. Since $I_n \cap S$ is infinite, we see $S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \neq \emptyset$. Therefore, x_0 is a limit point of S. □

Using pretty much the same idea, the following can be proved.

Corollary 4.6. *Every bounded sequence has a convergent subsequence.*

Proof. For the purposes of this proof, if $I = [a, b]$ is a closed interval, let $I^L = [a, (a+b)/2]$ be the closed left half of I and $I^R = [(a+b)/2, b]$ be the closed right half of I.

Let a_n be a bounded sequence and choose $B > 0$ such that $\{a_n : n \in \mathbb{N}\} \subset I_1 = [-B, B]$. At least one of the two sets $\{n : a_n \in I_1^L\}$ or $\{n : a_n \in I_1^L\}$ must be infinite. If $\{n : a_n \in I_1^L\}$ is infinite, let $I_2 = I_1^L$. Otherwise, $I_2 = I_1^R$.

Assume that I_m has been chosen for some $n \in \mathbb{N}$ such that $\{n : a_n \in I_m\}$ is infinite. At least one of the two sets $\{n : a_n \in I_m^L\}$ or $\{n : a_n \in I_m^L\}$ must be infinite. If $\{n : a_n \in I_m^L\}$ is infinite, let $I_{m+1} = I_m^L$. Otherwise, $I_{m+1} = I_m^R$.

In this way, a nested sequence of closed intervals, I_n, has been inductively defined, where the length of I_n is $B/2^{n-2} \to 0$. An application of the Nested Interval Theorem yields $\{x\} = \bigcap_{n \in \mathbb{N}} I_n$. It suffices to find a subsequence of a_n converging to x.

To do this, let $b_1 = a_{m_1}$, where m_1 is an arbitrary positive integer. Assuming $b_n = a_{m_n}$ has been chosen, pick $b_{n+1} = a_{m_{n+1}}$ from I_{n+1} so that $m_{n+1} > m_n$. It is possible to do this because $\{n : a_n \in I_{n+1}\}$ is infinite. In this way, a subsequence b_n of a_n has been inductive defined. Since $|b_n - x| \leq B/2^{n-2} \to 0$, it's clear $b_n \to x$. □

Corollary 4.7. *If* $\{F_n : n \in \mathbb{N}\}$ *is a nested collection of nonempty closed and bounded sets, then* $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$.

Proof. Form a sequence x_n by choosing $x_n \in F_n$ for each $n \in \mathbb{N}$. Since the F_n are nested, $\{x_n : n \in \mathbb{N}\} \subset F_1$, and the boundedness of F_1 implies x_n is a bounded sequence. An application of Corollary 4.6 yields a subsequence y_n of x_n such that $y_n \to y$. It suffices to prove $y \in F_n$ for all $n \in \mathbb{N}$.

To do this, fix $n_0 \in \mathbb{N}$. Because y_n is a subsequence of x_n and $x_{n_0} \in F_{n_0}$, it is easy to see $y_n \in F_{n_0}$ for all $n \geq n_0$. Using the fact that $y_n \to y$, we see $y \in F'_{n_0}$. Since F_{n_0} is closed, Theorem 4.4 shows $y \in F_{n_0}$. □

Extra Credit 6. An uncountable subset of $\mathbb{R}$ must have a limit point.